THE DEVELOPMENT OF MATHEMATICAL REASONING IN ELEMENTARY

SCHOOL STUDENTS' EXPLORATION OF FRACTION IDEAS

BY

DINA YANKELEWITZ

A dissertation submitted to the

Graduate School of Education

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements

for the degree

Doctor of Education

Graduate Program in Mathematics Education

Approved by

____

Carolyn A. Maher, Chair

____

Keith Weber, Committee

____

Elena P. Steencken, Committee

____

Mary Mueller, Committee

New Brunswick, New Jersey

May, 2009

ACKNOWLEDGEMENTS

First and foremost, I would like to thank my advisor, Carolyn Maher, for her concern, commitment to her students, and continued assistance. The sound advice and constant encouragement that she provided throughout my studies provided the necessary impetus for the completion of my dissertation. She is a doctoral mentor par excellence.

My committee members provided assistance, advice, and stimulating conversation throughout the dissertation writing process and helped me clarify and refine my research. Keith Weber shared his expertise in the areas of reasoning and proof, and his detailed reading and commenting added a new dimension to the final product. Elena Steencken provided a fresh perspective on the longitudinal study, young children's learning, and the learning of fractions. Mary Mueller was an invaluable guide who shared her wealth of experience and knowledge as I conceptualized and carried out the study.

The staff at the Robert B. Davis Institute for Learning assisted in locating the data necessary for this work. Marjory Palius, Robert Sigley, and Patricia Crossley outdid themselves in their willingness to help.

Several graduate students were most helpful in verifying and critiquing my work. Special thanks to Manjit K. Sran, Anna Brophy, Maria Steffero, Kate O'Hara, and Nicole Bernabe as well as the students in the Introduction to Mathematics Education course in the fall semesters of 2007 and 2008 for their assistance in this regard.

Finally, I would like to thank my family for their support and encouragement throughout my studies. A monumental task such as this would have been doomed to failure from the start without the support system that my husband, parents, siblings, and children provided as I dealt with the day-to-day challenges of achieving my goals.

ABSTRACT OF THE DISSERTATION

The Development of Mathematical Reasoning in Elementary School Students' Exploration of Fraction Ideas

By DINA YANKELEWITZ

Dissertation Director:

Carolyn A. Maher

Reformers and educators across the U.S. and internationally have called for an increased emphasis on the learning and teaching of reasoning, justification, and proof in K-12 and post-secondary mathematics education. Numerous studies have established that students experience difficulty with these processes. Recently, however, analysis of videotape data of sixth grade student work in an informal mathematics program has identified the use of multiple forms of reasoning by middle school students. This qualitative study, drawing on data from seventeen sessions from a longitudinal study conducted by Rutgers University in a fourth grade class of twenty-five students in a suburban/rural school in New Jersey, identified and traced the development of the forms of reasoning and argumentation used by elementary school students as they were introduced to fraction as number concepts and as they used Cuisenaire® rods and other manipulative materials to make sense of number relationships.

The following research questions guided this study:

1. What forms of reasoning and argumentation are elicited as students work on tasks involving the building of fraction ideas?

2. How does students' reasoning change as they revisit tasks introduced previously in the study and as they progress in their development of mathematical understandings?

Data for the study included forty-six videotapes, students' collected written work, and researcher field notes that were recorded during the seventeen 60-80 minute class sessions. The methodology of Francisco, Powell, and Maher (2003) was used for video data analysis. The video data was transcribed, verified, and coded for forms of reasoning, and a storyline and narrative was constructed to describe the results. Supplementary document analysis was used to verify and ensure validity of results. Analysis of the data showed that students used varied forms of reasoning and argumentation. Several tasks were flagged as supportive of the elicitation of varied forms of reasoning, and factors of those tasks and of the environment that encouraged the development of reasoning in the students were explored. The study has implications for effective strategies for the development of mathematical reasoning in the elementary school and ways that argumentation and proof can be introduced during the early school years.

Table of Contents

List of Figures

List of Tables

CHAPTER 1: INTRODUCTION

1.1 Statement of the Problem

Reasoning and justification is a central goal of mathematics. The NCTM has placed reasoning and proof as one of their five process standards from prekindergarten through grade 12 (National Council for Teachers of Mathematics [NCTM], 2000). They stated that "systematic reasoning is a defining feature of mathematics" (p. 57). Logical and effective thought have always been defining features of the doing of mathematics. "Mathematics is a discipline of clear and logical analysis that offers us tools to describe, abstract, and deal with the world (and later, world of ideas) in a coherent and intelligent fashion" (Schoenfeld, 1982, p.32).

At the International Congress of Mathematicians, mathematicians and mathematics educators have called for an increased emphasis on reasoning and proof at all levels of the curriculum (Ball, Hoyles, Jahnke & Movshovitz-Hadar, 2002). According to the panelists, many difficulties inhibit these processes from being emphasized. For example, proof is often viewed as a ritual devoid of meaning for many students, and such a view of proof prevents students from reasoning effectively as they try to justify mathematical ideas. In addition, since many mathematics curricula emphasize algorithms and procedures in the elementary school grades, students are ill-equipped to reason effectively when they are confronted with process-oriented tasks in secondary school and at the undergraduate level. The panelists called for the implementation of "a culture of argumentation in the mathematics classroom from the primary grades up all the way through college" (p. 907).

Davis and Maher (1996) have pointed out the need for a shift in mathematics instruction from rote memorization to the construction of powerful mental models and have emphasized that "the goal of mathematics is to learn to think in a very powerful way" (p 73). They explain that, if one would ask children direct questions about the purpose and meaning of proof, the discussion will very likely be of little or no benefit to the students. However, if students are challenged with a well-structured task and asked to convince others that they have found all possible solutions, they will engage in learning a great deal about mathematical proof and will benefit immensely from the activity.

However, as Davis and Maher (1996) point out, schools do not always implement what is most beneficial for their students. One reason this may be the case is that teachers are ill equipped with resources that can help them encourage the development of reasoning in their students. With more research on effective ways that reasoning and justification can be introduced into the mathematics curriculum, perhaps the large-scale change that has been demanded can be effected and more students will be able to learn to reason mathematically.

1.2 The Longitudinal Study

This study is grounded in an extensive body of research that has been conducted as part of the longitudinal study of the development of students' mathematical thinking and reasoning. The longitudinal study, conducted in two stages by Robert B. Davis and Carolyn A. Maher, began in the 1980's. It was situated in urban, suburban/rural, and working-class school districts and was partially funded by NSF grants MDR 9053597, directed by Robert B. Davis and Carolyn Maher, and REC9814846, directed by Carolyn Maher, as well as by grant 93-992022-8001 from the N.J. Department of Higher

Education. The primary setting of the longitudinal study was in the working-class community of Kenilworth, New Jersey, where a focus group of students was followed as they worked on open-ended strands of mathematical tasks over a period of twelve years. However, a smaller, three and one half year cross-sectional study was also conducted as a part of this larger study, and was situated in Colts Neck, a suburban/rural community in New Jersey, in the urban district of New Brunswick, and in the working-class district of Kenilworth. At all sites, nearly all sessions were videotaped, usually by multiple cameras, field notes were recorded, and student work was preserved. Analysis of many of these sessions have been conducted by numerous researchers over the years, and students' reasoning patterns in the areas of combinatorics, algebra, probability, and fraction concepts have been traced longitudinally.

Data for this study was drawn from the database of videotapes that was recorded in the fourth grade class in Colts Neck in the academic year 1993-1994. During the first half of the year, over a period of four months and twenty-five sessions, the primary focus of the sessions was that of constructing fraction ideas. Of these sessions, seventeen were focused primarily on building basic fraction concepts such as the concept of fraction as number, equivalence of fractions, comparison of fractions, and division of fractions.

In 1993, Robert B. Davis attempted to work with students in the fifth grade at the Harding School in Kenilworth to build strong representations of fraction concepts. However, these students, who had already been introduced to procedural methods of fraction operations, resisted his attempts. Davis then decided to work with the fifth grade in Colts Neck, New Jersey, and Carolyn A. Maher, who had planned on working with that fifth grade class, arranged with Mrs. Joan Phillips, the fourth grade teacher in Colts

Neck, to agree to allow Maher and her colleague, Amy Martino, to work with her students to build these ideas. In Colts Neck, although students prior to the fourth grade were introduced to strong ideas related to fraction as operator, fraction operations were only taught at the end of the fourth grade and during the fifth grade year. The students in this class, therefore, would be experiencing the learning of fraction operations and basic fraction as number concepts for the first time during this intervention.

The present study builds on the existent literature that focuses on this first series of sessions that took place in the fourth grade at the Colts Neck site. Steencken (2001) traced the growth of children's fraction ideas during the first seven sessions of the school year. She found that individual students' ideas about fractions traveled across the classroom community and students helped one another to build durable representations of the fraction concepts in the strand. She also identified the pivotal mathematical ideas that were the foundation of the lessons, including fraction as number, the creation of an assimilation paradigm for the concept of fractions, fraction equivalence, and fraction comparison. The students constructed these ideas through the use of a number of external representations, including building physical models, drawing pictures, and developing appropriate notation (Steencken & Maher 2002, 2003).

Reynolds (2005) analyzed six of the subsequent sessions that focused on comparison of fractions. She analyzed student conjectures and found that as students worked to justify the conjectures that they formed, they built a strong understanding of the mathematical concepts that were the basis of their investigations. In addition, she found that students used the ideas of others to make generalizations. Further, they built powerful representations of proportionality, inverse relationships, and the concept of the unit.

Bulgar (2002) analyzed four sessions toward the end of the unit on fractions that was conducted in Colts Neck. Her study focused on the sessions during which students constructed ideas related to division of fractions. She found that students used three primary strategies to solve division of fraction problems. These were the use of natural number concepts (as students converted the meter lengths to centimeter equivalents and then performed division using natural numbers), the use of measurement strategies (by using concrete materials to model the problem and measuring to find the solution), and the use of fraction concepts (in a manner somewhat similar to the algorithmic method of dividing fractions).

Another body of research that is relevant to this study is that conducted by Mueller (2007) and Mueller and Maher (2008). Mueller's research was set in another study funded by the National Science Foundation that was conducted at Rutgers University. This was the Informal Math Learning study (NSF grant REC0309062), directed by Carolyn A. Maher, Arthur B. Powell, and Keith Weber, which was situated in the economically depressed location of Plainfield, New Jersey. In this urban setting, where most students in the local public school were minority students, the researchers used tasks similar to those implemented in the beginning sessions at Colts Neck with the goal of creating a classroom community in which students shared and justified their ideas. Mueller analyzed the forms of reasoning that were elicited as the sixth grade students worked on these tasks during the first five sessions of the study. Mueller also investigated the nature of the student-to-student collaboration as they constructed arguments. Mueller's findings will be discussed in greater detail in the review of the literature.

The present study builds on the existent research in ways partly inspired by suggestions of the researchers whose work has preceded it. Firstly, Mueller suggests that it would be useful if future research would investigate if the forms of reasoning that were identified in the analysis of the work of the sixth grade students in the first cohort of the IML study also manifest themselves as younger students work on similar tasks. This study investigated that question. In addition, Bulgar (2002, 2003) suggested that more research should be undertaken to analyze the work at Colts Neck and determine "how this teaching experiment can provide information that will lead to a situation wherein all children will have an equitable opportunity to build powerful mathematical ideas and to think like mathematicians" (p. 300). This study, which analyzed the forms of reasoning and argumentation used by the students in Colts Neck as they investigated mathematical ideas, made an attempt to uncover that information and pinpoint the conditions that were most conducive to the use of various forms of mathematical reasoning. In addition, by analyzing a larger set of data that spans the children's long-term exposure to conditions promoting their growth in fraction understanding, this study also shed light on the tasks and conditions which promoted this development and how it may have been specific to the strand of activities in which it was situated.

1.3. Research Questions

In light of the research that has been conducted in the area of study, the following research questions guided this investigation:

1. What forms of reasoning and argumentation are elicited as students work on tasks involving the building of fraction ideas?

2. How does students' reasoning change as they revisit tasks introduced previously in the study and as they progress in their development of mathematical understandings?

CHAPTER 2: REVIEW OF THE LITERATURE

2.1 Introduction

Much of the literature on understanding, reasoning, proof, and representation that is relevant to this study is theoretical rather than empirical. Thus, this literature review will first discuss some of the important ideas that have been put forth in areas related to reasoning and proof that will form the theoretical framework that will guide this study, and will then discuss the research that has been carried out to investigate student understanding and reasoning in the context of school mathematics.

2.2 Theoretical Framework

2.2.1 Constructivist Perspectives of Mathematics Education

Yackel and Hanna (2003) discuss the phenomenon of the increased emphasis on incorporating reasoning at all levels of mathematics education, and explain its occurrence as a result of "a better understanding of how individuals come to know" (p. 227). They explain that mathematics educators now ascribe importance of encouraging students' explanation and justification in mathematics as a way of furthering their mathematical knowledge and understanding.

One of the more recent influences on mathematics education that has contributed to this change has been the use of constructivist models of learning. This theory is grounded in the belief that a person's knowledge is composed of building blocks that form mathematical ideas (Davis, 1984). These building blocks originate in a person's experiences, and the mental images derived from previous experiences can be used to build mathematical ideas (Maher, 1998). Since experience is inherently personal and unique, students come to individual ways of knowing mathematics, and should be provided with the opportunity to come to know mathematics in their own way (Noddings, 1990). When teachers invite children to express their ideas and treats their students' ideas with respect, rather than use direct instruction to teach procedural mathematics, students are provided the opportunity to construct rich and durable mathematical ideas (Maher 1998; Maher, Davis, and Alston, 1992). In addition, as teachers listen to their students' thinking, they can more appropriately adjust the mathematical tasks that they will introduce, so as to encourage theirs students' optimum development (Maher & Martino, 1992).

In the following sections, we will explore some of the hallmarks of these recent theories of learning, and relate these ideas to the emphasis on and the importance of reasoning, justification, and proof in this new view of mathematics education.

2.2.2 Knowledge and Understanding

Schroeder & Lester (1989) state that understanding is a "fundamental tenet of good mathematics teaching" (p. 37) that received little attention in many traditional mathematical curricula. However, the new trends discussed in the previous section led to a change in the way mathematics educators view mathematical knowledge, as mentioned by Yackel and Hanna (2003).

Prominent among the influences on this changing view of mathematical knowledge and the emphasis on understanding in mathematics is that of Skemp. Skemp (1976) differentiates between two forms of mathematical knowledge: relational and instrumental understanding. By relational understanding, he refers to a grasp of mathematical concepts as well as an understanding of why the mathematics underlying those concepts works. Instrumental understanding, on the other hand, refers to knowledge of rules and procedures. He opines that, in contrast to instrumental mathematics, relational mathematics is adaptable to new situations and is easier to remember than memorized procedures (because the underlying rationale for the mathematics is understood). He also proposes that relational knowledge is an intrinsic motivator, as students want to learn in order to understand. Finally, he explains that relational schema encourage students to seek out new knowledge and explore new topics in mathematics. So, according to Skemp, helping students understand mathematical concepts is a goal and means of helping them learn mathematics.

Skemp (1979) extends this idea to show the importance of justification in mathematics. He denotes a third category of understanding, that of logical understanding. Although relational understanding is sufficient to convince oneself, attaining logical understanding enables one to convince others. He explains that mathematical justification or proof enables others to replicate the mathematical ideas set forth by the originator. Skemp posits that even very young children are capable of intuitive logical understanding, but that the mathematical process of justification and proof involves reflective logical understanding, or the metacognitive ability to explain the intuitions held about logical understanding. The act of justifying in this manner demonstrates to the audience that "the final authority for accepting or not accepting a statement lies in the mathematics itself" (p. 49).

In the same vein, Hiebert and Lefevre (1986) differentiate between conceptual and procedural knowledge. They define conceptual knowledge as "knowledge that is rich in relationships" (p. 3). They emphasize that conceptual knowledge must be linked to many other pieces of information and must be learned with meaning in order to be of use.

Procedural knowledge, on the other hand, may or may not. If knowledge of a procedure is isolated from meaning, it becomes useless. However, by linking that procedure with a rich conceptual framework, the knowledge can be used in appropriate contexts.

Similarly, Brown, Collins, and Duguid (1989) contrast authentic and school mathematical activity. The former is that which professionals who work in the field of mathematics do, and the latter is that which is taught in school. Drawing a comparison to learning vocabulary from dictionaries and the potential for incorrect understanding to be built by students, they posit that teaching mathematics in a manner that is distant from authentic mathematical activity does similar harm.

Davis was a key figure on the frontier of mathematics reform and was a leading proponent of the shift toward mathematical understanding. His collected works form the foundation of the theoretical framework for the present study. Davis and Maher (1997) explain that new knowledge is constructed from old knowledge, and that by carefully designing students' experiences, new ideas can be integrated accurately into the students' schema. Davis (1984) discusses the concept of "frames," which are his term for knowledge representation structures. An assimilation paradigm (a concept adapting Piaget's [1972] terminology) allows for students to create frames that will allow them to link new knowledge to mental schemes that they have previously built, and to modify those schemes to allow for the integration of the new idea. One example of an assimilation paradigm that was used in the longitudinal study was situated in the Pebbles in the Bag activity, which helped students understand positive and negative integer computation [for a complete description of this activity, see Davis and Maher (1997)]. One can argue that the candy bar metaphor that was introduced to the students in the present study is also an assimilation paradigm. Davis and Maher note that more general experiences are also important for students as they study mathematics, and that less structured open-ended tasks allow students to create new mathematical knowledge and expand their existing schemas to allow for more elaborate frames in their own way.

As Schroeder and Lester (1989) note, the works of Skemp, Hiebert and his colleagues, Brown et. al, and Davis share a commonality of thought regarding mathematical understanding: that understanding is linked to the ability to recognize and construct relationships between mathematical ideas, contexts, and problems. They explain that when the goal of mathematics is understanding, mathematics becomes "a way of thinking about and organizing one's experiences" (p. 39).

2.2.3 Reasoning

2.2.3.1 A Brief Overview of Reasoning in the Cognitive Sciences

If understanding is so necessary in the learning of mathematics, it may be helpful to explore the ways of thinking that encourage and exhibit this understanding. Reasoning, broadly defined, is the process of coordinating evidence, beliefs, and ideas to draw conclusions about what is accurate or true (Leighton, 2003). From a slightly different angle, Rips (1994) describes reasoning as a "mental process that creates new ideas from old ones" (p. 10). So, in essence, good reasoning ability is prerequisite to understanding.

Psychologists have attempted to explain how the mind represents the rules of inference and deduction. One school of thought (Braine, 1978; Rips, 1983) proposes a model of rule-based reasoning that is closely related to formal logic systems.

Braine (1978) argues that people use a system of natural logic as they reason that is significantly different from formal systems of modern logic. This system is governed by two concepts. First, "inference rule schemata" are the preferred mode over axiom schemata, the use of which is a hallmark of formal logic. An inference rule states what conclusion can be drawn from a given set of propositions. Inference rule schemata is a definition of a an inference rule that shows its form. The second is the use of connectives, such as "and" and "or," whose meaning is closely related to the standard English meaning of the words.

Similarly, Rips (1983), in his discussion of a program that he designed to test his theory about the use of natural, rule-based deduction in the process of deductive reasoning, states: “[D]eductive reasoning consists in the application of mental inference rules to the premises and conclusion of an argument. The sequence of applied rules forms a mental proof or derivation of the conclusion from the premises, where these implicit proofs are analogous to the explicit proofs of elementary logic” (p. 40).

Another school of thought (Johnson-Laird, 1983; Johnson-Laird & Byrne, 1991, 2002) proposes that the mind uses another system to reason. Pointing to the fact that many errors in deduction are made by the human mind, they assert that people construct mental models to deal with logical deduction, rather than use a rule-based system. They explain that a mental model of an assertion is a representation of a single possibility when the truth of an assertion is given, and that each metal model corresponds to the row of the truth table that can be constructed for the given assertion. However, mental models are only constructed for true conclusions based on the assertion, not false ones. Information about what is false is captured in "mental footnotes," which are easier to forget and less likely to be used as conclusions are drawn. By constructing one or more mental models, the mind draws conclusions based on the given assertions.

2.2.3.2 Mathematical Reasoning

Ball & Bass (2003) discuss the importance of reasoning in school mathematics. They posit that mathematical understanding is impossible without emphasizing reasoning. They explain that without reasoning, understanding mathematics would only be procedural or instrumental. For example, Benny, a sixth grader who was taught using Individually Prescribed Instruction (IPI), believed that doing mathematics was like a "wild goose chase," and consisted of trying to determine what the teacher's answer key contained (Erlwanger, 1973). As a result, he concocted fallacious procedures that he used to find solutions to the problems he was presented with. Ball and Bass conclude that knowledge that lacks justification can easily become unreasonable, as in Benny's case.

According to Ball & Bass (2003), reasoning in mathematics serves a number of important functions. Firstly, without conceptual understanding of mathematics, the knowledge is difficult to use or to be applied to new and varied situations. Additionally, when mathematics is learned as a reasonable discipline, rather than as a set of procedures, the knowledge that has been attained can easily be reconstructed even when the memory of the accompanying procedure has faded.

Ball & Bass (2003) make an important distinction between reasoning and sense-making. They describe sense-making as a process carried out by the individual for the individual, but reasoning as a set of norms that is shared by the community and are discipline-specific. Although they do not specify the community or norms that they refer to, they imply that reasoning is performed when ideas are examined to reach a shared conclusion about the principles under discussion.

They also distinguish between "reasoning of inquiry" and "reasoning of justification". The former refers to the reasoning that allows for discovery and exploration of new mathematical ideas, and the latter to the reasoning that functions to justify and prove mathematical claims.

This dichotomy between reasoning that involves discovery and exploration and that which is associated with justification and proof is discussed by Polya (1954). He differentiates between what he calls “demonstrative reasoning”, or the reasoning that is used to produce formal proofs, and “plausible reasoning”, which is used for mathematical discovery. Polya discusses the second, often neglected category of reasoning patterns that are elicited as mathematicians and students of mathematics engage in exploring mathematical ideas. Firstly, students use inductive or empirical reasoning as they experiment with new mathematical ideas and test their hypotheses. In addition, students make generalizations as they try to extend the results of their experiments to a class of objects. Students also engage in specialization, during which they focus on a specific case within the larger class of objects under scrutiny.

Lastly, Polya (1954) explains students use analogical reasoning to extend ideas from one area or structural relationship in mathematics to another partially related or unrelated idea. "Analogy is often vague. The answer to the question, what is analogous to what, is often ambiguous. The vagueness of analogy need not diminish its interest and usefulness; those cases, however, in which the concept of analogy attains the clarity of logical or mathematical concepts deserve special consideration" (p. 28) Polya discusses the value of clarified analogies in mathematics, and says that "two systems are analogous if they agree in clearly definable relations of their respective parts" (p. 13).

Polya (1954) then discusses three types of clarified analogy that is useful in mathematical exploration. The first is that of isomorphism. The second is what he calls “similarity of relations”, in which the structural relationship between two quantities or ideas is perceived as similar.

De Villiers (2003) also discusses the role of inductive and analogical reasoning in mathematics. He identifies some functions of these forms of reasoning when doing mathematics, including making a conjecture, verifying a statement or conjecture, providing global or local counterexamples, and furthering understanding. As noted earlier, he explains that mathematicians usually use these inductive, intuitive, or analogical modes of reasoning before investigating the matter through a deductive lens, and that these other forms of reasoning and verification are prerequisite to the undertaking of a more formal deductive proof.

Although inductive and analogical reasoning are useful when thinking about mathematical problems, these forms of reasoning have their limitations. De Villiers (2003) points out that one must always keep in mind that they are insufficient means of guaranteeing mathematical truth. In addition, he notes that these forms of reasoning do not explain, systematize, or justify mathematical ideas as deductive reasoning does. In addition, deductive reasoning often opens up new horizons of possible relationships between seemingly unrelated concepts and areas of mathematics.

Reid (2002) notes that the exploratory forms of reasoning described above, and the process of formulating, testing, and accepting or rejecting conjectures are typical of scientific argumentation, and that they are necessary for the development of mathematical reasoning and argumentation. However, he states that mathematical reasoning is characterized by certain behaviors that are indicative of a "mathematical emotional orientation" (p. 24), which include noticing patterns and regularities, recognizing that statements must be supported by evidence, and expecting all mathematical assertions to have reasons that explain why they are true. He further explains that this last expectation gives rise to the need for deductive reasoning, and is a crucial factor that distinguishes mathematical reasoning from scientific reasoning.

As Skemp (1979) states, Ball & Bass extend, and Reid contextualizes, logical understanding and the reasoning of justification involve convincing others of the truth of and the rationale behind the mathematical ideas that one builds. This ability to convince others through argumentation and justification forms the foundation of mathematical reasoning.

2.2.4 Argumentation, Justification, and Proof

2.2.4.1 The Nature and Role of Proof in Mathematics

Proof is a mathematical form of argumentation that plays an important role in advanced mathematics, and is, according to Balacheff (1991), "a goal of most mathematical curricula" (p. 175). The nature of proof has undergone significant rethinking in the mathematical community. Originally, mathematicians and teachers of mathematics emphasized rigor and form as important features of proof. Hanna (1983) takes a strong stance against the stress on rigorous proof as the core of mathematics and secondary school mathematics education, positing that mathematicians do not rely solely on rigor to establish the certainty of mathematical claims, and that they value understanding and significance more than rigor. She implores mathematics educators to promote proof as a tool for understanding when exposing young students to mathematics, as the discipline values it.

More recently, Hanna has decried the neglect of proof in reform curricula (Hanna 1995; Hanna & Jahnke, 1996). She attributes this decline in the teaching of proof in schools to the influence of constructivist views of education and the social values that have changed the role of the teacher as an authoritative transmitter of ideas to the moderator of student discovery and construction of knowledge. She states further that educators view proof as a formality instituted by mathematicians to establish their authority and the infallible nature of mathematics. She argues that this view of proof is false, and that proof, by its very nature, removes all external authority by its very transparency. As Manin (1977) has stated, “Every proof that is written must be approved and accepted by other mathematicians. In the meantime, both the result and the proof itself are liable to be refined and improved.” (p. 49). She states that it would be foolish to think that proof is authoritative because of its deductive nature, because that would be equivalent to "challenging the idea of rules of reasoning" (Hanna, 1995, p. 46), and that these rules are the basis of human rational thought. She opines that the removal of proof from school mathematics deprives students of an important facet of the discipline, one which promotes understanding and explanation of mathematical ideas.

De Villiers (1990) discusses the function of proof in mathematical undertakings. He puts forth that the emphasis on proof as a means of verification of an idea is too narrow to give full justice to the importance and use of proof. He suggests that conviction is usually a necessary prerequisite for proof, and that without first reaching that conviction through quasi-empirical or experimental methods, mathematicians wouldn't expend the effort in trying to deductively prove the truth of that conviction. This discussion is reminiscent of Bell (1976) and Lunzer (1979), who also explain that students more often come to conviction through analogical or empirical means, rather than deductive reasoning. Bell defines three purposes of proof: verification, illumination, and systematization. Bell also posits that students will not use formal proof methods until they "are aware of the public status of knowledge and the value of public verification" (p. 25). He proposes that cooperative investigations in mathematics classrooms create an environment most conducive to the development of this awareness.

This over-emphasis on the role of proof in verification of results is also discussed by Hanna (1989), who delineates a similar dichotomy in the types of proof processes that are undertaken. She differentiates between "proofs that prove" and "proofs that explain". Unlike proofs that merely establish the validity of a mathematical statement, a proof that explains shows why the statement is true, and reveals the underlying mathematical statements that justify it. She emphasizes that proofs that explain are wholly acceptable forms of proof, and that their use in mathematics education correctly reflects mathematical practice, since mathematicians are more interested in the mathematics underlying the proof than the correctness of the proof in its own right. She states that the goal of teaching mathematics is to enable students to understand mathematics and that ascribing importance to this method of proof will further that goal.

Dreyfus and Hadas (1996) further this idea by explaining that students will feel motivated to prove a mathematical idea if they want to explain why they arrived at empirical results to a mathematical problem. This motivation to explain an idea that originated from a mathematical discovery can help them understand the intellectual need for proof.

In addition to Hanna's delineation of the importance of proofs that explain, de Villiers (1990) proposes additional functions of proof, including systematization, discovery, and communication. The nature of deductive reasoning that enables it to be a medium that explains, systematizes, and communicates mathematical ideas is what is lacking in empirical and analogical modes of thought.

2.2.4.2 The Transition to Formal Proof

As will be discussed in the review of the research that has been conducted on proof, students have been found to experience great difficulty with constructing, understanding, and validating proofs.

Epp (2003) discusses the difficulties that students experience with proof. She highlights some causes of the difficulties that have been documented by extensive research. First, she points out that everyday logic is different in significant ways than mathematical logic. Second, she explains that many students have been influenced by earlier mathematics instruction, during which teachers used examples to show the truth of a statement, rather than a mathematical proof. These two confounding factors make it difficult to transition students to mathematically correct modes of thought. Although Epp provides some suggestions to assist mathematics educators in this transition, she admits, "Trying to change thinking habits, especially ones that have become ingrained over a period of years, is a very difficult task" (p. 893).

As a result of the difficulties that have been documented, mathematics educators have begun to investigate ways to ease students' transition to proof. This transitioning may then enable students to be introduced to proof in a way that will not counter the habits of mind that they have become accustomed to in earlier years of formal schooling.

2.2.4.2.1 Transitional Proof Forms. Alibert & Thomas (1991) suggest two forms of proof that can be used to introduce students to proof without resorting to more abstract, linear proofs (these forms of proof are also discussed in Movshovitz-Hadar 1988). One of them, the structural proof, involves the breakdown of the proof into levels, with each level targeting one idea of the proof. These main ideas, or pivots, are then used to draw the conclusion. In this way, the rationale for the steps of the proof are made explicit, and students can more readily see that choices were made as the author formed the proof, rather than being presented a short, difficult to understand, and more authoritative proof.

The second alternate form of proof that is suggested by Alibert and Thomas is one that is more relevant to the use of justification by younger students. That is the form of generic proof. "Such a proof works at the example level but is generic in that the examples chosen are typical of the whole class of examples and hence the proof is generalizable. This may be contrasted with the more general nature of formal proof which does not make use of examples but consequently requires a higher level of abstraction. While there may be no replacement for the formal proof from the purely logical point of view, the generic proof may sometimes be preferable if it results in improved understanding on the part of the students" (pp. 216-217).

Harel and Tall (1991) argue that generic reasoning contains elements of generalization and abstraction. They explain that there are three kinds of generalizations. The first, expansive generalization, involves the expansion of an existent scheme without reconstructing it. The second, reconstructive generalization, involves the adaptation or reconstruction of a scheme in order to widen its applicability. The third form of generalization, termed disjunctive generalization, involves the creation of a new scheme that can include the new context. The researchers explain that the process of formal abstraction, one that is very important to the advanced mathematician, is a reconstructive generalization. Reconstructive generalization requires significant cognitive effort to master, and they suggest that generic abstraction is a way to ease the way for students to attain an ability to use formal abstraction. They argue that the use of generic examples involves both generalization, because the examples become embedded in a larger class of elements that are typified by the generic example, and abstraction, as the more general concept is abstracted from the example or examples that are used. They explain further that the use of generic abstraction allows for the easing of the cognitive strain that is normally imposed on students as they attempt to use formal abstraction in mathematics.

Balacheff (1988) differentiates between pragmatic proofs, or proofs that involve actions or showing, and conceptual proof, which do not involve action but "rest on formulations of the properties in question and relations between them" (p. 217). Balacheff's pragmatic proofs are similar in nature to Semadeni's "action proofs". This form of proof is a result of an internalization of a series of actions that convince one that a general statement is true. A student uses a generic case to think about the properties of a mathematical object in a concrete way, and internalizes the actions that are performed until it is a convincing mental model that can be used to justify statements (Semadeni, 1984). Moving from pragmatic to conceptual proofs involves thinking about the generic quality of the situations that have been originally considered directly.

2.2.4.2.2 Stages of Proof Development. Balacheff (1988) also describes the steps that young students undergo as they progress in their cognitive understanding of proof. His first stages is that of naïve empiricism, which is seen when students become convinced of a truth after verifying a number of cases. The next is the crucial experiment, in which the student decides to perform an experiment that will reject one of the two hypotheses that have been formed. In this case, however, the accepted hypothesis has not been verified; rather, the student chooses that hypothesis because the experimental data rejected the alternative.

The first two stages mentioned do not, in Balacheff's words, "establish the truth of an assertion" (p. 218), but the third and fourth do. The third stage is defined by use of a generic example to show the veracity of a mathematical statement. Balacheff notes that a generic example "involves making explicit the reasoning for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of its class" (p. 219).

The fourth stage is that of the thought experiment. This term is borrowed from Lakatos (1976) and signifies a substantiation of conjectures through the proposal of supporting conjectures. This is a more abstract verification than the generic example, as the thought experiment proves the assertion without using a specific example as part of the argument.

2.2.4.2.3 Informal/Preformal Proof as a Transitional Method. Balacheff (1988) makes a distinction between what he calls "proof" and "mathematical proof". The latter is that used in the larger mathematical community, while the former is the more informal lines of reasoning used by students in a classroom community. Although this distinction is important, researchers have suggested that the encouragement of informal proof forms may be crucial when attempting to enable students to transition to the modes of thinking required to construct and understand formal proofs.

Blum & Kirsch (1991) advocate an emphasis on preformal proof methods in school mathematics. They define preformal proofs as a series of correct but informally presented arguments that lead to a valid but informal conclusion. They argue that there are three levels of attempts at proof, namely, non-proofs, preformal proofs, and formal proofs, and that the only unacceptable proof modes are those which fall into the first category, which are purely experimental or based on heuristic arguments. They stress that preformal proofs are valid and are based on sound but more intuitive reasoning, and that, if formalized, must be correctly expressed as a formal mathematical argument.

Epp (1998) and others (Leron, 1985; Thompson, 1996; Antonini, 2003) discuss the difficulties that students experience with proof by contradiction and other forms of indirect proof. Epp explains that it is difficult for students to take the incorrect side of an argument in an effort to show that the opposite side is correct. Similarly, Leron discusses the mental strain that is imposed as the student attempts to treat a false world as if it were real. However, Antonini (2003, 2004) points out that indirect argumentation arises spontaneously in students when they are introduced to problems that lend themselves to that form of solution.

Similarly, Thompson (1996), in a discussion about the difficulties that students find with indirect proof in advanced mathematics, suggests that one way to assist students in their understanding of and ability to formulate proofs is to give younger students opportunities to work with indirect proof in informal ways. Citing some problems that can be used in this endeavor, she notes that, when introducing certain kinds of problems, students naturally use indirect reasoning to solve them. Thompson opines that these problems can be introduced as early as middle school, and posits that if the use of these informal proofs are encouraged, teachers will be better able to teach formal indirect proof by developing connections between the informal and formal proof structure.

Harel (2002), in his study of students' use of mathematical induction, found that during traditional instruction, students were only empirically convinced about the truth of their solutions. In a teaching experiment, he exposed students to mathematical induction first through implicit, rather than explicit recursion problems, and found that they used a transitional form of induction to solve them. He suggests that by introducing students to mathematical induction in this way, students will naturally use quasi-induction to solve them. He opines that mathematical induction is an abstraction of this transitional heuristic, and that this transition will better enable students to recognize the validity of proof by mathematical induction.

2.2.4.3 Argumentation and Justification

Balacheff (1991) differentiates between argumentation and mathematical proof in social contexts. He defines the goal of argumentation as "to obtain the agreement of the partner in the interaction" (p. 188), but explains that it allows for any form of reasoning that will succeed in convincing others. He explains that mathematical proof differs from argumentation in that it has to "fit the requirement for the use of some knowledge taken from a common body of knowledge on which people (mathematicians) agree" (p. 189). Students who are given the task of solving a problem with the goal of getting it done will use argumentation to convince others that they are correct, but those who are asked to establish the veracity of a mathematical statement will tend to try to use a form of mathematical proof.

Toulmin (1969) has constructed a model of argumentation composed of four basic components. The first is the claim, or the conclusion of the argument. The second is the data, or the evidence provided that is used to explicitly substantiate the claim. Third, warrants are used to establish an implicit causality between the data and the claim, and to show that drawing the conclusion that one has from the data that was provided is appropriate. Finally, a backing is often provided to provide legitimacy to the warrant. Toulmin notes that argumentation used informally, which he terms “substantial arguments”, differs significantly from analytical arguments. Substantial arguments are those in which the conclusion logically follows from the presentation of the other three components, but an analytical argument contains the additional component that the backing contains the essence of the conclusion in an explicit or implicit form. Toulmin notes that mathematical arguments are analytical, while those used in everyday argumentation is usually substantial.

Toulmin’s model has been used by researchers in mathematics education as they have attempted to trace the nature of students’ mathematical argumentation. Krummheuer (1995), for example, has noted that young students’ informal argumentation in classrooms is often substantial in nature rather than analytical, and this may be a key distinction between students’ informal argumentation and that used in formal proof. Other researchers have analyzed students arguments and challenges, noting the instances in which students challenge one another’s warrants as well as the data that are used to form the basis of their mathematical claims (Maher, Powell, Weber, & Lee, (2006); Weber, Maher, Powell, & Lee, in press).

Francisco and Maher (2005) discuss the importance of emphasizing justifying in school mathematics, rather than formal proving. They explain that the rigor and form expected of proof is difficult for young children to do well, and that by encouraging students to justify their solutions in a convincing manner, they will engage in proof-like activities without having to struggle with writing formal proofs. In addition, when students are encouraged to convince others of the truth of their claims as they do mathematics, proof-making becomes an integral part of the mathematical process and problem solving activity, rather than a peripheral one.

2.2.4.4 Proof in the Elementary School

Stylianides (2007) defines proof as it applies to elementary school mathematics. Stylianides proposes that proof in school mathematics is a sequence of assertions that fulfills three conditions: Firstly, it must use statements that are accepted by the classroom community as true, which may be definitions, axioms, or theorems. Secondly, it employs forms or modes of argumentation that are accepted by the community as reasonable or within their conceptual reach, and lastly, that it is communicated using representations or forms of expression that are understood by the community. Examples of these modes of argumentation are the application of rules of inference, the use of definitions to draw a conclusion, the proposal of arguments by contradiction and cases, and the use of counterexamples to counter a claim. He emphasizes that what may constitute a proof in a high school class may not be valid in an elementary school classroom, if it uses terms or forms of reasoning that are outside of the students' experience or ability. He justifies this definition by discussing the necessity of matching students' exposure to a discipline with experiences that are true to the discipline. Students who are exposed to mathematics in elementary school should get a sense of what mathematicians do. By allowing them to use forms of argumentation to the best of their cognitive ability, but to retain the accepted modes of argumentation and methods of proof that are used by mathematicians, students gain an authentic mathematical experience. In addition, a consistent notion of proof can be supported throughout the students' years of schooling since their exposure to mathematical argumentation remains static in terms of the "rules of play". At the same time, their understanding of argumentation develops as the modes of argumentation, the representations as well as the accepted mathematical claims become increasingly sophisticated. Stylianides argues further that this definition allows for analysis of classroom argumentation, since any argument that is offered in the discussion can be compared to what can be expected as a valid proof given the situation. The validity of the argument as a proof in context can then be determined.

Stylianides' definition of proof in the elementary school must be examined in light of the theory that has been developed about the social contexts that lend themselves to argumentation in mathematics. The classroom environment and the expectations set within that classroom are all-important in developing a robust ability to argue and justify mathematical ideas, as will be explored in the sections that follows.

2.2.5 Problem Solving Environments

Another approach to help students learn mathematics with understanding is expressed in the use of problem solving as a medium for mathematics learning (Schoenfeld, 1992, Schroeder & Lester, 1989). According to the *Principles and*

Standards for School Mathematics (NCTM, 2000), “Problem solving means engaging in a task for which the solution method is not known in advance. In order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understandings” (p. 51).

Schroeder and Lester (1989) discuss the importance of using problem solving as a means to develop understanding. They explain that rich problems give students the opportunity to develop their understanding of the relationships inherent in mathematical ideas.

Davis and Maher (1990) suggest that the traditional curriculum is far removed from actual mathematical practice. Rather than memorizing algorithms, which children are trained to do in the bulk of their mathematical studies, mathematicians analyze problems and create effective algorithms based on their understanding of those problems. By teaching via problem solving, then, students can engage in more creative, discipline-true tasks and learn to understand and appreciate mathematics.

Francisco and Maher (2005) point out that an important aspect of problem solving environments is the ability of students to “think together” (p. 369). Although the majority of research on collaborative learning has emphasized the value of the group’s cumulative knowledge, Francisco and Maher point out that group work is especially valuable when students work together to make sense of a problem and then challenge, question, and convince one another of the correctness of the claims presented.

Francisco and Maher also point out that a successful problem solving activity should be carried out as a part of a strand of similar tasks that encourage students to build durable understandings about a predetermined mathematical topic. The problems in the strand should be of comparable difficulty and should be structured in a similar way, so that students can use heuristics and knowledge that they have constructed previously in the strand to tackle new problems. They also stress the importance of introducing more complex tasks at first, so that students are challenged to find ways of tackling and organizing the problem, and then to introduce the more basic tasks, which will then be relatively simple for students to solve.

When using problem solving as a medium for teaching mathematics with understanding, the environment must be monitored to allow for such learning to take place. Nunokawa (2005) delineates four types of support that a teacher can introduce to ensure that students will be successful problem solvers. The teacher must consider the intended goals of learning that the problem solving sessions should obtain, and must match the experience to those goals. For example, if a teacher wants students to gain a deeper understanding of the problem situation, the teacher should make the problem motivating, should allow for empirical means of discovery, and should provide tools that allow students to more easily gain a strong grasp of the problem. However, if a teacher would like to teach "via" problem solving, as Schroeder and Lester encourage, students should be provided with a problem that is a bridge between old and new knowledge, and the teacher should introduce appropriate values into the classroom environment. This can be done by carefully constructing the problem so that it gives rise to a need for certain mathematical tools, such as generalizations, and establishing norms in the classroom that communicate to students the value of constructing knowledge and making sense of mathematics. This latter support for the goal of the problem-solving session will be discussed further in the next session.

2.2.6 The Social Context for Argumentation

There has been much discussion in the field of mathematics education regarding the social conditions that enable the effective learning of mathematics and promote reasoning and justification. Importantly, Yackel & Cobb (1996) have advanced the notion of sociomathematical norms, or, “normative aspects of mathematics discussions specific to students’ mathematical activity” (p. 459). They distinguish this term from social norms, by explaining that an example of a social norm is that students must explain their solutions, whereas a sociomathematical norm is the shared understanding of what is acceptable as a mathematical explanation. These norms are “taken-as-shared,” in that all students understand what is expected of them when they contribute to a mathematical discussion.

Cobb, Yackel, and Wood (1995) stress that these norms must not be taught by listing them or establishing them at the start of the teaching cycle. Rather, the teacher should capitalize on opportune moments after a student has either followed or transgressed a norm that they are trying to establish and use those moments to show students the value and importance of those norms.

Maher and Martino (1999) discuss the role of the teacher in encouraging students to justify their solutions. They posit that students will not naturally justify their solutions to each other in a small group setting; rather, the teacher should interject questions that prompt students to explain how they arrived at a solution and why it is true. These questions can take the form of “Can you explain your solution to me?” or “I don’t understand, can you show me why this works?” (p. 56). In addition, if a child offers an incomplete explanation, the teacher should encourage them to expand on their thinking and explain further. Teacher questioning can also encourage students to generalize their solutions. In this way, the teacher plays an active role in creating the mathematical community in the classroom.

When a teacher establishes sociomathematical norms in the classroom by motivating students to justify their solutions in a mathematically acceptable manner, as Maher and Martino have noted, this notion becomes "taken-as-shared" by the students (Yackel & Cobb, 1996). In the review of the research, we will examine how students can be acculturated into an environment where they ultimately learn to justify their solutions with minimal teacher intervention, due to the taken-as-shared understanding of what is acceptable as a mathematical solution.

Mueller (2007) has proposed that students co-construct arguments when they work in small groups and are encouraged to evaluate one another's reasoning. She emphasized that with time, students increase their comments about the reasoning of others, and challenge, correct, build on, and refine arguments that are offered by their fellow students. She concludes that by creating an environment in which students are provided an opportunity to justify their solutions and explain their thinking, students learn to work together to compose sound and strong arguments about mathematics.

These ideals of classroom environments in which children attempt to convince one another of the validity of their assertions can be linked to the world of proof as experienced by professional mathematicians. As Alibert and Thomas (1991) state, "[A] proof is a means of convincing others whilst trying to convince oneself" (p. 215). Maher and Davis (1995) explain,

> Our theoretical and practical work in schools is based on the recognition that individual learning takes place within a community whose members have access to the thinking of others. Our research views the relationship between the individual child and the other members of the community as fluid. A child's way of knowing may influence or be influenced by the representations produced by other members of the community. (p. 88)

Maher and Davis continue to explain this idea by using a metaphor to portray the relationship between the individual child's thinking and the influence of other members of the community's thinking on the individual child. They liken the individual ideas to ribbons, and the interconnections between these ideas as the bows that comprise a complete idea as expressed by the classroom community. This metaphor highlights the constant interplay between individual and collective thought, and how the two influence each other to allow children to reason in sophisticated and accurate ways.

The present study will be conducted with this knowledge. In particular, the study aims to explore how "[d]ifferent forms of reasoning co-exist within this community and different forms of representation are used to transmit ideas, explore and extend these ideas, and then act upon them" (p. 88).

Maher and Davis' mention of representation deserves a closer look. In the next section, we will discuss two meanings of representation, one of which can also be called mathematical tools.

2.2.7 Representation and Mathematical Tools

Davis (1984) discusses the role of representation in mathematical thinking. He explains that representations are mental models that allow human beings to associate the properties of a mathematical idea to the idea. He stresses that ideas are not stored in the mind in words or pictures, and that when we explore what these representations are, we are only approximating their true nature. This is because we can only see the expressions of these representations as via one or more of a number of tools that are available to students as they attempt to express their ideas.

Davis elaborates that doing mathematics involves a series of steps, through which the problem solver must cycle one or more times. First, a representation for the input data is built, as the student attempts to "make sense" of the nature of the problem. Then, one searches one's memory for knowledge that will assist in solving the problem, and maps the data representation with the knowledge representation that has been found. When the mapping seems accurate enough to be workable, the individual uses techniques associated with the knowledge representation to solve the problem.

Davis' reference to representation is similar to one of two meanings that have been ascribed to representation in mathematics education. Another use of the term "representation" has been termed by Goldin (2003) "external representation", which is also referred to by mathematics educators as mathematical tools. For example, Maher and Davis (1995) state "Among the various tools used by children to express their ideas are spoken and written language, physical models, drawings and diagrams, and mathematical notations" (p. 88). Goldin notes that the use of external representations allows for the standardization of mathematical understanding.

According to Maher and Martino (1996), students who are encouraged to build multiple external representations as they work on problems become sense-makers and active members of the mathematical community. The use of different tools to build and express ideas allow students to make connections between different internal representations and understandings and to better understand the mathematics that they are learning. In addition, the building and expression of these multiple external representations allow the observers (such as the teacher, the researcher, and fellow classmates) to better understand the students' ideas.

2.3 Review of the Research

2.3.1 Knowledge and Understanding

Davis (1984), Hiebert & Lefevre (1986), and Skemp (1976) have all differentiated between mathematics that is learned with conceptual understanding versus that which is learned with procedural understanding. A case study of one student who exemplified this dichotomy in a striking and telling way is that of Ling Chen, described in Alston and Maher (1989) and Davis and Maher (1990). Ling Chen, a high-achieving student who participated in an enrichment program for students in the summer preceding her entry into the sixth grade, was interviewed as she explored a problem involving finding one half of one third. She first used pattern blocks to build a model of her solution. Then, the interviewer asked her if she could "do this one with numbers". Ling then divided one third by one half, arriving at a solution of two thirds. Faced with a concrete representation of the problem that differed from her algorithmic solution, she said that she still thought that the solution was one sixth, and then proceeded to use the division algorithm to divide one half by one third, this time arriving at a solution of three halves. Ling then decided that the "invert and multiply" rule doesn't work in this case, and wrote that one third divided by one half is equivalent to one third multiplied by one half, and arrives at her "correct" solution of one sixth. The researchers concluded that it is often difficult for students to proceed directly from a problem that is stated to a correct algorithmic approach. They argued that in the doing of mathematics, children must first build a representation of the problem so that they can then retrieve a useful algorithm that they associate with that representation.

2.3.2 Reasoning

2.3.2.1 Forms of Reasoning

The research most relevant to the present study is that conducted by Mueller (2007), Mueller and Maher (2008). These studies analyzed the forms of reasoning that were exhibited by two focus groups of students in an informal after-school program for sixth grade students in an urban setting. Five sessions, all involving investigations with Cuisenaire rods, were conducted, and videotapes of the sessions were transcribed and coded for forms of reasoning. The forms of reasoning that were flagged were direct reasoning, reasoning by cases, reasoning using upper and lower bounds, and reasoning by contradiction. The analysis showed that all students constructed arguments, and all four forms of reasoning were flagged. In addition, the researchers found that students worked together and co-constructed arguments, by challenging, questioning, and building upon each other's ideas. The students reasoned effectively and used convincing arguments to make sense of the mathematical tasks that were posed (Powell, Maher, & Alston, 2004).

2.3.2.2 Conditional Reasoning

Hadar (1977, 1978) and Hadar and Henkin (1978) attempted to teach fifth grade students conditional reasoning skills. They devised a number of games that enable students to explore the validity of conditional statements in a problem-solving environment. The researchers found that the experimental group of students performed significantly better than the control group on a posttest of conditional reasoning. In particular, more than 80% of the experimental group improved performance of questions that were “undecidable”, or did not provide enough evidence for the students to establish the truth of the conditional statement. In contrast, 75% of the students in the control group performed consistently on these test items on both the pre- and posttest. The researchers concluded that carefully designed instruction can help students improve their conditional reasoning skills, and can help them recognize when there is not enough information to draw a logical conclusion.

Easterday and Galloway (1995) investigated the sentential logic skills of seventh, eighth, and twelfth graders, as well as those of preservice secondary mathematics teachers in 1976, 1986, and 1992. Although they found that the performance of the school-age students improved over the two and a half decades of the study, those of the preservice teachers decreased. They noted that "college students barely perform better than the children whom they may one day teach" (p. 435). They included five logical forms in their instrument: (I) Given the statement “If p then q”, and the establishment of p, students were asked if the truth of q was logical conclusion (which it is. (II) Given the statement “If p then q”, and told that q was true, they were asked whether they could conclude that p was true (which is not the case) (III) Given the same statement “If p then q” and told that p was false, they were asked what they could conclude about p (that it is false) (IV) Given the disjunction “p or q” and the told that p is true, they were asked what they can conclude about q (which is inconclusive); and (V) Given the disjunction “p or q” and told that q is false, they were asked to determine the truth value of p (which is true in this case). They found that all students scored better on Types I and V, but performed very poorly on Types II and IV. In addition, the seventh and eighth graders outperformed the twelfth grade and undergraduate students on the Type III problems. The authors concluded from this last result that "there is a negative relationship between mathematics training and performance on this form" (p. 434). This conclusion is difficult to accept, as it seems to ignore lurking variables. It is possible that it is a lack of exposure to this form of argumentation that caused older students to lose their ability to interpret it correctly, rather than the actual mathematics instruction. The authors did make a somewhat more plausible statement, however, stating that the results indicate a potential weakness in mathematics education. The longitudinal study conducted at Rutgers University, in contrast, showed that students' ability to reason improved over time, and it is likely that this is due to the students' exposure to environments and problems that encouraged them to reason effectively and convince others of the veracity of their reasoning (Francisco & Maher, 2005; Maher, 2002; Maher & Martino, 1996)

2.3.3 Argumentation, Justification, and Proof

Extensive research has been conducted investigating students' conception of proof and their ability to construct and understand proof. However, most of these studies have focused on secondary or post-secondary school students. Of the remainder, many of the studies on proof involving students younger than high-school age have been situated in the middles school. Few researchers have documented elementary school students' ability to construct proof-like arguments. In this section, research involving older students' understanding of proof will first be briefly outlined, and the research that has been conducted in the middle and elementary grades will then be discussed in greater detail, whose results will have more significant implications on the present study.

2.3.3.1 Research at the Secondary/Post-Secondary Level

Finlow-Bates (1994) investigated the conceptions of proof held by students in a first year undergraduate course designed to introduce fundamental topics in mathematics. The goal of the study was to investigate the value that undergraduates attribute to examples and to proofs. Six students were interviewed on a volunteer basis at the start of the course. The first interview was a pilot, and the problem was modified before interviewing the other students. The students were presented with a problem and four solutions. One was comprised only of examples, one was an informal proof, the third was an informal proof followed by examples, and the fourth was the same as the third with the order reversed. The students were asked to rank the solutions from best to worst.

Finlow-Bates found that all the students ranked the informal proof followed by examples as best and the examples followed by the informal proof as second. Four students ranked the informal proof third and the examples worst, while the fifth student reversed the order of the last two rankings. However, many students at first ranked the informal proof as best and then devalued it because it did not offer examples. Students explained that the proof followed by examples was best because it contained an explanation followed by verification of the explanation.

The researcher concluded that students considered the role of proof to be that of explanation, while examples were the part that convinced them of the truth of the statement. In addition, the study found that students had difficulty describing what the informal proof was, calling it "comments", "summary", and "notes". Finlow-Bates suggests that ways must be found to teach students the value of proof as that which is convincing, rather than just that which explains.

This study is important to note, as it shows the importance of emphasizing both roles of proof - that of convincing and that of explaining. As Hersh (1993) points out, proof has two simultaneous meanings, and neither should be ignored. Students should learn that proof should be used to convince as well as to explain.

The majority of research on proof conceptions at the undergraduate level is quite disturbing (Weber, 2003). Many studies (e.g. Coe & Ruthven, 1994; Martin & Harel, 1989) have documented the tendency of undergraduates to consider empirical evidence sufficient to be convinced of the truth of a statement, while others (e.g. Weber, 2001) have suggested that the inability of undergraduate students to properly construct proofs may stem from their lack of strategic knowledge, or the skill of choosing strategies and heuristics that are appropriate to use for the particular proof they are constructing. Selden and Selden (2003) have found that undergraduates are unable to validate proofs and have a limited conception of what constitutes a proof, and Harel and Sowder (1998) have identified proof schemes that undergraduates have that are at odds with traditional views of acceptable and unacceptable proofs.

Much of the research on proof at the high school level has documented similar tendencies as those of undergraduate students. For example, Healy and Hoyles (2000) surveyed high-attaining high school students and found that the majority of students used empirical means of justification when presented problems in algebra. Similar results are documented by Galbraith (1981) and Senk (1985).

Results from Maher's longitudinal study, however, show evidence of sophisticated use of proof strategies by high school students (Muter & Maher, 1998; Muter, 1999; Maher & Kiczek, 2000; Maher, 2002; Kiczek, 2000; Maher, Muter, &

Kiczek, 2006, Uptegrove & Maher, 2004). The disconnect between these two bodies of research will be discussed below.

2.3.3.2 Research in the Middle School

Balacheff (1988) investigated the proof forms that would be exhibited by middle school students. Twenty eight students thirteen and fourteen years of age worked on inventing and justifying a formula for the number of diagonals in a polygon. They worked in pairs, but each pair used only one pen to record their work. They were asked to write a note to a fellow student to explain how to find the number of diagonals of a polygon when the number of vertices is known. The students were told that they had as much time as they needed, and that, after they provided a response, the observer would suggest potential difficulties that a reader might encounter (in effect, providing a counter-example to the students' definition). The students would then work on revising their explanation to account for the counterargument.

Balacheff then analyzed the responses from the first part of the session, which occurred without intervention from the observer. He then described the types of proof observed during the session (according to his classification that was described in the theoretical framework). Naive empiricism was used most often by students and was classified as such when conjectures were formed based on a few cases. Often, students did not make claims as to the validity of the conjecture, but showed confidence that they were correct when they presented their ideas. Students used the crucial experiment to check their results, as well as to counter others' claims. The generic example was used to establish a true proposition about the problem ($fx(n) = (n-3) + (n-3) + (n-2) + ... + 2 + 1$), and, in all cases that the generic example was used, the students used crucial experiments to show that the proposition held. The thought experiment was used primarily to explain the proposition that had been developed, usually be means of generic example.

Balacheff concluded that the first two proof types are largely empirical in nature, and that a conceptual leap is involved when passing from the second type (the crucial experiment) to the third (the generic example). This third type is a transitional stage between pragmatic and conceptual proofs, as realized in the thought experiment. This last type involves an abstraction of the ideas used in the generic example that evidences a recognition of the relations between the mathematical objects in the problem and an application of these relations to a generalized argument.

Porteous (1990) investigated the extent to which children believe deductive proofs and empirical results. He first presented a questionnaire to 300 students aged eleven to sixteen years of age in England, containing a story about two children who thought about the fact that the sum of three consecutive integers is divisible by three. One child thought that this was true, first presenting empirical evidence, and later on in the questionnaire presenting deductive evidence. The second child argued that there was no way to know if this was true, and that there may be a set of three very large consecutive integers that would not follow this pattern. The students were asked to indicate which argument was correct, and, after reading about the empirical evidence, to explain why they thought the argument was correct. He found that 10% of students produced their own logical argument to prove the statement, and that older students did this more often than younger ones. He also found that 20% of boys and 26% of girls who did not produce their own argument were more convinced by the deductive argument than the empirical argument.

Porteous then interviewed fifty students in the same age group, first presenting a particular case and then a generalization of the problem first posed. Children of all ages in the study who deductively proved the generalization were able to transfer their understanding of the theorem to a second particular case. Children who merely tested the generalization for a number of cases, however, tended to test the particular case that was presented afterwards, showing that they did not fully believe the generalization. In addition, even after being presented a proof by the interviewer, the students tended to check the second particular case that was presented, rather than rely on the implications of the deductive proof that had been provided. Porteous concluded that, based on the results, deductive arguments are necessary to help children believe generalizations in mathematics, and that deductive reasoning should be encouraged in elementary school mathematics classes. More importantly, his findings show the importance of independent verification of mathematics by students, rather than mere presentation of proof by the teacher.

King (1973) conducted an investigation to determine if it was possible to provide instruction that enabled young students to understand proof. He coordinated a teaching experiment with ten sixth-grade students and trained a teacher to teach eleven lessons focusing on the teaching of six theorems. The teaching experiment was conducted using Bloom's mastery learning technique, in which students had as much time as they needed to learn a unit and were graded on a mastery or non-mastery basis. Both the experimental and control group were tested before and after the teaching experiment. Both groups scored 30% (or thereabout) on prerequisite skills and 0% on proof items on the pretest. The experimental group scored 96% on prerequisite skills and 97% on proof items on the posttest, while the results from the control groups showed no significant improvement from pretest results. King concluded that it is possible to teach proof concepts to sixth grade students.

2.3.3.3 Research in the Elementary School

Lampert (1990) analyzed a classroom discussion that took place with her fifth grade students. In an attempt to bring classroom mathematics learning closer to the professional mathematical activity that is the hallmark of the discipline, Lampert tried to create a classroom culture in which students would investigate properties of the mathematical objects that they were studying. The discussion began by students investigating the pattern inherent in the last digits of the squares of natural numbers (such as 1^2, 2^2, 3^2, 4^2, 5^2, and so on). After the students thought about the way the numbers formed a pattern (1, 4, 9, 6, 5, 6, 9, 4, 1, etc.), the teacher asked the students a second problem: What is the last digit in 5^4? 6^4? 7^4? Although initially, the students answered the first problem directly, a student then offered a generalization of the solution, arguing that any power of five would have a last digit of five. According to Stylianides (2007), this argument took the form of proof by mathematical induction, albeit a very informal one. The students then thought about the general pattern of the last digits of 7 raised to increasing powers. By the end of the discussion, fourteen out of the eighteen students had contributed mathematical ideas or arguments. Lampert concluded that the classroom discussion modeled the doing of mathematics that Lakatos (1976) and Polya (1954) advocate, and closely followed the patterns of discourse of the discipline.

Stylianides (2007) used his definition of proof as it pertains to elementary school classroom practice to judge the validity of arguments posed during three classroom episodes in a third grade classroom. These episodes were drawn from videotaped sessions of a yearlong teaching experiment conducted by Deborah Ball. In the first episode, the students investigated a problem involving finding the sum of any two of three denominations of coins: pennies, nickels, and dimes. They were encouraged to justify their arguments. When students used empirical claims to justify their solutions, the teacher encouraged them to think of a more convincing way to justify their claims. Stylianides concluded that the arguments offered did not meet the standards for proof.

During the second episode, the students tried to form number sentences that equaled ten. One students offered a solution of adding seven ones and three to equal ten. When asked to justify the solution, one student explained that it could be accounted for using the basic counting strategy, and proceeded to demonstrate. Another student offered a second solution, saying that the number sentence of one hundred divided by ten equals ten. When asked to justify the solution, the student responded that another classmate's mother had said that it was true. She also said that five times ten is fifty, and that fifty divided by five is ten, and that fifty plus fifty is one hundred, and five plus five is ten, so it followed that one hundred divided by ten is ten. However, the class could not understand this argument, since they hadn't learned about multiplication and division. The teacher therefore refrained from including this sentence from the list "until we have some way of showing that we know that it's right" (p. 308). Stylianides concluded from this second episode that the first solution and justification that was offered, rather than the second that was suggested, qualified as a proof because it was founded on statements that were accepted and understood by the classroom community. The second solution, however, used definitions and ideas that had not yet been accepted by the class as valid, due to their lack of exposure to these concepts. He opined that the second argument, then, could not qualify as a proof given the context.

During the third episode, a student proposed that there are infinitely many number sentences that equal ten. She used a generic example, saying that if one subtracts two hundred from two hundred and then adds ten, the result will be ten. She then said that numbers "go on and on" and that any number could be substituted for two hundred. She then concluded that there are infinitely many sentences that fit the requirement. The teacher then introduced variable notation to the students, explaining that the student had provided an excellent explanation, and that mathematicians had a method to make such an explanation more concise. The researcher then noted that both forms of representing the argument were valid for the context in which it was presented, since the classroom was able to understand the concepts on which it was founded.

Stylianides concluded that although the arguments that were presented were not as rigorous as the ideal mathematical proofs with which they were compared, many of them qualified as proof by the definition that he had formulated for use in primary grades.

Yackel & Cobb (1994) analyzed the nature of second grade children's arguments, and found that children used arguments to serve a number of distinct functions. Students used arguments to specify their understanding of the problem and explain their solution, to convince others of an error, to announce a mathematical discovery or generalization. One example of students' use of argumentation to articulate their interpretation of a problem situation occurred when the students worked on a problem in which the teacher flashed an array of sixteen dots arranged in rows of four on the board by turning on and off the overhead projector. Students explained their responses and indicated in their explanation that they had correctly interpreted the problem. Anita said, *I counted four and four right here and four right here and four right here (pointing to the rows) ... I saw four 4's and all that added up was 16 cause two fours and another two fours was 8 plus 8. And 8 plus 8 is 16.”* (p. 9). In another instance, a student made an addition error, and a second student used an argument to show the student that the calculation was incorrect. When asked by other students to explain what he said, he explained that he was trying to show that the first response was incorrect and why that was so. The researchers noted that Anita had attempted to explain to her classmates her way of thinking about the problem. Third, students used argumentation to discuss a mathematical discovery or generalization. Although many students who solved the addition problem of 8 + 9 by reasoning that eight plus eight equals sixteen and one more is seventeen, some students generalized this solution by stating *"When one of the numbers is one more and the other number stays the same the answer [sum] is always one more."* (p. 17). They found that students were successful at communicating these and other mathematical generalizations and used various forms of arguments, such as argument by contradiction, to challenge other’s erroneous arguments. They concluded that the NCTM’s call for an emphasis on mathematical reasoning at all ages is intended to encourage children’s sense-making of mathematics, which will motivate them to seek justification and to convince others of their ideas, and lay a foundation for future understanding of formal proof. They also contended that the concept of mathematical argumentation is a sociomathematical norm which is interactively constituted as children participate in classroom discussions.

Zack (1997) investigated the argumentation in her own fifth grade classroom in a private middle class school catering to students of heterogeneous ability and background as they worked on open-ended tasks. Her data included videotapes of small group work, students' math logs and other written work, and retrospective interviews. The class worked on finding the number of squares contained in a grid of various sizes (1 x 1, 2 x 2, and so on). They were asked to find the number of squares for the case of 60 x 60, and students worked to find a generalization of the solution in order to arrive at the answer for the larger case. Ten of the twenty-six students in the class found the "solution" by multiplying their result for the 10 x 10 case by six. However, three students searched for and found a more consistent pattern. Will first realized that the number of squares of certain sizes were inversely related to the number of squares contained within the square (for example, when considering the 4 x 4 case, there are 9 squares that are the size of four squares (2 x 2), and 4 squares that are the size of 9 squares (3 x 3). He then realized that the difference between the size of the squares constantly increased by two, and tried to use that information to solve the more complex problem. However, this was time-consuming, and his two partners realized that the solution could be found by summing the squares. These three students then refuted another pair of students' erroneous claim, that the solution was six times that of the 10 x 10 case. They used their knowledge of the generalization that they had formed and used arguments by contradiction to show that the incorrect solution could not be true. In addition, they explained that the solution would have to hold for other cases, and that the number of squares for the 4 x 4 case was not half that of the 8 x 8 case. With this argument, they showed an understanding of the importance of generalization in mathematical explanations. Later in the year, the teacher introduced to a smaller group of students the Anderson method of finding the solution for any size square (using the formula (n(n+1)(2n+1)). Although the students thought that this method was efficient and helpful, the majority of students felt that without evidence that it worked, and since the solution did not make sense to them, it was not as useful as the more tedious method that they understood. Some students emphasized that “if Johnston [Anderson] himself explained why it worked it would be more convincing” and that for a mathematical explanation to make sense to others, the originator should “show why” (p. 297). Zack concluded that, due to the establishment of norms of mathematical explanation, students had developed an emergent understanding of what proof should consist of and what criteria should be expected in a proof.

Reid (2002) analyzed the videotape data described above in Zack’s study. He described the reasoning patterns that he noted as they worked on the task, as well as the conjectures and the refutations of those conjectures that took place. Reid classified the students' reasoning as "stereotypical scientific reasoning" (p. 12), in which they observed patterns and then made and tested conjectures. They then either rejected the conjecture and returned to pattern observation or confirmed the conjecture and made a generalization based on that confirmation, which enabled them to continue exploring the problem. Reid noted that the reasoning that was observed among these groups of students was only partially mathematical. Although they noted patterns and expected that all statements made be supported, they did not have "the expectation that there be a reason why in mathematics, leading to explanations with a deductive basis" (p. 26). He further suggested that although the patterns of reasoning were noteworthy in that they provide a foundation for the development of mathematical reasoning, they were lacking this core factor that is needed to distinguish mathematical reasoning from scientific reasoning.

The Rutgers longitudinal study documented the development of proof-like arguments by young children in the third, fourth, and fifth grades. We will briefly summarize the research that has been conducted whose data is derived from videotaped sessions of students working on various open-ended tasks.

In the fourth grade, the students at the Kenilworth site were asked to find all possible combinations of towers 5-tall when selecting from cubes of two colors, red and blue. A case study of one student, Stephanie, has been followed as she explored this and related problems and formulated sound arguments to validate her findings. When Stephanie was first introduced to the problem, she and her partner Dana began by using a trial and error strategy to find combinations. She finally arrived at thirty-two different towers, arranged as pairs of opposites together with the pair of "upside-down" opposites. Stephanie concluded that "there's no way that you'd ever know if you had them all" (Martino, 1992, p. 233). The researcher then raised the idea that had been originated by another pair of students. This was the method of using a "staircase pattern" to organize the towers that were built. Stephanie began to consider this strategy to find all possible towers five cubes high that had two red cubes, and then thought about the situation of towers with only one red cube. She reasoned indirectly to show that she had found all the possible towers in each of those two categories. At the close of this activity, Stephanie began to think of an exhaustive method of finding the combinations of towers that were five cubes high (Martino, 1992).

Stephanie's work on problems involving towers continued throughout the fourth grade and beyond. Her growth in mathematical understanding and her development of mathematical proof is documented by Maher and Martino (1996a, 1996b, 2000), Maher

(1998), Martino and Maher (1997, 1999), and Maher and Speiser (1997). A brief summary of the episodes described in the literature cited above will be presented here.

In an interview, Stephanie shared her findings of classifying towers that were 6-tall by cases with classmates. She introduced a method of holding the color in one and then two positions of the tower constant while she changed the colors of other positions. According to Maher and Martino (1996b), this indicated an ability to coordinate more than one variable at a time.

At an interview three weeks later, Stephanie presented a proof by cases for building 4-tall towers when selecting from two colors. Then, in a group evaluation period, dubbed "The Gang of Four," Stephanie presented a proof by cases to the other students to account for all possible 3-tall towers, selecting from two colors. Stephanie's organization was interesting in that she separated the case of two blue cubes into towers that had two blue cubes in adjacent positions and those that did not. When her classmates pointed out that these two cases could be grouped into one broader group, Stephanie insisted on continuing her explanation as she had originally presented it. However, in a written assessment the next year, in fifth grade, Stephanie organized her cases as her classmates had suggested, producing a more "elegant" proof by cases (Maher & Martino, 1996a).

Milin's reasoning as he worked on the towers problem in the fourth grade was also documented by Martino (1992). They also used the strategy of building each tower along with its opposite. They finally arrived at a total of thirty towers, and concluded that since they had spent ten minutes trying to find a new combination and had not succeeded, there couldn't be any more. However, during a later interview, when asked to reflect on the strategies used by other students, he discussed the staircase method, and showed, using an argument by contradiction, that there were only five different combinations of towers five-tall that contained only one red cube.

Milin's progress as he worked on tasks involving towers has also been analyzed (Maher & Martino, 1996a, Alston & Maher 1993). During the class session in which the students participated in the task involving towers that were five cubes tall, Milin contributed to the class discussion about creating cases of towers by building staircases. He suggested building towers that had red cubes separated by one, two, or three yellow cubes, respectively, and noted that there were three possible combinations of towers in which the red cubes were separated by one yellow cube, two in which the red cubes were separated by two yellow cubes, and three in which the red cubes were separated by three yellow cubes.

In the first interview following this session, Milin used the cases he had described to find a subset of towers that were five-tall, and found the remainder by finding towers and their opposites. During this interview, he began to consider simpler cases, and said that there were four towers that could be built that were two cubes tall, and two that could be built that were one cube tall.

During subsequent interviews, Milin gradually built an argument by induction to show that the "doubling" pattern works indefinitely, and explained why the pattern worked. Finally, during an interview session, Milin presented his inductive argument to his classmates. When questioned by the researcher if his pattern held for towers taller than five-cubes high, he expressed his belief that it did. "We followed the pattern till five - Why can't it follow the pattern to six?" (Alston & Maher, 1993).

Milin's argument by induction was the stepping stone to the students' understanding of the general case, the possible combinations of towers n-tall. The progression of this understanding is documented by Maher and Martino (1997, 2000) and Maher (1998). Stephanie, during the interview preceding "The Gang of Four," noticed that the number of combinations possible for each successive tower height was double that of towers one shorter. For example, she noticed that there were eight possible towers that were three cubes tall, which was double the four possible towers that were two cubes tall. During that session, she calculated the number of towers that could be made that were ten cubes tall. However, although she repeated this claim during The Gang of Four, she didn't use it to justify her solution. Rather, she resorted to cases to show that she had accounted for all possibilities (Maher & Martino, 2000).

Once during the fourth grade and then again during the fifth grade, Stephanie participated in an activity in which she wrote a letter to a student who wasn't present that was to convince him that all possible towers that were three cubes tall had been found when selecting from two colors. During the fourth grade, Stephanie used a proof by cases to show that she found all the towers, and, when asked to generalized her strategy to show the number of towers that were possible for any height tower, she mentioned her strategy of doubling, and wrote that this procedure could be followed to find the number of towers that could be made of a specific height. Milin then suggested that the doubling pattern was a result of the fact that they were selecting from two colors, and that if there were three colors, one would multiply each result by three to get the total number of combinations for towers that were one cube taller. In the written assessment in fifth grade, Stephanie again used a proof by cases, but checked her solution by using the "doubling" method.

Later in the fifth grade, Stephanie and her partner, Matt, worked on the task entitled "Guess My Tower," which required them to reconstruct the combinations of towers that were four cubes high when selecting from two colors. Stephanie remembered her doubling rule, and declared that there were sixteen combinations. However, Matt insisted that there were only ten towers, and, upon encouragement by the researcher, Stephanie began to work with simpler cases (such as one-tall and two-tall towers) so that she could show that her pattern worked. Upon conferring with another group, Stephanie, Matt, Melissa, and Bobby arrived at a joint group of sixteen towers. The group then tried to find out what happened as they progressed from towers one cube high to towers two cubes high. When the researcher suggested that Milin proposed a rationale for the progression, Matt explained that at each successive level, two cubes could be added to each original tower since they were selecting from two colors. Stephanie then realized that Matt's inductive explanation justified her doubling pattern, and she enthusiastically explained the rationale to the rest of the group. At this point, Stephanie took ownership of the inductive justification as well as the proof by cases, highlighting a significant leap in her conceptual understanding and methods of justification (Maher & Martino 1997, 2000).

At another site of the Rutgers longitudinal study conducted by Maher and colleagues, this time at Colts Neck, students were introduced to the towers problems in the third and grade and findings from the sessions were documented by Martino and Maher (1999) and Maher (in press). There, a pair of students, Jackie and Meredith, worked on finding all towers that were four cubes tall. They first found sixteen towers by building towers and their opposites. Then, upon questioning by the researcher, they began to organize their towers by cases, but didn't organize them in a staircase by controlling one of the variables as the Kenilworth students had done. When considering towers with two cubes of each color, Meredith organized it into a tower with two red separated by no yellow cubes and its opposite, a tower with two red cubes separated by one yellow cube and its opposite, and a tower with two red cubes separated by two yellow cubes and its opposite. She then showed by contradiction that there was no other way to form a tower with two red cubes separated by two yellow cubes without making the tower larger then four cubes high. They were then asked to predict how many towers they would be able to make that were three cubes high, and they thought that there would be sixteen of those as well. However, upon experimentation, they found that there were only eight, and, when questioned about why this is so by the researcher, Meredith showed that each tower that is three cubes tall can have a yellow or a red cube added to it to make a two towers that are four cubes tall. Thus, Meredith used recursive reasoning, similar to that of Milin, Matt, and Stephanie in the Kenilworth focus group. The researchers concluded that these students used reasoning by cases, reasoning by contradiction, and recursive reasoning to build proof-like arguments about the problem.

Also at the Colts Neck site, in the fourth grade class, David used powerful proof forms to build arguments about fraction ideas (Maher & Davis 1995). His work will be further analyzed in the present study. Maher and Davis document his argument that established that when the blue rod is called one, there is no rod that can be named one half in the set of Cuisenaire® rods that was provided. He first used an argument using upper and lower bounds, and then an argument by cases, to show that the rod under question did not exist. Maher and Davis conclude that by formulating conjectures, discussing ideas with others, and inventing ways to convince both themselves and others of their ideas, students in the longitudinal study were successful at constructing proof-like forms.

Throughout the longitudinal study, students formulated and presented arguments that closely resemble the structure of proofs. The obvious disparity between the two areas of research, that of advanced students' inability to reason effectively about mathematics, and of young children constructing impressive justifications in mathematics, can be explained in light of the research on understanding and learning environments discussed in the theoretical framework. Yackel and Hanna (2003) call attention to the fact that the more discouraging studies of more advanced students must be understood in light of the fact that many of these students had never before been exposed to a relational view of mathematics, and had always been taught from the perspective of procedural understanding. In addition, they point out that the studies that showed evidence of sophisticated reasoning in children "demonstrate clearly that creating a classroom atmosphere that fosters this view of mathematics is a highly complex undertaking that requires explicit effort on the part of the teacher" (p. 234).

The longitudinal study conducted by Maher and her colleagues is important in that it provides evidence of sound reasoning produced by students in the elementary grades as well in high school and beyond. Their later success can be attributed, in part, to the use of "prototypes" for building ideas in mathematics that was introduced in the early years in study (Francisco & Maher, 2005, p. 366). In this way, students learned basic ideas in early years that they built upon and connected with more advanced ideas in later activities. These prototypes were “the building blocks for more complex forms of reasoning” (p. 267) that occurred in their high school years.

Dreyfus (1999) also notes the differences between results from research on high school, and undergraduate students' conceptions of proof and those from research conducted by Maher and Martino (1996), Lampert (1990), and Zack (1997). He opines that the latter studies were conducted in carefully planned environments that were intended to promote the learning of reasoning and proof. This statement, however, does not hold for the longitudinal studies at Rutgers University. He contends that "the studies only show that the transition to deductive reasoning is possible, not that it normally happens. He also posits that many students are never told what is considered a mathematical argument, and that college students must also be exposed to sociomathematical norms that encourage students to learn how to explain and justify properly in mathematics.

2.3.4 The Social Context for Argumentation

At the university level, Alibert and Thomas (1991) noted the disadvantage of presenting proof in a way that prevents students from experiencing the intellectual necessity of proof. They discussed an attempt made to change the environment in which teaching proof took place in a teaching experiment in a freshman course at a large university (Also documented in Alibert, Grenier, Legrand, & Richard, 1986). A classroom culture was formed to help students engage in scientific debate about conjectures in mathematics. First, the teacher led a discussion in which students would offer statements and they would be listed on the board without evaluation of their veracity. Then, the students discussed each statement in turn, and presented arguments or proofs to back the statement, or refuted the statement by providing a counterexample or another counterargument. Then, those that were validated would be labeled "theorems", while those that were refuted would be noted as false with the counterexamples recorded. From questionnaires that were completed at the end of the course, the researchers learned that these lessons helped students understand that mathematical errors are productive and that there was a need for proof to establish the truth of mathematical ideas.

Alibert and Thomas pointed out that certain techniques that were the hallmark of the series of lectures described. Specifically, they discussed that the teacher must stress to students that he/she is not the authority in mathematics; rather, the truth of mathematical statement is proven only when one shows that no counterexamples can be provided. In this way, students will learn that they can convince one another and that the teacher cannot make decisions for the class in this respect. The authors summarize the research by noting that: "The context in which students meet proofs in mathematics may greatly influence their perception of the value of proof. By establishing an environment in which students may see and experience first-hand what is necessary for them to convince others, of the truth or falsehood of propositions, proof becomes an instrument of personal value which they will be happier to use in the future." (p. 230).

Balacheff (1991) tried to create a social context in which students would be motivated to prove or argue about a solution in eighth and tenth grade classrooms. He found that the need to prove may arise in the context of social interaction, but may not necessarily occur. A key factor influencing the proving efforts of the students was the students' beliefs about the purpose of the activity. He concluded that if the students think the goal of the mathematical activity is “doing” the problem and arriving at a solution, they will focus on efficiency and reliability and will use argumentative behaviors. But if the students think the goal of the activity is knowing (and establishing the validity of that knowledge), they will focus on certainty and rigor and will use forms of mathematical proof.

Cobb, Yackel, Wood, & McNeal (1992) contrasted two classroom cultures in which a teacher worked with her students to build place value understanding. In the first, a third grade classroom, children quickly learned that the teacher would like them to learn specific procedures, and followed those procedures without understanding their rationale. Although the teacher did not intend for this to happen, the taken-as-shared norms in the classroom were such that students learned to follow the teacher's prompts and act accordingly. In the second classroom that was described, second grade students worked to share, explain, justify their solutions and challenge each others' solutions without prompting by the teacher. This, the authors argue, was a result of the taken-as-shared classroom norms that implied that students were expected to justify their own solutions and challenge others' arguments if their own were inconsistent with those presented. The teacher in the latter case acted as a representative of the mathematical community whose role was to moderate the discussion and situate the students' discussion within the context of generally accepted mathematical truths.

Yackel & Cobb (1996) discussed how establishing appropriate sociomathematical norms can help students understand what is a valid mathematical argument. They cite a case where a student changed her solution based on the teacher’s failure to validate it. The teacher then intervened, saying that just as the student would not be convinced if someone said her name was different than it really was, the students should be just as convinced of the truth of her mathematical claims, and should not rely on a teacher's validation of it to be sure that it is correct. The researchers note that, through this discussion, students were provided with a paradigm that they could refer to in the future as they learned about the expectations of the mathematical arguments in their classrooms.

Wood (1999) analyzed a series of fifty videotaped lessons in a second grade classroom. Logs of the videotapes and field notes were used to analyze the argumentation that the students used in the classroom discourse. First, Wood analyzed the patterns of interaction that were noted in the sessions that occurred toward the end of the first school year and found that a consistent pattern of interaction emerged. In general, when children argued about a mathematical idea, their discussion followed this pattern. A child would provide an explanation for his or her solution. Then, a child who disagreed with this solution would challenge the explanation. The first student would then offer a justification for the solution. If the challenger still did not agree, he or she would offer a rationale for the difference of opinion. The first student would counter the argument. This process would be continued, with other students offering explanations to back or challenge either of the solutions, until all the members of the class, including the teacher, were satisfied that the disagreement was resolved.

Wood then analyzed the set of sessions that occurred at the start of the second school year in the study. The purpose of this research was to determine how the teacher established an appropriate context for argumentation in her class that enabled her students to argue in the manner that she had found. Wood identified a number of expectations that the teacher established at the start of the school year. Firstly, she established a context for disagreement by conveying to her students appropriate ways to disagree with another's mathematical idea. She then enabled her students to experience situations that helped them learn how to participate in disagreements and how to participate in argumentation in general.

According to Mueller and Maher (2008), given a supportive environment, students have been found to work together to build strong arguments and use many form of reasoning. When students are encouraged to think about the arguments of others and participate in the discussion of mathematical ideas, they are successful at forming proof-like arguments and are active members of their mathematical communities. They posit that, in an environment where "the reasonableness of arguments was the measure for a student's success," sense-making became an expectation of the students.

These findings are similar to those of Maher (2005), who found that students who were exposed to an environment that encouraged them to convince each other of the truth of their arguments came to expect such standards from each other. As Powell (2003), a researcher who documented the work of the twelfth grade students in the longitudinal study recorded, students used language that had been modeled by the researchers throughout the twelve years of the study to see if their ideas made sense. For example, Michael, one of the students who participated in the study since its inception, asked the researchers if their solutions "convinced" them. This language reflected the students' way of thinking about truth and validity in mathematics.

2.3.5 Representation and Mathematical Tools

The case study of Ling Chen that is described earlier is a powerful example of the importance of creating careful representations of problem situations when doing mathematics (Alston & Maher, 1989; Davis & Maher, 1990). Davis and Maher (1990) document another case study which brings home a similar point. Brian, a sixth grade student, was videotaped as he worked with his partner, Scott, to solve a problem involving fractions. Scott used pencil and paper to solve the problem, arriving at an incorrect solution. Brian, on the other hand, first built a concrete representation of the problem using pattern blocks, and then modeled the actions described in the problem, arriving at a correct solution. The researchers concluded, as they had from the case of Ling Chen, that the building of accurate representations of a problem situation is essential to effectively doing mathematics.

Several studies have documented the power of representation to allow students to discover isomorphism and methods of proof (Maher & Martino, 1998; Kiczek, Maher, & Speiser, 2001). For example, Brandon, a fourth grade student at the Colts Neck site of the longitudinal study, discovered the isomorphism between two problems, one of which had been introduced four months before the second. The first problem was the towers problem, discussed earlier, and the second was the pizza problem, in which students are challenged to find all possible pizza choices when selecting from four toppings. Brandon used zeroes and ones to represent the presence or absence of a topping, and created a chart that showed the sixteen different combinations of pizza toppings. Then, when questioned whether the problem reminded him of any other problem he had worked on, he said that it had just occurred to him that the towers problem was similar, and proceeded to build towers and map them to the notation for the pizza combinations that he had written. Brandon’s representation that had been invented for the pizza problem allowed him to build an isomorphism to another problem that he had thought about previously (Maher & Martino, 1998; Greer & Harel, 1998).

Steencken and Maher (2002) discussed the teaching experiment that will be the focus of this study. They described the steps that the students took to build a resilient understanding of fraction concepts. In particular, they noted the various representations that were used by the students along their journey, such as concrete models, diagrams, and eventually more abstract notation. They also commented that even as students worked on complex problems later in the study, they often referred to the rods that they had worked with initially, even though they did not use the concrete materials at the time, to explain their ideas. This is evidence that the internal representations that were built by the students were strongly linked to the external representations that they had built during their early explorations with fraction ideas.

Powell, Maher and Alston (2004) noted in their analysis of sixth grade students using Cuisenaire rods that students were able to build their own ideas as well as reflect on the ideas of other students with the rods. The external representations that were formed allowed them to view fractions in a way that they never had done before. This study will investigate the representations that are built as fourth grade students use the same materials to build fraction ideas for the first time, and how these representation allow the students to build different forms of arguments to explain their ideas.

2.3.6 Fraction Schemes

Steffe (2002, 2003, 2004) outlines the fraction schemes that he constructed after studying the ways of thinking displayed by fourth and fifth grade students as they worked on various fraction tasks. His work during this study was grounded in his "reorganization hypothesis," which posited that fractional schemes are reorganizations of whole number schemes. Thus, he defines the splitting operation in terms of two whole number schemes, iterating and partitioning. The splitting operation is preceded by the development of a number of preliminary fractional schemes, such as the simultaneous partitioning scheme, in which a whole is partitioned into equal lengths, and the part-whole fractional scheme, which involves disembedding distinct segments from a partitioned whole. The development of the splitting operation is evidenced by the use of the equi-partitioning scheme, in which the size of the part is estimated based on the size of the whole and that part is iterated to confirm that the correct number of iterations constructs a whole that is equivalent to the original whole, and the partitive fractional scheme, in which a whole is partitioned into equal parts (and iterated for the same purpose as it was in the equi-partitioning scheme), and one of the parts is used to establish a relation between the part and the partitioned whole.

Steffe also discusses the importance of the development of the ability of recursive partitioning, which enables a student to produce a composite unit from another composite unit. For example, a student uses recursive partitioning when she divides each one-third length of a stick into four equal lengths, and concludes that each of these smaller lengths is one twelfth. Steffe argues that this ability develops as the inverse of the multiplying scheme, and that it is a prerequisite to the development of the unit fractional composition scheme, in which a student uses recursive partitioning to then find the name for the segment that is given, and the commensurate fractional scheme, which involves transforming one fraction into another fraction that both measure the same quantity.

Norton's (2008) research built on Steffe's by showing that the splitting operation can exist in a student's whole number system before fractional schemes, such as the partitive fractional schemes. His findings corroborated Steffe's reorganization hypothesis.

Confrey (1994), in her discussion of the teaching of exponents, points out that schools ignore the development of students' intuitive understanding of splitting as they emphasize counting and the concept of multiplication as repeated addition. Confrey cites evidence of young children's understanding of repeated halving, and describes the understanding of an eight year old child who explained that three splits would yield eight pieces rather than six. She notes that this evidences that children may be able to visualize repeated splitting before being exposed to repeated addition.

Unlike Steffe, Confrey (1994) does not link the development of the concept of splitting, and therefore of fractions, with that of counting and whole numbers. In fact, she emphasizes that splitting is a construct that is distinct from the counting scheme, and that development of this scheme can enable students to build a more sound understanding of ratio and proportion, exponential functions, and multiplicative rates of change. Noting the complexity of the relationship between counting and splitting, she calls for future research on the presence of these schemes in children and predicts that an interdependence of the two schemes may be established.

There is a significant body of research (e.g. Lamon 1993, 2007; Behr, Harel, Post, & Lesh, 1992) that delineates the connections between fractional schemes and the schemes necessary for a robust understanding of ratio and proportion. Although this research may be relevant when analyzing the developing understanding of the student’s reasoning about ratio and proportion, a review of this research is beyond the scope of this chapter.

2.3.7 Mathematical Tasks

The literature on mathematical tasks is scarce, but the existent research is highly relevant to the present study. Doyle (1988) studied the properties of the tasks introduced in two middle school mathematics classrooms and the overall nature of the tasks introduced within each class. He found that greater weight was ascribed to familiar, or routine tasks, rather than to novel tasks. In addition, although the students accomplished a great deal of work during instruction due to the large volume of tasks that they completed, they rarely were required to struggle to achieve a deeper understanding of the mathematics that they learned.

Henningsen and Stein (1997) investigated the factors associated with the support or decline of high-level cognitive engagement in mathematical tasks. Using data from observer field notes, narrative summaries, and video data, they analyzed the tasks implemented by four classroom teachers. They identified fifty-eight tasks that supported the doing of mathematics in the classrooms, and set out to pinpoint factors that were associated with the support or the decline of the cognitive activity that took place during the task implementation. The analysis showed that the appropriateness of the tasks for the students, as well as the supportive actions of teachers during task implementation were crucial for the maintenance of high-level thinking during the task. The supportive actions that were noted included scaffolding and encouraging students to make meaningful connections between mathematical ideas and to provide explanations for their ideas. The researchers also noted that a decline in cognitive engagement occurred when an inappropriate amount of time was allotted to a particular task. They concluded that the teacher plays an important role in the level of thinking that takes place in the classroom by selecting tasks, implementing them within an appropriate time frame, and supporting, but not reducing, the complexity of the cognitive activity demanded by the task.

Francisco and Maher (2005) also point to task complexity as an important factor that promotes students' reasoning. They show that, during the longitudinal study, tasks were presented that were challenging for the students at their grade level, rather than providing scaffolding to enable them to succeed at the task. They found that the students found novel strategies to solve the tasks due to their complexity. In addition to allowing students to use the knowledge that they have gained from previous tasks in the strand to tackle new challenges, the "strand approach" allowed the students multiple opportunities to work our cognitive difficulties that underlie a set of tasks. Rather than revisiting one task until those difficulties are overcome, the students were able to approach the difficulty from the varied perspectives of different, yet related tasks, and were thus better able to tackle the challenges inherent in the tasks.

Francisco and Maher (2005) also showed that the implementation of tasks as part of a strand of related tasks proved beneficial for the students in the longitudinal study.

Lesh, Hoover, Hole, Kelly, and Post (2000) encouraged the use of "model eliciting tasks". These tasks share six characteristics that they found to be useful in promoting the development of key ideas and understanding in mathematics at all levels of primary and secondary school. They described the characteristics in detail in their publications (Lesh et al., 2000; Lesh, Cramer, Doerr, Post, Zawojewski, 2003), and those characteristics will be summarized below.

First, the task should be personally meaningful, in that students understand that their ideas will be taken seriously, and that they will not be forced to assume their teachers' understanding of the problem situation. Second, the task should be structured in a way that ensures that students understand that they must build a model to represent their solution. Third, students should be able to self-evaluate solutions and should be able to judge when they have arrived at an adequate response and justification. Fourth, the task should require students to explain their thinking about their problem, rather than simply providing a solution. Fifth, the problem situation should be as simple as possible but still require the use of a model in the solution representation. In addition, the task or the solution should serve as a prototype that explain aspects of problems that are structurally similar to the task at hand. This aspect of the task can be compared to the concept of assimilation paradigm introduced by Davis (1984). Sixth, the model that is produced during the problem solving session should be one that can be manipulated and modified for use in other problem situations.

Lesh et al. (2003) also discussed the importance of two other kinds of tasks, which they called "model exploration" and "model adaptation" activities. Model exploration activities are designed to enable students to build a durable representation system for a conceptual system that is being targeted. Model adaptation activities allow students to use and extend the models to solve problems that would have been too difficult to solve without the use of the models that have been constructed.

2.4 Summary of Literature Review

The literature documents the difficulties that many students experience as they try to learn formal proof, and also provides evidence of effective reasoning used by students of all ages when provided the opportunity to work on rich tasks in a supportive environment. This study builds on the literature by identifying the forms of reasoning and argumentation that elementary school students use as they discuss their emergent understandings of fraction ideas in a whole class setting, and by examining the nature of the development of that reasoning over the course of the intervention.

CHAPTER 3: METHODOLOGY

3.1 Introduction

To answer the research questions, a qualitative study was conducted. A qualitative approach was chosen as opposed to a quantitative one for a number of reasons. First, this study attempted to provide a "detailed view" of the topic (Creswell, 2006), which is best be accomplished using qualitative measures. Second, this study involved careful analysis of data collected in a natural setting, rather than a controlled setting. This affected the number of variables present in the study, and thus lends itself more to a qualitative approach (Creswell, 2006). Lastly, the qualitative approach was chosen due to the way that it "allows for the discovery of new ideas and unanticipated occurrences" (Jacobs, Kanawaka, & Stigler, 1999, p. 718), which was desired in the present study.

The research was conducted as a nested case study, and was bounded in two ways: by time, or the number of sessions that were analyzed, and by place, the elementary school classroom in which it was situated. The case under study was the classroom activity of twenty-five fourth grade students as they investigated a strand of fraction tasks.

3.2 Setting

The present study was situated in the ongoing research study of the development of children's mathematical thinking and reasoning conducted at Rutgers University. The data were drawn from the longitudinal study of fourth grade children's mathematical behavior in the Conover Road School in Colts Neck, New Jersey. This location was the site of researcher intervention for a span of three and one half years. The series of interventions conducted at this site was situated within a larger twelve year longitudinal study conducted by the Rutgers researchers at three sites in New Jersey (see Maher, 2002, 2005 for an overview of this study).

The Conover Road School's curriculum calls for an introduction to fraction operations at the start of the fifth grade, as many schools in New Jersey do. The students in this fourth grade class, therefore, were not introduced to algorithmic methods of fraction operations in school, and their only experience with fraction concepts had been a rich introduction to ideas related to fraction as an operator in the third grade. Of the fifty sessions conducted during the school year of 1993-1994 by the researchers, the first twenty-five focused on fraction concepts and related areas. Each session was approximately sixty to ninety minutes in duration, and the twenty-five sessions that focused on fraction concepts took place between September 20 and December 15, 1993.

The goal of this part of the intervention was to investigate how students build concepts of fraction as number, how they develop representations to understand the concept of the unit, to compare fractions, and to build ideas of fraction equivalence. In addition, the study explored the ways that students construct their own ways to solve fraction problems without having learned the traditional algorithms to do so. Importantly, the students, rather than the teachers, provided closure to the tasks. When a question went unresolved during one session, the class would revisit the problem during the following session and work to resolve the issue. The researchers, however, did not offer solutions or tell students whether their ideas were right or wrong. In this way, students learned to rely on themselves and each other as the originators of the ideas that were built in the classroom and the arguments that validated them.

3.3 Sample

The class was composed of a heterogeneous group of twenty-five students, fourteen girls and eleven boys. The primary researchers who conducted the interventions were Carolyn A. Maher (T/R 1) and Amy M. Martino (T/R 2). The classroom teacher, Mrs. Joan Phillips (CT), was present during all the sessions. Occasionally, the principal of the elementary school, Dr. Judith Landis, Dr. Robert B. Davis, and other members of the research team at Rutgers University, attended the sessions and posed occasional questions as students worked on the tasks. All adults present in the room were told not to tell students if their ideas were correct or incorrect, and were only permitted to question students and to listen as they worked with their classmates.

Together with the classroom teacher, the researchers grouped the students in pairs, with the exception of one group of three. Students were encouraged to engage in discussion with other groups of students and to work closely with their partners. Some of the students were regrouped when necessary over the course of the intervention.

This study investigated the reasoning and argumentation used by the students during seventeen of the twenty-five sessions described above. Seven sessions aside from those listed took place between October 11 and December 2. Two of these sessions were a pencil and paper review of concepts developed during earlier sessions, four focused on rational number concepts, and one was a review session during which the children's parents were present in the classroom. Since these sessions were of a different nature than the remaining seventeen sessions, which all involved the building of models to understand fraction ideas, the data was not analyzed in this study.

Appendix A summarizes the dates of the sessions that were analyzed and the tasks that were introduced during each session. In addition, a list of the camera views that were available for each session is provided.

3.4 Tasks

After students worked on each task or group of tasks, students were invited to share their solutions in a whole-class setting, often building models of their work on the overhead projector. Students were encouraged to justify their solutions, and other class members were asked if they had any questions or alternative solutions.

During the majority of the sessions, students were provided with three-dimensional Cuisenaire® rods, which they used to build models of the fraction ideas that they investigated. Cuisenaire rods are a set of ten wooden rods of different colors, with the shortest rod measuring one centimeter and the longest measuring ten centimeters in length (see Figure 1 below for an image of these rods). Students used the attribute of length to model the fractions that were under study. A set of transparent, two-dimensional Cuisenaire rods was available for use at the overhead projector. In addition to working with individual rods, students made trains of rods by placing two rods end to end and reasoning about the train's length in comparison with the lengths of other rods or trains (see Figure 2 for an illustration). Students also made overhead transparencies to display their models. In addition, students often used paper and pen to record their solution strategies and explanations. Lastly, during the last three sessions of the unit, during which the students explored ideas related to division of fractions, students used ribbon, scissors, meter sticks, and strings to model the tasks that were posed.

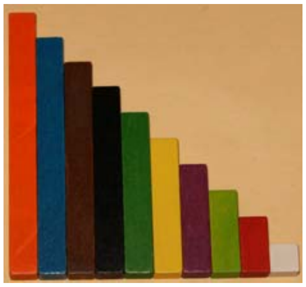


Figure 3.1. Staircase model of Cuisenaire® rods.

Figure 3.2. A train of two light green and one red rod alongside a brown rod

3.5 Data Collection

In keeping with case study methodology, multiple data sources - video recordings, students' written work, and researcher field notes - were used in the process of data collection; the data were triangulated to ensure validity of results (Creswell, 2006; Yin, 2003. The method of triangulation will be discussed in detail in the sections that follow.

3.5.1 Video recordings

The primary source of data was the database of video recordings that were taken during the sessions. During each session, between one and three video cameras captured various views of the class. Three cameras were used for the majority of the sessions. Two were operated by videographers; one was positioned at the side of the classroom opposite the door; the other at the front of the classroom. The third camera was stationary and was positioned at the back the room, and was used to record the models built by students at the overhead projector. However, during a few sessions, this third camera was also operated by a videographer and captured the activity of students as they worked in groups.

Video data were chosen as the primary data source for several reasons. Video provides the opportunity for in-depth study of student activity that it provides. Video data, by its very nature, is more "raw" than other forms of collected data, such as observational field notes, since the data contains the actual classroom occurrences and can be viewed and analyzed multiple times by the individual researcher as well as by other members of the research team (Jacobs, Kanawaka, & Stigler, 1999). In the case of the present study, this aspect of video data was used to ensure validity of results during the process of verification, (described in section 3.5), and to allow the researcher to view the video data multiple times in an effort to better understand the data.

In addition, the availability of the video data as part of a larger database of video provided a potential for increased reliability of the results of the study. By replicating the method of analysis on other data sets contained within the database, the results can be corroborated and verified in multiple ways.

Importantly, by viewing the video and analyzing the students' argumentation and exploration that is captured by the video, the researcher was able to analyze the forms of reasoning that the students use, thereby meeting the goal of this study.

Another way that the video data was used to inform the research was through the use of screenshots that capture student representations. This allowed for in-depth study of student models and presentations and provided a better understanding of the students' work (Jacobs, Kanawaka, & Stigler, 1999). The ability to view screenshots of the models together with the transcript of the activity on the video enabled the researcher to analyze the students' reasoning more effectively.

3.5.2 Students' Written Work

The second data source was a compilation of the students' work, which was, at times, recorded during the sessions, and, at other times, was assigned to students to complete after the sessions had taken place. The creation of these documents was initiated by the researchers during the implementation of the intervention, and thus can be considered researcher-generated documents (Merriam, 1998). As such, they were highly relevant to the study, as they were directly related to the intervention that was implemented by the researchers. In addition, the collected students' work were an invaluable data source, as it contains the in-class and after-class recordings of students models and justifications of the problems that were posed by the researchers. The reasoning used to justify their ideas were analyzed and considered in conjunction with the video data of their in-class work, thereby adding a new dimension to the data set.

The data derived from this source was used to supplement the video data and better enabled the researcher to construct a complete storyline documenting the students' mathematical understanding and reasoning. It allowed the researcher to determine whether the students were able to record their justification in written form, as opposed to their ability to present their arguments verbally to their classmates. Additionally, the written work was used to study the models and explanations of students that were not viewed on video in an effort to check the prevalence of the reasoning viewed on the video. In this way, an attempt was made to "establish a train of evidence" that allowed sound findings to be noted (Yin, 2003, p.34). These drawings of the models built during class sessions are a record of physical artifacts used by the students, which lent insight into the activity of the children who were not viewed. In addition, these records were used as evidence of the trace measure of the model-building activities of these students (Merriam, 1998), and provided a picture of the overall ability of the students to justify their solutions using the models that they had constructed. Representative samples of the students' work are included in Appendixes B-K.

3.5.3 Field Notes

The third data source consisted of field notes that were recorded by researcher Amy Martino. Dr. Martino recorded the tasks that were presented and the activities that took place during each session, the content of student presentations at the overhead projector, and the grouping of the students along with the rationale for any changes from the previous grouping. Any description of student presentations was accompanied by a sketch of the model they built using the Cuisenaire rods. Also, when a series of tasks was posed during one session, the list of tasks was typed and included in the body of field notes. Lastly, the field notes included comments recorded after researcher conferencing and discussion between sessions.

One limitation of the study was the inability of the researcher to observe the sessions in real time, and the study of the observational field notes acted as a partial substitute for in-person observation. This ensured the accuracy of the researcher's understanding of the classroom activity that was first assessed from the video data. In addition, an accurate understanding of the chronology of the sessions enabled the researcher to answer the third research question by tracing the changes in the students' reasoning over time.

3.6 Method of Analysis

Data analysis took place in several stages. First, the video data were analyzed and themes were identified. Then, the students' work was analyzed to add detail to the data derived from the video, and the forms of reasoning that were identified during the first stage were verified using this supplementary data. After emergent themes were identified in the data, a narrative was constructed to explain the findings, and the researcher field notes were used to ensure the accuracy of the narrative.

This study used some components of the analytical model for studying video data outlined by Powell, Francisco, and Maher (2003). This model is composed of several non-linear phases of analysis, which begin with attentive viewing of the video data and culminate in the composition of a narrative. The phases will be described as they pertained to the current study in the sections that follow.

3.6.1 Viewing

The video data were viewed by the researcher in order to become familiar with the content. During this phase, as Powell et al. (2003) suggest, the researcher "watch[ed] and listen[ed] without intentionally imposing a specific analytical lens on their viewing" (pp. 415-416). During this study, the videos were watched on chronological order. During the first viewing, no notes were taken, and the purpose of the viewing was to achieve some level of familiarity with the data. During subsequent viewings, transcription and the identification of critical events took place.

3.6.2 Identifying Critical Events

After viewing the video, the researcher identified *critical events* that occurred during the data that was used for further analysis. Events are considered critical when, according to Powell et al. (2003), they "demonstrate significant or contrasting change from previous understanding, a conceptual leap from earlier understanding" (p. 416). (This reference to a conceptual leap is distinct from Balacheff's use of the term.) During the present study, critical events were identified when students offered an argument or a line of reasoning during their mathematical explorations or as they justified their ideas and countered a researcher or classmates' claim.

As Tesch (1990) points out, an essential part of video data analysis is "breaking down the record" into segments that can be compared with one another and sorted in a way that helps the researcher make sense of the data. This process began during the phase of identified critical events, and was culminate as the researcher codes the data according to the methods described below.

3.6.3 Transcribing

After viewing the video data and beginning to identify critical events, the video was transcribed to enable a more detailed and thorough analysis of the data. A format to be used during transcription at the start of this phase to ensure uniformity of the documents. This format was created to suit the research purposes. First, line numbers were used to label full participant turns, rather than typed lines in the transcript. This more effectively showed the chronology of arguments as they were exchanged between students, and facilitated the selection of vignettes for further analysis and for inclusion and citation in the narrative. The video timestamps were recorded at regular intervals of about five minutes, as well as at the start of a new activity or interaction. For select sessions, timestamps were provided at the start of each line. A table format was used to facilitate the creation of different sets of coded transcripts, which was necessary due to the dual nature of the analysis, but was converted from this format for inclusion in this document. Brackets within the transcription indicate pauses, gestures, student or researcher activities, descriptions of models that were built by the researchers or the students, and references to screenshots that were indexed to supplement the transcript.

All transcripts were verified by at least one independent researcher. During the verification process, the video data was viewed and the transcript was checked for accuracy. The set of videotapes from the first seven sessions that were analyzed in this study were transcribed by an independent researcher, and this researcher, as well as another graduate student, independently verified the transcripts. In addition, two of the later sessions were transcribed and verified by graduate student researchers during a summer practicum. These sessions were verified once more by this researcher to ensure accuracy. The tapes from the remaining eight sessions were transcribed by this researcher and verified by an independent researcher.

For the majority of sessions, one transcript was created to include the data from all camera views that were available. Selections of the transcript that were derived from only one camera view is appropriately labeled. Transcripts from some sessions that consisted primarily of partner and small group work were organized by camera view and are appropriately labeled. All transcripts used for this analysis are included in Appendixes L-AC.

3.6.4 Coding and Identifying Themes

Following the transcription of the data, the researcher coded the data in an attempt to identify points in the data that can be used to answer the research questions. Coding involved drawing on a combination of typological and inductive methods of analysis (Hatch, 2002). A deductive coding scheme drawn from the literature (Mueller, 2007; Rowland, 2002; Smith, Eggen, & Andre, 2001; Polya, 1954; de Villiers, 2003) was first used to analyze the data and highlight occurrences of various forms of reasoning.

After the initial coding phase, the data was also studied inductively and codes that had not been previously identified were constructed to further categorize the data. In addition, the structure of the coding scheme was modified significantly and an additional review of the literature was undertaken to refine the scheme. During this phase, selected portions of the data were reviewed and corroborated by an independent researcher. Any differences in understanding were discussed and results of these discussion were reflected in the final composition of the coding scheme and narrative.

As part of the second phase of coding, the researcher analyzed the available documents and integrated the results of this analysis with the video data analysis. Document analysis (described in the next section) was used to analyze the students' work that was created during and after the activities recorded in the video tapes. This allowed the researcher to compare the written and verbal justifications of the students and more accurately answer the research questions.

3.6.5 Document Analysis

As discussed above, the collection of students' work and field notes was used to supplement the video data and to complete the analysis of the data. The justifications presented in the written student work was compared with the justifications offered during the class sessions as recorded on video and in the transcription, and similarities and differences in the forms of reasoning used were noted. Specifically, since the students' work was often recorded toward the end of an investigation, the students' final analysis and method of justification were found in this data source, and the researcher began to investigate whether the students refined their reasoning at a later stage, or if they used the same reasoning that was recorded in the video data. In this way, the development of the reasoning of the students was traced across the two data sources.

The researcher field notes were used to corroborate the narrative derived from other two forms of data (see below) and lent accuracy to the account of the class sessions. By using the documents to supplement the video data and aid in analysis, multiple data sources were used, thereby accomplishing triangulation of data and greater validity of results.

3.6.6 Constructing Storyline

After the data was coded, tables were constructed that summarized the forms of reasoning found during the sessions and allowed the researcher to identify themes within the video data. The table entries referenced the corresponding sections of the transcript and a summary of the reasoning that was displayed during the noted occurrence. The arguments were organized by task, the purpose, structure, form and sub-codes for each argument was noted, and the context of the argument (partner/researcher talk or whole class discussion) was indicated. This format was chosen to better enable the construction of a cohesive storyline and the composition of a descriptive narrative that takes the most relevant data into account.

In addition to tables that were used to summarize the results of each session, charts were constructed to trace the argumentation as it occurred during the session. These charts served a dual purpose. First, it enabled the researcher to better understand and describe the results of the session as the narrative was constructed. In addition, it was included at the end of each narrative as a visual description of the reasoning used during the session. Adapted from a graphic representation of argumentation originated by Chinn and Anderson (1998), the charts were used to show the nature of and relationship between the claims and counterarguments that were offered during each session. The structure of these charts is outlined in the introduction to the next chapter.

3.6.7 Composing Narrative

During the final phase, results from the coding and document analysis phase were strung together as the researcher tried to make sense of the results and began to identify an "emerging narrative about the data" (Powell et al., p. 430). The video data and the documents were be used to identify *traces*, or collections of events that lent insight into a student's or group of students' growth in mathematical understanding. In this study, the researcher attempted to identify traces of growth in the use of reasoning and argumentation as students progressed in their mathematical journey. As the narrative was composed, the field notes were used extensively to verify the accuracy of the account of the class sessions.

3.7 Discussion and Rationale of Coding Scheme

The arguments used by the students were analyzed using a four-level coding scheme. In addition, other forms of reasoning were noted as they were identified and described separately in the narrative. The scheme will be described and the forms of reasoning identified will be defined in the sections that follow.

3.7.1 Codes for Purpose of Arguments

The first step in the coding process was the identification of the purpose of the argument's formulation and presentation. Arguments that were used to justify a claim (Claim) were distinguished from arguments used to counter the claim of the researcher

(Counter-R) of a peer (Counter-argument number). Then, the structure of the arguments was examined.

3.7.2 Codes for Structure of Arguments

Two structures of argumentation were then used to classify virtually all arguments. These were direct and indirect reasoning. These two forms of reasoning were identified by viewing the argument as a whole and pinpointing the data and the conclusion of the argument, as well as the function that the argument served.

3.7.2.1 Direct Reasoning

Direct reasoning can be symbolized by the logical form $p \rightarrow q$ and is read "If p, then q". Direct reasoning is the most basic and common form of reasoning used in advanced mathematics (Smith, Eggen, & Andre, 2001). The validity of this form of reasoning can be ascribed to the *modes ponens* rule of logic (which states that if both $p \rightarrow q$ and p are true, it follows that q is also true.

3.7.2.2 Indirect Reasoning

The term indirect reasoning was used to label arguments that followed the form of reasoning by contradiction or by contraposition. Reasoning by contradiction, for the purpose of this study, is defined as follows. When trying to show that a statement is true (p), it is first assumed that the denial of the statement is true (~p), and that assumption is followed to arrive at a contradiction of a second statement (q and ~q). When both this second statement (q) and its denial (~q) are found to be true, it is concluded that ~p must be false, and that p is therefore true (Smith et al., 2001).An argument by contraposition is defined as one in which a statement $p \rightarrow q$ is shown to be true by showing $\sim q \rightarrow \sim p$.

For the purpose of this study, both forms of argument (those similar to reasoning by contradiction and contraposition) were coded as indirect reasoning. As noted by Antonini and Mariotti (2008), these forms of reasoning are inherently similar. One can verify this by substituting the p in the first description with q in the second, and will find that the q and ~q in the first are equivalent to the p and ~p in the second, and both forms of argument result in the same conclusion. Due to the informal nature of the reasoning used by the fourth grade students, indirect reasoning served the purposes of this research sufficiently and obviated the need to differentiate between the two forms of reasoning.

3.7.3 Codes for Forms of Reasoning within Arguments

The third classification of the reasoning found included forms of reasoning that were used to make up the direct or an indirect argument. These forms of reasoning were coded independently due to the flexibility of their use. For example, a student might use a direct argument and also use reasoning by cases to organize the argument and tackle each of *n* cases at a time. However, a student might also use an argument by cases to indirectly show that a statement is true. Therefore, the data was coded first for direct or indirect forms of reasoning, and was then further analyzed to identify forms or reasoning that are contained within the arguments.

3.7.3.1 Reasoning by Cases

Reasoning by cases, also known as the use of an argument by exhaustion, takes the logical form of $p_1 \rightarrow q, p_2 \rightarrow q, \ldots p_n \rightarrow q$. This form of reasoning organizes the argument by considering a set of finite, distinct cases, and arrives at the same conclusion after consideration of each case. This form of reasoning requires a systematization of all possibilities into an organized set of cases that can be analyzed separately (Smith et al, 2001).

3.7.3.2 Reasoning using Upper and Lower Bounds

When reasoning by upper and lower bounds (U/L), a student defines the upper and lower boundaries or limits of a class of numbers or mathematical objects. For example, for the set of numbers $\{1<x<4\}$, the upper bound of the set is 4 and the lower bound is 1, since all the numbers in the set are contained within the two bounds. After these bounds have been defined, the student reasons about the objects that are not contained between the bounds and draws conclusions based on this reasoning. This form of reasoning is often used to show that the set that is defined as all objects between the two identified bounds is empty.

3.7.3.3 Recursive Reasoning

Recursion is "a method of defining functions in which the function being defined is applied within its own definition. The term is also used more generally to describe a process of repeating objects in a self-similar way" ("Recursion," 2008). The most common use of recursive reasoning (recur.) in advanced mathematics is the proof by mathematical induction. This form of proof shows that for all natural numbers n, if n is contained in a set, then n + 1 is contained in the set. The proof then concludes that the set is equivalent to all natural numbers.

Recursive reasoning that is used informally in mathematical justification relies on the definition of basic cases and the determination of operations on these basic cases. All operations on any cases in the system can be derived from combinations of the base cases. In this way, the class of objects under study can be built from a few basic cases and rules. This form of reasoning is often used to show that a calculation is impossible ("Recursion," 2008), or to calculate a complex case by using a simpler one and the recursive definition. These recursive functions can then be used to justify a solution of the complex case.

3.7.3.4 Reasoning using the Generic Example

This form of reasoning has been described at length by mathematics educators and has been identified as a form of argumentation that can assist students in their journey to formulating valid proofs (Balacheff, 1988; Alibert & Thomas, 1991; Movshovitz-Hadar, 1988; Selden & Selden, 2007). Generic reasoning (gener.) occurs when a student reasons about the properties of a paradigmatic example that are representative of and can be applied to a larger class of objects in which it is contained and lends insight into a more general truth about that class. The consideration of the general application of these properties in turn verifies the claim made about the particular example (Rowland, 2002a, 2002b). Some researchers consider it to be a valid form of justification (Balacheff, 1988), and point out that it is easily understood by students at all levels, and is more intuitive than many other forms of proof (Alibert & Thomas, 1991).

3.7.4 Sub-codes

Lastly, the argument was coded for faulty (Faulty) or incomplete (Inc) reasoning. Occurrences of faulty reasoning were further analyzed to determine its nature and importance. Flaws in reasoning that were peripheral (-p) and relatively inconsequential to the argument were distinguished from flaws in the central argument. In addition, flaws that occurred only in the execution (-e) of a student's reasoning were distinguished from more serious structural flaws in the reasoning.

3.7.5 Other Forms of Reasoning

Some other forms of reasoning that students used as they justified their solution were identified. These forms of reasoning are not deductive in nature, and as a result, were traced separately in a separate narrative.

3.7.5.1 Generalization

Polya describes generalization as "passing from the consideration of a given set of objects to that of a larger set, containing the given one" (p. 12). This strategy is used to make a statement about a larger set of objects based on observations on a smaller set that is contained in the larger one. Although, as stated earlier, this cannot be used to deduce the validity of a statement, it can enable students to discover general properties in mathematics and test their observations to ascertain their validity.

3.7.5.2 Analogical Reasoning

As discussed in the theoretical framework, analogical reasoning is useful in mathematics when clarified analogies are used. Two useful forms of clarified analogies are isomorphism and the determination of a similarity of structure or relation between two mathematical propositions, functions, or operations (Polya, 1954). Polya explains that this second form of analogical reasoning is useful “if the relations are governed by the same laws" (p. 29). One common example of this form of reasoning is proportional reasoning, which has a sound mathematical basis and can be used to validate results due to the laws of numbers that are the source of the similarity of relations. Less conventional examples of analogies can be used to explore the properties of partially or completely unrelated mathematical ideas and establish similarity of structure between them.

The complete coding scheme used is summarized in Table 3.1 below.

Table 3.1.

Summary of Coding Scheme

1. Purpose of Argument
Justification of Claim
Counterargument to another's claim
2. Structure of Argument
Direct Reasoning
Indirect Reasoning
3. Form of Reasoning
Reasoning by Cases
Reasoning using Upper and Lower Bounds
Recursive Reasoning
Generic Reasoning
4. Subcodes
Faulty Reasoning Peripheral Execution
Incomplete Reasoning
Other Forms of Reasoning
Generalization
Analogical Reasoning

3.8 Validity

To ensure validity of results, a number of steps were taken throughout the process of data collection and analysis. In keeping with case study methodology, construct validity (Yin, 2003) was ensured through the identification of critical events that are directly related to the research questions and meet the objectives of the study. In addition, construct validity was established by triangulation of data (Yin, 2003). Triangulation of data with the use of researcher field notes, student work, and video recordings validated the accuracy of the storyline that was constructed. By using multiple forms of data, the data was supported by more than one source of evidence so that the results can be more convincing and accurate (Yin, 2003). Investigator triangulation (Patton, 1987) was used during the transcription and coding stages. During the transcription stage, all work was verified by at least one independent researcher and differences were discussed until arriving at a resolution. During the coding stage, large portions of the data were coded by independent researchers as well as by this researcher, and differences in coding were examined and resolved. In addition, an attempt was made to describe the data in an expressive manner. This description includes extensive vignettes drawn from the raw data in an effort to enable readers to independently arrive at similar conclusions from the description provided (Stake, 1995). By carefully documenting, describing, and verifying the research conducted, it is hoped reliability of results was established (Yin, 2003).

CHAPTER 4: RESULTS

4.1 Introduction

This study investigates the forms of reasoning and argumentation used by fourth grade students as they built fraction ideas over the course of seventeen class sessions. These sessions took place between September 20, 1993, and December 15, 1993.

The results of this study are presented in the following manner. The forms of reasoning used by the students as they justified their solutions to mathematical tasks are presented in section 4.2. A narrative of each session is provided, and the arguments of the students are presented, organized by task and in presented chronological order. Lines of reasoning are numbered and the numbers are included in boldface at the start of the paragraph in which the reasoning is described. The numbering system was used to trace the chronological order of the arguments as well as the repetition of arguments by students over the duration of each session. Different versions of a particular line of reasoning are distinguished by the used of lower-case letters after the number that is provided (e.g. 1a, 1b) and indicate the repetition of an earlier argument that similarly numbered.

Screenshots of the models built by the students at their desks and at the overhead projector (OHP) and the written work that they complete during the class sessions are noted at appropriate points in the narrative and are labeled according to the camera view from which it was taken (F, S, O) and the minute and second timestamp that indicate when the screenshot was captured. This labeling was used to enable researchers to easily locate these images when analyzing the video data in the future. For example, a screenshot captured twenty four minutes and thirteen seconds after the overhead camera was turned on for a session is labeled Figure O-24-13.

Mention is made of students work within the narrative as appropriate. All written work mentioned in this chapter is included in the appendices. When necessary, a separate discussion of the findings in the students' written work is included as a separate section after the session's narrative.

The tables included after each narrative summarizes the forms of reasoning that were used by the students during the session, as well as the kind of setting in which the reasoning was used. The type of argument is shown, and claims and counterarguments are distinguished. In addition, the structure of the argument (direct or indirect), any specific form of reasoning used (upper and lower bounds, cases, generic, and recursive), and faulty, flawed, or incomplete arguments are noted. Arguments offered during whole class discussions (WC) are shown, as well as those used during partner work or a student-researcher interaction (PR).

The charts placed after each narrative traces the patterns of argumentation that took place as the students worked on each task. Arguments are organized by task, and the tasks and arguments are numbered as they were in the text as well as in the table. Arrows were used to indicate arguments that supported a claim, while those that are marked with a short line indicate a counterargument. Curved arrows indicate an argument that were used to modify an earlier argument. Arguments are coded by color to show direct (blue) and indirect (red) arguments.

The second analysis contained in section 4.3 notes other forms of reasoning found during the sessions. In particular, instances of generalization and analogical reasoning are described.

4.2 Reasoning of Justification

4.2.1 Session 1: September 20, 1993

Task 1a: I claim that the light green rod is half as long as the dark green rod. What do you think?

Task 1b: What number name would we give the light green rod if I called the dark green rod one?

At the start of the session, after introductions were made, T/R 1 asked the students if they had ever used Cuisenaire rods. Most students raised their hands. T/R 1 then posed the first task to the students. She said, "I claim that the light green rod is half as long as the dark green rod. What do you think? What would you do to convince me" (line 1.0.1)?

1 The students proceeded to use the rods to solve the problem. T/R 1 asked Erin to respond. Erin, who hadn't used Cuisenaire rods before, replied that it was true, and reasoned directly by putting two light green rods next to the dark green rod and showing that they were equal in length (line 1.0.2).

Next, T/R 1 asked the students to name the light green rod if the dark green rod was called one, and the students responded correctly to the task. There is no evidence of the reasoning that was used to justify the solution offered.

Task 2: Someone told me that the red rod is half as long as the yellow rod. What do you think?

2 T/R 1 then posed the next task. She said, "Someone told me that the red rod is half as long as the yellow rod. What do you think" (line 1.0.11)? Danielle responded that the statement was not true, and justified her solution by saying, "Two red rods don't fit. You need to put more" (lines 1.0.21-1.0.21). This explanation was a simplistic version of indirect reasoning, in that it showed why the statement was not true by explaining that two red rods were not equal to the length of the yellow rod.

Task 3: Someone told me that the purple rod is half as long as the black rod. What do you think?

3, 4, 5 T/R 1 then asked the students if the purple rod was half as long as the black rod (lines 1.0.24-1.0.28). Alan and Erik, partners during this session, thought about the problem, and Alan said that it couldn't be true, because two purple rods were not equal in length to the black rod (lines 1.0.34-1.0.37). Meredith and Sarah, working together, told T/R 2 that the two purple rods were "too large" and that the black rod would need to be longer in order for the statement to be correct (lines 1.0.40-1.0.43). David, when asked during the whole class discussion to respond to the problem, replied similarly, saying "Two purples are too large" (line 1.0.46).

Alan, Sarah, Meredith, and David used reasoning by contradiction in a manner similar to Danielle as they explained their solutions to the fourth task. Assuming that, by the definition of one half, the purple rod could only be one half as long as the black rod if two purple rods were equal in length to the black rod. Showing that this was not the case, they justified their claim that the purple rod was not half as long as the black rod.

Task 4a: Someone told me that the red rod is one third as long as the dark green rod. What do you think?

Task 4b: If I called the dark green rod one, what number name would I give to the red rod?

6 The next task that was posed introduced a fraction other than one half. T/R 1 asked the students if the red rod was one third as long as the dark green or. Jackie, at the overhead, showed that three red rods are as long as the dark green rod (line 1.0.62). Michael explained that by lining up three red rods below the green rod, one can convince others that the statement is true (line 1.0.64). Jackie and Michael used direct reasoning to justify their solution.

7,8 T/R 1 then asked the students what number name they would give the red rod if the green rod was called one. Sarah showed T/R 2 directly that since there were three red rods lined up under the dark green rod, the red rod would be called one third. The students told T/R 1 that they had already built a model to show their solution. T/R 1 asked them to explain what they meant. Beth said that the red rod would be called one third. Beth elaborated by saying "Because if you put three on them it makes one whole" (lines 1.0.77). Beth and Sarah used direct reasoning to justify their solution.

Task 5a: Someone told me that light green is one third as long as blue. What do you think?

Task 5b: So if I call the blue rod one, what number name would I give to light green?

9 T/R 1 asked the students if the light green rod was one third as long as the blue rod. Jessica responded that this was true, since three light green rods equaled the blue rod in length, and the blue rod was called "one whole" (lines 1.0.81-1.0.83). T/R 1 then asked the students what the light green rod would be called if the blue rod was called one, and they chorused that it would be called one third. This response was not challenged for justification.

10,11 At this point, T/R 1 noted that in the previous task, the red rod was called one third when the dark green rod was called one, and in the seventh task, the light green rod was called one third when the blue rod was called one. T/R 1 asked the class if number names of the rods could change. The students answered that they could. She then asked the students if the color names of the rods could change. Although at first, some students responded that they could, they eventually came to a consensus that the color names of the rods were permanent. T/R 1 then asked the students to explain how the number names could change. Erik responded directly by using the two problems that they had just worked on. He said that the red rod could be called one third in one problem, but the light green rod could be called one third in a different problem (lines 1.0.103-1.0.107). Michael then explained further, using direct reasoning, that this happened when different rods were called one in the different problems (line 1.0.111).

T/R 1 asked the students to find all the rods and determine their color names. T/R 1 asked Kelly to describe how she had organized the rods, and Kelly explained that she had begun by finding the longest rod and had proceeded to organize them by length until she had found the smallest rod. She then listed the rods in that order.

12 T/R 1 mentioned that she had heard a student suggest that the red rod be called one if the dark green rod was called one. She asked the class it that was possible. Erik said, “No. Not now. Not if the dark green is one. Because if you're comparing the red to dark green it can't be one. But if you're comparing the red to something else it can be a one, it can be a whole” (line 1.0.137). Erik’s reasoning was incomplete, because he did not explain why it could not be so. Michael responded that the red rod could not be called one, and justified his response indirectly by saying, “Because the green is bigger and it takes three of the reds to make one green” (line 1.0.139). This justification was also incomplete, in that Michael did not make clear why this fact shows that the red rod could not be called one.

Task 6: What number name would I have to give to green if I wanted red to be one?

13a-c,14 T/R 1 asked the class what the dark green rod would be called if the red rod was called one. Erik, working with Alan at his desk, proposed that it would be called “three wholes” (line 1.0.142). He explained to Alan that since three red rods equal a dark green rod, and each red rod was called one, the dark green rod would be called three. Later, when T/R 1 asked the students to share their solutions in a whole-class discussion, Alan and then Erik used this explanation to justify their solution. Alan said, “Okay, if the red one is considered one, then the green one is a lot bigger. So it would have to be, it would take three whole ones to make another green so it should be considered three wholes” (line 1.0.174). Erik then repeated this reasoning (line 1.0.175). David used similar reasoning to justify the solution, saying that if the red rod was called one, “then the green would have to be two more wholes, so that would be three wholes” (line 1.0.177).

Task 7a: If I call brown one, what number name would I give to red?

Task 7b: Now I want to call the red rod one, what name would I give to the brown rod?

15,16 T/R 1 asked the class to name the red rod if the brown rod was called one. Danielle explained that it would be called one fourth and could be justified by placing four red rods next to the brown rod (lines 1.0.180-1.0.182). T/R 1 then asked the students a related question, requiring them to name the brown rod if the red rod was called one. Jacquelyn explained that if the red rod was one, the brown rod would be called four, because four red rods equal a brown rod (lines 1.0.184-1.0.186).

Task 8: What would I have to call one if I want to name the white rod one half?

17 T/R 1 challenged the students with one last problem. She asked them what rod would be called one if the white rod was called one half. Laura responded that the red rod would be called one, and Graham justified Laura's solution by saying that this could be shown by placing two white rods next to the red rod and seeing if they were the same length (lines 1.0.188-1.0.190).

Student-Created Tasks

18 T/R 1 then asked the students to work with their partners to come up with questions to challenge the class. Alan was first to present his challenge. He asked the class, to find the rod that would be called one if the red rod was called one fifth. Graham responded that the orange rod would be called one, and explained that five red rods were equal in length to the orange rod (lines 1.0.254-1.0.258).

19 Beth and Mark posed two related challenges. Beth asked the students, "If a [light] green was a whole, what would a blue be?" (line 1.0.261). Erik responded that it would be called "three wholes," but did not justify his solution (line 1.0.263). Mark asked the students to name the light green rod if the blue rod was called one. Jacquelyn responded that it would be called one third, and explained that three green rods equaled a blue rod, and so each light green rod would be called one third (lines 1.0.266-1.0.268).

20,21 Jacquelyn, Kelly, and Michael posed their problem to the class. They asked, "If white one is one whole what would the orange be?" (line 1.0.270). Erik responded that the orange rod would be called ten, since only ten white rods equaled an orange rod (lines 1.0.272-1.0.274). T/R 1 asked the class to name the white rod if the orange rod was called one. Jacquelyn responded that it would be called one tenth, using a line of reasoning similar to Erik (lines 1.0.276-1.0.278).

22 Meredith asked the class to find the rod that would be called one if the purple rod was called one half. Amy replied that the brown rod would be called one, because the purple rod was one half as long as the brown rod, and two purple rods equaled the brown rod (lines 1.0.282-1.0.284).

23a-c T/R 1 told the students to use the remaining class time to challenge their partners with problems using the rods. Erik asked Alan, "If a light green was one third, what would be a whole?" (line 1.0.287). Alan responded that the blue rod would be called one, but there was no evidence that he justified his solution. Alan asked Erik to find the rod that would be called one if the white rod was called one fifth (lines 1.0.291). Erik said that the yellow rod would be called one. T/R 2 asked Erik to justify his solution. Erik began by showing that five white rods equaled a yellow rod in length (line 1.0.299). He then showed how he also found the solution by using the staircase model he had built on his desk and counting up to the yellow rod, which he called five (line 1.0.301). He also said, "And I know that that's half of [the orange rod], and I know that yellow is half of orange, which is ten." (line 1.0.303). Thus, Erik used three lines of direct reasoning to justify his solution.

24 T/R 2 asked Alan and Erik, "If I call this [purple rod] two, what would one look like? Which rod would one be?" (line 1.0.312). Erik replied that the red rod would be called one, and, when asked to justify his solution, explained that the red rod was half as long as the purple rod, and "half of two is one" (line 1.0.318). He then showed that two red rods equaled the purple rod in length.

Erik asked Alan to find the rod that would be called six if the white rod was called three. Alan responded but there is no evidence that he justified his solution. Alan asked Erik to find the rod that would be called one if the purple rod was called one half. Erik named the brown rod one, but there is no evidence that he justified his solution.

25 Meredith and Sarah worked together, and Meredith asked Sarah what one would be if the white rod was one seventh. Sarah lined seven white rods against the black rod and concluded directly that the black rod would be called one (line 1.0.346).

26a,b T/R 1 approached Meredith and Sarah and asked them to find a rod that they could call one sixth. Meredith used faulty direct reasoning and said that the dark green rod could be called on sixth. T/R 1 asked Meredith what she was calling one. Meredith lined up six white rods against the dark green rod and counted them (line 1.0.358). T/R 1 asked her if she meant that the dark green was called one sixth. Meredith then changed her mind and said that the "one" was one sixth of the green (line 1.0.362). Upon questioning, she said that the white rod would be called one sixth. T/R 1 asked her what the dark green rod would then be called. Sarah, and then Meredith, replied that it would be called one. With this, T/R 1 called the session to a close.

Table 4.1

Forms of Reasoning, Session 1

	Student	Lines	Type	Structure	Form	Sub-codes	WC	PR
	Task 1a: I claim that the light green rod is half as long as the dark green rod. What do you think? *Task 1b: What number name would we give the light green rod if I called the dark green rod one?*							
1.	Erin	1.0.2	Claim	Direct			✓	
	Task 2: Someone told me that the red rod is half as long as the yellow rod. What do you think?							
2.	Danielle	1.0.21-1.0.23	Claim	Indirect			✓	
	Task 3: Someone told me that the purple rod is half as long as the black rod. What do you think?							
3.	Alan	1.0.34-1.0.37	Claim	Indirect				✓
4.	Sarah, Meredith	1.0.40-1.0.43	Claim	Indirect				✓
5.	David	1.0.46	Claim	Indirect			✓	
	Task 4a: Someone told me that the red rod is one third as long as the dark green rod. What do you think?							
6.	Jackie, Michael	1.0.61-1.0.64	Claim	Direct			✓	
	Task 4b: If I called the dark green rod one, what number name would I give to the red rod?							
7.	Sarah	1.0.69-1.0.72	Claim	Direct				✓
8.	Beth	1.0.75-1.0.77	Claim	Direct			✓	
	Task 5a: Someone told me that light green is one third as long as blue. What do you think? *Task 5b: So if I call the blue rod one, what number name would I give to light green?*							
9.	Jessica	1.0.81-1.0.83	Claim	Direct			✓	
10.	Erik	1.0.103-1.0.107	Claim	Direct			✓	
11.	Michael	1.0.111	Claim	Direct			✓	
12.	Erik, Michael	1.0.137-1.0.139	Claim	Indirect		Incomplete	✓	
	Task 6: What number name would I have to give to green if I wanted red to be one?							
13a.	Erik	1.0.142-1.0.158	Claim	Direct				✓
13b	Alan	1.0.167-1.0.174	Claim	Direct			✓	
13c.	Erik	1.0.175	Claim	Direct			✓	
14.	David	1.0.177	Claim	Direct			✓	
	Task 7a: If I call brown one, what number name would I give to red? *Task 7b: Now I want to call the red rod one, what name would I give to the brown rod?*							
15	Danielle	1.0.180-1.0.182	Claim	Direct			✓	
16.	Jacquelyn	1.0.184-1.0.186	Claim	Direct			✓	
	Task 8: What would I have to call one if I want to name the white rod one half?							
17.	Graham	1.0.190	Claim	Direct			✓	
	Task S1: If the red rod is considered one fifth, what would the orange rod be? [Alan]							
18.	Graham	1.0.254-1.0.258	Claim	Direct			✓	
	Task S2: If light green is one whole, what is blue? [Beth] If blue is one, what is light green? [Mark]							
19.	Jacquelyn	1.0.266-1.0.268	Claim	Direct			✓	
	Task S3: If white is one, what is orange? [Jacqueline and Kelly] If orange is one, what is white? [T/R 1]							
20.	Erik	1.0.272-1.0.274	Claim	Direct			✓	
21.	Jacquelyn	1.0.276-1.0.278	Claim	Direct			✓	
	Task S4: If purple is one half, what is one? [Meredith]							
22.	Amy	1.0.282-1.0.284	Claim	Direct			✓	
	(Task S5: If light green was one half, what would be a whole? [Erik]) *Task S6: If white is considered one fifth, what would one be? [Alan]*							

	Student	Lines	Type	Structure	Form	Sub-codes	WC	PR
23a -c.	Erik	1.0.296-1.0.303	Claim	Direct				✓
	Task S7: If I call purple two, what would one look like? [T/R 2] *(Task S8: If white is three, what is six? [Erik])* *(Task S9: If the purple rod is one half, what would be one [Alan]?)*							
24.	Erik	1.0.313-1.0.318	Claim	Direct				✓
	(Task S10: If you called one [white rod] a seventh. What would a whole be?[Meredith]) *Task S11: I want to find a rod that has number name is one sixth. Can you find it?(T/R 1)*							
25.	Sarah	1.0.346	Claim	Direct				
26a.	Meredith	1.0.354-1.0.360	Claim	Direct		Faulty		✓
26b	Meredith, Sarah	1.0.362-1.0.371	Claim	Direct				✓

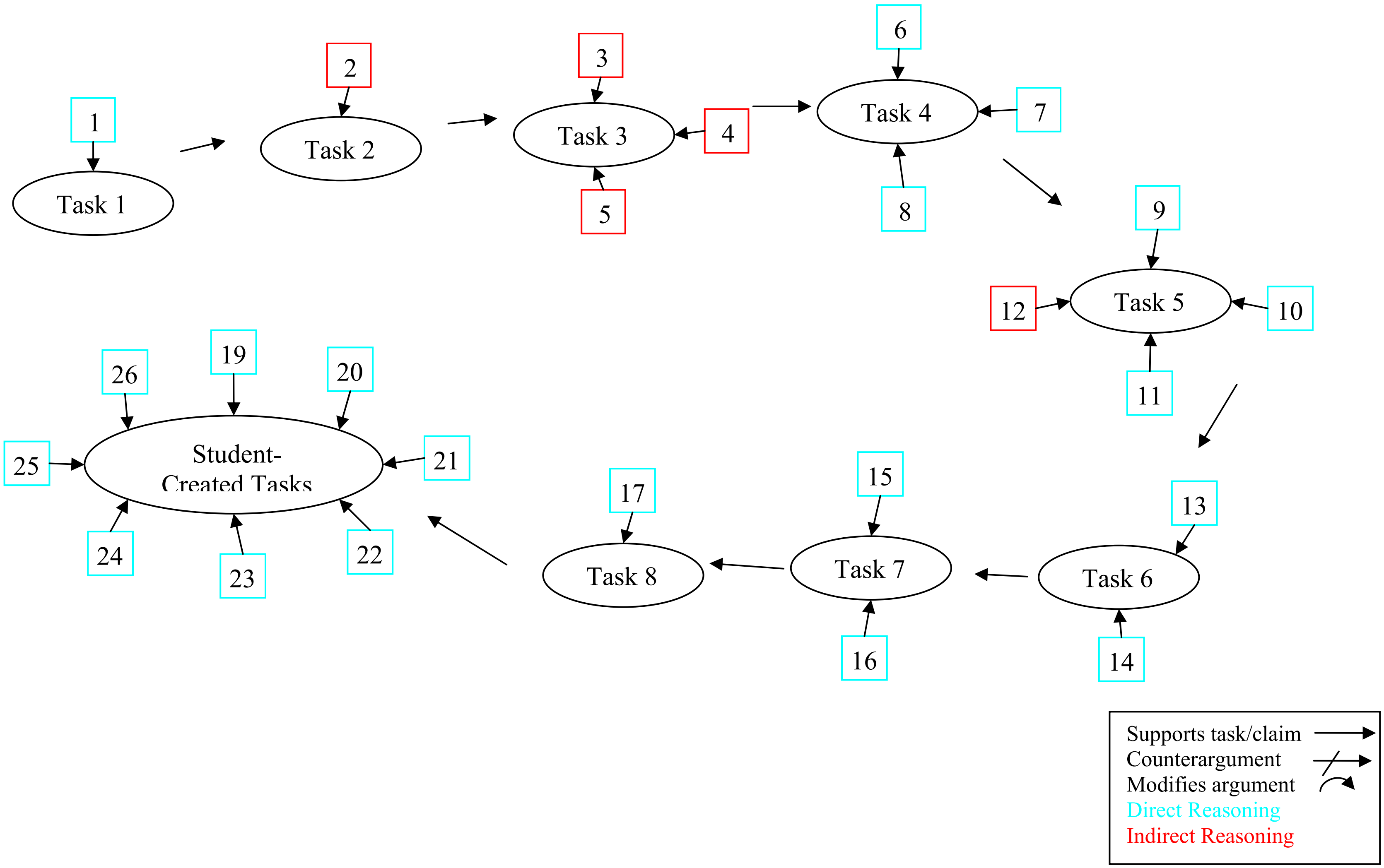


Figure 4.1. Argumentation Organized by Task, Session 1

4.2.2 Session 2: September 21, 1993

Task 1: If I call the yellow rod one half, what rod will I call one?

At the start of the session, T/R 1 asked the students to explain to a guest, Tom Purdy, what had happened the day before. Jessica and Michael provided general summaries of what had transpired, and Erik offered an example. He suggested a problem: "If we said that the blue rod would be one whole, um, we'd figure out what, we'd take all the blocks and try and figure out what would be half of it" (line 2.0.8). T/R 1 asked Erik what the solution to that problem was, and the students began to search for the correct rod. As the students tried unsuccessfully to find the rod that met Erik's specifications, a student suggested the yellow rod, but the class rejected that solution.

1 As the students tried to find the correct rod, T/R 1 posed another problem. "But suppose I wanted to call the yellow rod, I wanted to give it a number name one half. Can you tell me what I would have to call one?" (line 2.0.29). Brian2 offered a solution and explained why his solution was correct. Holding two yellow rods in his left hand and an orange rod in his right hand, he said, using direct reasoning directly, "Well, these two blocks equal up to this one whole" (line 2.0.39). Mr. Purdy asked Brian2 what the yellow rod would be called. Brian2 said that they would be called one half (lines 2.0.40-2.0.41). Due to the question posed by Mr. Purdy, Brian did not fully explain his solution the original task (which was to find the rod that would be called one when the yellow rod was called one half).

Task 2: If I call the blue rod one, what rod will I call one half?

T/R 1 then told the class that she was still worried about the problem that Erik had posed. This comment began a lively whole class discussion about this task, Task 2. First, Erik commented that he didn't think there was a rod that satisfied the conditions he had set (line 2.0.44). T/R 1 then asked David to share his thoughts with the class. David repeated Erik's comment, and, upon questioning by the researcher, began to explain why he thought no such rod existed. T/R 1 asked David to explain his thinking at the front of the room.

2 At the OHP, David placed a purple rod alongside a yellow rod. He then placed a white rod to form a purple and white train (Figure O-10-33). He then explained, "You see usually they are only one, with the shorter one, only one block apart" (line 2.0.58). He then created a train of two yellow rods and placed a blue rod next to the train, and did the same using a train of two purple rods (Figure O-11-01). He said, "[I]f you have two yellows, it would be too tall and if you have two purples, that'd be too short" (line 2.0.58). Finally, he said, "and then there's really nothing in between," lined up the rods in order of decreasing length (Figure F-11-56), pointed to the yellow and purple rods and said "And then here, there's nothing in between, right here, so there's no way that you can do that" (line 2.0.60). Using this staircase model, and showing that the yellow and purple rods could not be solutions to the task, David showed that no rod existed that was one half the length of the blue rod. This argument used indirect reasoning as well as reasoning using upper and lower bounds.


Figure O-10-33


Figure O-11-01


Figure F-11-56

105

3a, 4a Following David's argument, Jessica countered his claim that the problem could not be solved by offering a solution. Jessica used direct reasoning and explained that three greens equaled the blue rod. David countered her suggestion by explaining that "Erik wants the half" (line 2.0.63), implicitly suggesting that following Jessica's argument leads to a contradiction, since her method would produce thirds rather than halves.

5a,b Erik then countered David's claim that the rod under discussion did not exist by suggesting another solution to the problem. He said,

> I think you could do it, but they're… See, I figure if you take a yellow and a purple it's equal. They're not exactly the same, but they're both halves. Because the purple would be half of this even though the yellow is bigger because if you put the purple on the bottom and the yellow on top it's equal, so they're both halves, but only one's bigger than the other. So it equals up to the same thing.
> (Erik, line 2.0.64)

This argument used direct reasoning to show that the purple and yellow rod are equal in length to the blue rod. Erik's reasoning was faulty in that he assumed a definition for one half that was different than the mathematically correct definition. This erroneous definition was used by Erik several times during the discussion.

T/R 1 asked Erik to repeat his argument. This time, in addition to the explanation that he had already provided, he said, "Or maybe you could call this three quarters [holding the yellow rod] and you could call this one quarter [holding the purple rod]. And, but it would still equal up to the whole" (line 2.0.66). Erik's second explanation further indicated that he was defining "one half" as one of two parts which equal the whole. This argument used direct faulty reasoning as well.

6 T/R 1 asked David what he thought of Erik's argument. David replied that he hadn't thought of the problem in that way, since "I was thinking that… you would need the same" (line 2.0.68). David used indirect reasoning to counter Erik's argument. He explained that he had thought that the two halves would have to be identical, and that Erik's suggestion did not meet that condition, thereby disqualifying it as a valid solution.

7 T/R 1 then questioned Erik's definition of one half. She told Erik to imagine that the rods were bricks of gold, and asked Erik if it was a fair split to give Erik the purple rod and for her to keep the yellow rod. Erik replied that this was fair, thereby supporting his earlier claim.

8 The researcher then asked the class what they thought about this scenario. They responded in a chorus that they did not think this was a fair split. Kimberly countered Erik's claim that this was fair, explaining directly that "[T]he purple is smaller than the yellow and the person who got the yellow wouldn't have as much" (line 2.0.76).

9a Erik then modified his argument, focusing again on his version of the definition of one half to support his argument. He said, "Yeah, but you could call this three quarters and this one quarter and it would still be equal up to the whole. Then it just wouldn't be halves, it would be quarters. But it would still look like you're dividing it into halves, but you're really dividing into quarters" (line 2.0.77). With this argument, he still argued that the use of the purple and yellow train could provide a solution, but qualified his argument by explaining that it would "look like" halves, but would really be quarters. However, this reasoning still relied on his faulty understanding of what could be defined as one half.

3b, 4b At this point, Brian repeated Jessica's earlier suggestion. He said, "[Y]ou could at least split it into thirds" (line 2.0.79). David repeated his counterargument by explaining that they were trying to find halves and not thirds.

10, 11 Alan and Jessica each offered parts of an indirect argument to counter Erik's claim.

> Alan: When you're dividing things into halves, both halves have to be equal – in order to be considered a half.
> Jessica: …This isn't a half. Those two aren't both even halves.
> (lines 2.0.85-2.0.86)

9b T/R 1 asked Erik what he thought of these comments. She asked, "Can you divide things in halves and have them different sizes?" (line 2.0.91). In response, Erik repeated his modified argument. He said,

> Well, see. This isn't exactly dividing into halves. But I'm still using two blocks, but not… I'm dividing it in half still using two blocks, but one block is bigger than the other block. So it's like using three quarters and one quarter, but you're only using two blocks so it's almost like dividing it in half.
> (line 2.0.92)

This modified argument used direct faulty reasoning to explain that although the two rods weren't equal in length, the use of only two rods caused the solution to be "almost like dividing it in half".

12 Andrew then countered Erik's new argument. He said, "Well if he's saying, he's saying that he wants a half, but if he puts that, a purple and a yellow, he won't have a half. He would have three quarters and one quarter. And he wants a half" (line 2.0.94). This indirect argument showed the contradiction inherent in Erik's claim, and how that contradiction disqualified the validity of Erik's solution.

13 T/R 1 asked Alan to summarize the discussion that had taken place. Alan combined two of the arguments that had been offered during the conversation, saying, "You… can't divide that into halves, because you'd have to use rods that are of different sizes, but you could divide it into thirds using rods that are the same size which, which is the light green rods" (line 2.0.96). This summary used indirect reasoning for the first part of the argument, and direct reasoning for the second.

14 The last line of reasoning during this whole class discussion was offered by David. David used generic reasoning and reasoning by cases to show how the task could be generalized and how solutions to the general case could be found. He showed that the staircase model of rods could be reorganized into two cases, that of "even" rods and of "odd" rods. He showed that those in the "even" category were twice the length of another rod in the set, while those in the "odd" category were not. He then returned to the problem at hand, explaining that the blue rod was "odd", and that, as a result, there was no solution to the task (Figure O-17-20). By showing his solution using two different categories of rods, David used reasoning by cases, and by using the specific case of the blue rod to discuss the general properties of "odd" rods, David used generic reasoning to support his understanding of the problem.

Figure O-17-20

T/R 1 discussed Erik's position in the discussion. She asked Erik if what he meant was that he had found two rods that equaled the length of the blue rod, but that they were not the same length. Erik replied in the affirmative, and agreed that he thought two halves of one entity had to be the same size. After that clarification, T/R 1 posed a related challenge to the class.

Task 3: If you were designing a new set of rods and you wanted to call the blue rod one, can you tell me what that new rod might look like so that you would be able to call it a half?

T/R 1 asked the class to imagine that they were designing a new set of rods, and asked them what the rod that would be called one half would look like if the blue rod was called one. The students began to work on the task in their groups.

15, 16 Erik and Alan worked together on the task. Erik immediately said, “It can’t be anything ’cause you can’t divide nine equally” (line 2.0.121). Alan explained that the question was “if you could” (line 2.0.122), but Erik persisted, saying, “If this is ten [the orange rod], then this [the blue rod] is nine. It’s impossible to divide this evenly” (line 2.0.125). Thus, he used direct reasoning to establish the length of the blue rod. Alan again explained that the question asked the students to make a new rod. “You might be able to, like if you divide a blue rod in half you could that that length and make a new color and that would equal up to halves” (line 2.0.126). Alan also reasoned directly, saying, “If you cut this [the blue rod] down the middle, it would be four and a half” (line 2.0.130). Erik replied that “you can’t make a rod that’s four and a half”(line 2.0.131). He then continued his original train of thought, reasoning by contradiction that there was no rod that could be one half of the blue rod. He lined up nine white rods along the blue rod and counted from left to right, saying, “one two three four five, one two three four.” He then counted again, saying, “One two three four, one two three four five” (line 2.0.135). After showing that both counting methods did not yield two equal groups of white rods, he concluded, “It’s impossible to divide it in halves” (line 2.0.145).

17a Meanwhile, T/R 2 spoke with Sarah and Meredith about the task. She asked them to explain what the problem was and how they would solve it. Meredith reasoned directly and said that if the orange rod is ten [centimeters long], the blue rod is nine [centimeters long]. She then said that if the blue rod is split in half, each half would be four and a half [centimeters long]. She then called the purple rod four, and said that each half would be comprised of a purple rod and half a white rod (lines 2.0.174-2.0.178).

18 T/R 1 called the class together and asked Beth, Jackie, and Graham to share their solution at the OHP. Jackie, with some assistance from Graham, explained directly that if the blue rod was one, they could design a rod that was one half by using a purple rod and half a white rod. They said further that the smallest rod in the set would then be half the length of the white rod (lines 2.0.186-2.0.194).

19a-d Michael, Brian, and David extended their reasoning about the task. They reasoned recursively in an indirect manner, saying that the smallest rod in the set would then need to be divided in half as well. Michael first said, "If you're going to make a new rod, then you'd have to make a whole new set because there'd have to be a half of that rod, too" (line 2.0.199). Brian then elaborated, saying "No matter what there'll always be something that won't be equal to something, like… If you cut these little ones in half, then there wouldn't be something for the little ones to make a half out of them" (lines 2.0.204-2.0.206). David's argument was similar to Michael's. He said that he had thought that the white rod could simply be cut in half to create a rod that would be one half the length of the blue rod, but then "realized that you would have to make a whole set, and make a half for every one" (lines 2.0.211-2.0.213). Jacquelyn said that she agreed with Michael, "'Cause if you do that, um, it changes the whole pattern 'cause this has a set in pattern to it and the whole thing would change" (line 2.0.217). Jacquelyn's reasoning was incomplete.

17b Meredith offered a comment, saying, Well, you could just, if you do that then you'd have to cut the ones that are separate, the little blocks into halves, all of them, so then you could make it equal" (line 2.0.215). Although it appears that her reasoning was direct, it was incomplete, making it difficult to comprehend in context.

20 T/R 1 asked the students if anyone had another method of designing a rod that could be called one half if the blue rod was called one. James, at the OHP, showed the class a light green rod and told them to imagine another two. He used direct reasoning to explain that if the middle rod would be split in half, he would have two equal lengths that could each be called one half (line 2.0.228).

Task 4: If we call the orange 'two', what can we say about yellow?

T/R 2 began a new set of tasks with the students. Placing an orange and a yellow rod on the OHP, she asked the students what yellow would be called if the orange rod would be called two. The students worked in groups for one minute on the problem.

21, 22 Danielle explained her solution to Gregory. She lined up two yellow rods alongside an orange rod and used direct reasoning to explain that since the orange rod is called two, and that two yellow rods equal the length of the orange rod, each of the two yellow rods would be called one (line 2.0.239). Gregory then countered her argument and said that, "[w]hen the orange is one, we went like a half down" (line 2.0.240). CT asked Gregory if the orange rod was called one in the present problem. Gregory replied that it wasn't and agreed that the yellow rod would then be called one (lines 2.0.241-2.0.244). Gregory's first argument, which was a counterargument to Danielle's explanation, appears to have been both faulty and incomplete. His argument implied that in this case, one half should be subtracted from the value of the orange rod, just as it had been in the original problem. However, this was not articulated. He then used direct reasoning to agree to the correct solution.

23 Meredith, working with Sarah, said, "She called orange two. One half? Two? Then this would have to be one" (line 2.0.249). She used direct reasoning to arrive at the solution.

24 During the whole class discussion, Brian presented his solution at the OHP. He stated that the yellow rod would be called one, and explained, "You would put two yellows together and it would be the same size as that, and even if and that's like having, so if this [pointing to the orange rod] is considered a two. Then those two [pointing to the two yellow rods] would be considered like a regular orange, so it would be considered a one" (line 2.0.254). Brian also used direct reasoning to justify his solution.

25 After Brian presented his justification, Erik commented that he had another number name for the yellow rod. He asked T/R 2 if the orange rod had to be called two, and T/R 2 replied that for this problem, the orange rod was called two. Erik said, using direct reasoning, that if the orange rod were called one, each yellow rod would be called one half (lines 2.0.260-2.0.264).

Task 5: What if I change the name of the orange to 'six'… what number name would I call the yellow?

26a T/R 2 posed another problem. She called the orange rod six, and asked the class what number name the yellow would then be assigned. Kimberly suggested that it be called five. When asked to explain her reasoning, she said, "Look here [pointing to

Brian's model] before you said that [the orange rod] would equal two, and then Brian said that [yellow rod] would equal one. So now you're saying that that [orange rod] equals six, so I figured that if that equaled one before [yellow rod] it would equal five now" (line 2.0.270). Her direct reasoning was faulty in a way similar to Gregory's during the previous task.

27a,b T/R 2 asked the class if they agreed with Kimberly's argument. Alan challenged her solution, saying that in the previous task, the orange rod was two and the yellow rod was one. He then said that now, the orange rod was called six, and that half of six is three (line 2.0.272). Jessica echoed his argument, emphasizing the fact that half of six is three. Both Alan and Jessica used a direct argument to justify their solution (line 2.0.277).

26b T/R 1 mentioned that she was curious why Kimberly thought the yellow rod should be called five. Kimberly re-explained her thinking. She said, "Well, before you said that was two, the orange was two, and the yellow was one. So now you're saying it's six, so the yellow could be five" (line 2.0.279).

28, 29 T/R 1 used the rods on the overhead projector and asked Kimberly, "So you're saying if this [yellow rod] is five and this [yellow rod] is five, this [orange rod] is six?" (line 2.0.280). Kimberly then said that she had made a mistake, and T/R 1 encouraged her to explain her thinking. Kimberly said that she had forgotten that when adding, although one and one is two, five and five is not six (line 2.0.286). She used direct reasoning to counter and revise her original solution. T/R 1 then asked her what the orange rod would be called if the yellow rod was five. Kimberly used direct reasoning to explain that it would then be called ten (line 2.0.288).

Task 6: I'm going to call the orange and light green together one…Can you find a rod that has the number name one half?

T/R 2 posed a final task for the session that was similar in structure to Task 2. She called the orange and light green train one, and asked the class to find a rod that had the number name of one half. The students worked in groups for five minutes and then participated in a whole class discussion.

30 Brian built a model of an orange and light green train and placed two dark green rods below it, with a white rod separating to two dark green rods. He explained to T/R 2 that there was no rod that was one half the length of the train because "ten and three equals thirteen and thirteen is an odd number" (line 2.0.301). This part of his argument used indirect reasoning to show that the task was impossible given the rods they were provided. When asked to explain how that was connected to his solution, he explained directly that since thirteen was odd, the train couldn't be divided in half, and that the only way to find one half of the train would be to cut the white rod in half and put "put one half on one side and …put the other half on the other side" together with each of the dark green rods (lines 2.0.302-2.0.305).

31 Brian said that there was another way that the problem could be solved. He referred to the model that James had shared as they had discussed Task 2 (see item 20 in this session) and said "it would probably work" in this case as well. He built a train of four light green rods with a white rod in the center, and showed T/R 2 that this train could be divided in half by cutting the white rod in half in a manner similar to the way he had shown earlier (lines 2.0.307-2.0.311). In this case, as well, Brian used direct reasoning to show the correct method of finding a length that was one half the length of the orange and light green train.

32, 33 T/R 2 talked to Jessica and Laura about their solution. Jessica and Laura built a model of an orange and light green train alongside a train of two dark green rods and one white rod. The white rod in their model, unlike that of Brian's, was placed at the end of the train. Jessica told the researcher that they had to invent a new rod, since the dark green rod wasn't one half the length of the train. T/R 2 asked Jessica to show her what the new rod would look like. Jessica showed T/R 2 that the train of one dark green and one white rod would be called one half. At this point, Jessica's reasoning was flawed. Laura then offered her own direct argument, saying that the dark green rod along with half of one white rod would be called one half. Jessica agreed with Laura's suggestion, showing the researcher what each half would look like in her model (lines 2.0.313-2.0.325).

34 Meredith worked with Sarah on this problem. Meredith noted that the length of the train was "thirteen" (line 2.0.327), and set about looking for a rod that was "six" (line 2.0.329). After some trial and error, she found the dark green rod, and built a model identical to that of Brian's first and Jessica's second model. T/R 2 approached Meredith and Sarah and Meredith explained her solution. "I took two greens and I put a white in the middle, and if I cut the white in the middle in half, then you would have six and a half and six and a half" (line 2.0.358). Thus, Meredith used direct reasoning to provide a solution to the problem.

35, 36 During the whole class discussion, Andrew presented the model described above and used direct reasoning to show that the dark green rod and half the white rod would be equal in length to half the train of orange and light green, since two dark green rods and one white rod equaled the length of the train (line 2.0.363). T/R 2 asked the class why there was no rod that could be called one half of this train. Meredith offered an indirect argument that was partially flawed. She said, “Well, because you want to have seven and six, seven, but there are no rods that are really seven, and you need it to be thirteen.” When asked to explain her thinking further, she said that “Well, take two greens and a white… And there's no blocks that have half on them, and for the uneven numbers, for the odd numbers you need a half, because you can’t make it without it” (lines 2.0.366-2.0.370). Although Meredith was correct in explaining that the train was “odd” and that no rod in the set could be one half the length of that train, she erred in saying at first that, in order to have a rod that could be called one half, a rod was needed that was seven white rods long, and that such a rod did not exist.

37 T/R 2 asked Brian to explain what Meredith meant by “odd,” since he had also used that term in his earlier explanation. Instead of answering that question, Brian talked about the different ways that he had found a rod could be created that was the correct length to solve the problem. Brian had tried to find all the cases that could satisfy the given conditions. Figure S-47-12 shows the different models that Brian had built. Although this is not the typical argument by cases used in mathematics, Brian attempted to be exhaustive in finding solutions to the problem.

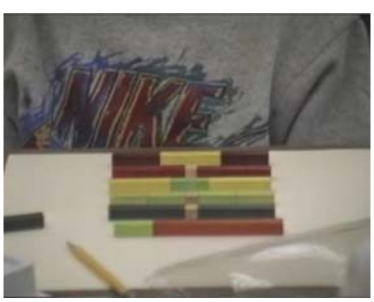


Figure S-47-12

38a,39 Meredith presented another solution to the problem (one that was built by Brian, but that had not been presented yet to the class). At the OHP, she and Sarah built a train of a yellow, light green, and yellow rod, and Meredith showed that if the light green rod is cut in half, "one and a half" could be added to each yellow rod to create a rod that could be called one half (line 2.0.374). Meredith's direct reasoning was incomplete in that she did not explain what she meant by "one and a half." T/R 2 asked the class what she meant, and Graham explained that Meredith was referring to the light green rod, and that "[y]ou… split it in the middle, and it would be one and a half on each side" (line 2.0.378). He raised the light green rod and gestured that it was cut in half.

38b T/R 2 asked Meredith what the orange and light green train would be called if the light green rod was three, as Graham and Meredith had implied. Meredith responded that it would be called thirteen. T/R 2 asked Meredith, "You were thinking of the whole length of the train as being thirteen of what? Thirteen blues, thirteen oranges, thirteen what?" (lines 2.0.281-2.0.283). Meredith said that she was referring to thirteen yellows. She said, "Well, if you cut that [light green rod] in the middle and then you just paint the light green of each piece yellow and you're making it thirteen and it will be equal to the train" (lines 2.0.384-2.0.388). Meredith's reasoning was still incomplete.

40 T/R 2 asked the class for assistance in explaining what Meredith meant. Erik volunteered that he thought he understood what she was trying to say. Erik explained that ten white rods equaled the length of the orange rod, and three rods equaled the length of the light green rod. He then reasoned directly that ten and three gives thirteen (line 2.0.393). As Erik spoke, Meredith modeled the equivalence of the light green rod and three white rods.

41, 42 Erik mentioned that he had another solution that he wanted to share with the class. At the OHP, he built a train of two light green rods and seven white rods. Then, he said,

> I figured you could take two light greens and put them there. And then after that I just took all these, the clear ones, and I figured, well, I put down seven. And I figured that they all equal, and if you have these two you would have three and then you could take one and put it on that and so it would be four, five, you would have three, four, and then four, five, five, six, six, and then seven.
>
> (line 2.0.397)

As Erik spoke, he motioned that he was assigning each of the white rods to one of two groups, and that each of the groups were comprised of a light green rod and three or four white rods. Upon questioning, he clarified that there would be a length of six white rods in one group and seven white rods in the other (line 2.0.401, see Figure O-53-49). T/R 2 then asked him what he would do with the extra white rod in the second group. At that, Erik rearranged his model, substituting three white rods for one light green rod. He then divided the light green and white rods into three groups, each containing a light green and a white rod, with the exception of one group, which contained a light green and two white rods (line 2.0.403, Figure O-54-20). However, T/R 2 asked him what he would do with the fourth white rod, and then asked him if he would agree that if they reverted to his first model of two light green rods and seven white rods, they could split the remaining white rod in half. Erik agreed that that could be done, and extended the researcher's line of reasoning, explaining that each half of the white rod could be placed in one of the two groups of rods (line 2.0.406-2.0.409). In this discussion, Erik used faulty reasoning, as well as direct reasoning as he agreed with T/R 2's suggestion.


Figure O-53-49


Figure O-54-20

43 At the close of the session, Brian2 shared an alternative solution. At his desk, he built a model of a purple rod, a yellow rod, and a second purple rod, and used direct reasoning to explain that if the yellow rod was cut in half, two equal length rods could be formed (line 2.0.411).

Table 4.2

Forms of Reasoning, Session 2

	Student	Lines	Type	Structure	Form	Sub-codes	WC	PR
	Task 1: If I call the yellow rod one half, what rod will I call one?							
1.	Brian2	2.0.39-2.0.41	Claim	Direct			✓	
	Task 2: If I call the blue rod one, what rod will I call one half?							
2.	David	2.0.48-2.0.60	Claim	Indirect	U/L		✓	
3a.	Jessica	2.0.62	Counter-2	Direct		Faulty	✓	
4a.	David	2.0.63	Counter-3	Indirect			✓	
5a-b.	Erik	2.0.64 2.0.66	Counter-2, Claim	Direct		Faulty	✓	
6.	David	2.0.68	Counter-5	Indirect			✓	
7.	Erik	2.0.71-2.0.73	Counter-6	Direct		Faulty	✓	
8.	Kimberly	2.0.74-2.0.76	Counter-7	Direct			✓	
9a.	Erik	2.0.77	Counter-8	Direct		Faulty	✓	
3b.	Brian	2.0.79	Counter-2	Direct			✓	
4b.	David	2.0.80-2.0.83	Counter-3b	Indirect			✓	
10.	Alan	2.0.85	Counter-9a	Indirect			✓	
11.	Jessica	2.0.86	Counter-9a	Indirect			✓	
9b.	Erik	2.0.92	Counter-10,11	Direct		Faulty	✓	
12.	Andrew	2.0.94	Counter-9b	Indirect			✓	
13.	Alan	2.0.96	Claim	Indirect			✓	
14.	David	2.0.99	Claim	Direct	Gener. Cases		✓	
	Task 3: If you were designing a new set of rods and you wanted to call the blue rod one, okay? Can you tell me what that new rod might look like so that you would be able to call it a half?							
15	Erik	2.0.121-2.0.135	Claim	Direct, Indirect		Faulty		✓
16.	Alan	2.0.121-2.0.135	Counter-15	Direct		Incomplete		✓
17a.	Meredith	2.0.170-2.0.178	Claim	Direct				✓
18.	Jackie, Graham	2.0.184-2.0.194	Claim	Direct			✓	
19a.	Michael	2.0.199	Claim	Indirect	Recur.		✓	
19b.	Brian	2.0.202-2.0.209	Claim	Indirect	Recur.		✓	
19c.	David	2.0.211-2.0.213	Claim	Indirect	Recur.		✓	
19d.	Jacquelyn	2.0.217	Claim	Direct		Incomplete	✓	
17b.	Meredith	2.0.215	Claim	Direct			✓	
20.	James	2.0.228	Claim	Direct			✓	
	Task 4: If we call the orange 'two', what can we say about yellow?							
21.	Danielle	2.0.237-2.0.244	Claim	Direct				✓
22.	Gregory	2.0.241	Counter-21	Indirect		Faulty, Inc.		✓
23.	Meredith	2.0.249	Claim	Direct				✓
24.	Brian	2.0.251-2.0.258	Claim	Direct			✓	
25.	Erik	2.0.264-2.0.268	Claim	Direct			✓	
	Task 5: What if I change the name of the orange to 'six'…what number name would I call the yellow?							
26a.	Kimberly	2.0.268-2.0.270	Claim	Direct		Faulty	✓	
27a.	Alan	2.0.272	Counter-26a	Direct			✓	
27b.	Jessica	2.0.276-2.0.277	Counter-26a	Direct		Incomplete	✓	

	Student	Lines	Type	Structure	Form	Sub-codes	WC	PR
26b.	Kimberly	2.0.279	Claim	Direct		Faulty	✓	
28.	Kimberly	2.0.280-2.0.286	Counter-25a	Indirect			✓	
29.	Kimberly	2.0.287-2.0.289	Claim	Direct			✓	
	Task 6: I'm going to call the orange and light green together one…Can you find a rod that has the number name one half?							
30.	Brian	2.0.299-2.0.305	Claim	Indirect, Direct				✓
31.	Brian	2.0.307-2.0.311	Claim	Direct				✓
32.	Jessica	2.0.313-2.0.315	Claim	Direct		Faulty		✓
33.	Laura, Jessica	2.0.316-2.0.325	Counter-32	Direct				✓
34.	Meredith, Sarah	2.0.357-2.0.361	Claim	Indirect				✓
35.	Andrew	2.0.363	Claim	Direct			✓	
36.	Meredith	2.0.366-2.0.370	Claim	Indirect			✓	
37.	Brian	2.0.372-2.0.373	Claim	Direct	Cases		✓	
38a.	Meredith	2.0.374	Claim	Direct		Incomplete	✓	
39.	Graham	2.0.376-2.0.378	Claim	Direct			✓	
38b.	Meredith	2.0.380-2.0.388	Claim	Direct		Incomplete	✓	
40.	Erik	2.0.390-2.0.395	Claim	Direct			✓	
41a.	Erik	2.0.397-2.0.401	Claim	Direct		Faulty	✓	
41b.	Erik	2.0.403	Claim	Direct	Cases	Faulty	✓	
42.	Erik	2.0.406-2.0.409	Claim	Direct			✓	
43.	Brian2	2.0.411	Claim	Direct			✓	

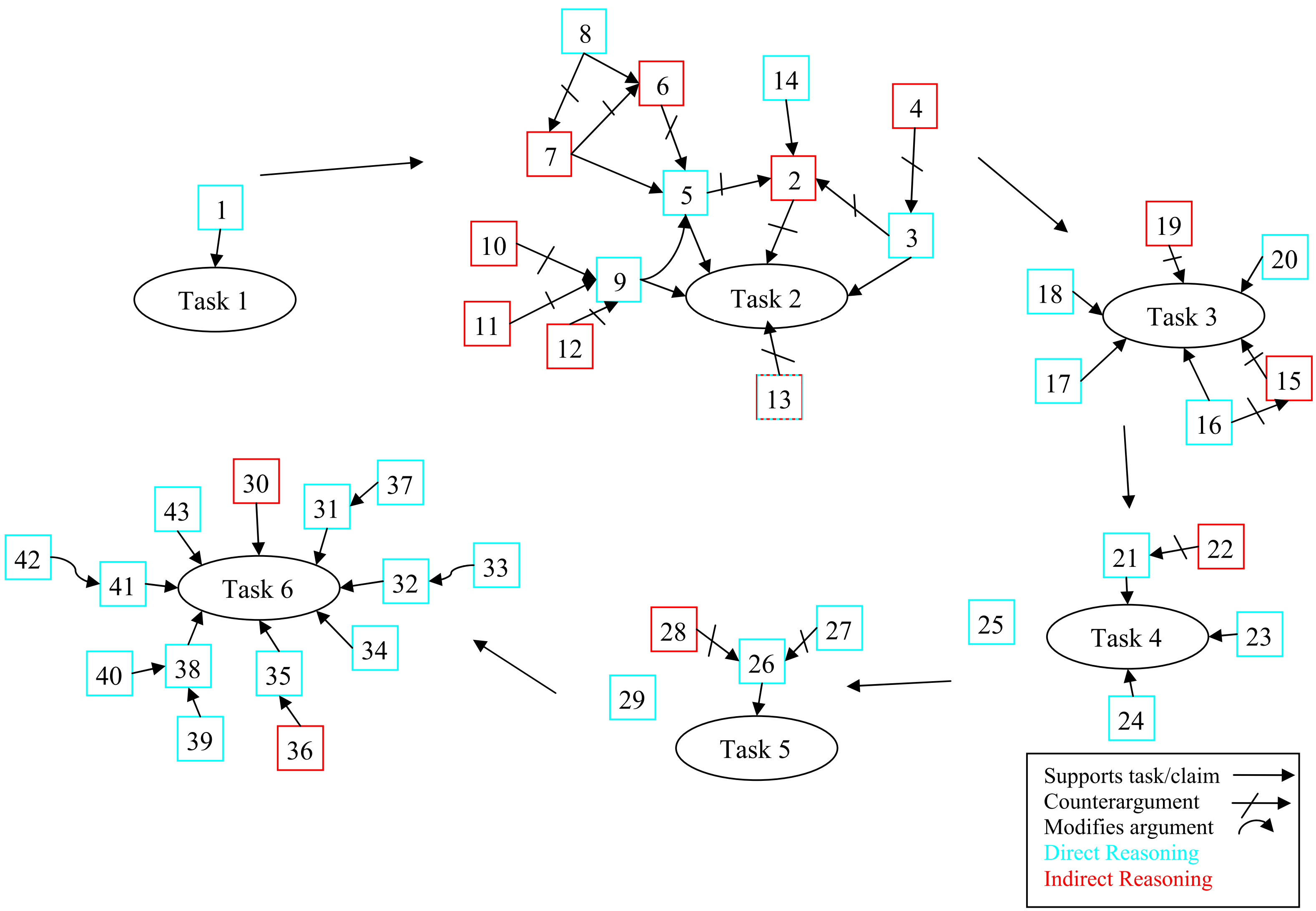


Figure 4.2. Argumentation Organized by Task, Session 2

4.2.3 Session 3: September 24, 1993

Task 1: If I gave the purple the number name one half, what number name would I give to the dark brown?

1 At the start of the session, T/R 1 asked the students to solve tasks similar to the ones that they had worked on earlier in the week. T/R 1 asked Audra to tell her what the brown rod would be called if the purple rod was called one half. Audra spoke with her partner, Sarah, and then explained directly that the brown rod would be called one if the purple rod would be called one half.

Task 2: What if I gave the purple rod the number name one? What number name would I give to the brown rod?

2 Then, T/R 1 posed another problem. She asked the students to tell her what the brown rod would be called if the purple rod was called two. Laura responded that it would be called two, and she explained her solution using direct reasoning. She held up two purple rods and showed that they equal the length of the brown rod, and said that each purple rod was called one. She counted the purple rods, saying, "[O]ne, two" (line 3.0.17) and concluded that the brown rod would then be called two as well.

Task 3: If we give the orange rod the number name two, can you tell me what number name we'd give to yellow?

3 T/R 1 then asked the class what the yellow rod would be called if the orange rod had the number name two. Jacquelyn replied that it would be called one. She said, "Because… if you take… two yellows and put them against the orange, they match up. And if, if this [the yellow rod] is one half of it… if orange is two, you would make this [yellow rod] into a one" (line 3.0.21). Her justification used direct reasoning quite similar to that of Audra and Laura above.

Task 4: If I call yellow and light green two, what number name would I give to red?

Task 5: I'd like you to make the yellow and light green one and then tell me what the number name would be for red also.

4,5a,b T/R 1 then posed a challenge. She asked the students to name the red rod if a train of yellow and light green was called two. Alan immediately built a model (see Figure F-13-22) of a brown rod, a yellow and light green train, and four red rods and raised his hand. He and Erik then discussed the solution.

Alan:	It would still be one half. Because
Erik:	If this is two
Alan:	Right. This is two. This would make one fourth
Erik:	One fourth
Alan:	This would make two
Erik:	One fourth. This is one fourth
Alan:	This would be one half of one, one half of two.
Erik:	No. It's one fourth. This is not a half of this. Two reds are a half
Alan:	Ok [mutters something inaudible]
Erik:	It takes four reds to make this. Two reds are half two reds are half. One red is one fourth.

(lines 3.0.24-3.0.33)

During this exchange, Alan used direct reasoning to show that if the yellow and light green train was two, the red rod would be called one half. Erik used faulty indirect reasoning to counter Alan's claim, saying that since two red rods are one half the length of the yellow and light green train, it was impossible to call one red rod one half as well. Erik concluded, using faulty direct reasoning, that one red rod would be called one fourth if two red rods were called one half.

Figure F-13-22

6a T/R 1 then posed a related question to the class. She asked the students to name the red rod if the yellow and light green train was called one. As soon as T/R 1 posed the problem, Alan replied that it would be one fourth.

Erik:	If they were one, what would you call the red?
T/R 1:	Yeah. And what if it's two what do you call the red?
Erik:	If it's two, you call the red [pause] one- ohh wait!
T/R 1:	Did you change your mind, boys? Hmmm. [to class] Let's see if you fall into the trap.
Erik:	Two. This would be one.
Alan:	Right
Erik:	Two of them [the red rods] would be one
Alan:	Right
Erik:	So this one would be a half.
Alan:	Right. One half would be red. And if this was one, this would be one fourth. One half and one fourth.
Erik:	[pause] Yeah.

(lines 3.0.40-3.0.50)

In this exchange, Erik realized that since the red rod would be called one fourth if the train was one, the original problem had a different solution. He reasoned directly, saying that if two red rods was called one, one red rod would be called one half. Alan repeated his original solution, and Erik agreed.

7a Jacquelyn and Brian2 worked together on this task. Jacquelyn built two identical models to represent the problems, while Brian2 built only one model (see Figure S-15-08). She told Brian2 that if the yellow and light green train was one, the red rod would be one fourth, and if the train was two, the red rod would be one half (lines 3.0.53-3.0.59). Although she reasoned directly using the rods, her justification was incomplete.

Figure S-15-08

8 During the whole class discussion, Sarah and Audra presented their solution to the class. Audra explained that they called the red rod one fourth when the train was called one, and one and one fourth when the train was called two. Audra did not provide an explanation as to the red rod was called one and one fourth, so although her reasoning was faulty, her justification was incomplete.

9 T/R 1 asked Audra to explain her other solution. Audra built a model of an orange and green train, and showed that the brown rod equaled the length of the train. She lined up four red rods against the train and said that normally, if the brown was called one, the number name for the red rod could be found by lining them up against the brown rod. She said that the same could be done for the train, since it was the same length as the brown rod, and concluded that the red rod would be called one fourth (line 3.0.82). Thus, she used direct reasoning to justify her solution.

6b Erik then said that he agreed with Audra's explanation. He said, "If the brown and the yellow and green they're equal and they're both called one, and four of the reds equal up to one, therefore that they'd have to be fourths, because there are four parts, they're fourths" (line 3.0.86). He used direct reasoning to justify the solution.

7b Brian2 told the class that he disagreed with Audra's second solution. He and Jacquelyn presented their solution at the OHP. He said that the red rod would be called one half if the train was called two. When asked by T/R 1 to justify his solution, he said that he wasn't able to do so. T/R 1 asked David to justify the solution of one half, since he had found the same.

10a,b David, at the OHP, explained that if the yellow and light green train was called two, the train of four red rods would also be called two. He then reasoned directly, saying that two red rods would then be called one, and divided the red rods into two groups of two (Figure O-28-10). He then removed a red rod from one of the groups and said "But then if you take away this [one red rod] this would be one half over there" (line 3.0.112). Then, he worked backwards, saying that if another red rod was added, the two red rods would then be called one, and that all four red rods together would be called two. Upon request by T/R 1, he repeated this explanation in a similar fashion.

Figure O-28-10

7c T/R 1 then asked Jacquelyn and Brian2 to restate what David had said. Brian explained, with some assistance from Jacquelyn, that each group of two red rods would be one half the length of the train, and would therefore be called one. He then showed that if another red rod was removed from the group of two red rods, the remaining rod would be called one half (lines 3.0.125-3.0.127). With this, Brian2 and Jacquelyn provided a complete direct justification of their original solution.

11 T/R 1 asked the class to think about what was difficult about the problem. Erik explained that the confusion resulted from forgetting that the train was called two rather than one. He then used direct reasoning to further justify the correct solution. “Because, see, if you have one there’d be two halves, but if you have two its two halves plus two halves which would be four halves. So you’d have- therefore, you'd have to call one of the reds one half” (line 3.0.140).

Task 6: Candy Bars and Pizzas

T/R 1 then led a whole class discussion. She told the class that she gave T/R 2 and Mr. Purdy half a chocolate bar each, and that Amy said that what she had done was unfair. After asking the class what she might have done that was unfair, she showed the class that she gave Mr. Purdy half of a large chocolate bar, while giving Amy half of a much smaller chocolate bar (lines 3.0.142-3.0.161).

12 The class then discussed how that related to the problems they had been working on. T/R 1 asked Jackie to show her how much one half of the three by four scored candy bar would be. Jackie motioned that the candy bar could be cut widthwise in half. T/R 1 asked the class why she hadn’t shown that it can be cut lengthwise in half. Graham replied that there are only three rows, and that four rows would be needed to cut the bar in half. Graham used an indirect argument to explain why Jackie had acted as she did (lines 3.0.162-3.0.171).

13 T/R 1 then asked Gregory how much one third of the candy bar would be. Gregory replied that one row would be one third of the candy bar. He then used direct reasoning and said that one third of the candy bar would be four pieces of chocolate (3.0.178-3.0.180).

T/R 1 then asked the students which was larger, one half or one third of the chocolate bar. The students replied that one half was larger. She asked the students to compare one half of the small bar and one third of the larger one, and they agreed that in that case, one third of the larger bar would be more than one half of the smaller bar. T/R 1 then told the class that this was an unfair thing that shouldn't be done in mathematics, and that they should establish a rule that candy bars, or what is called one, cannot be switched within a problem. After this discussion, T/R posed another task.

Task 7: Which is bigger, one half or one third, and by how much?

T/R 1 asked the class to compare one half and one third and determine which is larger and by how much. The class worked on this problem for the remaining ten minutes of the session.

14a Brian2 built two models to show the comparison of one third and one half. His first model was composed of three red rods and three light green rods, and the second one was composed of three purple rods and two dark green rods. He used direct reasoning to show that the two trains were equal in length, and that the purple rods could be called one third and the dark green rods could be called one half.

15 As Brian2 and Jacquelyn waited to share their model with the researcher, Brian2 decided to try to find other ways to build a model to show thirds and halves. He attempted to find other rods that would fit the conditions necessary to produce halves and thirds in one model. The two began to systematically try to find rods that would work, beginning with two orange rods and continuing with smaller and smaller rod lengths to find a match. Although there is no evidence that they tried all cases, they used an exhaustive approach to find all cases of models that could be built using the rods available (lines 3.0.248-3.0.257).

14b T/R 1 questioned Brian2 and Jacquelyn about their models. Brian2 used direct reasoning to explain that the purple rod was called one third and that the dark green rod was called one half, and that one half was larger than one third (lines 3.0.259-3.0.269).

16 T/R 1 then asked Brian2 how much larger one half was than one third. Brian showed, using his model, that the difference between one half and one third was the length of one red rod. T/R 1 asked him what number name the red rod should be given, and he replied that it would be called one fourth. This argument was faulty. When asked to prove that it was one fourth, Brian2 lined up six red rods against his original model, and told the researcher that he had changed his mind. However, his reasoning was incomplete in that he did not provide the researcher with a correct number name for the red rod (3.0.271-3.0.287).

17 During a brief whole class discussion, Laura and Jessica shared their model of an orange and red train, three purple rods, and two dark green rods (Figure O-57-26). Jessica explained that one half was larger than one third by the length of a red rod (line 3.0.294).

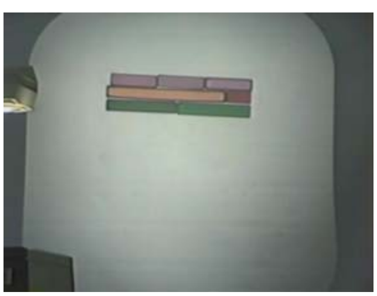


Figure O-57-26

18 T/R 1 asked the class what number name they would assign the red rod in Jessica's model. Alan suggested that it should be called one sixth. He justified his solution using direct reasoning and said, "Because we know already that… three reds would make a dark green and if there are two dark greens to make the orange… and the red rod then it would take six red rods to make the orange and the red rod" (line 3.0.298). With that, T/R 1 closed the discussion by noting that this was a point that would have to be explored further and that they would continue working on this problem during the next session.

19a As the students were dispersing after class, David called over T/R 1 to share his model. He built a balance beam using one vertical and one horizontal rod, placed two light green rods on end of the balance and three red rods on the other end of the balance. He told the researcher that he believed the balance would tip to the side of the light green rods when one light green rod and two red rods were removed. He also pointed out that the light green rod represented one half and the red rod represented one third (line 3.0.302). David used direct reasoning to explain his model and solution.

19b T/R 1 called over T/R 2 so that David could share his model with her as well. The other students also began to gather around David's desk to listen to his explanation. David rebuilt the model (Figure S-01-02-06) and explained,

> All right, I made a balance and the whole thing is dark green and the light green is a half and the reds are the thirds, but then what I'm doing is, um, I'm making a balance so when I take off that [one light green rod] and those two reds, then I think it will fall to this side and show that the half is bigger
>
> (line 3.0.306,)

He then demonstrated, and the experiment verified his prediction.

Figure S-01-02-06

20 T/R 1 asked David what rod he would put on the side of the red rod to ensure that it wouldn't tip. David replied that he would put a white rod on the red rod so that the balance would remain stable. When asked to explain why he chose the white rod, he showed a flat model of a train of a red rod and a white rod alongside the light green rod, and showed that the two lengths were equal using direct reasoning (lines 3.0.318-3.0.320). T/R 1 then asked the students what they would call the white rod, and encouraged them to think about the problem over the weekend.

Table 4.3

Forms of Reasoning, Session 3

	Student	Lines	Type	Structure	Form	Sub-codes	WC	PR
	Task 1: If I gave the purple the number name one half, what number name would I give to the dark brown							
1.	Audra	3.0.5-3.0.9	Claim	Direct			✓	
	Task 2: What if I gave the purple rod the number name one? What number name would I give to the brown rod?							
2.	Laura	3.0.11-3.0.17	Claim	Direct			✓	
	Task 3: If we give the orange rod the number name two, can you tell me what number name we'd give to yellow?							
3.	Jacquelyn	3.0.19-3.0.21	Claim	Direct			✓	
	Task 4: If I call yellow and light green two, what number name would I give to red? *Task 5: I'd like you to make the yellow and light green one and then tell me what the number name would be for red also.*							
4.	Alan	3.0.24-3.0.30	Claim	Direct				✓
5a.	Erik	3.0.25-3.0.33	Counter-4	Indirect		Faulty		✓
5b.	Erik	3.0.25-3.0.33	Claim	Direct		Faulty		✓
6a.	Erik, Alan	3.0.39-3.0.50	Claim	Direct				✓
7a.	Jacquelyn	3.0.53-3.0.59	Claim	Direct		Incomplete		✓
8.	Audra	3.0.73-3.0.75	Claim	Direct		Faulty, Inc.	✓	
9.	Audra	3.0.80-3.0.82	Claim	Direct			✓	
6b.	Erik	3.0.86	Claim	Direct			✓	
7b.	Brian2, Jacquelyn	3.0.96-3.0.109	Counter-8	Direct		Incomplete	✓	
10a, b.	David	3.0.112 3.0.117	Claim	Direct			✓	
7c	Brian2, Jacquelyn	3.0.125-3.0.127	Claim	Direct			✓	
11.	Erik	3.0.136-3.0.140	Claim	Direct			✓	
	Task 6: Candy Bars and Pizzas							
12.	Graham	3.0.169-3.0.171	Claim	Indirect			✓	
13.	Gregory	3.0.178-3.0.180	Claim	Direct			✓	
	Task 7: Which is bigger, one half or one third, and by how much?							
14a.	Brian2	3.0.246	Claim	Direct				✓
15.	Brian2, Jacquelyn	3.0.248-3.0.257	Claim	Direct	Cases?	Inc.		✓
14b.	Brian2, Jacquelyn	3.0.259-3.0.270	Claim	Direct				✓
16.	Brian2, Jacquelyn	3.0.271-3.0.287	Claim	Direct		Faulty, Incomplete		✓
17.	Jessica	3.0.290-3.0.294	Claim	Direct			✓	
18.	Alan	3.0.296-3.0.298	Claim	Direct			✓	
19a.	David	3.0.302	Claim	Direct				✓
19b.	David	3.0.306	Claim	Direct			✓	
20.	David	3.0.315-3.0.320	Claim	Direct			✓	

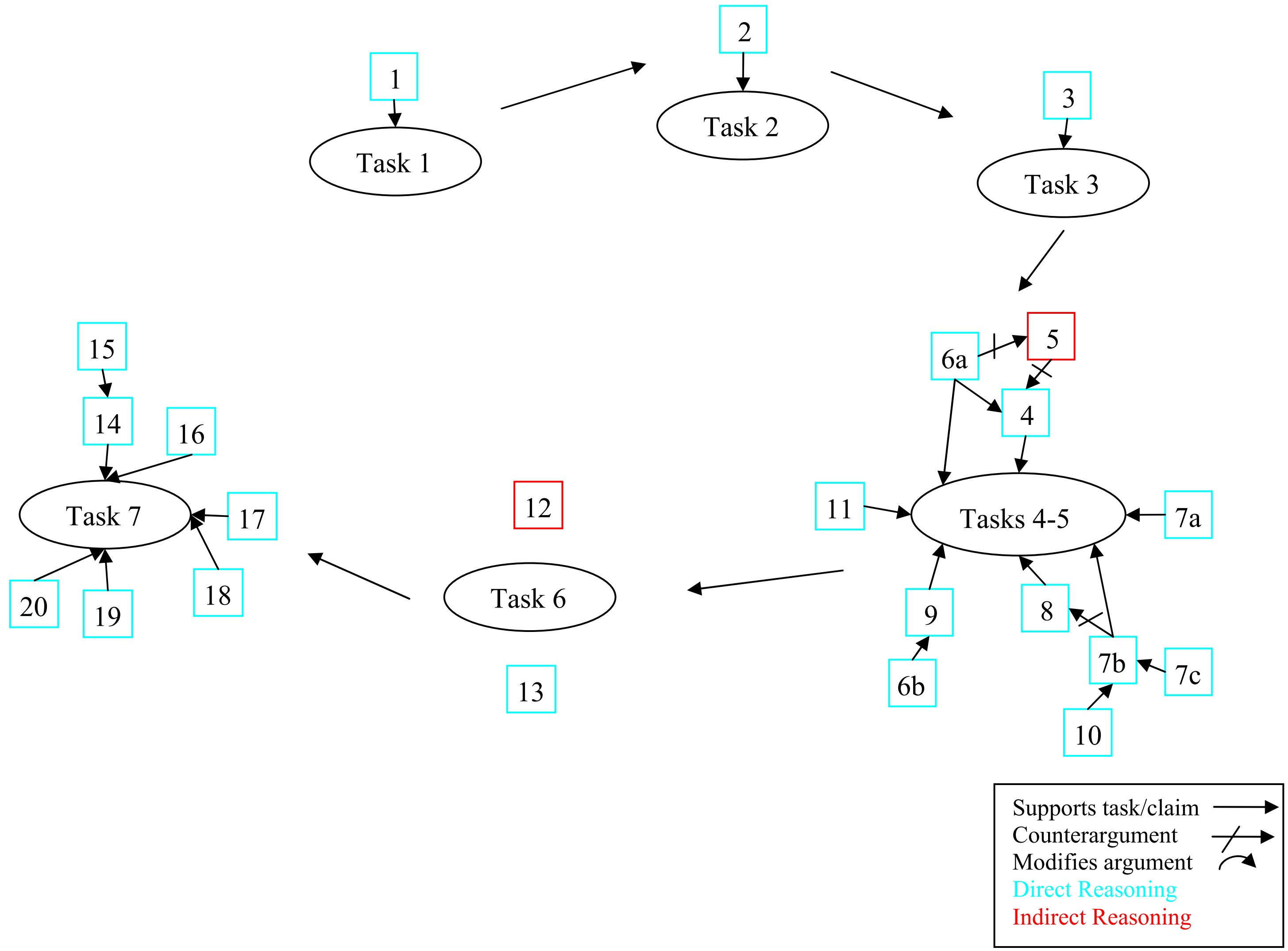


Figure 4.3. Argumentation Organized by Task, Session 3

4.2.4 Session 4: September 27, 1993

Task 1: If I give this orange rod a number name one, what number name would I give to white?

1 The fourth session was led by T/R 2. After posing the first task, which asked the students to name the white rod if the orange rod was named one, the students worked in their groups to find the correct number name. Brian explained to T/R 2, using direct reasoning, that if the orange rod is called ten, the blue rod would be called nine, and that the white rod would be called one. He concluded that in this problem, the white rod would be called one tenth (line 4.0.2).

2 During the whole class discussion, Kimberly said that she would call the white rod one tenth, and explained directly that ten white rods equaled the length of the orange rod (lines 4.0.4-4.0.6).

Task 2: If we are calling the orange rod the number name one, what would you call the number name for the red rod?

3 T/R 2 asked the class what the red rod would be called if the orange rod was called one. Gregory replied that it would be called one fifth, and justified his solution directly by saying that five red rods equal the length of the orange rod (lines 4.0.10-4.0.14).

Task 3: I'm calling the orange one, what number name would I give to two whites?

4 T/R 2 asked the class what two white rods would be called if the orange rod was called one. Mark called them one fifth, and when asked to explain how he arrived at this solution, he and Andrew built a model of an orange rod, a red rod, and two white rods. Andrew explained, "[T]he two whites equal up to the red, so, so if you put the reds and line them up to the orange, it makes one fifth" (line 4.0.21). Thus, Mark and Andrew used direct reasoning to justify their solution.

5a,b T/R 2 asked the class if they agreed with Mark and Andrew. Audra explained why she thought that one fifth was an appropriate number name for two white rods. She said, "Well, if you put two whites against the red rod, and you put two whites up against the red rod five times across it, you'd get the same amount of whites as would reds, so, and you got five reds before so you get five pairs of whites" (line 4.0.23). T/R 2 asked her to explain what she meant when she said that there would be the same number of whites as red. Audra reworded her direct justification, saying,

> If you put all the whites you could up against it [the orange rod] and you, and you double them up… If you put two whites together to make one block, it would be a red block. And if you did that as many times as you could you'd get five times."
> (line 4.0.29)

6 T/R 2 asked the students if anyone had a different solution. Meredith volunteered that she had called the two white rods two tenths. She built a model of an orange rod and ten white rods and said that since ten white rods equal the length of an orange. She then concluded that "two of ten is two tenths" (line 4.0.33). Meredith used direct reasoning to arrive at her solution.

7 Beth and Sarah were also asked by T/R 2 to explain their solution, which had been two tenths as well. Beth said that "since ten of these [white rods] all equal up to one orange, then if you took two of these it'd be two tenths because one would be one tenth and you just count one more and then you'd have two tenths" (line 4.0.35). Using direct reasoning, Beth said that one tenth and another tenth would be two tenths.

8a,b T/R 2 told the class that she was confused, because she believed the solution offered by Mark and Andrew as well as that offered by Meredith and Beth. Brian argued in opposition of Mark and Andrew's solution, saying, "Well even that, even that two white cubes equal up to one red cube, it's still not, it's still not, like um, like um imagining that this was another red cube so I think it really is two tenths because it really, really is two tenths" (line 4.0.39). Reasoning indirectly, he said that the two whites really look like two tenths and not one fifth. Meredith echoed his sentiments, saying, "Because there's two, there's only two, there's only two, there's not like, they're not joined together. If you want to join them together, you should use the red" (line 4.0.43). T/R 2 suggested that they leave the controversy for later discussion.

Task 4: I'm going to call the orange ten…and I'm wondering if you could tell me the number name for white.

9, 10 T/R 2 asked the students to name the white rod if the orange rod was called ten. The students worked on this problem in groups before joining in a whole class discussion. Meredith shared with T/R 2 that she believed that the white rod would be called one. She used direct reasoning, saying that since ten white rods equal one orange rod, each white rod would be called one if the orange rod was called ten. David agreed with her, using similar reasoning (lines 4.0.49-4.0.51).

11-14 Brian explained to T/R 2, after some discussion about the name of the orange rod, that the white rod would be called one (lines 4.0.53-4.0.68). Jessica and Beth used direct reasoning quite similar to that of Meredith and David as they explained their solutions to T/R 2 (lines 4.0.69-4.0.75). After Beth justified her solution, T/R 2 asked Beth what the number name for two white rods would be if the orange rod was called ten. Beth answered that they would be called two, and continued her logic by pointing to the next four white rods in succession, saying, "And then three and four and five and six" (line

4.0.77). Beth reasoned recursively, and extended her naming of the rods to include other numbers of white rods.

15, 16 Meanwhile, Alan and Erik discussed their ideas with Dr. Landis. Alan claimed that the white rod would be called one tenth, and said that this was because ten white rods equaled the orange rod (line 4.0.81). Erik countered Alan's faulty reasoning, saying that the white rod would be called one, since ten white rods equaled the orange rod (lines 4.0.83-4.0.95).

17 During a whole class discussion, Jackie offered a solution and direct justification that echoed many of the other students' reasoning. She explained that the white would be called one because ten white rods equal an orange rod (lines 4.0.100-4.0.106).

Task 5: If I take the same orange rod we've been working with, but I change the number name again. This time I'd like to call it… fifty. I'm wondering if anybody could tell me the number name for yellow.

18-20 The students were then asked to name the yellow rod if the orange rod was called fifty. Jessica explained to T/R 2 that it would be called twenty-five because half of fifty equaled twenty-five. She showed her model of two yellow rods and one orange rod and said that she used the model to see how many yellow rods were equivalent to the length of the orange rod (lines 4.0.113-4.0.117). David used the same direct reasoning as Jessica did to justify his solution, but also extended his reasoning, saying that if the orange rod were called one hundred, the yellow rod would be called fifty (lines 4.0.120-4.0.122). Alan, as well, justified his solution directly by saying that half of fifty was twenty-five (lines 4.0.134, 4.0.146).

21 Jacquelyn reasoned directly about the problem as well. She said, "Well, to make it even, if we had fifty cents, we have two quarters, we take half, um, fifty cents this would be twenty-five and twenty-five" (line 4.0.150).

22 T/R 2 then asked the class to share their solutions with each other. Michael used direct reasoning that was slightly different than that of the others.

> Michael: Well, I think I would call the yellow "twenty five" because twenty five plus twenty five equals fifty.
> T/R 2: That sounds, that sounds interesting. Okay, so what is twenty five and twenty five, what made you decide on using twenty five? Umm, why didn't you say ten plus ten plus ten plus ten plus ten equals fifty?
> Michael: Because it takes two yellow rods to equal one orange rod.
> (lines 4.0.154-4.0.156)

23 Beth then explained how she and Sarah had thought about the problem. She used indirect reasoning, saying, "[F]irst we thought it'd be umm twenty and thirty, but we knew we couldn't do that because they were exactly the same size, the yellows" (lines 4.0.158). She then explained that they had realized that two quarters equaled fifty cents, and that twenty-five could then be the solution.

Task 6: What number would I give to one white rod if we're still calling the orange the number name fifty?

24a,25 The students then worked to find the number name for the white rod if the orange rod was called fifty. Immediately, Sarah and Beth raised their hand and shared with T/R 2, as the other students worked in their groups, that the white rod would be called five. Sarah explained directly that if the white rods were counted by fives, ten white rods would amount to fifty (lines 4.0.163-4.0.165). T/R 2 asked Sarah why she didn't count by twos or tens. Sarah reasoned indirectly, saying that if they were to count by tens, only five white rods would be needed to reach fifty, and that if they would count by twos, "two, four, six, eight, ten, and it would go on and be more than ten" (line 4.0.167).

26a,27 Meredith explained to T/R 2 that she called the white rod five because "five times ten equals fifty" (line 4.0.171). Laura, also using direct reasoning, explained that she counted the ten white rods by five and reached fifty (line 4.0.181).

28 Brian used direct faulty direct reasoning to justify his solution. He called the white rod one fiftieth and explained that there are "if there are fifty of them in there, I guess I just call it one fiftieth" (line 4.0.199). When questioned by the researcher if there are fifty white rods, Brian replied in the negative, but said "but if we're calling this [orange rod] fifty, and there are each one in there, then it's pretty much called a fiftieth, I guess" (line 4.0.203).

29a,b Alan used direct reasoning similar to both Meredith and Laura. First, he counted the white rods by fives, and concluded that since he counted until fifty, the solution was five. Then, he explained to Dr. Landis that five times ten equal fifty. He then continued his train of thought, saying, "It takes ten white ones to make this and ten fives equals to fifty. So that's why I called this five" (line 4.0.216).

24b During the whole class discussion, Beth said, "If you count by fives to fifty, you'll have ten; you'll count, you'll have ten, you have to have five ten times to get to fifty" (line 4.0.229). Meredith the repeated her argument, saying that it was "sort of equivalent." She explained that five times ten equals fifty (line 4.0.231). T/R 2 asked her why she used a multiplication problem to help her solve the challenge. Meredith explained, "Because this is a ten rod [the orange rod]. It has ten ones. Ten times five equals fifty. So I said ten times what gives you fifty? And five, gives you ten times five equals fifty" (line 4.0.233).

30 Dr. Landis then shared Caitlin's reasoning with the class. She said,

> She was trying to say that she remembered when this was called a ten [holds up an orange rod] and when this was called a ten she remembered this little one was called what [holds up a white rod]? A one. So she said if this was a ten, this was a one. So she said if this was a fifty, this would be a five.
> (line 4.0.237)

Caitlin's direct reasoning, although articulated for the class by the principal, contained a different justification for the solution. Caitlin used beginning proportional reasoning when arriving at her solution.

31 Beth then offered another way of thinking about the problem. Using direct reasoning, she said that if each white rods was a nickel, she calculated "how many nickels would add up to ten and then, and then how many tens would add up to fifty" line 4.0.243).

Task 7: Which is larger, 1/2 or 1/3, and by how much?

32 T/R 2 then asked the students to revisit the problem they had been working on during the previous session: Comparing one half and one third and finding the number name difference between the two. Beth and Sarah built a model using a dark green rod, two light green rods, and three red rods. Beth explained to T/R 2 that one half is larger than one third and showed her that the light green rod was larger than the red rod. T/R 2 asked Beth how much larger one half was. Beth said, "I'd say one unit" (line 2.0.279) and held up a white rod. T/R 2 then asked them to give the white rod a number name. Beth said that the difference is one half of one third (line 2.0.281). T/R 2 then asked them how they had arrived at number names for the half and third, and how they could use that method to find out the number name for the white rod. Beth placed six white rods on her model (Figure S-47-19). T/R 1 asked them what the number name would then be. Sarah replied that it would be one sixth. When asked to explain, she pointed to the white rods and counted them. Beth added, “It is six and it makes up the green block” (line 2.0.291). Upon questioning, Sarah then summarized, saying that one half is larger than one third by one white cube, which is one sixth (line 4.0.293-4.0.297). In this exchange, Beth and Sarah used direct reasoning to justify their solution.


Figure S-47-19

33a,34a David showed T/R 2 the balance that he had shown during the previous session and repeated his direct argument (lines 4.0.306-4.0.312). T/R 2 asked how much larger one half was than one third. Meredith said that the difference was one half of the red rod. T/R 2 asked her what she could put if she didn’t want to break the red rod in half. Meredith placed a white rod on her model, showing directly that the light green was equivalent in length to the red and white train (lines 4.0.326-4.0.328).

35 T/R 2 then asked Jessica, Laura, and Brian to share their models. Jessica built two models, one using the orange and red train as one and the other using the dark green rod as one (Figure S-56-33). She explained, using direct reasoning, that the second model showed that the light green rod was one half and the red rod was one third (line 4.0.350). Brian then explained that his model, which was identical to Jessica’s larger model, showed that the orange and red train was one, the dark green rod was one half, and the purple rod was one third. Then, using direct reasoning, he said,

> The third is smaller because if, you have to make three of them, you have to make, to make it a third you have to have three of them in one, you have to have three of them in one whole, but there is less room for three of them so you, and you have more room for a half so half would be bigger.
>
> (line 4.0.354)

Figure S-56-33

34b,c,33b Meredith then showed T/R 2 a second way of showing that one half is larger than one third. She placed the red and green rods upright and placed an orange rod leaning on them, and showed that the rod slanted downward because the red rod was smaller. T/R 2 asked David and Meredith to tell her how much bigger one half was than one third. Meredith first said that it would be a difference of "two ones" (line 4.0.364). T/R 2 pointed out that the dark green rod had been called one. David said that the difference would probably be called one sixth. Meredith stacked two white rods next to an upright red rod, and said that it would be called two sixths (Figure S-1-00-29). Again, the students used direct reasoning to show the difference between the two fractions. However, Meredith's final line of reasoning was faulty, in that she assumed that the difference between the two quantities was two sixths instead of one sixth.

Figure S-1-00-29

34d,33c T/R 2 asked the David and Meredith to resolve this difference of opinion. Meredith said, “Like I said if you would separate ‘em. And you gave one of these [red], one of these [puts one red rod in front of David, one in front of T/R 2, and keeps one] and two more kids [places two green rods in separate places on her desk] then you’d have more” (line 4.0.372). T/R 2 asked her what the number name would be for the difference, and she replied that it would be one sixth for each third, concluding that since there were two light green rods, the solutions was two sixths (lines 4.0.379-382). David then said that he believed the solution was one sixth “because six of these add up to one” (line 4.0.383). Thus, Meredith reasoned incorrectly about the stated problem, while David used direct reasoning to arrive at and justify the correct solution.

34e T/R 2 then asked Meredith to compare only one half and one third. Meredith then reasoned directly that the difference would be one sixth (line 4.0.386).

36 Alan built a model of an orange rod, two yellow rods, and a light green rod (Figure F-44-27). He explained to Dr. Landis that the yellow rod was one half and the light green rod was one third, and that the yellow rod was larger. Dr. Landis asked him to show that the yellow rod was one half and the light green rod was one third. Alan said that the yellow rod was half the length of the orange rod, but since there was no rod that was one third of the orange rod, he used a light green rod, which was one third of the blue rod. Alan used faulty reasoning to build and justify his model.


Figure F-44-27

37, 38a Dr. Landis asked Erik if he agreed with Alan's model. Erik said, "[I]f you're using the orange rod for the halves, the halves are going to be a yellow. And if you're using the blue rod for your thirds, you can't compare them. Because the blue rod's smaller than the orange rod" (lines 4.0.414-4.0.416). Erik used indirect reasoning, saying that comparisons couldn't be made between fractions of rods of different sizes. Alan then argued that even if the halves and thirds were compared with the same rod that was called one, one half would still be larger than one third.

39a Erik then built a model of a dark green rod, two light green rods, and three red rods. He told Dr. Landis that he had found a model to compare the two fractions. He told Dr. Landis that one half was larger than one third (lines 4.0.446-4.0.458).

39b,c Dr. Landis asked Erik why this was true. Erik reasoned directly, saying,

> Because see, if you have one whole, and you want to divide it into halves, the halves have to be so big that you can only divide them into two parts. So, and if you wanted to divide it into thirds, they have to be big enough to divide into three parts. So if you only wanted to divide it into two parts, you have one whole, the whole has to be big enough to divide into two parts, two equal parts. So if you have two parts, two is less than three, but if you divide it into two parts, they have to be bigger than the thirds.
>
> (lines 4.0.460-4.0.472)

Dr. Landis asked him to explain further why the halves would be larger than the thirds. Erik said,

> The thirds would be smaller because two parts of one, like a circle or something, you cut it into two parts, they're going to have to be bigger, because it's two parts

> you're cutting it into. But if you're cutting it into three parts, the thirds are going to have to be bigger, I mean not bigger, smaller, because you're cutting it into three parts, and three parts, is, the number three is larger than two but if you're cutting something into two parts it's going to have to be larger than three.
> (line 4.0.482)

Erik concluded his argument and said, "So technically, if you're counting by numbers, the smaller number is the larger" (line 4.0.484). With this, Erik used the generic example of halves and thirds to make a generalized statement about fraction comparisons.

39d,e Dr. Landis asked if this statement was always true. Erik began to use the example of thirds and fourths, and Alan interjected and explained recursively,

> If you cut this into thirds, this, it would have three equal parts. If you would cut it into halves it would have two equal parts and if you cut it into fourths it would have four equal parts. The fourths would be smaller than the thirds and the thirds smaller than the halves
> (lines 4.0.513-4.0.515)

Erik then continued his train of thought, using the example of thirds and fourths to show that his rule held. He used generic reasoning to show that his generalization held for another example, and why that generalization was true. He said,

> What I'm saying is, see if you divide it into thirds and fourths, four is a larger number than three, but three, you're dividing it into, um, you're dividing it into three parts, so instead of dividing it into four parts you cut it four times into fourths and then, and that would be much smaller than the, a third. And if you divide it- if you cut it only three times, it'd be bigger. So therefore, four may be bigger than three, but the smaller the number, the larger the piece.
> (lines 4.0.520-4.0.526)

39f Dr. Landis then posed a related problem to Erik. She told Erik that a pizza was shared by eight people, and the same size pizza was shared by four people. She asked which pizza slices would be larger. Erik replied that each member of the group of four would get a larger slice. Dr. Landis asked Erik if his rule held for this problem as well. Erik, continuing his reasoning using his generic example, answered that it did, and repeated, "[B]ecause the smaller, the smaller the number, the bigger the pieces" (lines 4.0.456-4.0.458).

38b,c Dr. Landis then revisited the discussion that she had conducted with Erik and Alan about using two different rods to represent one. She asked the boys if they were allowed to do as Alan had originally done. First, Erik repeated his original indirect argument. Then, he changed his thinking and said,

> Wait, well, come to think of it, maybe you can compare. Because, yeah, I think you can compare. Because they may be smaller than each other, but one's dividing it into halves, like the orange rod you're dividing into halves, but the blue rod you're dividing into thirds, and the thirds are one smaller than halves and the blue rod is one smaller than the orange. So therefore they're equal.
> (line 4.0.581)

With this second argument, Erik incorrectly reasoned that two different rods could be used to represent one in the same problem.

39g,h Toward the end of the session, CT approached Alan and Erik, and they each repeated the explanation of why one half is larger than one third (lines 4.0.597-4.0.603).

Students' Written Work

During this session, the students worked to record their solutions to the last problem that they worked on. Appendix C contains selections of the students' written work from this session, which will be briefly described below.

Erik and Alan recorded the generic argument that Erik had proposed during the session, in addition to recording the model that they had built. Alan's justification consisted of the argument, "[B]ecause the smaller the number that you divide the rod, the bigger the piece will be."

Meredith used an explanation that referred to the Cuisenaire rod models but did not refer to a specific model that she had built. She said, "I think 1/2 is bigger than one third because if you take two rods that are the same size and you split one into halves and one into thirds and you put one 1/3 piece on top of one 1/2 piece, the 1/2 piece is bigger." Although Meredith's reasoning is not included in the video data, it appears that she also used some beginning generic reasoning to justify her solution.

Similarly, Michael wrote:

> I picked the dark green rod because we are doing the candy bar problem and dark green was the only one that had a third and half. We figured that 1/2 was bigger than 1/3 because 2 is less than 3 so it would only be 2 parts of a candy bar instead of three.

After writing this generic explanation, Michael drew a model of one, one half, and one third, ostensibly replicating a physical model that he had built.

Andrew's justification, although incomplete, suggests that his reasoning was also generic. He drew rods to show two halves and three thirds that were of the same length, and wrote, "1/2 is larger than 1/3 because it takes two halfs [sic] to make a hole [sic] and three 3rds to make a hole [sic]"

Other students, including Sarah and Amy, drew models that they had built and showed that one half was larger in the model. Audra drew a model of a chocolate bar used an explanation that was quite different than the arguments contained in the video data. She wrote, "If you have a chocolate bar with 6 pieces in it, and you divided it in half, you'd have 3 pieces in each half. If you divided it in 3rds you'd have 2 pieces in each third. So one half is bigger."

Table 4.4

Forms of Reasoning, Session 4

	Student	Lines	Type	Structure	Form	Sub-codes	WC	PR
	Task 1: If I give this orange rod a number name one, what number name would I give to white?							
1.	Brian	4.0.2	Claim	Direct				✓
2.	Kimberly	4.0.4-4.0.6	Claim	Direct			✓	
	Task 2: If we are calling the orange rod the number name one, what would you call the number name for the red rod?							
3.	Gregory	4.0.10-4.0.14	Claim	Direct			✓	
	Task 3: I'm calling the orange one, what number name would I give to two whites?							
4.	Mark, Andrew	4.0.19-4.0.21	Claim	Direct			✓	
5a-b.	Audra	4.0.23, 4.0.29	Claim	Direct			✓	
6.	Meredith	4.0.33	Claim	Direct			✓	
7.	Beth	4.0.35	Claim	Direct			✓	
8a.	Brian	4.0.39-4.0.41	Counter-4	Indirect			✓	
8b.	Meredith	4.0.43	Counter-4	Indirect			✓	
	Task 4: I'm going to call the orange ten…and I'm wondering if you could tell me the number name for white.							
9.	Meredith	4.0.51	Claim	Direct				✓
10.	David	4.0.49	Claim	Direct				✓
11.	Brian	4.0.53-4.0.68	Claim	Direct				✓
12.	Jessica	4.0.69-4.0.71	Claim	Direct				✓
13.	Beth	4.0.73-4.0.75	Claim	Direct				✓
14.	Beth	4.0.77	Claim	Direct	Recur.			✓
15	Alan	4.0.81-4.0.88	Claim	Direct		Faulty		✓
16.	Erik	4.0.83-4.0.95	Counter-15	Direct				✓
17.	Jackie	4.0.100-4.0.106	Claim	Direct			✓	
	Task 5: If I take the same orange rod we've been working with, but I change the number name again. This time I'd like to call it… fifty. I'm wondering if anybody could tell me the number name for yellow.							
18.	Jessica	4.0.113-4.0.117	Claim	Direct				✓
19a.	David	4.0.120	Claim	Direct				✓
19b.	David	4.0.122	Claim	Direct				✓
20a.	Alan	4.0.134	Claim	Direct				✓
20b.	Alan	4.0.146	Claim	Direct				✓
21.	Jacquelyn	4.0.150	Claim	Direct				✓
22.	Michael	4.0.154-4.0.156	Claim	Direct			✓	
23.	Beth	4.0.158	Claim	Indirect			✓	
	Task 6: What number would I give to one white rod if we're still calling the orange the number name fifty?							
24a.	Sarah	4.0.161-4.0.156	Claim	Direct				✓
25.	Sarah	4.0.167	Counter-R	Indirect				✓
26a.	Meredith	4.0.171-4.0.177	Claim	Direct				✓
27.	Laura	4.0.181	Claim	Direct				✓
28.	Brian	4.0.187-4.0.205	Claim	Direct		Faulty		✓
29a.	Alan	4.0.209	Claim	Direct				✓
29b.	Alan	4.0.212-4.0.216	Claim	Direct				✓
24b.	Sarah, Beth	4.0.227-4.0.229	Claim	Direct			✓	

	Student	Lines	Type	Structure	Form	Sub-codes	WC	PR
26b.	Meredith	4.0.231-4.0.233	Claim	Direct			✓	
30.	Caitlin*	4.0.235-4.0.237	Claim	Direct			✓	
31.	Beth	4.0.243	Claim	Direct			✓	
	Task 7: Which is larger, 1/2 or 1/3, and by how much?							
32.	Beth, Sarah	4.0.271-4.0.297	Claim	Direct				✓
33a.	David	4.0.306-4.0.312	Claim	Direct				✓
34a.	Meredith	4.0.326-4.0.328	Claim	Direct				✓
35.	Jessica	4.0.346-4.0.350	Claim	Direct				✓
36.	Brian	4.0.352-4.0.356	Claim	Direct				✓
34b.	Meredith	4.0.362	Claim	Direct				✓
34c.	Meredith	4.0.364-4.0.370	Claim	Direct		Faulty		✓
33b.	David	4.0.364-4.0.370	Claim	Direct				✓
34d.	Meredith	4.0.372-4.0.384	Claim	Direct		Faulty		✓
33c.	David	4.0.372-4.0.384	Claim	Direct				✓
34e.	Meredith	4.0.386	Claim	Direct				✓
37.	Alan	4.0.404-4.0.408	Claim	Direct		Faulty		✓
38a.	Erik	4.0.412-4.0.416	Counter-37	Indirect				✓
39a.	Erik	4.0.446-4.0.458	Claim	Direct				✓
39b.	Erik	4.0.460-4.0.472	Claim	Direct				
39c.	Erik	4.0.480-4.0.494	Claim	Direct	Gene.			
39d.	Alan	4.0.513-4.0.515	Claim	Direct	Recur			✓
39e.	Erik	4.0.520-4.0.526	Claim	Direct	Gene.			✓
39f.	Erik	4.0.534-4.0.560	Claim	Direct	Gene			
38b.	Erik	4.0.566-4.0.579	Counter-37	Indirect				
38c.	Erik	4.0.581	Claim	Direct		Faulty		
39g.	Alan	4.0.597	Claim	Direct				✓
39h.	Erik	4.0.599-4.0.603	Claim	Direct				✓

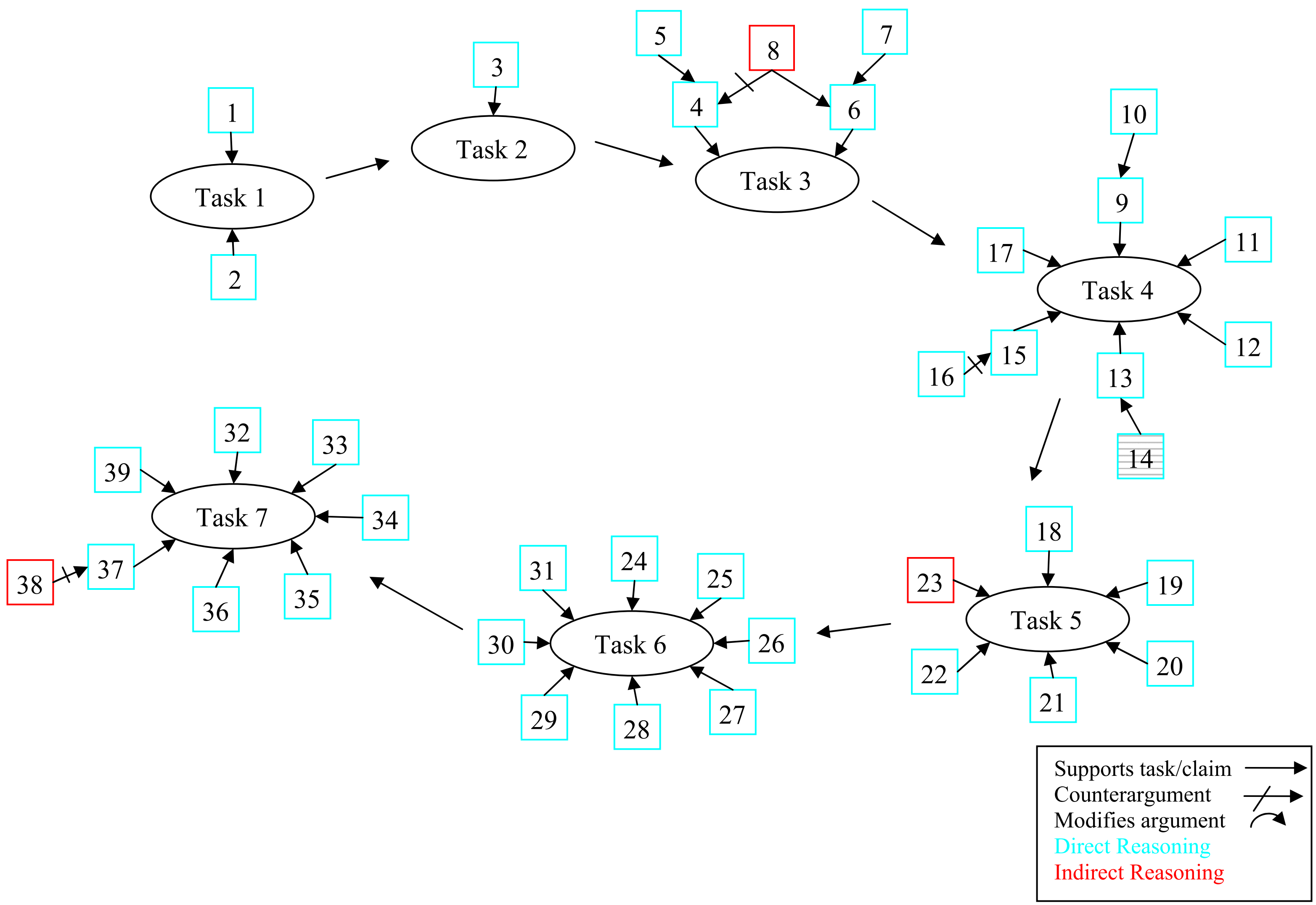


Figure 4.4. Argumentation Organized by Task, Session 4

4.2.5 Session 5: September 29, 1993

Task 1: Is 1/5=2/10?

At the start of the session, T/R 1 introduced Dr. Davis, who was visiting the classroom that day. She placed an overhead on the projector and told the class that they would be continuing a discussion that they had started in earlier session. She asked the students to read the problem that was written (Figure S-8-31) and what they remembered about the problem and the issue that it represented. The problem stated: Is 1/5=2/10?

1a Meredith built a model of an orange rod and two yellow rods at the OHP (Figure O-11-08). She said that the orange rod was called one, and that each yellow rod was called one half (line 5.0.12). T/R 1 asked her what that had to do with the question that had been asked. Meredith then said that the yellow rods were called two tenths, and placed five red rods on her model, saying that they equaled the length of the two yellow rods (line 5.0.18, Figure O-12-02). Meredith used faulty direct reasoning to relate her model to the problem posed.

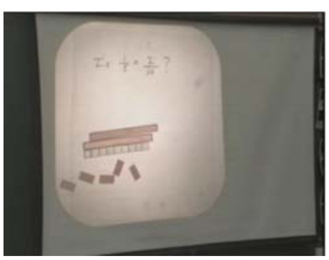


Figure S-8-31

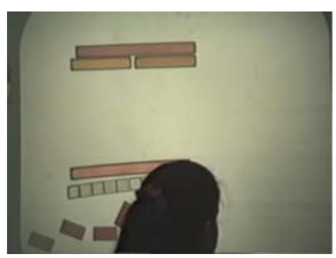


Figure O-11-08


Figure O-12-02

1b T/R 1 then asked her to repeat her reasoning to the class. Meredith placed two white rods on the OHP and called them two tenths (Figure O-12-52). Using the model of one orange rod and ten white rods that T/R 1 had placed on the OHP, she showed that the white rods are called tenths since ten white rods equaled the length of the orange rod, which was called one. She then moved the five white rods from her original model directly above the model of orange and white rods and said that the red rods were each called one fifth. Placing a red rod along the white rods at the side of the screen, she showed that the two lengths were equivalent (Figure O-13-21). She concluded her direct argument by saying that one fifth equals two tenths (lines 5.0.20-5.0.24).


Figure O-12-52

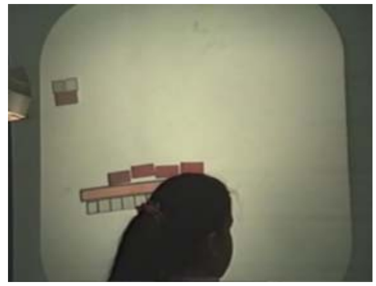


Figure O-13-21

2 T/R 1 asked the class if they agreed with Meredith's argument. Brian addressed the class at the OHP, saying that he agreed with her argument. He said, "Well, I agree because it's just like having one of these reds being a whole and one of these [a white rod] being a half. So it's just like saying, it's just like saying, two halves equal a whole. It's the same as being two tenths equal one fifth" (line 5.0.29). As Brian spoke, he used the model of one red rod and two white rods that Meredith had built. Here, Brian used direct reasoning to justify his solution.

3 Erik then presented his version of the direct argument to the class. He said that he agreed with Meredith, because ten white rods equal the orange rod, and five red rods equal the orange rod. He then said that two white rods were called two tenths and that one red rod was called one fifth, and that the length of the two were equal (line 5.0.31).

Task 2: What other number names can we give to one half of a candy bar?

T/R 1 then reminded the students about the candy bar story that they had discussed in an earlier session. She showed the students a drawing of the candy bar (Figure O-17-08) and asked the students to think of other number names that they could provide for one half of the candy bar. The students worked in their groups for a couple of minutes and then shared their ideas in a whole class discussion.

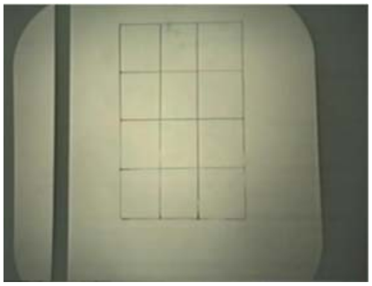


Figure O-17-08

4, 5a Jackie explained that another name for one half could be six twelfths, since there are twelve pieces of chocolate in total, and half of six is twelve. Danielle used direct reasoning as well to explain that "if she got a half, then the top two rows, um, is a half, and then that's two fourths" (line 4.0.48). T/R 1 asked Danielle to explain how she found two fourths. Danielle said, "Because there's four rows" (line 5.0.52).

6, 5b T/R 1 asked the class if they agreed with Danielle's solution. Some students indicated that they disagreed, and T/R 1 asked Brian to try to explain what Danielle said. Brian said,

> Well, I agree on two fourths because there's, because three times, because four times three equals twelve and if you split it in half there'd be three fourths,[he has turned the grid around]. Well, well, I agree because there are four thirds on there but, but when there's a half there are only, there's three fourths instead of four thirds.
>
> (line 5.0.57)

Brian's explanation was unclear, and it appears that he used faulty reasoning to justify Danielle's solution. T/R 1 said that she didn't understand what Brian had said, and Danielle offered another explanation for her solution. Using direct reasoning, she said, "If there's four fourths, and half of four is two, so two fourths would be a half" (line 4.0.61).

7 Brian offered an alternative number name for one half. He named it three sixths, and explained that he had found groups squares that could be called sixths. At the OHP, he showed directly that there were six pairs of squares. He then showed that there were three pairs on the top half of the grid (lines 5.0.64-5.0.66).

Task 3: Which is larger, 1/2 or 1/3, and by how much?

T/R 1 then reviewed the number names that had been provided by the students. On a transparency, she wrote: “1/2 = 6/12 = 2/4 = 3/6” (Figure O-27-07). She then asked the students if it was also true that one fifth was equal to two tenths, and the students responded in the affirmative.

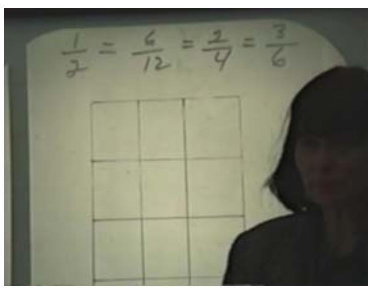


Figure O-27-07

T/R 1 then asked the students about the problem they had worked on during the previous session. Michael and Andrew volunteered that they had been comparing one half and one third. T/R asked the students which was bigger, and Laura replied that one half was bigger than one third. T/R 1 asked if the class agreed and there were murmurs of agreement. T/R 1 asked Laura and Jessica to present their solution to the problem at the OHP.

8 Jessica and Laura built a model of an orange and red train, two dark green rods, and three purple rods on the OHP (Figure O-31-38). They explained that the train was called one, the dark green rod was called one half, and the purple rod was called one third. Jessica said that, from the model, it was evident that the half was bigger than one third (lines 5.0.98-5.0.105). T/R 1 asked the students if they agreed. Audra said that she agreed, and, at the OHP, she used direct reasoning and said, “I agree because if you saw what the, um half, was here and then you saw what, no, what the half was here and then you saw what the third was there, and you saw that the half was bigger than the third” (line 5.0.109). T/R 1 then asked Jessica, Laura and Audra how much bigger one half was than one third. Audra placed two white rods next to the purple rod, showing that that train was equal in length to the dark green rod. Jessica placed a red rod next to another purple rod, saying that “It’s a red bigger” (line 5.0.112, Figure S-32-28). T/R 1 then asked them what number name they would give to the red rod. Audra and Jessica independently lined up three red rods against the model of the dark green rod and the purple and red train that they had used to compare the two rods, and said that the number name would be one third (lines 5.0.114-5.0.117, Figure O-34-49). Here, the girls used faulty reasoning, due to their use of the second model, to draw an erroneous conclusion.

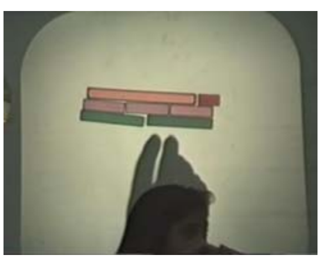


Figure O-31-38

Figure S-32-28

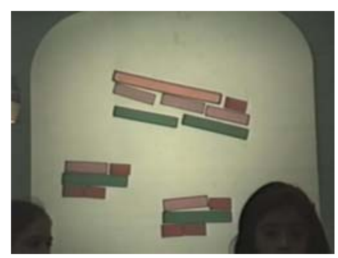


Figure O-34-49

9a T/R 1 asked the class if they agreed with the argument that had been presented. Kelly said that she agreed, and, at the overhead, built a train of two light green rods. She then placed a light green rod next to a red rod and said that one half was bigger than one third, indicating that the red rod was one third and the light green rod was one half (line 5.0.121, Figure O-36-23). Kelly, then, used direct reasoning, although incomplete, to show that one half was bigger than one third using a different model than what had been built earlier. Here, too, Kelly used faulty reasoning to justify her solution.


Figure O-36-23

10a,b T/R 1 asked Brian what he thought. Brian first verified that he had heard the first argument correctly. He asked whether the girls had said that one half was larger than one third by one third. He then said that he didn't agree, and that he thought the difference was one sixth. Using direct reasoning, he said that if the thirds were split in half they would become sixths.

> [W]hen they said it was one half bigger, if you split a third in half it'd make a sixth, like one, two, three, four, five six. Like… pretending they were split in half. If you split one of these in half and you have three of them up there… they'd make six and… when you split them in half right in the middle over there it's kind of like that, it's kind of like this, there was this was, that was the one third [points to a purple rod] and that was the one half [points to the dark green rod] on the bottom and so it's just like this and the red I'm pretending is like, is like, is a half of one of the purples and you see when I split it in half it's, it's one sixth and, and it equals, and it equals up to a green
>
> (line 5.0.137)

Brian indicated using direct reasoning that the red rod was one half of the purple rod, which would be called one sixth, and that the difference between one half and one third is one sixth. Brian then lined up a purple rod and two red rods and said that the reason the red rod would be called one sixth was "because two of these [red rods] equals, see they're two, they're two sixths, two halves of one purple and the purple is a third and the half of one third is sixth, there's sixths" (line 5.0.139, Figure O-40-19)

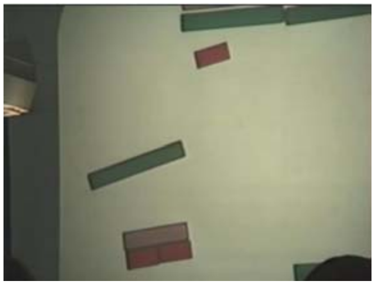


Figure O-40-19

9b,11 Jessica then said that Jackie and Kelly's argument was different than the original argument. Jackie repeated their argument, using a complete model of a dark green rod, two light green rods, and three red rods (Figure O-43-42). She said, "Well, we would call this dark green one and the reds one third and the light green one half, and we thought the, we thought one third was bigger by one of these white things" (line 5.0.145). In this explanation, Jackie used direct reasoning to show that one half was bigger by one third by one white rod. As she spoke, Kelly placed a light green rod alongside a red rod to show the comparison. After Jackie explained her model, Jessica said, "Oh, I think they're making a different size candy bar" (line 4.0.146). T/R 1 asked her if that was allowed, and she responded that it was not, because one half of the larger model would be a different size than one half of the smaller model. Thus she used indirect reasoning to arrive at this erroneous conclusion.

Figure O-43-42

9c T/R 1 asked Jackie to repeat her argument. She restated the argument, concluding that one half was larger than one third "by one which is the white one" (line 5.0.162). T/R

1 then acknowledged that Jackie had used a different model than Jessica and Audra, but asked them if they had said that the difference between the two fractions was one. They responded that they had, and T/R 1 asked them what they were calling the white rod. They replied that they called it one, and T/R 1 asked them if they were calling both the white and the green rods one. Jackie giggled and said no. In this part of her argument, Jackie used faulty direct reasoning to name the white rod (lines 5.0.151-5.0.174).

12,9d Erik volunteered that he knew what Jackie and Kelly meant to say. He said,

> I think they mean that they want to call this, the dark green one, one whole, and they want to call this, yeah, like you line all the whites up to it which I think should be six and they want to call it one sixth. I think that's what they're trying to say but they just, they're just not saying it. I think they just, they want to call it one sixth.
>
> (line 5.0.178)

As he spoke, Erik lined up five white rods against Jackie's model. T/R 1 told Erik that she didn't see sixths on the model, and Erik corrected the model so that it contained six white rods (Figure O-46-07). He concluded directly, "I think you meant to say not one whole but one sixth " (line 5.0.182). Jackie and Kelly agreed that that is what they meant to say, and, upon questioning by T/R 1, stated that they believed the difference was one sixth. T/R 1 asked them why it couldn't be called one, and they responded that it was because the dark green rod was called one (line 5.0.192)

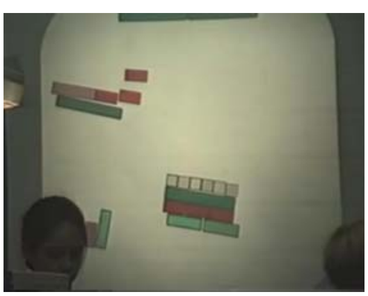


Figure O-46-07

13 Erik then said that the white rods could be called one. He said that if the dark green rod was called six, the light green rod would be called three. T/R 1 interrupted Erik's direct argument and asked him if this could be done within the same problem. Erik agreed that it could not (lines 5.0.194-5.0.198).

8d T/R 1 then asked Jessica to re-explain her original model. Jessica restated her faulty argument, saying that the model showed that one half was larger than one third by the red rod, which was one third (lines 5.0.200-5.0.202).

10b Brian then repeated his counterargument. He said that when one third is split in half, each part becomes one sixth. He then continued his argument using indirect reasoning. He placed a red rod on the second purple rod in Jessica's model and said, "[I]f you put one of these on top of it you might see that… that red is that much bigger than one of the halves" (line 5.0.206). He said that the red rod is called one sixth, and showed that the difference could not be one third, since the two thirds were larger than one half by one sixth (Figure O-50-08).

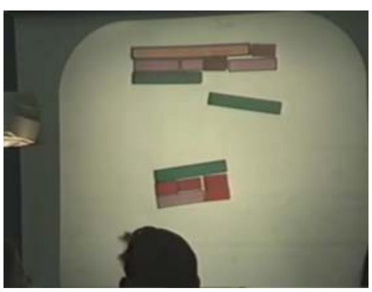


Figure O-50-08

14a-c,10c Erik then joined the argument. He stated his argument three times in succession, with slight modifications, along with some added commentary by Brian.

Erik: I don't think you can have an answer of a third because if you have one half [he goes to the overhead] and if you take the one half which would be the dark green, you have the one half and then these [purple rods] are the thirds. How could one half be bigger than the thirds by one third? Because, and you have the half and the thirds together that the half is almost as big as two thirds, but yet the two thirds aren't exactly, are not exactly, the green, the dark green is not, the dark green is not exactly as big as two, two thirds but, two thirds, it's the, but it's far enough so that the two thirds are not bigger than it by one third [Figure O-51-53].

Brian: I kind of agree with Erik. I think now I disagree with them [referring to the girls].

Erik: I don't really think that if you have this [a purple rod] that you could have one third bigger than it [Brian - yeah] because it's got to be one third and probably a third and a half.

Brian: Yeah, he's right.

Erik: It couldn't be, it couldn't be exactly a third.

Brian: Cause one third bigger, this would be one third bigger like that to the end over there [Figure O-52-40]. That would actually be like this [showing with the dark green and purple pieces], this would really be one third bigger and there's still some left over and there's still about [Figure O-52-51]

Erik: A half left over.

Brian: Yeah, there's still, there's still one more, there's still one more piece left, like about a sixth left [Figure O-53-04].

Erik: Cause it's like if you have, if you have the like dark green and it doesn't exactly equal up to, it doesn't exactly equal up. It's less than two thirds but it's more than one third. It's just about one third and a half. So it couldn't be exactly a third bigger than it and it couldn't be exactly two thirds or it couldn't be exactly one third bigger. It had to be one third and a half.

(lines 5.0.209-5.0.217)

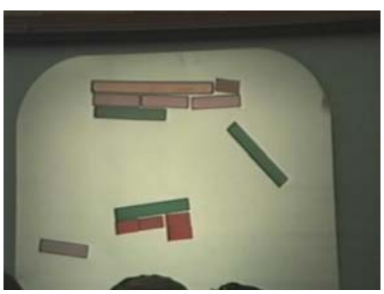


Figure O-51-53

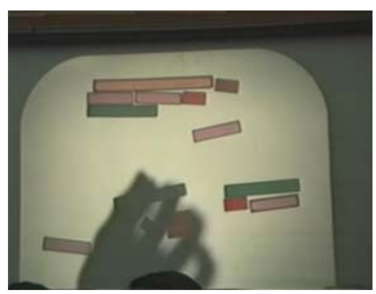


Figure O-52-40

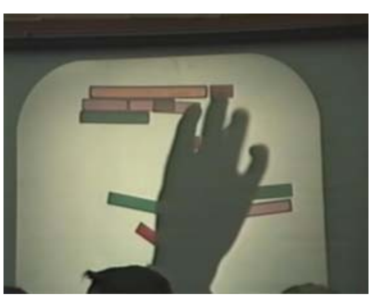


Figure O-52-51

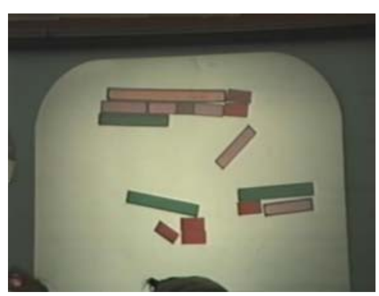


Figure O-53-04

Although Brian and Erik's language indicate that they perhaps misunderstood what "larger by one third" meant in the context of the problem, they used complex forms of reasoning to show that the difference between one half and one third was not one third, but rather one sixth. Erik, in the first and third versions of his argument, used indirect reasoning together with an argument using upper and lower bounds to show that one half was larger than one third but smaller than two thirds, and that, as a result, the difference between the two could not be one third. The second version of his argument used indirect reasoning to show that it could not be a difference of one third. At that point, he noted that the length of one half was equivalent to "a third and a half." Brian's argument also used indirect reasoning to show that one third could not be the solution. However, his models indicate that perhaps he was only peripherally countering the girls' argument with his statements.

15,14d Michael then joined the discussion. He used direct reasoning to state that one half was larger than one third by one sixth. He lined up six red rods above the orange and red train (Figure O-54-17) and said, "I think it should be called one sixth because… if you put six reds up to one orange with a red then… it would be the same size… so it would be called one sixth" (line 5.0.219). Brian then stated that he agreed with Michael, and Erik began to say that Michael was correct and offer an argument to support it, but T/R 1 asked Meredith to share her thoughts with the class.

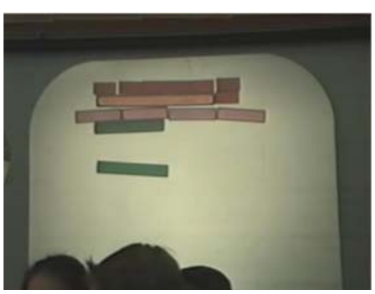


Figure O-54-17

16 Meredith used an indirect argument to show that one half was not larger than one third by one third. She lined up two purple rods along with a dark green rod (Figurre O-55-18) and said "[I]f you do call that a sixth… and if you put the dark green and two thirds… they said that it's a third bigger, if you did a third bigger, this is called a third and then you put it there, you see negative" (line 5.0.224). An intercom interrupted her explanation, and T/R 1 asked her to re-explain. Meredith placed a red rod next to the dark green rod (Figure O-55-27).

Meredith:	You said it was one third bigger, that can't be true because one third bigger
Erik, Brian:	Yeah
Brian:	It's about one sixth less. So it can't be a third bigger.
Erik:	And also, like
Meredith:	[removing one purple rod and placing the red rod next to the remaining purple rod.] So it's one sixth bigger [Figure O-55-55]

(lines 5.0.224-5.0.230)

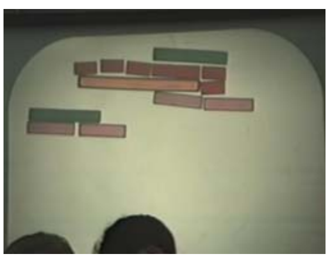


Figure O-55-18

Figure O-55-27

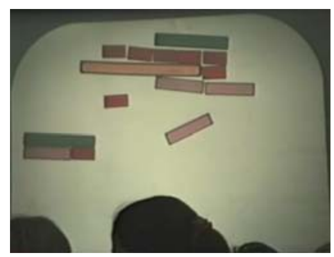


Figure O-55-55

Meredith used direct reasoning to show that the difference could not be two thirds due to the fact that two thirds was larger than one third by one sixth, and then reasoned directly, showing that one third was larger than one sixth by a red rod, which was called one sixth.

14e Erik then repeated his indirect argument using upper and lower bounds. He said,

> I think because if you have the light green, the light green, it's not bigger than, it's not bigger than the, it's not bigger than the umm third, it's not bigger than two thirds. It's bigger than one third, but it's not as big as two thirds so it's less than two thirds but more than one third. So it can't be a third bigger. And if you have that to make it two thirds large, there has to be a sixth.
>
> (line 5.0.231)

Erik placed a red rod next to the dark green rod to show that two thirds was larger than one half be one sixth (Figure O-56-29).

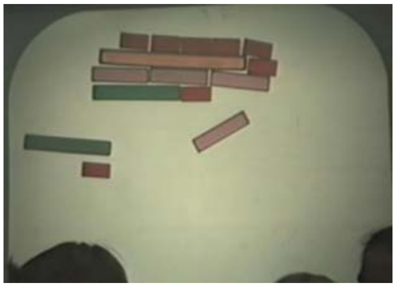


Figure O-56-29

17 T/R 1 told the class that they were each to write about one of the arguments that had been presented, and discuss whether or not they agreed with the argument. Jessica then said that she agreed with Brian and Erik, and explained why she had changed her mind. “Because I, I saw that um, it wasn’t the same as um, it can’t, it couldn’t be one third… Because… you’d have to add a red and that would be one sixth” (lines 5.0.237-5.0.239). Jessica used the indirect argument that had been articulated by some of the other students to explain why the difference could not be called on third. With this comment, T/R 1 brought the session to a close.

Table 4.5

Forms of Reasoning, Session 5

	Student	Lines	Type	Structure	Form	Sub-codes	WC	PR
	Task 1: Is 1/5=2/10?							
1a-b.	Meredith	5.0.12-5.0.24	Claim	Direct		a-faulty	✓	
2.	Brian	5.0.29	Claim	Direct			✓	
3.	Erik	5.0.31	Claim	Direct			✓	
	Task 2: What other number names can we give to one half of a candy bar?							
4.	Jackie	5.0.40	Claim	Direct			✓	
5.	Danielle	5.0.46-5.0.52	Claim	Direct			✓	
6.	Brian	5.0.57-5.0.59	Claim	Direct		Faulty	✓	
5b.	Danielle	5.0.61	Claim	Direct			✓	
7.	Brian	5.0.64-5.0.66	Claim	Direct			✓	
	Task 3: Which is larger, 1/2 or 1/3, and by how much?							
8a.	Jessica, Laura	5.0.98-5.0.112	Claim	Direct			✓	
8b.	Audra	5.0.107-5.0.112	Claim	Direct			✓	
8c.	Jessica, Audra	5.0.114-5.0.117	Claim	Direct		Faulty	✓	
9a.	Kelly	5.0.119-5.0.121	Claim	Direct		Incomplete	✓	
10a.	Brian	5.0.135-5.0.139	Counter-8	Direct			✓	
9b.	Jackie	5.0.145	Claim	Direct			✓	
11.	Jessica	5.0.146-5.0.147	Counter-9b	Indirect		Faulty	✓	
9c.	Jackie	5.0.151-5.0.174	Claim	Direct		Faulty	✓	
12.	Erik	5.0.176-5.0.182	Counter-9c	Direct			✓	
9d.	Jackie, Kelly	5.0.186-5.0.192	Counter-9c	Indirect			✓	
13.	Erik	5.0.194-5.0.196	Claim	Direct		Incomplete	✓	
8d.	Jessica	5.0.200-5.0.202	Claim	Direct		Faulty	✓	
10b.	Brian	5.0.204-5.0.206	Counter-8	Direct, Ind.			✓	
14a.	Erik	5.0.209	Counter-8	Indirect	U/L		✓	
14b.	Erik	5.0.211	Counter-8	Indirect			✓	
10c.	Brian	5.0.214-216	Counter-8	Indirect			✓	
14c.	Erik	5.0.217	Counter-8	Indirect	U/L		✓	
15	Michael	5.0.219	Counter-8	Direct			✓	
14d.	Erik	5.0.220-222	Counter-8	Direct		Incomplete	✓	
16.	Meredith	5.0.224-230	Counter-8	Indirect, Direct			✓	
10d.	Brian	5.0.228	Counter-8	Indirect			✓	
14e.	Erik	5.0.231	Counter-8	Indirect	U/L		✓	
17.	Jessica	5.0.235-5.0.239	Counter-8	Indirect			✓	

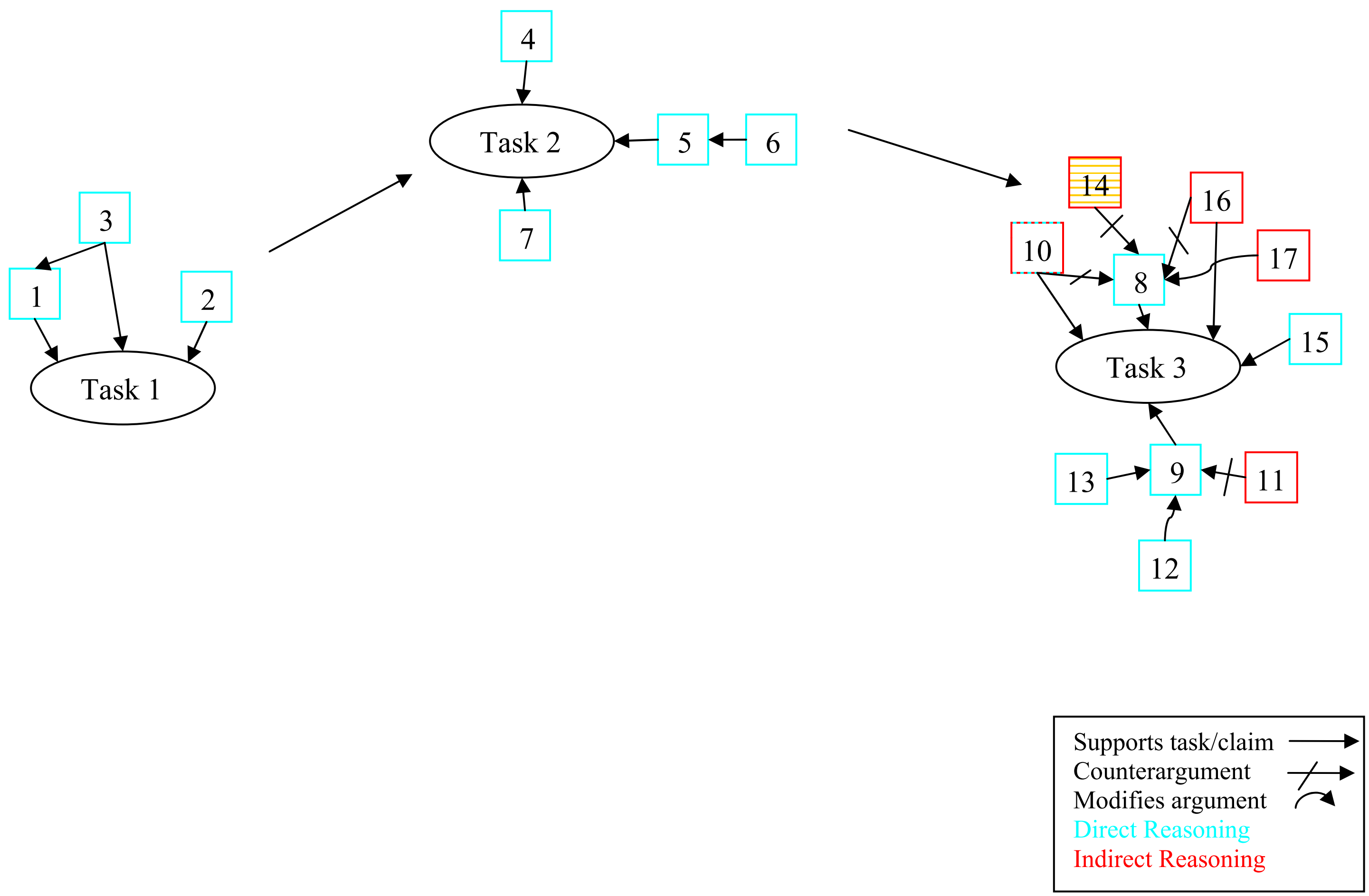


Figure 4.5. Argumentation Organized by Task, Session 5

4.2.6 Session 6: October 1, 1993

Task 1: Which is larger, one half or one third, and by how much?

1, 2 The sixth session began with a review of the problem that had been discussed during the previous session. T/R 1 asked the students if they felt confident to justify their solution to the problem: Which is larger, one half or one third, and by how much. Jessica built a model of an orange and red train, two dark green rods, three purple rods, and six red rods at the OHP. She named the rods one, one half, and one third, and one sixth, respectively. She then became flustered, said that she forgot what she was going to say, and asked Erik to help her out (line 6.0.24). Erik repeated the number names for the rods, and then recapped his argument using upper and lower bounds. He said that one half is larger than one third, but two thirds are larger than one half. He then showed directly that if a red rod is placed next to the purple rod, the length of that train will equal that of the dark green rod.

> I think that what you're trying to say that the orange and the red one, red rod is one and that the uh the green, the dark green is a half and then the purples are thirds and the reds are sixths. And then what I think is that if you take one of the dark greens which is the half… it's larger than one uh third but yet if you put another third… to the other third, that third is larger than it. So then, If you put it, like if you put one of the red rods… it's smaller, the third, the one third is smaller than the one half and one of these red ones, these, the reds are sixths if you put the red on top of… the purple it equals up to the exact same size as the dark green.
> (line 6.0.25)

3a Alan then presented another model to show the difference between one half and one third. He built a model using one dark green rod, two light green rods, three red rods, and six white rods, and showed directly that the difference between one half, or a light green rod, and one third, a red rod, is the white rod, which he named one sixth (Figure S-17-26).

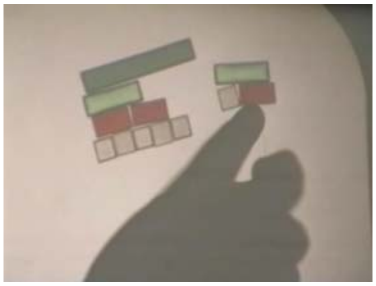


Figure S-17-26

4a, 5 Jessica challenged Alan's solution. Using faulty indirect reasoning, she said, "remember you said that it can be only be one size candy bar and that's like a whole different size candy bar he's making" (line 6.0.32). Kelly and Jackie said that they agreed with Alan and countered Jessica's argument. Jackie explained directly, "Well, because when you go to the store there's not just one size candy bar there's all different kinds of sizes so you can make a model with a different size," (line 6.0.36) and implicitly suggested that Alan had used a different size model, which was in line with the rules that had been set.

3b,4b,6 Michael then used direct reasoning to back Alan's argument. He said, "I agree with it because, um, it can be done because there's like six whites equal up to one green and then it takes one white plus a red to equal a light green which is half so that would be one sixth" (line 6.0.38). Jessica agreed that that was true, but repeated that it was a different size candy bar (line 6.0.39). Erik added to the argument using direct reasoning, saying that they had said that any one sixth was right, implying that Alan's solution was legitimate (line 6.0.40).

4c,7a T/R 1 asked the class what they thought why Jessica was confused. Erik said that the issue that Jessica had was that the sixth wasn't the same size, implicitly backing Jessica's indirect argument (line 6.0.42). Michael countered this, saying directly that the rod that was called a “whole” wasn’t the same size either, and implying that that resolved the problem (line 6.0.44).

4d Jessica repeated her argument. She said, “But because say if you wanted to give someone one sixth of that candy bar and then you were going to give someone one sixth of the other one, then the person with that size would get a smaller amount” (line 6.0.46).

8,4e Andrew attempted to clarify why Alan was correct. He reasoned directly that “it’s just a different size candy bar. If you just gave half of that to the person and the other half of that to another person you would still have the same size. You can’t switch the candy bars” (line 6.0.48). Erik then tried to back Jessica’s argument, noting that although Alan hadn’t switched candy bars within his solution, but that he had switched from the orange and red train that Jessica had used originally. He added that one half the length of the orange and red was larger than one half the length of the dark green rod, implying that that is where Alan’s error lay (line 6.0.50).

7b T/R 1 asked the class what Jessica was confused about. Michael repeated his earlier reasoning, saying, “What Jessica was confused about is, she didn’t think it would be right because they, you had a different size one sixth, but he also switched the whole, so the whole is smaller by one white” (line 6.0.54). With this summary, Michael integrated Andrew’s argument into his own and showed how Jessica’s reasoning was faulty.

9 T/R 1 then asked the class to determine the number name for the white rod in the first model that had been built, in which the orange and red train was called one. James, Laura, and Brian2 suggested names that they thought it might be called (lines 6.0.66-6.0.72). The students then worked in their groups to build the model and determine the number name for the white rod. Meredith and David, without building the model, named the white rod one twelfth (lines 6.0.79-6.0.81) using direct reasoning.

10 During the subsequent whole class discussion, James revised his solution. He lined twelve white rods against the original model and said that it would be called one twelfth. To directly justify his solution, he counted the white rods aloud (lines 6.0.88-6.0.90).

11,12 Beth and Sarah then presented two alternative direct justifications for the number name one twelfth. Beth said, “Well… a dark green is half of… orange and red and then the dark green has… six whites, and if you have two dark greens, six and six is twelve. And that’s why we think its twelve” (line 6.0.92). Sarah then offered a different justification.

Sarah:	Umm, I said that umm, you have six of these reds [Figure S-30-35]. If you times these by two you’d get twelve.
T/R 1:	If you times them by two, why would you times them by two Sara?
Sarah:	Because if you had if you put two next- two little ones right on the bottom of the red it would equal two. So you would go two times [Figure S-30-52].

(lines 6.0.94-6.0.96)

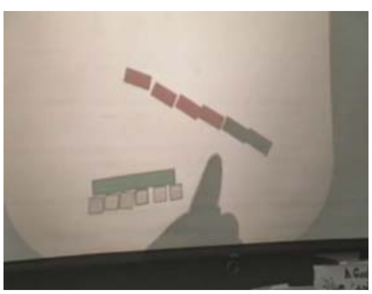


Figure S-30-35


Figure S-30-52

Task 1: Which is bigger, one half or one quarter, and by how much?

T/R 1 then asked the students to compare one half and one quarter and determine the difference between the two. The students worked in their groups to build models that would help them solve the problem.

13a,b Michael and Brian each built a model of two dark green and four light green rods. Brian said, “It’s not bigger. It’s not bigger! That’s weird, look!” (line 6.0.110). Michael then said that he knew why one fourth was not bigger than one half. He said, “[B]ecause there’s two at the end of the number and then one quarter has four on the bottom number. Like when you draw one quarter there’s one, and four on the bottom and when you draw one half, it’s one, and two on the bottom. So that would be two more… So it would only take two parts for this [pointing to two dark greens] and four parts for this” (lines 6.0.111-6.0.113). Michael used direct reasoning to explain why one fourth is smaller than one half. Brian said that he had never thought of it in that way. Then, Brian asked him to re-explain. Michael used his written notation for one half and one quarter and explained, “One, two. So there’s two on the bottom of this [pointing to denominator of one half] and four on the bottom of this [pointing to denominator of one fourth] So that would be, you’d have to divide this one [pointing to the one in the numerator of one fourth] into four parts and this one [pointing to one in the numerator of one half] into two. Two would be like this. Four, four would be like one, two, three, four [gestures with his hands]” (line 6.0.123).

14 Graham, whose partner was not present, spoke with T/R 1 about the solution to the problem. Using the same model as Michael and Brian, he explained directly, upon questioning, that one half was larger than one quarter by one quarter (lines 6.0.126-6.0.137)

15 Erik and Alan worked together to find a model that would show the difference between one half and one fourth. Alan built a model of similar to that which Michael, Brian, and Graham had built, and concluded directly, “One half is bigger than one quarter by one quarter” (line 6.0.152).

16a,b Alan then built another model that showed the difference between one half and one quarter. He used two orange rods and four yellow rods, and concluded “You can quarter a train of orange rods” (line 6.0.162-6.0.164). Erik then tried to build a model using the two brown rods. He found that that model could also be built to show fourths, and told Alan, “All you have to do is keep going down by two. Brown, you minus two, take that rod, and you can quarter that one. Brown, black then dark green!” (line 6.0.168). Erik used recursive reasoning to find models that could show halves and quarters. Alan explained their method to T/R 1, who asked them to repeat it to Dr. Davis.

14b,13c T/R 1 then asked the students to share their solutions as a class. Graham and Michael presented their solutions to the class. Graham said, “The orange and the red would be one and the dark greens would be a half and the light greens would be a quarter” (line 6.0.195). Michael then continued his direct argument, saying,

> [W]e think one half would be bigger than one quarter by one quarter because it takes two quarters to equal that. And why we think that is because four is um, two more than two, so it would take two fourths to equal two, two pieces. Because there’s four pieces and then they would have to put those two pieces together to make two pieces.
>
> (line 6.0.196)

17a Amy, Jacquelyn, and James then build another model at the OHP (Figure O-47-54). Amy explained that they had tried to find a model that did not include a train of rods. She said that the brown rod was called one, the purple rods were one half, the red rods were one quarter, and the white rods were one eighth (line 6.0.208). James then said, using faulty reasoning, that one half was larger than one quarter by one eighth (lines 6.0.211-6.0.218).

Figure O-47-54

18a Meredith countered the team's argument. She asked them if they were calling the white rod one eighth, and they answered that they were. She built a model of one purple rod and a train of one red and one white rod (Figure O-50-40). She then said that it couldn't be bigger by one eighth, "because there is still negative space" on the model that she had built. She added a second white rod onto her model and said that the difference could be two eighths (Figure O-51-03), and then substituted a red rod for the two white rods and said that an alternative name could be one quarter (Figure O-51-15). She concluded, "One quarter or one, um two eighths. It's the only way it could be bigger by" (line 6.0.224). Meredith used an indirect argument to show the contradiction in their solution.

Figure O-50-40

Figure O-51-03

Figure O-51-15

17b, 18b T/R 1 asked the three students what they thought of Meredith's argument. Jacquelyn said, "Well, I think we meant that all these put all together are one eighth" (line 6.0.227). Meredith asked if they meant that two white rods were one eighth. Jacquelyn clarified, "We thought, uh, all of these whites put together were one eighth" (line 2.0.230). Jacquelyn used faulty reasoning to revise their argument. In response, Meredith moved the train of eight white rods onto her model comparing the purple and red rods (Figure O-53-15). She asked rhetorically, “You think its bigger than one eighth and all these are one eighth? So that’s how much you think its bigger by?” (line 6.0.235). Jacquelyn laughed. Here, too, Meredith used indirect reasoning to show that their claim could not be true.

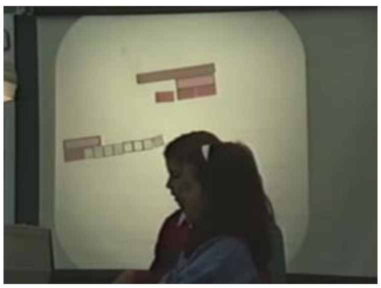


Figure O-53-15

19 Jacquelyn then revised her argument, and explained what Meredith had presented. She said that one half was larger than one quarter by two eighths or one quarter. T/R 1 asked her how she knew it was one quarter. She explained directly that the red rods were each one quarter and that two white rods viewed together can also be called one quarter (lines 6.0.244-6.0.252).

20 Danielle and Gregory presented their model to represent the solution. They built a model of an orange and dark green train, two brown rods, and four purple rods. Danielle explained directly that if one half, or the brown rod, was compared with one fourth, or the purple rod, it was larger by one purple rod, or one fourth (lines 6.0.260-6.0.264, Figure S-57-40).


Figure S-57-40

21 Andrew presented a fourth model. Using a purple rod, two red rods, and four white rods, he reasoned directly that one half was larger than one fourth by one fourth. He then said that he thought the solution would always be one fourth. T/R 1 asked him why he thought that. Andrew said that all the models that had been built “always had the room for one more fourth, and I think that because usually the fourths, or two of ‘em are equal up to the half, so then it would be a fourth” (line 6.0.269). As he spoke, he showed that on his model, two white rods equaled the length of the red rod. With this statement, Andrew used generic reasoning to justify why one half is always larger than one fourth by one fourth.

16c,22 Erik and Alan then presented their recursive method of building models to represent the solution.

Erik:	Well, I think that, we think that you could divide- I think that you could take, you could take rods and divide them equally into fourths I think six times. Well, and we also came to a theory that if, if you uh, yeah we also came to a theory that
Alan:	If you take an orange rod, go down two it would be a brown rod
Erik:	if you take an orange rod and go down two it will be the brown rod
Alan:	And you can make it into quarters, and then-
Erik:	Yeah you just divide two from each rod like you start with the orange rod divide by two and then the brown rod and you divide by two from the brown rod
Alan:	From the brown rod .
Erik:	And then whatever rod you get, divide two from that and keep going down.

(lines 6.0.271-6.0.277)

Although Erik confused division with subtraction, he explained that if the length of two white rods were subtracted from the length of the rod used previously, a model for the problem could be built. Alan and Erik displayed their work on the OHP (Figure O-1-05-53). Andrew then challenged the validity of their models, saying that “right on their problems that they have they don’t have a half” (line 6.0.285). Erik replied that he thought that other students had already shown the difference between one half and one fourth, and they were simply presenting a method of finding models.

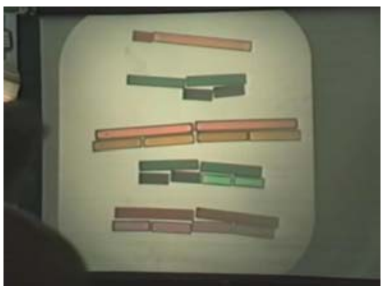


Figure O-1-05-53

T/R 1 then closed the session by asking the students to think about whether it is possible to build more than six models using the rods that they had. CT wrote the homework on the board, which asked the students to write about a classmate’s model for the problem they had been working on and to explain whether or not it was possible to have different solutions using different models. A sample of these submissions is presented in Appendix C.

Table 4.6

Forms of Reasoning, Session 6

	Student	Lines	Type	Structure	Form	Sub-codes	WC	PR
	Task 1a: Which is larger, one half or one third, and by how much?							
1.	Jessica	6.0.24	Claim	Direct		Incomplete	✓	
2.	Erik	6.0.25	Claim	Direct			✓	
3a.	Alan	6.0.28	Claim	Direct			✓	
4a.	Jessica	6.0.32	Counter-3	Indirect		Faulty	✓	
5.	Jackie	6.0.36	Counter-4	Direct			✓	
3b.	Michael	6.0.38	Counter-4	Direct			✓	
4b.	Jessica	6.0.39	Counter-3	Indirect		Faulty	✓	
6.	Erik	6.0.40	Counter-4	Direct			✓	
4c.	Erik	6.0.42	Counter-3	Indirect		Faulty	✓	
7a.	Michael	6.0.44	Counter-4c	Direct			✓	
4d.	Jessica	6.0.46	Counter-3	Indirect		Faulty	✓	
8.	Andrew	6.0.48	Counter-4	Direct			✓	
4e.	Erik	6.0.50	Counter-9	Indirect		Faulty	✓	
7b.	Michael	6.0.54	Counter-4	Direct			✓	
	Task 1b: What is the white rod called in model with the orange and red train that was built to solve the original problem?							
9.	Meredith David	6.0.75-6.0.81	Claim	Direct				✓
10.	James	6.0.88-6.0.90	Claim	Direct			✓	
11.	Beth	6.0.92	Claim	Direct			✓	
12.	Sarah	6.0.94-6.0.96	Claim	Direct			✓	
	Task 2: Which is bigger, one half or one quarter, and by how much?							
13a, b.	Michael	6.0.110-6.0.124	Claim	Direct				✓
14a.	Graham	6.0.136-6.0.137	Claim	Direct				✓
15a.	Alan	6.0.1446.0.152	Claim	Direct				✓
16a.	Alan	6.0.162-6.0.164	Claim	Direct	Recur			✓
16b.	Alan, Erik	6.0.168-6.0.174	Claim	Direct	Recur			✓
14b.	Graham	6.0.195	Claim	Direct			✓	
13c.	Michael	6.0.196	Claim	Direct			✓	
17a.	Amy, James	6.0.208-6.0.218	Claim	Direct		Faulty	✓	
18a.	Meredith	6.0.224	Counter-12a	Indirect			✓	
17b.	Jacquelyn	6.0.227-6.0.230	Counter-13a	Direct			✓	
18b.	Meredith	6.0.231-6.0.235	Couner-12b	Indirect			✓	
19.	Jacquelyn	6.0.236-6.0.252	Claim	Direct			✓	
20.	Danielle	6.0.260-6.0.264	Claim	Direct			✓	
21.	Andrew	6.0.267-6.0.269	Claim	Direct	Gener		✓	
16c.	Alan, Erik	6.0.271-6.0.277	Claim	Direct	Recur		✓	
22.	Andrew	6.0.285	Claim	Indirect		Faulty	✓	

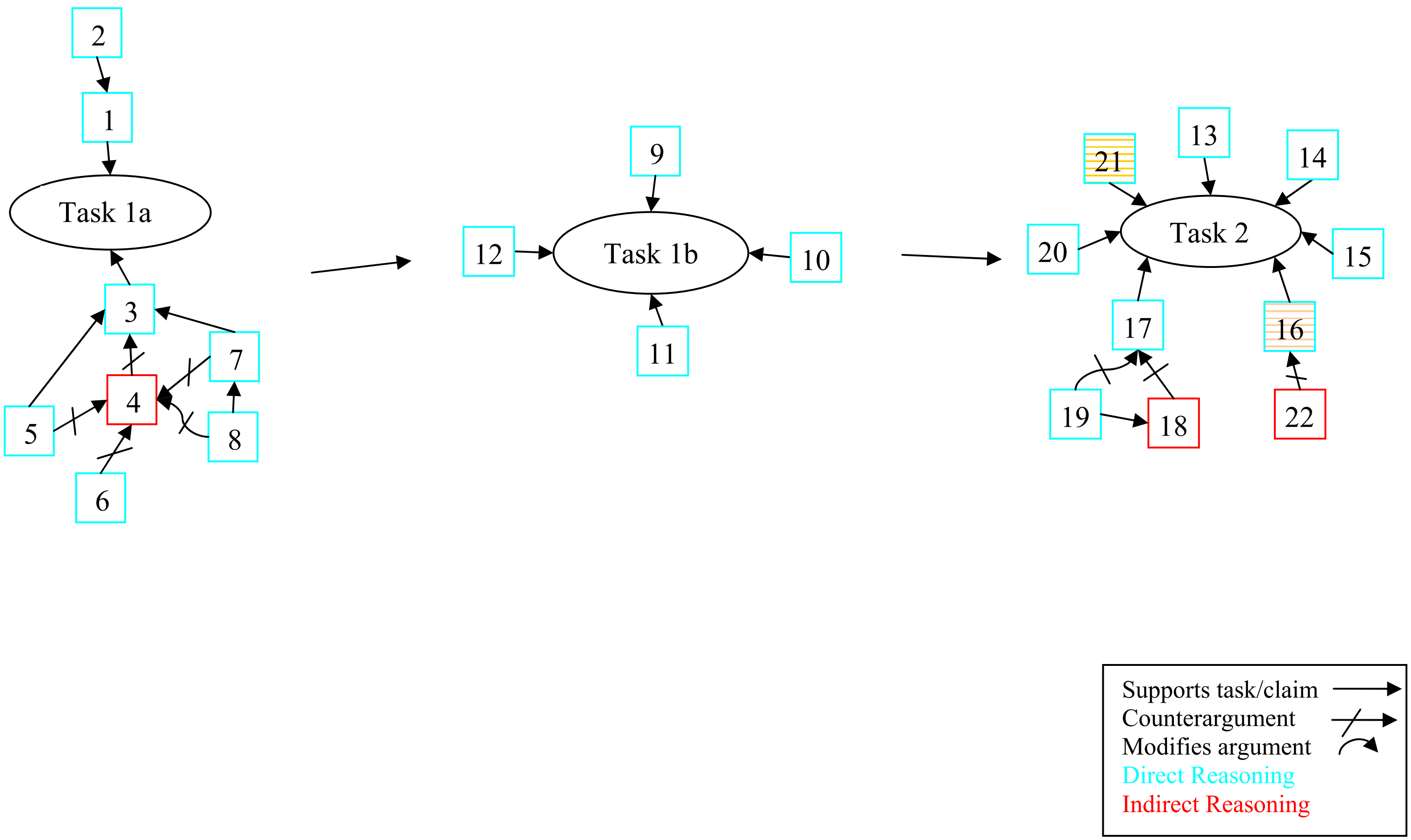


Figure 4.6. Argumentation Organized by Task, Session 6

4.2.7 Session 7: October 4, 1993

Class Discussion - The Fishing Boat

T/R 1 began this session with a discussion about a model that Mark had made for another class. The model was comprised of a boat, a fish, and two children. T/R 1 discussed the idea that if they all agreed to make the boat larger than the children and the children larger than the fish, they wouldn't each make identical models, but the sizes would be proportional. T/R 1 then asked the students what that had to do with the mathematics they were learning, and students replied that the models they built to compare fractions were often different in size, but the relationships between the fractions within each model was the same.

Task 1: Which is larger, one half or two thirds, and by how much?

1a T/R 1 then asked the students to build a model to show which is bigger, two thirds or one half, and by how much. The students worked with their partners to solve the problem. David and Meredith worked together, and Meredith built a model using a dark green rod, two light green rods, and three red rods. David followed suit, and they each lined up six white rods against their model. David concluded directly that "it's bigger by one sixth" (line 7.0.130).

2 Beth and Sarah built a model identical to that of David and Meredith, but they did not use white rods in their model. Beth showed that a red and white train was equal to a light green train and began to explain to T/R 1 that one half was "one unit" more than one third. However, T/R 1 asked them to remember what the problem task was, and Beth then compared two red rods with a light green rod, showing that the difference was one white rod. Thus, Beth and Sarah used direct reasoning to partially complete the task.

1b T/R 1 then spoke to David and Meredith about their solution to the task. Meredith and David said that two thirds is larger than one half by one sixth. Meredith justified the solution by showing that six “ones”, or white rods, were lined up against the dark green rod. T/R 1 asked Meredith to explain her language, and Meredith corrected it by saying that the white rod were called sixths and that six sixths equaled the dark green rod. She then showed directly that the difference between two red rods and one light green rod was one white rod, or one sixth (lines 7.0.145-7.0.166).

1c Five minutes after this exchange, T/R 2 approached David and Meredith. Meredith showed T/R 2 her original model along with a second model that she had built. Meredith explained that her second model showed that two purple rods, or two thirds, were larger than one dark green rod, or one half, by two white rods. Initially calling the two white rods “two sixths” (line 7.0.196), then changing the number name to one tenth, then one twelfth and then two twelfths (lines 7.0.198-7.0.202, Figure S-33-25). She then showed that in her original model, two thirds was larger than one half by one sixth (lines 7.0.204-7.0.208,). T/R 2 asked Meredith if there was anything else she could call the difference between two thirds and one half in the larger model, aside from two twelfths. Meredith said, “Um, yeah, well, maybe...” (line 7.0.212) and lined up six red rods against the larger model. She concluded, using direct reasoning, that the difference was one sixth (Figure S-35-22).

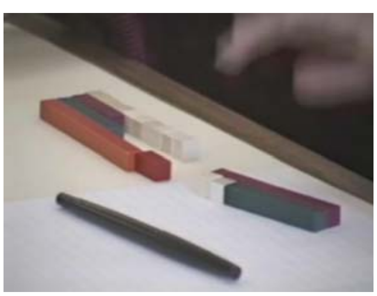


Figure S-33-25

Figure S-35-22

1d-f T/R 2 then questioned David about the models that he had built. David had three models on his desk, one using the purple rod as one, one using the dark green rod as one, and one using the orange and red train as one (Figure S-35-46). He began to explain his smallest model, composed of a purple rod, a light green and white train, and two red rods, but then changed the model so that it looked like his original model using the dark green rod, two light green rods, and a red rod. He then explained, “Alright, the dark green is one, and then the red is two thirds, and then the light green is one half, and then the white to the green is one sixth, so two thirds is bigger by one sixth” (line 7.0.222). He then showed T/R 2 his second model, which now was made up of an orange rod, a yellow and white train, and two light green rods. He said that the two light green rods were two thirds when the orange rod was one. When T/R 2 asked him to prove that statement, David lined up three light green rods against the orange rod and agreed that it was not the same length (lines 7.0.224-7.0.228). With that, he noticed the flaw in his reasoning and discontinued his argument. T/R 2 then asked him to explain his model using the orange and red train. David built a model similar to Meredith’s. However, he did not use white rods, but rather used red rods to show the difference between the two fractions. He explained, “Alright, then on this one, with the orange and the red, and then this [purple] is two thirds and that’s [dark green] one half, and then this is bigger by one sixth” (line 7.0.230). Here, David used direct reasoning to explain how his two models showed the solution.

Figure S-35-46

3a Alan and Erik worked together on this task. Alan first built a model using the dark green rod as one, and the two concluded that “[t]wo thirds are bigger by one sixth. And one half is one bigger than one third by one sixth” (line 7.0.239). He then built a model using the orange and red train and showed, using direct reasoning, that the difference between the two purple and dark green rods was a red rod, or one sixth. Erik agreed with his solution (lines 7.0.249-7.0.252).

3b Later, Erik and Alan explained their solution to T/R 1 and justified it using direct reasoning.

Erik:	Because if you have, we figured that, well, let me just see, right here, both models we have the halves and the thirds. Like, it was like the other problem, it was one half and one third. And we explained it, we said that one half was bigger than one third but smaller than two thirds. Like up here, there’s one half right there, and there’s the thirds, there’s the second third
T/R 1:	By how much?
Erik:	One sixth.
T/R 1:	But one half and two thirds.
Erik:	One- oh that’s exactly, that’s exactly what we meant. These are two thirds and that’s one half
Alan:	With one of the thirds, it would be a sixth. But if you added one, it would still be one sixth.

(lines 7.0.286-7.0.291)

4 T/R 3 worked with Gregory and Danielle on the task. Danielle built two models, one using the dark green rod as one and the other using an orange and a brown train, two blue rods, three dark green rods, and eighteen white rods. She showed directly that two thirds was larger by one half by three eighteenths (Figure F-38-58) when using the larger model, and that two thirds was larger than one half by one sixth when using the smaller model (lines 7.0.302-7.0.316). T/R 3 then questioned Danielle about the two solutions.

> T/R 3: Ok, so does that mean we have a different answer? No? This is different from the other one or the same?
> Danielle: It's different in a way and it's the same in a way
> T/R 3: How's it different and how's it the same?
> Danielle: Well, it's the same because the half is smaller and it's different because, um, this one, it only ta- the little box are only um, two three four, there's only six of them and here's there's eighteen, and this, the thirds are bigger by three eighteenths
> T/R 3: You mean, yeah, the two thirds are bigger by three eighteenths
> Danielle: and the two thirds over here is bigger by one sixth.
> (lines 7.0.317-7.0.322)

Meanwhile, Gregory tried to find another model. With some prompting, he built a model using the orange and red train as one (Figure F-44-17). Danielle concluded directly from his model that two thirds was still larger than one half by two twelfths. Gregory, as well, concluded that the difference was two twelfths by counting the white rods in his model.

T/R 3 then asked Danielle if there was any way that she could show the difference between two thirds and one half in her larger model without using white rods. Danielle used the light green rod to show the difference, and, lining light green rods up against the model, concluded that the number name for the light green rod would be one sixth (lines 7.0.346-7.0.355, Figure F-47-56). Upon questioning by the researcher, she showed that the difference between two thirds and one half in Gregory's model could be represented using the red rods, and reasoned directly that the red rod was called one sixth in that model (lines 7.0.356-7.0.353).

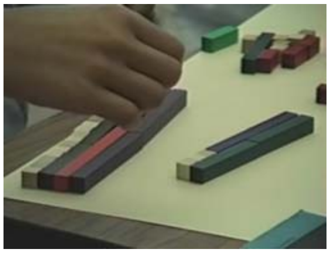


Figure F-38-58

Figure F-44-17

Figure F-47-56

5a,b Michael and Brian worked to build multiple models to show the difference between two thirds and one half (Figure O-35-30). T/R 1 asked them if the relationship held across the models they had built, if they expected that to happen and if they were convinced that it would. They responded in the affirmative. T/R 1 then asked them to write about it. Brian then began to explain their solution to Michael. Using the model of the dark green rod, he said,

Brian:	Because it takes six sixths to equal one whole [holding dark green rod]. And there are two sixths [holds two white rods and puts on top of the red rod], there are two sixths, in each, in each, in each third
Michael:	Hey! That may be right! Because a third for this one, a sixth for this one is one, [starts placing white rods alongside the second model that is 6 cm in length]
Brian:	And it takes, and it takes three sixths to equal up to one half. but, but, um
Michael:	And this would be red, it takes two of them to equal that. [Michael shows that two red rods equal a purple rod in the model that is twelve centimeters long. Figure O-40-29] Hey, that's neat!
Brian:	Sixths! That's what I did before
Michael:	It takes two sixths to equal a third! Wow! That's a neat thing to figure out fractions with.

Figure O-35-30

Figure O-40-29

Brian and Michael used generic reasoning to justify their solution to the problem and explain why their solution held for the many models that they built. Brian first used the six-centimeter long model to show that two white rods equal a red rod. Michael then verified his claim by lining the white rods against a second six-centimeter model, and showed that the same held true for the larger twelve centimeter model. Brian and Michael then worked to record their idea in writing.

Michael:	It takes, it takes two sixths to equal a third
Brian:	[inaudible] equal a third, no a half a half. That's what I wrote: 'Because it takes six sixths to equal one whole and, and
Michael:	And a sixth is always half of a third.
Brian:	Oh! [Michael laughs] And it takes, and there, and a sixth is a half of a third.
Michael:	Yeah!
Brian:	[after writing] Wait, what did I just say? Two sixths
Michael:	And two sixths, and it takes two sixths, two sixths, no, one sixth is half, is half of one third
Brian:	One sixth is … by one third, so it takes, so it takes three
Michael:	So it takes six, because there's two in every one so there's two four six. So it takes
Brian:	So it takes, should we write, so it takes three sixths to equal one half?
Michael:	No, it takes two, oh yeah, right, three equal one half.
Brian:	Two equal, two sixths equal one third, and three equal
Brian, Michael	One half
Brian:	So it takes three, oh I got it! So it takes three sixths to equal one half, but, so it takes three sixths to equal one half, but two thirds equal four sixths [Michael nods]. So it [back to writing] Wait, so it takes
Michael:	Three sixths to equal a half, but it takes
Brian, Michael	four sixths to equal two thirds. [laugh]
Brian:	And there's one extra. Yeah, and there's one sixth, look, look.
Michael:	Four, one two three four
Brian:	Look. See these, see these [Brian shows a light green rod and two red rods, and then places three white rods on the red rods]. Ok, now, you see there are three of them that are equal up to it, but
Michael:	There's one more
Brian:	Yeah
Michael:	To make two thirds
Brian:	Yeah, so there's one extra and it makes it bigger! So it takes three sixths to equal… and it takes, and it takes

Michael: Four
Brian: Four sixths, and it takes four sixths to equal two thirds, two thirds. And there's one [continues talking as he writes] Ok. This is what I wrote. So it takes three sixths to equal one half, but it takes four sixths to equal two thirds. But it needs, but it needs
Michael: But it needs?
Brian: Yeah, but it needs, but it needs four sixths. But it needs… to equal
(lines 7.0.419-7.0.445)

In this exchange, Brian and Michael used general terms to describe the pattern that they had noticed in the model they had built. For example, Michael explained that "a sixth is always half of a third" (line 7.0.421) and then continued to draw conclusions about the problem based on this general statement. As they discussed and wrote their general solution, they frequently referred to the model to show the correspondence between the general solution and the specific case of the model they had built.

6 Jackie explained her solution to T/R 1. She showed T/R 1, upon questioning, that in both the six- and twelve-centimeter models, the difference between the two fractions was one sixth. Jackie used direct reasoning to solve the task (7.0.397-7.0.414)

1g,7 T/R 1 then called the class together. At the OHP, Erin, Jackie, and Jessica had built the six and twelve centimeter models. T/R 1 asked the class to provide the solution to the problem, and they answered in unison that two thirds was larger than one half by one sixth. T/R 1 then asked if anyone had built a model that gave another solution. Meredith indicated that she had, and she was asked by T/R 1 to tell the class what she found. Meredith lined twelve white rods against the model using the orange and red train. Before she explained what she had done. Michael began to shake his head in disapproval. T/R 1 asked Michael what was wrong. Michael said, "No, they can't do that. Because um, the, the two thirds are bigger than the half by a red. So they can't use those whites to show it" (line 7.0.460). This argument, if anything, uses a simplified version of indirect reasoning. T/R 1 asked the class what rod they used to represent one sixth and they replied that they had used the red rod. T/R 1 said, “Well, she showed it's bigger by the two whites” (line 7.0.480). Michael replied, using indirect reasoning, “Yeah, but then she would have to call the two whites together one sixth” (line 7.0.481).

8ab,7c Erik then offered his view on the matter. He said, “Yeah, but see just the whites together. That'd be right, it would be two twelfths. But you have to combine them. You can't call them, you can call them separately, but you could also call them combined and if you combine them it would be uh, one sixth” (line 7.0.488). T/R 1 pointed out that Meredith was calling it two twelfths as well as one sixth. Meredith said, “There's two answers” (line 7.0.496). Michael and Erik responded “No, they're the same answer” (line 7.0.498). Erik explained directly, “No, they're the exact same thing, except she, she took the red and divided it into half, she divided it into halves, into half and called, and called each half one twelfth. They're the exact same answer except they're just in two parts” (line 7.0.499)

8c,9 T/R 1 then began to record the students’ arguments using mathematical notation at the OHP (Figure O-58-56). She wrote 1R=2W. Erik then continued his direct argument, saying,

> And since she's calling a white rod one twelfth and the other white rod one twelfth and the red rod is really one sixth. But, when she calls them two twelfths, the two twelfths are actually just two white rods put together to equal a red, so it should be really, it's really one sixth.
>
> (line 7.0.511)

T/R 1 then recorded 1/12 + 1/12 = 2/12, and Erik pointed out that it’s also one sixth. T/R 1 then said and wrote that Erik had also said that 1/2 of 1/6 = 1/12, and that 1/6 = 2/12. Erik agreed. Erik then said,

> But I don't really think you could call, call them two twelfths because two twelfths equal exactly to the same size as one sixth. Well, if you want to you could call them, I guess. But I think it would be easier just to call them one sixth, then wouldn't want to exactly call them one twelfth and another twelfth. I'd just call them one sixth. Therefore I think you just really call them one sixth.
> (line 7.0.517)

Brian then added to the direct argument and said,

> Brian: There's just half of one, there's just half of one.
> T/R 1: So you're saying that one half of the one sixth is another way of saying one twelfth.
> Brian: They're just two answers.
> (lines 7.0.522-7.0.524)

8d Jessica then continued Erik's argument, saying, "What Erik said is that two whites equal one red, so it would be the exact same thing" (line 7.0.526). The class agreed that all the statements that T/R 1 had recorded were true, and T/R 1 closed the session by asking the students to write about the different models that they had built. A representative sample of the written work is included in Appendix C.

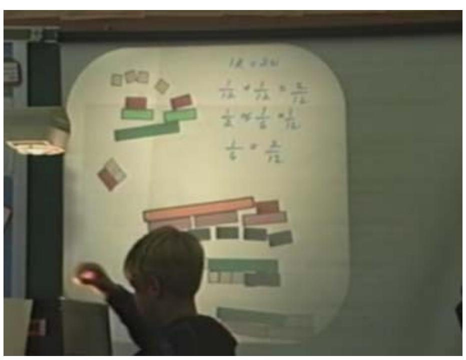
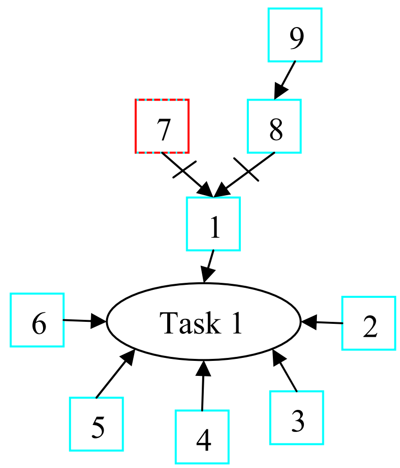


Figure O-58-56

Table 4.7

Forms of Reasoning, Session 7

	Student	**Lines**	**Type**	**Structure**	**Form**	**Sub-codes**	**WC**	**PR**
	Task 1: Which is larger, one half or two thirds, and by how much?							
1a.	David, Meredith	7.0.125-7.0.130	Claim	Direct				✓
2.	Beth, Sarah	7.0.131-7.0.143	Claim	Direct		Inc.		✓
1b.	Meredith, David	7.0.145-7.0.166	Claim	Direct				✓
1c.	Meredith	7.0.192-7.0.219	Claim	Direct				✓
1d.	David	7.0.220-7.0.222	Claim	Direct				✓
1e.	David	7.0.224-7.0.229	Claim	Direct		Faulty-e		✓
1f.	David	7.0.230	Claim	Direct				✓
3a-b.	Erik, Alan	7.0.234-7.0.252, 7.0.286-7.0.291	Claim	Direct				✓
4.	Danielle, Gregory	7.0.302-7.0.366	Claim	Direct				✓
5a-b.	Brian, Michael	7.0.385-7.0.396, 7.0.419-7.0.446	Claim	Direct	Gener			✓
6.	Jackie	7.0.397-7.0.415	Claim	Direct				✓
1g.	Meredith		Claim	Direct			✓	
7a.	Michael	7.0.460-7.0.472	Counter-1g	Indirect			✓	
7b.	Michael	7.0.481-7.0.484	Counter-1g	Indirect			✓	
8a.	Erik	7.0.484	Counter-1g	Direct			✓	
7c.	Michael	7.0.498	Counter-1g	Direct			✓	
8b.	Erik	7.0.496-7.0.499	Counter-1g	Direct			✓	
8c.	Erik	7.0.508-7.0.517	Counter-1g	Direct			✓	
9.	Brian	7.0.522-7.0.524	Counter-1g	Direct		Incomplete	✓	
8d.	Jessica	7.0.526	Counter-1g	Direct			✓	



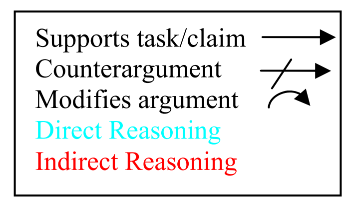


Figure 4.7. Argumentation Organized by Task, Session 7

4.2.8 Session 8: October 6, 1993

Task 1: Which is larger, one half or three fourths, and by how much?

This session was led by T/R 2, and, aside from an introductory whole class discussion, was comprised solely of group work on two tasks. The narrative for this session is organized by task and by group.

1a,b Michael and Brian worked together to compare one half and three fourths. Michael built a model of an orange and red train, two dark green rods, and four light green rods. He then compared the length of three light green rods and one dark green rod and concluded directly that three fourths is larger than one half by one fourth (lines 8.1.36-8.1.42). Two minutes later, T/R 2 asked Michael and Brian to explain their model. Michael repeated his direct argument, and Brian agreed with his justification (lines 8.1.60-8.1.75). Brian built a second model using the purple rod, two red rods, and four white rods (See Figure O-17-58 below for all three models that were built).

2 Michael tried to find another model, and he followed Brian's suggestion to use another length train that was "even" (line 8.3.95). Michael built a model using a blue and yellow train as one, and two black rods as halves. He then tried to find a rod that could be one fourth of the train. After trying the purple rods and seeing that they would not work, he used reasoning by upper and lower bounds to explain to Brian that the model didn't work.

Michael:	One less than this is gonna be [tries to use light green rods to make fourths] This can't be. Oh boy, this can't be done. Because there's not thirds to this, see, this doesn't work, this doesn't work. See this doesn't work, but the next size, Brian, you can't use this model
Brian:	What?
Michael:	You can't use this model, because if that doesn't work [purple rod] then this should [light green], but it doesn't, because this is the size

of this [shows that the light green rods were used for the model using the orange and red train - Figure O-16-21].
(lines 8.3.104-8.3.106)

1c Brian then built another model using a train of two blue and one red rod, two orange rods, and four yellow rods. He reasoned directly that this model also showed that three fourths was larger than one half by one fourth (Figure O-17-58, lines 8.3.111-8.3.131).

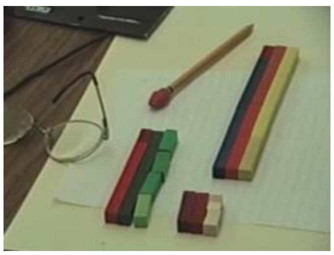


Figure O-17-58


Figure O-16-21


Figure O-22-34

1d Brian and Michael then decided to record their solution to the problem. Michael, with some assistance from Brian, found a fourth model (Figure O-22-34), and they each set out to record two models and explain why their solution made sense.

Brian: Ok three fourths is larger than one half by one fourth because, well, it takes two of em right here, look, here… well because it takes two of em [two white rods] to equal one half [the red rod], but the question is, but there are three of em

Michael: No, no, no, um if this is, this is a half and this is three. So it would be bigger by one fourth because it takes how many fourths does it take, it takes three fourths to equal um, Oh, this is confusing. It takes three fourths to equal

Brian: [interjecting] Why don't we just do what I said? It takes two fourths to equal one half, but the but but there's but but but it needs, but but it takes, but the question is three fourths, and so there's one fourth bigger [Figure O-34-34]

Michael: One fourth bigger? Yeah.

Brian: I guess it makes sense. [talking as he writes] is one half bigger, because it takes two fourths to equal one half,

Michael: [Figure O-33-46] I was gonna say because it takes two fourths to equal one half, but it takes three fourths to equal three fourths?
(lines 8.3.181-8.3.186)


Figure O-34-34

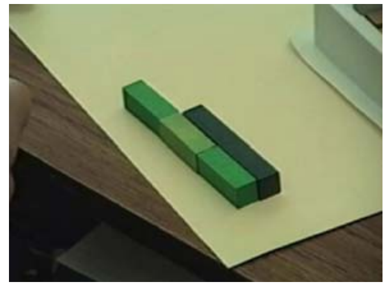


Figure O-33-46

As in the previous session, Brian and Michael used their models to write a general solution and justification to the problem. As they spoke, they referred to the models that they had built to ensure that their justification paralleled the specific model that they had used to think about the problem. Thus, they used generic reasoning when justifying their solution. T/R 2 then approached them and Brian read his justification to her, saying, “When you say it, it's very, very confusing” (line 8.3.193).

3a,b Kimberly and Audra worked together on the task. Kimberly built a model identical to Michael’s first model, and reasoned directly that three fourths was larger than one half by one fourth (line 8.1.45). She then set out to find another model, and built a model that was the same length using a different train as one (Figure S-14-16). T/R 2 questioned her about her two models, and Kimberly repeated her direct justification for her solution, showing the difference between three fourths and one half (lines 8.1.98-8.1.108). When questioned about her second model, she told T/R 2 that it was the same length as her first. T/R 2 encouraged her to find a model that was of a different length.

4,5 T/R 2 then worked with Mark and Laura. Two of Mark’s three models were identical to Kimberly’s, and Mark used direct reasoning to justify his solution using each model (lines 8.1.184-8.1.196). T/R 2 asked Mark if he thought that a different model might yield a different solution to the problem. Mark replied that if it was a flawed model, it might, but that otherwise he thought the solution would always be the same (lines 8.1.197-8.1.202). Laura also used direct reasoning to explain her solution to T/R 2. Laura's model used the purple rod as one, two red rods as one half, and four white rods as one fourth (lines 8.1.203-8.1.208).

6 Meredith shared her model with T/R 2. Using the orange and red train, she built a model to show halves, fourths, and twelfths. She reasoned directly that three fourths was larger than one half by three twelfths or one fourth. As she spoke, she showed T/R 2 that the length of three white rods was equivalent to that of one light green rod. T/R 2 questioned her about her second model, which used the purple rod as one, and she replied that in that model, the difference was one fourth (lines 8.1.244-8.1.258).

7a,8a Erik and Alan worked on the task together. Erik showed directly, using the orange and red train as one, that the difference between the two fractions was one fourth. Alan, using faulty reasoning, used the purple rods (which were equivalent to thirds in that model) to show that three purple rods were larger than one dark green rod (one half) by "two fourths, one half, or six sixths" (Figure F-14-44, line 8.2.82). Erik asked Alan why he was using thirds instead of fourths.

Figure F-14-44

7b Erik then explained his solution to Parish, a visiting researcher. He labeled his rods and explained that three fourths was larger than one half by one fourth (lines 8.2.87-8.2.94). Parish asked Alan if he agreed, and he said that he did.

7c,8b After this discussion, Alan built a second model to show the difference between three fourths and one half, this time using the brown rod as one (Figure F-20-36). T/R 2 then questioned Erik and Alan about their models, and Erik repeated his explanation (lines 8.2.135-8.2.137). Alan then used his new model to directly justify his solution of one fourth (lines 8.2.141-8.2.145).


Figure F-20-36

8c T/R 2 then noted that, although they had built different models, they had each arrived at the same solution. Alan stated, “Every time you make something like this, it will always be one fourth on this one if it's one fourth on that, and any other model that you make that can be like this it will always be one fourth” (line 8.2.146). He then used a model twice the length of his second model to show that the relationship held, and concluded that the difference between the two fractions would always be one fourth (Figure F-21-49, line 8.2.152). With this argument, Alan used partially generic reasoning to draw a general conclusion using the structure of a specific example, and referred to another example as he discussed that structure.


Figure F-21-49

9 CT worked with Amy, James, and Jacquelyn, and asked the group to justify their solution of one fourth. Using a model with black and yellow train that was twelve centimeters in length, James showed directly, with some assistance from Amy, that the difference between three fourths and one half was one fourth (lines 8.2.194-8.2.204)

10 Chris, a second visiting researcher, questioned Caitlin about her solution. Caitlin had built two models, one using the brown rod as one, and the other using the purple as one. She showed Chris, using direct reasoning, that the difference between three fourths and one half was one fourth (lines 8.3.165-8.3.170).

Task 2: Which is larger, two thirds or three fourths, and by how much?

The researchers introduced the second task to each group as they felt was appropriate. As a result, different groups worked for different lengths of time on each of the two tasks.

11a-c,12 Michael and Brian extended their model using the orange and red train to show thirds and twelfths in addition to fourths (Figure O-39-59). They used direct reasoning to show that the difference between two thirds and three fourths was one twelfth, and repeated their justifications to Parish and T/R 2 on separate occasions (lines 8.1.362-8.1.375, 8.1.387-8.1.392, and 8.1.442-8.1.481). Upon questioning by T/R 2, David used direct reasoning similar to Michael and Brian to show that the difference between the two fractions was one twelfth (lines 8.1.425-8.1.432).

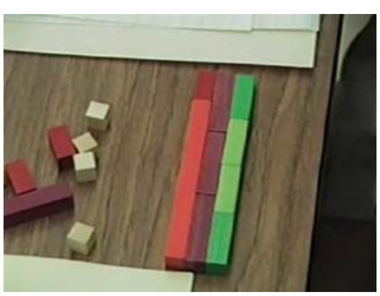


Figure O-39-59

13a-d Erik and Alan worked to extend their model using the orange and red train to show fourths and halves. They concluded that three fourths was larger than two thirds by one twelfth. Parish questioned Erik about his model. She asked Erik, "[H]ow much is three twelfths equal to?" (line 8.2.348). Erik replied that it was equivalent to one fourth. She asked him what he could call four twelfths, and he replied that it was equivalent to one third. Erik used direct reasoning with the model he built to answer these questions.

13c,d Alan and Erik then repeated their justification to T/R 2 (lines 8.2.403-8.2.420). She encouraged them to find another model that would show the solution to the problem. After much trial and error, they built a model that was twenty-four centimeters in length, using a train of two oranges and one purple rod, four dark green rods, three brown rods, and twelve red rods. They showed Parish that the difference between the two lengths was again one twelfth. Parish asked them what would happen if they lined white rods against this larger model. Erik and Alan used direct reasoning to name the white rods.

Alan:	No, that would be one twenty-fourths, because it takes two to make a red
Erik:	One twenty-fourth?
Alan:	Yeah.
Erik:	One twenty-fourth. I gotta see, wait, hold on, I just got a brain-something just popped into my brain.
Alan:	Yeah
Erik:	Two twenty-fourths
Alan:	Yeah, two twenty-fourths makes one twelfth and one twelfth is these.

(lines 8.2.522-8.2.528)

13e,f Erik then began to add white rods to the model to show that there were twenty-four. As he did that, Alan told T/R 2 that the difference between three quarters and two thirds was one twelfth or two twenty-fourths (lines 8.2.531-8.2.533). Alan then said "If you used three, you could still do the same answer as that, but you couldn't do it unless you had half of each of the little whites" (line 8.2.538). Although his reasoning was incomplete, it would appear that he was referring to using the light green rod as one twelfth, and that one twenty-fourth would not be able to be constructed unless there was a rod that was half the length of the white rod.

13g-i After Erik lined up the white rods, he concluded directly that the difference between the two fractions was one twelfth or two twenty-fourths. He repeated this solution to Parish soon thereafter, and again after he finished recording his solution.

14,15,16 Parish worked with Danielle and Gregory on this second task. Danielle showed her directly, using the twelve-centimeter long model, that three fourths was larger than two thirds, and began to line up white rods to show the difference between the two (lines 8.2.376-8.2.383). Gregory lined up six red rods against his model and concluded that the difference was one sixth (8.2.385). Danielle used indirect reasoning and said that Gregory's solution was incorrect, since the difference between three light green rods and two purple rods was only one white rod (lines 8.2.388-8.2.390). Gregory then said that the difference was one tenth, but the camera moved before he explained how he had arrived at that solution (lines 8.2.398-8.3.402).

17a,b Caitlin built a model using a blue and light green train as one. She reasoned directly that the purple rods would then be thirds and the light green rods would be fourths. When questioned by Chris, a visiting researcher, about the number name for the difference between the two, she lined white rods against her model and concluded, together with Graham, that the difference would be called one twelfth. Graham then repeated this line of reasoning to the researcher (lines 8.3.266-8.3.291).

Students' Written Work

T/R 2 asked the students to record their models so that T/R 1 could see what they had done. Most students did just that. A representative sample is included in Appendix C. Of special note are the explanations provided by Brian, Michael, and Andrew. Brian and Michael used generic reasoning to justify their solution to the task, and recorded their argument as described in the narrative. Andrew recorded four models to represent his solution. Above his first model, he wrote, “My first model looks like this. I think 3/4 is larger than 1/2 by 1/4.” He then wrote above his second model, “The second model is almost the same only biger (sic).” His third model contained the caption, “The 3rd model has the same thing of the 1st and the 2nd.” These comments are reminiscent of the generic reasoning that Andrew used in the sixth session when explaining that all models will show the same difference between one half and one fourth.

Table 4.8

Forms of Reasoning, Session 8

	Student	Lines	Type	Structure	Form	Sub-codes	WC	PR
	Task 1: Which is larger, one half or three fourths, and by how much?							
1a.	Michael	8.1.36-8.1.42	Claim	Direct				✓
1b.	Michael	8.1.62-8.1.74	Claim	Direct				✓
2.	Michael	8.3.104-8.3.106	Claim	Indirect	U/L			✓
1c.	Brian, Michael	8.1.121-8.1.130	Claim	Direct				✓
1d.	Brian	8.1.271-8.1.277, 8.1.284-8.1.297	Claim	Direct	Gener			✓
3a.	Kimberly	8.1.45	Claim	Direct				✓
3b.	Kimberly	8.1.97-8.1.107	Claim	Direct				✓
4.	Mark	8.1.185-8.1.195	Claim	Direct				✓
5.	Laura	8.1.202-8.1.207	Claim	Direct				✓
6.	Meredith	8.1.243-8.1.257	Claim	Direct				✓
7a.	Erik	8.1.61-8.1.81	Claim	Direct				✓
8a.	Alan	8.1.64-8.1.82	Claim	Direct		Faulty		✓
7b.	Erik	8.1.88-8.1.94	Claim	Direct				✓
7c.	Erik	8.1.135-8.1.137	Claim	Direct				✓
8b.	Alan	8.1.141-8.1.144	Claim	Direct				✓
8c.	Alan	8.1.146-8.1.154	Claim	Direct	Gener			✓
9.	Amy, James	8.1.192-8.1.204	Claim	Direct				✓
10.	Caitlin	8.3.165-8.3.170	Claim	Direct				✓
	Task 2: Task 2: Which is larger, two thirds or three fourths, and by how much?							
11a, b,c.	Michael, Brian	8.1.362-8.1.375, 8.1.387-8.1.392, 8.1.442-8.1.481	Claim	Direct				✓
12.	David	8.1.425-8.1.432	Claim	Direct				✓
13a, b, c, d, e.	Erik, Alan	8.2.219-8.2.257, 8.2.348-8.2.353, 8.2.403-8.2.420 8.2.505-8.2.508, 8.2.520-8.2.533	Claim	Direct				✓
13f.	Alan	538	Claim	Direct		Incomplete		✓
13g -i.	Erik	539	Claim	Direct				✓
14.	Danielle	8.2.376-8.2.383	Claim	Direct		Incomplete		✓
15a.	Gregory	8.2.385	Claim	Direct		Faulty, Inc		✓
16.	Danielle	8.2.388-8.2.390	Counter-15	Indirect				✓
15b	Gregory	8.2.398-8.3.402	Claim	Direct		Faulty, Inc		✓
17a, b.	Caitlin, Graham	8.3.266-8.3.291	Claim	Direct				✓

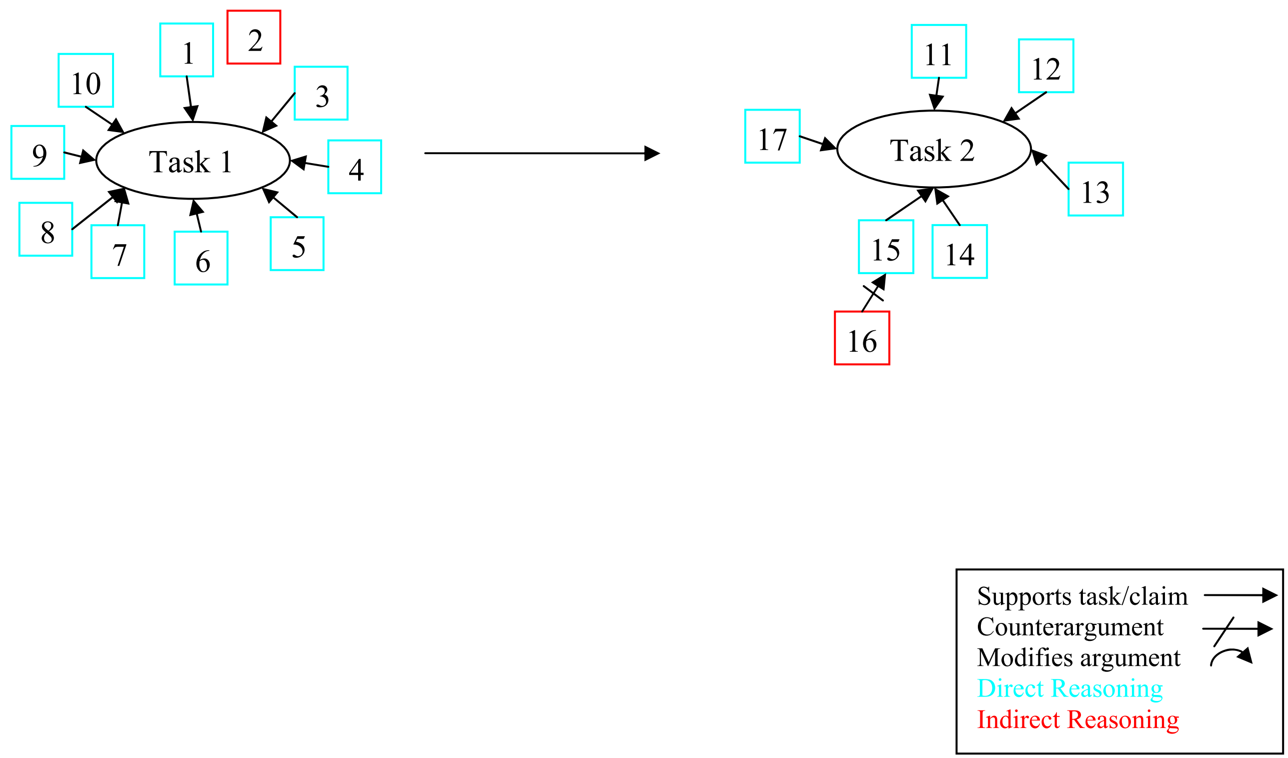


Figure 4.8. Argumentation Organized by Task, Session 8

4.2.9 Session 9: October 7, 1993

Task 1: Which is larger, two thirds or three fourths, and by how much?

During this session, T/R 1 provided the students an opportunity to revisit the task that they had been introduced to during the previous session: Which is larger, two thirds or three quarters, and by how much. The students were encouraged to build more than one or two models. The video data lent insight into the reasoning of four groups of students, whose ideas about this task will be traced below.

1 Erik and Alan began to work on the problem by building the models that they had found the day before. First, Erik built the twenty-four centimeter-long model, and T/R 1 asked him to reconstruct the other model that they had found. After some trial and error, they succeeded in building the twelve centimeter-long model, first by using a train of two yellow rods and a red rod, then modifying that train to an orange rod and a red rod (Figure F-26-26). Erik and Alan used direct reasoning to explain to T/R 1 that the first model showed that the difference between the two lengths was one twelfth or two twenty-fourths, and that the first model showed that the difference was one twelfth (lines 9.2.172-9.2.203).

Figure F-26-26

2 Alan then stated that the larger model was the only model that could show the difference in twenty-fourths. He explained indirectly that if one wanted the smaller model to contain twenty-fourths, the white rods would need to be split in half (line 9.2.204).

3 T/R 1 then challenged Erik and Alan to find another model. Erik proposed using a train of three orange rods. He then tried using a longer train of orange rods. During his exploration, he stated that the brown rods were equivalent to ten white rods, and that the orange rods were equivalent in length to twelve white rods. Alan used direct reasoning to show Erik the flaw in his assumption. He told Erik to line up white rods against the orange rod, and Erik concluded that the length of the orange rod was equivalent to ten white rods (lines 9.2.281-9.2.290).

4, 5a Erik then built a model using three orange rods and a dark green rod. Using direct reasoning, he found that the blue rods could then be called fourths, and showed Alan what he had found. Alan then challenged him to find thirds.

Alan:	Thirds. Erik, there's one prob. Using oranges, you can't third. You can't third, look, even if you subtracted two you couldn't third that. Because orange is twelve, there's five.
Erik:	Oranges are tens!
Alan:	I know, tens, you can make it into fourths but you couldn't third it.
Erik:	Wait you gave me, oh no.
Alan:	You just gave up
Erik	Yup.

(lines 9.2.326-9.2.331)

In this discussion, Alan used reasoning by upper and lower bounds to show Erik that no rod could be one third of the train that he had built. He said that the orange rod was ten white rods long, implying that the train was more than thirty white rods in length, and therefore none of the rods, the largest of which was the orange rod, could be called one third of the train. Erik agreed.

5b,6a Erik then checked what Andrew and Jessica were building. Andrew had built a train of two oranges and a red rod followed by another two oranges and a red rod. Erik used Alan’s argument, telling Andrew, “That's way too big, Andrew, I don't think you

204

can divide it into anything" (line 9.2.333). Andrew countered Erik's argument with a direct argument. He said, "Yeah, if you make two browns, two blues are thirds. If you can make a train for a whole you can make a train for a third and a fourth" (line 9.2.334). Erik then left Andrew to join T/R 1 and Alan at David and Meredith's table (see 9c below

6b Andrew then continued his direct justification as he spoke to Jessica. Pointing to the blue rods, he said, "See? Two of these are thirds, and that's a one third, third, third" (line 9.2.339). Moving the red rods to the end of the train and regrouping the four orange rods, he lined up eight dark green rods against his model.

7a Meanwhile, David and Meredith worked on building models to represent their solution. David first built two models to show halves and quarters, and T/R 1 pointed out that the problem was asking to find the difference between two thirds and three fourths. T/R 1 suggested that David listen to Meredith's solution, which she had shared with CT. Meredith had built a twenty-four centimeter-long model using a blue, black, and brown train. She lined up four dark green rods, three brown rods, and four red rods. She reasoned directly and said, "If you call all these, this one, and these fourths and these thirds, and you take twelve reds, you can call them twelfths, it would be bigger, if you take three thirds, three fourths would be bigger by one twelfth" (line 9.1.137). As she spoke, she lined two white rods against one red rod in her model. She then said, "Or it could be bigger by two twenty-fourths" (line 9.1.139, Figure S-24-44).

Figure S-24-44

7b,7c T/R 1 then asked Meredith to explain her second model to David. Before letting her do so, she told David and Meredith that their next challenge would be to think about what a third model would look like, and what they would call each of the rods in that model, even if they didn't build the model. Meredith then used direct reasoning to show David that in her twelve centimeter-long model, the difference between the two fractions was one twelfth (lines 9.1.145-9.1.147). Later, Meredith re-explained her two models to T/R 1, who encouraged them to think about the challenge that she had posed (lines 9.1.183-9.1.190).

8a Meanwhile, Brian and Michael worked to find models to represent the problem. Brian explained their model using an orange and red train to CT, and concluded directly that the difference between two thirds and three fourths was one twelfth (lines 9.1.155-9.1.163).

9a David and Meredith thought about what a third model would look like.

David: I think that this one [holding a red rod] might be one twenty-fourth, because

Meredith: No, because these are twenty fourths. These are twelfths. Well, if it was double the size of this

David: Yeah, I know, then this would be one twenty-fourth, and then this would be one, one forty-eighth, or something, yeah one forty-eighth…then we might be using something like this, and this would be something like one twelfth or something.

(lines 9.1.197-9.1.199)

Thus, David conjectured, and Meredith rationalized using direct reasoning, that, in a third model that was twice the size of her first one, the red rods would be called one twenty-fourth, and that the white rods would be called one forty-eighth. However, he incorrectly reasoned that the light green rod would be called one twelfth. This faulty part of the argument was only peripheral to the central line of reasoning.

8b Michael then found that the twenty-four centimeter-long model could show the difference between two thirds and three fourths. He told T/R 1 that the difference would be one twelfth, and showed directly that the red rod was that difference (lines 9.1.221-9.1.230).

9b,9c David then repeated his conjecture to T/R 1. T/R 1 called Alan and Erik over to David's desk and asked him to repeat his conjecture in their presence. David said, "Well, before, we had this other one, um, where the whites were one twenty-fourth and the reds were one twelfth. But then if we double that, then the reds would be one twenty-fourth, the whites would be one forty-eighth, and then the light green would be one twelfth" (line 9.1.273). T/R 1 suggested that David, Meredith, Alan, and Erik combine supplies and try to build a model to test David's conjecture on the floor at the front of the room. Erik then told T/R 1 about Andrew's model and his way of finding thirds and fourths, and joined the others on the floor.

6c Andrew then showed T/R 1 his model. T/R 1 called Brian and Michael over to Andrew's desk to hear his reasoning. Andrew used direct reasoning to explain the components of his model.

Andrew:	[Figure F-42-07] I took two browns and minded them as thirds, one third, and then two browns is one third, and two greens is one fourth, and then the purple would be one twelfth.
Brian:	Oh! I get it - Ahah! I think I have one now - look! Those are eight, this is twenty four, Mike, twenty-four, look, Mike, I have one!
T/R 1:	So how many twenty-fourths would it be with reds?
Andrew:	Twenty four, so the red would be one twenty-fourth.
T/R 1:	Ok, would the difference be one twenty-fourth?
Andrew:	No, the difference is, let's see, three fourths, the difference is one twelfth.
T/R 1:	One twelfth. What is the difference in twenty-fourths?
Andrew:	Um, two twenty-fourths.
T/R 1:	Two twenty-fourths, ok? Now could you subdivide it smaller than the red?

Andrew: Yeah, you could divide it into smaller by taking, by taking two whites and putting them up against everything.
T/R 1: Ok, you know how many of those there'll be?
Andrew: Well, there'd be, let's see, two times twenty-four is... it would be forty-eight.
T/R 1: Forty-eight? Ok. So in forty-eights, what would your answer be?
Andrew: Four. [Figure S-44-41]

(lines 9.1.324-9.1.337)


Figure F-42-07


Figure S-44-41

6d T/R 1 asked Andrew to record his model and to ensure that Jessica understood his model. Andrew explained, using direct reasoning, that since two white rods were equivalent to one red rod, and two times twenty-four was forty-eight, the white rods would be called forty-eighths (lines 9.1.342-9.1.345).

5c,5d David, Meredith, Erik, and Alan worked to build models on the floor. David built a model to test his conjecture, while Erik continued working on the thirty-six centimeter model that he had begun building previously. He made a train of three orange and one dark green rod and lined up four blue rods. Alan continued to use his indirect argument to show that Erik's model would not work, saying, "You can't third something like this. You'd need colossal rods" (line 9.2.379). He used his implicit upper bound argument again, saying, "Using oranges, if you use three oranges, you won't be able to third it. You won't be able to third it" (line 9.2.389).

4b,4c Erik then showed Alan that he had found a way of showing thirds in his model. He lined up three blue rods and nine white rods against the model (Figure F-48-54) and showed that three white rods could be added to each blue rod to complete the third. “I thirded it. One two three and then plus nine other of those, which would be one two three four five six seven eight nine. So it's just like making a new rod” (line 9.2.423). Erik was asked by a visiting researcher to re-explain. He said, “Ok. I have the three of 'em, and then I put nine other ones which would equal another blue, so if I thirded it, I would add one to there, one to there, and one to there, which would be three. And then four five six seven eight nine. So it's like adding another blue, but I'm making a new rod” (line 9.2.426). Erik used direct reasoning to support his solution.

Figure F-48-54

4d,4e The researcher asked Erik if there was another way to arrange the rods that he had used to show thirds. Meredith placed three white rods after each blue rod (Figure F-50-13), and Erik explained, using direct reasoning, what she had done.

> Ohhh! See, there are there to that, three to that, and three to that, so it's like, it's a blue plus one would be an orange, plus another would be a new rod, plus another would be a new rod, and if you have another one, it'd, you'd, you're just making new rods. Because if you add one of those to that, it'd be an orange, but then you add another two it'd be bigger than an orange.
>
> (line 9.2.437)

Figure F-50-13

4f,9d Soon afterwards, Erik repeated the explanation to CT (line 9.2.457). Then, he looked at what David was doing and wondered aloud about it. Then, he remembered that David had made a conjecture, and repeated it.

Erik:	I don't, I don't really understand what Dave's doing. That's the only problem. Actually, no, I do. He's calling two browns, two blacks, and two blues a one.
Meredith:	Yeah, cuz that was twice the other
Erik:	Yeah, and then the light greens are the twelfths and those
David:	I think that'd be sixteenths though
Erik:	Yeah, and the reds would be the twenty-four- the twenty-fourths. The reds would be the twenty-fourths and the whites would be the forty-eighths. Because he doubled everything.

(lines 9.2.461-9.2.465)

At this point, the other students understood David's direct argument, and David corrected his previous statement regarding the number name for the light green rod. Erik said, "Isn't this basically what we came here for" and demolished his thirty-six centimeter model, turning to concentrate on David's work.

9e,9f David then explained to CT why he had built the model and what he had found about the light green rod (lines 9.2.488-9.2.501). David and the others then worked to perfect the model (Figure F-57-57), which was comprised of the train of two blue, two black, and two brown rods, sixteen light green rods, twenty-four red rods, and twelve purple rods. Toward the end of the session, David showed the model to T/R 1. Pointing to the red rods and counting by two, David and Erik found directly that the white rods would indeed be called forty-eighths (lines 9.2.556-9.2.560).


Figure F-57-57

9g,9h David noted that he was surprised that the purple was one twelfth, rather than the light green. T/R 1 asked the students if they could think of other number names for the purple rod aside from one twelfth. David first said that it would be four twelfths, but then Erik, using direct reasoning, said that it would be called four forty-eighths, since four white rods equaled the length of the purple rod. David then said that that was what he had meant. Erik then suggested that it be called two twenty-fourths (lines 9.2.581-9.2.602). T/R 1 then asked if there were any other number names. Meredith proposed that it would be one sixteenth and one forty-eighth. Erik then said that the purple rod was half the brown rod, but did not complete his reasoning. T/R 1 asked her what number name that would be, but the session ended before they had a chance to continue this discussion. T/R 1 suggested that they continue to think about this question (lines 9.2.607-9.2.627).

Table 4.9

Forms of Reasoning, Session 9

	Student	Lines	Type	Structure	Form	Sub-codes	WC	PR
	Task 1: Which is larger, two thirds or three fourths, and by how much?							
1.	Erik, Alan	9.2.172-9.2.203	Claim	Direct				✓
2.	Alan	9.2.204	Counter	Indirect				✓
3.	Alan	9.2.279-9.2.290	Counter	Direct				✓
4.	Erik	9.2.303	Claim	Direct				✓
5a.	Alan	9.2.326-9.2.331	Counter-4	Indirect	U/L			✓
5b.	Erik	9.2.333	Counter	Indirect	U/L			✓
6a.	Andrew	9.2.334	Counter-5b	Direct				✓
6b.	Andrew	9.2.339	Claim	Direct				✓
7a.	Meredith	9.1.137-9.1.139	Claim	Direct				✓
7b.	Meredith	9.1.145-9.1.147	Claim	Direct				✓
8a.	Brian	9.1.155-9.1.163	Claim	Direct				✓
7c.	Meredith	9.1.183-9.1.190	Claim	Direct				
9a.	David	9.1.197-9.1.199	Claim	Direct		Faulty-p		✓
8b.	Michael	9.1.221-9.1.230	Claim	Direct				✓
9b.	David	9.1.263-9.1.265	Claim	Direct		Faulty-p		✓
9c.	David	9.1.273	Claim	Direct		Faulty-p		✓
6c.	Andrew	9.1.324-9.1.337	Claim	Direct				✓
6d.	Andrew	9.1.339-9.1.345	Claim	Direct				✓
5c.	Alan	9.2.377-9.2.379	Counter-4	Indirect				✓
5d.	Alan	9.2.385-9.2.391	Counter-4	Indirect	U/L			✓
4b,c.	Erik	9.2.419-9.2.426	Claim	Direct				✓
4d.	Meredith	9.2.430-9.2.436	Claim	Direct				✓
4e.	Erik	9.2.437	Claim	Direct				✓
4f.	Erik	9.2.457	Claim	Direct				✓
9d.	Erik, David, Meredith	9.2.461-9.2.465	Claim	Direct				✓
9e.	David	9.2.488-9.2.501	Claim	Direct				✓
9f.	David, Erik	9.2.556-9.2.560	Claim	Direct				✓
9g.	David, Erik, Meredith	9.2.581-9.2.601	Claim	Direct				✓
9h.	Erik, Meredith	9.2.608-9.2.627	Claim	Direct		Incomplete		✓

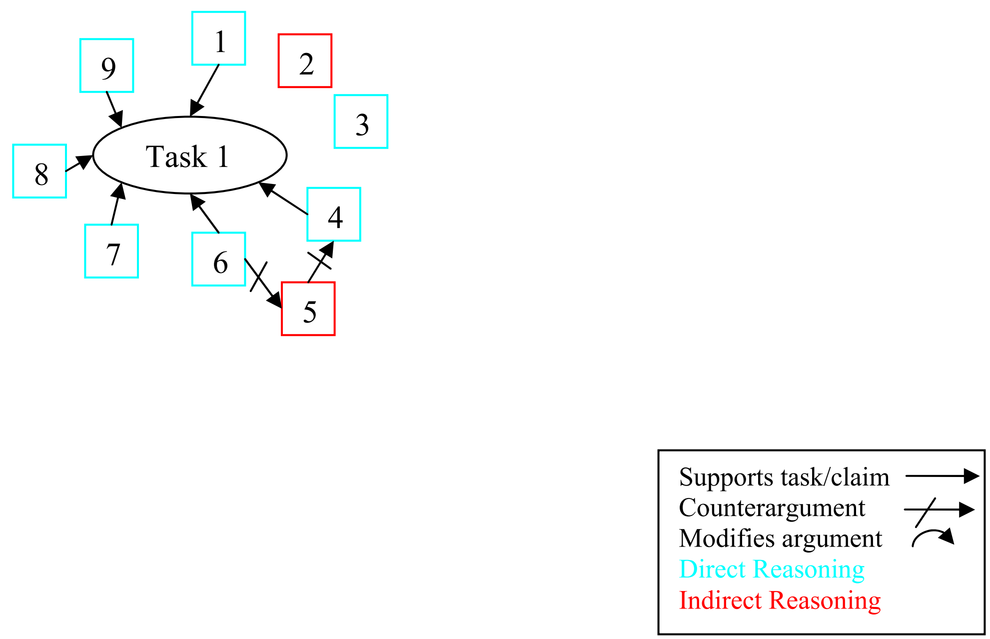


Figure 4.9. Argumentation Organized by Task, Session 9

4.2.10 Session 10: October 8, 1993

Task 1: Which is larger, two thirds or three fourths, and by how much?

1a In this short, forty minute session, T/R 2 provided the students an opportunity to briefly discuss the task that they had been working on during the previous session. She asked them what they had been doing, and Andrew and Jessica described that they had attempted to make large models and use trains for thirds and fourths. T/R 2 asked the students if they had all recorded their large models, and Andrew said that he had. T/R 2 asked the students how many models they thought could be built. Erik made a conjecture, saying that there would be a lot of models, and justified it by using the recursive reasoning that he and Alan had used to build models to show halves and fourths during the sixth session.

> Well, because see, what me, Alan and I figured, is if you start with one rod, and you can divide one rod that's a large number into thirds and fourths, then you just count down by two, because we think that even numbers you can divide into fourths and thirds, but odd numbers you can't, so it was like, if we started with the orange rod… you could probably divide it into thirds and fourths. And then just go down two and then just keep going down until whatever number you get and then you'll just keep going down and you should be able to.
> (line 10.2.23)

With this, Erik attempted to generalize the doubling pattern that he had noticed. However, his reasoning was faulty, since that pattern only applied to a narrow set of fractions.

2 Alan interjected, repeating his argument using upper and lower bounds that he had used during the previous session. He said, "We also realized that the bigger, like if you put … four, you couldn't third that unless you made a new rod using two others to be bigger than the orange" (line 10.2.24). He then re-explained, saying, "Like four oranges you can't third it without making a new rod. But three oranges you could call that a whole and have three more oranges as the thirds" (line 10.2.28).

1b,3,1c T/R 2 asked the class what they thought of Erik's original conjecture. Erik then modified his conjecture, saying that most even rods could be used to build models to show thirds and fourths (lines 10.2.30-10.2.32). Michael then noted that he agreed with the conjecture, but that he had found that some of the even rods could not be divided into thirds and fourths (lines 10.2.34-10.2.36). Erik then made a faulty claim, saying that the dark green rod was an example of an even rod that could be divided into thirds and fourths (10.2.38).

4a T/R 2 asked Erik what he meant when he called a rod even. Erik replied, "Well, a rod that if you put all of the whites up to it… all the whites real tight, and you determine if you can divide it in half" (line 10.2.42-10.2.44). Thus, Erik used direct reasoning, stating that if the number of white rods that equal the length of the train could be divided in half, the rod could be called an even rod. David added to this definition.

> I want to comment that an even rod, is, before, when I got up there, maybe about like a week ago, um, I said that like the white would be one, the reds would be two, so the reds are even, and then the light greens are three… they're odd, and then the purple is even.
>
> (lines 10.2.46-10.2.48)

T/R 2 suggested that the students who hadn't finished recording the large models they had built to represent the difference between two thirds and three fourths should work on that, and that those who had already recorded their models would begin working on another task. David, Erik, and Meredith attempted to reconstruct their large model on the floor. However, David recalled the model incorrectly, and instead of building a model using two brown rods, two black rods, and two blue rods as one, they built a model using two black rods, two blue rods, and two orange rods as one. They then lined up purple and light green rods against the train (Figure F-21-35). A visiting researcher asked the group to explain what they were doing.

Figure F-21-35

5a David described that Meredith had originally built a model using one black, one blue, and one orange rod, and that, on that model, the red rods were called one twelfth and the white rods were called one twenty-fourth. He said that he had predicted that if the model was doubled, the red rods would be called one twenty-fourth and the white rods would be called one forty-eighth. He then added that the light green rods were called one seventeenth, and that he had thought they would be called one twelfth. Erik corrected that statement, saying that the light green rod would be called one sixteenth (lines 10.2.85-10.2.92). Thus, although David and Erik's model was faulty in its execution, David and Erik used direct reasoning to explain what the model was supposed to show.

6 David added that this model couldn't show thirds or halves (line 10.2.92). Meredith extended this argument, saying "You would need a new model, maybe. If you put ten up to it, it won't do it" (line 10.2.93). Here, David and Meredith used reasoning by upper and lower bounds similar to Alan's argument during the previous session.

Erik then noted that the sixteenths were imprecisely lined up. The researcher commented that he was not quite convinced that the light green rods were sixteenths, and mentioned that the model looked different than it had the day before. Meredith noted that a white rod was needed to make the length of the light green train equivalent to the train that was called one (line 10.2.113).

The group began to discuss how to modify the model to make it work. Erik suggested substituting a brown rod for a black rod, and Meredith suggested using another blue rod instead of the second black rod. She then suggested removing the purple rods from the model. Erik objected, saying that there would be no twelfths if the purple rods were removed. David then said that the light green rods could be removed since they were not needed for the problem. Meredith then suggested replacing a black rod with an orange rod. Erik then tried to measure the length of the model using the meter stick, finding that its length was "fifty-two and a half" centimeters long (line 10.2.148).

5b The group then digressed from the activity at hand and began to make a long train using all their rods. The researcher suggested that they get back to work since the session would soon come to a close. They again built their flawed model, this time lining up purple and red rods. T/R 2 asked them to explain their model.

Erik:	[Figure F-36-17] Well, we have, as the whole we have two oranges, two blues and two blacks, because David said that Meredith made an original model that was one orange, one blue and when black, and then-
David:	[joins in] One orange, One blue, and one black, and then, well, she had um, the reds were one twelfth and then the whites were one twenty-fourth, and then
Erik:	We did, we doubled… two oranges two greens and two blacks
David:	Instead of one orange one blue and one black.
Erik:	The purples would be the twelfths, the reds would be the um twenty-fourths

(lines 10.2.180-10.2.186)


Figure F-36-17

7 The visiting researcher told them that he was not convinced that the purple rods were one twelfth in this model. Erik counted the purple rods and realized that they had made a mistake. Using direct reasoning, he and David concluded that the purple rods were called one thirteenth in the model they had built. David added that the light greens in this model were one seventeenth instead of one sixteenth (lines 10.2.193-10.2.202). However, the session closed before they determined the root of that mistake.

Task 2: Compare one half or two fifths. Which is larger, and by how much?

8a Meanwhile, Alan worked alone at his desk on a new task that T/R 2 had posed. He tried to build models to find the difference between one half and two fifths. Alan showed T/R 2 two models that he had built. The first was composed of an orange rod, five red rods, two yellow rods, and ten white rods. Reasoning directly, he said that one half is larger than two fifths by one tenth, showing that the difference between the length of one yellow rod and two red rods was one white rod (Figure F-17-22). He then showed T/R 2 his second model, which used a train of brown and red instead of the orange rod, and explained that that was the only difference between the two models (lines 10.2.59-10.2.61).

Figure F-17-22

8b T/R 2 asked Alan if he could try to find a model that was of a different length. Using a model that he had built with two blue rods and four purple rods, Alan quickly constructed a model using two orange rods, five purple rods, another set of two orange rods, and ten red rods (Figure F-19-01). He showed T/R 2 that the difference between an orange rod and two purple rods was one red rod, directly verifying his original solution (10.2.69-10.2.71).

T/R 2: Can I ask you a question now? Why did you choose the two oranges to be one? You seemed to come up with that pretty quickly.

Alan: Because up here, I knew that this was ten, and two tens would be twenty, and I knew that that would work, so it takes two of those to complete it using a double ten. So one of those [points to red rods] filled in the gap. Probably if you used another one [takes a third orange and gestures to show that a fourth orange rod would be placed along with the first three] another two, you could fill in that with more purples and using more reds, too.

(lines 10.2.72-10.2.73)

With this argument, Alan used recursive reasoning to explain that he had derived the second model from the first, and predicted what a third model would look like. T/R 2 encouraged him to test his prediction and build the third model.

Figure F-19-01

8c Ten minutes later, T/R 2 returned to Alan and questioned him about the third model that he had built (Figure F-30-58). Using direct reasoning, he showed that two of four orange rods would be called one half. He explained that since he had doubled the length of his model, the purple rods were now called tenths, and that the brown rods were called fifths and the red rods were twentieths. Alan showed that the twenty white rods that had been placed on his model would now be forty white rods, and that they would be called fortieths (line 10.2.154).

Figure F-30-58

8d,8e Alan then extended his argument. Placing five blue rods on his model (Figure F-31-18) he said,

> You can't make the model any bigger than this, you would have to use one blue. It wouldn't be the exact size. So you can't make a model any bigger than this, without making a train, making all these uneven. So basically, this is the only model you can make that's even without using trains, like this one here, that would make all of these unequal.
>
> (line 10.2.154)

Figure F-31-18

Reasoning recursively, he said that the next size model would need another four orange rods added to the train to be called one (line 10.2.156). Although Alan did not complete his argument using upper and lower bounds, it appears that he intended to say that this was the largest model that could be built using the pattern that he had found, since five blue rods would be shorter than the length of the eight orange rods. Alan’s written work is included in Appendix C.

Table 4.10

Forms of Reasoning, Session 10

	Student	Lines	Type	Structure	Form	Sub-codes	WC	PR
	Task 1: Which is larger, two thirds or three fourths, and by how much?							
1a.	Erik	10.2.23	Conjecture	Direct		Faulty	✓	
2.	Alan	10.2.24-10.2.28	Claim	Indirect	U/L		✓	
1b.	Erik	10.2.29-10.2.32	Counter-1a					
3.	Michael	10.2.34-10.2.36	Counter-1a	Indirect			✓	
1c.	Erik	10.2.38-10.2.40	Claim	Direct		Faulty	✓	
4a.	Erik	10.2.41-10.2.44	Claim	Direct			✓	
4b.	David	10.2.46-10.2.48	Claim	Direct			✓	
5a.	David, Erik	10.2.85-10.2.92	Claim	Direct		Faulty-e		✓
6.	David, Meredith	10.2.92-10.2.93	Claim	Indirect	U/L			✓
5b.	Erik, David	10.2.180-10.2.194	Claim	Direct		Faulty-e		✓
7.	Erik, David	10.2.196-10.2.202	Counter-5	Direct				✓
	Task 2: Compare one half or two fifths. Which is larger, and by how much?							
8a.	Alan	10.2.59-10.2.61	Claim	Direct				✓
8b.	Alan	10.2.69-10.2.73	Claim	Direct	Recur			✓
8c.	Alan	10.2.154	Claim	Direct	Recur			✓
8d.	Alan	10.2.154-10.2.158	Claim	Indirect	U /L	Incomplete		✓
8e.	Alan	10.2.156-10.2.158	Claim	Direct	Recur			✓

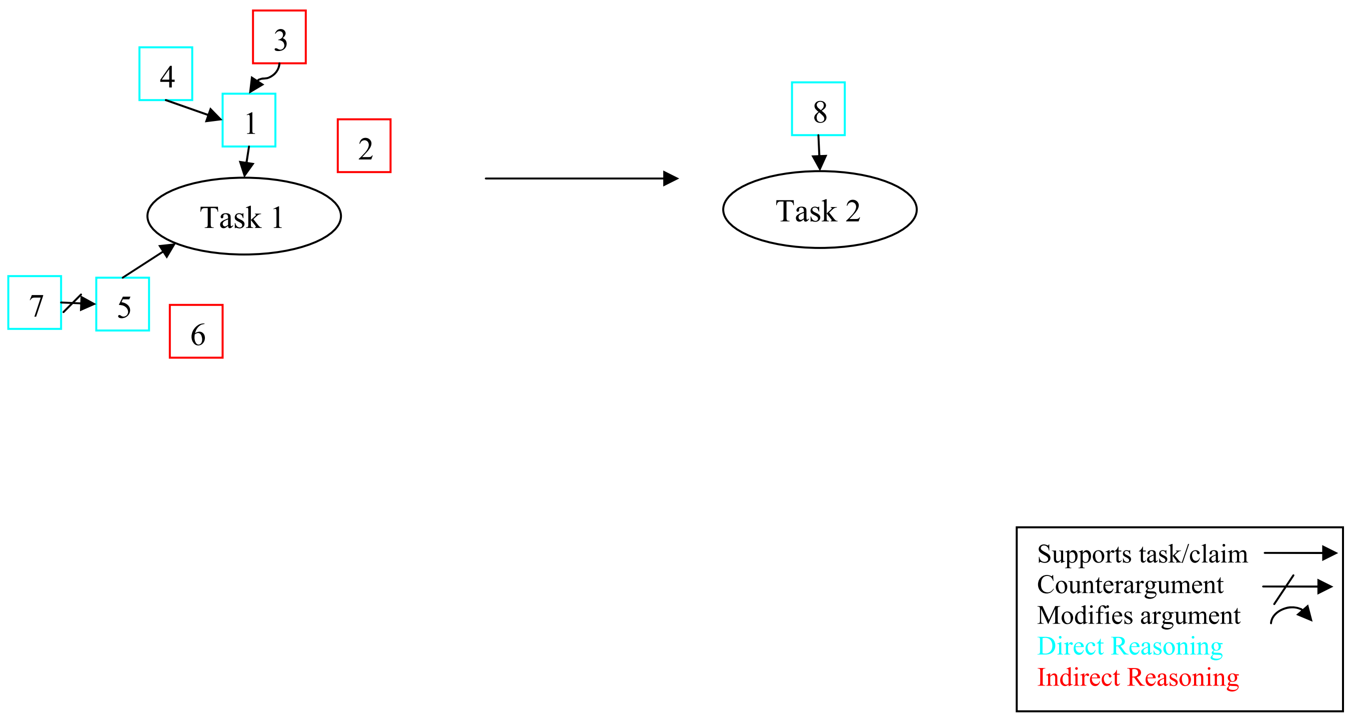


Figure 4.10. Argumentation Organized by Task, Session 10

4.2.11 Session 11: October 11, 1993

Task 1: Which is larger, two thirds or three fourths, and by how much?

1 At the start of the session, T/R 1 asked the students if they remember working on comparing two thirds and three quarters. She mentioned that she had seen students build more than one model, and asked the students how many models they thought were possible to build. Michael said, “Um, if you know what you're doing and you know what strategy, you could probably build, you could probably build one for every single rod” (line 11.0.4). When questioned by the researcher to explain further, he said, “You could, you could build a thing, you could build fractions of every single rod if you know what you're doing and you have a strategy or a secret that, that you know will work” (line 11.0.6). Although Michael’s explanation was incomplete, he conjectured that if one knew the “secret” behind the models, he could build models using all the rods.

2 Amy commented that they had found six models, and T/R 1 asked them if they thought those were all the models possible. Amy and her partners said that they thought there were more. Meredith then reasoned directly, saying, “[S]ay you had a white rod, and you divided the white rod, maybe you could make more models that way, if you divided the white rods” (line 11.0.14). T/R 1 asked her if she meant that more models could be made if there were more rods of different sizes, and she replied that she did (lines 11.0.15-11.0.16).

3a T/R 1 asked the class if there was a smallest model that they could build to represent the problem using the rods that were available. Beth and Sarah built a model using a light green and white train, two red rods, and four white rods at the OHP. Erik noted that the light green and white train could be substituted with a purple rod. T/R 1 asked them if that model could be used the problem under discussion, and Beth noted that the model did not contain thirds. Then, James, Amy and Jackie built a model of an orange and red train, three purple rods, and four green rods at the OHP. James said that the three fourths were larger than two thirds by one twelfth, showing directly that twelve white rods equaled the length of the train (Figure O-15-38).


Figure O-15-38

4, 3b T/R 1 asked the students if they were convinced by James' explanation. Jessica said that she didn't think the model they had built was the smallest one (line 11.0.38). T/R 1 asked the three students if there was a smaller one, and commented that if there wasn't, they should be able to show that. James then restated his direct argument, saying, "Well, we just put twelve whites on there and it takes one white to equal the two pinks, to the three, oh yeah, purple to the three greens. So that's why we think it's one twelfth" (line 11.0.40).

5 T/R 1 then asked the students to explain what she meant by a smaller model. Erik answered that it's smaller in size. T/R 1 asked the students if they could convince her that the model that James had built was the smallest length possible. Amy, a member of James' group, offered her justification.

> We say that there was no more, that you can't get a smaller one because every one you use equals up to an orange and a red, and the secret is that every one has three purples and four greens. And so you can't possibly make one smaller because you won't be able to fit, it won't work because every one you make equals up, equals up to the orange and red.

(line 11.0.46)

This argument was faulty in that it did not justify the claim that was made. Amy and her partners had worked to build models of the same length, and explained what they had done, but did not justify their claim that this model was the smallest possible. Other students also noted that there were other trains that could equal the orange and red train in length.

6 Erik then offered a counterargument to Jessica's claim that there was a smaller model that could be used to show the difference between two thirds and three fourths.

Erik:	Well, see, I agree that, that, I agree with them just at the part that there's no, there's no other smaller. I think, because at their model, they use the twelfth as the white ones, and there's no rod smaller than the white rod. So, therefore, if you make it a rod smaller than it, they can't, you can't divide it into twelfths.
T/R 1:	Ok. Did you hear what he said? Yeah!
Erik:	Because the twelfths right here are the smallest rod possible.
T/R 1:	Ok, so
Erik:	Unless you made a new rod.
T/R 1:	So unless we use Meredith's idea of creating new rods that had, that were smaller than the white rods, then you could make a smaller model, Erik?
Erik:	Yeah.

(lines 11.0.58-11.0.64)

Here, Erik indirectly countered Jessica's claim that there existed a smaller model that could represent the solution.

7, 8 T/R 1 asked the students what the model that was next in size would look like. Brian said that it would equal the length of twenty-four white rods. T/R 1 asked what the white rod would be called in that model. Michael conjectured incorrectly that it would be one twelfth, but did not explain why he thought so. Meanwhile Brian concluded directly that it would be one twenty-fourth (lines 11.0.74-11.0.91). Alan and Erik said that they had built that model, and proceeded to build it on the overhead (Figure O-24-48) using a train of two oranges and a purple rod as one.

Figure O-24-48

9 T/R 1 asked the students what the white rod would be called in Erik and Alan's model. The students answered that it would be called one twenty-fourth. She also asked them how that model can be used to solve the original problem that had been posed. Erik and Alan showed directly that the difference between two thirds and three quarters in this model was equal in length to a red rod or two white rods, and that the solution to the problem was one twelfth or two twenty-fourths.

10,**11,12** Amy then suggested, using direct reasoning that purple rods could be placed on the model to show one sixth. Andrew, Erik, and Alan challenged Amy, asking her why the purples would be necessary on this model. Erik, arguing partially using upper and lower bounds, asked her why they would be necessary, since the solution was one twelfths and "the purple would be too big" (line 11.0.205). Amy sighed, and T/R 1 suggested that perhaps Amy was answering a different question, and saying that there were other ways to make trains. T/R 1 then said that she could ask the students to compare one sixth or three quarters, and Erik and Alan conceded that it would then be necessary to show sixths in the model (lines 11.0.215-11.0.219).

13,14 T/R 1 asked David to share the theory that he had formulated with the class. David, with some assistance from Erik, said that Meredith had originally built a model using a train of one orange, one blue, and one black rod as one, and that the white rod was one twenty-fourth and the red rod was one twelfth (lines 11.0.223-11.0.229). As David spoke, T/R 1 built the model that he had described at the OHP (Figure S-34-45). The students then saw a discrepancy in David's model, and Alan and Erik showed the contradiction in David's model.

> Alan: Then the reds couldn't be twelfths.
> Erik: Yeah, then the reds couldn't be twelfths and the whites couldn't be twenty-fourths.
>
> (lines 11.0.236-11.0.237)


Figure S-34-45

15a T/R 1 asked Andrew and Jessica to share the model that they had built with the class. Referring to the twenty-four centimeter-long model that he had replicated on his desk (Figure F-36-39). He used direct reasoning to explain the model to the class.

> Well, I made a model that had the white was one forty-eighth and the purples were twelfths and the white was, I mean the red was twenty-fourths and I took two browns as the thirds and two dark greens as the fourths and they I called them the fourths and then the whole was four oranges and two purples.
>
> (line 11.0.240)

T/R 1 asked Andrew if one brown was called one third. Andrew and Jessica clarified that he had considered two brown rods placed end to end to be one third (lines 11.0.241-11.0.252).

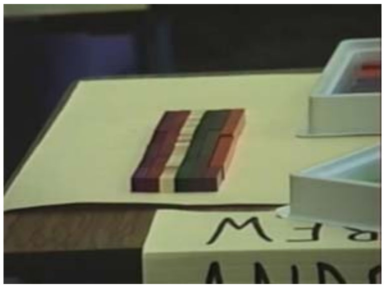


Figure F-36-39

15b,c,d T/R 1 asked Andrew and Jessica to build the model using large Cuisenaire rods at the front of the room (Figures F-40-26, F-44-40). They did so, and repeated their direct justification for their solution three times to different groups of students (lines 11.0.258-11.0.262, 11.0.268-11.0.288, and 11.0.290-11.0.298).


Figure F-40-26


Figure F-44-40

16 T/R 1 asked David if Andrew and Jessica's model was linked in any way to the theory that he had tested. David and Erik responded that it did, since they had thought that the white rods would be forty-eighths and that the red rods would be twenty-fourths, just as Andrew's model had shown (lines 11.0.304-11.0.307).

17a,b Alan and Erik then noted that there was a relationship between the two larger models that had been built.

> Alan: Since whites are doubles, they're forty-eighths
> Erik: So, in other words we doubled everything.
> Alan: Yeah. You basically just added, like, there originally were just two oranges, now there are four oranges and an extra purple. Now there are six, there are six browns.
>
> (lines 11.0.308-11.0.310)

T/R 1 continued Alan's direct train of thought by describing the models that had been built, saying that one model contained an orange and red train and the other a train of two orange rods and a purple rod. She asked the class if the models were related in any way. Alan repeated that "basically it's just doubled" (line 11.0.318). When questioned further by the researcher, Alan said that "it's doubled because it now it has four oranges and two purples or a brown" (line 11.0.320).

18,19 T/R 1 pointed out that the first model (composed of the orange and red train) that James had built did not contain any purple rods. Alan responded in a manner that indicated that he had misunderstood the question that T/R 1 had posed. He said that there were no purple rods on the twenty-four centimeter model since they hadn't been placed on the model, but that if they had been placed, they would have been the twelfths. James then explained why the smallest model that had been built did not contain a purple rod. He said "[W]hy the red's there, it's two reds make a purple and that, that means the two oranges and the red make two oranges and a purple" (line 11.0.336). Thus, James showed, using direct reasoning, that Alan's pattern could be used to show that the second model was twice the length of the first.

17b Alan then continued his original train of thought, explaining again that the third model was twice the length of the second. He concluded, "So it's basically doubled, each of the length is doubled" (line 11.0.337)

20,21 T/R 1 then repeated her question. She asked how the second model was double the length of the first. Kimberly said, using direct reasoning, "Well, they used a purple and the red, two reds make a purple, so now if they have a purple, they doubled the red" (line 11.0.343). T/R 1 asked the class to predict what the third model should look like if there indeed was a doubling pattern inherent in the models. Danielle stated that the train would be composed of four orange rods and two purple rods, and Erik and Alan confirmed that that was the case (lines 11.0.348-11.0.352. Here, in addition to the direct reasoning that had been used by the other students, Danielle used recursive reasoning to make a prediction about the third model based on the pattern that had been noted.

22,21b T/R 1 continued to question the students about this pattern. She asked them to predict what the next model would look like. Brian said that the length of the train would equal forty-eight white rods. T/R 1 pointed out that the third model that had been built was that length (lines 11.0.355-11.0.360). Brian conceded that he was mistaken, and Andrew predicted the length of the next model, saying that the train would be composed of eight orange and two brown rods (11.0.367-11.0.371). Andrew, as well, used recursive reasoning as he drew this conclusion.

23,24a,b,25 Alan challenged Andrew's reasoning by using his argument by upper and lower bounds that he had used in previous sessions.

> Alan: You can't double that. You can't double that model because if you did, then you wouldn't be able to third it.
> Erik: You wanna make a bet - all you had to do is train it - you just train it!
> Alan: Right because if you doubled that it would be eight oranges and two browns, now is there any rod that could third that?
> Erik: Well if you use a train
> Andrew: Yeah
> Erik: If you use a train, just like in Andrew's theory.
> (lines 11.0.373-11.0.378)

Andrew predicted that the thirds would be three brown rods and that the fourths would be three dark greens (line 11.0.382). David repeated Erik's direct argument, saying that thirds could be made using trains of rods (line 11.0.400). Erik and Alan then concluded,

> Alan: What I meant, what I meant is, you can't third it just using one rod.

T/R 1: Ok, Alan.
Erik: Exactly. You can't third it using one rod, but you can third it using trains.
T/R 1: Ok, so
Alan: You could double that, but you would have to use two rods to make it

(lines 11.0.401-11.0.405)

With that, they used direct reasoning to describe how one third could be represented in the large model that Andrew had described.

21c T/R 1 asked if a model could be built that was bigger than the one Andrew had described. Erik reasoned directly, "If you doubled that, it would be sixteen oranges and, sixteen oranges and four browns!" (line 11.0.407). With that, T/R 1 closed the session, and asked the students to think and write about the question: Is there a biggest model?

In this session, the students made a move to generalizing the doubling pattern that they had noticed, and predicting what the next sequence of rods would be used to build ever larger models.

Table 4.11

Forms of Reasoning, Session 11

	Student	**Lines**	**Type**	**Structure**	**Form**	**Sub-codes**	**WC**	**PR**
	Task 1: Which is larger, two thirds or three fourths, and by how much?(number of models)							
1.	Michael	11.0.4-11.0.6	Conjecture	Direct		Incomplete	✓	
2.	Meredith	11.0.14-11.0.16	Claim	Direct			✓	
3a.	James	11.0.32-11.0.34	Claim	Direct			✓	
4.	Jessica	11.0.36-11.0.38	Counter-3	Indirect		Faulty	✓	
3b.	James	11.0.40	Claim	Direct			✓	
5.	Amy	11.0.46-11.0.48	Counter-4	Direct		Faulty	✓	
6.	Erik	11.0.58-11.0.64	Counter-4	Indirect	U/L		✓	
7.	Brian	11.0.75-11.0.91	Conjecture	Direct			✓	
8.	Michael	11.0.87	Conjecture	Direct		Faulty, Incomplete	✓	
9.	Erik, Alan	11.0.176-11.0.187	Claim	Direct			✓	
10.	Amy	11.0.191-11.0.195	Claim	Direct			✓	
11.	Erik, Alan	11.0.196-11.0.211	Counter-10	Indirect?	U/L		✓	
12.	Erik	11.0.215-11.0.219	Claim	Direct			✓	
13.	David	11.0.223-11.0.229	Claim	Direct		Incomplete	✓	
14.	Erik, Alan	11.0.236-11.0.237	Counter-13	Direct			✓	
15a -d.	Andrew, Jessica	240-242,258-262, 268-288, 290-298	Claim	Direct			✓	
16.	David	11.0.304-11.0.307	Claim	Direct			✓	
17a.	Alan	11.0.308-11.0.310	Claim	Direct			✓	
17b	Alan	11.0.318-11.0.320	Claim	Direct			✓	
18.	Alan	11.0.322-11.0.334	Claim			Faulty	✓	
19.	James	11.0.336	Claim	Direct			✓	
17c	Alan	11.0.337	Claim	Direct			✓	
20.	Kimberly	11.0.343	Claim	Direct			✓	
21a.	Danielle	11.0.348-11.0.352	Claim	Direct	Recur		✓	
22.	Brian	11.0.355-11.0.360	Conjecture	Direct		Faulty	✓	
21b	Andrew	11.0.367-11.0.372	Claim	Direct	Recur		✓	
23.	Alan	11.0.373-11.0.401	Counter-22	Indirect	U/L	Faulty	✓	
24a.	Erik	11.0.374-11.0.403	Counter-23	Direct			✓	
24b	David	11.0.400	Counter-23	Direct			✓	
25.	Alan, Erik	11.0.401-11.0.405	Claim	Direct			✓	
21c.	Erik	11.0.407	Claim	Direct	Recur		✓	

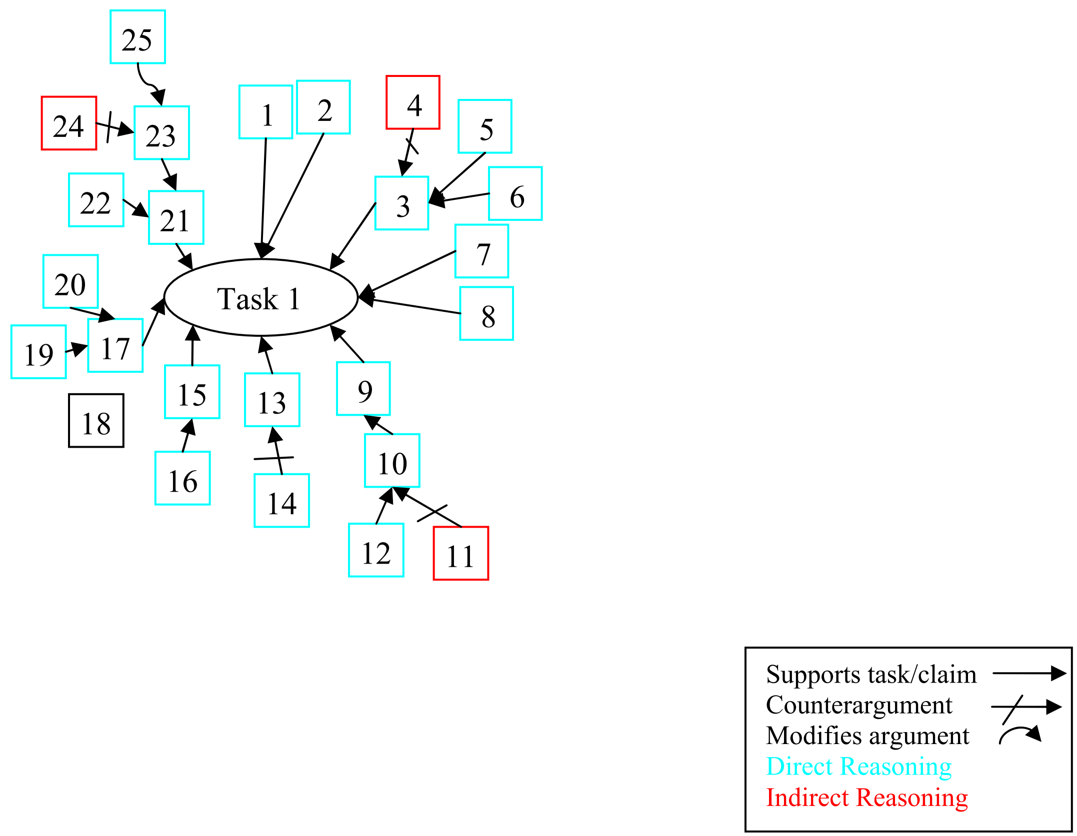


Figure 4.11. Argumentation Organized by Task, Session 11

4.2.12 Session 12: October 29, 1993

Task 1: Which is larger, one fourth or one ninth, and by how much?

1 The session began with a review of what had occurred the day before in math class. The students discussed the activity that had been conducted, during which the class had been divided into three groups and had each received ten pieces of chocolate. The students had to decide how to divide the chocolate into equal portions for each member of the group. Andrew explained what had happened. “[W]e said there was nine people, so we had to give a whole piece of candy to each person and then we had one left over so we would have to, and there's nine people, so if we divided it into ninths there would um be enough, for everyone” (line 12.2.14).

2 Jessica used similar direct reasoning to explain what had happened in her group of eight people, and said that since there were two pieces of chocolate left over after each member had received one apiece, they divided the remaining pieces into fourths so that they could share them equally, and that each student received one and one fourth pieces of chocolate (lines 12.2.24-12.2.26).

3 T/R 1 asked the class which group members received more chocolate, those in Andrew’s group or those in Jessica’s group. Using direct reasoning, Michael responded that the members of Jessica’s group had received more chocolate than those in Andrew’s group (line 12.2.28). T/R 1 asked the students if they could determine how much more chocolate that was.

4a Meredith offered a solution to the researcher’s challenge. She said, “Yeah, if we got one ninth and they got one fourth, then um, nine minus four equals five, so they got um one fifth bigger” (line 12.2.31). T/R 1 asked Meredith to re-explain her solution, and

Meredith said, “Nine minus four equals five so they got one fifth more” (line 12.2.43). By incorrectly assuming that fractions could be dealt with in the same way as whole numbers, Meredith used faulty direct reasoning to provide and justify her solution.

5 T/R 1 asked Kimberly, who had been a part of the third group, what had happened in their case. She replied that they had each received one and one fourth pieces, since there had been eight members of the group (line 12.2.46-12.2.51). Kimberly used direct reasoning similar to Andrew and Jessica to arrive at her solution.

T/R 1 asked the students if they believed what Meredith had said. All students in view raised their hands to indicate that they did. T/R 1 gave the students an analogical problem. She asked the students what the difference between one half and one quarter would be. The students answered that it would be one quarter. T/R 1 then said “But you would have told me a half more, think of the way you did that problem.” (line 12.2.56). Meredith smiled.

6 Meredith then justified the first part of her original claim using direct reasoning. She said that she knew that the members of Kimberly and Jessica’s groups had received more chocolate than those in Andrew’s group. She justified this statement by saying that “if you take a one rod and you divide it into ninths and fourths, the fourths are going to be larger because they're less. So they're going to be larger. So each person is going to be getting a larger piece” (line 12.2.75).

4b T/R 1 asked Meredith to imagine one ninth and one fourth. She then asked Meredith if she could imagine the rod which was called one fifth. Meredith said that she thought the yellow rod was the fifth rod. When T/R 1 asked Meredith if the rod that was called one fifth would be the difference between one fourth and one ninth, Meredith continued her faulty line of reasoning and said that "if you put the four and the five together it would equal up to the ninth rod" (line 12.2.85).

T/R 1 then suggested that the class work on the problem using the rods. Dr. Landis entered the room and expressed interest in hearing about the problem they were working on. The students began to work on the problem with their partners.

Brian and Meredith worked together on this task, and continued to use Meredith's faulty argument to show and justify their solution to the problem.

4c First, Brian and Meredith built identical models of a blue rod and yellow and purple train (Figure 10-29-01). T/R 1 asked Meredith what she had done. Meredith said, "[T]his is ninths, this rod has nine white little things, and this has five white ones in it, and this has four white ones, I added the five plus the four and it equaled a nine rod" (line 12.1.98).

Figure 10-29-01

T/R 1 asked Meredith "But the question, if you have the one fifth and the one fourth do you get one ninth?" (line 12.1.99). Meredith answered that that was the case, and T/R 1 challenged her to build a model that would show it.

4d Thirteen minutes later, T/R 1 returned to Brian and Meredith to hear their justification. Brian re-explained their original solution to T/R 1. Brian said that nine was smaller than one fourth by one fifth. T/R 1 told Brian to write what he had said, and as

Brian wrote, he said, “I mean one ninth is smaller than one fourth by one fifth” (line 12.1.112).

T/R 1 asked Brian to explain how his model showed his solution. Brian’s model was composed of a blue rod, nine white rods, and a purple and yellow train (Figure S-30-37). T/R 1 asked Brian how he would convince her that something was one fourth of something else. Brian replied that four white rods equaled the length of the purple rod. He then said that the white rods were ninths and the yellow rod was one fifth. T/R 1 asked him how the yellow rod was a fifth. Placing the yellow rod on the blue rod, she said, “Again, if this is a fifth, how could this be a fifth if this is one?” (line 12.1.141, Figure 10-29-03). Brian then said that he was not calling the blue rod one any longer. Brian’s reasoning here was faulty in a way similar to Meredith. T/R 1 told him that she had to know what one was in order to understand his solution, and suggested that he and Meredith work to build a model to show one ninth and one fourth.


Figure S-30-37


Figure 10-29-03

Brian and Meredith spent about ten minutes working to find a model that could show both fourths and ninths. When T/R 2 approached them to question them about their progress, Brian said that they had tried to use a model that was of an odd length so that there would be ninths, “but we can't have fourths because four is an even number” (line 12.1.249).

7 Dr. Landis questioned Alan and Kimberly about the task. Using a vertical version of Brian and Meredith's models (Figure 10-29-04), he explained that the yellow rod was the fifth, the purple rod was the fourth, and the blue rod was the ninth. He then said, reasoning in a faulty manner, that one fourth was smaller than one ninth by one fifth (lines 12.1.114-12.1.117).

Figure 10-29-04

8a Dr. Landis then pointed out that Alan had said earlier (not contained in the video data) that the purple rod was called four (line 12.1.148). At this, Alan changed his line of reasoning, but his new argument was faulty as well. He lined up nine white rods alongside a purple and yellow train and said that the white rod would be one ninth (line 12.1.149). Next, he lined four white rods alongside a purple rod, and said that the white rod was now one fourth (Figure S-36-14). He then said that the white rod was both one fourth and one ninth. Dr. Landis asked if the two fractions were equal. Alan replied that they were the same size but that they had different number values (line 12.1.155). Dr. Landis asked him what that meant. He re-explained that one ninth was smaller in "number value" but that in Cuisenaire rods, they were the same size (lines 12.1.157-12.1.165). By arguing that two fractions could be of the same physical length but different in "number values," Alan was not subscribing to accepted mathematical rules of numerical equivalence.


Figure S-36-14

9 Dr. Landis asked Kimberly, “Is that possible to be the same size with the rods but to be different with numbers?” (line 12.1.166). Kimberly said, “Maybe… If you have different size wholes” (lines 12.1.167-12.1.169). She then elaborated.

> I think that if you had, it could be a ninth on this [yellow and purple train], because it equals nine, but if I took this and this [dark green and yellow train], it would probably be a higher number, because this is bigger. So they can be different number names and be the same size, but they have to be different models on the top.
>
> (line 12.1.172)

Thus, Kimberly modified Alan’s argument using direct reasoning.

8b Dr. Landis asked Alan what he thought of Kimberly’s comment. He restated his previous faulty argument. Dr. Landis told him that she was still puzzled “because you're telling me that they're not the same size but then you're showing me with your model that they look the same” (line 12.1.189). She encouraged Alan to work on the problem some more.

8c,8d Alan made a train of two blue rods and lined up nine red rods alongside it. Removing one red rod from this new model and a white rod from his model of a purple rod and four white rods, he told Kimberly, “Two blues will be a whole, there [red rod] would be one ninth, but this would still be one fourth [white rod], and this would still be a higher number name than this” (line 12.1.196). Soon afterwards, T/R 1 approached Alan, and he showed her the model of a the two blue rods as well as one with two purple rods and four red rods and explained that in the former, the red rod was one ninth, whereas in the latter, the red rod was one fourth.

Throughout these discussions, Alan used direct reasoning to draw conclusions based on the models that he had built. However, this reasoning was faulty in that he used two different models and showed that one fourth and one ninth were the same length using the rods.

10 T/R 1 told him that she did not understand why he had switched candy bars during his justification. Alan replied, “What I'm just meaning is these are just models to show my hypotheses” (line 12.1.209). He then said,

> I know, these are just to explain the way I'm thinking. I'm thinking that the fourth is bigger than the ninth because if you took two of the same models and you divided it into fourths, those pieces would be bigger. If you divided it into ninths, those pieces would be smaller.
>
> (line 12.1.211)

Here, Alan used direct reasoning to explain why one fourth was larger than one ninth. T/R 1 encouraged him to find one model that could show the difference between the two.

Alan and Kimberly worked to build a model that could show both ninths and fourths. Alan told T/R 1 that the task was impossible because it involved one odd and one even number. T/R 1 suggested that he try build a model for thirds and fourths and see if that was possible. Later, Alan told T/R 2 that the task was impossible because only multiples of the blue rod could be divided into ninths and that one or two blue rods could not be divided into fourths.

11a Michael and Erik worked together on the task. After a considerable amount of work, they expressed their frustration to T/R 1. She suggested that they build the models for comparing one third and one fourth and for comparing one half and one third.

Michael found that the differences there were one twelfth and one sixth respectively using direct reasoning (lines 12.1.289-12.1.295). T/R 1 asked Michael and Erik to build the two models at the OHP.

12a,b Meanwhile, James built a model using a train of three oranges and a dark green rod, nine purple rods, four blue rods, and thirty six white rods. He explained to Dr. Davis that the purple rods were ninths, the blue rods were fourths, and that the difference between the two was five thirty-sixths. Dr. Davis then asked him about the smaller model that he had built using a blue rod and a train of a purple rod and five white rods. At first, James said that "I just think that the blue is bigger than the purple by one fifth 'cause it takes five whites to equal up to the blue, the one fourth" (line 12.2.321). After using this faulty sub-argument, he was questioned further by the researcher, and he restated his direct argument that the white rods were thirty-sixths and that this smaller model showed the difference to be five thirty-sixths (lines 12.2.322-12.2.339).

4e Erik and Michael built their two models showing the differences between one half and one third and one third and one fourth at the OHP. Then, they noticed that Kelly and Graham had built a model identical to James' that showed the difference between one fourth and one ninth (Figure 10-29-06). Meredith, too, examined their model. She argued against the validity of their model by continuing her original line of reasoning, saying, "Well they don't have fifths" (line 12.2.367).

Figure 10-29-06

13,14 In response, T/R 1 told Meredith that they were comparing fourths and ninths, not fifths. T/R 1 asked Graham and Kelly what the white rods were called, and Kelly said that the difference between one fourth and one ninth was one fifth, and showed that five white rods could be added to the purple rod so that it equaled the length of the blue rod (line 12.2.379). Graham said that the white rod would be called thirty-sixths, and that the difference between the two lengths was five thirty-sixths. Kelly agreed that that was correct (lines 12.2.380-12.2.403). With this, the students used direct reasoning to support their solution based on the model that they had built.

4f Meredith then continued her argument, saying that the difference was also one fifth. T/R 1 asked here where one fifth was in the model. She said that if there were one, that would be the solution.

15a,b Kelly countered Meredith's claim, saying that "there's no one fifth" (line 12.2.407). T/R 1 asked Meredith to imagine one fifth based on what one fourth was in the model. Graham countered her argument, saying that if there would be one fifth, it would be too big to be the difference between one fourth and one ninth (line 12.2.413). In a sense, Graham used upper and lower bounds to support Kelly's counterargument.

4g T/R 1 placed five yellow rods on Graham's model (Figure 10-29-07) and asked Meredith if the yellow rod could be called one fifth. Meredith countered her challenge by placing a yellow rod on the five white rods that showed the difference between one fourth and one ninth (Figure 10-29-08) and saying, using direct reasoning, "Well it does have five here" (line 12.2.416). T/R 1 reminded her that the five white rods were called five thirty-sixths and not one fifth, and Meredith agreed. T/R 1 recommended that Meredith think about what was causing the confusion.


Figure 10-29-07


Figure 10-29-08

16 The class was then called together for a discussion. Michael presented the two models that he had built at T/R 1’s suggestion.

Michael:	I also figured, um, is that you, it's so hard, like if you had you had to make a model with one fourth and one eighth in it, we could make a ton of them, but it's hard to make a model that has an odd number, which is one ninth, and a even number, which is one fourth. So I figured that that was really hard and there was only like two models or so of it and it was really hard to find you would have to make trains or something like that.
T/R 1:	Ok, so where did that leave you. You told me there couldn't be any models when you had an odd and even.
Michael:	I know. But then we figured that it had to be, because there was no other way to do it.
T/R 1:	But you built two models here and you're comparing fractions where, you have an odd and even number on
Michael:	Well, I didn't really, I was just building, I was just trying to get an idea from these old models and I didn't get one, but I guess Dr. Maher did, so she wanted us to come up and say what we were thinking, I was just trying to get an idea from it.
T/R 1:	When you compare this top one, what numbers were you comparing when you built this model here? [Figure 10-29-09]
Michael:	One third and one fourth.
T/R 1:	And what did you find?
Michael:	We found that it worked.
T/R 1:	What worked?
Michael:	That an odd and an even can go into a whole.

(line 12.2.430-12.2.444)

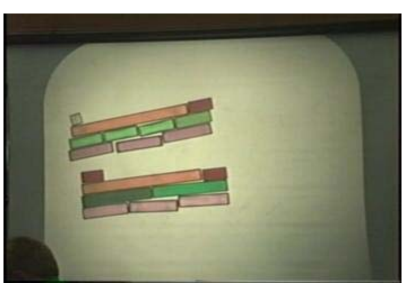


Figure 10-29-09

So, although Michael was not able to solve the harder problem based on what he had learned from building the old models, he admitted that his experiment had contradicted his assumption that one model cannot be built to compare an odd and an even fraction.

11b T/R 1 asked Michael what the differences between the two pairs of fractions were, and he replied that they were one sixth and one twelfth, respectively. Here, Michael used direct reasoning to arrive at these responses.

12c James presented his model to the class. He justified his solution using direct reasoning and said,

> First… I tried nine yellows and four oranges, for the ninths and the fourths, and I found out they weren't equal so I tried something else. I lowered its size so orange and uh the orange and the yellow and we got blue as the fourths and purple as the ninths and they were equal. So I just had to find a whole for that and I found out it was I just took three oranges and one dark green so then I had then I put up thirty-six whites on up to the whole and there, it took five whites to make the purple equal to the blue, so I think the answer would be five thirty-sixths.
> (line 12.2.450)

17 Jackie, Beth, and Erin then shared that they had built models similar to Graham, Kelly, and James to represent the problem. They concluded directly that the difference between one fourth and one ninth were five thirty-sixths (12.2.456-12.2.471)

The session closed with a discussion about what made the problem difficult. Kimberly said that the fact that the fractions contained odd and even numbers as denominators was the cause of the difficulty. T/R 1 asked the students what their solutions to the difference between one half and one third, one third and one quarter, and one fourth and one ninth were and recorded the differences using fraction notation on a transparency. Michael commented, “Oh, it sort of went up by six I guess” (line 12.2.489). T/R ended the session by telling the students to think about how the numbers were related to the models they had built.

Table 4.12

Forms of Reasoning, Session 12

	Student	Lines	Type	Structure	Form	Sub-codes	WC	PR
	Task 1: Which is larger, one fourth or one ninth, and by how much?							
1.	Andrew	12.2.10-12.2.14	Claim	Direct			✓	
2.	Jessica	12.2.24-12.2.26	Claim	Direct			✓	
3.	Michael	12.2.28	Claim	Direct			✓	
4a.	Meredith	12.2.31-12.2.43	Claim	Direct		Faulty	✓	
5.	Kimberly	12.2.47-12.2.51	Claim	Direct			✓	
6.	Meredith	12.2.61-12.2.75	Claim	Direct			✓	
4b.	Meredith	12.2.83-12.2.85	Claim	Direct		Faulty	✓	
4c.	Meredith	12.1.98	Claim	Direct		Faulty		✓
4d.	Brian	12.1.110-12.1.113, 12.1.118-12.1.147	Claim	Direct		Faulty		✓
7.	Alan	12.1.115-12.1.117	Claim	Direct		Faulty		✓
8a.	Alan	12.1.149-12.1.165	Claim	Direct		Faulty		✓
9.	Kimberly	12.1.166-12.1.172	Claim	Direct				✓
8b, c, d.	Alan	12.1.174-12.1.186, 12.1.196, 12.1.201-12.1.209	Claim	Direct		Faulty		✓
10.	Alan	12.1.211	Claim	Direct				✓
11a.	Michael	12.1.289-12.1.295	Claim	Direct				✓
12a.	James	12.2.317-12.2.339	Claim	Direct				✓
12b	James	12.2.321	Claim	Direct		Faulty		✓
4e.	Meredith	12.2.367	Counter-14	Direct				✓
13.	Kelly	12.2.379	Claim	Direct		Faulty		✓
14.	Graham, Kelly	12.2.371-12.2.401	Claim	Direct				✓
4f.	Meredith	12.2.404-12.2.406	Counter-14	Direct		Faulty		✓
15a.	Kelly	12.2.407	Counter-4	Indirect				✓
15b	Graham	12.2.413	Counter-4	Indirect	U/L			✓
4g.	Meredith	12.2.416	Counter-14	Direct		Faulty		✓
16.	Michael	12.2.430-12.2.444	Counter	Indirect			✓	
11b	Michael	12.2.446	Claim	Direct			✓	
12c	James	12.2.450	Claim	Direct			✓	
17.	Erin, Beth, Jackie	12.2.456-12.2.471	Claim	Direct			✓	

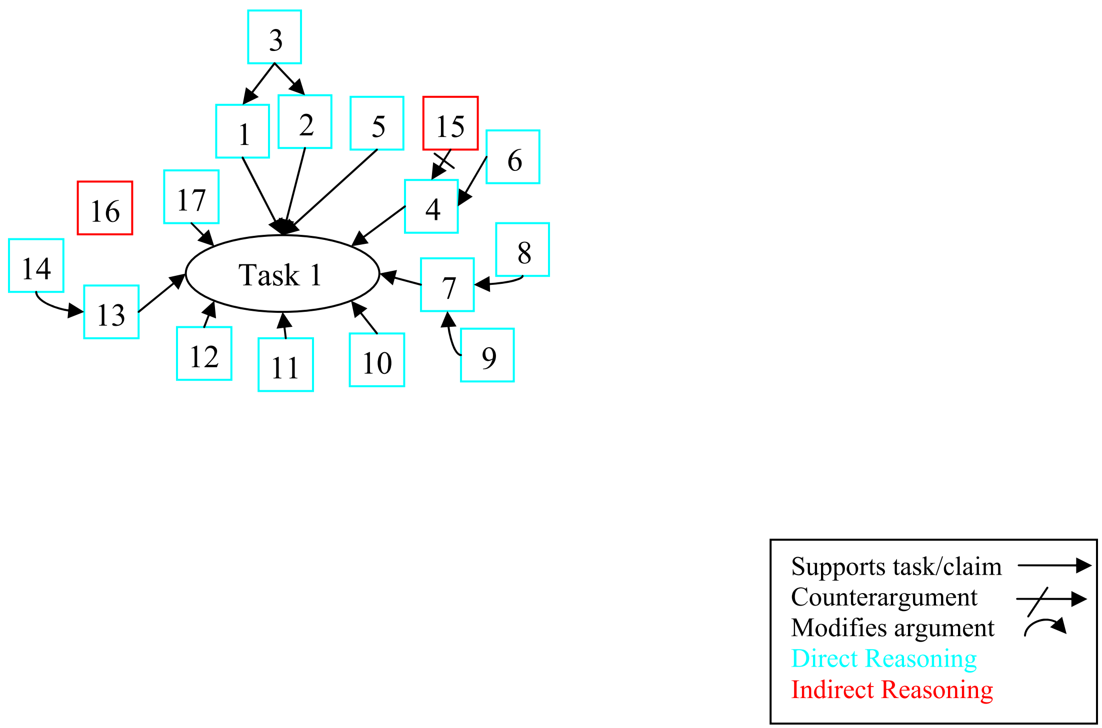


Figure 4.12. Argumentation Organized by Task, Session 12

4.2.13 Session 13: November 1, 1993

Task 1: Which is larger, one fourth or one ninth, and by how much?

1 The session on Monday, Nov. 1, began with Dr. Davis' introduction of Prof. Gunnar Gjone to the class. T/R 1 asked the class to tell Dr. Gjone about what they had done during the previous session. Graham explained, with the assistance of other students, that the class had worked on comparing one fourth and one ninth. T/R 1 asked the class if they remembered what the solution had been. Mark said that the difference had been five thirty-sixths, and James explained, using direct reasoning, that "we had the thirty-six whites and it took five whites to equal one fourth to one ninth" (line 13.0.12).

2a T/R 1 reminded the class that all the students had believed that the solution was one fifth before they had built the model to represent the problem. Meredith explained why she had thought so.

Meredith:	Well, um, well, I thought, well, if you put the uh, blue, which was the nine, which had nine ones in it, and the uh four rod and the five rod, the five equals up to the nine, if you put it up to the fours.
T/R 1:	You're saying if you took the blue, and what number name are you giving that?
Meredith:	Um, blue, I would call it nine.
T/R 1:	You are going to give it nine? And what would you give the other rods?
Meredith:	Um, the four rod which was I think the purple rod.
T/R 1:	You're saying you're calling the purple four, is that what you said?
Meredith:	Yeah, and um, the yellow would be five, and it would equal up to it. I thought, that's what I thought at first [T/R 1 models Meredith's solution on the OHP - Figure O-19-51].

(lines 13.0.20-13.0.26)

Here, Meredith used direct reasoning about whole numbers to justify why she had thought that the solution was one fifth.


Figure O-19-51

3, 2b Erik countered Meredith's argument. Using indirect reasoning and an element of reasoning using upper and lower bounds, he said,

> I think that it doesn't make sense, because how could the blue rod be one ninth of one model and the purple rod be one fourth when the blue rod is larger than the purple rod? Maybe if you made a super gigantic train then maybe the blue rod would be the ninth but I would think that the purple rod or the yellow rod will probably be the ninths and the blue rod will probably be the fourths.
> (line 13.0.28)

Meredith then said that she changed her answer and that "[t]he five rod equals up the same as the five thirty-sixths" (line 13.0.31).

4 Using direct reasoning, Jessica commented that the reason that different groups had divided the chocolate differently was because there was an odd number of students in the class. This was faulty reasoning, since the fact that there was an odd number of students was not the cause of these differences (since there had been three groups).

5 T/R 1 asked the students how the bars of chocolate could be divided equally among all twenty-five students. Andrew said that each student would receive one and one fifth pieces of chocolate. He explained directly, "Well, there was three candy-bars and each one had rectangles in them. So I took, um, twenty-five of them and circled it and put one. And then the five left, if you divided them up into five, five, ten, fifteen, twenty, twenty-five. So each person would get one and one fifth" (line 13.0.41). T/R 1 asked Andrew if he could draw a picture to show what he meant. Andrew said that he had drawn a picture of three candy bars that each contained ten pieces of chocolate. He said that each person got one piece, and that five pieces of chocolate from one bar remained. He finished, “And then there's five, so it's like one candy bar, only smaller, so you divide them into fifths, and then five ten fifteen twenty twenty-five, plus five times five is twenty-five so each one gets one and one fifth” (line 13.0.53).

6 T/R 1 asked the students if one and one fifth was more or less than one and one quarter. Danielle said that it was less than one and one quarter, and explained that one fifth was a bigger number, and “if it's a bigger number you get less” (line 13.0.57). Although she did not complete her explanation, Danielle used direct reasoning to justify her solution.

7 Brian extended her direct explanation. He said,

> [I]f it's a fifth it has to take, there has to be five of 'em in one whole, and if there are um, quarters, it only needs, it only needs four of 'em to go into one whole. So… five is a bigger number and… it needs more to fill up one whole… so it's less.
>
> (line 13.0.61)

8 T/R 1 verbally listed the numbers one, one half, one third, one fourth, and one fifth. She asked the students if they would be able to tell her which numbers were bigger and which were smaller. David replied, using direct reasoning and gesturing with his hands as he spoke. He said that if one would be a certain size, one half would be the largest, then one third, “because you would have to fit like three pieces in there” and that one fourth would be smaller than one third (line 13.0.63). T/R 1 asked him to sketch the rods that he was imagining at the OHP, and David did so (Figure O-28-10).

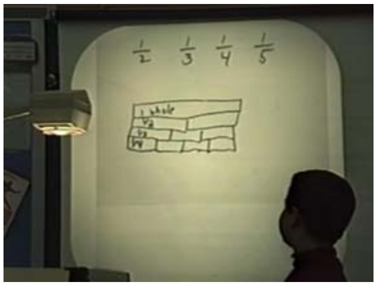


Figure O-28-10

9 Meredith noted that David’s sketch did not included one fifth. T/R 1 asked Meredith if it would be to the left or to the right of one fourth, and Meredith stated directly that it would be to the left of one fourth (lines 13.0.75-13.0.79).

For the remainder of the session, the students worked on ordering fractions on a number line without using Cuisenaire rods. This part of the session will not be included in the analysis.

Table 4.13

Forms of Reasoning, Session 13

	Student	Lines	Type	Structure	Form	Sub-codes	WC	PR
	Task 1: Which is larger, one fourth or one ninth, and by how much?							
1.	James	13.0.1213.0.14	Claim	Direct			✓	
2.	Meredith	13.0.20-13.0.26	Claim	Direct			✓	
3.	Erik	13.0.28-13.0.30	Claim	Indirect	U/L		✓	
2b.	Meredith	13.0.31	Claim	Direct		Faulty-p	✓	
4.	Jessica	13.0.33-13.0.37	Claim	Direct		Faulty	✓	
	Task 2: I want you to think about…sharing those three bars of candy so everybody got the same amount exactly							
5.	Andrew	13.0.39-13.0.53	Claim	Direct			✓	
6.	Danielle	13.0.55-13.0.59	Claim	Direct		Incomplete	✓	
7.	Brian	13.0.61	Claim	Direct			✓	
	Task 3: If I were, if I were to say things like this to you, one half, one third, one fourth, one fifth, right? If I were talking about these numbers, do you know which are bigger and which are smaller?							
8.	David	13.0.63-13.0.73	Claim	Direct			✓	
9.	Meredith	13.0.75-13.0.79	Claim	Direct			✓	

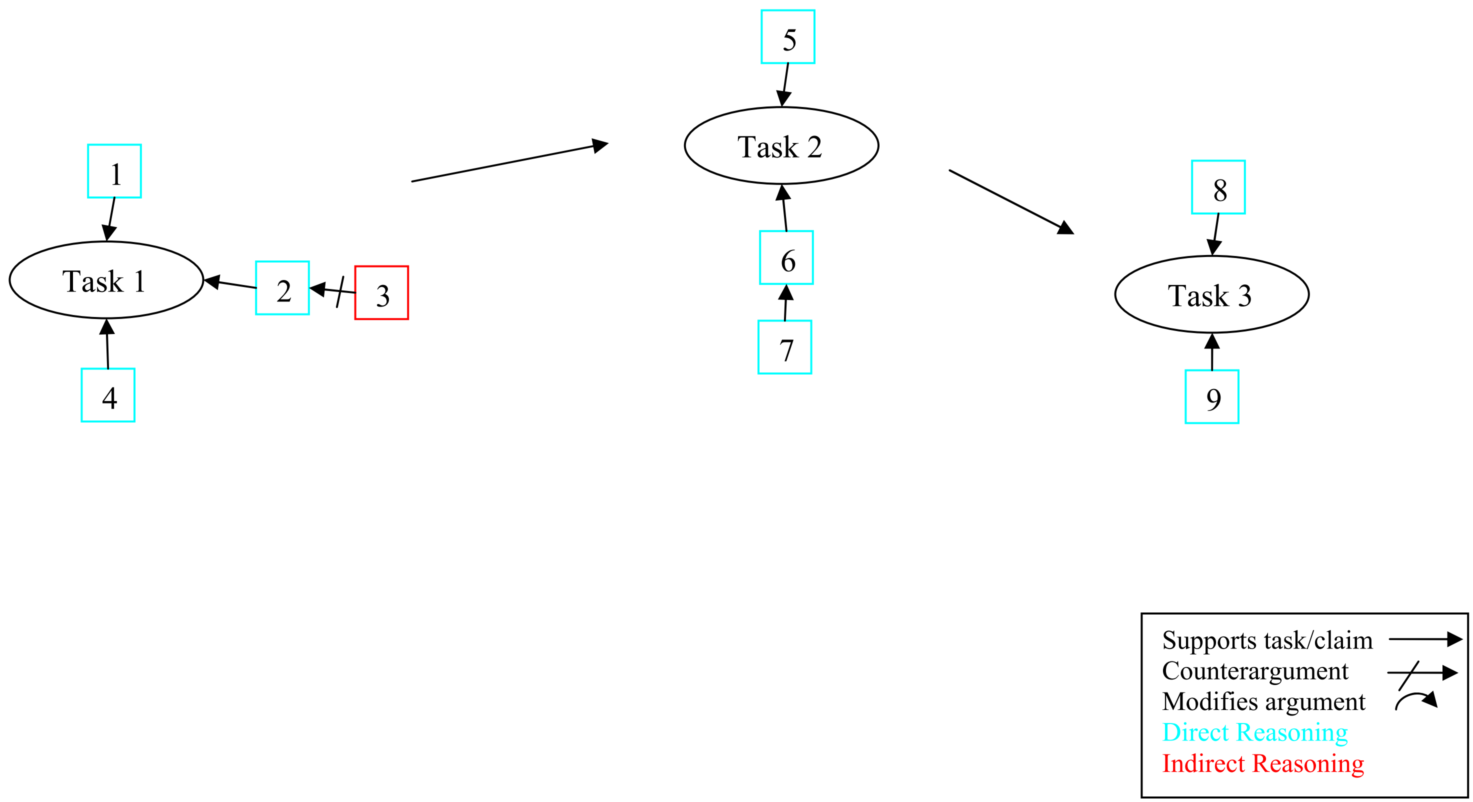


Figure 4.13. Argumentation Organized by Task, Session 13

4.2.14 Session 14: December 2, 1993

Task 1a: What number name would I call the white rod if the orange and red train was called one?

1 T/R 2 started the session by revisiting ideas that had been raised during a recent parents' visit to the class. She asked the students what one white rod would be called if the orange and red train was called one. Danielle answered that it should be called one twelfth, and justified her solution by saying that twelve white rods equal the length of the orange and red rod (lines 14.0.6-14.0.8).

Task 1b: How many whites are in a red and orange train

Task 1c: How many _______ are in ________?

T/R 2 then placed a transparency on the OHP that contained two questions, "How many whites are in a red and orange train?" and "How many _______ are in ________?" - Figure S-5-40. She asked the students to think about the first question, and then to think about how to replace the color names in the first questions to compose a second question with number names. The students worked with their partners for a short time on this problem.

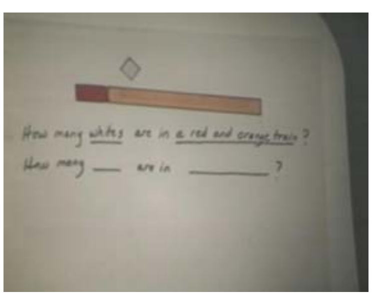


Figure S-5-40

2 Danielle spoke with CT about her solution. She told CT that there were twelve whites in a red and orange train, and then reasoned directly that the next line should read, "How many twelfths are in a whole" (line 14.0.24).

3 Meanwhile, Amy explained to T/R 2, using direct reasoning that the whites would be called one twelfth and the train would be called one (line 14.0.30).

4, 5 During the subsequent whole class discussion, David also explained directly that the whites should be called one twelfth and the red and orange train should be called one (14.0.33). T/R 2 asked Graham how many twelfths are in one, and he answered, using direct reasoning, that there are twelve twelfths in one (line 14.0.35).

6 Erik noted that the question could be phrased differently. Reasoning directly, he said, "For that equation, well, you could put how many 1/12's there are in 1, you can also put how many 1/12's are there in 12/12's" (line 14.0.39)

Task 2a: If I call the dark green rod 1, now it's not the orange and red train, it's the dark green rod that's going to be 1, what number name would I give to the white rod?

7 T/R 2 asked the students to build a model to show what the white rod would be called if the dark green rod was called one. Erin called the white rod one sixth, and explained, using direct reasoning, that this was true because six white rods equaled the length of the dark green rod (lines 14.0.47-14.0.49).

Task 2b: How many whites are in the dark green rod

Task 2c: How many ______ are in ________?

8, 9 T/R 2 then extended the problem, and asked the students to answer two questions similar to the previous set. Placing a transparency on the OHP, she asked the students how many white rods are in the dark green rod, and then to change the color names in that sentence to number names. Using direct reasoning, Meredith replied that there are six white rods in the dark green rod and that the sentence could be rewritten as, "How many 1/6's are in 1 whole" (line 14.0.53).

Task 2d: Is there another number name for one here?

10-13 Building on Erik's idea during the previous task, T/R 2 asked the students if there was another number name for one in this problem. Brian explained directly that it could be called six sixths (line 14.0.55). Michael then said that it could be called "plain six and that's the same thing" (line 14.0.57). The students immediately began to contest Michael's faulty argument, and Erik said, reasoning indirectly, "No, cause if you called it six it would be six wholes" (line 14.0.62). Michael then said that his statement was not true, and explained why that was so using indirect reasoning. He said, "You couldn't do that because you'd have to call the one sixth one whole" (line 14.0.64). Erik agreed, saying, "That would be right. That would be right because you can call the one whole six wholes and then each of the white ones could be one whole each" (line 14.0.65). Thus, he modified Michael's argument and used direct reasoning to explain how the problem could be changed to allow for Michael's solution. T/R 2 reminded him that that could not be done in the same problem.

Task 2e: Can we write this now as a number sentence including these numbers?

14 T/R 2 challenged the students to use the numbers to construct a number sentence. The students worked on this task with their partners for six minutes. Brian, working with Danielle, immediately concluded, "One divided by one sixth equals six" (line 14.0.70), and Danielle agreed with his direct reasoning.

15, 16a Michael and Meredith worked together. First, they each used direct reasoning to find a number sentence. Michael suggested the sentence of 'one whole minus five sixths equals one sixth' as a possible sentence (line 14.0.73). Meredith then suggested, “One is equal to six sixths” (line 14.0.74), but asked Michael if it had to be a division sentence.

16b T/R 2 visited Michael and Meredith, and Meredith showed T/R 2 the model that she had built, wherein she had written an equal sign between a model of a dark green rod and a model of six white rods (Figure S-18-17). T/R 2 asked her if she could write a number sentence using division that could describe the two models. Meredith thought for moment and then rearranged the models so that the six white rods were placed directly underneath the dark green rod (Figure S-19-01). She said, “I know. One divided by one sixth equals six sixths” (line 14.0.92). Here, Meredith used direct reasoning to find a division sentence using the rods. However, her reasoning was faulty due to her conclusion of six sixths rather than six.


Figure S-18-17


Figure S-19-01

17,16c Michael challenged her reasoning. He said, “I would agree with it that it would be one divided by one sixth equals six but not six sixths” (line 14.0.103). Meredith then changed her argument and said, “And one divided by one sixth equals six because there’s six, one sixths in one” (line 14.0.104). Michael and Meredith used direct reasoning to find the correct number sentence that described the situation.

18-20 Amy and Jackie worked together on the task. They first reasoned incorrectly that one divided by one sixth equals one third (lines 14.0.114-14.0.118). However, they did not explain how they arrived at their solution. Then, they changed their solution. Amy used faulty reasoning and concluded that one divided by one sixth equals one sixth, and Jackie used different faulty reasoning to conclude that one divided by one sixth equals one (lines 14.0.129-14.0.131).

21,22a T/R 2 asked the class to share their solutions. Mark provided the number sentence of one divided by one sixth equals sixth, and T/R 2 asked the class if they agreed with this solution. They did, and T/R 2 asked Michael to explain why he thought this sentence made sense.

Michael:	It works because division you see how many times you can get a number into a number. So you can get one sixth, umm you can get six times you can get one sixth into one with no remainders. So that would leave that that would be six.
T/R 2:	So you're saying then that one sixth.
Michael:	You can have six of them.
T/R 2:	It goes into one, if you were lining them up.
Michael:	Six times.

(lines 14.0.150-154)

23a,22b Erik challenged Michael, asking, "How can six go into one six times?" This challenge was accompanied by an explanation that that would be impossible "unless it was negative" (line 14.0.158). Erik's argument was faulty due to this peripheral comment. Michael countered Erik's objection directly and explained that he meant one divided be one sixth would give six.

23b,c Erik reworded his challenge, saying, "I know, but I'm saying if you were taking like six and one, you couldn't put six sixths into one" (line 14.0.160). T/R 2 noted that Michael was not making that point, and Meredith explained that Erik was merely trying to say that six sixths was equivalent to one (line 14.0.165).

Task 1d: Remember this one, the red and orange train? Can we rewrite this as a number sentence now? The question is how many 1/12's are in 1?

24a,b T/R 2 then revisited the series of problems that centered around calling the orange and red train one. She asked the class to think about rewriting the sentence as a number sentence and gave the students some time to work on this task. CT questioned Danielle, who reasoned directly that since the orange and red train was one and the white rod was one twelfth, one divided by one twelfth would yield twelve (lines 14.0.170-14.0.174). Then, during the whole class discussion, Danielle shared her number sentence with the class (line 14.0.179).

Written Task 1: If we give the red the number name 1, what number name would we give to white? How many whites are in a red? Write a number sentence to describe this relationship.

T/R 2 asked the students to work on and write about problems that were distributed in written form. For each set of problems, the students were asked to draw a model and to write a number sentence to describe their model. The first problem called the red rod one and asked for the number name for white. The students were then asked to write a number sentence describing how many whites are in a red.

25a,b Danielle explained to CT, using direct reasoning that if the red rod was called one, the white rod would be called one half (lines 14.0.183-14.0.187). Soon afterwards, T/R 2 questioned Danielle and her partner, Brian, about the task. Brian explained that he had written a division problem to describe the model of two white rods and one red rod. Danielle explained, using direct reasoning, that one half plus one half equals the whole, and that the whole is composed of two halves (lines 14.0.194-14.0.200).

25c,d T/R 2 then questioned Brian and Danielle to complete the problem. Brian said that there are two whites in a red (line 14.0.204), and Danielle said that the question “How many whites are in a red" could be changed to "How many halves are in a whole" (line 14.0.208). Both Danielle and Brian used direct reasoning to extend their understanding of the model and its description. Then, Danielle used direct reasoning to write a number sentence to describe the model, formulating the sentence, "One divided by one half equals two" (line 14.0.219). Brian and Danielle then recorded their solutions (See Appendix C) .

26 T/R 2 asked Michael about his written solution to the first problem. Michael had written '1 ÷ 1/2 = 1/2,' and began to defend his solution, but then realized that he had performed the subtraction operation and changed his solution to two (line 14.0.234). Michael used direct reasoning to arrive at this solution.

27 T/R 2, working with Amy, asked her to explain how her number sentence, which read '1 ÷ 1/2 = 2,' worked. Amy explained directly that "two halves go into one" and that "one half goes into one twice" (line 14.0.320).

Written Task 2: If we give the brown the number name 1, what number name would we give to white? What number name would we give to purple?

28a,b The second task asked the students to name the white rod and the purple rod if the brown rod was called one, and to build a model and write a number sentence to describe the model. First, Danielle reasoned directly that the white rod would be called one eighth by counting the white rods that she lined up against the brown rod (line 14.0.222). She then explained to T/R 2 that there were eight whites in a brown and eight one eighths in one. Reasoning directly, she concluded that the number sentence would read, "One divided by one eighth equals eight" (line 14.0.257).

29 T/R 2 questioned Meredith about her written solution to the task. Meredith reasoned directly that the sentence "How many purples are in brown" could be rewritten as '$1 \div 1/2 = 2$' (line 14.0.228).

30 James explained to T/R 2, using direct reasoning, that there are eight whites in brown because eight white rods are equivalent in length to the brown rod. Similarly, he explained that there are two purples in a brown because two purple rods equal the length of the brown rod (lines 14.0.243-14.0.247).

31 As Amy worked on the task, she reasoned aloud that the appropriate number sentence for model would be "1 divided by 1/8 equals 8" (line 14.0.326).

Written Task 3: If we give the orange and yellow train the number name 1, what number name would we give to white? What number name would we give to light green? What number name would we give to yellow?

32 Toward the end of the session, Brian explained his understanding of the solution for the third task to T/R 2. This task asked the students to name the orange and yellow train one, and to then name the white, light green, and yellow rods. He explained that fifteen white rods lined up to the train, and that five light green rods or three yellow rods were equivalent in length to the train as well. Reasoning directly, he named the white rod one fifteenth, the light green rod one fifth, and the yellow rod one third. He recorded the number sentence '$1 \div 1/15= 15$' (see Appendix C) and verbally explained to T/R 2 that the second number sentence would read "one divided by one fifth equals five" (line 14.0.269). When asked to explain his reasoning, he said, "There would be five fifths in one whole" (line 14.0.273).

33a,b Amy also worked on the third task. Lining up and counting fifteen white rods against the orange and yellow train, she recorded the number name for the white rod as one fifteenth. She then lined up five light green rods and labeled it one fifth (line 14.0.348). After finding the number name for the yellow rod, she worked to write number sentences to describe the model. Reasoning directly, she concluded aloud, "Now I need a number sentence. 1 divided by 1/15 equals 15. 1 divided by 1/5 equals 5. 1 divided by 1/3 equals 3" (line 14.0.371). Later, she explained her reasoning to a visiting researcher. When asked to translate her number sentences into words, she said, "1/15 goes into one 15 times. 1/5 goes into one 5 times. 1/3 goes into one 3 times" (line 14.0.392).

34 Amy also used direct reasoning to solve the fourth task, which asked the students to name the white, red, and black rods when a train of blue and yellow was called one. She lined up white rods, red rods, and then black rods to form lengths equivalent to the train, and then named the rods one fourteenth, one seventh, and one half, respectively. She then wrote number sentences for each as she had done in the previous problems (lines 14.0.377-14.0.380, 14.0.399).

Table 4.14

Forms of Reasoning, Session 14

	Student	Lines	Type	Structure	Form	Sub-codes	WC	PR
	Task 1a: What number name would I call the white rod if the orange and red train was called one?							
1.	Danielle	14.0.6-14.0.8	Claim	Direct			✓	
	Task 1b: How many whites are in a red and orange train *Task 1c: How many ________ are in ________?*							
2.	Danielle	14.0.11-14.0.24	Claim	Direct				✓
3.	Amy	14.0.30	Claim	Direct				✓
4.	David	14.0.33	Claim	Direct			✓	
5.	Graham	14.0.35-14.0.37	Claim	Direct			✓	
6.	Erik	14.0.39-14.0.43	Claim	Direct			✓	
	Task 2a: If I call the dark green rod 1, now it's not the orange and red train, it's the dark green rod that's going to be 1, what number name would I give to the white rod?							
7.	Erin	14.0.47-49	Claim	Direct			✓	
	2b How many whites are in the dark green rod? *2c How many _______ are in ________?*							
8.	Meredith	14.0.51-14.0.53	Claim	Direct			✓	
	Task 2d: Is there another number name for one here?							
9.	Brian	14.0.55	Claim	Direct			✓	
10.	Michael	14.0.57	Claim	Direct			✓	
11.	Erik	14.0.62	Counter-10	Indirect		Faulty	✓	
12	Michael	14.0.64	Counter-10	Indirect			✓	
13.	Erik	14.0.65	Claim	Direct			✓	
	Task 2e: Can we write this now as a number sentence including these numbers?							
14.	Brian	14.0.70	Claim	Direct				✓
15.	Meredith	14.0.71-74	Claim	Direct				✓
16a.	Michael	14.0.73	Claim	Direct				✓
16b.	Meredith	14.0.92	Claim	Direct		Faulty		✓
17.	Michael	14.0.103-14.0.110	Counter-15b	Direct		Incomplete		✓
16c.	Meredith	14.0.104-14.0.108	Claim	Direct				✓
18.	Amy, Jackie	14.0.113-14.0.118	Claim	Direct		Faulty		✓
19.	Amy	14.0.130	Claim	Direct		Faulty		✓
20.	Jackie	14.0.131	Claim	Direct		Faulty		✓
21.	Mark	14.0.146	Claim	Direct			✓	
22a.	Michael	14.0.150-154	Claim	Direct			✓	
23a.	Erik	14.0.156-14.0.158	Counter-22	Indirect		Faulty-p	✓	
22b.	Michael	14.0.159	Counter-23	Direct			✓	
23b.	Erik	14.0.160	Counter-22	Indirect			✓	
23c.	Meredith	14.0.165	Claim	Direct			✓	
	Task 1d: Remember this one, the red and orange train? Can we rewrite this as a number sentence now? The question is how many 1/12's are in 1?							
24a.	Danielle	14.0.170-14.0.174	Claim	Direct				✓
24b.	Danielle	14.0.179	Claim	Direct				✓
	Written task 1: If we give the red the number name 1, what number name would we give to white? How many whites are in a red? *Write a number sentence to describe this relationship.*							
25a.	Danielle	14.0.183-14.0.187	Claim	Direct				✓

	Student	Lines	Type	Structure	Form	Sub-codes	WC	PR
25b.	Brian, Danielle	14.0.194-14.0.200	Claim	Direct				✓
25c.	Brian, Danielle	14.0.204-14.0.210	Claim	Direct				✓
25d.	Danielle	14.0.219	Claim	Direct				✓
26.	Michael	14.0.230-14.0.236	Claim	Direct				✓
27.	Amy	14.0.320	Claim	Direct				✓
	Written Task 2: If we give the brown the number name 1, what number name would we give to white? What number name would we give to purple?							
28a.	Danielle	14.0.222	Claim	Direct				✓
28b.	Danielle	14.0.253-14.0.258	Claim	Direct				✓
29.	Meredith	14.0.227-14.0.228	Claim	Direct				✓
30.	James	14.0.243-14.0.247	Claim	Direct				✓
31.	Amy	14.0.326	Claim	Direct				✓
	Written Task 3: If we give the orange and yellow train the number name 1, what number name would we give to white? What number name would we give to light green? What number name would we give to yellow?							
32.	Brian	14.0.261-14.0.273	Claim	Direct				✓
33a.	Amy	14.0.348, 14.0.371	Claim	Direct				✓
33b.	Amy	14.0.382-14.0.392	Claim	Direct				✓
	Written Task 4: If we give the blue and yellow train the number name 1, what number name would we give to white? What number name would we give to red? What number name would we give to black?							
34.	Amy	14.0.377-14.0.380, 14.0.399	Claim	Direct				✓

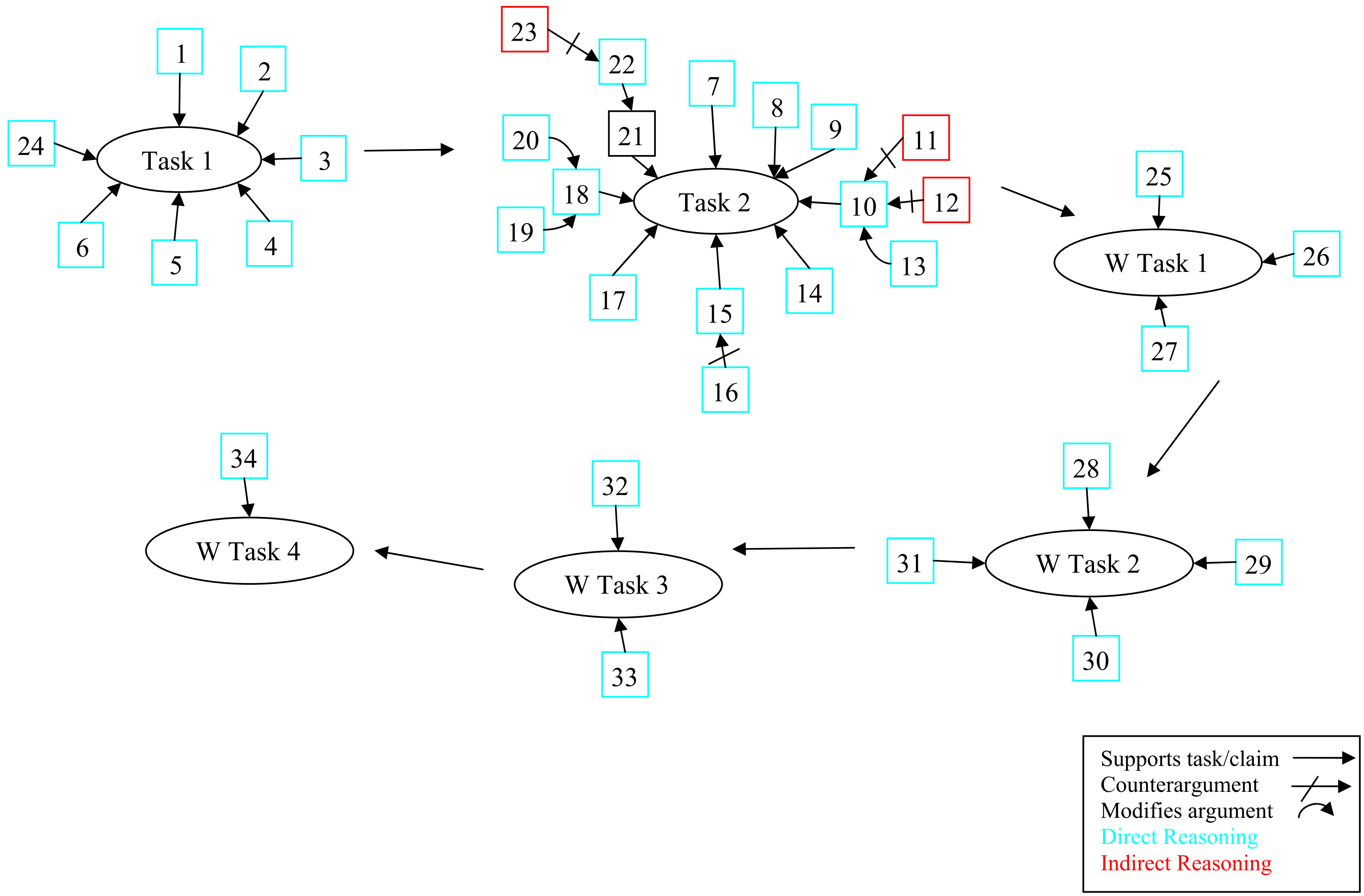


Figure 4.14. Argumentation Organized by Task, Session 14

4.2.15 Session 15: December 9, 1993

Ribbons and Bows Activity

This session marked a departure from those preceding it as it centered around a different activity. Instead of using Cuisenaire rods to represent fractions, the students were provided ribbons, meter sticks, and string, and were asked to form bows of specific fraction lengths from different lengths of ribbon (See Appendix D for a copy of the worksheet used during this session). For the first set of tasks (tasks 1a-d), the students were asked to find the number of bows that could be made from white ribbon, which measured one meter in length, if the bows were to be one half, one third, one fourth, and one fifth meters long. For the second set of tasks (tasks 2a-d), the students were asked to find the number of bows of those lengths that could be made from blue ribbon, which measured two meters in lengths. The last part of that set of tasks (task 2e) asked the students to also consider making bows that were two thirds of a meter in length from the blue ribbon. The third set of tasks (tasks 3a-f) asked the students to form bows from gold ribbon, which measured three meters in length. Finally, the fourth set of tasks (tasks 4a-f) required the students to consider using six-meter-long red ribbon to make bows. After T/R 1 introduced the three visiting researchers (V1, V2, and V3) and the activity, the students worked with their partners for the majority of the session, and attempted to solve the tasks posed on the worksheet. The students worked at different paces and their work will be traced by group over the duration of the session. The chart at the end of the narrative outlines the argumentation and is organized by task.

1,2 V1 worked with Danielle and Brian on the tasks. She questioned them about their understanding of the second task, where they were required to form bows from the two meter long blue ribbon. Both Brian and Danielle used direct reasoning to explain that four bows could be made that were one half of a meter in length. Brian explained that one meter was approximately equal to three feet, and that two meters would then be equivalent to six feet. He then said that half a meter would be one and a half feet in length, and that four bows of that length would equal the total length of six feet of blue ribbon (lines 15.1.61-15.1.67). Danielle said that she had thought about it differently. She said, "I just thought if it is two meters and each is a half, two halves are in a whole… And then there's two meters so I got four" (lines 15.1.73-15.1.75). Brian agreed that that was a simpler method.

3a V1 asked Brian if he could use Danielle's method to solve the third set of tasks, which involved the three-meter-long gold ribbon. He and Danielle said that there would be six bows formed that were one half of a meter in length. Then, he continued to the fourth set of tasks, which asked the students to find the number of bows that could be formed from red ribbon, which was six meters long. Brian reasoned directly, "There's two in one meter, which is approximately three feet, there are two halves. In another one there are two halves. Another one there are two halves, etc. And so if you keep counting by two up to six meters, that's be twelve meters" (line 15.1.88).

4, 5 Meanwhile, V3 worked with Jessica and Laura on the second set of tasks. Jessica used direct reasoning, saying that since in the previous task, two bows could be formed that were one half meter in length from one meter of ribbon, four bows would be made from two meters of ribbon that were one half meter in length. Then, Jessica said and Laura said that six bows could be formed that were one third meter in length. Jessica offered an incomplete justification, saying, "Because you're doubling, you're doubling, like last time was three, and three plus three is six. And now I think that would be eight, the next one. And that would be ten" (line 15.1.131).

6 V3 then questioned them about the last task in the second set of tasks, which asked the students to find the number of bows that were two thirds of a meter in length that could be formed from two meters of ribbon. Jessica and Laura determined, using direct reasoning, that the length of two thirds of a meter would be twice that of one third of a meter. Jessica said that since one third of a meter was thirty-three centimeters in length, two thirds of a meter would be sixty-six centimeters in length (lines 15.1.145-15.1.159). However, the camera moved before they completed this task.

7 Dr. Landis questioned Andrew and James about their understanding of the first set of tasks. Andrew explained how he knew the solution to the second task in the first set, which asked the students to find the number of bows that were one third of a meter in length that could be made from the white ribbon, which was one meter long. Andrew reasoned directly that since there were three thirds in a meter, three bows could be formed (line 15.1.26). Andrew and James went on to measure the length of one third of a meter so that they could make sure that they were correct. V2 asked them what they were doing.

> Well, Rutgers usually makes us prove what our answer is. So we had to do three, divided the ribbon into thirds, first third would be there, second would be there… We're proving. We are proving that, we wanted to make sure that when you divide it into thirds, there's no left over or anything.
>
> (lines 15.1.41-15.1.49)

8a-f Later, Dr. Landis returned to Andrew and James and questioned them about the second set of tasks. Andrew and James explained that four bows that were one half meter in length could be formed from the blue ribbon. They used direct reasoning to justify their solution, and between the two, repeated their justification in different ways six times as they talked with Dr. Landis. They explained that the blue ribbon is equivalent to the length of two white ribbons, and one white ribbon is one meter long. They then said that there are two half meters in one meter, and that since the blue ribbon is the length of two white ribbons, the blue ribbon is equivalent in length to four half meters (lines 15.1.162-15.1.220).

9,10,11 Andrew and James then discussed the next task with Dr. Landis. They explained directly that six bows could be formed that were one third of a meter in length from the blue ribbon, since three bows could be made from one meter of ribbon, and then three bows could be made from the second meter of ribbon (lines 15.1.226-15.1.234). They used similar direct reasoning to justify their solution for the next task, which involved bows that were one fourth of a meter in length (lines 15.1.237-15.1.239), as well as for the fourth task in the set, which involved bows that were one fifth of a meter in length (lines 15.1.241-15.1.244).

12a-c Andrew and James then worked on the last task in the set, which involved bows that were two thirds of a meter in length. Andrew used direct reasoning to solve this task.

Andrew: So one third is thirty-three,
James: Sixty-six,
Andrew: Thirty-three, sixty-six, so that's two. Three, you actually have two meters left over I mean, two thirds left over.
Dr. Landis: What do you mean, you have two thirds leftover?
Andrew: Because if you want to make, take two thirds and there's three thirds so take two thirds plus two thirds plus one third and one third you have two more thirds.
Dr. Landis: Oh, that's interesting. Say this again and let's see if we can follow him. What did you just say? Say it again.
Andrew: [gesturing with hands] There's three thirds so there's two thirds and one third and one third, that's two thirds and you still have one two thirds left over.

Dr. Landis:	Can you kind of show me a picture of that here and I want, James, I want to see if you understand what he's saying. This is real interesting.
Andrew:	[while drawing picture] So then there's one third and two thirds is two thirds so then here's the half [of the blue ribbon]. So you only have one third so then you have to get the other third [indicates first third of second meter]. This is two thirds so then you have two more thirds left over.

(lines 15.1.297-15.1.305)

Here, Andrew justified his solution three times using direct reasoning.

12d,e Dr. Landis asked James to explain Andrew's reasoning. James said, "There are two meters. Yeah, And there are six meters is in each, and it would be two thirds is one, two thirds is again and two thirds left" (lines 15.1.307-15.1.309). Although James' explanation was peripherally faulty in that he referred at to the six thirds as six meters, James used direct reasoning in his discussion of Andrew's justification. During the subsequent discussion, Andrew repeated his explanation for a fourth time (line 15.1.321).

13,14 Dr. Landis then asked Andrew and James how many ribbons would be cut that were two thirds of a meter in length, based on Andrew's description.

Andrew:	Four.
James:	I think three.
Andrew:	Why three if you have two thirds and two thirds?
James:	[pointing at drawing] You have two third and two thirds and then there are six and this is two thirds and this is two thirds would be one, two, three, yeah, four. One, two, no, three, one
Andrew:	I know but two thirds.
James:	Andrew I know but half is.
Andrew:	So two thirds of this, two thirds of the white then you have two thirds of the white ribbon then two thirds of the white ribbon, right? And there if you have one more third and one more third of the white and there if you have one more third and one more third of the white and then you have two thirds left over going that way or going that way.
Dr. Landis:	So how many ribbons could you cut that are two thirds long?
Andrew:	Four.

(lines 15.1.336-15.1.344)

James used direct reasoning to explain that three bows would be formed from the ribbon. Andrew used direct reasoning, but his reasoning was faulty. Although his justification was incomplete, making it difficult to determine from the data how he arrived at this solution, it would appear from his comment to James that he added two thirds and two thirds and concluded that there were four bows. Dr. Landis encouraged Andrew to think about the problem some more.

15, 16 Brian2, Erin, and Caitlin worked together on the tasks. T/R 2 questioned them about the first set of tasks. First, Erin used direct reasoning to explain that there would be three bows formed from the white ribbon that were one third of a meter long, since there are three thirds in one meter (lines 15.2.15-15.2.17). Then, Brian2, who had left the group while Erin provided her solution, offered a similar direct explanation (lines 15.2.22-15.2.26.

17 T/R 2 then questioned them about the next task in the set. Caitlin and Erin used direct reasoning to explain that since there are four fourths in a meter, there would be four bows made that were each one fourth of a meter in length (lines 15.2.30-15.2.32).

18, 19 Later, Caitlin offered her solution for the second and third set of problems. She reasoned directly that there would be six half-meter-long bows formed from the gold ribbon, which measured three meters in length (line 15.2.60). Then, using faulty additive reasoning, she continued the pattern in the third set of tasks as she had in the first, saying that there would be seven, eight, nine, and ten bows made from each of the lengths that were listed on the worksheet (lines 15.2.60-15.2.68).

20a Brian2 countered Caitlin's argument by explaining his understanding of the problem. He reasoned directly that the second set of problems followed a doubling pattern, and the third set of problems followed a tripling pattern. However, his justification of why this was so was incomplete (lines 15.2.69-15.2.77). Caitlin agreed with Brian2's explanation.

20b T/R 2 approached the group, and Caitlin explained directly, with some input from Brian2, why Brian2's explanation was correct.

T/R 2:	So, what happens here? What should this one be? What should two meters and making bows one third meter be?
Caitlin:	This one… this one would go up by two so that would be six, and this one would go up by three.
T/R 2:	That's interesting. How did you discover that?
Caitlin:	Well, because this is one meter so keep on going up one and this is two, so you go up two and then this is three so you go up three.
Brian2:	You had to times this by two or you'd have to time three by three and then you'd get the answer.

(lines 15.2.94-15.2.98)

21,22 The group then worked on the task involving two thirds of a meter bows that were to be cut from the blue ribbon. Brian2 and Erin each used faulty reasoning to justify their solutions. Brian2 said that since there were "two meters and you have to divide it into two thirds then that'd be six" (line 15.2.109). Erin offered a different solution. She said, "Because if you take one third, it's six bows, but two thirds you take two of those two meters so you'd have double that so you'd have six and six, twelve" (line 15.2.117). After hearing these justifications, T/R 2 suggested that they test their ideas using the materials.

23 Later in the session, V2 worked with Brian2, Erin, and Caitlin on this task. She suggested that the students determine a length that would be one third of the meter stick. They did so, and Caitlin reasoned directly that two thirds of a meter would be twice that length (lines 15.2.250-15.2.256). The group then cut a length of string that was equivalent to two thirds of a meter, and used that string to determine, using direct reasoning, that there would be three bows made of that length from two meters of ribbon (lines 15.2.272-15.2.281). V2 then questioned them further about the task.

V2:	Ok if you have one third, you get how many bows?
Brian2, Erin:	You get six bows.
V2:	Ok and now you got, you need more or less ribbon for two thirds than one third?
Erin:	Two thirds is larger than one third.
V2:	Ok, so if you need..
Erin:	So it'd have to be.
Brian2:	You would make bigger bows. You would make bigger bows.
V2:	Right, and so you make bigger bows, they each are going to be what? Are you going to make as many?
Brian2, Erin:	No.
V2:	Oh, so that like sort of makes sense.
Erin:	Yeah.

(lines 15.2.286-15.2.296)

25a Amy and Jackie worded as partners during this session. Amy used the ribbon to measure the number of bows that could be made from the blue ribbon that were one half and one third of a meter in length. She found solutions of four and six, respectively. Jackie predicted the solutions for the one third, one fourth, and one fifth meter-long bows after noticing the doubling pattern. However, she did not complete her justification of the pattern as she worked.

25b Later, Jackie explained to T/R 1 that she had noticed patterns in the first two sets of tasks (lines 15.2.150-15.2.152). As she justified her solution for the second set of problems, she said,

> One half would be four because… we doubled it because it was two meters. And that would be one would be six bows, and then eight bows, ten and twelve. And … we thought that all of them whatever that number how many meters it would be we thought that we would have to go by twos
>
> (line 15.2.154)

Here, Jackie used direct reasoning to explain why she though the doubling pattern worked, but again did not complete her justification.

26 Jackie and Amy then worked on the task involving two thirds of a meter. They determined, upon questioning by T/R 1, what two thirds of a meter would look like when using the white ribbon. T/R 1 then asked them to determine how many of those lengths would equal the length of the blue ribbon, and to be sure to justify their solution. Amy used the white ribbon and attempted to fold the blue ribbon in a way that would show the solution, but used a one third meter length of ribbon instead of two thirds of a meter. As a result, her reasoning was faulty and she concluded that there would be six bows of that length that could be cut from the blue ribbon (lines 15.2.203-15.2.205).

27, 28 Jackie challenged Amy's solution. She used indirect reasoning and said that the solution for the one third meter long bows was six, and that therefore that could not be the solution for two third meter long bows (lines 15.2.206-15.2.210). She then used faulty direct reasoning and said, "We have to double it because this is going to be one third instead of two thirds" (line 15.2.232). Jackie concluded that the solution would be twelve bows.

29, 30 The students then joined in a whole class discussion. They first reviewed the first set of tasks, and Kelly provided a justification for the first task. She reasoned directly that that since there were two half meters in one meter, two bows could be formed from one meter that were one half of a meter in length (line 15.1.349). Caitlin then used similar direct reasoning to explain that there were there one third meter lengths in a meter, and that three bows of that length could be made (lines 15.1.351-15.1.355).

31,32 Danielle offered a direct justification for the next task in the set. She said that by folding the white ribbon in half twice, one fourth meter lengths could be measured, and that would provide a solution of four bows that could be formed that were of that length (lines 15.1.367-15.1.369). V2 said that she was not convinced, and Kelly suggested that if four of those lengths were measured on the meter stick and equal the length of the stick, they could determine that the length was indeed one fourth of a meter. Graham extended her direct reasoning and said that four of the one fourth meter lengths would equal the length of one meter stick (lines 15.1.375-15.1.377).

The students talked about the pattern that they had noticed in the tasks and solutions, and T/R 1 recorded the results of the tasks on a transparency. She asked the students to predict the number of bows that could be made that were one tenth of a meter in length, and Graham predicted that the solution would be ten (lines 15.1.385-15.1.402).

8g The class then discussed their solutions to the second set of tasks. Andrew provided a direct justification for the first task in the set. He explained that two white ribbons equaled the length of one blue ribbon, and that there were two half meters in one half of the blue ribbon, and another two half meters in the other half of the blue ribbon. He then said that that would give four half meter long bows (line 15.1.408).

3b Brian then offered a solution to the second problem, which asked for bows that were one third of a meter in length. He explained directly,

> Well, there's two meters and in one meter there are three thirds and in the other meter there are three thirds. In the other meter there are three thirds. So you add them. In one meter there are three third and the other meter there are three thirds. If you add the two meters together it'd be three thirds and three thirds which is six.
>
> (line 15.1.414)

20c Brian2 offered another way of looking at this task. He said that he multiplied the denominator of the fractions in the second set of tasks by two, and concluded directly that three times two is six (line 15.1.418).

33, 34 T/R 1 then asked the class what they had found for the two third meter long bows. Andrew suggested that the solution was four bows, while Brian and Erin thought that there were three. T/R 1 asked the students how they would convince her of their solution. Kelly said that she folded the blue ribbon into three parts and found that three bows could be made. Her direct justification was incomplete. T/R 1 asked the class how they would convince her that each of those lengths measured two thirds of a meter. Kelly suggested that they measure it. Mark began to measure it on the meter stick and justify why Kelly's solution was correct, but did not complete his justification. T/R 1 suggested that they end the session and think about how they would convince her of their solution (lines 15.1.422-15.1.447).

35 As the other students were wrapping up from the session, Andrew called over T/R 1 to explain his solution to her. He repeated his justification for how he found the two third meter lengths as he had explained it to Dr. Landis. T/R 1 asked him how many two third meter lengths he had indicated during his explanation. Andrew counted and found that there were three lengths that were two thirds of a meter in length. T/R 1 asked him to think about how to explain his revised solution in a convincing manner (lines 15.1.438-15.1.460).

Table 4.15

Forms of Reasoning, Session 15

	Student	Lines	Type	Structure	Form	Sub-codes	WC	PR
1.	Brian	15.1.61-15.1.69	Claim	Direct				✓
2.	Danielle	15.1.73-15.1.75	Claim	Direct				✓
3.	Brian	15.1.86-15.1.88	Claim	Direct				✓
4.	Jessica	15.1.106-15.1.122	Claim	Direct				✓
5.	Jessica	15.1.126-15.1.131	Claim	Direct		Incomplete		✓
6.	Jessica, Laura	15.1.149-15.1.159	Claim	Direct				✓
7.	Andrew	15.1.26	Claim	Direct				✓
8a-f.	James, Andrew	15.1.162-15.1.220	Claim	Direct				✓
9.	James, Andrew	15.1.226-15.1.235	Claim	Direct				✓
10.	Andrew	15.1.237-15.1.239	Claim	Direct				✓
11.	James, Andrew	15.1.241-15.1.244	Claim	Direct				✓
12a.	Andrew	15.1.297-15.1.301	Claim	Direct				✓
12b	Andrew	15.1.303	Claim	Direct				✓
12c.	Andrew	15.1.305	Claim	Direct				✓
12d	James	15.1.307-15.1.309	Claim	Direct		Faulty-p		✓
12e.	Andrew	15.1.321	Claim	Direct				✓
13.	Andrew	15.1.336-15.1.342	Claim	Direct		Faulty		✓
14.	James	15.1.337-15.1.341	Claim	Direct		Incomplete		✓
15.	Erin	15.2.15-15.2.17	Claim	Direct				✓
16.	Brian2	15.2.22-15.2.26	Claim	Direct				✓
17.	Caitlin, Erin	15.2.30-15.2.32	Claim	Direct				✓
18.	Caitlin	15.2.60	Claim	Direct				✓
19.	Caitlin	15.2.60-15.2.68	Claim	Direct		Faulty		
20a.	Brian2	15.2.69-15.2.77	Counter-19	Direct		Incomplete		✓
20b	Caitlin, Brian2	15.2.93-15.2.98	Counter-19	Direct				✓
21.	Brian2	15.2.107-15.2.109	Claim	Direct		Faulty		✓
22.	Erin	15.2.117	Claim	Direct		Faulty		✓
23.	Caitlin	15.2.251-15.2.256	Claim	Direct				✓
24.	Brian2, Erin	15.2.279-15.2.294	Claim	Direct				✓
25a.	Amy, Jackie	15.2.34-15.2.56	Claim	Direct		Incomplete		✓
25b	Amy, Jackie	15.2.150-15.2.154	Claim	Direct		Incomplete		✓
26.	Amy	15.2.203-15.2.205	Claim	Direct		Faulty		✓
27.	Jackie	15.2.206-15.2.210	Counter-27	Indirect				✓
28.	Jackie	15.2.230-15.2.232	Claim	Direct		Faulty		✓
29.	Kelly	15.1.349	Claim	Direct			✓	
30.	Caitlin	15.1.351-15.1.355	Claim	Direct			✓	
31.	Danielle	15.1.367-15.1.369	Claim	Direct			✓	
32.	Kelly, Graham	15.1.375-15.1.377	Claim	Direct			✓	

	Student	Lines	Type	Structure	Form	Sub-codes	WC	PR
8g.	Andrew	15.1.406-15.1.408	Claim	Direct			✓	
3b.	Brian	15.1.412-15.1.414	Claim	Direct			✓	
20c.	Brian2	15.1.418-15.1.420	Claim	Direct			✓	
33.	Kelly	15.1.428-15.1.432	Claim	Direct		Incomplete	✓	
34.	Mark	15.1.434	Claim	Direct		Incomplete	✓	
35.	Andrew	15.1.438-15.1.460	Claim	Direct				✓

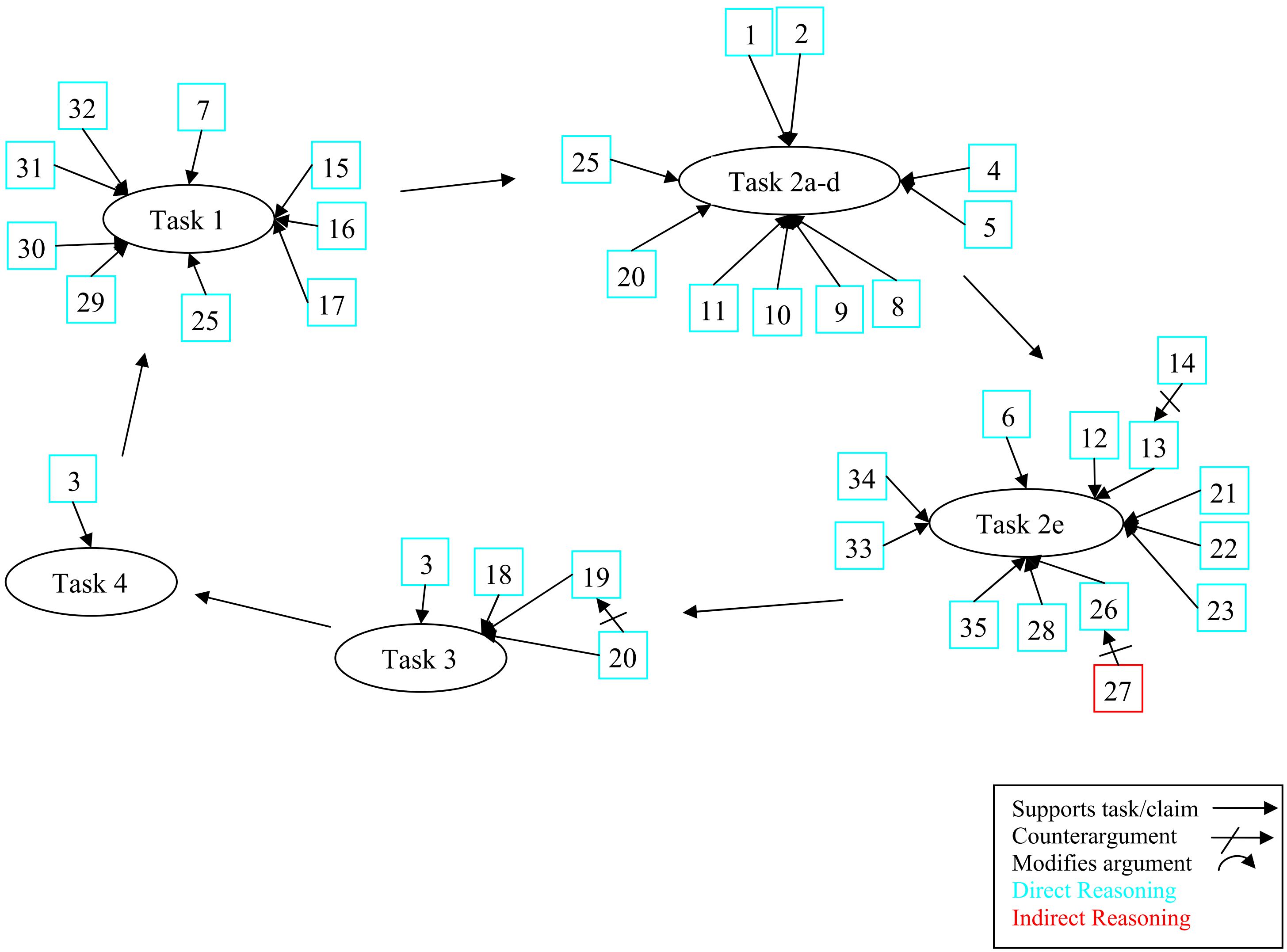


Figure 4.15. Argumentation Organized by Task, Session 15

4.2.16 Session 16: December 14, 1993

Task 1: Finding the number of one third meter long bows from three, nine, twenty-seven, and eighty-one meters of ribbon

1a,b The session began with a whole class discussion about making bows that were one third meter in length. T/R 1 asked the students to imagine that they had three meters of ribbon and that they were forming bows that would be one third meter long. Jessica, using direct reasoning, said that she would form these bows by cutting on the thirty three mark, since "thirty-three plus thirty-three plus thirty-three is ninety nine and that's… and then a hundred, around like thirty three and like a half almost" (lines 16.2.25-16.2.27). Her reasoning, a bit imprecise and thus flawed, communicated that she was attempting to find three equal lengths of ribbon that together were one meter long. Alan then said, using direct reasoning, that each bow would be thirty-three and one third long, "because if you take two more thirds you can get it to a hundred" (line 16.2.31). T/R 1 asked where the students had arrived at numbers such as thirty-three and one hundred, and the students explained that these were measurements in centimeters. Alan elaborated that the meter stick was marked until ninety-nine centimeters, but that the extra length at the end brought the count up to one hundred (lines 16.2.37-16.2.41)

2 Graham then noted that there were ten decimeters in one meter. Reasoning directly, he explained that this was the case because there are ten centimeters in a decimeter and that there are ten times ten centimeters in a meter (16.2.43-16.2.249).

T/R 1 asked the students how many bows they could make that were one third meter in length if they had three meters of ribbon. She asked them to speak with their partners to arrive at a solution. As the students worked on the problem, she expanded the task to include finding the number of bows that could be made from nine, twenty-seven, and eighty-one meters of ribbon. The students worked on these tasks for the majority of the session.

3 Amy and Jackie began by considering the problem without using meter sticks. Amy, reasoning directly, said that if the bows were a meter long, there would be three bows, but if they would be one third meter long, there would be nine bows (lines 16.2.60-16.2.64). They then joined Brian and Danielle, who were working with CT on the task (see 12 below)

4a,b Andrew and James represented the original problem of three meters of ribbon by placing three pens end to end and used direct reasoning to solve the task. Andrew reasoned that each meter would be divided into thirds, and James counted "Three, six, nine" (line 16.1.62). Later, they explained to T/R 1 that there were nine bows in three meters of ribbon (lines 16.1.77-16.1.79).

5 Jessica and Laura used direct reasoning to explain to T/R 1 how they had arrived at the answer of nine bows when using three meters of ribbon. Jessica explained that three bows could be made from one meter of ribbon, and Laura and Jessica then said that when there three meters of ribbon were involved, the three meters would multiplied by three to arrive at nine bows (lines 16.1.65-16.1.75).

6 Meanwhile, Jackie and Amy joined Danielle, Brian, and Erin, who were talking to CT about the first task. Brian showed that there would be three bows made from one meter of ribbon. Then, using direct reasoning, Danielle said that there would be six bows made from two meters of ribbon, and nine bows from three meters of ribbon (lines 16.1.87-16.1.117).

7a,b,8 T/R 1 spoke with Michael and Sarah about the task. Michael reasoned directly that if there were nine meters of ribbon, there would be twenty-seven bows, and justified his solution by adding three nine times. Shortly afterwards, he explained to T/R 2 that Sarah used multiplication but that he had used addition to find and check the solution, and justified the two methods by explaining that adding three nine times is the same as multiplying nine by three (line 16.2.129). T/R 2 asked them where they had found the number three. Michael said, "Because there's three thirds that make up a whole, or a meter" (line 16.2.134). T/R 2 asked Sarah to explain how she was thinking about the problem, and she said that since the problem asked for one third meter lengths, the number of meters of ribbon in question would be multiplied by three. Michael elaborated by indicating one third of a meter on the meter stick and said that there would be three of those lengths in the meter (lines 16.2.136-16.2.140).

9a Michael and Sarah began to work on ribbon lengths of ever increasing size. Using direct reasoning, they multiplied twenty-seven by three to arrive at eighty-one, and then multiplied eighty-one by three to arrive at two hundred forty-three. They continued in this manner for a good portion of the session. Although this activity is reminiscent of recursive reasoning, the recursive nature of the activity only generated more problems, rather than showed a recursive pattern inherent in the mathematics of the task (lines 16.2.143-16.2.196).

10 During this activity, T/R 2 questioned Michael and Sarah about their understanding of the task. She asked them what they would do if they were making bows that were one half a meter in length. They replied that they would multiply by two. Sarah explained while gesturing with her hands, that there were two halves in one meter.

Michael elaborated that if there were three meters of ribbon, six bows that were one half meter in length could be made. T/R 2 asked them what their strategy would be if they were making one fourth meter long bows. Sarah explained that they would then multiply by four (lines 16.2.267-16.2.269). Both Michael and Sarah used direct reasoning during this discussion.

11,12 Erik, Erin, and Brian went to the hallway, where they began to consider measuring and cutting the ribbon for the task involving nine meters of ribbon. However, they thought that the task involved finding the number of meters in length that each bow would be if they were making bows that were one third the length of nine meters. Erin and Erik concluded that each bow would be three meters in length. Reasoning directly, he explained to CT and subsequently to T/R 1 that the three bows would be cut at three meters, six meters, and nine meters, and would thus each be one third the length of nine meters of ribbon. T/R 1 asked Erik how many bows could be made if each bow was one third meter in length. Erik replied using direct reasoning, and added three eight times to arrive at twenty-four. T/R 1 cautioned him to check his arithmetic, and Erik modified his solution to twenty-seven bows (lines 16.2.367-16.2.370). T/R 1 asked the students to record their solutions to both problems.

9b As Michael continued to multiply his ever increasing lengths of ribbon by three to find the number of bows that could be made, Sarah shared the strategy that she and Michael had found with Jackie, Danielle and Brian. She explained directly that the number of meters of ribbon was multiplied by three to find the number of bows (lines 16.2.398-16.2.408). Jackie, Danielle, and Brian then began to use this strategy to work on the task.

Alan and Kimberly worked on the task together. Alan attempted to use multiplication to find the product of three and twenty-seven but used an incorrect algorithm. He then used addition to find the correct solution of eighty-one bows. T/R 1 approached Alan and asked him about his solution. After he showed T/R 1 the correct solution, T/R 1 questioned him about his attempted multiplication. Alan told T/R 1 about his algorithm but noted that it didn't work. T/R 1 asked Kimberly, who had used the traditional multiplication algorithm, how she had arrived at a solution.

> Kimberly: I did twenty-seven times three.
> T/R 1: And how did you do it?
> Kimberly: I times twenty, I times three times seven, I got twenty-one, so I carried the two, then I did three times two and added the two to my answer.
>
> (lines 16.1.138-16.1.140)

13a T/R 1 asked Kimberly how it worked, but Kimberly did not explain why the algorithm produced the correct solution. T/R 1 asked Alan and Kimberly if they thought they would get the same solution if they multiplied three by seven and then by twenty separately. Alan said that he thought it would, while Kimberly said that she didn't think so, but that she wasn't sure. Using direct reasoning, he showed that seven times three gave twenty-one, twenty times three was sixty, and Kimberly noted that the addition of the two products indeed gave the same solution. Alan added the two numbers using pencil and paper and showed that it yielded eighty-one (lines 16.1.166-16.1.171).

13b,c T/R 1 encouraged Alan and Kimberly to think about how to explain the procedure for multiplying. Alan worked on inventing a procedure based on the method he had used, and explained the procedure to Kimberly.

> What I'm doing is, you have your twenty-seven, so you take off the seven, and you get and you only have twenty. So then you do twenty times three and you get sixty, which brings me to step two. You don't have two, so you have the seven. So

> you do seven times three and that equals twenty-one. So you add the sixty and the twenty-one and you get eight one.
>
> (line 16.1.193)

Here, Alan used direct reasoning to explain this procedure to Kimberly. Kimberly asked him to re-explain, and he did (line 16.1.199).

13d T/R 1 returned to ask if they had worked on what she had asked them. Alan restated his procedure (lines 16.1.223-16.1.225), and T/R 1 challenged him further to think about Kimberly's original method and try to make sense of it.

14 Kimberly attempted to explain why the algorithm worked.

> But what I learned is you put the one there, and then you carry the two like you do in adding but you times the number so I times three times two and then whatever you got as your multiplication answer you added that number to that and you put, and then once you got there you got your answer.
>
> (line 16.1.251)

Although Kimberly directly described the steps that she had taken to find the solution, T/R 1 and Alan did not find her justification complete, and asked Kimberly to explain why she was carrying the two and what it meant.

15 T/R 1 spoke with Beth, Laura, and Jessica, who explained that they had found the solution of eighty-one by adding twenty-seven three times (line 16.1.282). T/R 1 asked them if there was another way to solve the problem. Laura suggested using multiplication, and Beth explained that twenty-seven times three was the same as adding that number three times. Laura carried out the traditional algorithm, and T/R 1 asked her about the procedure, and asked the girls if they could explain why they were carrying two. Jessica said that they were carrying two tens, and Beth explained that the two was being placed in the tens column. Here, Jessica and Beth used direct reasoning to justify why the algorithm made sense (lines 16.1.320-16.1.338).

16a Meanwhile, T/R 2 probed Alan and Kimberly's understanding of the task. She asked them what they would do if they were making bows that were one fourth of a meter in length. At first, Kimberly simply stated that she would multiply by four, and Alan agreed. Kimberly said, "[I]t's sort of like, you're just using regular numbers" (line 16.1.368). Alan said, reasoning directly, that if there were eighty-one meters of ribbon, there would be three hundred and twenty-four bows that were one fourth of a meter in length (line 16.1.373).

16b T/R 2 asked Kimberly to explain her thinking further. Kimberly said, "Twenty-seven times three is eighty-one but if …you have one meter and it was times four by fourths you get four bows, and if it was by thirds you get three bows so the third or the fourth would be three or four" (line 16.1.400). Here, Kimberly used direct reasoning to explain why she would multiply by four.

16c T/R 2 asked Kimberly if she could explain where she got the number twenty-seven from. Kimberly replied directly that the previous problem had asked about nine meters of ribbon, which made twenty-seven bows, and that T/R 1 had then asked about twenty-seven meters of ribbon (lines 16.1.407-16.1.410).

13e T/R 1 encouraged Alan to speak with Beth about the multiplication algorithm. After she showed him what she had done, Alan repeated his invented procedure (line 16.1.437).

16d-f T/R 2 approached Alan, Beth, and Jessica, and Kimberly joined the group. She questioned them about their method of multiplying by three. To probe their understanding, she posed a new problem.

T/R 2: Ok. New problem, the problem is I have seven meters of ribbon.
Alan: Seven.

T/R 2:	Ok, and I want to make bows that are a third of a meter each. How many bows would I get?
Jessica:	You'd get twenty-one. Because seven times three is twenty-one.
Kimberly:	Right
T/R 2:	Ok, but you're multiplying by three again and we didn't start with three meters, so I don't understand. We started with seven meters.
Alan:	Right, so that would be seven times seven.
T/R 2:	So is that where the three is coming from? That's what I don't understand.
Alan:	And you'd get forty-nine.
Jessica:	No.

(lines 16.1.473-16.1.482)

Here, Jessica used direct reasoning, while Alan used direct faulty reasoning to solve this new problem. Alan, though, immediately changed his thinking.

Alan:	Actually, the fraction that you have, the second digit in fraction is the number you multiply the number of meters that you have. That means if I had seven and I wanted to divide it into fourths, you go seven times four equals twenty-eight.
T/R 2:	So when you say the second number of the fraction, you mean the number on the bottom in the fraction?
Alan:	So the second number of the fraction, like it, one fourth,
T/R 2:	Ok, I see, you have a slash line it's the second number.
Alan:	The second number on the right side of the slash. And then you multiply by the meters that you've got and then you get your answer of how many bows can be made out of em.

(lines 16.1.494-16.1.498)

18a-c T/R 1 called the class together to review the different strategies that they had used to solve the first part of the task, finding the number of bows that could be made from three meters of ribbon. Sarah and Michael used direct reasoning to explain that they had added three three times to find the solution of nine (lines 16.2.444-16.2.446). Jackie then said that she had multiplied by three to find the solution (line 16.2.448), and Jacqueline explained directly that there were three thirds in every meter, and that was why she had multiplied by three (line 16.2.466). Andrew echoed Jacqueline's explanation (line 16.2.472)

19 T/R 1 asked Erin and the other students who had gone out to the hallway to explain what they had done. Erin explained that they had worked on finding the number of bows in nine meters of ribbon, and she noted that they had found that there were twenty-seven bows that could be made (lines 16.2.478-16.2.487).

20,21 T/R 1 then asked Erik and David to share their solution to the problem of the number of bows that could be made from nine meters of ribbon that were three meters in length. Erik explained directly that three times three was nine, and that there would therefore be three bows, and David said that one could also find the solution by adding three three times to find that three bows would take up nine meters in length. He explained further,

> Because you have three meters and then, um, alright one bow would take up three so there'd be six meters left another bow would take up three so then there would be uh three meters left and then there'd be a third one and there wouldn't be, there wouldn't be any ribbon left.
>
> (line 16.2.512)

T/R 1 closed the session by asking the students to write about what they had done, and that if they had thought about why a procedure had worked, they should write about that as well.

Students' Written Work

All the students who submitted written explanations of their class work used direct reasoning to justify their solutions. Alan wrote his three-step multiplication algorithm. Kimberly provided an explanation very similar to that which she gave during the session as to why the standard multiplication algorithm works. Beth, in addition to recording the number of bows that she found for each length of ribbon, wrote, "I think you can carry numbers, but if the number is going in the 10's column it has to have a ten value. Same with 100's, 1000's, ect."

Several students wrote a justification for why multiplying the number of meters of ribbon by three yielded the number of bows that could be made. Michael wrote, "I think this works because it takes 3 1/3 to equal a whole (or meter) and then you have a certain amount of meters and you times that by 3." Audra wrote, "…For example, 3 meters of ribbon and we have to divide them by 3rds, would be nine. Because there are 3 meters and there are 3 3rds 3x3=9." Laura's written description was noteworthy, in that she justified each solution that she had found by explaining how many one thirds there are in the number of meters that were given.

David wrote that he had "worked on the 9 meter problem. Each bow was 3 meters so 9-3=6, 1 bow, 6-3=3, 2 bows, and 3-3=0, 3 bows. We also could have just said 3x3=9 but that's how you can prove it." David also made a drawing to illustrate his first justification.

Erik's written solution differed slightly from his explanation in class. Erik wrote after class:

> In math I measured the 9 meter ribbon and if each bow is 3 meters long you can make 3 bows. If each bow is 1/3 of a meter you can make 27 bows. If you have 9 meter ribbon and divide it into 3 parts each part would be 3 meters. If you divided 9 meter ribbon into 27 parts each part would be 1/3 of 1 meter.

Erik provided a new view of the problem in this written explanation by noting that if the nine meter ribbon were divided into 27 parts, the lengths of those parts would be 1/3 of a meter long.

Table 4.16

Forms of Reasoning, Session 16

	Student	Lines	Type	Structure	Form	Sub-codes	WC	PR
	Task 1: Finding the number of one third meter long bows from three, nine, twenty-seven, and eighty-one meters of ribbon							
1a.	Jessica	16.2.23-16.2.27	Claim	Direct			✓	
1b.	Alan	16.2.31	Claim	Direct			✓	
2.	Graham	16.2.43-16.2.49	Claim	Direct			✓	
3.	Amy	16.2.60-16.2.64	Claim	Direct				✓
4a.	Andrew, James	16.1.59-16.1.63	Claim	Direct				✓
4b.	James	16.1.77-16.1.79	Claim	Direct				✓
5.	Jessica, Laura	16.1.65-16.1.75	Claim	Direct				✓
6.	Brian, Amy, Jackie, Danielle	16.1.86-16.1.117	Claim	Direct				✓
7a.	Michael	16.2.72-16.2.78	Claim	Direct				✓
7b.	Michael	16.2.119-16.2.140	Claim	Direct				✓
8.	Sarah, Michael	16.2.136-16.2.138	Claim	Direct				✓
9a.	Michael, Sarah	16.2.143-16.2.196	Claim	Direct				✓
10.	Michael, Sarah	16.2.254-16.2.269	Claim	Direct				✓
11.	Erik, Erin	16.2.323-16.2.359	Claim	Direct				✓
12.	Erik	16.2.367-16.2.369	Claim	Direct				✓
9b.	Sarah	16.2.398-16.2.408	Claim	Direct				✓
13a.	Alan, Kimber	16.1.166-16.1.171	Claim	Direct				✓
13b	Alan	16.1.190-16.1.193	Claim	Direct				✓
13c.	Alan	16.1.199	Claim	Direct				✓
13d	Alan	16.1.223-16.1.225	Claim	Direct				✓
14.	Kimberly	16.1.251,16.1.272	Claim	Direct		Incomplete		✓
15	Beth, Jessica	16.1.320-16.1.338	Claim	Direct				✓
16a.	Alan, Kimber	16.1.365-16.1.381	Claim	Direct				✓
16b	Kimberly	16.1.400	Claim	Direct				✓
16c.	Kimberly	16.1.405-16.1.413	Claim	Direct				✓
13e	Alan	16.1.437	Claim	Direct				✓
16d	Jessica	16.1.476	Claim	Direct				✓
16e.	Alan	16.1.479-16.1.481	Claim	Direct		Faulty		✓
16f.	Alan	16.1.484-16.1.498	Claim	Direct				✓
17.	Michael, Sarah	16.2.444-16.2.446	Claim	Direct			✓	
18a.	Jackie	16.2.448	Claim	Direct			✓	
18b	Jacquelyn	16.2.466-16.2.469	Claim	Direct			✓	
18c.	Andrew	16.2.472	Claim	Direct			✓	
19.	Erin	16.2.478-16.2.488	Claim	Direct			✓	
20.	Erik, David	16.2.503-16.2.512	Claim	Direct			✓	
21.	David	16.2.512	Claim	Direct			✓	

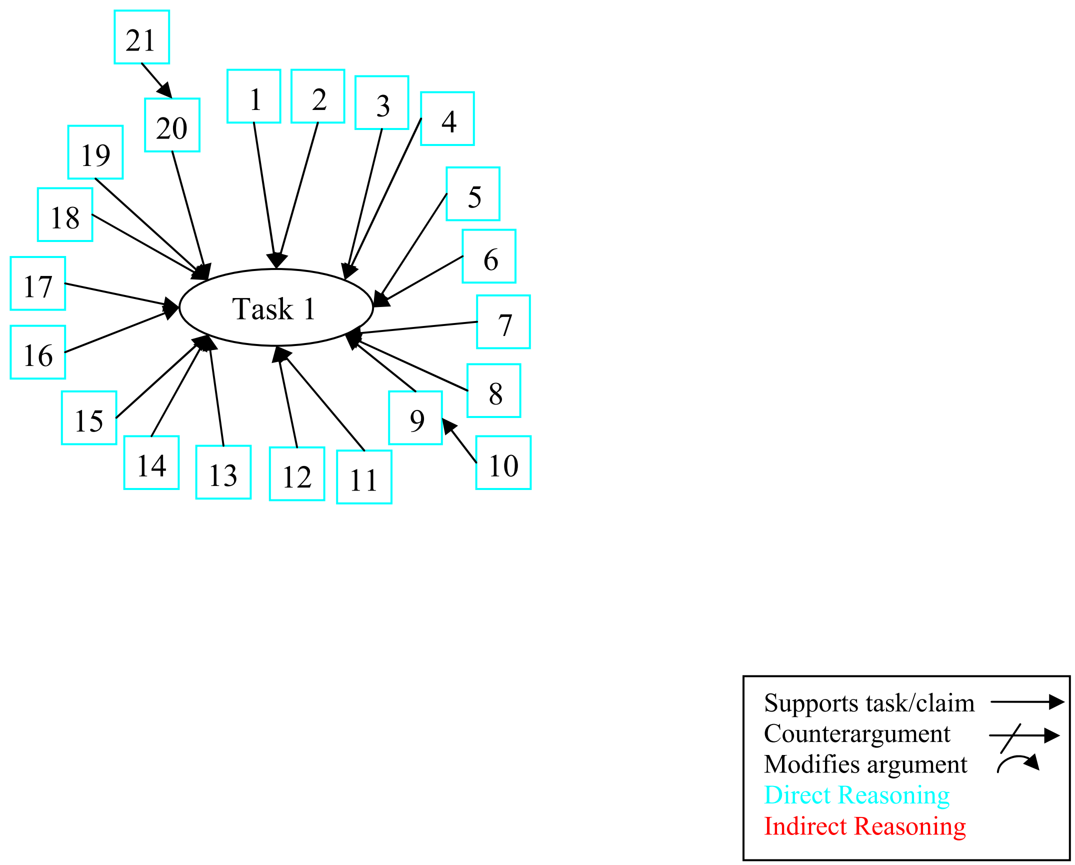


Figure 4.16. Argumentation Organized by Task, Session 16

4.2.17 Session 17: December 15, 1993

Task 1a: How many bows can you make from ribbon that's a third of a meter in length if you have nine meters of ribbon.

The session began with a whole class discussion about the problems that the students had worked on the day before. Afterwards, the students worked with their partners on a series of tasks, and the session culminated in a whole class discussion. Throughout the session, the students used direct reasoning exclusively. At times, the reasoning was faulty or incomplete, and those instances will be noted individually.

1a,b First, T/R 1 asked to students to tell her about the different ways that they had determined how many bows they could make that were one third meter in length when they had nine meters of ribbon. The students said that the solution was twenty-seven bows. Alan explained that this could be found by multiplying nine times three (line 17.2.68). T/R 1 asked Michael to elaborate, noting that they were making bows that were one third of a meter in length, and that it was unclear how Alan had used the number three. Michael clarified that three bows could be made from each meter of ribbon, and that there were nine meters of ribbon. T/R 1 asked Michael if he meant that there was three nine times, and he replied affirmatively (lines 17.2.70-17.2.74).

2 Audra then told the class about a different method that she had used to find the solution. She explained, “We did the three three times and we added it up to nine” (lines 17.2.84). Although her explanation was incomplete, T/R 1 recorded her solution as an addition problem that added three nine times. T/R 1 asked her if that was what she had done and she replied that she had.

Task 1b: I still have my nine meters of ribbon, but now I'm making ribbons that are three meters long

3 T/R 1 asked the students to consider cutting the nine meters of ribbon into three meter long bows, and asked them if these bows would be larger or smaller than those that were one third meter long. Michael replied that the one third meter bows would be smaller than the three meter bows.

4 T/R 1 noted that some students did not seem convinced, and asked the class to help them understand what these bows would look like. Brian explained that the nine meters would be divided into thirds to make bows that were three meters in length. T/R 1 asked him where the first cut would be made, and Brian replied that it would be on the three meter mark. Audra then said that the second cut would be on the six meter mark. T/R 1 asked the students if there would be a third cut, and they replied that there would not, but that there would be three pieces of ribbon and three bows that would be formed (lines 17.2.94-17.2.106). During this discussion, the students worked together to reason directly about the task.

T/R 1 recorded the results of this discussion at the OHP. She then posed another series of tasks to the students, and recorded them on a transparency. She asked the students to imagine that they had twelve meters of ribbon and were making half meter bows, two meter bows, one third meter bows, six meter bows, and two third meter bows. The students worked on these problems for approximately twenty minutes.

Task 2a: So now I'm starting with twelve meters of ribbon, [and making] one half meter bows… How many bows am I going to make?

Task 2b: I want to know how many you can make that are two meters [long].

5 Erin and Jackie worked together on the first two problems. Immediately, Erin offered her solution to the first problem.

Erin:	Twenty-four
Jackie:	No, we're doing one half. I think, see, two halfs make a whole, well, my fingers are halfs, so two, this is one whole, this is another whole, this is another whole, this is another whole, and this is another whole.
Erin:	But each one's going to have two, each whole would have.
Jackie:	Yeah, yeah, but two halves. But there would be six wholes.
Erin:	I'm not agreeing really.

(lines 17.2.116-17.2.120)

T/R 2 asked Erin and Jackie to tell her what they were thinking about the problem. Erin provided a direct argument for her solution, saying that it would be twenty-four because twelve times two was twenty-four. T/R 2 asked her why she multiplied, and Erin replied that "each meter's gonna have two bows in it and there's twelve meters they're gonna have you're gonna double the twelve so… you get twelve two times" (lines 17.2.130-17.2.132).

6 Jackie then told T/R 2 what she thought about the problem. She showed T/R 2 the picture she had drawn to represent the problem (see Figure F-29-08), and explained, "All these lines are halfs. So, um, if you group this, this would be one whole, this would be one, two, this would be another, and it would be six, because all these are whole are one" (line 17.2.136). T/R 2 asked Jackie if the lines represented one meter, and she explained that they represented half a meter. T/R 2 then asked if there were twelve meters of ribbon in her diagram. Jackie replied that there were twelve half-meters (lines 17.2.137-17.142). Jackie's faulty reasoning stemmed from her representation of twelve half meters instead of twelve meters, and she concluded that there were six meters in total.

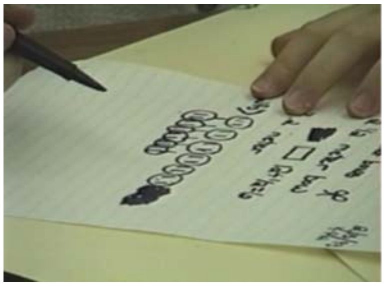


Figure F-29-08

7 T/R 2 asked Erin and Jackie to think about the second problem that T/R 1 had posed, which changed the length of the bows from half a meter to two meters. Erin used direct reasoning to explain that there would then be six bows. She said, “[T]here's gonna be less bows ‘cause each is two meters, each bow is gonna be two meters” (line 17.2.148). T/R 2 asked her how she had arrived at the solution of six. Erin explained, “Because half of twelve is going to be six and if you are counting up to twelve go [counts on fingers] two four six eight twelve that's six” (line 17.2.152).

8 Jackie then offered her own solution to the problem. Her first argument was faulty and was again based on a diagram that she made to represent the problem (Figure F-31-44). She drew twelve vertical lines and wrote the number “2” atop each one. She then explained that each line represented two meters. T/R 2 asked her how many lines would be needed if each line was two meters, and Jackie replied that twelve lines were needed.

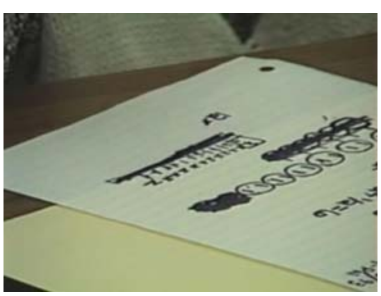


Figure F-31-44

9 T/R 2 suggested that Jackie draw a diagram that would show how she would cut the ribbon as if it were actually there. Jackie drew a horizontal line and wrote “twelve meters" above it. T/R 2 asked her where she would cut the ribbon so that each bow would be two meters in length. Jackie began to draw vertical lines to mark off lengths of the ribbon.

T/R 2 asked her where the first two meter length was, and she drew a horizontal line above the first line that she had drawn until the first vertical mark (Figure F-33-34). Jackie then continued to extend her second horizontal line, and T/R 2 counted the two meter marks as they were reached. Jackie then counted how many two meter lengths she had marked off, and found that she had ten (Figure F-33-58).

T/R 2 asked her to write the number "2" near each segment that represented that length. She then asked Jackie how many two meter lengths would be needed if there was a total of twelve meters of ribbon. Jackie replied that there would be six, and wrote six twos in her diagram (Figure F-34-35). She then concluded directly that six bows would be made.

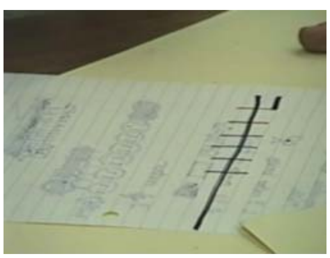


Figure F-33-34

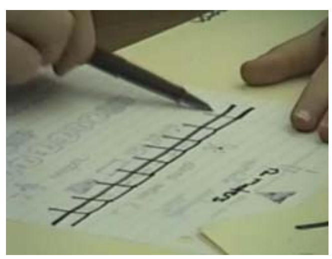


Figure F-33-58

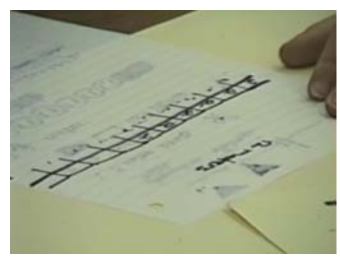


Figure F-34-35

10 T/R 2 suggested that the two students now revisit the first problem that they had been working on. Jackie worked on the problem again, drawing a new model to represent her solution. She drew a horizontal line and made vertical tick marks to represent half a meter. Then, she drew a circle around each pair of vertical marks to show each meter length "cause two halves make a whole" (line 17.2.178). She marked off twelve pairs of vertical tick marks, but it is unclear what she concluded as the solution (Figure F-40-04).

Figure F-40-04

11 Erin used two diagrams to justify her earlier solution to the problem. First, she drew twelve squares and divided each square vertically in half (Figure F-40-12). She told Jackie, “Now I have twelve squares here and I split them into all halves. Now we have to count each half. Two four six eight ten twelve fourteen sixteen eighteen twenty twenty-two twenty-four. There’s twenty-four halves” (line 17.2.181). Jackie asked her, “Are we counting up the halfs or the meters?” (line 17.2.182), and Erin replied that they were counting the halves. From this exchange, it may be interpreted that Jackie’s faulty reasoning about this problem stemmed from her misconception that they were counting the number of meters, rather than the number of bows.

Erin drew a second diagram consisting of twelve circles that were likewise divided vertically in half (Figure F-41-17). She said, “Ok, now I split those all in half it goes, one two three four five six seven eight nine ten eleven twelve thirteen fourteen fifteen sixteen seventeen eighteen nineteen twenty twenty-one twenty-two twenty-three twenty-four” (line 17.2.185).

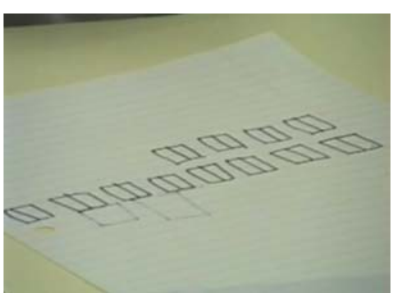


Figure F-40-12

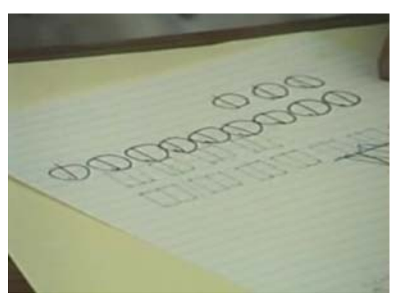


Figure F-41-17

Jackie suggested that they record both solutions on the overhead transparency that they were provided, since she wasn't sure which one was correct. Erin recorded her solution of twenty-four bows, and Jackie recorded her second diagram and wrote that there were "6 1 wholes" since "I added 1/2 twelve times" (See Appendix B for Jackie and Erin's written work).

Task 2c: How many bows can be made that are two third meters long if we have twelve meters of ribbon?

12 Toward the end of the session, T/R 1 called the class together and asked Alan and Kimberly to share their solution to the last problem that had been posed. This problem asked how many two third meter long bows could be made from twelve meters of ribbon. Alan had recorded his representation of the problem on an overhead transparency (Figure O-53-09, see Appendix C for Alan and Kimberly's written work), and he explained his solution to the class.

> This entire thing is twelve meters. The long line is the divider of each meter. The brackets are dividing the thirds up so there are two thirds, there are two thirds, there are two thirds, there are two thirds, and if you count up how many two thirds there are, you'll eventually get down to eighteen, and that's how many bows you can make of two thirds out of twelve meters.
>
> (line 17.2.211)

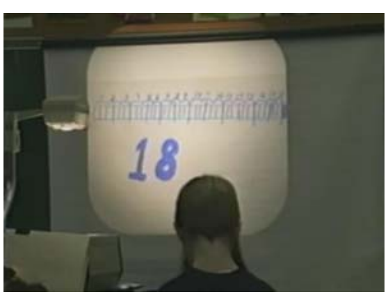


Figure O-53-09

T/R 1 asked Alan to explain his solution for a second time for the students that did not follow, and then asked Kimberly to rephrase what Alan had said. T/R 1 then asked the students to explain why they thought Alan and Kimberly had constructed the model the way they did. The students explained that they had marked the thirds so that they would know where to place the brackets, and that they had used the brackets so that they could keep track of each two third meter length. Kimberly explained that the brackets were numbered so they wouldn't lose track of the number of bows that could be made.

13 T/R 1 closed the session by asking the students to think about a different task that they had worked on, determining the number of one third meter bows that could be made from twelve meters of ribbon. Graham replied that there were thirty-six bows, and Andrew explained that this could be found by multiplying twelve times three (lines 17.2.282-17.2.284). T/R 1 asked the students to think about whether the secret worked for the problem that Alan and Kimberly had worked on, and said that perhaps they would discuss this question during the next session.

Table 4.17

Forms of Reasoning, Session 17

	Student	**Lines**	**Type**	**Structure**	**Form**	**Sub-codes**	**WC**	**PR**
	Task 1a: How many bows can you make from ribbon that's a third of a meter in length if you have nine meters of ribbon.							
1.	Alan	17.2.68	Claim	Direct		Incomplete	✓	
1b.	Michael	17.2.70-17.2.74	Claim	Direct			✓	
2.	Audra	17.2.78-17.2.84	Claim	Direct			✓	
	Task 1b: I still have my nine meters of ribbon, but now I'm making ribbons that are three meters long							
3.	Michael	17.2.90	Claim	Direct			✓	
4.	Br, Au, Kim, Erin	17.2.94-17.2.106	Claim	Direct			✓	
	Task 2a: So now I'm starting with twelve meters of ribbon, [and making] one half meter bows… How many bows am I going to make? *Task 2b: I want to k now how many you can make that are two meters [long].*							
5.	Erin	17.2.116-17.2.118, 17.2.124-17.2.132	Claim	Direct				✓
6.	Jackie	17.2.117-17.2.119, 17.2.136-17.2.142	Claim	Direct		Faulty		✓
7.	Erin	17.2.146-17.2.154	Claim	Direct				✓
8.	Jackie	17.2.156-17.2.162	Claim	Direct		Faulty		✓
9.	Jackie	17.2.164-17.2.174	Claim	Direct				✓
10.	Jackie	17.2.176-17.2.180	Claim	Direct		Incomplete		✓
11.	Erin	17.2.181-17.2.185	Claim	Direct				✓
	Task 2c: How many bows can be made that are two third meters long if we have twelve meters of ribbon?							
12a.	Alan	17.2.211	Claim	Direct			✓	
12b	Alan	17.2.219	Claim	Direct			✓	
12c.	Kimberly	17.2.223-17.2.229	Claim	Direct			✓	
13.	Graham, Andrew	17.2.282, 17.2.284	Claim	Direct			✓	

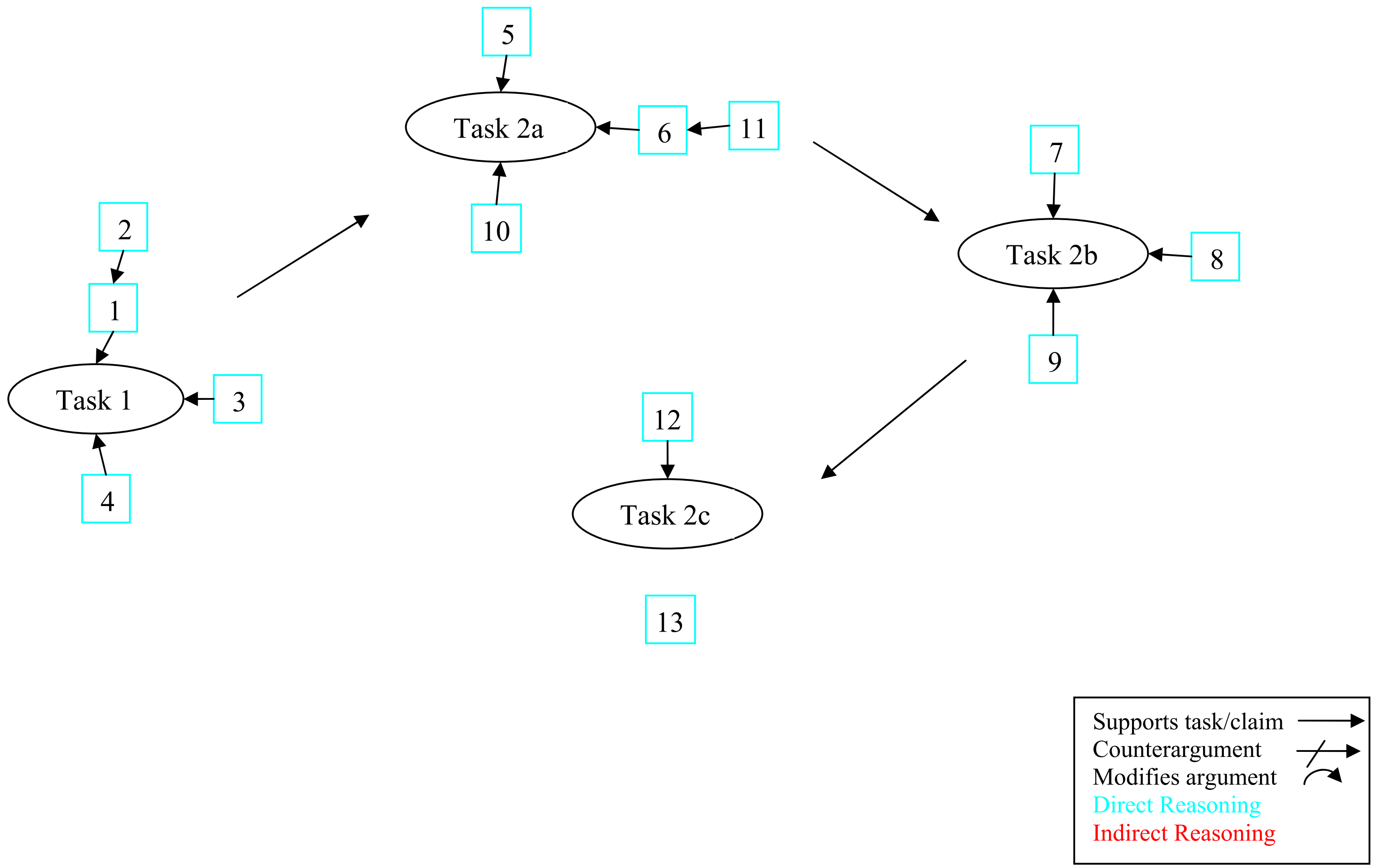


Figure 4.17. Argumentation Organized by Task, Session 17

4.3 Other Forms of Reasoning

As was noted in Chapter 3, the forms of reasoning described in this section are less deductive in nature than those described in the main body of the narrative. The development of this reasoning is therefore traced separately in the paragraphs below.

4.3.1 Generalization

As the fourth grade students built fraction ideas, they began to conjecture and make generalizations. At times, they justified these generalizations, and often used generic or recursive reasoning to do so. At other times, they simply noted a pattern or made a prediction based on the tasks that they were studying. In this section, the generalizations that were made by students will be identified as they occurred chronologically.

Session 2

The first instance of generalization in the data occurred during Session 2, as the students were working to find a rod that was one half the length of the blue rod. David presented a generic argument to show that the blue rod belonged to a larger class of rods for which rods that were one half their length did not exist in the given set of rods. With this argument, David made a generalization about the odd and even nature of the rods in the set.

Session 4

During this session, the students built models to compare one half and one third. Erik and Alan worked together on the task, and Dr. Landis asked Erik why one half was larger than one third. Rather than simply showing the difference in the model, Erik made a generalization about the relative size of fractions. Over the course of the session, he refined his generalization, and one version of his explanation is repeated here.

> What I'm saying is, see if you divide it into thirds and fourths, four is a larger number than three, but three, you're dividing it into, um, you're dividing it into three parts, so instead of dividing it into four parts you cut it four times into fourths and then, and that would be much smaller than the, a third. And if you divide it- if you cut it only three times, it'd be bigger. So therefore, four may be bigger than three, but the smaller the number, the larger the piece.
> (lines 4.0.520-4.0.526)

This argument was generic in that it showed why one third is larger than one fourth and used that argument to show a general truth about fraction relationships. Erik made a generalization about the relationship between the denominator of the fraction and the size of the fraction.

Session 6

The students worked on building models to show the difference between one half and one fourth. During this session, Andrew used generic reasoning to explain that any model would give the same solution for the comparison problem that they were solving. He explained that just as in his model, one half was equal in length to two fourths, all models that would be built would have that characteristic.

Erik and Alan found a way to generate many models that could show that difference. They shared the pattern with the class and explained that by using a train of two orange rods as halves the yellow rods could be called fourths. They then said that the next model could be found by finding the rod that was two centimeters shorter than the orange rod, and using that as one half (lines 6.0.271-6.0.277). This pattern was later generalized and will be discussed below.

Session 9

During this session, the students worked to compare two thirds and three fourths. Meredith built two models to show the difference, one measuring twelve centimeters in length and the other measuring twenty-four centimeters in length. T/R 1 challenged David and Meredith to predict what a third model would look like. David conjectured that, in this larger model, the white rod would be called one forty-eighth and the red rod would be called one twenty-fourth. He also said that the train that was called one in the twenty-four centimeter model would be doubled in the third model. With this discussion, David began to make a generalization about the strategy that could be used to build models to compare fractions.

Session 10

During this session, T/R 2 gave the students an opportunity to continue working on the task from the previous session. At the start of the session, Erik suggested using the pattern that he had noted in session 6 as a way to build models.

> Well, because see, what me, Alan and I figured, is if you start with one rod, and you can divide one rod that's a large number into thirds and fourths, then you just count down by two, because we think that even numbers you can divide into fourths and thirds, but odd numbers you can't, so it was like, if we started with the orange rod… you could probably divide it into thirds and fourths. And then just go down two and then just keep going down until whatever number you get and then you'll just keep going down and you should be able to.
> (line 10.2.23)

This generalization was a faulty one, in that the pattern could not be used for all fraction comparisons.

David, Meredith, and Erik spent the session trying to reconstruct the model that they had built the day before to test David's theory. David repeated his conjecture about the structure of the model several times during the session, and the video data lends evidence to the possibility that David's conjecture was based on a generalized doubling pattern that he had noticed.

Alan worked on a new problem during this session, and built models to compare one half and two fifths. Alan used recursive reasoning to build the models, and also noted the doubling pattern that was inherent in the models that he had built.

Session 11

This session provided the students an opportunity to discuss, as a whole class, the doubling pattern that was used to generate large models. The students analyzed the composition of the models that had been found and then predicted what larger models would look like. In this way, they were encouraged to use generalizations to think about the way models could be made and whether or not there existed a largest model for any problem.

Session 13

During this session, the students discussed the sizes of unit fractions. Asked to compare one half, one third, one fourth, and one fifth, they correctly showed which was largest and which was smallest. Danielle made a generalization about the relative size of fractions. She said that "if it's a bigger number you get less" (line 13.0.57). This comment was similar to the explanation that Erik provided during session 4. Brian then justified Danielle's statement, explaining that one fifth was smaller than one fourth.

> [I]f it's a fifth it has to take, there has to be five of 'em in one whole, and if there are um, quarters, it only needs, it only needs four of 'em to go into one whole. So… five is a bigger number and… it needs more to fill up one whole… so it's less.
>
> (line 13.0.61)

Session 14

The students began to think about division of fractions, and they first thought about writing number sentences using unit fractions. Danielle, during this session, verbalized the general rule for finding the solution to a division problem involving unit fractions. She asked T/R 2, "[S]o it's like every number, it's that, it's one divided by the fraction and then just the plain number?" (line 12.0.79). T/R 2 noted that Danielle had asked a very good question, but did not answer the question directly.

Session 15

This session was the first during which the students worked on the ribbon and bows task. One group, consisting of Brian2, Caitlin, and Erin, noticed and discussed the patterns that they noticed in the tasks. Caitlin first noticed that the first set of tasks (involving the 1 meter long white ribbon) could be solved by increasing each successive denominator by one. Brian2 then told the group that the second set of tasks (which involved the 2 meter long blue ribbon) could be solved by multiplying the denominator of each fraction by two, and that the third group of tasks was similar and could be solved by multiplying the denominator of the fraction by three. During the whole class discussion at the end of the session, Brian2 pointed out this pattern to the class.

Amy and Jackie also noticed the pattern that could be used to solve the problems. They discussed their findings with T/R 2.

Amy:	We got a pattern. In the three we did so far, we got patterns.
T/R 1:	Oh, ok and the patterns helped you solve it?
Amy, Jackie:	Yes.
T/R 1:	So that's... Ok…That's neat. And what is, what is the pattern that you think you see?
Jackie:	Well, it started in the first two patterns. It started in the first two problems, two, three, four, and five.
T/R 1:	Oh, interesting. How did you get the two, three, four, and five?
Jackie:	Well, because one half would be two, one third would be three, one fourth would be four and one fifth would be five.

T/R 1: Ok, I see that pattern, ok. Did you find the pattern?
Jackie: One half would be four because we um, we doubled it because it was two meters. And that would be one would be six bows, and then eight bows, ten and twelve. And since it was… we thought that all of them whatever that number how many meters it would be we thought that we would have to go by twos.
(lines 15.2.146-15.2.154)

Although these generalizations were useful, they did cause some confusion as the students attempted to solve more difficult tasks. Both groups of students attempted to use the doubling pattern to solve the problem that asked the students to make bows that were two thirds of a meter in length from the blue ribbon. Jackie decided that the number of bows in this problem would be twice the number of one third meter long bows, and concluded that the solution was twelve. Caitlin, as well, suggested that the solution would be twelve, and Erin offered an explanation as to why that would be true. She said, "Because if you take one third, it's six bows, but two thirds you take two of those two meters so you'd have double that so you'd have six and six, twelve" (line 15.2.117).

Session 16

The sixteenth session began with a review of what had occurred during the previous session, and Jessica, Michael, and Brian discussed the patterns that had been identified.

Jessica: Well, I noticed that after a while like it started making a pattern.
T/R 1: Ok. You want to say a little bit more about that?
Jessica: Well, um, I forget what pattern but I think it was going like it started going in three, six, nine, like… like when it said when you had like different size ribbons and every time it got like …like three times bigger and it kept doing it in all different kinds of patterns.
Michael: Yeah, because at first it went two, three, four, five
Jessica: And then it went…
Michael and the second one went, uh, the second one went four, eight, something like four, six, yeah
T/R 1: I don't remember any two, four, six or four, eight.

Michael: No, it's four, it's four, six, eight, ten… and then there was that odd, and then there was that two thirds one.
T/R 1: Ok, let's, let's, let's hold out… Brian what were you just saying?
Brian: Well, if we, remember we had the three meters, you would always like times the number by three. Like you go three, six, nine?
(lines 16.2.6-16.2.15)

The session continued with an exploration of a series of related problems. The students were asked to find the number of one third meter long bows that could be made from three, nine, twenty-seven, and eighty-one meters. They quickly recognized the pattern and extended the task to find ever larger numbers of bows that could be made from increasing lengths of ribbon.

Session 17

During the last session in the series of fraction investigations, the students discussed the pattern that they had noticed when solving division of unit fraction tasks. Andrew explained that the denominator of the fraction was multiplied by the number of meters of ribbon that was provided. The session closed with a question by T/R 1, who asked the students if 12 divided by 2/3 followed the same pattern. However, this question was not answered during the session.

4.3.2 Analogical Reasoning

In addition to the forms of reasoning noted previously, the students often used analogies to draw conclusions or make conjectures about the mathematical ideas that they investigated. In this section, we will trace the occurrence of analogical reasoning chronologically.

The researchers introduced analogies and metaphors to assist the students' understanding of some basic fraction concepts. These metaphors were often extended by the students later in the study and the metaphors were used to explain their understanding of the mathematics. These instances will also be noted.

This section will not differentiate between proportional reasoning and more simplistic forms of analogical reasoning. That discussion is beyond the scope of this analysis.

Session 2

The first half of the second session centered around a whole class discussion during which the students thought about finding a rod that could be called one half when the blue rod was called one. Erik suggested using the purple rod as one half and the yellow rod as the other half. David pointed out that he thought that the two halves had to be identical, and Erik challenged that claim.

Erik: You don't really.

T/R 1: You don't need the same? In other words, I could call this a half [the yellow rod] and I can call this a half [the purple rod]. Suppose this is a brick of gold and we're going to share it, Erik. And I'm going to take the yellow half and you get the purple half. Fair?

Erik: Yeah.

T/R 1: Do the rest of you agree? Do you like that? [Chorus of no's] Beth? No. Beth doesn't like that. Kimberly? Does it matter? Erik doesn't care. Do you care?

Erik: Well, well I mean-

Kimberly: Yes, cause the pink is, the purple is smaller than the yellow and the person who got the purple wouldn't have as much.

(lines 2.0.71-2.0.76)

In this part of the discussion, T/R 1 introduced the metaphor of a brick of gold, and Kimberly used that metaphor to reason about the problem, saying that the person who got the piece that was the size of the purple rod would not have as much as the one who took the piece that was the size of the yellow rod.

During this session, T/R 2 posed two related tasks. She asked the students to name the yellow rod if the orange rod was called two, and then asked them to name the yellow rod if the orange rod was called six. Both Alan and Kimberly used their understanding of the first task to reason about the second. Kimberly said that since in the previous problem, the yellow rod was called one, it should be called five in this problem. Alan suggested that it be called three, since in the previous problem it had been called one, and "half of three is six" (line 2.0.272). While Kimberly used analogical reasoning to arrive at an incorrect solution, Alan used the same form of reasoning in a more accurate manner and successfully identified the similar structure of the two problems.

After the discussion about the blue rod, the students worked to design a rod that was half the length of the blue rod. Then, at the end of the session, T/R 2 asked the students to find the rod that was one half the length of the orange and light green train. Brian presented his solution during the whole class discussion, and showed the many ways that he had found to design a new rod that could be one half of the train. He said,

> Well, like what we did last time with, when Mrs. Maher was talking about, about if we split the gold equally, what you could do is, well, I thought of a lot of ways. So like, once I have the white cube right in the middle, you split that in half, right in the middle. That's what we did last time.

(line 2.0.372)

Here, Brian used the analogy of the brick of gold to extend and explore the task that was posed.

Session 3

During this session, T/R 1 introduced two metaphors, candy bars and pizzas, to assist the students in understanding the importance of refraining from changing units when working with fractions. The candy bar analogy was used numerous times during later sessions as students reasoned about the tasks.

Session 4

The students used analogical reasoning as they thought about two tasks during this session. T/R 2 asked the class to name the yellow rod if the orange rod was called fifty. Jacquelyn said, "“Well, to make it even, if we had fifty cents, we have two quarters, we take half, um, fifty cents this would be twenty-five and twenty-five" (line 4.0.150). Similarly, Beth explained that two quarters equaled fifty cents, and that that realization had assisted her in arriving at the solution of twenty-five.

The next task challenged students to name the white rod when the orange rod was called fifty. Caitlin reasoned that previously, they had found that the white rod was called one when the orange rod was called ten, and said that she used that information to name the white rod five now that the orange rod was called fifty. Although no further explanation was provided, it appears that this may also be a sophisticated use of analogical reasoning.

Beth also used analogical reasoning as she solved this task. She said that she compared the white rod to nickels and found that ten nickels would equal fifty.

Session 5

During the fifth session, the class was asked to compare one half and one third, and a whole class discussion ensued. Jessica built a twelve-centimeter model at the OHP, and reasoned that one half was larger than one third by one third. Kelly then said that she agreed and showed that she used a six-centimeter model, arriving at the same conclusion. Jessica noted that Kelly had switched candy bars. Although Jessica had changed units during her own justification, she challenged Kelly's use of a different size model to show the difference between the two fractions, and used the candy bar metaphor to explain why she thought it was incorrect.

Session 6

Jessica's use of the candy bar analogy was replayed during the sixth session. This time, after Jessica built a twelve centimeter model at the overhead to show the difference between one half and one third, Alan built the six centimeter model and correctly showed the comparison between the two fractions. Jessica then challenged Alan's model, and a discussion centering around the candy bar metaphor took place.

T/R 1:	What do you think? Do you agree Jessica?
Jessica:	No
T/R 1:	Jessica doesn't agree?
Jessica:	I think he's like remember you said that it can be only be one size candy bar and that's like a whole different size candy bar he's making
T/R 1:	Now hold on, Alan, uh, ok Jessica disagrees. Kelly?
Kelly:	Well, me and Jacqueline agree
T/R 1:	Jackie and Kelly agree. Why do you agree?
Jackie:	Well, because when you go to the store there's not just one size candy bar there's all different kinds of sizes so you can make a model with a different size.
…	…
Jessica:	Yeah but it's, I think it still could be one sixth, but it's just a different size candy bar
Erik:	Yeah I know we said any one sixth is right.

T/R 1:	It can be one sixth either way. What do you think Jessica was confused about then?
Erik:	Yeah the sixth isn't the same size.
T/R 1:	Does it matter? This is a model where um
Michael:	Yeah because the whole is not the same size.
T/R 1:	Jessica
Jessica:	But because say if you wanted to give someone one sixth of that candy bar and then you were going to give someone one sixth of the other one, then the person with that size would get a smaller amount.
…	…
Andrew:	Well, um, that's right because if um, it's just a different size candy bar. If you just gave half of that to the person and the other half of that to another person you would still have the same size. You can't switch the candy bars.
T/R 1:	Okay you say as long I whatever I do, I do it in the same candy bar, that's fair but what I can't start doing is switching. Did anybody switch a candy bar here? In this problem where's the switch? In this problem?
Erik:	Well they didn't switch a candy bar in that problem but from the problem that Jessica, that Jessica did, he switched the candy bar, they switched the candy bar from the orange and the red to the dark green and if you're giving someone half of the orange and red and someone else half of the dark green the person getting half of the orange and the red is getting a bigger piece.

(lines 6.0.29-6.0.50)

T/R 1 asked Erik if that had been done in this problem, and Erik admitted that it had not. As can be seen from the data, the candy bar was used as an extended analogy to convey ideas about the mathematics that they were exploring.

Session 7

T/R 1 began session 7 by introducing another metaphor using the diorama that Mark had built of two children, a fish, and a boat. This metaphor was used to help the students think about the importance of keeping the relative sizes between models the same, and laid a foundation for proportional reasoning to develop. There is no evidence that the students used this analogy to reason about the fraction ideas that were introduced during later sessions. However, during the discussion, the students worked to map the analogy of the fishing boat to their attempts at building models to compare fractions. T/R 1 asked them how the discussion was related to their model-building attempts in mathematics. Meredith said, “Well if you have the same question asked and you do it right then you're going to wind up with the same answer and some of the models could be bigger and some of them could be smaller” (line 7.0.100). Later, Michael elaborated.

> Well, it's sort of like um, you can't, the fish has to be smaller than the people and the people have to be smaller than the boat… So… that just helps us understand what we’re talking about with the Cuisenaire rods when we are using different sized boxes to make different sized, um, halves and quarters, um, but, they’re basically you can call it the same thing as you would then just the small one with the small one if you call the box a whole, and the boat a half it would equal a quarter. You could still do that in Audra's model or any box.
> (line 7.0.112)

Session 15

Another instance of analogical reasoning took place during the fifteenth session. The students worked on dividing fractions during the ribbons and bows task. As Andrew worked to find out how many bows one half of a meter in length could be made from the two meter long blue ribbon, he said, “This is almost like rods” (line 15.1.171). However, he did not elaborate how the task was similar to the rods tasks.

CHAPTER 5: CONCLUSIONS

5.1 Introduction

In this section, an overview of the findings is presented, and patterns that were noticed are described. The findings are discussed in light of the relevant literature, and limitations and implications of the study are outlined.

5.2 An Overview of the Findings

5.2.1 Purpose and Structure of Arguments

5.2.1.1 General Findings

In all, the children used 364 arguments during the seventeen sessions. Of the total number, 309 arguments contained at least one version that was a justification of a claim, while sixty-two arguments contained at least one version that was a counterargument. One argument was both a counterargument as well as a justification of a claim. Figure 5.1 shows the number of arguments that fell into each of the two categories.

Of the total number of arguments, 319 contained at least one version that was a direct argument, while fifty-one contained at least one version that was indirect. Three arguments were composed of both direct and indirect arguments. Figure 5.2 shows a visual representation of these findings.

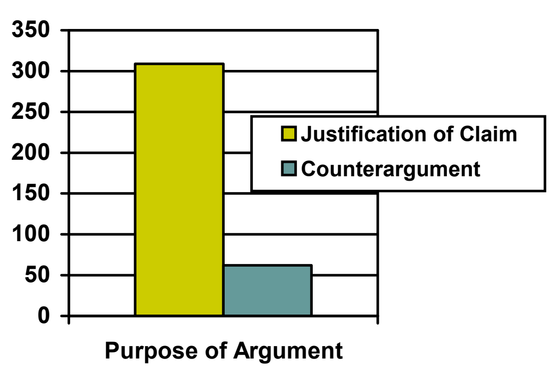


Figure 5.1. Results of first coding set

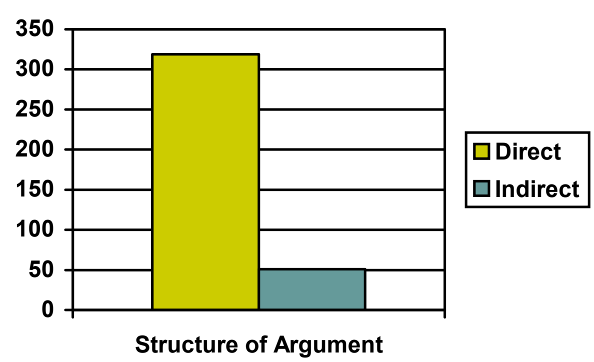


Figure 5.2. Results of second coding set

Thirty-five of the indirect arguments were offered as counterarguments, while only sixteen indirect arguments were offered as justifications of a claim. On the other hand, only twenty-nine direct arguments were offered as counterarguments, while 291 were justifications of claims (See Figures 5.3 and 5.4 for a display of these results). This suggests that the counterarguments used tended to be indirect, and that the majority of indirect arguments were elicited as counterarguments to the claims or ideas of others. In addition, it suggests that direct arguments were the preferred means of justification of a claim or a solution, and that indirect arguments were rarely used for this purpose.

An explanation for these patterns may be that when a student countered another's claim, the student was able to envision a scenario that was different than her/his original way of thinking. Once this scenario was envisioned by a student, it became easier to reason about this different way of thinking and to show that a contradiction was inherent in the inferences that could be drawn from the assumptions that were made.

It should be noted that close to half of the counterarguments were direct arguments, and slightly more than half were indirect arguments. This is important, as it evidences that counterarguments did not always take an indirect form of reasoning. However, the number of indirect arguments that were offered in the context of a counterargument was unusually large when compared with the number that were used to justify a claim, as has been noted above.

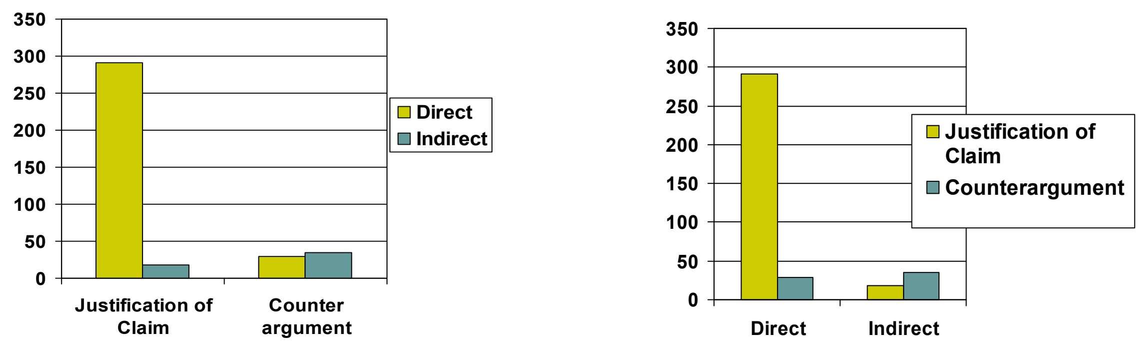


Figure 5.3. Classification of purpose and structure of arguments and their relationship.

5.2.1.2 Revisiting of Tasks

Over the course of the seventeen sessions, three tasks were revisited at later times. The first was the series of tasks that established the equivalence of one fifth and two tenths. During the fourth session, of the three tasks that focused on this concept, one of the eight arguments was an indirect counterargument, while the remaining seven were direct justifications of claims. During the fifth session, this task was revisited, and three direct justifications of claims were presented in a whole class setting. Then, during the seventh session, the class discussed the equivalence of one sixth and two twelfths, and one direct justification, two direct counterarguments, and one indirect counterargument was offered.

The second task that was revisited required that the students to compare one half and one third. This was the first fraction comparison task presented in the strand, and the students worked on this problem during the third, fourth, fifth, and sixth sessions.

During the third session, the students worked on the problem for approximately fifteen minutes, first with their partners and then in a whole class discussion. Seven direct arguments were used. However, no indirect arguments or counterarguments were used during this session. During the fourth session, the students worked on this task for twenty minutes with their partners. Seven direct arguments and one indirect counterargument took place. During the fifth session, a fifteen minute whole class discussion took place, and three direct arguments, five indirect counterarguments, and four direct counterarguments were used. During the sixth session, a twelve minute whole class discussion took place, and seven direct arguments, four direct counterarguments, and one indirect counterargument was used.

A similar pattern was followed as the students worked on and revisited a third task, this one requiring them to compare two thirds and three fourths, which was the fifth fraction comparison task in the strand. They worked on the task during the eighth, ninth, tenth, and eleventh sessions.

During the eighth session, the students worked on the task with their partners for varying lengths of time. Six direct arguments were used and only one indirect counterargument was flagged. During the ninth session, the students worked on the task for the entire class session. Most students worked with their partners, and some students worked in small groups of three or four students. During this session, five direct arguments, two direct counterarguments, and two indirect counterarguments were flagged. During the tenth session, most of the students worked on the task for the whole class session, while some worked on a sixth comparison task. During the whole class discussion, two direct arguments, one indirect argument, and one indirect counterargument was presented. During the small group work that followed, one direct argument, one indirect argument, and one indirect counterargument was used. Finally, during the whole class discussion that took place during the eleventh session, thirteen direct arguments, three direct counterarguments, and four indirect counterarguments were presented.

These findings suggest that indirect argumentation, as well as the use of counterarguments, may be elicited after students work on a task for an extended period of time with varied opportunities to explore and discuss the task. Importantly, it is possible that these forms of argumentation are encouraged by the pattern of academic work that was implemented during these sessions. By first working on the tasks in small groups, the students had ample time to build durable representations of their solutions. Then, during the whole class discussions that took place during later sessions, students were more comfortable and familiar with the task. Perhaps because of the sense of ownership that they now attained, they were more able to be active participants in discourse as they shared and discussed their understanding of the task. These factors could contribute to students use of counterarguments and, as a result, of indirect argumentation.

5.2.2 Forms of Reasoning Found

5.2.2.1 Generic Reasoning

There were six distinct occurrences of generic reasoning during the seventeen sessions. All occurrences of generic reasoning were found as students offered direct justifications of claims. David offered a generic argument during the second session, when he explained that the blue rod belonged to the set of odd rods and that there were no rods that were one half the length of odd rods. In session four, Erik and Alan used generic reasoning to explain why one half was larger than one third by explaining why smaller denominators resulted in a larger quantity. During session six, Andrew offered a generic explanation to show that one half was always larger than one fourth, no matter which model was built to represent the difference. In session seven, Brian and Michael used generic reasoning as they formulated a written solution showing the difference between one half and two thirds. Two of the instances took place during session eight. One occurred as Brian and Michael wrote a justification for the difference between one half and three fourths. The second occurred when Alan explained to T/R 1 that the difference between three fourths and one half was always one fourth, no matter which model was built.

All instances of generic reasoning were found as students offered direct justifications of claims. Five of the six instances occurred as students compared fractions. Two of these occurred as students recorded their justifications in written form. This may have occurred because students attempted to record their ideas in a more general and abstract manner. For example, when Brian and Michael attempted to explain their solution to comparison of fraction problems, they tried to explain why their solution made sense in mathematical terms and how it applied to any model built to compare the fractions, rather than explaining their model alone. This pattern is important to note, as it suggests that encouraging students to record their justifications may foster the abstraction and generalization of mathematical arguments

5.2.2.2 Reasoning by Cases

Students reasoned using cases on four occasions. Three instances were found during the second session. The first took place as David offered his classification of odd and even rods. The second was found when Brian reasoned by cases to find multiple ways to build a rod that was called one half when the orange and green train was called one. The third instance was one where Erik used faulty reasoning and attempted to use the rods that existed in the set of Cuisenaire rods to solve the same problem. The fourth instance occurred during the third session, as Brian2 and Jacquelyn attempted to find multiple models to show the difference between one half and one third. Although the video does not evidence that they completed their train of thought, the video data does suggest that their approach was exhaustive.

All instances of correct reasoning by cases occurred as students worked to directly justify claims. Erik's use of cases ultimately showed that his claim was not true. Of interest is that all instances of reasoning by cases took place during the first three sessions, and as students worked on the introductory problems in the strand. This might be explained by the fact that the students began to explore the rods during these sessions, and used random methods before attempting to reorganize their findings in a more logical way.

5.2.2.3 Recursive Reasoning

For all sessions, six instances of recursive reasoning were found. First, in the second session, Michael, Brian, and David used recursive reasoning to show that it was impossible to form a set of rods that, for each of the rods in the set, there existed another rod in the set that was one half its length. Then, during the fourth session, Beth used recursive reasoning as she named increasing numbers of white rods when the orange rod was called one. Also during the fourth session, Alan used recursive reasoning as he explained to Dr. Landis that one third was smaller than one half and that one fourth was smaller than one third. During the sixth session, Erik and Alan reasoned recursively as they found a way to build multiple models to show the difference between one half and one fourth. During the tenth session, Alan reasoned recursively as he explained how multiple models could be built to show the difference between one half and two fifths. During the eleventh session, Kimberly, Andrew, and Erik reasoned recursively as they explained how to build multiple models to show the difference between two thirds and three fourths.

All occurrences recursive reasoning were found as students offered direct justifications of claims. Four of the six instances took place as students worked to compare fractions. In addition, all instances were found as students extended the original problem or attempted to find multiple representations of their solution to the problem. This may have occurred due to the nature of this form of reasoning. Recursive reasoning is required as students attempt to generalize or discuss a pattern that they notice in a task. These patterns are more likely to be noticed as students work to build multiple models to solve a task.

5.2.2.4 Reasoning Using Upper and Lower Bounds

Twelve arguments contained reasoning using upper and lower bounds. The first occurred during the second session, when David showed that no rod could be called one half when the blue rod was called one. The remaining eleven instances occurred between sessions five and thirteen. During the fifth session, Erik used upper and lower bounds to counter the claim that one half was larger than one third by one third. In session eight, Michael used upper and lower bounds to show that a model could not be used to compare one half and three fourths. During the ninth session, Alan and Erik reasoned that models longer than a train of three orange rods could not show one third since the orange rod was the largest in the set. During the tenth session, Alan used this argument but modified it by explaining that the only way one third could be found was by making a new rod. David used a similar argument during this session, this time reasoning about the alleged forty-eight centimeter model that he attempted to reconstruct. Also during the tenth session, Alan reasoned that any model that was constructed to show the difference between one half and two fifths that was larger than forty centimeters would not show fifths unless a new rod was constructed. During the eleventh session, Erik used upper and lower bounds to show that the twelve centimeter model was the smallest model that could show the difference between two thirds and three fourths. During that session, Erik and Alan used upper and lower bounds to explain why it was unnecessary to use the purple rods in the twenty-four centimeter model. Finally, during this session, Alan used upper and lower bounds as he reasoned that a model using a train of eight orange rods and two brown rods could not be used to show thirds and fourths.

All twelve instances in which reasoning using upper and lower bounds was found occurred during the presentation of indirect arguments. Six of these indirect arguments were justifications of claims, and the remaining six were counterarguments. Eleven of the twelve instances occurred as the students worked to compare fractions.

5.2.2.5 Other Patterns Noted

One striking feature of the reasoning used during the sessions is that none of the four forms of reasoning that were the subject of this study were found during the last four sessions. Interestingly, few counterarguments and indirect arguments were noted during the last four sessions, which focused on division of fractions. Only direct arguments were offered during the last two sessions, and the two preceding those were marked by only five indirect counterarguments and two direct counterarguments. One reason that this may be the case is that the students had already build a strong understanding of fractions during the previous sessions, and once they worked on the division of fraction tasks, first using the rods and then as they worked on the ribbons and bows task during the fourteenth and fifteenth sessions, they quickly solved the problems posed during the final two sessions and used direct reasoning to arrive at correct solutions.

An indication that this may have been the case is suggested by a comment made by Andrew during the fifteenth session. As he explained a solution to Dr. Landis, he said, “This is almost like rods” (line 15.1.171). The rods had become an assimilation paradigm (Davis, 1984) for the students and they were able to solve other tasks because of the representations that they had built during the intervention.

In the later sessions, the students noticed a pattern in the solutions to the problem tasks and formulated a procedure, which they justified over the course of the sessions, and applied the procedure to solve many of the tasks that were posed. As a result of the more procedural nature of their solution strategies, they used direct reasoning to justify their ideas. This finding is important, as it suggests that the use of procedures to solve mathematical problems may foster the occurrence of direct reasoning, and thus leave little opportunity for students to use other forms of reasoning. Although it is important for students to formulate procedures as they work on mathematical tasks, limiting their activity to procedural work alone may limit the variety of forms of reasoning that students use.

5.3 Discussion

5.3.1 Informal Reasoning and Argumentation

The reasoning and argumentation used by the students during their investigations were informal in nature, as would be expected from fourth grade students. However, as is noted in the literature, this informal reasoning is crucial for the development of formal reasoning in later years of mathematics learning.

Harel (2008) explains that mathematical induction is an abstraction of quasi-induction, which is first introduced by engaging students with implicit recursion problems. During this study, there is no evidence that proof by mathematical induction was constructed and used by the students and could be explained by the nature of the tasks they were given.

However, the students reasoned recursively as they attempted to justify their solutions to the tasks that they were presented. This form of recursive reasoning may lay a foundation for the development of the ability to use and understand the importance of proof by mathematical induction. Further study is needed.

Similar ideas have been put forth about indirect proof. Thompson (1996), in her discussion of indirect proof, says:

> Given the difficulties identified by the research, what can we as teachers do to increase the likelihood that students can be successful with this proof technique? First, we should give students an opportunity to study indirect proof in more informal ways… If such indirect proofs are encouraged and handled informally, then when students study the topic more formally, teachers will be in a position to develop links between this informal language and the more formal indirect-proof structure. (p. 480)

Similarly, Antonini (2003, 2004) points out that by encouraging students to produce indirect argumentation spontaneously, it can become a way of thinking that can enable them to eventually write indirect proof. In this study, students were found to spontaneously reason indirectly, and used that indirect reasoning frequently during argumentation.

It is important to note that there are significant differences between indirect argumentation and indirect proof. This has been pointed out by Antonini and Mariotti (2008), and ways to ultimately bridge those differences will need to be explored so that young students can be trained to eventually master indirect proof.

Alibert and Thomas (1991) and Harel and Tall (1991) promote the importance of generic reasoning as a method of transition to formal proof. The use of generic reasoning by the fourth grade students allowed them to think about the general properties of the mathematical models that they built. These opportunities to think abstractly about mathematics lays the foundation for the use of formal abstraction when doing advanced mathematics.

5.3.2 Contributing Factors

What factors influenced the elicitation of varied forms of reasoning as were found in this study? As was noted, the study did not evidence a consistent growth in the variety students' reasoning over the course of the seventeen sessions of the intervention. Rather, specific tasks and strands of tasks were found to foster the use of more varied argumentation, and others were found to encourage the use of direct reasoning alone. Francisco and Maher (2005) have shown that task design can often be linked to specific methods of problem solving and patterns of reasoning used by students. In this section, aspects of task and environment will be explored as possible contributors to the results.

5.3.2.1 Task

One task, in particular, elicited multiple forms of reasoning and an unusual number of direct and indirect counterarguments. That task was initiated by Erik during the second session, and centered on finding a rod that could be called one half when the blue rod was called one. Similar to the results of Mueller's (2007) study of sixth grade students, the greatest variety of arguments was found as the students worked on this task. This can be explained by the nature of the task. Many tasks in the strand required students to show the existence of a rod or to build a model to represent fractions that could be constructed using the given set of rods. This task required students to show that a rod did not exist, and were thus encouraged to show that assuming that the rod existed led to a contradiction. A full discussion of the aspects of this task that may have led to the elicitation of many forms of reasoning can be found in Maher et al. (2009).

Most occurrences of recursive reasoning, generic reasoning, and reasoning using upper and lower bounds were found as students worked on fraction comparison tasks. One reason for this may be that these tasks can be classified as model-eliciting, model exploration, and model adaptation tasks (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). During the strand of fraction comparison tasks, the students built models, explored their properties, and later adapted them to solve new problems or extend their solution to the original problem, they began to use these other forms of reasoning. Generic and recursive reasoning occurred most often as students explored the models that they had built and as they adapted the models during the course of their activity.

5.3.2.2 Environment

Doyle's (1988) research suggested that students who work on many familiar tasks, rather than few non-routine tasks, during each class session, may not be provided the opportunity to engage in sense-making during their classroom work. He also suggests that if performance on non-routine tasks is not ascribed importance in the classroom system of accountability, students will not engage in them as fully as they will with the familiar tasks for which they are held accountable. The findings from this study support Doyle's interpretations. The students in this intervention worked on non-routine tasks, and were challenged to think about and construct their understanding of fraction as number concepts. They were not taught procedures to solve these tasks and many required them to experiment and explore the nature of the mathematical objects under discussion. In addition, they were not graded for their work on the tasks that they explored, but were only required to convince themselves, their classmates, and the researchers that their mathematical solutions were valid. Students were motivated to work together on the problems under the conditions established for the study.

Henningsen and Stein (1997) noted the importance of the allotment of appropriate amount of time to task exploration and showed the correlation between that and successful engagement of students in high level thinking as they worked on the task. In addition, they pointed out that teachers must support high-level cognitive activity by maintaining, and not reducing the cognitive demands of the task through their explanations and assistance.

From the analysis of the reasoning used by the students as they revisited tasks, it may be suggested that, by being allowed ample time to explore the tasks over several class sessions, first in small groups and later during whole class discussions, the students were encouraged to participate in argumentation and expand and refine their reasoning about the task. Reasons for this have been discussed earlier.

Throughout the intervention, the students provided justifications for their solutions. The researchers implemented the need for justification as a sociomathematical norm. This was initiated during the first session and maintained for the duration of the study. The arguments presented by the students suggest that they understood that a justification was expected as they engaged in doing mathematics. In the first written assignment completed by the students during the intervention, Meredith wrote: "Rutgers really worked us hard because every time someone came up with an answer Rutgers would say: "Convince us." So we did!" (See Appendix C).

During the fifteenth session, Andrew, as he discussed the process that he took to solve a task with a visiting researcher, explained that he was taking additional steps to ensure that his solution was correct by saying, "Rutgers usually makes us prove what our answer is" (line 15.1.41). He then went on to clarify how he was attempting to do so. Reid (2002) noted the lack of this perceived need to justify solutions in the fifth grade subjects of his research. From this study, it appears that these fourth grade students did develop a "mathematical emotional orientation" that is deemed so essential by Reid for their mathematical maturity, and that this was encouraged by the actions of the researchers during the intervention, as is suggested by Henningsen and Stein (1997). From the data, it is evident that these fourth grade students were effectively introduced to the importance of justification as "a vehicle to promote mathematical understanding," which Hanna (1995, p. 42) argues is crucial for the effective use of proof in the classroom.

5.4 Limitations

Due to the qualitative nature of this study and the conditions established for students , it is difficult to generalize results. As a case study, it must be replicated to establish external validity of the findings of this research (Yin, 2003). However, Mueller's (2007) study has already shown similar results at a different grade level, in a different context, and with a different population. It may be helpful to study the reasoning elicited by this set of tasks, as well as with task constructed with similar properties, and analyze the reasoning used by students in the elementary and middle grades as they work on these tasks.

A limitation of this study lies in the coding scheme used for analysis. It may be useful to analyze the data using a more refined scheme to give deeper insight into the nature and level of maturity of the argumentation used by the students. By providing a scheme that enables arguments to be sub-classified, students' development of arguments can be traced over the sessions and could lend insight into individual student growth over time.

5.5 Implications

This study was designed to identify the forms of reasoning that young students exhibit without formal training and to identify the factors that tend to elicit varied forms of reasoning. Both sets of findings were intended to assist mathematics educators in developing a more complete understanding of how to encourage students to exercise their ability to reason effectively, an ability which will enable them to become active members of their mathematical community. Furthermore, this study investigated the efficacy of the mode of learning that was introduced during the longitudinal study, and was designed to pinpoint aspects of the learning environment that may have contributed to the elicitation of reasoning and the effective argumentation used by the students.

5.5.1 Implications for Further Research

This investigation opens up many areas of further research. First, it would be helpful to use other systems of argumentation analysis, such as Toulmin's (1969), to analyze the discourse used by the students, both in their small group and partner work, as well as during whole class discussion. This may shed light on factors that contributed to the high-level mathematical activity that was observed and may suggest ways that such reasoning can be encouraged in the classroom.

It may also be productive to study the forms of fractional reasoning that were elicited during the intervention. Although this was beyond the scope of this study, it may be useful to look for connections between the forms of fractional reasoning that is found and the forms of argumentation that were used.

Another area that can be explored further is the occurrence of faulty reasoning that was flagged. It may be productive to study the tasks that the students worked on as they reasoned in a faulty manner and attempt to identify factors that may have contributed to their erroneous thinking. It would also be of interest to look for connections between the fraction ideas that underlie the tasks and the errors in reasoning that were made.

Importantly, it is crucial to study more extensively the nature of the tasks and the environment that contributed to the elicitation of reasoning in these students. Then, it would be helpful to attempt to design and implement tasks that are similar in structure to those which were found to successfully elicit varied forms of reasoning in a carefully planned environment and investigate the reasoning that students of different ages use to solve the tasks. This can allow for the development of resources that can be used by teachers and mathematics educators to promote the development of reasoning in students.

5.5.2 Implications for Practice

This study highlighted the ability of young students to reason effectively and use varied forms of argumentation, given an appropriate environment and challenged with well-designed tasks. As has been discussed, the findings suggest that aspects of task and environment that were identified should be replicated to produce these results.

The findings show a connection between the complexity of classroom argumentation, particularly in a whole class setting, and the use of indirect reasoning. This suggests that argumentation in a whole class setting should be encouraged in the classroom, and that this may enable teachers to develop students' ability to use indirect reasoning as they do mathematics.

In addition, the results showed that students used varied forms of reasoning while working on model-exploration tasks for extended periods of time. By revisiting tasks and providing students opportunities to discuss their ideas with their partners as well as during whole class discussions, students were able to reason effectively and offer arguments to back their claims. This finding may assist teachers in their attempt to encourage students to use varied forms of reasoning, as is recommended by *Principles and standards for school mathematics* (NCTM, 2000) as important for the development of reasoning and proof in mathematics.

Importantly, aspects of the environment that was created during the study may be replicated to encourage the elicitation of reasoning and justification. As noted above, these factors included the practice of the researchers to expect the students to justify their solutions, the importance that these justifications were ascribed in the accountability system of the class, and the researcher's effectiveness in supporting, rather than reducing, the complexity of the tasks that were presented.

As noted in the introduction, effective reasoning is crucial for student success in mathematics. Ultimately, the results of this study may enable mathematics educators to attempt a larger effort to encourage students to develop their reasoning ability in all areas of the mathematics curriculum.

REFERENCES

Alibert, D. & Thomas, M. (1991). Research on mathematical proof. In D. Tall (Ed.), *Advanced mathematical thinking*, (pp. 215-230). Dordrecht, The Netherlands: Kluwer Academic Publishers.

[Alibert, D. Grenier, D. Legrand, M., Richard, F. (1986). Introduction du débat scientifique dans un cours de premiere ânée du deug a l'université de Grenoble 1. *Rapport de l'ATP 'Transitions dans le système éducatif'* No. 122601 du MEN.]

Alston, A., & Maher, C. A. (1993). Tracing Milin's building of proof by mathematical induction: A case study. In B. Pence (Ed.), *Proceedings of the Fifteenth Annual Meeting for the North American Chapter for the Psychology of Mathematics Education* (Vol. 2, pp. 5-11). Pacific Grove, CA.

Antonini, S. (2003). Non-examples and proof by contradiction. In N. A. Pateman, B. J. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of PME and PMENA* (Vol. 2, pp. 49–55). Honolulu.

Antonini, S. (2004). A statement, the contrapositive and the inverse: intuition and argumentation. In M. Johnsen Høines, & A. Berit Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 47–54). Norway: Bergen.

Antonini, S. & Mariotti, M. A. (2008). Indirect proof: What is specific to this way of proving? *ZDM Mathematics Education, 40*, 401-412.

Balacheff N. (1988) Aspects of proof in pupils' practice of school mathematics. In: Pimm D. (ed.) *Mathematics, Teachers and Children* (pp.316-230). London : Hodder and Stoughton.

Balacheff, N. (1991). The benefits and limits of social interaction: The case of mathematical proof. In A. J. Bishop, E. Mellin-Olsen, & J. van Dormolen (Eds.), *Mathematical knowledge: Its growth through teaching* (pp. 175-192). Dordrecht, The Netherlands: Kluwer.

Ball, D.L., & Bass, H. (2003). Making Mathematics Reasonable in School. In J. Kilpatrick, W. G. Martin. & D. Schifter. (Eds.), *A research companion to principles and standards for school mathematics* (pp 227-236). Reston, VA: National Council of Teachers of Mathematics.

Ball, D. L., Hoyles, C., Jahnke, H. N., & Movshovitz-Hadar, N. (2002). The teaching of proof. In L. I. Tatsien (Ed.), *Proceedings of the International Congress of Mathematicians*, (Vol. III, pp. 907-920). Beijing: Higher Education Press.

Behr, M., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio and proportion. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 296-333). NY: Macmillan Publishing.

Bell, A. (1976). A study of pupil's proof-explanations in mathematical situations. *Educational Studies in Mathematics, 7,* 23-40.

Blum, W., & Kirsch, A. (1991). Preformal proving: Examples and reflections. *Educational Studies in Mathematics, 22,* 183-203.

Braine, M. D. S. (1978). On the relation between the natural logic of reasoning and standard logic. *Psychological Review, 85,* 1-21.

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher, 8*(1), 32-42.

Bulgar, S. (2002). *Through a teacher's lens: Children's constructions of division of fractions.* Unpublished doctoral dissertation, Rutgers, the State University of New Jersey, New Brunswick.

Bulgar, S. (2003). Children's sense-making of division of fractions. *Journal of Mathematical Behavior, 22*(3), 319-334.

Chinn, C. A., & Anderson, R. C. (1998). The structure of discussions that promote reasoning. *Teachers College Record, 100,* 315–368.

Cobb, P. Yackel, E., & Wood, T. (1995). The teaching experiment. In P. Cobb and H. Bauersfeld, (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 17-24). Hillsdale, NJ: Lawrence Erlbaum Associates.

Cobb, P., Wood, T., Yackel, E. & McNeal, B. (1992). Characteristics of classroom mathematical traditions: An interactional analysis. *American Educational Research Journal, 29,* 573-604.

Coe, R. and Ruthven, K. (1994). Proof practices and constructs of advanced mathematics students. *British Educational Research Journal, 20*(1)*,* 41-53.

Confrey, J. (1994). Splitting, similarity, and rate of change: a new approach to multiplication and exponential functions. In: G. Harel, & J. Confrey (Eds.), Multiplicative reasoning in the learning of mathematics (pp. 291-330). Albany, NY: SUNY Press.

Creswell, J. W. (2006). *Qualitative inquiry and research design: Choosing among five traditions.* Thousand Oaks, CA: Sage Publications.

Davis, R.B. (1984). *Learning mathematics: The cognitive science approach to mathematics education.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Davis, R.B. & Maher, C.A. (1990). What do we do when we do mathematics? In R.B. Davis, C. A. Maher & N. Noddings (Eds.), *Constructivist Views on the Teaching and Learning of Mathematics: Journal for Research in Mathematics Education Monograph No. 4* (pp.65-78). National Council of Teachers of Mathematics. Virginia: Reston.

Davis, R. B. & Maher, C. A. (1996). A new view of the goals and means for school mathematics. In M. C. Pugach & C. L. Warger (Eds.), *Curriculum Trends, Special Education, and Reform: Refocusing the Conversation*

Davis, R. B. & Maher, C. A. (1997). How students think: The role of representations. In L. D. English (Ed.), *Mathematical Reasoning: Analogies, Metaphors, and Images* (pp. 93-115), Hillsdale, NJ: Lawrence Erlbaum Associates.

De Villiers, M. D. (1990). The role and function of proof in mathematics. *Pythagoras, 24,* 17-24.

De Villiers, M. D. (2003). The role of experimentation in mathematics. Paper presented to the 9th National Congress of the Association for Mathematics Education of South Africa, Cape Town, South Africa.

Doyle, W. (1988). Work in mathematics classrooms: The context of students' thinking during mathematics instruction. *Educational Psychologist, 23*(2), 167-180.

Dreyfus, T., & Hadas, N. (1996). Proof as an answer to the question why. *Zentralblatt fur Didaktik der Mathematik (International Reviews on Mathematical Education), 96*(1), 1-5.

Dreyfus, T. (1999). Why Johnny can't prove (with apologies to Morris Kline). *Educational Studies in Mathematics, 38*, 85-109.

Easterday, K. E., & Galloway, L. L. (1995). A comparison of sentential logic skills: Are teachers sufficiently prepared to teach logic? *School Science and Mathematics, 95*, 431-436.

Epp, S.S. (1998), A Unified Framework for Proof and Disproof, *The Mathematics Teacher,* v. 91(8) pp. 708-713.

Epp, S.(2003) The role of logic in teaching proof in the *Mathematical Association of America, Monthly 110*.

Erlwanger, S.H. (1973). Benny's conception of rules and answers in IPI mathematics. *Journal of Children's Mathematical Behavior*, 1(2), 7-26.

Finlow-Bates, K. (1994). First year mathematics students' notions of the role of informal proofs and examples. In J. da Ponte & J. Matos, (Eds.), *Proceedings of the Eighteenth Annual Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 2, pp. 344-351). Lisbon.

Francisco, J. M. & Maher, C. A. (2005). Conditions for promoting reasoning in problem solving: Insights from a longitudinal study. *Journal of Mathematical Behavior, 24,* 361-372.

Galbraith, P. (1981). Aspects of proving: A clinical investigation of process. *Educational Studies in Mathematics, 12*, 1-28.

Goldin, G. A. (2003). Representation in School Mathematics: A Unifying Research Perspective. In J. Kilpatrick, W. G. Martin and D. Schifter (Eds.), *A Research Companion to Principles and Standards for School Mathematics* (pp. 275-285). Reston, Virginia: The National Council of Teachers of Mathematics.

Greer, B. & Harel, G. (1998). The role of isomorphism in mathematical cognition. *Journal of Mathematical Behavior, 17*(1), 5-24.

Hadar, N. (1977): Children's conditional reasoning: An intuitive approach to the logic of implication. *Educational Studies in Mathematics, Vol. 8*(4), pp. 413-438.

Hadar, N., & Henkin, L. (1978): Children's conditional reasoning: Towards a reliable test of conditional reasoning ability. *Educational Studies in Mathematics, Vol. 9,* pp. 97-114.

Hadar, N. (1978): Children's Conditional Reasoning: A design for research on children's learning of conditional reasoning and research findings. *Educational Studies in Mathematics, Vol. 9*, pp. 115-140.

Hanna, G. (1983). Rigorous proof in mathematics education. Toronto: Ontario Institute for Studies in Education.

Hanna, G. (1989). Proofs that prove and proofs that explain. In G. Vergnaud, J. Rogalski, & M. Artigue (Eds.), *Proceedings of the Thirteenth Meeting of the International Group for the Psychology of Mathematics Education* (Vol. II, pp. 45-51), Paris: PME.

Hanna, G. (1995). Challenges to the importance of proof. *For the Learning of Mathematics, 15*(3), 42-49.

Hanna, G. & Jahnke, H. N. (1996). Proof and proving. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 877-908). Dordrecht, The Netherlands: Kluwer.

Harel, G. (2001). The development of mathematical induction as a proof scheme: A model for *DNR*-based instruction. In S. Campbell & R. Zazkis (Eds.), *Learning and teaching number theory* (pp. 185–212). Westport, CT: Ablex.

Harel, G. & Sowder, L.: 1998, 'Students' proof schemes: Results from exploratory studies' in A.H. Schoenfeld, J. Kaput & E. Dubinsky (editors), *Research in*

Collegiate Mathematics Education, (Vol. 3, pp. 234-283). Providence, RI: American Mathematical Society.

Harel, G. & Tall, D. O. (1991). The General, the Abstract, and the Generic in Advanced Mathematics, *For the Learning of Mathematics*, **11** 1, 38-42.

Hatch, J. A. (2002). *Doing qualitative research in education settings.* Albany, NY: State University of New York Press.

Henningsen, M. & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education, 28(5),* 524-549.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics, 24,* 389-399.

Hiebert, J. & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: an introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Jacobs, K. J., Kawanaka, T., Stigler, J. W. (1999). Integrating qualitative and quantitative approaches to the analysis of video data on classroom teaching. *International Journal of Educational Research, 31,* 717-724.

Johnson-Laird, P. N. (1983). *Mental models: Towards a cognitive science of language, inference and consciousness.* Cambridge, England: Cambridge University Press.

Johnson-Laird, P. N., & Byrne, R. M. J. (1991). *Deduction*. Hillsdale, NJ: Erlbaum.

Johnson-Laird, P. N., & Byrne, R. M. J. (2002). Conditionals: A theory of meaning, pragmatics, and inference. *Psychological Review, 109,* 211-228.

Kiczek, R.D. (2000). *Tracing the development of probabilistic thinking: Profiles from a longitudinal study.* Unpublished doctoral dissertation, Rutgers, the State University of New Jersey, New Brunswick.

Kiczek, R.D. Maher, C.A., & Speiser, R. (2001) Tracing the origins and extensions of Michael's representations In A.A. Cuoco & F.R. Curcio (Eds.), *The role of representation in school mathematics, 2001 yearbook.* (pp. 201-214). Reston, VA: National Council of Teachers of Mathematics.

King, I. L. (1973). A Formative Development of an Elementary School Unit on Proof. *Journal for Research in Mathematics Education, 4*(1), 57-63.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229-269). Hillsdale, NJ: Erlbaum.

Lakatos, I. (1976). *Proofs and refutations*. Cambridge: Cambridge University Press.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Towards a theoretical framework for research. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629-667). Reston, VA: NCTM.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27,* 29-63.

Leighton, J. P. (2003). Defining and Describing Reasoning. In J. P. Leighton & R. J. Sternberg (Eds.), *The Nature of Reasoning*. New York, NY: Cambridge.

Lesh, R. A., Cramer, K., Doerr, H. M., Post, T. & Zawojewski, J. S. (2003). Model development sequences. In R. A. Lesh, & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, teaching and learning* (pp. 35-58). Mahwah, NJ: Lawrence Erlbaum Associates.

Lesh, R., Hoover, M., Hole, B., Kelly, A. & Post, T. (2000). Principles for developing thought-revealing activities for students and teachers. In R. A. Lesh, & A. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 591-646). Mahwah, NJ: Lawrence Erlbaum Associates.

Leron, U. (1985), A Direct Approach to Indirect Proofs, *Educational Studies in Mathematics,* 16,821-325.

Lunzer, E. A. (1979). Formal reasoning: A reappraisal. In A. Floyd (Ed.), *Cognitive Development in the School Years.* London, Croom Helm.

Maher, C. A. (1998). Constructivism and constructivist teaching: Can they coexist? In O. Bjorkqvist (Ed.), *Mathematics teaching from a constructivist point of view* (pp. 29-42). Finland: Abo Akeademi, Pedagogiska fakulteten.

Maher, C.A. (2002). How students structure heir own investigations and educate us: What we have learned from a fourteen year study. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the Twenty-sixth Annual Meeting of the International Group for the Psychology of Mathematics Education* (PME26) (Vol. 1, pp.31-46). Norwich, England: School of Education and Professional Development, University of East Anglia.

Maher, C.A. (2005). How students structure their investigations and learn mathematics: Insights from a long-term study. *Journal of Mathematical Behavior,* 24(1), 1-14. Maher, C.A. & Alston, A. (1989). Is meaning connected to symbols? An interview with Ling Chen. *Journal of Mathematical Behavior 8*, 241-248.

Maher, C. A. (in press). Children's reasoning: Discovering the idea of mathematical proof. In M. Blanton and D. Stylianou (Eds.), *Teaching and learning proof across the grades,* New Jersey: Lawrence Erlbaum Associates.

Maher, C.A. & Davis, R.B. (1995). Children's exploration leading to proof. *Mathematical Science Institute of London University of London*, 87-105.

Maher, C. A., Davis, R. B., & Alston, A. (1992). Teachers paying attention to students' thinking. *Arithmetic Teacher, 39*(9), 34-37.

Maher, C. A. & Kiczek, R. D. (2000). Long term building of mathematical ideas related to proof making. *Proceedings from the 9th International Congress on Mathematics Education*, 82-87. Japan.

Maher, C. A., Muter, E. M. & Kiczek, R. D. (2006). The development of proof making by students. In P. Boero (Ed.), *Theorems and proof in schools: from history, epistemology and cognition to classroom practice* (pp. 197-208). Rotterdam: Sense Publishers.

Maher, C.A. & Martino, A. M. (1992). Teachers building on students' thinking. *Arithmetic Teacher, 39*(7), 32-37.

Maher, C.A. & Martino, A. M. (1992). Teachers building on students' thinking. *Arithmetic Teacher, 39*(7), 32-37.

Maher, C. A., & Martino, A. M. (1996a). Young children invent methods of proof: The "Gang of Four". In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning*, (pp. 431-444). Hillsdale, NJ: Lawrence Erlbaum Associates.

Maher, C. A., & Martino, A. M. (1996b). The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education, 27*(2), 194-214.

Maher, C. A., & Martino, A. M. (1997). Conditions for conceptual change: From pattern recognition to theory posing. In H. Mansfield (Ed.), *Young children and mathematics: Concepts and their representations.* Durham, NH: Australian Association of Mathematics Teachers.

Maher, C. A. & Martino, A. M. (1998). Brandon's proof and isomorphism. In C. A. Maher, *Can teachers help children make convincing arguments? A glimpse into the process.* (pp. 77-112). Rio de Janeiro, Brazil, Universidade Santa Ursula

Maher, C. A., & Martino, A. M. (2000). From patterns to theories: Conditions for conceptual change. *Journal of Mathematical Behavior, 19*, 247-271.

Maher, C. A., Mueller, M., & Yankelewitz, D. (2009). A comparison of fourth and sixth grade students' reasoning in solving strands of open-ended tasks. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, C. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1. Thessaloniki, Greece: PME.

Maher, C. A., Powell, A. B., Weber, K., Lee, H. S. (2006). Tracing middle school students' arguments. In S. Alatorre, J. L. Cortina, M. Sáiz, and A. Méndez (Eds.), *Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 403-410). Mérida, México: Universidad Pedagógica Nacional.

Maher, C.A., & Speiser, R. (1997). How far can you go with block towers? Stephanie's intellectual development. *Journal of Mathematical Behavior, 16*(2), 125-132.

Martino, A. M. (1992). Elementary students' construction of mathematical knowledge: Analysis by profile. Unpublished doctoral dissertation. Rutgers University.

Martino, A. M. & Maher, C. A. (1999). Teacher questioning to promote justification and generalization in mathematics: what research practice has taught us. *Journal of Mathematical Behavior, 18*(1), 53-78.

Manin, Y. I. (1977). *A Course in Mathematical Logic* (N. Koblitz, Trans.). New York: Springer-Verlag.

Martin, W. G. & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education*, 20(1), 41-51.

Merriam, S. (1998). *Qualitative research and case study applications in education*. San Francisco, CA: Jossey-Bass Publishers.

Movshovitz-Hadar, N. (1988). Stimulating presentation of theorems followed by responsive proofs. *For the Learning of Mathematics*, 8(2), 12-19.

Mueller, M. (2007). *A study of the development of reasoning in sixth grade students.* Unpublished doctoral dissertation, Rutgers, The State University of New Jersey, New Brunswick.

Mueller, M. & Maher, C.A. (2008). Learning to reason in an informal after-school math program. Paper presented at the 2008 Annual Meeting of the American Educational Research Association.

Muter, E. M. (1999). *The development of student ideas in combinatorics and proof: A six year study.* Unpublished doctoral dissertation, Rutgers, The State University of New Jersey, New Brunswick

Muter, E. M. & Maher, C. A. (1998). Recognizing isomorphism and building proof: Revisiting earlier ideas. In S. Berenson, K. Dawkins, M. Blanton, W. Coulombe, J. Kolb, K. Norwood, and L. Stiff (Eds.), *Proceedings of the 20th Conference of* *the North American Group for the Psychology of Mathematics Education, (2)*, 461-467. Raleigh, NC: North Carolina State University.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Noddings, N. Constructivism in mathematics education. In R.B. Davis, C. A. Maher & N. Noddings (Eds.), *Constructivist Views on the Teaching and Learning of Mathematics: Journal for Research in Mathematics Education Monograph No. 4* (pp.65-78). National Council of Teachers of Mathematics. Virginia: Reston.

Norton, A. (2008). Josh's operational conjectures: Abduction of a splitting operation and the construction of new fractional schemes. *Journal for Research in Mathematics Education, 39*(4), 401-430.

Nunokawa, K. (2005). Mathematical problem solving and learning mathematics: What we expect students to obtain. *Journal of Mathematical Behavior,* 24, 325-240.

Patton, M. Q. (1987). *How to use qualitative methods in evaluation*. Thousand Oaks, CA: Sage.

Polya, G. (1954). *Mathematics and plausible reasoning* (Vols. 1 & 2). Princeton: Princeton University Press.

Porteous, K. (1990), What do Children Really Believe?, *Educational Studies in Mathematics*, 21(6), 589-598.

Powell, A.B. (2003). *"So let's prove it!": Emergent and elaborated mathematical ideas and reasoning in the discourse and inscriptions of learners engaged in a combinatorial task.* Unpublished doctoral dissertation, Rutgers, The State University of New Jersey, New Brunswick.

Powell, A. B., Francisco, J. M., & Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *The Journal of Mathematical Behavior*, 22(4), 405-435.

Powell, A. B., Maher, C. A., & Alston, A. S. (2004). Ideas, sense making, and the early development of reasoning in an informal mathematics settings, In D. E. McDougall and J. A. Ross, (Eds.). *Proceedings of the 26th Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education, (2),* 585-591. Toronto, Ontario: OISE-UT.

Recursion. In *Wikipedia Encyclopedia*. Retrieved online August 12, 2008 at http://en.wikipedia.org/wiki/Recursion

Reid, D. (2002). Conjectures and refutations in grade 5 mathematics. *Journal for Research in Mathematics Education, 33,* 5–29.

Reynolds, S.L. (2005). *A Study of Fourth-Grade Students' Explorations into Comparing Fractions*. Unpublished doctoral dissertation, Rutgers, The State University of New Jersey, New Brunswick.

Rips, L. J. (1983). Cognitive processes in propositional reasoning. *Psychological Review, 90,* 38-71.

Rips, L. J. (1994). *The psychology of proof: Deductive reasoning in human thinking*. Cambridge, MA: MIT.

Rowland, T. (2002a). Generic proofs in number theory. In S. Campbell and R. Zazkis (Eds.) *Learning and teaching number theory: Research in cognition and instruction.* (pp. 157-184). Westport, CT: Ablex Publishing.

Rowland, T. (2002b). Proofs in number theory: History and heresy. In *Proceedings of the Twenty-Sixth Annual Meeting of the International Group for the Psychology of Mathematics Education,* (Vol. I, pp. 230-235). Norwich, England.

Schoenfeld, A.H. (1982). Problem-solving Research and Mathematical Education. In F. K. Lester & J. Garofalo (Eds.), Mathematical Problem Solving: Issues in Research (pp. 27-37). Philadelphia: The Franklin Institute Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (pp. 334-370). New York: MacMillan.

Schroeder, T. L. & Lester, F. K. (1989). Developing understanding in mathematics via problem solving. In P. R. Trafton (Ed.), *New Directions for Elementary School Mathematics, 1989 Yearbook.*, Reston, VA: National Council of Teachers of Mathematics

Selden, A. and Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education*, 34(1), 4-36.

Semadeni, (1984). Action proofs in primary mathematics teaching and in teacher training. *For the Learning of Mathematics, 4*(1), 32-34

Senk, S.L. (1985). How well do students write geometry proofs? *Mathematics Teacher, 78,* 448-456

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20-26.

Skemp, R. R. (1979). Goals of learning and qualities of understanding. *Mathematics Teaching, 88*, 44-49.

Smith, D., Eggen, M., & St. Andre, R. (2001). *A transition to advanced mathematics, 5th ed.* Pacific Grove, CA: Wadsworth Group.

Stake, R. (1995). *The art of case study research*. Thousand Oaks, CA: Sage.

Steencken, E. (2001). *Tracing the growth in understanding of fraction ideas: A fourth grade case stu*dy. Unpublished doctoral dissertation, Rutgers, The State University of New Jersey, New Brunswick.

Steencken, E.P. & Maher, C.A. (2002). Young children's growing understanding of fraction ideas. In B.H. Litwiller (Ed.), *2002 NCTM Yearbook: Making sense of fractions, ratios, and proportions.* (pp.49-60). Reston, VA: National Council of Teachers of Mathematics.

Steencken, E.P. & Maher, C.A. (2003). Tracing fourth graders' learning of fractions: early episodes from a year-long teaching experiment. *The Journal of Mathematical Behavior* 22(2), pp. 113-132.

Steffe, L.P. (2002). A new hypothesis concerning children's fractional knowledge. *Journal of Mathematical Behavior, 20,* 263-307.

Steffe, L.P. (2003). Fractional commensurate, composition, and adding schemes: Learning trajectories of Jason and Laura: Grade 5. *Journal of Mathematical Behavior, 22*, 237-255.

Steffe. L.P. (2004). On the construction of learning trajectories of children: The case of commensurate fractions. *Mathematical Thinking and Learning, 6,* 129-162.

Stylanides, A. J. (2007) Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 38*(3), 289-321.

Tesch, R. (1990). *Qualitative research: Analysis types and software tools.* New York: Falmer Press.

Thompson, D.R. (1996), Learning and Teaching Indirect Proof, *The Mathematics Teacher* v. 89(6) pp. 474-82.

Toulmin, S. (1969). The uses of arguments. Cambridge: Cambridge University Press.

Uptegrove, E. & Maher, C. A. (2004). Students building isomorphisms. In M. J. Hoines and A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education (4),* 353-360. Bergen University College, Bergen, Norway.

Weber, K. (2001). Student difficulties in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 48*(1), 101-119.

Weber, K. (2003). Students' difficulty with proof. In A. Selden and J. Selden (eds.) *Research Sampler, 8*. Available from the MAA website on the teaching of learning of mathematics: http://www.maa.org/t_and_l/sampler/research_sampler.html

Weber, K., Maher, C. A., Powell, A. B. & Lee, H. S. (in press). Learning opportunities from group discussions: Warrants become the objects of debate. *Educational Studies in Mathematics.*

Wood, T. (1999). Creating a Context for Argument in Mathematics Class. *Journal for Research in Mathematics Education, 30*(2), 171-191.

Yackel, E. & Cobb, P. (1994, April). *The development of young children's understanding of mathematical argumentation.* Paper presented at the meeting of the American Educational Research Association, New Orleans.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*, 458-477.

Yackel, E. & Hanna, G. (2003). Reasoning and proof. In J. Kilpatrick, W. G. Martin. & D. Schifter. (Eds.), *A research companion to Principles and Standards for school mathematics* (pp 227-236). Reston, VA: National Council of Teachers of Mathematics.

Yin, R. K. (2003). *Case study research: Design and methods*. Thousand Oaks, CA: Sage.

Zack, V. (1997). You have to prove us wrong: Proof at the elementary school level. In E. Pehkonen (Ed.), *Proceedings of the Twenty-First International Conference on the Psychology of Mathematics Education, Vol. 4* (pp. 291-298). University of Helsinki, Finland.

Appendix A

Table of Sessions, Dates, Tasks, and Camera Views

Session	Date of Session	Tasks		Camera Views
1.	Sept. 20, 1993	1a	I claim that the light green rod is half as long as the dark green rod. What do you think?	S, F, OHP
		1b	What number name would we give the light green rod if I called the dark green rod one?	
		2	Someone told me that the red rod is half as long as the yellow rod. What do you think?	
		3	Someone told me that the purple rod is half as long as the black rod. What do you think?	
		4a	Someone told me that the red rod is one third as long as the dark green rod. What do you think?	
		4b	If I called the dark green rod one, what number name would I give to the red rod?	
		5a	Someone told me that light green is one third as long as blue. What do you think?	
		5b	So if I call the blue rod one, what number name would I give to light green?	
		6	What number name would I have to give to green if I wanted red to be one?	
		7a	If I call brown one, what number name would I give to red?	
		7b	Now I want to call the red rod one, what name would I give to the brown rod?	
		8	What would I have to call one if I want to name the white rod one half?	
		S1	If the red rod is considered one fifth, what would the orange rod be? [Alan]	
		S2	If light green is one whole, what is blue? [Beth] If blue is one, what is light green? [Mark]	
		S3	If white is one, what is orange? [Jacqueline and Kelly] If orange is one, what is white? [T/R 1]	
		S4	If purple is one half, what is one? [Meredith]	
		S5	If light green was one half, what would be a whole? [Erik]	
		S6	If white is considered one fifth, what would one be? [Alan]	
		S7	If I call purple two, what would one look like? [T/R 2]	
		S8	If white is three, what is six? [Erik]	
		S9	If the purple rod is one half, what would be one [Alan]?	
		S10	If you called one [white rod] a seventh. What would a whole be?[Meredith]	
		S11	I want to find a rod that has number name is one sixth. Can you find it?(T/R 1)	

Session	Date of Session	Tasks		Camera Views
2.	Sept. 21	1	If I call the yellow rod one half, what rod will I call one?	S, F, OHP
		2	If I call the blue rod one, what rod will I call one half?	
		3	If you were designing a new set of rods and you wanted to call the blue rod one, okay? Can you tell me what that new rod might look like so that you would be able to call it a half?	
		4	If we call the orange 'two', what can we say about yellow?	
		5	What if I change the name of the orange to 'six'…what number name would I call the yellow?	
		6	I'm going to call the orange and light green together one…Can you find a rod that has the number name one half?	
3.	Sept. 24	1	If I gave the purple the number name one half, what number name would I give to the dark brown	S, F, OHP
		2	What if I gave the purple rod the number name one? What number name would I give to the brown rod?	
		3	If we give the orange rod the number name two, can you tell me what number name we'd give to yellow?	
		4	If I call yellow and light green two, what number name would I give to red?	
		5	I'd like you to make the yellow and light green one and then tell me what the number name would be for red also.	
		6	Candy Bars and Pizzas	
		7	Which is bigger, one half or one third, and by how much?	
4.	Sept. 27	1	If I give this orange rod a number name one, what number name would I give to white?	S, F, OHP
		2	If we are calling the orange rod the number name one, what would you call the number name for the red rod?	
		3	I'm calling the orange one, what number name would I give to two whites?	
		4	I'm going to call the orange ten…and I'm wondering if you could tell me the number name for white.	
		5	If I take the same orange rod we've been working with, but I change the number name again. This time I'd like to call it… fifty. I'm wondering if anybody could tell me the number name for yellow.	
		6	What number would I give to one white rod if we're still calling the orange the number name fifty?	

Session	Date of Session	Tasks		Camera Views
		7	Which is larger, 1/2 or 1/3, and by how much?	
5.	Sept. 29	1	Is 1/5=2/10?	S, F, OHP
		2	What other number names can we give to one half of a candy bar?	
		3	Which is larger, 1/2 or 1/3, and by how much?	
6.	Oct. 1	1a	Which is larger, one half or one third, and by how much?	S, F, OHP
		1b	What is the white rod called in model with the orange and red train that was built to solve the original problem?	
		2	Which is bigger, one half or one quarter, and by how much?	
7.	Oct. 4	1	Which is larger, one half or two thirds, and by how much?	S, F, OHP
8.	Oct. 6	1	Which is larger, one half or three fourths, and by how much?	S, F, OHP
		2	Which is larger, two thirds or three quarters, and by how much?	
9.	Oct. 7	1	Which is larger, two thirds or three fourths, and by how much?	S, F
10.	Oct. 8	1	Which is larger, two thirds or three fourths, and by how much?	F
		2	Which is larger, one half or two tenths, and by how much?	
11.	Oct. 11	1	Which is larger, two thirds or three fourths, and by how much?(number of models)	S, F, OHP
12.	Oct. 29	1	Which is larger, one fourth or one ninth, and by how much?	S, F, OHP
13.	Nov. 1	1	Which is larger, one fourth or one ninth, and by how much?	S, F, OHP
		2	I want you to think about…sharing those three bars of candy so everybody got the same amount exactly	
		3	If I were, if I were to say things like this to you, one half, one third, one fourth, one fifth, right? If I were talking about these numbers, do you know which are bigger and which are smaller?	
14.	Dec. 2	1a	What number name would I call the white rod if the orange and red train was called one?	S, F, OHP
		1b	How many whites are in a red and orange train	
		1c	How many _______ are in ________?	
		2a	If I call the dark green rod 1, now it's not the orange and red train, it's the dark green rod that's going to be one, what number name would I give to the white rod?	

Session	Date of Session	Tasks		Camera Views
		2b	How many whites are in the dark green rod?	
		2c	How many ______ are in ________?	
		2d	Is there another number name for one here?	
		2e	Can we write this now as a number sentence including these numbers?	
		1d	Remember this one, the red and orange train? Can we rewrite this as a number sentence now? The question is how many 1/12's are in 1?	
		W1	If we give the red the number name 1, what number name would we give to white? How many whites are in a red? Write a number sentence to describe this relationship.	
		W2	If we give the brown the number name 1, what number name would we give to white? What number name would we give to purple?	
		W3	If we give the orange and yellow train the number name 1, what number name would we give to white? What number name would we give to light green? What number name would we give to yellow?	
		W4	If we give the blue and yellow train the number name 1, what number name would we give to white? What number name would we give to red? What number name would we give to black?	
15.	Dec. 9	1	Ribbons and Bows. Students were given ribbon lengths of 1, 2, 3, and 6 meters. The students were asked how many bows of specific lengths could be made from a length of ribbon.	S, F, OHP
16.	Dec. 14	1	Finding the number of one third meter long bows from three, nine, twenty-seven, and eighty-one meters of ribbon	S, F, OHP
17.	Dec. 15	1a	How many bows can you make from ribbon that's a third of a meter in length if you have nine meters of ribbon.	F
		1b	I still have my nine meters of ribbon, but now I'm making ribbons that are three meters long	
		2a	So now I'm starting with twelve meters of ribbon, [and making] one half meter bows… How many bows am I going to make?	
		2b	I want to know how many you can make that are two meters [long].	
		2c	How many bows can be made that are two third meters long if we have twelve meters of ribbon?	

Appendix B

Transcripts

Session 1, Sept. 20, 1993 (Front, Side, and OHP)

Line	Time	Speaker	Transcript
1.0.1	0:14:53	T/R 1:	[Introducing the task, speaking slowly, repeating question several times] I claim that the light green rod is half as long as the dark green rod. What do you think? How many think that's true? What would you do to convince me? What would you do to convince me that that's true? Do you want to think about that for a minute with your partner? I think Andrew has already decided how to convince me. I think Caitlin also has decided. And Brian and Graham. [Approx. 1 min. given to class as children raise their hands when ready.] Okay, Erin, you've never done this before. Can you tell me?
1.0.2		Erin:	It's true. [She puts two light green rods next to the dark green rod.] Take two light green rods and put them both together.
1.0.3		T/R 1:	So what you would do is you would put two light green rods together. How many of you did that? Does that make sense? So you would convince me. So we would give the light green rod the number name - what do you think? What number name would we give the light green rod if I called the dark green rod one? What number name would we give the light green rod? Talk to your partner and see if you agree.
1.0.4	0:17:00	Meredith:	[to Sarah] One half.
1.0.5		T/R 1:	This would be half of the green rod? You all think about that for a minute. If I called the dark green rod one, what number name would I give the light green rod? Why don't you talk to your partner and see if you agree. [T/R 1 walks around the room, talks to children, attempting to see who agrees.]
1.0.6		T/R 1:	Alan and Erik, do you agree? [Both boys mutter in agreement.] Did you talk? [Boys nod in agreement.] Whisper.
1.0.7		Alan:	[Whispering] One half.
1.0.8		T/R 1:	How many of you think you have a number name for the light green rod? If you think you have a number name in your group would you raise your hand? Ok, Kelly?
1.0.9		Kelly:	One half
1.0.10	0:18:03	T/R 1:	Kelly thinks one half. How many of you agree?
1.0.11		T/R 1:	[Addressing whole class] Is there anyone who disagrees? So if I call the dark green rod one, I would call the light green rod one half? Okay, that's interesting. Someone told me, someone told me that the red rod is half as long as the yellow rod. What do you think?
1.0.12		Erik:	Which red rods?

1.0.13		Alan:	These little ones. [He is holding up red rods to show Erik.]
1.0.14		T/R 1:	Someone told me that the red rod is half as long as the yellow rod. What do you think?
1.0.15		Meredith:	No.
1.0.16		Sarah:	No.
1.0.17		Alan:	[To Erik and to himself] No. Look. [He holds up the red rods to show Erik.]
1.0.18		Erik:	Nope.
1.0.19		Alan:	There is enough to fit more here. There is enough to fit another light green in here.
1.0.20	0:19:01	T/R 1:	Danielle, what do you think?
1.0.21		Danielle:	No.
1.0.22		T/R 1:	Danielle thinks no. What can you do to convince me that that's not true? Can you speak nice and loud, Danielle?
1.0.23		Danielle:	Put the two red red rods next to the yellow one. And there's more space.
1.0.24		T/R 1:	So it doesn't work, does it? How many of you agree that the red rod is not half as long as the yellow rod? How many of you agree that whoever told me that I shouldn't believe it? And you can all convince me? And you all did the same thing in convincing me, right, you put the two red ones and it didn't come out to be a half, that's what you did? What a smart class this is! I'd better be careful who I listen to. Ok. Someone told me that the purple rod
1.0.25		Alan:	[He hold up the purple rod] This one
1.0.26		T/R 1:	Is half as long as the black rod.
1.0.27	0:20:07	Erik:	No.
1.0.28		T/R 1:	What do you think?
1.0.29			*SIDE VIEW*
1.0.30		Erik:	No.
1.0.31		Alan:	Nope.
1.0.32		T/R 1:	[To Erik] The black rod.
1.0.33		Erik:	Oh, the black rod. [He puts back the blue rod, which he has used by mistake; he takes out a black rod.]
1.0.34		Alan:	It would take another light green to make a whole and that's not half. [He is holding up the black rod with the purple rod in one hand and with his other hand, he takes the light green and puts it together with the purple rod to show that the train of purple and light green is equal in length to the black rod.]
1.0.35		Erik:	Yeah, it is, look. [Erik puts two purple rods in a train next to the black rod.]
1.0.36		Alan:	That is not as long as the black, it would take another light green one.
1.0.37		Erik:	Oh.
1.0.38			*FRONT VIEW*
1.0.39		T/R 2:	What do you think over here?

1.0.40		Sarah:	[pointing to the rods] It's too big.
1.0.41		T/R 2:	It's too large?
1.0.42		Sarah:	Yeah.
1.0.43		T/R 2:	[Meredith says something, inaudible] So you'd need to make it longer.
1.0.44			*BOTH VIEWS*
1.0.45		T/R 1:	[To the class] What do you think? David?
1.0.46		David:	No, two purples are too large.
1.0.47		T/R 1:	How many of you agree with David?
1.0.48		T/R 1:	Can you find dark green? Are you ready for this one? Someone told me, that the red rod is one third as long as the dark green rod. What do you think?
1.0.49		Erik:	Yep.
1.0.50		Alan:	Yep.
1.0.51		Erik:	Mmm hmm
1.0.52		Alan:	Mmm hmm. Cause two of these makes
1.0.53		T/R 1:	Discuss it with your partner.
1.0.54		Erik:	Yeah, I think so.
1.0.55		Alan:	Umm, cause if you did it like this
1.0.56			[Andrew has the dark green rod on the table and is putting three red rods; Erik points to the red rod in his staircase]
1.0.57			*FRONT VIEW*
1.0.58		T/R 2:	[to Jacquelyn] So you've got three of these, and you're matching it up? [Jacquelyn nods]
1.0.59			*BOTH VIEWS*
1.0.60	0:21:49	T/R 1:	Jackie?
1.0.61		Jackie:	Yeah.
1.0.62		T/R 1:	Jackie thinks so. How many of you agree with Jackie? [Most of the students raise their hands.] What would you do to convince me? You want to come up here and convince me, Jackie, on the overhead? [Jackie goes to the overhead.]That the red rod is one third as large, as long as the dark green.
1.0.63		T/R 1:	What do you think, guys? Michael?
1.0.64		Michael:	You put the red rods, um, right below the green rod.
1.0.65		T/R 1:	How many did that? Okay, what a smart class. You sure everyone didn't work with this last year? What a smart class. That's lovely, thank you, Jackie. [Jackie has put the dark green rod with three red rods on the overhead.] [On the overhead there is a dark green rod with three red rods below it. T/R 1 takes one red rod away.]
1.0.66		T/R 1:	If I had to give another name, a number name for the red rod, if I called the dark green rod one, what would I call the red rod? What number name would I give to it? If I called the dark green rod one, what number name would I give to the red rod? [Most of the students have their hands raised.] How

			many of you think you know a number name, for the red rod? Would you talk to your partner a little bit and see what you think?
1.0.67		Students:	We already did.
1.0.68			*FRONT VIEW*
1.0.69		T/R 2:	Sarah, what do you think?
1.0.70		Sarah:	One third.
1.0.71		T/R 2:	How does that work?
1.0.72		Sarah:	[to T/R 2] Because there's three of them. [She points to the red rods under the dark green rod.] So there is a third.
1.0.73			*BOTH VIEWS*
1.0.74		T/R 1:	You did already? [Most hands are still raised.] Beth, you want to tell us?
1.0.75		Beth:	One third.
1.0.76		T/R 1:	How many think one third? [All hands go up.] You all agree. Can you tell me why you would give it the number name one third?
1.0.77		Beth:	Because if you put three on them it makes one whole.
1.0.78		T/R 1:	Okay, so if it makes three, it would make one whole. Do you agree with that?
1.0.79		Beth:	Umm.
1.0.80		T/R 1:	So we give it the number name one third. Okay, very good. Someone told me that light green is one third as long as blue. What do you think? Someone told me that light green is one third as long as blue. What do you think? What do you think, Jessica?
1.0.81		Jessica:	It is.
1.0.82		T/R 1:	Jessica thinks it is. And how would you convince me, Jessica?
1.0.83	0:24:44	Jessica:	Because you could, if you have, you have three of these you could put it up to the blue, and it's one whole.
1.0.84		T/R 1:	So if I call the blue rod one, what number name would I give to light green, everybody?
1.0.85		Students:	One third.
1.0.86		T/R 1:	I would give it the number name one third. Now notice, if I called the dark green rod one what number name would I give to the red?
1.0.87		Students:	One third.
1.0.88		T/R 1:	So are the number names always the same?
1.0.89		Students:	[Tentatively] No
1.0.90		T/R 1:	Are the color names always the same?
1.0.91		Students:	No.
1.0.92		T/R 1:	Does this have another name other than blue? Am I ever going to call this something else other than blue?
1.0.93		Students:	No, oh yeah.
1.0.94		T/R 1:	What am I going to call it?

1.0.95		Erik:	Dark blue.
1.0.96		T/R 1:	Well, dark blue. [The question is again raised.]
1.0.97		Erik:	Color names, no.
1.0.98		T/R 1:	Erik thinks that the color names don't change. Right, do we agree that the color names are always the same?
1.0.99		Students:	Umm.
1.0.100		T/R 1:	Do the number names change?
1.0.101		Students:	Yes.
1.0.102		T/R 1:	Okay, tell me why the number names change. Give me an example of why the number names change. Erik?
1.0.103	0:26:02	Erik:	You could think, you could say that, you could take an orange block and a blue block and that would be, that would be three thirds of it.
1.0.104		T/R 1:	Is it?
1.0.105		Erik:	Wait. No, wait, hold on. Would, no, I mean, if you ta, if you take a blue rod and you could call it one whole and you would take an, a diff, a smaller rod and
1.0.106		T/R 1:	Which one?
1.0.107		Erik:	Um, well, oh, you take a light green, and that'll be a third of it.
1.0.108		T/R 1:	Okay, so I called the light green one third and I called the red one third. Why could the light green be a third and why could the red be a third? How is that possible for both of these to be a third?
1.0.109		Erik:	[Raising his hand] Oh, oh, I know.
1.0.110		T/R 1:	Michael?
1.0.111		Michael:	Because there is a different size whole.
1.0.112		T/R 1:	Because there is a different size whole. There is a different number name I gave for what I called one, okay? So when the blue is one, can you see that the light green becomes a third? But can you see when the dark green is one, do you see that the red becomes a third? So do we have permanent number names? (Students: No) No the number names change, that's very important when using these rods. Do the color names change? (Students: No) No the color names don't change. Every one of these rods has a permanent color name, but the number name changes with the problem. So let's be sure that we're familiar with the color names so that we can be sure that we're calling them the same thing, um, for the rods, we have a color name for this one, we're calling this one blue or dark blue, right, we agree with that? And we're calling this one light green, why don't you pull out all of them and see if it is true that there are ten of them, and then we'll do some problems with number names. [Many students build staircases.]
1.0.113			*FRONT VIEW*

1.0.114		Jacquelyn:	Which is the biggest?
1.0.115		Michael:	Orange is the biggest? There's ten.
1.0.116			*BOTH VIEWS*
1.0.117		T/R 1:	How many think there are ten (colors)? [long pause] As I look around the room I see that you all, some of you did the same thing, some of you did them differently? Kelly, your arrangement? You want to say something about your arrangement?
1.0.118	0:29:56	Kelly:	[Describing the staircase that she built] I put the tallest, then the second to tallest-
1.0.119		T/R 1:	Which is the longest rod?
1.0.120		Kelly:	Um, the orange
1.0.121		T/R 1:	You all agree?
1.0.122		Students:	Mmm hmm
1.0.123		T/R 1:	And then what did you do?
1.0.124		Kelly:	And then I put down the blue, then the brown, then the black, then black, then dark green, then yellow, then like the pinkish-red
1.0.125		T/R 1:	We call this purple.
1.0.126		Kelly:	Um, light green, then red, and white.
1.0.127		T/R 1:	How many agree? Ten rods? Ten color names? You agree. Now, you remember we said when we were the dark green rod one, what number name were we giving to red? Michael?
1.0.128		Michael	One third.
1.0.129		T/R 1:	One third. And, why, Michael, one more time?
1.0.130		Michael	Because three of the red ones equal one of the green ones
1.0.131		T/R 1:	Ok. I heard someone say in the back of the room, I was listening, someone asked a question, someone said, look, we have three red ones making the green one, why can't we call the red one one? I heard someone say that, somebody was very confused by that. And we said that, you told me this is a third, is that possible or is it not possible, can you convince me that it's not possible for the red rod to have the number name one?
1.0.132		Erik:	Oh, I can convince you that it's possible.
1.0.133		Michael:	It's possible because two whites equal a red.
1.0.134		T/R 1:	That wasn't the question. I said if you called the dark green is called one, what number name do we give the red. You told me one third, and everybody agreed.
1.0.135		Michael:	Oh, yeah.
1.0.136		T/R 1:	Now I'm asking you, if I called dark green one, and I called red one third could red now be one?
1.0.137		Erik:	No. Not now. Not if the dark green is one. Because if you're comparing the red to dark green it can't be one. But if you're comparing the red to something else it can be a one, it can be a whole.

1.0.138		T/R 1:	Michael, what do you think?
1.0.139		Michael:	Well, red can't be a whole, red can't be one. Because the green is bigger and it takes three of the reds to make one green.
1.0.140		T/R 1:	Ok, suppose I said I wanted the red to be one, what would that tell you about the dark green? If I wanted to make it so the red were one, what would I have to make the dark green to make the red one? Gregory, is that your hand up, or are you thinking? Do you understand my question? Suppose I were to change to problem, and I wanted now red to be one, a brand new problem, what number name would I have to give to green if I wanted red to be one? [Pause as students work on this task.] You want to talk about it with your partner for a minute?
1.0.141			*SIDE VIEW*
1.0.142		Erik:	[To Alan] three wholes?
1.0.143		Alan:	Can't give it a name because it can't be put into, into two. Because look, [Alan points to the dark green rod, which is part of a staircase on his desk.]
1.0.144		Erik:	But the dark green is bigger.
1.0.145		T/R 1:	Okay, suppose I want the red to be one, what number name would I have to give dark green for red to be one?
1.0.146		Erik:	[To Alan] Three wh, three wholes. Because if this is a one [He holds up a red rod]
1.0.147		T/R 1:	[She comes to their desks.] Hi, Erik. What Erik?
1.0.148	0:34:11	Erik:	[Still to Alan] It would be three wholes. Because if this is a one [He is holding up a red rod.]
1.0.149		T/R 1:	Right.
1.0.150		Erik:	It takes three to figure this out. [He takes three red rods and puts the next to the dark green rod.] And if all the reds are one.
1.0.151		T/R 1:	So dark green would have to be three.
1.0.152		Erik:	And there are three, that'll be three wholes.
1.0.153		T/R 1:	Do you agree, Alan?
1.0.154		Alan:	[Nodding] Umm.
1.0.155		T/R 1:	Does that make sense, Alan?
1.0.156		Alan:	Umm, yes, I guess so.
1.0.157		T/R 1:	okay.
1.0.158		Alan:	Three wholes [Alan nods to Erik.]
1.0.159		T/R 1:	Okay, are you ready to argue that, Erik?
1.0.160		Erik:	Yeah.
1.0.161			*FRONT VIEW*
1.0.162		Jacquelyn:	She's saying she wants this to be one
1.0.163		Michael:	She saying this is one whole, so what happens to this?
1.0.164		Kelly:	It could be one half. [Michael looks skeptical]
1.0.165			*BOTH VIEWS*

1.0.166		T/R 1:	Okay. [to the class] I'm hearing two answers. Some students say three, some say one third. [Brian?] There are two different answers. I would like to hear both answers. Ah, let's see, who wants to give me an answer? [Erik and Alan raise their hands.] I heard, I heard a couple of different answers. Only this table wants to give me an answer? I heard a different one. Okay, Alan?
1.0.167	0:36:12	Alan:	Three wholes.
1.0.168		T/R 1:	Nice and loud, Alan.
1.0.169		Alan:	Three wholes.
1.0.170		T/R 1:	Do you want to tell me why you think so? Are you all hearing what Alan says?
1.0.171		Erik:	I know. I know why.
1.0.172		Alan:	Okay, because the, the
1.0.173		T/R 1:	Ah, you need to talk much louder.
1.0.174		Alan:	Okay, if the red one is considered one [He points to the red rod in Erik's model] then the green one is a lot bigger. So it would have to be, it would take three whole ones to make another green so it should be considered three wholes.
1.0.175		Erik:	[Continuing] Well, I think, well, if you, if you say that this would be one [He holds up towards the teacher one red rod]. This is one, and it takes three of the one, the one wholes to equal up one of these [He points to a dark green rod on his desk]. And it that's one whole, umm, one whole plus one whole plus one whole would equal three wholes. So the green would have to be three wholes.
1.0.176		T/R 1:	Does that make any sense? Do you understand what Erik is saying? What do you think, David?
1.0.177		David:	Well, I thought the same thing. If red is one, then the green would have to be two more wholes so that would be three wholes.
1.0.178		T/R 1:	Does anybody think anything else? What I'm hearing David say, and Alan and Erik, that if red is one, green is three. How many of you agree with that - if red is one, green is three? [All students seen raise hands] How many of you aren't sure? How many of you disagree? Well, we'll see if you understand. I'll give you another one, ok?
1.0.179		T/R 1:	How about this one. Let's look at brown and let's look at red. If I call brown one, what number name would I give to red? Danielle?
1.0.180	0:38:55	Danielle:	Fourths
1.0.181		T/R 1:	How many of you think one fourth? What would you do to convince me?
1.0.182		Danielle:	Well, I put the four blocks up to the brown.
1.0.183		T/R 1:	Ok. So you would put four like this, and you would give the red the number name one fourth. Is there anybody who

			disagrees with that? Now let me change the problem. Now I want to call the red rod one, what name would I give to the brown rod? Jacquelyn?
1.0.184		Jacquelyn:	Four.
1.0.185		T/R 1:	Why would you call it four? You'd call it four. Why would you call it four?
1.0.186		Jacquelyn:	If this is one [raising the brown rod] and if you line them all up. And if you add them all up it would be four wholes
1.0.187		T/R 1:	How many of you agree with that? Now I'm wondering, some of you looked at me like, ahah, I see what she's doing. Are we changing the number names for the rods? Yeah, we're changing the number names for the rods. So we have to always say to ourselves, what's one? Right? I need to know what one is before I can tell you the number names for the others, isn't that true? Now I'm going to have you make up a problem for me. I want to call the white rod one half. What would I have to call one, if I make the white rod one half? What would I have to call one if I want to name the white rod one half? Talk to your partner to be sure you agree, and if you and your partner agree, in this case, we have two partners with Jacquelyn, Kelly, and Michael, raise your hand at your table. [repeats question] Laura?
1.0.188		Laura:	The red.
1.0.189		T/R 1:	Laura says I'm going to call the red rod one. How many agree? And what would you do to convince me? What would you do, Graham, to convince me?
1.0.190		Graham	You would take a white rod and stick it right on the end of a red one.
1.0.191	0:42:10	T/R 1:	I think you have the idea. Make up a problem for me. At your table with your partner make up an idea for me and the rest of the class. When you think you have one, be careful how you are going to ask it. Practice how you are going to ask the problem and then raise your hand.
1.0.192			*SIDE VIEW*
1.0.193		Erik:	[Erik puts five red rods next to an orange rod.] Ha, ha, ha, hm.
1.0.194		Alan:	No, that's one fifth.
1.0.195		Erik:	I know
1.0.196		Alan:	Oh, yeah. [He starts to set up Erik's problem.] We are out of reds. Oh, well.
1.0.197		T/R 1:	Try to get a hard one and try to stump us. [Alan raises his hand.]
1.0.198		Erik:	Yes, I got it. [He puts two purple rods next to the orange rod.]
1.0.199		Alan:	No, those won't make it.
1.0.200		Erik:	What makes thirds?

1.0.201	Alan:	Thirds, thirds out of a, thirds out of this? [He is pointing to an orange rod.] Probably the greens.
1.0.202	Erik:	Light green,
1.0.203	Alan:	Light green would make thirds out of the orange. [Alan puts light green rods next to the orange rod.]
1.0.204	Erik:	Yeah.
1.0.205	Alan:	No, it wouldn't.
1.0.206	Erik:	Yeah, it would.
1.0.207	Alan:	No, it doesn't. Try it.
1.0.208	Erik:	Then what does?
1.0.209	Alan:	I know what makes thirds.
1.0.210	Erik:	What?
1.0.211	Alan:	There's got to be one.
1.0.212	T/R 1:	[T/R 1 approaches their desks.] Oh, this is an interesting one. [She points to the orange rod with five red rods next to it.]
1.0.213	Erik:	[To T/R 1] Which one makes thirds? What makes
1.0.214	T/R 1:	[To Alan] This would be an interesting problem, what would you ask me here, Alan?
1.0.215	Alan:	If, if the red rod was considered one fifth, what
1.0.216	T/R 1:	Or if the orange rod is considered, if the red rod is one fifth, what would the orange rod be?
1.0.217	Alan:	Umm.
1.0.218	T/R 1:	Good problem, that is a good one to ask. Okay, good problem.
1.0.219	Erik:	No, but what makes it?
1.0.220	Alan:	Nothing can divide twelve into thirds except
1.0.221	Erik:	Red.
1.0.222	Alan:	No. [He counts on the five red rods next to the orange rod] Two, four, six eight, ten. Ten divided into thirds. No, ten can't be divided into thirds.
1.0.223 0:44:45	Erik:	But nine can.
1.0.224	Alan:	Nine can, but there is no nine rod. Oh, yeah there is.
1.0.225	Erik:	Eleven, this is twelve though. [Alan holds up the orange rod.]
1.0.226	Alan:	No, it isn't, look [Alan counts on the five red rods next to the orange rod] Two, four, six, eight, ten. The orange rod is ten.
1.0.227	Erik:	Okay, ten. So that's ten, this must be nine. [He holds up a blue rod.] And this divided into thirds must be
1.0.228	Alan:	It takes
1.0.229	Erik:	Light green
1.0.230	Alan:	It takes green to divided the nine into thirds.
1.0.231	Erik:	Blue [the "nine" Alan is referring to]
1.0.232	Alan:	No, we are doing this one. I'm doing this one, the one I made up.
1.0.233	Erik:	[Simultaneously] I'm doing this one [the three light green and blue model]. Yeah.

1.0.234			*FRONT VIEW*
1.0.235			Jacquelyn, Kelly and Michael build the same model of ten white rods lined up next to an orange rod. No question is heard.
1.0.236		Meredith:	[to Sarah] How many colors would it take to make up this blue? And you can use different colors.
1.0.237		Sarah:	Everyone else will have different answers.
1.0.238		Meredith:	Maybe we can ask: How many different ways can you have [T/R 1 approaches]
1.0.239		T/R 1:	Meredith and Sarah, let's hear what you're going to ask.
1.0.240		Sarah:	How many different
1.0.241		Meredith:	How many different ways are there to find out how many colors it takes.
1.0.242		T/R 1:	Ok, that's an interesting problem but remember, we're talking about names, number names. We want to talk about number names for the colors. That's really a hard problem, I think we'll need a double period to solve that one, Meredith. We only have ten minutes. So I want you to make it hard, but not impossible.
1.0.243			*BOTH VIEWS*
1.0.244		T/R 1:	O.k. I'm ready to hear some questions because you can maybe keep thinking of more. Are you all ready to listen to the questions and we'll try to go around and if we don't finish you can finish tomorrow - try to remember what you've done? O.k are you all ready to listen. O.k. Alan has one for us, I'd like to hear Alan's, Alan you want to come up here and ask us and build it again?
1.0.245	0:46:13	Alan:	If one red rod was considered one fifth, what would a whole be considered?
1.0.246		T/R 1:	Okay, do you understand the question? One more time, ask the question. That's really a hard question.
1.0.247		Alan:	If the red rod would be one fifth, what would one, what would one be? [Alan gestures to Graham to respond.]
1.0.248		Erik:	Me?
1.0.249		T/R 1:	If the red rod was one fifth, what would we call one?
1.0.250		Graham:	The orange rod.
1.0.251		Alan:	Umm.
1.0.252		T/R 1:	Nice and loud, Erik.
1.0.253		Erik:	No, he called on Graham.
1.0.254		Graham:	The orange rod.
1.0.255		T/R 1:	Oh, Graham.
1.0.256		Graham:	The orange rod.
1.0.257		T/R 1:	Can you prove it?
1.0.258		Graham:	Five red ones make up an orange rod.
1.0.259		T/R 1:	O.k. what do you think? How many of you agree with that? That was a hard one. If I call the red one one fifth, what

would I call the orange? Ok Graham, but you didn't let it stump you. O.k. Beth has one for us. You want to come up here Beth? And Mark has one right after that. Why don't you come up too Mark and you can both ask together and help each other find them.

1.0.260 Next to present – Beth and Mark are partners

1.0.261 Beth: If a green was a whole, what would a blue be?

1.0.262 T/R 1: Did you all hear the question. If she calls the light green rod one, what number name will we give to blue?

1.0.263 Erik: three wholes.

1.0.264 T/R 1: Just say "three". Give it the number name three.

1.0.265 Mark: If blue was one whole, what would a light green be?

1.0.266 Jacquelyn: One third.

1.0.267 T/R 1: Why?

1.0.268 Jacquelyn: Put three light greens up to the blue. Each is one third.

1.0.269 T/R 1: Ok very nice, thank you Mark. Who has another one? A really really hard one? I think Jacquelyn and Kelly have one - do you have different ones or the same one? Try to stump us now. Uh oh, a hard one.

1.0.270 Jacquelyn and Kelly: If white one is one whole what would the orange be?

1.0.271 T/R 1: If we call the white rod one, what number name would we give to the orange?

1.0.272 Erik: Ten wholes, or ten.

1.0.273 T/R 1: How would you convince me?

1.0.274 Erik: Because if you took the orange block and you took white ones, you would need just only ten, well.

1.0.275 0:51:20 T/R 1: I have one for you. If I call the orange rod one, what number name would I give to the white rod? [groans] Oh, Michael has his hand up. You people have the number name for it. You understand the question, if I call the orange rod one, what number name would I give to the white rod? If you know it, raise your hand. Ok, Jacquelyn, you tell me.

1.0.276 Jacquelyn: One tenth.

1.0.277 T/R 1: How many think one tenth? Why one tenth? Ok, Jacquelyn?

1.0.278 Jacquelyn: Because if you line them up on the side and add them all up it will be ten.

1.0.279 T/R 1: So if you add one tenth up, how many times would you be adding one tenth? Ten times? Is that true, Sarah, is that what you were going to say? O.k., what a class. You have a real hard one, Erik? Let's hear from Meredith and then we'll hear from you. Meredith gave us one before and I said no that'll probably take a week and we only have a few minutes.

1.0.280 Meredith: If I call the purple is a half, what would a whole be?

1.0.281 T/R 1: Ok, if purple were one half, what would one be?

1.0.282 Amy: Brown.

1.0.283		T/R 1:	What would you do to convince us? You're really getting to know these!
1.0.284		Amy:	I would take I took the purple and tried to match up all the rods up to purple and brown, it was half way up to brown, so I took another purple and it was a whole.
1.0.285	0:54:00	T/R 1:	O.k. you are really very wonderful. We have about five more minutes where we can do some problem solving and then put these things away. This is what I would like you to do. I would like you to take a turn to make a problem that will challenge your partner, and to ask your partner that problem, and then make your partner convince you, and then you'll switch roles....
1.0.286			*SIDE VIEW*
1.0.287	0:54:52	Erik:	I have a problem. If a light green was one third, what would be a whole?
1.0.288		T/R 1:	What would one be?
1.0.289		Alan:	Blue
1.0.290		T/R 1:	Can you show it?
1.0.291	0:55:20	Alan:	[T/R 2 comes] If the white one was considered one fifth, what would be considered one? [He holds up a white rod.]
1.0.292		Erik:	What?
1.0.293		Alan:	If this was one fifth
1.0.294		T/R 2:	It’s a good one.
1.0.295		Alan:	What would be one?
1.0.296		Erik:	Yellow.
1.0.297		Alan:	Right.
1.0.298		T/R 2:	Are you going to let him get off that easily?
1.0.299		Alan:	He knows it anyway. [Erik starts to put five rods next to the yellow rod.]
1.0.300		T/R 2:	Just in case you couldn’t remember it in your head, you should always be able to go back and prove it.
1.0.301		Erik:	Umm, and also I did it, I just counted up five [Erik counts up one the staircase on his desk.]
1.0.302		T/R 2:	Oh.
1.0.303		Erik:	And I know that that’s half of [He points to the orange rod], and I know that yellow is half of orange, which is ten.
1.0.304		T/R 2:	Clever. So you’re using [She accidentally knocks Erik’s staircase], Oh, I’m sorry. [She straightens them]
1.0.305		Erik:	That’s okay.
1.0.306		T/R 2:	You’re using the staircase then to help you, so you don’t have to do all that. That’s very clever.
1.0.307		Erik	Should we think of another problem and do it?
1.0.308		T/R 2:	Yeah, why not?
1.0.309		Erik	Ok, let's see
1.0.310		T/R 2:	I have one for you.
1.0.311		Alan:	Okay.

1.0.312	0:56:20	T/R 2:	Okay, let me see. No, that's too easy, let's see. [She takes a purple rod.] Okay. If I call this two, what would one look like? Which rod would one be?
1.0.313		Erik:	That's two.
1.0.314		T/R 2:	That's two.
1.0.315		Erik:	Then one would be red.
1.0.316		Alan:	Umm.
1.0.317		T/R 2:	Okay, why?
1.0.318		Erik:	Well, you see if that's two [He points to the purple rod in his staircase] This would be a half of it and half of two is one.
1.0.319		T/R 2:	Umm.
1.0.320			[Alan puts a red rod next to the purple rod that T/R 2 has selected.]
1.0.321			And you take another one.
1.0.322		Erik:	[He puts another red rod next to Alan's red rod.]
1.0.323			I have one for you, Alan.
1.0.324		T/R 2:	Clever. Okay, go ahead. [She leaves them.]
1.0.325	0:57:00	Erik:	If this is three [He holds up a white rod], what is six? If this is three what is six?
1.0.326		Alan:	If that little thing is three, what is six?
1.0.327		Erik:	Yeah.
1.0.328		Alan:	This? [He holds up a light green rod.]
1.0.329		Erik:	This [He is shaking his head 'no' and holds up a white rod again.]
1.0.330		Alan:	No, if that, if that, if that
1.0.331		Erik:	This is three, what is six?
1.0.332		Alan:	If that was considered a
1.0.333		Erik:	Hold on, I've got to check. [He checks Alan's answer, while covering his model with one hand.]
1.0.334		Alan:	Three of something
1.0.335		Erik:	Oh, whoops, you were right. Sorry, sorry.
1.0.336		Alan:	All right now
1.0.337		Erik:	I was thinking it was that [He points to yellow in his staircase.]
1.0.338		Alan:	Let me bump you off with one.
1.0.339		Erik:	Like you can.
1.0.340		Alan:	If this [holding up purple] would be considered one half, what would be one?
1.0.341		Erik:	Probably. What is that - purple? A brown.
1.0.342			*FRONT VIEW*
1.0.343		Meredith:	[to Sarah] If you called one [holding white rod] a seventh. What would the whole be?
1.0.344		Sarah:	Brown.
1.0.345		Meredith:	Nope.
1.0.346		Sarah:	[She places a black rod on her desk, and counts white rods as she places them on top of the black rod] One, two… Black.

1.0.347	Meredith;	Yup.
1.0.348	Sarah:	If red is one third, what would one be? [T/R 1 joins the girls]
1.0.349	T/R 1:	I want to find a rod that has number name is one sixth. Can you find it?
1.0.350	Meredith:	Oh, I found it, I found it, I found it! Ok, I found it. Because I just said that black was seven.
1.0.351	Sarah:	Yeah.
1.0.352	Meredith:	We found it.
1.0.353	T/R 1:	What would one sixth be?
1.0.354	Meredith:	Green. Because I figured out-
1.0.355	T/R 1:	Green would be one sixth?
1.0.356	Meredith:	Yeah, because I found out, I asked her the question, if the one was considered one seventh
1.0.357	T/R 1:	I'm not convinced that the green is one sixth. What's one?
1.0.358	Meredith:	[lines up white rods against dark green] … five, six.
1.0.359	T/R 1:	I want to know what's one sixth. You told me that green is one sixth.
1.0.360	Meredith:	Yeah.
1.0.361	T/R 1:	Is that what you mean? Green is one sixth?
1.0.362	Meredith:	No the one is. The one's one sixth of green.
1.0.363	T/R 1:	Ok, so what number name, which rod has the number name one sixth?
1.0.364	Meredith:	The white.
1.0.365	T/R 1:	The white. You said green. That's how easy it is? You have to be so careful how you say it. So what number name is green?
1.0.366	Sarah:	One.
1.0.367	Meredith:	Six.
1.0.368	T/R 1:	Sarah said one. Now, is it one or one sixth?
1.0.369	Meredith:	Oh! one.
1.0.370	T/R 1:	Ok, don't change your mind so fast, Sarah, right? Sarah says one, ok. If I wanted the green to be six, if I wanted the green to have the number name six, what would white be?
1.0.371	Meredith:	One.
1.0.372	T/R 1:	You have to be very careful, you see? You have to go back and forth, ok?

1.0.373 59:00 S End of class.

Session 2, Sept. 21, 1993 (Front, Side, and OHP)

Line	Time	Speaker	Transcript
2.0.1	05:21 S	T/R 1:	What went on here yesterday? …Somebody want to help him [Tom Purdy] out? Thank you, Jessica.
2.0.2		Jessica:	Um. We did activities with rods and we um had to see like which rods were bigger and we had to…um, we did math problems with them.
2.0.3		T/R 1:	Okay. Somebody want to sum up a little bit more? Michael?
2.0.4		Michael:	Um. We, well, we, what we did is was, we called one 'one' and then we had to decide the littler one, what it would be called, one thirds, one fourth, or a half of that, the bigger block.
2.0.5		T/R 1:	You know, I don't want to embarrass Mr. Purdy, but you have to go very slow for him. He often needs an example.
2.0.6		Tom:	I need to see it.
2.0.7		T/R 1:	He needs to see it. He just… that's the way he learns. Can you help him a little better then, Michael? Maybe make up and example for him or somebody? We really need to help him out…Erik, you want to help while Michael is thinking of something else?
2.0.8		Erik:	Well, let's [He picks up a blue rod.] If we said that the blue rod would be one whole, um, we'd figure out what, we'd take all the blocks and try and figure out what would be half of it. [He holds the purple rod next to the blue rod.] And let's, I figured that the purple block would be half of it. So, well, no, not exactly, but…
2.0.9		T/R 1:	Mr. Purdy goes through the same thing, Erik.
2.0.10		Erik:	But if we call this one whole [holding up the blue rod], we'd figure out which block would be one half of it
2.0.11		Tom:	Uh huh.
2.0.12		Erik:	And which block would equal up the two blocks of… these two blocks of it, that would equal up to one of these we'd call that one half of the whole block. So, that's basically what we did.
2.0.13		T/R 1:	You're not going to help solve it for him? [to Tom]
2.0.14		Tom:	I was going to say, did you find it? Or
2.0.15		Erik:	Oh, oh.
2.0.16		Tom:	I mean I don't be- I mean, you're making me believe you can't do it.
2.0.17		Erik:	Well, yeah we did, but
2.0.18		Tom:	You're making me believe maybe you can't do it.
2.0.19		Erik:	No. We did find it. I just can't remember which one it was. [He holds up two dark green rods, end to end, next to the blue rod and discards them when he sees that two dark green rods are not equal in length to the blue rod.] I think it was

			the... [He measures two yellow rods, end to end, to the blue rod.]
2.0.20		T/R 1:	Maybe some of you can help Erik out.
2.0.21		Erik:	I think it was the dark green.
2.0.22		Tom:	You're saying the blue one is one [Children are working with the rods.]
2.0.23	S	Student	Try the yellow.
2.0.24	S	Alan:	The little green one was the thirds. The yellow was the half. No. The yellow is the halves of the orange one.
2.0.25	S	Erik:	I don't think there is one.
2.0.26		Tom:	I think you picked a good one.
2.0.27		T/R 1:	Erik, Erik. Suppose I wanted, suppose I wanted to call the yellow one "one half". Suppose I wanted to do that.
2.0.28		Student:	Found it! [Erik turns quickly as this is said – off camera view].
2.0.29		T/R 1:	But suppose I wanted to call the yellow rod, I wanted to give it a number name one half. Can you tell me what I would have to call one?
2.0.30	F	Meredith:	Oh, oh!
2.0.31		T/R 1:	I think you need to get your rods and build it for me.
2.0.32	F	Meredith:	Oh.
2.0.33		T/R 1:	If I wanted to call the yellow one half, can you show me
2.0.34	S	Alan:	Easy
2.0.35		T/R 1:	What would I have to call one?
2.0.36	S	Alan:	It's orange. It's easy. See? The orange one. [Erik is working with his rods.]
2.0.37	F	Meredith:	Oh! Yes I can, yes I can, yes I can. Oh!
2.0.38		T/R 1:	Brian, you want to tell Mr. Purdy?
2.0.39	0:08:38	Brian2:	Well, these two blocks equal up to this one whole. [He holds up two yellow rods in his left hand and an orange rod in his right hand.]
2.0.40		Tom:	Those two blocks equal up to one whole. So how much is each one? Each one of the yellows?
2.0.41		Brian2:	One half.
2.0.42	0:08:57	T/R 1:	So you are going to call the yellow one half? I'm still worried about Erik's problem. Erik wants to call this one [She holds up a blue rod] and Erik is trying to call something one half. Don't you want to help Erik out?
2.0.43	F	Meredith:	No.
2.0.44	S	Erik:	I don't think there is one.
2.0.45	S	Alan:	A little green makes a third out of that. Look I can do it
2.0.46		T/R 1:	If you call the dark blue one "one"
2.0.47		Alan:	One, two, three.
2.0.48	0:09:15	T/R 1:	David, what do you think? What does David think? I can't hear you, David. Hold on.
2.0.49		David:	I don't think that you can do

2.0.50		T/R 1:	Why, David? Slowly and loud.
2.0.51	0:09:25	David:	I don't think that you can do that because if you put two yellows that'd be too big, but then if you put two purples that's uh, that's uh, that'd be too short and
2.0.52		T/R 1:	What about something between purple and yellow?
2.0.53		David:	I don't think there is anything.
2.0.54		T/R 1:	Why not? [David pauses.]
2.0.55			Show us what you have there, David. Why do you think there isn't any? Cause I think you built it to show us. Can you show us your yellow and your purple?
2.0.56		David:	Well, I was thinking. Cause there's usually, the tall one… [inaudible]
2.0.57		T/R 1:	David, why don't you come up here and explain your reasoning. David doesn't think it's possible because Mr. Purdy said, "Well, maybe it's not possible." So let's, let's see. Let's help him out a little. Here's the two yellows and here's the two purples. What's, what's your reasoning? Let's listen to what David has to say.
2.0.58	0:10:20	David:	[F - Meredith builds some erect models on her desk as David explains] [He comes to the overhead and puts a blue rod onto it. He places a yellow rod and a purple rod, end to end, with one white rod - Figure O-10-33] All right. You see usually, um, they are only one, with the shorter one, only one block apart [Figure O-11-01]. Like that and so these, but then if you have for the blues, like if you have two yellows, it would be too tall and if you have two purples [He puts two yellow rods, end to end, next to the blue rod and then two purples next to another blue rod - Figure F-11-56]
2.0.59		T/R 1:	Do you need another purple? Here
2.0.60		David:	That'd be too short and then there's really nothing in between 'cause if you do [He builds a 'staircase' of rods, beginning with the longest, orange rod, then places blue, etc. until he reaches the shortest rod, the white one.] And then here [between the yellow and the purple rods], there's nothing in between, right here, so there's no way that you can do that.
2.0.61		T/R 1:	Are you all convinced? Jessica? Jessica has a question for you, David.
2.0.62	0:12:00	Jessica:	But if you put three greens to it you could
2.0.63	0:12:04	David:	Yeah, but Erik said, Erik wants the half. [inaudible]…'cause I figured that out, too.
2.0.64	0:12:11	Erik:	I think you could do it, but they're… See, I figure if you take a yellow and a purple it's equal [to the length of the blue rod - Figure S-12-24]. They're not exactly the same, but they're both halves. Because the purple would be half of this even though the yellow is bigger because if you put the purple on the bottom and the yellow on top it's equal, so they're both

			halves, but only one's bigger than the other. So it equals up to the same thing.
2.0.65		T/R 1:	Did you all hear what Erik said? Erik, do you want to say that one more time? How many heard what Erik said? How many would like to hear it a second time? Ok, Erik, would you say that one more time to David and the rest of us?
2.0.66	0:12:54	Erik:	If this would be one whole [the blue rod], you could take the yellow to be and you could call it one half [holding a yellow rod next to the blue]. But if you took another yellow it would be too big. But if you took a purple with the yellow, and put it on top of yellow, it equaled to the blue. So, the purple would be a half and the yellow would be a half, except that the yellow would just be one bigger than the other. Or maybe you could call this three quarters [holding the yellow rod] and you could call this one quarter [holding the purple rod]. And, but it would still equal up to the whole. [F - As Erik speaks, Jessica models the blue rod and the yellow and purple train at her desk]
2.0.67		T/R 1:	What do you think, David?
2.0.68		David:	I didn't think of that. [Erik chuckles. David places a yellow and a purple rod end to end, next to a blue rod.] Like that. Cause I was thinking that, um, that you would need the same.
2.0.69		T/R 1:	You think you would need the same?
2.0.70		David:	Yeah, but that might
2.0.71		Erik:	You don't really.
2.0.72		T/R 1:	You don't need the same? In other words, I could call this a half [the yellow rod] and I can call this a half [the purple rod]. Suppose this is a brick of gold and we're going to share it, Erik. And I'm going to take the yellow half and you get the purple half. Fair?
2.0.73		Erik:	Yeah.
2.0.74		T/R 1:	Do the rest of you agree? Do you like that? [Chorus of no's] Beth? No. Beth doesn't like that. Kimberly? Does it matter? Erik doesn't care. Do you care?
2.0.75		Erik:	Well, well I mean-
2.0.76		Kimberly:	Yes, cause the pink is, the purple is smaller than the yellow and the person who got the purple wouldn't have as much.
2.0.77		Erik:	Yeah, but you could call this three quarters and this one quarter and it would still be equal up to the whole. Then it, just wouldn't be halves, it would be quarters. But it would still look like you're dividing it into halves, but you're really dividing into quarters.
2.0.78	14:07 F	T/R 1:	What do you think, Brian?
2.0.79		Brian:	Well, you could, you could use say, if there, if there was three people – you could at least split it into thirds, you could at least split it into thirds.

2.0.80		T/R 1:	Is that, is that the question?
2.0.81		David:	Well, no. It's not. You see we're trying to do it in halves.
2.0.82		T/R 1:	[To Brian] We're trying to work on halves.
2.0.83		Brian:	Oh.
2.0.84		T/R 1:	Okay. Alan.
2.0.85	0:15:00	Alan:	When you're dividing things into halves, both halves have to be equal – in order to be considered a half.
2.0.86	0:15:09	Jessica:	[inaudible] this isn't a half. Those two aren't both even halves.
2.0.87		T/R 1:	Erik?
2.0.88		Erik:	Yeah?
2.0.89		T/R 1:	What do you think of that?
2.0.90		Erik:	Well
2.0.91		T/R 1:	Can you divide things in halves and have them different sizes? I think that's what Jessica is asking and Alan and David.
2.0.92		Erik:	Well, see. This isn't exactly dividing into halves. But I'm still using two blocks, but not… I'm dividing it in half still using two blocks, but one block is bigger than the other block. So it's like using three quarters and one quarter, but you're only using two blocks so it's almost like dividing it in half.
2.0.93		T/R 1:	Andrew? What do you think about that, Andrew?
2.0.94	0:15:43	Andrew:	Well if he's saying, he's saying that he wants a half, but if he puts that, a purple and a yellow, he won't have a half. He would have three quarters and one quarter. And he wants a half.
2.0.95		T/R 1:	It seems to me we have some differences here, don't we? Um. How many of you agree with Erik? [no hands are raised, children giggle] How many of you disagree with Erik? [all hands are raised, more giggling]. Hm, okay, what's the issue, do you think, here in the disagreement? Can somebody summarize the issue? Alan, do you want to try again?
2.0.96	0:16:22	Alan:	Um. You can't, if you're div, you can't divide that into halves, because you'd have to use rods that are of different sizes, but you could divide it into thirds using rods that are the same size which, which is the light green rods.
2.0.97		Erik:	But I didn't want thirds.
2.0.98		T/R 1:	[inaudible] can be very helpful to Mr. Purdy. Because I think, go ahead, David. What do you think?
2.0.99	0:16:45	David:	[at OHP, pointing to the rods on the OHP] I think that some of these that you can't do like this would be odd. [David moves the white rod to one side.] this could be even. [David begins a new group with the red rod.] This would be odd. [He moves the light green rod next to the white rod.] Be

		even. [He moves the purple rod next to the red rod. Continuing in this manner, he moves the yellow, black and blue rods next to the white and light green rods. He moves the dark green, brown and orange rods next to the red and purple rods.] This, be, you see, then when you get up to here, blue would be odd, but like with brown, you could take these two [He places two purple rods next to the brown rod.] and put them together and that would be even. Take the orange, put the yellows, with the orange and that would be even [He does this as he is speaking - Figure O-17-20].
2.0.100	T/R 1:	Okay, let me see, I think that we have. Maybe, Erik, the way we can resolve this is, I don't think I'm hearing you say, Erik, that you want to call yellow one half and purple one half. I don't, I don't hear you say that. You're not saying that, are you?
2.0.101	Erik:	No [agreeing that he is not saying that].
2.0.102	T/R 1:	You're saying that you agree with the rest of the class that if you call something one half of something
2.0.103	Erik:	Yeah
2.0.104	T/R 1:	They have to be the same size.
2.0.105	Erik:	Yeah, yeah.
2.0.106	T/R 1:	Right?
2.0.107	Erik:	Yeah.
2.0.108	T/R 1:	You are in essence answering a different question, maybe?
2.0.109	Erik:	Yeah.
2.0.110	T/R 1:	Where you were saying, "Well, if I call this one, there are other rods that make up one and maybe they're not the same size." I think you're very generous, Erik. Not as generous as Beth and Kimberly. And if we're talking about bricks of gold, letting me have the larger one if we're sharing one half. I, I really appreciate your generosity. I know Mr. Purdy wouldn't be so generous. Is that right, Mr. Purdy?
2.0.111	Tom:	That's right.
2.0.112	T/R 1:	That's right. But I do appreciate your generosity, so we'll have to talk later about some, some sharing. Um. We could go into business together, Erik. But I think that what we're saying from this is the point that David is making and Alan and some of you have expressed very nicely, that if we are calling a rod one half, okay, if we call a rod one half, of, let's say, a rod that we called one, was given a number rod one, there are two conditions that have to be satisfied. Can you tell me what those conditions are? And I think one more time as a summary because you're saying that purple could not be considered one half because one of the conditions isn't met, right? I mean, they're both [two purple rods] the same size.

2.0.113		David:	Um, hm. But they don't, um, if you put like that [He puts two purple rods together.], they don't, uh, they're not as big as the blue.
2.0.114		T/R 1:	Do you agree? Do you all see the second condition that's not met? See the space in here? Or if you can put them like this, see that space? And I think that David has made another very powerful, interesting argument that I'd like you to think about. He claims that that's missing, right? And that there couldn't be another rod in between to do it, right? That's interesting, now, you know, suppose you had to manufacture these rods and make another color. Okay? Here we have a purple rod that's too small, right? To qualify to be a half. Do you agree the purple's too small? And here we have a yellow one, right? That's too big, right? To qualify, do you see that? If you were designing a new set of rods and you wanted to call the blue rod one, okay? Can you tell me what that new rod might look like so that you would be able to call it a half? [pause] Do you understand my question? We have rods here with ten, we have ten colors, don't we? You told me that yesterday.
2.0.115		Students:	Yeah.
2.0.116	0:20:29	T/R 1:	Right? And you all told me that if I wanted to call blue one in terms of the box you have, right? You can't find a rod that you could give a number name one half. Isn't that what you all told me? [mumbles of agreement] That's a problem. Because, um, there's another school that wants to have rods where they want to call blue one and have another rod that they can give a number name one half. Okay? Now can you tell me what the design of that rod might begin to look like? Why don't you talk to your neighbor and think about that problem? Do you understand my problem?
2.0.117		Students:	Yeah.
2.0.118	0:21:06	T/R 1:	We know it can't be purple and we know it can't be yellow. What do you think, David? You've convinced me that's there's nothing in between. [T/R 1 and David confer at the OHP. Their conversation cannot be heard.]
2.0.119			*SIDE VIEW*
2.0.120			[Erik and Alan as partners begin immediately:]
2.0.121	0:21:10	Erik:	It can't be anything 'cause you can't divide nine equally. You see if this is
2.0.122		Alan:	If you could
2.0.123		Erik:	No you can't. This is ten.
2.0.124		Alan:	If you could make a rod.
2.0.125		Erik:	If this is ten [the orange rod], then this [the blue rod] is nine. It's impossible to divide this evenly.

2.0.126		Alan:	Different rods. You might be able to, like if you divide a blue rod in half you could that that length and make a new color and that would equal up to halves. Which would mean it would be like [noise]
2.0.127		Erik:	It's impossible. You can't divide it in half. You can't divide it in half, Alan.
2.0.128		Alan:	Right, you could divide it in half if you had [inaudible] parts. You could divide it in half, but having equal parts, but you couldn't have equal numbers.
2.0.129		Erik:	[inaudible]
2.0.130		Alan:	If you cut this [the blue rod] down the middle, it would be four and a half, [inaudible] the same length.
2.0.131		Erik:	Four and a half. You can't make a rod that's four and a half.
2.0.132		Alan:	Um, hm. So you can't divide into anything.
2.0.133		Erik:	Except thirds.
2.0.134		Alan:	Except thirds. Or, or singles.
2.0.135	0:22:12	Erik:	You can't divide it into halves. "Cause I put this up here and there are nine of these and one, two, three, four, five. One, two, three, four [pause] four, one two, there four five. One, two, three, four. One, two, three, four. One, two, three, four, five [He is counting the two groups of white rods next to the blue rod - Figure S-22-24].
2.0.136		Alan:	Over here you have thirds.
2.0.137		Erik:	You can divide it into thirds, but you can't divide it into halves.
2.0.138		Alan:	You can divide it into thirds. You can divide it into ninths.
2.0.139		Erik:	But you can't divide it into halves.
2.0.140		Alan:	You can't divide it into anything else but thirds and ninths.
2.0.141		Erik:	Exactly, you're right.
2.0.142		Alan:	Just thirds or ninths. That's all you can do. That's productive reasoning.
2.0.143		Erik:	What?
2.0.144		Alan:	Productive reasoning. So there can be only thirds and ninths. And the umm singular rods. And you can't divide it into halves.
2.0.145		Erik:	Exactly. It's impossible to divide it in halves.
2.0.146		Alan:	That can't be done.
2.0.147		Erik:	It's impossible, Alan. You can't divide it into halves.
2.0.148		Alan:	It's been proven.
2.0.149		Erik:	Exactly. [noise]
2.0.150	0:23:30	Alan:	Mind handing some over my way? All right um, what I'm going to do right now is make out of everything, I'm going to halve or third every color,
2.0.151		Erik:	You can't halve every color, you can third every color.
2.0.152		Alan:	[singing] I can third every color. I can halve every color.
2.0.153		Erik:	Except blue.

2.0.154		Alan:	You can third.
2.0.155		Erik:	You can third. You can third.
2.0.156		Alan:	And ninth.
2.0.157		Erik:	and ninth.
2.0.158		Alan:	Now black.
2.0.159			*FRONT VIEW*
2.0.160			[Brian G. and Jacqueline built staircases independently.]
2.0.161			[Sarah and Meredith sat quietly; T/R 2 approached the partners.]
2.0.162		T/R 2:	You two seem very quiet over here. What do you two think about this? What would the new rod you design be like? Do you understand the question? Sarah, do you understand the question? [Sarah nods her head] What are you being asked to do?
2.0.163		Sarah:	I know. [to Meredith] Let's see if you do.
2.0.164		Meredith:	You don't know.
2.0.165		Sarah:	Yes I do. Can you do it?
2.0.166		Meredith:	Let's see you do it.
2.0.167		T/R 2:	My am I getting the feeling neither of you are going to be able to tell me. [Sarah and Meredith laugh]
2.0.168		Sarah:	We aren't good talkers
2.0.169		T/R 2:	Oh, yes you are.
2.0.170		Meredith:	She's asking us to find a rod that will make up a whole, that will make blue. Find one that will fit.
2.0.171		T/R 2:	Ok, so,
2.0.172		Sarah:	Inaudible
2.0.173		T/R 2:	That's uneven, ok, so you think if we were, if we were able to go into a workshop with wood and build a new rod, what would that rod look like? If we could go and build and make a rod any way we'd like, what would that rod look like? How would you describe it? Would you describe it in terms of these other rods here?
2.0.174		Meredith:	This [orange] is ten, this [blue] is nine. And if you split this in half, it would have to be four and a half and four and a half.
2.0.175		T/R 2:	Alright. Why is that?
2.0.176		Meredith:	Because half of nine is four and a half.
2.0.177		T/R 2:	Ok. So…
2.0.178		Meredith:	And then, and you would have to make one, say, you had four, and then you make, this is a four and this is a four [purple rods]. You'd have to have a half on this four and a half on this four. [model is not in camera view] [T/R 1 starts talking, T/R 2 and Meredith speak but inaudible]
2.0.179			*WHOLE CLASS*
2.0.180	0:24:10	T/R 1:	Okay, I'd like us, if you don't mind, if we can stop for a minute and I'm going to ask Beth, Graham, and Jackie to

			come up and pose their solutions. I heard a few of your solutions. I know David has a solution I heard already, up front. I'd like to hear some other possible solutions. You can clear off there [the OHP] what you don't need.
2.0.181	F	T/R 2:	What do you think, Sarah? [Sarah nods yes] I'm anxious I want to make sure you share that.
2.0.182	F	Meredith:	You don't need the brown, all you need is
2.0.183		Beth:	I got it, right here.
2.0.184	0:25:00	Jackie:	[Places purple then white then purple rods in a line on OHP - Figure O-27-44]. We thought that to make a new rod we would make, um, we would cut this white one in half and attach it
2.0.185		T/R 1:	Could you speak nice and loud? Cause I'm a student back here and I can't hear you. Do you want to try and talk really loud?
2.0.186	0:25:44	Jackie:	We thought of, to cut the white one in half and add it to one rod [purple] and then add it to the other rod [purple]. And we thought the color would be light pink.
2.0.187		Graham:	[To Jackie] And the smallest one would be a half 'cause it was the white one.
2.0.188		Jackie:	And the smallest one would be a half 'cause it was the white one.
2.0.189		T/R 1:	Did you all hear what they said? No, they, Kimberly didn't hear you, dear.
2.0.190		Jackie:	We thought to cut the white one
2.0.191		T/R 1:	You can come in front and talk nice and loud, I know you can Graham.
2.0.192		Jackie:	We could cut the white one in half and add it to the purple rod and add one one half to one purple rod and the other to the other one and we thought that we could call the color light pink.
2.0.193		T/R 1:	And you said something else, what would your smallest rod be?
2.0.194		Jackie:	Oh, yeah. Our smallest rod would be half of the white one.
2.0.195		T/R 1:	What are you going to call that? [some giggling] You're the designers. What are you going, it's not going to be white, what do you think? You want to help them out? You could have other consultants to this design. Why don't you call on someone for help and consulting? Graham? Beth?
2.0.196		Graham:	We cut the clear one [the OHP version of the white rod] in half to like make this. Then you'd, then you would have to cut like a reg- a regular one in half to be your smallest one [F - model on Brian's desk - staircase, with the top filled in by white rods, Figure F-26-18]
2.0.197	F	Meredith:	[whispers to Sarah] They took my answer.
2.0.198		T/R 1:	I see some hands up. Why don't you see if…?

2.0.199	0:26:55	Michael:	If you're going to make a new rod, then you'd have to make a whole new set because there'd have to be a half of that rod, too.
2.0.200		T/R 1:	What do you think, Graham? What do the rest of you think? Do you think there would have to be a whole new set? There are some other people who have opinions. Why don't you go, who's going to, why don't you take it, Jackie? You call on people, okay?
2.0.201		Jackie:	Um, Brian.
2.0.202	0:27:27	Brian:	No matter what, there'll always be something
2.0.203		T/R 1:	Nice and loud, Brian, I can't hear you.
2.0.204	0:27:32	Brian:	No matter what there'll always be something that won't be equal to something, like
2.0.205		T/R 1:	Can you say a little more about that, Brian? Nice and loud.
2.0.206	0:27:45	Brian:	If you cut these little ones in half, then there wouldn't be something for the little ones to make a half out of them.
2.0.207			[laughter]
2.0.208		T/R 1:	Did you all hear what Brian said? That's, Brian, I, we didn't hear back here. Kimberly and I are trying hard. Can you turn around and say it nice and loud?
2.0.209	0:28:05	Brian:	If you cut one of these in half then there wouldn't be a half of the litt, of the ones that you, of the halves of these.
2.0.210		T/R 1:	What do you think about that? David, you had your hand up. Do you have a different point?
2.0.211	0:28:24	David:	Well, before I told you. I thought that, uh, to cut it in half, too, but then I realized that, uh, that you would have to make a whole set.
2.0.212		T/R 1:	Yeah.
2.0.213		David:	And make a half for every one.
2.0.214		T/R 1:	Okay, that's what we heard, um, Michael tell us. Meredith has something to say to the group.
2.0.215	0:28:43	Meredith:	Well, you could just, if you do that then you'd have to cut the ones that are separate, the little blocks into halves, all of them, so then you could make it equal.
2.0.216		T/R 1:	What do you think, Jacquelyn?
2.0.217		Jacquelyn:	Um, it, I agree with Michael. 'Cause if you do that, um, it changes the whole pattern 'cause this has a set in pattern to it and the whole thing would change.
2.0.218		T/R 1:	It's an interesting question, isn't it? It's an interesting question. So in other words, when you designed a solution, you're telling me, for the problem where you're making now a pink rod, is that what you're calling it?
2.0.219		Jackie:	Yes.
2.0.220		T/R 1:	You're creating a pink rod. And as I understand it, the pink rod is made up of purple and half a white. Is that what you said? Um. You solved the problem of having a rod that you

			can call one half when you call the blue rod one, right? But then, as some of you pointed out, then your smallest rod is then, with this new design, your smallest rod is then,
2.0.221		Meredith:	Half
2.0.222		T/R 1:	Half of the
2.0.223		Meredith:	White
2.0.224		T/R 1:	White rod, right? And what are you going to call that? Let's give that a name. Let's give that a name. Can you give that a name? It's not white any more. It's half of white. What color name shall we give it?
2.0.225		Jackie:	Light blue.
2.0.226		T/R 1:	Pardon? Light blue? Okay, so your smallest rod is going to be light blue. But I heard some other people say, like Brian in particular, and some others, Meredith, that, okay, you've solved that problem, but you could expect new problems. Yeah. That's interesting. Well, that's something to think about. You did a really nice job. Did anybody have another way to make the argument? James? [James goes to OHP]
2.0.227	F		[Sarah has a model of P-W-P built on her desk.]
2.0.228	0:30:43	James:	Well, I thought that if you had a blue rod as one, you could take light green, imagine there are two others here. Then you could split the middle one in half and you could call that a light blue rod.
2.0.229		T/R 1:	Is that okay? That's another way, huh? Does anybody have another way? …Do you think there's still another way?
2.0.230		Meredith:	Mmm hmm.
2.0.231		T/R 1:	Do you think there are other ways than this? That's really good? Now, Mr. Purdy, did that help you?
2.0.232		Tom:	Yes.
2.0.233		T/R 1:	That's great. Ok, well that was very very helpful. What do you think, Dr. Martino? You want to give them some more problems? [Dr. Martino says something] You want to do some more? She's going to challenge you with some other ones. Um, uh oh, here we go, Mr. Purdy pay attention, because she's going to really challenge you.
2.0.234		T/R 2:	Alright. Let's try something a little different now. Ok. Now, if we call the orange "two", what can we say about yellow? Think about it for a minute, and you want to talk to your partner?
2.0.235			*SIDE VIEW*
2.0.236			[Erik and Alan discuss, but conversation inaudible.]
2.0.237	0:32:54		[CT asked Danielle to explain. She said "one". CT noticed that Danielle's partner, Gregory, was frowning.]
2.0.238		CT:	He's not convinced; he's a little shaky. Can you explain to him why you'd call it one?

2.0.239		Danielle:	Because if this is two [yellow rods next to an orange rod]. Then these [yellow rods] should be one. Because see if these are two, there's two of them. We call that [orange rod] two. We call these one [yellow rods] because this [orange rod] is called two.
2.0.240		Gregory:	When the orange is one, we went like a half down
2.0.241		CT:	[To Gregory] Is this [orange rod] still one?
2.0.242		Gregory:	No, it's two.
2.0.243		CT:	So then
2.0.244		Gregory:	Yeah, one.
2.0.245			*FRONT VIEW*
2.0.246		Meredith:	You're using all the yellows
2.0.247			[Kimberly raises her hand. She has built a model of two yellow rods under the orange rod. Sarah raises her hand. Meredith returns to her desk]
2.0.248		Sarah:	I have them. I only have two! [Meredith goes to back of room].
2.0.249		Meredith:	Oh! She called orange two. One half? Two? Then this would have to be one.
2.0.250			*WHOLE CLASS*
2.0.251	0:33:51	T/R 2:	[Class called together by T/R 2] Ok. I'm anxious to hear some answers to this, hear what people have come up with. I hear, I hear a couple of different things here and I think that's something- let's see if we can get some answers up here and discuss them. Uh, let's see. Who haven't we heard from? Let's see. Brian, what do you think, now when we call this, we give this the number name two, the orange, what number name are we going to give to yellow?
2.0.252		Brian:	One.
2.0.253		T/R 2:	Why one? You want to come up here. You can come up here and show us. [Brian goes to overhead.]
2.0.254		Brian:	You would put two yellows together and it would be the same size as that, and even if and that's like having, so if this [orange rod], is considered a two. Then those two [yellow rods] would be considered like a regular orange, so it would be considered a one.
2.0.255		T/R 2:	Okay, so you'd consider each of these [yellow rods] a one, is that what you're saying?
2.0.256		Brian:	No, that like together they would equal the same as that [orange] so it would be a one.
2.0.257		T/R 2:	O.k. So the number name you're giving yellow then was what?
2.0.258		Brian:	One, one.
2.0.259		T/R 2:	Okay, alright, one. What do you think about that? Does anyone want to come- Who agrees with that, that you give

			the yellow the number name one? Ok. Does anybody disagree with that? I heard, I heard some-
2.0.260	0:34:53	Erik:	I have another name. You can call it another name.
2.0.261		T/R 2:	Ok, what would you call it, Erik?
2.0.262		Erik:	Well, see, do you have to call the orange two?
2.0.263		T/R 2:	Well, I've arbitrarily picked that I'm calling the orange two.
2.0.264		Erik:	Well you could call it one, and if you call it one, then two yellows would be a half. If that would be considered, if the orange would be considered two, then you'd call those [yellow rods] one. But if you can call it [orange] one, you could call those [yellows] halves.
2.0.265		TR/ 2:	That's interesting, what if I call the orange…uh
2.0.266		Brian:	[at overhead]. There might be other ways. You can split them, you can maybe split it into thirds, and call that a one but we don't have enough thirds-
2.0.267	0:35:38	T/R 2:	Okay, yeah, you probably could.…Let me ask you another question then, I'm going to ask this to everybody, too. What if I change the name of the orange to six. What would I call the yellow- what number name would I call the yellow? Let's see, uh, somebody I haven't had a chance to talk with, James, is your hand up? Kimberly?
2.0.268		Kimberly:	Five.
2.0.269		T/R 2:	Five. That's interesting. Can you come up and tell us about that? [Kim goes to the overhead.]
2.0.270		Kimberly:	Look here[pointing to Brian's model] before you said that [the orange rod] would equal two, and then Brian said that [yellow rod] would equal one. So now you're saying that that [orange rod] equals six, so I figured that if that equaled one before [yellow rod] it would equal five now. [F - Sarah has built a model on her desk 36:10]
2.0.373	0:36:54	T/R 2:	That's interesting. What do you think about that, some of these other folks? Did you all hear Kimberly's argument here? She's saying when you call this one, the number name two, the orange, that the yellows were each one, ok, they had the number name of one. She's saying, so if I call this six now, she'd call that five. What do you think? [Meredith and others shake their heads negatively.] Ok, I see some people are shaking their heads and I want to hear why. Uh, let's see. Alan?
2.0.272	36:25 F	Alan:	[Goes to the overhead] You said that the orange rod was six. And before you said that this was two and this [yellow rod] was one. So now if you're calling this [orange rod] six, and half of six would be three. So that's
2.0.273		T/R 2:	Okay. So we have another argument. What do you all think about Alan's argument? He's calling this [yellow rod] three,

			the number name three when I call this [orange rod] the number name six? Meredith?
2.0.274		Student:	Yeah.
2.0.275		Meredith:	[inaudible]
2.0.276		T/R2:	You agree with that? Jessica?
2.0.277		Jessica	I agree with him because like half of six is three so that would
2.0.278		T/R 1:	I'm curious how Kimberley thought of five? Can you help me understand why you think five?
2.0.279		Kimberly:	Well, before you said that was two, the orange was two, and the yellow was one. So now you're saying it's six, so the yellow could be five.
2.0.280		T/R 1:	The yellow is five... That's where I am confused. So you're saying if this [yellow rod] is five and this [yellow rod] is five, this [orange rod] is six?
2.0.281		Kimberly:	Ok, I made a mistake, I-
2.0.282		T/R 1:	You didn't mean that? What did you mean, Kimberly?
2.0.283		Kimberly:	Well, I made the mistake. I figured it out now.
2.0.284		T/R 1:	Tell me what you were thinking. I'm curious about what you were thinking.
2.0.285		T/R 2:	That's what I want to know.
2.0.286		Kimberly:	I made the mistake thinking from before, I forgot that adding one and one is two, but five and five isn't six, so, I made that mistake.
2.0.287		T/R 1:	If you want this to be five, what would you have to call the orange?
2.0.288		Kimberly:	Ten.
2.0.289		T/R 1:	You'd have to call orange ten. Do you agree with that? [students: Yeah] What a class! You're going to have trouble stumping them, Dr. Martino.
2.0.290	0:38:33	T/R 2:	I know, this is tough! Okay, let's try another one. Umm, okay if we call [long pause]let's see …
2.0.291		T/R 1:	I think we're going to have to consult to give you a problem hard enough. You're just getting too good for us and… how about this one? Suppose we made a train, ok, I'm going to take Erik's idea from earlier, and I'm going to call the orange and [light] green together, one. You like that? I'm calling this one. The orange and green train together, one. Now I didn't work this problem, but, I'm curious; can you find a rod that has the number name one half?
2.0.292			*SIDE VIEW*
2.0.293	0:39:33	Alan:	Erik, look! Erik, look, this is the biggest I can find, you see?
2.0.294		Erik:	I'm trying to figure it out.
2.0.295		Alan:	You'd have to have this
2.0.296		Erik:	I'm trying to figure it out.
2.0.297		Alan:	There's no way to call something half [in this train].

2.0.298		Erik:	How do you know?
2.0.299	0:40:00	Brian:	[Brian G and Erin as partners. Brian raised his hand and T/R 2 joined them. Brian built the following model: O-LG train with G-W-G train directly beneath.] It's like the other problem we had. You split that [white rod] in half and then put one side on one side [green rod] and then take the other half on the other side [of the other green rod]. It's like what we did last time.
2.0.300		T/R 2:	There was no rod that worked perfectly when you take two of them?
2.0.301		Brian:	No, because ten and three equals thirteen and thirteen is an odd number.
2.0.302		T/R 2	What does that have to do with it?
2.0.303		Brian:	Well, with thirteen, you can't split thirteen in half equally. Except you take a one and you split it in half and you put one side on, you put one half on one side and you put the other half on the other side, like what we did last time.
2.0.304		T/R 2:	Oh, that's interesting.
2.0.305		Brian:	And you would change the color.
2.0.306		T/R 2:	Okay, so we are going to develop a new rod again. We'd have to go back to the workshop and make a new rod. Ok. This is wonderful. What do you think, Erin? Do you agree with what Brian said? Have you checked and made sure that there aren't any pairs of the same that would fit here? [Erin nods affirmatively.]
2.0.307	0:41:00	Brian:	You could probably do it another way. That's what James did and I thought it would probably work again. Maybe it would work, it would probably work. When he was using the blue with the nine, he was using these others [light green rods], so I thought [He places four light green rods (12cm) under the train of orange and light green (13 cm).] No, no. Oh yeah you could do this like we just did. [He places one white rod between the light green rods. His train is LG-LG-W-LG-LG] Yeah, I think so, yeah.
2.0.308		T/R 2:	Okay, so show me where one half would be. One half of that [orange and light green] train.
2.0.309		Brian:	Well, right there [he points to the white rod] would be the half of one.
2.0.310		T/R 2:	Down the center there.
2.0.311		Brian:	Yeah.
2.0.312		T/R2:	Nice thinking, Brian. Let me see what some other people have come up with.
2.0.313	0:41:50	Jessica:	[T/R 2 left Brian and Erin and joined Laura and Jessica. T/R 2 questioned the girls about Jessica's model, a train of G-G-W beneath the train of O-LG] …for one half. So you'd had

			to like invent a new rod. So like, here's the dark greens. And you'd have to, that doesn't work, so you'd have to put a white.
2.0.314		T/R 2:	Ok, so what would one half of this green and orange train be? How much of this, in other words? Can you show me?
2.0.315		Jessica:	[Stacking one green rod on top of the other: G then G-W on bottom]. Well
2.0.316		T/R 2:	So what do you think, Laura? Do you know what I'm asking her? I want to be able to see the one half in my head.
2.0.317		Jessica:	This [holding up a train of green and white rods] would be one half.
2.0.318		T/R 2:	Okay, that [green rod] and the white?
2.0.319		Jessica:	Well, it's sort of in thirds, but if you, if you like say if this [orange and light green train] was one, then this [green-green-white train] would be two. And you have to like pretend that this [G-G-W train] was one whole right here.
2.0.320		T/R 2:	What do you think, Laura?
2.0.321		Laura:	Well, I think that like one of these and half of this one [white] would be half.
2.0.322		T/R 2:	Okay, so
2.0.323		Jessica:	Yeah, half of the white.
2.0.324		T/R 2:	Okay, so if I imagine that I had a saw, a small saw and I could cut that [white rod] in half, then you'd take a green and the half of that.
2.0.325		Jessica:	Yeah, so then half of this and this would be one half and half that would be the other half, and that would be one half.
2.0.326			*FRONT VIEW*
2.0.327		Meredith:	It's thirteen. Do we have a seven? What's seven?
2.0.328		Sarah:	Green doesn't work.
2.0.329		Meredith:	Oh, ok. This is easy. One, we need a six. Six
2.0.330		Sarah:	Blue [picks up blue rod, puts it down, inaudible]
2.0.331		Meredith:	Green, no, purple, I think purple.
2.0.332	40:18 F	Sarah:	No purple's right here [pointing to her staircase, Figure F-40-18]. One, two three four five six [pointing to brown rod].
2.0.333		Meredith:	It's not a six, see watch. This is a ten, right? Ten, nine eight. This is eight, two less than this [holding a brown rod]. That's one less
2.0.334		Sarah:	I know that.
2.0.335		Meredith:	Yellow
2.0.336		Sarah	No, yellow won't work.
2.0.337		Meredith:	Yeah? Oh, yeah, um.
2.0.338		Sarah:	Oh
2.0.339		Meredith:	Oh yeah, it will.
2.0.340		Sarah:	No it won't
2.0.341		Meredith:	Give me back the yellow
2.0.342		Sarah:	No it won't!

2.0.343	41:08	Meredith:	I’ll show you. I’ll prove it to you! Yellow. Watch. [She puts Y-W-Y under the train of O-LG - Figure F-41-08. Sarah chuckles. Meredith dismantles her model.] What’s highest, the next highest after yellow?
2.0.344		Sarah:	No no no, you took my thing apart.
2.0.345		Meredith:	It’s not that, it was purple, then red, this
2.0.346		Sarah:	No green.
2.0.347		Meredith:	This
2.0.348		Sarah:	No we already tried that.
2.0.349		Meredith:	Dark green.
2.0.350		Sarah:	Dark green doesn’t work.
2.0.351		Meredith:	Who said it doesn’t. Watch.
2.0.352		Sarah:	I tried it.
2.0.353	41:53 F	Meredith:	Yes it does, remember halves? [Meredith’s model: DG-W-DG. Figure F-41-53]
2.0.354		Sarah:	Yes, I do. [Both girls raise their hands.]
2.0.355		Meredith:	Oh, oh! It works. [T/R 2 joins them.]
2.0.356			*BOTH VIEWS*
2.0.357	0:43:27	T/R 2:	Can I join you? Have you come up with an idea?
2.0.358	0:43:33	Meredith:	Yeah, since, I didn’t end up doing halves with the whites, I took two greens and I put a white in the middle, and if I cut the white in the middle in half, then you would have six and a half and six and a half.
2.0.359		Sarah:	And cut that up.
2.0.360		Meredith:	And then you would have it.
2.0.361		T/R 2:	Okay, so you’re telling me, I see what you’re doing. Ok. You're going to cut this right down the middle to give me a half.
2.0.362			[To the class] Ok. I want to hear from some people now. I hear some wonderful thinking here. This was not an easy one. Ok, is there somebody who would like to share a solution with me to show me one half of this train which has an orange and a green? To show me something that is one half as long as this orange and green train. Ok, let’s see, uh, Andrew? Could you come up and show? If you worked with Mark the two of you can come up and show us? I'd like everybody to watch what they do because if you have a different way of thinking about it, I'd like to see that also.
2.0.363	0:44:41	Andrew:	Well we thought that if we had to find one half of that, we took two dark greens and the white one. And we said if we split the white one in half, then it would be half, because if you put the white one there, it would equal up the the train that you made. [Andrew places a train of green, white, and green on OHP next to orange and green - Figure O-46-42]
2.0.364		Meredith:	[whispering] Quick, give me the yellow, I need the yellow. I have a solution, I have a solution.

2.0.365		T/R 2:	Ok, do you all follow what Andrew said here? Erik, did you have a comment on that? You had something different. First of all, what do you think of this? Does this work? Looks like we're into inventing our own rods again, right? Making up a rod, a new rod here. Why do you think that works? I mean, why do you think that that works? You have any ideas? Meredith?
2.0.366	0:45:44	Meredith:	Well, because you want to have seven and six, seven, but there are no rods that are really seven, and you need it to be thirteen. So, those two blocks and half of that would equal up to it, and it would help-
2.0.367		T/R 2:	Ok, can you say a little bit more about that?
2.0.368		Meredith:	Well, take the two greens and take a white. And you do that.
2.0.369		T/R 2:	So you're showing the two greens and the white that are up here. Ok, it's just like our picture up here
2.0.370		Meredith:	And there's no blocks that have half on them, and for the uneven numbers, for the odd numbers you need a half, because you can't make it without it.
2.0.371	0:46:40	T/R 2:	Ok, Brian said something like that too, about the numbers being odd. Brian, what did you want to add?
2.0.372		Brian:	Well, like what we did last time with, when Mrs. Maher was talking about, about if we split the gold equally, what you could do is, well, I thought of a lot of ways. So like, once I have the white cube right in the middle, you split that in half, right in the middle. That's what we did last time.
2.0.373		T/R 2:	Great! Ok, well you came up with several different ways. I see one of the ways that Brian has is, he used light greens, all light greens and one white, right? Ok. That would be another way to do it, wouldn't it? That's really very nice. [Brian built 5 models for the orange and light green train: G-W-G, LG-LG-W-LG-LG, R-R-R-W-R-R-R, P-Y-P - Figure S-47-12] Ok you two can take a seat. Does anybody else have anything they want to add to this before I begin a new problem? [Sarah and Meredith raise their hands] Ok, let's see, Sarah? [Sarah and Meredith go to the overhead.] You can take these off if you want to. Oh, you need another light green, you know I don't think we have any for the overhead, so maybe we could just use one of the regular - why don't we try those, ok? [talk about the rod looking black on the overhead] Well, we can pretend that it's a light green, can't we? Ok, go ahead.
2.0.374		Meredith:	[She builds a train of Y-LG-Y - Figure O-48-31] If that's a light green, then you could just make a yellow and add one and a half to the yellow and one and a half to the other yellow.
2.0.375	0:48:32	T/R 2:	What do you mean, one and a half? Does anyone know what Meredith means? I don't want you to tell me yet. Does

			anyone knows what Meredith means by adding one and a half to the yellow on each side? Where did she get one and a half from? I see a couple of hands, let's see. Graham.
2.0.376		Graham:	The light green
2.0.377		T/R 2:	The light green, ok. How does this become one and a half? What piece of it [the train] becomes one and a half? I don't understand.
2.0.378		Graham:	You like split it in the middle, and it would be one and a half on each side. [He holds up the light green rod and shows cutting through the middle of it.]
2.0.379	0:49:08	T/R 2:	Oh, okay, all right. So if I cut that [Light green rod] down the middle, I see, okay. Well, if we're calling this light green three, what are you calling this train with the light green and the orange together?
2.0.380		Meredith:	Well, the yellow is I think the yellow's um, I think yellow is about five long, and the green in the middle [Counting cm in the train] Ten [two yellow rods], eleven, twelve, and then thirteen [for the light green rod], thirteen yellows.
2.0.381		T/R 2:	You were thinking of the whole length of the train as being thirteen of what?
2.0.382		Meredith:	Thirteen
2.0.383		T/R 2:	Thirteen blues, thirteen oranges, thirteen what?
2.0.384		Meredith:	Thirteen yellows.
2.0.385		T/R 2:	Thirteen yellows?
2.0.386		Meredith:	If you turn the light green into yellows.
2.0.387		T/R 2:	I don't understand.
2.0.388		Meredith:	Well, if you cut that [light green rod] in the middle and then you just paint the light green of each piece yellow and you're making it thirteen and it will be equal to the train.
2.0.389		T/R 2:	Do you understand my question, though? She keeps saying thirteen for the train that I made with the orange and the green. I don't understand where she's getting the number thirteen from. Why thirteen?
2.0.390		Erik:	Wait, she's getting thirteen from the number of the whole train?
2.0.391		T/R 2:	Well she keeps saying that the length of this is thirteen.
2.0.392		Erik:	Yeah, I know, I know where she's getting it.
2.0.393	0:50:47	Erik:	Well, see, If you take one of the orange rods and take all these little things [white rods] and you put it up to it, it will equal ten. And then if you do the same thing with the light green rod, it'll equal three. And if you have ten and three it's thirteen [As Erik speaks, Meredith lines three white rods on top of the green rod - Figure O-51-14].
2.0.394		T/R 2:	Oh! So then what you're saying is if you line up the little white cubes along the, uh, the train with the orange and the

			green there'd be thirteen of them? I understand, ok, I understand what you're saying, that's wonderful.
2.0.395		Erik:	Yeah, thirteen
2.0.396		T/R 2:	Do we have a minute to do another one, or do we have to clean up.
2.0.397	0:51:45	Erik:	I have another solution. [Some talk by T/R 2… He goes to the overhead and puts two light green rods under the orange and light green train. He adds seven white rods to the right of the light green rods - Figure O-53-00]. I figured you could take two light greens and put them there. And then after that I just took all these, the clear ones [white rods]; and I figured, well, I put down seven. And I figured that they all equal, and if you have these two you would have three and then you could take one and put it on that and so it would be four, five, you would have three, four [He motions that he is adding one W to the LG , one W to the other LG, etc.], and then four, five, five, six, six, and then seven.
2.0.398		T/R 2:	Ok, alright. So you figured then that you can put, have seven on each of our halves?
2.0.399		Erik:	Yeah, of the halves, and then like you're making a new rod.
2.0.400		T/R 2:	So there'd be seven and seven? What do you think about that? He's saying
2.0.401		Erik:	Yeah, well no, well, I mean, not seven and seven, seven and six. It's an odd number of white, the clears, so it wouldn't be seven one would be seven and one would be six
2.0.402		T/R 2:	Ok, so, in other words, one of these could go here with this group, one of these goes here, back and forth like this. Ok, what happens to this guy, though? [pointing to the white rod to the far right] How can I be fair in making my two halves the same size? What could I do?
2.0.403		Erik:	What you could do, I think what you could do is, hmm, you could take this [white]. And you could replace those two, those three with a light green, yeah, one of the light greens like that. [He moves three whites and places a light green in his model.]
2.0.404		T/R 2:	Uh huh, oh, but I have one for this guy, and one for this guy, one for this guy, and what about this guy? [She points to the remaining W on the far right.]
2.0.405		Erik:	Oh, what this guy would go
2.0.406		T/R 2:	I think we ran into the same problem, didn't we? Would you agree that if we went back to this model, Erik, where we had these [She rearranges the rods] and we were divvying them up. Would you agree that maybe I could take this one [white rod] and saw it in half, if I had a saw?
2.0.407		Erik:	Yeah.
2.0.408		T/R 2:	And then what could I do with it, if I sawed it in half?

2.0.409		Erik:	Then you could put half here and half here [pointing to the two columns of rods]
2.0.410		T/R 2:	Ok, ok, I think we are almost out of time, aren't we? We probably need to clean up. Unless we have a minute? Ok, yes, ok, what do you have to share with us? Brian?
2.0.411	54:55 F	Brian 2:	[Brian, directly in front of T/R2, raises his hand and shows his model of P-Y-P - Figure F-54-55] If you take two purples and you put them on the sides and you put yellow in the middle and you cut it in half, then there'd be, then it would be equal.
2.0.412		T/R 2:	Ok, that's sort of like the solutions that other people were talking about, but you used different colors in order to show that, right? That's really nice. Ok I think we're going to clean up for today, and one thing that we're hoping that you all do, hopefully we're coming back on Friday, but what we're hoping you'll all do …is write about what we worked on the past few days, etc.
2.0.413	0:56:42 S		End of class.

Session 3, Sept. 24, 1993 (Front, Side, and OHP)

Line	Time	Speaker	Transcript
3.0.1	9:01 S	T/R 1:	You remember we did lots of activities with this? Maybe Audra can tell us. [Figure O-9-17] Audra, if I gave the purple the number name one half, what number name would I give to the dark brown?
3.0.2			[David and Meredith build models that are erect – one dark brown rod next to two purple rods.]
3.0.3		Meredith:	One half.
3.0.4	09:35	T/R 1:	We have a consultation. Sarah [Audra's partner], you can help Audra decide. You can, sort of, discuss it with her, see if you agree. Okay, your consultation is over. Audra?
3.0.5		Audra:	One half.
3.0.6		T/R 1:	If we gave the purple the number name one half, we're going to give the dark brown the number name one half?
3.0.7		Audra:	No, I mean, um, the dark brown would be one.
3.0.8		T/R 1:	The dark brown would have the number name one. How many of you agree with that? [All visible hands are raised. Audra is smiling.] And why is that, Audra? Can you tell me how you would convince me?
3.0.9	10:25	Audra:	Because I put the purple rods up against the brown rod and I got two purple rods.
3.0.10		T/R 1:	How many of you agree with that? [All visible hands are raised.] What if I gave the purple rod the number name one? What number name would I give to the brown rod? (repeats question) Laura?
3.0.11		Laura:	Two.
3.0.12		T/R 1:	Laura says two. How many of you agree with Laura? [All visible hands are raised.] You want to tell us why, Laura?
3.0.13		Laura:	Because if you put
3.0.14		T/R 1:	Nice and loud, Laura.
3.0.15		Laura:	Because if you put one of these to-
3.0.16		T/R 1:	I can't hear you. This machine is very noisy up here. You have to really talk loud.
3.0.17	11:05	Laura:	If you put two of these [purple rods] together and each of these was one, then you- one, two. And that [the brown rod] would make that two.
3.0.18		T/R 1:	Did you all hear her? Did you hear her, Andrew? So do you all agree with that? You're pretty good at doing this? How many of you feel pretty good at doing that? [Hands go up.] Okay, that's neat. All right. [At overhead] And you remember if we give the orange rod the number name two, can you tell me what number name we'd give to yellow? You know how to do that? [Meredith raises hand] You remember how to do that? Jacquelyn?

3.0.19	11:56	Jacquelyn:	[At her desk] You name the, um, yellow one.
3.0.20		T/R 1:	Why?
3.0.21		Jacquelyn:	Because, um, if you take the yellow, two yellows and put them against the orange, they match up. And if, if this [the yellow rod] is one half of it, it'd be, well, if orange is two, you would make this [yellow rod] into a one.
3.0.22	12:52	T/R 1:	You all agree? How many of you agree? Wow, that's fantastic. Ok, that's move along then. Suppose I had a train now. Remember we talked about a train? And I'm making my train with one yellow, and one light green. One yellow and one light green. [Some conversation between T/R 1 and T/R 2 takes place about whether the next problem had already been examined.]Ok, good. Then, the yellow and the light green?… We've called the one with the yellow and light green, I'm going to make that two. If I call yellow and light green two, what number name would I give to red [Figure O-13-28]? Remember, you have to convince me why. And when you think you have an answer, why don't you discuss it with your partner and see that your partner agrees with you.
3.0.23			*FRONT VIEW*
3.0.24		Alan:	[Alan builds a model and immediately raises his hand. Alan's model - Figure F-13-22] It would still be one half. Because
3.0.25		Erik:	If this is two
3.0.26		Alan:	Right. This is two. This would make one fourth
3.0.27		Erik:	One fourth
3.0.28		Alan:	This would make two
3.0.29		Erik:	One fourth. This is one fourth
3.0.30		Alan:	This would be one half of one, one half of two.
3.0.31		Erik:	No. It's one fourth. This is not a half of this. Two reds are a half
3.0.32		Alan:	Ok
3.0.33		Erik:	It takes four reds to make this. Two reds are half two reds are half. One red is one fourth.
3.0.34		T/R 1:	Do you agree with that? Do you think it's one fourth?
3.0.35	S 13:36	Meredith	Four. Oh, one fourth! [David and Meredith's hands go up.]
3.0.36			*WHOLE CLASS*
3.0.37	14:13	T/R 1:	If you think you have done the problem and you're waiting for someone to come around, [T/R 1, T/R 2, and Dr. L. circulate among the children, asking questions about their answers. Some conversation is heard. camera focuses on Brian, who is thinking quietly.] I'd like you to make the yellow and light green one and then tell me what the number name would be for red also. Remember the problem I asked you to do, though, if yellow and light green were two, not one. I want you to do both problems.
3.0.38			*FRONT VIEW*

3.0.39		Alan:	[After T/R 1 asks second question.] This would be one fourth.
3.0.40		Erik:	If they were one, what would you call the red?
3.0.41		T/R 1:	Yeah. And what if it's two what do you call the red?
3.0.42		Erik:	If it's two, you call the red [pause] one- ohh wait!
3.0.43		T/R 1:	Did you change your mind, boys? Hmmm. [to class] Let's see if you fall into the trap.
3.0.44		Erik:	Two. This would be one.
3.0.45		Alan:	Right
3.0.46		Erik:	Two of them would be one
3.0.47		Alan:	Right
3.0.48		Erik:	So this one would be a half.
3.0.49		Alan:	Right. One half would be red. And if this was one, this would be one fourth. One half and one fourth.
3.0.50		Erik:	[pause] Yeah.
3.0.51			[camera focuses on Graham, Beth and Jackie talking to T/R 2 - no sound]
3.0.52			*SIDE VIEW*
3.0.53		Jacquelyn:	[Brian2 and Jacquelyn build two physically identical models - Figure S-15-08]. You can call it one fourth
3.0.54		Brian2:	[raising hand] One fourth.
3.0.55		Jacquelyn:	This is two, and this is, this-
3.0.56	14:48	T/R 1:	[to class] Let's see if you fall into the trap.
3.0.57		Jacquelyn:	is one. This is one fourth, one fourth, and this is one half. [raises hand.] One fourth one half.
3.0.58		Brian2:	One fourth and one half?
3.0.59		Jacquelyn	This is one whole and this is two
3.0.60		T/R 1:	[She talks with David and Meredith.] Do you agree, Meredith?... One half, the number name one half. You're changing your mind, from what you said before?
3.0.61		Meredith:	[pointing to the yellow and green train in David's model] Yeah, cuz I thought you meant if that one was one whole.
3.0.62		T/R 1:	Yeah, you see, I didn't say that, I said two. Ok, you're going to be able to explain and argue for your solution? [they nod] Ok. I could also ask you, suppose I, well, let's let it go with that.
3.0.63			*WHOLE CLASS*
3.0.64		T/R 1:	How many of you changed your mind from what you first thought? [Some hands go up.] Some of you changed your mind. Ahah! Okay, if you're ready to discuss your answer, raise your hand. I have two problems on the table. One problem: if the yellow and green I'm calling two, what number name would I give to red? That's the first problem. The second problem is, if the yellow and green train is called one, what number name am I giving to red? So I am wondering if you have answers to each of those problems and

			be able to convince all of us that your answer is indeed one that you can support. How many of you are ready for discussion? Now if you haven't talked to your partner, you need to do that. And if you and your partner need to talk to someone else, you could even quietly go to another table if you'd like. Quietly. [Only some hands are raised. Some children are talking to their partners, more time is given.]
3.0.65	17:10	Brian:	I think it'd be fourths. [Erin's comment is inaudible]. One eighth. [Pause]. This is two, um…a half, I mean, yeah, I mean this is like, this was a half and these would be eighths…no, one [cannot see his model]. [Both Brian and Erin are not speaking; they seem to be deep in thought.] I'm not sure, I'm not sure. [Camera focuses on Meredith's models - Figure S-19-09]
3.0.66	20:10	T/R 1:	Okay. Could someone tell Dr. L what are the problems I've given you? Can someone explain to Dr. L… You want to call one someone, Dr. L?
3.0.67		Dr. L:	Alan!
3.0.68		Alan:	Ok. We made a train and if this was considered two, what would the red one be, the red rod be?
3.0.69		Dr. L.:	Uh, huh! Boy, that's some problem.
3.0.70		Alan:	And the other one was if this [train] was considered one, what would the little red rod be?
3.0.71		Dr. L.:	Okay, was anyone able to figure that out? Yeah? [Some hands are raised.] Audra, what did you come up with?
3.0.72		T/R 1:	Can you up and show us, maybe? [Sarah and Audra go to the OHP. Audra builds a model of the train Y - LG] Dr. Landis, Audra's going to show us at the overhead.
3.0.73	21:09	Audra:	Well, first we put the red rods [she places four red rods underneath the train] up to the yellow and the green rod and then we said if the yellow and the green was two, what would we call the red rods? And we thought that we would call it one and one fourth. And then if it [the train] was one, we would call it one fourth.
3.0.74		Dr. L:	Okay, so if it was one you said you'd call it one fourth and if it was two, what did you say?
3.0.75		Audra:	It would be one and one fourth.
3.0.76		Dr. L.:	One and one fourth, I don't know if I understand that. [To the class] Do you all agree with that? Did you come up with the same names for that? No?
3.0.77		T/R 1:	How many of you agree that if we call the yellow and the green one two
3.0.78		Audra:	Two, it would be
3.0.79		T/R 1:	The red would be one and one fourth. How many of you agree with that? [No hands are raised.] Ok, you're not having people agree with that, so you are going to have to convince

			them, Sarah and Audra. What would you do to convince the class that it would have the number name one and one fourth? But before we ask you to convince them, I'm curious about the other. If you call the yellow and green together one, what did you call the red rod?
3.0.80	22:25	Audra:	One fourth.
3.0.81		T/R 1:	How many of you agree with that? So we have some people agreeing with that. So you're going to have to convince them. They agree with your second solution, but not your first [one]. So let's hear the arguments. Okay, you're all listening? Because if you don't agree with the arguments that Audra and Sarah are going to give you, you have to come up with a different argument.
3.0.82		Audra:	Well, because see, the yellow and the green was the same size as the brown, so if we put the reds up against the, no, wait, no. See, because there's, if there was one, we would- if it was brown we would normally call it one. And if we put the reds up against it we would all call it one fourth, so we thought if we called the yellow and the green one, it would be the same thing as the brown [Figure O-23-31].
3.0.83		T/R 1:	How many of you agree with that argument for calling the red one fourth when the yellow and the green [train] together are one? How many of you agree with the argument that Audra just gave us? Do you disagree with her argument? You don't know? How many of you don't know, how many of you agree, how many of you disagree? Cuz what she said...this is the same as brown. Is that what I heard you say? [T/R 1 places the brown rod above the yellow and green train - Figure O-23-57.] If you said if you call the brown one that would be like yellow and green being called one. And then you argue that red would therefore have the number name one quarter. Erik?
3.0.84		Erik:	I agree.
3.0.85	24:10	T/R 1:	You agree?
3.0.86		Erik:	Yeah, because see if the brown and the yellow and green they're equal and they're both called one, and four of the reds equal up to one, therefore that they'd have to be fourths, because there are four parts, they're fourths.
3.0.87		T/R 1:	Would you raise your hands if you agree with the argument that Sarah and Audra and Erik gave us? Up high, so I can tell. Now, there are some hands that aren't up; does that mean that you disagree or you're not sure? Brian?
3.0.88		Brian2:	We disagree.
3.0.89		T/R 1:	Nice and loud, Brian.
3.0.90		Brian2:	We thought the um, the two, we called that one half.

3.0.91	24:53	T/R 1:	Okay, we're talking about when we call it [the train] one. You're talking about the other. You agree that when we call it one, that this is a fourth?
3.0.92		Brian2:	Yes
3.0.93		T/R 1:	Okay, now the second part you disagree. Now you give your argument for when you call the yellow and green two. All of you disagree with the argument of calling the red, one quarter when you called the yellow and green together one. And I like the brown rod up there to show you that's another way to call it [the train] one. That's very nice. Some of you didn't do this. That's something new that Sarah and Audra introduced that I think is very nice. But now let's hear the other argument. How did you get one and one quarter when you called the brown rod two now? [Audra and Sarah are quiet. They seem unsure.] You're not sure you have an argument?
3.0.94		Audra:	No.
3.0.95		T/R 1:	Do you want to pull back your argument and listen to other people's? That's fair enough, sure you can sit down. Let's have someone else. If you don't agree with one and one quarter and if you don't have an argument, does someone have something else. Now Brian, you want to come up here because you said you had a different argument… And Jackie? I like to hear your argument and see if you convince Sarah and Audra who want to be convinced. [Brian2 and Jacquelyn come to the overhead.]
3.0.96		Brian2:	Well, we thought the two [he moves the train of yellow and green] would be called a half.
3.0.97		T/R 1:	The two what, Brian?
3.0.98		Jacquelyn:	The two would be a half.
3.0.99		T/R 1:	The two of what?
3.0.100		Jacquelyn:	When this [the train] is two, these [the red rod] would be called a half.
3.0.101		T/R 1:	You're saying a red would become a half?
3.0.102		Jacquelyn:	Yeah.
3.0.103		T/R 1:	Hmm, that's an interesting idea. So when yellow and green become two, the reds, how could you, how could you convince us? Because I see your teacher there, Mrs. P. saying, how did they get that? Right? She wants to know how did you figure that out? That's an interesting idea. How many of you agree? A few of you agree with this argument. Now you've gotta help Audra and Jackie prove it. We're listening. Can you convince us?
3.0.104		Jacquelyn:	Not really.
3.0.105		T/R 1:	Brian?

3.0.106	Brian2:	Jackie thought of the two. So she should be able to explain it.
3.0.107	Jacquelyn:	Well, this was called two [the train] and this would be called, all of these would be called one half [the red rods]. Because uh, um… [She sighs and strums on the overhead projector.]
3.0.108	T/R 1:	You don't remember how you did that?
3.0.109	Jacquelyn:	Yeah, I forgot.
3.0.110	T/R 1:	David, do you want to help them out? You want to come up here and help them out? Because you also called it one half, didn't you, you and Meredith.
3.0.111		[David goes to the overhead projector.]
3.0.112 26:06	David:	Um, yeah, and um, so if this is called two [the yellow and light green train] and then this would be two too [four red rods]. So then this would be one [indicating the two red rods] and this would be one [David separates the red rods into two groups of two rods - Figure O-28-10]. But then if you take away this [one red rod] this would be one half over there [the red that is remaining] and put another one half that would be one and another, then that would make up to be two [realigns the four red rods to equal the length of the yellow and light green train. Jacquelyn nods].
3.0.113	T/R 1:	Did you all follow what, what David said? David, you're going to have to do it again. I think some people had a little trouble following it. All right. Michael, did you follow it?
3.0.114	Michael:	Yeah.
3.0.115	David:	All right, so…
3.0.116	T/R 1:	You can help say it another way. It might help other people follow it so let's give David another chance and then maybe Michael can help him out, and Meredith.
3.0.117	David:	Alright, so if this is two [the yellow and light green train], then this would be a half because if you put another one and another one that'd be two [He aligns four red rods]. And if you take away these [two red rods] that would be one and took away that [He takes away another red rod], leaving one red rod], that would be a half of [inaudible].
3.0.118	T/R 1:	How many of you understand? How many of you followed what David said? Raise your hand if you followed what David said. So more hands came up now, so more people are agreeing. Is that what you were thinking, Jackie?
3.0.119	Jacquelyn:	Yeah. I just couldn't it out.
3.0.120	T/R 1:	You couldn't get it out. You want to try it again now that David helped you the way he was thinking?
3.0.121	Jacquelyn:	No.
3.0.122	T/R 1:	Who wants to give it a try at another way of saying it? You want to give it a try? Go ahead, Brian.
3.0.123	Brian2:	Well,

3.0.124	T/R 1:	Because I liked Audra's trick of finding out what one was in the other problem. Remember Audra and Sarah came up with the brown rod. Do you remember that? I wonder if you can use your little trick of coming up with brown rod to help explain this idea to people who aren't catching it. If you think you understand it, maybe that might help some people.
3.0.125 30:03	Brian2:	If you take these two [two red rods], that would be one half. And this would be another one half.
3.0.126	Jacquelyn:	These would be one.
3.0.127	Brian2:	These are ones. And then if you take one away then this would be a half [the red rod].
3.0.128	T/R 1:	Yeah that's sort of what I heard David say, same argument as David, ok. But you were beginning to say something else [T/R 1 goes to the overhead and moves two of the red rods]. The temptation I noticed, and some of you did this in the beginning, you wanted to call, many of you wanted to call the two reds a half and the other two reds a half. And then you changed your mind Jackie was there shaking her head.
3.0.129		And I walked around and I saw lots of people doing that. I wonder what you were thinking when you wanted to call this a half. Is it okay to call this [two red rods] a half and call this [the other two red rods] a half sometimes? And is it okay to do it this time?
3.0.130	Students:	No
3.0.131	T/R 1:	What's the difference? Jakki, want to talk about that a little bit?
3.0.132	Jacquelyn:	Well if this, if well, it's because of this one because it's two. If we call this, both of these one half, it [the train of yellow and light green] would be one.
3.0.133	T/R 1:	Oh, so you're saying its okay to call it one half if we call this one?
3.0.134	Jacquelyn:	Yeah.
3.0.135	T/R 1:	You all agree with that? If the yellow and green together are one, then it's okay to call the two reds one half. How many of you agree with that? To give it the number name one half. What do you think is so confusing here? When we called it the other name two then this had to be one and this had to be one you said because one plus one is two right? But what, what's confusing here? Because there's something that a lot of people got confused about and I'm wondering if you could understand what the confusion is. That would help. Thank you, you can sit down. That was very nice. Erik?
3.0.136 31:50	Erik:	I think the confusion is, they think, that they just, they think, they have the temptation of calling, since there are four red blocks, they think they're gonna call it one fourth 'cause they forget that the yellow and the green are two.

3.0.137		T/R 1:	What are they thinking that the yellow and the green are when they do that?
3.0.138		Erik:	One.
3.0.139		T/R 1:	They are thinking that the yellow and the green are one when they do that.
3.0.140		Erik:	Because, see, if you have one there'd be two halves, but if you have two its two halves plus two halves which would be four halves. So you'd have- therefore, you'd have to call one of the reds one half.
3.0.141		T/R 1:	Wow, that's something to think about isn't it? How many of you understood Erik's argument? Raise your hand if you understood Erik's argument. A couple of you seem to understand it. What do you think Michael? What's your comment on this? I thought some people, how many of you fell into the trap? When I asked you that problem right away, I said, call the yellow and green two, what number name would you give red? How many of you called it first one-quarter? How many of you fell into that trap in the very beginning? {Many students raise their hands.] I think mostly everybody fell into that trap, right? And then when I asked you what the yellow and green, if that were given the number name one, then you said, oh wait a minute, right? That's very interesting. Um, I kind of knew you'd fall into that trap.
3.0.142	0:33:18		How many of you like chocolate? [Hands are raised]. Pretty much everybody. Right. Um, you know, we were talking about sharing things that people like and I was talking to, um, Tom earlier and I was talking to Amy. And, um, I said you know, Mrs. H. was nice enough to bring some candy because I thought we would look at some nice fraction problems and I said, well if we share these. So Tom said, I want one-half a candy bar and Amy said, I want one-half a candy bar. So I said okay, you each can have a half and I gave them each a half and they were so angry with me. They looked at me and said, well Tom was happy, but Amy said to me I don't really like what you just did. That didn't seem really fair. Now, how could that be? It seemed fair to me. I gave one-half to Tom and I gave one-half to Amy. Didn't I do the right thing? Meredith doesn't think so, and- Mark, what do you think?
3.0.143		Mark:	You probably gave Tom the, uh, bigger, a bigger half than Amy.
3.0.144		T/R 1:	Can a half be a bigger half? You told me when I called this one, the red rod, right; this is one half and this is one half, how could one be a bigger half? They're the same size. The two reds make a half and the two reds make a half. Is that

			what you were thinking I did Meredith? Gave a bigger half? And does that make sense to give a bigger half?
3.0.145	33:25	Mere:	Mm, hmm [positive response].
3.0.146		T/R 1:	How?
3.0.147		Mere:	Well, say you gave, this was one [indicates a yellow and light green train] and then you gave this much to Tom [yellow rod] and this much to Amy [the light green rod]. That wouldn't be a fair cut.
3.0.148		T/R 1:	I agree with that but I wouldn't call that a half. Why wouldn't, why wouldn't I call this, if I called this one I wouldn't call green a half and I wouldn't call yellow a half. If I did, Dr. L. wouldn't let me come back. She'd say stay out of that class, what are you teaching these students? Would I have called it a half? David?
3.0.149		David:	No, because it wasn't even.
3.0.150		T/R 1:	What do you mean by that, David?
3.0.151		David:	Well, um, the half should be even so that the other side is the same as it is. So the yellow is bigger than the green and the half should be the same size.
3.0.152		T/R 1:	So all of you know what a half would look like, wouldn't you? Does this have a half by the way? Can you find me a rod that would be called a half, if this was my candy bar? If I called my candy bar dark brown right instead of yellow and green, it's the same size isn't it? I want to use, uh, Sarah and Audra's trick and I want to call yellow and green dark brown. Is that okay? Okay? So according to David, David's thinking that I would have in mind giving the purple to Tom and the purple to Amy. I would know they should be the same size. [OHP - yellow and green train, brown rod, and two purple rods] Brian?
3.0.153		Brian:	Two purple would make a half.
3.0.154		T/R 1:	Each of those would be one half?
3.0.155		Brian:	Yeah.
3.0.156		T/R 1:	Do you agree with that? What could I have done David? So I didn't violate that condition. What could I have done to make Amy so annoyed at me? Like I thought she wasn't going to stay. Audra, what do you think?
3.0.157		Audra:	Because, see, you had the red. Well, if the candy bar was this size [holding up a train of green and yellow - Figure S-37-37] and you were to divide it in half and then Amy probably got a piece like this [green rod] and..
3.0.158		T/R 1:	But I didn't do that. I really made the halves of the candy bar the same size, so I didn't trick her that way. What else could I have done that could have made her feeling badly about this? Do you want me to tell you? How many of you want to know what I did? Tell me if what I did was right or

		wrong. [She holds up a large candy bar.] I gave Tom Purdy half of this candy bar, right down the middle, right? You like that? [students giggle] I gave him half didn't I? [students agree] Yeah, and Dr. Martino, what did I do? [T/R 1 holds up a small candy bar.] Right down the middle, right two equal parts. Right? I can't imagine. I gave them each a half. Why should she [Amy] be annoyed with me? [Children are giggling.] Did anyone ever pull that on you? You wouldn't pull that on a younger brother or sister would you?
3.0.159	Students:	Yes!
3.0.160	T/R 1:	You would! Ok tell me why Dr. Martino was annoyed with me. What do you think, Caitlin?
3.0.161 39:04	Caitlin:	Cause they're weren't the same size chocolate bar.
3.0.162	T/R 1:	They weren't the same size. How may of you agree they weren't the same size? They weren't the same size chocolate bars, right! That's right! They weren't the same size chocolate bar. What does that have to do with what we're doing here, if anything? Brian?
3.0.163	Brian:	We're working with halves.
3.0.164	T/R 1:	That's true, we're working with halves. Audra.
3.0.165	Audra:	Because, see, since we're working with halves you took these two together [She indicates a yellow and light green train] and you called it, called it two and it would be like one candy bar and the other candy bar. And, if you put the reds on top, um, I think, someone, they said that if it was a half, if you put two reds on top of the green and it isn't 'cause the two reds is bigger than one green, than one light green and it can't be half. [As she speaks, T/R 1 adds four red rods to the model on the OHP - Figure O-40-21] Just like the chocolate bar couldn't be a half.
3.0.166	T/R 1:	That's very interesting, what I'm hearing Audra say. Let me try to say it with candy bars. [T/R 1 uses the large candy bar to find one half of it; uses the smaller candy bar to find one half of it] Audra is saying it's sort of like what I did. If I call this candy bar one; one candy bar right? Then I could call this half a candy bar, agreed? You all agree with that, if I call this one? If I call this one [holds up a small candy bar], then I can call this a half [points to two of the four squares of the small candy bar]. She's saying what I'm doing is a little bit like taking a piece of this candy bar and taking a piece of this candy bar and mixing up my ones. Is that sort of what you're saying Audra? Is that allowed when you're comparing things? Are you allowed to mix up your ones? 'Cause then I could say to you, is it fair to compare different sizes? Cause then I could say to you, what do you think is bigger, one half or one third. What do you think is bigger,

		one half or one third? You could even think of this candy bar here. Can you all imagine half of it? [Mmm hmmm] How do you imagine a half of a candy bar, Jackie?
3.0.167	Jackie:	Cut it right down the middle [she motions].
3.0.168	T/R 1:	Okay Jackie says she would cut it right down the middle, here, right, for a half. Why wouldn't she cut it long ways for a half very easily? Why couldn't she do that? Why not Graham?
3.0.169 42:10	Graham:	'Cause there's three of them [three sections across the bar].
3.0.170	T/R 1:	Cause there's three of them.
3.0.171	Graham:	And you would need to have four so I could cut it in half.
3.0.172		[The candy bar is scored in a three by four grid pattern.]
3.0.173	T/R 1:	[T/R 1 asks what about a third of the candy bar.] Okay, so if I cut it in half, do you see how many pieces she would get here? How many of these little chunks she would get? How many?
3.0.174	Jackie:	Six
3.0.175	T/R 1:	So you would get six out of a total of twelve, right?
3.0.176	Jackie:	[Nods.]
3.0.177	T/R 1:	But if you got a third, can you tell me what you would get? Can you all see? Gregory? If I gave you a third of this candy bar; we shared it among all three of you here? Which part would you get, which part would Dr. Landis get, and which part would Danielle get? To be fair Gregory. How could we share this three equal ways? [starts talking, inaudible] Gregory, nice and loud. [T/R 1 drops the candy bar.] Gregory, nice and loud so everyone can hear you back. there.
3.0.178	Gregory:	Well, Danielle will have one row and Dr. Landis will have one row, then I would have one row.
3.0.179	T/R 1:	And how many wedges would that be for you?
3.0.180	Gregory:	Um, four.
3.0.181	T/R 1:	Four out of twelve, right? So when you got half, Jackie said you got six out of twelve [this is melting in my hands] and when we got a third, you got how many out of twelve?
3.0.182	Gregory:	Um, four.
3.0.183	T/R 1:	Four out of twelve. Who got more? The person who got one half or the person who got a quarter, a third? Who got more? What do you think Amy?
3.0.184	Amy:	The person who got a half.
3.0.185 43:49	T/R 1:	How many of you agree? The person who got half a candy bar got more than the person who got a third. You all agree with that, you all understand that? [All children agree.] And no one could convince you otherwise and you'd make sure you got your fair share. I know that you would get your fair share if you were negotiating among yourselves. I'm not

sure with a younger brother or sister how that would work. However, as this is melting, let’s switch candy bars. So of that candy bar, right, you'd pick what? How many of you would pick a half of it? How many of you would pick a third of it? You’d all pick a half, right? There’s no question. However, is it possible, is it possible, if I were talking about different candy bars? Like these here, right, different size candy bars. Could you imagine if it were possible that a third could be more than a half? How many of you could imagine that? Suppose I gave you half of this candy bar, right, the little one. Suppose I gave you a third of the big one. What would you want? Andrew?

3.0.186	45:17	Andrew:	I would want the third
3.0.187		T/R 1:	Of the big one.
3.0.188		Andrew:	Yeah
3.0.189		T/R 1:	How many of you would want the third of the big one? You could sit and study these candy bars. So what’s wrong if I say to you. You say I want a half and I could be a very clever older sister and say okay I’ll give you a half and I go back and get my little candy bar, right, and I say you got a half. And then your little brother or sister says you got a third and how come you have more? You’re confusing your little brother or sister. How come you have more? Michael?
3.0.190		Michael:	Because you, uh, you were gonna split up a big one, so, but then she ran back and got the little one and split that in half and gave that half to him and then she took the big one and got a third of it and ate that piece.
3.0.191		T/R 1:	Sort of a little dishonest, is she? So it's sort of dishonest to switch candy bars, isn't it, right? Isn't that right? Now, we have to be very careful, because, you know, this often happens, you order a pizza pie, you go out with a group of friends, and you say, ok, um, I would like one third of that pizza pie. And someone else says, ok, that's fine, um, I'm going to take one half. And there's this little pie that gets delivered, did you ever see these personal pizzas, these small ones, right? And you get one half of this little pie, and then they have this big pie, and you get one third. Who has the better buy? David?
3.0.192		David:	The guy with one third.
3.0.193		T/R 1:	The guy with one third. So what’s the question you should always be asking yourself when you're comparing fractions? Meredith?
3.0.194	47:02	Mere:	Which one’s bigger.
3.0.195		T/R 1:	Which?
3.0.196		Mere:	Which thing is bigger.

3.0.197		T/R 1:	Which thing is bigger. Which object is bigger. Are you allowed to compare different things when you compare fractions? Is that really fair? Is that really fair?
3.0.198		Students:	No.
3.0.199		T/R 1:	No, it's not really fair. I mean it's sort of a kind of tricky kind of thing people might do, but you know what, you could get tricked into that. If you don't think carefully about when someone is asking you to compare. Because when we really ask the question, which is bigger one half or one third, what are we, what are we assuming? What's sort of the common understanding about that? When I ask you that question? Michael?
3.0.200	47:48	Michael:	Well, normally half is bigger than one third, but if you got a bigger piece of a candy bar or pizza, you got a big pizza, and you get and you get half or one third of that, then that'll be more than just a little pizza that you get half of.
3.0.201		T/R 1:	Okay. We don't want to fall into that trap; can we have an agreement in this class? And maybe you want to think about this the rest of your life in mathematics. When we compare fractions, it's the same thing. We're not gonna play tricks on each other. If I ask you, which is bigger, a half or a third, we mean of the same object. Okay? You're not allowed to think a half of one cake and a third of another cake. We're talking about either cakes that are exactly the same size or candy bars that are exactly the same size. Do you understand that? We're not allowed to switch. Do we agree on that? Because once we have those rules then maybe we can talk to each other and argue. Now, I have one last problem for you to do. I think we still have 10 minutes. We're gonna go til ten after. One last problem. I wanna see if you fall into the trap. If we agree on we must keep whatever it is we're comparing the same unit. So if I'm comparing one half and one third, what I call one has to be the same for one half and it has to be the same for one third. Is that clear here? [Mmm hmm.] Alright. I want you to draw me a model with your, make me a model with your Cuisenaire rods and to show me which is bigger one half or one third. And I want you to tell me, show me which is bigger and I want you to tell me how much bigger and be able to convince me. Which is bigger, one half or one third?
3.0.202		Erik:	[some argument about who gets rods] Let's see…
3.0.203		T/R 1:	You want to put these in the middle, Erik, so that you and Alan can share them? [similar talk]
3.0.204		Erik:	One third and one half
3.0.205		Alan:	One third. [some more arguing] The blue can be divided into thirds.

3.0.206		Erik:	You think I care? You don't need that
3.0.207		Alan:	[inaudible, David borrows yellow rods from Erik and Alan]
3.0.208		Erik:	See, Alan, you messed it all up!
3.0.209		T/R 1:	Ok, please remember the ground rules. As I walk around and I watch what you're doing, are we allowed to switch candy bars?
3.0.210		Students:	No.
3.0.211		T/R 1:	Are we allowed, if we're making a half and third, are we allowed to have different size candy bars?
3.0.212		Students:	No.
3.0.213		T/R 1:	No. We have to not- we are not allowed to switch candy bars. Remember that's the rule from now on in mathematics.
3.0.214		Alan:	Get the model of a half
3.0.215		Erik:	What?
3.0.216		Alan:	Get the model of a half.
3.0.217		Erik:	No! So do I. Unfair info, this is no model of a half.
3.0.218		Alan:	A half would be bigger nevertheless.
3.0.219		Erik:	What?
3.0.220		Alan:	A half would be bigger nevertheless.
3.0.221		Erik:	There's no half of a blue, then why did you pick the blue? [Alan puts a purple rod up to his model of a blue rod and three green rods. Erik grabs it back] And give this back - you don't need it.
3.0.222		Alan:	There's nothing else that can be divided into thirds and halves.
3.0.223		Erik:	Yeah, I'm sure there isn't.
3.0.224		Alan:	You'd have to make your own rod for each one 'em.
3.0.225	53:38	Dr. Landis	[Sitting with Danielle and Gregory, who have built a model of a green rod, three red rods, and two light green rods - Figure F-54-34. Conversation is inaudible.]
3.0.226		Erik:	You don't need the blue. We're not using the blue. We're using the brown.
3.0.227		Alan:	The brown can be divided into thirds?
3.0.228		Erik:	Can blue?
3.0.229		Alan:	Yes.
3.0.230		Erik:	Can blue be divided into halves?
3.0.231		Alan:	No. Can brown be divided into thirds?
3.0.232		Erik:	It doesn't matter. You bet it can. If I just find the right rod. [Erik experiments silently.] Ok, what rod are we going to use then? It can't be divided into anything.
3.0.233		Alan:	Your own rod. [hums]
3.0.234		Erik:	What are you doing? Get off! [Some arguing about who owns the rods. Alan constructs a balance, David complains that he's copying him.]
3.0.235			*SIDE VIEW*

3.0.236			[David begins to construct a balance beams with rods for their model. Meredith builds an upright staircase model - Figure S-51-23. Some of their interaction is not transcribed. The other students work on the problem as they had worked on the others]
3.0.237		Meredith	Let's do the model that I did before. Remember?
3.0.238		David:	[To Meredith] That's nine remember. You can only make it with the even bars.
3.0.239		Brian2:	[To Jakki] Now let's get three of these [light greens] and make a half out of that, if we can.
3.0.240		Jacquelyn:	Wait, let me do the red ones.
3.0.241		T/R 1:	Ok, please remember the ground rules. As I walk around and I watch what you're doing, are we allowed to switch candy bars?
3.0.242		Students:	No.
3.0.243		T/R 1:	Are we allowed, if we're making a half and third, are we allowed to have different size candy bars?
3.0.244		Students:	No.
3.0.245		T/R 1:	No. We have to not- we are not allowed to switch candy bars. Remember that's the rule from now on in mathematics.
3.0.246		Brian2:	And you take two light greens and that would be equal, and they'd be equal. And if you take the purple and the dark greens and you made the purple a third and then the greens a half, then they'd be equal. [Brian attempts to make thirds and halves "equal" so that they both represent the same whole.]
3.0.247		Jacquelyn:	That's what you're gonna tell the class (laugh). We can raise our hands. [They raise their hands.]
3.0.248	54:07	Brian2:	Wait, let's, in the meantime, let me try to figure out another way. How about three of these [white rods]. I don't think there's any other ways.
3.0.249		Jacquelyn:	Only three reds, and two light green, and three purples, two dark greens
3.0.250		Brian2:	Wait, wait, wait, wait, if we take three blacks, and we take two oranges, and that'd be, it's not equal
3.0.251		Jacquelyn:	How many ways can you get?
3.0.252		Brian2:	It's not equal!
3.0.253		Jacquelyn:	Oh!
3.0.254		Brian2:	What's smaller than this [black rod]?
3.0.255		Jacquelyn:	Dark green.
3.0.256		Brian2:	Ok. What's-
3.0.257		Jacquelyn:	And smaller than that? Yellow. I'm trying yellow. What half of an orange? This [She has a model of one orange and two yellows]. What's third of an orange? Oh, do you have one more purple? No, that wouldn't work either.
3.0.258	55:23	T/R 1:	What do you have here?

3.0.259		Brian2:	We found out if you take two dark greens and you make each one a half and you make these [purple] a third, they'd be equal.
3.0.260		T/R 1:	So which is bigger?
3.0.261		Brian2:	They're equal in these colors [indicates the length of the whole – which train is he using?]
3.0.262		T/R 1:	What number name is this [a dark green rod]?
3.0.263		Brian2:	A half.
3.0.264		T/R 1:	What number name is this [purple]?
3.0.265		Jacquelyn:	One third.
3.0.266		Brian2:	A third.
3.0.267		T/R 1:	Which is bigger a half or a third?
3.0.268		Brian2:	The half.
3.0.269		T/R 1:	The half is bigger
3.0.270		Jacquelyn:	Oh yeah.
3.0.271		T/R 1:	Right, by how much?
3.0.272		Jacquelyn:	By an inch.
3.0.273		Brian2:	No, by a….
3.0.274		T/R 1:	By a red. And what number name would you give the red then?
3.0.275		Brian2:	A quarter.
3.0.276		T/R 1:	Remember what you called one.
3.0.277		Jacquelyn:	A quarter.
3.0.278		T/R 1:	What number name, prove to me that red is a quarter. [Jacquelyn moves closer to Brian to see what he is doing.] If this is red, that's a half [the dark green rod]. Prove to me. Sure it's a quarter?
3.0.279		Jacquelyn:	Oh.
3.0.280		T/R 1:	Change your mind?
3.0.281		Jacquelyn, Brian2:	Yeah.
3.0.282		T/R 1:	Okay, great. Okay, can you explain that?
3.0.283	56:16	Brian2:	Maybe…
3.0.284		Jacquelyn:	Not real-
3.0.285		Brian2:	[T/R 1 walks away as the students continue] Okay, so what would these be? [Jacquelyn counts the red rods.] They're six.
3.0.286		Jakki:	One fourth. And one half. A half plus a fourth. Oh God.
3.0.287		Brian2:	Wouldn't they be one third? A third? No. [Jacquelyn sighs as Brian2 continues thinking.] What would we call that? Two of these white ones.
3.0.288			*WHOLE CLASS*
3.0.289		T/R 1:	Ok, uh, I know we're not going to have enough time to hear from all these wonderful solutions I'm seeing, but I'm hoping that on Monday if you don't forgotten what you've done you can start thinking about it, and Dr. Landis said she may stop by here on Monday, and Dr. Martino will be here, and we

			hope you'll share your solutions and write about it, but let's hear a real quick one from, from Laura and from Jessica?
3.0.290	57:27	Jessica:	[Laura and Jessica use a train of three purples and a train of two dark greens as their model - Figure O-57-26] Well this, this here would be one third, the pink would be one third. And one half would be the green, the dark green.
3.0.291		T/R 1:	So which is bigger?
3.0.292		Jessica:	The one half.
3.0.293		T/R 1:	Okay now the next question I asked, the dark green is bigger. You can all see that right? How much bigger is it? So the next question, how much bigger?
3.0.294		Jessica:	[Jessica shows that the red rod fills in the space] It's the size of the red.
3.0.295		T/R 1:	It's that much bigger. It's a red bigger. Okay so the question is what number name would you give to the red? We know it's bigger by a red. What number name would you give to the red? [reconstruct original model] So you're saying it's a red bigger. You're saying a half is bigger than a third because the green is a half and the, you called it, the pink is a third and it's a red bigger. What number name would you give to the red and why? You don't have to tell me that now. Do you think you know? Why don't you think about that. We're going to have to stop I'm afraid because of time. How many of you think you know what number name you would give to the piece that's bigger? How many of you think you've answered that problem? Gregory did you figure that out yet? James, did you figure that out? Okay, this is what I want you to think about this weekend. If you had to give this a number- What did you call one? [Laura and Jessica point to the orange and red train] What did you call one here? One candy bar. Okay this is one, right? So the question I'm asking you is if that's [the orange and red train] 'one', what would be the number name would you give to red? [Figure O-59-20] Do you understand the question? How many of you think you know the answer to that? Alan?
3.0.296	57:46	Alan:	One sixth.
3.0.297		T/R 1:	Alan thinks one sixth. Why do you think so Alan?
3.0.298		Alan:	Because we know already that, that, three reds would make a dark green and if there are two dark greens to make the orange and the, and the red rod then it would take six red rods to make the orange and the red rod.
3.0.299		T/R 1:	So you think one sixth. You think one sixth. That's something we'll have to think about. I'm afraid we have to stop. But, we're gonna start Monday with this problem and we're going to ask you to build it again and come up with your solution, and I think I saw about four different solutions

		and I would like you to be ready to come up and share them. I want to thank you for a fun week and I hope you have a great weekend.
3.0.300		*SIDE VIEW*
3.0.301		[The class is getting ready to leave and David is setting up a balance.]
3.0.302	David:	So when I take off these, I think that it'll fall over here. And the green is the whole thing, and the light green is a half and the reds are a third,
3.0.303	T/R 1:	That is very very nice. David, will you share this on Monday, you think you can build this all again? You'll remember what you just built there? This is ver- I've never seen this before. This is quite different. [To Meredith] Will you help remember? Maybe we can start with having them share this balance that you've made? Because Dr. Martino didn't see this, and I think she'll want to see it.
3.0.304	David:	Well, she's making something else and I'm making, I made the balance. She's making another-
3.0.305	T/R 1:	That's lovely. Dr. Martino, if we can get her here. [T/R 1 calls T/R 2 to listen to David. The whole class begins to gather around David.]
3.0.306	David:	[Figure S-01-02-06] All right, I made a balance and the whole thing is dark green and the light green is a half and the reds are the thirds, but then what I'm doing is, um, I'm making a balance so when I take off that [one light green rod] and those two reds, then I think it will fall to this side and show that the half is bigger. [He places two light green rods on one side and the three red rods on the other. When he removes the one light green rod and the two red – leaving one light green rod (a half) and one red rod (a third) – his structure falls to the side of the light green rod (signifying that side of the balance is heavier - bigger)].
3.0.307	T/R 1:	It did fall to that side, didn't it? So your prediction was right. Okay, now the question I'm going to ask you, when you work on this balance, what would you have to have put there to stop it from falling? What other, what other rod could have been put on the left side so that it wouldn't fall when you took that off? Do you understand my question? What did you take off?
3.0.308	David:	I took off the two reds and a light green.
3.0.309	T/R 1:	Okay, now if you didn't want it to collapse, right? It, it, you said it fell to the right the way you had it built, okay?
3.0.310	David:	Um.
3.0.311	T/R 1:	And the red were on the right side? Is that correct and the greens were on the other side, or was it the other way?
3.0.312	David:	Well, the reds were on the left side.

3.0.313		T/R 1:	On the left side. So you took the two reds from the left side and the green from the right.
3.0.314		David:	Mmmm hmm
3.0.315		T/R 1:	Okay. What would you have had to put on that other side so it wouldn't tip once you took the two reds and the green off? Do you understand my question?
3.0.316		David:	Um, let's see.
3.0.317		T/R 1:	What would you have guessed it should have been?
3.0.318		David:	Um, maybe a little white - these? [He has a light green rod next to a red rod and equals the length of both by adding a white rod to the red rod.]
3.0.319		T/R 1:	A little white? Okay, we could try that experiment on Monday, right? That's a good guess. Why did you guess that? I think you went looking for something specific. Why were you looking for that one?
3.0.320		David:	Well, 'cause when I went like this [he lined up the red and light green] I just saw there was one space in between and I knew the white is that space.
3.0.321		T/R 1:	Okay, what number name would you give to white? That's the next question.
3.0.322		David:	Um.
3.0.323		T/R 1:	You don't have to answer it now. When you called this one half and when you called this one third. What number name would white have had to have? You have any idea? You can think about it. You have any idea Meredith? What number name would white have if green were one half and red were one third? Anybody? Does anybody have a clue what number name white would have for the model that was built here? What do you think?
3.0.324		Andrew?	One fourth?
3.0.325		T/R 1:	You think one fourth? Anybody else? You're not sure? Well, that's something to think about this weekend, ok? Great. Well thank you very much.
3.0.326		T/R 2:	That was wonderful.
3.0.327		T/R 1:	See you next week.
3.0.328	1:04:54		End of class

Session 4, Sept. 27, 1993 (Front, Side, and OHP)

Line	Time	Speaker	Transcript
4.0.1	3:00 S	T/R 2:	Let's see what you can do with this. Ok we have got an orange. Now, if I give this orange rod a number name one, what number name would I give to white? Ok, if you think you know, I think a lot of you think you know in your head, can you also show me? Can you do something with the rods and show me? Put it out in front of you. I want to give everybody a chance to think because some of us, it takes a little longer for brain to click in on Monday. [Hands raise immediately.]
4.0.2	4:11	Brian:	Well, because this is, the whole orange rod, in Cuisenaire rods, orange is like ten and the next color would be nine. Those [the white rods] are each, those are each a one cube, so I called it a tenth.
4.0.3	4:25	T/R 2:	Oh so you are thinking about it that way. I want to come back to that. Looks like everyone has something. So let me get a volunteer to tell me about this. We are starting out slowly but it's gonna get, it's gonna go faster. Let's see, I haven't had a chance to speak with Kimberly. Kimberly, can you tell me the number name we call the white rod?
4.0.4	4:46	Kimberly:	One tenth.
4.0.5	4:50	T/R 2:	One tenth. Ok? How many people agree with that [Hands raise immediately] Interesting, ok, looks like everybody's hand's up. Does anybody disagree with that? Just in case I missed a hand. Ok. Can you tell us about that? Would you like to come up here and show us? Ok. We have to be able to- we really have to be able to justify and convince other people what we did.
4.0.6	5:09	Kimberly:	I took ten of the white ones and just put them against there [the orange rod]. And I put them there and I counted them to see if it really came out to ten [agreement from class].
4.0.7	5:20	T/R 2:	Ok so you used ten white ones. Is that what everybody here did? Ok, alright. You don't have to finish lining them up then. We will believe that since everybody, it's established that everybody has that, I will accept that. Ok. Now let me ask you another question. Same orange rod now, ok, I am still calling it one, what do you think I'd call the red rod? What would be the number name for the red rod? [Students working on the problems and hands raise]
4.0.8	6:00	T/R 2:	Ok, I see people making, building an argument that's good. Ok. I think a lot of you are starting to know these so quickly. Ok. I see a bunch of people with hands up. Let's give everybody else a chance to build their model.

4.0.9	6:40	T/R 2:	Alright. Ok. Let's hear now, let's hear from somebody. I heard a very nice explanation up at the front here, ok, from Gregory. I want to see if the rest of you agree with it or if any one has done anything differently. Gregory if we are calling the orange rod one, the number name one, what would you call the number name for the red rod?
4.0.10	7:00	Gregory:	One fifth
4.0.11	7:02	T/R 2:	Ok, did everyone hear what he said, [Can you hear way in the back, Mark and Andrew, can you hear what he said?] Could you say it a little louder, Gregory?
4.0.12	7:09	Gregory:	One fifth
4.0.13	7:10	T/R 2:	One fifth he says. Ok. And can you tell us why that works?
4.0.14	7:13	Gregory:	Because five of the red blocks equal up to one orange block.
4.0.15	7:17	T/R 2:	Ok. He says he used five red blocks in order to get that same length as the orange, ok, so he's calling it one fifth. Ok, interesting. Alright, I have another one for you. Same orange rod, we're still calling it the number name one. What number name would I give to two of the whites? Two white rods?
4.0.16		Erik;	Two of the white rods? That's a red rod?
4.0.17		T/R 2:	Yeah, I'm calling the orange one, what number name would I give to two whites? Ok, I see some hands, some people think they know this. If you're not sure, you can talk to your partner. [Meredith whispers to David]
4.0.18	8:30	T/R 2:	Ok, I am hearing some interesting arguments here. I'm hearing some people that have some very interesting ideas about this. Um, let me see. Let me call on, uh, Mark. What are you going to call the two white rods?
4.0.19	8:50	Mark:	One fifth
4.0.20	8:52	T/R 2:	One Fifth! Ok. How did you come up with one fifth? Would you like to come up and show us? Andrew, would you like to come also? [Mark and Andrew come to the OHP]. I hear a couple of answers here and I'm confused, so I want to hear this.
4.0.21	9:10	Andrew:	Alright, the reason why we called it one fifth is we put the red one up to two whites, and you said, "what would you call the two whites?" and the two whites equal up to the red, so, so if you put the reds and line them up to the orange, it makes one fifth.
4.0.22	9:30	T/R 2:	Interesting! What do you all think about that? Do you all agree with that, that we could- what do you think? First of all I want to hear agrees. How many people agree that that looks like it would work? Ok. You all agree with that? Um, Audra you agree with that, you think that works? Jessica? Do you agree with that that it works? Ok, Is there any comment either of you would like to add to that to help us understand? Or are you just agreeing at this point? Audra, go ahead.

4.0.23	10:00	Audra:	Well, if you put two whites against the red rod, and you put two whites up against the red rod five times across it, you'd get the same amount of whites as would reds, so, and you got five reds before so you get five pairs of whites.
4.0.24	10:24	T/R 2:	Ok. I think I'm just not understanding your language. Would I be using five whites to get the same length as the orange?
4.0.25	10:33	Audra:	[shakes head]
4.0.26	10:34	T/R 2:	Ok, explain a little further to me. Cause you said the same number of reds and whites and that's the only thing that confused me.
4.0.27		CT:	Audra, explain what you just said.
4.0.28		T/R 2:	Yeah, because I think you have the idea, I think I'm just not understanding it.
4.0.29	10:48	Audra:	If you put all the whites you could up against it [the orange rod] and you, and you double them up… If you put two whites together to make one block, it would be a red block. And if you did that as many times as you could you'd get five times [Figure F-11-48].
4.0.30	11:22	T/R 2:	Ok. Ok. So you are saying I have what looks like the same, if I take two whites and think about them together, it's like a red? Ok, and that's what you are saying? I thought that's what you are saying? [many answer 'yes] Okay. That's very interesting. I've heard some other answers to this. I want to hear more that. Meredith you have your hand up?
4.0.31	11:30	Meredith:	I think it's two tenths.
4.0.32	11:32	T/R 2:	Ok that's what I heard Sarah and Beth say too. And I want to hear more about this argument as well. Ok. Um, why don't you, you two can sit down now. That's interesting. Ok. I am believing what I hear Mark and Andrew are saying but I want to hear what Meredith is saying too. Because I may believe that as well. Meredith you want to come up? Sarah and Beth do you want to come up? [They go to OHP].
4.0.33	11:57	Meredith:	Well, take the red away and you put this [white rod] up to the orange. When we did it before, we said that, we said that orange has ten whites, it measures up to ten whites. And if you put the whites up it would have ten. And two of ten is two tenths.
4.0.34	12:22	T/R 2:	Ok. Well now, what do you think? Do you want to add to that? Is that basically what you were arguing or do you want to add to that some more?
4.0.35	12:29	Beth:	Yeah. Um, since ten of these [white rods] all equal up to one orange, then if you took two of these it'd be two tenths because one would be one tenth and you just count one more and then you'd have two tenths.

4.0.36	12:55	T/R 2:	Ok, that's interesting. Sarah, do you want to add anything to that? That's was basically what you were telling me too. David how about you?
4.0.37	13:02	David:	Um, well, I think the same as Meredith.
4.0.38	13:04	T/R 2:	You think the same as Meredith. Ok, now I'm really confused because I believe Mark and Andrew and I believe Meredith, and David and Sarah and Beth. What do you all think about that? Some people are telling me that two of the white rods, the number name, if we put two of 'em together, would be two tenth, some of you are telling me it would be one fifth. How can that be? Brian?
4.0.39	13:33	Brian:	Well even that, even that two white cubes equal up to one red cube, it's still not, it's still not, like um, like um imagining that this was another red cube so I think it really is two tenths because it really, really is two tenths.
4.0.40	13:58	T/R 2:	Because you can see two there, is that what you're saying?
4.0.41	14:00	Brian:	Yeah, yeah, it, it also is one, it also is one fifth, but what you are seeing right here really is two tenths not a fifth.
4.0.42	14:11	T/R 2:	Do you think it's possible that those could both be, could each of these be a number name for the two whites if I am calling the orange a "one?" What do the rest of you think of that? Is it possible that those could both be number names that would work? ... Not too sure? Okay, this is a real dilemma. You see why I was confused. Ok, I'm hearing what Brian is saying though. Brian is saying he can see how it could be one fifth if we changed it to a red, if we took the two whites and changed it to red, but when he looks at the two whites, he thinks of two tenths, right?
4.0.43	14:46	Meredith:	Because there's two, there's only two, there's only two, there's not like, they're not joined together. If you want to join them together, you should use the red.
4.0.44	14:54	T/R 2:	Interesting. Okay. Alright. Okay, I think for now, I think we're going to leave this controversy and come back to it later on, but this is really very, very interesting. I want you to continue to think about it too and maybe if you come up with a reason why both of these seem to work or if you have an argument why one way or the other, maybe you can write to me about it in those math journals. I had such a good time, I read the math journals this weekend, which you all wrote, and some of that stuff really was absolutely wonderful. I learned so much about the way you think about things by reading that. So maybe that's something to write about even later this week. Think about that some more. Unless anyone else have anything to add to this. This is great. Ok, why don't you all have a seat. I have another problem for you. … This is a little bit different, but I am still going to go with the

			orange here. Ok, this time though, I'm going to call the orange "ten." The number name "ten." And I'm wondering if you could tell me the number name for white. Okay, I hear some little mumblings "oh this is easy, this is easy." I'm going to want to hear this. If you think you know, please raise your hand. Some of us will be around to kind of hear what you're thinking about it.
4.0.45			*SIDE VIEW*
4.0.46	16:26 S	Beth:	This is a ten, and ten of these equal to one.
4.0.47		:	[Sarah and Beth discuss the problem but the recording is inaudible.]
4.0.48	16:45	T/R 2:	It gets to be a really tough when I change the number name, doesn't it? You have to rethink the whole thing. Okay, think, why don't you discuss and think a little bit more about what we might call this piece [white rod], ok, of this, and I'll come back… I want to hear something thinking of some other folks. Okay, what do we think over here, you two?
4.0.49	17:05	Meredith:	This is only one because if you call the orange a ten and the one's equal to a ten when you call this [white rod] a one, then they're going to change to a one.
4.0.50	17:16	T/R 2:	So you'd give it a number name of one? So do you feel that by showing me this, you've proven that? That's interesting. David, what do you think? Are you in agreement with that or do you think it's something different?
4.0.51	17:27	David:	Well, I think I agree with Meredith. Because, if this is ten, then this would be one then because if you add ten of these up… then umm, there would be ten of these. So it's ten.
4.0.52	17:45	T/R 2:	Okay, alright, I'll buy that. Okay, let me talk to a couple of other people, then we'll get somebody to come up and tell us about this. Brian, what do you think?
4.0.53	17:52	Brian:	Well, umm, ten wholes because well these are originally tenths, and this is considered if this is considered ten, then this would be, this would be like switched around ten wholes.
4.0.54	18:11	T/R 2:	Okay.
4.0.55	18:12	Brian:	So it's like switched around.
4.0.56	18:15	T/R 2:	Okay, so if we are going to call this [orange] "ten," and the number name for this [white rod] would be…
4.0.57	18:18	Brian:	Ten wholes. One's. One.
4.0.58	18:22	T/R 2:	I'm confused now. Okay, this piece here, this one piece here, what are we going to call? The one white rod? Would it be called ten?
4.0.59	18:32	Brian:	Yeah. Well, if this was one, then that would be ten, but when it's switched around, this is… all, all ten, all ten tenths when you put them on here they're like they would be like ten wholes, even though this would be considered… Well,

			[chuckles] if this is supposed to be ten, you put then of these up then these would be ten wholes.
4.0.60	19:06	T/R 2:	Okay, but what, so then what would the name for one of them be? You said if I put then of them up, it's ten wholes.
4.0.61	19:12	Brian:	One. One whole. One whole… no… I don't know how to say it, but I think I know it. I don't know how to say it.
4.0.62	19:25	T/R 2:	You said something like one.. if this would be…
4.0.63	19:28	Brian:	Yeah, yeah, one. Well, if you put all of them up, there would be ten wholes, but…
4.0.64	19:37	T/R 2:	Let's just call them "ten". There would be ten of these. Okay, because we're giving them number names now. We don't want to say "wholes" or anything like that. We don't have any particular item, like pies, we are thinking about or anything. Just call it "ten."
4.0.65	19:49	Brian:	Ok, but, but didn't you call this [orange] a "ten?"
4.0.66	19:51	T/R 2:	Yeah, I called that a ten. That was my question. I asked you what you would call this [white rod].
4.0.67	19:56	Brian:	Oh, then this would be… one… I guess.
4.0.68	19:59	T/R 2:	That sounds like a good name for it, doesn't it? Okay, now let me just hear what these ladies over here have come up with.
4.0.69	20:04	Jessica:	Umm, we got, since umm, if you have, like if you put all these [white rods] up to this [orange], umm you would get umm ten, I think. Wait… Ten. And, and this one [white], just that would be one.
4.0.70	20:21	T/R 2:	So that's the number you would give it then?
4.0.71	20:23	Jessica:	Yeah because if you have ten, then [white] that would be one of them [orange].
4.0.72	20:27	T/R 2:	Okay. Alright. I'll buy that. Okay, I think a lot of people have come up with something. Let me just… What did you come up with?
4.0.73	20:38	Beth:	If ten of these [white] equals one of these [orange], then one of these [white] could be one whole because, and then ten of these would ten wholes.
4.0.74	20:51	T/R 2:	Okay, we'll just call them one and ten. We won't say wholes for now.
4.0.75	20:54	Beth:	Okay.
4.0.76	20:56	T/R 2:	Ok? So this would be one then. If I'm calling this ten, you're saying this would be one. So what would this be [puts two white blocks against the orange block]?
4.0.77	21:01	Beth:	That would be… two. And then three and four and five and six.
4.0.78		T/R 2:	Nice thinking.
4.0.79			*FRONT VIEW*
4.0.80	17:14 F	Dr. L.:	Got your hand up?

4.0.81		Alan:	I think that it would be one tenth, because it takes ten ones to make [points to orange rod]
4.0.82		Dr. L.:	Mmm hmm. [to Erik] What do you think?
4.0.83		Erik:	Well, I think it'd be one whole, because she said the orange would be one- ten, and if this is, this is one tenth it would be one whole, the white would be one whole.
4.0.84		Dr. L.:	Ok, we'd better find out what she said. Hmmm. That's interesting.
4.0.85		Erik:	Alan, what do you think?
4.0.86		Alan:	I think it's one tenth
4.0.87		Dr. L.:	[to Alan] Yeah, so you heard a different question than he heard, huh? Right? What was the question you heard Amy say, Alan?
4.0.88		Alan:	If this was, if the orange was one, I mean ten, what would this be? If this would be one
4.0.89		Erik:	You said it's ten.
4.0.90		Dr. L.:	Ok, so you heard the same question as Erik? If the orange was ten, what would the little white one be? You both are saying the same thing? Yeah? Why couldn't it be a tenth?
4.0.91		Erik:	Because if this was ten, if this was one, this would be one tenth, because if takes of the white ones to equal up to the orange rod.
4.0.92		Dr. L.:	Mmm hmm.
4.0.93		Erik:	But if this is ten wholes, this would have to be one, because it takes ten of them to equal this.
4.0.94		Dr. L.:	Mmm hmm, mmm hmm.
4.0.95		Erik:	And so it would have to be.
4.0.96		Dr. L.:	Mmm hmm. Interesting. Interesting, let's hear from someone behind you. [talks to Graham, conversation inaudible, Erik and Alan start to balance rods, CT talks to Jackie and Kelly, inaudible]
4.0.97			*WHOLE CLASS*
4.0.98	21:09 S		
21:47 F	T/R 2:	Okay.	Alright, I've gotten to hear the thinking of a lot of people and I hope I'm not interrupting anybody who's still thinking about this. Is there anybody who would like to tell me what they think about this?
4.0.99	21:22 S	CT:	Call on my girlfriend here. Jackie.
4.0.100	21:29	Jackie:	Umm. The white would be one.
4.0.101	21:30	T/R 2:	The white would be one. Okay. I heard a lot of people say they would call the white "one." Why would you call the white "one?"
4.0.102	21:38	Jackie:	Well because it takes ten ones to make up an orange.
4.0.103	21:42	T/R 2:	Do you all agree with that? Did you hear what Jackie said? Okay, Danielle didn't hear what you said, Jackie. Can you say it again please?

4.0.104 21:48 Jackie: It takes ten one's to make up an orange.
4.0.105 21:57 T/R 2: Did you hear that? Do you agree with that? She said it takes ten of these [white rod] to make one of these oranges. Okay, and if I'm calling the orange "ten," you're calling the white the number name…
4.0.106 22:09 Jackie: One.
4.0.107 22:11 T/R 2: Okay, is there any disagreement about that? See I sort of switched it on you. It took you a minute to rethink that one. Okay, you all knew, but I really did switch that one on you. Okay, let me ask you another question about that. Okay, my next question is if I call, okay, now we're really getting into a big number here. If I take the same orange rod we've been working with, but I change the number name again. This time I'd like to call it… fifty.
4.0.108 22:43 Class: Fifty??? [gasp]
4.0.109 22:46 T/R 2: I'm wondering if anybody could tell me the number name for yellow.
4.0.110 *SIDE VIEW*
4.0.111 22:48 : BREAK IN RECORDING - side view
4.0.112 23:09 T/R 2: [directed toward Meredith and David] That's an interesting argument. I think I would listen to that one.
4.0.113 23:15 Jessica: This is half of it [orange] and there's this too. So if that's fifty, half of fifty is twenty five.
4.0.114 23:24 T/R 2: Do you agree with that Laura?
4.0.115 23:25 Laura: Yes [nod]
4.0.116 23:26 T/R 2: Okay, is that the way you thought about it? You lined these up along these here?
4.0.117 23:28 Jessica: Yeah, I lined these up along that, and I was going to see.
4.0.118 23:33 T/R 2: Interesting. Okay. Alright.
4.0.119 23:44 Dr. L: What did you think David?
4.0.120 23:45 David: Well, I think that it's twenty five. If this [orange] is fifty, this would be half, this [yellow] would be half of the orange.
4.0.121 23:55 Dr. L.: That's kind of what you [Meredith] said, huh?
4.0.122 23:57 David: [inaudible] If this would be a hundred, this would be fifty.
4.0.123 24:28:00 T/R 2: [to Sarah and Beth] That's really good thinking. Okay, so then you think then it's twenty five. And did you have the same argument for that, Sarah, for why that works? The same as what Beth said?
4.0.124 24:37:00 Sarah: [Nod] Mmm hmm
4.0.125 24:38:00 T/R 2: Do you want to add anything to that?
4.0.126 24:40:00 Sarah: It's true [giggle]
4.0.127 24:41:00 T/R 2: It works and it's true. Okay.
4.0.128 24:43:00 T/R 2: [To class] Is there anybody else who hasn't had a chance to talk to Dr. Landis, Mrs. Phillips, or me?
4.0.129 24:58:00 : Meredith & David make balancing towers out of the rods. Brian does as well.

4.0.130 25:19:00 Brian: [After balancing his tower] I got it! I got it! I got the balance beam!

4.0.131 *FRONT VIEW*

4.0.132 [camera focuses on Michael's model, an orange rod and two yellow rods. Then, it focuses on Gregory's models, an orange rod and two yellow rods, and a model of an orange rod and five red rods. Alan and Erik talk about balances]

4.0.133 Dr. L.: What'd you come up with, Alan?

4.0.134 Alan: Um, twenty-five, because this is fifty, and the yellow is half of this, half of fifty is twenty-five.

4.0.135 Dr. L.: That's what I heard them saying at the other table. Erik, you agree with that?

4.0.136 Erik: Yep.

4.0.137 Dr. L.: Yeah? Well, it doesn't sound like that was too hard for you.

4.0.138 Erik: Not at all, not at all.

4.0.139 [to Gregory and Danielle] What are you doing up here? What do you think?

4.0.140 Danielle: [inaudible]

4.0.141 Dr. L.: And what was your reasoning?

4.0.142 Gregory: The yellow was half of orange

4.0.143 Dr. L.: Ok.

4.0.144 Gregory: And you divided that [inaudible, Dr. Landis nods]

4.0.145 25:40 F T/R 2: [To class] Is there anybody else who hasn't had a chance to talk to Dr. Landis, Mrs. Phillips, or me? Alan.

4.0.146 Alan: I think it's twenty-five, because if this is fifty, half of it would be twenty-five.

4.0.147 T/R 2: Ok, interesting, is Erik thinking like that too?

4.0.148 Erik: Yeah.

4.0.149 T/R 2: Ok. Alright. Let's see if I can hear from Graham [approaches Graham and Jacquelyn] What do you think?

4.0.150 Jacquelyn: [also on side view] Well, to make it even, if we had fifty cents, we have two quarters, we take half, um, fifty cents this would be twenty-five and twenty-five

4.0.151 T/R 2: Mmm hmm, ok I agree with that.

4.0.152 *WHOLE CLASS*

4.0.153 25:51 S

26:44 F T/R 2: Ok, I want to hear some arguments now. Okay, I've had a chance to listen to a lot of people and I think that everybody's had a chance to sort of practice their argument on one of us here. Okay, let's see. We're calling the orange rod the number name "fifty." How about the yellow rod? What number name will we give it? I would love to hear from somebody that I have not had a chance to hear from. Okay, let's see here, is there anybody here I haven't had a chance to hear from? Michael.

4.0.154	26:19:00	Michael: Well, I think I would call the yellow "twenty five" because twenty five plus twenty five equals fifty.
4.0.155	26:27:00	T/R 2: That sounds, that sounds interesting. Okay, so what is twenty five and twenty five, what made you decide on using twenty five? Umm, why didn't you say ten plus ten plus ten plus ten plus ten equals fifty?
4.0.156	26:37:00	Michael: Because it takes two yellow rods to equal one orange rod.
4.0.157	26:42:00	T/R 2: Oh, okay. Does anyone want to add to that? ... That's very nice. I want to hear, I heard people say it slightly differently, but that's the, that's really a very important idea. Beth.
4.0.158	26:54:00	Beth: Umm, me and Sarah thought, first we thought it'd be umm twenty and thirty, but we knew we couldn't do that because they [yellow rods] were exactly the same size, the yellows. So then we started thinking in cents and we thought of two quarters equal umm fifty and so one twenty five and twenty five equals fifty.
4.0.159	27:19:00	T/R 2: Interesting. Did anyone think about it like that in terms of money? I heard some people say that where they thought about it like two quarters being like fifty cents. Jackie, you thought about it that way. Anyone else think about it that way? That's very nice. Oh, I thought I was going to stump you with that one. Okay, let me ask you another question. Let's keep the name fifty [for the orange] for a minute. The number name "fifty" for this [orange]. Okay, now this is going to be a little tougher. What number would I give to one white rod if we're still calling the orange the number name "fifty?"
4.0.160		*SIDE VIEW*
4.0.161	27:56:00	Sarah: [Whisper] Five, ten, fifteen, twenty … Oh!
4.0.162	28:04:00	T/R 2: Okay, I want to hear, now take a little time, think about it and talk to your partners.
4.0.163	28:07:00	Sarah: We would call these little [white] ones five, ten, fifteen, twenty, twenty five, thirty, thirty five, forty and then [adds two more] fifty. And you get fifty out of ten.
4.0.164	28:23:00	T/R 2: That's interesting. How did you come up with it so quickly?
4.0.165	28:25:00	Sarah: I did [smile].
4.0.166	28:26:00	T/R 2: You just thought of it? What made you think of five's? Why didn't you start counting by two's or ten's?
4.0.167	28:31:00	Sarah: Because. Ten's, because ten's would only need five. So it would be ten, twenty, thirty, forty, fifty. So for ten's it would be five. And then for two's, two, four, six, eight, ten, and it would go on and be more than ten.

4.0.168 28:47:00		T/R 2: And five just popped into your head as being the right one?
4.0.169 28:48:00		Sarah: Yeah
4.0.170 28:49:00		T/R 2: Very nice. That was nicely done. Let me hear what some other people are saying.
4.0.171 28:55:00		Meredith: I think it is five because, cause five times ten equals fifty.
4.0.172 28:59:00		T/R 2: [To David] What do you think?
4.0.173 29:01:00		David: I think that it's five too.
4.0.174 29:02:00		T/R 2: You think five works?
4.0.175 29:03:00		Meredith: Yes.
4.0.176 29:04:00		T/R 2: You all thought of that so quickly. How did you think of it so quickly? That's what I'm wondering.
4.0.177 29:08:00		Meredith: We know our, we know our division? Ha ha, we know out multiplication tables.
4.0.178 29:14:00		T/R 2: Do you fell that is pretty convincing? Do you really feel you could convince people that we would be calling the white rod the number name "five?"
4.0.179	Meredith:	Uh huh.
4.0.180	T/R 2:	Okay, I'm going to hear from some other folks. Let me go over here first and then I'll come back to you Brian. Laura and Jessica always get to see me last. Here, let me come over and see then next.
4.0.181 29:33:00		Laura: Five. You count by fives. Five, ten fifteen, twenty, twenty five, thirty, thirty five, forty, forty five, fifty.
4.0.182 29:41:00		T/R 2: Ok, that's interesting. What made you think of five so quickly? I'm asking people that around here.
4.0.183 29:46:00		Jessica: Well we tried, like, we just though, umm, like… well, we tried umm two's, and that didn't work. So then we just thought five [giggles].
4.0.184 29:57:00		T/R 2: And it worked?
4.0.185 29:59:00		Jessica: Yeah.
4.0.186 30:00:00		T/R 2: Very nice. Brian, let's hear what you have here.
4.0.187 30:03:00		Brian: I got the answer, but I don't know how to say it… explain it really.
4.0.188 30:06:00		T/R 2: Well do you want to try? It's a good chance to practice talking to me.
4.0.189 30:11:00		Brian: Well, I just found out, five, I don't know, I can't find out why. I think, well, five times ten equals fifty, so …
4.0.190 30:28:00		T/R 2: Does that have something to do with it? Why times ten? Why are you telling me five times ten? That's interesting.
4.0.191 30:34:00		Brian: Well, well because there are ten of them here [white rods] and… hm, I'm not sure, I'm not sure. Before I said one fiftieth, but I think, Dave explained something to me and I

thought, I guess I just thought it was right. But, I guess I'll just go with one fiftieth.

4.0.192 31:00:00 T/R 2: You're going to go with that?
4.0.193 31:01:00 Brian: Yeah, one fiftieth.
4.0.194 31:02:00 T/R 2: One fiftieth?
4.0.195 31:03:00 Brian: Yeah, but I'm not that sure.
4.0.196 31:05:00 T/R 2: Ok. It just seems like a good name to call it because you're calling this [the orange rod] fifty?
4.0.197 Brian: Yeah
4.0.198 T/R 2: Is that what you're thinking?
4.0.199 31:12:00 Brian: Yeah, and if there are fifty of them in there, I guess I just call it one fiftieth.
4.0.200 31:18:00 T/R 2: Are fifty of what in there? Fifty of what kind of what?
4.0.201 31:24:00 Brian: Um….Fifty…
4.0.202 31:28:00 T/R 2: Fifty of these here [white rods]?
4.0.203 31:30:00 Brian: Well no, but, but if we're calling this [orange rod] fifty, and there are each one in there, then it's [white rod] pretty much called a fiftieth, I guess.
4.0.204 31:41:00 T/R 2: Hmm, ok so that's the number name you've given it.
4.0.205 31:44:00 Brian: Yeah.
4.0.206 T/R 2: Ok.
4.0.207 *FRONT VIEW*
4.0.208 Erik: One whole - no.
4.0.209 Alan: Five. Five ten fifteen twenty twenty-five thirty thirty-five forty forty-five fifty. And the orange is a ten. Five.
4.0.210 Erik: Yeah.
4.0.211 [camera focuses on Graham's desk, where he has arranged rods vertically in order of height. Dr. Landis speaks with Caitlyn and Brian2, CT speaks with Mark and Andrew]
4.0.212 Alan: [Dr. L. approaches Alan]. Twenty-five.
4.0.213 Dr. L.: How come? How did you get it?
4.0.214 Alan: Because ten fives equals to fifty.
4.0.215 Dr. L.: Ah hah.
4.0.216 Alan: It takes ten white ones to make this and ten fives equals to fifty. So that's why I called this five.
4.0.217 Dr. L.: Ah hah. How about you? What did you, what did you call it?
4.0.218 Erik: I got the same thing.
4.0.219 Dr. L.: And how did you get it? Did you get it the same way or a different way?
4.0.220 Erik: Uh, well actually I think I got it the same way.
4.0.221 Dr. L.: You think you got it the same way?
4.0.222 Erik: Yup.
4.0.223 Dr. L.: Ok.

4.0.224	Erik:	[Alan and Erik begin to play pool with the white and orange rods]
4.0.225		*WHOLE CLASS*
4.0.226 31:50 S		
32:43 F T/R 2:	[To class]	I got to hear a lot of interesting arguments. Is this a good time for me to ask people what they came up with or are there still a lot of people working here? People pretty much look like they've thought about it some. I'd like to hear some of your ideas. Ok for the number name of this. Remember this is called fifty, the orange. Ok, let's see, um, Sarah. Beth, you come up with her? Please everybody listen to this and see if this is an argument that's like yours, because if yours is different, you know, I'd like to hear it.
4.0.227 32:27:00		Sarah: If we call the white cube five, then it would equal up to fifty.
4.0.228 32:33:00		T/R 2: Ok, so you're going to call the white the number name five. How many people called it five? Interesting. Ok, it looks like a lot of people did. Did anybody call it something different? Ok let's hear their argument maybe this will convince us one way or another that this works.
4.0.229 32:52:00		Beth: If you count by fives to fifty, you'll have ten; you'll count, you'll have ten, you have to have five ten times to get to fifty. And then…um…
4.0.230 33:20:00		T/R 2: Is there anything else you want to say about that? Or does that basically say it? … What do you think? Would anybody like to add to that? They said they counted by fives to get up to fifty. Ok. And that it worked. That it brought them up to fifty, when they counted each of those as a five, those white ones. [calls on Meredith]
4.0.231 33:51:00		Meredith: Well, I did it sort of equivalent. I did five times ten equals fifty.
4.0.232 33:58:00		T/R 2: Five times- ok, Meredith said she did five times ten equals fifty. Ok, why did you, why did you decide to do that, to use a multiplication problem to help you?
4.0.233 34:06:00		Meredith: Because this is a ten rod [the orange rod]. It has ten ones. Ten times five equals fifty. So I said ten times what gives you fifty? And five, gives you ten times five equals fifty.
4.0.234 34:31:00		T/R 2: Interesting. All right. Ok does anyone have any other arguments that they want to add to that?
4.0.235 34:36:00		Dr Landis: Caitlin started to say something and, and I found it fascinating and I wonder if anyone else could follow what she was starting to say. Listen carefully, because I'm not sure I got it totally, but I was fascinated by it.
4.0.236 34:49:00		T/R 2: I would like to hear what she's saying.

4.0.237 34:50:00 Dr. Landis: She was trying to say that she remembered when this was called a ten [holds up an orange rod] and when this was called a ten she remembered this little one was called what [holds up a white rod]? [Some students respond "one"]. A one. So she said if this was a ten, this was a one. So she said if this was a fifty, this would be a five. Now I don't know how she figured that out but it was pretty interesting because it's the same answer that you were getting, huh?

4.0.238 35:17:00 T/R 2: What do you think about that? Does anyone have any idea of how that works? That sounds like a different way of going about it to get, to get five. What do you think? Did you do that? Sarah says that she did something like that.

4.0.239 Dr. L.: Really?

4.0.240 T/R 2: Can you tell us a little bit more about that?

4.0.241 35:38:00 Sarah: It's the same thing.

4.0.242 35:40:00 T/R 2: It's the same thing that she did. Which was what? So that I understand. [giggle] Ok. All right.

4.0.243 35:51:00 Beth: Another way you can think of it, if you want to think of it as cents again, you can think of it as nickels and then, this, um, how I thought of it, I thought of it to how many nickels would add up to ten and then, and then how many tens would add up to fifty.

4.0.244 36:17:00 T/R 2: Ok. That's interesting. Ok. You know what I'd like to do at this point? I'd like to move away from this, I'm getting tired of the orange rod. [Student - me too] Let's move away from this. It was very interesting. We went back and watched Dr. Maher watched the tape of all of you working on Friday on that last problem we were working on. Does anyone remember the problem that we were working on when we were building models at our desks? It had to do with comparing fractions. Does anyone remember that? Ok, a couple of people are nodding their heads. Does anyone feel confident enough that they remember it to share it with us again? Ok Meredith thinks she does. I saw a couple people nodding. How about, anybody else? Remember what we worked on? It goes back I know it was Friday. It's ancient history already. David and Meredith feel ready. Erik, do you remember too? Ok, let's see anybody else. We need somebody to refresh our memories about the problem because I want to think about it some more. Uh, let's see… David.

4.0.245 37:27:00 David: Well, um, I think we were, I think that we were trying to see which was bigger… I think like um one third or, um, uh, one half or something.

4.0.246 37:53:00	T/R 2: We have some dispute as to what the numbers are but we were doing a problem which is bigger and Beth?
4.0.247 38:00:00	Beth: We used chocolate to see if, to see um, which one was bigger and a smaller bar and a bigger bar and if one third- which would be bigger one third or one half and if you were talking about a bigger bar it would be one half because the little bar, I mean it would be one third for the big bar cause the littler bar's half was smaller than one third.
4.0.248 38:36:00	T/R 2: Do you remember that story with the candy bars? And remember we agreed on some rules for the candy bars. When we were comparing one half and one third what was the rule we agreed on in order to be fair? Ok, what was something that we agreed that we would do if you want to think of it in terms of candy bars that's fine. Ok that might even help to think about it because you feel more invested if it's something like a candy bar. What were the rules? Does anybody remember what we said if we're going to compare one third and one half in order to be fair what is it that we have to do? We have to set up some ground rules right? Amy?
4.0.249 39:13:00	Amy: You can't switch the object.
4.0.250 39:15:00	T/R 2: Ok can you say a little bit more about that?
4.0.251 39:18:00	Amy: If we are using chocolate bars and you have two chocolate bars you can't change to the other one. If you're using like a big one, you can't change to a little one. You have to stay with the one you were using in the beginning.
4.0.252 39:30:00	T/R 2: Unless of course you were tricking your little brother or sister, but we agreed we weren't going to do that here. We agreed that we were going to use the same size thing to make our comparisons. So if it's a medium sized chocolate bar and we're taking the half and the third, we agreed that we would stick with the medium sized chocolate bar and not switch to a littler one or a bigger one, right? Ok. Now just to get you back on track there I'd like you to think about that again. I see some people actually building some things already. But we said that the problem we were exploring was which is bigger a half or a third. The important thing was we said we wanted to be able to convince other people that our argument works. Alright, so I'd like you to, maybe you could talk with your partner and work on that for a little while and we'll be around to talk to you, Dr. Landis and Mrs. Phillips and myself. Because once you think you have a good argument what I'm going to want you to do is I'm going to want you to write that argument. … If you use blocks maybe actually trace them onto a piece of paper. We're going to give you pens and paper for this, and

actually show us what your argument looks like on paper. But first we want to hear these arguments so why don't you take a couple of minutes to think about that problem again. Which is bigger a half and a third- or a third? Then we can talk about the difference between those two sizes. Take a little time to talk about it.

4.0.253		*SIDE VIEW*
4.0.254	40:50:00	Sarah: [Raises hand, T/R 2 approaches. inaudible]
4.0.255	40:54:00	T/R 2: [to Sarah] I want to see what you have built to show me that.
4.0.256	40:58:00	Sarah: A half would be this [She has two yellow rods next to an orange rod]. And then a third would be probably [looks at different size rods] oops, one third, nope not that one.
4.0.257	41:16:00	T/R 2: Ok, experiment with that. I want to see a model.
4.0.258	41:19:00	Sarah: [To Beth] We can't find any [inaudible]
4.0.259	:	David and Meredith are building a balance beam. [difficult to hear]
4.0.260	41:34:00	David: [To Meredith] Try something else.
4.0.261	42:13:00	T/R 2: Your hands aren't as steady today as they were on Friday.
4.0.262	:	[They are attempting to balance rods on top of each other, but their structures keep falling down.]
4.0.263	42:43:00	T/R 2: It looks like you're using light greens and reds, David?
4.0.264	42:47:00	David: What I did was, light greens were one half, and the reds are one third and the dark green was one whole.
4.0.265	43:02:00	T/R 2: Ok so this is, we're calling one [dark green], you're calling these [light green] each a half and you're calling this [red] a third. So when you go to compare them how do you do that?
4.0.266	43:13:00	David: Well, I made a balance to see which was bigger.
4.0.267	43:18:00	T/R 2: Ok, do you want me to let you get your balance together and then I'll come back?
4.0.268	43:24:00	David: I'll get it together soon.
4.0.269	43:25:00	T/R 2: Ok.
4.0.270	:	[T moves back to Sarah and Beth.]
4.0.271	43:30:00	Beth: [Mid sentence] … dark green. You'd use two light [green] cubes to make one half of it and then you take three reds and you'd have one third of it.
4.0.272	43:42:00	T/R 2: Interesting. Ok, so ….
4.0.273	43:45:00	Beth: And then we found a different way if you balance it and you put two greens on one side and three reds on the other side and they balance.
4.0.274	43:59:00	Sarah: It's hard to make it balance.
4.0.275	44:02:00	T/R 2: So you're making it balance sort of like David's doing but it's a little different.

4.0.276	:	[They attempt to make similar balancing structures, but they keep falling down.]
4.0.277	44:27:00	Beth: Oh I know how to do this. I know. One half would be this big [holds up light green] and one third would be this big [holds red next to light green].
4.0.278	44:36:00	T/R 2: So you're comparing them. How much bigger is- by how much?
4.0.279	44:43:00	Beth: I'd say one unit [She holds up a white rod].
4.0.280	44:46:00	T/R 2: Ok, I have tougher question for you to think about. … Let me ask you something now. Ok, Beth says that she's calling this one half [light green] and that's a third [red]. And I asked her what the difference between the two was. What number name are we going to give to this [white]? You gave this a number name of half and this one the number name of a third, what number name are you going to give to this little guy [the white rod]? And can we prove it? Is there a way you could prove it?
4.0.281	45:39:00	Beth: It is half of one third.
4.0.282	45:42:00	T/R 2: It is half of one third? Hmm. Ok. Well, what number name can we give this if we had to give it actual fraction names, well, how did you find that this is a third and this is a half?
4.0.283	45:57:00	Beth: First we saw how many equal one third of the green and that was two so we knew that was a half. And then, and then it might not work unless there is something of three that fits on the, on the dark green and that would be the red, cause you can fit three.
4.0.284	46:30:00	T/R 2: Can I ask you something then, can we do the same thing for the white rods? To find out what the number name might be? Would that work?
4.0.285	46:39:00	Beth: Yeah.
4.0.286	46:40:00	T/R 2: What would we have- where would we have to place them in order to know what the number name might be? [Beth adds six white rods to her model of a dark green, two light greens, and three reds- Figure S-47-19]
4.0.287	47:18:00	T/R 2: So what number name would we give to the white? If you know that, you're calling this one, you told me, this a half, this a third. What might you call the white rod? Go ahead, Sarah
4.0.288	47:30:00	Sarah: One sixth, one sixth.
4.0.289	47:38:00	T/R 2: Ok you say that very uncertainly, but why do you think that?
4.0.290	47:43:00	Sarah: 1, 2, 3, 4, 5, 6 (she points to the six white rods and counts).
4.0.291	47:46:00	Beth: It is six and it makes up to the green block.

4.0.292	47:48:00	T/R 2:	Sure, right? And we are calling this one, right? Ok, so what is the difference between a half and a third then?
4.0.293	47:55:00	Sarah:	A half is much larger than a third
4.0.294	47:56:00	T/R 2:	By how much?
4.0.295	48:00:00	Sarah:	By uh, by this! (holding up a white cube)
4.0.296	48:02:00	T/R 2:	Which is how much? Which is what number name?
4.0.297	48:05:00	Sarah:	One sixth!
4.0.298	48:07:00	T/R 2:	Ok, do you think you can write that up for me and trace your rods and explain it? Ok, I am going to give you pens and paper. This is great. don't worry if you need more sheets of paper, it is up there. Put your names on it and the date.
4.0.299	48:27:00	Sarah:	Let's draw pictures.
4.0.300		T/R 2:	Put your names on this, too. And the date.
4.0.301	48:33:00	Beth:	First we can draw this balance, then this….first draw this because this is the thing that she wants us to do.
4.0.302	48:44:00	Sarah:	I'm not going to be exact. It is not a perfect fit. It is just an inch away.
4.0.303	49:39:00	T/R 2:	Ok, that is a good point, is there another solution to this?
4.0.304	49:43:00	David:	I'll do a see-saw
4.0.305	49:45:00	T/R 2:	Ok, while you are working on that Meredith, David will show me this.
4.0.306	49:50:00	David:	This is what I did on Friday and the dark green will be the whole thing and the light green will be the half.
4.0.307	49:56:00	T/R 2:	The whole thing, what number name is that?
4.0.308	49:58:00	David:	Umm, that'd be one.
4.0.309		T/R 2:	Ok.
4.0.310	50:05:00	David:	And the red would be one third. What I did before, when I was ready, I took off two of the reds and one of the, the light green.
4.0.311	50:16:00	T/R 2:	Well, why did you take off two of these off (pointing to the reds)?
4.0.312	50:19:00	David:	Well, because then there is only one piece. I thought it would fall to the right. And on Friday, that's what happened.
4.0.313	50:27:00	T/R 2:	You predicted that would happen. Do you want to try it?
4.0.314	50:30:00	David:	I am not sure if it will because of these two pieces.
4.0.315	50:35:00	T/R 2:	Because of the double support?
4.0.316	50:47:00	T/R 2:	Why do you think it didn't fall this time? I remember when you did this on Friday.
4.0.317	50:50:00	David:	Because I have more support and probably you need it a little wobbly to fall.
4.0.318	50:54:00	T/R 2:	So then can I ask you which is bigger a half or a third?

4.0.319 David: A half.
4.0.320 51:00:00 T/R 2: Can I ask you another question? Meredith I want you to think about this also. David showed me the model and he has said that a half is bigger than a third. And I am asking by how much. How much bigger is it?
4.0.321 Meredith: Well, it's bigger by…
4.0.322 51:26:00 T/R 2: Can you use the rods to show me?
4.0.323 Meredith: (Explaining but recording is difficult to hear. She is balancing 3 reds and 2 light green on a orange see-saw)
4.0.324 52:06:00 T/R 2: Ok, there is a difference there. How much bigger do you think that half is than that third? If we had to give a number name for, for what's missing there. In other words, you are getting a smaller piece if I give you a third of that candy bar rather than if I give you a half. I don't understand that, can you explain that.
4.0.325 52:28:00 T/R 2: David, are you listening to this too, cause I really want to understand this. Maybe you can help me.
4.0.326 Meredith: If you put another rod there it's going to be cutting it there and another rod there and cutting this one in half and taking this off, breaking this in half, and put one on this, it would be the same. But since we put it here and there, it's not the same.
4.0.327 53:11:00 T/R 2: Interesting. Rather than break one of these red ones in half, what else could we put there to make them the same length.
4.0.328 53:20:00 Meredith: [places a white rod in the missing area]
4.0.329 53:24:00 T/R 2: So a half is this much bigger?
4.0.330 53:37:00 T/R 2: Ok, David what do you think now? We're looking at this, we are comparing the half and the third and we see if we put a white rod up here it makes the third the same size as the half. You see that? Could we come up with a name for that? What the difference is between the two? One is bigger than the other, obviously. What would be a good number name for that?
4.0.331 54:04:00 David: Um, maybe…
4.0.332 54:08:00 T/R 2: Or could we go back and figure out what the number name would be for that?
4.0.333 54:13:00 David: If this was the whole thing, like one [light green] then this [red] could be three fourths.
4.0.334 54:19:00 T/R 2: Ok, but is this the whole thing? Remember, this is how much of the bar?
4.0.335 54:24:00 David: This is a half (green).
4.0.336 54:26:00 T/R 2: And this was established was a third (red). You showed me that, right here. Ok. The question is, can we come up with a number name for the difference between a half and a third.

4.0.337		David:	Let's see, maybe… we could do something like uh…
4.0.338	55:06:00	T/R 2:	It really is something to think about, because it must have a number name.
4.0.339	55:10:00	David:	Like 1.5 or something [starts talking about balance again, Figure S-55-53]
4.0.340	55:51:00	T/R 2:	Ok, well I want to you to think about this question again, about what we would call this difference between the two. Ok? Think about that, maybe you two can think about that.
4.0.341	56:02:00	T/R 2:	[to Brian, Jessica, and Laura] Ok, I want to get to the both of you, we are running out of time. Why don't the three of you come together here, pull your chair over here Brian.
4.0.342		Jessica:	Well, we got two answers.
4.0.343		T/R 2:	Ok, I want to hear about this.
4.0.344		Jessica:	Well, she didn't have enough.
4.0.345		T/R 2:	Who would like to share?
4.0.346	56:19:00	Jessica:	Well, first we got this one. This would be one half and this would be one third and that would be one third.
4.0.347	56:28:00	T/R 2:	Ok, show me, show me. Hold up, one half and one third.
4.0.348	56:33:00	Jessica:	One half and one third [Figure S-56-33].
4.0.349	56:36:00	T/R 2:	And how do you know that?
4.0.350	56:39:00	Jessica:	Cause you take, cause this, these two, that is one half if that green (dark) and so is that. So, and then that would be one of the half's and this is all one thirds of it. And this would be one of the thirds.
4.0.351	56:56:00	T/R 2:	Ok, and Brian, you worked on both of these two, can you explain this model to me?
4.0.352	57:00:00	Brian:	Well, this is…I have to put this together…this was the whole candy bar.
4.0.353	57:06:00	T/R 2:	You guys made a train. That is neat. Ok.
4.0.354	57:10:00	Brian:	And this would be the half, and this would be the third. And uh, the third is smaller and the half is bigger. The third is smaller because if, you have to make three of them, you have to make, to make it a third you have to have three of them in one, you have to have three of them in one whole, but there is less room for three of them so you, and you have more room for a half so half would be bigger.
4.0.355		T/R 2:	Oh, ok.
4.0.356	57:39:00	Jessica:	Yeah, that's what we got two, this would be the whole and that would be the half and that would be the third.
4.0.357	57:44:00	T/R 2:	I wanted you all to write these solutions. Do you think you could remember them because we are probably going to have to turn it into an assignment?
4.0.358	57:51:00	Jessica:	Could we just maybe just draw something?

4.0.359	57:52:00		T/R 2: Yes. [brings paper] We are just about out of time but if you want to maybe start it, if you want to jot down an idea
4.0.360		Jessica:	Could we trace them and color it?
4.0.361		T/R 2:	Yeah if you want to trace them that's fine. [talks to Danielle and Gregory about their drawings]
4.0.362	58:48:00		Meredith: I have another model, I have another model to show which one is bigger. If you put one red one there and a green one there and then you put an orange rod. It goes up because this is larger.
4.0.363	59:02:00		T/R 2: But my next question to both of you to think about, because I think you both really understand this, is what's the difference between those two? What would be the number difference between those two, between the half and the third?
4.0.364	59:17:00		Meredith: If you take off the one, it would be a difference of two ones. But when you take, but if you take this one off…two of these, and one of these off.
4.0.365	59:49:00		T/R 2: There is definitely a difference there, you can see it.
4.0.366	59:59:00		Meredith: And then, it's a difference of two.
4.0.367	1:00:01	T/R 2:	Two what? What are we calling these ones? I am confused; I thought we were calling this [dark green] one?
4.0.368	1:00:08	David:	And then probably we would call this, we would call that sixths. One sixth.
4.0.369	1:00:19	T/R 2:	What would we call that?
4.0.370	1:00:22	Meredith:	A difference of two sixths. [Figure S-1-00-29]
4.0.371	1:00:24	T/R 2:	There is a difference of two sixths between a half and a third? What do you think David? I heard one sixth and I heard two sixths and I want to know before I go, then, what it is.
4.0.372		Meredith:	(Meredith is moving pieces around) Like I said if you would separate 'em. And you gave one of these [red], one of these [puts one red rod in front of David, one in front of T/R 2, and keeps one] and two more kids [places two green rods in separate places on her desk] then you'd have more.
4.0.373		David:	You sure would
4.0.374	1:01:18	T/R 2:	And how much more? But you were starting to build a model there. What do you think David? Meredith thinks two sixths. This difference between a red and a …because she told me if I put two of them here it should balance? Is that what you are telling me Meredith? If you put two little white ones in there.
4.0.375		CT:	Boys and girls, continue this as homework…
4.0.376	1:01:48	T/R 2:	Ok.
4.0.377		Meredith:	[Inaudible]
4.0.378	1:01:59	T/R 2:	So what's the difference here? Between the red and the green? Right, but if, now I took it away again then what would be the number name for the difference.

4.0.379	1:02:11	Meredith:	One in each handful, one in each third.
4.0.380	1:02:19	T/R 2:	Ok, one of those white blocks. What is the number name for that, I forgot, what are we calling the white block?
4.0.381		David:	Um, one sixth.
4.0.382	1:02:25	Meredith:	Because there's two one sixths.
4.0.383	1:02:34	David:	I was calling this one sixth because six of these add up to one, one whole.
4.0.384	1:02:48	Meredith:	And if you only had one sixth, if you said it was only one sixth then you wouldn't have one for that. So it has to be two sixths.
4.0.385	1:02:58	T/R 2:	What if the question is though that instead of asking you to look at the whole candy bar then I am just asking you to compare the amounts of a half and a third of the candy bar?
4.0.386	1:03:09	Meredith:	One half? Then the difference would be one sixth for each half.
4.0.387	1:03:14	T/R 2:	I understand. That is very nice. Thank you. Ok, we're going to be writing about this. Try to remember this for tomorrow when we will be writing about this, ok?
4.0.388			*FRONT VIEW*
4.0.389	41:37 F	Erik:	Wait, which is bigger, a half and, or a third?
4.0.390		Alan:	Yeah, which is bigger a half or a third
4.0.391		Erik:	A half, no? Which is bigger, a half or a third?
4.0.392		Alan:	That's what she said.
4.0.393		Erik:	It's easy
4.0.394		Alan:	That's what she said. That's what she said.
4.0.395		Erik:	Which is bigger, a half or a third. There. Half [Erik's model - a brown rod and two purple rods. red rods and takes them away]. No. [Puts down green rods, takes them away] Wait a minute.
4.0.396		Alan:	We need to make a train again.
4.0.397		Dr. L.:	What do you mean, to make a train?
4.0.398		Alan:	[using a model of an orange and yellow train?] This can be halved and third [exchanges the yellow rod for a purple rod]
4.0.399		Dr. L.:	What are you measuring?
4.0.400		Erik:	I'm trying to see which is a third
4.0.401		Dr. L.:	Trying to see what?
4.0.402		Erik:	I'm trying to divide it into thirds and halves
4.0.403		Dr. L.:	Ah hah, ok. [to Alan, who is balancing rods] Well, how can you convince me that's a half and that's a third?
4.0.404		Alan:	Well the bigger half it will fall to the side with that bigger half.
4.0.405		Dr. L.:	I see you're using two different size rods but how do I know
4.0.406		Alan:	There's the half
4.0.407		Dr. L.:	That's a half

4.0.408	Alan:	And there's the thirds. Now that would be a third of the blue rod, no, but you can't make the orange rod into thirds [Alan's model - Figure F-44-27].
4.0.409	Dr. L.:	Oh. Well then what do you do? Are you allowed two different rods? Can you use the blue rod to get your thirds and can you use the orange rod to get your halves? Yeah, you think so? What do you think, Erik?
4.0.410	Erik:	What?
4.0.411	Dr. L.:	What Alan is saying is he's going to use the orange rod to get his halves and then he's going to use the blue rod to get his thirds and then he's going to compare the third with the half.
4.0.412	Erik:	You can't compare the third with the half. See if you're using the orange rod-
4.0.413	Alan:	Either way if you make your own
4.0.414	Erik:	Alan, if you're using the orange rod for the halves, the halves are going to be a yellow.
4.0.415	Alan:	Right
4.0.416	Erik:	And if you're using the blue rod for your thirds, you can't compare them. Because the blue rod's smaller than the orange rod
4.0.417	Alan:	Nevertheless, even if you make a new rod to make the orange into thirds, it still wouldn't be as big as the yellow.
4.0.418	Dr. L.:	Wait, Graham is saying what?
4.0.419	Graham:	You can't do it.
4.0.420	Dr. L.:	You can or you can't?
4.0.421	Graham:	You can't
4.0.422	Erik:	You can't do it. I don't think you can
4.0.423	Graham:	[inaudible]
4.0.424	Dr. L.:	No, what what Alan is saying is he wants to use his orange rod to get his halves and he wants to use his blue rod to get his thirds
4.0.425	Erik:	And then he's saying he wants to compare but you can't compare a larger rod with a smaller rod
4.0.426	Alan:	I know but even if you made thirds with the orange rod, they wouldn't be as big as the halves
4.0.427	Erik:	So that is the answer right there! You don't have to use the blue rod to compare with the orange rod.
4.0.428	Dr. L.:	Can you show me thirds with the orange rod?
4.0.429	Alan:	You'd have to make a new rod.
4.0.430	Erik:	Exactly.
4.0.431	Alan:	Just like we've been doing.
4.0.432	Erik:	You can't divide, I don't know if you can divide a single rod into a half and a third. Can you divide any rod into thirds and halves?
4.0.433	Alan:	You can divide any rod you want into halves
4.0.434	Erik:	I know but you can't divide it - no you can't

4.0.435	Alan:	You can’t divide any rod into thirds
4.0.436	Erik:	You can’t divide any rod you want into halves
4.0.437	Alan:	You can’t divide the red rod into thirds.
4.0.438	Erik:	Wait a minute, I think I just got a rod, and you can divide into both things. Yep! I got it
4.0.439	Alan:	What?
4.0.440	Erik:	The dark green rod. See, look. Half and a half is the light green [Figure F-47-12]
4.0.441	Alan:	They’re not equal, how can you tell which is big? Well, anyway the half would be bigger.
4.0.442	Erik:	They’re both the same. [to Dr. L.] I figured out a rod that can do both. You can divide into halves and thirds.
4.0.443	Dr. L.:	Which one?
4.0.444	Erik:	Because I, I did this- I used this one
4.0.445	Dr. L.:	Ok
4.0.446	Erik:	And I experimented a lot - I studied the rods
4.0.447	Dr. L.:	Ok.
4.0.448	Erik:	And said maybe you can use the light greens and I put the light greens up against it. And then I go ok, you can use it, you can do that as a half.
4.0.449	Dr. L.:	Ok.
4.0.450	Erik:	And then I studied again I go maybe the ones can do it but then I looked at it again and I go no and then I’d go well maybe one larger than that and then I go oh the red rod. So I put the red rods up against it and I divided it into thirds.
4.0.451	Alan:	Oh yeah.
4.0.452	Dr. L.:	Ok, so then your reds are your thirds
4.0.453	Erik:	Yup
4.0.454	Dr. L.:	and your greens
4.0.455	Erik:	Light greens are your halves
4.0.456	Alan:	Now I get it
4.0.457	Dr. L.:	So which is larger a half or a third?
4.0.458	Erik:	A half
4.0.459	Dr. L.:	Why?
4.0.460	Erik:	Because see, if you have one whole,
4.0.461	Dr. L.:	Right
4.0.462	Erik:	And you want to divide it into halves
4.0.463	Dr. L.:	Right
4.0.464	Erik:	The halves have to be so big that you can only divide them into two parts.
4.0.465	Dr. L.:	Ok.
4.0.466	Erik:	So, and if you wanted to divide it into thirds, they have to be big enough to divide into three parts.
4.0.467	Dr. L.:	Ok.

4.0.468	Erik:	So if you only wanted to divide it into two parts, you have one whole, the whole has to be big enough to divide into two parts.
4.0.469	Dr. L.:	Ok.
4.0.470	Erik:	Two equal parts
4.0.471	Dr. L.:	Ok.
4.0.472	Erik:	So if you have two parts, two is less than three, but if you divide it into two parts, they have to be bigger than the thirds
4.0.473	Dr. L.:	Well, that's kind of neat. Do you follow what he said?
4.0.474	Alan:	Yeah. I have it here [points to his own model]
4.0.475	Dr. L.:	But you're not comparing the size of these two, let me see if I could say what Erik said. I think what I heard you say was when you're dividing it into halves you're getting two equal parts
4.0.476	Erik:	Yeah
4.0.477	Dr. L.:	And when you're dividing it into thirds
4.0.478	Erik:	You're getting three equal parts
4.0.479	Dr. L.:	Ok. So then what was your reason. Why is the half going to be larger than the third?
4.0.480	Erik:	Well, because if you divide something into halves, it's only two parts, and if you cut something in halves, and you cut something in thirds-
4.0.481	Alan:	I know. The thirds would be smaller
4.0.482	Erik:	The thirds would be smaller because two parts of one, like a circle or something, you cut it into two parts, they're going to have to be bigger, because it's two parts you're cutting it into. But if you're cutting it into three parts, the thirds are going to have to be bigger, I mean not bigger, smaller, because you're cutting it into three parts, and three parts, is, the number three is larger than two but if you're cutting something into two parts it's going to have to be larger than three.
4.0.483	Dr. L.:	That's kind of neat! I like what you're saying.
4.0.484	Erik:	So technically, if you're counting by numbers, the smaller number is the larger
4.0.485	Dr. L.:	Say that again, the smaller number
4.0.486	Erik:	Technically, well, technically if you're counting by numbers
4.0.487	Dr. L.:	Ok
4.0.488	Erik:	using Cuisenaire rods
4.0.489	Dr. L.:	Ok
4.0.490	Erik:	And you're cutting into halves and thirds
4.0.491	Dr. L.:	Right
4.0.492	Erik:	It's gonna be the smaller the number, the larger the half- the larger the piece
4.0.493	Dr. L.:	Isn't that interesting. Does that always work?
4.0.494	Erik:	Yeah.

4.0.495	Dr. L.:	What do you think, Alan?
4.0.496	Erik:	Because if you're dividing something into thirds and fourths.
4.0.497	Alan:	Because the halves
4.0.498	Dr. L.:	Well, hold on- follow what he's saying
4.0.499	Erik:	If you're dividing something into thirds and fourths,
4.0.500	Dr. L.:	Thirds and fourths - which one do you think will be the larger piece?
4.0.501	Erik:	I don't know if I can divide this into fourths.
4.0.502	Alan:	Thirds
4.0.503	Dr. L.:	You think thirds
4.0.504	Erik:	Well yeah, because
4.0.505	Alan:	If you had to chop it [motioning with his hands] three times you have smaller pieces
4.0.506	Erik:	[speaking over Alan] See, if you have thirds you have to divide it into three parts
4.0.507	Dr. Landis	If you chop it three times
4.0.508	Erik:	Three times you div- you have bigger pieces, because if you divide it four times
4.0.509	Alan:	If you chopped it three times, ok, you'd have four pieces. But if you made it into thirds
4.0.510	Erik:	No you wouldn't have four pieces [Alan nods] If you chopped this into three pieces you wouldn't have four
4.0.511	Alan:	I'm not - I know
4.0.512	Erik:	But that's what you said
4.0.513	Alan:	If you cut this into thirds, this, it would have three equal parts.
4.0.514	Erik:	Yeah I know.
4.0.515	Alan:	If you would cut it into halves it would have two equal parts and if you cut it into fourths it would have four equal parts. The fourths would be smaller than the thirds and the thirds smaller than the halves
4.0.516	Erik:	Exactly
4.0.517	Dr. L.:	The fourths is smaller than the thirds and the thirds is smaller than a half
4.0.518	Erik:	And, and what I was thin-
4.0.519	Dr. L.:	But I want to know
4.0.520	Erik:	What I'm saying is, see if you divide it into thirds and fourths,
4.0.521	Dr. L.:	Ok
4.0.522	Erik:	Four is a larger number than three
4.0.523	Dr. L.:	Ok.
4.0.524	Erik:	But three, you're dividing it into, um, you're dividing it into three parts, so instead of dividing it into four parts you cut it four times into fourths and then, and that would be much smaller than the, a third. And if you divide it- if you cut it

		only three times, it'd be bigger. So therefore, four may be bigger than three
4.0.525	Dr. L.:	Right
4.0.526	Erik:	But the smaller the number, the larger the piece.
4.0.527	Dr. L.:	The larger the piece. So if I had a pizza pie,
4.0.528	Erik:	Yeah
4.0.529	Dr. L.:	And I had eight people that were sharing it, right?
4.0.530	Erik:	Yeah
4.0.531	Dr. L.:	And then I have that same pizza pie but this time I'm sharing it with four people
4.0.532	Erik:	Uh huh.
4.0.533	Dr. L.:	Which would be, who would get more pizza? When I-
4.0.534	Erik:	If you're sharing it with four people each person would get a piece, a fourth.
4.0.535	Dr. L.:	Uh huh
4.0.536	Erik:	And if you're sharing it with eight people, they'd get, well actually, I think a large pizza serves four people.
4.0.537	Dr. L.:	Ok
4.0.538	Erik:	So they'd get one slice. And if you're serving eight people
4.0.539	Dr. L.:	Right
4.0.540	Erik:	They'd each get a half a slice
4.0.541	Dr. L.:	So who would be eating more pizza?
4.0.542	Erik:	The fourth. Well, well, the four- the people who are having the four people would get a bigger slice
4.0.543	Dr. L.:	Mmm hmm so they'd be eating more pizza
4.0.544	Erik:	Yeah.
4.0.545	Dr. L.:	So does that follow what you figure out
4.0.546	Erik:	Yeah, because
4.0.547	Dr. L.:	Interesting
4.0.548	Erik:	the smaller, the smaller the number, the bigger the pieces
4.0.549	Dr. L.:	Isn't that interesting, you gotta share that with Amy. I find that fascinating. You agree with what he said?
4.0.550	Alan:	Mmm hmm.
4.0.551	Dr. L.:	Really interesting, really interesting, that'd be. I see you finally found a rod that divided into halves and thirds, right? Before you were trying to do it with this one and this one.
4.0.552	Erik:	I was trying the blue rod, the brown rod
4.0.553	Dr. L.:	You were trying to divide this into halves and this into thirds, right?
4.0.554	Erik:	Yes
4.0.555	Dr. L.:	Now, here you're using the same rod to divide into halves and thirds. Could you have used different rods?
4.0.556	Erik:	I don't know. I think this is the rod- I've tried almost every rod
4.0.557	Alan:	They wouldn't be equal halves
4.0.558	Dr. L.:	They can't be equal halves? What do you mean?

4.0.559	Alan:	You can't divide this [orange rod] into thirds
4.0.560	Dr. L.:	Right
4.0.561	Alan:	And you can't divide this [blue rod] into halves
4.0.562	Dr. L.:	Ok
4.0.563	Alan:	But you can divide this into thirds and you can divide this into halves
4.0.564	Dr. L.:	Ok, ok.
4.0.565	Alan:	You can do both.
4.0.566	Dr. L.:	Could we, you can do both. Well, would it have been ok to use two different rods and then get a half of this one and a third of this one and compare the half and third? Would that have been ok?
4.0.567	Erik:	No, I don't think so
4.0.568	Dr. L.:	Erik says no. What do you think?
4.0.569	Erik:	Because if you cut this into halves
4.0.570	Dr. L.:	Yeah
4.0.571	Erik:	You're dividing it into yellows
4.0.572	Dr. L.:	Ok
4.0.573	Erik:	They'd be half
4.0.574	Dr. L.:	Right
4.0.575	Erik:	And if you're dividing this into thirds
4.0.576	Dr. L.:	Uh huh
4.0.577	Erik:	this rod is smaller than the orange rod
4.0.578	Dr. L.:	Ok
4.0.579	Erik:	so therefore you can't compare
4.0.580	Dr. L.:	Kind of the chocolate bars? Is that kind of like the chocolate bars?
4.0.581	Erik:	Wait, well, come to think of it, maybe you can compare. Because, yeah, I think you can compare. Because they may be smaller than each other, but one's dividing it into halves, like the orange rod you're dividing into halves, but the blue rod you're dividing into thirds, and the thirds are one smaller than halves and the blue rod is one smaller than the orange. So therefore they're equal.
4.0.582	Dr. L.:	When we had different size chocolate bars, were we able to compare a half of one chocolate bar with a third of another if they were different sizes?
4.0.583	Erik:	Well,
4.0.584	Dr. L.:	Were we able to do that?
4.0.585	Erik:	I think you can compare, you could probably a third of a big one to a half of a small one.
4.0.586	Dr. L.:	You guys think about that one. I'm not sure I remember. And I'm not sure about that one. You gotta think about that.
4.0.587	Alan:	Erik, you think we should go get a paper and write our answer? [they go to get paper]

4.0.588	CT:	[Dr. Landis and CT confer. CT approaches Erik and Alan] Ok, I'm hearing great stuff, I'm hearing great stuff from everybody. What's the story, fellows?
4.0.589	Erik:	Well, I think you can divide this into halves and thirds
4.0.590	CT:	Alright, let's see
4.0.591	Erik:	Because if you have this and you take the light green rod, it divides into two parts, and then if you take the red rods, it divides into three parts. And I also think
4.0.592	CT:	And the question being.
4.0.593	Erik:	And I think that the halves would be bigger than the thirds
4.0.594	CT:	Because
4.0.595	Erik:	See because if you divide-
4.0.596	CT:	Alan wants to explain
4.0.597	Alan:	If you divide this more times, you have smaller pieces. Because you'd have to have it bigger to have equal pieces to divide it into thirds. So the halves would be bigger because you're only, um, you're only dividing it once, down the middle, but here you're dividing it twice so there are going to be smaller pieces
4.0.598	CT:	I want you to put that in to Dr. Maher, ok? Make sure your name is on here, make sure 4ph is on, because you're part of my math class, and put the date and then put the question.
4.0.599	Erik:	Mrs. Phillips? I also think that even though when you're counting by numbers, three, if you're, the number three is larger than two, but when you're dividing it into this, when you're dividing something into thirds and halves, I think that the smaller the nu-, the smaller the value of the number, the larger it actually is, when you're dividing-
4.0.600	CT:	And that goes along with how many times you have to divide in, what you [Alan] were saying
4.0.601	Erik:	See if you divide by twos, I mean by halves and by thirds, the thirds, the thirds in the numbers would be three, and that may be a larger number than two, but yet when you're dividing something into halves and thirds, the half would be larger because you're dividing it down the middle of something, and that's only two parts. So two parts and three parts, two parts would be bigger.
4.0.602	CT:	Alright
4.0.603	Erik:	And also, therefore, it's two and three, three is bigger than two, but when you're using fractions, the smaller the number, the larger it is in value
4.0.604	CT:	Right because all that time you were doing division [to Alan] and you were saying you divided less but you had a bigger number. Fine, this is very interesting. Remember to give an answer.

4.0.605	Alan:	[camera focuses on Michael, who is writing his solution, then on Erik.] The smaller the number
4.0.606	Erik:	I messed up
4.0.607	Alan:	What?
4.0.608	Erik:	I messed up on this.
4.0.609	Alan:	You didn’t make them equal?
4.0.610	Erik:	No.
4.0.611	Alan:	Well. [Erik throws his paper out.] Mine aren’t perfect, but I did it. Because the smaller the number, [writes] the smaller the number that you divide the rod, the big- bigger the, the bigger the piece. [repeats]
4.0.612	Erik:	Messed up again. [discussion about how many times Erik messed up]
4.0.613	CT	Boys and girls, continue this as homework…
4.0.614	Erik:	What’d you write?
4.0.615	Alan:	I wrote: Because the smaller the number that you divide the rod, the bigger the pieces will be
4.0.616	Erik:	Copied me. Exactly what I said and you didn’t even figure that out. [reads Alan’s paper aloud] You copied me. [some more arguments]
4.0.617 1:03:11	S	End of class

Session 5, Sept. 29, 1993 (Front, Side, and OHP)

Line	Time	Speaker	Transcript
5.0.1	6:37 S	T/R 1:	Well, good morning everyone.
5.0.2		Students:	Good morning.
5.0.3	6:41	T/R 1:	I am so glad to be back. It seems like I have been away for a long time. But, I want to thank all of you for writing about what you are doing when I am not here. I really like reading what you are solving and it helped me understand some of what I missed. I also looked at some of the tapes. And that also was very helpful to me. Um, there is a colleague I work with very closely with at Rutgers who is visiting with us today who was away and I also showed him some of the tapes and he really wanted very much to come and visit the classroom and get to know you also. This is Dr. Robert Davis back here, some of you went and introduced yourself. And, how many of you recognized Dr. Davis? How many of you have seen him in the building before? Last year or another time? So, you don't recognize him. Well, we have another friend who is going to be very much interested in the way you do mathematics and the way you think about mathematics and will be very interested in hearing about the way you think, so we are really happy about that. Um. We have so many things to share today and to catch up particularly since Dr. Davis is here for the first time and since I wasn't here earlier this week. And I know that Dr. Martino shared so much with me but I, I still was just so interested in some of the questions you were asking. And, I thought maybe we would start with the one that I understand there was some discussion about and see if we could think about that together. Do you know what that question is? Do you remember that? Can you read that statement? What does that say, Michael?
5.0.4			*Task 1 is 1/5 = 2/10?*
5.0.5	08:35	Michael:	It says, is one fifth equal to two tenths? [Figure S-8-31]
5.0.6	8:40	T/R 1:	How many of you agree that is what that says? That's my question. Is one fifth equal to two tenths? [quiet in the room]
5.0.7	8:48	T/R 1:	Do you remember talking about any of those ideas at all? What do you remember about that, Meredith?
5.0.8	8:54	Meredith:	Um?
5.0.9	8:55	T/R 1:	You want to tell us?
5.0.10	8:56	Meredith:	We had used our Cuisenaire Rods.
5.0.11	8:59	T/R 1:	Come up and show us what the issue is. I have some of this up here. Because, Dr. Davis also wasn't here.

[Meredith moves to take position at the overhead to explain].

5.0.12	9:11	Meredith:	Say this is called one. [She positions an orange rod on the screen.] Then you … took this …. [Then she positions what looks to be two yellow rods of equal length under the orange rod. Both rods together are equivalent to the length of the orange rod.] This would be half of the orange rod [pointing at one yellow rod] … and this would be um …. called the one [pointing at the orange rod] and these would be one [pointing at the orange rod] and these would be called the two halves [pointing at the yellow rods - Figure O-11-08].
5.0.13	9:34	T/R 1:	Ok, so what name would you give the yellow rod?
5.0.14	9:37	Meredith:	A half
5.0.15		T/R 1:	One half. Ok.
5.0.16	9:42	Meredith:	And, this would be called one [pointing at the orange rod]
5.0.17	9:45	T/R 1:	And that would be called one. Can you tell me what that has to do with my question? Is one fifth equal to two tenths?
5.0.18	9:49	Meredith:	Um. Two tenths. You call this two tenth. This is usually a tenth [she places two yellow rods on the projector] and then you put this up and you call this two tenth and then you take this and you put it up to them [she places five red rods near the yellow rods] and it's five- they equal up to it [Figure O-12-02].
5.0.19	10:22	T/R 1:	Can you say that so the class hears you and see what they think? I am not- did you all hear what Meredith said? I am not- I didn't hear either. But maybe if you can tell the class what your thinking, let's see what they think.
5.0.20	10:38	Meredith:	Well if you have the orange rod and you put it up to the yellow rod, the yellow rods, it's two halves and that would be called two tenths. Uh, yeah. Two tenths… wait two tenths would be ……. two ones … [she places 2 white rods on the screen] That is what two tenths would be, 'because these ones are tenths [pointing at the white rods and looking at T/R 1 - Figure O-12-52].
5.0.21	11:30	T/R 1:	Why don't you tell the class and-
5.0.22	11:31	Meredith:	These ones are tenths [pointing at the white rods] 'cause this orange rod is one and when you put the ones up to it there's ten of them and two of them would equal two tenths. And this [she moves the red rods directly above the orange rod] is one fifth to the orange rod. And if you take one of them [she moves one red rod above the two white rods] it's equal to two tenths [Figure O-13-21].
5.0.23	11:58	T/R 1:	So what is your conclusion if I ask you the question, is two tenths equal to one fifth?
5.0.24	12:02	Meredith:	Yes.
5.0.25	12:04	T/R 1:	You think it is. Ok, let's have some discussion; Thank you, Meredith. Let's see what other people think. Do we have some other discussion about that? Brian?

[Brian comes to the overhead.]

5.0.26	12:19	T/R 1:	By the way, how many of you agree with what Meredith said? How many aren't sure? How many of you disagree? Ok, so we have some not sures and agree, so let's see Brian which category are you in? Do you agree or are you not sure?
5.0.27	12:34	Brian:	I agree.
5.0.28	12:35	T/R 1:	Ok, you want to explain why you agree?
5.0.29	0:12:36	Brian:	Well, I agree because it's just like having one of these reds being a whole and one of these [a white rod] being a half. So it's just like saying, it's just like saying, two halves equal a whole. It's the same as being two tenths equal one fifth.
5.0.30	12:59	T/R 1:	What do you think? Erik is making a face. Erik do you want to, do you want to say what you think out to Brian? [Erik goes to the overhead.]
5.0.31	0:13:12	Erik:	Well, I kind of agree with Meredith, because if you take the orange rod, it takes ten of the white rods to equal up to the orange and five of the red ones. Then if you take two of these [points to white rods] which is two tenths and this [red rod] is one fifth. Because, well, it's, it takes five of them to equal up to the orange rod and, and if you put two tenths next to it they equal up to each other and you'll have one fifth and one tenth- and two tenths together and they both equal up to the same amount.
5.0.32	13:55	Brian:	[whispering] Like what I said kind of. [Erik and Brian chuckle]
5.0.33	13:56	T/R 1:	Ok, anybody else? Any other discussion? Thank you gentlemen. [Brian and Erik return to their seats.]
5.0.34	14:01	T/R 1:	Any other discussion? Anybody else have something to say about that? [No response] How many of you believe that one tenth- two tenths and one fifth represent the same length? Do you think it would be okay to give them the same number name? How many of you think it would be ok? We can call this two tenths or we can call this one fifth. How many of you agree with that? [several hands go up] Yeah. Makes sense doesn't it?
5.0.35	:		*Task 2: What other number names can we give to one half of a candy bar?*
5.0.36	0:14:42	T/R 1:	Do you remember the candy bar? Remember. Did you get any? [Students say yeah.] This was the little candy bar, it looks something like this. If you excuse my sketch. Do you remember it looks like this. It was broken up into three columns and then four rows. Do you remember that? Remember I was giving half of this little candy bar to Dr. Martino ……..? Remember how appreciative she was? Remember that? We said we were giving her half the candy

			bar, didn't we? Right? Okay? Could someone have told me another name, another number name for how much of the candy bar I gave her? Do you understand my question? If this is my candy bar and I gave her that much, right? One number name we said was one half, didn't we? Can someone think of another name that very exactly tells me how much of the candy bar I gave Dr. Martino? And if you think you know, why don't you discuss it with your neighbor and see if you agree. [discussion] Ok, so discuss it. You have to be able to prove it and say why. [Groups working together - Figure O-17-08].
5.0.37	0:17:08	T/R 1:	Ok, Are you ready to share your ideas? I heard a couple of different number names. How many of you think you have another number name that tells me how much of the candy bar Dr. Martino got? We already agree that one number name was one half. Right? That represented how much of the candy bar. How many of you think you have a different number name? Ok, Jackie. [Jackie stands]
5.0.38	17:37	Jackie:	Um, well we thought it was six twelfths, because-
5.0.39	17:42	T/R 1 :	Jackie, I am sorry I can't hear, it's so loud here
5.0.40	0:17:44	Jackie:	Um, we thought is was, um, six twelfths because there are um twelve pieces in all and she got six, and six makes half.
5.0.41	17:55	T/R 1:	What do you think?
5.0.42		Erik:	I have another one.
5.0.43	17:58	T/R 1:	What do you think? Did you all hear, uh, what Jackie said? How many of your heard what Jackie said? Raise your hand if you heard what she said. [several hands go up] How many of you agree with what Jackie said? [several hands go up] That another number name you said Jackie, was…[asking the student to repeat]
5.0.44		Jackie:	Six—
5.0.45	18:19	T/R 1:	--Six twelfths. How many of you agree with that? [several hands go up] Does anyone disagree with that? [all hands go down quickly] Does someone have a different number name? I heard some other number names as I walked around. [Danielle raises her hand] Danielle?
5.0.46	00:18:30		Danielle: Um, I thought, um, it would be, um, two fourths.
5.0.47	18:37	T/R 1:	You thought two fourths? How did you think that?
5.0.48	00:18:39		Danielle: Because if she got a half, then the top two rows, um, is a half, and then that's two fourths.
5.0.49	18:50	T/R 1:	Then we can think of this as two fourths. What do you think about that? How many of you think that's another number name? Two fourths? Some people aren't sure. Danielle why don't you come up and show them what you're thinking? I am not sure if people were following you. [Danielle walks up to the projector] Danielle thinks that

			another number name for how much of the candy bar we gave Dr. Martino is two fourths so we're interested in knowing how Danielle was thinking about that.
5.0.50	00:19:17		Danielle: Well, this row and this row is a half [points to top two rows together] and then these are two fourths. [pointing to the top two rows again]
5.0.51	19:25	T/R 1:	Why are they two fourths? Can you help the class understand that?
5.0.52	19:28	Danielle:	Because there's four rows.
5.0.53	19:31	T/R 1:	Because there are four rows and we if talk about the first two rows that's two fourths? What do you think? How many of you think that's another number name for how much of the candy bar we gave her. How many of you agree with that? How many of you disagree? If you disagree, why do you disagree? [Kelly and Mark seem confused.]
5.0.54	19:47	T/R 1:	Mark? Not sure why. [Mark shrugs his shoulders] Kelly was your hand up for disagree. No. Is it that you don't see that it's two fourths? You don't see that it's two fourths? Can someone help Kelly and Mark then, they don't disagree, but they don't see it. Brian?
5.0.55	20:15	Brian:	I just found out another way that a half can be… [block noises and mumblings cut off the last part of what he said] [Brian says he found another number name and is asked to wait until the answer of 2/4 is explained. He tries to explain.]
5.0.56	20:22	T/R 1:	You have another way that a half would be. Ok, remember that and hold on to that. I would like someone to help the people who don't quite see that that's two fourths. Can some one try to explain that? Who wants to pretend to be a teacher or explainer, if you agree? Somebody want to give it a try? Nobody wants to try? Brian?
5.0.57	00:20:41		Brian: If I agree? Ok. [Brian proceeds to the front of the room to use the screen] Well, I agree on two fourths because there's, because three times, because four times three equals twelve and if you split it in half there'd be three fourths,[he has turned the grid around]. Well, well, I agree because there are four thirds on there but, but when there's a half there are only, there's three fourths instead of four thirds.
5.0.58	21:31	T/R 1:	Wait. Hold up. Where are your four thirds?
5.0.59	21:36	Brian:	Well…well, well when you said … to split it in half [he covers six of the twelve blocks] like that there are only two fourths left over, yeah.. there are only two fourths when it's a whole there are four thirds … I can't explain it too well…
5.0.60	21:56	T/R 1:	I am not sure I understand what you are saying. Can some else help? Danielle do you want to go up there and ….?

5.0.61	00:22:02		Danielle:No, I just have something to say …. Cause, If there's four fourths, um, and half of four is two, so two fourths would be a half.
5.0.62	22:22	Brian:	But I thought about it the other way around.
5.0.63	22:23	T/R 1:	What do you mean by the other way around, um, Brian?
5.0.64	00:22:44		Brian: Well, instead of, well she went two fourths like going across, like cause those are fourths and that's a fourth and there are two of them. I have another way. It's, I have three, I think three sixths.
5.0.65	22:43	T/R 1:	Three sixths?
5.0.66	22:45	Brian:	Because well, because a whole, there's a whole, if there's [looks at overhead and counts with fingers] yeah it's three sixths. Because there are six of them, there, there, there I found groups of sixths. There's one sixth, another sixth, another sixth, another sixth, another sixth, another sixth and if you split them in half, there's three of them on top
5.0.67	23:22	T/R 1:	That's very interesting. What do you think about that? Brian found another way. That's very interesting, Brian. So, he is saying that these two little pieces of candy, and these two, and these two and these two, and these two and these two, I don't know if that is what you said Brian, but something like that.
5.0.68	23:39	Brian:	Yeah.
5.0.69	23:40	T/R 1:	You can talk about taking this candy bar and sharing it in six pieces where we would have two wedges to be one piece. Do you follow that? That's neat. And so you are saying that, if you got three of those, right? One of the two, one of the two, one of the two, then you still get a half of candy bar. So, what do you think? You're saying that you think that three sixths is another name for one half? What do you think? Ok, let's write this down. Thank you very much Brian, that's interesting.
5.0.70	24:12	T/R 1:	So what do we have here? The first question, does one half equal two tenths? Do you believe that? No, I am sorry. We said one fifth equals two tenths. When we went to our candy bar we said one half equaled? One half of the candy bar. This [one fifth equals two tenths] was with the rods.
5.0.71	24:49	T/R 1:	One half equals six twelfths. Do you agree to that? Six twelfths? What else? What's another number name? Brian?
5.0.72	25:05	Brian:	We did two fourths?
5.0.73	25:06	T/R 1:	Nice and loud.
5.0.74	25:07	Brian:	We did two fourths? [louder]
5.0.75	25:08	T/R 1:	Two fourths….Do we have another number name for half of the candy bar? Jessica?
5.0.76	00:25:18		Jessica: No, I was going to say one half.
5.0.77	25:22	T/R 1:	We have one half. Are these all the ones we had? Danielle.

5.0.78	25:25	Danielle:	There are six twelfths
5.0.79	25:26	T/R 1:	Six twelfths, I think we have. We said six twelfths, two fourths, we said one half, Brian?
5.0.80	25:35	Brian:	Three sixths
5.0.81	25:36	T/R 1:	Brian says three sixths. Remember that? Is that correct? Three sixths. How did we get three sixths? Do you remember? [Brian shakes his head yes. Figure O-27-07]
5.0.82	25:46	T/R 1:	Any others? What do you think about that? Let's go back to the original question here. You agree then that if we call this one and we call the red rod what number name?

[mumblings of one fifth from the students]

5.0.83	26:10	T/R 1:	One fifth and we call the two whites together what number name did we give it? The two whites together.
5.0.84	26:16	Brian:	Two tenths…
5.0.85	26:17	T/R 1:	Two tenths. I can also give it the number name…..? What else can I call the two white ones besides two tenths? What did we decide? [Quiet] What other number name can I give the white ones besides two tenths, if I call the orange rod one. Michael? [Figure O-28-29]
5.0.86	26:48	Michael:	One fifth
5.0.87	26:49	T/R 1:	I can call it one fifth. Alright. And what other number name can I call the red one.
5.0.88	26:55	Michael:	Two tenths
5.0.89	26:56	T/R 1:	Two tenths, is that right? [Michael: Mmm hmm]
5.0.90		:	*Task 3- Which is bigger one half or one third and by how much?*
5.0.91	00:27:00		T/R 1: Very interesting. Ok. Do you have any comments or questions about this? I see some beautiful things you are making. Some beautiful pieces of architecture. Ok, well maybe we will leave this go. Tell me what you did the last time Dr. Martino was here. What was the problem you were working on? [Michael raises his hand] Anybody want to tell me and tell Dr. Davis? You were working on a problem I think in class together I think you were in groups, weren't you? Do you all want to think for a moment and maybe discuss with your partner to help you remember what you were working on? [Michael's hand is still up] Michael?
5.0.92	27:44	Michael:	We were working on the candy bar problem. Like, with like which is bigger a half or one third and we were using candy bars to show that.
5.0.93	27:55	T/R 1:	Ok, so you were working on which is bigger, one half or one third. Andrew?
5.0.94	00:28:00		Andrew: Yeah, we were working on, we had to write about um and we had to do an example on it, and um to see if which is bigger, one half or one third.

5.0.95	28:11	T/R 1:	How many of you worked out which is bigger? One half or one third? [several hands go up] How many of you think they are the same? [all hands go down] How many of you think one is bigger? [several hands go up again] Which is bigger? One half or one third. Laura.
5.0.96	00:28:31		Laura: One half.
5.0.97	29:32	T/R 1:	You say one half is bigger. What do the rest of you think? Do you think one half is bigger? [several students provide affirmation] Do you think you can convince Dr. Davis that that's the case? [several hands go up] Can you convince Dr. Davis that one half is bigger than one third. By the way, do you know how much bigger? How many of you think you know how much bigger it is? Ok, that's the second question. Ok, I really would like someone to come up. Jessica maybe and Laura can come up to the overhead and show Dr. Davis how you decided which is bigger. And see if you can convince us of your result. [Jessica and Laura come to overhead - Figure O-31-38].
5.0.98	00:29:32		Jessica: Well, um, one third would be just this piece here [she points to the purple rod] and one half of that would be [she sets up two dark green rods] and one half would be this [one dark green rod] and one third is bigger than one half cause this [purple rod] would be one third and then this bigger piece [dark green rod] would be one half of that. And-
5.0.99	30:18	T/R 1:	Can you tell me what number name you're calling the orange and the red rod?
5.0.100	30:22	Jessica	Um, one.
5.0.101	30:29	T/R 1	You're calling the orange and red rod one? Can you say that again, what number names gave to each of those rods so I can hear from back here?
5.0.102	00:30:36		Jessica [whispers to Laura] You say. Um, this would be, this, we're counting this as one whole [orange and red train] and I think this [dark green rod] has two and this [purple rod] has, wait, um. Um [giggling], um I can't we called it, yeah, [Laura helps her out] I think this one was one-
5.0.103		Laura:	That was one third
5.0.104		Jessica:	this was one third, and this was one half.
5.0.105		Laura:	One half.
5.0.106	31:21	T/R 1:	What do the rest of you think? What do you think? Audra what do you think of what… the two young ladies built up there?
5.0.107	00:31:34		Audra: I agree because-
5.0.108	31:37	T/R 1:	Want to speak to the class [asking her to go up front]
5.0.109	00:31:46		Audra: [comes to overhead] I agree because if you saw what the, um half, was here and then you saw what, no, what

			the half was here and then you saw what the third was there, and you saw that the half was bigger than the third.
5.0.110	32:07	T/R 1:	How many of you agree with the argument that a half is bigger than a third with the argument that was made here? Ok, did you figure out how much bigger?
5.0.111		Audra	It's two, two [places two white rods next to purple rod]
5.0.112	00:32:16		Jessica: It's a red bigger, but [Figure S-32-28]
5.0.113	32:30	T/R 1:	Ok, you're saying it's a red rod bigger or two white ones but that's what I see you have built there, but I would like you to tell me what number name you have for how much bigger it is.
5.0.114	00:32:40		Audra: Um, wait, it's one third bigger, I think [organizing the dark green and red blocks together].
5.0.115	00:33:07		Jessica : I think it's one third bigger too because if you put the red to the green
5.0.116		Audra:	You'd see that there's three
5.0.117		Jessica:	You need three and if you put the purple one to it also and then it takes one third of them. [Showing the purple differs from the dark green by one red block - Figure O-34-49]
5.0.118	00:33:25		T/R 1: Okay so these young ladies have proved that one half is bigger than one third and it's- one half is one third bigger than one third. What do you think? That one half is one third bigger than one third. What do you think about what they just proved? Now you were all watching their argument up there and have they convinced you? [Child in front row has his hand up] I don't know if they have convinced Dr. Davis. Um, but I am wondering if they have convinced you? Kelly. What do you think, do you agree with this? [Kelly stands up, and comes to the front of class]
5.0.119		Kelly:	Um, yes.
5.0.120		T/R 1:	You agree with them.
5.0.121	00:34:03		Kelly: Well, if you have, um, a red, if you hold, um, well, if you have, um, well we used these and we went like, and then we like held reds up and we showed that um, that um, one half is bigger by, because this part is smaller, and this is supposed to be one, one third so that's how we did it [Figure O-36-23].
5.0.122	34:57	T/R 1:	Brian you are making a face, what do you think? Do you agree with them?
5.0.123	35:03	Brian:	Not really.
5.0.124	35:04	T/R 1:	Brian doesn't agree with you
5.0.125	00:35:07		Audra or Jessica I think that's like changing the problem because we are using the dark greens and she [Kelly] is using the light greens.
5.0.126		T/R 1:	Oh, hmm.

5.0.127		Jessica:	If you take this and it has three thirds [Brian builds the model on his desk]
5.0.128	35:21	T/R 1:	Let me make sure I understand this. You're calling this one, right? And you're calling this one third,
5.0.129		Jessica:	And we're calling this
5.0.130		T/R 1:	And you're calling this one third
5.0.131		Jessica:	One third
5.0.132		T/R 1:	Right? And you're calling this one half? And you're saying one half is bigger than one third by one third? [Girls agreeing with her as she demonstrates what they said]
5.0.133	35:36	Jessica and Audra:	Yeah, you can put three of these, three reds up to one green and then it would take one, one third of the red to make um, to go there, like.
5.0.134	35:55	T/R 1:	Ok, I would like all of you—How many of you agree? How many of you disagree? Now if you disagree you have to say why you disagree because they are saying that one third is smaller than a half, one half is bigger than a third, it's one third bigger than a third, that's what they are saying. Now either you have to agree, or disagree or not know. How many of you aren't sure? [several hands go up] A few of you aren't sure, but some of you disagree. And if you disagree we have to say what's wrong with their argument. There must be something wrong with their argument if you disagree. Or maybe their argument is right because I'm very confused. Brian what do you think?
5.0.135	00:36:38	Brian:	Well, when they said one third is bigger than one half by one third. I think they said, is that what they said? Well, I don't really agree, because well if you split, if you split one of the thirds in half which would make [counting the blocks], which would make a sixth. I think it's a sixth bigger. Like, well, [holds his rods], um should I go up there?
5.0.136	37:11	T/R 1:	Sure, ladies can you make a little space here for Brian. Maybe you need to have a little conference here, we have some disagreement.
5.0.137	37:21	Brian:	[He goes to the overhead.] Well, see for um, when they said it was one half bigger, if you split a third in half it'd make a sixth, like one, two, three, four, five six. Like, like pretending they were, like pretending they were split in half. If you split one of these in half and you have three of them up there they'd make, they'd make six and any way, and when you split them in half right in the middle over there it's kind of like that, it's kind of like this, there was this was, that was the one third [points to a purple rod] and that was the one half [points to the dark green rod] on the bottom and so it's just like this and the red I'm pretending is like, is like, is a half of one of the purples and you see when I split it in half

it's, it's one sixth and, and it equals, and it equals up to a green [Figure O-39-51].

5.0.138 00:38:30 T/R 1: I'm hearing you say Brian that the number name for red is one sixth and the reason why is—

5.0.139 38:32 Brian: Well, I mean a red, I'm considering a red one sixth [Dr Maher: yeah] because two of these [red rods] equals, see they're two, they're two sixths, two halves of one purple and the purple is a third and the half of one third is sixth, there's sixths [Figure O-40-19].

5.0.140 .00:38:57 T/R 1: So you're giving a red the number name one sixth and I understand the young ladies up at the overhead are giving red the number name one third and can red have the number name one third and one sixth at the same time? That's my question.

5.0.141 Brian: Well, what I mean is-

5.0.142 00:39:13 T/R 1: I heard what you said Brian, I just wish everyone would listen here because your going to have to decide and write about this in a few minutes and you're going to have to decide of the arguments which you agree with; Brian's argument or the argument of the other people. And you need to know the arguments of both people so you can write about them and tell me which do you believe and why. And if you don't believe an argument you have to tell why you don't believe it, and if you believe an argument you have to be able to prove it. So we have two different arguments at the table and it's very important that you listen so you understand what the arguments are. [camera pans to Graham who appears to have made the Eiffel Tower with his block, this may be because he is bored or does not understand what is going on]

5.0.143 00:39:50 Jessica: Kelly and Jackie have something else that like goes with this like—

5.0.144 39:55 T/R 1: Ok we will hear Kelly and Jackie and we will hear Brian's again. Brian said it and I know some of you heard it, I heard it. But I would like you all to listen to these arguments.

5.0.145 00:40:02 Jackie: Well, we would call this dark green one and the reds one third and the light green one half, and we thought the, we thought one third was bigger by one of these white things. [Her model is using 6 cm as the unit.]

5.0.146 40:21 Jessica: Oh, I think they're making a different size candy bar

5.0.147 40:25 T/R 1: Is that allowed?

5.0.148 40:28 Jessica: Um, no.

5.0.149 40:30 T/R 1: Why not? What's wrong with that? In what way it is not fair?

5.0.150 00:40:33 Jessica: Because if say you give someone half of this one [12cm?] and then one half of that one [6cm?] and this is bigger than [takes a light green and dark green rod in hand].

5.0.151 40:50	T/R 1:	Ok so what do you ladies think? Are you making different size candy bars? What are calling the candy bar when you started the problem? What was one? What did you call one if you're thinking of candy bars when you began the problem?
5.0.152 00:41:00		One of the girls: The dark green…
5.0.153 41:02	T/R 1:	Is that what you built when you went up there, you said the dark green is one? Is that what you said?
5.0.154 41:41:06		One of the girls: Yeah…[the girls look at each other in agreement]
5.0.155 41:10	T/R 1:	Ok then use the—okay if your calling dark green one then I want to hear your argument which is bigger a half or a third and by how much?
5.0.156 00:41:14		Jackie: Okay, we think that a half is bigger than the third.
5.0.157 41:18	T/R 1:	Okay you think a half is bigger than one third and you're calling the dark green one? Did you change your mind?
5.0.158 41:23	Jackie:	Yeah, and we think light green is a half [of the 6 cm model].
5.0.159 41:25	T/R 1:	Well show me your argument now and tell me which is bigger a half and a third and by how much?
5.0.160 41:31	Jackie:	Okay, this is, this is a half [light green] and the red is a third.
5.0.161 41:35	T/R 1:	Can you show me why that's a half?
5.0.162 00:41:36		Jackie: Because if you put these all together they equal up to the one…[Showing that three reds, two light greens both equal the dark green which is one] and we think the light green which is a half is bigger than the red by, by one which is this white one. [Showing the difference between red and LG is a white. Figure O-43-42]
5.0.163	T/R 1:	Ok, I see that you switched what you made, um, your model, uh, but you showed me that one half is still bigger than a third and you still believe that. But what number name did you give to white? You said it was a white rod bigger but I didn't hear what number name you gave to white. I thought I heard you say it's one bigger
5.0.164	Jackie:	Yeah.
5.0.165	T/R 1:	Did you say that?
5.0.166	Jackie:	Yeah, the green, the light green is one bigger than the red. And the red is one bigger, the light green is one bigger
5.0.167	T/R 1:	And what number name are you calling the white?
5.0.168	Jackie:	One
5.0.169	T/R 1:	You all agree with that?
5.0.170	Jackie:	Actually, I used this to um, to tell that the light green is one white bigger.
5.0.171 42:46	T/R 1:	Ok, and the number name you are giving to the white you're saying is one,
5.0.172	Jackie:	Yeah.
5.0.173	T/R 1:	you called the green one and your calling the white one?
5.0.174	Jackie:	No. [giggling]

5.0.175		T/R 1:	That's what I thought I heard you say. [asking the class] You hear my question? Is everybody hearing my question. You said you called the light green one, you said you called the red one third, and you said you called the light green one half. Right? And now the white one, right… [puts the white and red together next to the light green] The white one which tells you how much bigger it is, you said you're calling it one. So your calling this one and this one [pointing to the white and dark green].
5.0.176	00:43:32		Erik: [from his seat] I think I know what they mean.
5.0.177	43:32	T/R 1:	Erik, what do they mean I'm so confused.
5.0.178	43:34	Erik:	[walks to the overhead] I think they mean that they want to call this, the dark green one, one whole, and they want to call this, yeah, like you line all the whites up to it which I think should be six and they want to call it one sixth. I think that's what they're trying to say but they just, they're just not saying it. I think they just, they want to call it one sixth [Figure O-45-54].
5.0.179	43:58	T/R 1:	I don't see six of them up there.
5.0.180	00:44:04		Erik: Well however many are up there that what they are trying to say.
5.0.181		Jessica:	Yeah because I think they meant
5.0.182		Erik:	I think you meant to say not one whole but one sixth [Figure O-46-07].
5.0.183	44:16	T/R 1:	Is that what you meant to say?
5.0.184	44:17	Girls:	Yeah.
5.0.185	44:18	T/R 1:	So you're saying then you all agree, that's what, you all really wanted to call the little white one, one sixth and not one? When you call the light green one? So I'm a little concerned now? Are you agreeing with Brian or disagreeing with Brian that the number name that you would give for how much bigger one half is than one third? Is how much? One half is how much bigger than one third?
5.0.186	44:45	Girls:	Um, one, one sixth.
5.0.187	44:46	T/R 1:	Is it one or one sixth?
5.0.188	44:48	Girls:	One sixth.
5.0.189	44:49	T/R 1:	You're sure it's one sixth?
5.0.190	44:50	Girls:	Yea.
5.0.191	44:51	T/R 1:	Why can't it one?
5.0.192	44:52	Girls:	Because that's be um, the dark green.
5.0.193		T/R 1:	The dark green is one? I understand when-
5.0.194	00:45:01		Erik: But I think you can call it one because you can make the dark green bigger—
5.0.195	45:05	T/R 1:	But they didn't, they called the dark green one, Erik —

5.0.196	45:06	Erik:	[continuing]:You could call, you can call the dark green one six, the dark green rod six and then you could call the light green rod three—
5.0.197	45:15	T/R 1:	But can you do that in the same problem?
5.0.198	45:18	Erik:	No [slumps down in his chair]
5.0.199	45:20	T/R 1:	Yeah you can't change the rules in the problem, now I want to go back—[maybe recognizing Erik]that's very very nice and Erik that was very helpful to me and to the folks up there but I still want to go back to the problem Brian was helping them with the problem up there, I still wonder if we can solve this one because you started with this other one and you said that the orange and red [together] are one, right? Isn't that what you said?
5.0.200	45:43	Jessica:	This is one whole, and then this is one third and this is one half. [pointing to the three different rod lengths]
5.0.201	45:45	T/R 1:	Right, and you said it's bigger by the red, right? And the question was, what number name do you give to the red? Now if you really understood what mistake you made here maybe you'll figure out what mistake you made up there.

[girls whisper to each other]

5.0.202	00:45:58	Jessica: Well, we and we, um, named, well, three reds equal up to um, one greens and then you put the purple next to it and you need one more red, you need a red to go next to the purple, so it would be one third.
5.0.203	00:46:34	T/R 1: Well how can you build a model and say that one half is bigger than a third by a sixth and build another model that says one half is bigger than a third by a third? How is that possible? I am so confused. Brian, it's just his face tells me that he is so unhappy with that. Do you believe that Brian? They're still telling me that one half is bigger by—one half is bigger than one third by one third. Can anyone tell me what's going on here? I am so confused.
5.0.204	00:47:16	Brian: I don't- I still don't think so, well, because, well, well, see like I said before when you split the ahh, when you split the thirds in half and they make sixths, it's still like [He goes to the overhead.]

5.0.205	47:48	T/R 1:	So Brian is giving the red rod a different number name, he's not calling it a third he's calling it a sixth. They don't believe that though, they still want to call it a third. Someone has to-
5.0.206	48:02	Brian:	See, well, because when you put it right there you see that, you see that there's one of these, if you put one of these on top of it you might see that, that it's that much that, that red, that red is that much bigger than one of the halves because one of these reds I'm calling is, is, is a sixth and anyway a half of one of these, a half of one of the thirds. But when you

put it on top of one of the thirds it's that much bigger than one of the halves [Figure O-50-08].

5.0.207 00:48:51 Jessica: Well, I think they might both be answers.

5.0.208 48:54 T/R 1: You think it can be a third and a half? How many think they could be a third and a half? How many of you don't think it could be a third and a sixth? How many of you disagree?

5.0.209 49:10 Erik: I don't think you can have an answer of a third because if you have one half [he goes to the overhead] and if you take the one half which would be the dark green, you have the one half and then these [purple rods] are the thirds. How could one half be bigger than the thirds by one third? Because, and you have the half and the thirds together that the half is almost as big as two thirds, but yet the two thirds aren't exactly, are not exactly, the green, the dark green is not, the dark green is not exactly as big as two, two thirds but, two thirds, it's the, but it's far enough so that the two thirds are not bigger than it by one third [Figure O-51-53].

5.0.210 00:50:15 Brian: I kind of agree with Erik. I think now I disagree with them [referring to the girls].

5.0.211 50:19 Erik: I don't really think that if you have this [a purple rod] that you could have one third bigger than it [Brian - yeah] because it's got to be one third and probably a third and a half.

5.0.212 50:30 Brian: Yeah, he's right.

5.0.213 50:31 Erik: It couldn't be, it couldn't be exactly a third.

5.0.214 50:34 Brian: Cause one third bigger, this would be one third bigger like that to the end over there [Figure O-52-40]. That would actually be like this [showing with the dark green and purple pieces], this would really be one third bigger and there's still some left over and there's still about [Figure O-52-51]

5.0.215 00:50:56 Erik: A half left over.

5.0.216 51:02 Brian: Yeah, there's still, there's still one more, there's still one more piece left, like about a sixth left [Figure O-53-04].

5.0.217 51:05 Erik: Cause it's like if you have, if you have the like dark green and it doesn't exactly equal up to, it doesn't exactly equal up. It's less than two thirds but it's more than one third. It's just about one third and a half. So it couldn't be exactly a third bigger than it and it couldn't be exactly two thirds or it couldn't be exactly one third bigger. It had to be one third and a half.

5.0.218 51:37 T/R 1: Michael wanted to say something for a long time and has been very patient.

5.0.219 00:51:39 Michael:

BREAK
In side

VIDEO

Umm, I think it should be called one sixth because [he goes to the overhead] because if you put six reds up to one orange [arranges six reds under the orange w/ red rod train] with a red then it would equal, there would be, there would be, it would be the same size just, so it would be called one sixth because reds like that [Figure O-54-17].

5.0.220 Brian: Yeah, I agree with Michael and Erik

5.0.221 T/R 1: So, so Brian, Michael is offering another way of thinking about that red as being one sixth. You thought about the red as being one sixth to make a half of a third and Michael is saying that red is one sixth

5.0.222 Erik: Yeah, Michael is right because it takes three sixths to equal one half, and if-

5.0.223 T/R 1: I see Meredith is wanting to say something.

5.0.224 Meredith: I agree with Erik, Michael and Brian because if you do call that a sixth, a sixth, and if you put the dark green and two thirds, you said it was, you said it was, um, they said that it's a third bigger, if you did a third bigger, this is called a third and then you put it there, you see negative, [Figure O-55-18, interrupted by intercom. Meredith placed a red rod next to the dark green rod - Figure O-55-27]

5.0.225 T/R 1: I'm sorry, Meredith, could you start again

5.0.226 Meredith: You said it was one third bigger, that can't be true because one third bigger

5.0.227 Erik and Brian: Yeah

5.0.228 Brian: It's about one sixth less. So it can't be a third bigger.

5.0.229 Erik: And also, like

5.0.230 Meredith: So it's one sixth bigger [Figure O-55-55]

5.0.231 52:43 Erik: And also yeah, and also, I think because if you have the light green, the light green, it's not bigger than, it's not bigger than the, it's not bigger than the umm third, it's not bigger than two thirds. It's bigger than one third, but it's not as big as two thirds so it's less than two thirds but more than one third. So it can't be a third bigger. And if you have that to make it two thirds large, there has to be a sixth [Figure O-56-29].

5.0.232 53:19 T/R 1: Well that is really something, uh I think --

5.0.233 53:23 Michael: It's sort of like one sixth in both cases.

5.0.234 53:27 T/R 1: Well you find that you are consistent, you do get one sixth when you use both models. I am really interested I, uh, hearing about what all of you are thinking about these arguments [To the children at the overhead] You can sit down now. Thank you very very much, that was very very helpful. What I am going to ask you to do in the next class, Mrs. Phillips, is if they can have some time tomorrow where they could have their rods, or the next chance you get to… so

			maybe the substitute would let them do that. If they could have their rods, and I'd like you to write about that there are two arguments here. Right? There are two models that were built and I would like you to try to tell me what you believe were the arguments and if you had to persuade somebody who was confused, if you could try to think what the confusion was, or you can tell me what all the things you learned about this model. You had some wonderful arguments that you gave up here, and there were so many ideas that I know that it's hard to catch all the ideas to listen, but maybe if you had the rods and you talked to each and you worked together maybe that would be helpful. What do you think Jessica?
5.0.235	00:54:37		Jessica: Well, I think now that I agree with Brian and Erik.
5.0.236	54:42	T/R 1:	You now agree with Brian and Erik? Why did you change your mind Jessica?
5.0.237	54:43	Jessica:	Because I, I saw that um, it wasn't the same as um, it can't, it couldn't be one third.
5.0.238	54:52	T/R 1:	Why couldn't it be one third?
5.0.239	54:53	Jessica:	Because you'd, it, you'd have to add um, a red and that would be one sixth.
5.0.240	54:59	T/R 1:	Okay now that's very interesting, if you could write about that, and why you changed your mind when they gave their arguments, that would be really interesting I would really like reading when you write about these things, you do so nicely. And what you also might want to do is if you didn't understand peoples' arguments because you know maybe sometimes when someone's talking up here it's hard to catch it all, maybe you can get them aside and talk to them privately. And say you know I don't really quite understand what you were doing, can you help me. I'm sure Jessica and Laura would be very happy to show you their models and show you the way they were first thinking about it. And maybe how their thinking changed, right, and why? I think that would be very interesting. I think we need to talk about this problem some more. It's a very important problem. What do you think? How many of you have enjoyed working this problem? [several hands go up] How many of you have found it very hard work? [A couple hands stay up] Some of you haven't? Dr. Davis what do you think?
5.0.241	55:58	Dr. Davis:	Um, I think that this is one of the most interesting discussions in mathematics I have ever heard. You people were sensational. I'm gonna be curious about how it all comes out when you write about it, to see if you really all agree…But you're doing some very good mathematics. I'm very impressed.

5.0.242	56:16	T/R 1:	The thinking in this class is absolutely wonderful and I'm just so impressed at your thinking and the way you're writing about your ideas. So I just can't wait—Is it possible Mrs. Phillips that they could write or do you have something else planned for tomorrow.
5.0.243	56:29	CT:	Umm, yes, tomorrow your question is, uhh do you believe that one half is larger than one third and by how much and if you didn't understand the uhh explanation you will have the Cuisenaire rods in front of you and you will work it out. You'll have time to talk to your neighbors about it and you will write a discussion that I will love to see, and that I know that all our Rutgers friends will love to see. You're being given the paper now so that you have it. You write in what everybody?
5.0.244	57:01	Student:	Pen.
5.0.245	57:03	CT:	Pen, pen, you know where the pens are, alright. This is not homework but this is paper ready for you to go tomorrow, your notebook paper. It'll be the first thing on the agenda. Any questions? Yes, sir.
5.0.246	57:18	Brian:	Could we do it for homework? Start on it for homework if we would like too.
5.0.247	57:23	CT:	Alright, alright, if you fully understand, surely.
5.0.248	57:26	T/R 1:	And then you could help other people understand your thinking and be available – [cuts out] Thank you very very much, I can't wait to come back on Friday. See you then, thank you.
5.0.249	57:30	:	End of class

Session 6, Oct. 1, 1993 (Front, Side, and OHP)

Line	Time	Speaker	Transcript
6.0.1	4:55 S	T/R 1:	Well, it does look like everyone is here. Good Morning. I understand your parents were playing with the rods yesterday? [Mmm hmm] Did you hear that? [Mmm hmm] How did they do?
6.0.2	5:10	Michael:	They were okay not as good as us
6.0.3	5:11	T/R 2:	It went very nicely, we had people building towers
6.0.4	5:14	T/R 1:	Is that what they were doing?
6.0.5	5:15	T/R 2:	That is what they were doing, you know some of them
6.0.6	5:16	Michael:	They weren't as good as us
6.0.7	5:17	T/R 1:	Not as good as you were, Michael? Yeah what do you expect, right? Did any of them solve as of the problems with the rods that you know of?
6.0.8		Sarah:	No
6.0.9		T/R 1:	Sarah? No? So your parents were working with them yesterday? Did they ever see them before?
6.0.10		Students:	No
6.0.11	5:39	T/R 1:	No? Now did you tell them about them?
6.0.12		Student:	Yeah.
6.0.13		T/R 1:	So are they learning? Are you helping them to learn about them? Is it hard? Yes, Meredith and Michael says no? What do you mean?
6.0.14	5:51	Michael:	Well, it's hard to get my Dad to learn
6.0.15	5:53	T/R 1:	No, your Mom learns easier? Is that true for others? Do you think so? I wonder why that is? Wonder why that is .that's an interesting question, isn't it? I thought today might be a good idea, a good day to begin to share, uh, your thinking. You wrote some lovely uh, explanations of your solution. And do you remember what I asked you to do as a last challenge on Wednesday? Michael?
6.0.16	6:24	Michael:	You asked us, um, one half is bigger than one third by how much?
6.0.17	6:26	T/R 1:	How many of think you know the solution to that problem? You're pretty confident about it? Raise your hand if you know the solution to that and you're pretty confident? Very confident. Raise your hand if you're still a little shaky about it? Sort of by maybe, some of you are still a little shaky. Raise your hand if you think your understand the two arguments that were going on Wednesday. Interesting, so it looks as if we have, uh, in this classroom sufficient expertise to maybe, umm, start to resolve these issues. What do you think? Think it might be a good time to try to see, um, what the confusion might be and maybe to try to help people, umm, see what other people are thinking? You think that's a

			good idea? Could we maybe do that today? How many people would like to do that today? Straighten out the issues and deal with them. Do you think by the time we leave we hope everyone will know a little bit more than when they came in? [students: Yeah] Would that be nice if that were the case? Alright, umm, the issue then, let's see if we can state what the issues are. First of all there was a problem posed. The problem was comparing two fractions? Right, comparing? If one were bigger than the other and by how much? And the two fractions we were comparing were as Michael said
6.0.18	7:59	Michael:	One half and one third
6.0.19	8:02	T/R 1:	One half and one third, and when we were comparing them we asked by how much some of you came up with a solution to that which was?
6.0.20	8:10	Michael:	One sixth
6.0.21	8:11	T/R 1:	One sixth. And you were able to make an argument to show that, right? Um, and we had a couple of arguments being presented for that and we had a couple of models that were being built with the rods to make the argument. Now I have really another goal for today that I want you to think about the goals so that I am asking you to do some more writing and Mrs. Phillips is asking you to do some more writing this weekend. You can sort of maybe think about what you will be writing about. Many of you have written your solution to that problem and we've read them, Dr. Martino and I, and we have really enjoyed them and we are really pleased about what you have been writing. Umm but what I would like you to think about another solution. One different than the one you've written that someone else has proposed. Do you understand? That's reasonable. Or, you may choose somebody else's solution that you don't agree with that you find something wrong with it that you write about that. That you feel I'm not convinced and the reason I am not convinced is because so and so is arguing this and I am having trouble with this part of the argument because or however you wish to say it. Cause what mathematicians do is they argue about a certain kind of reasoning and it has to make sense. And if a certain kind of reasoning doesn't make sense then its their job to show what about it doesn't make sense. So you're going to take another role, take the role of taking an argument that makes sense if you can or showing that some argument doesn't make sense. Do you understand? Now some have you have shown an argument to make sense but you haven't taken on the second task of showing why another argument might not make sense. So we're going to

push you a little bit more to reason like a mathematician. Because what you do is really you're doing mathematics and we'd like you to do it much like mathematicians do mathematics. Okay, so is it clear what we might hope will come out of today? [Students: Mmm hmm] We'll give it a try. Now where are the hands of those who really believe that they can prove that one half is larger than one third by one sixth? Where are the people who believe it can be proved? Raise your hand so I can really be sure and you have no doubt and you feel very very strongly about that. Okay, umm, and you're willing to come up and argue your position. You're willing to come up and do it. Ok, hands up again so I can be sure who those people are? Ok, Jessica why don't you come up and do it for us.

6.0.22 11:29 T/R 1: Some of you may be explaining Jessica's argument when you write to us this weekend

6.0.23 12:36 Jessica: Umm there's not enough reds for there .

6.0.24 12:48 Jessica: [Figure O-13-35] Well, umm, well, I, I have this I counted as my one whole and that was my one half but then this my one third it's not it doesn't I have my this is one sixth. This is one sixth. The red is one sixth and I, and I didn't think, um, one third was right now because one third is smaller, umm, one, this is one third and I put that here because I wanted to show how that this red here how it takes two reds. Wait I forget what I was going to say. I forget all what I was going to say. [Figure S-13-57] Umm, Erik

6.0.25 14:18 Erik: I think that what you're trying to say [Jessica speaks over him: I forget all what I was trying to say. I know what I mean] that the orange and the red one, red rod is one [Jessica: "yeah"] and that the uh the green, the dark green is a half and then the purples are thirds and the reds [Jessica: "are sixths"] are sixths. And then what I think is that if you take one of the dark greens which is the half it equ- it's larger than one uh third but yet if you put another third onto the um, the um dark green, I mean not to the dark green, to the uh, purple to the other third, that third is larger than it. So then, If you put it, [Jessica: "right next to it"] like if you put one of the red rods, well it's sm- like I said it's smaller, the third, the one third is smaller than the one half and one of these red ones, these, the reds are sixths if you put the red on top of the pink, uh the purple it equals up to the exact same size as the dark green."

6.0.26 15:51 Jessica: Yeah, that's what I was trying to say.

6.0.27 00:15:53 T/R 1: Okay, um, thank you. Alan, Alan has a different way by the way how many did it the same way that Jessica did it? You didn't agree with him? Let's hear what he has to say and maybe change his mind. Let's hear what he has to

			say alright, so, we had one argument to choose but we have Alan and he has a different one.
6.0.28	00:16:33		Alan: Okay I used the dark green and um and I used the light greens for the halves and the red rods for the thirds and then I took the white rods and put them up against the reds [Figure O-17-55]. And those would be the halves. Those would be the thirds and those would be the sixths. So I took that out because that would be a half and that would be a third and one of these would be one sixth so I put that up to here and it took one sixth to complete, so it's, the half is bigger than a third by one sixth [Figure S-17-26].
6.0.29	00:17:43		T/R 1: What do you think? Do you agree Jessica?
6.0.30	00:17:45		Jessica: No
6.0.31	00:17:46		T/R 1: Jessica doesn't agree?
6.0.32	00:17:47		Jessica: I think he's like remember you said that it can be only be one size candy bar and that's like a whole different size candy bar he's making
6.0.33	17:57	T/R 1:	Now hold on, Alan, uh, ok Jessica disagrees. Kelly?
6.0.34	18:02	Kelly:	Well, me and Jacqueline agree
6.0.35	18:07	T/R 1:	Jackie and Kelly agree. Why do you agree?
6.0.36	00:18:09		Jackie: Well, because when you go to the store there's not just one size candy bar there's all different kinds of sizes so you can make a model with a different size.
6.0.37		T/R 1:	So you can make the argument with different size candy bars. Ok Michael?
6.0.38	00:18:25		Michael: I agree with it because um it can be done because there's like six whites equal up to one green and then it takes one white plus a red to equal a light green which is half so that would be one sixth [Figure S-18-43].
6.0.39	00:18:43		Jessica: Yeah but it's, I think it still could be one sixth, but it's just a different size candy bar
6.0.40	00:18:49		Erik: Yeah I know we said any one sixth is right.
6.0.41	00:18:51		T/R 1: It can be one sixth either way. What do you think Jessica was confused about then?
6.0.42	00:18:56		Erik: Yeah the sixth isn't the same size.
6.0.43	00:18:58		T/R 1: Does it matter? This is a model where um
6.0.44	00:19:02		Michael: Yeah because the whole is not the same size.
6.0.45		T/R 1:	Jessica
6.0.46	00:19:06		Jessica: But because say if you wanted to give someone one sixth of that candy bar and then you were going to give someone one sixth of the other one, then the person with that size would get a smaller amount.
6.0.47	00:19:16		T/R 1: Okay I think I see what the confusion is. Um, does anyone else see what the confusion is here? (T/R 1 listing out names of students who think they see what the confusion is: Beth, Andrew, Michael Erik, Brian, Jackie, Kelly, Mark) Ok,

			lots of people I think back here see what the confusion is. Uh, who's going- well, let me see what we agree on. We had a model that was built by Jessica and Erik and you agreed that that proved that one half was bigger than a third by one sixth and then we had a second model that Alan built, right? And you agree that Alan proved that one half was bigger than one third by one sixth. You agree with that? Um, so what's the problem, is there, is there a problem? What do you think?
6.0.48	00:20:14		Andrew: Well, um, that's right because if um, it's just a different size candy bar. If you just gave half of that to the person and the other half of that to another person you would still have the same size. You can't switch the candy bars.
6.0.49	00:20:31		T/R 1: Okay you say as long I whatever I do, I do it in the same candy bar, that's fair but what I can't start doing is switching. Did anybody switch a candy bar here? [Erik answers Yes] In this problem where's the switch? In this problem? [someone answers No]"
6.0.50	00:20:46		Erik: Well they didn't switch a candy bar in that problem but from the problem that Jessica, that Jessica did, he switched the candy bar, they switched the candy bar from the orange and the red to the dark green and if you're giving someone half of the orange and red and someone else half of the dark green the person getting half of the orange and the red is getting a bigger piece.
6.0.51	00:21:06		T/R 1: That's true but are we doing that?
6.0.52		Erik:	No.
6.0.53		T/R 1:	No, you're right if we do that it would be the wrong this to do but I think, um, as Andrew said we really didn't do that. Once we made a new one as long as we're in this one, whatever we share is from this one, then it's fair. And if we make the other one whatever we share from the other one is fair and we didn't switch we just showed it with the different candy bar. I think that is what Jackie was saying that you can show your model several ways. Michael?
6.0.54	00:21:40		Michael: What Jessica was confused about is, she didn't think it would be right because they, you had a different size one sixth, but he also switched the whole, so the whole is smaller by one white.
6.0.55	00:21:58		T/R 1: Ok, so it was what you called "one" that changed. In this problem Alan called one this "green" right? The dark green and when he called dark green then what became one sixth? Andrew?
6.0.56	22:15	Andrew:	The white
6.0.57	22:16	T/R 1:	The white. In the other model what was white? Jessica's model? One was, Beth?
6.0.58	22:24	Beth:	Orange and red

6.0.59	22:25	T/R 1:	Orange and red. So was it okay to call white one sixth now? No, what did you have to call one sixth now? Beth?
6.0.60	22:39	Beth:	Twelve I mean wait.
6.0.61	22:40	T/R 1:	Umm, what became ones sixth. I think you answered it what would you call white I think is the question you might. [long pause] What are you thinking Beth? [long pause] Sarah and Beth want to talk a minute? My question wasn't perhaps clear let me ask it again. When we call the orange and red one. What number name- what rod what rod had the number name one sixth?
6.0.62	24:00	Beth:	Oh, um, a red
6.0.63	24:02	T/R 1:	A red okay I didn't think I asked that clearly. So you all agree with that? Ok. White then would have what number name when I call the orange and red one? White would have what number name? How many of you think you know the answer to that? What number name would I give to white when I call the orange and red one? How many of you think you know the answer to that? You might want to look at your model, you might build a model build an orange and red one. Ok, hw many think you know the answer to that some of you have built a- James?
6.0.64	25:12	James:	[inaudible] aren't really sure
6.0.65	25:15	T/R 1:	What do you think?
6.0.66	25:15	James:	I think its one sixth
6.0.67	25:17	T/R 1:	You think the white is one sixth?
6.0.68	25:19	James:	Yeah
6.0.69	25:20	T/R 1:	Okay so James isn't sure but he thinks its one sixth, Brian?
6.0.70	25:24	Brian2:	Is it one twelfth?
6.0.71	25:25	T/R 1:	Brian thinks its one twelfth? Ok, Laura
6.0.72	25:28	Laura:	One tenth
6.0.73	25:29	T/R 1:	Laura thinks it's one tenth. We have three possibilities here now. Wow, one sixth, one twelfth, one tenth. Okay take a minute, talk to someone next to you and see if you agree. Unless we have a different is there another? Okay, what's the number name for the white when you have an orange and red?
6.0.74			*SIDE VIEW*
6.0.75	00:25:56		Meredith: Yo, yo! [tapping David on the back] One sixth. Count the ones! [Pointing to the overhead.]
6.0.76	00:26:07		David: No, We're working with orange and red.
6.0.77	00:26:10		Meredith: No, We're working with that. We're working with that model. [Pointing to the overhead where there are six whites]
6.0.78	00:26:15		T/R 1: Orange and red is one.
6.0.79	00:26:17		David: We are working with orange and red.
6.0.80	00:26:21		Meredith: Oh, Orange and red? Then it's twelve.

6.0.81	00:26:22		David: Yes, I know. One twelfth. [they talk about their sculptures.]
6.0.82			*FRONT VIEW*
6.0.83			[Erik has built a sculpture using almost all the blocks, leaving Alan with very few. Alan tells Erik that he should build the model and they argue a bit]
6.0.84			*WHOLE CLASS*
6.0.85	27:04	T/R 1:	Okay, um, James changed his mind, so since he changed his mind I am going to let him tell us when he changed his mind. He thinks he's going to argue for why he's changing his mind.
6.0.86	27:28	James:	Okay.
6.0.87		T/R 1:	If we call the orange and red one.
6.0.88		James:	Now I think it might be one twelfth, cause orange, and orange and red equals twelve white ones. So umm [Figure O-28-52]
6.0.89		T/R 1:	Let's see, are there twelve of them there? I see one, two , three, four, five, six, seven, eight, nine, ten. Looks like you made an argument for ten. It's hard - let me help you by holding this still.
6.0.90	28:37	James:	And its one twelfth cause there's one two three four five six seven eight nine ten eleven twelve [Figure O-29-35].
6.0.91	28:47	T/R 1:	How many of you agree with that? How many of you still disagree? How many of you think its one tenth? How many of you think it's one sixth? How many of you think its one twelfth [all visible hands raised]? I heard, I heard another comment from Sara and Beth and I would like you to share. Thank you James
6.0.92	20:16	Beth:	Well, um, a dark green is half of one orange, I mean orange and red and then the dark green has six blocks, six whites, and if you have two dark greens six and six is twelve. And that's why we think its twelve [Figure S-30-02].
6.0.93		T/R 1:	But I heard Sarah something even different than that?
6.0.94	30:11	Sarah:	Umm, I said that umm, you have six of these reds [Figure S-30-35]. If you times these by two you'd get twelve.
6.0.95	30:19	T/R 1:	If you times them by two, why would you times them by two Sara?
6.0.96		Sarah:	Because if you had if you put two next- two little ones right on the bottom of the red it would equal two. So you would go two times [Figure S-30-52].
6.0.97	30:58	T/R 1:	Ok, that's interesting. Anybody have any other comments before we leave this problem. Do you agree now that we have two different ways of showing? That one half is bigger than one third by one sixth? Do you think you could write about those two different ways? How many think you could write about those now? Some of you are still not sure? Erik

			you could write about those two different ways can't you? Can't why?
6.0.98	31:28	Erik:	Um
6.0.99	31:29	T/R 1:	Who thinks they can write about them two different ways, and why, Kelly?
6.0.100	31:36	Kelly:	I think I can write about them because now I understand which or how is what one goes cause now there's not just one different kind there's all different kinds.
6.0.101	31:52	T/R 1:	Anybody else what to say why he or she thinks, Brian?
6.0.102	31:57	Brian:	Well I understand the problem and I feel that I can write about it.
6.0.103	32:02	T/R 1:	Okay, both of them? Both of the models?
6.0.104	32:06	Brian:	Well, not both of them but the one, the twelfth I can write about it
6.0.105	32:13	T/R 1:	What do you mean the one twelfth
6.0.106	32:14	Brian:	The twelfths, the twelve of them equal the red and the orange rod
6.0.107	32:22	T/R 1:	Anybody else want to comment about the way you feel about this problem? Okay well a good thing to do is try another problem. Want to try build another one?
6.0.108	32:47	T/R 1:	Ok, let's try this one, which is bigger, one half or one quarter and by how much? Which is bigger one half or one quarter? And whichever is bigger by how much? Do you understand the problem? Work with your partner, and build a model and see if you can solve it.
6.0.109			*SIDE VIEW*
6.0.110	33:33	Brian:	Well, well, It's not bigger. It's not bigger. That's weird look.
6.0.111	00:33:38		Michael: Oh I know why, because there's two at the end of the number and then one quarter has four on the bottom number. Like when you draw one quarter there's one, and four on the bottom and when you draw one half, it's one, and two on the bottom. So that would be two more.
6.0.112	00:33:57		Brian: Oh yeah!
6.0.113	00:34:00		Michael: So it would only take two parts for this [pointing to two dark greens] and four parts for this [Figure S-34-02].
6.0.114	00:34:04		Brian: Oh yeah, I never thought of that! [Brian has an identical model on his desk]
6.0.115	00:34:07		Michael: It's a weird way to think about it.
6.0.116	00:34:13		Brian: So what's bigger?
6.0.117	00:34:14		Michael: So the orange would be the whole
6.0.118	00:34:18		Brian: So, but what should I write? What should I write?
6.0.119	00:34:22		Michael: Orange and the red is the whole [Brian and Michael add an orange and red train to their models- Figure S-34-44].
6.0.120	00:34:28		Brian: With the thirds, yeah, um, so can I explain it?
6.0.121	00:34:39		Michael: Sure

6.0.122 00:34:41 Brian: Well, well first why don't you explain it to me a little bit? It's wait, one

6.0.123 00:34:45 Michael: One, two So there's two on the bottom of this [pointing to drawing of one half] and four on the bottom of this [pointing to drawing of figure one fourth] So that would be, you'd have to divide this one into four parts and this one into two. Two would be like this. Four, four would be like one, two, three, four.

6.0.124 00:35:03 Brian: I think I get it. I think I'm getting it now.

6.0.125 T/R 1: [T/R 1 works with Graham. Figure S-35-18] How much bigger? [Graham moves a second light green rod to the model of a dark green and light green] It's that much bigger? That one has a number name?

6.0.126 Graham: Two quarters

6.0.127 T/R 1: And this one has a number name? One of these-

6.0.128 Graham: A half

6.0.129 T/R 1: Two of them have a number name a half, but what does one of them have a number name of? What do you call one of these [light green rods]?

6.0.130 Graham: [inaudible]

6.0.131 T/R 1: You call this [dark green] one half and you call this [light green]

6.0.132 Graham: One quarter

6.0.133 T/R 1: Which is bigger?

6.0.134 Graham: A half

6.0.135 T/R 1: By how much? It's this much bigger [showing the empty space]

6.0.136 Graham: One quarter

6.0.137 T/R 1: By a quarter. Are you ready to share that?

6.0.138 Graham: [inaudible]

6.0.139 T/R 1: Why don't you go build it? Get someone to help you do it. See if you can build it, ok? If you need a partner to go up there maybe someone here will help you. [Graham asks Michael to come up with him] Is Mike's model the same? Ok, why don't Mike and Graham, Graham you do it and Michael can help you. Ok? You want to go build it up there?

6.0.140 Michael: Is it the same?

6.0.141 Graham: Yeah. [Michael, Graham, and Brian go to OHP]

6.0.142 *FRONT VIEW*

6.0.143 Erik: One half or one quarter?

6.0.144 Alan: Now I get that the quarter is. Look, here's a quarter. You can't make this into quarters [dark green rod]. A quarter is four parts. But you could make this [orange and red train] into a quarter.

6.0.145 Erik: Ah hah

6.0.146 Alan: By taking

6.0.147	Erik:	Actually you can't make it into a quarter
6.0.148	Alan:	What?
6.0.149	Erik:	I don't think you can, well, actually you can, these, these will probably [takes light green]
6.0.150	Alan:	Oh yeah yeah yeah yeah
6.0.151	Erik:	two three four
6.0.152	Alan:	Now we eliminate that [moves aside red and white rods]. One half is bigger than one quarter by one quarter [Figure F-34-03].
6.0.153	Erik:	Exactly! [laughs] That was easy!
6.0.154	Alan:	Hey, now I quartered it, so I can put these [red rods] back on.
6.0.155	Erik:	There we go! That's yours, where's mine. There it is!
6.0.156	Alan:	There we go! A whole model and only sized that [holds a green rod]
6.0.157		*WHOLE CLASS*
6.0.158 37:00	T/R 1:	How many of you think you have a solution? How many of you think you now the answer to that problem and you can prove your answer? Raise your hand if you think you have a solution and you can prove your answer. And you know you have a solution. Ok, I see two different solutions possibly, or two different arguments you have to convince us they're correct. So if you're done and you're waiting you might want to think about a second one. Ok.
6.0.159 37:36	T/R 1:	So have many do you have David, how many arguments can you make, how many models can you build? Okay David said he could build two or three. I see Jessica has two and Andrew has two some of you are building a few models
6.0.160 38:05	T/R 1:	Ok, I see one model up there and there's another one maybe Amy and, uh, James you could build your model when their finished?
6.0.161		*FRONT VIEW*
6.0.162	Alan:	Hey, there's another thing you can quarter! Look! There's two ways [two orange rods and four yellow rods]
6.0.163	Erik:	Oh!
6.0.164	Alan:	You can quarter a train of orange rods
6.0.165		*WHOLE CLASS*
6.0.166 38:51	T/R 1:	Ok, um, I really, I saw a new one, Gregory has one I haven't seen yet. Um so I see three of them so far. I see four of them so far Alan has another one I didn't seen. Four different models. I am seeing if I can see another one that I haven't seen. I see four different models I see five different models! Andrew has one I haven't seen and Jessica. Five different models! I wonder if you can argue your models. Five of them. Let's see if you can find one that I haven't seen yet.
6.0.167		*FRONT VIEW*

6.0.168	Erik:	I wonder if you can quarter this. [As T/R 1 speaks] I got another one! [whispering] All you have to do is keep going down by two. Brown, you minus two, take that rod, and you can quarter that one. Brown, black then dark green!
6.0.169	Alan:	Dark green can't be quartered, no it can't
6.0.170	Erik:	Two dark greens
6.0.171	Alan:	We got it! We have an answer! [to T/R 1] We have four different models
6.0.172	T/R 1:	Four? So you're going to explain how you got your different models, Alan?
6.0.173	Alan:	We're subtracting by two. Two down from the orange would be the brown.
6.0.174	T/R 1:	Could you explain that to Dr. Davis back there? Whisper that to him. Tell him what you're doing to get your models. [To Dr. Davis] I want you to hear this.
6.0.175	Erik:	[coming back to their seats] -four already. So two from the brown would be yellow
6.0.176	Alan:	Two from the brown would be two yellows
6.0.177	Erik:	Yellows- reds
6.0.178	Alan:	What, no.
6.0.179	Erik:	Yeah.
6.0.180	Alan:	No. You can't quarter the yellow. That's just the point you can't
6.0.181	Erik:	Hold on. Oh yeah, you're right. Purple!
6.0.182	Alan:	Purple, purple. That's it, that's it.
6.0.183	Erik:	Purple's reds, then.
6.0.184	Alan:	Yeah, purple. Two reds for a purple. Definitely, definitely.
6.0.185	Erik:	Two minus purple would be red! Red
6.0.186	Alan:	Here's what we'll do. We'll put all our fractions in this box top so they won't break.
6.0.187	Erik:	We'll just put it on the table. We're ready, oh no we're not.
6.0.188	Alan:	Yes we are.
6.0.189		*WHOLE CLASS*
6.0.190 41:15	T/R 1:	Ok, ok, I think we're almost ready. We have a few ideas here I've seen about four different, five different models.
6.0.191 41:27	Erik:	We have like six.
6.0.192 41:29	T/R 1:	You have six different models. Okay, now it's listening time because, how many have one model? How many have two models? How many have three models? How many have four models? How many have five models [Erik and Alan raise their hands]? How many have more than five models? So, okay, wow! Let's get .
6.0.193 41:48	Erik:	I think we may. We have six or seven models
6.0.194 41:49	T/R 1:	Six or seven models okay you all can listen right? Because there are only two people here that claim they have more than five everyone can listen and we'll learn something or

		we'll argue that they're wrong, right? You either have to refute what they are saying or you have to say "gee I can learn something from what they've done" That's the important thing. So listening is important. Let me make a friendly suggestion to you. Listening with rods means the following when someone is giving an explanation what you should be doing is at your seat building what they are building up there. That's how you're going to follow their reasoning if it makes sense. And if at any point it doesn't make sense you have to stop them. Your job is to say "hey wait a minute, how did you do that? Why did you give that rod that number name? I don't understand that." Do you understand, your job you're the audience, in a sense, you're the jury. You can not let them get by with saying something unless you're convinced. And you can't just sit there if you're not convinced, not being convinced. Your job is to be convinced. Is this clear what your job is? Okay let's get started. Up front you're all listening very carefully to the team. Graham built a model and Graham lost his partner so he wanted some more partners but give it a try Graham. Let's hear it.
6.0.195 43:27	Graham:	[Figure O-43-24] The orange and the red would be one and the dark greens would be a half and the light greens would be a quarter.
6.0.196 43:51	Michael:	And um, it- we think it will be bigger, we think one half would be bigger than one quarter by one quarter because it takes two quarters to equal that. And why we think that is because four is um, two more than two, so it would take two fourths to equal two, two pieces. Because there's four pieces and then they would have to put those two pieces together to make two pieces.
6.0.197 44:31	T/R 1:	Alan you have a question or a comment? Any questions or comments, uh, for the team up on top. They added some extra ideas what do you think, Andrew?
6.0.198 44:45	Andrew:	Is the green and the light green supposed to be, uh, are you going to have the fourths to it?
6.0.199 44:52	Michael:	What do you mean?
6.0.200	Andrew:	There's no fourths what rod is the fourths?
6.0.201 45:00	Michael:	One fourth is a quarter, its another name for a quarter
6.0.202 45:02	Andrew:	It's looks like you have a dark green there
6.0.203 45:07	Michael:	This is just, it's one of the regular ones because we ran out of the greens
6.0.204 45:08	Andrew:	Yeah, but do you have four fourths?
6.0.205 45:10	Brian:	Yeah
6.0.206 45:12	Brian:	One, two, three, four

6.0.207	45:15	T/R 1:	Can you separate those two on top a little bit because they're a different color and they look like does that help Andrew? That's a very good observation. I was wondering about that myself. I was glad you asked that question. Do we have any other questions from the audience? We'd like questions from the audience anything that was presented by that team that you want to question anything they said anything they showed. We'd like to thank you very nicely done up front. Now do we have another way, I'd like to hear from that team in that corner, Jackie and Amy and James. Thank you gentlemen. [Some more talk about jury and audience]. Come on Amy.
6.0.208	47:22	Amy:	[Figure O-47-54] Okay, um, we decided, we tried the orange and we couldn't, we just , we didn't want to make a train. We wanted to use one color and we couldn't find any thing to make a quarter of that so we went down to the blue we couldn't find a half of that, then we went down to the black, I mean brown, and then we found a half of that and a quarter for that and so we used brown and we took two purples and we put those underneath the brown, then we took, then we found red were half of purples so we put the reds underneath the purples and then we had to see how many whites would equal up to all the, would equal up to a brown so we kept putting them on and so we found eight.
6.0.209	48:08	T/R 1:	So what did you decide? What is bigger and by how much?
6.0.210	48:14	Amy:	James figured that out
6.0.211	48:15	James:	One eighth
6.0.212	48:16	T/R 1:	Are you convinced what James did?
6.0.213	48:17	Amy:	Yes
6.0.214	48:19	James:	It's one eighth, yeah, we think its one eighth
6.0.215	48:21	T/R 1:	Okay so James says that in this model one half is bigger than a quarter or one quarter is bigger than a half?
6.0.216	48:30	James:	No, one half is bigger than one quarter
6.0.217	48:33	T/R 1:	By one eighth.
6.0.218	48:34	James:	Yeah
6.0.219	48:36	T/R 1:	Okay class. Do you agree? Oh, we have some disagreement what's your disagreement? Let's start, Kelly, and I am going to hear from Gregory in just a minute. Kelly, you disagree?
6.0.220	48:51	Kelly:	Well, me and Jackie have another one.
6.0.221	48:56	T/R 1:	I'm talking about this one.
6.0.222	48:56	Kelly:	Oh.
6.0.223	48:58	T/R 1:	They're claiming that, the team before just showed that one half was bigger than a quarter by a quarter. Isn't that right? I think that's what Graham did and his team, right? Now a new team claims that one half is bigger than a quarter by an eighth!? Is it possible that different models can give you

			different, different answers? Some of you think different models can give you different answers? That's interesting- Alan says no. Okay let's hear from Meredith she hasn't talked in a while. Then we'll hear from some others.
6.0.224	00:49:45		Meredith: Well you said its bigger by an eighth, are you calling this an eighth? [mmm hmm] Ok. Take these two. This is an eighth; it is not bigger by an eighth because there is still negative space [Figure O-50-40]. You're calling that an eighth, it's not equal. But if you take another one, it could be bigger by two eighths [Figure O-51-03] and, or it could be bigger by one quarter [Figure O-51-15]. One quarter or one, um two eighths. It's the only way it could be bigger by. [some of them laugh]
6.0.225	50:45	T/R 1:	What do you think up front? Amy, James, Jacquelyn? What do you think about what Meredith is saying?
6.0.226	50:52	Amy:	Um
6.0.227	50:55	Jacquelyn:	Well, I think we meant that all these put all together are one eighth. I think that's what we meant.
6.0.228		James	Yeah [Jacquelyn laughs]
6.0.229	51:07	Meredith:	Both these put together are one eighth?
6.0.230	51:15	James:	No. We thought, uh, all of these whites put together were one eighth. That's what we thought
6.0.231	51:23	Meredith:	But the question was: Is one half bigger, is one half bigger than one quarter?
6.0.232	51:32	Jacquelyn:	And we said one half.
6.0.233	51:34	Meredith:	So you think all these are one eighth? And that is bigger than 1/8? The um, because that's the question. The question is, is one half bigger than one quarter, right?
6.0.234	51:51	Jacquelyn:	Right
6.0.235	51:52	Meredith:	You think its bigger than one eighth and all these are one eighth? [Meredith - Figure O-53-15. Jacquelyn laughs, Meredith smiles.] So that's how much you think its bigger by?
6.0.236	51:57	Jacquelyn:	No, we, I think we got the question wrong.
6.0.237	52:20	T/R 1:	What question do you think you were answering, uh, Miss Jackie, you think you were answering a different question I'm hearing you say
6.0.238		Jacquelyn:	Yeah
6.0.239	52:30	T/R 1:	That's what I am hearing you say, Okay. Let's think about the question we asked before that lets go to Meredith. What do you think about what Meredith said. Given Meredith said the question was which is bigger and by how much? What do you think of Meredith's question, have you changed your mind now to the answer given Meredith's question. one half bigger than one quarter?
6.0.240	52:57	Jacquelyn:	That is what we said

6.0.241 52:56 T/R 1: By how much?
6.0.242 Jacquelyn: I think that is where we got a little wolbby (sic)
6.0.243 T/R 1: Okay, so you see it's bigger by how much
6.0.244 Jacquelyn: Two
6.0.245 53:07 T/R 1: Two what?
6.0.246 53:09 Jacquelyn: Two eighths
6.0.247 53:10 T/R 1: Two eighths? Or Meredith suggested another name for that would be
6.0.248 53:15 Meredith: One quarter
6.0.249 T/R 1: One quarter and you see how she got the other name for that? [Jacquelyn nods.] How did she get the other name for that? Tell the rest of the class who may not have- who may have missed it.
6.0.250 52:28 Jacquelyn: Well, this was one quarter and all these were one quarter and these two together make up a quarter [Figure O-54-33].
6.0.251 53:54 T/R 1: Ok, so you are changing your answer to one quarter or two eighths.
6.0.252 53:57 Jacquelyn, James: Yeah
6.0.253 T/R 1: Okay well you did add something to this we said that one half is bigger than one quarter by one quarter and for the rest of the class who might not of though about this very interesting discussion is that one half is bigger than a quarter by two eighths. So we are sort of happy you introduced the eighths because it gave us another way to think about how much bigger one is than the other. That was very helpful to us and I have to thank you for that and Meredith. But now I wonder what could someone in the audience tell me what question you think that team was answering. Because Jacquelyn suggested she thought they were answering a different question. And Jessica whispered something, her hand was up because she thought she knew what question they were answering. Do you think you know Jessica or not?
6.0.254 54:44 Jessica: Umm I am not sure, well, for that problem?
6.0.255 54:50 T/R 1: Yes
6.0.256 54:52 Jessica: What they mean?
6.0.257 54:53 T/R 1: Yes
6.0.258 54:55 Jessica: Umm well I don't get first they changed, first they had an answer, then they changed it and well, they didn't, yeah they did, they changed it. And, well, I'm not really sure.
6.0.259 55:15 T/R 1: You're not really sure what the other question was, anybody? See I can tell you what I think it might have been but I may be wrong. I think what we're so used to ah what we decide what one is we begin to get number names for the other rods like one half in this case, like one quarter in this case, right? And, and usually people like to know what other number names they can make once they called something one. So I

			think you said ”oh gee there’s the white I can also give that a number name – that’s an one eighth” Maybe that’s the question you answered: What’s a number name for white? But that wasn’t the question that was asked. You see the difference here? Ok, I would like to hear Gregory and Danielle’s solution because it’s a little bit different and then I want to hear the comment about, we have two different models we ended up with the same answer but some of you seem to think that you could build a different model and not necessarily have the same answer that’s a very important question. If we don’t get to talk about that today I want everyone to think about that this weekend. A very important question. Is it possible to get one answer with one model and a different answer with another model. That’s a very important question. Now, we have two models, we did end up with the same answer or another name to call, another number name for a quarter. Did anyone do the model that Danielle and Gregory are building? This is one I didn’t see, I walked around it’s a different one than I had seen. Did anyone else build Danielle and Gregory’s model. See how we think about these things differently? Brian did you have that one? Brian had that one too, ok, he was the only other person who had that one. Okay, Danielle, Gregory let’s hear your thinking.
6.0.260	57:11	Danielle:	[Figure O-57-43] Well, we think that the, um the half which is the brown, um, was bigger than the fourth because if you take the brown, the half and the fourth you could see that the half is bigger.
6.0.261	57:28	T/R 1:	How much bigger?
6.0.262	57:31	Danielle:	By a purple.
6.0.263	57:33	T/R 1:	And what number name is that?
6.0.264	57:34	Danielle:	A fourth [Figure S-57-40].
6.0.265	57:35	T/R 1:	What do you think class? Does that model, does it give you that same answer? The new model? [Yeah, mm hmm.] Okay. We have some other models we have some other models but I’d like to know, umm, Andrew you want to go give model? This one I’ve seen a lot. I don’t seem to see it up there.
6.0.266	58:22	T/R 1:	How many of you built this one? Raise your hand if you’ve built this model. If you haven’t built it you may want to build it while Gregory is building it- Andrew is building it. Ok lets all listen to Andrew’s, um, solution now and see if you agree.
6.0.267	58:45	Andrew:	[Figure O-59-23] It’s bigger by um one fourth because, there and, put that there. I think they’re all if you have the fourths on your um, your problem and I think that it would always be one fourth would be as much as, more. The half would be more than it by one fourth if you always have a fourth.

6.0.268	59:26	T/R 1:	Why do you think that?
6.0.269	59:29	Andrew:	Well, because out of all the people that came up here, they always had the room for one more fourth, and I think that because usually the fourths, or two of 'em are equal up to the half, so then it would be a fourths.
6.0.270	59:50	T/R 1:	Okay, what is that the rest of you are saying? I hear, uh, some of you whispering that before Alan had a theory and Erik had a theory what do you think of Andrew's theory? Erik?
6.0.271	1:00:11	Erik:	Well, I think that, we think that you could divide- I think that you could take, you could take rods and divide them equally into fourths I think six times. Well, and we also came to a theory that if, if you uh, yeah we also came to a theory that
6.0.272		Alan:	If you take an orange rod, go down two it would be a brown rod
6.0.273		Erik:	if you take an orange rod and go down two it will be the brown rod
6.0.274	1:00:44	Alan:	And you can make it into quarters, and then-
6.0.275	1:00:45	Erik:	Yeah you just divide two from each rod like you start with the orange rod divide by two and then the brown rod and you divide by two from the brown rod
6.0.276	1:00:55	Alan:	From the brown rod .
6.0.277	1:00:55	Erik:	And then whatever rod you get, divide two from that .and keep going down.
6.0.278	1:01:01	T/R 1:	Alan, did you want to add something?
6.0.279	1:01:04	Alan:	[take their models up to the board - Figure F-58-47] We have a lot of them we need to remember
6.0.280	1:01:13	T/R 1 :	You might not have enough time to build them all I would like if you just talk about them, I don't think you'll have time to get them all up there. David? Will your answer change when you change your model?
6.0.281	1:01:27	David:	Um I don't think so cause it might just be in a different size but it might be the same thing but they kind of changed it around a little bit.
6.0.282	1:01:38	T/R 1:	Ok, I want you to think and write about why you think the model won't change. Those of you who might think differently. Ok, so I want to hear generally what you're learning about the models you're building when your comparing a half and a quarter. And I'd like really everyone to think about it from the few models. I'd like you to think of more than one. If you thought about the problems for one or two think about three or four. I really am pleased at what I see. One, two, three, four, five, six models Meredith has here [Figure F-1-02-57].
6.0.283		Erik:	That's what we have too
6.0.284	1:02:15	T/R 1:	And you have six models and you have six models, Brian and Michael. Of course my question then is could there be more

			than six? Why or why not, can you prove it with the rods you have. That's my next question. Can there be more than six. Why or why not? Okay, what do you think Andrew?
6.0.285	1:02:36	Andrew:	Well, you really can't tell, if um, you can't tell if the half is bigger than the fourth, cause right on their problems that they have they don't have a half.
6.0.286		Erik:	Well, we didn't do the halfs we just did the fourths. Because all you really had to do was the fourths.
6.0.287		Andrew :	But the question was is one half bigger than a fourth, by how much?
6.0.288		Erik:	But we think before we went up we heard other people go up and they explained that they thought one half was bigger than one third by one fourth so we just figured that we'd just do the problem just to show and even if we had to do the fourths or the halves half of that would be actually, would probably, I'm not sure but we didn't really have them because we thought that people answered the question. That they'd be bigger by one fourth [Erik and Alan's work on the OHP - Figure O-1-05-53].
6.0.289	1:03:34	T/R 1:	Okay, well I'd like to thank you all for such a wonderful job. I saw every single person in this room thinking very very hard today .and really explaining I want you to think about one more question, especially those of you who have built six of these, right? Especially those of you who built six models. The question I want you all to think about it is it possible to make more than six models? Because I'm wondering if you had to explain this to a younger child, let's say in the third grade and they came up with a model would it be one that you would be familiar with? Is it possible for a third grader to come up with a model with these particular rods that you haven't seen? You understand? And why or why not? Because you may have to be helping some of your third graders later in the year. Third grade fellow students. I want you to imagine all possible things they can build, ok? This was superb thinking and I want to thank all of you. I hope you have a wonderful weekend and I'm going to see you on Monday, I think I have a really good problem for you on Monday [Figure S-1-05-31 - homework questions].
6.0.290	1:04:48		End

Session 7, Oct. 4, 1993 (Front, Side, and OHP)

Line	Time	Speaker	Transcript
7.0.1	2:33 S	T/R 1:	Good morning! Are you all as awake as I am?
7.0.2	2:39	Meredith:	Yeah.
7.0.3	2:40	T/R 1:	I don't know if that is good or bad, Meredith. Let me shut this [the overhead projector fan] off. [Holding up Mark's diorama - Figure O-5-07] I was thinking when I was looking at Mark's model, and I noticed many of you made models, also for projects for another class. I was thinking about this model because we were talking about models the other day, weren't we. Remember that? And I, remember I asked you to think about something about the models that you built. Remember what I asked you to think about? Does anyone remember, Andrew?
7.0.4	3:12	Andrew:	Um, is one-half bigger, uh is one half bigger than one fourth, by how much?
7.0.5	3:21	T/R 1:	Does anyone else remember anything in our discussion about models? Andrew remembered something. Is your hand up Audra?
7.0.6		Audra:	No.
7.0.7	3:32	T/R 1:	Audra’s yawning. Does anyone remember anything in our discussion about models? We talked about models, we asked some questions about them. Think for a minute. Do you remember Meredith?
7.0.8	3:45	Meredith:	Um, what’s bigger, one half or one quarter and by how much?
7.0.9	3:50	T/R 1:	That's what Andrew said. Right, which is bigger. But we also were talking about models in general. We asked ourselves some questions about models. Did you all build the same model?
7.0.10	4:04	Students:	No.
7.0.11	4:04	T/R 1:	To answer that question?
7.0.12	4:05	Students:	No, no.
7.0.13	4:07	T/R 1:	Some of you built different models. [Erik raises his hand] Erik?
7.0.14	4:10	Erik:	Some of us built the same models and some of us built different.
7.0.15	4:12	T/R 1:	Some of you built different models, and I asked you a question about that. Do you remember?
7.0.16	4:18	Erik:	[Raising his hand] Oh!
7.0.17	4:19	T/R 1:	Erik?
7.0.18	4:20	Erik:	Could you get different answers
7.0.19		Michael:	Using barred models?
7.0.20		Erik:	if you use different models.

7.0.21	4:27	T/R 1:	Yeah, can you get different answers, right Michael and Erik? - if you use different models. What did you think? How many of you thought you shouldn’t get different answers? [Some hands are raised] How many of you are not sure? [Few more hands are raised] It's okay not to be sure. Have you been thinking about that at all since then? Maybe not much. Michael, have you been thinking about that a little bit?
7.0.22	4:48	Michael:	Um, well, I figured that it couldn't be because our answer that we got, me and Brian, was that it was bigger by one fourth because it will always take two, it will always take four quarters to equal up
7.0.23	5:03	Erik:	Yeah, because four is an even number and you can divide it by two.
7.0.24	5:04	Michael:	In half
7.0.25	5:08	Erik:	So there will always be one fourth and two fourths, three fourths, four fourths and two fourths is always going to be a half, a half in fourths.
7.0.26	5:16	T/R 1:	What do you think about that? There are a lot of good ideas in what you are saying. [Picking up Mark's diorama] I was thinking that maybe it would help you, it sort of helped me to look at Marks' model. Sometimes it helps to look at a model that's a little different. Maybe this is a model that doesn't use the Cuisenaire rods, but in a sense it's a model. Um, I found out a little bit from Mark about a book he read, he was telling me. This [pointing into diorama] was supposed to be a sea monster and this was supposed to be [again pointing into the diorama] two friends. And I looked at, I looked at what he built here[still pointing to diorama] to represent some of the story and I thought by looking at this model that I couldn't really tell of the boy and the girl who was taller by looking at them, I wasn't really sure, and I didn't know really if Mark cared about that. But I looked at the sea monster, okay, and I looked at the boat, okay, and I was thinking about their sizes a little bit, right? What are you, why are you smiling about Mark?
7.0.27	6:27	Mark:	Uh, well I wasn't thinking about the sizes. I made the sea monster bigger than the boat.
7.0.28	6:33	T/R 1:	Did you want the sea monster to be bigger than the boat?
7.0.29	6:36	Mark:	No.
7.0.30	6:36	T/R 1:	You really didn't. What about the boat and the children?
7.0.31	6:42	Mark:	Those too. The children are bigger.
7.0.32	6:43	T/R 1:	The children are bigger than the boat. Did you want that? [Mark, still smiling, shakes his head sideways, indicating negation.] No.
7.0.33	6:48	Michael:	Maybe he was trying to focus on the children and instead of just the boat.

7.0.34	6:52	Erik:	Yeah, he was probably trying maybe to make them look bigger, like you're looking at the children, not the boat
7.0.35	6:55	T/R 1:	I, I
7.0.36	6:58	Erik:	like he doesn't, he just put the boat in
7.0.37	7:01	T/R 1:	Yeah
7.0.38	7:03	Erik:	Cause they're at the dock. Yeah, but he wasn't just focusing on the boat.
7.0.39	7:04	T/R 1:	Maybe the boat wasn't intended to be so close, but that he could make it, you know, the dock, not as far out as those things. Or maybe he didn't think about it, all those things. That wasn't what he was focusing on, but I think suppose changed this, suppose we took this story and made it a math problem. Suppose we changed it for a different purpose. And I said to all of you, I want you to go and make me a model of two children, right, and they're sitting at a dock and they're fishing, and they just caught a fish, right? Let's not make it a sea monster and let's change it a little bit, they're fishing and then their boat is docked somewhere, do you understand? If I asked you to do that and it mattered now what sizes they were. What would you expect to be the largest object and the next and the next? What would you expect if you were really worrying about the size, you had two children at a dock and you have a boat and you have a fish, now we're not going to go with sea monsters. Mark?
7.0.40	8:09	Mark:	Uh, the boat's the biggest.
7.0.41	8:10	T/R 1:	The boat's the biggest. Do you agree?
7.0.42	8:12	Students:	Mm, hmm [nods of affirmation from various students].
7.0.43	8:13	T/R 1:	You think the boat's the biggest. okay.
7.0.44	8:15	Mark:	And then the children, um, and then the fish.
7.0.45	8:20	T/R 1:	Would be the smallest. You all agree with that?
7.0.46	8:23	Students:	Um, hmm.
7.0.47	8:24	T/R 1:	Ok. Is there anyone who disagrees with that? Now suppose we said okay. We're all agreeing that that's our assignment and you're all supposed to go home and do that. Would you all make the same model?
7.0.48	8:37	Students:	No
7.0.49	8:38		
	T/R 1:	But, now what would be the same about all of your models? What wouldn't change about all of your models, Beth?	
7.0.50	8:45	Beth:	We'd have the same idea.
7.0.51	8:46	T/R 1:	And what's that idea that would be the same?
7.0.52	8:48	Beth:	That two people, fish and...[inaudible]... and the boat [inaudible]
7.0.53	8:58	T/R 1:	Okay, is that enough? Because we have that here.
7.0.54	9:02	Beth:	Their sizes are, the boat's biggest and then the children

7.0.55	9:12	T/R 1:	And then, and then the fish. Okay so you agree with that, that there are certain things that all of you would have in your model. You have these four principal players right or things. The boat, two children and you have the fish. What size will you make them will that necessarily be the same? Meredith?
7.0.56	9:33	Meredith:	Well, maybe not because everybody can't have the same because they don't have, they're not like copying each other
7.0.57	9:41	T/R 1:	Yeah. You make your children. some of you might use little dolls or something, right or bigger dolls, or
7.0.58	9:47	Meredith:	You're not measuring the same.
7.0.59	9:48	T/R 1:	You wouldn't measure them the same. But one thing that would be the same is the relative, one thing you have to be careful each of you in your own models would be the sizes in relation to the other sizes, right? And if somebody came in now with a fish bigger than the boy, [laughing] that would have missed the point, right?
7.0.60	10:06	Michael:	No, because a fish could be bigger than a boy.
7.0.61	10:10	T/R 1:	That's true, ok, that's true. But we really mean two children fishing at a little dock, not out in the ocean somewhere where we expect the fish to be smaller, but you're right, you're absolutely right Michael, it could be. But we'd have to agree on some things, on some constraints, here. Obviously if we changed it and we were deep sea fishing right, and we could be catching some whales or whatever. Some very, very big fish. That would change things. Now, what does that have to do with the models you made and some of the comments that Michael and Erik made about the models you made? What does that have to say about it? Or anything? Thank you very much Mark [returns his diorama to Mark at his desk]. I'm glad I saw that lovely model. Does it have anything to do with the models we make in order to make an argument. Would you expect one model to come up with something different than the other? Would it look different?
7.0.62	11:13	Students:	Yes
7.0.63	11:15	T/R 1:	Maybe. Would the relationships that you're suppose to show change?
7.0.64	11:16	Students:	No
7.0.65	11:18	T/R 1:	No. And that's the important thing to remember. That your model that you make should not be changing, right, your argument. But suppose Mark had his model and suppose Danielle made a model, ok? And Danielle decided to make a very little model okay a little tiny model? She doesn't like to carry big things to school. And let's suppose that Audra made a big model, right? She got some help. Could I take the fish in Danielle's model, the little fish in Danielle model, and

			swap it, or let's take Audra's big fish, can I put it in Danielle's little box. No. Well, it depends on how big the fish is.
7.0.66	12:18	T/R 1:	Danielle's little box is really a little box, so, it's, um, you know, about this size [she holds horizontally a thermos bottle approximately 10 inches long]... and Audra's is like that [with her hands she shapes in air a box approximately two feet by two and one half feet] and so Audra's fish is maybe about this big [she holds two pens together in a straight line as these dashes are formed --] and Danielle's fish is about this big [she holds her thumb and forefinger approximately one inch apart]. Would it be okay to put Audra's fish in Danielle's box? No.
7.0.67	12:44	Michael:	It would look like a shrimp!
7.0.68	12:46	T/R 1:	It would look like a shrimp. Why wouldn't it be okay? What would probably happen if you did that? Graham?
7.0.69	12:52	Graham:	Well it wouldn't fit.
7.0.70	12:54	T/R 1:	It wouldn't fit in it. That's exactly right, it probably wouldn't even fit in. Maybe it would but it might not, right? And what would happen, Meredith?
7.0.71	13:02	Meredith:	Well you could put the Audra's fish and you could put Danielle's fish into Audra's box, because it’s small and it could fit in.
7.0.72	13:14	T/R 1:	It could be a shrimp [laughing]
7.0.73	13:19	Meredith:	but you can't put Audra's fish into Danielle's box because it's [the box] too small
7.0.74	13:27	T/R 1:	Ok, it raises some interesting questions doesn't it? We're sort of, you know, making up some hypothetical things and imagining some things. But do you get the idea? That once you've built your model and you decide what you are going to call one, right? You've chosen to make your other principal players in relationship to that one, right? So in this case if, if your one is going to be the size of this little stage, if you like [gestures in the air a rectangle approximately one and one half feet by one foot], your players are made the boy the girl the fish the boat in relationship to this stage isn't it.
7.0.75	14:06	Erik:	Mm, hmm [agreeing]
7.0.76	14:07	T/R 1:	But if you've made your one a much bigger stage, if you like[gestures a rectangle approximately three feet by two feet] your players are going to be in relationship to that stage, isn't that right? And as long as you stay within your stage, right, you show your relationships and if they may or may not work when you switch stages right? And that’s like switching candy bars right? Isn't that right?
7.0.77	14:24	Erik:	Yup.
7.0.78	14:33	T/R 1:	So I want you to think about that for models. Would you expect if you were building a brand new model that what you

			showed to be true with your first model, should it still work? Should it still work with the new model, the relationships you showed with your old model? Would you expect it to work if your-
7.0.79	14:45	Meredith:	Maybe, maybe.
7.0.80	14:50	T/R 1:	[As Michael is shaking his head side to side in negation] Michael changed his mind, he doesn't expect it to work Before he said it should work, and now he saying it may not work. So tell me what you're thinking.
7.0.81	15:02	Michael:	Well, your old model, say your old model, you decided it was too little and you couldn't see all the figures in it. So you make a bigger model and you try to take the fish from that little model because you decide that you don't want to make another one, you put it in and you wouldn't be able to see it there.
7.0.82	15:19	T/R 1:	Okay, but that's not my question now. Suppose in your little stage you showed the people and the boat and the fish, right? And you showed the fish were smaller than the people who were smaller than the boat. Right?
7.0.83		Michael :	Yeah, mmm, hmm [agreeing]
7.0.84		T/R 1:	Would you expect, let's say in Audra's model, which is a different model that her fish was smaller than the people and smaller than the boat?
7.0.85	15:44	Michael:	You'd have bigger people, bigger boat and a bigger fish.
7.0.86	15:50	T/R 1:	But should those relationships hold?
7.0.87	15:51	Michael:	Yeah.
7.0.88	15:52	Others:	Yes [simultaneous to Michael's reply]
7.0.89	15:54	T/R 1:	Yeah is that right?
7.0.90	15:55	Michael:	Yeah.
7.0.91	15:55	T/R 1:	Or if we had sort of a medium size model like Mark's and he were trying to make these fit, would you expect the fish to be smaller than the people than the boat?
7.0.92	16:04	Michael:	Yeah.
7.0.93	16:05	T/R 1:	So in each of your models would have those relationships holding, right?
7.0.94	16:07	Students:	Yeah.
7.0.95	16:08	T/R 1:	But they wouldn't all be built the same way and they wouldn't all be the same size
7.0.96	16:11	Erik:	So it'd be, it's standard that the fish would be smaller than the boat and the people, except the fish would be different sized and the people different sized and the boat different sized.
7.0.97	16:20	T/R 1:	Right. Is that like what you're doing when you make models to compare fractions?
7.0.98	16:23	Students:	Yeah.
7.0.99	16:24	T/R 1:	In what way is it the same or different? [some students raise their hands] Meredith?

7.0.100	16:28	Meredith:	Well if you have the same question asked and you do it right then you're going to wind up with the same answer and some of the models could be bigger and some of them could be smaller.
7.0.101	16:40	T/R 1:	What do the rest of you think? How many of you agree with what Meredith said? [some hands are raised]How many of you disagree? [no additional hands raised, at least in what was visible] How many of you are still not sure? [more hands are raised] You know we have to help the people who are not sure to understand. they don't disagree, but they're still not following. Can someone help? Let’s talk about this a little bit more to help them? Who wants to give it a try? [Meredith's hand goes up] Or the people who aren't sure want to tell us what they are confused about. Do you want to talk a little bit? Audra? Jackie? What bothers you and then maybe the people here will try to help, ok? Do you know what the question is? What do you think the question is?
7.0.102	17:33	Jacquelyn:	Um, is the mod- is different models bigger than others and um…
7.0.103	17:44	T/R 1:	Do you want to say that one more time?
7.0.104	17:47	Jacquelyn:	You can use different models that are, they're the same.
7.0.105	17:51	T/R 1:	Is that the question? What do you think, Audra, is that your question? You think it’s a different question? Maybe we are answering a different question. Let's see what Audra thinks the question is and then we can hear from those of you who can try to help.
7.0.106	18:06	Audra:	[hesitantly] It's that we um, it's about, ah, there are different sizes of, just like the candy bar that we did before. Um, you were asking, um if we thought what sizes can fit into each box, what sizes should be that we are going to get confused that the fish can fit into a box.
7.0.107	18:48	T/R 1:	Who else is confused, what you think the problem is? There are a some other people who are confused, or aren't sure. Laura? Are you in this category of not being sure? [Laura nods affirmatively] What do you think the question is?
7.0.108	19:04	Laura:	I'm not sure.
7.0.109	19:06	T/R 1:	You’re not sure what the questions is. Okay, well, that's a start. Maybe if we got the question, if we understood the question, that might help us. Who's going to try with the question? [Erik's hand is raised.] Go ahead Erik, give it a try, because we also have some people coming in
7.0.110	19:18	Erik:	It has to do with the model that Mark made. Can the fish, the fish should be smaller than the people in the boat, but the people should be bigger than the boat, or, no, they should be bigger than the fish, but they shouldn't be bigger than the boat either. And how does that, how do those models [pause]

			help us understand the models we're building [models of fractions built with Cuisenaire rods]?
7.0.111	19:52	T/R 1:	Jackie? Michael do you want to add to that?
7.0.112	19:59	Michael:	Well, it's sort of like um, you can't, the fish has to be smaller than the people and the people have to be smaller than the boat, cause the people have to go in the boat and the people have to be able to pull the fish out of the water and if it was bigger than it they might have a little trouble getting it out. [laughter] So um, so then, um, its sort of like so, that just helps us understand what we're talking about with the Cuisenaire rods when we are using different sized boxes to make different sized, um, halves and quarters, um, but, they're basically you can call it the same thing as you would then just the small one with the small one if you call the box a whole, and the boat a half it would equal a quarter. You could still do that in Audra's model or any box.
7.0.113	21:11	T/R 1:	Does that help Laura, Audra, or would you like to ask Michael a question? Does anybody want to add to that? We've heard from Michael and we've heard from Erik. Meredith, you were going to say something earlier? [Meredith mutters]. Oh, it was said already?
7.0.114	21:28	Meredith:	Yes.
7.0.115	21:29	T/R 1:	Does anybody want to add to that? Sarah, Beth, okay, well it's something to think about isn't it, as we make, uh, different models. Um I remember that you wrote about the models that you worked on and I, I'm looking forward to reading them and, um, knowing more about they way you think about them. Let's try a different one. Ok, let's try a different one. Let's see what happens here. So this is the problem I would like you to think about. I'm wondering which is bigger, one half or two thirds. [pauses] Now before you model it you might think in your head, before you begin to model it what you is bigger and if so, if one is bigger, by how much. Why don't you work with your partner and see if you can figure it out.
7.0.116			*SIDE VIEW*
7.0.117			[David builds a model with one orange, two yellow, and a purple]
7.0.118	22:40	David:	Try the purples. Get third purples. It doesn't work, try the greens
7.0.119	22:51	Meredith:	What was it? Two thirds?
7.0.120	22:55	David:	It would be like brown or something like that.
7.0.121	22:58	Meredith:	Ok
7.0.122	23:01	David:	We're not doing the one third, we're doing two thirds. That is one third

7.0.123	23:10	Meredith:	First we've got to find out what a third of it is. What's a third of an orange?
7.0.124	23:15	David:	One third? [He places two green rods instead of the purple rod]
7.0.125	23:33	Meredith:	What's third of an orange? Let's start a different model. [She begins to make a different model] The green. The green, half of it is the light green [David places a third light green rod next to the original two]
7.0.126	23:45	David:	[Demolishes his original model and begins to build the same as Meredith] Alright, yeah, I was thinking of that way before
7.0.127	23:48	Meredith:	And you can take the take the red, and the light green, and put it up to it [Meredith has a model of one dark green rod, two light green rods, and three red rods], it's, she asked, is one half bigger than, what did she ask? What did she ask?
7.0.128		David:	She asked, which is bigger, one half or two thirds?
7.0.129	24:16	Meredith:	One half or two thirds? Now take six of the ones [she takes six white rods]
7.0.130	24:23	David:	Yeah, I know, and put 'em up to there, and that would be one sixth. Hey, wait a minute, hey wait, maybe that's it, yeah it's bigger by one sixth
7.0.131	24:40	T/R 1:	[To Beth and Sarah] What do you think? Which is bigger?
7.0.132	24:42	Beth:	One half [Beth's model is the same as Meredith's]
7.0.133	24:44	David:	I think one half is
7.0.134	24:48		This is one half [pointing to the light green rod], this is a half and this is a third [pointing to the red rod], and a half is one unit more than [she places a white rod next to the red rod, Figure S-24-58]
7.0.135	24:58	T/R 1:	Could you tell me the problem I asked you to solve?
7.0.136	25:01	Sarah:	How much, is one half, what?
7.0.137	25:18	Beth:	Oh, by two thirds, it's if two thirds is bigger than one half, because we did, we did one third
7.0.138	25:26	T/R 1:	You compared one third and a half and what was the question I asked you to solve?
7.0.139	25:31	Beth:	Two thirds and a half
7.0.140	25:31	T/R 1:	So now you did two problems, ok. So, ok, so the question is, what's the questions, Sarah, that I asked you to solve?
7.0.141	25:41	Sarah:	You wanted us to figure out if one half or two thirds is larger
7.0.142	25:48	T/R 1:	Ok, and by how much?
7.0.143	25:52	Beth:	[Beth puts a white rod next to the light green rod and compares it to the two red rods] It's one
7.0.144	26:09	T/R 1 :	Yes, David and Meredith? What do you have?
7.0.145	26:11	Meredith and David:	Well
7.0.146		David:	we think
7.0.147		Meredith:	two thirds
7.0.148		Meredith and David:	is bigger than
7.0.149	26:15	T/R 1:	You both agree?

7.0.150	26:16	Meredith:	one half by one sixth. Cause if you put six ones up to a whole
7.0.151	26:27	David:	dark green
7.0.152	26:28	Meredith:	If you put it up to a whole
7.0.153	26:28	T/R 1:	I'm sorry, what's the number name for dark green
7.0.154	26:32	David and Meredith:	One
7.0.155	26:33	T/R 1:	Ok.
7.0.156	26:35	Meredith:	And you put six ones up to the dark green
7.0.157	26:39	T/R 1:	Hold on, I'm a little confused. Tell me again. Six ones? You called this [dark green] one? What are you calling these [white]?
7.0.158	26:45	Meredith:	One sixth
7.0.159	26:46	T/R 1:	One sixth.
7.0.160	26:46	David:	And then these, this would be
7.0.161	26:50	Meredith:	We're calling them each sixths,
7.0.162	26:51	T/R 1:	Ok
7.0.163	26:54	Meredith:	So there's six sixths
7.0.164	26:54	David:	This would be [red] one third, and this [light green] is one half of dark green, and then it would be bigger by one sixth, because
7.0.165	27:01	T/R 1:	Do you both agree with that?
7.0.166	27:02	Meredith:	Yeah, mm hmm.
7.0.167	27:03	T/R 1:	Ok, could you write that up? Uh, let me get you some paper, I want you to write that up. And see if you can make me - if it works for other models because some students don't believe that it works for other models and I think you two believe that it works for other models
7.0.168	27:15	Meredith:	Mmm hmm, yeah
7.0.169	27:16	T/R 1:	So can you try to find some so that you can try to convince them that it should work for other models, and come up with a way of explaining it to the class, ok? That if it works, if you really believe it, that if it works for one it works for others, and then write it up, let me get you some paper.
7.0.170			[David builds a model of an orange and two yellow rods again, then tries to line up purples, discards them, and tries green]
7.0.171	27:39	Meredith:	It's not good for brown [David and Meredith get up to get pens]
7.0.172	28:18	David:	I'll make another
7.0.173	28:35	Meredith:	Oh! Purple
7.0.174	28:36	David:	I already tried purple. I just tried it, it doesn't work.
7.0.175	28:41	Meredith:	Oh, dark - this green. Oh! We need halves! [pause] Let's try brown, oh no.
7.0.176	29:05	David:	[Holding the blue rod] It's an uneven number, it's an odd number

7.0.177	29:13	Meredith:	Oh! [She looks at a model from another table] Orange and red! Connected. Remember? Orange and red connected [Meredith builds a train of orange and red] And then you take, let's see, what's, the two dark greens and three purples
7.0.178	29:59	T/R 3:	[Talking to Gregory and Danielle] What was this?
7.0.179	30:00	Gregory:	One half [Gregory's model is a dark green rod, two light greens, three reds, and six white rods]
7.0.180		T/R 3:	This is one half. And so - [BREAK IN TAPE]
7.0.181	30:18	T/R 3:	Is there some other way you can show using some rods or anything? Is there another model you could build to show the same thing? Which is bigger, two thirds or one half?
7.0.182		David:	That's the one half
7.0.183		Meredith:	No. The question is, what's bigger, one half-
7.0.184		David:	I know, but then we're going to put this up and put the other red, I just don't have the one whole
7.0.185		Meredith:	You don't understand. [David's model- Figure S-31-29] Seven, ten, I need them. Go get another box. [Meredith finishes her second model]
7.0.186	32:36	T/R 2:	[approaches Meredith and David] Ok I see you had your hands up over here?
7.0.187	32:37	Meredith:	Yeah
7.0.188	32:41	T/R 2:	Let me come around and see what you're doing.
7.0.189		Meredith:	We found two answers
7.0.190		David:	Well, I have three.
7.0.191		T/R 2:	You have two solutions... three. Ok, let me hear about one of these models. Here, which one do you want to tell me about?
7.0.192	32:47	Meredith:	That one. [Meredith refers to her model consisting of a train with one orange rod and one red rod which represents her unit. Beneath the train she has 2 dark green rods; below, she has 3 purple rods; and again, below, she has 12 white rods. Figure S-32-55]
7.0.193	32:49	T/R 2:	Yeah, that's an interesting looking one, tell me about it.
7.0.194	32:53	Meredith:	Now if you call this rod one...
7.0.195	32:55	T/R 2:	The orange and red together?
7.0.196	32:56	Meredith:	Yeah, and you take the two dark green rods, those are the halves... [She takes a dark green rod and a purple rod from her model and places the purple rod beneath the dark green.] And you take two thirds, and put it up to it, and you take... two sixths, it's bigger than two sixths. [She puts two white rods next to the dark green rod. Figure S-33-25] And in this one, it you take this [Indicating a white rod]...
7.0.197	33:28	T/R 2:	Can we go back to that one again?
7.0.198	33:30	Meredith:	I mean it's bigger than one tenth, I mean twelfth, one twelfth, one twelfth. [She puts the white rods back to make a train of 12.]

7.0.199	33:34	T/R 2:	How does that work? I'm confused about that. I'm confused about the little white rods, I am following you right up to that point.
7.0.200	33:41	Meredith:	If you put the white rods up to here [She moves the orange and red train under the train of 12 white rods], there's twelve of them, and then you call them twelfths, because there are twelve of them.
7.0.201	33:52	T/R 2:	All right, okay.
7.0.202	33:55	Meredith:	And then you take the two thirds, and you take two twelfths, and then you put it up to the thirds [She moves two white rods over to her model representing two thirds and one half] and it is bigger by two tens... two twelfths.
7.0.203	34:12	T/R 2:	By two twelfths, okay.
7.0.204	34:13	Meredith:	If you use this model... [Meredith refers to the original model in which 1 dark green rod represent her unit. Beneath them, she placed 3 red rods, then 6 white rods, and then one dark green rod.]
7.0.205		T/R 2:	Uh hmm.
7.0.206		Meredith:	And if you use this model [referring to the 1 dark green rod as unit model]
7.0.207		T/R 2:	Uh hmm.
7.0.208		Meredith:	[Meredith then removes 1 light green rod and 2 red rods from her original model] And you call these [white rods] sixths, and you put this one [white rod] up to it [light green rod] and it is bigger than one sixth. [Meredith, with her original model, has indicated that the difference between one half - length of 1 light green rod- and two thirds -length of a train of two red rods- is one sixth -the length of a white rod. Figure S-34-41]
7.0.209	34:34	T/R 2:	Okay, so here it was bigger by two twelfths and here it was bigger by one sixth.
7.0.210	34:37	Meredith:	Yeah.
7.0.211	34:28	T/R 2:	That's interesting. Could we call the difference between the two thirds and the one half in this model [a train of orange and red as the unit] another number name besides two twelfths?
7.0.212	34:53	Meredith:	Um, yeah, well, maybe...
7.0.213	34:56	T/R 2:	You said two of those little white ones were two twelfths, right?
7.0.214	35:02	Meredith:	[Meredith places 6 red rods below her larger model as she speaks] Yeah, and maybe since two of these little white ones equals up to one of these [She puts 1 red rod on top of 2 white rods in the train, showing that a red rod is the same length as a train of 2 white rods.] or it's one fifth, [she starts lining up red rods against her original mode] oh, I mean one sixth, the reds [Figure S-35-22].

7.0.215	35:19	T/R 2:	Oh, that's interesting, that's kind of interesting, then, so if you then used the reds to describe the difference, you could call this one sixth, the difference.
7.0.216	35:	Meredith:	Uh hmm.
7.0.217		T/R 2:	And over here [Meredith's second model with dark green as the unit.] one of the whites you say is one sixth?
7.0.218		Meredith:	Yeah.
7.0.219	35:37	T/R 2:	Oh, that’s interesting, two different models. Okay. David, did you say you had another one? These are wonderful.
7.0.220	35:41	David:	What I did was the purple was one and then these [red] were the two thirds, no wait a minute, I think this one was [David originally has a model of one purple, two reds, and a train of light green and white, Figure S-35-46]
7.0.221		T/R 1:	[To class, as David is remodeling and speaking] Now if you’re writing about your solution, I’m hoping you’re writing about more than one model, if you found more than one, and I’d also like you to answer the question, does it hold up for different models, is that what you expected? Why or why not?
7.0.222			[David’s model is now similar to Meredith’s original model] Alright, the dark green is one, and then the red is two thirds, and then the light green is one half, and then the white to the green is one sixth, so two thirds is bigger by one sixth
7.0.223	36:35	T/R 2:	Ok, very nice
7.0.224	36:37	David:	And then I had the same thing down here [points to his model of one dark green, two light greens and a train of yellow and white], and that shows that shows that the light green is, I think I messed this up a little, ok, I think it’s like this [puts down and orange instead of the dark green], and then the light green was two thirds, um, two thirds - [Figure S-37-05]
7.0.225	37:05	T/R 2:	How would you prove that - that the light green is two thirds of the orange?
7.0.226	37:10	David:	Like this, and that [puts a third light green down]
7.0.227	37:17	T/R 2:	Is that the same length?
7.0.228	37:27	David:	Ok [shows that it isn’t and laughs]. Alright.
7.0.229	37:30	T/R 2:	Well, that model is all botched up. I wonder if you modify it to make a model that would work. Well, this one worked, right? [points to the model with the orange and red train as one] With an orange and a red.
7.0.230	37:38	David:	Alright, then on this one, with the orange and the red, and then this [purple] is two thirds and that’s [dark green] one half, and then this is bigger by one sixth
7.0.231	37:48	T/R 2:	Ok, do you think there are any more models? You’ve already come up with two that I’ve seen that work. I’ll let you think about this some more and if you come up with any more I want you to call me back. Um, while you’re thinking

			about it, you may want to take some time to report what you've done. Before you even begin you may want to put your models down. But I want you to continue to think about that question, both of you, ok, if there's any more models that will work.
7.0.232			[Meredith and David begin to draw their models, Meredith's model - Figure S-46-22]
7.0.233			*FRONT VIEW*
7.0.234	24:46 F	Erik:	One half, where's the dark green, one half or two thirds.
7.0.235		Alan:	This time you [inaudible]
7.0.236		Erik:	This time I what?
7.0.237		Alan:	Two thirds are bigger. Look
7.0.238		Erik:	Exactly
7.0.239		Alan:	Two thirds are bigger by one sixth. And one half is one bigger than one third by one sixth [Alan's first model - Figure F-25-15]. But also, making a train model,
7.0.240		Erik:	Oh no
7.0.241		Alan:	Create a chain reaction using the theory of relativities
7.0.242		Erik:	Ok, it's bigger by
7.0.243		Alan:	Who's using up all the twosies?
7.0.244		Erik:	It can't be done. Can't be done.
7.0.245		Alan:	A half is not bigger than two thirds.
7.0.246		Erik:	Oh this is the exact-
7.0.247		Alan:	This is one half
7.0.248		Erik:	This is the exact same problem we had before except it's one third, remember?
7.0.249		Alan:	It's only one sixth [Alan's second model - Figure F-26-10]
7.0.250		Erik:	This is easy. One half is larger than one third but smaller
7.0.251		Alan:	It's still one sixth
7.0.252		Erik:	Of course. It's larger by one little sixth. [looks for pencil] Ok. There! I did it.
7.0.253		Alan:	I did it. I know another way to figure it out. Create a balance.
7.0.254		Erik:	Make the balance like this.
7.0.255		Alan:	This would be a half, this would be two thirds. Determine which is bigger. Two thirds are bigger.
7.0.256		Erik:	But you have to do it like this. Ok, here we go. Ok, now, one half, uh, give me two more reds please. Two thirds. Let me support this. Perfect! It stays! It's equal. [Figure F-28-44]
7.0.257		Alan:	No, they're not equal. Look here. Those are halves
7.0.258		Erik:	It's equal
7.0.259		Alan:	These are thirds
7.0.260		Erik:	The balance is equal. But if I do it like this, with the orange, it's very, very different. Two thirds is bigger.
7.0.261		Alan:	Ok. Look. These are two thirds. Which is bigger? See? This is bigger [uses train model].
7.0.262		Erik:	Well, one half...

7.0.263	Alan:	Erik,
7.0.264	Erik:	Yeah?
7.0.265	Alan:	Look. This is two thirds.
7.0.266	Erik:	Yeah, I know.
7.0.267	Alan:	That is one half. Which is bigger, the two thirds or the half?
7.0.268	Erik:	Two thirds. Of course!
7.0.269	Alan:	You're right!
7.0.270	Erik:	Now I can easily make a train model.
7.0.271	Alan:	You can easily quarter it.
7.0.272	Erik:	Could I have the purples? Thank you, three purples, that's all I needed.
7.0.273	Alan:	We still haven't [inaudible]
7.0.274	Erik:	What? Dark green! Oh no, that's a black. Let's see, where's another dark green, where's another dark green, ah! There we go!
7.0.275	T/R 1:	Gentlemen? What do you think?
7.0.276	Alan:	He used up my example.
7.0.277	Erik:	I have it right here!
7.0.278	T/R 1:	Ok, is it possible to make another example, Alan?
7.0.279	Alan:	Yeah I guess.
7.0.280	T/R 1:	Would it still work?
7.0.281	Alan:	Yeah.
7.0.282	T/R 1:	You're sure it would work?
7.0.283	Erik:	Just like we did! Two after the other can be third-
7.0.284	T/R 1:	By the way, which is bigger?
7.0.285	Alan:	Ok. We figured out by taking out
7.0.286	Erik:	Because if you have, we figured that, well, let me just see, right here, both models we have the halves and the thirds. Like, it was like the other problem, it was one half and one third. And we explained it, we said that one half was bigger than one third but smaller than two thirds. Like up here, there's one half right there, and there's the thirds, there's the second third
7.0.287	T/R 1:	By how much?
7.0.288	Erik:	One sixth.
7.0.289	T/R 1:	But one half and two thirds.
7.0.290	Erik:	One- oh that's exactly, that's exactly what we meant. These are two thirds and that's one half
7.0.291	Alan:	With one of the thirds, it would be a sixth. But if you added one, it would still be one sixth.
7.0.292	T/R 1:	Ok, could you write it up and any others you can find, gentlemen? And be ready-
7.0.293	Erik:	What do we do, just diagram them?
7.0.294	Alan:	Yeah, just diagram them
7.0.295	Erik:	Just diagram which ones we did?
7.0.296	Alan:	Yeah we just

7.0.297		Erik:	And then write about them.
7.0.298		Alan:	Yeah, I'll just, we just have to diagram one of 'em
7.0.299		Erik:	No, I'll diagram both
7.0.300		Alan:	Yeah, same here. I diagrammed both on the one third bigger than one half by how much, you know? I did both on that one. I did those over the weekend.
7.0.301		Erik:	I'm going to trace them just to get the exact size. You're writing?
7.0.302	36:13 F	Danielle:	Well, we've got, we've got that whole, this is the whole, we have the three thirds, and we then the half
7.0.303		T/R 3:	And what we supposed to figure out after we did that?
7.0.304		Danielle:	Which is bigger a half or two thirds?
7.0.305		T/R 3:	Oh, I want to know. Is it still the same or does it change when your model changes?
7.0.306		Danielle:	Two thirds is still bigger.
7.0.307		T/R 3:	How much? [Danielle begins to line up white rods] Let's line 'em up. Two thirds is bigger, but now I want to know by how much. Can you figure that? [Gregory passes white rods to Danielle. Talk about getting white rods] You need some more whites. Uh, how many more do you think you need? A bunch? Takes a lot, doesn't it? How many do you think…
7.0.308		Danielle:	Eighteen [Figure F-38-23]
7.0.309		T/R 3:	Hmm. So how much larger?
7.0.310		Danielle:	It's bigger
7.0.311		Gregory:	One eighteenth
7.0.312		T/R 3:	[Danielle begins to dismantle her model to show the comparison] You can use some more of these if you want
7.0.313			[T/R 1 talks to class about writing about more than one solution]
7.0.314		Danielle	[Figure F-38-58] It's bigger by three eighteenths.
7.0.315		T/R 3:	My goodness, tell me, help me remember what it was over there.
7.0.316		Danielle:	It was bigger by one, one sixth.
7.0.317		T/R 3:	Ok, so does that mean we have a different answer? No? This is different from the other one or the same?
7.0.318		Danielle:	It's different in a way and it's the same in a way
7.0.319		T/R 3:	How's it different and how's it the same?
7.0.320		Danielle:	Well, it's the same because the half is smaller and it's different because, um, this one, it only ta- the little box are only um, two three four, there's only six of them and here's there's eighteen, and this, the thirds are bigger by three eighteenths
7.0.321		T/R 3:	You mean, yeah, the two thirds are bigger by three eighteenths
7.0.322		Danielle:	and the two thirds over here is bigger by one sixth

7.0.323 T/R 3: Mmm hmm. And so you think that you get a different answer if you have different models? As to how much bigger? I agree with you, you're saying that two thirds is still bigger, but it it bigger by a different thing?

7.0.324 Danielle: Well, [long pause]

7.0.325 T/R 3: [to Gregory] You're still looking for another way to do it? We found one way over here, we found this way, it seems to me there ought to be something in between, is that what you're thinking? Hmm, I wonder if there's another way. Hmm, she used the orange and the brown, is there something smaller than the brown that you could put together that would work, no add onto the orange? She added the brown to the end of the orange and that got hers to work. This, is there something smaller than this brown that would work attached to this? You tried that one, it didn't work. Let's try this one and see if it can work. Why don't you try the orange and the red. [to Danielle] I'm still concerned about, about whether the three eighteenths is a different answer from the one sixth. You said here that if you have two thirds and a half, oh, there, you said over here [to Gregory] now you have to see if you can do it with thirds, is that right? [to Danielle] Hmm. Look, we have a different model over here, even. So now we have three. I wonder if it's going to be the same as yours, or if it's going to be the same as this one. Is two thirds still bigger, Greg, is two thirds still bigger than a half, on this model too, or did it change? [they get another box of rods] Ok, Danielle, what do you think about this time? [Gregory's two models - Figure F-44-17]

7.0.326 Danielle: Well, um, two thirds

7.0.327 T/R 3: What is two thirds? Can you build a two thirds and a one half for him separate so we can then compare?

7.0.328 Danielle: Here's the two thirds, and here's the half

7.0.329 T/R 3: What's the difference?

7.0.330 Danielle: and it's bigger by two [counts Gregory's white rods] twelfths. It's, um, it's bigger by two twelfths

7.0.331 T/R 3: Oh, so is he getting a different answer from that, too, or are they the same? How are the answers, I don't understand, what do you think about this?

7.0.332 Danielle: One, two three

7.0.333 T/R 3: Over here it was how much?

7.0.334 Danielle: This one was bigger by three eighteenths

7.0.335 T/R 3: And this one?

7.0.336 Danielle: Was bigger by… how much?

7.0.337 Gregory: Two twelfths. One two three four five six seven eight nine ten eleven twelve.

7.0.338 T/R 3: And your original one was

7.0.339	Danielle:	It was bigger by one sixth.
7.0.340	T/R 3:	Oh, so what do you think?
7.0.341	Danielle:	I think they're all different, but then all the same. Cause they're the same because the thirds are always bigger than the half
7.0.342	T/R 3:	The two thirds are always bigger than the half?
7.0.343	Danielle:	And, um, and they're different because these are all, the whites
7.0.344	Gregory:	they're different sizes
7.0.345	Danielle:	They're all different, like one, two, uh three, they're different. So they're different like that.
7.0.346	T/R 3:	Mmm hmm. Is there any other way that you could show that difference here than with the whites? It's the only way you could show it there, isn't it? I don't mean for you to change your model, I mean, is there any other way that you could show me what that difference looks like without using the whites? Or this difference here?
7.0.347	Danielle:	You could use a light green
7.0.348	T/R 3:	What would that be?
7.0.349	Danielle:	That would be one [starts to line up light green rods - Figure F-47-56] That would be one sixth.
7.0.350	T/R 3:	Hmm. And what did you say it was over here, with the little one?
7.0.351	Danielle:	Um, that's one sixth
7.0.352	T/R 3:	Mmm hmm. So if you used the light green
7.0.353	Danielle:	It could be one sixth
7.0.354	T/R 3:	It could be one sixth. And if you used the whites
7.0.355	Danielle:	It would be three eighteenths
7.0.356	T/R 3:	Mmm hmm. What about for this one?
7.0.357	Danielle:	What problem-
7.0.358	T/R 3:	It was this one here [pointing to Gregory's model using the orange and red]. Uh, Gregory, I want you to watch and see if you agree with what Danielle is doing here. [Danielle lines up red rods on Gregory's model]
7.0.359	Danielle:	[After lining up and counting six red rods, Danielle shows that he two white rods that show the difference between one half and two thirds is equal in length to the one red rod- Figure F-49-02] And then that would be one sixth too.
7.0.360	T/R 3:	Mmm, over each of 'em?
7.0.361	Danielle:	That would be one sixth, that would be one sixth, and that one would be one sixth.
7.0.362	T/R 3:	But you have, had two, two different names for the answer if you did it this way it was
7.0.363	Danielle:	It was two twelfths
7.0.364	T/R 3:	And, and, uh, Gregory, for this one over here, where she had the three, what was the name for that one?

7.0.365 Gregory: Three eighteenths.
7.0.366 T/R 3: Yeah, it was.
7.0.367 *OHP VIEW*
7.0.368 Camera picks up some conversation/desk work from Audra, Michael and Brian, and Graham
7.0.369 T/R 2: [Michael and Brian build a model using the dark green rod as one. T/R 2 speaks to them about their model but most of conversation is inaudible] But while you're doing that, could you also be thinking about another model, at least, that could show this? Ok? [Brian and Michael begin to build another model]
7.0.370 T/R 1: [Graham has the model using the dark green rod on his desk. He builds the next largest model] What do you think? Which is bigger?
7.0.371 Graham: Um, two thirds.
7.0.372 T/R 1: You agree with that, Caitlyn? You agree with him? What did you build here? Can you show me what you built? Can you show me what one is? That's one, ok. [continues to speak to Caitlyn, but her side of the conversation is inaudible. Meanwhile, Graham builds a third model using the yellow and white as one]… Can you write that up for me? And I see Graham is building some more models and I know he's going to want to draw them for me. That's very lovely. See if you can find another. Write that one up first. [to Graham, pointing to his third model - Figure O-35-54] Ah! That looks like this one. Oh, no it's not, it looks a little different, but yet it looks the same. That's interesting. Why? [Graham shows that the dark green rod and the yellow and white train are equal in length, inaudible] That's another name for one. You're giving me two different names for one, right? Two different ways of building one?
7.0.373 T/R 1: [Michael's models: Figure O-35-30. T/R 1 talks to Audra and Kimberly about comparing the smaller rods to the train they called one] Gentlemen, gentlemen, wow, you are so busy working here, Brian and Michael. You know what I'm going to ask you. How many models did you make?
7.0.374 Brian and Michael: One, two three four five. Five [Figure O-37-56]
7.0.375 T/R 1: Did the relationship hold?
7.0.376 Michael: Yeah we got one sixth.
7.0.377 T/R 1: Did you expect that to happen?
7.0.378 Brian and Michael: Yes
7.0.379 T/R 1: Yes and you're absolutely convinced that will happen?
7.0.380 Brian and Michael: Yes. [laugh]
7.0.381 T/R 1: Ok. You can write about that?
7.0.382 Brian and Michael: Yes.
7.0.383 T/R 1: That's good. I can't wait to read it.

7.0.384		[T/R 1 tells class about writing more than one solution]
7.0.385	Brian:	Because it takes six sixths to equal one whole [holding dark green rod]. And there are two sixths [holds two white rods and puts on top of the red rod], there are two sixths, in each, in each, in each third
7.0.386	Michael:	Hey! That may be right! Because a third for this one, a sixth for this one is one, [starts placing white rods alongside the second model that is 6 cm in length]
7.0.387	Brian:	And it takes, and it takes three sixths to equal up to one half. [Figure O-40-12], but, but, um
7.0.388	Michael:	And this would be red, it takes two of them to equal that. [Figure O-40-29] Hey, that's neat!
7.0.389	Brian:	Sixths! That's what I did before
7.0.390	Michael:	It takes two sixths to equal a third! Wow! That's a neat thing to figure out fractions with.
7.0.391	Brian:	So what are we going to write?
7.0.392	Michael:	Well, first let's write our answer. And then we'll write why.
7.0.393	Brian:	I just did that. Um, what could we write? Because
7.0.394	Michael:	Because, well, just write what you just said to me.
7.0.395	Brian:	Does that make sense?
7.0.396	Michael:	Yeah. They'll know it makes sense. You think these people are teachers for nothing? [laugh]
7.0.397	Jackie:	Yeah, but it could also be, if you put whites up against it, it could be one twelfth.
7.0.398	T/R 1:	Ok, the white would have the number name one twelfth. But what would the red have a number name?
7.0.399	Jackie:	One sixth.
7.0.400	T/R 1:	One sixth. Now I'm going to ask you the question. What did you call one, what did you call one in this model?
7.0.401	Jackie:	Orange and red
7.0.402	T/R 1:	The orange and red. And what did you call one third? [Jackie points] What did you call two thirds [points again]? What did you call one half?
7.0.403	Jackie:	Dark green
7.0.404	T/R 1:	Ok, and which did you say is bigger?
7.0.405	Jackie:	[inaudible, points]
7.0.406	T/R 1:	Two thirds. By how much?
7.0.407	Jackie:	By one seventh.
7.0.408	T/R 1:	By one seventh? And why did you say one seventh?
7.0.409	Jackie:	[points to red rods, inaudible]
7.0.410	T/R 1:	Because the red has the number name one seventh?
7.0.411	Jackie:	[nods] Mmm hmm.
7.0.412	T/R 1:	One two three four five six.
7.0.413	Jackie:	I mean one sixth
7.0.414	T/R 1:	You mean one sixth. Do you agree with that, Erin?
7.0.415	Erin:	Yes.

7.0.416	T/R 1:	What number name does red have in this model?
7.0.417	Jackie:	One third
7.0.418	T/R 1:	One third. Why? Why is it one third here and why is it one sixth here?
7.0.419	Michael:	It takes, it takes two sixths to equal a third
7.0.420	Brian:	[inaudible] equal a third, no a half a half. That's what I wrote: 'Because it takes six sixths to equal one whole and, and
7.0.421	Michael:	And a sixth is always half of a third.
7.0.422	Brian:	Oh! [Michael laughs] And it takes, and there, and a sixth is a half of a third.
7.0.423	Michael:	Yeah!
7.0.424	Brian:	[after writing] Wait, what did I just say? Two sixths
7.0.425	Michael:	And two sixths, and it takes two sixths, two sixths, no, one sixth is half, is half of one third
7.0.426	Brian:	One sixth is … by one third, so it takes, so it takes three
7.0.427	Michael:	So it takes six, because there's two in every one so there's two four six. So it takes
7.0.428	Brian:	So it takes, should we write, so it takes three sixths to equal one half?
7.0.429	Michael:	No, it takes two, oh yeah, right, three equal one half.
7.0.430	Brian:	Two equal, two sixths equal one third, and three equal
7.0.431	Brian and Michael:	One half
7.0.432	Brian:	So it takes three, oh I got it! So it takes three sixths to equal one half, but, so it takes three sixths to equal one half, but two thirds equal four sixths [Michael nods]. So it [back to writing] Wait, so it takes
7.0.433	Michael:	Three sixths to equal a half, but it takes
7.0.434	Brian and Michael:	four sixths to equal two thirds. [laugh]
7.0.435	Brian:	And there's one extra. Yeah, and there's one sixth, look, look.
7.0.436	Michael:	Four, one two three four
7.0.437	Brian:	Look. See these, see these [Brian shows a light green rod and two red rods, and then places three white rods on the red rods. Figure O-47-33]? Ok, now, you see there are three of them that are equal up to it, but
7.0.438	Michael:	There's one more
7.0.439	Brian:	Yeah
7.0.440	Michael:	To make two thirds
7.0.441	Brian:	Yeah, so there's one extra and it makes it bigger! So it takes three sixths to equal… and it takes, and it takes
7.0.442	Michael:	Four
7.0.443	Brian:	Four sixths, and it takes four sixths to equal two thirds, two thirds. And there's one [continues talking as he writes] Ok. This is what I wrote. So it takes three sixths to equal one half,

		but it takes four sixths to equal two thirds. But it needs, but it needs
7.0.444	Michael:	But it needs?
7.0.445	Brian:	Yeah, but it needs, but it needs four sixths. But it needs… to equal
7.0.446		*WHOLE CLASS*
7.0.447	T/R 1:	[Jackie, Erin, and Jessica have built two models at the OHP - Figure O-49-55] I'm going to have to have to stop you for a minute, I know that, I hate to do this because I know you're working all so hard, but I would like to spend ten minutes, uh, just have us think about a few things, and you can finish this, is that ok, Mrs. Phillips if they can finish writing this up for us when we come back on Wed? Ok, so you really have today and maybe some time tomorrow to finish writing this up. I, I would like all of you though to sort of give me your attention for a minute, um, because I'm wondering about a few things. I need you to help me straighten out some things in my head, and if you can help me straighten them out in my head, you may be helping out other people straighten them out also. So I'm going to ask you some, some important questions, are you all listening to my questions? If you could stop what you're doing for a moment, I know it's hard, and listen to my questions. How many of you made one model and absolutely are convinced, that you know by your model, which is bigger one half or two thirds? How many of you did that with one model and you are absolutely convince with your model you know which is bigger, one half or two thirds. Would you please raise your hand if you made one model, you could have made more than one but you made at least one. If you made at least one model, girls, and you're absolutely convinced [All visible hands raised] No one could persuade you otherwise that you know which is bigger, one half or two thirds. Alright, so tell me, which is bigger?
7.0.448	Students:	Two thirds.
7.0.449 47:59	T/R 1:	Again?
7.0.450 48:36	Students:	Two thirds!!
7.0.451	T/R 1:	And you also know how much bigger. How many of you are convinced you know how much bigger and no one can persuade you otherwise that two thirds is not only bigger than a half but it is how much bigger?
7.0.452	Students:	One sixth.
7.0.453 48:40	T/R 1:	How many of you believe one sixth? [All visible hands raised] That's what I thought Walking around I thought that…that is what I believe that everyone has done. How many of you made a second model?
7.0.454	Meredith:	Oh, Oh! [eagerly]

7.0.455		T/R 1:	You could have made more than two, but you made at least two models. And in your second model you got a different answer. You got an entirely different answer, you no longer have two thirds bigger than one half, you showed something else. Are you watching? Some of you got a second model that showed something different. Meredith? Let's listen to what Meredith says. Girls [Jackie, Erin, and Jessica], why don't we stop that for just a moment and then we'll make some more. Listen, listen. You [Meredith] think you showed something else in your second model? [Meredith goes to the overhead projector and places twelve white rods beneath the two dark green rods in the girls' second model - two of the other girls help her find and position the white rods - Figure O-53-03] I am really confused. I have no ideas what Meredith is going to do. Because I thought I understood this and I thought she was going to tell me she got the same answer. Did you think that? And now she's telling me no. [Meredith is smiling.] I'm going to get so confused. You are all going to have to help me. [Pause.] How many of you built a second model that looks like that model up there that Meredith is fiddling with? How many of you have a model that looks like that? [Many hands are raised.] By the way, what was one in that model? What did you call one in that model? Amy?
7.0.456		Amy:	Ah, the orange and red.
7.0.457		T/R 1:	How many of you called 'one' orange and red in that model? Yeah, you did that model. Did you get to have two thirds bigger than a half?
7.0.458	50:53	Michael:	No. [Michael raises his hand, shaking his head from side to side, signifying dissent.]
7.0.459	50:56	T/R 1:	Amy got two thirds bigger than a half in that model, how many of you got two thirds bigger than a half in that model, where the orange and the red were one. Michael didn't, Meredith did. You didn't Michael?
7.0.460	51:04	Michael:	No, they can't do that. [He begins to stand.] Because um, the, the two thirds are bigger than the half by a red. So they can't use those whites to show it.
7.0.461	51:08	T/R 1:	Oh, but you're saying that, you're saying that two thirds, what's a third?
7.0.462	51:21	Michael:	A third is the purple [He begins to approach the overhead projector.]
7.0.463	51:31	T/R 1:	And what's two thirds? Just tell us.
7.0.464	51:37	Michael:	[He returns to his seat.] Um, two thirds is two purples.
7.0.465	51:39	T/R 1:	Did you all do that? Did you get two thirds to be two purples? [She addresses Michael] And what did you get to be one half?

7.0.466	51:42	Michael:	Uh, dark green.
7.0.467	51:43	T/R 1:	[She addresses the class.] Did you get dark green to be a half? [Mutterings of assent occur.] And you got two thirds to be bigger than one half?
7.0.468	51:49	Michael:	[Politely impatient] Yes.
7.0.469	51:51	T/R 1:	By how much?
7.0.470	51:56	Michael:	[Deliberately, again almost impatiently] By one sixth.
7.0.471	51:57	Meredith:	Or, or two twelfths.
7.0.472	51:58	Michael:	[Shaking his head sideways] No.
7.0.473	52:00	:	[Mutterings in the classroom of no.]
7.0.474	52:02	T/R 1:	Tell us Meredith. Aha! How many of you got one sixth? [Most hands are raised.] And what rod did you use to represent one sixth? What color rod?
7.0.475		Students:	Red.
7.0.476	52:04	T/R 1:	How many of you used a red rod to represent one sixth in that model and you showed it was bigger by one sixth? And Meredith says she did it a little differently and she didn't get one sixth. And what did you get Meredith?
7.0.477	52:15	Meredith:	Two twelfths.
7.0.478	52:15	T/R 1:	What do you think about that?
7.0.479	52:26	Students:	Well, in a way. No. Uh, uh [negatively].
7.0.480	52:29	T/R 1:	Well, she showed it's bigger by the two whites, she shows two whites bigger [Figure O-55-01].
7.0.481	52:32	Michael:	Yeah, but then she would have to call the two whites together one sixth.
7.0.482	52:35	Erik:	Yeah, exactly.
7.0.483	52:40	Michael:	She's calling the whites, one white one sixth.
7.0.484	52:44	Erik:	Yeah, she said
7.0.485	52:45	T/R 1:	She's calling one white one sixth?
7.0.486	52:46	Meredith:	No I'm not, I'm calling it one twelfth.
7.0.487	52:50	T/R 1:	She's calling one white one twelfth.
7.0.488	52:52	Erik:	Yeah, but see just the whites together. That'd be right, it would be two twelfths. But you have to combine them. You can't call them, you can call them separately, but you could also call them combined and if you combine them it would be uh, one sixth.
7.0.489	52:53	T/R 1:	Ok, but she didn't combine them and she's calling the two whites together, again, Meredith?
7.0.490	52:53	Meredith:	One twelfth, two twelfths.
7.0.491	53:11	T/R 1:	Two twelfths. She’s calling [Michael, still seated, shakes his head sideways in dissent and fingers some rods.] Do you all agree that one white has a, the number name for the white rod is one twelfth? Someone told me that when I was walking around, it might have been Audra. And some other people told me a white would be one twelfth? Is that true? And two white rods would be

7.0.492	53:15	Students:	Two twelfths.
7.0.493	53:17	T/R 1:	Two twelfths. And one red would be
7.0.494	53:31	Students:	One sixth.
7.0.495	53:32	T/R 1:	One sixth. So, so what is Meredith saying here?
7.0.496	53:35	Meredith:	There's two answers.
7.0.497	53:37	T/R 1:	Are there two answers?
7.0.498	53:41	Michael:	[simultaneously with Erik] No, they're the same answer.
7.0.499	53:43	Erik:	No, they're the exact same thing, except she, she took the red and divided it into half, she divided it into halves, into half and called, and called each half one twelfth. They're the exact same answer except they're just in two parts.
7.0.500	53:44		[Note all written notation will be enclosed in […] as it is recorded by the teacher. Figure O-58-56]
7.0.501	53:47	T/R 1:	[Joins the four girls at the overhead projector] Let me write this down. This, what you are saying here is so important, here. Let me see if I can write this down. You're saying that you're calling the red, you're giving red the number name, right? The length of the red, right? We'll give it the number name, what did you say?
7.0.502		Students:	One sixth.
7.0.503	54:02	T/R 1:	[R one sixth] One sixth. And two whites, can I write two ‘w ‘ for two whites?
7.0.504_54:23		Students:	Yeah.
7.0.505	54:25	T/R 1:	And you're calling two whites
7.0.506	54:29	Students:	Two twelfths.
7.0.507	54:31	T/R 1:	[2W two twelfths] Two twelfths. But what Erik just told me, right?, is something about red and white.
7.0.508	54:33	Erik:	Yeah. A red, one red equals, one red rod up here, one red equals two of the white ones.
7.0.509	54:36	T/R 1:	[1R = 2W] So we're talking about the length of the red rod, the length of the red rod is the same as the length of the two white rods? [On the overhead projector, Meredith builds a model with one red rod as the base and places two white rods directly above it.] Is that true? Do you all agree to that?
7.0.510	54:43	Students:	Yeah. Yes.
7.0.511	55:03	Erik:	And since she's calling a white rod one twelfth and the other white rod one twelfth and the red rod is really one sixth. But, when she calls them two twelfths, the two twelfths are actually just two white rods put together to equal a red, so it should be really, it's really one sixth. Because two whites, two whites
7.0.512	55:12	T/R 1:	She says one white is a twelfth [1/12] and then if you put it together with another one twelfth [+1/12], she's saying you get two twelfths [= two twelfths].
7.0.513	55:12	Erik:	And it's one sixth, it's one sixth.

7.0.514 55:39 T/R 1: And you're saying if you have, if you take one half that's all right? [1/2] If you're taking one half of one sixth [of one sixth], you're saying you get one twelfth [= 1/12]
You're saying that. That's the two things I'm hearing. Right? And you're saying that [one sixth], the length of one sixth is the same as the length of two twelfths. [= two twelfths]
Is that what you are saying?
7.0.515 55:47 Erik: Yeah.
7.0.516 55:48 T/R 1: All those things, are they true?
7.0.517 56:15 Erik: Yeah. But I don't really think you could call, call them two twelfths because two twelfths equal exactly to the same size as one sixth. Well, if you want to you could call them, I guess. But I think it would be easier just to call them one sixth, then wouldn't want to exactly call them one twelfth and another twelfth. I'd just call them one sixth. Therefore I think you just really call them one sixth.
7.0.518 56:16 Student: Well, maybe you can call them
7.0.519 56:18 Erik: Well you can call them, if you want to, but
7.0.520 56:46 T/R 1: Well, we have different number names for these rods
7.0.521 56:47 Student: they're not different
7.0.522 56:49 Brian: There's just half of one, there's just half of one.
7.0.523 56:53 T/R 1: So you're saying that one half of the one sixth is another way of saying one twelfth.
7.0.524 57:00 Brian: They're just two answers.
7.0.525 57:01 T/R 1: Well, you're saying if you took a twelfth, a rod that has length one twelfth, and another rod that has length one twelfth and put them together, right? That rod would have length two twelfths. Isn't that what you said?
7.0.526 57:09 Jessica: What Erik said is that two whites equal one red, so it would be the exact same thing.
7.0.527 57:09 T/R 1: Or a rod that has length one sixth, that would be the red one in this problem, would also have length two twelfths. Is that what you said when you talk about the lengths of the rods? So are all of these [pointing to the recorded notations on the overhead projector] true statements?
7.0.528 57:19 Students: Yeah.
7.0.529 57:26 T/R 1: That's amazing. Look at all the fancy mathematics you're doing, that's amazing. That's something for us to think about, ok? So Meredith is still saying that, "I don't disagree with you when I say that it's a red bigger in this model," right?
7.0.530 57:38 Meredith: Um, hmm.
7.0.531 57:39 T/R 1: I'm just going to give this red a different number name. I could give it the number name one sixth, if I think about it when I compare it to the rod I call one, the orange and red, I could give it the number name one sixth. Or, if I'm thinking about the white rods, right? I could give it the number name

			two twelfths. And that's very interesting. Does that contradict what you're doing? Or does it still work, what you're doing? It still works, Meredith thinks. That's something to think about, isn't it? That's very interesting, thank you for sharing that. Well, I think we've run out of time. Um, there's a lot of things to write about. We have a whole lot of new ideas, don't we? I really hope that you write to me about your different models and I hope when you write to me and show me as many models as you can. That you will also, you will also, think about, in your models. What is different about each of those models? Write a statement about each of those models that makes it a different model, okay? And then, what is alike about all of those models? Is that a good question, Meredith?
7.0.532	57:55	Meredith:	Mm, hmm.
7.0.533	57:56	T/R 1:	You can think about that question and write to me about it, I'd really like to know what you're thinking… What is different and what is alike. I can't wait to read what you write to me
7.0.534	59:19		End of class

Session 8, Oct. 6, 1993 (Front, Side, and OHP)

Session 8, October 6, 1993 Side View

Line	Time	Speaker	Transcript
8.1.1	1:09 S	T/R 2:	It's good to be back today. How have you all been? Good? You see Dr. Maher is not here today and, uh, our job is going to be to take what we're working on today and to be able to put it in some sort of a written form that Dr. Maher can read tonight so that when she comes back tomorrow to teach the lesson she understands what she did today so that's going to be part of your job today. I want to introduce a couple of new people who've came today and I've got to tell you the reason that they came today is because they saw some of the videotapes of this class from the past couple of weeks and they were so interested in what you were doing that they wanted to come see for themselves today and they'll be friends uh that you can talk to about what you're doing uh this is Parish in front of the room.
8.1.2	1:58	CT:	Hello Parish
8.1.3	2:00	T/R 2:	And this is Chris in the back who a lot of you introduced yourselves to. They'll be walking around and talking to you because they're curious about what you're doing as will Mrs. Phillips and myself today. Um I want to take us back to where we were on Monday. Does anybody remember what we were doing on Monday? That was the last time we were in. (pause) What have we been doing?
8.1.4	2:23	Students	Oh
8.1.5	2:25	T/R 2:	Yeah, come on, ok, it’s clicking. I can see it clicking out there. Um let's see um Andrew
8.1.6	2:33	Andrew:	We, um we divide, um we got, um we had a whole a half and a third and then we had fourths and then we, we took the half and the third and to see how, is two thirds bigger than a half by how much. And we figured it out that it would be by a fourth,
8.1.7	2:58	T/R 2:	Hmmm.
8.1.8	2:59	Andrew:	I mean by a sixth.
8.1.9	3:01	T/R 2:	That's interesting ok, did you all hear what Andrew said he said that you were comparing, um,
8.1.10	3:07	Students:	Fractions.
8.1.11	3:08	T/R 2:	Fractions, you were comparing a half and two-thirds, did you say? And trying to figure out which was bigger and by how much? Oh, that's interesting and, and you said you came to a decision that it was a difference of how much?
8.1.12	3:21	Andrew:	One-sixth

8.1.13	3:22	T/R 2:	One sixth what do the rest of you think of that? do you remember that?
8.1.14	3:24	Student:	Uh huh.
8.1.15	3:27	Erik:	Well, I don't know. I remember, I remember the same thing but then I remember that Meredith had another argument about calling 'em two twelfths other than one sixth.
8.1.16	3:38	T/R 2:	Mmm hmm, is that ok?
8.1.17	3:39	Erik:	Well I think so but it would be easier if you just called it one sixth because she called the twelfths the white ones I believe and well she said two white ones would equal up to a red one which is one sixth and she called them each one twelfth but I think it'd just be easier to call it one sixth.
8.1.18	4:02	T/R 2:	Ok that's interesting let me just put that up here for a minute. I remember that method. Ok now we said… [students start talking amongst themselves] ok uh ok, just to get you back up here for a minute now um some people proposed that one sixth was a possible difference between two thirds and a half but Erik says that Meredith said two twelfths? Is that what she said? Ok now if we're focusing on with the rods if we're focusing on the length that it makes in other words when we line them up either the two little white ones or the red one, right when we're lining those up, we're focusing on length would they be equal, the two twelfths and one sixth do you think?
8.1.19	5:01	Erik:	Length? Yes. Lengthwise yeah.
8.1.20	5:02	T/R 2:	If we're using length as our focus?
8.1.21	5:04	Students:	Yes
8.1.22	5:05	T/R 2:	Ok alright so we could make a statement like this, then, we could say that they were equal if we're looking at length. Alright? Ok that's interesting. Does anybody else have comments from the other day, about what we were working on? That was about the size of it and then you were asked to write about that and I just got those and I'm going to read through those. Well what I'd like to do is start off today with a little more challenging problem. You're gonna compare fractions again and you're gonna use your rods. Um, the only thing I would ask you to do today is that when you build your models and we'll be reminding you as we walk around the room ok please record them on paper ok so that we can keep track of your models and this way Dr. Maher can read them tonight and see what you've worked on. Ok, the first problem I want you to think about is the following and you can discuss with your partner and you have to come up with a justification you have to come up with an argument.
8.1.23		Erik:	Oh, that's easy.

8.1.24	6:07	T/R 2:	Can someone read the problem to me? Ok so we're all focusing on this problem, uh, lets see, David?
8.1.25	6:14	David:	Which is larger, three fourths or one half?
8.1.26	6:19	T/R 2:	Ok. Which is larger, right, three fourths or one half and by how much? Ok, so that's what we're looking at. Now, look, I've asked you to do something a little differently today, ok? You're gonna build models but I'd like you to build all of you with your partner build more than one model today for each problem that we give you. I'd like you to try to think of different models, different ways to show me and to justify your argument ok? So we're going to compare those and see what the difference is by how much. Did, do you have a question, Erik?
8.1.27	6:47	Erik:	No
8.1.28	6:49	T/R 2:	Do you want to make a statement or do you want to tell me the answer yet?
8.1.29	6:53	Erik:	No, I was just gonna answer, I was just gonna wait, I guess.
8.1.30	6:56	T/R 2:	Ok. Alright does anybody have any questions about what we're going to do? I'd like everybody working with somebody so I think I'm going to ask David and Erin if they wouldn't mind sitting together [talks to David and Erin about moving].
8.1.31	7:22	Erik:	Can we just do the singles?
8.1.32	7:23	Alan:	No.
8.1.33	7:24	Erik:	Why not
8.1.34	7:25	Alan:	We can [inaudible] but we can't [inaudible]
8.1.35			*Michael and Brian are working together*
8.1.36	7:43	Michael:	Fourths are the green, dark, light green
8.1.37	7:46	Brian:	They are? Um, oh yeah, yeah, yeah, they are. I was going to try that, but I didn't, I didn't.
8.1.38	7:55	Michael:	Now just take three of them. It's bigger than one half by one fourth. See this is one fourth and this is three of them, see? It's bigger by one fourth. No, wait, maybe its one…
8.1.39	8:28	Brian:	How 'bout… why don't we just use this, why don't we just use this one like we did last time
8.1.40	8:35	Michael:	See, this, see? Its fourths, its one fourth
8.1.41	8:39	Brian:	No we can't make them
8.1.42	8:41	Michael:	It's bigger by one fourth, but by, and so, by how m-
8.1.43			*Kimberly and Audra work together*
8.1.44			[Kimberly has built a model using the orange and red train as one, the dark green as one half, the purple as one third, and the light green as one quarter.]
8.1.45	8:48	Kimberly:	The two thirds is bigger, no, the three fourths is bigger by one fourth. Three fourths is bigger by a fourth now how about another model, so…
8.1.46			*T/R 2 works with Brian and Michael*

8.1.47			**missing**
8.1.48	9:48	T/R 2:	What do you think over here. Did you come up with one model yet?
8.1.49	9:51	Brian	Yeah we came up with this and um and last time what we did what we got it wasn't a fourth bigger and when-
8.1.50	10:00	T/R 2:	What were we comparing the last time
8.1.51	10:02	Michael:	We were comparing two thirds
8.1.52	10:04	T/R 2:	And what
8.1.53	10:05	Brian:	And a half, and we did, we did this. I did this last time. We made this model
8.1.54	10:12	Michael:	Yeah we found out that this is always going to half of a third, like one sixth, like no matter what size we had it
8.1.55	10:19	Brian:	Oh I maybe that these are even
8.1.56	10:24	Michael:	You're saying you can call three fourths two thirds?
8.1.57	10:29	Brian:	No, no I mean like the one whole maybe the one whole is an even number that's probably why cause it's an even number
8.1.58	10:40	T/R 2:	Can you tell me about this model that you built
8.1.59	10:43	Michael:	yeah it is because it's twelve, it's twelve-
8.1.60	10:45	Brian:	Yeah and this is four, and this is four and it's one fourth bigger so I guess when it's an even number it's one fourth bigger.
8.1.61	10:55	T/R 2:	Can you tell me about the model you've done here for three, for comparing three fourths and one half
8.1.62	11:02	Brian:	Yeah, well the model here
8.1.63	11:04	Michael:	Well this is half, the dark green, the fourths are the light green, and this is the one, this is the one and
8.1.64	11:11	T/R 2:	ok so the orange and red is the one
8.1.65	11:13	Michael:	Yeah so and then we took this away we took three of them and then we said ok it's bigger, it's bigger by two,
8.1.66	11:22	Brian:	One
8.1.67	11:23	Michael:	-three fourths is bigger than one half by one fourth cause, yeah right there
8.1.68	11:30	T/R 2:	That's the same length as one of your fourths then
8.1.69	11:33	Michael:	And to prove that it takes four of these to equal the- that
8.1.70	11:39	T/R 2:	You agree with that, Brian?
8.1.71	11:40	Brian:	Yeah
8.1.72	11:42	T/R 2:	You agree completely with that argument? Ok. Alright so you're telling me then that the difference between three fourths and one half is… how much?
8.1.73	11:49	Michael:	One fourth
8.1.74	11:50	T/R 2:	One fourth, ok. And which one is bigger?
8.1.75	11:4	Michael:	The dark, the light greens, the fourths.
8.1.76	11:58	T/R 2:	Which was the three fourths? Ok, alright, so that's a model you can build to show me that and that does justify it can you build me another model for that same problem?
8.1.77	12:07	Brian:	Ok let's try… **missing**

8.1.78	12:28	T/R 2:	Ok
8.1.79	12:28	Michael:	So I'm going to try to find a half of this, let's see.
8.1.80	12:30	T/R 2:	Alright, well, why don't you see if you can come up with another model now. That's, that's really wonderful. It's very good.
8.1.81	12:38	Brian:	ummm….
8.1.82	12:40	Michael:	I think I found one
8.1.83	12:43	Brian:	What about this one? Wait..
8.1.84	12:47	Michael:	Nope, that's not it, it needs to be one bigger than this.
8.1.85	12:51	Brian:	You're taking all my pieces! Oh, wait, this is the same as this too.
8.1.86	12:55	Michael:	I wonder if this is the same. Nope this one isn't.
8.1.87	13:00	Brian:	Let me try this, this is a nine and five
8.1.88	13:05	Michael:	That's not the same
8.1.89	13:06	Brian:	Fourteen, it's fourteen, it's still even. You want to try it?
8.1.90	13:10	Michael:	Sure, ok, now we just have to find, I found a half, that's the black, I just can't
8.1.91	13:16	Brian:	The half is a black?
8.1.92	13:17	Michael:	Yeah
8.1.93	13:18	Brian:	It is?
8.1.94	13:19	Michael:	mmm hmmm
8.1.95	13:20	Brian:	Oh. Dang, you took the blacks
8.1.96	13:24	Michael:	Um, you can get an extra bag up there from back of the class
8.1.97	13:29	Brian:	Ok [gets up and returns with more rods]
8.1.98	13:42	Kimberly:	…The red and the orange.
8.1.99	13:43	T/R 2:	[To Kimberly] The combination of the red and the orange, ok, alright, and, then these other pieces were what number names?
8.1.100	13:50	Kimberly:	[Figure S-14-16] This was, that was a half the dark green's the half,
8.1.101	13:53	T/R 2:	Mmm hmmm
8.1.102	13:54	Kimberly:	Those are the thirds,
8.1.103	13:55	T/R 2:	Mmm hmmm
8.1.104	13:56	Kimberly:	And th… the light green are fourths
8.1.105	13:58	T/R 2:	Ok, and then you were comparing… a half and three fourths.
8.1.106	14:03	Kimberly:	Right
8.1.107	14:04	T/R 2:	Ok, show me that again.
8.1.108	14:06	Kimberly:	Here's the half, one fourth, [mmm hmmm] two fourths, three fourths [mmm hmmm]. So if I take this one away, two fourths and a half are the same size, so it's bigger by a fourth.
8.1.109	14:26	T/R 2:	Very nice, ok, it looks like you cam up with another model over here. I noticed something [Kimberly tries to continue explaining], before you start, I noticed something interesting about your two models
8.1.110	14:38	Kimberly:	They're pretty much the same thing. The only thing I changed was that.

8.1.111	14:41	T/R 2:	Ok
8.1.112	14:41	Kimberly:	And they still
8.1.113		T/R 2:	You changed that color
8.1.114	14:42	Kimberly:	[at the same time] And they're still the same answer
8.1.115		T/R 2:	Ok
8.1.116	14:44	Kimberly:	And this one I'm working on another one
8.1.117	14:46	T/R 2:	You're working on a third one, yeah, you know what I would love for you to do, see if you can find me a model where one is a different length from this length that you're using
8.1.118	14:55	Kimberly:	Ok, that's what I'm trying to figure out.
8.1.119	14:57	T/R 2:	This is really nice, Audra, how are you doing? You're working on developing a third model here on the side?
8.1.120	15:04	Audra:	Ok, so I think I found another length, thirteen, fourteen, a half of fourteen is seven,
8.1.121	15:20	T/R 2:	These look very nice. Try to find a third one that's a different length and once you do find a third one you can call me back over here and then you can start to record these while you're waiting so that these ones that you have, very nice, ok
8.1.122	15:33	Michael	These are the halves and the whole
8.1.123	15:37	Brian:	And these are the wholes, this is the whole, the one
8.1.124	15:40	Michael:	No it's [inaudible]
8.1.125	15:41	Brian:	I know I know, I need some extras, look
8.1.126	15:45	Michael:	[laughs]
8.1.127	15:46	Brian:	One whole, two halves, and, look, it's bigger by one fourth
8.1.128	15:54	Michael:	Yay!
8.1.129	15:56	Brian:	So that's eighteen, though, that's eighteen, this is twenty!
8.1.130	16:00	Michael:	[laughs]
8.1.131	16:01	Brian:	Twenty, wow!
8.1.132	16:02	Michael:	[laughing] You can definitely get long. Let's see how long we can go
8.1.133	16:11	Brian:	um, uh, what about this one, I'm going to try this one
8.1.134	16:14	Michael:	I'm trying this one
8.1.135	16:15	Brian:	K, what's a half of the brown? What's a half of the- Oh, hey,
8.1.136	16:20	Michael:	Half the brown
8.1.137	16:23	Brian:	Think of a half… no
8.1.138	16:27	Michael:	It has to be one bigger than that - orange - nope
8.1.139	16:32	Brian:	No
8.1.140	16:34	Michael:	[laughs]
8.1.141	16:35	Brian:	phooey
8.1.142	16:36	Michael:	[laughs] - Too big
8.1.143	16:37	Brian:	Man, that was such a good model. Oh! Twelfths, is this.. are these twelfths? Does this equal twelve? Yeah, yeah it is. Uh, ok,
8.1.144	16:55	Michael:	Let's try blacks
8.1.145	16:57	Brian:	I need a uh

8.1.146		Michael:	[makes some noises]
8.1.147	17:06	Brian:	Ok.
8.1.148	17:10	David:	Can we borrow a red?
8.1.149	17:12	Brian	Sure, we have a million of them.
8.1.150	17:29	Audra:	Yeah we need
8.1.151	17:30	Kimberly:	That’s a five.
8.1.152	17:30	Audra:	We made another one.
8.1.153	17:32	Kimberly:	Yeah a purple right?
8.1.154	17:34	Audra:	A purple and then four reds
8.1.155	17:35	Kimberly:	Four reds, ok. So the purple are the thirds, right?
8.1.156		Audra:	Ok.
8.1.157	17:40	Kimberly:	What's the thirds? Ok, here is the halves
8.1.158	17:44	Audra:	I don't use the halves, I mean the thirds, because
8.1.159		Kimberly:	ok
8.1.160	17:47	Audra:	There’s no reason to use them because you would get mixed up.
8.1.161	17:51	Kimberly:	But what are the thirds, I found thirds, no I didn't
8.1.162	17:57	Audra:	That’s not thirds.
8.1.163	17:58	Kimberly:	This may not have thirds, so I'll just go to the fourths.
8.1.164	18:04	Audra:	Blue, I don’t think we can do any more with the blue.
8.1.165	18:07	Kimberly:	We can't, unless we do
8.1.166	18:10	Audra:	Wait, I think we can
8.1.167	18:12	Kimberly:	Did you try two
8.1.168	18:13	Audra:	You need dark green though. Let me show, I think I can do it, maybe
8.1.169	18:18	Kimberly:	Good.
8.1.170	18:20	Audra:	Dark green
8.1.171	18:32	Kimberly:	Odd number… [inaudible] We might be able to do it. You could, try the green. Let me try something. I found one, Audra, look, I found another one
8.1.172	18:53	Audra:	Nine
8.1.173	18:54	Kimberly:	Look
8.1.174	18:55	Audra:	I know. Nine, ten, eleven, twelve, thirteen, fourteen, fifteen. No.
8.1.175	19:04	Kimberly:	Audra, Audra, look, one two three four. The little ones. You know… And we've gotta write all this down. [missed]
8.1.176	19:19	Audra:	I know
8.1.177	19:21	Kimberly:	The more we have the better.
8.1.178	19:26	Audra:	I'm gonna try with the black
8.1.179	19:29	Kimberly:	No it's uneven, so
8.1.180	19:30	Audra:	What’s the next size? Black I think is uneven
8.1.181	19:35	Kimberly:	It is,
8.1.182	19:36	Audra:	Yeah because it’s next. Ok. Now, let me count, what's next? Black,
8.1.183	19:47	T/R 2:	Oh, you have a lot of models here. Ok, can you.. ok, this one looks interesting, why don't you tell me about this one?

8.1.184	19:58	Mark:	This one, this one is supposed to be, like I didn't have another one of these so that’s-
8.1.185	20:04	T/R 2:	I can imagine that that's a dark green. Ok, so tell me what you're calling one, and what you're calling a half and what you're calling three fourths
8.1.186	20:13	Mark:	[Figure S-20-26] This is one, the big green and blue
8.1.187	20:18	T/R 2:	k blue and green together
8.1.188	20:20	Mark:	And this is half, each dark green one, and these are fourths, yellow and light green.
8.1.189	20:30	T/R 2:	Mmm hmmm, ok, so where, show me, show me what you're comparing now
8.1.190	20:35	Mark:	So
8.1.191	20:36	T/R 2:	Can you pull out the pieces that you're comparing?
8.1.192	20:49	Mark:	Three fourths is bigger than one half by one fourth, if you take off that, this is even so
8.1.193	21:02	T/R 2:	Ok, so the difference is then, you're saying, one fourth. Ok that's interesting, how about for this model?
8.1.194	21:10	Mark:	Well, this model's like the same, this one whole, these are one half, and these are fourths. So it's bigger by one fourth.
8.1.195	21:28	T/R 2:	Ok so the difference here was a fourth and the difference here is a fourth. How about up here?
8.1.196	21:33	Mark:	This is, just like the same as this, but just, like, these equal up to that. I just changed the colors.
8.1.197	21:48	T/R 2:	Oh I see, yeah, so it’s the same length that model. These two models have the same length but this one has a different length. Ok, so you got one fourth as a difference? Do you think it makes a difference, do you think that um, when you build a model, do you think that um, what do you think that's gonna happen with the difference? Does it matter what type of model you build? Is that gonna change the difference, or...
8.1.198	22:09	Mark:	If you build like a wrong model, it might change the answer, so
8.1.199	22:13	T/R 2:	Mmm hmm, but if you build a model that um that where you can justify to me what you're calling one and what you're calling a half and what you're calling a fourth, do you think that um, the differences will be, will always be the same or do you think they'll be different?
8.1.200	22:31	Mark:	I think they would be the same
8.1.201	22:32	T/R 2:	You think so?
8.1.202	22:33	Mark:	Yeah.
8.1.203	22:35	T/R 2:	Laura, you have a model to share with me? You have one that looks a little different. Can you tell me about this model?
8.1.204	22:43	Laura:	[Figure S-22-51] These are the fourths,
8.1.205	22:45	T/R 2:	Mmm hmm
8.1.206	22:47	Laura:	And this is the half,

8.1.207 22:48	T/R 2:	Ok
8.1.208 22:49	Laura:	And I think that it's bigger by one fourth.
8.1.209 22:50	T/R 2:	Ok, so here's another model Mark that Laura did which is smaller than yours, but she still gets a difference of one fourth. Ok, that's very interesting. These are nice, now you've got three of them that work here, and I think Mark shared with me what how it works with these two. Can you record these now, on paper for me?
8.1.210 23:09	Laura:	Ok.
8.1.211 23:10	T/R 2:	Ok? [talks about getting markers] And uh, while you're recording, do me a favor. Besides tracing the rods and putting, you know, labeling the one half and the one, could you also put the color name in each rod as you're tracing 'em and write it in there?
8.1.212 23:38	Brian:	Should we color these? We could probably color these?
8.1.213 23:43	Michael:	I'm not sure.
8.1.214 23:46	Brian:	Yeah.
8.1.215 23:55	Michael:	This is P
8.1.216 23:56	Brian:	Red and blue
8.1.217 24:02	Michael:	P for purple, G for green, O, D, P, ok. I'm done.
8.1.218 24:35	Brian:	Blue, and this is blue and red, yeah, blue and red.
8.1.219 24:44	Michael:	Ok, And now, I'm going to go on to this one. Orange, ok. [makes some noise]
8.1.220 25:00	Brian:	Now what do I do?
8.1.221 25:01	Michael:	Draw your other model.
8.1.222 25:02	Brian:	I drew my two models.
8.1.223 25:05	Michael:	I'm not done with my two models.
8.1.224 25:07	Brian:	I did my long one, I already drew mine.
8.1.225 25:45	T/R 2:	I see a hand over there.
8.1.226 25:46	Brian:	I did my two models. Should I write about them?
8.1.227 25:49	T/R 2:	Um, if you want, actually, if you want to explain them, what would you write if you want to write about them?
8.1.228 25:55	Brian:	Uh, about this one, hmmm, I don't know
8.1.229 26:01	T/R 2:	You know what would help me, if you can write what the problem was up at the top here, what we're comparing, and then maybe write what the difference was between the three fourths and the one half. Those two pieces of information would be very helpful. Ok, then I want to give, when Michael's done recording, I want to give you two a second problem to think about, ok?
8.1.230 26:20	Brian:	Ok.
8.1.231 26:21	T/R 2:	Alright.
8.1.232 26:22	Michael:	I'm done.
8.1.233 26:25	Brian:	Write the question on the top.

8.1.234	26:26	T/R 2:	Yeah, put the question, and then, put what the difference was between the half and the three fourths, ok? That's so we remember.
8.1.235	26:34	Brian:	But, can I write that on the bottom here?
8.1.236	26:35	T/R 2:	Sure.
8.1.237	26:36	Brian:	Should I write, should I, I'll write the question here
8.1.238	26:38	T/R 2:	I'd put the question there and the …
8.1.239	26:42	Brian:	And write it's bigger down there, on the bottom?
8.1.240	26:45	T/R 2:	Sure. Looks very nice. Then I'll come back.
8.1.241	27:02	T/R 2:	Does anybody want to share one of your models?
8.1.242	27:04	Student	Ok, um…
8.1.243	27:14	T/R 2:	So tell me about one of these.
8.1.244	27:33	Meredith:	[Figure S-28-18] Ok, this is, I put the orange and the red together and called it a one
8.1.245	27:37	T/R 2:	Ok so that's one, and then
8.1.246	27:38	Meredith:	Mmm hmmm, these are the halves, these are the fourths, and these are the twelfths, I made the twelfths.
8.1.247	27:47	T/R 2:	So, what happens when we compare those two?
8.1.248	27:57	Meredith:	I take three of the twelfths, it's bigger by three twelfths or it could be bigger by one fourth.
8.1.249	28:09	T/R 2:	Neat! Ok, so if we're focusing on the length of the rods, you can either call the difference, you're saying, three twelfths, or one fourth
8.1.250	28:22	Meredith:	yep
8.1.251	28:23	T/R 2:	Interesting! Ok, that's very nice, Ok so I'm going to let you record that now.
8.1.252	28:27	Meredith:	Ok.
8.1.253	28:29	T/R 2:	I just have one other question for you. I see you have a second model here
8.1.254	28:32	Meredith:	Yeah.
8.1.255	28:33	T/R 2:	Ok, what was the difference between the three fourths and the one half in the second model?
8.1.256	28:39	Meredith:	Well, since the, um, as you see the, um, quarters are smaller than the quarter here, the, I mean the uh fourths here, it's gonna have a smaller value to a bigger value. You take three thirds, it's bigger by one fourth.
8.1.257	29:05	T/R 2:	Ok. So that one's a fourth, and you said this one, when it was up here, you could call it either three twelfths or one fourth?
8.1.258	29:13	Meredith:	Yeah.
8.1.259	29:14	T/R 2:	Interesting. Ok. Alright. Very nice. Ok, I'll let you record that, When you're done recording, let me know and what I'd like to do is uh, I'd like to give you a second problem to think about. Dave, you can help me out when you're recording, put the color names in here, so that Dr. Maher will know - see that's what - that's very nice Erin, yes. That will really

			help us. Now let me give Meredith a chance to trace. [Meredith laughs].
8.1.260	29:44	Michael:	By how much
8.1.261	29:47	Brian:	And one half, oops, again, I keep missing, by oh by…
8.1.262	30:05	Michael:	Two, oh yeah, how do you make a two again?
8.1.263	30:10	Brian:	Because
8.1.264	30:17	Michael:	Question, answer! Three fourths and one
8.1.265	30:27	T/R 2:	I'll let you finish this
8.1.266	30:29	Brian:	Um… uh, Mike I need help
8.1.267	30:43	Michael:	What?
8.1.268	30:44	Brian:	[giggles] I need help with this.
8.1.269	30:46	Michael:	Ok I'll be right there - just gotta finish, three fourths is bigger than one half…
8.1.270	30:57	Brian:	I can't think, well I know one I can think of now.
8.1.271	31:03	Michael:	Ok
8.1.272	31:03	Brian:	Ok three fourths is larger than one half by one fourth because, well, it takes two of em over here, look, here… well because it takes two of em to equal one half, but the question is, but there are three of em
8.1.273	31:23	Michael:	No, no, um say this is a half and this is three. So it would be bigger by one fourth because it takes how many fourths does it take, it takes three fourths to equal um, Oh jeez, this is confusing. It takes three fourths to equal
8.1.274	32:01	Brian:	[interjecting] Why don't we just do what I said? It takes two fourths to equal one half, but the but but there's but but but it needs, but but it takes, but the question is three fourths, and so there's one fourth bigger.
8.1.275	32:16	Michael:	One fourth, so this, I guess
8.1.276	32:20	Brian:	How about that?
8.1.277	32:22	Michael:	Ok I guess it makes sense
8.1.278	32:26	Brian:	[talking as he writes] is one half bigger, because it takes…
8.1.279	32:42	Audra:	Dark green is small. Ok, find a piece of paper, oh no.
8.1.280	33:02	Kimberly:	Audra is in trouble
8.1.281	33:04	Audra:	What
8.1.282	33:06	Kimberly:	Audra may run out of paper
8.1.283	33:08	Audra:	Yes. Making mistakes
8.1.284	33:11	Kimberly:	Ok. Try not to make any more, otherwise you'll be out of paper.
8.1.285	33:30	Brian:	[writing?] The question is, three fourths, and so
8.1.286	33:37	T/R 2:	Yeah, he's writing it, he's writing out his explanation in words. But just about what you were saying
8.1.287	33:42	Brian:	so
8.1.288	33:44	T/R 2:	About a half and two fourths in the second one
8.1.289	33:46	Michael:	It takes two fourths equal to one half, to equal a half, and - I got it!
8.1.290		Brian:	

8.1.291	34:13	Michael:	Q-u-e-s-t…What'd you put for the rest
8.1.292	34:26	Brian:	But the question is three fourths, so there is one fourth left - pretty confusing
8.1.293	34:32	Michael:	But the question is three fourths,
8.1.294	34:35	Brian:	Because it takes two fourths to equal one half, but the question is three fourths, and so there is one fourth left - very confusing!
8.1.295	34:44	Michael:	Alright, but the question is three fourths, and so
8.1.296	34:47	Brian:	[to T/R 2] When you say it, it's very very confusing. [T/R 2 laughs]
8.1.297	34:51	Michael:	So there
8.1.298	34:52	Brian:	Three fourths is larger than one half because one fourth, by one fourth because it takes two fourths to equal one half, but the question is three fourths, and so there is one fourth left
8.1.299	35:02	T/R 2:	I understand that.
8.1.300	35:05	Michael:	That's because you're a math, a doctor in math!
8.1.301	35:08	Brian:	What do you mean, if like, um, my mom, my mom would have read that, she wouldn't
8.1.302	35:15	T/R 2:	[laughs] Ok, you're ready to think about another question?
8.1.303	35:18	Michael:	Yeah.
8.1.304	35:22	T/R 2:	Ok [intercom interrupts]
8.1.305	36:01	Michael:	They picked up all that
8.1.306	36:03	T/R 2:	I want to ride on the fire truck! [students laugh]. And I'd go with the kindergarteners I guess. Um, ok I want you to think about, another problem.
8.1.307	36:14	Brian:	Do I have enough room to write?
8.1.308	36:16	T/R 2:	You could have another sheet of paper. Ok, this time I want you to compare. [talk about room on sheets] This time I want you to compare two thirds and three fourths.
8.1.309	36:34	Michael:	Two thirds and three fourths.
8.1.310	36:34	Brian:	Ok.
8.1.311	36:35	T/R 2:	Decide which one is bigger, and by how much, if in fact one is bigger.
8.1.312	36:39	Brian:	I'm going to use my big model that I made
8.1.313	36:40	Michael:	Ok, so we should put, I'm going to put my name
8.1.314	36:45	T/R 2:	In fact you will want to put those two fractions down so that you remember what they are.
8.1.315	36:48	Brian:	I'm going to use my big model that I made
8.1.316	36:49	T/R 2:	Ok.
8.1.317	36:50	Michael:	I know I made, we, we, me and him made this huge model. I made another one. I made one of thirty. This one's..
8.1.318	36:59	Brian:	We made thirty - three of those, but we couldn't make fourths.
8.1.319	37:07	T/R 2:	Ok, so the problem is two thirds, compare two thirds and three fourths, which is bigger and by how much
8.1.320	37:12	Brian:	Two thirds

8.1.321	37:16	Michael:	Wait a minute, we have to change our -
8.1.322	37:17	Brian:	Three fourths
8.1.323	37:18	Michael:	We have to change this
8.1.324	37:20	Brian:	Oh, why don't we just make this one, the old one?
8.1.325	37:26	Michael:	Two thirds [makes noise]
8.1.326	37:34	Brian:	But we can't, we can't make fourths with this.
8.1.327	37:36	Michael:	Yes we can.
8.1.328	37:37	Brian:	Can we?
8.1.329	37:38	Michael:	Yeah
8.1.330	37:39	Brian:	Oh yeah, yeah
8.1.331	37:39	Michael:	We can use the light greens
8.1.332	37:45	Brian:	[Figure O-39-59] Yeah, Hang on, ok, k, what was it, three fourths compared to… wait, what was it?
8.1.333	38:01	Michael:	It was, which, um, which is bigger, two thirds or three fourths, by how much? Two thirds is bigger
8.1.334	38:08	Brian:	By two thirds,
8.1.335	38:10	Michael:	No, not by two thirds
8.1.336	38:13	Brian:	No, no, wait, wait
8.1.337	38:14	Michael:	No! Wait! Three fourths is bigger than two thirds, see?
8.1.338	38:19	Brian:	I know, I know
8.1.339	38:20	Michael:	By one sixth!
8.1.340	38:22	Brian:	Two thirds-
8.1.341	38:23	Michael:	By one sixth, see?
8.1.342	38:25	Brian:	Wait, wait, wait, what was the question? Two thirds and three fourths?
8.1.343	38:31	Michael:	No, which is bigger, two thirds or three fourths?
8.1.344	38:33	Brian:	Let me write it down, let me just write it down.
8.1.345	38:38	Michael:	Which is bigger, two thirds or three fourths
8.1.346	38:59	Brian:	Ok so it's two thirds
8.1.347	39:02	Michael:	or three fourths
8.1.348	39:03	Brian:	Two thirds
8.1.349	39:04	Michael:	by how much
8.1.350	39:06	Brian:	or three fourths
8.1.351	39:08	Michael:	Yeah, [writing] by how much? Ok, I'm done. Look at this.
8.1.352	39:27	Brian:	Question mark
8.1.353	39:36	Michael:	Oh! Ok, so it's bigger by
8.1.354	39:38	Brian:	Wait a minute let me make two thirds, let me make two thirds
8.1.355	39:41	Michael:	What the… It's bigger by one twelfth
8.1.356	39:44	Brian:	Why did you make that model? Ok, now it's three fourths, let me just copy this down.
8.1.357	40:01	Michael:	Don't copy it down yet. We may be wrong
8.1.358	40:05	Brian:	No, no no, I'm copying down two thirds and three fourths
8.1.359	40:10	Michael:	Ok, ok, so will I.
8.1.360	40:17	Brian:	Good we have… Ok [pause] Ok, now three fourths.
8.1.361	41:15	Michael:	Three fourths. [other students talking, Jessica's model]

8.1.362	42:14	Brian:	[Figure O-39-59] How is it big.. how much is it bigger by?
8.1.363	42:16	Michael:	It's bigger by a little white thing. But what do we call the white thing?
8.1.364	42:21	Brian:	A twelfth
8.1.365	42:23	Michael:	A twelfth?
8.1.366	42:24	Brian:	Yeah.
8.1.367	42:26	Michael:	A twelfth
8.1.368	42:27	Brian:	Yeah, yeah, wait, yeah, that is twelve
8.1.369	42:29	Michael:	Yeah, it's a twelfth
8.1.370	42:31	Brian:	And those are the thirds, and these are the fourths.
8.1.371	42:38	Michael:	Jeez. We're getting all these different answers - I thought they'd be, I thought we'd get the same answer
8.1.372	42:44	Brian:	What about yesterday, did you write "yes I think it's possible to get different answers for different models" so did I but I didn't write down, but I didn't write down what we did here. I wrote yes and I did a different one. And then, I was just about to say no. I was just about to say that. Um, by one twelfth.
8.1.373	43:09	Michael:	See? One, two, three, four, five, six, seven, eight, nine, ten eleven, twelve.
8.1.374	43:20	Brian:	I'm putting that right there.
8.1.375	43:30	Michael:	By one twelfth, right?
8.1.376	43:32	Brian:	Good.
8.1.377	43:46	Michael:	Ok, now the white dude
8.1.378	43:50	Brian:	Look, I just, like, put my model right on top of what I draw.
8.1.379	43:57	Michael:	There, there's my model
8.1.380	44:00	Brian:	Oh wait wait wait.
8.1.381	44:06	Michael:	What are doing the pointer for? [intercom interrupts]
8.1.382	44:18	Brian:	Which step. A twelfth extra.
8.1.383	44:28	Michael:	Oh, One twelfth extra?
8.1.384	44:30	Brian:	Yeah, so look, I just put my um, I just put the Cuisenaire rods right on top of what I just did.
8.1.385	44:38	Michael:	Oh, I made it too small.
8.1.386	44:41	Brian:	Look, I just put my Cuisenaire rods right on top. Look, see?
8.1.387	45:00	T/R 2:	How are we doing?
8.1.388	45:02	Brian:	I just did um, I just figured out that three-fourths is bigger than two-thirds
8.1.389	45:09	Both	By one twelfth
8.1.390	45:10	Brian:	Cuz one twelfth is like extra, it's like right there, see
8.1.391	45:14	T/R 2:	Oh
8.1.392	45:15	Brian:	And I put it right there and I pointed to it, and I wrote one twelfth extra.
8.1.393	45:20	T/R 2:	Ok, so I can compare the two fractions here. What was the whole here, what were we calling the one here, the whole train?

8.1.394	45:25	Brian:	[interjecting] The whole is, this is the whole, well this was the whole and there was one fourth, it used to have been a fourth right here but I guess we could change the
8.1.395	45:41	T/R 2:	So the train
8.1.396	45:43	Brian:	It'd be nine, right there.
8.1.397	45:49	T/R 2:	That was one
8.1.398	45:50	Brian:	Yeah
8.1.399	45:51	T/R 2:	That's what you're calling one
8.1.400	45:52	Brian:	Well, I just did that right now. Cuz the whole was really supposed to be this.
8.1.401	45:56	T/R 2:	Ok, so this was one
8.1.402	45:58	Brian:	Yeah, this was supposed to be one, but it said two thirds, so we took this one out and we put that in there to make it equal up to the three fourths
8.1.403	46:12	T/R 2:	Ok so then this was one.
8.1.404	46:13	Brian:	Yeah
8.1.405	46:14	T/R 2:	Ok, so you want to add that as well, maybe you can even trace it in on the top here, or…
8.1.406	46:20	Brian:	I guess I could put it on the bottom.
8.1.407	46:21	T/R 2:	Or the bottom, and label it one. K, so this was one, purples turned out to be thirds, and greens turned out to be fourths.
8.1.408	46:30	Brian:	Well but what should I do, should I just put another one here, like that? Because, cuz I drew something under it
8.1.409	46:41	T/R 2:	No that's ok, I understand what you did here. All I need to see now is what one was. What you called one
8.1.410	46:47	Brian:	Oh, oh.
8.1.411	46:48	T/R 2:	That's all I need to understand about your problem
8.1.412	46:50	Brian:	Should I use this? Should I use this? Even that is one whole to these, or should I use the one right here?
8.1.413	46:57	T/R 2:	I don't know, it's a good question, What do you think?
8.1.414	47:00	Brian:	How it was originally the one whole only I had to take this one out to make it two thirds, that's what I was thinking
8.1.415	47:07	T/R 2:	What do you think? What's your instinct what we should be using?
8.1.416	47:10	Brian:	Well, I think I probably should use this cuz this is changing the one whole, because, because that, we just took, we just took out um one third to make the, to make this problem, and this wasn't the real third, this wasn't, I mean the real whole anyway, so I guess I should just use this.
8.1.417	47:30	T/R 2:	Yeah you'd be changing the problem wouldn't you?
8.1.418	47:32	Brian:	Yeah
8.1.419	47:33	T/R 2:	Ok, since all of your fraction names came from what your number name one was, you want to go back to that.
8.1.420	47:38	Brian:	So should I just copy this down?
8.1.421	47:40	T/R 2:	That would help me, yeah, that would help me to remember, and remember, put the colors inside, too, so I can remember

8.1.422	47:45	Brian:	Oh, should I put it just on the side?
8.1.423	47:48	T/R 2:	You could just put it on the side. Yeah. I know they don't always fit. Ok. And you might want to also explain to Michael um why you're adding that.
8.1.424	48:03	Brian:	Ok, um, um, I'm adding, this was originally the one, the one, this was originally the one whole, you just trace that on the bottom.
8.1.425	48:32	T/R 2:	Have you come up with a model for me yet? Ok. Can you tell me about it, David?
8.1.426	48:51	David:	[Figure S-49-26] Ok. This is three fourths the light green, and the purples are two thirds, and the, then it's bigger by, so then it's bigger by, um, one twelfth.
8.1.427	49:19	T/R 2:	Ok, why one twelfth?
8.1.428	49:23	David:	Because, um, I put all these up here
8.1.429	49:25	T/R 2:	mmmm hmmm
8.1.430	49:27	David:	And there's twelve in all.
8.1.431	49:29	T/R 2:	Mmmm hmmm
8.1.432	49:31	David:	So then, one of these, it'd take twelve of these to make this whole thing, so this would be one part of the one twelfth.
8.1.433	49:42	T/R 2:	Ok. Alright. I will agree with that. You think you have a chance to record that before we uh, go? Ok.
8.1.434		put in erik and tr2 from other camera	Also classroom teacher conversation
8.1.435	50:12	Jessica:	Well there's twelve that go up to this one and … two thirds, three fourths is bigger than [inaudible]
8.1.436	50:36	Kimberly:	Not on one size and now the other side, and I don't go for the real size, I just draw it. I just draw it.
8.1.437	50:56	T/R 2:	So which one is bigger and by how much?
8.1.438	51:27	V1:	You've done the three quarters and two thirds one
8.1.439	51:30	Michael:	We're doing it
8.1.440	51:33	Brian:	We did it. Yeah, I just finished mine, I think.
8.1.441	51:35	V1:	You, did? Oh, you're very neat.
8.1.442	51:39	Brian:	Ok, um, it said two thirds and three fourths.
8.1.443	51:44	V1:	mmm hmmm
8.1.444	51:45	Brian:	and um and three fourths was bigger by one twelfth and um
8.1.445	51:50	V1:	and how did you know that was a twelfth
8.1.446	51:53	Brian:	well because I put mine [side comment]
8.1.447	51:57	Brian:	We didn't have one
8.1.448	51:59	Michael:	Yeah we have a couple
8.1.449	52:00	V1:	Oh, I didn't take all of them, oh you even have…. Here's a whole bunch of them hiding under there
8.1.450	52:06	Brian:	So you put this right there
8.1.451	52:08	V1:	right
8.1.452	52:09	Brian:	and
8.1.453	52:10	V1:	and you said it was one white square bigger
8.1.454	52:11	Brian:	Yeah

8.1.455	52:12	V1:	And then how‘d you know how much a white square was?
8.1.456	52:14	Michael:	Because we put it up to the one whole which was
8.1.457	52:18	Brian:	This was the one whole, this was the one whole
8.1.458	52:20	Michael:	And we lined twelve of them up
8.1.459	52:21	Brian:	These were the thirds
8.1.460	52:23	Michael:	Well, Here’s another one.
8.1.461	52:25	Brian:	These were the thirds, I mean, um, yeah, the thirds,
8.1.462	52:27	V1:	Right
8.1.463	52:28	Brian:	And these were the fourths
8.1.464	52:30	V1:	The light green ones were the fourths
8.1.465	52:33	Brian:	Yeah and so and this is and all this and this is and the one whole and the um like when we line them up down into steps the orange was a ten and we added and the red was a two and we add that together and that was a twelve
8.1.466	52:50	V1:	Ok.
8.1.467	52:51	Brian:	That was twelve,
8.1.468	52:52	V1:	Makes sense.
8.1.469	52:53	Brian:	So this is, if you take twelve of these, all the way in here, put them against here, twelve of them, you could see that there are twelve of them there and it equals up to the one whole
8.1.470	53:12	V1:	So one of them
8.1.471	53:14	Brian:	Yeah, and they're all twelfths
8.1.472	53:16	V1:	I see, because twelve of them equal the whole one
8.1.473	53:18	Brian:	Yeah.
8.1.474	53:19	V1:	I see, ok, and then since it's only one little triangle bigger,
8.1.475	53:23	Brian:	Yeah
8.1.476	53:24	V1:	One little square, uh,
8.1.477	53:26	Brian:	Cube
8.1.478	53:27	V1:	Thank you, one cube bigger, then that's one twelfth
8.1.479	53:29	Brian:	Yeah
8.1.480	53:30	V1:	Ok, that makes sense to me.
8.1.481	53:31	Brian:	And then I made the whole down here.
8.1.482	53:34	V1:	Now, can you make another model for this?
8.1.483	53:36	Brian:	Uh, yeah, I think so.
8.1.484	53:38	V1:	Ok, you're just going with the flow, huh, you're like yeah, sure why not, I can make another model, sure I can do anything. Oh, ok, perfect. I’m very impressed. [To Michael] Have you drawn that?
8.1.485	54:00	Michael:	Yeah
8.1.486	54:01	V1:	And you agree with him, right?
8.1.487	54:02	Michael:	Yeah.
8.1.488	54:03	V1:	You're in total agreement with him
8.1.489	54:04	Michael:	yes
8.1.490	54:05	V1:	You'll go wherever he goes. [Michael says yes again and laughs]. Ok. Now try and get another model

8.1.491	54:09	Brian:	Ok, um, think of another model. Would this be the same? Would this be the same length? Could we do this, even though it's the same?
8.1.492	54;23	Michael:	Mmm, I doubt it. Why don't we try…
8.1.493	54:32	Brian:	How about the black, try the black
8.1.494	54:33	Michael:	The dark green is gonna be the thirds
8.1.495	54:38	Brian:	Wait, wait, how about this?
8.1.496	54:46	Michael:	We need some fourths. How about the browns?
8.1.497	54:54	Brian:	How 'bout…how 'bout the browns? How about this one? Um… Here,
8.1.498	55:10	Michael:	I'm trying to figure something. Nope, that won't work either
8.1.499	55:16	Brian:	Look - it works! This works
8.1.500	55:22	Michael:	So what's gonna be the, those? Those can't be the fourths.
8.1.501	55:27	Brian:	I know
8.1.502	55:28	T/R 2:	Those of you who are finishing up recording something for me so that I can share these with Dr. Maher, please make sure that your name is on each page that you've done and make sure you've written what the problem or the question was at the top of the page, these look wonderful I'm going to share these with her this afternoon when I see her. [CT says great] So just finish up what you're working on now, because I think we do probably have to- we do have to clean up at this point. We can talk about these tomorrow.
8.1.503	56:01	CT:	If you have turned in your work, I'll take your markers.
8.1.504	56:20	Michael:	I need black.
8.1.505	56:23	Brian:	We're done.
8.1.506	56:24	Michael:	No we're not.
8.1.507	56:25	Brian:	Yes we are.
8.1.508	56:27	Michael:	But we need two models.
8.1.509	56:29	Brian:	What? We do?
8.1.510	56:30	Michael:	Yeah - [reading from board] please make more than one model to justify your work
8.1.511	56:33	Brian:	No, no, that's talking about this
8.1.512	56:39	Michael:	Oh, ok.
8.1.513	56:40	Brian:	See look, cuz um she just said, she said would you guys, would you guys like to do another problem, it wasn't up there so she didn't [inaudible]
8.1.514	56:58		clean up

Session 8, October 6, 1993, Front View

Line	Time	Speaker	Transcript
8.2.1	5:05	T/R 2:	It's good to be back today. How have you all been? Good? (Jackie- Good)You see Dr. Maher is not here today and, uh, our job is going to be to take what we're working on today and to be able to put it in some sort of a written form that Dr. Maher can read tonight so that when she comes back tomorrow to teach the lesson she understands what she did today so that's going to be part of your job today. I'd want to introduce a couple of new people who've came today and I've got to tell you the reason that they came today is because they saw some of the videotapes of this class from the past couple of weeks and they were so interested in what you were doing that they wanted to come see for themselves today and they'll be friends uh that you can talk to about what you're doing uh this is Parish in front of the room.
8.2.2		CT:	Hello Parish
8.2.3		T/R 2:	And this is Chris in the back a lot of you introduced yourselves to they'll be walking around and talking to you because they're curious about what you're doing as will Mrs. Phillips and myself today. Um I want to take us back to where we were on Monday does anybody remember what we were doing on Monday? That was the last time we were in. What have we been doing?
8.2.4		Students	Oh
8.2.5		T/R 2:	Yeah, come on, ok, it’s clicking. I can see it clicking out there. Um let's see um Andrew
8.2.6		Andrew:	We, um we divide, um we got, um we had a whole a half and a third and then we had fourths and then we, we took the half and the third and to see how, is two thirds bigger than a half by how much. And we figured it out that it would be by a fourth,
8.2.7		T/R 2:	Hmmm.
8.2.8	6:57	Andrew:	I mean by a sixth.
8.2.9		T/R 2:	That's interesting ok, did you all hear what Andrew said he said that you were comparing, um,
8.2.10		Students:	Fractions.
8.2.11	7:05	T/R 2:	Fractions, you were comparing a half and two thirds did you say? And trying to figure out which was bigger and by how much? Oh! that's interesting and, and you said you came to a decision that it was a difference of how much?
8.2.12		Andrew:	One sixth
8.2.13	7:17	T/R 2:	One sixth what do the rest of you think of that? Do you remember that?
8.2.14		Student:	Uh huh.

8.2.15	7:20	Erik:	No, I have another one. I remember, I remember the same thing but then I remember that Meredith had another argument of her calling 'em two twelfths other than one sixth.
8.2.16		T/R 2:	Mmm hmm, is that ok?
8.2.17		Erik:	Well I think so but it would be easier if you just called it one sixth because she called the twelfths the white ones I believe and well she said two white ones would equal up to a red one which is one sixth and she called them each one twelfth but I think it'd just be easier to call it one sixth.
8.2.18	9:40	T/R 2:	Ok that's interesting let me just put that up here for a minute. I remember that method. Ok now we said… [students start talking amongst themselves] ok uh ok, just to get you back up here for a minute now um some people proposed that one sixth was a possible difference between two thirds and a half but Erik says that Meredith said two twelfths? Is that what she said? Ok now if we're focusing on with the rods if we're focusing on the length that it makes in other words when we line them up either the two little white ones or the red one, right when we're lining those up, we're focusing on length would they be equal the two twelfths and one sixth do you think?
8.2.19		Erik:	Length? Yes. Lengthwise yeah.
8.2.20		T/R 2:	When we're using length as our focus?
8.2.21		Students:	yes
8.2.22		T/R 2:	Ok alright so we could make a statement like this, then, we could say that they were equal if we're looking at length. Alright? (Eric- but if we were looking at…) Ok that's interesting does anybody else have comments from the other day, about what we were working on? That was about the size of it and then you were asked to write about that and I just got those and I'm going to read through those. Well what I'd like to do is start off today with a little more challenging problem. You're gonna compare fractions again and you're gonna use the rods, um, the only thing I would ask you to do today is that when you build your models and we'll be reminding you as we walk around the room ok please record them on paper ok so that we can keep track of your models and this way Dr. Maher can read them tonight and see what you've worked on. Ok, the first problem I want you to think about is the following and you can discuss with your partner and you have to come up with a justification you have to come up with an argument.
8.2.23		Erik:	Oh, that's easy.
8.2.24		T/R 2:	Can someone read the problem to me? Ok so we're all focusing on this problem, uh, lets see, David?
8.2.25		David:	Which is larger three fourths or one half?

8.2.26		T/R 2:	Ok. Which is larger, right, three fourths or one half and by how much? Ok, so that's what we're looking at. Now, look, I've asked you to do something a little differently today, ok? You're gonna build models but I've like you to build all of you with your partner build more than one model today for each problem that we give you I'd like you to try to think of different models, different ways to show me and to justify your argument ok? So we're going to compare those and see what the difference is by how much. Did, do you have a question, Erik?
8.2.27		Erik:	No, I was just
8.2.28		T/R 2:	Do you want to make a statement or do you want to tell me the answer yet?
8.2.29		Erik:	No, I was just gonna answer, I was just gonna wait, I guess.
8.2.30	10:52	T/R 2:	Ok. Alright does anybody have any questions about what we're going to do? I'd like everybody working with somebody so I think I'm going to ask David and Erin if they wouldn't mind sitting together [talks to David and Erin about moving].
8.2.31	11:00	Alan:	Ok.
8.2.32		Erik:	Ok. Let's see.
8.2.33		Alan:	The other day we were doing the um, brown, and we were making
8.2.34		Erik:	Yeah, but can't we just do singles?
8.2.35		Alan:	No.
8.2.36		Erik:	Why not?
8.2.37		Alan:	You can't quarter the browns.
8.2.38		Erik:	But you can quarter the dark greens
8.2.39		Alan:	But you can't fourth the dark greens
8.2.40		Erik:	How do you know?
8.2.41		Alan:	Because you can only halve and third the dark greens.
8.2.42		Erik:	One, two, three
8.2.43		CT:	Before you start, put your names on your paper, we had an anonymous and we don't know who it was. Alan, your name on your paper
8.2.44		Alan:	Both of em?
8.2.45		Erik:	You only got two?
8.2.46		Alan:	I got [inaudible]
8.2.47		Erik:	Halves
8.2.48		Alan:	Ok, um, the dark greens, the dark greens could third this.
8.2.49		Erik:	No it can't
8.2.50		Alan:	Oh yeah you're right. Again, black,
8.2.51		Erik:	You you took all the purples
8.2.52		Alan:	There were only four in here.
8.2.53		Erik:	Why did you take 'em all?
8.2.54		Alan:	Can't help it.

8.2.55		Erik:	Yeah right. Just give me f- just give me three.
8.2.56		Alan:	Wow, [makes noise] But that would be equal as an orange.
8.2.57		Erik:	What? But you can't, you can't uh third oranges
8.2.58		Alan:	What?
8.2.59	12:57	Erik:	You can't third oranges you can only halve them, divide 'em
8.2.60		Alan:	I suggest this.
8.2.61		Erik:	Ok, I guess you do suggest that. [Places and orange and red train] Then you divide them equally halve by by thirds, I should say, the purples.
8.2.62		Alan:	De purples And zen
8.2.63		Erik:	But then you can't halve them if you do it like this, oh yeah you can. I think we need another thing.
8.2.64		Alan:	No we don't. And here are the quarters, or the fourths, look. [Alan's model: An orange and red train, two dark greens, two purples because there were no more, and four light greens]
8.2.65		Erik:	Ok, what are the quarters? Uhh. [Alan places a third purple rod down.]
8.2.66		Alan:	It's, I call it borrowing. Only in a different way.
8.2.67		Erik:	Ok, there's one model. Now, what is the question? Three fourths or one half? Ok, one two three, oh yeah three fourths are definitely bigger by
8.2.68		Alan:	A half
8.2.69		Erik:	One
8.2.70		Alan:	Three fourths are bigger than a half
8.2.71		Erik:	No.
8.2.72		Alan:	by one half
8.2.73		Erik:	[Figure F-14-25] No. Three fourths, three fourths by one half by one fourth. See, look, see look, one half, one two three. Three fourths
8.2.74		Alan:	Yet again I have to show you my logic.
8.2.75		Erik:	You don't need to.
8.2.76		Alan:	See? How much more would it take to make it? It would either take two fourths or one half.
8.2.77		Erik:	Look at this.
8.2.78		Alan:	Or six sixths.
8.2.79		Erik:	You're weird. Ok. Look at this. One two three, three fourths go there.
8.2.80		Alan:	Here's the logic in mine.
8.2.81		Erik:	It's one, it's one fourth.
8.2.82		Alan:	[Figure F-14-44] It, look, this is one fourth, is it one fourth bigger? I don't think so. It would either be two fourths, one half, or six sixths.
8.2.83		Parish:	Wait, what's your argument?
8.2.84		Erik:	He thinks that it would be one half bigger, but it shows here that it'd be one two, this.
8.2.85		Parish:	Wait, show me which ones you're holding.

8.2.86	Erik:	I think that, Alan you're doing it with thirds! They're fourths! Why are you doing it with thirds?
8.2.87	Parish:	Wait, just explain to me what you're doing.
8.2.88	Erik:	What I'm doing is I'm doing three fourths, and they end here. And then,
8.2.89	Parish:	Ok, what are you holding?
8.2.90	Erik:	But then one half is smaller
8.2.91	Parish:	Where's your one half?
8.2.92	Erik:	The one half's are there.
8.2.93	Parish:	Alright. The dark green ones.
8.2.94	Erik:	The dark greens. And if you put this one up there it'll equal three fourths. So I think three fourths is larger than one half by by one, one fourth.
8.2.95	Parish:	And you don't agree with him?
8.2.96	Alan:	No
8.2.97	Parish:	Well, wait, what's wrong with what he says.
8.2.98	Alan:	Here's what's wrong. These are the three fourths, I mean oh yeah the three fourths
8.2.99	Parish:	Wait, which are you,
8.2.100	Alan:	Oh, it's the thirds.
8.2.101	Erik:	That's what I was trying to tell you.
8.2.102	Alan:	Oh yeah.
8.2.103	Parish:	So now do you agree with him?
8.2.104	Alan:	Yeah. I guess so.
8.2.105	Parish:	You guess so?
8.2.106	Alan:	Yeah
8.2.107	Parish:	You're not sure?
8.2.108	Alan:	I agree.
8.2.109	Erik:	So now I just have to figure out another one.
8.2.110	Parish:	I think you did a very good explanation. But now you have to go.
8.2.111	Erik:	One more.
8.2.112	Alan:	I guess
8.2.113	Parish:	But wait, did you draw that one?
8.2.114	Erik:	Oh no, oh yeah, I have to draw
8.2.115	Parish:	You'd better draw that one. So Dr. Maher can see it.
8.2.116	Erik:	I'd prefer to trace.
8.2.117	Alan:	Bingo, exactly on the margin!
8.2.118	Erik:	What?
8.2.119	Alan:	Bingo!
8.2.120	Erik:	What?
8.2.121	Alan:	All I need to do is draw a straight line
8.2.122	Erik:	Alan, what are you doing?
8.2.123	Alan:	I'm trying to draw straight lines.
8.2.124	Erik:	You don't have to draw straight lines. It can be crooked.
8.2.125	Alan:	Not let's see dividing it here.

8.2.126	Erik:	I'll just take this dark green.
8.2.127	Alan:	Ok. [Camera moves to Amy's group and shows Amy and James' models.]
8.2.128	CT:	You have a different model?
8.2.129	Student:	Yeah.
8.2.130	CT:.	And see if you come to the same conclusion
8.2.131	Alan:	There, here's the model. There. [The sound picks up discussions of other groups but it is hard to make out. Now let's make another one. [makes noises]
8.2.132 19:39	T/R 2:	How are you two doing over here?
8.2.133	Alan:	Fine
8.2.134	T/R 2:	Can you tell me about what you've done.
8.2.135	Erik:	Well, I saw that that the three fourths and the one half, the one half will definitely be smaller than the three fourths,
8.2.136 19:52	T/R 2:	Mmmm hmmm
8.2.137	Erik:	But again like we did before, I put that on top of the other half, and it equaled up to the same as the uh fourths. So I figured that one half would be smaller than three thirds by one, smaller than three fourths by one fourth.
8.2.138 20:12	T/R 2:	Ok. Do you agree with that, Alan? You have a different model you want to show me?
8.2.139	Alan:	Yeah, I wrote down that other model that
8.2.140	T/R 2:	And now you're working on this one.
8.2.141	Alan:	[Figure F-20-36] Yeah, here's mine, with the brown, this would be the half, and here are the three thirds. Now it would take one more to fill in the gap so it's one third bigger. I mean a fourth, one fourth bigger.
8.2.142	T/R 2:	Alright, so you're calling the red ones one fourth, the number name one fourth.
8.2.143	Alan:	Yeah, mmm hmmm.
8.2.144 20:32	T/R 2:	Ok, that's very interesting, so yours came out to be a difference of a fourth and your model came out to be a difference of
8.2.145	Erik:	Yeah, a difference of a fourth
8.2.146	Alan:	Yeah, but that's the conclusion using past information. Every time you make something like this, it will always be one fourth on this one if it's one fourth on that, and any other model that you make that can be like this it will always be one fourth. Generalization
8.2.147	T/R 2:	Oh, that's interesting.
8.2.148	Erik:	Alan, I'm not going to do this one. this one
8.2.149	T/R 2:	That's an interesting theory. You think that works for
8.2.150	Alan:	[Figure F-21-49] Because if you did this, you could, these would be the halves, and you could imagine there being one whole rod there. Now to fourth this, it would take, may I borrow one of these? Continues argument

8.2.151	Erik:	No, oh yeah, I guess.
8.2.152	Alan:	May I borrow one of these? And if it would be one there, it would be one half and three thirds. And this might equal up to a purple, that would be one fourth right there. It would always be one fourth of that.
8.2.153 20:40	T/R 2:	Ok, so you think it doesn't matter whatever model you build the difference
8.2.154	Alan:	Yeah, it will always come out to the same answer.
8.2.155	T/R 2:	Interesting, that's an interesting theory. Ok. I like that. Ok, so why don't you record these, it looks like your developing another one. When you're done recording these, uh, I want you to think about a new problem
8.2.156 22:10	Alan:	A new problem? For the rest?
8.2.157	T/R 2:	Yeah. And now I'd like to make sure you've got your models down here, it would be nice if you'd come up with a couple like the next one, I want you to compare, is I want you to compare two thirds and three fourths, I want you to think about which is bigger and by how much. Ok?
8.2.158	Alan:	Oh, hold on a sec. We have to keep one model over here. Let's see.
8.2.159	Erik:	You don't, actually, Alan, you don't need to do thirds, all you have to really do is halves and fourths
8.2.160	Alan:	Right
8.2.161	Erik:	Because you're only comparing thirds, halves and fourths.
8.2.162	T/R 2:	For this problem?
8.2.163	Erik:	No, not for that problem, but for this one, because actually here I made the thirds but I had to take them out, but the thirds, you don't really need to make the thirds, because all you're comparing really is three fourths, the half, the fourths and the half's.
8.2.164	T/R 2:	That's true, isn't it, so it is sort of inefficient to spend a lot of time doing that, I agree with you, now as you're recording, can you do one more thing to make it even clearer than it is, these are really wonderful,
8.2.165	Erik:	Uh huh. recording
8.2.166	T/R 2:	Can you each put the color names in each of these two so that we remember what the color names were
8.2.167	Erik:	Oh, just write 'em in?
8.2.168	T/R 2:	Yeah, if like if this was orange, put in like an "o" or an "or" you can write out orange, either way. You can either abbreviate or write it out, whatever you can fit. For each color ok, because that will help us to remember exactly what you did. Ok, I'll be back in a little bit.
8.2.169	Alan:	Mmm, sure.
8.2.170	Erik:	What were the thirds? Purples?
8.2.171	Alan:	No, the fourths were purples.

8.2.172	Erik:	Fourths were purples? Ok, then what was the thirds? What were the thirds? [camera focuses on Gregory's writing]
8.2.173	Alan:	Thirds?
8.2.174	Erik:	Yeah. For this one.
8.2.175	Alan:	I didn't write down the thirds. I only did the fourths and the halves.
8.2.176	Erik:	Oh yeah, I guess I don't need thirds.
8.2.177	Alan:	But the thirds would have been the yellows.
8.2.178	Erik:	No way. Would have been the...
8.2.179	Alan:	No, can't be the yellows. Oh now I know what the thirds were.
8.2.180	Erik:	The, um,
8.2.181	Alan:	The thirds were the purples. Look,
8.2.182	Erik:	Then what was the fourths?
8.2.183	Alan:	I measured it.
8.2.184	Erik:	Oh.
8.2.185	Alan:	It comes out, wow wow. [inaudible] hyper spaz hyper spaz
8.2.186	Erik:	Now I have to do the other one. What do you mean hyper-spas?
8.2.187	Alan:	Never mind. Ok, two thirds, [hums]
8.2.188	Erik:	One brown,
8.2.189	Alan:	Ah hah. [Hums]. Ok. The two thirds, oh, three fourths is bigger. Two thirds, [makes noise, takes a white rod from the next table.
8.2.190	Danielle:	Alan!
8.2.191	Alan:	I need it. [inaudible]
8.2.192	Amy:	James got it.
8.2.193	CT:	Alright, James. [to James] Where's your whole.
8.2.194	James:	... These are the halves and these are the fourths. And the three fourths is bigger than half by a fourth.
8.2.195	CT:	By what?
8.2.196	James:	A fourth.
8.2.197	CT:	Why do you say a fourth?
8.2.198	James:	[Figure F-27-28] Cuz this, light green's a fourth, if you just take another one and put it right in here to equal. [He has taken one of the two dark green rods in the model and three of the light green rods in the model and places a fourth light green rod next to the dark green rod.
8.2.199	CT:	I see, and I see you have three models down and the conclusion you came you all three models?
8.2.200	James:	Fourths.
8.2.201	CT:	A fourth? It's greater by a fourth? What is greater by a fourth?
8.2.202	James:	Uh, three fourths
8.2.203	CT:	Three fourths is greater by a fourth than?
8.2.204	Amy:	Than a half.

8.2.205	CT:	Than a half. Are you going to agree with this, miss, uh,
8.2.206	Amy:	Jackie
8.2.207	CT:	Jackie? [Jacquelyn nods] Good. [inaudible] Are you trying to figure, now remember you have to go write your conclusions down. Show your three models. That's super. That's great. But before you start working on a fourth model, draw your conclusions so that people can look at it and say "Yes, these people know their beans".
8.2.208	Jacquelyn	Ok, oh guys wait we gotta make sure of something.
8.2.209	James:	What?
8.2.210	Jacquelyn	It's not the same size
8.2.211	Amy.	: It's not. I checked that, I checked it before I even drawed. Hold on, I'll be—there are three lines yellow g, g d, l., w.
8.2.212	Erik:	Alan, Alan should I do this one, too? recording
8.2.213	Alan:	Yeah
8.2.214	Parish:	Yeah?
8.2.215	Erik:	Ok. Two thirds,
8.2.216	Alan:	One model got wacky.
8.2.217	Erik:	Well, wait
8.2.218	Parish:	He's straightening up his models, right? [Erik sighs] Now can you build, now that you've done that, can you build another model for that?
8.2.219	Erik:	Wait, Alan! I actually think that you can use the same model you did for this problem as for, as for this problem. Because see, all y- yeah, really all you need is, cuz you're only comparing the thirds and the fourths, but all you really need is divide it into fourths, thirds and then you can use the same model. Because look, that is
8.2.220	Parish:	Well, you used this model, right, I mean you made the whole the same both times.
8.2.221	Erik:	But you can also, I think that you can tell, you can tell the answer, that all you have to do is draw it with the thirds and then you can tell the answer with the same models.
8.2.222	Alan:	Yeah, you just have to put the thirds in there, and it would be the same answer as that, then you'd have to draw the twelfths, twelfths
8.2.223	Erik:	No, you wouldn't have to draw the twelfth rod, what for?
8.2.224	Alan:	For the
8.2.225	Erik:	What for?
8.2.226	Alan:	[Figure F-35-28] These are how many? Look, those are two thirds and those are three fourths. That fits there so two thirds is smaller than three fourths
8.2.227	Erik:	Hold on, hold on. Hold on, hold on.
8.2.228	Parish:	That's your whole too?
8.2.229	Erik:	Yeah, let me just take some of these. Ok.
8.2.230	Alan:	Exactly

8.2.231	Erik:	Ok, exactly.
8.2.232	Parish:	Ok, you guys, where are your whole rods?
8.2.233	Erik:	The whole, right there.
8.2.234	Parish:	Ok.
8.2.235	Erik:	And then it says two thirds and three fourths.
8.2.236	Parish:	Ok.
8.2.237	Erik:	Two thirds, three fourths.
8.2.238	Parish:	Which one's bigger?
8.2.239	Erik:	Three fourths.
8.2.240	Parish:	Three fourths yeah
8.2.241	Erik:	By, one white one which would probably have to place
8.2.242	Parish:	Well, how do you know it's one white one bigger, because Alan told you?
8.2.243	Erik:	Well, because, no, one two three [giggles] and then two and then all you have to do is go like that, add that onto the thirds.
8.2.244	Parish:	I see, fair enough, I buy it.
8.2.245	Erik:	And then, you have to
8.2.246	Parish:	Find out how much those little white ones are?
8.2.247	Erik:	Yeah, just place it, now you need
8.2.248	Parish:	We can always get more.
8.2.249	Erik:	Now you need twelve. And now I only need three or four.
8.2.250	Alan:	I did, I was going to borrow three but I had to give back to them. [Parish hands Erik more white rods]
8.2.251	Erik:	Ok, there we go!
8.2.252	Parish:	So how much is one white one?
8.2.253	Erik:	.Two.. three four five six seven eight nine ten eleven twelve. One twelfth.
8.2.254	Parish:	One twelfth. So which one is bigger?
8.2.255	Erik:	Uh, three fourths.
8.2.256	Parish:	And how much bigger?
8.2.257	Erik:	One twelfth.
8.2.258	Parish:	Ok, so why don't you draw that model and we'll try to do another model. recording
8.2.259	Alan:	I'll do it on the other pa-
8.2.260	Erik:	Alan, you already drew the orange one, didn't you?
8.2.261	Alan:	Yeah, but I need to do this on this paper, because there's another problem.
8.2.262	Erik:	Oh, you didn't do this problem yet?
8.2.263	Parish:	Yeah he did.
8.2.264	Alan:	Yeah, it's on here.
8.2.265	Parish:	He just hasn't drawn it yet.
8.2.266	Alan:	Yeah I didn't draw it yet.
8.2.267	Erik:	Oh I'm going to draw it, I'm going to draw it too.
8.2.268	Alan:	Four pieces of paper.
8.2.269	Erik:	Yeah I know

8.2.270	Parish:	Killing a lot of trees, aren't you.
8.2.271	Erik:	Yeah.
8.2.272	Alan:	Four, yeah
8.2.273	Erik:	I'm going to do on this one, because I still have room. Oh my, Alan, look how much room you have on that paper. You have some room on the other one. Put your name on it. Put your name on it.
8.2.274	Alan:	It's there.
8.2.275	Erik:	Oh, on this paper. Sorry.
8.2.276	Alan:	Thanks.
8.2.277	Erik:	You're welcome. [laughs]
8.2.278	Alan:	A black ink splatter, thanks.
8.2.279	Erik:	You're welcome, man. You see, I told you you can use the same thing, you can use the same models. Hah hah, nanananana off topic, brother bashing
8.2.280	Alan:	Nanana. What are you talking about? Ok, let us see. Um,
8.2.281	T/R 2:	Oh, ok, and we're recording at this point. Ok, well I'll come back when that's built, I think, because I want to hear about this, but I'll let you work right now.
8.2.282	Erik:	Ok
8.2.283	Alan:	Yet again they came around to collect the taxes.
8.2.284	Erik:	Alan!
8.2.285	Alan:	You steal from me, I steal from you
8.2.286	Erik:	I didn't steal from you!
8.2.287	Alan:	You loan me I loan you.
8.2.288	Erik:	You don't need the halves, you don't have to draw the halves, you know.
8.2.289	Alan:	There.
8.2.290	Erik:	Three fourths, what are the thirds? Oh, purples. Do we have to do two models for this, Alan?
8.2.291	Alan:	Mmmm hmmm.
8.2.292	Erik:	No.
8.2.293	Alan:	Yeah.
8.2.294	Erik:	We have to do, oh no, it's going to be impossible to draw those twelfths.
8.2.295	Alan:	No it isn't.
8.2.296	Erik:	It's going to be hard.
8.2.297	Alan:	Two thirds
8.2.298	Erik:	Yeah, we're only doing two, but we can come up with more. [speaking to someone else] We're just doing two. We can come up, we know we can come up with more.
8.2.299	Parish:	How many did you come up with?
8.2.300	Jessica:	Three so far.
8.2.301	Parish:	Three? For the quarter and a half?
8.2.302	Erik:	Alan, didn't we one time come up with four? Like six?
8.2.303	Alan:	Seven

8.2.304	Erik:	Oh yeah! Yeah, that was it.
8.2.305	Alan:	But that was for a different one.
8.2.306	Erik:	Yeah that was for a different problem
8.2.307	Alan:	Which is bigger, two thirds or three fourths.
8.2.308	Erik:	I don't know I don't care, ok?
8.2.309	Alan:	[laughs] I don't know I don't care, ok?
8.2.310	Erik:	Yep.
8.2.311	Alan:	Somebody left his pencil here!
8.2.312	Erik:	Could it be me?
8.2.313	Alan:	Oh vell, let's see.
8.2.314	Erik:	Yah
8.2.315	Alan:	Yah
8.2.316	Erik:	Alan did you know everyone in my brother's school hates your brother?
8.2.317	Alan:	What?
8.2.318	Erik:	Everyone in your brother's school hates your brother. They do. Do you like your brother? Do you like your brother? Do you?
8.2.319	Alan:	Well, you know that everyone in this school hates you, so hah!
8.2.320	Erik:	Not
8.2.321	Alan:	Yeah.
8.2.322	Erik:	Not.
8.2.323	Alan:	You're disliked by everyone. You don't even have a girlfriend
8.2.324	Erik:	Like you do? So what, you don't have to have one. So?
8.2.325	Alan:	For practical reasons.
8.2.326	Erik:	And I suppose you have any friends.
8.2.327	Alan:	Mmm hmmm.
8.2.328	Erik:	Like who? Alan, did you, are you doing twelfths yet? You didn't do twelfths yet?
8.2.329	Alan:	Erik?
8.2.330	Erik:	What?
8.2.331	Alan:	Have you ever seen the original Star trek movies?
8.2.332	Erik:	No.
8.2.333	Alan:	Like Star trek 4?
8.2.334	Erik:	Have you seen Star trek 7, the lost uh country?
8.2.335	Alan:	Starter 7?
8.2.336	Erik:	Yeah.
8.2.337	Alan:	They played it already.
8.2.338	Erik:	Yeah, the lost country, where they go to that new like ice planet.
8.2.339	Alan:	William Shatner, he's 61 you know
8.2.340	Erik:	Well this is like 7 or 8 it's like the lost colony or something else
8.2.341	Alan:	Well I saw 4, did you see the one where they go [continue in this vein, intercom interrupts]

8.2.342		Erik:	Hey James, James, on your computer, did you beat the bishop thing yet? It's impossible, isn't it? Ok, let's see,
8.2.343		Alan:	[continue with star trek, camera moves to Danielle and Gregory]
8.2.344	41:30	Danielle:	Get that.
8.2.345		Gregory:	Huh?
8.2.346		Danielle:	Get that.
8.2.347		Erik:	If it was two twelfths it would probably be one sixth.
8.2.348		Parish:	Well, look what you've done, how much is three twelfths equal to? Incomplete
8.2.349	42:29	Erik:	Three twelfths is equal to one fourth.
8.2.350		Parish:	And what about four fourths? I mean four twelfths, sorry.
8.2.351	42:36	Erik:	Four twelfths is equal to, one third.
8.2.352		Parish:	So maybe Meredith's argument didn't make sense.
8.2.353		Erik:	Yeah
8.2.354	42:49	Parish:	Did you label it?
8.2.355		Alan:	Yeah
8.2.356		Parish:	Great.
8.2.357		Erik:	Ok, I did one diagram. Now I just have to
8.2.358		Parish:	Now you need another one.
8.2.359		Erik:	Can you go in the back or should I use a different paper? Because this paper.
8.2.360		Parish:	I think you want to use a different paper.
8.2.361		Erik:	Ok.
8.2.362		Jackie:	Oh, because we wrote on the back.
8.2.363	43:10	Parish:	Are you writing with pencil or with pen?
8.2.364		Jackie:	Pen.
8.2.365		Parish:	Well, can you read it? If you can read it it's ok.
8.2.366		Erik:	Well, I don't want to go on the back.
8.2.367		Parish:	No I don't think you can read that
8.2.368		Erik:	I don't want to use this one because that's there and then that can't read through.
8.2.369	43:22	Parish:	So why don't you use that one right there?
8.2.370		Erik:	Yeah, I'll use this one for the other diagram. Ok. Alan, what other diagram is it? I know, the browns. And then we just do that, no.
8.2.371		Alan:	[more about Star trek, camera moves]
8.2.372		Parish:	So you found, what did you find so far.
8.2.373		Gregory:	I found fourths.
8.2.374		Parish:	Greens are what?
8.2.375		Gregory:	And the purples are the.
8.2.376	44:45	Parish:	Ok, show me which ones you're whole. Which one's your whole? The middle one?
8.2.377		Danielle:	[Figure F-46-26] Mmm hmm.
8.2.378		Parish:	And show me where your quarters are.
8.2.379		Danielle:	Here.

8.2.380		Parish:	And where are your purples? I mean what are your purples then?
8.2.381		Danielle:	They're the one thirds.
8.2.382		Parish:	They're the thirds. So show me how much three fourths is [three light greens]. Ok, and show me how much two thirds is. [two purples]. So, how much is, which one is bigger?
8.2.383	78:13	Danielle:	Um, these, three fourths.
8.2.384		Parish:	Three fourths? And how much bigger?
8.2.385		Gregory:	By one sixth?
8.2.386		Parish:	You think, now show me how much three quarters is.
8.2.387		Gregory:	Um,
8.2.388		Danielle:	It's not going to work.
8.2.389		Parish:	What's not going to work?
8.2.390	45:55	Danielle:	That, using the reds, because it's only bigger by one of these small ones.
8.2.391		Parish:	It's only bigger by one of the small ones. How do you know that? And so what are you doing now?
8.2.392		Danielle:	I'm putting white ones on so I could see, so I could see, um, how many of these would be one of that number, that's how much bigger the thirds.
8.2.393		Parish:	One of which number?
8.2.394		Danielle:	The number that these all are.
8.2.395		Parish:	The whole ones?
8.2.396	46:33	Danielle:	There, these
8.2.397		Parish:	Ok, let me find, I'll find [to another group] Can I borrow some of your white ones?
8.2.398	46:40	Gregory:	One tenth, it's bigger by one tenth.
8.2.399		Danielle:	What are you doing?
8.2.400		Gregory:	It's bigger by one tenth.
8.2.401		Parish:	Here you go. Now you thought it was one sixth?
8.2.402		Gregory:	No, by one tenth because of the red.
8.2.403		Alan:	Dark greens are the halves
8.2.404		Erik:	The halves, but we didn't use the halves really, we didn't diagram the halves.
8.2.405		Alan:	And then those were the thirds, the light greens
8.2.406		Erik:	No, the purples were the thirds, the purples were the thirds, the light greens
8.2.407		Alan:	The light greens were the fourths
8.2.408		Erik:	were the fourths
8.2.409	47:19	Both:	And the whites were the twelfths.
8.2.410		Erik:	We did the, we did the twelfths because what we did was, let's see, Alan can I use your model for a second? Well, because we said that the question was two thirds or three fourths.
8.2.411		T/R 2:	Mmm hmm
8.2.412		Erik:	The three fourths, three fourths.

8.2.413	Alan:	Now
8.2.414	Erik:	Three fourths would be larger than the two thirds by one twelfth.
8.2.415	Alan:	Whoops!
8.2.416	Erik:	Because, wait, wait, wait, well, because the three and then the two if you put this at the end of it, that would equal [intercom interrupts] Second grade? And then we just, and then I just can't think of another diagram.
8.2.417	T/R 2:	Ok, and the difference was how much?
8.2.418	Erik:	One twelfth.
8.2.419	T/R 2:	One twelfth, ok.
8.2.420	Alan:	We've spent a lifetime on this.
8.2.421	T/R 2:	And you haven't come up with another one.
8.2.422 48:28	Erik:	No, I can't think of one.
8.2.423	T/R 2:	I'm going to make one suggestion. Think big.
8.2.424	Erik:	Oh, two browns.
8.2.425	Alan:	Two oranges.
8.2.426	Erik:	Yeah, the yellows fourth it. Remember we did that?
8.2.427	T/R 2:	I'll give you your rods back, think big.
8.2.428	Erik:	Alan, remember we did that?
8.2.429	T/R 2:	See if you can come up with another one before we have to leave today.
8.2.430	Alan:	Where are the yellows?
8.2.431 48:49	Erik:	One two, three, I have the half and I, no I have the fourths, all we need is the thirds.
8.2.432	Alan:	I'll keep this model, you make the other one.
8.2.433	Erik:	Third it.
8.2.434 49:05	Alan:	Bingo, dark greens.
8.2.435	Erik:	Bingo, browns third it, I mean blacks
8.2.436	Alan:	Uh, right, blacks blacks blacks.
8.2.437	Erik:	No
8.2.438	Alan:	I told you dark greens third it
8.2.439	Erik:	Browns maybe.
8.2.440	Alan:	Look, see this?
8.2.441	Erik:	Yeah, it's the dark greens, I bet.
8.2.442	Alan:	I know what it is.
8.2.443	Erik:	What is it?
8.2.444	Alan:	It's third two oranges, would mean you'd have to use the blacks.
8.2.445	Erik:	No, the blacks don't work.
8.2.446	Alan:	What we should do is another problem.
8.2.447	Erik:	No it's the same problem. The blacks don't work, Alan.
8.2.448	Alan:	You're right, but what can third? Make a train out of the orange again, look, add a
8.2.449 49:56	Erik:	Add a white! No, because then we have to train the whole, these uh, yellows. One bigger than the yellows would be dark

			greens, wouldn't it? Yeah. Dark greens. So, we add a white onto the oranges, change those to dark greens, two three,
8.2.450		Alan:	Imagine that.
8.2.451		Erik:	No,
8.2.452		Parish:	Well, you can make the oranges bigger.
8.2.453		Erik:	Yeah, but then we can divide it into thirds. I know the half for this, I know how to halve this, nana, and I know how to fourth this, I just don't know how to third it.
8.2.454		Alan:	Make another train, look.
8.2.455		Parish:	Well, maybe you can make it even bigger.
8.2.456		Alan:	Add a yellow onto the two oranges and then fourth it using one up from the yellows.
8.2.457		Parish:	Oh, that's a good idea.
8.2.458		Alan:	One up from the yellows
8.2.459		Erik:	Is a dark green.
8.2.460		Alan:	Using the yellow you can fourth it.
8.2.461	51:15	Erik:	Fourth it using these, one two three, another dark green
8.2.462		Parish:	Does it work?
8.2.463		Alan:	Yeah
8.2.464		Erik:	No it doesn't
8.2.465		Alan:	A light green! Make the light green train! Put a light green there and then third it.
8.2.466		Erik:	Purple purple
8.2.467		Alan:	Purple, right, put in a purple.
8.2.468		Parish:	Ok.
8.2.469		Erik:	Got it. There's the fourths.
8.2.470		Parish:	So now you've got quarters, now you need to get what?
8.2.471		Alan:	Fourth it! Third it!
8.2.472		Erik:	Third it!
8.2.473	51:47	Alan:	Third it! Black it!
8.2.474		Erik:	Black it! Yeah.
8.2.475		Parish:	[laughs] Black it.
8.2.476		Erik:	Whatever.
8.2.477		Alan:	Green it! Blue it, yellow it, red
8.2.478		Erik:	No, these don't third it.
8.2.479		Alan:	Blue
8.2.480		Erik:	Blue, yes blue it.
8.2.481		Alan:	The blue might be able to third it.
8.2.482		Erik:	Probably will. Yup. No.
8.2.483		Alan:	Brown.
8.2.484		Erik:	Yep, hold on let me just get this straight, the browns.
8.2.485		Alan:	The browns will do it, I can tell.
8.2.486		Parish:	You can tell, without even touching it you can tell, that's an amazing visual ability, very impressive.
8.2.487		Erik:	Perfect! It'll work.

8.2.488		Parish:	Alright, so show me which one's bigger, three quarters or two thirds.
8.2.489	52:28	Erik:	Oh, no we have to do the twelfths. Reds. I think.
8.2.490		Parish:	You think reds this time?
8.2.491		Erik:	Yep.
8.2.492		Alan:	Mmm hmm.
8.2.493		Erik:	One, two.
8.2.494		Alan:	You've got plenty of reds up there
8.2.495		Erik:	I know, one two three four,
8.2.496		Parish:	You need some reds from them?
8.2.497		Erik:	Five, six, seven, eight, nine,
8.2.498		Parish:	You're making him do all the work.
8.2.499	52:52	Erik:	Ten, your visual talent did not work
8.2.500		Alan:	Here.
8.2.501		Erik:	One two three four five six seven eight nine ten
8.2.502		Alan:	Eleven twelve
8.2.503		Erik:	Eleven twelve. Perfect
8.2.504	53:05	Alan:	Perfecto perfecto
8.2.505		Erik:	Now, the what is it? Three fourths or two thirds?
8.2.506		Parish:	You show me three fourths
8.2.507		Erik:	One two three and then of course by
8.2.508		Alan:	By a twelfth. Yup. That's another model.
8.2.509		Parish:	Sounds pretty good, now wait a minute, I'm going to ask you another question, keep that other model.
8.2.510		Erik:	But how are we going to fit this on the paper? It's going to be way too big!
8.2.511		Parish:	Turn the paper sideways.
8.2.512		Erik:	Ahhh, never thought of it! Never thought of it that way. [bangs on desk] Thank you, Uh oh, I don't think it still fits, unless we go from there. And add a purple.
8.2.513		Alan:	Well, it just fits.
8.2.514		Erik:	It's huge.
8.2.515		T/R 2:	Did thinking big help?
8.2.516		Erik:	Uh, yeah, we thought real big
8.2.517		T/R 2:	Ok, so you're calling one two oranges and a purple?
8.2.518		Alan:	Hey, maybe we can use three oranges!
8.2.519		T/R 2:	Does this one work? Oh, here it is, oh here it is, here it is.
8.2.520	54:18	Parish:	I wanted to ask them, what if you line up the whites!
8.2.521		Erik:	Uh yah yah yah yah yah
8.2.522		Alan:	No, that would be one twenty-fourths, because it takes two to make a red
8.2.523		Erik:	One twenty-fourth?
8.2.524		Alan:	Yeah.
8.2.525		Erik:	One twenty-fourth. I gotta see, wait, hold on, I just got a brain- something just popped into my brain.
8.2.526		Alan:	Yeah

8.2.527		Erik:	Two twenty-fourths
8.2.528		Alan:	Yeah, two twenty-fourths makes one twelfth and one twelfth is these.
8.2.529		Erik:	They gave me a brain buster here but I can figure it out.
8.2.530		T/R 2:	Alan, while he's lining those up, so which was bigger, which fraction was bigger and by how much
8.2.531		Alan:	Three fourths
8.2.532		T/R 2:	By?
8.2.533		Alan:	One twelfth. Or two twenty-fourths.
8.2.534		T/R 2:	Are two twenty-fourths and one twelfth the same length of the Cuisenaire rods?
8.2.535		Alan:	Mmm hmm. But wait, you couldn't make the twenty-fourths with anything else!
8.2.536	55:12	Erik:	I know, exactly, but hey, it's the same answer.
8.2.537		T/R 2:	Interesting
8.2.538		Alan:	If you used three, you could still do the same answer as that, but you couldn't do it unless you had half of each of the little whites. what does this mean?
8.2.539	55:23	Erik:	[Figure F-55-34] One two three four five six seven eight nine ten eleven twelve thirteen fourteen fifteen sixteen seventeen eighteen nineteen twenty twenty-one twenty-two twenty-three twenty-four. So it's either, ok, it's either one two three, one two, one two, it's either two twenty-fourths, it's either two twenty-fourths or one twelfth.
8.2.540		Alan:	Why did you just do that, Erik?
8.2.541		Erik:	Why, look at this. Yeah, two twenty-fourths.
8.2.542	56:03	T/R 2:	Without building, because it's getting to be a lot with the rods, can you think of any other model with the rods, in other words, something that you might call one that might work?
8.2.543		Alan:	Another? Well, four oranges rod
8.2.544		Erik:	How many did we have before, well, we have one whole, fourths, thirds, twelfths, and twenty-fourths.
8.2.545		Alan:	Twenty-fourths
8.2.546		Erik:	Twenty-fourth
8.2.547		Alan:	Right, Twenty-fourths are the little whites.
8.2.548		Erik:	Twenty-fourths, twelfths, thirds, fourths, whole. Now the problem is fitting it on our paper. Do we have to do all the twenty-fourths?
8.2.549		Alan:	Hold it
8.2.550		Jacquelyn:	We just made ours
8.2.551		Alan:	Look.
8.2.552		Erik:	This is going to be impossible. We have twenty-four whites
8.2.553		Alan:	Just enough.
8.2.554		T/R 2:	Do your best for somebody to record this, because, to try to draw, make a sketch of it or something so you remember it.
8.2.555		Alan:	We'll remember it.

8.2.556	T/R 2:	We should, and I want to make sure we compare
8.2.557	Erik:	No we won't. I'll just diagram it.
8.2.558	Alan:	I'll use pencil
8.2.559	T/R 2:	That's fine
8.2.560	Erik:	There's only one way to fit.
8.2.561	Alan:	Oh, perfect, so we'll remember it.
8.2.562	Erik:	How many layers, five, she told us to think fast, to think big.
8.2.563 57:22	Alan:	I'm not doing a really, you know, advanced drawing, I'm just sketching it out so we'll remember it.
8.2.564	Erik:	I'm doing the exact drawing. That was real weird.
8.2.565	Alan:	Yeah you're telling me!
8.2.566	Erik:	Because what if you divided with fourths, with the white ones, that'd be twenty-fourths, I"m like, wait a minute, another answer!
8.2.567	Alan:	Yeah. Basic sketch. Just write up two models.
8.2.568	Erik:	I can't label this Alan. Orange orange purple. I'm just going to put one
8.2.569	Parish:	So what did you find, when you put the whites up?
8.2.570	Erik:	The twenty-fourths? Two twenty-fourths.
8.2.571	Parish:	Two twenty-fourths. And that's the same thing as what?
8.2.572	Erik:	One sixth.
8.2.573	Parish:	One-
8.2.574	Erik:	I mean one twelfth.
8.2.575	Parish:	One twelfth.
8.2.576	Alan:	Erik, the info is seen on this paper.
8.2.577	Erik:	No, I have the info, diagram, exactly, the exact size of it.
8.2.578	Parish:	Have you drawn this? You haven't drawn this?
8.2.579	Erik:	So I have, I have the exact info, I'm doing the ex- I'm doing the exact info because I have the exact size and the exact shape.
8.2.580	Alan:	Erik, you don't need to draw it. I've already just got a basic sketch of them already. I didn't just, you know, make it really advanced, I just, you know, sketched out the two oranges,
8.2.581	Erik:	I'm making mine
8.2.582	Alan:	Oh, yeah.
8.2.583	Erik:	Alan, mine are advanced, I have the exact info right here.
8.2.584	Alan:	Aright.
8.2.585 59:10	Erik:	Ok, Mark, I'll give you a number to think big for - twenty-four. We have twenty-fourths.
8.2.586	T/R 2:	Um, those of you who are finishing up recording something for me so that I can share these with Dr. Maher, please make sure that your name is on each page that you've done and make sure you've written what the problem or the question was at the top of the page, these look wonderful I'm going to share these with her this afternoon when I see her. [CT says great] So just finish up what you're working on now, because

			I think we do probably have to- we do have to clean up at this point. We can talk about these tomorrow.
8.2.587	59:59	CT:	If you have turned in your work, I'll take your markers.
8.2.588		Erik:	I need a brown for the thirds, Alan, how are we going to do the ones, the twenty-fourths.
8.2.589		Alan:	Have no fear. One two three four five six seven eight nine ten eleven twelve thirteen fourteen fifteen sixteen seventeen eighteen nineteen twenty twenty-one twenty-two twenty-three twenty-four.
8.2.590		Parish:	You want to label 'em and tell me which one is bigger?
8.2.591		Erik:	[laughs] You gotta label 'em Alan.
8.2.592		Parish:	You know what, you don't have to label all of them.
8.2.593		Alan:	O, O, R, no O, O.
8.2.594		Parish:	Ok, I wouldn't label each twenty-fourth, though, don't you think that would take a long time?
8.2.595		Erik:	Yeah.
8.2.596		Alan:	DG, DG
8.2.597		Erik:	What I'm going to do is I'm not going to label orange orange purple
8.2.598		Parish:	No you don't have to do that, you just have to label one of them
8.2.599		Erik:	When I do the whites, I'm just gonna do one white, I'll put wh in one of them. Alan, that doesn't look good, because, look at that one and look at that one, that looks like a twelfth.
8.2.600		Alan:	Uh oh, B B B. Now R R R R R [continues]
8.2.601		Erik:	We need to show the whites. [sings]
8.2.602		Alan:	Twenty-four twenty-fourths.
8.2.603		Erik:	We have to do all the twenty-fourths?
8.2.604		Alan:	By two twenty-fourths or
8.2.605		Erik:	Twenty-four white things. Think how long that's going to take
8.2.606		Alan:	Or one twelfth. One twelfth. Yo, I think we have the
8.2.607		Parish:	Why don't you clean up the rods while he's finishing and you're not
8.2.608	59:53	Alan:	Yeah.
8.2.609		Parish:	[after half a minute of clean up] So which one's bigger, a third or a quarter? Wait, how much is a red, you just haven't told us that?
8.2.610		Erik:	I did over here.
8.2.611		Parish:	Oh, ok.
8.2.612		Erik:	I'll just do it over here.
8.2.613		Parish:	So which is bigger, three quarters or two thirds? [Erik talks as he writes] Is larger than what?
8.2.614		Erik:	Is larger than, let's see, two thirds, by two thirds by one twelfth or twenty-four twenty-fourths.

8.2.615	Parish:	Wait a minute, hold on. It's bigger by twenty-four twenty-fourths?
8.2.616	Erik:	By one twenty-fourth. No, two, two, wait a minute. Wait a minute, I don't know, three fourths, two twenty-fourths. There!
8.2.617	Parish:	There you go, very nice!
8.2.618 1,05,05		End

Session 8, October 6, 1993, OHP View

Line	Time	Speaker	Transcript
8.3.1		T/R 2:	It's good to be back today. How have you all been? Good? You see Dr. Maher is not here today and, uh, our job is going to be to take what we're working on today and to be able to put it in some sort of a written form that Dr. Maher can read tonight so that when she comes back tomorrow to teach the lesson she understands what she did today so that's going to be part of your job today. I want to introduce a couple of new people who've came today and I've got to tell you the reason that they came today is because they saw some of the videotapes of this class from the past couple of weeks and they were so interested in what you were doing that they wanted to come see for themselves today and they'll be friends uh that you can talk to about what you're doing uh this is Parish in front of the room.
8.3.2		CT:	Hello Parish
8.3.3		T/R 2:	And this is Chris in the back who a lot of you introduced yourselves to. They'll be walking around and talking to you because they're curious about what you're doing as will Mrs. Phillips and myself today. Um I want to take us back to where we were on Monday. Does anybody remember what we were doing on Monday? That was the last time we were in. (pause) What have we been doing?
8.3.4		Students	Oh
8.3.5		T/R 2:	Yeah, come on, ok, it's clicking. I can see it clicking out there. Um let's see um Andrew
8.3.6		Andrew:	We, um we divide, um we got, um we had a whole a half and a third and then we had fourths and then we, we took the half and the third and to see how, is two thirds bigger than a half by how much. And we figured it out that it would be by a fourth,
8.3.7		T/R 2:	Hmmm.
8.3.8		Andrew:	I mean by a sixth.
8.3.9		T/R 2:	That's interesting ok, did you all hear what Andrew said he said that you were comparing, um,
8.3.10		Students:	Fractions.
8.3.11		T/R 2:	Fractions, you were comparing a half and two-thirds, did you say? And trying to figure out which was bigger and by how much? Oh, that's interesting and, and you said you came to a decision that it was a difference of how much?
8.3.12		Andrew:	One-sixth
8.3.13		T/R 2:	One sixth what do the rest of you think of that? do you remember that?
8.3.14		Student:	Uh huh.

8.3.15	Erik:	Well, I don't know. I remember, I remember the same thing but then I remember that Meredith had another argument about calling 'em two twelfths other than one sixth.
8.3.16	T/R 2:	Mmm hmm, is that ok?
8.3.17	Erik:	Well I think so but it would be easier if you just called it one sixth because she called the twelfths the white ones I believe and well she said two white ones would equal up to a red one which is one sixth and she called them each one twelfth but I think it'd just be easier to call it one sixth.
8.3.18	T/R 2:	Ok that's interesting let me just put that up here for a minute. I remember that method. Ok now we said… [students start talking amongst themselves] ok uh ok, just to get you back up here for a minute now um some people proposed that one sixth was a possible difference between two thirds and a half but Erik says that Meredith said two twelfths? Is that what she said? Ok now if we're focusing on with the rods if we're focusing on the length that it makes in other words when we line them up either the two little white ones or the red one, right when we're lining those up, we're focusing on length would they be equal, the two twelfths and one sixth do you think?
8.3.19	Erik:	Length? Yes. Lengthwise yeah.
8.3.20	T/R 2:	If we're using length as our focus?
8.3.21	Students:	Yes
8.3.22	T/R 2:	Ok alright so we could make a statement like this, then, we could say that they were equal if we're looking at length [Figure O-7-13]. Alright? Ok that's interesting. Does anybody else have comments from the other day, about what we were working on? That was about the size of it and then you were asked to write about that and I just got those and I'm going to read through those. Well what I'd like to do is start off today with a little more challenging problem. You're gonna compare fractions again and you're gonna use your rods. Um, the only thing I would ask you to do today is that when you build your models and we'll be reminding you as we walk around the room ok please record them on paper ok so that we can keep track of your models and this way Dr. Maher can read them tonight and see what you've worked on. Ok, the first problem I want you to think about is the following and you can discuss with your partner and you have to come up with a justification you have to come up with an argument.
8.3.23	Erik:	Oh, that's easy.
8.3.24	T/R 2:	Can someone read the problem to me? Ok so we're all focusing on this problem, uh, lets see, David?
8.3.25	David:	Which is larger, three fourths or one half?

8.3.26 T/R 2: Ok. Which is larger, right, three fourths or one half and by how much? Ok, so that's what we're looking at. Now, look, I've asked you to do something a little differently today, ok? You're gonna build models but I'd like you to build all of you with your partner build more than one model today for each problem that we give you. I'd like you to try to think of different models, different ways to show me and to justify your argument ok? So we're going to compare those and see what the difference is by how much. Did, do you have a question, Erik?

8.3.27 8:50 Michael: Three fourths, are these fourths?

8.3.28 Brian: [negative mmm mm]. Thirds

8.3.29 Michael: See? Now three of

8.3.30 Brian: I have ano- I have a new way. Look! Mike, I have a new way! This, instead of that! It's the same [places a brown and purple train and shows that it is the same length as the orange and red train.] Ok.

8.3.31 Michael: It's bigger by, it's bigger by three fourths, one fourth, it's bigger by a quarter, because here's the half

8.3.32 Brian: There's a half, now what are the fourths on this? What are the fourths?

8.3.33 Michael: Fourths are the green, dark, light green

8.3.34 Brian: They are? One, two, Um, oh yeah, yeah, yeah, they are. I was going to try that, but I didn't, I didn't.

8.3.35 Michael: Now just take three of them. It's bigger than one half by one fourth. See? This is one fourth and this is three of them, yeah, see? It's bigger by one fourth. No, wait, maybe its one, one two, three

8.3.36 Brian: How 'bout… why don't we just use this, why don't we just use this one that we did last time [uses dark green rod, two light green rods]

8.3.37 Michael: See, this, see? Its fourths, its one fourth

8.3.38 Brian: No we can't make, but we can't make

8.3.39 Michael: It's bigger by one fourth, but by, and so, by how - the two fourths is bigger, the three fourths is bigger, but it's bigger by one fourth.

8.3.40 Brian: Let me make my model first, ok? Let me just make my model [Brian has completed the model using the orange and red train] Ok, now. It's bigger by, um

8.3.41 Michael: One fourth. Because it takes four of these to equal one of these

8.3.42 Brian: Wait, oh yeah, it's bigger by one fourth

8.3.43 Michael: Yeah

8.3.44 Brian: But that isn't what we got last time

8.3.45 Michael: I know

8.3.46 Brian: That's weird

8.3.47	Michael:	That's because it's a different problem. It's three fourths, not two thirds.
8.3.48	Brian:	Oh. Well, at least I got it right in the paper.
8.3.49	T/R 2:	What do you think over here. Have you come up with one model yet?
8.3.50	Brian	Yeah we came up with this and um and last time what we did what we got it wasn't a fourth bigger and when-
8.3.51	T/R 2:	What were we comparing the last time?
8.3.52	Michael:	We were comparing two thirds
8.3.53	T/R 2:	And what
8.3.54	Brian:	And a half, and we did, we did this. I did this last time to help me. I made this model, a small model
8.3.55	Michael:	Yeah we found out that this is always going to half of a third, like one sixth, like no matter what size you had it
8.3.56	Brian:	Oh I maybe that these are even
8.3.57	Michael:	You're saying you can call three fourths two thirds?
8.3.58	Brian:	No, no I mean like the one whole maybe the one whole is an even number that's probably why cause it's an even number
8.3.59	T/R 2:	Can you tell me about this model that you built
8.3.60	Michael:	[Figure O-13-31] yeah it is because it's twelve, it's twelve-
8.3.61	Brian:	Yeah and this is four, and this is four and it's one fourth bigger so I guess when it's an even number it's one fourth bigger.
8.3.62	T/R 2:	Hmm, can you tell me about the model you've done here for three, for comparing three fourths and one half
8.3.63	Brian:	Yeah, well the model here
8.3.64	Michael:	Well this is half, the dark green, the fourths are the light green, and this is the one, this is the one and
8.3.65	T/R 2:	ok so the orange and red is your one
8.3.66	Michael:	Yeah so and then we took this away we took three of them and then we said ok it's bigger, it's bigger by two,
8.3.67	Brian:	It's bigger by one
8.3.68	Michael:	-three fourths is bigger than one half by one fourth cause, yeah right there
8.3.69	T/R 2:	That's the same length as one of your fourths then
8.3.70	Michael:	And to prove that it takes four of these to equal the- that [begins to line light green rods above orange and red train]
8.3.71	T/R 2:	You agree with that, Brian?
8.3.72	Brian:	Yeah
8.3.73	T/R 2:	You agree completely with that argument? [yeah] Ok. Alright so you're telling me then that the difference between three fourths and one half is… how much?
8.3.74	Michael:	One fourth
8.3.75	T/R 2:	One fourth, ok. And which one is bigger?
8.3.76	Michael:	The dark, the light greens, the fourths.

8.3.77	T/R 2:	Which was the three fourths? Ok, alright, so that's a model you could build to show me that and that does justify it can you build me another model for that same problem?
8.3.78	Brian:	Ok let's try… I did one right here
8.3.79	Michael:	No, but that's the same thing, that's the same thing as here because that's the same length.
8.3.80	Brian:	Oh. Oh, ok.
8.3.81	T/R 2:	Is this the same model or a different model [indicating Brian's small model using the purple rod as one] here?
8.3.82	Michael:	That, that's part of this model, see this is gonna, that's, that's the whole, but it's the same size as that [referring to the brown and purple train on Brian's desk]
8.3.83	Brian:	Yeah, right here [Brian shows that the lengths of the two trains are equivalent]
8.3.84	T/R 2:	Ok
8.3.85	Michael:	So I'm going to try to find a half of this, let's see.
8.3.86	T/R 2:	Alright, well, why don't you see if you can come up with another model now. That's, that's really wonderful. It's very good.
8.3.87	Brian:	ummm….
8.3.88	Michael:	I think I found one
8.3.89	Brian:	What about this one? Wait..
8.3.90	Michael:	Nope, that's not it, it needs to be one bigger than this.
8.3.91	Brian:	You're taking all my pieces! Oh, wait, this is the same as this too. [makes a train of blue and light green]
8.3.92	Michael:	I wonder if this is the same. Nope this one isn't.
8.3.93	Brian:	Let me try this, this is a nine and five
8.3.94	Michael:	That's not the same
8.3.95	Brian:	Fourteen, it's fourteen, it's still even. You want to try it?
8.3.96	Michael:	Sure, ok, now we just have to find, I found a half, that's the black, I just can't
8.3.97	Brian:	The half is a black?
8.3.98	Michael:	Yeah
8.3.99	Brian:	It is?
8.3.100	Michael:	mmm hmmm
8.3.101	Brian:	Oh. Man, you took the blacks
8.3.102	Michael:	Um, you can get an extra bag up there from back of the class
8.3.103	Brian:	Ok [gets up and returns with more rods]
8.3.104	Michael:	One less than this is gonna be [tries to use light green rods to make fourths] This can't be. Oh boy, this can't be done. Because there's not thirds to this, see, this doesn't work, this doesn't work. See this doesn't work, but the next size, Brian, you can't use this model
8.3.105	Brian:	What?
8.3.106	Michael:	[Figure O-16-21] You can't use this model, because if that doesn't work [purple rod] then this should [light green], but it

		doesn't, because this is the size of this [shows that the light green rods were used for the model using the orange and red train].
8.3.107	Brian:	Ok, um, why don't we use this model that I did last time [using purple as one- Figure O-13-06]? That's a nice little model. And how about this, how about this? Let me try this. This, ok, I got this [two orange rods and four yellow rods], I remember I thought of this one. A long one.
8.3.108	Michael:	Yeah, it's a long one.
8.3.109	Brian:	Very long.
8.3.110	Michael:	But I'm working on a different one, that doesn't work
8.3.111	Brian:	Ok. So far I have got, uh, three! How about this one [one orange and two yellows], oh yeah
8.3.112	Michael:	That doesn't work I just tried that
8.3.113	Brian:	You can't make fourths.
8.3.114	Michael:	[pointing to 20cm model] But what's fourths?
8.3.115	Brian:	There
8.3.116	Michael:	The fourths?
8.3.117	Brian:	For this?
8.3.118	Michael:	Yeah, I'll make one too and see if I can
8.3.119	Brian:	[pointing to yellows] Those, right there! One two, three, four
8.3.120	Michael:	So you're calling this [orange rods] one? What's the whole? What's the half
8.3.121	Brian:	These [orange] are the half and I can't make the wholes yet.
8.3.122	Michael:	The whole
8.3.123	Brian:	And these are the whole. This is the whole, the one [two blue rods]
8.3.124	Michael:	No it's not [points to empty space]
8.3.125	Brian:	I know, I know, I need some extra, look!
8.3.126	Michael:	[laughs]
8.3.127 17:58	Brian:	[Figure O-17-58] One whole, two halves, and, look, it's bigger by one fourth
8.3.128	Michael:	Yay!
8.3.129	Brian:	So that's eighteen, though, that's eighteen, this is twenty!
8.3.130	Michael:	[laughs]
8.3.131	Brian:	This is twenty, wow!
8.3.132	Michael:	[laughing] You can definitely get long. Let's see how long we can go.
8.3.133	Brian:	um, uh, what about this one, you want to try this one
8.3.134	Michael:	I'm trying this one
8.3.135	Brian:	K, what's a half of the brown? What's a half of the- Oh, wait,
8.3.136	Michael:	Half the brown
8.3.137	Brian:	Think of a half… no
8.3.138	Michael:	It has to be one bigger than that - orange - nope
8.3.139	Brian:	No

8.3.140	Michael:	[laughs]
8.3.141	Brian:	phooey
8.3.142	Michael:	[laughs] - Too big
8.3.143	Brian:	Man, that was such a good model. Oh! Twelfths, is this.. are these twelfths? Does this equal twelve? Yeah, yeah it is. Uh, ok,
8.3.144	Michael:	Let's try blacks
8.3.145	Brian:	I need a uh
8.3.146	Michael:	[black black black etc.]
8.3.147	Brian:	Ok.
8.3.148	David:	Can we borrow a red?
8.3.149	Brian	Sure, we got a million of them.
8.3.150	Michael:	Uh, oh, this one doesn't work, yes it does!
8.3.151	Brian:	I think we have to draw ours down now. We have to draw it down now
8.3.152	Michael:	I made one
8.3.153	Brian:	We have to draw it down now. Uh, oh, this is not going to fit, Oh, no this is not going to fit. Hmm, it doesn't fit. This doesn't fit! Wait, does it? [trying to fit models on paper] Yeah it does. yes it fits! This one won't go, this was won't, do it sideways!
8.3.154	Michael:	What's half of this, what's half of this.
8.3.155	Brian:	I'm doing this one now
8.3.156	Michael:	You do this one, and this one, we've only go two models!
8.3.157	Brian:	What?
8.3.158	Michael:	We each have to draw the same model.
8.3.159	Brian:	I have three models.
8.3.160	Michael:	Which ones, where?
8.3.161	Brian:	Oh no, no, no, no, no, I have two
8.3.162	Michael:	Which ones, where are your models?
8.3.163	Brian:	That and that, and that's yours, and this one, oh no no no, why don't you find that one with the brown, I remember finding that one before, I do. [Michael finds another model - Figure O-22-34]
8.3.164		[Brian and Michael record, camera moves to Caitlin and Graham]
8.3.165 21:41	Chris:	Ok, you think three fourths is bigger than one half by how much?
8.3.166	Caitlin:	By-
8.3.167	Chris:	By three fourths? Can you show that with a diagram? Which one, one or three, is it one fourth or three fourths?
8.3.168	Caitlin:	One fourth. Um, right here, they would be the same if you put one fourth in there.
8.3.169	Chris:	Ok, very nice. Can you write that out?
8.3.170	Caitlin:	Should I make that into a one? [Caitlin changes the three to a one on her written work]

8.3.171	Chris:	But you also need to prove how you came to that conclusion [Caitlin starts writing] You write just what you told me. Very good, Graham, ok you want to get to work on the next problem, ok, now I want to know, what's bigger, two thirds or three fourths? [talks to Caitlin about writing the problem down]
8.3.172 32:14	Brian:	Because
8.3.173	Michael:	Question, answer! Three fourths and one
8.3.174	T/R 2:	I'll let you finish this
8.3.175	Brian:	Um… uh, Mike I need help
8.3.176	Michael:	What?
8.3.177	Brian:	[giggles] I need help with this.
8.3.178	Michael:	Ok I'll be right there - just gotta finish, three fourths is bigger than one half…
8.3.179	Brian:	I can't think, well I know one I can think of now.
8.3.180	Michael:	Ok
8.3.181	Brian:	Ok three fourths is larger than one half by one fourth because, well, it takes two of em right here, look, here… well because it takes two of em [two white rods] to equal one half [the red rod], but the question is, but there are three of em
8.3.182	Michael:	No, no, no, um if this is, this is a half and this is three. So it would be bigger by one fourth because it takes how many fourths does it take, it takes three fourths to equal um, Oh, this is confusing. It takes three fourths to equal
8.3.183	Brian:	[interjecting] Why don't we just do what I said? It takes two fourths to equal one half, but the but but there's but but but it needs, but but it takes, but the question is three fourths, and so there's one fourth bigger [Figure O-34-34]
8.3.184	Michael:	One fourth bigger? Yeah.
8.3.185	Brian:	I guess it makes sense. [talking as he writes] is one half bigger, because it takes two fourths to equal one half,
8.3.186	Michael:	[Figure O-33-46] I was gonna say because it takes two fourths to equal one half, but it takes three fourths to equal three fourths?
8.3.187	Brian:	What are you writing? He's writing his explanation in words, but about what you were just saying about one half equals two fourths. It takes two fourths equal to one half, to equal a half, and - I got it!
8.3.188	Michael:	What'd you put for the question - but the question
8.3.189	Brian:	But the question is three fourths, so there is one fourth left - pretty confusing
8.3.190	Michael:	But the question is three fourths,
8.3.191	Brian:	Because it takes two fourths to equal one half, but the question is three fourths, and so there is one fourth left - very confusing!

8.3.192	Michael:	Alright, but the question is three fourths, and so
8.3.193	Brian:	[to T/R 2] When you say it, it's very very confusing. [T/R 2 laughs]
8.3.194	Michael:	So there
8.3.195	Brian:	Three fourths is larger than one half because one fourth, by one fourth, because it takes two fourths to equal one half, but the question is three fourths, and so there is one fourth left
8.3.196	T/R 2:	I understand that.
8.3.197	Michael:	That's because you're a math, a doctor in math!
8.3.198	Brian:	What do you mean, if like, um, my mom, my mom would have read that, she wouldn't
8.3.199	T/R 2:	[laughs] Ok, you're ready to think about another question?
8.3.200	Michael:	Yeah.
8.3.201	T/R 2:	Ok [intercom interrupts]
8.3.202	Michael:	They picked up all that
8.3.203	T/R 2:	I want to ride on the fire truck! [students laugh]. And I'd go with the kindergarteners I guess. Um, ok I want you to think about, another problem.
8.3.204	Brian:	Do I have enough room to write?
8.3.205 37:25	T/R 2:	You could have another sheet of paper. Ok, this time I want you to compare. [talk about room on sheets] This time I want you to compare two thirds and three fourths.
8.3.206	Michael:	Two thirds and three fourths.
8.3.207	Brian:	Ok.
8.3.208	T/R 2:	Decide which one is bigger, and by how much, if in fact one is bigger.
8.3.209	Brian:	I'm going to use my big model that I made
8.3.210	Michael:	Ok, so we should put, I'm going to put my name
8.3.211	T/R 2:	In fact you will want to put those two fractions down so that you remember what they are.
8.3.212	Brian:	I'm going to use my big model that I made
8.3.213	T/R 2:	Ok.
8.3.214	Michael:	I know I made, we, we, me and him made this huge model. I made another one. I made one of thirty. This one's..
8.3.215	Brian:	We made thirty - three of those, but we couldn't make fourths.
8.3.216	T/R 2:	Ok, so the problem is two thirds, compare two thirds and three fourths, which is bigger and by how much
8.3.217	Brian:	Two thirds
8.3.218	Michael:	Wait a minute, we have to change our -
8.3.219	Brian:	Three fourths
8.3.220	Michael:	We have to change this
8.3.221	Brian:	Oh, why don't we just make this one, the old one?
8.3.222	Michael:	Two thirds [makes noise]
8.3.223	Brian:	But we can't, we can't make fourths with this.
8.3.224	Michael:	Yes we can.

8.3.225	Brian:	Can we?
8.3.226	Michael:	Yeah
8.3.227	Brian:	Oh yeah, yeah
8.3.228	Michael:	We can use the light greens
8.3.229	Brian:	[Figure O-39-59] Yeah, Hang on, ok, k, what was it, three fourths compared to… wait, what was it?
8.3.230	Michael:	It was, which, um, which is bigger, two thirds or three fourths, by how much? Two thirds is bigger
8.3.231	Brian:	By two thirds,
8.3.232	Michael:	No, not by two thirds
8.3.233	Brian:	No, no, wait, wait
8.3.234	Michael:	No! Wait! Three fourths is bigger than two thirds, see?
8.3.235	Brian:	I know, I know
8.3.236	Michael:	By one sixth!
8.3.237	Brian:	Two thirds-
8.3.238	Michael:	By one sixth, see?
8.3.239	Brian:	Wait, wait, wait, what was the question? Two thirds and three fourths?
8.3.240	Michael:	No, which is bigger, two thirds or three fourths?
8.3.241	Brian:	Let me write it down, let me just write it down.
8.3.242	Michael:	Which is bigger, two thirds or three fourths
8.3.243	Brian:	Ok so it's two thirds
8.3.244	Michael:	or three fourths
8.3.245	Brian:	Two thirds
8.3.246	Michael:	by how much
8.3.247	Brian:	or three fourths
8.3.248	Michael:	Yeah, [writing] by how much? Ok, I'm done. Look at this.
8.3.249	Brian:	Question mark
8.3.250	Michael:	Oh! Ok, so it's bigger by
8.3.251	Brian:	Wait a minute let me make two thirds, let me make two thirds
8.3.252	Michael:	What the… It's bigger by one twelfth
8.3.253	Brian:	Why did you make that model? Ok, now it's three fourths, let me just copy this down.
8.3.254	Michael:	Don't copy it down yet. We may be wrong
8.3.255	Brian:	No, no no, I'm copying down two thirds and three fourths
8.3.256	Michael:	Ok, ok, so will I.
8.3.257	Brian:	Good we have… Ok [pause] Ok, now three fourths. [go to side view for the rest]
8.3.258 42:53	Graham:	[camera moves] Ok, now, three fourths.
8.3.259	Caitlin:	I found fourths for the purple but there won't be thirds. Here's thirds for the green. Do you have any dark greens over there? Do you have any dark greens [Caitlin builds a model of a yellow rod, and three red rods] Do you have a dark green? Jessica, do you have a dark green? [Some discussion w/ graham about number of models, builds a

		model with one dark green and three reds. Graham's model: four yellows, three dark greens, and two dark greens]
	Chris:	How are you guys doing over here? Any luck?
8.3.260	Caitlin:	No
8.3.261	Chris:	Don't worry about it. Believe it or not, you're pretty close there, yeah, you had the right idea but then you started going in a different direction. Don't worry, it's going to hit you like that.
8.3.262	Caitlin:	Instead of the orange and the red we could try the blue with the green.
8.3.263	Graham:	That's the same thing. Yes it is. A blue with a green?
8.3.264	Caitlin:	Well, maybe it would work.
8.3.265	Graham:	See, ok. Orange and red.
8.3.266	Caitlin:	I think I got it. I got thirds for this, now I need fourths. Ok, let's see will this be fourths? [lines up light greens] Yes, got it.
8.3.267	Graham:	You got it?
8.3.268	Caitlin:	Yeah.
8.3.269	Graham:	Yay
8.3.270	Chris:	Hey, were you looking over there?
8.3.271	Caitlin:	No, I just saw this [holds up model of blue and light green]
8.3.272	Chris:	I was teasing. See, that's just it. You have to use two different colors. Now
8.3.273	Graham:	Then you could do, the, uh, then you could use the orange and the red for that. They're the same size.
8.3.274	Chris:	Mmm hmm. So which is bigger?
8.3.275	Caitlin:	Three fourths.
8.3.276	Chris:	By how much?
8.3.277	Caitlin:	Uh, one of these? [holds up a white rod]
8.3.278	Chris:	Ok, but how much is that?
8.3.279	Caitlin:	That would be, hold on. Take 'em all and put 'em on the top of each other. [starts lining up white rods]
8.3.280	Chris:	I'll be right back.
8.3.281	Caitlin:	They are called I think they might be called twelfths
8.3.282	Graham:	They're called one twelfth.
8.3.283	Caitlin:	I thought so. Plus this is equal to-
8.3.284	Chris:	How much did you say it is?
8.3.285	Graham and Caitlin:	One twelfth
8.3.286	Chris:	Why do you say that?
8.3.287	Graham:	Well, cause twelve of these
8.3.288	Chris:	ok
8.3.289	Graham:	Equal up to this.
8.3.290	Chris:	Equals up to what?
8.3.291	Graham:	The one whole.
8.3.292	Chris:	One whole? Ok, that sounds good to me. Can you write that down? Can you write out that model? That looks pretty good.

8.3.293	Graham:	Then we have to do another model, right?
8.3.294	Chris:	Well, just worry about getting this one on paper first, ok? Sounds good, though. You got good work there.
8.3.295	Graham:	Here, I want to put the white ones on top. [they start writing and continue until end of session.]
8.3.296		[if don’t have students’ work, check the end of this tape]

Session 9, Oct. 7, 1993 (Front and Side)

Session 9, October 7, 1993 Side View

Line	Time	Speaker	Transcript
9.1.1	00:14:53		T/R 1: Well, good morning! [students answer good morning]. I saw how hard you were working yesterday, I looked at tapes last night and early this morning, and I feel very close to you. You had breakfast with me this morning some of you, and you had, um, I guess, some dinner with me and one of my colleagues who was visiting, and it was really wonderful to watch the way you were solving those problems. Um, and I read your papers, so did Dr. Martino, and uh, I was so impressed at how hard you were all working and the wonderful wonderful thinking that you shared with me in the pictures you drew and the models you made. Yesterday I was working with a group of thirty teachers - that's why I couldn't be here - um, Mr. Purdy was there in the afternoon, he was here in the morning, and I was showing them some of your work and weren't they impressed?
9.1.2		Purdy:	They were very impressed.
9.1.3		T/R 1:	They were very impressed, and your teacher Mrs. Phillips knows some of these other teachers and they said "Oh my goodness, those students are doing such wonderful mathematics!" They were so pleased. So I'm glad to be here, today, I need to tell you, I'm going to be gone for a couple of weeks, um, we have to go to a conference in Califonia, Dr. Martino and I, and uh, we're leaving next week. Dr. Martino will be here Monday, and then it will be two weeks before we come back. Um, so while we're gone, and the other mathematics you're doing with Mrs. Phillips, I hope you'll continue to write to me about what you're doing and to Dr. Martino, so, we sort of can still feel close to what's going on when we're not here. So would you do that [Students nod and say Mmm hmmm]? Would you be writing [CT says "Sure"] and then I, we won't be able to wait until we come back. Um, and then we'll be here for a little while again. Ok? Um, I was watching and reading and I was really interested in some of the questions that you were sort of thinking about as you were making your models and I noticed that everyone made a few models in the problems you were solving, isn't that right? You all were making a few models and I know I know Erik was making a model and he's worried about how he can get it one his paper, right? And, cuz it was a large one on his desk, and I'm kind of thinking, um, how are they gonna get it one the overhead when they share it with us,

			right? That's gonna be a problem. But I thought, you know, we can always get a couple of pieces of paper and paste them together if you had to, that's ok. You know, you can fold them or something. So, we'll figure out ways to record even if some of your models do get bigger. Um, what I was going to ask you to think about, um, one of the problems a little bit before we even shared and that was the problem that I think everyone did work on, uh, the second one, which was larger, three quarters or [students say two thirds] two thirds. Did everyone here work on that problem? Somebody might have been ab- raise your hand if you worked on that problem. [All students shown raise their hands] Which is larger, three quarters or two thirds? Ok, and how many of you built more than one model to show a soluitons to that problem? [a few students raise their hands]. How many of you built three models? [No hands are raised] Some of you built two models, were working on two models? Yes, I'm really interested in this. Um, do you remember anything about the problem? I know you don't have the rods yet, but I want you to try to imagine in your mind if you can remember what you did when you solved the problem, which is larger three quarters or two thirds? By the way, do you remember which was larger? [students say mmm hmm] You do remember [mmm hmmm, yeah]. How many of you remember which is larger? [some students raise their hands] Can you think about it in your minds, what you built? I'm kind of curious, what helps you remember, Sarah?
9.1.4		Sarah:	Uh, that two thirds is larger
9.1.5		T/R 1:	She remembers that two thirds is larger. [Erik: I remember something] Erik?
9.1.6		Erik:	I remember that two, wait, three fourths is larger than two thirds by one twelfth or two twenty-fourths.
9.1.7		T/R 1:	Erik remembers it differently. Anybody else? Anybody else remember it? You're not so sure? Michael, what do you remember?
9.1.8	8:42	Michael:	I agree. I agree with Erik, um, because, that's, I remember three fourths being bigger than it because the four, wait I had three light greens and then only two purples and the three light greens were larger.
9.1.9		T/R 1:	Hmm, it could be we need our rods. It's hard for me to remember these. You think that will help? [students say yes]. Ok. Could you give out these for me, Jackie to the tables? What are you thinking, Meredith, while we're giving these - Erik [inaudible] Alan. [Students distributes sets of Cuisenaire rods]. Meredith?
9.1.10	9:00	Student in Back	Oh, I notice.

9.1.11		Meredith:	I remember the greens [holds a light green rod] were three fourths, the fourths and the purples [picks up a light green rod] were the um, thirds. And if you took a third, that's four, and two thirds are eight and you have three of these [light green rods] which are three, this is nine, and this is eight, so three fourths will be more.
9.1.12		T/R 1:	You agree with that, David?
9.1.13		David:	Yeah?
9.1.14		T/R 1:	You remember that?
9.1.15		David:	Yeah.
9.1.16		T/R 1:	They don't have their matts on the table, um, [inaudible]
See:			
9.1.17		Voices:	Why do you need to make that big thing?
9.1.18		Erik:	The dark greens. Dark greens are the fourths.
9.1.19		Jackie:	[Jackie has built a model of an orange and red train, four light green rods, and three purple rods. She also shows two purple rods next to a light green rod.] It wasn't.
9.1.20		T/R 1:	You remember, Jackie?
9.1.21		Jackie:	Two thirds, wait, two thirds and one fourth?
9.1.22		T/R 1:	Three fourths
9.1.23		Jackie:	Oh.
9.1.24	11:53	T/R 1:	Two-Thirds and three-fourths.
9.1.25	11:56	T/R 1:	Do you think this will work, Erin and Jackie?
9.1.26		Danielle:	Jackie, what was the problem?
9.1.27		Jackie:	Two thirds and four, three fourths, which is bigger?
9.1.28		T/R 1:	[To Erik and Alan] I have a question for both of you. I've watched [inaudible] in the tapes for breakfast this morning, so I feel very close to your solution, Erik, and Alan, but I have another question. While you're building this, I'd like you to build the other model you also made.
9.1.29		Erik:	That was
9.1.30		Alan:	Oh, yeah, the two browns, remember?
9.1.31		Erik:	Yeah
9.1.32		Alan:	One brown, two, I think it was the two of those.
9.1.33		Erik:	One of those
9.1.34		Alan:	Yeah, one
9.1.35		Erik:	Something like that.
9.1.36		T/R 1:	Ok, I'd like you to build the other model, and then I want to ask you a question about your two models. Try to remember what
9.1.37		Alan:	Yeah it was the two browns
9.1.38		T/R 1:	Why do you think it was the two browns?
9.1.39		Alan:	Because two browns would be able to third it and fourth it. So, let's see. One,
9.1.40		Erik:	Don't take any whites, though. I need all the whites possible.
9.1.41		Alan:	I know.

9.1.42		T/R 1:	We can get some more.
9.1.43		Erik:	Plus there are probably no whites left in there.
9.1.44		Alan:	Let's see,
9.1.45		Erik:	There are two whites, don't take any of them. Now we know that there's twenty four...
9.1.46		T/R 1:	Ok, build the other model and then when you're done, call me back
9.1.47		Alan:	One th- uh,
9.1.48		Erik:	One third
9.1.49		Alan:	I need um,
9.1.50		Erik:	Yeah, no
9.1.51	14:00	Alan:	Give me two dark greens, no, three, make it three, um, blacks that might do it. Yeah, three blacks thirded this. Three
9.1.52		Erik:	No, cuz blacks are bigger than dark greens.
9.1.53		Alan:	Oh yeah, dark greens, get me three dark greens
9.1.54		Erik:	No, dark greens don't work.
9.1.55		Alan:	Those are two browns? Oh yeah. Oh, now I remember, it was a train of two browns and a red.
9.1.56	14:29	Erik:	Yeah, that's what I remember - don't take a red, no, not from there! [Erik has built a model of an orange and red train, three puple rods, four light green rods, six red rods, and twelve white rods]. Greg, can you spare us some
9.1.57		Alan:	I'll just take it.
9.1.58		Erik:	Ok. Here, so brown, two browns, a red, and yellows were the thirds, I think.
9.1.59		Alan:	No, fourths.
9.1.60		Erik:	No.
9.1.61	14:55	Alan:	Purples were the, no, dark greens thirded it.
9.1.62		Erik:	Could you spare us three, uh, three dark greens, Greg? We need-
9.1.63		Alan:	I can't get any rods these days. We low on supplies. Oh great.
9.1.64		Erik:	There's nothing left in the boxes, there's like absolutely nothing in the boxes!
9.1.65		Alan:	There are none up there.
9.1.66		Erik:	Oh, there's another dark green.
9.1.67		Alan:	Oh, good good good
9.1.68		Erik:	We need two.
9.1.69		Alan:	Uh, I think that might do.
9.1.70		Erik:	I don't know. Where's the half?
9.1.71		Alan:	[mimicking] I don't know.
9.1.72		T/R 1:	There may be some more boxes in the back.
9.1.73		Erik:	More boxes in the back?
9.1.74	15:48	Brian:	Five, fifths.
9.1.75		Michael:	I want to make another model
9.1.76		T/R 1:	There were some bags.

9.1.77		Michael:	Ok, let's try the blues [student in background: I got it!] Three, one two,
9.1.78		Brian:	I can't think of one!
9.1.79		Michael:	This is hard.
9.1.80		Brian:	This is hard, I can't think of any.
9.1.81		Michael:	I found one, I think. [Michael builds a model of a blue rod and three light green rods.] There's thirds, now I need fourths, so that should be red [Michael lines four reds up against his model, but that train is shorter than the others]. No..
9.1.82	16:19	Michael	I thought four this.
9.1.83	16:30	T/R 1:	Did you do it another way, Brandy. Three-quarters and two-thirds Three-quarters and two-thirds.
9.1.84		Brian:	Oh, I have one,
9.1.85		Michael:	One went under the desk. Nope we used up all our dark greens.
9.1.86		David:	This one.
9.1.87		Michael:	Ok.
9.1.88		Brian:	I have it, wait! Wait!
9.1.89		Michael:	Make it one shorter - make it the next size down - black - make it black. Here, I'll give you some.
9.1.90		Brian:	No, it's not going to be thirds.
9.1.91		CT:	Gents, when you start to write, you know, what problem are you doing?
9.1.92		Michael:	Um, we're doing, we're doing, which is larger three fourths or two thirds?
9.1.93		CT:	Ok, make sure when you start to write that you have your name down and your problem.
9.1.94	17:31	Brian:	I only started doing this yesterday. Can I do it over? I did.
9.1.95		T/R 1:	Can I make a suggestion, gentlemen?
9.1.96		Erik:	uh huh. I think it was one brown rod and red.
9.1.97	18:17	T/R 1:	My suggestion is, you have the answer to your question if you carefully study what you built here. If you carefully study this, and study what you did here, you may have the answer to it. If you think about how you built your one here, that should help you, just think about it. [turns attention to another student] Yes, sir?
9.1.98		Alan:	Hold it
9.1.99		Erik:	[makes noise]
9.1.100		Alan:	There. Subtract two from each of those things. What would you get? Two from the purple would be a red, two from an orange would be a blue, two from a brown, would be a - right! So two browns and a red must be the answer, right?
9.1.101		Erik:	No.
9.1.102		Alan:	Oh.
9.1.103		Erik:	Just try one brown

9.1.104	Alan:	One brown.
9.1.105 18:52	Erik:	Let's see what does it. Sorry.
9.1.106	Alan:	Oh, wait, wait, wait, wait!
9.1.107 19:15	Erik:	Light greens, you take a part [inaudible]. No, it's one brown and a red. The puples wouldn't make a third. Wait.
9.1.108 19:21	Alan:	Fourths, maybe we could try a red? It's a brown. Four blacks. Yeah, exactly!
9.1.109	Erik:	Ok. Didn't we have, we need, wait, maybe is was two browns and a red. Two browns and a red, wait then two from a brown would be a black, wouldn't it? No
9.1.110	Alan:	No, dark green, d.g.
9.1.111	Erik:	Wait, yeah, wait
9.1.112	Alan:	Yeah, dark green, get me three dark greens. Alright
9.1.113	Erik:	We did this already now what's the fourths? Ok, fourths there are dark greens, two from the dark greens would be a, a
9.1.114	Alan:	A light
9.1.115	Erik:	Purple
9.1.116 20:37	Alan:	Purple would fourth this. You see? One, two, three, four. I already know that one. I already know it's the same. Look.
9.1.117	Erik:	And it's the same, and it's gotta be a - the light green's smaller,
9.1.118	Alan:	Hold it, look at this. Two browns, which would equal up to ten, wouldn't it?
9.1.119	Erik:	No.
9.1.120	Alan:	Yes, two down from the brown
9.1.121	Erik:	Is bigger.
9.1.122 21:09	T/R 1:	Can you tell me, I want you to think real hard about it, if you look at the models, do you see any relationships among them, or between them? If you look at one model and you look at another and you look at another, do you see any connections?
9.1.123	David:	Well...
9.1.124	T/R 1:	Do you understand my question?
9.1.125	David:	Yeah, I think so, on the second question, well, um, both my models were, um, like in this shape, like that.
9.1.126	T/R 1:	Ok, that's neat. I haven't seen that model. Maybe you can build that one on the overhead when we're finished.
9.1.127	David:	Cuz I think this was...
9.1.128 22:03	T/R 1:	Ok, that's interesting. Ok, so if you were comparing three quarters and two thirds, how would you do it with that model?
9.1.129	David:	[Figure S-22-01] Um, wait, this would be one whole, this is one half, and one of these would be one fourth.
9.1.130	T/R 1:	Ok, that's one half and one fourth. But we're doing three quarters and two thirds.
9.1.131	David:	Well, cuz this was I think was on the second question.
9.1.132	T/R 1:	Right, but now we're doing three quarters and two thirds.

9.1.133	David:	Let's see, um [starts playing with rods as he thinks]
9.1.134	T/R 1:	You know Meredith, that's very interesting what you're telling Mrs. Phillips. I couldn't help but overhearing that, and I'm also talking to David here, but I have a question for you and David to think about, Ok?
9.1.135	Meredith:	[interjecting] Mmm hmmm
9.1.136	T/R 1:	Uh, I probably want you to tell David what you just told Mrs. Phillips. I sort of was listening on the side. Because then I have another question, a challenge for both of you. Why don't you tell David what you just did so David catches up? He was doing a different problem, right, David?
9.1.137	Meredith:	[Figure S-24-44, Meredith has built a model of a blue, brown, and black train, four dark green rods, and three brown rods, and has also included four red rods. As she speaks, she adds two white rods to the model] If you call all these, this one, and these fourths and these thirds, and you take twelve reds, you can call them twelfths, it would be bigger, if you take three thirds, three fourths would be bigger by one twelfth. Or it would be bigger by-
9.1.138	T/R 1:	Just listen to the rest of what she's says, David, for a minute.
9.1.139	Meredith:	Or it could be bigger by two twenty-fourths.
9.1.140	T/R 1:	By two twenty-fourths or by one twelfth. Well, David may need to think about that a little bit, but I noticed that you have a different model here, and I'm going to let you explain that to David, also, but before you do, you can share that with David. I want you to tell me, this is my question to David also, Meredith, do you see any connections between these two models, ok? And now first of all explain it to David and tell him if you see any connections, and then see if you can even imagine a third model and how that would be connected, but it's important that David understands both of these first. Ok? So I'll leave
9.1.141	David:	I think I have some, um, models,
9.1.142	T/R 1:	I think so too.
9.1.143	David:	But I just can't remember them.
9.1.144 24:20	T/R 1:	But why don't you work with these? You don't have to build new ones. You should get a little closer to Meredith here and work with these because she has them built and use these two and see if you can imagine a third one even if you can't build it. But, Meredith, can you share this with David and then I'll be back because then I'll want to hear from it in a little bit.
9.1.145	Meredith:	[Meredith has built a second model of an orange an red train, four light green rods, three purple rods, and twelve white rods] This [orange and red train] is called the one, these [light green rods] are the fourths, and these [purple rods] are the

		thirds, and these [white rods] are twelfths. It's, if you take, two thirds, three fourths,
9.1.146	David:	Yeah, I know, I made the same model
9.1.147	Meredith:	It's bigger by one twelfth. Easy.
9.1.148 25:05	David:	That's what I kept on doing but what I'm saying is this. I kept on making the same shape when I did my models, like that. [David has the model of a purple rod, a red rod, and three white rods on his desk]
9.1.149	Meredith:	Why don't you just work with me because we don't have really a lot of cubes?
9.1.150	David:	I had a lot of models I just can't really remember any of them. I was working on the second question.
9.1.151	Meredith:	I need ones. [laughs]. Can I use these ones? I really need ones. I need twenty-four ones.
9.1.152	David:	[points to the white rods that are on Meredith's desk. inaudible. Meredith completes her first modelby placing eight more red rods and twenty-two white rods in her model.]
9.1.153	Meredith:	Do you have any more reds over there?
9.1.154	David:	Yeah.
9.1.155	Brian:	Three, those are the four, fourths
9.1.156	CT:	A'right.
9.1.157	Brian:	And these are the three thirds.
9.1.158	CT:	Right, ok. I understand that so far.
9.1.159	Brian:	And so, so, they only asked for two thirds, so I took out one third
9.1.160	CT:	Right
9.1.161	Brian:	And they only asked for three fourths, so I took out one right there. And then they said, how, how much bigger is it, and I said by one twelfth, and I put it right there, and that's how I got it.
9.1.162	CT:	You're calling this one twelfth.
9.1.163	Brian:	Well, well, it, it.takes it takes twelve of these to equal up to one, to equal up to all that. So it's one twelfth. [Figure S-27-03]
9.1.164	CT:	How did you know this? Did you guess it or did you,
9.1.165 27:04	Micheal	We experimented.
9.1.166	CT:	You experimented with that?
9.1.167	Brian:	Yeah, yeah.
9.1.168	CT:	And it came out to twelfths?
9.1.169	Brian:	I was just going to say that.
9.1.170	CT:	Oh, wow, you people have three models, do you not?
9.1.171	Brian:	Yeah.
9.1.172	CT:	Oh, wait a minute. Or do you? You have one, two, and this is the same one.
9.1.173	Michael:	This is the same one. Yeah.

9.1.174	CT:	So you have two models, and they're asking you for one more.
9.1.175	Michael:	We want each to have two different models.
9.1.176	CT:	Excuse me?
9.1.177	Michael:	Brian wants to have two models of his own, and I want to have two models of my own.
9.1.178	T/R 1:	Kimberly, do you have some extras...
9.1.179	David:	You don't have to fill it up, all you have to do is put it in there.
9.1.180	T/R 1:	What do you need? Ones have become precious I don't see any ones in here. Some. Alright, we'll make a mess. One, two, three, four, five, you have a friend who also... here's some more, ok? You know, a suggestion I have, Alan and Erik, if you can find another table who's solving the same problem, maybe you can combine
9.1.181	Erik:	Well, we need a lot more pieces. Well,
9.1.182	T/R 1:	[speaking to other students]... smaller model. Maybe you can all come together. Maybe, uh, Meredith and David can help you. Ok, what do you have here, David? [to Erik] Here! [Erik says oh good]. Ok, where are we?
9.1.183	Meredith:	If you call this a one, these fourths, these thirds, and these twelfths, and these twenty-fourths. And you take three thirds, two thirds- three fourths and two thirds, it's bigger by one twelfth or two twenty-fourths [Figure S-30-10].
9.1.184	T/R 1:	Mmmm hmmm. Ok, I see that, you see that, too, David, and you showed us this one too, but now that's not my question, ok? I'm, I'm asking you a different question. You found in this model that three quarters was bigger than two thirds by
9.1.185	Meredith:	One twelfth.
9.1.186	T/R 1:	One twelfth, right? You found in this model that three quarters was bigger than two thirds by
9.1.187	Meredith:	Two twelfths, two
9.1.188	T/R 1:	Two twelfths - by one twelfth?
9.1.189	Meredith:	Yeah, one twelfth
9.1.190	T/R 1:	Or two twenty-fourths, right? Is that right? One twelfth or two twenty-fourths. So here, this was the difference, in your little model, and here this was the difference in the bigger model, correct? I'm asking you to imagine, ok, so, this is, I'm going to pull this out for a minute, this was your one twelfth, right? And this was your one twelfth or, two twenty-fourths, right? If you were to build a bigger model, can you predict, can you predict without building it, what your comparisons might look like? Can you predict it in your minds and maybe sketch it or...

9.1.191	31:50	David:	It would probably be a much, much bigger, because if the model is, say, this big, it would need more reds and more whites than these, 'cause these are small.
9.1.192		T/R 1:	Can you predict how many more reds and how many more whites? I need you to think about that.
9.1.193		Meredith:	It depends how big the, uh, model is
9.1.194		T/R 1:	Ok, that's, that's fair enough. So can you imagine one a certain size and able to predict how many reds and whites. You understand my question? That's a real good question for both of you to think about.
9.1.195		Meredith:	[inaudible, laughs]
9.1.196		T/R 1:	But, to be able to justify your answer, why don't you talk to each other about it, and see what you each think, and try to uh, convince each other first, and then you can try to convince me. Ok?
9.1.197		David:	I think that this one [holding a red rod] might be one twenty-fourth, because
9.1.198		Meredith:	No, because these are twenty fourths. These are twelfths. Well, if it was double the size of this
9.1.199		David:	Yeah, I know, then this would be one twenty-fourth, and then this would be one, one forty-eighth, or something, yeah one forty-eighth. Question, then we might be using something like this, and this would be something like one twelfth or something.
9.1.200		Kimberly:	... three quarters, and if you take one of the twelfths and you put it down here, it fits.
9.1.201	33:18	T/R 1:	Hmmm, that's really interesting.
9.1.202		Audra:	And this is the same here.
9.1.203		T/R 1:	Is that another model there, Audra?
9.1.204		Audra:	[Figure S-34-05] Yeah, yeah, cuz this is the same here cuz this is the same size, because there's the same purples and it will fit twelve.
9.1.205		T/R 1:	Ok, now, those models look to me the same. You have four greens and you have three purples, it's just that here you have, is it really different, though?
9.1.206		Kimberly:	Not really
9.1.207		T/R 1:	What number name is this?
9.1.208		Kimberly:	That's a whole and that's a whole.
9.1.209		T/R 1:	What number name did you give it?
9.1.210		Kimberly:	A whole, one
9.1.211		T/R 1:	What number name?
9.1.212		Kimberly:	One
9.1.213		T/R 1:	If this has the number name one and this has the number name one, uh, are they different or the same? Are the lengths the same or different?
9.1.214		Kimberly:	They're the same.

9.1.215	T/R 1:	Can you make one where the lengths, for what you pick one is going to be different? Are these models, I guess, this is my question to you, are they really different?
9.1.216	Kimberly:	No
9.1.217	T/R 1:	You see what I'm saying? Can you think of another?
9.1.218	Kimberly:	Audra, can you help me, That one?
9.1.219	Audra:	You don't need halves
9.1.220	Kimberly:	I know but it's easier for me to find it
9.1.221	T/R 1:	And, what rod would you use to represent one twelfth in that model?
9.1.222	Michael:	In this one?
9.1.223	T/R 1:	Yeah.
9.1.224	Michael:	Hmmm, probably, this one, let's see, just a second.
9.1.225	T/R 1:	Just think backward.
9.1.226	Michael:	Just a second, I'll try and measure.
9.1.227	T/R 1:	That's very interesting, Brian and Michael. That's very very interesting. It's the red.
9.1.228	Brian:	I know, I know [Michael shows that a red rod represents the difference between two thirds and three fourths.]
9.1.229	T/R 1:	So you think you're going to use red to represent one twelfth
9.1.230	Michael:	I think. I also came up, I just came up with just came up with the- oh, here it is. [Michael has a second model of an orange and yellow train, three yellow rods, and is trying to place purple rods next to this model]. Nope, I didn't get up to another model.
9.1.231	Brian:	Yes, I think I have fourths, Mike, Mike, wait, Mike, wait, I have one - I think I have one.
9.1.232	Michael:	We already tried that one [Brian groans]
9.1.233	Brian:	I'm frustrated
9.1.234	Michael:	[laughs] I never thought this problem would be this hard.
9.1.235		[conversation between T/R 1 and Erik - view not on camera]
9.1.236	T/R 1:	You might want to study Andrew's model to see what you have to do to make it bigger.
9.1.237 36:19	Erik:	We did - we did two oranges and-
9.1.238	T/R 1:	Right, but I want you to make one bigger than that.
9.1.239	Erik:	I can divide it into thirds, but I can't divide it into fourths.
9.1.240	T/R 1:	Maybe you gotta make it bigger.
9.1.241	Meredith:	Ok, let's try to go to thirty. Let's maybe try to go to thirty. This is twenty-four, we need to make it six more. What is six?
9.1.242	David:	[counting out white rods from Meredith's small model] One, two, you don't really need this anymore.
9.1.243	Meredith:	[stopping David] I do
9.1.244	David:	You don't really need that one.
9.1.245	Meredith:	Well, I have an idea.
9.1.246	T/R 1:	Yeah.

9.1.247	Meredith:	Well, say we called it thirty.
9.1.248	T/R 1:	Thirty.
9.1.249	Meredith:	Yeah, um model. Thirty of the ... thirty ones, and
9.1.250	T/R 1:	You're using thirty white ones to make your train, is that what you're telling me?
9.1.251	Meredith:	Yeah
9.1.252	T/R 1:	Using thirty white rods to call one? Will it work?
9.1.253	Meredith:	No not thirty white ones, you would add a six block, which would be, I think would be this six [a yellow rod], yeah so this is six. That would make thirty and you would call the oranges thirds
9.1.254	T/R 1:	Would that work?
9.1.255	Meredith:	And some of
9.1.256	T/R 1:	Well, try building that and tell me if that works.
9.1.257	Meredith:	Ok.
9.1.258	T/R 1:	That's, that's something to try. Why don't you try?
9.1.259	David:	I also thought of, um,
9.1.260	T/R 1:	Can you get over there to help Meredith? Are you in an awkward situation where the blocks are down there? Would it be easier for you to put your chair here, do you think?
9.1.261	David:	Well, I was also thinking about the other one. It was, um, it was, um, twice the size of that [pointing to Meredith's larger model] Then,
9.1.262	T/R 1:	Hold on, let's hear what David says.
9.1.263	David:	Then this, then the red would be, um, one twenty-fourth, the whites would be, I think that would be one forty-eighth.
9.1.264	T/R 1:	Oh, so you're saying that if it would be twice the size.
9.1.265	David:	And then this [light green] would be one twelfth.
9.1.266	T/R 1:	That's very interesting. That's an interesting theory. Why don't you test the theory with Michael and Alan, I think they would like to hear this theory. Would you like to hear - I think David has a theory - why don't you come over here. They have an interesting - David has an interesting theory, I don't know if Meredith heard it, tell them his theory, now listen carefully, Jackie, you want to hear this theory?
9.1.267	Erik:	They [pointing to Andrew's table] already had a theory, I heard it.
9.1.268	T/R 1:	Ok, let's hear David's theory.
9.1.269	David:	Well, before
9.1.270	T/R 1:	You know this model, gentlemen, don't you?
9.1.271	Alan:	Yeah.
9.1.272	T/R 1:	Ok, listen to what he's saying with this model. Meredith? Ok, I'm ready to listen.
9.1.273	David:	Well, before, we had this other one, um, where the whites were one twenty-fourth and the reds were one twelfth. But then if we double that, then the reds would be one twenty-

		fourth, the whites would be one forty-eighth, and then the light green would be one twelfth.
9.1.274	T/R 1:	You may have to say that again. Alan is making a face.
9.1.275	Erik:	I just I
9.1.276	Alan:	No, meaning
9.1.277	T/R 1:	You're thinking that's possible?
9.1.278	Erik:	I heard what Andrew said was
9.1.279	T/R 1:	I would suggest that all of you get your blocks together and pick a spot on the floor over there
9.1.280	Erik:	But I heard-
9.1.281	T/R 1:	And take some mats
9.1.282	Erik:	But I overheard Andrew's - Andrew's doing, what he's doin' is he's makin' a train for the wholes and he said if you could make a train for one whole, why can't you make a train for the thirds and the fourths?
9.1.283	T/R 1:	Interesting question. Let me make a suggestion. If you put floormats on the floor, over there by Chris, who's running the camera-
9.1.284	Erik:	He'll have to look straight down.
9.1.285	T/R 1:	And took all your - he'll manage - and take all your rods, all your boxes, why don't you try building David's model and see if it works.
9.1.286	David:	Um, but
9.1.287	T/R 1:	You can destroy this, because someone else has it. You will use someone else's and you help them, ok Meredith? Because you'll need the blocks.
9.1.288	David:	What was yours before? Was it like two blues... no
9.1.289	T/R 1:	Remember what this is, though.
9.1.290	David:	No, one blue, one black, and um, one
9.1.291	Meredith:	No, one blue, one brown and one black.
9.1.292	T/R 1:	You might want to spread your mats on the floor and make a big model together, but you should put your mats on the floor - all four mats. You'll work right here. Um, you guys need to watch Andrew, and Jessica.
9.1.293 40:29	Andrew:	Erik, I made it. [Andrew has built a model of a train of four orange and one brown rod, six brown rods, and eight dark green rods]
9.1.294	Erik:	What? Let's see if you could divide by ones.
9.1.295	Andrew:	Let's see if I can get it to twelfths.
9.1.296	Erik:	Ten, twenty, thirty, forty, fourtieths, forty forty
9.1.297	Jessica:	Well, it worked!
9.1.298	Andrew:	Maybe these.
9.1.299	Erik:	Forty eighths! Hey you're - that's the same one we're gonna do!
9.1.300	Andrew:	Really?
9.1.301	Jessica:	Really? Well, how come you didn't do it yet?

9.1.302		Andrew:	Two, four, six, eight, ten, twelve, I got it. I got the twelfths [Andrew adds twelve purple rods to his model]
9.1.303		Jessica:	The twelfths?
9.1.304		Andrew:	Yep! The twelfths are purples. Well, I got the biggest model
9.1.305		Jessica:	He's doing something different. He's counting this as one third.
9.1.306		Andrew:	I'm counting two browns as one third, and two greens as one fourth.
9.1.307		T/R 1:	That's interesting.
9.1.308		Andrew:	And purples would be
9.1.309		T/R 1:	Is that ok to do, Jessica?
9.1.310		Jessica:	Yes.
9.1.311		T/R 1:	That's a way to do it! Ok, that's a different way.
9.1.312		Andrew:	One twelfth
9.1.313		T/R 1:	What would the purples be?
9.1.314		Andrew:	Twelfths.
9.1.315		T/R 1:	Ok, neat!
9.1.316		Andrew:	Now we need two more.
9.1.317		Jessica:	That needs about three pieces of paper.
9.1.318		Andrew:	There!
9.1.319		T/R 1:	Did you make another one, Sarah?
9.1.320		Andrew:	There!
9.1.321		T/R 1:	Brian, look what Andrew's doing! What do you think he's doing, Michael?
9.1.322		Andrew:	See, these two are thirds, and these two are fourths.
9.1.323	43:00	T/R 1:	Come on Michael, let's wait for Michael to tell Michael what you've done.
9.1.324		Andrew:	[Figure F-42-07] I took two browns and minded them as thirds, one third, and then two browns is one third, and two greens is one fourth, and then the purple would be one twelfth.
9.1.325		Brian:	Oh! I get it - Ahah! I think I have one now - look! Those are eight, this is twenty four, Mike, twenty-four, look, Mike, I have one!
9.1.326		T/R 1:	So how many twenty-fourths would it be with reds?
9.1.327		Andrew:	Twenty four, so the red would be one twenty-fourth.
9.1.328		T/R 1:	Ok, would the difference be one twenty-fourth?
9.1.329		Andrew:	No, the difference is, let's see, three fourths, the difference is one twelfth.
9.1.330		T/R 1:	One twelfth. What is the difference in twenty-fourths?
9.1.331		Andrew:	Um, two twenty-fourths.
9.1.332		T/R 1:	Two twenty-fourths, ok? Now could you subdivide it smaller than the red?
9.1.333		Andrew:	Yeah, you could divide it into smaller by taking, by taking two whites and putting them up against everything.
9.1.334		T/R 1:	Ok, you know how many of those there'll be?

9.1.335	Andrew:	Well, there'd be, let's see, two times twenty-four is... it would be forty-eight.
9.1.336	T/R 1:	Forty-eight? Ok. So in forty-eights, what would your answer be?
9.1.337	Andrew:	Four. [Figure S-44-41]
9.1.338	T/R 1:	Four of them. Would you write that up? in words, what you just said, I'm going to ask you to share that in a minute? But I'd like you to besure Jessica understands what you've just done. Because you just told me the answer in forty-eighths.
9.1.339	Jessica:	Yeah, because-
9.1.340	T/R 1:	What do you think, Jessica?
9.1.341	Jessica:	You're putting rods up to it now?
9.1.342	Andrew:	Yeah, and if you take two whites and you put them up to the reds-
9.1.343	Jessica:	And then that's
9.1.344	Andrew:	-they would be yeah, twenty-four times two equals forty eight
9.1.345	Jessica:	[simultaneously] forty eight.
9.1.346	T/R 1:	Is that interesting? That's just, just really neat. So, I would like you to write up your solution to that one [inaudible] Um, you could do this one, you could do this one, and you could [inaudible]
9.1.347	Erik:	Alan you're stealing go to front camera for accurate transcript and coding
9.1.348	Alan:	No I'm not!
9.1.349	Erik:	Alan, you're stealing from us!
9.1.350	Alan:	Us?
9.1.351	Erik:	Oh! Oh! And the thirds... And the thirds, the thirds can easily be done by the blues, oh I've got a good idea. The thirds, and how much room do we have left? We have one blue left! Which is nine! One two three, four five six, seven eight nine!
9.1.352	Meredith:	You need the brown rod.
9.1.353	Erik:	It all works out
9.1.354	Alan:	You know you could try use three blues, and the light green, then use the oranges to third it. Then you could fourth it easily.
9.1.355	Erik:	Mmmm.
9.1.356 46:33	Meredith:	I already have the thirds. See? I took off the purple and the, I took off the green
9.1.357	Alan:	Do you have any blues?
9.1.358	Meredith:	Yeah, but three of these.
9.1.359	Alan:	And the light green.
9.1.360	Erik:	I did it!
9.1.361	Alan:	Easily your thirds can be used.
9.1.362	Erik:	Hello! Alan! I did it!
9.1.363	Alan:	You fourthed it too?

9.1.364	Erik:	Yeah! One two three four. I thirded it, one two three, and then plus nine more of those, which will be one two three, four five six, seven eight nine.
9.1.365	Alan:	Now look at this!
9.1.366	Erik:	So it's just like making a new rod.
9.1.367	Alan:	Fourthing it.
9.1.368	V1:	Can you run that by me again? That's a little-
9.1.369	Erik:	Ok.
9.1.370	V1:	I'm not quite following it.
9.1.371	Erik:	Dave, could you move for a second? Ok. What I have, the three, and then I put nine other ones, which would equal another blue
9.1.372	V1:	Ok.
9.1.373	Erik:	So if I thirded it, I would add one to there, one to there, then one to there, which would be three, and then four five six seven eight nine, so it's like adding another blue but making a new rod.
9.1.374	Meredith:	What, this?
9.1.375	Erik?	I'm making a blue rod.
9.1.376	V1:	Ok, can you set that up
9.1.377	Meredith:	It can't be it [see Meredith's model]
9.1.378	Erik:	Well, the thirds
9.1.379	V1:	Just a little bit, use the same pieces but can you set it up a little differently
9.1.380	David:	Oh, I have an idea, put the three next to that, and the three next to that, and the three next to that.
9.1.381	Erik:	What?
9.1.382	David:	I'll show you what I mean. Can I have some more [rods]?
9.1.383	Alan:	Wow, wow, wow, wow, hey! Someone drew on this!
9.1.384	Erik:	Alan, most of them all have muck [?]
9.1.385	Erik:	How can we make it bigger than him? He did the exact same thing.
9.1.386	Meredith:	There!
9.1.387	Erik:	Ohhhhh!
9.1.388	Meredith:	There! Get it?
9.1.389	Erik:	Ohhhhhh! See there are three to that, three to that, and three to that, so it's like, it's a blue, plus one would be an orange, plus another would be a new rod, plus another would be a new rod. And then if you have another one, you'd just, you're just making new rods. Cuz if you add one of those to that, it'd be an orange. But then you add another two, it'd be bigger than an orange. [describe the evolving model]
9.1.390	V1:	I got you.
9.1.391	Erik:	Hmmm. Or you could just take the oranges and do that.
9.1.392 49:01	Alan:	No just take out those uh,
9.1.393	?:	no those would uh

9.1.394	Erik:	you could take out an orange
9.1.395	Meredith:	You could take an orange and use two ones
9.1.396	Alan:	You could take out three
9.1.397	Meredith:	Orange-
9.1.398	Alan:	Erik, you could take out three six nine and put a blue in there
9.1.399	Meredith:	Orange and six purples. Wait! Wait a second! Aren't these nines? Weren't these nines, right? And these are tens, right?
9.1.400	Erik:	Yes, tens.
9.1.401	Meredith:	But, if they're tens, [inaudible] see what I mean? You put the, uh, put the
9.1.402	Erik:	You know why the blue's bigger than em? Because they have the three whites added to em.
9.1.403	Meredith:	No the whites, the orange is bigger.
9.1.404	Erik:	Of course, because the orange is ten, those are nine.
9.1.405	CT:	I don't want to break your train of thought, but what's happening here?
9.1.406	Erik:	Well, see, we took the three oranges and the dark green to be one, and then the four blues to be um, the fourths, and down here, we took three blues, and then, uh, nine whites, and we took three whites which would go to that one, so we're making a new rod because if you had one it'd be an orange. If you had two other ones it'd be bigger than an orange so we're making a new rod there and we do the same here and the same here, so we're making new rods for thirds.
9.1.407	CT:	Ok.
9.1.408	Erik:	Understand?
9.1.409	CT:	Yes, I do.
9.1.410	Erik:	[laughs] That's the only problem. Actually, no, I do! He was calling two browns, two blacks, and two blues, a one
9.1.411	David:	Yeah, because that was, that was the other problem.
9.1.412	Erik:	Yeah, and then the light greens are the twelfths and those are the
9.1.413	David:	I think that would be sixteen, though.
9.1.414	Erik:	Yeah, and the reds would be the twenty-four, twenty-fourths, the reds would be the twenty-fourths, and the white would be the forty-eighths.
9.1.415	T/R 1:	[maybe to someone else?] What did you get the difference to be?
9.1.416	Erik:	Because he, he just doubled everything.
9.1.417	Meredith:	What are the thirds? What are the fourths?
9.1.418	Erik:	Exactly.
9.1.419	David:	I'm just working on mine.
9.1.420	Erik:	He's working on that. David, that's basically what we came here for.
9.1.421	CT:	Yeah, I do, that's very interesting! Do you understand how you would get fourths and thirds out of that?

9.1.422	Erik:	David, isn't this basically what we came for?
9.1.423	Alan:	He's getting it lined up, trying to get it lined up.
9.1.424	Erik:	Yeah, he's messing up. So basically, we don't need this, all this. We could just push that aside, and work with David's. Isn't this basically what we came here for, David?
9.1.425	David:	Yeah, I know, that's why-
9.1.426	Erik:	[laughs] And everyone is trying to make another model!
9.1.427	David:	I know cuz I told-
9.1.428	CT:	Basically you came here for what?
9.1.429	Erik:	We basically came to discuss David's original model.
9.1.430	CT:	And then they built something else?
9.1.431	Erik:	Yeah, we forgot the whole point why we came here.
9.1.432	David:	Yeah, I told everybody, and then she said to go over there and build David's model.
9.1.433	Erik:	And we lost the point for some reason.
9.1.434	CT:	Ok, but I don't think David did this.
9.1.435	Erik:	No, David's like here, let me do this.
9.1.436	CT:	David, how about you explain to me what you're doing so [inaudible] your thinking.
9.1.437	David:	Well, before Meredith built this other thing and then she had the reds were one twelfth and then the whites were one twenty-fourth, but then
9.1.438	Erik:	We built that, me and alan built that and then they did it, and then
9.1.439 52:53	David:	Meredith did too, but then, uh, so then, uh, she thought to think of a bigger model, then I thought that then maybe the greens would be something like one twelfth, but then we figured out that would be sixteenths, then I put them up there
9.1.440	CT:	Alright
9.1.441	David:	And
9.1.442	Erik:	No it wouldn't this one still has some room. I think.
9.1.443	David:	No, it's just that this [inaudible]. Well um, and then I thought the reds would be one twenty-fourth and the whites might be one forty-eighth. Cuz I just doubled it.
9.1.444	CT:	Did it work out? Did it work out?
9.1.445	David:	What?
9.1.446	CT:	Did it work out? I mean, did you, did you find what you thought you would find?
9.1.447	David:	Well, not really, because this one was one sixteenth, um, one sixteenth.
9.1.448	CT:	And the reds came out to?
9.1.449	David:	I was working on that right now.
9.1.450	CT:	Oh, ok.
9.1.451	Erik:	What about the purples? How about the purples? The purples could come out to be.
9.1.452	David:	Yeah they might be the-

9.1.453		Erik:	I think the purples would be, the purples would probably be twelfths.
9.1.454		David:	Alright, so now,
9.1.455		CT:	This is so interesting, where are you going with this, though? Where are you going with this? I mean, this is very interesting, I'm enjoying this very much. You put a lot of work into it.
9.1.456		Alan:	This isn't going to fit on notebook paper.
9.1.457		CT:	We can take, listen, we can take this and paste it together and put your work on
9.1.458		Erik:	Well, it barely even fits on this!
9.1.459		CT:	Well, you have more than one piece there, so there's no problem. We can do that.
9.1.460	54:16	Erik:	I mean, if it doesn't fit on this, of course it can't fit on a single piece of notebook paper, but if we put a couple of pieces together
9.1.461		CT:	It's ok, we can set up a model. What should we?
9.1.462		David:	I think, maybe I counted wrong but that, but I counted it to be one twenty-third. Let me count again.
9.1.463		CT:	Look and see. See if you have it even.
9.1.464		Erik:	One two three, four, one two three
9.1.465		T/R 1:	They don't look lined up there, David. David, I'm not convinced they're lined up.
9.1.466		Erik:	Eleven twelve thirteen fourteen fifteen sixteen
9.1.467		Alan:	Dave, you have something wrong, you need another
9.1.468		Erik:	Twenty-three. You need to line them up.
9.1.469		Alan:	Here, you've got, yeah, you need another one of that.
9.1.470		T/R 1:	How about a ruler, would that help? The yardstick, behind the board there? A yardstick might help.
9.1.471		Erik:	Yeah [gets up].
9.1.472		T/R 1:	See it over there?
9.1.473		Alan:	Now, push, push, push the reds down.
9.1.474		Erik:	Just push em in, and then you can get one more.
9.1.475		Alan:	There.
9.1.476		Erik:	Now put one more on.
9.1.477		Alan:	Take a yardstick and flatten the whole thing out.
9.1.478		Erik:	What do you mean, flatten it out?
9.1.479		Alan:	It's all wavy.
9.1.480		Meredith:	Yo!!! I just worked [inaudible]
9.1.481		Erik:	No, I mean, it's not ok, cuz, no offense Meredith, but isn't this called the major model we're working on?
9.1.482		David:	That's what we're doing.
9.1.483		Meridith:	That's why we came over here.
9.1.484		Alan:	Ok. Pointless.
9.1.485		Erik:	Nine, ten, eleven, twelve, thirteen, fourteen fifteen, oops, sorry. I just think the purples

9.1.486	Alan:	Is that enough?
9.1.487	Erik:	One two three four five six seven eight nine ten
9.1.488	David:	This is going to be twelve.
9.1.489	Erik:	Eleven Twelve
9.1.490	David:	I know it. The purples
9.1.491	Erik:	Five six seven eight nine ten eleven twelve. There we go.
9.1.492	Meredith:	[Alan begins to straighten the model with the yardstick] No, that side's
9.1.493	Erik:	You don't really need- Wait a minute, now I just gotta do the thirds and fourths.
9.1.494	David:	Don't touch anything now.
9.1.495	Erik:	One two three four five six
9.1.496	David:	Don't touch anything. [David gets up and leaves view of camera for a minute and returns] I think the ones would be one forty-eighth
9.1.497	Erik:	Wait, four, eight twelve, just count by fours, cuz.
9.1.498 57:29	David and Erik:	Two four six eight ten twelve fourteen sixteen eighteen twenty twenty-two twenty-four twenty-six twenty-eight.
9.1.499	David:	Thirty.
9.1.500	Erik:	Two four six eight ten twelve fourteen sixteen eighteen twenty twenty-two twenty-four twenty-six twenty-eight thirty, thirty-two, thirty-four, thirty-six, thirty-eight, forty, forty-two, forty-four, forty-six, forty-eight. Yep, forty-eight.
9.1.501	T/R 1:	Are you surprised that it's forty-eight?
9.1.502	David:	No, that's what I thought it would be.
9.1.503	T/R 1:	That's what you guessed? In other words, you were able to build what you thought, what you predicted. Are you going to be able to write this up?
9.1.504	David:	Um, well, not draw, maybe not
9.1.505	T/R 1:	Maybe sketch it, maybe you want to take some notes on your diagram before it ends. What do you think, Meredith? You think you made another, you made a different model. Ok, you might want to take some notes to sketch it so you remember what you did. So you can start
9.1.506	David:	Cuz I thought the greens were the purples one twelfth.
9.1.507	Erik:	So I think what I'm gonna do
9.1.508	T/R 1:	So you think the purple's one twelfth - is there another name for that purple?
9.1.509	Erik:	Um, one, one
9.1.510	T/R 1:	Meredith knows how to find other names for these
9.1.511	Erik:	One twelfth
9.1.512	T/R 1:	That's one name, one twelfth. Is there another number name for the purple?
9.1.513	Erik:	One fourth, no. I mean, uh, what's it called. Wait,
9.1.514	T/R 1:	If you were using-

9.1.515		Erik:	One whole!
9.1.516		T/R 1:	If, let me ask you this
9.1.517		Erik:	One whole, one half
9.1.518		T/R 1:	Don't just guess cuz you're gonna have to prove it to me, Erik. This is my question, to, to Meredith, who likes to come up with different number names and Erik sometimes says on the tape, "I don't know why we have to have more names. I like to have lots of names, frankly. Um,
9.1.519		David:	Four twelfths.
9.1.520		T/R 1:	Ok, David thinks four twelfths
9.1.521		Erik:	One twelfth! One twelfth!
9.1.522		T/R 1:	We know it's one twelfth, we've proved it's one twelfth and you've proved it's one twelfth.
9.1.523		Erik:	Four twenty-eighths. I mean, four forty-eighths.
9.1.524		T/R 1:	Four forty-eighths.
9.1.525		Erik:	Because the whites would be, the whites would be forty-eighths, and then, and then it takes
9.1.526		David:	[interjecting]-I didn't mean-
9.1.527		Erik:	[continuing] Four whites to equal up
9.1.528		David:	Four twelfths.
9.1.529		Erik:	Four forty-eighths.
9.1.530		T/R 1:	You mean four forty-eighths.
9.1.531		Erik:	I said four forty-eighths.
9.1.532		T/R 1:	Meredith? You think that makes sense?
9.1.533		Erik:	Four forty-eighths or
9.1.534		Meredith:	One twelfth.
9.1.535		Erik:	One twelfth.
9.1.536		T/R 1:	So we have one twelfth, we have four forty-eighths. Any other names?
9.1.537		Erik:	Oh, wait! Oh, yeah! Two, two, two twenty-fourths!
9.1.538		T/R 1:	Two twenty-fourths.
9.1.539		Erik:	Two twenty fourths
9.1.540	59:56	T/R 1:	Ok, we have one twelfth, two twenty-fourths, four forty-eighths, anything else? How many different number names and different blocks.
9.1.541		Erik:	Well, does it have to be the same whole?
9.1.542		T/R 1:	What do you think?
9.1.543		Meredith:	It can also be bigger by, um,
9.1.544		Erik:	Two, or it can be thirds, halves, it could be a
9.1.545		T/R 1:	What are green? What's one green?
9.1.546		Erik:	Those are sixteenths.
9.1.547		Meredith:	One sixteenth and one forty-eighth.
9.1.548		T/R 1:	One sixteenth.
9.1.549		Meredith:	Or one forty-eighth.
9.1.550		T/R 1:	How did you get sixteenths?
9.1.551		Erik:	Because there are sixteen that line up to the answer.

9.1.552 Meredith: One sixteenth
9.1.553 T/R 1: Show me they're sixteen.
9.1.554 Erik and Meredith: One two three four five six seven eight nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen.
9.1.555 T/R 1: Ok, so the green is one sixteenth. But is the difference between three quarters and two thirds a green?
9.1.556 Erik: Is the difference between
9.1.557 Meredith: Oh, a green and blue, one forty-eighth.
9.1.558 T/R 1: So how would, what number name would you give for the differences between
9.1.559 Erik: Also, the, it also could be it would take two of them to equal up to a brown.
9.1.560 T/R 1: Well, these are the things I want you to think about and write about. Ok? I think these are good questions that are for you. We're up to seventh grade math already.
9.1.561 Erik: Seventh?
9.1.562 T/R 1: So I think you could work it out if you worked hard enough.
9.1.563 Meredith: Yeah, but I think if you took one sixteenth and one forty - eighth and you put it up to it, it
9.1.564 T/R 1: The difference? Oh, so what number name would you give to that?
9.1.565 Meredith: Uh, one forty eighth [laughs] I don't-
9.1.566 T/R 1: Well, think about it. [to class] Ok. I think we have to clean up
9.1.567 Class: Ohhh!
9.1.568 T/R 1: I know, I'm sorry, I really am, I hope maybe Mrs. Phillips will let you work on this tomorrow and actually finish writing up what you're doing and describing it for Monday. Is that possible, Mrs. Phillips, that maybe tomorrow they can continue this and finally summarize and write this up?
9.1.569 CT: Sure.
9.1.570 T/R 1: Oh, good work! You have to think about that! You have to think hard about it. No guessing, you have to be able to prove it to me, ok?
9.1.571 T/R1: Mrs. Phillips, can they take their papers and work on them tomorrow. On Monday.
9.1.572 1:02:36 Clean up.

Session 9, October 7, 1993, Front View

9.2.1	5:33	T/R 1:	Well, good morning! [students answer good morning]. I saw how hard you were working yesterday, I looked at tapes last night and early this morning, and I feel, uh, very close to you. You had breakfast with me this morning some of you, and you had, um, I guess, some dinner with me and one of my colleagues who was visiting, and it was really wonderful to watch the way you were solving those problems. Um, and I read your papers, so did Dr. Martino, and uh, I was so impressed at how hard you were all working and the wonderful wonderful thinking that you shared with me in the pictures you drew and the models you made. Yesterday I was working with a group of thirty teachers - that's why I couldn't be here - um, Mr. Purdy was there in the afternoon, he was here in the morning, and I was showing them some of your work and weren't they impressed?
9.2.2		Purdy:	They were very impressed.
9.2.3		T/R 1:	They were very impressed, and your teacher Mrs. Phillips knows some of these other teachers and they said "Oh my goodness, those students are doing such wonderful mathematics!" They were so pleased. So I'm glad to be here, today, I need to tell you, I'm going to be gone for a couple of weeks, um, we have to go to a conference in California, Dr. Martino and I, and uh, we're leaving next week. Dr. Martino will be here Monday, and then it will be two weeks before we come back. Um, so while we're gone, and the other mathematics you're doing with Mrs. Phillips, I hope you'll continue to write to me about what you're doing and to Dr. Martino, so, we sort of can still feel close to what's going on when we're not here. So would you do that [Students nod and say Mmm hmmm]? Would you be writing [CT says "Sure"] and then I, we won't be able to wait until we come back. Um, and then we'll be here for a little while again. Ok? Um, I was watching and reading and I was really interested in some of the questions that you were sort of thinking about as you were making your models and I noticed that everyone made a few models in the problems you were solving, isn't that right? You all were making a few models and I know I know Erik was making a model and he's worried about how he can get it one his paper, right? And, cuz it was a large one on his desk, and I'm kind of thinking, um, how are they gonna get it one the overhead when they share it with us, right? That's gonna be a problem. But I thought, you know, we can always get a couple of pieces of paper and tape them together if you had to, that's ok. You know, you can fold

			them or something. So, we'll figure out ways to record even if some of your models do get bigger. Um, what I was going to ask you to think about, um, one of the problems a little bit before we even shared and that was the problem that I think everyone did work on, uh, the second one, which was larger, three quarters or [students say two thirds] two thirds. Did everyone here work on that problem? Somebody might have been ab- raise your hand if you worked on that problem. [All students shown raise their hands] Which is larger, three quarters or two thirds? Ok, and how many of you built more than one model to show a solutions to that problem? [a few students raise their hands]. How many of you built three models? [No hands are raised] Some of you built two models, were working on two models? Yes, I'm really interested in this. Um, do you remember anything about the problem? I know you don't have the rods yet, but I want you to try to imagine in your mind if you can remember what you did when you solved the problem, which is larger three quarters or two thirds? By the way, do you remember which was larger? [students say mmm hmm] You do remember [mmm hmmm, yeah]. How many of you remember which is larger? [some students raise their hands] Can you think about it in your minds, what you built? I'm kind of curious, what helps you remember, Sarah?
9.2.4	9:36	Sarah:	Uh, that two thirds is larger
9.2.5		T/R 1:	She remembers that two thirds is larger. [Erik:I remember something] Erik?
9.2.6		Erik:	I remember that two, wait, three fourths is larger than two thirds by one twelfth or two twenty-fourths.
9.2.7		T/R 1:	Erik remembers it differently. Anybody else? Anybody else remember it? You're not so sure? Michael, what do you remember?
9.2.8		Michael:	I agree with Erik, um, because, that's, I remember three fourths being bigger than it because the four, wait I had three light greens and then only two purples and the three light greens were larger.
9.2.9		T/R 1:	Hmm, it could be we need our rods. It's hard for me to remember these. You think that will help? [students say yes]. Ok. Could you give out these for me, Jackie to the tables? What are you thinking, Meredith, while we're giving these - Erik [inaudible] Alan. [Students distributes sets of Cuisenaire rods]. Meredith?
9.2.10	11:36	Erik:	Ok, what do we need?
9.2.11		Alan:	We need the uh, two oranges and the purple
9.2.12		Erik:	Yeah, I remember, two oranges and the purple. This was our last one, because I remember I was tracing on it,

9.2.13		Alan:	Oh, yeah,
9.2.14		Erik:	Two oranges, one purple, the browns I remember were the thirds.
9.2.15		Alan:	Yeah. And the halves were the
9.2.16		Erik:	We didn't need, we didn't need halves, remember?
9.2.17		Alan:	I know, but we did build em.
9.2.18		Erik:	I think it was the blacks, or the dark greens.
9.2.19	12:08	Alan:	Dark greens fourthed it.
9.2.20		Erik:	Yeah,
9.2.21		CT:	[hands out mats] Put the mats under because it's far too noisy
9.2.22		Alan:	Yeah, Erik, have a mat, it's too noisy. Have a mat.
9.2.23		Erik:	Ok, one purple, brown,
9.2.24		Alan:	Yeah, try the purple, the dark greens did fourth it.
9.2.25		Erik:	They did, I know.
9.2.26		Alan:	Yeah. And then the twenty-
9.2.27		Erik:	No, twelfths were the reds.
9.2.28		Alan:	Twelfths were the reds, and then the whites were the twenty-fourths.
9.2.29		Erik:	Oh, they're copying us, they're doing twenty-fourths!
9.2.30		Alan:	Hey! Somebody's copying.
9.2.31		Erik:	Oh crap, we don't have any more reds! Seven we have eight nine ten, we just need two more
9.2.32		Alan:	[To group of three] Can you spare two red rods? Can we have some? Here we go!
9.2.33		Erik:	Two three four five six.
9.2.34		Alan:	Do you have twenty-four twenty-fourths?
9.2.35		Erik:	Probably not.
9.2.36		Alan:	Oh, I think you overdid it, you overdid it,
9.2.37		Erik:	What?
9.2.38		Alan:	Well, maybe not.
9.2.39		Erik:	What do you mean, overdid it?
9.2.40		Alan:	Well, get out twenty-four ones.
9.2.41		Erik:	I think we need twenty-four ones.
9.2.42		Alan:	Mmm hmm
9.2.43		Erik:	One, two three four five six seven, I'll just take out as many as possible
9.2.44		T/R 1:	[To Erik and Alan] I have a question for both of you. I've watched you do this in the tapes at breakfast this morning, so I feel very close to your solution, Erik, and Alan, but I, I have another question. While you're building this, I'd like you to build the other model you also made.
9.2.45		Erik:	That was
9.2.46		Alan:	Oh, yeah, the two browns, remember?
9.2.47		Erik:	Yeah
9.2.48		Alan:	One brown, two, yeah it was the two
9.2.49		Erik:	One of those

9.2.50	Alan:	Yeah, one
9.2.51	Erik:	Something like that.
9.2.52	T/R 1:	Ok, I'd like you to build the other model, and then I want to ask you a question about your two models. Try to remember what
9.2.53	Alan:	Yeah it was the two browns I think.
9.2.54	T/R 1:	Why do you think it was the two browns?
9.2.55	Alan:	Because two browns, you would be able to third it and fourth it. So, let's see. One, two
9.2.56	Erik:	Don't take any whites, though. Because I need all the whites possible.
9.2.57	Alan:	I know.
9.2.58	T/R 1:	We can get some more.
9.2.59	Erik:	Plus there are probably no whites left in there.
9.2.60	Alan:	Let's see,
9.2.61	Erik:	There are two whites, don't take any of them. I need twenty-four of em. Now we know that there's twenty four...
9.2.62	T/R 1:	Ok, build the other model and then when you're done, call me back.
9.2.63	Erik:	Twenty-eight whites and one fifth.
9.2.64	Alan:	I need the um
9.2.65	Erik:	Yeah, no
9.2.66	Alan:	Give me two dark greens, no, three, make it three, um, blacks that might do it. Yeah, three blacks thirded this.
9.2.67	Erik:	No, no, cuz blacks are bigger than dark greens.
9.2.68	Alan:	Oh yeah, dark greens, get me three dark greens
9.2.69	Erik:	No, dark greens don't work.
9.2.70	Alan:	Those are two browns? Oh yeah.
9.2.71	Erik:	Maybe.
9.2.72	Alan:	Oh I know. Oh, now I remember, it was a train of two browns and a red.
9.2.73	Erik:	Yeah, that's what I remember - don't take a red, no, not from there! [Erik has built a model of an orange and red train, three puple rods, four light green rods, six red rods, and twelve white rods - Figure F-22-14]
9.2.74	Alan:	Greg, can you spare some of the red? Oh never mind. I'll just take it. We don't ask. [laughs]
9.2.75	Erik:	Ok. Here, so brown, two browns, a red, and yellows were the thirds, I think.
9.2.76	Alan:	No, fourths.
9.2.77	Erik:	No.
9.2.78	Alan:	Purples were, no, dark greens thirded it.
9.2.79	Erik:	Maybe, uh yea I guess. Could you spare us three, uh, three dark greens, Greg? We need-
9.2.80	Alan:	I can't get any rods these days. We're low on 'em. We're low. We low on supplies. Oh. Oh great.

9.2.81		Erik:	There's nothing left in the boxes, there's like absolutely nothing in the boxes!
9.2.82		Alan:	There are none up there.
9.2.83		Erik:	Oh, here's another dark green!
9.2.84		Alan:	Oh, good good good
9.2.85		Erik:	We need two.
9.2.86		Alan:	Uh, I think that might do.
9.2.87		Erik:	I don't know. Where's the half?
9.2.88		Alan:	[mimicking] I don't know, know.
9.2.89		T/R 1:	Alen There may be some more boxes in the back.
9.2.90		Erik:	More boxes in the back? Aren't there also some bags?
9.2.91		Alan:	Bags of Cuisenaire rods?
9.2.92		Erik:	We need
9.2.93	17:18	Alan:	Sheesh, we're wasting trees, three pieces of paper? Wow.
9.2.94		Erik:	David, can you spare us three dark greens? Or two, one rather. Got 'em.
9.2.95		Alan:	Got ‘em. Oh good, we got three. Let's see if that thirds it. Hey, come on no peeking, no peeking, you have eyeballitis.
9.2.96		Erik:	Yah.
9.2.97		Alan:	Ok, it works.
9.2.98		Erik:	Ok, let's see, fourths should be,
9.2.99		Alan:	Fourth would be the purples.
9.2.100		Erik:	Yeah, that's what I was thinking. Two, three
9.2.101		Alan:	[makes noise]
9.2.102		Erik:	No, ok.
9.2.103		T/R 1:	If you don't have enough of the little rods, you can imagine them, or what you could do, besides imagining them you could take some of them off, once you put twenty-four we believe you, right? Here are some more of them.
9.2.104		Alan:	Now let's see. What fourths this?
9.2.105		Erik:	We're trying to figure out. It wasn't the purple but
9.2.106		Alan:	It can't be. Oh, now I remember the combo.
9.2.107		Erik:	What was it? No way, no way, no!
9.2.108		Alan:	It has to be. The yellows did have some part in this.
9.2.109		T/R 1:	Can I make a suggestion, gentlemen?
9.2.110		Erik:	Uh huh. I think it was one brown plus a red.
9.2.111	19:17, Fig 1		T/R 1: My suggestion is, you have the answer to your question if you carefully study what you built here. If you carefully study this, and study what you did here, you may have the answer to it. If you think about how you built your one here, that should help you, just think about it. [turns attention to another student] Yes, sir.
9.2.112		Alan:	Hold it
9.2.113		Erik:	[makes noise]

9.2.114	Alan:	There. Subtract two from each of those things. What would you get? Two from the purple would be a red, two from an orange would be a blue, two from a brown, would be a
9.2.115	Erik:	A brown.
9.2.116	Alan:	Yeah, Right! So two browns and a red must be the answer, right?
9.2.117	Erik:	No.
9.2.118	Alan:	Oh.
9.2.119 20:12	Erik:	Just try one brown
9.2.120	Alan:	One brown.
9.2.121	Erik:	Let's see what does it, sorry.
9.2.122	Alan:	Oh, wait, wait, wait, wait!
9.2.123	Erik:	Light greens would take a part in it. No, it's one brown and a red. The purples wouldn't take a part. Wait…
9.2.124	Alan:	Fourths, maybe we could try a red? Yeah, exactly!
9.2.125	Erik:	Four Blacks. One, two, three… Let's see, we don't need halves, we need, wait, maybe it was two browns and a red. Two browns and a red, then two from a brown would be a black, wouldn't it? No
9.2.126 21:20	Alan:	No, dark green, d.g.
9.2.127	Erik:	Wait, yeah, wait
9.2.128	Alan:	Yeah, dark green, get me three dark greens. Alright
9.2.129	Erik:	We did this already now what's the fourths? Ok, fourths there are dark greens, two from the dark greens would be a, a
9.2.130	Alan:	A light, purple. Purple would fourth this. You see? One, two, three, four.
9.2.131	Erik:	And it's the same, and it's gotta be a - the light green's smaller,
9.2.132	Alan:	Hmmm…Hold it, look at this. Two browns, which would equal up to ten, wouldn't it?
9.2.133	Erik:	No.
9.2.134	Alan:	Yes, two down from uh, the uh brown. So this is ten, twelve. Half of twelve would be six. We need something that, these are four each.
9.2.135	Erik:	Those are six.
9.2.136	Alan:	Right, now all we need to do is divide twelve.
9.2.137	Erik:	It's not twelve, it's not twelve, that is a, that's a, two down from ten would be eight. Eight, twelve,
9.2.138	Alan:	Twenty-two. That's twenty-two
9.2.139	Erik:	It can't be twenty-two.
9.2.140	Alan:	Twenty-two divided into four parts
9.2.141	Erik:	No wait, no wait. Eight sixteen eighteen, it would be eighteen, because eight sixteen, seventeen, eighteen. Eighteen divided by six
9.2.142	Alan:	Would equal
9.2.143	Erik:	Wait

9.2.144		Alan:	Eighteen divided by six would equal two.
9.2.145		Erik:	No, no, no, no, no
9.2.146		Alan:	No, twelve divided by six would equal two.
9.2.147	23:41	Erik:	But,
9.2.148		Alan:	Oh,
9.2.149		Erik:	That's impossible.
9.2.150		Alan:	Impossible? But totally unexpected. B-L-A-C-K-S. Get me blacks.
9.2.151		Erik:	Oh, I have three, or four. [hands blacks to Alan]
9.2.152	24:34, Fig 2	Alan:	There [Alan has buit a model of two browns and a yellow and three black rods]
9.2.153		Erik:	What are you doing? That's not what we…
9.2.154		Alan:	Sure
9.2.155		Erik:	No it was, No, it was two yellows and a red! Remember? It was two yellows and a red?
9.2.156		Alan:	Oh, yeah… No! It was an orange.
9.2.157		Erik:	No it wasn't
9.2.158		Alan:	Look: two yellows and a red would equal an orange and a red.
9.2.159		Erik:	No it wouldn't
9.2.160		Alan:	Yeah it would
9.2.161		Erik:	No it was like that and then the light greens
9.2.162		Alan:	Were the fourths
9.2.163	25:02, Fig 3 & 4	Erik:	Told ya!
9.2.164		Alan:	Hold it, let me see. Look, there's a way you can eliminate these two yellows. There we go! That was an adventure.
9.2.165		Erik:	Just put these along with this. [Erik moves this new model of an orange and red train, four light green rods, and three purple rods, next to his other model - Figure F-26-26]
9.2.166		Alan:	We have this model. You busted it!
9.2.167		Erik:	No I didn't, I can make it again.
9.2.168		Alan:	Well, you'll back the other model, because we might have, we do have enough. Good. Erik, come on, Dr. Maher is here. We done. We done.
9.2.169	26:20	T/R 1:	Gentlemen, gentlemen.
9.2.170		Alan:	Ok, that's the second one.
9.2.171		T/R 1:	Oh, what do we have here? Tell me what we have here.
9.2.172		Both:	An orange and a red
9.2.173		Alan:	And purples for thirds
9.2.174		Erik:	And three purples
9.2.175		Alan:	And light green for fourths.
9.2.176		T/R 1:	Ok, right.
9.2.177		Alan:	And, um, here how I used to figure it out.
9.2.178		Erik:	Twelfths! Oh no, those are singles
9.2.179		T/R 1:	Honestly, Erik, I could imagine if you explained to me what I'm supposed to imagine.

9.2.180		Alan:	Ok.
9.2.181		T/R 1:	Ok? I'll try real hard, but I'll try to imagine
9.2.182		Alan:	Suppose there are twelfths under that.
9.2.183		T/R 1:	I can imagine that.
9.2.184		Alan:	And you took out two of those purples and three light greens
9.2.185		T/R 1:	I could imagine
9.2.186		Alan:	It would take one of those twelfths to fill in the gap between the, between the um um
9.2.187		Erik:	See?
9.2.188		Alan:	Two thirds and three fourths
9.2.189		T/R 1:	I see that.
9.2.190		Erik:	And we came to up here
9.2.191		Alan:	So Three fourths is bigger than two thirds by one twelfth
9.2.192	Fig 5	Erik:	And what we came to up here, two thirds and three fourths, it would be bigger by one twelfth or-
9.2.193		Both:	Two twenty-fourths.
9.2.194		Erik:	Because two of 'em equal up to a red like the orange and the
9.2.195		T/R 1:	Why is it a red here and why is it a white here?
9.2.196		Alan:	Well
9.2.197		Erik:	Well, because, see each model is different
9.2.198		T/R 1:	In what way?
9.2.199		Erik:	Because this model is bigger than this model
9.2.200		Alan:	Erik! You could put the reds on that model and make it sixths!
9.2.201		Erik:	But then it would be- so why would we need sixths on that model?
9.2.202		Alan:	Oh yeah, you're right. So either it's one twelfth or one twenty-fourths
9.2.203		Erik:	Two twenty-fourths
9.2.204		Alan:	Two twenty-fourths on this one. This is probably the only model that can get the twenty-fourths cuz you can't, you'd have to halve each white to get twenty-fourths there.
9.2.205	28:07	Erik:	But what if you get three three, um uh, three oranges together
9.2.206		Alan:	We tried that already
9.2.207		Erik:	No we didn't we could get like fiftieths.
9.2.208		T/R 1:	You think it would be fiftieths if there would be three oranges?
9.2.209		Erik:	Well, I don't know exactly but it would be a lot.
9.2.210		T/R 1:	Do you still expect that you would get the same answer?
9.2.211		Erik:	Well, we can divide it.
9.2.212		Alan:	Looking at this it would not be fiftieths.
9.2.213		T/R 1:	Why not?
9.2.214		Alan:	I'm imagining a this (takes another orange) instead of the purple there.
9.2.215		T/R 1:	Instead of the purple?

9.2.216	Alan:	It would take another six of those so it would only be thirtieths
9.2.217	T/R 1:	I'd like you to try that other model.
9.2.218	Alan:	Three oranges?
9.2.219 28:42	T/R 1:	Well whatever you think it is, um, I'd like you find a third model and I think Dr. Martino said to think big. I'd like you to find a third model thinking big.
9.2.220	Alan:	Ok
9.2.221	Erik:	We could think real big.
9.2.222	T/R 1:	And see what you come up if you work on that.
9.2.223	Erik:	Dr. Martino said the key is think big, so
9.2.224	T/R 1:	Well, maybe, we'll see if it works.
9.2.225	Erik:	So now were gonna think real big!
9.2.226	Alan:	Yeah, four of 'em
9.2.227	Erik:	Three, give me three of these. Let me just put these back…
9.2.228	Alan:	Four of 'em that would be right!
9.2.229 29:07	Erik:	Fiftieths, I told ya.
9.2.230	Alan:	Four of 'em, make four, then it would be two yellows
9.2.231	Erik:	Friar tuck, may I have them? I think Friar Tuck's going to have to go around
9.2.232	Alan:	Two four six eight, there would be eighths
9.2.233	Erik:	Alan, Friar Tuck's have to go around, ok?
9.2.234	Alan:	Uh, what do you need?
9.2.235	Erik:	I'm probably going to need whites.
9.2.236	Alan:	How many?
9.2.237	Erik:	Well, it's going to be fiftieths, and we only have twenty-eight.
9.2.238	Alan:	Ok.
9.2.239	Erik:	So we're going to need about fifty thousand. We're going for three.
9.2.240	Alan:	I think Erik you better go.
9.2.241	Erik:	No
9.2.242	Voice:	You don't need fifty singles. We trust you on that.
9.2.243	Alan:	Ok.
9.2.244	Voice:	Because otherwise no one's going to have any.
9.2.245	Alan:	Right.
9.2.246	Erik:	I know what the thirds are.
9.2.247	Alan:	What?
9.2.248	Erik:	Oranges
9.2.249	Alan:	Oranges?
9.2.250	Jessica:	Are you figuring out the big one again?
9.2.251	Erik:	No
9.2.252	Alan:	No, we're trying to…
9.2.253	Erik:	Three oranges.
9.2.254	Alan	Erik, use the yellows. Think big.

9.2.255	T/R 1:	A suggestion I have, Alan and Erik, if you can find another table who's solving the same problem then maybe you can combine
9.2.256 30:12	Erik:	Well, we need a lot more Cuisenaire Rods. Well, let's work with three and then we'll do four.
9.2.257	Alan:	Right.
9.2.258	Erik:	Ok, what would be the thirds. Thirds would easily be the oranges. One two three.
9.2.259	T/R 1:	Well, just build your big model and we could use Meredith and David's smaller model. And then you could come together to put all your models together.
9.2.260	Alan:	And then show them on the overhead?
9.2.261	T/R 1:	Yes.
9.2.262	Alan:	Ok.
9.2.263	T/R 1:	So work on the big model. See what you can do.
9.2.264	Alan:	Erik,
9.2.265	Erik:	we need oranges. [to next group] Do you have three oranges we can borrow?
9.2.266	T/R 1:	Here
9.2.267	Erik:	Oh, good. I'll just pour them into the little - Ah!
9.2.268	Alan:	Ok,
9.2.269	Erik:	Now we need,
9.2.270	Alan:	Ok, perfect! There are thirds
9.2.271	Erik:	Right, now fourths, would be two smaller than an orange, a brown, no, yeah! Three, no that's too big. Two smaller, what's two smaller than a brown. Not a black, but a yellow, no, not a yellow
9.2.272	Alan:	Yes,
9.2.273	Erik:	No
9.2.274	Alan:	A dark green - look it look it for your answer.
9.2.275	Erik:	The dark green would be the fourths?
9.2.276	Alan:	Mmm hmmm. Believe it or not, they are. They might be the fifths.
9.2.277	Erik:	They're the fifths. Then what would be the-
9.2.278	Alan:	Blues would be the
9.2.279	Erik:	This would only be thirty. This would only be thirty because ten twenty thirty.
9.2.280	Alan:	Thirty plus twelve. Forty-two
9.2.281	Erik:	Wait a minute. Since we got these two packs, couldn't we have, Alan, couldn't we have like, um, Alan, couldn't we have, ten twenty thirty forty fifty sixty, wait, ten twenty thirty forty fifty sixty seventy if we all put them
9.2.282	Alan:	Erik, those aren't tens, those are twelves
9.2.283	Erik:	Yeah those are tens.
9.2.284	Alan:	You know what tens are? The browns.
9.2.285	Erik:	Look at this.

9.2.286	Alan:	Prove it.
9.2.287	Erik:	Look at this
9.2.288	Alan:	Put ten up to that.
9.2.289	Erik:	Ok.
9.2.290 33:03	Alan:	Ten. Put ten. Put ten up to that. [Erik does so] Maybe it is ten. Ok, ten twenty thirty forty fifty, it would have to be ten,
9.2.291	Erik:	Ten twenty thirty forty fifty sixty seventy
9.2.292	Alan:	Here we go again.
9.2.293	Erik:	Let's just start with thirty.
9.2.294	Alan:	Yeah, let's eliminate two of the tens.
9.2.295	Erik:	Ok, what would be the fourths?
9.2.296	Alan:	Fourths of that
9.2.297	Erik:	Brown could be in here somewhere
9.2.298	Alan:	Nope, nnnnope
9.2.299	Erik:	Blues
9.2.300	Alan:	Nope. Too big. Eeew! Erik, wipe those rods off immediately. Erik, you're thinking. Hold it...
9.2.301 34:21	Erik	Blacks
9.2.302	Alan	Blacks blacks blacks blacks, right right right, go go go go go. Yup, told you. They're one short. Oh
9.2.303	Erik:	Four long? No. Hah. Alan. Whoops, never mind, that's a five. We didn't forget how to make a big one. We're just experimenting. Perfect! Now just do that, one two three, [noise] No, one larger than this would be the [noise. Erik has built a model of three oranges and a dark green] I got the fourths. [blue - Figure F-35-50]
9.2.304 35:47, Fig 6		Alan: Now make the thirds.
9.2.305	Erik:	Ok, what if we did this? I bet I could make the thirds
9.2.306	Alan:	I think uh, yo, Erik, I think we were just tipped. Erik, come here, go go go. Go go. Alright.
9.2.307	Erik:	Bigger than a dark green, well, how much bigger do I need it then, how much bigger can it get?
9.2.308	Alan:	Erik, hold it, the thirds.
9.2.309	Erik:	I am trying to do something.
9.2.310	Alan:	Thirds thirds thirds. Wait a second, three oranges would have to be the thirds.
9.2.311 37:00	Erik:	What? What?
9.2.312	Alan:	[looking at model that Jessica and Andrew built] That would be two oranges and a yellow. Two oranges and a purple
9.2.313	Erik:	We already did that.
9.2.314	T/R 1:	How are you gentlemen doing, did you get another new model?
9.2.315	Alan:	Yeah
9.2.316	Erik:	Not exactly, actually. You see
9.2.317	T/R 1:	You might want to study, you might want to study Andrew's model to see what you have to do to make it bigger.

9.2.318	Erik:	Well, that's the exact same thing we did. We did two oranges and a purple.
9.2.319	T/R 1:	Yeah, but I want you to make one bigger than his.
9.2.320	Erik:	We're trying, but we can only divide it into one two three four fi- fifths. I can divide it into thirds, but I can't divide it into fourths.
9.2.321	T/R 1:	Well, maybe you gotta make it bigger. See my problem? This is a good challenge for you two. Study that, yeah.
9.2.322	Erik	Those are twelfths.
9.2.323	Alan:	Make six of those and it would be ten greens.
9.2.324	Erik	We want thirds and fourths, not tens.
9.2.325	T/R 1:	I wonder if Meredith and David made any progress. Meredith and David [walks away]
9.2.326 38:33	Alan:	Thirds. Erik, there's one prob. Using oranges, you can't third. You can't third, look, even if you subtracted two you couldn't third that. Because orange is twelve, there's five.
9.2.327	Erik:	Oranges are tens!
9.2.328	Alan:	I know, tens, you can make it into fourths but you couldn't third it.
9.2.329	Erik:	Wait you gave me, oh no.
9.2.330 39:12	Alan:	You just gave up
9.2.331	Erik	Yup.
9.2.332	Alan	Hold on a sec, look, look, you take that off, you could use that
9.2.333	Erik:	That's way too big, Andrew, I don't think you can divide it into anything
9.2.334	Andrew:	Yeah, if you make two browns, two blues are thirds. If you can make a train for a whole you can make a train for a third and a fourth.
9.2.335	Erik:	Ohhh!
9.2.336		[taken from other view, but can be heard partially here] That's very interesting. That's an interesting theory. Why don't you test the theory with Michael and Alan, I think they would like to hear this theory. Would you like to hear - I think David has a theory - why don't you come over here. They have an interesting -
9.2.337	Erik:	So do they have a theory.
9.2.338	T/R 1:	David has an interesting theory, I don't know if Meredith heard it, tell them his theory, now listen carefully, Jackie, you want to hear this theory? [Andrew has built a model of four oranges and two purples, and six browns.]
9.2.339	Andrew:	[to Jessica] See? Two of these are thirds, and that's a one third, third, third. [Andrew has originally made his train of "one" as two oranges followed by a red and then that pattern repeated. He now moves the reds to the end. He then takes

			eight green rods and puts them down] Erik, I figured out the thirds, I just need the fourths.
9.2.340	41:21	Erik:	You did? How did you figure out the thirds?
9.2.341		Alan:	Ok, Erik. Get ready for big city play dude, big time.
9.2.342	41:51, Fig 7		Andrew: Erik, I made it!
9.2.343		Erik:	Wow, now divide it into twelfths and see what you can divide by - [Erik joins Andrew. Camera focuses on David, Meredith, Erik and Alan on the floor.]
9.2.344		Alan:	OK, Here are the rods
9.2.345		Erik:	I'm working here. You could do it! You could do it! Andrew did the same, Andrew did the same model. They did the same model. Ok, this is what you do. You do three oranges, you do actually you do three oranges and two purples. Three oranges
9.2.346		Alan:	Four oranges
9.2.347		Erik:	Three oranges!
9.2.348		Alan	Ok
9.2.349		Erik	Three oranges and two purples.
9.2.350		Alan:	Two purples would just be a brown.
9.2.351		Erik:	An then… a brown? No it wouldn't, yeah it would, and then you could make a train for the thirds. [talk about whose mat is who's] Ok, and then the browns, two browns would make a train for one third
9.2.352		Alan:	Right,
9.2.353		Erik:	And then,
9.2.354		Alan:	Woo, woo, woo, woo
9.2.355	43:40	Erik:	And then two more browns would make another train for thirds. One, two, I know. No, wait, no, it wouldn't be browns, it would be blacks.
9.2.356		David:	Could I just do what I was thinking of? Could I just
9.2.357		Erik:	No they're not, look, it'd be blue and a purple or a blue and a
9.2.358		David:	Could I just make something?
9.2.359		Alan:	Hey you're robbing me.
9.2.360		Erik:	Everyone's robbing you, remember?
9.2.361		David:	Could I have some
9.2.362		Erik:	Me and David will work together.
9.2.363		David:	Could I have some?
9.2.364		Meredith:	I'm working with Erik.
9.2.365		David:	Can't I just use some of the blocks over there? I brought it over Meredith.
9.2.366		David:	Can I have some?
9.2.367		Alan:	Check it out
9.2.368		Erik:	No, I'm just kidding
9.2.369		Alan:	Check it out.
9.2.370		Erik:	Now divide it into thirds
9.2.371		Alan:	Hmmm

9.2.372	Erik:	You can't
9.2.373	Meredith:	I have and idea.
9.2.374	Alan:	Wow, something just popped into my head
9.2.375	Erik:	Me too
9.2.376	David:	Something just popped into my head.
9.2.377 44:56	Alan:	The bigger you make the model, you can't third it.
9.2.378	Erik:	No no no no no, can I have these?
9.2.379	Alan:	[continuing] You can’t third something like this. You'd need colossal rods.
9.2.380	Erik:	Like the ones over there?
9.2.381	David:	I know something, alright?
9.2.382	Alan	Impossible. That’d just like one dark green.
9.2.383	David	Can I um do something?
9.2.384	Erik:	Hold on, let me do something [start fighting over rods] Could I have the blue
9.2.385	Alan:	Erik! You can't third that big orange model.
9.2.386	Erik:	You want to make a bet? I bet I can.
9.2.387	Alan:	You can't
9.2.388	Erik:	I bet I can. Oranges
9.2.389	Alan:	Because if you use more. Using oranges, if you use three oranges, you won't be able to third it. You won't be able to third it!
9.2.390	Erik:	This is what I was thinking. One, two three. Oh contraire… It needs to be, let's see, how much smaller?
9.2.391 46:05	Alan:	Look you can't third it, you fourthed it but you can't third it.
9.2.392	Erik:	Ok, let's see, four one two, easily how you can do it.
9.2.393	Alan:	Third it then.
9.2.394	Erik:	What?
9.2.395	Alan:	Third it then.
9.2.396	Erik:	What do you mean?
9.2.397	David:	Who took my thirds? I was using them.
9.2.398	Erik:	Me! I think, no it wasn't me. It was Alan.
9.2.399	Alan:	Make three blues and train it. Then you could use those
9.2.400	Erik:	What do you think I was thinking of? Give me my rods back. Stop!
9.2.401	David:	Meredith, can I have my rods? I brought them over.
9.2.402	Erik:	Alan, you're stealing from, no no no, Alan you're stealing from us! No.
9.2.403	Meredith:	Oh, oh! Did you just take one of my blues
9.2.404	Alan:	No
9.2.405	Erik:	Yeah. And for the thirds the thirds can easily be done by the blues
9.2.406	David:	I have an idea.
9.2.407	Erik:	I've got a good idea.
9.2.408	David:	I've got a better idea.

9.2.409	Erik:	The thirds, and then how much room do we have left? We have one blue left which is nine. One two three four five six seven eight nine.
9.2.410	David	Just listen out.
9.2.411	Meredith:	Me need a brown rod
9.2.412	Erik:	It all works out.
9.2.413	Alan:	You know what you could try? Use three blues and the light green then use the oranges to third it then you could fourth it easily
9.2.414	Erik:	Now.
9.2.415	David:	I already have a third. See just put down the purple and I took off the green. Here's what I made.
9.2.416	Alan:	Look it. [to Meredith] You have any blues?
9.2.417	Erik:	Yeah but she's not going to give them to you.
9.2.418	Alan:	And the light green. Easily your thirds can be used.
9.2.419	Erik:	Perfect, I did it! Hello Alan, I did it!
9.2.420	Alan:	You fourthed it too?
9.2.421	Erik:	Yup! One two three four
9.2.422	Voice:	Can you third that?
9.2.423	Erik:	I thirded it. One two three and then plus nine other of those, which would be one two three four five six seven eight nine. So it's just like making a new rod.
9.2.424	Alan:	Fourthing it might be.
9.2.425	Voice:	Can you run it by me again? I'm not quite following that.
9.2.426 48:37	Erik:	Ok. I have the three of 'em, and then I put nine other ones which would equal another blue, so if I thirded it, I would add one to there, one to there, and one to there, which would be three. And then four five six seven eight nine. So it's like adding another blue, but I'm making a new rod. [Erik's model is -Three orange rods and a dark green rod, a train of four blue rods, and a train of three blue rods and nine white rods - Figure F-48-54]
9.2.427	Voice:	Ok, can you set that up in a different way?
9.2.428	Erik:	Well, in thirds
9.2.429	Voice:	Use the same pieces, but can you set it up a little differently?
9.2.430	Meredith:	Oh, I have an idea, put the three next to that, and then the three next to that and the three next to that.
9.2.431	Erik:	Huh?
9.2.432	Meredith:	I'll show you what I mean. [Meredith places three white rods after each blue rod.
9.2.433	Erik:	How can we make it bigger than him? He did the exact same thing.
9.2.434	Meredith:	There!
9.2.435	Erik:	Ohhhh!
9.2.436	Meredith:	There! Get it?

9.2.437	49:54	Erik:	[Figure F-50-13] Ohhh! see, there are there to that, three to that, and three to that, so it's like, it's a blue plus one would be an orange, plus another would be a new rod, plus another would be a new rod, and if you have another one, it'd, you'd, you're just making new rods. Because if you add one of those to that, it'd be an orange, but then you add another two it'd be bigger than an orange.
9.2.438		Voice:	I got you.
9.2.439	50:25, Fig 8	David:	Told you I got it.
9.2.440		Meredith:	Or you could just take the oranges and do that.
9.2.441		Erik:	No, those were uh
9.2.442		David	You could take out the three six nine
9.2.443		Meredith:	You could take out the orange and use two ones.
9.2.444		Erik:	I think the greens would be sixteenths, not
9.2.445		David	You could take out the three six nine and put a blue in there
9.2.446		Meredith:	Orange and six ones. Oh, wait a second! Aren't these nines? Aren't these nines, right?
9.2.447		Erik:	Yeah the blues are nines.
9.2.448		Meredith:	And these are tens, right?
9.2.449		Erik:	Yes, those are tens.
9.2.450		Meredith:	But, if they're tens, why are they bigger than these?
9.2.451		Erik:	Huh?
9.2.452		Meredith:	See what I mean? You put the, put the four orange
9.2.453		Erik:	You know why the blues bigger than 'em? Because they have the three whites added
9.2.454		Meredith:	But the orange is bigger!
9.2.455		Erik:	Of course, the orange are ten, those are nine.
9.2.456	51:26	CT:	I don't want to break your train of thought, but what's happening here?
9.2.457		Erik:	Well, see, we took the three oranges and the dark green to be one, and then the four blues to be um the fourths, and down here we took three blues and then uh nine whites and we took three whites which would go to that one, so we're making a new rod, because if you add one it would be an orange, but if you add to other ones, it would be bigger than an orange, so we're making a new rod there and we do the same here and the same here, so we're making new rods for thirds.
9.2.458		CT:	Ok.
9.2.459		Erik:	That's basically what we're doing.
9.2.460		CT:	You have to.
9.2.461		Erik:	I don't, I don't really understand what Dave's doing. That's the only problem. Actually, no, I do. He's calling two browns, two blacks, and two blues a one.
9.2.462		Meredith:	Yeah, cuz that was twice the other
9.2.463		Erik:	Yeah, and then the light greens are the twelfths and those
9.2.464		David:	I think that'd be sixteenths though

9.2.465		Erik:	Yeah, and the reds would be the twenty-four- the twenty-fourths. The reds would be the twenty-fourths and the whites would be the forty-eighths. Because he doubled everything.
9.2.466		Meredith:	Where are the thirds? Where are the fourths?
9.2.467		Erik:	Exactly.
9.2.468		David:	I'm just working on this.
9.2.469		Erik:	He's working on that. Ok!
9.2.470		Meredith:	He's just working on the model
9.2.471	52:47	Erik:	Dave, isn't this basically what we came here for?
9.2.472		CT:	[talks to other students]
9.2.473		Erik:	Dave, isn't this basically what we came here for?
9.2.474		David:	Why did you do that, Alan?
9.2.475		Alan:	I'm getting it lined up. Trying to get it lined up.
9.2.476		Erik:	So we don't need this, basically we don't need all this. We can just push that aside and work with Dave's. Isn't this basically what we came here for, David?
9.2.477		David:	Yeah, I know, that's why
9.2.478		Erik:	Everyone's just trying to make another model.
9.2.479		CT:	Basically you came here for what?
9.2.480		Erik:	Basically we came here to discuss David's original model.
9.2.481		CT:	And then you built something else.
9.2.482		Erik:	Yeah.
9.2.483		David:	Yeah, cuz I told everybody and then she said to go over there and build Dave's model, and then
9.2.484		Erik:	We lost the point for some reason.
9.2.485		CT:	Ok, but I don't think David did. Did you?
9.2.486		Erik:	No, David's did, but David's like, here let me do this.
9.2.487		CT:	David, how about you explain to me what you're doing so I can understand your thinking.
9.2.488	53:40	David:	Well, before Meredith built this other thing and then she had the reds were one twelfth and then the whites were one twenty-fourth, but then.
9.2.489		Erik:	We built that, me and Alan built that originally.
9.2.490		David:	Yeah, and Meredith, Meredith did too, and then um, uh, so then she, she thought to think of a bigger model, then I thought that then maybe the greens would be something like one twelfth but I figured out that would be sixteen when I put them up there, and
9.2.491		CT:	Right.
9.2.492		Erik:	No it wouldn't because you still have some room.
9.2.493		David:	No,
9.2.494	54:20	Erik:	I think
9.2.495		David:	No it's just that that piece is hanging out, um, then I thought the reds would be one twenty-fourth and the whites might would one forty-eighth because I just doubled it.
9.2.496		CT:	Did it work out?

9.2.497		David:	What?
9.2.498		CT:	Did it work out? Did you find what you thought you would find?
9.2.499		David:	Um, well, not with the greens, that turned out to be one sixteenth
9.2.500		CT:	The greens turned out to be what, sweetheart?
9.2.501		David:	One sixteenth.
9.2.502		CT:	And the reds came out to?
9.2.503		David:	I'm working on that right now.
9.2.504		CT:	Oh, ok. I'm sorry.
9.2.505		Erik:	What about the purples? The purples, the purples might come out to be
9.2.506		David:	Yeah they might be one-
9.2.507		Erik:	I think the purples would do that.
9.2.508		David:	Maybe it would be something else.
9.2.509	55:03	Erik:	The purples would be one twelfth.
9.2.510		David:	Alright, so now
9.2.511		CT:	This is so interesting, where are you going with this, then?
9.2.512		David:	What?
9.2.513		CT:	Where are you going with it? I mean, this is very interesting, I'm enjoying it very much, you put a lot of work into it.
9.2.514		Alan:	This isn't going to be able to fit on notebook paper.
9.2.515		CT:	We can take, listen, we can take this and paste it together and put your work on it.
9.2.516		Erik:	Well, it barely even fits on this!
9.2.517		CT:	Well, you have more than one piece there, so there's no problem there, don't worry about that.
9.2.518		Erik:	I mean, if it doesn't fit on this, of course it can't fit on a single piece of notebook paper, but if we put a couple of pieces together it'd fit.
9.2.519		CT:	It's ok, we can set up a model. What should we?
9.2.520		David:	I think, maybe I counted wrong but that, but I counted it to be one twenty-third. Maybe I'll count again.
9.2.521		CT:	Ok, let's see. See if you have it even.
9.2.522		Erik:	One two three, four, one two three
9.2.523	55:59	T/R 1:	They don't look lined up to me, David. David, I'm not convinced they're lined up.
9.2.524		Erik:	Eleven twelve thirteen fourteen fifteen sixteen
9.2.525		Alan:	Dave, you have something wrong, you need another
9.2.526		Erik:	Twenty-three. You need to line them up.
9.2.527		Alan:	Here, you've got, yeah, you need another one of that.
9.2.528		T/R 1:	How about a ruler, would that help? A yardstick, behind the board there? A yardstick might help.
9.2.529		Erik:	Yeah [gets up].
9.2.530		T/R 1:	See it over there?
9.2.531		Alan:	Now, push, push, the reds down.

9.2.532		Erik:	Just push em in, and then you can get one more.
9.2.533		Alan:	There.
9.2.534		Erik:	Now put one more on, just put one more on.
9.2.535	56:36, Fig 9	Alan:	Take a yardstick and flatten the whole thing out.
9.2.536		Erik:	What do you mean, flatten it out?
9.2.537		Alan:	It's all wavy.
9.2.538		Meredith:	Yo!!! I just worked [inaudible]
9.2.539		Erik:	No, I mean, it's not ok, cuz, no offense Meredith, but isn't this called the major model we were supposed to be working on?
9.2.540		David:	That's what we're doing. That's why we came over here.
9.2.541		Alan:	Ok. Pointless. Use the purple!
9.2.542		Erik:	One two three four five six seven eight nine, ten, eleven, twelve, thirteen, fourteen fifteen, oops, sorry. I just think that the purples
9.2.543		David:	We need the purples
9.2.544		Alan:	I know, I'm giving them to you. Is that enough?
9.2.545		Erik:	One two three four five six seven eight nine ten
9.2.546		David:	This is going to be twelve. I know it.
9.2.547		Erik:	Eleven Twelve
9.2.548		David:	I know it. The purples
9.2.549		Erik:	Two three four
9.2.550		David:	Ok, let me do it.
9.2.551		Erik:	five six seven eight nine ten eleven twelve. There we go. Now we can just knock all those.
9.2.552	57:52	Meredith:	[Alan begins to straighten the model with the yardstick] No, that side's
9.2.553		Erik:	You don't really need- Wait a minute, now I just gotta do the thirds and the fourths.
9.2.554		David:	Don't touch anything now.
9.2.555		Erik:	One two three four five six
9.2.556	Fig 10	David:	[Figure F-57-57] Don't touch anything. You can just count. [David gets up and leaves view of camera for a minute and returns] alright, let's see I think the ones would be one forty-eighth
9.2.557	58:44	Erik:	Wait, four, eight twelve, just count by fours, cuz.
9.2.558		David and Erik:	Two four six eight ten twelve fourteen sixteen eighteen twenty twenty-two twenty-four twenty-six twenty-eight.
9.2.559		David:	Thirty.
9.2.560		Erik:	Two four six eight ten twelve fourteen sixteen eighteen twenty twenty-two twenty-four twenty-six twenty-eight thirty, thirty-two, thirty-four, thirty-six, thirty-eight, forty, forty-two, forty-four, forty-six, forty-eight. Yep, forty-eight.
9.2.561	59:23	T/R 1:	Are you surprised that it's forty-eight?
9.2.562		Erik:	No, not really
9.2.563		David:	No, that's what I thought it would be.

9.2.564	T/R 1:	That's what you guessed? So in other words, you were able to build what you thought, what you predicted. Are you going to be able to write this up?
9.2.565	David:	Um, well, not draw it, maybe not
9.2.566	T/R 1:	Maybe sketch it, maybe you want to take some notes on your diagram before it ends. What do you think, Meredith? You think you made another, you made a different model. Ok, you might want to take some notes to sketch it to you remember what you did. So you can start
9.2.567	Erik:	But how would we sketch it?
9.2.568	David:	Well I was surprised because I thought the greens were the purples one twelfth.
9.2.569	Erik:	So I think what I'm gonna do
9.2.570	T/R 1:	So you think the purple's one twelfth - is there another name for that purple?
9.2.571	Erik:	Um, one, one
9.2.572	T/R 1:	Meredith always like to have other names for these
9.2.573	Erik:	One twelfth
9.2.574	T/R 1:	I know, that's one name, one twelfth. Is there another number name for the purple?
9.2.575	Erik:	One fourth, no. I mean, uh, what's it called. Wait,
9.2.576	T/R 1:	If you were using-
9.2.577	Erik:	One whole!
9.2.578	T/R 1:	If, let me ask you this
9.2.579	Erik:	One whole, one half
9.2.580	T/R 1:	Don't just guess cuz you're gonna have to prove it to me, Erik. This is my question, to, to Meredith, who likes to come up with different number names and Erik sometimes says on the tape, 'I don't know why we have to have more names.' I like to have lots of names, frankly. Um,
9.2.581	David:	Um, wait a minute, um, four twelfths?
9.2.582	T/R 1:	Ok, David thinks four twelfths
9.2.583	Erik:	One twelfth! One twelfth!
9.2.584	T/R 1:	We know it's one twelfth, we agreed it's one twelfth, and you've proved it's one twelfth.
9.2.585	Erik:	Four twenty-eighths. I mean, four forty-eighths.
9.2.586 1:1:50	T/R 1:	You think four forty-eighths?
9.2.587	Erik:	Because the whites would be, the whites would be forty-eighths, and then, and then it takes
9.2.588	David:	[interjecting]-I didn't mean- four twelfths I mean four forty-eighths
9.2.589	Erik:	[continuing] Four whites to equal up
9.2.590	David:	Four twelfths.
9.2.591	Erik:	Four forty-eighths.
9.2.592	T/R 1:	You mean four forty-eighths.
9.2.593	Erik:	I said four forty-eighths.

9.2.594		T/R 1:	Meredith? You think that makes sense?
9.2.595		Erik:	Four forty-eighths or
9.2.596		Meredith:	One twelfth.
9.2.597		Erik:	One twelfth.
9.2.598		T/R 1:	So we have one twelfth, we have four forty-eighths. Any other names?
9.2.599		Erik:	Oh, wait! Oh, yeah! Two, two, two twenty-fourths!
9.2.600		T/R 1:	Two twenty-fourths.
9.2.601		Erik:	Two twenty fourths
9.2.602		T/R 1:	Ok, we have one twelfth, two twenty-fourths, four forty-eighths, anything else? How many different number names and different blocks.
9.2.603	101:27	Erik:	Well, does it have to be the same whole?
9.2.604		T/R 1:	What do you think?
9.2.605		Meredith:	It can also be bigger by, um,
9.2.606		Erik:	Two, or it can be thirds, halves, it could be a
9.2.607		T/R 1:	What are the green called? What's one green?
9.2.608		Erik:	Those are sixteenths.
9.2.609		Meredith:	One sixteenth and one forty-eighth.
9.2.610		T/R 1:	One sixteenth.
9.2.611		Meredith:	And one forty-eighth.
9.2.612		T/R 1:	How did you get sixteenths?
9.2.613		Erik:	Because there are sixteen all lined up to the answer.
9.2.614		Meredith:	One sixteenth
9.2.615		T/R 1:	Show me the sixteen.
9.2.616		Erik and Meredith:	One two three four five six seven eight nine, ten, eleven, twelve, thirteen, fourteen fifteen, sixteen.
9.2.617	1:02:09	T/R 1:	Ok, so the green is one sixteenth. But is the difference between three quarters and two thirds a green?
9.2.618		Erik:	Is the difference between
9.2.619		Meredith:	A green and one forty-eighth.
9.2.620		T/R 1:	So how would, what number name would you give for the differences between
9.2.621		Erik:	Also, the, it also could be it would take two of them to equal up to a brown.
9.2.622		T/R 1:	Well, these are the things I want you to think about and write about. Ok? I think these are good, good questions that are for you. We're up to seventh grade math already so.
9.2.623		Erik:	Seventh?
9.2.624		T/R 1:	So I think you could work it out if you worked hard enough.
9.2.625		Meredith:	Yeah, but I think if you took one sixteenth and one forty - eighth and you put it up to it, it equals
9.2.626		T/R 1:	The difference? Oh, so what number name would you give to that, then?
9.2.627	1:03:01	Meredith:	Uh, one forty eighth [laughs] I don't-
9.2.628		T/R 1:	Well, think about it. [to class] Ok. I think we have to clean up

9.2.629		Class:	Ohhh!
9.2.630		T/R 1:	I know, I'm sorry, I really am, but I hope maybe Mrs. Phillips will let you work on this tomorrow and actually finish writing up what you're doing and describing it for Monday. Is that possible, Mrs. Phillips, that maybe tomorrow they can continue this part of summarizing and write this up?
9.2.631		CT:	Sure.
9.2.632	1:03:45	T/R 1:	Oh, good work! You have to think about that! You have to think hard about it. No guessing, you have to be able to convince me, ok?
9.2.633	1:03:57		End of class.

Session 10, Oct. 8, 1993 (Front)

Line	Time	Speaker	Transcript
10.2.1	00:16	Andrew:	So, everybody started doing that and like combining, um, two like browns and saying that was fourth of the…
10.2.2	00:27	T/R 2:	Ok, so you started to make, you started to use trains [yeah] to make your number one and then you were able to work from there. What else did you find out? That's interesting. What problem were you working on? Somebody tell me what the problem was that you were working on here? Amy.
10.2.3	00:44	Amy:	Um, which was larger, two thirds or three fourths?
10.2.4	00:48	T/R 2:	Okay, two thirds or three fourths, and what did Dr. Maher want you to do with this? What did she ask you to do, yesterday? I'm trying to build this in my own mind as to what she asked you. Uh, let's see, Jessica.
10.2.5	01:01	Jessica:	Um, well, actually, we just really tried to bring more problems up than we did last time, and like some people were making big ones, and like, combining like, two browns and like making that equal one.
10.2.6	01:18	T/R 2:	Ok, so they were making like bigger models? Erik, did you want to add to that?
10.2.7	01:22	Erik:	Well, we also had to, I think she gave us, like the question that Amy said, that she gave us that question and we had to make two models to explain it.
10.2.8	01:30	T/R 2:	Mmm hmm.
10.2.9	01:33	Erik:	That's basically all she said.
10.2.10		T/R 2:	And that's basically what she said? Did she ask you to think any more about this or write about it or- Meredith?
10.2.11	01:42	Erik:	Yeah she told us to write about it.
10.2.12	01:42	Meredith:	Well, she told us to um, like, draw our models on a piece of paper.
10.2.13	01:47	T/R 2:	Ok, ok, has everybody here had an opportunity to do that?
10.2.14	01:51	Erik and other students	Yeah.
10.2.15		T/R 2:	Ok. What I was wondering is, I heard that, I heard that yesterday there were some, some people came up with some really big models. Were you able to draw those?
10.2.16	02:03	Erik:	Well, we didn't draw 'em, we couldn't draw 'em, too big.
10.2.17		Andrew:	I drew mine.
10.2.18	02:07	T/R 2:	Ahh, neat. Ok, so Andrew drew one. What he did was he taped some paper together. That's very clever. So maybe we can actually record some of those bigger ones today for people who did that, uh, what I'm wondering is, now that you've had a chance to build some models, and you came up with some different ones, from what I'm hearing, how many models do you think it's possible to build for comparing those two fractions?

10.2.19	02:33	Erik:	Comparing what, two thirds and three fourths?
10.2.20	02:35	T/R 2:	Yeah.
10.2.21		Erik:	A lot.
10.2.22		T/R 2:	A lot, you want to say more about why that's so?
10.2.23		Erik:	Well, because see, what me, Alan and I figured, is if you start with one rod, and you can divide one rod that's a large number into thirds and fourths, then you just count down by two, because we think that even numbers you can divide into fourths and thirds, but odd numbers you can't, so it was like, if we started with the orange rod, we could prob-, you could probably divide it into thirds and fourths. And then just go down two and then just keep going down until whatever number you get and then you'll just keep going down and you should be able to Faulty Conjecture
10.2.24	03:15	Alan:	We also realized that the bigger, like if you put three of these together, that if you put four, you couldn't third that unless you made a new rod using two others to be bigger than the orange. U/L
10.2.25	03:30	T/R 2:	Oh.
10.2.26	03:31	Alan:	So the big- so if the mod- the bigger the model or, this is the biggest model you can get without having not being able to third it. The bigger the model, then you can't third it.
10.2.27	03:43	T/R 2:	Oh.
10.2.28		Alan:	Like four oranges you can't third it without making a new rod. But three oranges you could call that a whole and have three more oranges as the thirds.
10.2.29	03:51	T/R 2:	Ok, what do you think about this theory that, uh, that Erik and Alan have about the even numbers? They said that they think they can divide even numbers into thirds and fourths, and uh, Erik said that
10.2.30	04:02	Erik:	Most of them, most of them, not all. Modifies conj
10.2.31		T/R 2:	Oh, most of them, ok, alright. So not all of them.
10.2.32		Erik:	I think most of them.
10.2.33	04:09	T/R 2:	Ok, that's interesting. Does anyone else have any other ideas about, about, the number of models you could build? You know some people here saying a whole lot. What do you think? Do you agree with that? Do you disagree with that? If you agree with it, why do you agree with it? Michael, do you want to say something?
10.2.34	04:30	Michael:	Um, I agree with Erik and Alan, um, we, we, um, we came up basically with the same thing, but we also found out that some of the ev- some of the even numbers didn't work modifies conjecture
10.2.35	04:41	T/R 2:	Mmm hmm.
10.2.36	04:42	Michael:	You couldn’t divide them into thirds and fourths.
10.2.37		T/R 2:	Mmm hmmm.

10.2.38		Erik:	One of the even numbers we found that you could divide into thirds and fourths is the dark green rod. faulty
10.2.39	04:50	T/R 2:	Oh, that one works, so that's an even number one that works.
10.2.40		Erik:	Mmm hmmm.
10.2.41		T/R 2:	What do you mean by an even number rod?
10.2.42		Erik:	Well, a rod that if you put all of the whites up to it.
10.2.43		T/R 2:	Mmm hmmm
10.2.44		Erik:	All the whites real tight, and you determine if you can divide it in half.
10.2.45		T/R 2:	Ok, ok, David, did you want to add, you want to add to that? Or you want to comment on that?
10.2.46	05:08	David:	I want to comment that an even rod, is, before, when I got up there, maybe about like a week ago, um, I said that like the white would be one, the reds would be two, so the reds are even, and then the light greens are three,
10.2.47	05:24	T/R 2:	I see..
10.2.48	05:25	David:	They're odd, and then the purple is even.
10.2.49	05:28	T/R 2:	Because of the number of whites you could put alongside of it to show? Is that why you're saying that, in other words? Why you're giving it a name two because you can use two whites to show it? Ok that's interesting. Well, what I'd like to do is, is there, first of all, is there anyone here who had not had an opportunity to record their models [hands raised]. Ok, there are some people who haven't. I'd like to give those people an opportunity to record, I think it was two or three models? Ok, if you have done that, I want to give you something else to try, ok? And if we have time then the other people who finish recording can try it as well, but what I'd like to do is, if your models were big
10.2.50	06:07	Erik:	Yeah
10.2.51	06:08	T/R 2:	We've got tape and we've got paper, we've got blank white paper up here, we've got notebook paper, um, you could, you could tape some sheets together like Andrew's done. I think that's a really neat idea. Ok? I want to come over and talk to Andrew about that. Um, for those of you who have, for those of you who need to do this now, who need to work on your models, please raise your hand, and um, you've got rods, and if you need markers, I have markers here, I'll come around with, and paper. Ok? The rest of you, I want to have you work on something else. [students gather to receive materials]. Ok, you need paper? Ok, is there anybody in here who has had a chance to record all their models and would like to begin thinking about something else? Ok, I'll be around. [camera focuses on David, Meredith, and Erik on the floor, then moves to James, Jacqueline, and Amy. T/R 2 is overheard posing the new problem to a student in the

			background]. Which is bigger, one half or two fifths, and by how much? Decide which is bigger, and by how much, ok? And please, when you get a model, call me over.
10.2.52	11:32	Amy:	[James builds a twenty-four cm model using orange, blue, and yellow] What are we doing? Which one?
10.2.53	11:34	James:	Any one you want. After we do... after we both after we do both of these models we'll do another problem, k?
10.2.54		Jacquelyn:	What's next?
10.2.55	12:26	James:	I don't know.
10.2.56	12:32	Amy:	An orange and a purple, two oranges and a purple.
10.2.57	13:59	Alan:	[camera moves to Alan, who has built a model of an orange, five red rods, two yellow rods, and ten white rods. Figure F-12-47] One that has a lot of reds. [gets another box of rods, works, T/R 2 joins.] I'm trying to work on my third model.
10.2.58	16:54	T/R 2:	You already got two!
10.2.59		Alan:	[Discussing model with one orange rod, five red rods, two yellow rods, and ten whites]the one is the orange, there are the fifths, there are the halves, and there are the tenths. Ok, so, um, if you took out two of those [two red rods] that would be two fifths, and that would be a half [a yellow]. And one of those would fill in the gap [white rod], so it would be one tenth, so it would be a half is bigger than two fifths by one tenth [Figure F-17-22].
10.2.60	17:21	T/R 2:	Neat, ok.
10.2.61		Alan:	And then down here it's basically the same size as the orange, I just made a train of a brown and a red, [inaudible, Alan's second model is a brown and red train, two yellow rods, and five red - Figure F-17-36]
10.2.62	17:29	T/R 2:	Ok, do you think now that you can, I mean you're working on this one I can see, do you think you can come up with one that's a different total length than, one's that's different from these two, these two have the same length?
10.2.63	17:44	Alan:	I'm working on that one down here.
10.2.64		T/R 2:	You're working on that one, it looks like you're working on a bigger model here.
10.2.65		Alan:	Mmm hmm. [Alan's third model is two blue rods, four purple rods, but he dismantles it and builds one with two trains of two orange rods, five purple rods, and ten red rods]
10.2.66	19:00	T/R 2:	Does that one work?
10.2.67		Alan:	Yup.
10.2.68		T/R 2:	A working model here? Tell me about that one.
10.2.69		Alan:	[Figure F-19-01] Ok, the two oranges make the whole.
10.2.70	19:07	T/R 2:	So we're calling this, the two oranges together one. Ok.
10.2.71		Alan:	And these, the five purples are the fifths, and the two oranges again are just the halves, now down here, the reds are the tenths. And again if you remove that [two purples and an

			orange] it would take one of those [red] to fill in the gap, so it's bigger by, one half is bigger than two fifths by one tenth.
10.2.72	19:35	T/R 2:	Can I ask you a question now? Why did you choose the two oranges to be one? You seemed to come up with that pretty quickly.
10.2.73	19:43	Alan:	Because up here, I knew that this was ten, and two tens would be twenty, and I knew that that would work, so it takes two of those to complete it using a double ten. So one of those [red rods] filled in the gap. Probably if you used another one [takes a third orange and gestures to show that a fourth orange rod would be placed along with the first three] another two, you could fill in that with more purples and using more reds, too.
10.2.74	20:07	T/R 2:	Interesting!
10.2.75		Alan:	And it could make more.
10.2.76	20:09	T/R 2:	Ok, so you did ten, you called it ten and twenty because of the little white ones. That's an interesting theory, could you kind of test that one out for me and, see, see if you could build a bigger model?
10.2.77	20:22	Alan:	I'm trying to build a bigger model.
10.2.78		T/R 2:	If you need more stuff, I've got more rods
10.2.79	20:27	Alan:	I can use from these models [dismantles first model]
10.2.80		T/R 2:	Ok. I've also got more rods up there. Ok, I'm going to come back, [camera moves to the floor]
10.2.81	20:47	David:	You my best friend
10.2.82		Erik:	Before he com- oh he's here.
10.2.83	21:02	V1:	Erik, can you tell me about this?
10.2.84		Erik:	Well,
10.2.85		David:	We were working on this thing, cuz before she made um one that had one orange one blue and one black
10.2.86		Meredith:	We told you about it yesterday!
10.2.87	21:12	V1:	Well, we're gonna make sure we got it down.
10.2.88		David:	And then I thought that maybe if we double it. Well, first, first we had the reds and they were one twelfth
10.2.89	21:24	Erik:	Purples
10.2.90		David:	[Figure F-21-35] No, we had the reds were the one twelfths, and we had the whites there were one twenty-fourths. So then, um, so then I thought if we doubled it, then the purples would be uh, one twelfth, and then, then I thought the greens might be kind of half of it, so, well, maybe not half but the greens were one seventeenth.
10.2.91	21:53	Erik:	Sixteenth.
10.2.92		David:	Oh, sixteenths, then, um, so then the reds are one twenty-fourths and the whites are one forty-eighths. And, you can't really make anything like halves, or like one-thirds.

10.2.93	22:12	Meredith:	You would need a new model, maybe. If you put ten up to it, it won't do it.
10.2.94		V1:	Ok, so, now hold on a minute.
10.2.95		Erik:	Well, we did this before, we did this a couple of days ago, and sixteenths fit. But it's a little off.
10.2.96	22:26	V1:	You said those greens are what?
10.2.97		Erik:	They're sixteenths.
10.2.98	22:30	V1:	But I'm not convinced about that end part.
10.2.99		Erik:	Yeah, I know, we had it yesterday.
10.2.100		David:	I'm not sure if they're lined up right though.
10.2.101	22:52	Erik:	Couldn't we do the browns? Could we do the browns as opposed to blacks?
10.2.102		V1:	I don't know this looks a little different than yesterday.
10.2.103		Erik:	Can we do the browns as opposed to the blacks? We can have the dark greens
10.2.104		Meredith:	We were going to do the red.
10.2.105	23:21	T/R 2:	How's this big model coming?
10.2.106		Erik:	Not too good
10.2.107		David:	Not too good.
10.2.108		Erik:	We had it, we had it better yesterday.
10.2.109		T/R 2:	What happened?
10.2.110		Erik:	It was fine yesterday but now it doesn't work.
10.2.111		Meredith:	Oh I see what's wrong!
10.2.112	23:31	T/R 2:	What do you think's messing things up?
10.2.113		Meredith:	[Figure F-23-30] It needs a one.
10.2.114		Erik:	But can't we, can't we, can't we trade in one of the blacks for a brown?
10.2.115	23:49	David:	But then that wouldn't fit.
10.2.116		Erik:	Yeah it would.
10.2.117	23:51	David:	It would mess everything up though, Erik. The purples wouldn't fit, the greens wouldn't fit, the whites would fit, but maybe not the reds.
10.2.118	24:02	Meredith:	No, if we trade it for a... no let's trade it for a blue [Figure F-24-36].
10.2.119	24:08	T/R 2:	Oh, I see, you're calling, I see, you're calling one of those, that top train, with the oranges and the blues and the blacks?
10.2.120	24:16	Meredith:	Because then if you put another green here.
10.2.121	24:19	Erik:	Oh, yeah! But,
10.2.122		Erika and David:	What about the purple?
10.2.123		Meredith:	Just take the purples out, you don't need the purple.
10.2.124		David:	Well, then that's going to mess everything up, Meredith.
10.2.125		Erik:	Then what will be the twelfths? No yeah, then what would be the twelfths?
10.2.126		Meredith:	We don't need the twelfths!
10.2.127		Erik:	Yeah we do.
10.2.128		David:	Because that's the whole thing.

10.2.129	Erik:	That's the whole question. That's the whole answer. It's either three twenty-
10.2.130	Meredith:	Well, where's the two thirds?
10.2.131 24:40	Erik:	Well, we don't really know.
10.2.132	Meredith:	[laughing] But the question is which is bigger, two thirds or three fourths.
10.2.133	David:	Well, Erik, um, remember, fourths, if green was one twelfth then that would be it, but like I said before that I thought that well we don't really need the greens.
10.2.134 24:58	Erik:	Wait wait wait wait wait. This isn't the model we did before. The model we did before I believe was three oranges and like something else
10.2.135 25:05	David:	No it wasn't, cause I remember your original model was an orange a blue and a black, and then I thought if we doubled it.
10.2.136 25:17	Erik:	What if we did just, an, two oranges, two blues, one black and one blue. That one's not totally messing it up.
10.2.137 25:30	Meredith:	Except the purples
10.2.138	Erik:	Purple we could figure out-
10.2.139 25:36	Meredith:	Wait! Wait. I've an idea. Take away this, put on this [an orange instead of a black - Figure F-25-49]
10.2.140	Erik:	Oh no
10.2.141 25:43	Meredith:	And then put a one there. Then you could put one here, it would fit better.
10.2.142	Erik:	Then put a red [some inaudible conversation]
10.2.143 26:23	David:	Do you really need the green?
10.2.144 26:25	Erik:	No, not really.
10.2.145	David:	So should we just take it out?
10.2.146	Erik:	Yeah, cuz I mean it's giving us too big of a problem, and we don't need it. I don't know why we put it on.
10.2.147 26:35	David:	I just did that because I thought the green ones were the twelfths.
10.2.148	Erik:	[Figure F-27-48] Yeah, I know. This is a- oh let's measure it! It is approximately, fifty-three. No, it's fifty-two. No it's fifty-two and a half.
10.2.149 27:46	Meredith:	No it isn't. Watch. It needs to be equal.
10.2.150	David:	Erik, it starts like that.
10.2.151 28:05	Erik:	No it doesn't start at one, it starts at zero. [take away meter sticks, mess it up, fix it] Yes. [start putting reds on model]
10.2.152 29:48	Meredith:	Another one, another one, another one, another one.
10.2.153 30:16	T/R 2:	Ok, Alan, now you tried to make it with four orange. Tell me about this model and tell me just what you told me before.
10.2.154 30:26	Alan:	[Figure F-30-58] Ok, originally I had two oranges, and that was, that would only use twenty of the whites, but if you added another two of them on it would be forty of the whites. So the whites down here are the fortieths. And the purples

would take five to use for the two, and another five over here, so that would be the tenths. And now, these, if you put two oranges together, the two oranges each would be the halves. These would be the twentieths [reds], and the browns would be the fifths. Now there should be, I think nineteen more on here to complete the fortieths. You can't make the model any bigger than this, You would have to use one blue. It wouldn't be the exact size. [places five blue rods - Figure F-31-18]]. So you can't make a model any bigger than this, without making a train, making all these uneven. So basically, this is the only model you can make that's even without using trains, like this one here, that would make all of these unequal.

10.2.155 31:41	T/R 2:	So, if I wanted to continue my train with oranges, you're saying, I would have trouble showing
10.2.156	Alan:	Another four, no, another four oranges to fit five more of the browns on, so it would be a yard long probably.
10.2.157 31:57	T/R 2:	Oh my goodness, can you imagine the size of that? If you wan- if you wanted to make a train, though, where you were adding a different color rod on the end of this train of four oranges, do you think you could come up with other models?
10.2.158 32:09	Alan:	Well, it could be, but this is the, basically the only equal model using, you know, tenths twentieths, fortieths. ... for the whole.
10.2.159 32:21	T/R 2:	That's interesting, that's really interesting Ok, could you, I almost hate to ask this but could you, we have a couple of minutes left, could you try to trace this so we don't lose what you did here? Uh, or maybe you can draw a sketch of it, okay? And just label a sketch of it. That would be easier for you, let me get you some paper. [leaves to get paper] Alan, you know what would be really, you know what would be helpful to me, I want to make sure that you get the information down [inaudible]. What you used, in other words, oranges to make the ones, and purples to make the tenths, can you write that?
10.2.160 33:20	Erik:	[camera moves. Erik, David and Meredith a building a really long model.] We can use every single rod possible. You could take off the purples because we don’t need them. We’re making this gigantic-
10.2.161 33:47	V1:	Ok, but, uh, Erik, is this going to help you solve the problem?
10.2.162 33:52	Erik:	Yeah.
10.2.163	V1:	How?
10.2.164	Erik:	We're making it big.
10.2.165	V1:	How do you think this will help you solve the problem? Are you going to let that equal your whole?
10.2.166	Erik:	Uh huh.

10.2.167 34:08	V1:	That's going to be a pretty big whole.
10.2.168	Erik:	They have a big whole down there.
10.2.169	V1:	Do you think this will help you solve the problem that we want you to solve?
10.2.170	David:	I don't think it will.
10.2.171 34:18	V1:	David doesn't think so.
10.2.172	Erik:	Neither do I, really.
10.2.173	Meredith:	But we just want to do it anyway.
10.2.174 34:23	V1:	Put it this way, you know you probably could but it's gonna take you a while and we're running out of time. So can you show me what you got with the model that you had on here before? [Erik destroys the long model]. Cause we're gonna, we're gonna have to clean up in a couple of minutes, ok? So
10.2.175 35:05	Erik:	Uh oh
10.2.176 35:16	V1:	So what do you have here?
10.2.177	Erik:	Well,
10.2.178	T/R 2:	So how's it coming over here?
10.2.179 35:43	V1:	Well, let's see.
10.2.180	Erik:	[Figure F-36-17] Well, we have, as the whole we have two oranges, two blues and two blacks, because David said that Meredith made an original model that was one orange, one blue and when black, and then-
10.2.181 35:57	David:	[joins in] One orange, One blue, and one black, and then, well, she had um, the reds were one twelfth and then the whites were one twenty-fourth, and then
10.2.182	Erik:	We did, we doubled
10.2.183	David:	Put that down, we don't need that, alright?
10.2.184	Erik:	We doubled two oranges two greens and two blacks
10.2.185 36:22	David:	Instead of one orange one blue and one black.
10.2.186	Erik:	The purples would be the twelfths, the reds would be the um twenty-fourths
10.2.187 36:31	V1:	I'm not convinced about this, okay?
10.2.188	T/R 2:	Yeah, wait a minute.
10.2.189 36:36	V1:	Wait, you say how much are the purples?
10.2.190	David:	Alright, here wait a minute. Alright, Meredith made this model with one orange, one blue and one black,
10.2.191	Meredith:	Yeah, and it had thirds and it had fourths.
10.2.192 36:47	David:	And then the, so then reds-
10.2.193	V1:	So how much are the purples?
10.2.194 36:54	Students:	The purples are one twelfth.
10.2.195 36:59	V1:	I'm not sure, I don't see that.
10.2.196 37:01	Erik:	Well, see, one two three four five six seven eight nine ten eleven twelve, thirteen! We made a mistake.
10.2.197	V1:	There's thirteen of them. What does that tell you?
10.2.198 37:18	Erik:	They're one thirteenths.
10.2.199	V1:	Thirteenths.

10.2.200	David:	Because you see yesterday
10.2.201	Erik:	Yesterday this whole thing came out perfect.
10.2.202	David:	Yesterday we had one twelfth, the greens were one sixteenth, and now they're one seventeenth.
10.2.203 37:31	Meredith:	Hah ha ha ha ha you were wrong.
10.2.204	V1:	Well, what, how could that be?
10.2.205 37:50	Erik:	Why do we need oranges on top?
10.2.206	CT:	...Kindly if you have work with your name on it that you want to share with Dr. Martino, give it to her please.
10.2.207 38:21	V1:	I don't know. What do you, what do you guys think what do you think happened? Because, you know, I see thirteen things here.
10.2.208 38:29	Meredith:	I don't think they know how to count.
10.2.209	Erik:	I think Meredith sabotaged it. [inaudible, laughter]
10.2.210	David:	Well, I think I think, yesterday, maybe it was three blues
10.2.211 38:38	Erik:	No it was smaller.
10.2.212	V1:	It looks pretty, well, let's get it - this is the model you guys just had, right?
10.2.213	Meredith:	No, we had one that was straighter.
10.2.214 38:54	V1:	Ok, well, let's even out the ends. Okay? Now that looks pretty straight to me. Okay, now, these are all even, but I see, yeah there's thirteen, aren't there?
10.2.215	Meredith:	We don't need twelfths.
10.2.216	Erik:	That's the whole point!
10.2.217 39:23	Meredith:	What's the point of twelfths? The point is two thirds and three fourths.
10.2.218	Erik:	The answer is one twelfth.
10.2.219 39:29	David:	Meredith, I made this thing to show that when you double it. To show that when you double it. The reds were one twelfths, now the reds aren't one twelfths, now the reds are, uh,
10.2.220 39:44	Erik:	So you're trying to show that with different models, thirds, that they're twelfths, that if the numbers will change, no they're changing size but they don't change in answer!
10.2.221 40:04	V1:	Ok, guys, you gotta start putting the stuff away. I'm afraid we need a little bit more work on that model. End of class

Session 11, Oct. 11, 1993 (Front, Side, and OHP)

Line	Time	Speaker	Transcript
11.0.1	3:24	T/R 1:	Well, good morning! I surprised you, I came back! Yeah! I just couldn't stay away. I heard such really wonderful things happened on Friday and I watched a tape on Thursday, so I had to rearrange my schedule so I could be here, and I'm just so happy to be here. Um, I was watching the tapes, you know I do that, and I was reading your papers, and I did that, and I was also talking to some of the people who were here when I wasn't here, and trying to figure out some of the things you were doing. And I understand that, uh, let's see, I think it was on Thursday, that you were working on all sorts of problems, everybody was working on something, I know this group here with Amy James and Jackie, they discovered a secret, they told me, do you remember? Yes, and then I know that there was a group here, I think that was Alan, Erik, David, and Meredith, who were testing a theory? Is that right, you were testing a theory? So we have a secret that we want to hear about, we have a theory that was being tested, and then I know that, um, Andrew's not here?
11.0.2		CT:	Yes he is but [some excuse]
11.0.3	3:45 (S)	T/R 1:	Ok, but he and Jessica had also built a model, right, that was rather interesting for one of the problems, that they were sharing with Brian and Michael. Remember that, and also with Erik was over there, I noticed talking about their model and some other people came over. And um, there were all kinds of interesting things happening, uh, and I read some of the papers of the different kinds of thinking you were doing, and um, Sarah had built a new model that I hadn't seen for one of her problems and Kimberly and Audra and Erin and Jackie, and oh, just such exciting things, I mean, how can we get to all of this? And, um, back there, Graham and Kelly were building their models as well. So I thought that, you know, maybe what we'll do is we'll start with some of the things that confused me and maybe you can help straighten me out. Um, I know that one of the things that happened was when the group built a model on the floor, I looked at that tape, um, on Thursday, and the group built a model to test David's theory which I'm sure David and his friends will share with you, and then the next day, when I think it was David, Erik, and Meredith trying to build it again they had some trouble. Uh, and I wondered if they figured it out, I didn't finish watching that tape, but I watched the one the day before. Some of you are already working on different problems by now, now we have Andrew back he can tell us

his theory, so I thought you know, maybe we would start with that and then, you know, try to have you all contribute and share and, wouldn't it be nice to know what other people are doing now? Aren't you interested in what other groups are doing and the way they're thinking about some of these problems? I get kind of curious, you know, once I've worked on something after a while, I wonder I say, I wonder how other people are thinking about this, I wonder what they're doing, what do you think, Alan? Do you get sick of it after a while or do you want to be curious about it. You know, it depends, I suppose. Um, do you remember the problem that, uh, I think everybody in this class has now spent a bit of time working on the problem, which is larger, three quarters or two thirds, and, if you decide which one is larger you were asked, by how much. Do you remember, how- Everybody here has worked on it, isn't that true? Raise your hand if you've worked on that problem? I believe everybody, yeah. Some of you started a new one, but everybody has worked on that. Ok, the question, how many of you have built two models for that problem? For that problem, which is larger, three quarters or two thirds, how many of you built two models? [Some students raise hands] Ok, how many of you have built more than two models? [Three raised hands are seen]. So we have some, you know, really differences here and some time to share, and Gregory you built more than two also and Danielle, you've got two models. Um, how many you think there are? How many models do you think you can build? Michael.

11.0.4	6:52	Michael:	Um, if you know what you're doing and you know what strategy, you could probably build, you could probably build one for every single rod.
11.0.5		T/R 1:	What do you mean, one for every single rod, tell me what you mean by that.
11.0.6	7:14	Michael:	You could, you could build a thing, you could build fractions of every single rod if you know what you're doing and you have a strategy or a secret that, that you know will work.
11.0.7		T/R 1:	What can be such a secret?
11.0.8		Michael:	That's what I was trying to figure out.
11.0.9		T/R 1:	That's what you were trying to figure out? Does anybody else have any ideas about that? Those of you who built three models, do you think that's all? Can you build more? This table here, Jackie and James, Amy, you think there can be more? [mmm hmm] How many do you think?
11.0.10		Amy:	Well, we got six
11.0.11		T/R 1:	You think that's it?

11.0.12 Jessica, Amy, James: [Shake heads] No.

11.0.13 T/R 1: No? Anybody else? Any thoughts? [Meredith raises her hand] Meredith? What do you think?

11.0.14 8:00 Meredith: Well, maybe if you, say you had a white rod, and you divided the white rod, maybe you could make more models that way, if you divided the white rods.

11.0.15 T/R 1: So you're thinking if you had more, different size rods?

11.0.16 Meredith: Yeah, yeah.

11.0.17 T/R 1: Like if you took the white rod and designed one smaller than the white rod? [mmm hmm] Ok. Anybody else, what do you think? There was a theory, I know David's looking at what we built here, David you had a theory that you were testing about uh building some models. There were some models that everyone here built, maybe we should put two of them on the overhead, if you can help me do that, to compare which is bigger three quarters or two thirds. Um, to decide with the rods you have, what was the smallest model you can build? With the rods you had available, not being able to cut them or make them different. Did you find- is there a smallest model, when you compared three quarters or two thirds? [pause] Can you remember? What, what do you think, Mark?

11.0.18 Mark: um, I think, (Pause) I'm not that sure.

11.0.19 T/R 1: You're not sure, you want to think a little bit more? Sarah, what do you think? I see Sarah and, uh Beth? What do you think?

11.0.20 9:42 Sarah: Um, there's a smaller one.

11.0.21 T/R 1: What do you think it is? You want to come up and build it really quickly? [Sarah and Beth get up] Maybe while they're, they're building the small one, you can think of the next one. I don't think all will fit on the overhead, but at least we'll get some of these models up, we might have something to talk about. [Sarah and Beth build a model of a light green and white train, two red rods, and four white rods - Figure S-10-49.] Ok, what do you think?

11.0.22 Erik: I think it's, they're right, but one green and uh, and a one I think equals one purple, because if you would put that to that it would just equal one purple and put the

11.0.23 T/R 1: Ok, tell me what you're showing up here, uh, Sarah and Beth. You can put a purple up there, too. Erik is suggesting above it.

11.0.24 Sarah: Yeah [She places a purple rod above the model - Figure S-11-27]

11.0.25 T/R 1: Ok, but tell us what you're showing. We're trying to show three quarters and two thirds, is that right? [mmm hmm] Uh,

		can you show us how that shows three quarters and two thirds? [Sarah and Beth whisper to each other]
11.0.26	Beth:	Oh, we don't have, [to T/R 1] We don't have thirds in there.
11.0.27	T/R 1:	oh, ok.
11.0.28	Beth:	We only have the half and the whole.
11.0.29	T/R 1:	Ok, so you have three quarters and a half. Ok. Can you build one for three quarters and two thirds? Does someone have any other suggestions? Alan? Did your hand go up or did you just wave it? Anybody else remember how you did that? Kelly, you think you have something, you want to come and help? Graham, you can come and help.
11.0.30	Kelly:	I don't have fourths in my one that I made
11.0.31	T/R 1:	You don't have fourths? How about here? Amy? James? Jackie? [Amy James and Jackie come to OHP] Oh we're starting to get think- we're starting thinking now. [Amy, James, and Jackie build the model using orange and red train as one, purple as one third, and light green as one quarter.] Ok, you want to tell the class about what you have there?
11.0.32	James:	[Figure O-12-55] Ok, um, well um, we had this model and we uh this is two thirds and three fourths and we think that three fourths is bigger by one twelfth. [He separates two purple rods and three light green rods from the original model to show the comparison].
11.0.33	T/R 1:	Questions, anybody? What's one twelfth?
11.0.34	James:	Because, um, twelve whites equal up to this. Let me show, here. [He puts two white rods on his model]
11.0.35	T/R 1:	Are you all convinced?
11.0.36	Jessica:	But didn't you say to make a smaller one?
11.0.37	T/R 1:	Well I said you make the smallest one you can make?
11.0.38	Jessica:	But I don't think that's the smallest one.
11.0.39	T/R 1:	Can you make a smaller one? It's an interesting question. If you don't think there's a smaller one you should be able to show it, or at least-. [James and his partners continue placing the twelve white rods on the OHP]. What do you think?
11.0.40	James:	[Figure O-15-38] Well, we just put twelve whites on there and it takes one white to equal the two pinks, to the three, oh yeah, purple to the three greens. So that's why we think it's one twelfth.
11.0.41	T/R 1:	How many of you agree with that? How many of you agree with that model? How many of you found that same model when you worked it out? Raise your hand if you found the same model. [All students visible raise hands] Ok, it looks as if everyone did. I have a question that Jessica is raising, I'm listening to Jessica next to me. Jessica says she thinks there's a model that you can build that's smaller, now, when I use the

			word smaller, you use the word smaller, what do you think we mean? What do you think, Erik?
11.0.42		Erik:	Smaller in size-wise? Like size for the thirds, the fourths, and, and the whole. Smaller by size.
11.0.43		T/R 1:	Ok, so what what we called one in this problem was what, what did we call one? Brian?
11.0.44		Brian:	Well, um, the orange and the red.
11.0.45		T/R 1:	Yeah that train we called one, right? And I guess the question is, that train has a particular length, right? You can see the length of that train? Is it possible to build a model to show the comparison of three quarters and two thirds with a train that has lengths smaller than that, now if you think it isn't, you've gotta convince me with some argument.(Oh I know) Oh, Erik thinks he has an argument to convince me there's not a smaller one because Jessica doesn't believe it. Right, Jessica? Jessica seems to think there's one that can be made with a train that has length shorter than the one up there. So if you think you have an argument, Erik thinks he does, raise your hand. I want you to think about this. You might want to talk to your partner about your argument and see if your partner buys it. You don’t have to talk to your partner. Ok, we'll let you guys go first and Erik is listening to see if he agrees. Ok.
11.0.46	17:41	Amy:	We say that there was no more, that you can't get a smaller one because every one you use equals up to an orange and a red, and the secret is that every one has three purples and four greens. And so you can't possibly make one smaller because you won't be able to fit, it won't work because every one you make equals up, equals up to the orange and red.
11.0.47		T/R 1:	So you're telling me the six models that you made
11.0.48		Amy:	Were the same length.
11.0.49		T/R 1:	Were all the same length. In all of your cases, the, the, what you called one had the same length as the orange and red. That's very interesting.
11.0.50		James:	Well, well, we could make another model to show that.
11.0.51		T/R 1:	We believe you. Does anybody need to have that shown, what they would've done? How many of you did that too? You found different ways to show one, that had the same length as orange and red,
11.0.52		Beth:	Yeah, Sarah just built another model that uses the exact same length as the other one.
11.0.53		T/R 1:	And what did, what was, how did she make her train?
11.0.54		Jessica:	Yeah, that's what I was going to do.
11.0.55		Beth:	Blue, light green, and then the half is dark green, and the third is purple and the fourths were the dark greens- were the light greens.

11.0.56		Jessica:	See, that's what I did, I was gonna make one that was the same exact size.
11.0.57		T/R 1:	Ok, so many of the models you made were really models where your, what you called one, that train, really had the same length as the orange and red. Is that all you can make? How do you know that, that, that there's not one smaller? Erik? I'm still not convinced that there's not one smaller. They didn't convince me. How would you convince me?
11.0.58	19:22	Erik:	Well, see, I agree that, that, I agree with them just at the part that there's no, there's no other smaller. I think, because at their model, they use the twelfth as the white ones, and there's no rod smaller than the white rod. So, therefore, if you make it a rod smaller than it, they can't, you can't divide it into twelfths.
11.0.59		T/R 1:	Ok. Did you hear what he said? Yeah!
11.0.60		Erik:	Because the twelfths right here are the smallest rod possible.
11.0.61		T/R 1:	Ok, so
11.0.62		Erik:	Unless you made a new rod.
11.0.63		T/R 1:	So unless we use Meredith's idea of creating new rods that had, that were smaller than the white rods, then you could make a smaller model, Erik?
11.0.64		Erik:	Yeah.
11.0.65		T/R 1:	The rest of you agree with that? So, so then, ok, I'll buy that, how many of you buy that argument? That seems reasonable. So you've made the smallest one already. Jessica, is that reasonable to you?
11.0.66		Jessica:	Yeah.
11.0.67		T/R 1:	Ok, my next question is, can you make one that shows the comparison of three quarters and two thirds, that's bigger than this? Are there others?
11.0.68		Students:	Yeah.
11.0.69		Michael:	I know, I just did one.
11.0.70		T/R 1:	Ok, um, you think there are others, ok. You have another one?
11.0.71		Michael:	Yeah.
11.0.72		T/R 1:	I would like one, that uh, is the next smallest.
11.0.73		Michael:	Next sma-
11.0.74		T/R 1:	Can you predict something about the one that would be next smallest? I mean next largest, I'm sorry. The one that's the next largest. Brian, what's your prediction?
11.0.75		Brian:	I think it would be twenty-four.
11.0.76		T/R 1:	You think what would be twenty-fourths? What rods?
11.0.77		Brian:	Well, the next, the next, the next larger one will be, I think the whole will be twenty-four.
11.0.78		T/R 1:	But we, we call the whole one.
11.0.79		Brian:	Yeah, I know, but what I mean

11.0.80	21:01	Michael:	No, no, it would take twenty four ones to equal a whole.
11.0.81		T/R 1:	What would be, what would be twenty-fourths?
11.0.82		Brian:	Like the, there would be, there would take twenty-four white cubes to equal up to a whole
11.0.83		Student:	I, I also have a strategy
11.0.84		T/R 1:	Wait, wait a second, you're saying twenty-four white ones would equal your train.
11.0.85		Brian:	Yeah, yeah.
11.0.86		T/R 1:	That you're going to call one. So then what would one white one be called in that next model do you think?
11.0.87		Michael:	One twelfth
11.0.88		Brian:	I think, um
11.0.89		T/R 1:	What would that white one be called?
11.0.90		Michael:	Well it's not gonna, we're not gonna. Let me see this [Michael begins to build the model]
11.0.91		Brian:	Um, one twenty-fourth I think.
11.0.92		T/R 1:	Brian thinks then white ones in the next train would be one twenty-fourth. You think that too?
11.0.93		Erik:	Yeah, Alan and I made that same model. We made the same model that was, I think it was two oranges and like one purple, yeah it was two oranges and one purple and then it had the thirds
11.0.94		T/R 1:	Ok, why don't the rest of you sit down and let's have Erik and Alan make that model. Did you make it too, Michael? Is that what you had?
11.0.95		David:	I made that also
11.0.96		T/R 1:	Ok, you watch what they're doing, and seeing if -leave that other one up there ohhh! Ok, can you leave the other one up there maybe while you're making that?
11.0.97		Erik:	Sure we can.
11.0.98		T/R 1:	Keep the other one up there. Erik: Just move this over.
11.0.99		T/R 1:	James, why don't you make the other one too, so it doesn't go away on the bottom.
11.0.100	22:14	Brian:	Kaitlin, can I borrow some oranges. [whispering, inaudible] 2 more, no [starts to count something in his model] 6 more
11.0.101	23:20 (F)	T/R 1:	The rest of you could be making these if you haven't already, in your seats just so you have them in front of you. I would suggest you to make both models in your seats and keep them in front of you.
11.0.102			*SIDE*
11.0.103		Brian:	Oh, you need a lot. I need some, I need some oranges to make this big one that I made last time.
11.0.104		Michael:	I know.
11.0.105		Brian:	I need a lot of them. Nobody can give me any!
11.0.106		Michael:	[laughs and gets up to look for rods]
11.0.107		Erik:	Ok, I got it, I got it.

11.0.108	Alan:	I'll get the ones up.
11.0.109	Erik:	You want me to help you because there's a lot of them?
11.0.110	Alan:	Yeah [laughs]
11.0.111	Erik:	A lot of those little.
11.0.112	Alan:	Hold it, let's just straighten this out.
11.0.113	Erik:	[whispers, inaudible]
11.0.114	Alan:	Oh yeah, you do, you're right. All Mondays.
11.0.115	Erik:	We don't need dark greens.
11.0.116	Alan:	I know.
11.0.117	Erik:	The whites were the twenty- what were the twelfths? Reds.
11.0.118 23:43	Alan:	Twelfths, You do the twelfths - Sadly to say I don't think there are enough ones.
11.0.119	Erik:	Oh I think we will.
11.0.120	Alan:	We might. Don't give up hope now old chum.
11.0.121	Erik:	[counting] Oh I found another white one.
11.0.122	Alan:	Never mind, we got to the point where I think we have too many.
11.0.123	Erik:	Four five six seven eight nine.
11.0.124	Alan:	Oh here.
11.0.125	Erik:	Ten, I need, ok, perfect, I got twelve, I'll put them up here.
11.0.126	T/R 1:	You can use this as a ruler if you'd like.
11.0.127	Alan:	Wow.
11.0.128	T/R 1:	Here, this might work better.
11.0.129	Alan:	those are big.
11.0.130	Erik:	I wonder if they work on this.
11.0.131	T/R 1:	No, just as a ruler, they won't work on it.
11.0.132	Erik:	Did they just turn black?
11.0.133	Alan:	Yeah, you just push on that side and I'll push on this one.
11.0.134	Erik:	No these look like they're coming out, they're like, they you can't see them.
11.0.135	Alan:	Ok, so we do - Erik, move your arm. We need one more, two more ones down at the edge.
11.0.136	Erik:	Uh, let's see.
11.0.137	T/R 1:	You know what the problem is, some of this you can't see so let's just do it this way.
11.0.138	Erik:	I don't if we - do we have enough reds?
11.0.139	Alan:	Yeah, no, we need one more.
11.0.140	Erik:	One more?
11.0.141	T/R 1:	Ok. That's beautiful.
11.0.142	Erik:	Let's, there we go. Ok.
11.0.143	Alan:	Ok. Erik, you talk.
11.0.144	Erik:	Just, I don't think we need this. [laughs]
11.0.145	Alan:	What?
11.0.146	Erik:	I don't believe we need this right here.
11.0.147	T/R 1:	Ok, leave that there, in fact if I had more whites I would even. But what you can do is show it here, right? Two thirds.

		Right by the twelfths up here. That's not bad. Let's talk about this.
11.0.148		*FRONT*
11.0.149 25:50(F)	Amy:	Wait, no, twelve plus two, wait twelve plus twelve, twenty-four, twenty-four here. You've got twenty-four plus twenty-four is forty-eight.
11.0.150	James:	I need those. I need another red. I need thirty-six whites.
11.0.151	Amy:	Thirty-six? Why?
11.0.152	Jacquelyn:	Why? Why are you saying thirty-six?
11.0.153	Amy:	I need the reds, I'm trying to prove a point here.
11.0.154	Jacquelyn:	You can have all the reds you want. I'm going to sit here and watch you guys.
11.0.155	James:	We're in battle
11.0.156		*BOTH VIEWS*
11.0.157	T/R 1:	[Erik and Alan's model - Figure O-24-48. to class] Ok. Now I know, I know you're building these models and I know you don't have enough rods so I know that you have to uh, share some of your uh rods and sometimes you can only build one model on a desk, and I know some of you are able to imagine the models now too. How many of you could imagine what it looks like, even though you haven't quite built it? Raise your hand if you could imagine what it looks like. [A few children raise their hands] I'm kind of curious, what do you imagine that you don't have there, Jessica? I see that you built a model that has two oranges and a purple that you're calling one.
11.0.158	Jessica:	Well, I imagine the white ones.
11.0.159	T/R 1:	And you're imagining the whites, and how many do you imagine are there?
11.0.160	Jessica:	Twenty-four.
11.0.161	T/R 1:	You're imagining twenty-four. And Andrew, I see, built it. And how many do you have there, Andrew?
11.0.162	Andrew:	Um, twenty-four whites.
11.0.163	T/R 1:	Andrew has twenty-four. And can you see on the overhead how many whites, those of you who don't have enough? Can you see? I know it's hard, I have trouble counting when it's not nearby when there's so many little pieces. But you built it too, Amy, how many do you have, Amy?
11.0.164 27:08	Amy:	Twenty-four.
11.0.165	T/R 1:	Twenty-four? Yes?
11.0.166	Student:	Twenty-four.
11.0.167	T/R 1:	Twenty-four also. How many of you are convinced that with the model, that Alan and Erik have up on the overhead, how many of you are convinced of the number of white cubes there? [Two hands visible are raised]. How many of you are convinced how many are up there? Raise your hand if you're convinced how many. Because if you're not, you may want to

go up and count them to be sure or go to someone else who built the model and count them.

11.0.168 Michael: I, I built the model.

11.0.169 T/R 1: How many of you have a model that shows that there are twenty-four? Raise your hand if you have a model nearby. Ok, so Sarah and Beth, you built one, you don't have enough?

11.0.170 Beth: No.

11.0.171 T/R 1: I'm sure Andrew will lend you a few if you want to complete your model. I guess you need some whites and some reds. Ok, they don't need quite that many. Ok, do you believe there are - Sarah and Beth?

11.0.172 Beth: Yeah [Sarah nods]

11.0.173 T/R 1: Are you convinced, Kelly and Graham? You're convinced? Let's see, Michael and Brian I know you're convinced, I saw you had that built. Kimberly and Audra, are you convinced? [Audra nods] What about Gregory and Danielle - are you convinced [mmm hmmm] You're convinced also? Erin and Jackie? Ok, I know David and Meredith are convinced, and ok, so it sounds as if everyone is convinced that that's the case. So now, let's talk about, um, Erik's theory, Erik says now that the white one on this model, the larger model, where we called one the train that was made up of two orange and one purple, right? That particular train that he built? That he's now going to give the white one the number name, what, class?

11.0.174 Students: One twenty-fourth.

11.0.175 T/R 1: One twenty-fourth. So you agree with Brian's conjecture. Right? Brian says one twenty-fourth. How many of you agree with Brian's conjecture? [All students visible raise hands] The white one in that model has the number name one twenty-fourth. Now how does that help you solve the problem what is the difference between two thirds and three quarters, gentlemen who are up there? We know the white one has number name one twenty-fourth in that model.

11.0.176 29:13 Erik: Uh, see what we did here was we have the fourths and the thirds.

11.0.177 Alan: Yes, mmm hmmm.

11.0.178 Erik: And then the twelfths and they, they said that the twelfths would do it.

11.0.179 Alan: mmm hmmm

11.0.180 Erik: So the twelfths would be the reds which is one, which is two whites, and then people think the twelfths would be the answer, but if you take two of the twenty-fourths

11.0.181 Alan: It would equal up to a red rod.

11.0.182 Erik: It would equal up to a red rod.

11.0.183 Alan: Which would be equal to twelfths.
11.0.184 Erik: Which would be one twelfth. So, see, we think, I think that the answer is either two twenty-fourths or one twelfth.
11.0.185 Alan: Mmm hmmm.
11.0.186 T/R 1: How many of you agree with what they said?
11.0.187 29:46 Alan: So there are two answers. Both the same
11.0.188 T/R 1: You agree the answer is either two twenty-fourths or one twelfth. Does anyone think the answer is one twenty fourth? Ok, that's very interesting, that's very nice, gentlemen, thank you that's lovely. That was very helpful. And then I remembered - let's leave that up there, you can sit down. I have another question. I remember then some people had written - do you have a question, Amy?
11.0.189 Amy: No, I just want to say something.
11.0.190 T/R 1: Ok.
11.0.191 Amy: Um, that, but two reds equal up to a purple.
11.0.192 T/R 1: So?
11.0.193 Amy: So you could put six purples down to make
11.0.194 Erik: What do you mean six purples?
11.0.195 Amy: Becau- six purples, because look. [holds up a purple and two reds]
11.0.196 Andrew: Ok, but why would you need the purples?
11.0.197 Erik: Why would we need the purple? It's only one twelfth.
11.0.198 Amy: I know, but you could also do it that way. You could also put six.
11.0.199 Alan: Why would we need that?
11.0.200 Erik: We only need one twelfth. It's either two twenty-fourths or one twelfth.
11.0.201 Alan: They're both the same answer.
11.0.202 Erik: They're both the same answer.
11.0.203 Amy: I know.
11.0.204 Alan: [inaudible drowned out by Erik]
11.0.205 Erik: But why would we need the purple. The purple would be too big.
11.0.206 Amy: Six purples equal up to the whole train you made.
11.0.207 Erik: But why do we need - we don't need sixths. We only need thirds and fourths.
11.0.208 Alan: Yeah why would we need that?
11.0.209 Erik: And twelfths and twenty-fourths. That's all we need.
11.0.210 Amy: Ok [sighs]
11.0.211 Erik: I don't think we need sixths.
11.0.212 T/R 1: So it sounds like Amy is answering a different question. I think I hear the que- I hear what Amy is saying because I heard her say it earlier. Amy is saying there are other ways you can uh make trains, right? That have length [Amy nods] of two orange and one purple. And I know that this table

spent a lot of time doing that, of course what Erik and Alan and James are saying that's true, but it really isn't necessary or related to solving this problem. Do you agree with that, Amy? [Amy nods and says mmm hmmm] But that's an interesting thought that there is another way and when you think about all those purples there, what number name would you give to a purple then?

11.0.213 Amy and Erik: One sixth.

11.0.214 T/R 1: So I think that Amy is saying look! On this same train I could show one sixth! Which almost suggests I could ask you another problem. [Erik laughs]

11.0.215 Erik: Which is if you asked another, Which is Which is if you asked a question with a sixth, then you

11.0.216 Alan: Then you do it

11.0.217 Erik: Then you could do it.

11.0.218 T/R 1: Yeah, but I could ask you a question which is bigger a sixth or three quarters. Don't do that now!

11.0.219 Erik: Then, then you do a sixth!

11.0.220 T/R 1: Don't do that now, but I could ask you that question, couldn't I, which is bigger a sixth or three quarter?

11.0.221 Erik: Yeah.

11.0.222 T/R 1: And you should be able to answer it with this model. [Erik says yeah] Let's hold that question for a minute. I don't want to lose the question but I don't want to lose what we're talking about so we'll put that aside. I remember last Thursday when I walked around the room then I said could you make another model and a lot of you said "oh you know I don't have enough. I don't have enough of these blocks so I said can you imagine it and I remember talking and I know Andrew actually made the model when David had a theory that he shared with um Erik and Alan and Meredith, right, David? And so he shared a theory and I remember Erik said hey wait a minute that's what Andrew built! And then Jessica said that they already built what the theory was, that's what I heard, so I'd like to hear um, David's theory again, if you don't mind, David, if you think you can remember your theory and Andrew I want you to listen very carefully and Jessica and the rest of you I want you to listen carefully to David's theory because it really has to do with if I were to make another model, is it possible do you think to make another model if we had more blocks, it is a possible thing to do? [Student says yes]. How many of you think we can [Most/all students visible raise hands]. Ok. How many of you think we can make another model? Some of you aren't sure, how many of you aren't sure? Meredith's not sure? Erik's not sure? Danielle's not sure? Audra's not sure. Ok. How many of you are sure we can make another model? [All other students raise their hands.] Ok, that looks like that's James and Alan and Andrew and Jessica and Beth and Sarah, Kelly, Graham, Brian, Michael, Caitlin, did I leave anybody out? David is sure. Ok. Let's listen to David's theory and see if we could convince those or else they have to show us our theory doesn't work.

11.0.223 David: Well, first, um, Meredith made um, a model which had one orange, one blue, and one black.

11.0.224 T/R 1: Ok, she made a model with an orange a blue and a black. That's what you told me?

11.0.225 David: Yeah. And then she had, um, the whites, I think they were something like

11.0.226 Erik: Twenty-fourths.

11.0.227 David: Yeah, one twenty-fourth and the reds were one twelfth and, um,

11.0.228 34:28

35:26(F) Erik: Just like the one up there.

11.0.229 David: Yeah.

11.0.230 T/R 1: So you're saying that if i had an orange, a blue and a black, that the model should look like the one up here [Figure S-34-45].

11.0.231 Erik: Just about.

11.0.232 T/R 1: But it doesn't.

11.0.233 Erik: well...

11.0.234 T/R 1: Right? See what happens?

11.0.235 Erik: But then, then the one, then the, the uh, um, I don't know

11.0.236 Alan: Then the reds couldn't be twelfths.

11.0.237 Erik: Yeah, then the reds couldn't be twelfths and the whites couldn't be twenty-fourths.

11.0.238 Alan: Right, it would either take one [inaudible]

11.0.239 T/R 1: Andrew, what do you think? Andrew and Jessica, what do you think?

11.0.240 Andrew: [Refers to twenty-four cm model on desk - Figure F-36-39] Well, I made a model that had the white was one forty-eighth and the purples were twelfths and the white was, I mean the red was twenty-fourths and I took two browns as the thirds and two dark greens as the fourths and they I called them the fourths and then the whole was four oranges and two purples.

11.0.241 T/R 1: Now, you're telling me that you used browns, two browns to be

11.0.242 Jessica: One, like one, one third.

11.0.243 Andrew: Yeah.

11.0.244 T/R 1: One brown was one third, two browns was two thirds?

11.0.245 Andrew: No

11.0.246 T/R 1: Is that what you're telling me?

11.0.247 Erik: No
11.0.248 Andrew: Two browns was one third
11.0.249 Erik: Two browns was one third.
11.0.250 Andrew: I took two browns and put them together
11.0.251 T/R 1: Two browns to be one third!
11.0.252 Andrew: Yeah.
11.0.253 T/R 1: Oh, ok, that's not going to fit. But maybe, um, you want to come up here and do that? [Andrew and Jessica come to front of class.] Ok, here you go. Why don't you build that right here. Do it up front here, uh, why don't you come all the way around, Jessica. Ok, let's see what they're doing here because, um, it looks to me as if you need a bunch of rods to do this. [They work for about two minutes to build the model of a train of four oranges and two purples, six brown rods and eight dark green rods, and twelve purple rods, twenty-four red rods, and white rods - Figure F-40-26]
11.0.254 Andrew: It might not be enough.
11.0.255 T/R 1: Now, I want all of you to see what Jessica and Andrew are building, and, now you all can't come up at one time, so I'm gonna, if it's ok with Mrs. Phillips, I'm gonna ask you in little groups to go up there and take a look at their model and so um we can be able to talk about it and then some of you maybe can look at it from where you're sitting. I know that Gregory and Danielle are very fortunate - they have front row seats. I think, can you see Alan and Erik?
11.0.256 Erik: Not really.
11.0.257 T/R 1: Not really. So some of you are going to have to go up in a minute to see what they're doing. So why don't we start at least with Erin and Jackie, why don't you sort of kind of come up and see what they're building. Be careful there's a cord here too. Ok, if you've seen what they're building and you think you understand how they built their model, then if you can sit down and then I'd like another group to come up and see.
11.0.258 Jessica: [Figure F-44-40, Explaining to Erin and Jackie] And what we did is, this was our whole and this was like, these, like, um, two, [takes two brown rods and holds them together] two browns equal one third, like we counted two as one. And those were our thirds, and two greens was one, and one two three four, so those were our fourths, and
11.0.259 Andrew: Purples were the
11.0.260 Jessica: Purples were the twelfths and the reds were the
11.0.261 Andrew: Twenty-fourths
11.0.262 Jessica: And the white ones were forty-eighths.
11.0.263 T/R 1: Ok, it looks like it's going to be a little harder than I think, I think that you're gonna need an explanation when you go up.

And so I think maybe we should have, rather than have you do this a lot of times, maybe we should have a few explanations. Maybe we should have more people up here. Um, some of you can come around while we can hear Andrew and Jessica - would you mind doing this a couple of times?

11.0.264 Jessica: Yeah.

11.0.265 T/R 1: So why don't you come around, a few of you can come around the table and listen to their explanation. I think that's the best way. Some of you can come behind the table, I think.

11.0.266 Andrew: So what we did was, we, um.

11.0.267 T/R 1: Ok, just a second, let's wait till as many people, uh, can

11.0.268 Andrew: We had, the, um, whole, was four oranges and two purples, and then we, our strategy was we took two browns and we put em together and they were the third. And then we took two dark greens and put them together and they were the fourths. And then the purple was the twelfth

11.0.269 Jessica: Twenty-twelfth.

11.0.270 Andrew: And the red was the twenty-fourth and the white was the forty-eighth.

11.0.271 Jessica: Forty-eighths.

11.0.272 Andrew: And,

11.0.273 Jessica: Cuz, what we did really to figure it out.

11.0.274 T/R 1: Well, what's the difference?

11.0.275 Andrew: The difference is one twelfth.

11.0.276 T/R 1: Can you show us?

11.0.277 Jessica: Yeah.

11.0.278 Andrew: Ok, we don't have enough, we'll just take them over here. Two thirds, that's all we need, and then

11.0.279 Jessica: So, um, do we need anything?

11.0.280 Andrew: Here, we need brown.

11.0.281 Jessica: We need some of those.

11.0.282 Andrew: Ok, well, we came up with the conclusion of that

11.0.283 Jessica: That two thirds.

11.0.284 43:06 Andrew: That two thirds - that four, three fourths [As Andrew speaks, Jessica points to relevant rods on model] were bigger than two thirds by one twelfth

11.0.285 Jessica: By one twelfth [holds up purple rod].

11.0.286 Andrew: Or two twenty-fourths [Andrew and Jessica place two red rods and four white rods, to complete the model of a purple, two reds, and four whites]

11.0.287 Jessica: Or

11.0.288 Andrew: Or four forty-eighths.

11.0.289 T/R 1: Ok, now, I'm wondering if some of you can pull aside and maybe the rest of you can come up [Side view: break in tape Front view: and I'd like Jessica and Andrew to say it nice and loud so in back of the room, the people who are sitting could hear. Now those of you who think you understood their argument um, you can go back to your seat. The rest of you can come and sit on the floor, but if they can turn around and try to share it so the people in the back of the room could understand, would you mind going through it one more time particularly the different names for the way you've represented the difference? Ok, hold on, I'd like you all, now if you're back in your seats, if you want to try to stand up and look while they're explaining it, that's ok. Ok, are we all ready? Hold on a minute, Jessica, I think we're having a little discussion about uh solutions there.], Ok, can you nice and loud for the people back there show them what you're doing?

11.0.290 Jessica: Well, what we did was we made a model and we counted um

11.0.291 Andrew: Four oranges

11.0.292 Jessica: four oranges and two purples as our whole and for our thirds we counted, we counted two oranges as one, I mean two browns as one. [holds up two brown rods end to end] And we had

11.0.293 Andrew: that was our third.

11.0.294 Jessica: That was our thirds, and for our fourths we counted two greens as one [holds up a train of two dark green rods], two dark greens as one.

11.0.295 Andrew: Purples were our twelfths, the reds were the twenty-fourths [Jessica says twenty fourths] and the whites were forty-eights.

11.0.296 Jessica: Forty-eighths. And we think that, we think that, three um, fourths are bigger than two thirds by either, um, one forty- I mean four forty-eighths um, two twelfths, or, um

11.0.297 Andrew: No, two twenty-fourths.

11.0.298 Jessica: Two twenty-fourth or what's that? One twelfth.

11.0.299 T/R 1: What do you think about that, Michael?

11.0.300 Michael: I guess I agree with it, it's what I came up with.

11.0.301 T/R 1: You came up with the same model, didn't you?

11.0.302 Michael: Yeah

11.0.303 T/R 1: Did anyone else come up with that same model? That's very lovely. Thank you so much, Andrew, and does anybody have a question to ask Andrew and Jessica before they're finished? Does anybody have a question? Does anybody have a comment? You sure you don't want to ask them any of that? Sarah what do you think?[Sarah says no.] Is that interesting [Sarah says mmm hmmm]? It's very interesting Ok, um, I'm going to ask you to sit down and I want to thank you very much for making that model for us. But I guess I'm asking the question, uh, to Meredith and James and to Erik and Alan right now, uh, does this have anything to do with your theory and the theory you tested? Meredith and David and Erik and Alan - does this model have anything to do with the theory you tested, David?

11.0.304	David:	Uh, yes because we thought that the ones would be one forty-eighth
11.0.305	Erik:	And and the
11.0.306	David:	And then the reds would be, um, one
11.0.307	Erik:	Twenty-fourth and the purple, well originally, we thought that the light greens would be, well David thought that the light greens would be twelfths, but then we tried it and they would become the sixteenths, so then we tried the purple, yeah the sixteenths and we tried the purple and then that was the twelfths.
11.0.308	Alan:	Since whites are doubles, they're forty-eighths
11.0.309	Erik:	So, in other words we doubled everything.
11.0.310	Alan:	Yeah. You basically just added, like, there originally were just two oranges, now there are four oranges and an extra purple. Now there are six, there are six browns.
11.0.311	T/R 1:	So let's see, on this model here we had an orange and a red, and then on that model there we have two orange and a purple and in this orange here we have four orange and two purple. All of these represent one, is that a surprise?
11.0.312	Alan:	It could have been two purples changing into a brown.
11.0.313	T/R 1:	It could have been two purples changing into a brown
11.0.314	Alan:	Yeah.
11.0.315	T/R 1:	That's true.
11.0.316	Alan:	And
11.0.317	T/R 1:	I guess my question is what you called one in each of these models? Are they related in any way? The lengths? All of these you called one, are the lengths related to each other, if you study each of the models you built. You see this one here you called the orange and red one, isn't that right, and here you called one two orange and purple, right?
11.0.318	Alan:	So basically it's just doubled. That's
11.0.319	T/R 1:	What do you mean by that "basically it's doubled", Alan? That's an interesting idea. In what way is it doubled?
11.0.320	Alan:	Um, ok, it's doubled because it now it has four oranges and two purples or a brown, so
11.0.321	T/R 1:	But the first one doesn't have any purples.
11.0.322	Alan:	Well, that's because this had nothing to do with the first problem because of the first question, but
11.0.323	T/R 1:	I'm not sure I understand what you're saying.
11.0.324	Alan:	Had there have been sixths.
11.0.325	Erik:	I know.
11.0.326	T/R 1:	We didn't have sixths, we had twelfths here.
11.0.327	Alan:	Mmm hmm.

11.0.328	Erik:	I think I know what he's saying.
11.0.329	Alan:	Right, there you have twenty-fourths and the whites are forty-eighths this time. Now, up there, there are no purples, because they weren't put on. But had they have been, on the bottom, which they are, they are twelfths, because
11.0.330	Erik:	Purples? In that
11.0.331	Alan:	Purples are twelfths.
11.0.332	Erik:	In that model they became twelfths, but over there they would be the sixths. Like Amy said, if
11.0.333	Alan:	Right, because if you double each of them, it would come out to twice the number.
11.0.334	Erik:	Exactly!
11.0.335	T/R 1:	James?
11.0.336	James:	Uh, I think um, that um, because there are two oranges and two purples I agree with Alan that it's double but why the red's there, it's two reds make a purple and that, that means the two oranges and the red make two oranges and a purple.
11.0.337	Alan:	Yeah, cuz if you took the two oranges out of that model and a purple, and then two more oranges and a purple, and you put them on top of each other, they'd be equal. But if you put em side to side you'd have four oranges and two purples, or the two purples could be a brown. So it's basically doubled, each of the length is doubled.
11.0.338	T/R 1:	I wonder if the rest of you see this, I'm saying, this is an orange and it's not a purple, it's an orange and a red, right? Now, how does this get doubled to be this? I see there are two oranges, instead of one orange, I see the one orange length got doubled, instead of one orange there's two, right? Isn't that true? But how did the red get doubled?
11.0.339	Alan:	The red-
11.0.340	T/R 1:	I'm confused, how did the red get doubled here?
11.0.341	Alan:	The red
11.0.342	T/R 1:	I see the orange got doubled here because there are two oranges, right? From one orange to two oranges, I don't know how did the red get doubled? I don't see that. Jessica? Kimberly.
11.0.343	Kimberly:	Well, they used a purple and the red, two reds make a purple, so now if they have a purple, they doubled the red.
11.0.344	T/R 1:	Is that what you were going to say?
11.0.345	Jessica:	Yeah.
11.0.346	Alan:	I was going to say something different
11.0.347	T/R 1:	So you're tellling me that instead of the one orange and one red, we have two oranges and two reds in this model. But they just called it a purple rather than two reds. Do the rest of you see that? [mmm hmm] Ok, so this model is doubled of this, now you have to convince me that this model is double

of this, so instead of two oranges and a purple, what should we have now if it's doubled? Don't look. What would you expect we would have then if it's doubled? Danielle.

11.0.348 Danielle: Um, four oranges and two purples.
11.0.349 T/R 1: Let's see. Do we have four oranges and two purples?
11.0.350 Erik: One, two, three, four, yup, or four oranges and one brown.
11.0.351 T/R 1: Or four oranges and one brown.
11.0.352 Alan: Yep
11.0.353 T/R 1: Ok, this is the question I ask you. If I were to make another model, Andrew's hand is up, Andrew knows my question, what do you think my question is, Andrew?
11.0.354 Andrew: If you were gonna make another model, what, um, the doubles be?
11.0.355 T/R 1: Ok, what would my one look like in terms of rods? Brian!
11.0.356 Brian: Um, forty-eight.
11.0.357 T/R 1: What would I call one? Imagine in your head what I would call one?
11.0.358 Brian: Forty-eight? Cuz there would be, well, cuz there would be forty-eight whites equal up to one and then.
11.0.359 T/R 1: Well, we have forty-eight whites going up to one here, don't we?
11.0.360 Brian: Oh!
11.0.361 T/R 1: In this model.
11.0.362 Erik: So we have to double that?
11.0.363 Alan: But, no!
11.0.364 T/R 1: I don't know, I'm asking you, that's my question, Andrew what do you think?
11.0.365 Erik: Well you're saying what-
11.0.366 Alan: No, it can't
11.0.367 Andrew: Well, the whole would be eight orange rods and
11.0.368 Alan: It can't be done
11.0.369 T/R 1: Eight orange rods, I'm listening.
11.0.370 Erik: Eight orange rods and two browns
11.0.371 Andrew: And two browns.
11.0.372 T/R 1: And two brown rods.
11.0.373 Alan: You can't double that. You can't double that model because if you did, then you wouldn't be able to third it.
11.0.374 Erik: You wanna make a bet - all you had to do is train it - you just train it!
11.0.375 Alan: Right because if you doubled that it would be eight oranges and two browns, now is there any rod that could third that?
11.0.376 Erik: Well if you use a train
11.0.377 Andrew: Yeah
11.0.378 53:00 Erik: If you use a train, just like in Andrew's theory.
11.0.379 Alan: Well, if you train the rod, but that would make it not equal.
11.0.380 Andrew: It would probably be-

11.0.381 Alan: Up there, it's just plain, except for the whole.
11.0.382 Andrew: It would probably be three browns would be the thirds and three dark greens would be the fourths.
11.0.383 Alan: Right, but that would be using more than one rod to make another rod to fit, fit the same thing.
11.0.384 Erik: Yeah, so you can do that! Just like, you, Andrew said, you can use a train to make a third and a fourth. Cuz he, like, I, I overheard, they said that if you can use a train to make a whole why can't you use it to make a third and a fourth?
11.0.385 Andrew: Yeah.
11.0.386 T/R 1: David?
11.0.387 Andrew: And a half
11.0.388 Alan: But then it wouldn't be equal.
11.0.389 Erik: Yeah they would! Cuz the third could be, like in that model, Andrew used the two browns, that's equal!
11.0.390 Alan: But in that model, the three browns don't have anything attached on so it's totally equal
11.0.391 Erik: So? They just doubled it!
11.0.392 Alan: But if you added something on
11.0.393 Erik: We just doubled, we doubled that model to equal that model.
11.0.394 Andrew: Yeah, and I doubled the brown - two browns,
11.0.395 Erik: Yeah, exactly.
11.0.396 Andrew: So in the next model
11.0.397 T/R 1: David, what do you think? Did you want to say something?
11.0.398 David: Um, I agree with Erik
11.0.399 T/R 1: What part of what Erik said?
11.0.400 David: Well, Alan didn't think that you could uh third it, but like Erik said that you can train it and put the other blocks onto the other one
11.0.401 Alan: What I meant, what I meant is, you can't third it just using one rod.
11.0.402 T/R 1: Ok, Alan.
11.0.403 Erik: Exactly. You can't third it using one rod, but you can third it using trains.
11.0.404 T/R 1: Ok, so
11.0.405 Alan: You could double that, but you would have to use two rods to make it
11.0.406 T/R 1: Ok, so you think you can double it and you think you can imagine - can you make one bigger than that?
11.0.407 Erik: If you doubled that, it would be sixteen oranges [laughter] and, sixteen oranges and four browns!
11.0.408 T/R 1: Ok, the question I want to leave you all to think about, I'd like you to uh, first I'd like to thank you for the wonderful models you built, but the question I'd like you to think about is, uh, is there, is there a biggest model?
11.0.409 Erik: Thirty-two oranges! [laughs]

11.0.410	T/R 1:	Is there a biggest model? And if you don't have enough, uh, rods, you could imagine, we could write to Cuisenaire and we can have them ship us buckets and buckets and buckets and buckets
11.0.411	Erik:	Or we could combine all our stuff.
11.0.412		We could start by that but my question to all of you is there a biggest model? Why or why not? And I'd like you to write to me about, about that. Would you do that? Would you write to me? Maybe Mrs. Phillips can let you combine and build together, that might take a little while and a camera. Ok, I think we have to stop, I'll see you in two weeks, and if you could
11.0.413 57:00		[end of class]
11.0.414 58:22	Erik:	I wonder how many oranges
11.0.415 58:17(F)	Alan:	No, I wonder how many oranges it would take to get from here to California

Session 12, Oct. 29, 1993 (Front, Side, and OHP)

Session 12, October 29, 1993, Side View

Line	Time	Speaker	Transcript
12.1.1	15:38	T/R 1:	Well, good morning. [students Good morning] What did you do yesterday in math? [students raise hands] Ah. All these people are going to tell me. Amy?
12.1.2	15:47	Amy:	We did, we figured out the chocolate, we divided chocolate.
12.1.3	15:52	T/R 1:	Oh. Did you all agree?
12.1.4	15:53	Students:	Yeah.
12.1.5	15:56	T/R 1:	You agree? Was that an easy decision?
12.1.6		Andrew:	Yeah
12.1.7		T/R 1:	No discussion, or, or differences?
12.1.8	15:59	Andrew:	Well, a little
12.1.9	16:02	T/R 1:	How did that work.
12.1.10	16:04	Andrew:	Well, we um like divided us into groups, the class into groups and um, and our, in my group, there was like nine people, so each person got like, um one and one ninth.
12.1.11	16:21	T/R 1:	How did you decide that? How much did you have to start with?
12.1.12	16:25	Andrew:	We had uh, ten pieces.
12.1.13	16:28	T/R 1:	Ten pieces. I see, how did you do one and one ninth? I'm curious.
12.1.14	16:32	Andrew:	Well, we um, we said there was nine people, so we had to give a whole piece of candy to each person and then we had one left over so we would have to, and there's nine people, so if we divided it into ninths there would um be enough, for everyone. D/?/i
12.1.15	16:53	T/R 1:	Is that hard to do?
12.1.16	16:56	Andrew:	Yeah, a little.
12.1.17	16:57	T/R 1:	But you did it?
12.1.18	16:59	Andrew:	Yeah.
12.1.19	17:00	T/ R 1:	And you all felt good about it?
12.1.20	17:02	T/R 1:	Oh, and you were in that group too, Graham, huh?
12.1.21	17:02	Graham:	Yeah.
12.1.22	17:04	T/R 1:	What about another group? What did another group do? You were in a different group? Jessica, what did you do?
12.1.23	17:08	Jessica:	Well, my group, we like had uh, eight people in our group, so well, we each got one whole piece and then we had two pieces left over so then we divided each of the two pieces into fourths. D/?/i
12.1.24	17:22	T/R 1:	And, so, how, how much did each person get?
12.1.25	17:26	Jessica:	One and one fourth.
12.1.26	17:29	T/ R 1:	You got one and one fourth. Did you all think that was fair, in that group? [mmm hmmm] Did the people in Andrew's

group get the same amount as the people in Jessica's group? [no] Who got more, the people in Andrew's group or the people in Jessica's group? Michael?

12.1.27 17:46 Michael: The people in Jessica's group.

12.1.28 17:50 T/R 1: The people in Jessica's, now, of course I could ask you how much more, you think you could you figure that out? You don't have to tell me that right now.

12.1.29 17:53 Michael: Yeah

12.1.30 17:56 Meredith: Yeah, if we got one ninth and they got one fourth, then um, nine minus four equals five, so they got um one fifth bigger, than we… D/?/f

12.1.31 18:08 T/R 1: Say that again?

12.1.32 18:11 Meredith: See, um, we had, each of us had one and one ninth.

12.1.33 18:16 T/R 1: Let's see, let's see, Andrew's group had nine people, right? Each person,

12.1.34 Andrew: Got one and one ninth

12.1.35 T/R 1: And, in Jessica's group, eight people and each person got, you said,

12.1.36 Jessica: One and one fourth

12.1.37 T/R 1: One and one fourth

12.1.38 18:35 Meredith: And

12.1.39 T/R 1: And so, you're telling me,

12.1.40 Jessica: But there was another group.

12.1.41 18:36 T/R 1: Maybe we'll hear about the other group and we'll come back to this, but I also didn't want to lose what Meredith said, what Meredith said was the people in Jessica's group got more than the people in Andrew's group. [Meredith laughs] and I, I kind of asked how much more

12.1.42 18:55 Meredith: Nine minus four equals five so they got one fifth more.

12.1.43 19:01 T/R 1: So you're claiming, this is Meredith's claim

12.1.44 19:02 Meredith: One fifth. They got five more, whatever.

12.1.45 19:09 T/R 1: [writing on overhead transparency, figure 10-29-01] That each person in Jessica's group got how much more did you say Meredith? Got one fifth more than each person in Andrew's group. How many of you believe that? [all students raise hands]. Ok, you're gonna have to then convince me. But we'll let that hold for a minute. But who's the other group? [there was three groups] Ok, who was, who was in a different group? A group other than Andrew's and Jessica's group? Kimberly? Ahah. How many in your group, Kimberly?

12.1.46 20:04 Kimberly: There, we each got one and one fourth.

12.1.47 20:08 T/R 1: How many people in your group?

12.1.48 20:10 Kimberly: Eight

12.1.49 20:11 T/R 1: Eight people? And in your group got

12.1.50 20:12 Kimberly: One and one fourth.

12.1.51	20:12	T/R 1:	One and one fourth. So the people in Jessica and Kimberly's group, right? You're claiming you got more. And the difference you claim is
12.1.52	20:26	Kimberly:	Five, one fifth.
12.1.53	20:34	T/R 1:	Wow, that's a good question. I don't know you got one fifth. Um, it's sort of like saying to me, if I got a half, and Amy got a quarter, right? Who got more? I got more, right? I got a half, ok, and Amy got a quarter, but by your theory, you would tell me that I got, how much more?
12.1.54	21:34	Meredith:	One fourth
12.1.55	21:34	T/R 1:	But you would have told me a half more, think of the way you did that problem.
12.1.56	21:19	Meredith:	Oh
12.1.57	21:21	T/R 1:	Is that right, Meredith? Right? Did I get a half more [Meredith laughs]. You all know I didn't get a half more. I got how much more?
12.1.58	21:21	Meredith:	Oh.
12.1.59	21:23	T/R 1:	A quarter more. But if I used your method of figuring out how much more I'd be subtracting the four and the two, and I'd end up with a half more. That sort of doesn't make sense, does it? You still believe that it's a fifth more for sure? How many of you are not so sure? Not so sure [all students raise hands] It's a good question. Hmmm, I don't know. Well, we ought to keep this question in mind and uh, we ought to try to answer it, don't you think? Ok? I guess maybe another way to ask that question might be, might be what? You tell me what you think the question is. What are the fractions that are of interest in this problem? What decides who got more, the people in Andrew's group or the people in Jessica and Kimberly's group? Meredith?
12.1.60	22:24	Meredith:	I know that Jessica and Kimberly's group got more than Andrew's group did.
12.1.61	22:48	T/R 1:	You know they got more, right?
12.1.62	22:50	Meredith:	Right.
12.1.63	22:51	T/R 1:	And what number tells you?
12.1.64	22:59	Meredith:	Well, um, if they got, uh, one fourth, and one rod was like, the one rod, and you had ninths and you had fourths, if you had the fourths they would take
12.1.65	23:11	T/R 1:	You're talking about ninths and fourths, is that right?
12.1.66	23:12	Meredith:	Yeah.
12.1.67	23:20	T/R 1:	You all agree that it's ninths and fourths that's at issue here? And it's not the one piece? So let's focus on the ninth, right, and let's focus on the fourth. So which one you're claiming is bigger?
12.1.68	23:25	Students:	The one fourth.
12.1.69	23:26	T/R 1:	The one ninth is smaller

12.1.70 23:28 Students: Yeah.
12.1.71 23:28 T/R 1: Did you ever see that symbol, smaller than? [Figure 10-29-02]
12.1.72 23:30 Students: Yeah
12.1.73 23:31 T/R 1: One ninth is smaller than one fourth? Ok? So I'm sorry, Meredith.
12.1.74 23:41 Meredith: And, um, if you take a one rod and you divide it into ninths and fourths, the fourths are going to be larger because they're less. So they're going to be larger. So each person is going to be getting a larger piece.
12.1.75 23:50 T/R 1: Ok, so you've convinced me that if I could imagine a rod that I call one, and I imagine four pieces and I think of one of those pieces, that's going to have what number name?
12.1.76 24:01 Meredith: Fourths.
12.1.77 24:08 T/R 1: And if I kept that same rod and I imagine nine pieces, one of those pieces will have the number name.
12.1.78 24:10 Meredith: Ninths.
12.1.79 24:12 T/R 1: One ninth. And you could imagine in your head without the rods and telling me that the one ninth, that the one fourth is
12.1.80 24:16 Meredith: Bigger than
12.1.81 24:18 T/R 1: Bigger than the one ninth, or the one ninth is smaller than the one fourth. The question I'm asking is the difference one fifth? Now, could you imagine the fifth rod, what that looks like?
12.1.82 24:31 Meredith: Um, I think it would be the yellow rod, I'm not sure, I think it was the yellow rod that was the fifth.
12.1.83 24:37 T/R 1: Whatever you're thinking, you could imagine a fifth, you could imagine a fourth, you could imagine a ninth, do you imagine in your head, is my question, do you imagine in your head that the, if you'd compare the one fourth rod and the one ninth rod, the difference would be the one fifth rod, do you think that makes sense to you, as you're imagining this in your head?
12.1.84 25:02 Meredith: Ummm, if you put the four and the five together it would equal up to the ninth rod. A? f
12.1.85 25:10 T/R 1: You think so? [mmm] I think we ought to get out the rods.
12.1.86 25:11 CT: Yeah.
12.1.87 25:12 T/R 1: I think we ought to get out the rods, what do you think? How many of you want to work on this? How many of you want to know how much more the people in Andrew's and Je- uh, Andrew's group, uh Jessica's group and Kimberly's group got than the people in Andrew's group. [Students raise hands, Dr. Landis enters with rods] Can somebody tell Dr. Landis the problem because she doesn't know any of the story of any of what happened and how this all came to be, could someone be so kind as to tell Dr. Landis the whole story?

Kimberly do you want to give it a try? Dr. Landis? Do you want to hear what's going on?

12.1.88 Dr. Landis: I do want to hear it, yes!

12.1.89 25:48 Kimberly: First of all, they went to California and they brought back candy bars and we, yesterday, we had three groups cuz there were three candy bars and in my group and in Jessica's group got one and one fourth, but in Andrew was in his group got one and one ninth. So we're trying to figure out how much more my group and Jessica's group got than Andrew's group.

12.1.90 26:18 Dr. Landis: That’s an important problem, yeah.

12.1.91 26:20 T/R 1: Meredith conjectured one fifth.

12.1.92 26:23 Dr. Landis: Ok, Yeah, good

12.1.93 26:25 Alan: Now we gotta break out the rods.

12.1.94 26:26 T/R 1: Ok.

12.1.95 26:31 Meredith: Now we put the fifth up to the fourth

12.1.96 26:33 Brian: No, here. [Meredith and Brian raise hands]

12.1.97 27:06 T/R 1: What do you think?

12.1.98 27:08 Meredith: [Meredith has built a model of a blue rod and train of a purple and yellow rod.] We call this um, this is ninths, this rod has nine white little things, and this has five white ones in it, and this has four white ones, I added the five plus the four and it equaled a nine rod. [Figure 10-29-23] D/?/f

12.1.99 27:25 T/R 1: I believe that. But the question, if you have the one fifth and the one fourth do you get one ninth?

12.1.100 27:30 Meredith: Yeah.

12.1.101 27:36 T/R 1: Can you show me what one fifth is and show me what one fourth is [Meredith makes a train of an orange and a red]

12.1.102 27:48 Brian: I don't think it fits by one fifth

12.1.103 27:50 Meredith: Oh, you can't even do that it's crazy. Let's see, let's get the ninths out. [Meredith places white rods next to her blue rod]

12.1.104 28:52 Erik: What would the, what would the ninth be? Reds?

12.1.105 28:58 Michael: I think the light greens might be.

12.1.106 29:00 Erik: [Erik has built a model of two orange rods, four yellow rods] I'm going to try the reds. One two three four five six seven eight, uh I need, give me all your reds.

12.1.107 29:20 Michael: I just need one more red. Dave, can we borrow one red? [Figure 10-29-24]. If that's one over, the red's one over. Then what would it be? Oh yeah! It is two more reds! See? use the two whites

12.1.108 30:00 Erik: Those are the tenths! Tenths.

12.1.109 30:08 Michael: I'm going to try light greens.

12.1.110 30:19 Brian: ... that um one fourth, I mean, nine is smaller than one fourth by one fifth, um, because

12.1.111 30:37 T/R 1: Why don't you write that down, what you just said?

12.1.112 30:42 Brian: [Brian writes: 9 is smaller than and pauses] I mean one ninth is smaller than one fourth by one fifth.

12.1.113 31:30 T/R 1: Ok, so you want to write that one ninth is smaller. [Brian begins to write again on the next line - 1/9 is smaller than 1/4 by 1/5] You want to explain that to Dr. Landis? She looks very confused.

12.1.114 32:08 Dr. Landis: [to Alan] I'm confused. Are you confused too, Kimberly?

12.1.115 32:14 Alan: Here's the fourth, here's the ninth, now if you take out the fifth, if you take out one of those from the nine, you have a fifth. So this would be smaller than one ninth by one fifth.

12.1.116 32:36 Dr. Landis: I'm confused. This is, you said, what was this?

12.1.117 32:14 Alan: One fourth [purple], now here's the ninth [yellow stacked on top of the purple, Figure 10-29-25], you could take this [yellow] and put it on here [purple] and it would be nine too D/?/f

12.1.118 32:44 T/R 1: [to Brian] So this is one ninth, I see one ninth, I don't see one fourth.

12.1.119 32:47 Brian: Oh, this is one fourth [purple rod]

12.1.120 32:50 T/R 1: How can this be one fourth if this is one?

12.1.121 32:56 Brian: Ohh. You mean it's

12.1.122 32:58 T/R 1: How could this be one fourth, if this is one? I don't understand that. Meredith, help me. How can the purple be one fourth if the blue is one?

12.1.123 33:05 Meredith: Well, I'm not really calling the blue one, I couldn't find a fourth of the blue, but if, if these are fourths of something.

12.1.124 33:17 T/R 1: No, well you have to tell me what it is, I need to know what you're calling the blue, because I'm confused unless you tell me what number name the blue is.

12.1.125 33:26 Meredith: Blue is, OK

12.1.126 33:29 Brian: Well, I guess-

12.1.127 33:30 T/R 1: Brian is calling the blue one and I could believe that the white is one ninth, I could imagine that.

12.1.128 33:35 Brian: We mean, is um, I guess we mean the um, like one of them, all of them is just like saying, well nine ninths you put together.

12.1.129 33:45 T/R 1: I understand that, I really, I understand that, I understand that you're calling this one,

12.1.130 33:46 Brian: Yeah.

12.1.131 33:46 T/R 1: And I understand that you're calling this one ninth. I have trouble now when you're calling this one fourth. [Points to purple in model, 10-29 Side 30,37]. You understand that? Well, what do you mean by a fourth? How would you convince me that something is a fourth?

12.1.132 34:04 Brian: Well, well um, that would be four of those. D/?

12.1.133 34:06 T/R 1: That's right, how could that be a fourth? See where I'm having trouble?

12.1.134 34:11 Brian Well, it's like, put four of these against here, it's like a fourth [four white rods and a purple]
12.1.135 34:16 T/R 1: I agree that this is a fourth of this, but you called this one
12.1.136 34:21 Brian: Well, I guess, I guess I was wrong.
12.1.137 34:24 T/R 1: You want to think about that a little bit more, you and Meredith?
12.1.138 34:27 Brian: I guess I mean that one of them put together will just be one, will, like um, like um, this is, these are ninths, this is fifth, and…
12.1.139 34:40 T/R 1: Wait, wait, wait, I get mixed up again. What's a fifth?
12.1.140 34:45 Brian: One of these.
12.1.141 34:50 T/R 1: Again, if this is a fifth, how could this be a fifth if this is one? [Figure 10-29-27]
12.1.142 34:51 Brian: I, I'm not calling it one anymore.
12.1.143 34:54 T/R 1: Well, you have to always tell me what one is. Otherwise I don't understand what you're doing. What are you doing, Meredith?
12.1.144 35:01 Meredith: Well, I'm trying to make a model so that.
12.1.145 35:01 T/R 1: So you can show me what the names are?
12.1.146 35:05 Meredith: Yeah. [Meredith's model is composed of four purple rods and an orange and blue rod.
12.1.147 35:06 T/R 1: I see, well, why don't you work on that? Why don't you two talk a little bit?
12.1.148 35:33 Dr. Landis: Ok, that's what they're saying but what you're telling me a minute ago was that this was four.
12.1.149 35:37 Alan: No, this was, oh wait a minute, hold it, this would be a fourth, but you couldn't do that cuz you'd have, look. That would be one ninth. [Alan holds up one white rod. He has built a model of nine white rods and a train of a purple and yellow, Figure 10-29-28]. This would be one fourth, one fourth and one ninth would be the same size. D/?/f
12.1.150 36:00 Dr. Landis: Oh, that's interesting.
12.1.151 36:01 Alan: If you take out that, and you'd have four here, [Alan builds a model of a purple rod and four white rods]
12.1.152 36:07 Dr. Landis: Kimberly, you'd better listen too, because I think I'm going to need your help.
12.1.153 37:09 Alan: Then if you eliminated one, that would be one fourth and if you eliminated one that would be one ninth but they'd still have the same value [Figure S-36-14].
12.1.154 36:18 Dr. Landis: So now you're saying that they're equal.
12.1.155 36:20 Alan: No, they're just, that's the same size, but they don't have the same number value. They're just the same size.
12.1.156 36:26 Dr. Landis: They're the same size but they don't have the same number value. What does that mean?
12.1.157 36:29 Alan: That means that this one is one ninth and that is smaller than one fourth.

12.1.158 36:37 Dr. Landis: You're telling me that this is one ninth [points to one white rod]
12.1.159 36:40 Alan: Right
12.1.160 36:42 Dr. Landis: And this is one fourth [points to one white rod]
12.1.161 36:42 Alan: Right
12.1.162 36:44 Dr. Landis: And you're saying that this one's smaller, to me they look the same.
12.1.163 36:46 Alan: In numbers. In number value.
12.1.164 36:49 Dr. Landis: In numbers they're smaller, but in the cubes, in the Cuisenaire rods
12.1.165 36:50 Alan: They're the same size.
12.1.166 36:56 Dr. Landis: They're the same. Is that possible to be the same size with the rods but to be different with numbers?
12.1.167 36:57 Kimberly: Maybe.
12.1.168 36:57 Alan: Yeah.
12.1.169 36:59 Kimberly: If you have different size wholes
12.1.170 37:02 Alan: If you take out one from there and then you take out one from there it's the same size
12.1.171 37:06 Dr. Landis: Kimberly just said that if they have different size wholes it would work.
12.1.172 37:28 Kimberly: [Figure S-37-17] Right, I think that if you had, it could be a ninth on this [purple and yellow train], because it equals nine, but if I took this and this [dark green and yellow train], it would probably be a higher number, because this is bigger. So they can be different number names and be the same size, but they have to be different models on the top
12.1.173 37:31 Dr. Landis: That's interesting, what do you think about what she's saying?
12.1.174 37:49 Alan: Well, I still think that about my number values. This could be a fourth and this could be a ninth because basically a number value the smaller the fraction, like, one fourth, it's bigger. But one ninth means you're dividing it into more pieces. So that would mean this had a smaller number value. D/?
12.1.175 37:55 Dr. Landis: Ok, hold on, ninth, you're dividing it into smaller pieces, so what does that mean about the pieces?
12.1.176 37:58 Alan: So that means that these [points to the purple rod], you can only divide it into four
12.1.177 38:00 Dr. Landis: Right.
12.1.178 38:00 Alan: If you had different size rods that are smaller than this [points to white rod] you could divide this into ninths and it'd still be equal to that, but now you're dividing it into fourths, so this has a higher number, fraction value than the ninth.
12.1.179 38:10 Dr. Landis: Which is bigger, though, a fourth or a ninth, do you think?
12.1.180 38:12 Alan: A fourth.

12.1.181 38:13 Dr. Landis: Which do you think, Kimberly?
12.1.182 38:14 Kimberly: A fourth.
12.1.183 38:15 Dr. Landis: But I don't see them looking like they're -
12.1.184 38:18 Alan: Yeah.
12.1.185 38:18 Dr. Landis: And that's why I'm confused, you two talk about it a little because.
12.1.186 38:22 Alan: You could make a different model, but they'd still be the same value. Like, this would be the fourth [points to white rod], this would be the ninth [points to white rod], if you made another model that divided these up into ninths, this would still be a ninth [white rod] and this would still be a fourth [white rod]. Then you could make a model for this, you could have a fourth that was bigger, this would still be a fourth and this would still be a ninth.
12.1.187 38:42 Dr. Landis: I don't know, I'm just puzzled.
12.1.188 38:45 Alan: Well,
12.1.189 38:45 Dr. Landis: Because you're telling me that they're not the same size but then you're showing me with your model that they look the same. That kind of confuses me.
12.1.190 38:52 Alan: [counting out red rods] two three four five six seven eight nine.
12.1.191 38:57 Dr. Landis: Ok, if you have some- I'm going to walk around. Call me back if you kind of, uh, think of something else. Because I'm puzzled.
12.1.192 39:05 Alan: Ok, let's see. I need another blue.
12.1.193 39:09 Kimberly: Blue?
12.1.194 39:10 Alan: Two blues
12.1.195 39:10 Kimberly: I just gave one of our oranges to Graham
12.1.196 39:17 Alan: [Figure S-39-23] Two blues will be a whole, there [red rod] would be one ninth, but this would still be one fourth [white rod], and this would still be a higher number name than this.
12.1.197 39:47 Kimberly: Do you have any spare blues?
12.1.198 39:49 Erik: Yeah, how many? I have like twenty, if you wanted to. [camera shifts]
12.1.199 39:55 Kimberly: Actually, I need four.
12.1.200 39:55 T/R 1: Where I'm confused here is that you're telling me, what's one?
12.1.201 39:57 Alan: Two blues are one
12.1.202 39:59 T/R 1: mmm hmmm
12.1.203 40:00 Alan: [Figure S-41-04] The two purples are one. Here, you divide it into four.
12.1.204 40:03 T/R 1: Did you start out with different size candy bars when you shared?
12.1.205 40:09 Alan: No.
12.1.206 40:10 T/R 1: Why did you make the candy bars different?
12.1.207 40:10 Alan: Because they're one fourth and one ninth up.

12.1.208 40:12 T/R 1: Yeah, but you still started with the same size candy bar, I don't understand why you're switching your candy bars. Obviously you should get differences if you switch your candy bars.

12.1.209 40:20 Alan: What I'm just meaning is these are just models to show my hypotheses.

12.1.210 40:23 T/R 1: Yeah, but you're changing your candy bars, and you're not allowed to do that.

12.1.211 40:26 Alan: I know, these are just to explain the way I'm thinking. I'm thinking that the fourth is bigger than the ninth because if you took two of the same models and you divided it into fourths, those pieces would be bigger. If you divided it into ninths, those pieces would be smaller. D

12.1.212 40:41 T/R 1: So do that for me. Show me, make me the same model and show me.

12.1.213 40:46 Alan: Let's see.

12.1.214 40:47 T/R 1: When you have it, call me.

12.1.215 40:48 Alan: Nine plus nine is eighteen, let's see. I can't wait to go to art. Dark green please? Dark green please?

12.1.216 41:20 Erik: Take us up to a red, give me a red. Take us up to a blue, we have that, then we need a purple

12.1.217 41:46 Kimberly: over here, what are those two things? Wait a minute, wait a minute.

12.1.218 42:56 Alan: Alright, here goes. This is the ninth, those are ninths, but you can't make ninths, you can't make fourths out of two nines. Two nines would be eighteen, no you can't fourth it. [takes purple rods] One two three, no that would be too small. Now what's the next after a purple? A yellow. E/GE/i

12.1.219 43:51 Brian: But, how, how are we going to make ninths?

12.1.220 43:54 Meredith: We can't. Because it's even. [They have four yellow rods and five purple rods, Figure 10-29-29] D/?/f

12.1.221 43:59 Brian: We have to make an odd number. How about twenty-five? Last time I thought of sixty.

12.1.222 44:13 Meredith: Don't forget the fifths.

12.1.223 44:20 Brian: Yeah, um, the fifths.

12.1.224 44:26 Meredith: Because if you're going to do twenty-five, the fifths, the fourths,

12.1.225 44:33 Brian: Oh, uh,

12.1.226 44:35 Meredith: Very hard. You can't make ninths for it

12.1.227 44:41 Brian: We're trying to make ninths.

12.1.228 44:42 Meredith: How about we call this the ninths? This is the ninth, this is one. call this one [blue rod and nine white rods]

12.1.229 44:56 Brian: Yeah, but we can't make fourths and fifths out of them.

12.1.230 45:05 Alan: Nine is an odd number, and fourths are an even number.

12.1.231 45:07 T/R 1: How about half and thirds?

12.1.232 45:10 Alan: Half and thirds, I know, but if you had nine here, you count by twos you can't get to nine. And you can't make a model with fourths and ninths at the same time. E/GE/i

12.1.233 45:17 T/R 1: Maybe you shouldn't be counting by twos. I hear what you're saying. What would happen with the third and the fourth? You don't count by threes or fours? What happens when you work with thirds and fourths? How does it work with thirds and fourths?

12.1.234 45:38 Alan: The thirds and fourths, you're just thirding and fourthing it.

12.1.235 45:42 T/R 1: Well, compare, compare thirds and fourths to see how that works.

12.1.236 45:45 Alan: Ok.

12.1.237 45:46 T/R 1: If you want to. See how that works. See if that'll help for fourths and ninths. Ok?

12.1.238 45:55 T/R 2: You have to make a comparison. It's a tough one, huh?

12.1.239 46:00 Meredith: We're trying.

12.1.240 46:01 T/R 2: What have you tried?

12.1.241 46:04 Brian: Well, we've tried to make this. We've tried making one of these, it's like this.

12.1.242 46:08 Meredith: Well here's something, it shows that you started.

12.1.243 46:08 Brian: You split that in half and you turn it to red, and you put one of the halfs of that and you put it on that side, take one of the halfs and you put it on that side. That's what we've tried. D/R?

12.1.244 46:18 Meredith: But we've tried doing this.

12.1.245 46:19 Brian: Yeah.

12.1.246 46:25 T/R 2: [inaudible]

12.1.247 Brian: Yeah, and it didn't work

12.1.248 46:32 Meredith: And we tried this.

12.1.249 46:35 Brian: Yeah, and we tried an odd number to make the ninths, but we can't have fourths because four is an even number. D/?/f

12.1.250 46:43 T/R 2: I think you want to think, I think you really want to think big in terms of models here, because I was walking on the other side of the room and I saw somebody come up with a model where they were able to use the rods right here to show that, to show the one fourth and the one ninth. So I think you want to think some more about this, ok? It's possible, is what I'm saying.

12.1.251 47:11 Brian: I was thinking of like a sixty or something.

12.1.252 47:12 T/R 2: You made a sixty?

12.1.253 47:12 Brian: Yeah

12.1.254 47:15 T/R 2: Did it work.

12.1.255 47:17 Brian: But it only had um fourths and um

12.1.256 47:20 T/R 1: They did do ninths and fourths

12.1.257 Michael/ Erik: We were so close. We did do ninths, and we came, we came so close

12.1.258 Dr. Landis: They are getting real close. What they said, you know what they just said, which I thought was real interesting, they said if your problem was eighths and fourths, they could build a million models. But this is more

12.1.259 47:21 T/R 2: I'd be interested to see what

12.1.260 48:02 T/R 1: Do a third and a fourth. Compare a third and a fourth, you've done that one, right. You've done a third and a quarter maybe if you do that it will help you understand how to do a fourth and a ninth.

12.1.261 48:08 Michael: The quarters are these.

12.1.262 48:10 T/R 1: Is there enough?

12.1.263 48:11 Michael: These are the quarters

12.1.264 48:14 T/R 1: He's thinking, he's thinking.

12.1.265 48:14 Erik: My brain is very scrambled I have no idea what's going on.

12.1.266 48:28 T/R 1: If you believe that you can do a third and a quarter, right? So what's the difference, between a third and a quarter? What's the difference?

12.1.267 48:36 Michael: One.

12.1.268 48:39 T/R 1: One what? What number name would you give to that white rod?

12.1.269 48:43 Erik: One twelfth.

12.1.270 48:45 T/R 1: One twelfth. You can do a third and a quarter, you said it's a twelfth, not one, right? Does that help you?

12.1.271 48:52 Michael: Oh, I get it!

12.1.272 48:53 T/R 1: You do?

12.1.273 48:54 Michael: Yeah.

12.1.274 48:56 T/R 1: What do you get?

12.1.275 48:57 Michael: Sort of like, um,

12.1.276 49:01 T/R 1: What's a half and a third?

12.1.277 49:03 Michael: A half and a third? The half is a half

12.1.278 49:07 T/R 1: If I compared a half and a third, you wouldn't get one, would you?

12.1.279 49:09 Michael: No.

12.1.280 49:10 T/R 1: What would you get?

12.1.281 49:11 Michael: One twelfth.

12.1.282 49:13 T/R 1: Show me. You said a quarter and a third, right, you got a twelfth, now compare a half and a third.

12.1.283 49:21 Michael: A half and a third?

12.1.284 49:35 Erik: I think I have a conclusion.

12.1.285 49:40 Michael: Oh I've got an idea!

12.1.286 49:40 Erik: I do too, I do too. I think I got it somehow.

12.1.287 49:44 T/R 1: Try this one.

12.1.288 49:45 Erik: Dr. Maher, I think I have it somehow.

12.1.289 49:50 Michael: It's a sixth!

12.1.290 49:51 Erik: I think I got it, Michael I think I got it.

12.1.291 49:55 Michael: Sixth, and then the thirds...

12.1.292 49:58 Erik: Mike, I think I got it.
12.1.293 50:01 Michael: It's one, what? It's one sixth. Look at that! It's one sixth.
12.1.294 50:08 T/R 1: What do you have with a half and a third, it's not one, huh? And a quarter and a third what did you say it was?
12.1.295 50:18 Michael: One twelfth.
12.1.296 50:20 T/R 1: Now we have a fourth and a ninth.
12.1.297 50:23 Erik: I think I just figured it out somehow.
12.1.298 50:25 T/R 1: What did you figure out, Mi- Erik?
12.1.299 50:28 Erik: Well what I did was I
12.1.300 50:29 T/R 1: You're really trying to confuse me, you two. Michael's now sitting by Erik, and Erik's now sitting by Michael [laugh] and now you want to know why I call you by the wrong names. Ok, tell me what you figured out.
12.1.301 50:39 Erik: Well I don't know, I did this thing, I don't know if it's, kind of tricky but - I said like this, three blues and then for the ninths I did light greens, and then for the fourths I did three browns and then I took a light green because I figured that evenly odd, even, odd it could take one and then put it and
12.1.302 51:04 Michael: My first thing was that I got, that I could get ninths out of was eighteen so I made an eighteen rod and I couldn't get fourths out of it.
12.1.303 51:08 T/R 1: Just think what you just did. You did very important things. You know what do me a favor, if you would build the two models you just made for comparing a half and a third and a quarter and a third on the board.
12.1.304 51:30 Alan: In one model. So that means up here there are ninths. You can't fourth that. It's not even. Now here
12.1.305 51:38 T/R 2: Yeah, but a third wasn't even.
12.1.306 51:41 Alan: Right, but you can't do it! Because you can't ninth that, nor can you ninth that, none of those, nor can you ninth two oranges in any way. Nor can you ninth any other combination, you can only ninth one blue or two blues. But you can't fourth a blue. E/GE/f
12.1.307 51:52 T/R 2: Are you sure about that?
12.1.308 52:06 Alan: Pretty much. I guess. What? You too? You too?
12.1.309 52:12 T/R 2: I'm just thinking back to some other models that we've built over the past month and I'm not
12.1.310 52:17 Kimberly: I though I was the one driving him crazy, not him driving me crazy. [T/R 2 laughs].
12.1.311 52:26 T/R 2: See, let me ask you another question, do you have to use blues in order to show ninths? Is there anything else you can use?
12.1.312 52:32 Alan: Yeah, no but I know but if you try singular things a singular blue you could ninth but you have to either add another rod onto some other rod to ninth cuz you can't use the two, the four, the two, the three, the four, the five, the sixth, the

seventh the eighth, ... you can't do it, that equals up to twenty. You can't ninth twenty.

12.1.313 53:21 T/R 1: We have Graham and Kelly they have a model they want to share

12.1.314 53:24 Dr. Landis: Great I'd love to hear about that!

12.1.315 54:29 Jessica: [to Graham and Kelly] Did you get ninths yet? Did you get fourths?

12.1.316 54:37 : [Michael and Erik build the following models on the OHP: an orange and red train with four light green rods, three purple rods, and a white rod, and an orange and red train with two dark green rods, three purple rods, and a red rod. Figure 10-29-20]

12.1.317 58:48 T/R 1: [talking to Erik, Michael, Kelly, Graham, David, and Meredith] How is that model related if at all to these models?

12.1.318 58:53 Erik: Well they don't have fifths, and they don't have this.

12.1.319 58:56 T/R 1: No, they don't

12.1.320 58:57 Michael: They have ninths - like that! One two three four five six seven eight nine.

12.1.321 59:02 T/R 1: They're comparing ninths and fourths, Meredith, why do they need fifths. Your theory is they need fifths. Now they're comparing ninths, this is fourths, right? [Lays down blue rod] And this is ninths, is that correct? [Lays down purple rod]

12.1.322 Graham: Yeah [Figure 10-29-16]

12.1.323 T/R 1: You're comparing fourths and ninths and it's this, ok? [Graham hands teacher the yellow rod, and she shows that the yellow and purple are the same length as the blue rod]

12.1.324 59:21 Meredith: That's my method.

12.1.325 T/R 1: Well, so what number name are you going to give this? [Talking about the yellow rod]

12.1.326 Graham: This?

12.1.327 T/R 1: Wait a minute, let me see what you have here. This is one two three four and this is one two three four five six seven eight nine.

12.1.328 Graham: What, the white ones? What would we give the white ones?

12.1.329 T/R 1: You're saying how much is the difference? Do you have any more white ones? Can you get some or borrow some? Ok, let's see. Meredith?

12.1.330 Kelly: It's bigger by one fifth because you see. [Points to blue rod with one purple and five whites next to it] D/?/f

12.1.331 59:23 T/R 1: What number name is this?

12.1.332 Graham: Thirty-fifths, one thirty-fifth.

12.1.333 T/R 1: Thirty-fifths.

12.1.334 Graham: Yeah. [T/R 1 straightens out model, Figure 10-29-17]

12.1.335 T/R 1: One two three four five six seven eight nine. And how many of them are there here? Counting? [Graham counts]

12.1.336 Graham: Thirty-six.

12.1.337 Meredith: What?
12.1.338 Graham: Thirty-six.
12.1.339 Voice: Thirty-six?
12.1.340 Graham: Yeah.
12.1.341 Voice: So what would that white one be?
12.1.342 Graham: One thirty-six. Ok, he's right.
12.1.343 T/R 1: So what do you have here? What did you come up with, Kelly?
12.1.344 Kelly: One thirty-sixth.
12.1.345 T/R 1: How many? What's the difference?
12.1.346 1:02:27 Graham: Well, there's thirty-six. [In addition to the larger model of a train of three orange rods and a dark green rod, nine purple rods, four blue rods, and thirty-six white rods, there is a small model of a blue rod, a purple rod, and five white rods]
12.1.347 1:02:30 T/R 1: There's thirty-six of these?
12.1.348 1:02:30 Graham: Yeah, the whites.
12.1.349 1:02:32 T/R 1: And what's the difference between the two? How many of the thirty-sixths?
12.1.350 1:02:38 Graham: Five.
12.1.351 1:02:39 T/R 1: So, the difference between one ninth and one quarter is how much?
12.1.352 1:02:40 Graham: Five
12.1.353 Kimberly: Thirty-sixths.
12.1.354 T/R 1: Five thirty-sixths.
12.1.355 Meredith: And one fifth.
12.1.356 T/R 1: Well, where's the one fifth?
12.1.357 Meredith: Well, if you had one
12.1.358 Kelly: There's no one fifth.
12.1.359 1:02:49 T/R 1: Do you think that this is five thirty-sixths. If you could imagine one fifth in here,
12.1.360 1:02:56 Meredith: Yeah.
12.1.361 1:02:56 T/R 1: Right?
12.1.362 1:02:56 Meredith: Uh huh
12.1.363 1:02:56 T/R 1: You could imagine one fourth, it's the blue, right? Is this [yellow rod] one fifth? For one fifth, [T/R 1 places five yellow rods on the model] Could that be one fifth? Is that big enough to be one fifth?
12.1.364 Kelly: I don't think it's one fifth.
12.1.365 1:03:14 Meredith: Well, but it does have uh five thirty-sixths in there.
12.1.366 1:03:19 T/R 1: It's this length, but this has the number name, what, what, the yellow has what number name?
12.1.367 1:03:29 Meredith: Five thirty-sixths.
12.1.368 1:03:30 T/R 1: Five thirty-sixths. Not one fifth, right?
12.1.369 1:03:33 Meredith: Uh huh.
12.1.370 1:03:33 T/R 1: Think about what is causing the difficulty, ok, Meredith? [to class] Ok, is this a good time maybe to pull together for a few

minutes and do some sharing? [no] Is this a good time? [to Kelly] Keep your model here. [to class] Ok. Is it possible, can, can I have your attention for a minute, we have a little bit of extra time thanks to Dr. Landis, uh, she's given us a little extended time, but we have some interesting ideas here and I think it's really important to share our ideas, I see some wonderful models another model here, right, with, um, Mark and Audra, right? You have another model. I guess, um, I was very interested in listening to your ideas as I walked around and I heard um our, does anyone, did anyone change their mind what they thought the difference between, uh, one quarter and a ninth were? Did anybody change their mind? Some of you changed your minds? How many of you still aren't sure about that difference [some students raise their hands]. Ok, so, so we had a theory, let's call it Meredith's theory, but she may have changed her mind she may not have, but Meredith's theory seems to suggest that if you wanted to find the difference between one fourth and one ninth that it's one fifth. That was the theory that we were testing, right? Now, if you used that same theory and I asked you what the difference was between one quarter and a third, and you applied that theory, what would you have said the difference was between a quarter and a third?

12.1.371 1:05:48 Meredith: A quarter and a third?.

12.1.372 T/R 1: Using that same theory.

12.1.373 Meredith: A quarter and a third would be, well, how big would the third be.

12.1.374 1:06:06 T/R 1: Ok, well one of the gentlemen here who have built the models up here, can you all kind of listen for a minute to what Michael and Erik and um James have built

12.1.375 1:06:15 Students: James?

12.1.376 1:06:16 T/R 1: I'm sorry, not James, it's David.

12.1.377 1:06:21 Michael: Um, uh, well, what me and Erik, me and Erik started building models like these to try and help us figure out how to one fourth and one ninth, and Dr. Ma- and um, and then we were on the edge of trying to find it out and then we had another model we started just we lost the idea of that was that we had before and

12.1.378 1:06:53 T/R 1: Do you want to tell us what that idea was?

12.1.379 1:06:54 Michael: Well, that idea was, try to get, try to um find the number and divide, um, divide it and see if it equals nine, then you've got a ninth, but we found that every single one that we tried there wasn't a fourth if there was a ninth, and if there was fourth there wasn't a ninth. So, um, we, we, um, we decided to try a new idea it turns out when we, uh, when we tried the new idea, the first time we tried it we were wrong.

12.1.380 1:07:28 T/R 1: What was that new idea?
12.1.381 1:07:31 Michael: Well, I don't really remember what we were thinking.
12.1.382 1:07:34 T/R 1: Was it the odd and even?
12.1.383 1:07:40 Michael: Yeah, I think so, yeah, what I also figured, um, is that you, it's so hard, like if you had you had to make a model with one fourth and one eighth in it, we could make a ton of them, but it's hard to make a model that has an odd number, like one ninth, and a even number, which is one fourth. So I figured that that was really hard and we made only like two models or so of it and it was really hard to find to get a train to something like that. D/?
12.1.384 1:08:11 T/R 1: Ok, so where did that leave you. You told me there couldn't be any models when you had an odd and even.
12.1.385 1:08:17 Michael: I know. But then we figured that it had to be, because there was no other way to do it.
12.1.386 1:08:22 T/R 1: But you built two models here and you're comparing fractions where, you have an odd and even number on
12.1.387 1:08:28 Michael: Well, I didn't really, I was just trying to get an idea from these old models and I didn't get one, but I guess Dr. Maher did, so she wanted us to come up and say what we were thinking, I was just trying to get an idea from it.
12.1.388 1:08:46 T/R 1: When you compare this top one, what numbers were you comparing when you built this model here? [Continuing figure 10-29-20]
12.1.389 1:08:50 Michael: One third and one fourth.
12.1.390 1:08:52 T/R 1: And what did you find?
12.1.391 1:08:53 Michael: We found that it worked.
12.1.392 1:08:53 T/R 1: What worked?
12.1.393 1:08:54 Michael: That an odd and an even can go into a whole.
12.1.394 1:09:02 T/R 1: So, you mean you compared the quarter and the third, what did you find to be that difference?
12.1.395 1:09:07 Michael: The difference would be, the difference would be one twelfth. But in this model with the half and the third it would be one sixth. D/?
12.1.396 1:09:18 T/R 1: Ok, so you could do that. Ok, um, alright, now James did James has some idea here let's here what James says and we all know that Graham and, why don't you sit down? Thank you very much, gentlemen. And let's, let's hear what James' idea is and then we'll hear if Graham and Kelly agree. Where did Graham go?
12.1.397 1:09:40 James: [at OHP] can I take this off?
12.1.398 1:09:40 T/R 1: Yeah, sure.
12.1.399 1:10:26 James: [James put an overhead transparency on OHP, Figure 10-29-21] Well, like, I got a huge model for this problem. First, but by experimenting I tried nine yellows and four oranges, for the ninths and the fourths. and I found out they weren't equal

so I tried something else. I lowered its size so orange and uh the orange and the yellow and we got blue as the fourths and purple as the ninths and they were equal. So I just had to find a whole for that and I found out it was I just took three oranges and one dark green so then I had then I put up thirty-six whites on up to the whole and there, it took five whites to make the purple equal to the blue, so I think the answer would be five thirty-sixths. D/?

12.1.400 1:10:38 T/R 1: Anybody do anything like that?

12.1.401 1:10:40 Erik: Well I guess I sort of

12.1.402 1:10:43 T/R 1: Oh, Erin, Jackie, Beth, what did you do? Did you do something like that?

12.1.403 1:10:47 Erin and Beth: Uh, yes.

12.1.404 1:10:48 T/R 1: Just tell us about it.

12.1.405 1:10:48 Jackie: Um, well, we did the same thing we have the same fourths and the same ninths

12.1.406 1:11:52 Beth: But we have a different whole.

12.1.407 1:11:58 T/R 1: So you called one and you used different rods to show your one?

12.1.408 1:12:01 Beth: Yeah.

12.1.409 1:12:02 T/R 1: Ok, and so, uh, can you move aside a little bit, Erik, so the class can see? Uh, so your model here, it looks very much the same as James' model

12.1.410 1:12:12 Jackie: Except we have, instead of three oranges and one dark gree we have one dark green, one orange, one red, um, one black, one brown, and a light green.

12.1.411 1:12:27 T/R 1: Ok, so what rod did you give the number name one quarter?

12.1.412 1:12:36 Erin: Um blue.

12.1.413 1:12:37 T/R 1: The dark blue? And what rod did you give the number name one ninth?

12.1.414 1:12:40 Girls: Purple

12.1.415 1:12:41 T/R 1: Did you do the same thing?

12.1.416 1:12:41 James: Yeah.

12.1.417 1:12:45 T/R 1: Did anybody else do that? You did that and you did that and you did that and you did that? Ok, and so what number name did you give the white one?

12.1.418 1:12:52 Girls: Thirty-sixths, one thirty sixth.

12.1.419 1:12:54 T/R 1: One thirty-sixth? And what did you find the difference was between the ninth and the quarter?

12.1.420 1:12:59 Jackie: Five thirty-sixths.

12.1.421 1:13:02 T/R 1: How many of you got five thirty-sixths? I see. I see. Ok, what do you think? So, so you can actually see, what makes this problem so hard? What makes it so hard?

12.1.422 1:13:21 Kimberly: The odd number and the even number.

12.1.423 1:13:25 T/R 1: Pardon?

12.1.424 1:13:25 Kimberly: The odd number and the even number.

12.1.425 1:13:28 T/R 1: The odd and the even number? What about that makes it hard? You have a four and a nine.
12.1.426 1:13:32 Kimberly: Because it's harder to make a model when you have an even number for one and an odd for the other.
12.1.427 1:13:39 T/R 1: Ok, now have you learned anything on the models that you've seen today that might help you get some ideas for how to pick that number? If you remember that Erik and Michael when they compared a half and a third, what was your difference?
12.1.428 1:13:57 Michael: A half and a third?
12.1.429 1:13:58 Erik: A half and a third was
12.1.430 1:13:59 Michael: Was one sixth.
12.1.431 1:14:00 T/R 1: When you compared a half and a third it was one sixth. And when you compared a third and a quarter?
12.1.432 1:14:06 Erik: It was, it was, one one twelfth.
12.1.433 1:14:07 T/R 1: It was one twelfth. And when you compared a quarter and a ninth?
12.1.434 1:14:17 Michael: A quarter and a ninth?
12.1.435 1:14:18 T/R 1: One fourth and one ninth?
12.1.436 1:14:19 Michael: Oh.
12.1.437 1:14:21 T/R 1: It became, who did it here? You did it here, Erin and Beth you got five thirty-sixths.
12.1.438 1:14:33 Michael: Oh, it sort of went up by six I guess. E/GE/f
12.1.439 1:14:37 T/R 1: It's something to think about, isn't it? It's something to think about, right? Well we have here, thank you very much, and Kelly and Graham and all of those wonderful models, I'm going to keep this, that's lovely, thank you. How many of you believe the difference is five thirty-sixths, raise your hands. If you don't believe it, if you need to walk over to these models before we put them aside and see what they've done. When, we compared one half and a third, we got one sixth. When we compared a third and a quarter, right? We got one twelfth. When we compared a quarter and a ninth we got five thirty-sixths. [Writes on transparency: 1/2- 1/3 = 1/6, 1/3 - 1/4 = 1/12, 1/4 - 1/9 = 5/36, Figure 10-29-22.] Is there anything in these numbers that relate to the model you built? That's my question. We'll let you think about that. If you haven't built the model, I really think we have enough people here, we have Kelly and Graham, we have the table in the back, what do you think? Ok, so we can think about them. I'm wondering if there's anything that might give you a clue to building your models in the future. Maybe you ought to try to build some more and study these a little bit. It's something to think about. Ok, I'm going to see you on Monday, good! We get to talk some more. Thank you very much and thank you

	for staying longer, I appreciate, Mrs. Phillips, the extra time. A really good job.
12.1.440 1:16:40	Clean up

Session 12, October 29, 1993, Front View

Line	Time	Speaker	Transcript
12.2.1	5:32	T/R 1:	Well, good morning. [students Good morning] What did you do yesterday in math? [students raise hands] Ah, All these people are going to tell me. Amy,
12.2.2		Amy:	We did, we figured out the chocolate, we divided chocolate.
12.2.3		T/R 1:	Oh. Did you all agree?
12.2.4		Students:	Yeah.
12.2.5		T/R 1:	You agreed! Was that an easy decision?
12.2.6		Andrew:	Yeah.
12.2.7		T/R 1:	No discussion, or, or differences?
12.2.8		Andrew:	Well, a little
12.2.9		T/R 1:	How did that work.
12.2.10		Andrew:	Well, we um like divided us into groups, the class into groups and um, and our, in my group, there was like nine people, so each person got like, um one and one ninth.
12.2.11		T/R 1:	How did you decide that? How much did you have to start with?
12.2.12		Andrew:	We had uh ten pieces.
12.2.13		T/R 1:	Ten pieces. I see, how did you do one and one ninth? I'm curious.
12.2.14		Andrew:	Well, we um, we said there was nine people, so we had to give a whole piece of candy to each person and then we had one left over so we would have to, and there's nine people, so if we divided it into ninths there would um be enough, for everyone. D/?/i
12.2.15		T/R 1:	Is that hard to do?
12.2.16		Andrew:	Yeah, a little.
12.2.17		T/R 1:	But you did it?
12.2.18		Andrew:	Yeah.
12.2.19		T/ R 1:	And you all felt good about it?
12.2.20		Graham:	Yeah.
12.2.21		T/R 1:	Oh, and you were in that group too, Graham, huh?
12.2.22		Graham:	Yeah.
12.2.23		T/R 1:	What about another group? What did another group do? You were in a different group? Jessica, what did you do?
12.2.24		Jessica:	Well, my group, we like had uh, eight people in our group, so well, we each got one whole piece and then we had two pieces left over so then we divided each of the two pieces into fourths. D/?/i
12.2.25		T/R 1:	And, so, how, how much did each person get?
12.2.26		Jessica:	One and one fourth.
12.2.27		T/ R 1:	You got one and one fourth. Did you all think that was fair, in that group? [mmm hmmm] Did the people in Andrew's group get the same amount as the people in Jessica's group?

[no] Who got more, the people in Andrew's group or the people in Jessica's group? Michael?

12.2.28 Michael: The people in Jessica's group.

12.2.29 T/R 1: The people in Jessica's, now, of course I could ask you how much more, you think you could you figure that out? You don't have to tell me that right now.

12.2.30 Michael: Yeah

12.2.31 7:53 Meredith: Yeah, if we got one ninth and they got one fourth, then um, nine minus four equals five, so they got um one fifth bigger, than we… D/?/f

12.2.32 T/R 1: Say that again?

12.2.33 Meredith: See, um, we had, each of us had one and one ninth.

12.2.34 T/R 1: Let's see, let's see, Andrew's group had nine people, right? Each person,

12.2.35 Andrew: Got one and one ninth

12.2.36 T/R 1: And in Jessica's group, eight people and each person got, you said,

12.2.37 Jessica: One and one fourth

12.2.38 T/R 1: One and one fourth.

12.2.39 Meredith: And

12.2.40 T/R 1: And so, you're telling me,

12.2.41 Jessica: But there was another group.

12.2.42 T/R 1: Maybe we'll hear about the other group and we'll come back to this, but I also didn't want to lose what Meredith said, what Meredith said was the people in Jessica's group got more than the people in Andrew's group. [Meredith laughs] and I, I kind of asked how much more

12.2.43 Meredith: Nine minus four equals five so they got one fifth more.

12.2.44 T/R 1: So you're claiming, this is Meredith's claim

12.2.45 Meredith: Yeah.

12.2.46 T/R 1: [writing on overhead transparency, figure 10-29-01] That each person in Jessica's group got how much more did you say Meredith? Got one fifth more than each person in Andrew's group. How many of you believe that? [all students raise hands]. Ok, you're gonna have to then convince me. But we'll let that hold for a minute. But who's the other group? [there was three groups] Ok, who was, who was in a different group? A group other than Andrew's and Jessica's group? Kimberly? Ahah. How many in your group, Kimberly?

12.2.47 Kimberly: There, we each got one and one fourth.

12.2.48 T/R 1: How many people in your group?

12.2.49 Kimberly: Eight

12.2.50 T/R 1: Eight people? And in your group got

12.2.51 Kimberly: One and one fourth.

12.2.52 T/R 1: One and one fourth. So the people in Jessica and Kimberly's group, right? You're claiming got more. And the difference you claim is

12.2.53 Kimberly: Five, one fifth.

12.2.54 T/R 1: Wow, that's a good question. I don't know you got one fifth. Um, it's sort of like saying to me, if I got a half, and Amy got a quarter, right? Who got more? I got more, right? I got a half, ok, and Amy got a quarter, but by your theory, you would tell me that I got, how much more?

12.2.55 Students [including Meredith]: One fourth

12.2.56 T/R 1: But you would have told me a half more, think of the way you did that problem.

12.2.57 Meredith: Oh

12.2.58 T/R 1: Is that right, Meredith? Right? Did I get a half more [Meredith laughs]. You all know I didn't get a half more. I got how much more?

12.2.59 Meredith: Oh.

12.2.60 T/R 1: A quarter more. But if I used your method of figuring out how much more I'd be subtracting the four and the two, and I'd end up with a half more. That sort of doesn't make sense, does it? You still believe that it's a fifth more for sure? How many of you are not so sure? Not so sure [all students raise hands] It's a good question. Hmmm, I don't know. Well, we ought to keep this question in mind and uh, we ought to try to answer it, don't you think? Ok? I guess maybe another way to ask that question might be, might be what? You tell me what you think the question is. What are the fractions that are of interest in this problem? What decides who got more, the people in Andrew's group or the people in Jessica and Kimberly's group? Meredith?

12.2.61 12:40 Meredith: I know that Jessica and Kimberly's group got more than Andrew's group did.

12.2.62 T/R 1: We know they got more, right?

12.2.63 Meredith: Right.

12.2.64 T/R 1: And what number tells you?

12.2.65 Meredith: Well, um, if they, uh, one fourth, and one rod was like, the one rod, and you had ninths and you had fourths, if you had the fourths they would take

12.2.66 T/R 1: You're talking about ninths and fourths, is that right?

12.2.67 Meredith: Yeah.

12.2.68 T/R 1: You all agree that it's ninths and fourths that's at issue here? And it's not the one piece? So let's focus on the ninth, right, and let's focus on the fourth. So which one you're claiming is bigger?

12.2.69 Students: The one fourth.

12.2.70 T/R 1: The one ninth is smaller

12.2.71 Students: Yeah.

12.2.72 T/R 1: Did you ever see that symbol, smaller than? [Figure 10-29-02]

12.2.73 Students: Yeah

12.2.74 T/R 1: One ninth is smaller than one fourth? Ok? So I'm sorry, Meredith.

12.2.75 Meredith: And, um, if you take a one rod and you divide it into ninths and fourths, the fourths are going to be larger because they're less. So they're going to be larger. So each person is going to be getting a larger piece. A?

12.2.76 T/R 1: Ok, so you've convinced me that if I could imagine a rod that I call one, and I imagine four pieces and I think of one of those pieces, that's going to have what number name?

12.2.77 Meredith: Fourths.

12.2.78 T/R 1: And if I take that same rod and imagine nine pieces, one of those pieces will have the number name.

12.2.79 Meredith: Ninths.

12.2.80 T/R 1: One ninth. And you could imagine in your head without the rods you're telling me that the one ninth is, that the one fourth is

12.2.81 Meredith: Bigger than

12.2.82 T/R 1: Bigger than the one ninth, or the one ninth is smaller than the one fourth. The question I'm asking is the difference one fifth? Now, could you imagine the fifth rod, what that looks like?

12.2.83 Meredith: Um, I think it would be the yellow rod, I'm not sure, I think it was the yellow rod that was the fifth.

12.2.84 T/R 1: Whatever you're thinking, but you could imagine a fifth, you could imagine a fourth, you could imagine a ninth, and do you imagine in your head, my question, do you imagine in your head that the, if you'd compare the one fourth rod and the one ninth rod, the difference would be the one fifth rod, do you think that, does that make sense to you, as you're imagining this in your head?

12.2.85 Meredith: Ummm, if you put the four and the five together it would equal up to the ninth rod. A? f

12.2.86 T/R 1: You think so? [mmm] I think we ought to get out the rods.

12.2.87 CT: Yeah.

12.2.88 15:09 T/R 1: I think we ought to get out the rods, what do you think? How many of you want to work on this? How many of you want to know how much more the people in Andrew's and Je- uh, Andrew's group, uh Jessica's group and Kimberly's group got than the people in Andrew's group. [Students raise hands, Dr. Landis enters with rods] Can somebody tell Dr. Landis the problem because she doesn't know any of the story of any of what happened and how this all came to be, could someone be so kind as to tell Dr. Landis the whole story? Kimberly do you want to give it a try? Dr. Landis? Do you want to hear what's going on?

12.2.89 Dr. Landis: I do want to hear it, yes!

12.2.90 15:46 Jessica: It's not, it's not going to be an orange

12.2.91 Andrew: I'm going to make a whole model

12.2.92 Jessica: It's a yellow, that's a fifth, you don't have to make a whole model.

12.2.93 Andrew: To figure it out you do. Here's fourths and ninths would be one two three [Andrew's model is an orange and red train, and four light green rods, Figure 10-29-03]

12.2.94 Jessica: Well that's ninths, cuz it's sixths [Jessica's model is a yellow rod and four white rods, Figure 10-29-04]

12.2.95 Andrew: What are you doing?

12.2.96 Jessica: Nothing. [Andrew lines up six red rods, Figure 10-29-05]. This one doesn't show ninths

12.2.97 Andrew: Gotta make it bigger. Orange and a purple

12.2.98 Jessica: You do? You have to make it bigger if it doesn't work? Now I need the green.

12.2.99 Andrew: I need the purples.

12.2.100 Jessica: I need the purples. [Jessica makes an orange and purple train, first places green rods, then removes them, Figure 10-29-06]

12.2.101 Andrew: I have to make even bigger!

12.2.102 Jessica: Wait, first you make it small

12.2.103 Andrew: Let's try two more bigger. How about the brown, that's good

12.2.104 Jessica: Look you can make it like this. And then you have it, look. Andrew.

12.2.105 Andrew: I don't want to hear it.

12.2.106 Jessica: Well, it's the same thing as the greens, I think it is. Yeah it's the same thing. Could I have another box? Ok.

12.2.107 Andrew: One, two, three, four, five, six, seven, eight. [Andrew has built an orange rod next do a dark green rod with eight red rods beside it, Figure 10-29-07] No! Ahh. I'm gonna die! I am really, really, really, really gonna die. You know how far I've gotten up to?

12.2.108 Jessica: Look at this purple, it looks red. No it doesn't.

12.2.109 19:15 Andrew: Make it go smaller. We need to do something smaller

12.2.110 Jessica: This is working, I know what I'm going to do. Like this is more than nine, oh reds.

12.2.111 Andrew: I can't find the fourths. Maybe these are fourths.

12.2.112 Jessica: Five, six, seven, eighths, ninths, didn't work.

12.2.113 Andrew: I have ninths, I have thirds. If there's thirds, there has to be fourths, but I cannot find fourths. [Andrew has an orange next to a brown, nine red rods, and three dark green rods, Figure 10-29-08]

12.2.114	Jessica:	These don't work. There's a million purples. Oh, you just have to add a red.
12.2.115	Andrew:	I'm just doing two oranges and a brown. I've had it!
12.2.116	Jessica:	You're not going to get, oh yeah, you are.
12.2.117	Andrew:	I want fourths. Fourths is going to be the browns. One, two
12.2.118	Jessica:	Fourths. How do you come up with fourths?
12.2.119	Andrew:	Three, four.
12.2.120	Jessica:	Three four
12.2.121	Andrew:	Yes! I have fourths. Fourths are
12.2.122	Jessica:	Brown
12.2.123	Andrew:	No
12.2.124	Jessica:	Blue
12.2.125	Andrew:	No
12.2.126	Jessica:	It could be blue, no
12.2.127	Andrew:	Black. Fourths are blacks. One two three. There's the fourths. [Andrew has two orange rods and a brown rod, with four black rods next to it, Figure 10-29-09] Now, I'm going to find the ninths.
12.2.128	Jessica:	Ninth I think I know what the ninths are
12.2.129	Andrew:	Two four six eight. Four, oh thanks a lot, Jessica. You could use reds to substitute the ninths.
12.2.130	Jessica:	These are one, two three four, these are sevenths. [laughs]
12.2.131	Andrew:	Oh, good, give me those two in your hand
12.2.132	Jessica:	I need 'em I'm trying something
12.2.133	Andrew:	You don't need 'em
12.2.134	Jessica:	Ok, one. I'm trying to see something.
12.2.135	Andrew:	What are they? Come on, I'll find out for you. Just, fine, have it your way.
12.2.136	Jessica:	If I put reds that's fourteenths.
12.2.137	Andrew:	Do I care? Sevenths, one two three four five six seven. And two more of these, is four of these, and a half, a half, a half
12.2.138	Jessica:	One two three, four, fifths, sixths
12.2.139 23:19	T/R 2:	How's it going? I'm sort of watching what you're doing. What have you tried here?
12.2.140	Andrew:	Well we've tried
12.2.141	Jessica:	Eighths, oh I think I've
12.2.142	Andrew:	like um
12.2.143	Jessica:	got it.
12.2.144	Andrew:	We've tried making models
12.2.145	Jessica:	We're trying to make one model that has
12.2.146	Andrew:	Fourths and
12.2.147	Jessica:	And ninths
12.2.148	Andrew:	Ninths
12.2.149	T/R 2:	And what have you tried? What are some of the things you've tried to call one?

12.2.150 Andrew: Well we tried to call this one, [orange and red] this [orange and purple,
12.2.151 Jessica: I think I've got it.
12.2.152 Andrew: this, and now we're working on this.
12.2.153 T/R 2: Oh, that's interesting. Ok.
12.2.154 Jessica: I got, I th- I just got, I thought I got ninths here, the light green, but then I counted one two three four five six seven eighths ninths, and then I have that little space. [Jessica has two orange rods and a brown rod next to four black rods and nine light green rods, Figure 10-29-10]
12.2.155 T/R 2: Something hanging over there, ok.
12.2.156 Andrew: So then it's going to be impossible. So then you need
12.2.157 T/R 2: So you have a plan? What are you going to try next?
12.2.158 Jessica: Oh, purples. No but I did.
12.2.159 Andrew: Cut this [the brown rod that is part of the two orange and one brown train] in half this is a purple
12.2.160 Jessica: I can add
12.2.161 Andrew: So let's try a purple
12.2.162 Jessica: And then put browns there maybe, wait, yes. It's working. Then I could put browns there.
12.2.163 T/R 2: Ok, I'll let you experiment some more. Let me know if you come up with one that works, ok?
12.2.164 24:41 Jessica: Ok, so I think I got one. That
12.2.165 Andrew: How about just two oranges
12.2.166 Jessica: I'm doing two oranges, a brown, and a white.
12.2.167 Andrew: Fine.
12.2.168 Jessica: And I think it's working, wait
12.2.169 Andrew: Of course it's, oh no it's not going to work
12.2.170 Jessica: Oh it's not working. [laughs] I just add something every time. Brown, oh it didn't work. Just do two oranges.
12.2.171 Andrew: Maybe I can find fifths.
12.2.172 Jessica: I found fifths. Fifths are um purples.
12.2.173 Andrew: Six [Andrew's model is two orange rods, four yellow rods, and six green rods, Figure 10-29-11] eight nine
12.2.174 Jessica: Fourths, I think, what are ninths? Maybe greens, are greens ninths?
12.2.175 Andrew: No.
12.2.176 Jessica: You added something? I'm just going to take the greens away and try to get ninths.
12.2.177 Andrew: I found fifths.
12.2.178 Jessica: I found fifths and fourths
12.2.179 Andrew: No, no I didn't find fifths.
12.2.180 Jessica: Fifths are um, purples
12.2.181 Andrew: Ok.
12.2.182 Jessica: Ninths. Reds.
12.2.183 Andrew: I think, no I can't

12.2.184	Jessica:	Oh you can make like two reds one.
12.2.185	Andrew:	So
12.2.186	Jessica:	So then you could do it.
12.2.187 27:20	Andrew:	It's not working. Let's try four oranges
12.2.188	Jessica:	One, two, three, four, five. Fifths. I have two fifths. [Camera goes to James, who has nine purple rods in a row and is trying to match other colors to be the same length, Figure 10-29-12]
12.2.189	Andrew:	I'm just doing an old problem that we made like a year ago.
12.2.190 27:46	T/R 1:	What are you trying to do, Andrew and Jessica? That's interesting.
12.2.191	Both:	We're trying to figure out
12.2.192	Jessica:	We're trying to figure out a problem that has both of them in it
12.2.193	T/R 1:	That has both of what in it?
12.2.194	Jessica:	Like, um, both one ninth and one fourth.
12.2.195	T/R 1:	Very good.
12.2.196	Jessica:	And we got one fourth
12.2.197	T/R 1:	Ok
12.2.198	Jessica:	We got one fifth and we still, we need one ninth. And I was just trying to count that as one, then one, and one.
12.2.199	T/R 1:	Ok, so that's what you're working on. Ok that's a good thing to work on
12.2.200	Jessica:	And one and one, but that doesn't work, that's this again.
12.2.201	T/R 1:	Ok, ok. Good
12.2.202	Andrew:	One fourth, two fourths,
12.2.203	Jessica:	You're doing that one again? This doesn't work.
12.2.204	Andrew:	Back to the old biggies.
12.2.205	Jessica:	How many oranges was that? Four.
12.2.206	Andrew:	I need. Yippee. Fourths
12.2.207 29:08	Dr. Landis:	Andrew, what are you doing?
12.2.208	Jessica:	It's not working. We're building
12.2.209	Andrew:	I made like this um big model
12.2.210	Jessica:	A long time ago so we're trying it again
12.2.211	Andrew:	A long time ago, I'm trying it again because I want to figure out how much is one ninth, how much is one fourth bigger by, bigger than one ninth.
12.2.212	Dr. Landis:	Ok, ok.
12.2.213	Jessica:	Is there a brown there?
12.2.214	Andrew:	I have the fourths, two of these equal a fourth. [Andrew has four oranges and two purples in the model, and he has six browns lined up next to it, Figure 10-29-13] Two browns I'm putting together and they're fourth. So now
12.2.215	Dr. Landis:	Say that again. Two browns, what are you doing?
12.2.216	Andrew:	Two browns
12.2.217	Dr. Landis:	Yeah

12.2.218 Andrew: Like two browns are the fourths, and I remember that I put two of these together, they would be the thirds
12.2.219 Dr. Landis: Right
12.2.220 Andrew: So I have thirds but I don't need thirds
12.2.221 Dr. Landis: Uh huh
12.2.222 Andrew: These two are one fourth
12.2.223 Dr. Landis: Oh I see
12.2.224 Andrew: So how many fourths do you have there?
12.2.225 Andrew: One two three, I need one more. Two more
12.2.226 Jessica: We need a lot more browns
12.2.227 Dr. Landis: Do you think maybe, you're running out of rods, do you think if you work together to build one model that would help?
12.2.228 Jessica: Yeah ‘cause we're running out, I need a lot more.
12.2.229 Andrew: I need
12.2.230 Jessica: We have like three boxes
12.2.231 Dr. Landis: Oh, you have another box? Oh. But it doesn't have the colors? Do you want another box to work separately or do you want to build the same model?
12.2.232 Jessica: Well, we're building the same one
12.2.233 Dr. Landis: Ok, I mean do you want to work together to build one model or do you want me to get you some more rods so you can get your own?
12.2.234 Jessica: Oh there's bags of rods over there.
12.2.235 Dr. Landis: There are more? Ok.
12.2.236 31:18 Jessica: [Comes back with rods] I got one
12.2.237 Andrew: Oh my, I have no browns left. I found a brown! It has beads in it.
12.2.238 Jessica: I know
12.2.239 Andrew: You could have just got a bag.
12.2.240 Jessica: Well this has a lot, the others didn't have any browns.
12.2.241 Andrew: These were the fourths. Done with that! Done done done!
12.2.242 Andrew: Yeah I don't need any more.
12.2.243 Dr. Landis: You have enough, ok?
12.2.244 Andrew: The browns were the thirds.
12.2.245 Jessica: Oh the browns were the thirds.
12.2.246 Dr. Landis: Do you have enough of what you need to?
12.2.247 Jessica: Um, yes.
12.2.248 Dr. Landis: Ok.
12.2.249 Andrew: Now I need greens.
12.2.250 32:43 Dr. Landis: You said you needed some more of these colors
12.2.251 Andrew: Thank you
12.2.252 Dr. Landis: I'll leave it here.
12.2.253 Jessica: Ok. Andrew, can you put this on the other side of the desk.

12.2.254 Dr. Landis: What you can do is you can put this in here and that way you won't have so many containers. How's that? You want to get rid of this?
12.2.255 Jessica: Um yeah, we don't need one.
12.2.256 Andrew: Green in half [inaudible]
12.2.257 Jessica: It didn't work, purples, four browns, three browns and a purple it was.
12.2.258 Andrew: Three browns and a purple? It was two purple
12.2.259 Jessica: Oh yeah, something and two purples. Like three oranges and two purples it was
12.2.260 Andrew: Two purples, two purples equals one brown. And that's not equaling up.
12.2.261 Jessica: Yeah but that is only three oranges.
12.2.262 Andrew: Oh!
12.2.263 Jessica: Does that make sense?
12.2.264 Andrew: That's why! You took it off.
12.2.265 Jessica: That was mine
12.2.266 Andrew: Hey that was mine.
12.2.267 Jessica: Here's another one.
12.2.268 Andrew: Oh. Let's move it down this way.
12.2.269 Jessica: Purples... it was four and two purples, right?
12.2.270 Andrew: Four and one brown
12.2.271 Jessica: Or two purples
12.2.272 Andrew: Or two purples. Just making it look smaller. This doesn't work anyway. It works.
12.2.273 Jessica: I have no clue.
12.2.274 Andrew: Alright, we needed sixths, fifths.
12.2.275 Jessica: Fifths. Um, one two three fourths, um now it's browns.
12.2.276 Andrew: Why would browns be fifths?
12.2.277 Jessica: I don't know.
12.2.278 Andrew: Regular browns?
12.2.279 Jessica: What are browns gonna be?
12.2.280 Andrew: Maybe, maybe.
12.2.281 Jessica: One
12.2.282 Andrew: Hey you're taking out of my bin. One, two, three four
12.2.283 Jessica: Ok, browns are thirds.
12.2.284 Andrew: Five. No they're not.
12.2.285 Jessica: Yeah they are.
12.2.286 Andrew: I'm not counting by two. Count by ones.
12.2.287 Jessica: Six
12.2.288 Andrew: Count by ones.
12.2.289 Jessica: One two three four five six seven eight
12.2.290 Andrew: Ones
12.2.291 Jessica: I am.
12.2.292 Andrew: One two three four five
12.2.293 Jessica: Six. Ok, so

12.2.294 Andrew: I can't figure this out.
12.2.295 Jessica: One two three four five six
12.2.296 Andrew: We did have sixths
12.2.297 Jessica: Seven eighths,
12.2.298 Andrew: We did have twelfths
12.2.299 Jessica: One two three four five six. Because we were counting by two as one.
12.2.300 Andrew: Did we have tenths?
12.2.301 Dr. Davis: Carolyn, you want to bring the mike?
12.2.302 Andrew: Did we have tenths.
12.2.303 Jessica: Tenths? Yes reds.
12.2.304 Andrew: Reds were tenths.
12.2.305 Jessica: No green. No purples. Purples [mike moves to James]
12.2.306 : [most of this conversation is inaudible]
12.2.307 38:00 James: I took the ninths [purples] and yellows So I worked out like this. [inaudible] is bigger than one ninth.
12.2.308 Dr. Davis: Oh, so what's the answer?
12.2.309 James: Five whites equal up to a blue and the ninth is the purple and the blue
12.2.310 Dr. Davis: So the blue is one fourth and the purple is one ninth?
12.2.311 James: Yeah.
12.2.312 Dr. Davis: What's the white rod? [inaudible] What you call this?
12.2.313 James: Uh, one thirty-sixth.
12.2.314 Dr. Landis: One thirty-sixth, he said. One thirty-sixth.
12.2.315 : [another mike is brought over]
12.2.316 Dr. Davis: Can you explain that again?
12.2.317 39:50 James: Ok, first I tried nine yellows, and I tried to equal up the orange with the nine yellows, four oranges to equal the nine yellows, and the oranges were too small, so then I put nine purples right here, and then I put this [holds up orange rod] at a lower level in size, and then I took blues, and that equaled up to the nine purples. Then I just had to make a whole and my whole right now is three oranges and a dark green. [Figure 10-29-14] D/UL
12.2.318 Dr. Davis: Alright, that's very nice. And so the white rod is?
12.2.319 James: One, uh, thirty-sixths. They equal five
12.2.320 Dr. Davis: And what did you do over here? [pointing to model with a blue rod next to five white rods and a purple rod]
12.2.321 James: Well, I, I just think that the blue is bigger than the purple by one fifth cuz it takes five whites to equal up to the blue, the one fourth. D/?/f
12.2.322 Dr. Davis: Now, let me get this straight. The purple rod is, what name do you give to that?
12.2.323 James: One ninth.
12.2.324 Dr. Davis: One ninth, I understand that because nine of them are as long as your [inaudible]

12.2.325 James: Uh huh.
12.2.326 Dr. Davis: And what name do you give this?
12.2.327 James: One fourth because there are four.
12.2.328 Dr. Davis: One fourth, and what name do you give to the white rods?
12.2.329 James: One thirty-sixths.
12.2.330 Dr. Davis: So, then, how much is this? This, this [one of the white rods from the difference model] would be how much?
12.2.331 James: One thirty-sixth
12.2.332 Dr. Davis: This would be how much? [adds another white rod]
12.2.333 James: Two thirty-sixths
12.2.334 Dr. Davis: Yeah.
12.2.335 James: Oh, so it's five thirty-sixths.
12.2.336 Dr. Davis: Sounds right to me. Ok, so you can say did you solve that problem that you set out to do? Say what the problem was again, ok?
12.2.337 James: Um, how much bigger is one fourth than one ninth?
12.2.338 Dr. Davis: Yeah. And your answer is?
12.2.339 James: Five thirty-sixths.
12.2.340 Dr. Davis: I think that's gorgeous.
12.2.341 Dr. Landis: Yeah, I'm impressed too.
12.2.342 T/R 1: Can you write that up on an overhead for me and draw a picture, James?
12.2.343 James: Uh, yeah.
12.2.344 Dr. Davis: Thanks.
12.2.345 42:35 Andrew: Yes, it fits
12.2.346 Jessica: But how do you make ninths? With um
12.2.347 Andrew: Easy, you get a four. Anything but ninths. Hmm [Andrew's model is two orange rods, with four yellow rods beside it] ninth would be
12.2.348 Jessica: The fifth would be green, right?
12.2.349 Andrew: Yes it is. [Andrew places down light green rods, Figure 10-29-15]
12.2.350 Jessica: One two three four five six seven.
12.2.351 Andrew: We need something, how about red?
12.2.352 43:53 [camera moves to Kelly's desk with a model identical to James' - inaudible]
12.2.353 Kelly: We know what it is [some talk about copying]
12.2.354 Jessica: It's for the ninths and the fifths
12.2.355 Erik: One two three four five six
12.2.356 Michael: One two three four five six seven eight nine.
12.2.357 Erik: Wait, one two, I don't like you. One two three four five six seven eight nine one two three four not fair
12.2.358 Michael: We almost solved that, me and Erik were right at the edge of it, and then we sort of went into space with another idea.
12.2.359 Erik: [some nasty comments]
12.2.360 Meredith: Do you have any one rods?

12.2.361 Graham: Yes.
12.2.362 Meredith: If you called them, if you made a new model and made them halfs, and then
12.2.363 Michael: Graham you know, can I just use
12.2.364 Graham: We don't need these, we don't need these
12.2.365 48:41 T/R 1: How is this model related if at all to these models?
12.2.366 Michael: I don't know... Oh!
12.2.367 Erik: Well they don't have fifths, and they don't have this.
12.2.368 T/R 1: No, they don't.
12.2.369 Michael: They have ninths - like that! One two three four five six seven eight nine.
12.2.370 T/R 1: They're comparing ninths and fourths, Meredith, why do they need fifths. Your theory is they need fifths. Now they're comparing ninths, this is fourths, right? [Lays down blue rod] And this is ninths, is that correct? [Lays down purple rod]
12.2.371 Graham: Yeah [Figure 10-29-16]
12.2.372 T/R 1: You're comparing fourths and ninths and it's this, ok? [Graham hands teacher the yellow rod, and she shows that the yellow and purple are the same length as the blue rod]
12.2.373 Meredith: It's my method
12.2.374 T/R 1: Well, so what number name are you going to give this? [Talking about the yellow rod]
12.2.375 Graham: This?
12.2.376 T/R 1: Wait a minute, let me see what you have here. This is one two three four and this is one two three four five six seven eight nine.
12.2.377 Graham: What, the white ones? What would we give the white ones?
12.2.378 T/R 1: You're saying how much is the difference? Do you have any more white ones? Can you get some or borrow some? Ok, let's see. Meredith?
12.2.379 Kelly: It's bigger by one fifth because you see. [Points to blue rod with one purple and five whites next to it] D/?/f
12.2.380 T/R 1: What number name is this?
12.2.381 Graham: Thirty-fifths, one thirty-fifth.
12.2.382 T/R 1: Thirty-fifths.
12.2.383 Graham: Yeah. [T/R 1 straightens out model, Figure 10-29-17]
12.2.384 T/R 1: One two three four five six seven eight nine. And how many of them are there here? Counting? [Graham counts]
12.2.385 Graham: Thirty-six.
12.2.386 Meredith: What?
12.2.387 Graham: Thirty-six.
12.2.388 Voice: Thirty-six?
12.2.389 Graham: Yeah.
12.2.390 Voice: So what would that white one be?
12.2.391 Graham: One thirty-six. Ok, he's right.

12.2.392 T/R 1: So what do you have here? What did you come up with, Kelly?
12.2.393 Kelly: One thirty-sixth.
12.2.394 T/R 1: How many? What's the difference?
12.2.395 Graham: Well there are thirty-six.
12.2.396 T/R 1: There are thirty-six of these?
12.2.397 Graham: Yeah, the whites.
12.2.398 T/R 1: And what's the difference between the two? How many of the thirty-sixths?
12.2.399 Graham: Five.
12.2.400 T/R 1: So the difference between one ninth and one quarter is how much?
12.2.401 Graham: Five
12.2.402 Kimberly: Thirty-sixths.
12.2.403 T/R 1: Five thirty-sixths.
12.2.404 Meredith: And one fifth.
12.2.405 T/R 1: Well, where's the one fifth?
12.2.406 Meredith: Well, if you had one
12.2.407 Kelly: There's no one fifth.
12.2.408 T/R 1: Do you think that, do you think this is five thirty-sixths, if you could imagine one fifth in here,
12.2.409 Meredith: Yeah, uh huh.
12.2.410 T/R 1: Right? You could imagine one fourth, it's the blue, right? Is this one fifth?
12.2.411 Meredith: Uh...
12.2.412 T/R 1: If it were one fifth.
12.2.413 Graham: It would be too big. D/UL
12.2.414 T/R 1: [places five yellow rods down, Figure 10-29-18] Would that would be one fifth? Is that big enough to be one fifth?
12.2.415 Kelly: I don't think it's one fifth.
12.2.416 Meredith: Well it does have five here [places a yellow rod on the five white rods, Figure 10-29-13]
12.2.417 T/R 1: It's this length but this has the number name, what, the yellow has what number name?
12.2.418 Students: Five thirty-sixths
12.2.419 T/R 1: Five thirty-sixths. Not one fifth, right?
12.2.420 Meredith: Uh huh.
12.2.421 53:33 T/R 1: Think about what is causing the difficulty, ok, Meredith? See other camera view for end of transcript

Session 13, Nov. 1, 1993 (Front, Side, and OHP)

Line	Time	Speaker	Transcript
13.0.1	10:37	T/R 1:	Well, good morning! [students: good morning]. It is Monday, they were like that last Monday too. You know we have a visitor, another visitor, and maybe Prof. Davis could say a few words about our visitor.
13.0.2		Dr. Davis:	Ok, you know well what country the city of Oslo is in?
13.0.3		Student:	Norway.
13.0.4		Dr. Davis:	That is right, well that is where he's from, Professor Gunnar Gjone from Oslo, Norway, is here to see what we're doing.
13.0.5		T/R 1:	So that's quite a long distance, isn't it? [mm hmm]. Ok, um, it is Monday morning, that's true and I know you all had a wonderful weekend [yeah] Yes, it was a very special weekend, wasn't it? [yeah]. Too bad it rained but I bet you made the rest of it. But it is Monday and I'm wondering if you could think really hard and sort of help remind us what were were doing Friday. Do you remember how it all happened, or was it Friday? I think so. It was something you had done on Thursday led to something you were doing on Friday, remember? [students oh] Oh look we have three people, four people, five people remembering what we did on Friday, I know it takes a while. You could look to your partner if your partner [inaudible]. More people are remembering. Ok, there are still some people who aren't remembering, I can't believe James doesn't remember. I think James remember. See? Someone might help James remember, you helping James remember? Oh, he says, ok. Ok, who wants to tell our visitor what happened. Who wants to tell our visitor? Graham, your hand was up first, you want to tell us?
13.0.6		Graham:	Well we had candy bars on Tuesday, and we, one group got one and one ninth the other groups got, I forget what it was.
13.0.7		Students:	One fourth
13.0.8		Graham:	One fourth. And then we had to make a problem, like, we used our rods to see who got more and by how much.
13.0.9		T/R 1:	Ok, and? And how does that story end? Did it? Who got more and by how much? Can someone tell us the rest of that story? Mark?
13.0.10		Mark:	The people, the groups that got one fourth, they got, um, they got more by five thirty-sixths, I think.
13.0.11		T/R 1:	Five thirty-sixths more! How many of you remember that? Five thirty sixths more [most students raise hands] How many of you believe that? Ok, you all seem to believe it but you all don't quite remember it [laughs] Do you remember how you did it? Do you remember how you were able to

		show that they got more by five thirty-sixths? Does anybody want to sort of kind of review how you went to show that? One quarter was larger than one ninth by five thirty-sixths? Can you kind of remember it in your head without the rods, how that worked, James?
13.0.12	James:	Um, well, we had the thirty-six whites and it took five whites to equal one fourth to one ninth. So, one ninth to one fourth, so it took five thirty-sixths [inaudible]
13.0.13	T/R 1:	To show the difference?
13.0.14	James:	Yeah.
13.0.15	T/R 1:	How many of you remember that? [most hands raised] You know what I'm curious about? Some of you said one fifth. In fact, everyone in this class thought the difference would be one fifth before you did the activity, do you remember that? [mmm hmm] I asked you. I'm kind of curious, what made you think one fifth? Brian?
13.0.16	Brian:	Well, It's the same as kind, me and Meredith kind of thought it was the same as like, nine minus four equals five.
13.0.17	T/R 1:	So it would be in whole numbers.
13.0.18	Brian:	Yeah.
13.0.19	T/R 1:	Does it work that way with fractions? [Brian shakes head slightly] What do you think, Meredith?
13.0.20	Meredith:	Well, um, well, I thought, well, if you put the uh, blue, which was the nine, which had nine ones in it, and the uh four rod and the five rod, the five equals up to the nine, if you put it up to the fours.
13.0.21	T/R 1:	You're saying if you took the blue, and what number name are you giving that?
13.0.22	Meredith:	Um, blue, I would call it nine.
13.0.23	T/R 1:	You are going to give it nine? And what would you give the other rods?
13.0.24	Meredith:	Um, the four rod which was I think the purple rod.
13.0.25	T/R 1:	You're saying you're calling the purple four, is that what you said?
13.0.26	Meredith:	Yeah, and um, the yellow would be five, and it would equal up to it. I thought, that's what I thought at first [T/R 1 models Meredith's solution on the OHP - Figure O-19-51].
13.0.27	T/R 1:	So what's wrong, what's wrong with that? Five plus four is nine, five plus four is nine, I believe that, that works. Erik, you were going to say something?
13.0.28	Erik:	Well, I think that it doesn't make sense, because how could the blue rod be one ninth of one model and the purple rod be one fourth when the blue rod is larger than the purple rod? Maybe if you made a super gigantic train then maybe the blue rod would be the ninth but I would think that the purple

		rod or the yellow rod will probably be the ninths and the blue rod will probably be the fourths.
13.0.29	T/R 1:	But that's not what I heard Meredith say. I heard Meredith call the blue rod
13.0.30	Erik:	No, I just don't think the way Meredith explained it, the way she thought before, made much sense.
13.0.31	Meredith:	Yeah, I changed my answer. The uh five rod equals up the same as the five thirty-sixths.
13.0.32	T/R 1:	Mmm hmm, so you think the five thirty-sixths um somehow is related, that's an interesting idea, we'll have enough of these on here. Yeah, how's that, is that better? Ok, so that's a start that can get you very confused is that right? [mm hmm] If you call the blue rod nine and you can say that the white rod is one and the pink rod is four and the yellow rod is five and you've proved five plus four is nine. You actually proved five plus four is nine, but it sort of doesn't quite work that way with the fractions, does it? What do you think? Ok, well that was very interesting, so, um, I was just wondering, when you saw the big model that was built and you saw that the person who got one quarter of the candy bar got five thirty-sixths more than the person who got the ninth of the candy bar, is that much of a difference, do you think?
13.0.33	Jessica:	Probably not, I just realized, um, I think that um, well, there's twenty-five people in the class
13.0.34	T/R 1:	Yes,
13.0.35	Jessica:	And, that's an odd n umber, wait.
13.0.36	T/R 1:	Mmm hmm
13.0.37	Jessica:	Yeah, and um, so like you couldn't have like all even groups so that's why I think some people got like one ninth and one fourth.
13.0.38	T/R 1:	I wondered if there would have been a way, I want you to solve this I want you to think about, of sharing those three bars of candy so everybody got the same amount. Exactly. If you can think about a way, think about that, Andrew, any ideas?
13.0.39	Andrew:	Well, I um, what I did one day, for homework, to divide it equally, so I came up with the answer that everyone got one and one fifth.
13.0.40	T/R 1:	How did you do that?
13.0.41	Andrew:	Well, there was three candy-bars and each one had rectangles in them. So I took, um, twenty-five of them and circled it adn put one. And then the five left, if you divided them up into five, five, ten, fifteen, twenty, twenty-five. So each person would get one and one fifth.
13.0.42	T/R 1:	That's an interesting conjecture, isn't it? You hear that, what Andrew said? How many of you followed what Andrew

said? [few hands are raised] I wonder if there's a way to um, to test that, that it would have been um, ok, can you draw a picture to show us, is there a way

13.0.43 Andrew: Well, yeah.

13.0.44 T/R 1: Andrew, how did you show that?

13.0.45 Andrew: Well, um, I drew three candy bars.

13.0.46 T/R 1: Can we try imagine what we're doing, the three candy bars

13.0.47 Andrew: And each has ten pieces in it.

13.0.48 T/R 1: Alright, ten, ten, ten. Can you all imagine that? [mmm hmmm] Ok.

13.0.49 Andrew: And then I took two candy bars and five pieces of the other one and it's twenty-five

13.0.50 T/R 1: Everybody gets one of those thirty pieces and there are how many left over?

13.0.51 Andrew: Five.

13.0.52 T/R 1: Five? Do you all follow that? How many of you follow so far? Those thirty pieces, everybody got a piece so there's five left over, ok.

13.0.53 Andrew: And then there's five, so it's like one candy bar, only smaller, so you divide them into fifths, and then five ten fifteen twenty twenty-five, plus five times five is twenty-five so each one gets one and one fifth.

13.0.54 T/R 1: Very interesting. What do you think about that? Would that have been fair, do you think? To get one and one fifth, as compared to some people getting one and one quarter and some people getting one and a ninth? What do you think? Is one and a fifth more or less than one and a quarter? What do you think? Is one and a fifth more or less than one and a quarter? That's what happened, that those of you who got one and a quarter now you got one and a fifth, would you have gotten more or less? Is one and a fifth more and less? Danielle?

13.0.55 Danielle: Um, less.

13.0.56 T/R 1: Ok, how many of you think it's less? [most hands raised] Why?

13.0.57 Danielle: Because, that's, that's a bigger number, and so um if it's a bigger number you get less.

13.0.58 T/R 1: Which is a bigger number, Danielle?

13.0.59 Danielle: Five.

13.0.60 T/R 1: Five? Ok, what do you think about that? What do you think, Brian?

13.0.61 Brian: Well, I agree with that. And because, if you have a bigger number, then you need to take, like, say, um see, it's one and one fifth and if it's a fifth it has to take, there has to be five of 'em in one whole, and if there are um, quarters, it only needs, it only needs four of 'em to go into one whole. So, so, so, so

		um, five is a bigger number and and it needs more to fill up and it needs more to fill up one whole so it's so it's less.
13.0.62	T/R 1:	If I were, if I were to say things like this to you, one half, one third, one fourth, one fifth, right? If I were talking about these numbers, do you know which are bigger and which are smaller? How many of you think you know, which of these numbers are bigger and which are smaller [most hands seen are raised] And you think you can explain why? Imagine a model, David what do you think?
13.0.63	David:	Well, I think that if you have about this big, one half would be in the middle, the biggest, and then one third, that would be kind of smaller because you would have to fit like three pieces in there [gestures with hands as speaks] and then one fourth would be even smaller than one third because
13.0.64	T/R 1:	Can you come up - do you all hear what David is saying? [mmm hmmm]. Sure.
13.0.65	David:	So, then maybe like,
13.0.66	T/R 1:	You want to draw your rods, call something one and draw it, show me?
13.0.67	David:	Uh, maybe uh, orange.
13.0.68	T/R 1:	You can sketch it, sure, you can sketch it.
13.0.69	David:	Um, like if this is the one whole [draws a rod and writes 1 whole inside], um then one half would be there [draws a line in the center beneath the rod and then draws the two half rods and then you have to put
13.0.70	T/R 1:	Can you mark one half like put it underneath?
13.0.71	David:	Oh, wait a minute
13.0.72	T/R 1:	Just, draw the number one half, where you want to show one half.
13.0.73	David:	One half [writing] then one third [draws thirds and labels] and then one fourth [does the same] and then [Figure O-28-10]
13.0.74	T/R 1:	Ok, and then one fifth, thank you very much. Anyone have a question to ask David before he sits down about what he drew up here? Can you imagine this with the rods? Thank you David.
13.0.75	Meredith:	Where's the one-fifth?
13.0.76	T/R 1:	You want to see the one-fifths? What do you think it would be, Meredith?
13.0.77	Meredith:	Like, the whole would be divided into fifths.
13.0.78	T/R 1:	You would divide it into fifths? So where would it be, would it be to the right of the quarter or to the left of a quarter?
13.0.79	Meredith:	To the left
13.0.80	T/R 1:	You mean something like this, maybe? 00,26,17,23 I'm going to call this zero and I'm going to call this one [draws a number line with two marks at ends labeled zero to the left

and one to the right]. I wonder who wants to come up here and mark where the number one half would be, Michael?
[The rest of the session is not relevant to this study]

Session 14, Dec. 2, 1993 (Front, Side, and OHP)

Line	Time	Speaker	Transcript
14.0.1	00:00:00	T/R 2:	I want to take a minute before we begin. I thought it would be interesting to come back, especially after the last time which was really neat with all your parents in the room, it was quite a crowd. But we thought we'd come back today and sort of think about what we started to talk about the day your parents were here. We started to talk about a little bit about dividing and dividing with fractions. So Dr. Maher was putting things like this up on the overhead. [T/R 2: writes $1 \div \frac{1}{2}$ on the overhead.] And I was really wondering how many people were following along with that. It's a tough idea. Some people are saying "yeah we're following along." I think it would be worthwhile for us to go back over this today and spend a little time thinking about the ideas behind this. Do you think that might be a worthwhile thing to do? For those of you who were kinda like oh yeah last week my goodness I kind of remember that but you know it would be really hard for me to explain that to somebody. Erik you had your hand up, did you want to make a comment?
14.0.2	00:01:12	Erik:	Um yeah. I just wanted to explain it to you.
14.0.3	00:01:15	T/R 2:	Oh you wanted to explain this. Ok. Well I'd be interested in hearing.
14.0.4	00:01:19	Erik:	If anyone did not remember.
14.0.5	00:01:20	T/R 2:	I would be interested in hearing. Maybe the best way though for you to help us is that as I start and go along and present you with a problem for maybe to help us to go back over this and to understand. Ok? So maybe we'll just hold off on any explanations right yet cause I want to get all of you thinking about the same problem again. Ok I want you to think about this train. [T/R 2: puts a red and orange train on the overhead.] Can everybody take out the rods and make this train with the red and the orange rod? We've spent a lot of time thinking about this train, haven't we? We've spent a lot of time building models using this train. Now in the way of review, can anybody tell me if I give this train the number name 1? Ok I'm going to call that train 1. What number name would I give to one of the little white rods? [She puts a white rod below the red and orange train.] And if you think you know, can you build me the model to show me so that you can explain it to us? Remember the red and orange have the number name 1 and I want to know what number name you might give to the

white that would make sense. [Approx. 1 min. given to class as children raise their hands when ready.] I'm hearing some interesting things, and I don't think we need to dwell on this one. I think a lot of people really are anxious to tell me how this works. Is there somebody who feels they can explain how this works? They built a model and they can explain how this works and what number name they gave for white. Ok let's see. Danielle.

14.0.6 00:04:15 Danielle: I would call it 1/12.

14.0.7 00:04:18 T/R 2: She would call it 1/12 she says. How many people agree with that? [Several students in view raise their hands.] This looks pretty encouraging. You can put your hands down. 1/12 you're saying, does anyone disagree first of all with 1/12? No, nobody does. Maybe I should have asked that first. Ok, Danielle, why do you think 1/12?

14.0.8 00:04:38 Danielle: Because the red and the orange that's the whole and 12 white ones make up the whole.

14.0.9 00:04:47 T/R 2: Ok. So if we call red and orange 1, we're calling it the number name 1, you're saying that it takes 12 of those little white ones to equal up to the length of the orange and the red? [Danielle nods]. And so you would give this the name 1/12? [Danielle nods]. Do you agree with that? Does that seem reasonable? Ok well now what we can do is maybe we can answer a question or two about this train. [T/R 2: writes two questions on the overhead. The first is, "How many <u>whites</u> are in <u>a red orange</u> train?" and the second question is, "How many _______ are in ________?" - Figure S-5-40] Now this is what we've been answering right? How many white are in the red and orange train? Can we now replace these color names, for the train and for the white, with number names in that sentence? Can we change the color names of white and the train with red and orange to number names at this point now? Can we rewrite this with numbers in that sentence? A couple people are saying that they can. I would like you all to think about for a minute, maybe even to discuss it with your partner what you might call these. Danielle has told us part of this; you just have to put it into the sentence now.

14.0.10 *SIDE VIEW*

14.0.11 CT: Read me, read me what it says there.

14.0.12 Danielle: How many whites are in a red and orange train?

14.0.13 CT: Well, you said…

14.0.14 Danielle: Uh, twelve

14.0.15 CT: Ok, twelve, go ahead. How many

14.0.16 Danielle: How many blank are in

14.0.17 CT: What would you call one of these [white rods]

14.0.18		Danielle:	A twelfth
14.0.19		CT:	Ok. So how many blanks are in… You said how many twelfths are in
14.0.20		Danielle:	A whole?
14.0.21		CT:	Are in one, right, you have it!
14.0.22		Danielle:	I do?
14.0.23		CT:	Ok, say it again. Read the second line
14.0.24		Danielle:	How many twelfths are in a whole?
14.0.25		CT:	Are in one? Aren't you calling this one? [Danielle nods] Ok, wait, maybe I'm wrong, what did you say?
14.0.26		Brian:	How many twelfths are in one?
14.0.27		CT:	Do you agree with her? [Brian nods] That's what you said.
14.0.28			*FRONT VIEW*
14.0.29	:06:13	T/R 2:	[to Amy and Jackie] What do you think?
14.0.30	:06:14		Amy: I would think that the white would be 1/12 and the red and orange would be 1, a whole. [Approximately 1 minute given to the students.]
14.0.31			*BOTH VIEWS*
14.0.32	00:07:20	T/R 2:	I think we're ready to talk about this one, ok? I've heard some very nice thinking on this. All we're doing is substituting in number names for these color names at this point. Now that I've defined what an orange and a red is, I've said that it was 1. Right, I'm calling orange and red train 1. Can somebody tell me what number names I can put in here to make the same sentence? It's just putting in number names now. I've heard some people tell me this already. Who feels confident that they could tell me what we're going to call these and how we're going to say this sentence? Ok, let's see, I haven't had a chance to… David.
14.0.33	00:08:00	David	The white would be 1/12 and the red and orange train would be 1 whole.
14.0.34	00:08:08	T/R 2:	Ok. So I could say maybe 1/12's or something like that. How many 1/12's are in one [whole (David adds)]. I'm just going to call it the number 1. Alright so we could rewrite this as this right? [T/R 2: fills in the blanks in the second question so that it says, "How many 1/12's are in 1?"] We could rewrite it with numbers. Can anybody answer that question now? A couple people already did when they were talking about it they answered it for me, but I'd like you to think about that for a minute. You can talk to your partner again if you'd like. They question is how many 1/12's are in 1? [Approx. 1 min. given to class as children; raise their hands when ready.] No tricks here. There really are no tricks here. This is something I want you all to be clear on though before we move on. Ok? I know that you know this. Ok, let's see, I don't see any hands over here ladies. Do you

			think you could answer this question? Think about it ok. If you have an idea, raise your hand. Ok. Let me hear from Graham.
14.0.35	00:09:30	Graham:	There is 12 twelfths.
14.0.36	00:09:33	T/R 2:	Ok. So then you're telling me that how many 1/12's are in 1 is 12.
14.0.37	00:09:41	Graham:	Yeah.
14.0.38	00:09:42	T/R 2:	Ok. Alright, so Graham's answered the question by saying that there are 12 of them there. Do you agree with that? If you agree with that, raise you hand. Ok, that's great. Now Erik, did you have something that you wanted to add?
14.0.39	00:09:54	Erik:	For that equation, well, you could put how many 1/12's there are in 1, you can also put how many 1/12's are there in 12/12's.
14.0.40	00:10:05	T/R 2:	Oh ok. So I could also rewrite this you're saying then as 12 over 12.
14.0.41	00:10:12	Erik:	Yeah.
14.0.42	00:10:13	T/R 2:	Is that the same thing 1 and 12 over 12? Are they the same thing?
14.0.43	00:10:16	Erik:	Yes.
14.0.44	00:10:18	T/R 2:	Ok, Erik says that 1 and 12 over 12 represent the same number or the same amount. What do you think about that? Do you agree with that? Are they equal to each other? If you have an idea about that, raise your hand.
14.0.45	00:10:35	Jessica:	What did he say about 1/12?
14.0.46	00:10:37	T/R 2:	He said that the number 1 and 12 over 12 or 12/12's, he said those are really the same thing. Do you agree with that? Jackie says yes that she agrees with that. David, do you agree with that? Mark, do you agree with that? Laura? Ok. We have some agreement here. Ok, that's very interesting. Thank you Erik for adding that. That was one problem to think about. Let's try another, ok? My goal is that we try a couple of these together to establish some ground rules and then you're going to work with your partner on some and then maybe make some of these up for me. Let's try another one. I'd like everybody to take out a dark green rod. Alright. I want to ask you a question about that dark green rod. Ok, my question to you is, if I call the dark green rod 1, now it's not the orange and red train, it's the dark green rod that's going to be 1, what number name would I give to the white rod? Please build the model. [Approximately 1 minute given to the students.] Remember we want the number name for the white rod. I think you can all tell me. I really do. We're getting to be pros with these rods. We really know what we're doing here. Uh, let's see. Erin, can you tell me?

14.0.47	00:12:25	Erin	1/6.
14.0.48	00:12:26	T/R 2:	Ok. Erin says she'd call it 1/6. Does anybody disagree with that? No? Ok, so then I'm assuming that you all agree with what Erin is saying. She says that the white rod has number name 1/6. Erin can you tell us why?
14.0.49	00:12:41	Erin	Because 6 of the white rods make up the dark green rod.
14.0.50	00:12:46	T/R 2:	Alright. I will agree with that. That's the same length, isn't it? They are the same length when you line them up. 6 of these under here. [T/R 2: points to the white rod under the dark green rod.] Ok, let me ask you a question then. I'm going to ask you the same question I did before, but this time with the dark green. Remember, the dark green here is equal to 1 and we said now that the white is 1/6. Ok, how many whites are in the dark green rod and can somebody rewrite this sentence for me with numbers [Figure S-13-19]? Meredith.
14.0.51	00:13:26	Meredith:	Six white rods
14.0.52	00:13:27	T/R 2:	Ok. So you're answering the question and telling me that there are six. Can somebody rewrite this sentence for me? Meredith, would you like to do that since you started this?
14.0.53	00:13:37	Meredith:	How many 1/6's are in 1 whole?
14.0.54	00:13:41	T/R 2:	Could we give 1 another name here? Remember Erik's rule for doing that if we wanted to? I like 1, 1 is fine, but I'm wondering if there's another number name. Brian.
14.0.55	00:13:56	Brian:	6/6's.
14.0.56	00:13:57	T/R 2:	Yeah. We could call it 6/6's right, since we've established that that really shows us the same amount.
14.0.57	00:14:05	Michael:	Or you can just call is plain 6 and that's the same thing.
14.0.58	00:14:10	T/R 2:	Plain 6, are you sure about that?
14.0.59		Michael:	Yeah, yeah, 'cause then you could
14.0.60		Students:	No, no.
14.0.61		Michael:	Oh, no, no, no
14.0.62	00:14:11	Erik:	No, cause if you called it 6 it would be 6 wholes.
14.0.63	00:14:15	T/R 2:	What do you think Michael?
14.0.64	00:14:16	Michael:	You couldn't do that because you'd have to call the 1/6 one whole.
14.0.65	00:14:18	Erik:	That would be right. That would be right because you can call the one whole six wholes and then each of the white ones could be one whole each.
14.0.66	00:14:28	T/R 2:	Ok, but you're answering a different question from the way I defined the rule. If you change the rule, what you've done here is you've said well now we can call this 6 instead and then what would I call the white and you're saying I'd call the white 1 whole. Yeah, but you're changing my problem. Ok, so I don't appreciate that Michael. Haha. But you're correct, you'd have to change the model in order for that to

			work. Ok so there's 6 of these, 6 1/6's are in 1. Ok. Can we write this now as a number sentence including these numbers? Going back to that division we were working with last week, does anybody think they could write this situation here, the question I'm asking here in a number sentence that would work? A couple people think they might be able to.
14.0.67	00:15:20	CT:	They should talk among themselves about it.
14.0.68	00:15:21	T/R 2:	Yeah. Yeah. I see two people with their hands up but I'd like everybody to think about that maybe a little bit. Maybe write it in words here. How could we rewrite this as a math sentence, as a number sentence? [Students discuss their answers with their partners. They are given approximately 6 minutes.]
14.0.69			*SIDE VIEW*
14.0.70		Brian:	[to Danielle] One divided by one sixth equals six [Danielle nods, Brian raises hand]
14.0.71		Meredith:	One whole minus, oh, I have an idea. One whole minus six sixths is zero. One whole
14.0.72		Student:	One whole minus-
14.0.73		Michael:	five sixths equals one sixth.
14.0.74		Meredith:	No, no no no no. I have an idea. No. No, give me a pencil. Look, this is one. One is equal to six sixths. Does it have to be division? Does it have to be a division problem?
14.0.75		Michael:	One divided by six is not six sixths
14.0.76		Meredith:	I didn't say it was.
14.0.77		Brian:	[T/R 2 speaks to Brian] One divided by, I think if you're doing a division problem, you could do one divided by one sixth equal six, because it was like what we did when our parents were here, we did one divided by one eighth equals eight, so it was like well, what, how many eighths are in one?
14.0.78		T/R 2:	Ok, so that's like the question we were asking here. Do you agree with that, Danielle?
14.0.79		Danielle:	Uh, yeah but I, so it's like every number, it's that, it's one divided by the fraction and then just the plain number?
14.0.80		T/R 2:	What do you think?
14.0.81		Brian:	Well, uh,
14.0.82		T/R 2:	It's a good question, isn't it? She's thinking in general terms does that always work? I think that's something we need to do some more exploring with to see if that works, so we'll do some more problems to see if that's the case.
14.0.83		Brian:	It's just like saying six divided by uh, it would be like the same thing, kind of. Six divided by one sixth equals what? And then you check it.

14.0.84	T/R 2:	Ok, that's a very good question, though, Danielle. I think, I think maybe it will become more evident as we do more problems. I don't want to just give you the answer on that, I want you to think about it some more.
14.0.85	Meredith:	… One whole
14.0.86	T/R 2:	Alright, ok, so that would work. So these are your sixths.
14.0.87	Meredith:	[Figure S-18-17] Yeah, mm hmm [gestures to show that there is an equal sign between the two models]
14.0.88	T/R 2:	And that's one, ok. And how many, ok, so there's six of them. Ok, yeah, ok. Can you write a sentence now using, um, division, that would also describe the situation? Think about that. [walks away]
14.0.89	Michael:	One…
14.0.90	Meredith:	Oh, yeah! Wait, of course I can.
14.0.91	Michael:	One whole… nah, six sixths divided by
14.0.92	Meredith:	[Figure S-19-01] One divided by one sixth equals [laughs]. I know. One divided by one sixth equals six sixths.
14.0.93	Michael:	What? That doesn't make sense!
14.0.94	Danielle:	How do you spell division? D-I-V-I-S-I-O-N, something like that.
14.0.95	Meredith:	No, you're not taking away when you divide
14.0.96	T/R 2:	Are we taking away when you divide? That's a good question.
14.0.97	Michael:	No, you're doing subtraction.
14.0.98	Meredith:	No, see. Do we have a pen? Do you have a pen in that desk?
14.0.99	T/R 2:	How is it related to subtraction, Michael?
14.0.100	Meredith:	See, look
14.0.101	Michael:	It would be, I would, it would be
14.0.102	T/R 2:	I would agree with you that it is related
14.0.103	Michael:	I would agree with it that it would be one divided by one sixth equals six but not six sixths.
14.0.104	Meredith:	And one divided by one sixth equals six because there's six, one sixths in one.
14.0.105	Michael:	Yeah, that's what I mean
14.0.106	Meredith:	So six divided by one, if you do six times one
14.0.107	Michael:	But she said six sixths, I would agree with six, but not six sixths.
14.0.108	Meredith:	equals six, minus equals zero [Figure S-20-49]. It's uh, one divided by one sixth.
14.0.109	T/R 2:	That would work, then, you're saying. Ok. Alright
14.0.110	Michael:	But not six sixths.
14.0.111	T/R 2:	Let's see what other people think. Let's put this on the table, I think.
14.0.112		*FRONT VIEW*
14.0.113	Jackie:	Well, one minus
14.0.114	Amy:	One divided

14.0.115 00:15:37 Jackie: 1 divided by 1/6.
14.0.116 Amy: Equals, one third
14.0.117 Jackie: Yeah.
14.0.118 00:15:47 Amy: Ok. One third.
14.0.119 What do the rest of you think.
14.0.120 Jacquelyn I just got thrown off the track.
14.0.121 T/R 2: You're thrown off the track. Well, I think this is something we may need to discuss.
14.0.122 Amy: [This exchange may be a bit inaccurate - hard to hear] One divided by one sixth equal six
14.0.123 Jackie: Yeah, how many, no, one. Because how many ones are in six?
14.0.124 Amy: One sixth
14.0.125 Jackie: No, one.
14.0.126 Amy: You're confusing me, Jackie
14.0.127 Jackie: Alright, so one goes in one sixth
14.0.128 Amy: Yeah
14.0.129 Jackie: See, there's one sixth [writing on back of name sign] And one, that's one, one sixth. [They get up to get paper.]
14.0.130 Amy: One [inaudible] one sixth, equals one sixth
14.0.131 Jackie: One.
14.0.132 Amy: One sixth.
14.0.133 Jackie: One.
14.0.134 Amy: One sixth.
14.0.135 Jackie: One.
14.0.136 Amy: One sixth.
14.0.137 Jackie: One.
14.0.138 Amy: One sixth.
14.0.139 Jackie: One.
14.0.140 Amy: One sixth. [Amy has written: $1 \div \frac{1}{6} = \frac{1}{6}$. They keep arguing in this manner]
14.0.141s Jessica: We think it's one divided by one sixth equals [inaudible, leaves the two. Rest of their conversation is inaudible]
14.0.142 *BOTH VIEWS*
14.0.143 :20:59 F
21:00 S T/R 2: Ok. Can I get everyone's attention? I have some people that say wow, you just really threw me right off the track and I have some people that are asserting that they're pretty sure that they have an idea of how to do this, so I think that we need to discuss it now. What we want to do is take this second sentence here "how many 1/6's are in 1?" and change it completely to a number sentence, ok? We started to do that when your parents were here last week, but we got into it I think at varying degrees, some of us really got further with

		it than others. So I think we need to kind of discuss and talk. Who thinks that they have an idea for what might be a number sentence that would describe this? Ok, I've heard some ideas? Gregory? Mark and Gregory, you can report together, however you want on your discovery.
14.0.144 00:21:45	CT:	Do you want them up there?
14.0.145 00:21:47	T/R 2:	Why don't you tell me first, and then if we need to build a model we may have you come up and do that.
14.0.146 00:21:51	Mark:	Well we have 1 divided by 1/6 equals 6.
14.0.147 00:21:59	T/R 2:	1 divided by 1/6 equals 6. [T/R 2: writes $1 \div \frac{1}{6} = 6$ on the overhead.] Ok. Did anyone else come up with that for a number sentence to describe the question how many 1/6's are in 1? We have several pairs, looks like about maybe 8 or 9 people who came up with the same sentence. Do we have any other ideas to put out on the table here, the things that might be a possible number sentence? Ok, alright. So this is what's being proposed. I want somebody to come up and tell me about this now, about how this describes this sentence.
14.0.148	Jessica:	They had one, I don't know if they want to do it [referring to Amy and Jackie, they laugh.
14.0.149	T/R 2:	Oh, we've got some modest folks here. I don't want to put anybody on the spot. If you'd like to come up though, just raise your hand. Uh, let's see. Ok, 3 or 4 people are volunteering. How about, anybody else? Some of you are just going hmmm, I don't know about this. Ok let's see, how about, I'd like to hear from Michael I think, because he was working with Meredith and they were arguing about this. I want to hear what Michael has to say. I've heard what Meredith has to say. [Michael walks up to the overhead]. If this works, how does it work?
14.0.150 00:23:20	Michael:	It works because division you see how many times you can get a number into a number. So you can get 1/6, umm you can get 6 times you can get 1/6 into 1 with no remainders. So that would leave that that would be 6.
14.0.151 00:23:42	T/R 2:	So you're saying then that 1/6.
14.0.152 00:23:42	Michael:	You can have 6 of them.
14.0.153 00:23:46	T/R 2:	It goes into 1, if you were lining them up.
14.0.154 00:23:48	Michael:	6 times.
14.0.155 00:23:49	T/R 2:	6 times. Does that make sense?
14.0.156 00:23:53	Erik:	How can 6 go into 1 six times?
14.0.157 00:23:55	Michael:	No, I said
14.0.158 00:23:57	Erik:	1 whole but if you're dealing with numbers, it wouldn't make sense unless it was negative.

14.0.159	00:24:05	Michael:	No, I mean, no, 1 divided by 1/6, when you ha- you would get 6.
14.0.160	00:24:13	Erik:	I know, but I'm saying if you were taking like 6 and 1, you couldn't put 6/6's into 1.
14.0.161	00:24:19	Michael:	No, I never said that.
14.0.162	00:24:21	T/R 2:	Erik, I don't think that's what Michael said though.
14.0.163		Erik:	I know.
14.0.164		T/R 2:	Ok, You're doing another problem. You're taking this to a challenge problem here I think.
14.0.165	00:24:32	Meredith:	He's just trying to say that there's 6/6's is 1 whole.
14.0.166	00:24:36	T/R 2:	I think Erik knows that. Erik is really taking us on to another problem to think about I think. But let's get back to this one, does this make sense? For those of you who really weren't too sure how to begin, does this make sense to write it this way? Ok. How many 1/6's go into 1? Ok, and you guys are telling me there's 6 of those. Ok I'll agree with that. Ok, let's try one more. You can take a seat Michael, thank you. Actually, can we go back to that first one for just a minute, and maybe write this one as a number sentence? Remember this one, the red and orange train? Can we rewrite this as a number sentence now? The question is how many 1/12's are in 1?
14.0.167			[Stands, raises hand] Oh, oh, I can do it! [Students are given approximately 2 minutes.] I see a couple people. I see the same people who can tell me a number sentence this time. I think more of you could tell me, so I'm going to wait until more of you feel like you want to talk to me today. I see a couple more hands. I think more of you can tell me what the number sentence for this would be, especially after doing the last one and seeing how that worked. Does anyone want to try? Someone different? I see Mark's hand, I see Allen's hand, Graham. Ladies? Any ladies want to tell me how this might work? Jessica, Laura, I see some more hands. I'm getting happier. Ok let's see, more hands. A couple more hands I would like to see come up.
14.0.168			*SIDE VIEW*
14.0.169		CT:	[As T/R 2 continues talking below] Ok, what do you call this?
14.0.170		Danielle:	One
14.0.171		CT:	One. What do you call this?
14.0.172		Danielle:	One twelfth.
14.0.173		CT:	If I do one divided by one twelfth, what do I get?
14.0.174		Danielle:	Um, twelve?
14.0.175		CT:	Do you have twelve parts here?
14.0.176		Danielle:	Um, yeah.

CT: One divided by one twelfth. Both hands [Danielle raises both hands.]

14.0.177 *BOTH VIEWS*

14.0.178 T/R 2: I know I'm being a bit of a pain, but I really want to see you all participating today. It's important that you all understand. Oh, Amy's hand's up now. Jackie's hand's up. Brian, how about you? Do you think that you could do it? [Brian's neighbor raises his hand for him]. Alright, let's hear from some folks. Ok, a number sentence, uh, how about, well I would like to hear from Brian and Danielle. There seemed to be a lot of discussion going on over there, I want to hear from them. Danielle, do you want to start? Tell me how you'd write the number sentence.

14.0.179 00:27:42 Danielle: I would write it 1 divided by 1/12 equals 12. [T/R 2: writes $1 \div \frac{1}{12} = 12$ on the overhead.]

14.0.180 00:27:50 T/R 2: Ok. Is there anyone who does not agree with that? All of you had your hands up, is this what you were thinking of telling me? [Students nod, say yes]. I believe that. Ok, I really believe that you can do this. Ok, what I'd like to do now is we have approximately 25 minutes left, I'd like to give you some problems to work on on your own, with your partner. The only thing is I want you to do a couple things for me, ok? The problems are all on one sheet. There are four of them, but there's some that have several questions. I'd like you to, I'm going to give you some sheets that are just like blank sheets for your name and then you have today's date on them. This will keep me organized, so I know what day we worked on these. If you could just put all your work and build your models and trace them, you know with your Cuisenaire rods on these blank sheets and just number and label them. Instead of trying to write on this sheet here and cramming them all in, if you can write on the blank sheets for me, you'll get some of these. That would be really helpful. I want you to do a couple things. When I give you a problem, like the question I asked you "if this is 1, what would this be", I'd like you to build a model to show the different parts, the model like you've been doing with the rods, and I would like you to actually write this question for each one, "how many blanks are in blank? " Ok, and actually fill in the number amount, and then write me a number sentence to go with it, so you'll have a model or a drawing, you'll have a question following that model, and you'll have a number sentence. I can leave this one up here as an example for you. And if you can do all of that on the yellow sheets we'll have some really nice things to look at, and I'll be able to keep organized and read all of your work to see what you're thinking. Do you have any questions about what you're being asked to do? These problems are very much like what I did up here at the overhead projector, ok? When you think you've really mastered this and got a bunch of them, raise your hand. I'd love to hear what you're thinking about this. The problem sheets are coming around, and the blank papers are coming around also. And you have rods and there's markers up here. You probably want to write in the dark pen. Ok, I'm going to leave the markers right up here. [Students go up to the front of the room to get supplies.]

14.0.181		*SIDE VIEW*
14.0.182	CT:	What are you calling this?
14.0.183	Danielle:	One
14.0.184	CT:	One. Now what are they asking you?
14.0.185	Danielle:	Um, what number name would you give to white?
14.0.186	CT:	Ok, so you put it here. So you know that, what number name would you give the white?
14.0.187	Danielle:	A half?
14.0.188	CT:	Alright! Why don't you put 1a. Go ahead. That's alright. Oh, you don't have to write it, you don't have to write that. Just put one half. Just put your answer. Did she say? I'm sorry, yes, I'm sorry you're absolutely right? It says what number name would you give to white? So I'd say white is one half.
14.0.189	Danielle:	Because I thought we'd have to write how many
14.0.190	CT:	Oh, ok, go ahead, I'm getting you confused. [to T/R 2] I'm getting Danielle all confused. [camera focuses on Meredith and Michael working silently, returns to Danielle] I'm sorry you were right. They want a picture first, that's alright, keep going, stick your picture up there, and then they want it just the way you have it there. They want you to follow that format. They want you to follow that format through number four. Now, remember you guys are supposed to be talking. [speaks with other students]
14.0.191	Danielle:	[some talk about what to write]
14.0.192	Meredith:	Now I'm going to my next problem.
14.0.193	Danielle:	Like the division problem?
14.0.194	Brian:	I'm doing an addition
14.0.195	T/R 2:	How are we doing?
14.0.196	Danielle:	Do we have to write a division problem?
14.0.197	T/R 2:	Well, we have to write something that describes the situation
14.0.198	Brian:	I wrote an addition problem.

14.0.199	T/R 2:	Ok, maybe you can explain to me how that works. You have one half plus one half
14.0.200	Danielle:	Equals the whole, and then the whole is two halves.
14.0.201	T/R 2:	Ok, alright, that, that would be one way to describe what you have here, but does that answer my question? My question was to you how many whites are in red? How many whites are in red? Two. Ok, so does this number sentence answer my question? It is a number sentence which certainly describes your model, but does it answer my question, is what I'm asking Brian? I think Danielle's skeptical.
14.0.202	Brian:	I thought you wanted a number sentence.
14.0.203	T/R 2:	I want a number sentence that describes exactly the question that I'm asking. You're right, this describes this picture, but does it describe the question that I'm asking. How many whites are in red. How many whites are in red?
14.0.204	Brian:	Two. Should I write that? How many whites are in red?
14.0.205	T/R 2:	Mmm hmm What do you think, Danielle? Right, we have red, two whites. We could, we could say that in numbers now, couldn't we? How many, how would we say it, how many what are in what? If we used numbers
14.0.206	Brian:	How many whites are in red?
14.0.207	T/R 2:	Yeah, change it into numbers now? Danielle?
14.0.208	Danielle:	How many halves are in a whole?
14.0.209	T/R 2:	How many halves are in one? How many halves are in one?
14.0.210	Danielle:	Two.
14.0.211	T/R 2:	Ok.
14.0.212	Danielle:	But, see, I wrote this as a question. Should I just
14.0.213	T/R 2:	Wait wait wait, no no no no, don't erase what you have here. What did you want to change it to?
14.0.214	Danielle:	Just a half.
14.0.215	T/R 2:	What you have is fine here. That's ok, no yours is fine. Maybe you can answer the question for me, over here, how many halves are in one. Ok, alright, now what you need to think about is, how you can describe this question as a number sentence, and would it be a half plus a half or would it be something else? Think about that. Ok?
14.0.216	Brian:	Just a plain number sentence?
14.0.217	Danielle:	Should we use division?
14.0.218	T/R 2:	What I'm, what I'm saying to you is, you keep telling me that there are two one halves in one, right? Where does the two come into play with a half and a half? Where is that in your number sentence? Is it describing the question, is what I'm asking. Because you keep telling me two is the solution, right, to this question. [Danielle thinks.] We're just writing it with numbers this time, right? That's all we're doing.

We’re writing it with words, we’re writing it with numbers. Do you agree that describes this question, do you really agree with me, Brian? Ok, now I just want to make sure that Danielle has thought about this and has considered this. What do you have here, Danielle?

14.0.219 Danielle: One divided by one half equals two.

14.0.220 T/R 2: Ok. We can just say two, because there are two whites in a red, two whites equal up to a red. And that’s completely with numbers, right? Ok. That describes the question, doesn’t it? And you’ve answered it with words up here and you’ve answered it with numbers. Now you’d like to try the next one? And you can use this one up here as a model, just do the rest like this.

14.0.221 T/R 2: [Danielle starts working on next problem, builds a model of a brown rod and eight white rods.] Before you change this, what number name are you going to give to white, if you’re calling the brown one?

14.0.222 Danielle: [points and counts silently] One eighth.

14.0.223 T/R 2: Ok. Ok, now

14.0.224 Danielle: No, white [changes what she wrote]

14.0.225 T/R 2: Ok. Now can you write me a sentence about that in words, how many blank are in blank, what would you write?

14.0.226 Danielle: Because what I’m going to do is I’m going to write what these are, and then I’m going to write the sentence.

14.0.227 T/R 2: Ok, how are you doing, Brian? Let me just peek over your shoulder, you’re building your models first, and then you’re going to go back and do the questions? Good. How’s it going back there? [To Meredith] You’ve worked on a bunch of these, do you want to share one of these with me, as to what you did, maybe one of the ones you’ve already done. How about this one, the second one, if you give brown the number name one, what number name would you give to white? An eighth, and you showed that here, and what number name would we give to purple? A half. Ok, now, can you write me a number sentence that describes how many one eighths are in eight. Or how many one eighths are in one, I mean. Changing the problem, ok, alright. Can you do that for this one? How many purples are in brown or how many one halves in one?

14.0.228 Meredith: How many purples are in brown. [writes $1 \div \frac{1}{2} = 2$]

14.0.229 T/R 2: Ok, very nice thinking, good, so I’ll let you continue. How are you doing, Michael.

14.0.230 Michael: Good, uh, I did number one I’m on number two right now.

14.0.231 T/R 2: Can I see what you’ve done here? Ok, If we give the red the number name one, what number name would we give to

		white? Ok, half ok, I'd call it a half because two of them equal up to it. Ok, and here's my sentence, oh that's nice, you showed me that this is the red rod and you start using the white rod
14.0.232	Meredith:	That's what I did.
14.0.233	T/R 2:	Ok. My one - my question is to you, are you sure about that?
14.0.234	Michael:	Well, one divided by one half is one half, no that wouldn't be, that would be, that would be by subtracting so it would have to be 2, this wouldn't be one half it would be equals two
14.0.235	T/R 2:	Sure, because if you translated it back into words again and you're saying how many one halfs are in one, right?
14.0.236	Michael:	It would be two
14.0.237	T/R 2:	Alright, excellent, alright you continue.
14.0.238	Michael:	Ok.
14.0.239	James:	[inaudible]
14.0.240	T/R 2:	Is that the same answer
14.0.241	James:	Yeah
14.0.242	T/R 2:	And James, can you write me a complete number sentence for that now? With all numbers, to describe this question, how many one eighths are in one? How would you write that with numbers? [James pauses, thinking of what to write] Maybe just tell me, what would you write in terms of numbers, if you wanted to rewrite this all in numbers without any words, and you wanted to describe to me how many whites are in a brown, or how many one eighths are in one.
14.0.243	James:	Um, that would be eight because eight [lines up white rods beneath brown rod] Well I'm not going to have enough whites but eight of the whites equal up to here. And I used the reds [lines up four red rods] because the length would be eight, there would be eight whites here and so there would be eight in a whole brown because two whites-
14.0.244	T/R 2:	So you think that there are eight of them. Ok, if you're using, if you're working with how many whites are in a brown, how many eighths are in one
14.0.245	James:	Yeah.
14.0.246	T/R 2:	Ok, alright, I'll agree with that, and I see you did the purple also, and you're calling the purple what number name?
14.0.247	James:	One half
14.0.248	T/R 2:	Ok, looks good, ok, I'll let you continue, I understand now. [to Brian] How are we doing? Build your models? Ok we gotta think about those number sentences and writing them for each, ok because I want one for each. Ok, now while

		you're working on that let me talk to Danielle about number two here. What did you do?
14.0.249	Danielle:	I, I made, um, a box with the names
14.0.250	T/R 2:	Oh, that's nice sort of like a key to show me what everything means.
14.0.251	Danielle:	And then, and then I drew the models and the two purples are the halves, and the browns and the whites [points as she talks] is the one eighths and I gave the white the name one eighth and the purple one half because there's two purples and then one eighth because-
14.0.252	T/R 2:	Ok, so let me ask you the question now, how many whites are in brown?
14.0.253	Danielle:	Eight
14.0.254	T/R 2:	How many one eighths are in one?
14.0.255	Danielle:	Eight
14.0.256	T/R 2:	Ok, can you write this as a number sentence now, to describe this? How many one eighths are in one?
14.0.257	Danielle:	One divided by one eighth equals eight. [writes]
14.0.258	T/R 2:	Ok, how about, now that tells me about the white in the model, how about, um, how many purples are in brown? [Danielle writes $1 \div \frac{1}{2} = 2$] Is this starting to make sense? Is it starting to pull together in all the different ways? That's very nice, ok, I'll let you continue.
14.0.259		[camera roves catching students' work]
14.0.260	T/R 2:	Brian, you're all done? Can you describe maybe number three to me, explain to me what you did here?
14.0.261	Brian:	Well, uh, I made the orange and yellow train as one whole and then I put [T/R 2 interrupts to tell s/o about handing in work.] I used the orange and yellow train as one, and I put as much as, and I put, and I put whites up against the orange and yellow train, and I saw that there was fifteen, and then, and then it asked for I think for these [light greens] I think, [interruption again], then it asked for light greens, and I, and I uh, put them against and found that it was fifths, it took five of them, and I think it asked for the yellows, but I didn't really see that, and I thought
14.0.262	T/R 2:	Well, maybe you can describe one of these to me, though, Brian. How about the whites, ok, why don't you tell me, because I'm interested in this number sentence you wrote
14.0.263	Brian:	The whites, the whites would be fifteen, and it would take fifteen of them to equal up to the orange and yellow train
14.0.264	T/R 2:	So this tells me, then, how many fifteenths are in one.
14.0.265	Brian:	Yeah. [nods]
14.0.266	T/R 2:	Ok, what would the number sentence be for the light greens to answer that, to answer the question?

14.0.267	Brian:	Should I write it?
14.0.268	T/R 2:	Do you want to tell me first? Before you write it in stone there?
14.0.269	Brian:	One, one, one divided by, one divided by one fifth equals five.
14.0.270	T/R 2:	Ok, and in words, what would that mean?
14.0.271	Brian:	Uh, there are, there are, there are five, there are five fifths in
14.0.272	T/R 2:	[interruption, Brian gets up] Wait wait wait, I know you have to go, but answer that question for me.
14.0.273	Brian:	There would be five fifths in one whole and for the yellow it would be three thirds, one divided by three
14.0.274	T/R 2:	Ok, so it'd give three, ok, so here's, nice. Ok, Danielle, how far did you get?
14.0.275	Danielle:	I got until number three.
14.0.276 100:00 S	T/R 2:	Ok, a lot of people didn't finish, that's fine [collects papers from students]
14.0.277		*FRONT VIEW*
14.0.278 00:29:58	Amy:	[One camera focuses on Amy and Jackie.] They say sheet 1.
14.0.279 00:30:13	Jackie:	We don't use pencils anymore?
14.0.280 00:30:15	Amy:	You're not supposed to. You're supposed to use a pen on this.
14.0.281 00:30:20	Jackie:	I just want to write my name on this.
14.0.282 00:30:24	Amy:	If we give….[becomes inaudible].
14.0.283 00:30:47	Jackie:	[Jackie goes to get pens from the front of the room and then returns with an handful of pens.] Amy, I'm looking at all of these and seeing which one works the best.
14.0.284 00:30:51	Amy:	I would look at the erasers. Jackie, are you doing this?
14.0.285 00:30:58	Jackie:	Ha ha. Yeah, but I just want to see which one is the best. [Jackie tries several pens out on her paper. The teacher asks her to move her binder, so she moves it off the floor.] Oh, that's easy. That doesn't work.
14.0.286 00:31:17	Amy:	[Amy turns to the student to her right]. But you guys, you're supposed to make/draw a model.
14.0.287 00:31:23	Jackie:	Doesn't work. [Jackie returns the extra pens to the front of the room and then returns to her seat.]
14.0.288 00:31:36	Amy:	How many whites are in red? Two whites… One divided by,
14.0.289 00:31:56	Michael:	[The camera focuses in on Michael who is playing with his rods. He is stacking them. He stacks them so high that they fall over, and then he must pick them off the floor.]
14.0.290 00:32:09	Amy:	[The camera focuses back in on Amy and Jackie.] Would you help me with this thing? One whole and a 2. One divided by ½ equals two whites or one red. Ok. 1 divided by 2, 1 divided by ½ equals…
14.0.291 00:33:10	Jackie:	Do we have to prove it?

14.0.292 00:33:12 Amy:	Yeah. You have to do this. You have to write a number sentence to describe this relationship.
14.0.293 00:33:20 Jackie:	I messed up. [She erases something she wrote.]
14.0.294 00:33:22 Amy:	You don't have to erase. Well what would it be?
14.0.295 00:33:37 Jackie:	1 divided by ½.
14.0.296 00:33:42 Amy:	Don't erase.
14.0.297 00:33:44 Jackie:	Equals 2.
14.0.298 00:33:45 Amy:	You can't erase anything.
14.0.299 00:33:46 Jackie:	Why?
14.0.300 00:33:55 Amy:	[The camera zooms in on Amy's paper. This is what she has written next to number 1.] Ok. Number 2. If you give a brown number name 1. Are you doing number 2? You're not supposed to erase.
14.0.301 00:34:10 Jackie:	If we give the brown number name 1. [Jackie drops some rods and picks them up.]
14.0.302 00:34:45 Amy:	What number would we give the whites? What number name would we give the purple? First we've got to do the whites. [Amy builds a model with the rods.] Are you doing this Jackie? [Amy's model consists of 2 red rods next to 8 white rods next to 1 brown rod.] You would give the white. Wait a second, you have to draw this.
14.0.303 00:35:17 Jackie:	Well I will. I will.
14.0.304 00:35:18 Amy:	You're supposed to draw it first Jackie. [Amy draws her model on her paper.]
14.0.305 00:35:33 Jackie:	[Jackie begins to build the model. She lines up 8 white rods next to the brown rod]. 1, 2, 3, 4, 5, 6, 7, 8. [Jackie write stuff down on her paper.]
14.0.306 00:36:05 Amy:	Now we have to give what number you call the purple.
14.0.307 00:36:09 Jackie:	I didn't draw anything.
14.0.308 00:36:12 Amy:	Well I'm done the first one already.
14.0.309 00:36:17 Jackie:	Oh no. And I can't erase.
14.0.310 00:36:20 Amy:	Cross it out.
14.0.311 00:36:23 Jackie:	I guess. [Jackie crosses out some things on her paper.]
14.0.312 00:36:24 Amy:	That's the most logical thing to do Jackie, wouldn't you say?
14.0.313 00:36:27 Jackie:	Now my paper's messy.
14.0.314 00:36:29 Amy:	So what? Mine's even messier. [Jackie traces the rods to draw the model on her paper.] 1/2. 1, 2, 3, 4, 5, 6, 7, 8. 1/8.
14.0.315 00:37:10 T/R 2:	How are we doing over here?
14.0.316 00:37:11 Amy:	Ok.
14.0.317 00:37:12 T/R 2:	Are you clear on what's happening?
14.0.318 00:37:13 Amy:	Mm-hmm.
14.0.319 00:37:16 T/R 2:	Ahh. Ok. So this tells me then. How do you translate this back into words now, this number sentence? What does that tell me?

14.0.320 00:37:28 Amy:	[The camera focuses in on her paper. We can see her number sentence $1 \div \frac{1}{2} = 2$.] 1 is. 1/12 is. Wait. 2 halves go into 1. ½ goes into 1 twice.
14.0.321 00:37:49 T/R 2:	Ok. Alright. So you're saying then that it takes 2 of these ½'s to give you that length. I agree with you.
14.0.322 00:38:19 Amy:	Now we have to write the number sentence for this.
14.0.323 00:38:21 Jackie:	For what?
14.0.324 00:38:23 Amy:	What are you doing? You're supposed to see how many purples go into brown. Purples would be twice, not ¼. ½. [Amy writes ½ on Jackie's paper. Jackie crosses out where she wrote ¼.] Haha. Jeez Jackie.
14.0.325 00:38:58 Jackie:	There. Thank you. [Jackie crosses out some more stuff on her paper.]
14.0.326 00:39:04 Amy:	Now we have to write 1 divided by. We would have to like write 2. 1 divided by, it's ½, equals 2. 1 divided by 1/8 equals 8. I'm on number 3.
14.0.327 00:39:28 Jackie:	I'm on number 2.
14.0.328 00:39:32 Amy:	This is simple.
14.0.329 00:40:00 Kimberly	[The camera focuses in on Kimberly and Erin. Kimberly built a train with a red rod and a yellow rod. She then lines up white rods next to this train.] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15. Ok. Time to trace.
14.0.330 00:40:59 Jackie:	[The camera focuses back in on Amy and Jackie.] I caught up.
14.0.331 00:41:03 Amy:	I gotta find this ruler. [Amy pulls out a ruler and uses it to help her trace the rods. She measures the length of a train made up of a red rod and a yellow rod and draws it on her paper.]
14.0.332 00:41:34 Jackie:	I need a new paper.
14.0.333 00:41:37 Amy:	We've got some right here Jackie. [Next to the red and yellow train on her paper, Amy writes 1.] What number name would we give the whites?
14.0.334 00:42:21 Jackie:	4 [Meaning she's on problem #4].
14.0.335 00:41:25 Amy:	That's not 4, that's 3. Erase that. Now she thinks you put 4 instead of 3.
14.0.336 00:41:40 Jackie:	Should I? [Jackie erases something on her paper.]
14.0.337 00:42:47 Jessica:	[Jessica asks Jackie a question but it's inaudible.]
14.0.338 00:42:54 Amy:	Oh, number 2 is easy. You make two equations.
14.0.339 00:43:01 Jackie:	We had to make an equation?
14.0.340 00:43:02 Amy:	Yeah. A number sentence. See on number 1, we did that. [She points to Jackie's paper.] You have to do that for every single one.
14.0.341 00:43:14 Jackie:	Oh, what was it? What is it? I know, I know, never mind.
14.0.342 00:43:25 Amy:	The first one is…
14.0.343 00:43:28 Jackie:	1 divided by 1/12 equals 12.

14.0.344 00:43:35 Amy: Where did you get 1/12?
14.0.345 00:43:38 Jackie: You gave it to me.
14.0.346 00:43:39 Amy: No I didn't.
14.0.347 00:43:40 Jackie: ½.
14.0.348 00:43:42 Amy: ½ not 1/12. [The camera zooms in on Amy's paper. She is drawing in the white rods underneath the sketch she made of the red and yellow train.] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13. [She lines up white rods underneath an orange rod and underneath a yellow rod. She counts that there are 15 white rods so she writes 1/15 next to the white rods in her sketch.] 1/15. What number name would we give the light green? [Amy lines up light green rods underneath the red and yellow train. She draws the light green rods below the white rods on her sketch.] 1, 2, 3, 4, 5. 1/5. [Amy writes 1/5 next to her sketch of the light green rods. Her pen runs out of ink, so she gets up to get a new pen.]
14.0.349 00:46:18 CT: Jackie, how are you doing? Now some people said that they had the fractions so easy, what did you get for number 2?
14.0.350 00:46:23 Jackie: That was easy.
14.0.351 00:46:26 CT: That was easy. Tell me what did you get for the brown? What did you call that fraction?
14.0.352 00:46:29 Amy: We had 2.
14.0.353 00:46:32 CT: What did you call the white on the brown?
14.0.354 00:46:34 Jackie: Umm, 1/8.
14.0.355 00:46:38 CT: What did you call the purple on brown?
14.0.356 00:46:42 Jackie: ½.
14.0.357 00:46:43 CT: Well I bet you think you're so smart. Did you put the equations down for her to see?
14.0.358 00:46:50 Jackie: I put this one down.
14.0.359 00:46:55 CT: 1 equals ½. Oh ok. Where's the one for this one?
14.0.360 00:47:06 Amy: This is going to be a three.
14.0.361 00:47:07 CT: A three part one.
14.0.362 00:47:09 Amy: Number 3 is going to be a three parter. Hey, probably number four will be a four parter.
14.0.363 00:47:13 CT: A four part one. Are you having any trouble with this?
14.0.364 00:47:20 Jackie: No.
14.0.365 00:47:21 CT: Do you understand your beans?
14.0.366 00:47:22 Jackie: Yes.
14.0.367 00:47:23 CT: Are you sure?
14.0.368 00:47:24 Jackie: Yes.
14.0.369 00:47:25 CT: It's looking good.
14.0.370 00:47:29 Jackie: Look how messy this is. I wish I could erase.
14.0.371 00:47:26 Amy: [Amy draws the yellow rods underneath where she sketched the light green rods. She writes 1/3 next to these rods in her sketch.] Now I need a number sentence. 1 divided by 1/15

	equals 15. 1 divided by 1/5 equals 5. 1 divided by 1/3 equals 3. I'm on number 4.
14.0.372 00:48:21 Jackie:	I'm on number 4 too.
14.0.373 00:48:23 Amy:	I'm on 4, you're still on 3.
14.0.374 00:48:27 Jackie:	No, I'm on 4. I'm on 4. I am. See. Oh no, I'm on 3. I'm not done yet. I'm on 4 too.
14.0.375 00:49:00 Amy:	They gave you a blue and a yellow.
14.0.376 00:49:26 Jackie:	I'm not done yet.
14.0.377 00:49:27 Amy:	[Amy draws the blue and yellow train.] 1. [Jackie's binder falls on the floor.] Nice one Jackie. [Jackie picks up her binder and the two laugh about it.] What would you give the whites? Same old business. [Amy lines up white rods next to the blue and yellow train.] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14. [Amy puts the rods back in the tray. She draws the white rods below the blue and yellow train in her sketch.] 1/14. 4. What number name would we give the reds?
14.0.378 00:52:15 Jackie:	Finally on 4. What did you get? What did you have to do?
14.0.379 00:52:19 Amy:	Read.
14.0.380 00:52:20 Amy:	[Amy lines up red rods under the blue and yellow train.] 1, 2, 3, 4, 5, 6, 7. [Amy puts the rods back in the tray and draws the red rods underneath the blue and yellow train on her paper.] I'm almost finished. I'm on number 4.
14.0.381 00:53:13 T/R3	Can I see what you did for number 3? I haven't seen anyone's number 3 yet.
14.0.382 00:53:19 Amy:	Well, we did what they said. This is the orange and red train.
14.0.383 00:53:26 T/R3	Did you call that 1?
14.0.384 00:53:27 Amy:	Yes. And this, there are 15 whites, there are 5 light greens, and there are 3 yellows. And I did 3 equations, 1 equation for each.
14.0.385 00:53:40 T/R3	Ok. So then the whites are going to be called what number name?
14.0.386 00:53:44 Amy:	1/15.
14.0.387 00:53:45 T/R3	And how about the light green?
14.0.388 00:53:46 Amy:	1/5.
14.0.389 00:53:47 T/R3	And the yellow?
14.0.390 00:53:48 Amy:	1/3.
14.0.391 00:53:51 T/R3	Alright. I see you wrote me an equation for each. [The camera zooms in on Amy's paper. She sketched a model of the rods. Underneath this model she wrote three equations: $1 \div \frac{1}{15} = 15$, $1 \div \frac{1}{5} = 5$, $1 \div \frac{1}{3} = 3$.] Can you do this for me? You did it so well before. Can you translate this back into words? What does this first one mean?

14.0.392 00:54:00 Amy:	15. 1/15 goes into one 15 times. 1/5 goes into one 5 times. 1/3 goes into one 3 times.
14.0.393 00:54:12 T/R3	That's very nice. Let me let you continue.
14.0.394 00:54:17 Jackie:	Wait a minute. There's only supposed to be 14 right?
14.0.395 00:54:21 Amy:	In what?
14.0.396 00:54:22 Jackie:	In this.
14.0.397 00:54:25 Amy:	You will find out. You should see. You should actually check before you do.
14.0.398 00:54:33 Jackie:	Who cares? I just made it a little bigger. [Jackie continues writing on her paper.] What number name will we give to red?
14.0.399 00:55:38 Amy:	1/7. 1/2. 1 divided by 1/7 equals 7. 1 divided by ½ equals 2. I finished.
14.0.400 00:56:54 Jackie:	I hate making this all over again. I finished. [The students finish their work, collect their materials, and put everything away. They hand their work to T/R 2:. The students talk amongst themselves.]

Session 15, Dec. 9, 1993 (Front, Side, and OHP)

Session 15, Dec 9, 1993, Side View

Line	Time	Speaker	Transcript
15.1.1	00:07:27	T/R 1:	Here’s what we are trying to do today. If you notice we have some displays and this is thanks to Mrs. Demming who is actually in the Cedar school right now doing another kind of filming and she made these displays for us. Maybe the best way is to consider the activity we have for you, of a problem, that we’re going to ask you to help us to solve. So let me tell you a little bit about the problem then I am going to say go to it- right? That is what you like to do. We have just received a shipment of ribbon that are going to be used to make bows for the holiday. Ok? You are to consider this examples of your shipment of ribbon. [Holds up ribbon.] Now what’s interesting about this ribbon is you’ll notice that this ribbon comes in different lengths. On your yellow paper you can see that red ribbon comes in packages of length that is what do you see? 6 meters. Do you all see that? [Students agree]. Gold ribbon? What’s the length of gold ribbon? 3 meters. What about blue ribbon? [students answer] 2 meters. White ribbon? [students answer] 1 meter. Ok. Now from these bows, from this ribbon we can make bows and depending upon how much ribbon we use the bows you see can the bows can be different sizes, they can be different shapes, right? Now your problem is going to be to find out how many bows of certain lengths that we can make from the ribbon that’s in your package. That’s what you have to figure out. Now before you start figuring it out, let me just point out a few things to you. If you need to try to figure out these bows, the lengths of these bows, you might not at first want to cut up your ribbon. You might want to use your ribbon so that you know how much you have to deal with. You might want to cut up the string so you can test your ideas. And if you need to do that you might have to take the string, you might have to measure it, of the length of that ribbon that you have to figure out your problems. Now, now look at the first problem, see the first problem? Do you think you understand what the first problem is asking? How many of you think you understand? You want to talk to your partner for a minute to be sure you understand what the first problem is asking? Chat with your partner. [Students given 20 seconds to talk] Ok, we have a question. Someone is asking a question. Where is the first problem? Who can help answer the

		question- where is the first problem? Andrew you want to give it a try? Ok, let's see what Andrew has to say.
15.1.2	00:10:30 Andrew:	By the number one, it says white ribbon it has one meter, and one meter on the bottom where it says. And then it says make a ribbon length of bows and number of bows. So you have to find out look under the ribbon length of bow so it will be one meter and one half how many bows can you get out of one half a meter. So you would have to figure out that.
15.1.3	00:10:59 T/R 1:	So if you were to take your white ribbon, can you imagine the white ribbon being this long? Can you imagine that? [Holds ribbon up] You might test it if you don't believe it. You might take your white ribbon to see if it's this long. One meter right? Ok, but now with the white ribbon the first question can you see Danielle on there, the white ribbon? What is the length of ribbon you are going to use to make your bow? What does it say on the chart?
15.1.4	00:11:26 Michael:	You have to use ribbon length of the bow. You have to use one half of a meter.
15.1.5	00:11:31 T/R 1:	One half of a meter to make your first bow. Right? Now, if you are making your meter, I don't want you to answer out, if you are making your ribbon from a bow one half of a meter in length, I want to know how many bows you could make? Ok, think about it. This is the length of your white ribbon. [holds ribbon] Ok.
15.1.6	00:11:54 Erik:	One ribbon is a half a meter
15.1.7	00:11:55 T/R 1:	One ribbon is a half a meter.
15.1.8	00:11:57 Erik:	And then you have to figure out how many bows, how many of those bows you make?
15.1.9	00:12:03 T/R 1:	That's right. From one meter ribbon and each bow is a half meter in length.
15.1.10	00:12:06 Erik:	And how many bows of that kind.
15.1.11	00:12:11 T/R 1:	Now our guests will walk about and chat with you if you have any questions. They might, they might not ask you questions, they might ask you a question instead. That's the kind of visitors we have. But they might come around so if you aren't clear about something, you might want to chat with them. Ok? So go to it. [12:27] [Students getting materials, etc]
15.1.12	00:13:00 V1:	What are you guys doing? [student say something] How long is that one?
15.1.13	00:13:05 Brian:	The blue ribbon is about 2 meters.
15.1.14	00:13:10 V1:	How long is the white one supposed to be?
15.1.15	00:13:11 Danielle:	One meter.
15.1.16	00:13:13 V1:	Is that one meter?
15.1.17	00:13:16 Danielle:	I don't know.

15.1.18 00:13:18 V1: How can you tell? [kid 4 leaves seat]
15.1.19 00:13:23 V1: Do you know how big a meter is supposed to be?
15.1.20 00:13:25 Brian: Three feet [can't understand]
15.1.21 00:13:28 V1: Well, not quite 3 feet is it? It's near 3 feet- but feet and meters are different. You know how they are different?
15.1.22 00:14:22 Andrew: [is measuring] 99, well actually 100
15.1.23 00:15:02 Dr. Landis: What are you trying to figure out guys?
15.1.24 00:15:03 Andrew: We're trying to figure out the second problem in number one. One meter is three thirds. So we have to find three thirds of this one meter ribbon and then we'll know how much bows we can make. I think I know already.
15.1.25 00:15:24 Dr. Landis: What do you think you know?
15.1.26 00:15:25 Andrew: I think its three because when you divide thirds up it's one-third, two-thirds, three-thirds. So it would be like one ribbon, one ribbon, one ribbon [cannot understand but pointing to ribbon.] so divide it into thirds but we need to make sure.
15.1.27 00:15:53 Dr. Landis: What do you think of what Andrew said?
15.1.28 00:15:56 James: Yeah, I agree with that but we're just making sure.
15.1.29 00:15:59 Dr. Landis: Ok, you think it makes sense, but you're just kind of checking it.
15.1.30 James: Yeah.
15.1.31 00:16:10 Andrew: I know that fourths is 25. It would be right here. 28, 40, 42
15.1.32 00:16:30 James: I think I got 30 and a half. [reading off measurements to Andrew]
15.1.33 00:16:33 Andrew: According to him [measuring] 30 and a half? 30 60, 61, plus 30 is only 91 and a half. What did you get? 30 and a half? I think its 32. 32 and 32 is 64. Then 32 is 90.
15.1.34 00:17:43 James: 32 and half. So there will be 65. 32 and 65 is 97. 33. 33, 66, 99, 8 and half. So it would be 100. 33.
15.1.35 00:17:43 V2: What was the question?
15.1.36 00:17:50 Andrew: We want three bows.
15.1.37 00:17:53 V2: You were doing something very interesting just now. What was the number you came up with?
15.1.38 00:17:57 Andrew:, James: 33
15.1.39 00:17:59 V2: What was that? What was it?
15.1.40 James: Centimeters
15.1.41 00:18:05 Andrew: Well, Rutgers usually makes us prove what our answer is. So we had to do three, divided the ribbon into thirds, first third would be there, second would be there, and third
15.1.42 00:18:21 V2: So what was that though actually-this 33 that you came up with?
15.1.43 Andrew: One third.
15.1.44 00:18:27 James: One third, yeah, one third of a
15.1.45 Andrew: That's sixty fi- sixty-

15.1.46	00:18:33	V2:	What's one third—what was it? Like, here are the answers you gave- you can make 3 bows. That 33-what was it?
15.1.47	00:18:40	James:	That was like in centimeters.
15.1.48	00:18:42	V2:	Right, but what was that number? You're telling me its 33 centimeters. Well what is that?
15.1.49	00:18:48	Andrew:	We're proving. We are proving that, we wanted to make sure that when you divide it into thirds, there's no left over or anything so we can actually [break in video 19:01]
15.1.50	00:19:43	V1:	[camera focuses on V1 with Brian and Danielle] So which one do you want to do next? You can do anyone you want.
15.1.51	00:19:55	Brian	I'm doing this one. Three meters and I used this. I got 39 plus 39 plus 39 and.
15.1.52	00:20:07	V1:	Although, is that really accurate? Is that exact?
15.1.53	00:20:16	Brian:	Two meters?
15.1.54	00:20:17	V1:	Is it exactly 114 inches?
15.1.55	00:20:25	Danielle:	Would this be 4?
15.1.56	00:20:30	V1:	Yeah, explain, is it four? Does that make sense?
15.1.57	00:20:39	Danielle:	Yeah
15.1.58	00:20:40	V1:	So Brian, Danielle just said if you have 2 meters of ribbon and it has to be a half meter length bow, she says there's 4 ribbons, 4 bows to make. Do you buy that?
15.1.59	00:20:55	Brian:	Uh, well, which one is she doing?
15.1.60	00:20:57	V1:	Blue ribbon, the first one
15.1.61	00:21:10	Brian:	4. Yeah, 4.
15.1.62	00:21:12	V1:	Do you see why?
15.1.63	00:21:15	Brian:	Oh, yeah, I know why. I think I know why.
15.1.64	00:21:18	V1:	Yeah, why?
15.1.65	00:21:21	Brian:	Well, one meter is say approximately three feet without any inches. Say it was just plain 3 feet then, and if there was, two meters would be, two meters would be, approximately about 12 feet, no, 6 feet, 6 feet put together.
15.1.66	00:21:46	V1:	Approximately.
15.1.67	00:21:48	Brian:	Yeah approximately, not exactly. And if it was 6 feet and each one was a half, then it'd be, then half of three would just be like saying one and a half, another one and a half, and that'll fill up three. Then for the other three, one and a half, one and a half.
15.1.68	00:22:07	V1:	So how many is that all together?
15.1.69	00:22:09	Brian:	4
15.1.70	00:22:10	V1:	Is that the way you did it? Is that the way you thought about it?
15.1.71	00:22:12	Danielle:	No.
15.1.72	00:22:13	V1:	How did you do it?
15.1.73	00:22:14	Danielle:	I just thought if it is two meters and each is a half, two halves are in a whole.
15.1.74	00:22:22	V1:	So two halves make a whole?

15.1.75	00:22:23	Danielle:	Yeah. And then there's 2 meters so I got 4.
15.1.76	00:22:29	V1:	So you have two wholes. Now, do you see what she did? You went one extra step. You went to feet, then you did it with feet, then you came back. Do you have to do that?
15.1.77	00:22:41	Brian:	No.
15.1.78	00:22:44	V1:	Yeah, do you see how to do it more quickly? Try it for the gold ribbon
15.1.79	00:22:52	Brian:	It would be six. Six. No, wait, it'd be eight.
15.1.80	00:22:59	Danielle:	But it says there's two, four, six.
15.1.81	00:23:03	Brian:	Yeah, it'd be six. Six meters.
15.1.82	00:23:06	V1:	Make sense? [nods]
15.1.83	00:23:08	Brian:	Yeah, I get it. Six meters.
15.1.84	00:23:13	Danielle:	What about the rest of it?
15.1.85	00:23:17	V1:	You'll get to that. You're on a roll now.
15.1.86	00:23:21	Brian:	It'd be twelve.
15.1.87	00:23:26	V1:	So, is there, is there a rule here?
15.1.88	00:23:30	Brian:	There's two in one meter, which is approximately three feet, there are two halves. In another one there are two halves. Another one there are two halves, etc. And so if you keep counting by two up to six meters, that's be twelve meters.
15.1.89	00:23:49	V1:	That makes sense. Ready to go on?
15.1.90		Jessica:	[Jessica rolls out the blue ribbon with Laura] This keeps going.
15.1.91	00:24:25	V3:	Aha, so how do you do that? It's one, so it's one.
15.1.92	00:24:32	Jessica:	One meter, and it's two meters.
15.1.93	00:24:34	V3:	How do you know if it's true or not?
15.1.94	00:24:36	Jessica:	Well it says two meters but do we have to get another one of these?
15.1.95	00:24:42	V3:	Do you need to? Can you just use this one to measure?
15.1.96	00:24:46	Jessica:	This you could….
15.1.97	00:24:49	V3:	Can you find some way to do that?
15.1.98	00:24:49	Jessica:	Yes, I think.
15.1.99	00:24:54	V3:	Ok, I hold this. [measuring blue ribbon] Ok. I think its short but this one is too long. But it's not tight. Ok let's put together and then you try to stretch it.
15.1.100	00:25:24	Jessica:	It's three [cannot understand]. We'll have to pretend that's right.
15.1.101	00:25:33		V3: Yes, ok, now we have two meters.
15.1.102	00:25:38	Jessica:	Ok so ..
15.1.103	00:25:39	V3:	So the first questions is
15.1.104	00:25:41	Jessica:	One half of a meter. One half of one meter or two meters?
15.1.105	00:25:47	V3:	So now we use one half meter we can make how many bows we can make?
15.1.106	00:25:52	Jessica:	Two, well, would it be four?

15.1.107 00:25:56	V3:	Compare with the answer in the first question. Last time we had-
15.1.108	Jessica:	Two.
15.1.109	V3:	-one meter we can make two. Now we have two meters.
15.1.110 00:26:06	Laura:	Four, yeah.
15.1.111 00:26:11	V3:	The reason being?
15.1.112 00:26:14	Jessica:	Well you have to double that because now that there's double that now it's two meters so
15.1.113 00:26:21	V3:	And in two meters how many half meter we have?
15.1.114 00:26:25	Jessica:	A half meter we have two
15.1.115 00:26:30	V3:	In one meter, how many half meters we have?
15.1.116 00:26:35	Jessica:	Well we can make two, we have one [measuring]
15.1.117 00:26:41	V3:	I said in one meter, how many half meters we have?
15.1.118 00:26:47	Jessica:	Is that like?
15.1.119 00:26:50	V3:	Half meter, 50
15.1.120 00:26:54	Jessica:	[gesturing with hands] One, two, one, no that would be two
15.1.121 00:27:00	V3:	Good. Now in two meters, how many half meters we have?
15.1.122 00:27:05	Jessica:	Four.
15.1.123 00:27:06	V3:	Good. So that's why. That's exactly. In the first case we only have one meter. Now we have two meters right?
15.1.124 00:27:13	Jessica:	Yup
15.1.125 00:27:15	V3:	So the answer is four. [writing on papers] What about the second one? The one we spent lots of time last time.
15.1.126 00:27:23	Jessica:	Three meter. That would be, I think that would be five…
15.1.127 00:27:29	V3:	Why five?
15.1.128 00:27:32	Jessica:	Well, because we just added one bow to that side, oh, no, now it'd be six
15.1.129 00:27:36	Laura:	Six!
15.1.130 00:27:37	V3:	Ok, so why's that?
15.1.131 00:27:41	Jessica:	Because you're doubling, you're doubling, like last time was three, and three plus three is six. And now I think that would be 8, the next one. And that would be 10.
15.1.132 00:28:07	V3:	What about the last one?
15.1.133 00:28:13	Jessica:	Oh they added one this time, so
15.1.134 00:28:17	V3:	Yeah, so this one says, it needs two-thirds meters to make one. Ok let me ask you Jessica, this is two meters right? [folds ribbon in half] If I fold it this like this and then I measure it like this, this is half of two meters right?
15.1.135 00:28:43	Jessica:	That would be one meter.
15.1.136 00:28:47	V3:	Yes. But if I do this, I make into three equal lengths, then I measure it, what that would be?
15.1.137 00:28:58	Jessica:	I think it'd be a half. Well, it would be… [playing with ribbon]
15.1.138 00:29:20	V3:	So now you have four folds. Right?
15.1.139 00:29:21	Jessica:	Four? We need three right?
15.1.140 00:29:25	V3:	Right, here we say two-thirds.

15.1.141	00:29:38	Jessica:	How would you make three? That's four.
15.1.142	00:29:46	V3:	That's right. Suppose, let's just use this because this is one meter, right? [folding white ribbon] So here's one meter if I do this. This is not very easy to do but what I mean is. Can you hold one end? Just hold this. Ok, now roughly we have three. We have cut one meter into three equal lengths, right?
15.1.143	00:30:18	Jessica:	Put it up against this and see what it would come to. Not exactly.
15.1.144	00:30:28	V3:	Yeah, but roughly.
15.1.145	00:30:29	Jessica:	Thirty-three?
15.1.146	00:30:30	V3:	Remember what we did last time about one-third of a meter.
15.1.147	00:30:35	Jessica:	Yeah, I don't-
15.1.148	00:30:38	V3:	Right, right. So see this is exactly what's happening when we did beginning, I mean, like what you did. If you do this by half. This is exactly come to fifty. But then what I just did, a third.
15.1.149		Jessica:	It would come to thirty-three.
15.1.150		V3:	It comes to a third. So how can you get two-thirds from here?
15.1.151	00:31:09	Jessica:	Well that came to --.. so then you get [cannot understand]
15.1.152	00:31:23	V3:	What is two-thirds?
15.1.153		Laura:	Two one thirds [inaudible]
15.1.154		V3:	Add another thirty-three. Very good. So how much we get out of that? We said thirty-three - we said is close to a third. So what is two-thirds? Add them together.
15.1.155	00:31:51	Laura:	That would be, um, sixty-six.
15.1.156	00:31:54	V3:	Just what you said right now. Add two one – [cannot understand]
15.1.157	00:31:58	Jessica:	Sixty-six.
15.1.158	00:32:02	V3:	Sixty-six right.
15.1.159		Jessica:	And that would equal up to here.
15.1.160		V3:	Right, right, exactly.
15.1.161	00:32:15	Dr. Landis:	[mid-sentence] which is two meters long. Why are you going to get four bows? Because I'm confused. Convince me.
15.1.162	00:32:23	Andrew:	It's double.
15.1.163	00:32:24	James:	Its one half meter. This is two meters. [pointing to blue ribbon] so one half of one meter would be two. And this is two meters so another half of a meter would be plus the other half would be four. So that would explain why four bows.
15.1.164	00:32:41	Andrew:	Yeah, what he's saying is you have this. This acts as four [blue ribbon].
15.1.165	00:32:46	Dr. Landis:	Why acts as four?
15.1.166	00:32:47	Andrew:	This.

15.1.167 00:32:48 Dr. Landis: Why does this act as four?
15.1.168 00:32:50 James: Because, this acts as two [white ribbon].
15.1.169 00:32:51 Andrew: Well yeah, this
15.1.170 00:32:53 Dr. Landis: Why does this act as two?
15.1.171 00:32:55 Andrew: Because we're saying this is a half. You cut this [white ribbon] in half. You have two parts. So you put that up to it and it's two. And then you need – this [white ribbon] is a half of that [blue ribbon]. So you need one more and that's four. So then this can act as three- as cutting it into thirds. Is putting six-thirds up to two-thirds. But this two meters is the whole so actually this. If two meters is the whole then this is the half. This is almost like rods. We don't have every one though. This is the whole. This is the half. We have another one a half.
15.1.172 00:33:46 Dr. Landis: Ok, Let me go back cause I think you're telling me a lot of things and I just want to be sure I understand. Um, if you have the blue ribbon and we want to figure out how many bows we could make, and each one is a half a meter how did you figure out first you told me it was two and then you said no, it's four.
15.1.173 00:34:07 James: It's four because
15.1.174 00:34:08 Andrew: It could be either way.
15.1.175 00:34:09 Dr. Landis: It could be either? What do you think about that, James? Could it be either?
15.1.176 00:34:10 James: I don't know. That one would be half of two meters.
15.1.177 00:34:19 Andrew: So one half of two meters not one half of one meter.
15.1.178 00:34:22 Dr. Landis: Ok. I think what it's saying is the ribbon length of the bow is a half a meter so each ribbon is gonna be a half a meter long, ok? And the question is how many bows can you get out of that blue ribbon if each bow is gonna be a half a meter long?
15.1.179 00:34:38 James: Four cause this is going to be two meters long.
15.1.180 Andrew: But its saying it's a half of two meters.
15.1.181 00:34:41 Dr. Landis: Let me see if I can understand what James is saying.
15.1.182 00:34:44 Andrew: But its saying it's a half of two meters because the blue ribbon here is two meters is the whole so then this is half of two meters that would only be two.
15.1.183 00:34:54 Dr. Landis: Ok, what does this say to you? What is a half a meter?
15.1.184 00:34:59 James: One half of one of these [white ribbons]
15.1.185 00:35:03 Andrew: Yes, four. That would be four and this would be
15.1.186 00:35:07 James: This would be two. One of these would go up against them. And that would be two halves and another one you put on, and that would be four
15.1.187 00:35:19 Dr. Landis: Ok, so you're saying if you took your white ribbon. Let me see if I know what you're saying. If you took your

white ribbon and cut it in half, how long would each bow be?
15.1.188 00:35:30 Andrew: Each bow?
15.1.189 00:35:32 Dr. Landis: If you cut the white ribbon in half, how big would each bow be? How long is that white ribbon?
15.1.190 00:35:40 Andrew: 100 meters so it would be 50 [hold up white ribbon]
15.1.191 00:35:42 James: 100 centimeters
15.1.192 00:35:44 Andrew: Centimeters [measuring white ribbon] So it would be right.
15.1.193 00:35:52 James: From there to there. Then you cut it in half it would be two.
15.1.194 00:36:05 Dr. Landis: Ok, if you cut the white ribbon in half it would be two and each one. If this is a meter, how long would each ribbon be?
15.1.195 00:36:13 James: It would be 50 centimeters.
15.1.196 00:36:14 Dr. Landis: Fifty centimeters, ok. And in terms.
15.1.197 00:36:19 Andrew: These two are halves.
15.1.198 00:36:21 Dr. Landis: Halves of what?
15.1.199 00:36:22 Andrew: Of one meter.
15.1.200 00:36:23 Dr. Landis: So each one is a half of a meter. Ok, so now my question.
15.1.201 00:36:27 Andrew: You can fit four of them in the blue.
15.1.202 00:36:29 Dr. Landis: That's what I was trying to find out. How do you know you can fit four of them in the blue?
15.1.203 00:36:33 Andrew: Well you can fit two of these in the blue. Then you cut that in half. That's two. So you have two in one. Then you have the other side. That would be two and two is four.
15.1.204 00:36:47 Dr. Landis: Ok, I, I think I am following what you are saying. Let's see if James agrees.
15.1.205 00:36:51 James: Yeah, I do agree. [measuring white ribbon]
15.1.206 00:36:53 Dr. Landis: You do agree?
15.1.207 00:36:54 James: Yeah.
15.1.208 00:36:56 Dr. Landis: What are you just doing now?
15.1.209 00:36:57 James: I-
15.1.210 00:36:59 Dr. Landis: What did you do?
15.1.211 00:37:00 James: I put the – 50 centimeters.
15.1.212 00:37:03 Andrew: Yeah, so if you put that up to
15.1.213 00:37:06 Dr. Landis: Why don't you show me? Why don't you put it? Ok, why don't you show me what you just did? Because then I'll know how many ribbons you can make out of that blue…out of that blue ribbon. Each ribbon we want to be a half meter long. Right? Let's take a look and see what he's doing there. You might want to help him, James, ok? [laying ribbons on meter stick] Can I hold the other side for you so that it'll stay stretched out?
15.1.214 00:37:44 Andrew: Is the dot here, yeah, there's the crease.
15.1.215 00:37:52 Dr. Landis: You need another finger here?

15.1.216 00:37:58 Andrew: Yeah. So where the crease is, is where it would be cut. So that's two.
15.1.217 00:38:00 Dr. Landis: I am following you now. It's making more sense. Let's see.
15.1.218 00:38:04 Andrew: It's two.
15.1.219 00:38:09 Dr. Landis: Ok so out of this big blue ribbon we can make bows that are a half a meter long and how many bows did you get?
15.1.220 00:38:17 James: and Andrew: Four.
15.1.221 00:38:18 Dr. Landis: Four. Ok?
15.1.222 00:38:21 Andrew: And then with the third we were trying to figure out, to be a third of a bow.
15.1.223 Dr. Landis: Yeah.
15.1.224 Andrew: We thought it was sixty but.
15.1.225 00:38:31 Dr. Landis: Ok, what this is saying again and I think just to understand what it is asking is, they want to know if you start with this big blue ribbon again, how many bows could you make if each bow is a third of a meter long?
15.1.226 00:38:46 Andrew: Six.
15.1.227 00:38:46 Dr. Landis: You think six? What do you think?
15.1.228 00:38:48 James: Six.
15.1.229 00:38:49 Dr. Landis: Tell me why.
15.1.230 00:38:51 Andrew: Because there are three, three
15.1.231 00:38:53 Dr. Landis: You can tell me. You don't have to do it if you can explain it to me.
15.1.232 00:39:57 Andrew: It would be three thirds in this so if you put three [gesturing with the white ribbon] on one side
15.1.233 00:39:04 Andrew: and James: and three on the other side
15.1.234 00:39:06 Dr. Landis: Ok, so you think you're gonna get six ribbons that are a half of a meter… a third of a meter long, from that big blue ribbon?
15.1.235 James: Yeah.
15.1.236 Dr. Landis: How about from the blue ribbon if you wanted to make bows that are a quarter of a meter?
15.1.237 00:39:21 James: Eight.
15.1.238 00:39:23 Andrew: Eight. Because maybe we can, a half of the half is a fourth. Since there's two halves, it'd be two, four, four on one side, four on the other, so it would be eight.
15.1.239 00:39:47 James: It would be eight.
15.1.240 00:39:48 Dr. Landis: Ok you're both agreeing on that.
15.1.241 00:39:51 James: Yes. The same for fifths
15.1.242 00:39:54 Dr. Landis: What would be for fifths?
15.1.243 Andrew: It would be
15.1.244 39:56 Andrew: and James: Ten. Because it would be five on one side and five on the other.

15.1.245 40:01 Dr. Landis: Ok, ok, so you're convinced now that these should be different than what you said originally. Ok, you have two meters, you have that blue ribbon. Now you want to cut bows that are two thirds meters long.
15.1.246 40:16 Andrew: That's where we were.
15.1.247 40:18 Dr. Landis: That's where you were left.
15.1.248 40:24 James: Ok, alright, let's try this again.
15.1.249 40:32 Andrew: Put that over there.
15.1.250 40:33 Dr. Landis: You want me to hold it down here? I'll be your helper? You're gonna hold it for him?
15.1.251 40:38 James: I'll hold it right here.
15.1.252 40:41 Andrew: We were thinking it has to be somewhere between 71 and 60. Because we tried 60 and we tried 71 and 71 was over and 60 was less so we came down and we were trying 69. So it would be 69.
15.1.253 41:02 James: And then 138. 138 and 69 is? It's over. So we'll try 68
15.1.254 41:17 Andrew: 68 and 68 is 136. 68 is over. Wait no, yeah it over. So, it's about 105.
15.1.255 41:34 Dr. Landis: I'm confused what you're doing. What are you doing now? You are working with these numbers I didn't hear you working with the numbers before.
15.1.256 41:39 Andrew: Yeah. That's because the thirds, We didn't actually find the thirds yet. And this is saying two thirds. So, it's kind of harder. Because, the length of the third is 33. It would be 36. I mean 66. 66.
15.1.257 42:06 James: 32
15.1.258 42:07 Andrew: Yeah, 32. Only 98
15.1.259 42:11 James: 67. 67.
15.1.260 42:16 Andrew: 66 and a half.
15.1.261 42:19 James: 66 and half. 32. 33.
15.1.262 42:25 Andrew: Yeah, 33
15.1.263 42:25 James: 99
15.1.264 42:28 Andrew: Yeah.
15.1.265 42:31 Dr. Landis: What yeah, I'm confused. Confused again.
15.1.266 42:34 James: It's 199 not 200. We want it to be 200, right?
15.1.267 42:39 Andrew: Yeah, but this is 200. We're talking about. We counted this and then 120, 120, 133, plus 60 is 290.
15.1.268 42:56 James: 66 and a half and 33 and that makes 99 and a half.
15.1.269 43:06 Andrew: 199 and a half.
15.1.270 43:09 James: Hold on, let me try 67, 67, 134, yeah.
15.1.271 43:22 Andrew: Yeah, 134
15.1.272 43:24 James: That's 67
15.1.273 43:25 Andrew: It's one over.
15.1.274 43:28 Dr. Landis: What are you trying to figure out?
15.1.275 43:30 Andrew: We're trying to figure out a third of blue.

15.1.276 43:34 Dr. Landis: Tell me, what kind of ribbon are you trying to cut now from this big blue ribbon?
15.1.277 43:41 Andrew: We're trying to cut.
15.1.278 43:44 Dr. Landis: How long is going to be the ribbon you're going to be cutting?
15.1.279 43:47 Andrew: Two thirds of one meter.
15.1.280 43:51 Dr. Landis: Ok and want do we want to find out?
15.1.281 43:53 Andrew: If it's two thirds of one meter, that would be, that would be..
15.1.282 44:01 Dr. Landis: Just tell me what the question is so I know that you know what you're working on.
15.1.283 44:03 Andrew: It's two thirds of one meter.
15.1.284 44:06 Dr. Landis: What's two thirds of a meter?
15.1.285 44:07 Andrew: So it's two meters, and two thirds of a meter, the number of bows
15.1.286 44:15 Dr. Landis: James what are you trying to find out?
15.1.287 44:17 James: I think it's four.
15.1.288 44:19 Dr. Landis: You think it's four? But what's the question, just so I know you know what you're working on?
15.1.289 44:22 Andrew: It's how many..
15.1.290 44:23 James: Two thirds
15.1.291 44:25 Andrew: How many two thirds lengths bows can you make of two meters?
15.1.292 44:32 Dr. Landis: Ok so how many bows are you going to cut from this blue ribbon if each bow is going to be how big?
15.1.293 44:39 James: Two thirds
15.1.294 44:40 Dr. Landis: Two thirds of a meter, right?
15.1.295 44:42 James: Yeah.
15.1.296 Dr. Landis: Ok.
15.1.297 44:45 Andrew: So one third is 33,
15.1.298 44:49 James: 66
15.1.299 44:51 Andrew: 33, 66, so that's two. Three, you actually have two meters left over I mean, two thirds left over.
15.1.300 45:07 Dr. Landis: What do you mean, you have two thirds leftover?
15.1.301 45:09 Andrew: Because if you want to make, take two thirds and there's three thirds so take two thirds plus two thirds plus one third and one third you have two more thirds.
15.1.302 45:19 Dr. Landis: Oh, that's interesting. Say this again and let's see if we can follow him. What did you just say? Say it again.
15.1.303 45:24 Andrew: [gesturing with hands] There's three thirds so there's two thirds and one third and one third, that's two thirds and you still have one two thirds left over.
15.1.304 45:34 Dr. Landis: Can you kind of show me a picture of that here and I want, James, I want to see if you understand what he's saying. This is real interesting.
15.1.305 45:52 Andrew: [while drawing picture] So then there's one third and two thirds is two thirds so then here's the half [of the blue

ribbon]. So you only have one third so then you have to get the other third [first third of second meter]. This is two thirds so then you have two more thirds left over.

15.1.306 46:13 Dr. Landis: Do you follow what he's saying? What is he… tell me how you're hearing it. I think that's real interesting. What did he just say?

15.1.307 46:20 James: He's saying that there's six meters. There are two meters.

15.1.308 46:29 Dr. Landis: This blue ribbon is two meters, uh huh,

15.1.309 46:31 James: [pointing to picture] Yeah, And there are six meters is in each, and it would be two thirds is one, two thirds is again and two thirds left.

15.1.310 46:48 Dr. Landis: Ok. Now, you know what I heard um Andrew saying? This is what I heard him saying. I think I heard you saying that if you took this blue ribbon and again imagined it as two white ribbons is that what you said when you said I'm going to divide it in half, right, ok? And if you had it as two white ribbons. Ok? Can you picture that white ribbon over there if we took it and divided it into thirds?

15.1.311 47:19 James: Yeah.

15.1.312 47:21 Andrew: And then it would be you divide this. This one and this one around here would be a half. Like right there would be half because this is the whole now. So then it would be one third, two thirds, three thirds.

15.1.313 47:46 Dr. Landis: So if I just looked this from here over [the first meter], what would I be looking at? Would I be looking at one of these ribbons? Which ribbon am I looking at?

15.1.314 47:56 Andrew: The white.

15.1.315 47:59 James: This whole thing is the blue ribbon and this is the white.

15.1.316 48:02 Dr. Landis: Can you show it to me with the white and the blue ribbons? Because I think you're onto something that's real interesting. [laying ribbons on meter stick] Ok, I'll put this down here. Ok. So this is kind of what you pictured here, wasn't it Andrew? Where this is the whole blue ribbon and this now is that white ribbon ok and then you actually divided up that white ribbon. What did you divide it into over here?

15.1.317 48:42 Andrew: Thirds.

15.1.318 48:43 Dr. Landis: Thirds ok.

15.1.319 48:45 Andrew: It would be a third of 33, 66, 99.

15.1.320 48:49 Dr. Landis: Ok if you were trying to divide it using little numbers you would but you didn't use any little numbers here. You just divided it into thirds and, what did you end up with over here?

15.1.321 48:59 Andrew: Two… I ended up with um two thirds. That would be that would be one and one third but you had two thirds left over.

15.1.322 49:12	Dr. Landis:	Ok you're saying that if you take that white ribbon and divide it up into thirds how many thirds would you get if you took that white ribbon and divided it into thirds?
15.1.323 49:22	Andrew:	Three
15.1.324 49:25	Dr. Landis:	What do you think James?
15.1.325 49:26	James:	Yeah, three.
15.1.326 49:26	Dr. Landis:	You think three. And what about from your finger over to here, if you took this white ribbon and divided it up?
15.1.327 49:36	James:	Three
15.1.328 49:39	Dr. Landis:	Ok, now but what Andrew was saying which I think is real interesting is …you're trying to make your bows that are how long?
15.1.329 49:48	James:	Two thirds of a meter
15.1.330 49:50	Dr. Landis:	Two thirds of a meter so he's saying this would be 2/3 of a meter, right? Ok. And then what else… what else over here? How… how many ribbons could you make from this?
15.1.331 50:06	James:	I have to go.
15.1.332 50:07	Dr. Landis:	You're going to? Oh. Stay just one minute cause I'm just real interested 'cause I think that you have something down here. How many ribbons were you able to cut from this big blue ribbon?
15.1.333 50:21	Andrew:	Two
15.1.334 50:23	Dr. Landis:	You think you could only get two bows that are two thirds?
15.1.335 50:25	James:	No, three.
15.1.336 50:27	Andrew:	Four.
15.1.337 50:28	James:	I think three.
15.1.338 50:30	Andrew:	Why three if you have two thirds and two thirds?
15.1.339 50:32	James:	[pointing at drawing] You have two third and two thirds and then there are six and this is two thirds and this is two thirds would be one, two, three, yeah, four. One, two, no, three, one
15.1.340 51:02	Andrew:	I know but two thirds.
15.1.341 51:04	James:	Andrew I know but half is.
15.1.342 51:12	Andrew:	So two thirds of this, two thirds of the white then you have two thirds of the white ribbon then two thirds of the white ribbon, right? And there if you have one more third and one more third of the white and there if you have one more third and one more third of the white and then you have two thirds left over going that way or going that way.
15.1.343 51:37	Dr. Landis:	So how many ribbons could you cut that are two thirds long?
15.1.344 51:41	Andrew:	Four.
15.1.345 51:43	Dr. Landis:	From this blue ribbon?
15.1.346 51:45	James:	Yeah.

15.1.347 51:47 Dr. Landis: You think four? Think about that again, ok?

15.1.348 51:51 T/R 1: What I would us to do is I want to spend time talking about the way you're thinking, about what you discovered about your problems and how you can convince me that you can make so many bows. I'm curious. I think some of you can get some part time jobs working in some wrapping department so that we don't waste ribbons right and we'll know how many bows we can make and we know what to order. We don't want to waste anything. We want to try to have maybe an operation or a company that produces lots of these bows with very little waste. I am interested in the way you're thinking about it. I was walking around listening to all kinds of ideas. How many of you finished problem number one? [students raise hands] With the white ribbon? How many of you are convinced that your answers for number one are correct and can prove it to me? And nobody can persuade you that you're wrong. How many of you are absolutely convinced about that? How many of you still aren't sure you have answers but you're not totally convinced [two hands raised in view]? Looks like we have some work cut out for us, isn't that right? Ok, can someone tell us what that first problem is about in your own words? What we're asked to do in that first problem? And then tell us how you think you can you convince us that your solution is correct? Kelly.

15.1.349 53:23 Kelly: Well, you're supposed to like… there's one meter of white ribbon and then you have to split it into, um, a half so I got, um, two bows.

15.1.350 53:38 T/R 1: So you're telling me I have one meter of white ribbon and I split it in half, I have two bows? How many of you agree with that? [students raise hands] Ok, I'm convinced. What about the next one if I have one meter of ribbon and I have to make bows that are one third meter in length, how many of you think you know that answer to that one? And you're sure you have the answer to that one? What do you think is it? How many ribbons can you make? Caitlin.

15.1.351 54:08 Caitlin: Three

15.1.352 54:09 T/R 1: Caitlin thinks three. How many of you think three? [students raise hands] How many of you think something else [no hands in view raised]? Can you prove it?

15.1.353 54:18 Caitlin: Well, we have one meter so you take one and there's two in one.

15.1.354 54:30 T/R 1: I see that, how did you do that?

15.1.355 54:31 Caitlin: Then it's one third so you divide it into thirds and you would get three.

15.1.356 54:36	T/R 1:	Can anybody show me that? I was having a little bit of trouble. I was talking to Laura and Jessica a few minutes ago and asking them if they can convince me that if I have a ribbon a meter in length, and I'm making bows from ribbon one third meter in length that I could find three of them. Can someone show me that? Prove it to me? Why don't you- can you do that? Kelly can you come show me? Kelly thinks she can prove that. [Kelly goes to front and uses white ribbon to demonstrate]
15.1.357 55:14	T/R 1:	Can I hold it for you? [Kelly folds ribbon.] What did Kelly do here? Did you all see what she did? Who can tell me what Kelly did? Kimberly.
15.1.358 55:45	Kimberly	She folded it three times.
15.1.359 55:46	T/R 1:	She folded three times. Let's unravel it. Open it. It's three pieces, right? So, She did one fold, right,
15.1.360 55:56	Kimberly	Two, but it came out..
15.1.361 55:57	T/R 1:	What do you think Jessica and Laura? See that. You did that and sort of forgot how you did it. How many of you can do that? Can convince me, convince your partner, and convince each other that if the ribbon is one meter in length, and you are making bows of a third of a meter, you find how many? You believe that. How many of you absolutely believe that? Is that what you did Jackie? Why don't you go show that to Jackie. Jackie is not convinced. No, Jackie back there.
15.1.362 56:39	T/R 1:	That's a neat thing Kelly. Thank you. How many of you discovered that neat folding trick? Kimberly did it do too and Amy did it and I saw some other people. And Jackie.
15.1.363 56:53	Jessica:	We did it at first then we forgot how we did it.
15.1.364 56:53	T/R 1:	Yes, you can forget it can't you. Caitlin. Ok so how many can you make with they're one third meter in length? Class.
15.1.365 57:01	Class:	Three.
15.1.366 57:02	T/R 1:	How many of you are absolutely certain, convinced, nobody can persuade you otherwise? [students raise hands] Dr. Fransblau, I didn't see your hand up. Thank you Kelly. Alright. What about a quarter of a meter in length? How many a quarter a meter in length? Here's my meter, how many bows can you make that are a quarter a meter in length? Danielle.
15.1.367 57:32	Danielle:	Four.
15.1.368 57:33	T/R 1:	Four. Can you convince me? You're absolutely sure? You want to come up here? How about the rest of you in your seats? Can you convince me you're absolutely sure? [Danielle folds white ribbon in front of class.] Ok, neat. How many times did you fold it, Danielle?
15.1.369 58:04	Danielle:	Twice.

15.1.370 58:05	T/R 1:	Once, and then twice. What do you think Jessica? The quarters are easy, huh? Yeah. How many of you did that? See that? Gregory you did the same thing? Ok, great. So how many can you make if you have bows, Dr. Pearl?
15.1.371 58:24	V2:	Is it really a quarter of a meter when it's folded?
15.1.372 58:27	T/R 1:	She's not convinced.
15.1.373 58:28	V2:	I'm not sure.
15.1.374 58:30	T/R 1:	How can we convince her that it's a quarter of a meter once it's folded? Kelly, what do you think?
15.1.375 58:35	Kelly:	We can measure it.
15.1.376 58:36	T/R 1:	You can measure it. And what would you do to measure it? How would you convince Dr. Pearl? Graham.
15.1.377 58:43	Graham:	Well, you could take four of these [holds up white ribbons and folds] and put them up together on a meter stick and see if they fit together.
15.1.378 58:50	T/R 1:	Can you see that Dr. Pearl? How many of you think that's fair if she took four of these and put it on a meter stick that would convince you? How many can you make that are a quarter meter in length? Class.
15.1.379 59:03	Class:	Four.
15.1.380 59:04	T/R 1:	How many of you are absolutely convinced? [students raise hands] Ok, we have a new visitor here. This is Dr. Golden who works also with us at Rutgers. And boy, you are going to have to convince him. Pardon?
15.1.381 59:15	Dr. Landis:	Dr. Houser is here too.
15.1.382 59:16	T/R 1:	Dr. Houser. You all know him, don't you? He is a good friend of ours. Let's see if we can convince them. If you're making your bows now 1/5 m in length from a meter of ribbon, how many bows can you make? How many of you think you know? Erin.
15.1.383 59:38	Erin:	Five.
15.1.384 59:39	T/R 1:	Five. Erin thinks five. How many of you think it's something else? How many think five? What's happening here? What do you think? Any ideas? Brian. Caitlin?
15.1.385 59:55	Caitlin:	I think that each time it goes up one. Like one, two, it went up one for one meter. Then so two would be three, and three would be four and four would be five.
15.1.386 1:00:	T/R 1:	What do you think Brian?
15.1.387 1:00:18	Brian2:	How I got them was I divided all of them by one. And I got mostly when it said one third, I got three bows.
15.1.388 1:00:32	T/R 1:	You're telling me from what you said. Let me be sure I understand what you are saying. You're telling me if you have one meter and if you're dividing it by one half, what did you get?
15.1.389 1:00:52	Brian2:	I'd get two bows.

15.1.390 1:00:54 T/R 1: You got two bows. [writes on overhead] If you had one meter length and divided it by one third, what did you get?

15.1.391 1:00:59 Brian2: Three bows.

15.1.392 And what was the next one class?

15.1.393 1:01:01 T/R 1: And what was the next one class?

15.1.394 1:01:03 Class One fourth.

15.1.395 1:01:05 T/R 1: One divided by one quarter was?

15.1.396 1:01:07 Class: Four [Dr. Maher has written four equations on the overhead.

$1 \div \frac{1}{2} = 2 \qquad 1 \div \frac{1}{3} = 3 \qquad 1 \div \frac{1}{4} = 4 \qquad 1 \div \frac{1}{5} = 5$].

15.1.397 1:01:09 T/R 1: And what was the last one? [cannot understand] That's very interesting. What do you think, Kelly?

15.1.398 1:01:17 Kelly There's a pattern.

15.1.399 1:01:18 T/R 1: What do you mean?

15.1.400 1:01:20 Kelly: Well, each time you have one that like one half came out to two, then one third came out to three, one fourth came out… came out to four and one fifth came out to five so there's like a pattern because with the ones and so you like, you look over here and you could see because the one and the two you could really see that it comes out to two bows.

15.1.401 1:01:50 T/R 1: So what would you predict if were making bows that were one tenth of a meter in length from ribbon that was one meter long? Graham?

15.1.402 1:01:59 Graham: Maybe ten?

15.1.403 1:02:00 T/R 1: Graham would predict that if I had one divided by one tenth that would be ten. What do you think? Do you agree with his prediction, or not? Kimberly?

15.1.404 1:02:10 Kimberly I think that it would be ten but they would be very, very tiny.

15.1.405 1:02:14 T/R 1: They'd be very tiny? Anybody else, what do you think? Does anybody disagree with that idea? That's an interesting idea. Let's continue with the next step and see what we have. Did you finish the two meter ribbon? What's the two meter ribbon? So if we're making bows a half meter length from two meter ribbon, what do you think? Andrew.

15.1.406 1:02:50 Andrew: Well I think if you're making one half of a meter and two meters, there would be four bows.

15.1.407 1:02:57 T/R 1: How can you convince me that that's true?

15.1.408 1:02:59 Andrew: Well, the reason why it would be four bows, is because it says one half of one meter and there's two meters and so you have the blue rod and you take, see, you put a white rod up to it, I mean ribbon, and you put two, you put one white ribbon up to it and it equals one half.[holding up blue and white ribbons together] And then another one and it equals the other half so two whites go into a blue. So then, since

		this is one meter and it's asking one half of one meter and so you have a half of it is two on one half of the blue and then two on the other half side of the blue so that's four bows.
15.1.409 1:03:54	T/R 1:	What do you think? How many of you agree with what Andrew says? [students raise hands] Did anybody think about it of it in another way? Is there anybody who disagrees? Is there anybody who isn't sure? You're all sure? Wow. Well, what about so, you're telling me that two divided by one half--what did you say Andrew?
15.1.410 1:04:17	Andrew:	Four.
15.1.411 1:04:18	T/R 1:	You're saying two divided by one half is four. How many of you got that? [students raise hand] Two divided by one half is four? What about two divided by a third? What about that one? Two divided by one third. Brian.
15.1.412 1:04:30	Brian:	Six.
15.1.413 1:04:31	T/R 1:	How did you get six?
15.1.414 1:04:3	Brian:	Well, there's two meters and in one meter there are three thirds and in the other meter there are three thirds. In the other meter there are three thirds. So you add them. In one meter there are three third and the other meter there are three thirds. If you add the two meters together it'd be three thirds and three thirds which is six.
15.1.415 1:04:46	T/R 1:	What do you think? Do you all understand what Brian said - how many of you understood what Brian said? [students raise hands] Can someone try to say it one time for me so I understand it? I have to be sure because Dr. Goldin needs to hear it a few times. Right Dr. Goldin? Can someone help him with this? He just came in late. He hasn't been cutting ribbons. Who wants to.. that's an interesting way. Who else wants to say it for him? Andrew, it sounded very much like the way you did the other one.
15.1.416 1:05:13	Andrew:	Yeah.
15.1.417 1:05:14	T/R 1:	Let's hear someone else say it for me. Can someone else try to say it for me? Can we hear from someone else? Who wants to try? Well, how many of you believe that two divided by one third is six…is six? [students raise hands] Someone want to give a try saying it again? Good for you, Brian.
15.1.418 1:05:37	Brian2:	I did the same thing on all of them like I did on the white. I would times the two by whatever, like the one third. Say two times three is six.
15.1.419 1:05:44	T/R 1:	So you're telling me you got that because two times three is six.
15.1.420 1:05:47	Brian2:	Right

15.1.421 1:05:49	T/R 1:	You're seeing a pattern here you're telling me. What do you think –how many of you think there's some pattern here? If that's a pattern, how would do two divided by two thirds? By the way, how many ribbons can you make that two thirds of a meter long if you start with two meters of ribbon? How many of you have an answer for that one that you believe? Two divided by two thirds? How many of you have an answer for that? That you believe. Let's hear what some of the answers are. Erin.
15.1.422	Erin	I got three bows.
15.1.423	T/R 1:	Erin has three bows. Anybody have something else? Andrew.
15.1.424 1:06:06	Andrew:	I got four.
15.1.425 1:06:07	T/R 1:	Ok, so we have one answer of three. One answer of four. Anybody else? Brian?
15.1.426 1:06:15	Brian:	I have three.
15.1.427 1:06:17	T/R 1:	How many of you have three? How many of you have four? [students raise hands] How many of you have something else? How many of you aren't sure? Oh, we have a lot of unsure people here. Let's hear why you think three. Who's gonna tell me? Kelly.
15.1.428 1:06:30	Kelly:	Well I think three because well, it's just the same as the other one. You like, you take this and you divide it in three pieces [holding up blue ribbon] and you get three.
15.1.429 1:07:13	T/R 1:	You took that and divided it into three pieces. What's the length of each of those pieces?
15.1.430 1:07:24	Kelly:	Two thirds.
15.1.431 1:07:28	T/R 1:	How- are you all convinced of that? How can you convince us each piece is two thirds?
15.1.432 1:07:35	Kelly:	You can measure it.
15.1.433 1:07:38	T/R 1:	You can measure it. Anybody else? How else? What do you think?
15.1.434 1:07:44	Mark:	If this is two thirds, it would be right here. [measuring blue ribbon] This is one third and that's two thirds, then three thirds.
15.1.435 1:08:12	T/R 1:	Gregory, you like that Gregory?
15.1.436 1:08:20	Mark:	One, then two, wait, hold on.
15.1.437 1:08:41	T/R 1:	Maybe this is the place we should stop because we're running out of time. I guess what I would like you all to think about for the next time we come is how you can convince me, if you think you can, what two divided by two thirds is. Is that where we left off? In fact, I would like you to write me a little explanation of why you think it's three, or four, or whatever you think it is and how you could convince somebody of your answers. Ok? Do you want to continue working with this to finish the sheets? How many

			of you would like to continue working with this? How many of you want to make bows? [students raise hands] Ok, we can maybe figure out how much it would cost to make some of these bows because maybe what we can do is maybe sell them and donate it to some wonderful cause for the holidays. And we can figure out how much we spend for materials and then how much we can send them for so maybe cover our costs and donate our profits? What do you think? Would you like to do that? Ok, sounds wonderful. I want to thank you all and if you could put all your materials in your little plastic bags and, um, and put your names on your papers. Yes, thank you very much.
15.1.438	1:10:26	Andrew:	You have the blue, and you have, this is, and you take, you have a white
15.1.439	1:10:33	T/R 1:	Ok this is two meters, right?
15.1.440	1:10:34	Andrew:	Yeah, you have a white, so that is half of this.
15.1.441	1:10:37	T/R 1:	Ok, right, I got that.
15.1.442	1:10:38	Andrew:	The white is three and then the other white is three. So it's asking you how many two thirds of one meter, you would take a third [holding blue ribbon]
15.1.443	1:10:50	T/R 1:	You have six one third meters here. You have six one third meters, so how many two third meters do you have?
15.1.444	1:10:56	Andrew:	Well we have, so you have two meters,
15.1.445	1:11:02	T/R 1:	Let's see. Now imagine, this is the one. You have a one third, another one third, and another one third. Show me where two thirds would be.
15.1.446	1:11:09	Andrew:	Right there [points to ribbon]
15.1.447	1:11:10	T/R 1:	So where would be the next two thirds?
15.1.448	1:11:12	Andrew:	There.
15.1.449	1:11:14	T/R 1:	Where would be the last two thirds? Would you get three or four?
15.1.450	1:11:18	Andrew:	You get, so it would be one, [pointing to ribbon] two, wait, there's one, two, and three.
15.1.451	1:11:37	T/R 1:	You think three.
15.1.452	1:11:38	Andrew:	I thought, you see what I thought?
15.1.453	1:11:40	T/R 1:	I think you counted wrong. What did you think?
15.1.454	1:11:42	Andrew:	No, I thought two thirds, I thought you had to get two thirds. So you have two thirds, two thirds, so that's, um. I didn't count the last two thirds.
15.1.455	1:12:03	T/R 1:	But you got four, now you're saying three. So you counted an extra two thirds.
15.1.456	1:12:06	Andrew:	Yeah.
15.1.457	1:12:08	T/R 1:	Ok, you're convinced of that?
15.1.458	1:12:10	Andrew:	Yeah.
15.1.459	1:12:11	T/R 1:	Can you do it carefully to be sure that you can explain it maybe?

15.1.460	1:12:13	Andrew:	Yes.
15.1.461	1:12:14	T/R 1:	Ok, great. Thank you [end of video]

Session 15, Dec 9, 1993, Front View

Line	Time	Speaker	Transcript
15.2.1	07:27 S	T/R 1:	Here's what we are trying to do today. If you notice we have some displays and this is thanks to Mrs. Demming who is actually in the Cedar school right now doing another kind of filming and she made these displays for us. Maybe the best way is to consider the activity we have for you, of a problem, that we're going to ask you to help us to solve. So let me tell you a little bit about the problem then I am going to say go to it- right? That is what you like to do. We have just received a shipment of ribbon that are going to be used to make bows for the holiday. Ok? You are to consider this examples of your shipment of ribbon. [Holds up ribbon.] Now what's interesting about this ribbon is you'll notice that this ribbon comes in different lengths. On your yellow paper you can see that red ribbon comes in packages of length that is what do you see? 6 meters. Do you all see that? [Students agree]. Gold ribbon? What's the length of gold ribbon? 3 meters. What about blue ribbon? [students answer] 2 meters. White ribbon? [students answer] 1 meter. Ok. Now from these bows, from this ribbon we can make bows and depending upon how much ribbon we use the bows you see can the bows can be different sizes, they can be different shapes, right? Now your problem is going to be to find out how many bows of certain lengths that we can make from the ribbon that's in your package. That's what you have to figure out. Now before you start figuring it out, let me just point out a few things to you. If you need to try to figure out these bows, the lengths of these bows, you might not at first want to cut up your ribbon. You might want to use your ribbon so that you know how much you have to deal with. You might want to cut up the string so you can test your ideas. And if you need to do that you might have to take the string, you might have to measure it, of the length of that ribbon that you have to figure out your problems. Now, now look at the first problem, see the first problem? Do you think you understand what the first problem is asking? How many of you think you understand? You want to talk to your partner for a minute to be sure you understand what the first problem is asking? Chat with your partner. [Students given 20 seconds to talk] Ok, we have a question. Someone is asking a question. Where is the first problem? Who can help answer the question- where is the first problem? Andrew you want to give it a try? Ok, let's see what Andrew has to say.

15.2.2	10:30	Andrew	By the number one, it says white ribbon it has one meter, and one meter on the bottom where it says. And then it says make a ribbon length of bows and number of bows. So you have to find out look under the ribbon length of bow so it will be one meter and one half how many bows can you get out of one half a meter. So you would have to figure out that.
15.2.3	10:59	T/R 1:	So if you were to take your white ribbon, can you imagine the white ribbon being this long? Can you imagine that? [Holds ribbon up] You might test it if you don't believe it. You might take your white ribbon to see if it's this long. One meter right? Ok, but now with the white ribbon the first question can you see Danielle on there, the white ribbon? What is the length of ribbon you are going to use to make your bow? What does it say on the chart?
15.2.4	11:26	Michael::	You have to use ribbon length of the bow. You have to use one half of a meter.
15.2.5	11:31	T/R 1:	One half of a meter to make your first bow. Right? Now, if you are making your meter, I don't want you to answer out, if you are making your ribbon from a bow one half of a meter in length, I want to know how many bows you could make? Ok, think about it. This is the length of your white ribbon. [holds ribbon] Ok.
15.2.6	11:54	Erik::	One ribbon is a half a meter
15.2.7	11:55	T/R 1:	One ribbon is a half a meter.
15.2.8	11:57	Erik:	And then you have to figure out how many bows, how many of those bows you make?
15.2.9	12:03	T/R 1:	That's right. From one meter ribbon and each bow is a half meter in length.
15.2.10	12:06	Erik:	And how many bows of that kind.
15.2.11	12:11	T/R 1:	Now our guests will walk about and chat with you if you have any questions. They might, they might not ask you questions, they might ask you a question instead. That's the kind of visitors we have. But they might come around so if you aren't clear about something, you might want to chat with them. Ok? So go to it. [12:27] [Students getting materials, etc]
15.2.12	13:09 F	T/R 2:	So you've got this one meter, right… of ribbon and you want to make bows that are going to require a third of a meter of ribbon. Maybe some of those medium sized ones or something and you want to know how many you can make. Now, somebody said three, but I was curious why you said three. What was your… what were you thinking when you said three?
15.2.13	00:13:32	Caitlin:	I don't know, just because it said one and one, three, I was just thinking..[cannot understand]

15.2.14	00:13:40	T/R 2:	What do you think Erin?
15.2.15	00:13:42	Erin:	Well, if it's gonna, uh, for a third, a third of a meter. Then if you split the meter into thirds each part would be one third.
15.2.16	00:13:56	T/R 2:	Right. Ok. And how many parts would you have?
15.2.17	00:13:58	Erin:	Three.
15.2.18	00:13:59	Caitlin:	So you could have three.
15.2.19	00:14:02	T/R 2:	Ok. What do you think, Brian? Do you hear with their argument?
15.2.20	00:14:06	Brian2:	I wasn't here for
15.2.21	00:14:08	T/R 2:	Ok. What they're saying is that this is this is one meter of ribbon and I mean you have to believe that. I mean you may want to take a meter stick and test it later on and this is one meter of ribbon and they want to cut it into strips of one third of a meter so that we can make bows, um, how many bows can we make?
15.2.22	00:14:29	Brian2:	We have to use a third of a meter?
15.2.23	00:14:32	T/R 2:	Yeah, well that's what it's asking us. Yeah, one third of a meter for each bow and we've got this strip that's one meter.
15.2.24	00:14:38	Brian2:	You have to cut it in thirds.
15.2.25	00:14:40	T/R 2:	Ok. How many strips would that give us?
15.2.26	00:14:44	Brian2:	Three bows.
15.2.27	00:14:46	T/R 2:	Ok. Oh, so you agree, you're all agreeing then?
15.2.28	00:14:50	Caitlin:	Yes, so should we put it?
15.2.29	00:14:51	T/R 2:	So you can put that down. Sure. And if you feel that you want to test out any of these ideas there are meter sticks. There are, that you could measure out string with. If you want to test any of these ideas, you could make yourself a string that's a meter long and test it. But following the same logic that you're following now, what do you think is gonna happen for the next thing we do with the white ribbon?
15.2.30	00:15:10	Caitlin:	I think there's gonna be.
15.2.31	00:15:12	Caitlin: and Erin:	Four.
15.2.32	00:15:14	Erin:	Cause we're going one, two three,
15.2.33	00:15:25	T/R 2:	Ok well, make sure you're all in agreement on that, ok? If you are, then you can record that information, ok? Then you are going to be thinking about the blue ribbon which is a different length, ok? I am going to come back. I wanna see what you do with this. If you decide to fill this in, go right ahead. Then think about the blue and answer the same types of questions. This is nice. [camera focuses on other group]
15.2.34	00:16:36	Jackie:	Ok, so now one fourth. Four. It's just a number over here.
15.2.35	00:16:45	Amy:	You hold this side, a half. Yeah, four.
15.2.36	00:17:11	Jackie:	One fifth meter, five.
15.2.37	00:17:12	Amy:	Fifth, five, yeah
15.2.38	00:17:15	Jackie:	That was easy.

15.2.39	00:17:18	Amy:	Ok, now blue. Two meters, now we need. Two meters. String's right here. Hold that right there. [lining up meter sticks]
15.2.40	00:17:43	Jackie:	Do you have the scissors?
15.2.41	00:18:15	Amy:	Next we're gonna need three meters. Erin, [inaudible] Four.
15.2.42	00:18:22	Jackie:	It's just the double amount.
15.2.43	00:18:45	Amy:	Four. Now we have to find out this three.
15.2.44	00:19:07	Jackie:	Alright, one third meters, six. Six. I bet you it's six.
15.2.45	00:19:35	Amy:	Four, five, six. [measuring ribbon]
15.2.46	00:19:36	Jackie:.	Six. I told ya. [writing on paper] I bet you it's eight.
15.2.47	00:19:53	Amy:	Well, let's see, ok? Then we got two thirds. [counting] You think it's what, eight?
15.2.48	00:20:05	Jackie:	Eight, yeah.
15.2.49	00:20:11	Amy:	One, two, three, four, five, six, seven, hold on. I think I did something wrong.
15.2.50	00:20:22	Jackie:	Make it smaller.
15.2.51	00:20:31	Amy:	One, two, three, four, five, six, seven,
15.2.52	00:20:44	Jackie:	That's too small.
15.2.53	00:20:47	Amy:	Eight, ten, seven. You try it. You try it, huh? [gives to Jackie. She tries.]
15.2.54	00:21:37	Amy:	Just put eight down, we know it's eight. It's exact. We know it's eight.
15.2.55	00:21:43	Jackie:	Alright. That was eight I think.
15.2.56	00:21:51	Amy:	You're going to say ten.
15.2.57	00:22:43	Caitlin:	The next column cause you keep on doubling [pointing to worksheet] when you go to another thing.
15.2.58	00:22:49	Erin:	[cannot understand] As you double it…
15.2.59	00:22:50	Brian2:	I'm getting it wrong.
15.2.60	00:22:52	Caitlin:	It starts out on two, and then you add two more it's four, then you add two more it's six. Cause see, cause see three plus each, each three has a half in it. So the first one's gonna be one, two, the next one's gonna be three, four, and the next one's gonna be five, six. So you have to write six down here. [points to paper] Ok, so then you write six bows under five and then seven bows.
15.2.61	00:23:35	Brian2:	I don't get it. What's the last one Caity?
15.2.62	00:23:37	Caitlin:	And then eight bows.
15.2.63	00:23:41	Brian2:	Caity, did you read this? Two.
15.2.64	00:23:45	Caitlin:	I think that's it. Ok, now you go down to there and it's six. Six bows, seven bows. Are you writing this down? I don't think so.
15.2.65	00:24:07	Erin:	Because I wanna test it out. I don't want to just keep writing it.
15.2.66	00:24:10	Caitlin:	See. Look it. Look it. Look it. Do you think this [pointing to paper] would be six? This guy? If this is three meters and three meters and each one has two halves in it,

so then three. Don't write on my paper. Three and one would be one, two. The next one would be three, four and the next one would be five, six. Ok? So that's three meters like we did… so ok? You believe me?

15.2.67 00:24:55 Erin: Well, I'm just writing what I think.
15.2.68 00:25:00 Caitlin: Well, I'm gonna mess them up. Ok. Ok. Then there would be eight bows… eight bows, nine bows, ten bows.
15.2.69 00:25:17 Brian2: Caity, you have to times all these by three.
15.2.70 00:25:21 Caitlin: So, no, you just keep going up. That's how it is on these two other ones. So why wouldn't it be on the last one?
15.2.71 00:25:31 Erin: I think somebody was right.
15.2.72 00:25:35 Caitlin: How do you know?
15.2.73 00:25:38 Brian2: Because I know. Because I know.
15.2.74 00:25:41 Caitlin: How do you know?
15.2.75 00:25:43 Brian2: They all don't go in order. They don't go two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen.
15.2.76 00:25:49 Caitlin: Dr. Martino said this was right.
15.2.77 00:25:51 Brain2 Yeah, but that's with one. That was one, two and then with three you times these by two and these by three. You times these by two and you times these by three.
15.2.78 00:26:00 Erin: Maybe they have an extra sheet.
15.2.79 00:26:02 Brain2 I've been trying to tell you.
15.2.80 00:26:06 Caitlin: Ok. We messed up on this.
15.2.81 00:26:14 T/R 2: What happened?
15.2.82 00:26:15 Brian2:, Erin: She messed up.
15.2.83 00:26:16 T/R 2: Ok. What happened?
15.2.84 00:26:17 Caitlin: This would, this would be right though, right? Cause one, two, two, two but then I wrote five, six, seven, eight.
15.2.85 00:26:29 T/R 2: Oh, ok. What happened?
15.2.86 00:26:31 Brian2: She thought it would be the same as this.
15.2.87 00:26:33 Erin: Yeah, but it's..
15.2.88 00:26:35 Brian2: It would be in order.
15.2.89 00:26:37 T/R 2: You saw, ok, you were looking at a pattern there.
15.2.90 00:26:37 Brian2: I know but this is only one.
15.2.91 00:26:39 Caitlin: This is one.
15.2.92 00:26:41 T/R 2: Which one?
15.2.93 00:26:42 Brian2: You times these by one.
15.2.94 00:26:45 T/R 2: So, what happens here? What should this one be? What should two meters and making bows one third meter be?
15.2.95 00:26:49 Caitlin: This one… this one would go up by two so that would be six, and this one would go up by three.
15.2.96 00:26:54 T/R 2: That's interesting. How did you discover that?
15.2.97 00:26:58 Caitlin: Well, because this is one meter so keep on going up one and this is two, so you go up two and then this is three so you go up three.

15.2.98	00:27:07	Brian2:	You had to times this by two or you'd have to time three by three and then you'd get the answer.
15.2.99	00:27:14	T/R 2:	What about something like this last one and the two meter set though? The two thirds? That's one you hadn't done yet.
15.2.100	00:27:19	Erin:	Yeah, I skipped that because I wasn't sure on that one.
15.2.101	00:27:23	T/R 2:	So you have to go back and think about that one. Ok, so you'll want to, you'll want to correct this then Caitlin, right?
15.2.102	00:27:29	Caitlin:	Yeah, but it's in pen.
15.2.103	00:27:30	T/R 2:	Don't worry about that. Just put a line through it and put the proper number next to it. Doesn't matter. Was that one ok?
15.2.104	00:27:38	Caitlin:	[mumbles something]
15.2.105	00:27:40	T/R 2:	Yeah, don't worry about the neatness here. What's important is that you detected the error and you thought about it and know what to do. How about that two thirds one now, Brian? Hmm, Erin, what do you think? We have two meters of blue ribbon and we have to make bows that require a length of two thirds of a meter. Would it help to test?
15.2.106		Erin:	Uhh, fifteen bows.
15.2.107	00:28:17	Brian2:	That would be six bows.
15.2.108	00:28:19	T/R 2:	Ok, why six?
15.2.109	00:28:22	Brian2:	Two… two meters and you have to divide it into two thirds then that'd be six.
15.2.110	00:28:33	T/R 2:	Well, it's interesting because what you just, you all just told me was that when I have two meters.
15.2.111	00:28:36	Caitlin:	I think this one would be 12 down here.
15.2.112	00:28:39	T/R 2:	Ok, let's… let's hold on to this idea for a second. Brian, but you all just told me that if I make them a third of a meter in length ok we've got the two meters, that it's six bows. How can they both be six bows? How can both the one third lengths and the two thirds lengths give me, from the same piece of ribbon give me six bows? [Erin giggles]
15.2.113	00:28:59	Brian2:	You mean down here?
15.2.114	00:29:01	T/R 2:	No, no. Up here you told me that when I have two meters of ribbon and the lengths are a third of a meter, you get six bows. Ok, but now you're telling me that if I have two meters of ribbon and the lengths are two thirds meters, I get six bows. I don't understand.
15.2.115	00:29:17	Brian2:	There's two meters.
15.2.116	00:29:29	T/R 2:	What do you girls think? Caitlin said something where she thought it might be 12. She was looking at a pattern here. I think… because it went two… or it went four, six, eight, ten, but I don't buy that as an answer. I want to know if it works, why it works.

15.2.117 00:29:48 Erin:	I think… I think I know. Because if you take one third, it's six bows, but two thirds you take two of those two meters so you'd have double that so you'd have six and six, 12.
15.2.118 00:30:00 T/R 2:	That's interesting. Can you think of something doubling?
15.2.119 00:30:03 Brian2:	That's what I think too.
15.2.120 00:30:04 T/R 2:	Yeah? Can we test it out? Can we test these theories out? Some people think it's…. Well, first Brian said six. Now you're saying 12, all of you, but I really can't picture it and I'm wondering, can we test it out, maybe with string and actually make some cuts and decide if we get, in fact get 12 pieces? Can we make something that's two meters long and do that? You've got some string right here.
15.2.121 00:30:25 Caitlin:	Should we get another meter stick?
15.2.122 00:30:28 T/R 2:	Either that or you can go back and forth on this one. What you can do it take it to the end and turn, make a turn and come back. I'll get you a scissor.
15.2.123 00:30:41 Brian2:	What should I do?
15.2.124 00:30:49 Erin:	It's making another trip around. [measuring ribbon on meter stick]
15.2.125 00:30:58 T/R 2:	You're making a trip but you have to pull it tight in order for it to work. Ok, so let's do this Caitlin. Let's not be.. You what we're gonna with that, let's see before we make the cut, let's see if it still goes too the end. Does it go to the end? And I want Caitlin to put her thumb, actually we're dangling over the edge a little bit. Ok, that's good. Let's pull it tight. Ok, now let's make the cut there. Is it lined up? Brian, tell us, you're our expert. Is it lined up?
15.2.126 00:31:39 Brian2:	No.
15.2.127 00:31:40 T/R 2:	So what do we need to do?
15.2.128 00:31:43 Caitlin:	I know what, just pull it down, pull this piece down and we get more string. Pull it tight. Pull it. Pull the end.
15.2.129 00:32:01 V1:	Good, now we can just snip the ends off and we should have it. If we snip both those ends off right at the wood we should have it. Now, we have to make lengths that are two thirds of meters long, how are we going to do that now with the strip? How would you measure that?
15.2.130 00:32:20 Caitlin:	It's two thirds, we got two meters here.
15.2.131 00:32:24 T/R 2:	Where is two thirds of meters on here [pointing to meter stick]. Ok, well where's a half a meter? Ok, so about where do you think two thirds of a meter is?
15.2.132 00:32:42 Brian2:	There [mumbling, pointing to something]
15.2.133 00:32:43 Caitlin:	Right there. [pointing] There's two, two.
15.2.134 00:32:49 T/R 2:	How can we be sure? You need to be pretty accurate in this.
15.2.135 00:32:56 Caitlin:	One two three four five six. [counting one meter stick, mumbling] that's right here.

15.2.136 00:33:17 T/R 2:		Ten of those spaces. From where to where are you saying? Well you tell me from zero to 50 is a half, right? Is two thirds going to be more than that or less than that? Is two third more than a half or less than a half?
15.2.137 00:33:35 Caitlin:		Wait I think less.
15.2.138 00:33:38 T/R 2:		Two thirds. Brian, two thirds? Not one third, two thirds.
15.2.139 00:33:56 Brian		More.
15.2.140 00:33:57 T/R 2:		Ok more, how much more?
15.2.141 00:34:08 T/R 1:		Ok, Jackie and Amy what are you doing here?
15.2.142 00:34:08 Jackie:		Well, we figured out.
15.2.143 00:34:11 T/R 1:		Which problem?
15.2.144 00:34:12 Amy:		We are on problem four or five?
15.2.145 00:34:18 T/R 1:		What are you discussing with me?
15.2.146 00:34:22 Amy:		We got a pattern. In the three we did so far, we got patterns.
15.2.147 00:34:28 T/R 1:		Oh, ok and the patterns helped you solve it?
15.2.148 00:34:31 Amy: and Jackie:		Yes.
15.2.149 00:34:32 T/R 1:		So that's... Ok…That's neat. And what is, what is the pattern that you think you see?
15.2.150 00:34:35 Jackie:		Well, it started in the first two patterns. It started in the first two problems, two, three, four, and five.
15.2.151 00:34:42 T/R 1:		Oh, interesting. How did you get the two, three, four, and five?
15.2.152 00:34:44 Jackie:		Well, because one half would be two, one third would be three, one fourth would be four and one fifth would be five.
15.2.153 00:34:52 T/R 1:		Ok, I see that pattern, ok. Did you find the pattern?
15.2.154 00:35:00 Jackie:		One half would be four because we um, we doubled it because it was two meters. And that would be one would be six bows, and then eight bows, ten and twelve. And since it was… we thought that all of them whatever that number how many meters it would be we thought that we would have to go by twos.
15.2.155 00:35:26 T/R 1:		Ok, I understand that but the one I don't understand is this last one here. I don't understand how from two meters, right, and you- What are you being asked in this last one? What's the question?
15.2.156 00:35:43 Jackie:		How many bows would be in two thirds?
15.2.157 00:35:46 T/R 1:		How many what?
15.2.158 00:35:47 Amy:		Bows in two thirds. How many bows can you get out of two thirds?
15.2.159 00:35:51 T/R 1:		I don't think that's the question.
15.2.160 00:35:55 Jackie:		Two thirds.
15.2.161 00:35:59 T/R 1:		You're getting bows out of what? How many meters do you have to start with?
15.2.162 00:36:03 Amy:		Two.
15.2.163 00:36:03 Jackie:		Two.

15.2.164 00:36:04 T/R 1:	Two meters… ok.. So can you imagine two meters? So, show me the ribbon that's the two meter ribbon. The blue, ok, so I want to get bows out of that two meter ribbon, right, and what is the length of the bow that I want?
15.2.165 00:36:24 Jackie:	Two thirds.
15.2.166 00:36:25 T/R 1:	Two thirds. Can you show me a bow that's 2/3 of a meter? [Amy starts to unwind the blue ribbon] No, no. No, no. Show me with something else… a bow that would be… Can you show me a bow? Can you show me a ribbon that's a meter?
15.2.167 00:36:39 Amy:	Yeah.
15.2.168 00:36:41 Jackie:	The white.
15.2.169 00:36:43 T/R 1:	Ok. Can you show me a ribbon that's half a meter? From the white? How would you do that?
15.2.170 00:36:48 Amy:	You would..
15.2.171 00:36:51 Jackie:	Cut half of that.
15.2.172 00:36:52 T/R 1:	Ok. How could you show me a bow that's one third of a meter?
15.2.173 00:36:57 Jackie:	The gold ribbon.
15.2.174 00:36:59 T/R 1:	No, let's do it with the white. How would you show me a bow that's one third of a meter? [Amy folds the ribbon] What are you doing, Amy?
15.2.175 00:37:07 Amy:	I'm folding it over.
15.2.176 00:37:13 T/R 1:	So you're making thirds?
15.2.177 00:37:15 Amy:	[shakes head yes] Mmm hmm
15.2.178 00:37:16 T/R 1:	Ok, that's neat. Do you see what she's doing, Jackie? Do you agree with that? [Jackie shakes her head yes] If you can show me one third, I want you to actually show me one third and when you have one third I want you to call me cause if you can show me one third, I think you'll be able to show me two thirds, won't you? You think so? Is that a third? What do you mean by a third? Here's your meter. Here's your ribbon. What would one third be?
15.2.179 00:37:46 Michael:	I can't keep this straight. Every time I fold it out, it goes. I'm trying to make another two thirds and I can't because this thing's going.
15.2.180 00:37:58 T/R 1:	Why don't you mark it off on...Why don't you mark off two thirds on here and then if something's here that's long, you can just check it on there. Ok if this is a meter… if this is one meter, how would you get one third?
15.2.181 00:38:17 Jackie:	Well, you would take another ribbon.
15.2.182 00:38:18 T/R 1:	No, no. From this ribbon, how would you?
15.2.183 00:38:22 Amy:	You would cut it into parts.
15.2.184 00:38:23 T/R 1:	And how would I get three parts? That I know are the same size? They have to be the same size, don't they?
15.2.185 00:38:27 Jackie:	By folding it.

15.2.186 00:38:30 T/R 1:	By folding it. Can you do that together? Fold this so you have three parts, right, only three parts and they're all the same size. Can you do that?
15.2.187 00:38:35 Amy:	Mmm.
15.2.188 00:38:51 T/R 1:	I don't understand what you're doing. You're getting more than three parts. I'm confused. Now already, you have three parts already. You want this whole thing to be into three parts. You hold it, Amy, Jackie, how could you make three parts out of it? Ok, about here and here, so let's fold. Go ahead. [They fold the white ribbon] Now how could you be sure the parts are all the same?
15.2.189 00:39:15 Jackie:	By folding that over.
15.2.190 00:39:18 T/R 1:	Ok. Right. Now let's make sure they're exactly the same. Now do you know a third? Do you know what a third is?
15.2.191 00:39:28 Jackie:	Yes.
15.2.192 00:39:29 T/R 1:	Do you know what a third is?
15.2.193 00:39:30 Amy:	Mmm hmm.
15.2.194 00:39:32 T/R 1:	Ok. That's a third. Do you know what two thirds is? You can mark it if you want to, where the thirds are, here and here. Right? Let's mark it, put a little mark in here, with your pen, and here. Ok, now you know what? Show me a third.
15.2.195 00:39:56 Amy:	A third, here's a third [points to ribbon].
15.2.196 00:39:58 T/R 1:	Show me 2/3. Show me 2/3. [Jackie points to ribbon] Ok is that our two thirds? Now, two of them? Is that two of our thirds? I don't want three thirds. Three thirds would be the whole ribbon, right? I only want two of my thirds.
15.2.197 00:40:18 Jackie:	So you have to either cut that or..
15.2.198 00:40:21 T/R 1:	No, I don't want to cut it. I wanted two thirds. This is two thirds now. Do you agree? Do you believe it? Are you sure? Amy?
15.2.199 00:40:30 Jackie:	Yeah, this one [pointing to ribbon].
15.2.200 00:40:32 T/R 1:	Is two thirds…where it ends here is two thirds. Amy, are you convinced? [Nods] See what you did here? You marked one third. You marked another third, so that's 2 and this [the end of the ribbon] is three thirds. Right? Correct? Now, what's the question I'm asking you? There's the blue ribbon, right? How many ribbons this length can be made from ribbon that length? Isn't that right? You have two meters. The lengths are two thirds, right? How many ribbons of length two thirds can be made from ribbon two meters? Do you understand the question? How many of these can you make from that?
15.2.201 00:41:16 Jackie:	Two.

15.2.202	00:41:18 T/R 1:	You gotta prove it to me. When you've proved it to me and convinced me, then you can call me. [walks away] [puts white and blue ribbon together]
15.2.203	00:41:34 Amy:	I messed up. One, two, three, four.
15.2.204	00:42:05 Jackie:	Wait, twelve.
15.2.205	00:42:15 Amy:	Five, Six, six, six, six, six.
15.2.206	00:42:28 Jackie:	Wait. But that was six.
15.2.207	00:42:30 Amy:	That's three.
15.2.208	00:42:34 Jackie:	Well, this one's six.
15.2.209	00:42:38 Amy:	Look.
15.2.210	00:42:41 Jackie:	Remember when we doubled it?
15.2.211	00:42:44 Amy:	One, two, three. This is three. This is three, Jackie. This is one meter. This is a third. This is a third and this is a third. How many thirds?
15.2.212	00:43:01 Jackie:	Three.
15.2.213	00:43:02 Amy:	Then that's the answer, three.
15.2.214	Jackie:	That's wrong
15.2.215	00:43:08 T/R 2:	How are we doing?
15.2.216	00:43:09 Amy:	We're going back over our answers
15.2.217	00:43:11 T/R 2:	Ok. And how's it going? Are you able to verify your answers? You change your mind about any of them?
15.2.218	00:43:18 Amy:	Uh huh.
15.2.219	00:43:19 Jackie:	I still think it's twelve.
15.2.220	00:43:21 Amy:	Jackie, we already, we checked it.
15.2.221	00:43:24 T/R 2:	Better make sure.
15.2.222	00:43:26 Jackie:	Because we doubled the…we doubled all the answers right?
15.2.223	00:43:32 Amy:	Yeah, but Jackie we have to check them over.
15.2.224	00:43:44 Jackie:	When we doubled it, we doubled this number. So it would be six. Wait a minute.
15.2.225	00:43:58 Amy:	[folds ribbon] One, two, three, four, five, six.
15.2.226	00:44:05 Jackie:	I still think there's 12.
15.2.227	00:44:09 Amy:	There's six.
15.2.228	00:44:11 Jackie:	No, we have to double it.
15.2.229	00:44:13 Amy:	There are six here.
15.2.230	00:44:15 Jackie:	You have to double it though.
15.2.231	00:44:16 Amy:	No, this is two thirds, two meters. This is two meters long.
15.2.232	00:44:19 Jackie:	But that was six. We have to double it because this is going to be one third instead of two thirds. If we left it at six…
15.2.233	00:44:40 Amy:	Let's just go on to number four. Let's just forget it. Six meters, [girls are measuring, camera shifts to left]
15.2.234	00:46:07 V2:	These one ribbons, if they said here it's two thirds of a meter, what does two thirds of a meter look like?
15.2.235	00:46:12 Brian2:	Can we open this up and look?
15.2.236	00:46:14 V2:	Just show me on a ruler. What does two thirds, here's the whole meter. What does two thirds like? Well, what does

	one third look like? Just roughly, not exactly. Just show me, just point. Like what does half of it look like?
15.2.237 00:46:34 Caitlin:	Like this, this would probably be what it..
15.2.238 00:46:37 V2:	Ok. So you're saying like about that much.
15.2.239 00:46:41 Caitlin:	This would probably be one third.
15.2.240 00:46:43 V2:	Ok, now what would two thirds be? This is, here, when dividing into thirds.
15.2.241 00:46:50 Erin:	Maybe there.
15.2.242 00:46:51 V2:	That's one third and this is one third. And how much is this? Just from here to here, how much is this?
15.2.243 00:47:02 Caitlin:	One fourth?
15.2.244 00:47:03 V2:	Well, Ok, let's get the ribbon and see if we can, see if you can show me three lengths that altogether that will fill up the ruler. Show me three lengths, a particular length.
15.2.245 00:47:28 Erin:	You guys want to start all over again?
15.2.246 00:47:30 V2:	Let's get a feel of what we're talking about then I think it will all flow right out.
15.2.247 00:47:33 Caitlin:	Should we get new string?
15.2.248 00:47:35 V2:	Ok, she's showing me that. She thinks it's about this. Ok now how do we prove that will fill up the ruler? Somebody want to hold a finger there? Let's just swing it around.
15.2.249 00:47:53 Erin:	Now we have to get.
15.2.250 00:47:58 V2:	Ok, does that look about right? Maybe a little bit bigger. Say about that. Do you have a scissor here? [cuts ribbon] Do you want to check it to see if it does it? One third. Ok, I'll hold my finger there. And great. Ok now, what is this thing we got here? What is this length?
15.2.251 00:48:42 Brian2:	One
15.2.252 00:48:43 T/R 2:	One what?
15.2.253 00:48:44 Erin: and Caitlin:	One third.
15.2.254 00:48:46 V2:	Now suppose I wanted you to make me a string that is two thirds.
15.2.255 00:48:50 Brian2:	You would
15.2.256 00:48:51 Caitlin:	You would double this.
15.2.257 00:48:52 V2:	Make me another string that is two thirds. This is one third. And then I need a string that is two thirds. How would I do that? Can we use this to measure?
15.2.258 00:49:00 Erin:	Yeah.
15.2.259 00:49:03 V2:	We did all this work. This is one third. Can you make me another one third?
15.2.260 00:49:16 Erin:	I can try.
15.2.261 00:49:20 Caitlin:	I think it would probably maybe be up to here [on meter stick].
15.2.262 00:49:24 V2:	Ok. She's gonna make it a different way.
15.2.263 00:49:28 Brian2:	I think it would be

15.2.264 00:49:30 V2:	She's gonna make it a different way and we'll see if it's the same. She's gonna make me another one third and then all together we'll have how many thirds? How many thirds do I have now?
15.2.265 00:49:55 All	Two thirds.
15.2.266 00:49:56 V2:	Let's string them out and see where that takes us. And then we just match it to this other string. Ok, oops, let go. Whoops.
15.2.267 00:50:17 Brian2:	Can I just stick [cannot understand]
15.2.268 00:50:22 V2:	This is the one. This is the other. Ok so let's now if we cut this big one there. Ok so what do we got here? What's this long one?
15.2.269 00:50:52 Erin:	Two thirds.
15.2.270 00:50:53 V2:	Two thirds. Can we use this one as a measuring thing now? Now how many, we've got two meters right, did someone make a two meter length of string? The blue, where is the blue? The blue is two meters. What is the question now? What does this question ask? This is the blue string right? And this is what? One of those?
15.2.271 00:51:22 Brian2:	Yes.
15.2.272 00:51:22 V2:	Where? Ok, so what's the question now? What do we want to find out here?
15.2.273 00:51:27 Brian	How many bows can you make with two thirds?
15.2.274 00:51:31 V2:	And this is the two thirds. So open that.
15.2.275 00:51:34 Erin:	Someone's gotta hold one side. Its three pieces.
15.2.276 00:51:42 V2:	You only want one of those, right? [unfolding blue ribbons] What do we got now? Aren't you excited? How many so far?
15.2.277 00:51:59 Caitlin:	One.
15.2.278 00:52:23 V2:	What's it look like?
15.2.279 00:52:24 Brian2:	Maybe three.
15.2.280 00:52:32 V2:	Looks like it. [writes on paper]
15.2.281 00:52:35 Erin:	Yeah, I think so.
15.2.282 00:52:40 V2:	Ok, now is there something else there that would help you believe that you could get three bows if you had two thirds meters for each bow? There's some number there that I think might give you a hint.
15.2.283 00:53:04 Erin:	The two thirds cause there's three.
15.2.284 00:53:08 V2:	Yeah. But is there something else in there, some other number that might. Ok tell me about it. What do you think?
15.2.285 00:53:16 Brian2:	One third.
15.2.286 00:53:18 T/R	Ok if you have one third, you get how many bows?
15.2.287 00:53:23 Brian2:/Erin:	You get six bows.
15.2.288 00:53:25 V2:	Ok and now you got, you need more or less ribbon for two thirds than one third?
15.2.289 00:53:36 Erin:	Two thirds is larger than one third.

15.2.290 00:53:39 V2: Ok, so if you need..

15.2.291 00:53:41 Erin: So it'd have to be.

15.2.292 00:53:43 Brian2: You would make bigger bows. You would make bigger bows.

15.2.293 00:53:47 V2: Right, and so you make bigger bows, they each are going to be what? Are you going to make as many?

15.2.294 00:53:56 Brian2:/Erin: No.

15.2.295 00:53:57 V2: Oh, so that like sort of makes sense.

15.2.296 00:54:00 Erin: Yeah.

15.2.297 00:54:02 V2: This is twice as much ribbon for each bow, and you have half as many bows. [Brian2 and Erin nod. To Caitlin] You don't look as though you believe it. You look gone. Think about it. Those are the hardest kinds of problems. Everybody has trouble with that part of math.

See other camera view for end of transcript

Session 16, Dec. 14, 1993 (Front, Side, and OHP)

Session 16, December 14, 1993, Side View

Line	Time	Speaker	Transcript
16.1.1	00:07	T/R 1:	Now, I know Beth wasn't here, she's, she's, I… I understand that umm she knows about the activities some people have shared, uhhh but uh, let's see what can we tell Beth about what we did last time? Any, any discoveries that we made in our project? Anything we remembered about making these ribbons that would be an important kind of thing to have noticed? Jessica?
16.1.2		Jessica:	Well, I noticed that after a while like it started making a pattern.
16.1.3		T/R 1:	Ok. You want to say a little bit more about that?
16.1.4		Jessica:	Well, um, I forget what pattern but I think it was going like it started going in three, six, nine, like… like when it said when you had like different size ribbons and every time it got like …like three times bigger and it kept doing it in all different kinds of patterns, I thought.
16.1.5		Michael:	Yeah, because at first it went two, three, four, five
16.1.6		Jessica:	And then it went…
16.1.7		Michael	and the second one went, uh, the second one went four, eight, something like four, six, yeah
16.1.8		T/R 1:	I don't remember any two, four, six or four, eight.
16.1.9		Michael:	No, it's four, it's four, six, eight, ten… and then there was that odd, and then there was that two thirds one.
16.1.10		T/R 1:	Ok, let's, let's, let's hold out… Brian what were you just saying?
16.1.11		Brian:	Well, if we, remember we had the three meters, you would always like times the number by three. Like you go three, six, nine?
16.1.12		T/R 1:	Yeah, yeah Michael's asking the question I had which number. Let's use that as an example. I have ribbons three meters long and I'm making bows how long? For example. Michael?
16.1.13		Michael:	Uh, one half
16.1.14		T/R 1:	One half a meter long, so if I have, I could sort of imagine ribbon three meters long, three of these sticks long, that's how long, and I'm making bows a third of a meter long, how can I imagine a third of a meter? How could I imagine one third of a meter? You could imagine a meter, right? You can see a meter? How can you imagine a third? Can you all in your heads imagine a third? How many of you can, imagine a third? So what are you imagining when you

			imagine a third? Not everyone is imagining it. Beth, what do you imagine?
16.1.15		Beth:	[hems and haws]
16.1.16		T/R 1:	Is it longer than this? No? Is it shorter than this? [students yeah] Is it shorter than this length?
16.1.17		Beth:	Uhh huh.
16.1.18		T/R 1:	Ok, so it's shorter than this length. About how short, much shorter is it than this length? What are you imagining? You're the only ones who can imagine how much shorter it is? I think more of you can imagine. Can you imagine a third of a meter? I have some half hands up. Jessica, what do you imagine?
16.1.19		Jessica:	Well, I imagine if you like pull the ruler into like three pieces and then it would be like, like, up to the um I think wait, um thirty-three mark, I think.
16.1.20		T/R 1:	Well how, how did you decide on the thirty-three mark?
16.1.21		Jessica:	Well that's what I think because um, um, thirty-three plus thirty-three plus thirty-three is ninety nine and that's,
16.1.22		Michael	No, but there's a hundred…
16.1.23		Jessica	Yeah, and then a hundred, around like thirty three and like a half almost.
16.1.24		T/R 1:	What do you think? Jackie, your hand up partially?
16.1.25		Jackie:	Something around.
16.1.26		T/R 1:	Something around that.
16.1.27		Alan:	I think there, it's thirty-three and one third because if you take two more thirds you can get it to a hundred.
16.1.28		T/R 1:	What do you think, Jessica? Thirty-three and a third?
16.1.29		Jessica:	Yeah.
16.1.30		T/R 1:	That what you're imagining, so this…
16.1.31		Michael:	I'm, I'm imagining it just being cut into three equal halves
16.1.32	4:53	T/R 1:	Equal parts. Three equal parts. How many of you imagined it cut into three equal parts? [many hands raised] Ok, and Jessica and Alan were a little more explicit they were trying to actually tell me the… how long those parts are, right? And uh, and so you're telling me in this meter stick, because there…you're telling me there are a hundred meters here? A hundred centimeters here? A hundred what here?
16.1.33		Students	A hundred centimeters…
16.1.34		T/R 1:	A hundred centimeters? How do you know that?
16.1.35		Alan:	Because it only goes up to ninety-nine but there's an extra length that could be a centimeter.
16.1.36		T/R 1:	This piece over here?
16.1.37		Alan:	Mmm hmmm.
16.1.38		T/R 1:	I see, the numbers go to ninety-nine but it goes up to here, you're telling me. So you're telling me there are a hundred

		centimeters here and you're telling me that if you were to make three equal parts, Graham, what do you think?
16.1.39	Graham:	Well, there's ten decimeters.
16.1.40	T/R 1:	Ten decimeters, well.
16.1.41	Graham:	Well, that's ten centimeters, and then there's ten decimeters.
16.1.42	T/R 1:	How do you get ten decimeters?
16.1.43	Graham:	Well there's, well there's ten centimeters in a decimeter and there's ten of them on that so it would go to a hundred.
16.1.44	Michael:	What? Ten centimeters, plus ten centimeters, plus ten centimeters is [inaudible]
16.1.45	Graham	10 times
16.1.46	Michael	Oh.
16.1.47	T/R 1:	Ten times ten? Very interesting. Let's talk about that another time, what Graham is saying. Um, but, for now, you, you all can imagine a third? So what was the question that you posed to me? If we had three meter length ribbon is that what you said earlier, Brian? And we wanted to know how many ribbons one third of a meter long? And what did we decide? How many? We're going to hear Jessica's theory now.
16.1.48	Jessica:	Um
16.1.49	T/R 1:	We had three meters; I can imagine three of these, now I could imagine a ribbon a third of a meter, right? You helped me with that and in fact you were very precise about helping me with that, and how many bows can you make?
16.1.50	Jessica:	Um, I think you could have made, um, oh I forget, um,
16.1.51	T/R 1:	Why don't you all sit and talk to your partner for a minute and confer and see what you think.
16.1.52	Jessica:	What do you think, I think [inaudible]
16.1.53	Beth:	How many [inaudible]
16.1.54	Laura:	I think it's nine
16.1.55	Jessica:	I forget what I wrote on my paper.
16.1.56	Laura:	Three meters, so…
16.1.57	Jessica:	Yeah you can make three bows.
16.1.58	T/R 1:	We have three meters of ribbon, and we're making bows, we have three meters of ribbon to start with and our bows are to be one third of a meter in length. How many bows can I make from three meters of ribbon?
16.1.59	Andrew:	One third of a meter. So if this [using pens] is a meter and then.. [Figure S-7-13]
16.1.60	James:	and then this is a meter, then this is a meter
16.1.61	Andrew:	so this would be divided into thirds
16.1.62	James:	Three six nine.
16.1.63	Andrew:	Yeah.

16.1.64		CT:	You've got, you've got ribbon how long? Three meters. Alright, but a bow is going to be, what do you think one third of what?
16.1.65		Jessica:	I think I thought in order to make bows, I thought you can make three of them
16.1.66		T/R 1:	From one meter.
16.1.67		Laura:	Yeah.
16.1.68		Jessica:	From, from here to here.
16.1.69		T/R 1:	Mmm hmm.
16.1.70		Jessica:	For one meter, and like, like, what I was saying, like thirty three and it would be somewhere arond there
16.1.71		T/R 1:	Right, but now I have three meters of ribbon.
16.1.72		Jessica:	Three meters? Oh, three, nine
16.1.73		T/R 1:	So why don't you talk - nine? You agree? Laura?
16.1.74		Beth:	Yeah [Laura nods] because three times three is nine
16.1.75		Jessica:	Because three times three is nine.
16.1.76		T/R 1:	Ok, James?
16.1.77		James:	Um, Andrew [inaudible] um we think um it's nine and there's nine in three meter sticks.
16.1.78		T/R 1:	Ok.
16.1.79		James:	That's what we think.
16.1.80		T/R 1:	Ok, and you could persuade us, everybody, that that's the case?
16.1.81		James:	Yeah.
16.1.82		T/R 1:	Ok.
16.1.83	8:40	Danielle:	[standing with CT, Brian, Jackie, and Amy] It would be like [points with fingers]
16.1.84		Jackie:	Can I ask Mrs. Palmer …
16.1.85		Amy:	Can we ask Mrs. Palmer if we could borrow her meter sticks or no?
16.1.86		CT:	Well, well you're bothering her, ok, well, you've got a meter stick right here, help us with this. We're trying to figure out what one third of a meter is so we can figure out how many one thirds go into three meters. So you think right here is one,
16.1.87		Brian:	Yeah, cuz…
16.1.88		CT:	And then where would the other one be? [Brian points] Right here?
16.1.89		Brian:	Yeah.
16.1.90		CT:	Alright,
16.1.91		Brian:	And then the other one right there [pointing]
16.1.92		CT:	Alright so then this is the third of a meter, this is a third, between these two? And it's a third between Amy and myself, well how many to a meter then?
16.1.93		Students:	Three.
16.1.94		CT:	Three, well how many to three meters?

16.1.95 Erin: Nine.
16.1.96 CT: How did you get that?
16.1.97 Amy: Three times three, three times
16.1.98 CT: Well, is that right?
16.1.99 Brian: I think so
16.1.100 Danielle: Yeah.
16.1.101 CT: Well, here's here’s one meter stick and I have how many, how many bows from here?
16.1.102 Brian: You have three meters in one and then three meters
16.1.103 Danielle: But doesn't it
16.1.104 CT: Go ahead, go ahead, we don't know if we're right, so go ahead. Doesn't it what, hon?
16.1.105 Danielle: I don't know.
16.1.106 CT: Alright, well how many bows do you have here?
16.1.107 Danielle: Three.
16.1.108 CT: Three, and how many meters is it?
16.1.109 Danielle: One.
16.1.110 CT: Now let's just take a flight of imagination and keep this one here in your mind let's move this one, here's a second one. How many, how many uhh bows do you have in this one?
16.1.111 Danielle: Three.
16.1.112 CT: So how many do you have to make two?
16.1.113 Danielle and Amy: Six.
16.1.114 CT: And here, keep that in your mind. Here's one, here's one, you've got that in your mind. Here comes the third one, how many do you have here, in this third one?
16.1.115 Danielle: Three.
16.1.116 CT: How many is that all together?
16.1.117 Danielle: Nine.
16.1.118 CT: You're sure.
16.1.119 Danielle: Yeah.
16.1.120 CT: Are you really sure?
16.1.121 Danielle: [nods head]
16.1.122 CT: What do you think [to others]? What do you think? Well we’ll see.
16.1.123 Amy: [speaks, but inaudible]
16.1.124 CT: Do whatever you need, do, use whatever you need to use. If you think you know your means then use it.
16.1.125 Jessica: Can we take our papers back and start, um
16.1.126 T/R 1: Well, you won't need it yet.
16.1.127 Jessica: Ok.
16.1.128 Danielle: [this group is measuring meters of ribbon] Here's two meters, I mean one meter.
16.1.129 T/R 1: You mean twenty-seven three times is eighty one, now tell me what you did here.

16.1.130 Alan: I did twenty, I did seven times three and that equals this, never mind.

16.1.131 T/R 1: Tell me what you did, I want to know what you did Alan.

16.1.132 Alan: Well, it's wrong, anyway.

16.1.133 T/R 1: Well, what did you do, though?

16.1.134 Alan: I did, I multiplied seven times three and got twenty one

16.1.135 T/R 1: [interjecting] Twenty-one.

16.1.136 Alan: And then I put the two up there, added that, times it and got twelve, now it's wrong.

16.1.137 T/R 1: Ok, so that particular rule didn't work, now what did you do, Kimberly?

16.1.138 Kimberly: I did twenty-seven times three.

16.1.139 T/R 1: And how did you do it?

16.1.140 Kimberly: I times twenty, I times three times seven, I got twenty-one, so I carried the two, then I did three times two and added the two to my answer. [Figure S-12-53]

16.1.141 T/R 1: Why does that work?

16.1.142 Kimberly: Umm

16.1.143 T/R 1: Or does it work? I mean, Alan showed me here three times, if you have three times twenty-seven that means you have a twenty-seven three times, and he proved to me that it's eighty-one, how does, why does that work?

16.1.144 Kimberly: because you add that.[Figure S-13-13]

16.1.145 T/R 1: I don't understand why it works.

16.1.146 Alan: Because you're basically doing is writing, you have twenty-seven three times

16.1.147 T/R 1: Mmm hmm

16.1.148 Alan: And then you get your answer eighty-one.

16.1.149 Kimberly: It's just, you're just adding it faster.

16.1.150 T/R 1: I don't know why that works, that adding faster. See, Alan added faster and it didn't work. Does it always work?

16.1.151 Kimberly: No.

16.1.152 T/R 1: Doesn't always work.

16.1.153 Kimberly: But it does sometimes.

16.1.154 T/R 1: But Alan's was different.

16.1.155 Kimberly: Yeah.

16.1.156 T/R 1: I'm kind of curious about that. See if you can come up with a rule that works all the time. You know what it means, right?

16.1.157 Kimberly: Yeah.

16.1.158 T/R 1: Something to think about, right?

16.1.159 Alan: Yeah.

16.1.160 T/R 1: You said you have twenty seven three times. Would it work if you had twenty three times and seven three times?

16.1.161 Alan: Yeah

16.1.162 Kimberly: Uh, maybe, uh I don't think so.

16.1.163	T/R 1:	Kimberly isn't sure. But you think so
16.1.164	Kimberly:	I don't think so
16.1.165	T/R 1:	She doesn't think so, you do.
16.1.166	Alan:	Yeah, I do, cuz you have twenty [writes]
16.1.167	T/R 1:	Right.
16.1.168	Alan:	if you have seven three times
16.1.169	Kimberly:	Add them together, you'd have to add them together.
16.1.170	T/R 1:	Ok, right.
16.1.171	Kimberly:	You would have to add them together.
16.1.172	T/R 1:	And you add them together and what do you get. [Alan's paper has the addition of three twenties and the addition of three sevens, and then the sum of those two sums] Ok, so that worked, didn't it? Does that help you figure out a way to make it work every time? I bet you can invent a rule that works, Alan. If you think about what you did. Ok? You have to add them together. What do you mean you have to add them together, Kimberly? [Figure S-14-22]
16.1.173	Kimberly:	You would have to add those two answers together to get
16.1.174	T/R 1:	Why?
16.1.175	Kimberly:	Because, if you wanted to do it faster.
16.1.176	T/R 1:	Here.
16.1.177	Kimberly:	Because you wouldn't be able to get the answer for this if you were using this, and you would try to get the answer eighty-one, you wouldn't be able to get the answer unless you added the two answers together.
16.1.178	T/R 1:	But why?
16.1.179	Kimberly:	[shakes head] I don't know.
16.1.180	T/R 1:	You don't know. Well, that's what I was asking you to think about. [to class] Ok, just for a time out for a minute while you're working on this, for those of you who are finished with that problem, I asked you, how many ribbons one third meter in length can you make from three meters of ribbon, right? And then I said suppose you had nine meters of ribbon, how many ribbons can you make one third meter in length and then I said suppose you had twenty-seven meters of ribbon, how many ribbons can you make one third meter in length? So those are the problems you're working on, I just want to be sure you know all know the problems you're working on now.
16.1.181 15:51	Jackie:	Ok, now we have eighty-one, that's just extra.
16.1.182	Jessica:	Nine, you got nine, right?
16.1.183	Jackie:	We got nine.
16.1.184	Jessica:	So did I.
16.1.185	Amy:	Let's get some paper to write this down on.
16.1.186	Danielle:	I like your sweatshirt.

16.1.187	Brian:	[Jackie and Jessica are talking] Guys, this is math, you're on camera. You're on camera and you're like oh nice sweatshirt.
16.1.188	Amy:	You can do that out at the playground
16.1.189	Sarah:	Guys are you on the first one?
16.1.190	Alan:	... Seven times three, and you get the twenty-one. You add the sixty and the twenty-one and you get the eighty-one. Now you get it? [Figure S-17-00]
16.1.191	Student:	Neither do I.
16.1.192	Kimberly:	Got it.
16.1.193	Alan:	What I'm doing is, you have your twenty-seven, so you take off the seven, and you get and you only have twenty. So then you do twenty times three and you get sixty, which brings me to step two. You don't have two, so you have the seven. So you do seven times three and that equals twenty-one. So you add the sixty and the twenty-one and you get eight one.
16.1.194	Kimberly:	Ok. I think I got it. Alright, I don't get it.
16.1.195	Alan:	You still don't get it. Ok, I'll put it in a lot more words.
16.1.196	Kimberly:	Cuz I'm not sure about something! [takes Alan's paper] I think we're supposed to try make a rule for that.
16.1.197	Alan:	Not that one, it was wrong.
16.1.198	Kimberly:	Ok. I'm confused.
16.1.199	Alan:	Ok, you added twenty-seven, before you multiply you take off the seven and then you get twenty. And then you have twenty times three and that equals sixty. So then you go to step two. You don't have the two there anymore so you have the seven. You do seven times three and that equals twenty-one so you add your two answers and you get eighty one.
16.1.200	Kimberly:	Got it.
16.1.201	Alan:	Good. So,
16.1.202	Kimberly:	Ok, what do we have to do now? Ok, what do we have to do now?
16.1.203	Alan:	What do you want to do now?
16.1.204	Kimberly:	I don't know.
16.1.205	Alan:	We finished the problem, so
16.1.206	T/R 1:	How much ribbon do you have?
16.1.207	Michael:	We have six thousand five hundred and sixty one yards, um meters of ribbon.
16.1.208	T/R 1:	Did you have a calculator check your computation?
16.1.209	Erik:	And right now, we're tying ourselves down to get nine meters of ribbon! We're tying ourselves down.
16.1.210	T/R 1:	Literally, Erik, you're literally tying yourself down?
16.1.211	Erik:	Yay, we got-
16.1.212	T/R 1:	Well, did you figure it out, Alan?
16.1.213	Alan:	For the strategy, for the strategy.

16.1.214 T/R 1: Tell me
16.1.215 Alan: Twenty-seven meters.
16.1.216 T/R 1: Ok. Did you discuss it with Kimberly?
16.1.217 Kimberly: Yeah.
16.1.218 T/R 1: Did you both agree on this?
16.1.219 Kimberly: Yeah.
16.1.220 Alan: Finally, she got it.
16.1.221 T/R 1: Oh, I can't wait.
16.1.222 Kimberly: Yeah, I got confused.
16.1.223 Alan: Alright, so you have the twenty-seven before you multiply it and you take off the seven and you have twenty so then you multiply twenty three times and you get sixty.
16.1.224 T/R 1: Mmm hmm.
16.1.225 Alan: And so you go to step two. Then you don't have the two anymore and you only have the seven. And you multiply seven times three and you get twenty-one. So then step three you add sixty and twenty-one and get eighty-one.
16.1.226 T/R 1: Ok. Ok. Now, I'm curious, I was very intrigued by what Kimberly used some kind of procedure here that I don't quite understand but is there any way on the basis of what you did you could make sense of what she did?
16.1.227 Alan: Well
16.1.228 T/R 1: Can you, suppose someone…cause I don't understand why this procedure works, I understand what you explained to me, but, I want to know why this works cuz this seems to work too
16.1.229 Alan: It does work.
16.1.230 T/R 1: It does work. But why does it work is my question to you.
16.1.231 Alan: First can you explain the problem.
16.1.232 Kimberly: Ok, well, all it is is you have the twenty-seven but on his you took the seven away. And all I did was multiply the twenty-seven and
16.1.233 Alan: You did
16.1.234 Kimberly: I multiplied it together instead of having to multiply it separately and add them together.
16.1.235 T/R 1: Ok, but tell me how you multiplied it together.
16.1.236 Kimberly: Ok.
16.1.237 T/R 1: In other words tell me what you did what did you do when you multiplied.
16.1.238 Kimberly: Well, I put twenty seven times three equals [writes] so I, I times the seven and the three and I got twenty one
16.1.239 T/R 1: Ok, so
16.1.240 Kimberly: I carried the two, but…
16.1.241 T/R 1: Is that a two?
16.1.242 Alan: In multiplication you don't carry.
16.1.243 T/R 1: Is that a two?

16.1.244	Kimberly:	Well, I learned to do that.
16.1.245	T/R 1:	Ok, Kimberly, is that a two. This is a one, but my question is, is that a two?
16.1.246	Alan:	Yeah, but why do you have the two up there? Because two , because two added to two times three equals six, you had, hold it, two times two, two plus two…
16.1.247	Kimberly:	No, what I learned to do was
16.1.248	Alan:	times three equals twelve.
16.1.249	Kimberly:	What I learned to do was do was multiplication, then you add that to your multiplication answer.
16.1.250	T/R 1:	But what are you adding? I don't. When you say three times seven is twenty-one, write that down, three times seven is twenty-one, [Kimberly writes] now I always, I always learned that that twenty-one, that isn't a two, this is the one, but this isn't a two. This is two tens, I learned.
16.1.251	Kimberly:	But what I learned is you put the one there, and then you carry the two like you do in adding but you times the number so I times three times two and then whatever you got as your multiplication answer you added that number to that and you put, and then once you got there you got your answer.
16.1.252	T/R 1:	I understand that, but I want to know why it works.
16.1.253	Kimberly:	I don't know why.
16.1.254	T/R 1:	Alan?
16.1.255	Alan:	Well, what she is doing is she multiplied seven times three and got twenty-one. She carried the two and added those and multiplied it twice.
16.1.256	T/R 1:	Ok, well you gotta think about that one.
16.1.257	Alan:	But wait. But.
16.1.258 22:42	T/R 1:	I understand what you did here, it makes sense to me, I'm not so sure I understand that. I'm not saying I don't agree that it works, but I don't know why it works. [Figure S-22-43]
16.1.259	Alan:	Kim you might have to rephrase your number problem here. Because what you're doing is your doing seven times three is twenty-one, you're carrying that, and you only multiply every number by two.
16.1.260	Kimberly:	Maybe I should divide it into steps or something.
16.1.261	Alan:	If you can explain it that way.
16.1.262	Kimberly:	I think I can.
16.1.263	Alan:	Alright, put it in steps.
16.1.264	Kimberly:	Let me try.
16.1.265	Alan:	And then I'll read your.
16.1.266	Kimberly:	Alright, I have an idea, I gonna put it a little bit like that, ok? [writes] No, I keep making messing up. I'm gonna do it my own way. I keep making mistakes on this.

16.1.267 Alan: You want me to get a calculator?
16.1.268 Kimberly: No. Alright go use a calculator. No, I got it, I got it.
16.1.269 Alan: Where is x? Where is the times symbol?
16.1.270 Kimberly: Ok, ok, ok, I only have one little step, hold on, hold on.
16.1.271 Alan: Alright, step one, twenty-seven times three equals sixty one. What?
16.1.272 Kimberly: No, come here. Look, see this right here? If you, I brought that over. And then you do that, you do those and then you do that if you didn't, if that number didn't exist you'd have sixty-one, but then you take that and you add that two, but that two becomes a twenty and then you add it. So I, I can't explain this problem. I can't explain how I did it, I just know how to do it that way. [Figure S-26-41]
16.1.273 Alan: Wait, let's see. I know that twenty-seven times three equals eighty one [uses calculator]
16.1.274 Kimberly: Right.
16.1.275 Alan: It says right here.
16.1.276 Kimberly: Right.
16.1.277 Alan: And if you do twenty-seven times four it only equals, it equals one oh nine. Right there, I typed that in. Anyway, um, twenty-seven equals, so there's your eighty-one. Now the way you're doing it can't be done on the calculator.
16.1.278 Kimberly: I know. You're ignoring the two. Forget that two.
16.1.279 T/R 1: What did you times three? Twenty-seven?
16.1.280 Beth: Twenty-seven.
16.1.281 T/R 1: Ok show me how you did three times twenty-seven and got seventy-eight.
16.1.282 Laura: We just kept on adding.
16.1.283 T/R 1: You added? Well, rather than adding three is there another way you can do it?
16.1.284 Laura: Times
16.1.285 T/R 1: Yeah that's one way is there another way? What does three times twenty-seven mean? You said you could have twenty- you said you could add three twenty-seven times.
16.1.286 Jessica: Yeah.
16.1.287 T/R 1: That's twenty-seven times three. What does three times twenty-seven mean?
16.1.288 Jessica: Three times twenty-seven that's seventy-eight.
16.1.289 T/R 1: Show me.
16.1.290 Jessica: I did-
16.1.291 T/R 1: What does it mean to have three times twenty-seven?
16.1.292 Beth: Twenty-seven three times.
16.1.293 T/R 1: Ok, so why don't you have put twenty-seven three times, you could add twenty-seven three times. Ok, that's true.
16.1.294 Beth: Eighty one

16.1.295 28:25	T/R 1:	Eighty one. Ok, now what do you get when you get three twenty-seven times? Seventy-eight, is that possible?
16.1.296	Jessica:	Twenty-seven three times?
16.1.297	T/R 1:	You told me you got seventy-eight when you added three twenty-seven times. You kept adding threes.
16.1.298	Jessica:	[Beth laughs] No, I guess we counted wrong.
16.1.299	T/R 1:	Maybe you added twenty-six times?
16.1.300	Jessica:	Yeah, that would be eighty one, and that number would be eighty-one.
16.1.301	T/R 1:	Ok, now you said there was another way you could do it, three times twenty-seven, you said you could multiply it rather than add it three times? How do you do that? How do you multiply three times twenty-seven?
16.1.302	Jessica:	Uh,
16.1.303	T/R 1:	Can you show me how to do that? [Laura uses the standard multiplication algorithm.] [Figure S-29-19]
16.1.304	Beth:	What do we have to do?
16.1.305	T/R 1:	You said you could multiply three times twenty-seven. You know how to do that? [Beth begins to write] That says twenty-seven times three. Beth wrote three times twenty-seven you wrote, well, depends on how you how you read it, I guess. How do you do that? Do you know how to do that? Did you learn that? Three times twenty-seven. [Jessica writes the same as Laura][Figure S-30-29]
16.1.306	Jessica:	Yeah, and then you get eighty-one.
16.1.307	T/R 1:	You got the same answer? Laura?
16.1.308	Laura:	Yeah.
16.1.309	T/R 1:	How, how does, why does that work? How does that work? Three times twenty-seven, what did you do there?
16.1.310	Beth:	First, I did three times seven is twenty-one, put down a two
16.1.311	Jessica:	Carry the two.
16.1.312	Beth:	and then three times two is six, plus two is seven, I mean eight. And you get eighty-one.
16.1.313	T/R 1:	You said you carried- three times seven is twenty-one why don't you write that down, three times seven is twenty-one? [Beth does so] Now, when you say twenty-one, what does that mean, twenty-one? Does that mean two plus one? Or three? What does that mean, the twenty-one?
16.1.314	Jessica:	The twenty-one means that you're seven, fourteen, twenty-one, that you're taking the seven.
16.1.315	Beth:	You're taking the seven three times
16.1.316	Jessica:	Three times.
16.1.317	T/R 1:	Yeah, but what does twenty-one mean? What does twenty-one mean?
16.1.318	Laura:	Twenty-one means that two times

16.1.319	T/R 1:	That means two tens and one one, Laura? That means two tens and one one. Ok
16.1.320	Jessica:	Yeah, like if you have, like last year we were doing about these things, and they were like ten blocks in there and then we had two of them,
16.1.321	T/R 1:	Oh ok, yeah, two tens.
16.1.322	Jessica:	And then we had the one.
16.1.323	T/R 1:	Ok, so I'm confused when you say carry the two you're not carrying two of these?
16.1.324	Jessica:	No, we're carrying two tens.
16.1.325	T/R 1:	You're carrying two tens, you're carrying two tens, so how does this work? Three times seven is twenty-one ones, or two tens and one one. Right? So, how does that work? Why does that work? What do you think, Laura? What does this carrying the two mean?
16.1.326 32:25	Beth:	Because-
16.1.327	T/R 1:	Beth? What are you thinking?
16.1.328	Beth:	I was thinking it would be alright because this, this two is in the tens column
16.1.329	T/R 1:	That's a ten also, ok, so here you have three times two tens that's six tens and this is two more tens? That gives you eight tens. So your answer is eight tens and one one? Does that make sense? [Figure S-32-48]
16.1.330	Beth:	Yeah.
16.1.331	T/R 1:	Ok, I'm wondering if you can share that with Jessica who didn't hear what you just said because I, I might ask you later why does that work. Do you think you can explain it?
16.1.332	Beth:	Yeah.
16.1.333	T/R 1:	Because some people don't know why that works, so I want you to think about that.
16.1.334	Jessica:	And that-
16.1.335	T/R 1:	Ok, well, Beth, let's see if Beth can explain it to you and be sure you all agree and come up and write up why you think that works. Ok?
16.1.336	Beth:	Because two is in the tens column, and so is that so.
16.1.337	Jessica:	Yeah, I know. So it would be, so it's like you're carrying two tens.
16.1.338	Beth:	Yeah, and plus two tens. Now, let's keep doing this.
16.1.339	Jessica:	How many were we up to?
16.1.340	Beth:	Well, we have to change that to an eighty-one.
16.1.341	Laura:	How much, wait?
16.1.342	Jessica:	You have to change this.
16.1.343	Beth:	Change it to an eighty-one.
16.1.344	Jessica:	Now, let's go see how much eighty-one meters is outside.
16.1.345	Beth:	Why?
16.1.346	Laura:	Now I now I have to change this to an eighty-one.

16.1.347	Beth:	What?
16.1.348	Laura:	This.
16.1.349	Beth:	Yeah
16.1.350	Laura:	This.
16.1.351	Beth:	Now we gotta do this eighty-one times. Ok, keep working.
16.1.352	Laura:	What were we up to?
16.1.353	Jessica:	Ok, that's thirty-three down there. And that's thirty-three.
16.1.354	Beth:	It's thirty-three too! It's sixty six. Because it's thirty-three here and thirty-three here, sixty-six!
16.1.355	Jessica:	Fifty-seven, fifty-eight, fifty-nine, sixty!
16.1.356	Laura:	[Says something inaudible] Sixty, now I'm going to go to the next page. I guess, right? Ok.
16.1.357	T/R 2:	[inaudible]
16.1.358	Alan:	Two for every meter.
16.1.359	T/R 2:	Ok, and
16.1.360	Alan:	That means if you had [inaudible] divided by how many other meters you have, but I think you should do two times, wait, if you have eighty-one meters and you want to find out how many ribbons should be in that, you know that two ribbons can be made out of each meter
16.1.361	T/R 2:	Ok,
16.1.362	Alan:	So that means two times eighty-one and your answer is one hundred and sixty-two, which is obviously the answer you'd have to give.
16.1.363	T/R 2:	Mmm hmmm, Ok, what did Kim- what if instead
16.1.364	Alan:	If you had a thousand
16.1.365	T/R 2:	What if instead of a half a meter, what if they were um uh a fourth of a meter? Then what would you do?
16.1.366	Kimberly:	That would be times four.
16.1.367	T/R 2:	Ok, why does that work? Why does multiplying by two or three or four work?
16.1.368	Kimberly:	Uh, because uh that's the num- that's the, it's like a four, and if you're using um a fourth, and you use four, it's sort of like, you're just using regular numbers.
16.1.369	T/R 2:	Ok, so if I have one meter of ribbon, and they were a fourth of a meter, how many bows could I make?
16.1.370	Kimberly:	Four,
16.1.371	T/R 2:	Do you agree with that?
16.1.372	Kimberly:	eight, twelve…
16.1.373	Alan:	So that would be if you had eight one and then you'd have to multiply that by four you get three hundred and twenty-four bows
16.1.374	T/R 2:	Three hundred and twenty-four bows if I had eighty-one meters of ribbon?
16.1.375	Kimberly:	Ok,

16.1.376	Alan:	Three hundred and twenty-four meters would be the entire perimeter of this school.
16.1.377	T/R 2:	Wow, are you up to measuring that out?
16.1.378	Alan:	No actually this long hall is eighty-two so it would be only one meter less than that.
16.1.379	Kimberly:	So if you would do it again you would have one thousand, two hundred thirty-six.
16.1.380	T/R 2:	Ok
16.1.381	Alan:	Times four. If you use the answer of eighty-one times four, you'd get I don't know. You'd just keep going and then divide it by the number of
16.1.382	Kimberly:	In this [inaudible]
16.1.383	T/R 2:	What happened? The calculator's not going past a million?
16.1.384	Kimberly:	The calculator quit.
16.1.385	T/R 2:	You need a bigger calculator.
16.1.386	Kimberly:	No, the calculator quit, it said error.
16.1.387	T/R 2:	Error. This is all very interesting. Do you feel better about being able to explain this?
16.1.388	Kimberly:	I think so.
16.1.389	T/R 2:	Can you try it, can you practice on me before she asks you to explain this?
16.1.390	Kimberly:	I don’t think I can.
16.1.391	T/R 2:	Yes you can
16.1.392	Alan:	Oh, I could tell you how I could explain mine.
16.1.393	T/R 2:	Ok, I'm not concerned about that, but Kim’s nervous because what if she gets called on now? You listen too, ok, listen to her argument. Ok, Kim, why does this work?
16.1.394	Kimberly:	I don't know. I’m confused, that’s why I can’t do it.
16.1.395	T/R 2:	Well, you just told me some beautiful things about all the patterns and relationships here.
16.1.396	Kimberly:	Well, I’m confused. [to students approaching instructor] It quit!
16.1.397	T/R 1:	[to other students] Ok, How far did you go?
16.1.398	Danielle:	We went up to three point eight seven four two six four eight.
16.1.399	T/R 2:	Ok, that's a lot of bows, ok you two get to work on making those! [to Kimberly] Ok, tell me about this. You have twenty-seven times three is eighty-one
16.1.400	Kimberly:	Twenty-seven times three is eighty-one but if and if you have one meter and it was times four by fourths you get four bows, and if it was by thirds you get three bows so the third or the fourth would be three or four.
16.1.401	T/R 2:	Ok.
16.1.402	Kimberly:	[inaudible]
16.1.403	T/R 2:	Ok. Alright, if you had to say where the twenty-seven came from, do you remember?

16.1.404 T/R 1: [to Alan] Beth, Laura and Jessica may have figured out, Alan, why that, why Kimberly's algorithm works.

16.1.405 Kimberly: Well

16.1.406 T/R 2: I understand you got it by multiplying by 3.

16.1.407 Kimberly: We got an answer. She asked us what, what would you get how many bows would you get if you had three times nine and we got twenty-seven and she said how many bows would you get if it was three twenty-seven?

16.1.408 T/R 2: Ok, so this is how much ribbon you have.

16.1.409 Kimberly: Yeah.

16.1.410 T/R 2: You have twenty-seven meters of ribbon.

16.1.411 Kimberly: Right.

16.1.412 T/R 2: Ok, and tell me again why you're multiplying by three.

16.1.413 Kimberly: Because she said, how many ribbons can you make out of twenty-seven meters, out of, um, if you're making three thirds.

16.1.414 T/R 2: If you're making a third, ok, I understand that perfectly well, I think you're fine.

16.1.415 Kimberly: But I'm confused.

16.1.416 39:37 T/R 2: Are you still feeling confused about it? It's kind of a hard idea

16.1.417 Kimberly: Yeah

16.1.418 T/R 2: Yeah, isn't it? Yeah, I think that's what it is.

16.1.419 Kimberly: It's easy to learn it but it's hard to explain it.

16.1.420 T/R 2: It's hard, it really is hard but you know we always ask you to explain. ok.

16.1.421 Alan: [to beth] Carrying the two. Now what is your way of doing that? You multiplied the three times, what Kim did is she multiplied the three times the seven and then carried the two up there. Right, but

16.1.422 Beth: We carried because that's

16.1.423 Alan: Just show me how you did yours.

16.1.424 Beth: That's, I did the same thing as Kimberly.

16.1.425 Alan: I know, but

16.1.426 Beth: And it works because this two is ten and that two is ten and when you add them.

16.1.427 Alan: It's forty.

16.1.428 Beth: No, because you do three times two

16.1.429 Alan: Right, and that would be six

16.1.430 Beth: And then plus two more

16.1.431 Alan: Is eight

16.1.432 Beth: Right,

16.1.433 Alan: But wait, what you're doing is you're only multiplying that two and adding that twenty onto that, you're not multiplying that two. What you're doing is you're just adding that onto there. You're not multiplying that two.

16.1.434	Jessica:	You're not supposed to
16.1.435	Beth:	[shaking head] You're not supposed to, you're not.
16.1.436	Kimberly:	...you have, you learned that last year right? And you had Ms. Firestone right? Then I know why you're getting a different answer than him. He had [inaudible] and I had Warwick and you had Firestone. So maybe Warwick and Firestone taught the same thing, but [inaudible] didn't. So that's why Alan's confused and we know what we're doing.
16.1.437	Alan:	Well, I made up one of mine, and this is what I did. First you have your um twenty-seven, then you take off the seven and you only have twenty. So you multiply twenty by three and you get sixty. So then in the step two you only have your seven left so you multiply seven times three and you get twenty-one. You add sixty and twenty-one and you get eighty-one.
16.1.438	T/R 2:	It's a different way isn't it?
16.1.439	Jessica:	Very different.
16.1.440	T/R 2:	It seems to work. Have you tried it for any other numbers to see if it works?
16.1.441	Jessica:	We did, um, we did um, we can make out of nine meters we can make twenty-seven bows, out of twenty-seven meters you can make eighty-one bows, and then out of eighty-one meters we got two hundred and twenty-fourty, but now's it's even so now we think it's two hundred and fourty-three.
16.1.442	Kimberly:	Yeah, it is,
16.1.443	Jessica:	I got the same thing with the calculator. And we were doing it like this
16.1.444	Kimberly:	Yikes, yikes, yikes.
16.1.445	T/R 2:	Oh, ok.
16.1.446	Jessica:	So we must have made an error.
16.1.447	T/R 2:	What, can I ask you, I mean you probably have said this to Dr. Maher but I wasn't over here, why are you multiplying by three?
16.1.448	Jessica:	Well because she asked us
16.1.449	Alan:	The first problem that we had to do was if we had three meters,
16.1.450	T/R 2:	Does it have to do with that three meters of ribbon?
16.1.451	Alan:	Cause, you had to multiply it by the number of three, by the number of meters you had.
16.1.452	Jessica:	Because it's a pattern or something.
16.1.453	T/R 2:	Ok, it's a pattern, I'm real
16.1.454	Jessica:	It just seems to be working.
16.1.455 43:06	T/R 2:	I'm real confused though about why the three why, why multiply by three, why not multiply by two?
16.1.456	Alan:	Because the problem was to only have three meters.

16.1.457	Jessica:	Right.
16.1.458	Alan:	That wasn't the problem.
16.1.459	T/R 2:	Ok, what if I had
16.1.460	Alan:	Had it been two meters, this would have only been [inaudible]
16.1.461	T/R 2:	What if I had started with um six meters?
16.1.462	Alan:	Twenty-seven times six.
16.1.463	T/R 2:	Ok, and I wanted to make bows that were a third a meter
16.1.464	Alan:	And twenty-six.
16.1.465	T/R 2:	Six meters of ribbon
16.1.466	Alan:	Hold it
16.1.467	T/R 2:	Bows that were a third a meter in length each.
16.1.468	Alan:	Times three, nine, no. Ok, I got that too. I think it's
16.1.469	T/R 2:	Where did you get twenty-seven from?
16.1.470	Kimberly:	We just did, [inaudible] only we pulled out the three and put the six in, we just doubled the three. Right, all we did was we kept the twenty-seven but we just doubled the three.
16.1.471	T/R 2:	Ok, listen to this now, I want you to , I want you to start fresh, ok? I don't want you to think about any of the past stuff we've been working on today.
16.1.472	Alan:	Ok.
16.1.473	T/R 2:	Ok. New problem, the problem is I have seven meters of ribbon.
16.1.474	Alan:	Seven.
16.1.475	T/R 2:	Ok? and I want to make bows that are a third of a meter each. How many bows would I get?
16.1.476	Jessica:	You'd get twenty-one. Because seven times three is twenty-one.
16.1.477	Kimberly:	Right
16.1.478	T/R 2:	Ok, but you're multiplying by three again and we didn't start with three meters, so I don't understand. We started with seven meters.
16.1.479	Alan:	Right, so that would be seven times seven.
16.1.480	T/R 2:	So is that where the three is coming from? That's what I don't understand.
16.1.481	Alan:	And you'd get forty-nine.
16.1.482	Jessica:	No
16.1.483	T/R 2:	Ok, now you're saying something different here, ok, why?
16.1.484	Alan:	So you multiply the number of meters you got by the number, by the fraction you're making.
16.1.485	Kimberly:	The third is just like the three, it's like a regular number.
16.1.486	T/R 2:	Is it?
16.1.487	Kimberly:	It isn't, it's sort of used as a regular number but it's really a third.
16.1.488	Alan:	Yeah, what you do is you take the number of ribbon you have, and then make the fraction, the fraction like one third,

		the three, multiply the number of meters you have and then you get your answer of how many bows can be made out of them.
16.1.489	T/R 2:	Oh, you're using some sort of a rule here.
16.1.490	Alan:	Yeah cause say I had fifty, fifty meters, and I wanted a third of each of those meters. That would mean each meter gets three parts, so you multiply this by three, and I get a hundred and fifty, so that's how many bows you can get.
16.1.491	T/R 2:	So you're starting, the light is starting to go on for me, ok? I'm starting to see what you're doing.
16.1.492	Alan:	[interjects, inaudible]
16.1.493	T/R 2:	You'll have to say that again.
16.1.494	Alan:	Actually, the fraction that you have, the second digit in fraction is the number you multiply the number of meters that you have. That means if I had seven and I wanted to divide it into fourths, you go seven times four equals twenty-eight.
16.1.495	T/R 2:	So when you say the second number of the fraction, you mean the number on the bottom in the fraction?
16.1.496	Alan:	So the second number of the fraction, like it, one fourth, [uses calculator]
16.1.497	T/R 2:	Ok, I see, you have a slash line it's the second number.
16.1.498	Alan:	The second number on the right side of the slash. And then you multiply by the meters that you've got and then you get your answer of how many bows can be made out of em.
16.1.499	T/R 2:	Ok, you, are you all in agreement with that? That seems to work?
16.1.500	Others:	Yeah.
16.1.501	T/R 2:	Ok, I want you to think about something else then, ok? Let's go back to, [T/R 1 starts speaking], I guess we'll think about it later.
16.1.502	T/R 1:	Ok, I wonder if I could ask you to give me your attention for a moment. We have only a few minutes left I know you've been working very very hard, I know there have been some wonderful thinking and wonderful mathematics going on, I have some questions that may be. Ok, let's start with some things that I know we all know the answer to, you can answer it together if you all stop what you're doing for a moment we'll have more time to finish. First question, three meters of ribbon, how many bows one third of a meter in length can we make? Class.
16.1.503	Students:	Nine.
16.1.504	T/R 1:	Does anybody disagree? You're all absolutely convinced? How many of you are convinced? How many of you can prove it? How many of you know how to prove it? Ok, that looks like everybody, I think, Danielle, is your hand up?

		Your hand is not up. So Danielle, you don't know how to prove it?
16.1.505	Danielle:	Kind of.
16.1.506	T/R 1:	Kind of over here? Kind of. Sarah, how would you prove it?
16.1.507	Sarah:	Um, you go three-
16.1.508	T/R 1:	Nice and loud so they can hear you. We're listening to the proof, gentlemen.
16.1.509	Sarah:	You go three plus three plus three and that would equal nine. And
16.1.510	Jackie:	Or three times
16.1.511	Michael:	That's why because you have three meters and take… and you have three one thirds in each meter so three, three threes, and that equals nine.
16.1.512 49:05	T/R 1:	Jackie, Danielle, does that make any sense?
16.1.513	Jackie:	I think it's three meters times three meters equals nine meters.
16.1.514	T/R 1:	Danielle, do you agree or disagree?
16.1.515	Danielle:	Yeah, that's what I did.
16.1.516	T/R 1:	You think that's a good idea.
16.1.517	Michael:	Well, you can times it, but you can add it too.
16.1.518	T/R 1:	What confuses me is that you don't have three meters, you have a third of a meter, so you're telling me that you multiply by three. So how did you do this? What are some ways of doing this?
16.1.519	Michael:	Three times three.
16.1.520	T/R 1:	So you-, I'm asking you three meters of ribbon, and I'm making bows, I'm dividing it into one third meter length bows, and you're telling me that I can do that answer by multiplying it three times three and getting nine. How many of you did it that way? You said three divided by a third gave me three times three or nine? [some students raise hands] Some of you did it differently, some of you said three divided by a third is equal to three plus three plus three or nine? How many of you did it that way? A couple of you did it that way. How many of you did it the first way? Some of you raised your hands for one way, and only a couple- how many of you did it a different way then? How many people measured it out? How many of you took nine meters of ribbon and measured it out? [other hands raised] And how did you do it, to convince yourself, uh, yes? Erin?
16.1.521	Erin:	Uh, we took string and went out in the hallway and measured the nine meters out.

16.1.522	T/R 1:	So, you measured out nine meters, and how did you get umm, how did you measure out nine? You measured nine bows or nine meters?
16.1.523	Erin:	Nine meters.
16.1.524	T/R 1:	I'm confused, we started with three meters.
16.1.525	Erin:	Ok, um, I didn't have to um measure it out.
16.1.526	T/R 1:	You didn't have to measure that one, so that one you had the three meters, and what did you, what was the question you were asking, you didn't have to measure it, so how did you do it?
16.1.527	Erin:	Um, I did the first way, umm, three times three.
16.1.528	T/R 1:	How did you know to multiply it three times three?
16.1.529	Erin:	[laughs]
16.1.530	T/R 1:	Do you understand my question, how did you know to multiply three times three? Jackie?
16.1.531	Jacqueline:	Well, well, see, we had three meters so you put three down, and you're trying to divide it into thirds so you put another three down and then you times it and that would equal up to nine.
16.1.532	T/R 1:	Ok, so you're telling me that in the one meter, you have three thirds, is that what you're telling me?
16.1.533	Jacqueline:	Mmm hmm.
16.1.534	T/R 1:	How many of you did it that way, in one meter you have three thirds so in the nine meters you have a total of nine thirds - you have three one thirds, another three one thirds, and another three one thirds. You didn't do it that way.
16.1.535	Jacqueline:	No, I'm trying to think.
16.1.536	T/R 1:	Did anybody do it that way? I'm confused how you got your answer. I'm so confused. Andrew?
16.1.537	Andrew:	Well, me and James did three times three like that and we got the three and three because, um, you eventually have three meters and so one third, three, so you have three thirds of a meter so that's three thirds of a meter, so that's three times three meters equal nine meters, nine meters. Yeah.
16.1.538	T/R 1:	Ok, maybe, maybe… James? Do you agree with that?
16.1.539	James:	Yeah.
16.1.540	T/R 1:	Anybody else? Maybe we should move on to the next question. Now we have nine meters of ribbon and bows are a third of a meter. Is that when you measured it in the hall, Erin?
16.1.541	Erin:	Yeah
16.1.542	T/R 1:	So tell me what you did in the hall? You had nine meters of ribbon.
16.1.543	Erin:	Umm, and we measured it out, and um,
16.1.544	T/R 1:	So what did you do out in the hall we couldn't see you [Erin laughs] What were you doing out there?

16.1.545 53:27	Erin:	Well, um,
16.1.546	T/R 1:	So what's the question you measured out nine meters out there, and you're making bows, how long were the bows?
16.1.547	Erin:	One third.
16.1.548	T/R 1:	One third. Did you have one third meter string?
16.1.549	Erin:	Yeah.
16.1.550	T/R 1:	And how many of those one thirds?
16.1.551	Erin:	Twenty-seven
16.1.552	T/R 1:	There were twenty-seven of them. You measured it out, that's really neat. Anybody else measured it out like that? I saw some other people out in the hall measuring. In fact, we lost some people. Did you measure it out like that? What did you do, Mark?
16.1.553	Mark:	Well, we measured out um, yeah we measured twenty-seven meters.
16.1.554	T/R 1:	You ended up with twenty-seven of them?
16.1.555	Mark:	Yeah, we…
16.1.556	T/R 1:	Twenty-seven of those one thirds? And I know David and Erik you did something like that too.
16.1.557	Erik	We did, we did it with Erin, we did it with Erin
16.1.558	Graham:	We did it with twenty-seven meters
16.1.559	T/R 1:	Ok, so you said to me that nine divided by one third, right, when you measured it out you found out that that was twenty-seven, and some of you did it differently. Who did it differently, without measuring it? Those of you who did it without measuring it, Sarah, what did you do?
16.1.560	Sarah:	We timesed.
16.1.561	T/R 1:	You said nine divided by a third is the same as nine times three?
16.1.562	Sarah:	Yeah, and then,
16.1.563	T/R 1:	Or twenty-seven
16.1.564	Sarah:	Yeah and then we kept on timesing by three whatever the answer was.
16.1.565	T/R 1:	Ok, I know that time is running out but I have this other question I want to ask you. Um, when you have nine meters of ribbon, I think Erik and David did this, and now we're making our, our ribbons three meters in length, not one third of a meter in length. Do you understand my question? How many bows can you make?
16.1.566	Erik:	We're using nine meters, right?
16.1.567	T/R 1:	You have nine meters of ribbon and now your bows are three meters in length.
16.1.568	Erik:	Ok, you have nine meters of ribbon and your bows are three meters in length. If you have wait, yeah, if you have three meters all you have to do is multiply three times three and you get nine meters because you, if you have if each…

16.1.569	T/R 1:	Ok, so how many can you make?
16.1.570	Erik:	You can make three, three bows
16.1.571	T/R 1:	So you're saying if I have nine meters and I'm making them three meters in length we could make three bows.
16.1.572	Erik:	Yes.
16.1.573	T/R 1:	What do you think, class? David?
16.1.574	David:	I think the same thing, because, um, if each one takes up like a meter, um, nine divided by three, that, that would be three ribbons.
16.1.575	T/R 1:	Each one takes up three meters.
16.1.576	Erik:	Yeah, each one takes up three meters.
16.1.577	David:	Oh, yeah, wait a minute, um, it would be, it's like three times three would equal nine so uh nine divided by three equals three, um, [laughs] it's just because if you have three plus three plus three so you can if each one takes up three meters then you can make three bows out of nine. Because you have three meters and then, um, alright one bow would take up three so there'd be six meters left another bow would take up three so then there would be uh three meters left and then there'd be a third one and there wouldn't be, there wouldn’t be any ribbon left.
16.1.578	T/R 1:	Alright, I don't, I don’t know the way the rest of you think about that. Do you agree with that? If you have nine meters bow and the three meters in length, you could make three of them. I think we have to stop now. What I'd like you to do, many of you did different things, right? I would like you to write to us and tell us what you did and why you did it. I also would like, particularly, the table of Beth, Jessica, Laura, Kimberly and Alan to write up your, why your rule works. As best as you can explain why your rule works. Ok? So if you're using a particular rule of multiplying, if you can explain to me why that works, we're going to share that tomorrow, we're coming back tomorrow, and we can start sharing, so whatever you did to get your answers, I want you to write up a story to us to explain it to us. That's your assignment. What you did and why.
16.1.579	Erik:	So whatever answer you did? Whatever answers you did.
16.1.580	T/R 1:	And how. How you did it.
16.1.581 57:22		[End of Class]

Session 16, December 14, 1993, Front View

Line	Time	Speaker	Transcript
16.2.1	00:29	T/R 1:	Well, Hi
16.2.2		Students:	Hi.
16.2.3		T/R 1:	Um, let's see, what were we doing the last time?
16.2.4		Students:	We were making ribbons…
16.2.5		T/R 1:	Making ribbons? Yeah? What do you… Now, I know Beth wasn't here, she's, she's, I… I understand that umm she knows about the activities some people have shared, uhhh but uh, let's see what can we tell Beth about what we did last time? Any, any discoveries that we made in our project? Anything we remembered about making these ribbons that would be an important kind of thing to have noticed? Jessica?
16.2.6		Jessica:	Well, I noticed that after a while like it started making a pattern.
16.2.7		T/R 1:	Ok. You want to say a little bit more about that?
16.2.8		Jessica:	Well, um, I forget what pattern but I think it was going like it started going in three, six, nine, like… like when it said when you had like different size ribbons and every time it got like …like three times bigger and it kept doing it in all different kinds of patterns.
16.2.9		Michael:	Yeah, because at first it went two, three, four, five
16.2.10		Jessica:	And then it went…
16.2.11		Michael	and the second one went, uh, the second one went four, eight, something like four, six, yeah
16.2.12		T/R 1:	I don't remember any two, four, six or four, eight.
16.2.13		Michael:	No, it's four, it's four, six, eight, ten… and then there was that odd, and then there was that two thirds one.
16.2.14		T/R 1:	Ok, let's, let's, let's hold out… Brian what were you just saying?
16.2.15		Brian:	Well, if we, remember we had the three meters, you would always like times the number by three. Like you go three, six, nine?
16.2.16		T/R 1:	Yeah, yeah Michael's asking the question I had which number. Let's use that as an example. I have ribbons three meters long and I'm making bows how long? For example. Michael?
16.2.17		Michael:	Uh, one half
16.2.18		T/R 1:	One half a meter long, so if I have, I could sort of imagine ribbon three meters long, three of these sticks long, that's how long, and I'm making bows a third of a meter long, how can I imagine a third of a meter? How could I imagine one third of a meter? You could imagine a meter, right? You can see a meter? How can you imagine a third? Can

			you all in your heads imagine a third? How many of you can, imagine a third? So what are you imagining when you imagine a third? Not everyone is imagining it. Beth, what do you imagine?
16.2.19		Beth:	[hems and haws]
16.2.20		T/R 1:	Is it longer than this? No? Is it shorter than this? [students yeah] Is it shorter than this length?
16.2.21		Beth:	Uhh huh.
16.2.22		T/R 1:	Ok, so it's shorter than this length. About how short, much shorter is it than this length? What are you imagining? You're the only ones who can imagine how much shorter it is? I think more of you can imagine. Can you imagine a third of a meter? I have some half hands up. Jessica, what do you imagine?
16.2.23		Jessica:	Well, I imagine if you like pull the meter into like three pieces and then it would be like, like, up to the um I think um thirty-three mark, I think.
16.2.24		T/R 1:	Well how, how did you decide on the thirty-three mark?
16.2.25		Jessica:	Well that's what I think because um, um, thirty-three plus thirty-three plus thirty-three is ninety nine and that's,
16.2.26		Michael	No, but there's a hundred…
16.2.27		Jessica	Yeah, and then a hundred, around like thirty three and like a half almost.
16.2.28		T/R 1:	What do you think? Jackie, your hand up partially?
16.2.29		Jackie:	Something around.
16.2.30		T/R 1:	Something around that.
16.2.31		Alan:	I think there, it's thirty-three and one third because if you take two more thirds you can get it to a hundred.
16.2.32		T/R 1:	What do you think, Jessica? Thirty-three and a third?
16.2.33		Jessica:	Yeah.
16.2.34		T/R 1:	That what you're imagining, so this…
16.2.35		Michael:	I'm, I'm imagining it just being cut into three equal halves
16.2.36	4:53	T/R 1:	Equal parts. Three equal parts. How many of you imagined it cut into three equal parts? [many hands raised] Ok, and Jessica and Alan were a little more explicit they were trying to actually tell me the… how long those parts are, right? And uh, and so you're telling me in this meter stick, because there…you're telling me there are a hundred meters here? A hundred centimeters here? A hundred what here?
16.2.37		Students	A hundred centimeters…
16.2.38		T/R 1:	A hundred centimeters? How do you know that?
16.2.39		Alan:	Because it only goes up to ninety-nine but there's an extra length that could be a centimeter.
16.2.40		T/R 1:	This piece over here?
16.2.41		Alan:	Mmm hmmm.

16.2.42		T/R 1:	I see, the numbers go to ninety-nine but it goes up to here, you're telling me. So you're telling me there are a hundred centimeters here and you're telling me that if you were to make three equal parts, Graham, what do you think?
16.2.43		Graham:	Well, there's ten decimeters.
16.2.44		T/R 1:	Ten decimeters, well.
16.2.45		Graham:	Well, that's ten centimeters, and then there's ten decimeters.
16.2.46		T/R 1:	How do you get ten decimeters?
16.2.47		Graham:	Well there's, well there's ten centimeters in a decimeter and there's ten of them on that so it would go to a hundred.
16.2.48		Michael:	What? Ten centimeters, plus ten centimeters, plus ten centimeters is like [inaudible]
16.2.49		Graham	10 times
16.2.50		Michael	Oh.
16.2.51		T/R 1:	Ten times ten? Very interesting. Let's talk about that another time, what Graham is saying. Um, but, for now, you, you all can imagine a third? So what was the question that you posed to me? If we had three meter length ribbon is that what you said earlier, Brian? And we wanted to know how many ribbons one third of a meter long? And what did we decide? How many? We're going to hear Jessica's theory now.
16.2.52		Jessica:	Um
16.2.53		T/R 1:	We had three meters; I can imagine three of these, now I could imagine a ribbon a third of a meter, right? You helped me with that and in fact you were very precise about helping me with that, and how many bows can you make?
16.2.54		Jessica:	Um, I think you could have made, um, oh I forget, um,
16.2.55		T/R 1:	Why don't you all sit and talk to your partner for a minute and confer and see what you think.
16.2.56		:	[camera moves]
16.2.57	7:21	T/R 1:	Jackie, what are you thinking?
16.2.58		Jackie:	Three.
16.2.59		T/R 1:	You thought three; you can make three of them? [to class] We have three meters of ribbon, and we're making bows, we have three meters of ribbon to start with and our bows are to be one third of a meter in length. How many bows can I make from three meters of ribbon?
16.2.60		Amy:	There are nine! You have three meters, Jackie.
16.2.61		Jackie:	Three meters.
16.2.62		Amy	You have three meters.
16.2.63		Jackie	Yeah there's three meter sticks.
16.2.64		Amy:	And you have to put 'em into one…in third's. It could either be nine, or three. Cuz if you could, you could do it they could be a meter long, they could be a meter long, or

			they can be a third long, which is nine. Where are meter sticks? I'm getting a meter stick [gets up]
16.2.65		Erik:	And that equals ninety-nine because three point five is half of one, so three point five is, is a third of ten, a ten is one whole...
16.2.66		CT:	Ok, well we've got a meter stick right here, help us with this. We're trying to find out what one third of a meter is. So we can figure out how many one thirds go into 3 meters. So one would be right here, and where would the other one be? Right here, right. Alright, so this is a third of a meter? This is a third, between these two? This is a third between Amy and myself? Well how many in a meter then?
16.2.67	10:02	Danielle	3
16.2.68		CT:	Right, well then how many in three meters?
16.2.69		Erin:	9
16.2.70		CT:	How did you get that?
16.2.71		T/R 1:	Suppose I had nine meters of ribbon.
16.2.72		Michael:	Well, that would be nine times, no, no remember? It can't be nine times three, because that's, well, if we had nine of these there'd be three in this, well, well three is nine, so then there's twelve, then fifteen, then there's eighteen then there's twenty-one
16.2.73		T/R 1:	How about your way, Sarah?
16.2.74		Michael:	And twenty-four. And then-
16.2.75		T/R 1:	Sarah thinks twenty-seven.
16.2.76		Michael:	Twenty-seven.
16.2.77		T/R 1:	Do you agree?
16.2.78		Michael:	Yeah.
16.2.79		T/R 1:	You did it differently, so why don't you compare the way, the ways you did it? Ok, gentlemen, suppose you had nine meters of ribbon?
16.2.80		Andrew:	Nine meters?
16.2.81		Gregory:	Uh... Twenty-seven.
16.2.82		T/R 1:	Suppose you had twenty-seven meters of ribbon?
16.2.83		Michael:	Oh man!
16.2.84		Andrew:	Twenty-seven meters - oh!
16.2.85		T/R 1:	You can sit down.
16.2.86		Michael:	Oh, twenty-seven! That's twenty-seven times three.
16.2.87		Sarah	So you write…
16.2.88		Michael	twenty-seven times three.
16.2.89		T/R 1:	Ok Beth, suppose you had 9 meters of ribbon.
16.2.90		Michael	Oh. Ok.
16.2.91		T/R 2:	Use whatever you need to use. If you think you know your means, then use it.
16.2.92		Michael	Ok, twenty-seven, let's see so it's three…
16.2.93		Sarah	It's coming up nine

16.2.94	Michael	plus three… plus three…and that equals…
16.2.95	Sarah:	You're adding that up like it adds up to twenty-seven.
16.2.96	Michael:	Oh th- you want me to oh ok. [sighs] Three… plus three… plus three, ok, so that equals nine, now, now this one is three plus three plus three plus three plus three plus three, one, two, three, four, five. …
16.2.97	Graham:	This is a yardstick!
16.2.98	Michael:	No, this is a meter stick!
16.2.99	Student:	Can we use it?
16.2.100	Graham:	Can I use it?
16.2.101	Michael:	No, we're using it.
16.2.102		Can I see this side?
16.2.103	Michael:	See this is the centimeter side
16.2.104	Graham:	Decimeter.
16.2.105	Michael:	Decimeter.
16.2.106	Graham:	Decimeter.
16.2.107	Michael:	Three plus three plus three plus three plus three, that's five, six, seven, eight, nine, plus equals twenty-seven. Ok. I'm on eight… well you did yours nine times three, but I kept on adding three and three and three. Now, now we have twenty-seven times three. Ok. So it's seven, three times seven is twenty-one. One and two. Three times two is six
16.2.108	Sarah:	Six, seven eight.
16.2.109	Michael:	Eight
16.2.110	Both:	Eighty-one.
16.2.111	Michael:	[Sighs] No, no, no …
16.2.112	Sarah:	They're ahead of… Now what if we have to do eighty-one now times three?
16.2.113	Michael:	Oh man, weird! Pretty soon we're gonna be in the millions. I don't know if this is correct, let's see if this is correct, let's see, this is twenty-seven if we have nine, now ten is …
16.2.114	Sarah:	So we have eighty-one times three.
16.2.115	Michael:	No, but I don't know if twenty-seven times three…it does… twenty-seven times three is eighty one but I don't know if that's the correct answer.
16.2.116	Sarah:	Yeah!
16.2.118	T/R 2:	What are you doing? What are you two doing? I missed something I had to go get meter sticks.
16.2.119	Michael:	Ok, um, we're doing twent… how many… if we had a twenty-seven,
16.2.120	Sarah	Meters
16.2.121	Michael	twenty-seven meters of ribbon, um,
16.2.122	Sarah	This is what we had here
16.2.123	Michael	and we divide it into thirds, how, how many thirds can we get out of twenty-seven meters?
16.2.124	Sarah:	I have a different one from him because I timesed.

16.2.125	Michael:	She timesed, and I made sure with adding.
16.2.126	Both:	He added
16.2.127	T/R 2:	Ok. Ok so you were checking [talks about moving speaker] Over hear? Over hear? Like this? So we can here both of you I guess. Ok, alright, so how does that work? How does multiplying or adding work to do that to give you.
16.2.128 14:57	Sarah:	Because you have to add times three
16.2.129	Michael:	Because three plus three plus three plus three plus three plus three, like hers is nine times three and that means nine of these times three, that means that you're timesing three nine times.
16.2.130	Sarah:	He's doing it the harder way.
16.2.131	T/R 2:	I see
16.2.132	Michael:	It's just a shorter way of writing it. I mean you could go wheee
16.2.133	T/R 2:	Where did you get the…can I ask you a question? Where did you get the threes from? Why three?
16.2.134	Michael:	Because there's three thirds that make up a whole, or a meter.
16.2.135	T/R 2:	I'm sorry can you say that- Sarah, can you tell me?
16.2.136	Sarah:	For the one third that was on the umm board yesterday, and it said one third, one…how many bows can you make out of one
16.2.137	T/R 2:	Uh huh
16.2.138	Sarah:	One-third, so, you would, so you would times it by three.
16.2.139	T/R 2:	Ohh.
16.2.140	Michael:	See that's about, this is about one third of it (points to 1/3 of meter stick), so if you keep on going over, if you do, do this link three times then it's, then it's a, then it equals over, it equals… the whole thing.
16.2.141	T/R 2:	Ok, I understand, that's interesting.
16.2.142	T/R 1:	Ok, just for a time out for a minute while you're working on this, for those of you who are finished with that problem, I asked you, how many ribbons one third meter in length can you make from three meters of ribbon, right? And then I said suppose you had nine meters of ribbon, how many ribbons can you make one third meter in length and then I said suppose you had twenty-seven meters of ribbon, how many ribbons can you make one third meter in length? So those are the problems you're working on, I just want to be sure you all know what problems you're working on now.
16.2.143	Michael:	Ok, now we gotta do eighty-one times, times three equals, one times three equals three, [Sarah says something inaudible] one times three, let's see, nine is twenty-seven, so twenty-seven subtracted by three
16.2.144	Sarah:	Twenty-four.

16.2.145	Michael:	Is twenty-four. So it's two hundred and forty three.
16.2.146	Sarah:	Wow!
16.2.147	Michael:	Now we gotta do two hundred forty-three
16.2.148	Sarah:	Times three!
16.2.149	Michael:	Two hundred forty-three times three.
16.2.150	Sarah:	Oh boy!
16.2.151	Michael:	[laughs] Three times three is nine, no three times three, yeah, three times three is nine, three times four is
16.2.152	Sarah:	Twelve!
16.2.153	Michael:	Twelve, so we put down the two and carry the one, three times two is six, but take down the one
16.2.154	Sarah:	Seven hundred and twenty-nine!
16.2.155	Michael:	Seven hundred and twenty-nine.
16.2.156	Sarah:	Guys are you on the first one?
16.2.157	Danielle:	Yeah.
16.2.158	Jackie:	No, we're writing it down.
16.2.159	Sarah:	Whoa, do you know what we…
16.2.160	Michael:	We're already on seven hundred and twenty-nine
16.2.161	Sarah:	Seven hundred…we're on seven hundred and twenty-nine
16.2.162	Michael:	Times three.
16.2.163	Brian:	What?
16.2.164	Michael:	Seven hundred and twenty-nine times three
16.2.165	Andrew:	We're on two thousand one hundred and eighty-nine times three? [laughter]
16.2.166	Sarah:	Do you think he's just joking?
16.2.167	Michael:	I don't know. Three times nine is, you know that.
16.2.168	Sarah:	Three times nine is twenty…
16.2.169	Michael:	It's right down there. It's
16.2.170	Both:	Twenty-seven.
16.2.171	Michael:	Of course, so that's you put down the seven carry the two, three times two is six, that's eight, now three times seven is
16.2.172	Both:	Twenty-one.
16.2.173	Michael:	So it's two thousand one hundred and eighty-seven. [Sarah laughs] Two thousand one hundred and eighty-seven.
16.2.174	Sarah:	It's good when you're working with one number.
16.2.175	Michael:	[laughs] Times three. Is three times seven is twenty-one so put down the one, carry the two,
16.2.176	Sarah:	[Thinking to herself] twenty four, twenty-three, twenty-four
16.2.177	Michael:	three times eight is
16.2.178	Sarah:	Twenty-four
16.2.179	Michael:	Twenty-four
16.2.180	Sarah:	Plus two.
16.2.181	Michael:	Plus…
16.2.182	Sarah:	Plus two
16.2.183	Michael:	Oh, plus two is six, then we carry the… then we carry the another one,

16.2.184 Sarah: Five
16.2.185 Michael: Five, and then three times one is three five
16.2.186 Both: And three times two is six [Figure F-18-43]
16.2.187 Michael: six thousand five hundred and sixty one.
16.2.188 Sarah: He said he was on two thousand.
16.2.189 Michael: Ok, now let's look at this, I think he was on the problem that we just had before, two thousand one hundred and eighty-seven.
16.2.190 Sarah: Yeah.
16.2.191 Michael: Now we go six thousand five hundred and sixty one times three, three times one is three, three times six is
16.2.192 Sarah: Nine…nineteen thousand, six hundred and eighty-three.
16.2.193 Michael: What? [laugh] We're in the thousands.
16.2.194 T/R 1: Well, how much ribbon do you have?
16.2.195 Sarah: Whoa!
16.2.196 Michael: We have six thousand five hundred and sixty one yards, uh meters of ribbon.
16.2.197 T/R 1: Did you have a calculator to check your computation?
16.2.198 Erik: And now we're tying ourselves down trying to get nine meters of ribbon. We're tying ourselves down.
16.2.199 T/R 1: Literally, Erik, you're literally tying yourself down?
16.2.200 Michael: Oh, we're on six thousand five hundred and sixty one yar - meters of ribbon [laughs]
16.2.201 Erik: For what?
16.2.202 Michael For this.
16.2.203 20:00 Erik: Why are you doing that?
16.2.204 Michael: [laughs] Because I want to.
16.2.205 Erik: Ayeee!
16.2.206 Michael: [laughs] What's thre… you got…what did you get for this one? Six thousand, no you got nineteen thousand, nineteen thousand, nineteen thousand, six hundred and eighty-three, six hundred eighty three. Ok. Now the next problem, I'm already, I'm already have to go to the top again. Ok, the next problem is nineteen thousand six hundred and eighty-three times three.
16.2.207 Sarah: You didn't, you didn't do that?
16.2.208 Michael: No, I'm still on that. Three times three is nine, three times eight, oh what's three times eight, oh that's twenty-four. So this is four, put down, carry the two, three times six is, three times six is eighteen, eighteen ti- eighteen plus two is twenty, put down the zero carry the two, three times nine is twenty-seven, put down um and car- and that's twenty-nine, so I put down the nine and carry the two, three times one is three, plus the two is
16.2.209 Sarah: [laughs] I h- look what I have.
16.2.210 Michael: Five.

16.2.211	Sarah:	Look what I have now! Look at this.
16.2.212	Michael:	It's fifty nine thousand fi- fifty nine thousand, zero forty-nine.
16.2.213	Sarah:	Look what I got for the next one!
16.2.214	Michael:	Oh, man.
16.2.215	Sarah:	[To other group] This is what we're on.
16.2.216	Michael:	Fifty nine thousand, uhh four hundred and nine. What did you get for the answer - what did you get for this answer?
16.2.217	Sarah:	One thousand seven hundred and seven, one hundred and forty seven.
16.2.218	Michael:	So, so you got seven four one, then you got umm one, then you got, then you got one seventy seven, plus one seventy-seven, one forty seven. Ok, one seventy-seven times three ok let's get down to business here, three times seven of course is twenty-one.
16.2.219	Sarah:	Got it!
16.2.220	Michael:	Carry the two..
16.2.221	Sarah:	Whoa!
16.2.222	Michael:	uh oh, five, five thousand three hundred and one, no, five thousand three hund- no,
16.2.223	Both:	thirty one
16.2.224	Michael:	four hundred and forty one.
16.2.225	Sarah:	[to next group] You know guys, you can times it. Look! Look at that number! Jackie, what do you think of this number… but it will be crowded.
16.2.226	Michael:	Three times, three times four is
16.2.227	Sarah:	What?
16.2.228	Michael:	I'm just on this one.
16.2.229	Sarah:	What? What do you mean your on that one?
16.2.230	Michael:	I'm on the five hundred and whatever one
16.2.231	Sarah:	We're on this number.
16.2.232	Michael:	We're on five hundred thirty-one thousand four hundred and forty-one.
16.2.233	T/R 2:	My goodness, what are you doing here?
16.2.234	Michael:	We're timesing three by whatever number we get for the other one.
16.2.235	T/R 2:	Ok
16.2.236	Michael:	Because we figured that's what she asked us to do?
16.2.237	T/R 2:	Does that seem to work? Does that give you, what is that giving you, what are these numbers down here telling you?
16.2.238	Michael:	That, that, this is telling us how many threes, how many thirds we can get into this.
16.2.239	T/R 2:	Ok, ok, or how many bows you can make.
16.2.240	Michael:	Yeah.
16.2.241	T/R 2:	Is another way we can say it too, right? But you're right,
16.2.242	Sarah:	I'm in the millions! [laughs]

16.2.243	T/R 2:	Oh my goodness.
16.2.244	Michael:	Oh man, I'm not even doing it.
16.2.245	T/R 2:	I don't think I want to make all those bows [laughs]
16.2.246	Sarah:	I'll get a person who'll do it.
16.2.247	T/R 2:	[laughs]
16.2.248	Michael:	Five hundred and thirty one four, four
16.2.249	T/R 2:	This looks very interesting, can I ask you a question?
16.2.250	Michael:	Yes
16.2.251	T/R 2:	Ok, I don't want to interrupt what you're doing but I do have a question I'm curious about, ok?
16.2.252	Michael:	Mmm hmmm.
16.2.253	T/R 2:	If instead of we, instead of making bows that were a third of a meter, if we were making bows that were half a meter.
16.2.254	Sarah:	Times two.
16.2.255	T/R 2:	What would the pattern be?
16.2.256	Sarah:	Times one, a half
16.2.257	Michael:	No times two.
16.2.258	Sarah:	Yeah, times two.
16.2.259	T/R 2:	Ok, why?
16.2.260	Michael:	Because
16.2.261	Sarah:	It's half [gesturing with hands] half and half.
16.2.262	Michael:	the two you always take, you just add, you just add, umm like three times two is six, that means you could get um six out of three meters.
16.2.263	Sarah:	We're stopping there. We're stopping there.
16.2.264	Michael:	No, I'm not.
16.2.265	T/R 2:	Ok. Alright, would it be the same kind of a pattern if you were doing bows that were, um, let's say a fourth of a meter.
16.2.266	Michael:	[to himself] Times three, one million, one million five hundred ninety four thousand three hundred twenty three
16.2.267	Sarah:	Yeah, we think, well it won't be the same number, but we were doing like what were doing like…
16.2.268	T/R 2:	What would you do if it were, if it were a fourth of a meter?
16.2.269	Sarah:	I would go, um um, two hundred and forty-three times four.
16.2.270	Michael:	Ok! Times three [sighs and begins to sing] Merrily we play along, play along…
16.2.271	Sarah:	[T/R 2 nods and gets up] Are you still going? You're like the energizer bunny!
16.2.272	T/R 2:	Yeah, you guys are energetic, keep going.
16.2.273	Sarah:	You're in the millions!
16.2.274	Michael:	Yeah, I'm in the millions. Three times three is nine, three times two is six,
16.2.275	T/R 1:	I would be more comfortable if you checked your calculations with a calculator.
16.2.276	Michael:	Ok.
16.2.277	Sarah:	It's a lot of calculations to check!

16.2.278	Michael:	Where is everybody?
16.2.279	:	[camera moves to Caitlin's group]
16.2.280 25:49	Erin:	Whoooa! Ooh. We have eight. Cut this. Wait don't cut yet. [measures string] Seven, eight, nine, cut it.
16.2.281	Caitlin:	Let's go ask Mrs. Phillips if we can go take this out in the hall.
16.2.282	:	[camera moves out to where Gregory, Graham, and Mark are measuring ribbon]
16.2.283 28:37	CT:	What is your point in being out here? What are you trying to do?
16.2.284	Student:	I'm trying to get this…
16.2.285	Amy:	We're trying to measure it, because we can't measure it in the classroom.
16.2.286	Jackie:	Yeah, yeah it's hard…
16.2.287	Amy:	It's hard to measure
16.2.288	Student:	We measured it in the classroom.
16.2.289	Jackie:	We were having so much trouble; it took us like a half hour to [inaudible]
16.2.290	:	[Students discuss]
16.2.291 30:18	Caitlin:	Just tie it on to the door, should we tie it on to the door?
16.2.292	Mark:	It'll take up more room
16.2.293	CT:	What do you think? Are you, are you now measuring, or what are you doing?
16.2.294	Caitlin:	We're gonna measure it to the…
16.2.295	Mark:	That's nine meters?
16.2.296	CT:	Where's the meter stick? Don't run with the scissors, honey.
16.2.297		[Camera moves away and focuses on Graham's group]
16.2.298 31:43	CT:	Greg, where are you now? What's happening now?
16.2.299	Mark:	They're at that other end of the hall.
16.2.300	CT:	Yeah, but what's happening?
16.2.301	Mark:	[Speaks but inaudible.] Here wait…
16.2.302	Student	We know that this is two meters here, now we're going on three meters.
16.2.303	Student	This looks about three meters.
16.2.304	CT:	Why don't you work as a team with them instead of doing this?
16.2.305	Student:	Good idea!
16.2.306	CT:	So that you don't have to do that. Why don't you work as a team so you need another pair of hands?
16.2.307	Student:	Work as a team. We should work as a team!
16.2.308	CT:	They need another pair of hands and you two guys are out here.
16.2.309	Caitlin:	Why don't we just get tape and tape it down.
16.2.310	Student:	Five meters.
16.2.311	Caitlin:	Wouldn't it be better to just get tape and tape it down?
16.2.312	Student:	We're gonna help you guys.

16.2.313	CT:	Ok, watch there…
16.2.314	Student:	Watch our string.
16.2.315	Student:	Anyone need string?
16.2.316	Caitlin:	David, come here. Hold this down. Put this string down and then we're going to estimate someone's whole foot. Hold it down with your foot. Hold it down this way.
16.2.317	CT:	Alright, talk together so you know what you're doing and trying to accomplish.
16.2.318	Student:	Ok, what are we doing?
16.2.319	Student:	Do you think this is like a third?
16.2.320	Student:	This is nine meters.
16.2.321	Erik:	Half of nine meters would be four and a half…four and a… half is four and a half so we just have to… wait.
16.2.322	Student:	Here is four.
16.2.323	Erik:	Ok, I think two meters would be…no, yeah…yeah three meters, I think it's either two or three
16.2.324	Erin:	Three, it's three, it's three.
16.2.325	Erik:	Yeah, three six nine.
16.2.326	Erin	Three meters.
16.2.327	Erik:	Three six nine, three meters is one third, three six nine. It's 3! [dances in excitement]
16.2.328	Brian:	Three meters can't be one third.
16.2.329	Erin:	Yea it can.
16.2.330	Erik:	Yes it can.
16.2.331	Erin and Erik:	Three, six, nine.
16.2.332	Erin:	Three!
16.2.333	Erik:	Three six nine, thee six nine, three six nine, three six nine. [to CT] We figured it out what it would be. It would be three meters, to…three of those things would be one third, because all three of those meters is a third, three… because three six nine.
16.2.334	Student:	Ok, It's taped down. It's taped down so nobody…
16.2.335	Erik:	Three would be one, six, nine. Three six nine would be…. Because if you have three meters, three, and then another three meters, six, and then another three meters, nine.
16.2.336	CT:	What question are you answering?
16.2.337	Erik:	We're answering…we're answering the nine meters.
16.2.338	Erin:	The nine meters.
16.2.339	T/R 1:	If you have nine meters of ribbon, how many bows do you have, and what did you come out with?
16.2.340	Erin:	Um, three meters, right? Three meters would be one third.
16.2.341	T/R 1:	One third length bows, nine meters of ribbon, so how many bows can you make?
16.2.342	Erin:	Three.
16.2.343	T/R 2:	With nine meters of ribbon?
16.2.344	Erik:	With nine meters of ribbon, each bow is three meters long.

16.2.345	T/R 1:	Oh, each bow is three meters long! Oh, ok. So, that's alright, that's an interesting question.
16.2.346	Erik:	One third, then it'd be three meters.
16.2.347	Erin:	A third, three meters is…
16.2.348	T/R 1:	Oh, you're telling me if it's three meter length bows, you can make three of them? And if they're one third a, one third a meter length bows you could make…
16.2.349	Erin:	Three.
16.2.350 35:00	Erik:	You can make three, because one third is three meters.
16.2.351	T/R 1:	Now I'm confused. That's why you were confused.
16.2.352	Erik:	Ok see, if you have the nine, the whole nine meters.
16.2.353	T/R 1:	Ok.
16.2.354	Erik:	One third is three meters, because if you have three meters…
16.2.355	CT:	[to another student] Oh watch how your're holding those honey, hold them down.
16.2.356	T/R 1:	Can you show me a third of a meter? You have the nine meters, can you get me a piece of string that is a third of a meter? Roughly.
16.2.357	Erik:	You mean a third of a meter or just one third?
16.2.358	T/R 1:	Well is has to be one third of something. We're making ribbons one third of a meter. Are you asking a different question, Erik, you're asking one third of the nine meters are three meter length bows?
16.2.359	Erik:	I think the question you asked was nine meters and one third of nine meters is three.
16.2.360	T/R 1:	That's true, so if I asked you that question, that's a different question. Then I'm making my ribbons three meters long, right? Three meter long bows? And you said I can make three of them.
16.2.361	Erik:	Yeah.
16.2.362	T/R 1:	Ok, now if I'm making them one third of a meter long…
16.2.363	Erik:	One third of a meter.
16.2.364	T/R 1:	Not three meters, right. You see the difference?
16.2.365	Erik:	Not one third of nine meters
16.2.366	T/R 1:	One third of a meter
16.2.367	Erik:	One third of one meters would be three six nine twelve fourteen sixteen eighteen, twenty one… twenty four, twenty four meters
16.2.368	T/R 1:	Well I think it's twenty seven, you should check your arithmetic.
16.2.369	Erik:	Yea, yea twenty-seven.
16.2.370	T/R 1:	Ok, so if you're making a third of a meters long, you're make…I have nine meters, so you're saying twenty-seven? But if you're making them three meters long, one third of the nine meters, you have three. [Erik nods] Could you

		write that up? Ok, all of you? That's very nice. And write it up so you can explain it. Maybe on an overhead.
16.2.371		[Students head into class]
16.2.372	Erik:	Should we just write it down on paper or should we write it on the overhead?
16.2.373	T/R 1:	Um, you know what, we might not have enough time to share it. Thank you, here Erik.
16.2.374	Erik:	On the transparency?
16.2.375	T/R 1:	Yes and get your group to help you. [To Michael] Where's your group?
16.2.376	Michael:	They're writing on a transparency.
16.2.377	Sarah:	We're in the billions [they laugh] Well, my hands will get tired, my hands will get tired.
16.2.378	Michael:	Oh, man
16.2.379	T/R 1:	Did you check your work?
16.2.380	Both:	It doesn't go up that high
16.2.381	T/R 1:	Oh my.
16.2.382	Sarah:	It goes error
16.2.383	Michael:	There's errors.
16.2.384	T/R 1:	So what does that, what does that tell you? You need different calculators?
16.2.385	Michael:	No, see four thirty, thirty six it couldn't times that by 3.
16.2.386	T/R 1:	I see the problem.
16.2.387	Sarah:	I think we should stop, my hands are getting tired.
16.2.388	Michael:	No, I want to [continues talking about his calculations]
16.2.389	Sarah:	We're in the billions, your still on nine?
16.2.390	Brian:	Nine meters
16.2.391	Sarah:	You don't have to measure, you just have to whatever number, nine times three. We're timesing it, it just keeps getting higher and higher. You don't have to measure.
16.2.392	Michael:	Times six is eighteen. Bring down the eight carry the one. Three, four, bring down the one, so this is… We're in, I'm still in the, I'm in, I'm in the three billions.
16.2.393		[Sarah and Michael compare answers]
16.2.394	Sarah:	Thirty seven…
16.2.395	Michael:	Yea because you have to carry the one here.
16.2.396	Sarah:	Let me see. Wait, wait.
16.2.397	Jackie:	Sarah, did you get the strategy?
16.2.398	Sarah:	See, look you have too much in there. Look, one two three, you have to make (inaudible) [gets up and goes to Danielle, Jackie and Brian] You just times, nine times three, and whatever your answer you times by, you times by three
16.2.399	Jackie:	Three times three times three times three times three - like that?
16.2.400	Sarah:	Ok, nine. Nine times three, what is it?
16.2.401	Jackie:	Twenty seven

16.2.402	Sarah:	Twenty-seven. Twenty-seven times three?
16.2.403	Jackie:	Yeah but we have three times three is nine.
16.2.404	Sarah:	Don't do it like that, do it down. It doesn't matter. Times! Look up your times table.
16.2.405	Jackie:	But wait a minute! Wait! But what do we do with this?
16.2.406	Sarah:	It doesn't matter! You don't have to, you can scribble it out!
16.2.407	Jackie:	Ahhh! But now what do we do?? Do we do three or nine? Do we go nine times three?
16.2.408 40:57	Sarah:	Nine times three, twenty-seven. Twenty-seven times three. Whatever the blank is, times three.
16.2.409	Jackie:	Ok, nine times three… [goes back to Michael]
16.2.410	Michael:	Ok. Giving everybody our strategy.
16.2.411	Sarah:	I don't know where you are.
16.2.412	Michael:	I'm on [points]
16.2.413	:	[camera moves to Jackie, Brian, Danielle, and Amy]
16.2.414	Jackie:	Alright now, what's eight one times three? But but! Twenty-seven times three is eighty-one. Wait use a calculator.
16.2.415	Brian:	Wait what's eight times three.
16.2.416	T/R 1:	You look like you're all tangled in your work. Do you always get wrapped up in your work like this? I really believe you. I wouldn't want the job of unraveling you.
16.2.417	Jackie:	Focus!
16.2.418	Brian:	Nine... oh! Wait wait wait wait, three hundred twenty-nine
16.2.419	Jackie:	Twenty…I'm in the thousands. I'm higher than the thousands.
16.2.420	Amy:	Jackie look. Jackie. Jackie.
16.2.421		[Students speak, inaudible]
16.2.422	Brian:	What is, seven hundred twenty-nine times three? Time for a calculator.
16.2.423	Amy:	You can get one, there's some up there.
16.2.424	Brian:	[gets up to get a calculator]
16.2.425	Jackie:	I got a different answer than you!
16.2.426	Danielle:	Oh no.
16.2.427	Jackie:	Wait a minute.
16.2.428	Brian:	Ooh. Two one eight seven. [writes down answer] ..seven, times three. [calculates] Six five six one. Times three equals… whoa! One nine six eight three. One nine six eight three. [writes it down] One nine six eight three. One nine six eight three times three. [Figure F-45-29]
16.2.429	Jackie:	You don't have to do that. Now you just have to do times three times three times three. Like this.
16.2.430	Brian:	What?
16.2.431	Danielle:	See like you have the answer and then you just put times three and then you put then answer and then you do times three.

16.2.432	Jackie:	Nine times three, twenty seven. Three times three, twelve thirteen fourteen. Zero one. Zero times three. What's zero times three?
16.2.433	Danielle:	Zero
16.2.434	Jackie:	Zero. Nine times three, twenty seven….. times three equals. One seven seven one four seven.
16.2.435	Danielle:	Oh my. We're not using a calculator. So seven times three…twenty one. Four times three, eleven. No, [Figure F-47-56]
16.2.436	Jackie:	Four four, eight
		See other camera view for end of transcript

Session 17, Dec. 15, 1993 (Front)

Line	Time	Speaker	Transcript
17.2.1	06:20	T/R 1:	Ok, good morning! Um, well, um, Meredith, it's so good to see you back and Meredith is probably curious to know what we did and since you were able to write to us to tell us what you did we have some information, Michael, what did you want to say?
17.2.2		Michael:	Well, we've been working with bows and discussing like um well the length, the, how many bows we could fit into a certain amount of meters if the bow took up a certain amount of ribbon. And uh we were recently doing thirds and uh we we got we got like stuff like we came up with like you times nine times three or something like that to get answers.
17.2.3		T/R 1:	Anybody want to add to that? Thank you, Michael. Any other comments about that? Ok, I guess what I'm curious about is how much you could, um, predict about ribbons and bows, maybe without having the ribbons and bows in front of you, if you try to remember some of the things you did and as you try to explain to me your thinking, um, on some predictions, so I'd like you all to imagine, how many of you can imagine in your heads a meter? A ribbon that is a meter long? How many of you can imagine in your heads? If you can would you raise your hand [all students raise hands] How long that is? Is a meter longer than the width of this room?
17.2.4		Students:	No
17.2.5		T/R 1:	Is it shorter than the width of this room?
17.2.6		Students:	Yes.
17.2.7		T/R 1:	Is a meter about the length of this chalkboard
17.2.8		Students:	No
17.2.9		T/R 1:	Is it bigger.
17.2.10		Students:	No
17.2.11		T/R 1:	Is it smaller?
17.2.12		Students:	Yes
17.2.13		T/R 1:	Yes? Um, ok, that's very interesting, um, most of the hands were up but I guess, Brian yours wasn't up, no? You can't imagine the length of a meter? Can someone um help Brian without getting the meter stick and try to describe it? Caitlin?
17.2.14		Caitlin:	I think it's about this big.
17.2.15		T/R 1:	About that big. How many of you think that's reasonable? You want to hold your hand up, Caitlin so the rest of the class can see again? [Caitlin spreads arms] About that big? Is that reasonable? Does that help, Brian? Just keep your

hands like that let's test this. It's pretty good, actually [brings over meter stick] Not bad, huh? That's pretty good! Ok, Brian? So what's one kind of way that might help you remember? Let's see how Brian does. Is that going to be too long or too short do you think?

17.2.16		Student:	A little short.
17.2.17		T/R 1:	What do you think, Brian?
17.2.18		Erik:	Yeah, it's a little, it's going to be a little short.
17.2.19		T/R 1:	Hey, that's pretty good, isn't it? Brian got it exactly. So Brian, you can't imagine but that's a good way of doing it, so you could all imagine the length of a meter stick. And what do you do in your heads to imagine the length of two meters? What do you sort of do to imagine it in your heads to imagine the length? Graham?
17.2.20		Graham:	You double it?
17.2.21		T/R 1:	You double it. How many of you do that kind of in your heads? [hands raised] And if you wanted three meter sticks? I know that somebody figured out about the length of nine meters, right Mark, you were working on that. Weren't you also working on that Gregory out in the hall with your, yeah, and can you tell us, can you tell the class something that would give you an idea of the length of three meters-nine meters.
17.2.22		Mark:	Well, just uh if like eight meters plus one meter.
17.2.23		T/R 1:	But if you had to tell someone here is like part of the school about nine meters?
17.2.24		Mark:	Oh, I guess maybe probably as big as the chalkboard or a little bigger. Probably like from the chalkboard, from the end of the chalkboard to Danielle.
17.2.25		Erik:	No.
17.2.26		T/R 1:	Gregory? You measured out something in the hallway.
17.2.27		Gregory:	Oh, I know we measured the hallway.
17.2.28		T/R 1:	You measured the hallway and what did you find the length of the hallway to be?
17.2.29		Gregory:	Twenty-seven meters.
17.2.30		T/R 1:	Was that twenty-seven meters, ok. So you're thinking that that's about twenty-seven. What do you think, Danielle?
17.2.31		Danielle:	I think it's bigger than.
17.2.32	11:16	T/R 1:	You think it's bigger than the length of the wall in here. How many of you think it's bigger than the length of of this here? How many of you think that it's smaller?
17.2.33		Student:	Because this is eight meters.
17.2.34		T/R 1:	This is eight meters? How do you know that?
17.2.35		Students:	Because we measured it
17.2.36		Danielle:	When we were measuring the school.
17.2.37		T/R 1:	And you remember that?

17.2.38		Students:	Yes.
17.2.39		T/R 1:	Ok, so what would nine meters be then, about? What do you think?
17.2.40		Brian:	Well, it would be about from the chalkboard to there.
17.2.41		T/R 1:	From the chalkboard to whom?
17.2.42		Brian:	Graham.
17.2.43		T/R 1:	From the chalkboard to Graham? That would be about nine meters? Graham, you agree?
17.2.44		Graham:	No.
17.2.45		T/R 1:	Why not? You think it's too big, too small?
17.2.46		Graham:	Too small.
17.2.47		T/R 1:	You think it's too small.
17.2.48		T/R 1:	The classroom is only eight meters.
17.2.49		T/R 1:	The classroom which way? The width of the classroom?
17.2.50		Students:	Both
17.2.51		Erik:	It's eight by eight
17.2.52		Students:	It's eight by eight.
17.2.53		T/R 1:	It happens to be an eight by eight classroom?
17.2.54		Students:	Yeah.
17.2.55		T/R 1:	So that's interesting, I would have not thought that. That's very interesting. What do you think, Brian?
17.2.56		Brian:	Twenty-seven meter hall comes to this hall, coming down here, it's about from the door to um to the middle of to when you start going down there.
17.2.57		T/R 1:	David?
17.2.58	12:31	David:	I was going to say that um to the door, the outside door down to where you take a right[students talk simultaneously]
17.2.59		T/R 1:	Oh so all the way where the intersection is where you can make a right, Danielle?
17.2.60		Danielle:	If you go right to the bathroom then you will go up.
17.2.61		T/R 1:	If you go into the bathroom, ok that's very interesting, Michael?
17.2.62		Michael:	Also, I think about the width of it is like a little bit away from the wall and there um and all then to that wall.
17.2.63		T/R 1:	So if we started at that wall, how much out of the classroom would we have to go to get nine meters?
17.2.64		Students:	One meter.
17.2.65	13:29	T/R 1:	One meter. Right, you all agree? That's very, very good, so you can imagine these lengths which I think is wonderful that you can get, um, some idea. So the next thing to, can you imagine in your heads what a third of a meter is? How many of you can imagine in your heads about a third of a meter? I'm not going to ask you exactly but you have some idea what a third of a meter is, you know what we mean by a third of something, don't you? How many of you know

that? And you can imagine a third of a meter, right? That's real good, now I want to discuss with all of you, does anyone want to talk about that third of a meter? We did that yesterday in the beginning of class, Meredith wasn't here, but what we did, Meredith, remember some of you actually made string of a meter and you made your string into thirds and you had your third of a meter length, I remember that, right? Many of you did this and you did it last week. Um, I guess I wanted to discuss with you a problem that I believe um somebody was working on in the hallway, it might have been Erik, and um the problem had to do with, I remember I asked you how many bows can you make from ribbon that's a third of a meter in length if you have nine meters of ribbon, right? And I think everyone was able to solve that problem, right? You remember that everybody, raise your hands if you think you know the answer to that problem. And what did you get, Laura, what was that? How many, how many bows can you make if you start with nine meters and you have each bow to be a third of a meter in length? How many can you make? Talk to your partner for a minute, just for a minute, see if your partner agrees with you or not, Laura? Twenty-seven Laura says. How many of you think twenty-seven? And you can imagine there are twenty-seven of those one third meter lengths, right, in the nine meters of ribbon. So I'm imagining a ribbon along the floor going out one meter, and I'm imagining one third meter lengths and I'm counting them, right? How many of them will I count out if I count them?

17.2.66 Student: Twenty-seven.

17.2.67 T/R 1: Twenty-seven. But some of you didn't count them, some of you found an interesting way of getting your answer without counting, right, Alan? What was that?

17.2.68 Alan: It was multiplying nine times three and you get twenty-seven

17.2.69 T/R 1: Ok, but we wrote that problem as if we had nine meters divided by one third of a meter, you said that's the same as nine times three or twenty-seven. How many discovered that? How many of you discovered that yesterday? You you found the answer to be twenty-seven. I remember people did it several ways. Ok? Let's look at the different ways people thought about this. Some of you counted up how many one third meter lengths there were. Some of you did it by counting. And you, you took the one third meter and you counted them, how many of them when you counted them up?

17.2.70 Michael: You had, well you had three.

17.2.71		T/R 1:	Three for each meter, so for nine meters you had.
17.2.72		Michael:	Um, you had nine
17.2.73		T/R 1:	You had the three nine times
17.2.74		Michael:	Yeah.
17.2.75		T/R 1:	Is that right? Some of you did it that way. Does that make sense? So some of you had the three, like one third one third one third and you had it nine times. Well that's what we wrote here, didn't we? Nine times three, right? The three nine times? Ok? So that was a way to do it. Some of you did it, said well, what was another way? We have counting, we have this way, nine times three equals twenty-seven, what was the third way that someone did this problem? Audra.
17.2.76		Audra:	[inaudible]
17.2.77		T/R 1:	Yeah, how did you do that.
17.2.78		Audra:	You put the number down three times and you counted
17.2.79		T/R 1:	Yeah, but, but what you did is you said there are three times and there are three times again and there are three times again, right? How many times did you have three?
17.2.80		Audra:	Um,
17.2.81		T/R 1:	One two three four five six seven eight nine, is that right?
17.2.82		Audra:	Three.
17.2.83		T/R 1:	That added up to
17.2.84		Audra	We did the three three times and we added it up to nine.
17.2.85	18:20	T/R 1:	Ok so you added this up to nine, you added this up to nine, you added this up to nine, and all those nines gave you twenty-seven? That was another way, is that right? That some people did that problem? [Figure O-20-28] Ok? Did anyone do it any other way? Now, I'm wonder- I don't really, um, all these are wonderful ways, right? They're all great ways of doing it. Whatever way you're thinking about it, the important thing is, can you do a different problem and work it out with these ways? So I want you to imagine a different problem, this is the one I think Erik gave us. I still have my nine meters of ribbon, can you imagine that? My nine meters? But now I'm making ribbons that are three meters long. Remember that, Erik?
17.2.86		Erik:	Yes.
17.2.87		T/R 1:	Are my ribbons going to be bigger ribbons or smaller ribbons? Raise your hands if you think you know? I have nine meters of ribbon but not my ribbons are three meters long. Will my ribbons end up being bigger ribbons or smaller ribbons? That's what I want you to think about. Talk to your partner for a minute. Ok, let's see how many of you think it's going to be smaller? How many of you think the ribbons are going to be bigger? How many of you aren't

sure? There are some people who didn't raise their hands either way. We'll go one more time. How many of you think it's going to be smaller? How many of you think the ribbons are going to be bigger? How many of you aren't sure? Ok. So, we now have to help the people who aren't sure - no one thinks smaller

17.2.88		Erik:	Come to think of it, you could have, you could have three bows, well, I mean you could have each bow that's three meters long, and you can have one bow that you have to twist and everything using three meters of ribbon making it small and just like twisting it every which way.
17.2.89		T/R 1:	That's an interesting idea, ok, maybe that's I hadn't thought about it that way, Erik, but let's suppose we're making identical kinds of bows, we're making, can we talk about a standard bow in here? All the bows are going to look like this blue one. Like the blue one, so they're going to be all like the red one. So the question is, if I make my bows to be like this, would my one third meter bows that I'm making for my nine meters, be smaller or bigger than my three meter bows that I'm making from my nine meter ribbon?
17.2.90		Michael:	The one third meter bow would probably be smaller, unless you had two different kinds of ribbon.
17.2.91		T/R 1:	But we're not, though, we're keeping them the same.
17.2.92	21:22	Michael:	But if you did, then you could make this the three meters really complicated bows so it is going to be very little and that one basically
17.2.93		T/R 1:	Which is what Erik and David are saying, now some people aren't convinced of that, so what can we do to convince them that we are going to have bows that are bigger? Brian, you're not convinced, right? Ok. I'm imagining my nine meters on the floor. Are you imagining that? Ok. Now I'm going to cut that ribbon to make bows, right? But I'm going to cut it how long? How long will I cut it? Starting with nine meters, now I'm making three meter bows, right? So what's happening? How can we explain what's happening? What do you think, Brian?
17.2.94		Brian:	It would be in thirds.
17.2.95		T/R 1:	Ok, so can you, can you sort of tell me where one of the first cuts would be if I rolled this ribbon on the floor and it went out the room.
17.2.96		Brian:	Um, on three meters.
17.2.97		T/R 1:	On three meters, how many of you agree that the first cut would be on three meters? Do you imagine that? Is there anyone who can't imagine that? So my first cut's going to be

		on three meters. Where's my second cut going to be? Audra?
17.2.98	Audra:	Um, on six meters.
17.2.99	T/R 1:	On six meters. Is there a third cut?
17.2.100	Students:	No
17.2.101	T/R 1:	It's all done, right? Ok, so how many cuts will there be? How many cuts? Kimberly?
17.2.102	Kimberly:	Two.
17.2.103	T/R 1:	There are going to be two cuts, right? How many pieces of ribbon will there be? Erin?
17.2.104	Erin:	Three.
17.2.105	T/R 1:	Three. Ok, so how many bows will I make?
17.2.106	Students:	Three
17.2.107	T/R 1:	I'll make three bows. You know that. Nine divided by three is three, that's one way, the other way is if I do it by cutting, right? One cut, my second cut, my second cut gives me three cuts each three meters long, right? To give me my nine meters, is that right? [Figure O-26-08] Ok, that's very good. How many of you understand that? Ok, let's give you some more. Could you imagine twelve meters of ribbon? How many of you can imagine? Sort of? Is it going to be more than nine meters?
17.2.108	Students:	Yes.
17.2.109	T/R 1:	Right? It's going to go past that hallway, you think?
17.2.110	Students:	Mmm hmm
17.2.111	T/R 1:	Yeah, who's class is that on the corner?
17.2.112	CT:	Mrs. Warwick
17.2.113	T/R 1:	It's going into Mrs. Warwick's room, isn't it? If we're rolling out that ribbon, ok? So you could imagine twelve meters. Now I'm making my bows a half a meter in length.
17.2.114	Student:	A half meter!
17.2.115	T/R 1:	A half a meter in length. So now I'm starting with twelve meters of ribbon, one half meter bow, right? How many bows am I going to make? Talk to your partners. Ok, ok, if you've solved that one, you can also solve the one if you're making them two meters long, while you're waiting for the other people. I want to know how many you can make that are half meter bows, and how many you can make that are two meters. Ok, if you think you have a way of getting those answers, if you've found a sort of secret about how to do it, I'd like to know what your secret is. [Figure O-28-33]
17.2.116	Erin:	Twenty-four
17.2.117	Jackie:	No, we're doing one half. I think, see, two halfs make a whole, well, my fingers are halfs, so two, this is one whole, this is another whole, this is another whole, this is another whole, and this is another whole.

17.2.118	Erin:	But each one's going to have two, each whole would have.
17.2.119	Jackie:	Yeah, yeah, but two halves. But there would be six wholes.
17.2.120	Erin:	I'm not agreeing really.
17.2.121	T/R 2:	You're not agreeing really. Which one are you working on?
17.2.122	Erin:	Um, the one half meter one.
17.2.123	T/R 2:	The half meters? Ok, well, I'd like to hear both arguments, what do you think?
17.2.124	Erin:	I say that you can make twenty-four bows.
17.2.125	T/R 2:	Ok, Erin. Why?
17.2.126	Erin:	Because, if you take twelve times two you get twenty-four.
17.2.127	T/R 2:	Ok, and why did you decide to multiply?
17.2.128	Erin:	Because
17.2.129	T/R 2:	How does that work?
17.2.130	Erin:	[laughs] each meter's gonna have two bows in it and there's twelve meters they're gonna have you're gonna double the twelve so
17.2.131	T/R 2:	Ok
17.2.132	Erin:	You get twelve two times
17.2.133	T/R 2:	So you're saying then, um, you're multiplying the twelve meters by two because each one of the twelve meters is going to make two bows that are half a meter. Sounds interesting, ok, Jackie, what do you think?
17.2.134	Jackie:	Well, I think
17.2.135	T/R 2:	For the half meter bows?
17.2.136 29:12	Jackie:	[Figure F-29-08] All these lines are halfs. So, um, if you group this, this would be one whole, this would be one, two, this would be another, and it would be six, because all these are whole are one.
17.2.137	T/R 2:	Ok, so each of those lines is a meter of ribbon?
17.2.138	Jackie:	One half.
17.2.139	T/R 2:	Is half a meter of ribbon.
17.2.140	Jackie:	But you have to make twelve, twelve
17.2.141	T/R 2:	Ok, does that total up to what does your picture total up to twelve meters of ribbon? I guess that's what I'm asking.
17.2.142	Jackie:	Oh, there's twelve halfs though.
17.2.143	T/R 2:	Ok, then, so what would you do to fix that? Can I ask you another question? Let's stop thinking about the half meter one for a minute. Let's think about the two meter one, ok? If the bows were two meters, in length, ok? You have twelve meters of ribbon, this big long piece we're going to make these bows that are two meters, then how many bows do you think there are going to be? That may help us to decide
17.2.144	Erin:	Two meters?
17.2.145	T/R 2:	Uh hum! Try to picture it
17.2.146	Erin:	[laughs] I say six.
17.2.147	T/R 2:	Ok, why?

17.2.148	Erin:	Because, um, there's gonna be less bows cuz each is two meters, each bow is gonna be two meters
17.2.149	T/R 2:	They're big bows, you're saying,
17.2.150	Erin:	Yeah
17.2.151	T/R 2:	So you're going to make less bows, ok, why six? Why not, you know, five or four? Why six specifically?
17.2.152	Erin:	Because half of twelve is going to be six and if you are counting up to twelve go [counts on fingers] two four six eight twelve that's six.
17.2.153	T/R 2:	Ok, I see, so you're counting by twos you're figuring each is a two meter chunk
17.2.154	Erin:	Yeah.
17.2.155	T/R 2:	What do you think, Jackie? What are you working on? It looks interesting. [Erin laughs] Jackie's drawing us a picture, using a diagram there.
17.2.156	Jackie:	[Figure F-31-44] I think, I'm not sure of this, um, but this would be, this is two meters, and two meters would be one
17.2.157	T/R 2:	Would be one, in other words, would make one bow, is that what you mean?
17.2.158	Jackie:	Yeah, I think.
17.2.159	T/R 2:	Ok, look at all those twos, though. Explain to me why you put all those twos there. What made you decide to put that many and you know not more or less.
17.2.160	Jackie:	Well um, because each line would be two meters.
17.2.161	T/R 2:	Ok, and you get twelve meters of ribbon to start with, so how many of those twos should you have there?
17.2.162	Jackie:	Twelve.
17.2.163	T/R 2:	To get to twelve? Why don't you start from scratch? Why don't you draw yourself a picture of the twelve meters, something that you know, something that makes you think of the twelve meters, ok? That might help.
17.2.164	Jackie:	Alright, this is it. [draws straight line and writes twelve meters - Figure F-32-59]
17.2.165	T/R 2:	Ok, there's your twelve meters. Ok, now what are you going to do in order to make the bows that are two meters long, they have two meters worth of ribbon in them? What would you do if you actually had the ribbon in front of you and you were going to cut it? [Jackie draws six vertical lines on the horizontal one] Ok, now before you keep going, ok, show me where those lengths of ribbon are, where the two meter lengths of ribbon are, maybe you can mark it for me. [Jackie draws a line until the first vertical mark - Figure F-33-34] Ok, is that one?
17.2.166	Jackie:	Yeah.
17.2.167	T/R 2:	It's one two meter length? Ok. [As Jackie extends the second line] Another. Three four five six. Ok, is there any

		more or did we get to twelve? [Jackie keeps going - Figure F-33-58] Can you,
17.2.168	Jackie:	[points to the lengths that she has marked with the tick marks and counts silently] Ten
17.2.169	T/R 2:	Can you mark for me now, maybe make a number two to show me where the two meters are? For each two meter strip? Now, remember, we have a total of twelve meters. So how many of those are you going to mark to get to twelve meters?
17.2.170	Jackie:	Six.
17.2.171	T/R 2:	Ok, why don't you mark those? Ok, so two meter strips and you're going to have how many bows? [Figure F-34-35]
17.2.172	Jackie:	Six.
17.2.173	T/R 2:	Ok, which is I think is what Erin said, earlier on. You agree with her now?
17.2.174	Jackie:	Yeah.
17.2.175	T/R 2:	Ok. Now, I want you to go back and think about those half meter ones now, ok? Ok? I'll let you do that.
17.2.176 34:51	Jackie:	Ok, this will be twelve meters, just put another line. And my paper is a mess. Alright, this is um twelve meters again, alright this is twelve meters, now [Erin laughs] alright, [makes vertical lines, crosses it off] It's a mess. Alright, alright, see how alright over here, now we have to break this up into halfs.
17.2.177	Erin:	Yeah.
17.2.178 37:14	Jackie:	Each meter goes into halfs now. Alright, these are halfs. Alright, these are halfs. Now we have to do, we have to go up to, we have to count till twelve, we have to go up to twelve. So this would be one, this is, this would be one meter, cuz two halfs make a whole [draws another horizontal line to span two vertical ones - Figure F-36-45] This would be another meter. This will be another one, this will be another, Alright, now we have to go up to six. Alright, this, no we have to get up to twelve, one two three four five six seven eight [Figure F-37-38].
17.2.179	Erin:	Uh oh.
17.2.180	Jackie:	I'm going to get another paper. Ok. [writes heading] Ok. [Jackie draws a line with vertical tick marks - Figure F-39-37] Alright, here it is. Now we have to get up to twelve. So this would be one, this would be one, this would be one, this would be one, this would be one, this will be one, this will be one, one, one, one. And we have an extra. Now count one two three four fives six seven eight nine ten eleven twelve. We got twelve. [Figure F-40-04]
17.2.181	Erin:	[Erin makes twelve squares and then divides each section in half - Figure F-40-12] Now I have twelve squares here and

		I split them into all halves. Now we have to count each half. Two four six eight ten twelve fourteen sixteen eighteen twenty twenty-two twenty-four. [laughs] There's twenty-four halves.
17.2.182	Jackie:	Are we counting up the halfs or the meters?
17.2.183	Erin:	The halves. First you just draw, first I just drew a …
17.2.184	Jackie:	Ok, we could put down the answer, there's two answers cuz I'm not sure. Ok, one half, count the halfs there'd be twenty-four. Ok, what was my answer?
17.2.185	Erin:	Ok, now I split those all in half it goes, One two three four five six seven eight nine ten eleven twelve thirteen fourteen fifteen sixteen seventeen eighteen nineteen twenty twenty-one twenty-two twenty-three twenty-four. [Erin has made circles and split them in half - Figure F-41-17]
17.2.186	T/R 1:	When you've done will you write up how you've done it on the overhead?
17.2.187	:	[camera shifts to another group but other voices are not heard]
17.2.188 43:30	Erin:	Is there another pen?
17.2.189	Jackie:	This one I'm not sure, we'll just say we have two ways. Just say we're not sure, I think it's both.
17.2.190	Erin:	Ok, so we'll write both.
17.2.191	Jackie:	Ok, just write picture this as twelve meters.
17.2.192	Erin:	Just write twelve meters. [writing] Here is
17.2.193	Jackie:	Put picture this as twelve meters.
17.2.194	Erin:	Alright.
17.2.195 45:05	Jackie:	Alright, now you do your way.
17.2.196 45:35	T/R 2:	How are you doing? Worked out all the problems?
17.2.197	Jackie:	Uh, no, Dr. Maher told us to do the first problem.
17.2.198 45:40	T/R 2:	Great, okay let me let you do that then, I'll come back and talk to you when you're done with that.
17.2.199	Jackie:	Alright, you'll do it your way, I'll do it my way.
17.2.200	Erin:	Alright this is problem one, alright.
17.2.201	Jackie:	Just make it on the line like I did it before.
17.2.202	Erin:	Let's measure it. That's twenty-one.
17.2.203	Jackie:	Those are centimeters. Inches.
17.2.204	Erin:	That's twelve so I have to go down for the last one, two three four five six seven eight nine ten. So close. So close.
17.2.205	Jackie:	Just do a little under too. In the middle, just make it right there.
17.2.206	Erin:	Wait, I have a better idea, wait I have a better idea. First of all, we need [talk about doing it again, camera focuses on Caitlin her group working with the meter stick]
17.2.207	Brian:	You can't let it out.
17.2.208	Jackie:	Make it inches.

17.2.209	Erin:	Ok, there, there [a horizontal line with twelve vertical lines above it]
17.2.210 50:11	T/R 1:	Ok, I'm going to ask you to, we're going to need to stop for a moment, I'd like you all to just stop I know you're all in the middle of this and maybe Mrs. Phillips will let you um continue a little bit of this, it's up to her, but I would like us to do some sharing, because there are some ways people have been thinking about it. Ok, I'm interested in your sharing the way you're thinking about it, and maybe we'll have another couple of minutes to finish up. Ok, um, Alan and Kimberly wanted to share with you the way they were thinking about the twelve meters divided by bows of two thirds of a meter, right? We had twelve meters of ribbon and we're making the bows two thirds - can you kind of look to see because I'm going to ask you to write about what they did, I want to ask you to write about what they did and I want to make sure that what they did makes sense to you or doesn't make sense to you, because if it doesn't make sense their job is to uh either convince you or you convince them, so can you all give us your attention here for a minute? Ok, Kimberly and Alan, tell us what you did.
17.2.211 51:24	Alan:	[Figure O-53-09] This entire thing is twelve meters. The long line is the divider of each meter [inaudible] The brackets are dividing the thirds up so there are two thirds, there are two thirds, there are two thirds, there are two thirds, and if you count up how many two thirds there are, you'll eventually get down to eighteen, and that's how many bows you can make of two thirds out of twelve meters.
17.2.212	T/R 1:	Questions?
17.2.213	Andrew:	Well, me and James did uh the same thing that did the twelve and we got eighteen too.
17.2.214	T/R 1:	You did it the same way. Any other questions, comments?
17.2.215 52:32	Erik:	How are we going to be able to write what they did? I mean, if we write that cause we're going to have to diagram it, there's no way we're going to be able to write it.
17.2.216	Alan:	Should I explain it again.
17.2.217	Erik:	No, I know what you mean, but we'd have to diagram it to write it, we couldn't write it in words, we'd have to diagram it.
17.2.218	T/R 1:	Ok, now first of all, I heard somebody say they'd like to hear a second explanation. How many of you would like another explanation? Ok, now in your explanation, my suggestion is, go through each part, be sure people understand each part, Kimberly, and don't move onto the next part until each little part they understand. Fair enough? Ok, so one more time please?

17.2.219	Alan:	Ok, this, this is all twelve meters, the line, this line is what divides each meter up, in each meter there are three thirds. The bracket has two thirds under it, which means those are the two thirds to make your bow and if here are two thirds, here are two thirds, here are two thirds, here are two thirds, and you keep going on to the end until you get up to eighteen, and that's how many bows you can make out of two thirds each meter, um, of twelve meters of ribbon.
17.2.220	T/R 1:	Question?
17.2.221	Beth:	I agree with that, because in my book, we had the [inaudible] book, for two thirds, uh, I did the same thing like that, and that's how I got my answer.
17.2.222	T/R 1:	Ok, other comments? How many of you, um, understood this explanation? Raise your hand if you understood the explanation? How many of you would like the explanation broken down again? Raise your hand if you'd like it again. What I'd like you to do Alan is each part say how many of you know where I got the twelve? How many of you know where I got the one? Ok, break it down. Why don't you give it a try, Kimberly? Ok, go very slowly, Kimberly.
17.2.223 54:56	Kimberly:	This one here, all together is twelve meters. And these here, the long lines separate between 'em. And there are three meters in each meter and the brackets separate two meters and two thirds in each meter.
17.2.224	T/R 1:	I think I heard you say there are three meters in each meter. I don't think you meant to say that.
17.2.225	Kimberly:	I know, I didn't
17.2.226	T/R 1:	What did you mean then?
17.2.227	Kimberly:	I meant three thirds
17.2.228	T/R 1:	Three thirds of a meter in each meter
17.2.229	Kimberly:	And then the brackets separate two thirds in each meter
17.2.230	T/R 1:	How many big lines are there? How many big lines are there? Many of you said there are twelve meters, and the big meter marks off each meter. How many big lines would there be to mark off each meter? What do you think, Laura?
17.2.231	Laura:	Twelve.
17.2.232	T/R 1:	Laura thinks twelve. Someone think something else? Andrew? What do you think?
17.2.233	Andrew:	um, Eleven?
17.2.234	T/R 1:	Andrew thinks eleven.
17.2.235	Andrew:	Wait no ten.
17.2.236	T/R 1:	Andrew thinks ten. Brian.
17.2.237	Brian:	Thirteen.
17.2.238	T/R 1:	Brian thinks thirteen. James.
17.2.239	James:	I think eleven.
17.2.240	T/R 1:	James thinks eleven. David.

17.2.241	David:	Well, I think, I think ten.
17.2.242	T/R 1:	You think ten, Erik?
17.2.243	Erik:	I think eleven.
17.2.244 56:02	T/R 1:	Well, we just really aren't agreeing. Well, how can we find out? Let's actually count them. Can we point it out, let's count them we'll check it with Alan as we're doing it here. So here's the first one, let's count together, [students join] one two three four five six seven eight nine ten eleven. Would you have to cut the last one or is it cut for you already?
17.2.245	Student:	No
17.2.246	T/R 1:	Is the last cut, is the last piece of ribbon or is it cut for you already.
17.2.247	Andrew:	There is one more.
17.2.248	Kimberly:	Alan made a mistake.
17.2.249	T/R 1:	Ok, but if you're cutting this ribbon, how many cuts do you make?
17.2.250	Students:	Eleven.
17.2.251	T/R 1:	You make eleven cuts and how many pieces do you get when you cut it?
17.2.252	Students:	Twelve.
17.2.253	T/R 1:	Twelve, twelve meter lengths. Ok, do you all understand how we get twelve of those one-meter length? How many of you understand that? With the eleven cuts? But there are twelve marks, that's right what Laura said, if you count, if you're looking at marks. Now, what did they do after that? After they marked off these one meter lengths, what did they do next?
17.2.254	Student:	They put the two thirds in.
17.2.255	T/R 1:	Well, before they put the two thirds in, what did they do before they put the two thirds in, Meredith?
17.2.256	Meredith:	Brackets.
17.2.257	T/R 1:	Well, they did something before then, I think. Before they marked two thirds, what did they mark first, Andrew?
17.2.258	Andrew:	Well, the, um, the thirds.
17.2.259	T/R 1:	They marked the thirds first. Is that what you did, Alan and Kimberly? [mmm hmm] The marked the thirds first. Why do you suppose they marked the thirds first? Why do you think they did it that way? What would your guess be? What were they after? Meredith?
17.2.260	Meredith:	Well, so they could know where to put the brackets.
17.2.261	T/R 1:	So they know where to put the brackets. And what did the brackets show in this problem? Did the brackets show one third?
17.2.262	Student:	No
17.2.263	T/R 1:	What did the brackets show? Graham?

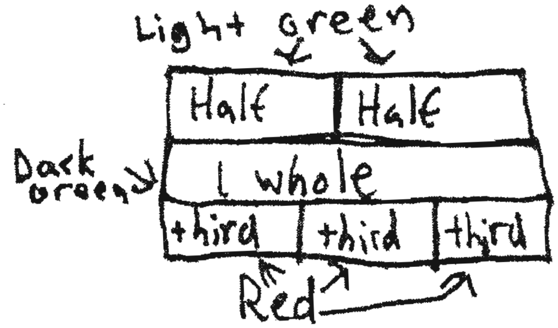
17.2.264	Graham:	Two thirds.
17.2.265	T/R 1:	Two thirds. Can you see that? They had to mark one third, and each meter they marked a third, you see how they did that? And then they marked two thirds, right? And then they put brackets. Now, what did they do after they marked all those two thirds off with the brackets? What did they do after that? Andrew?
17.2.266	Andrew:	They numbered them.
17.2.267	T/R 1:	They numbered them. Why do you think they numbered them? Why do you suppose they used the strategy of numbering them? That was kind of clever of them to number them, at first they didn't number them, and later on they came with those numbers. Why did you start numbering them, Kimberly?
17.2.268	Kimberly:	So that we can find out the answer, because we lost count a few times before we put the numbers.
17.2.269	T/R 1:	Oh, yeah, you lost count, I remember one time you said seventeen and sixteen and
17.2.270	Kimberly:	Yeah.
17.2.271	T/R 1:	A couple of you also lost count didn't you I noticed as I walked around. I noticed what you were doing the same thing, but some of you lost count. So the numbering was a very good strategy. And what did, how many numbers did they end up having when they counted two of the thirds?
17.2.272	Student:	Eighteen.
17.2.273	T/R 1:	Eighteen. How many of you are convinced that eighteen is the answer? How many of you think that you can write up or try to write what they did?
17.2.274	Erik:	I think I can diagram it, I don't think I can write about it.
17.2.275	Michael:	We did basically what they did
17.2.276	T/R 1:	Now, before you go I have just one question to ask you, it's Alan's birthday so we're not letting you off so easily. [inaudible] It's twelve divided by two thirds, now before when I asked you how many one third ribbon lengths when there were twelve meters of ribbon, what did you tell me the answer was? Everybody? If there are one third meter lengths, how many?
17.2.277	Student:	Seven.
17.2.278	T/R 1:	Twelve meters of ribbon, one third meter each.
17.2.279	Student :	Oh.
17.2.280	Graham:	Twenty-four
17.2.281	T/R 1:	No one third.
17.2.282	Graham:	Oh thirty-six.
17.2.283	T/R 1:	Thirty-six. One half meter, Graham, was twenty-four. You said thirty six, and some of you found a secret for finding it, what was that secret? Andrew?

17.2.284 59:56	Andrew:	Um, I said, I um multiplied either um one third two third six or two to twelve and I got the answer.
17.2.285	T/R 1:	Ok, so you said there was a rule like this you found the secret, twelve divided by one third, you found that by multiplying twelve times three and getting thirty-six. My question to you is does that secret work here? Twelve multiplied by three and a two. That's a big question mark. Maybe when we come back we'll think about that secret. I think our time is up.
17.2.286	CT:	Um, Dr. Maher, would you like them to explain how many two third meters you can make out of twelve meters?
17.2.287	T/R 1:	You can get out of twelve meters. And you can draw a picture, Erik, if you can try words, I'd like that, you can draw a sketch, any way you want to do it. Thank you.
17.2.288 1:00:56		End of class [focus on Jackie finishing her transparency]

Appendix C

Students' Written Work

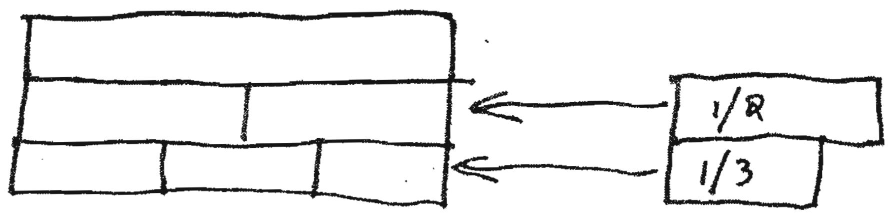
Erik 4-Ph 9-27-93



I think 1/2 is larger than 1/3 because if you have a whole and cut into halfs you have two equal parts, third you have three. If you cut something in half it is parts and two may be a larger number but in fractions 1/2 is larger than 1/3. it takes a larger piece to be a half because it is two and three you could make smaller. and thirds are three parts.

Alan 4Ph Sept. 27, 199

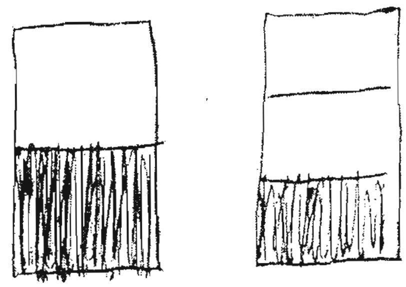
Which is bigger, 1/3 or 1/2



1/2 because the smaller the number that you divide the rod, the bigger the pieces will be.

Meredith

Today we had many challenging questions. Here is one of them.

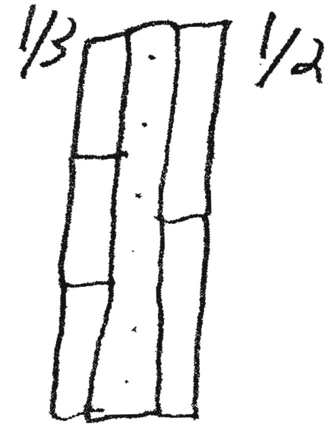


What is bigger 1/2 or 1/3? 1/2

I think 1/2 is bigger than 1/3 because if you take two rods that are the same size and you split one into halves and one into thirds and you put one 1/3 piece on top of one 1/2 piece, the 1/2 piece is bigger.

Michael Ph.4 9/27/93

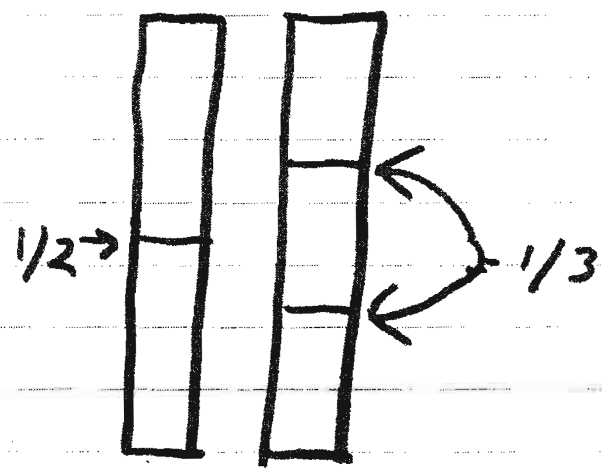
I picked the dark green rod ~~be~~ because we are doing the candy bar problem and dark green was the only one that had a third and half. We figured that 1/2 was bigger than 1/3 because 2 is less than 3 so it would only be 2 parts of a candy bar instead of three.



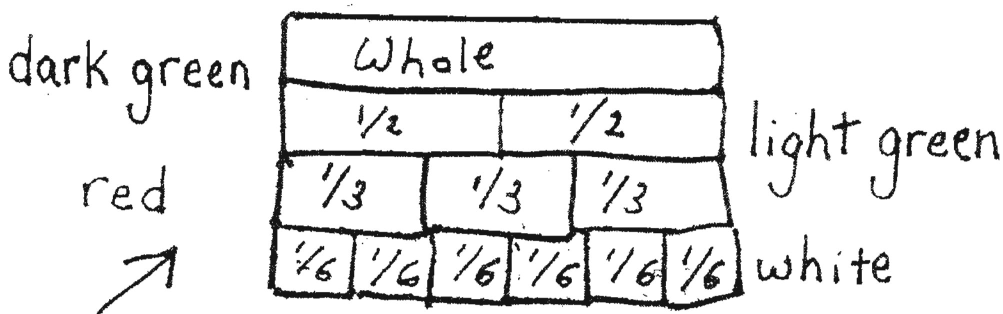
Which is larger 1/2 or 1/3?
Why? Andrew

P.H.

1/2 is larger then 1/3 because it takes two halfs to make a hole and three 3rds to make a hole.



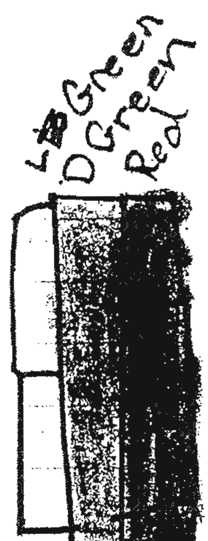
Sarah 9/27/93 Ph. 4



Which is larger 1/2 or 1/3? Why 1/2 is larger

Amy Ph. 4. 9/27/93

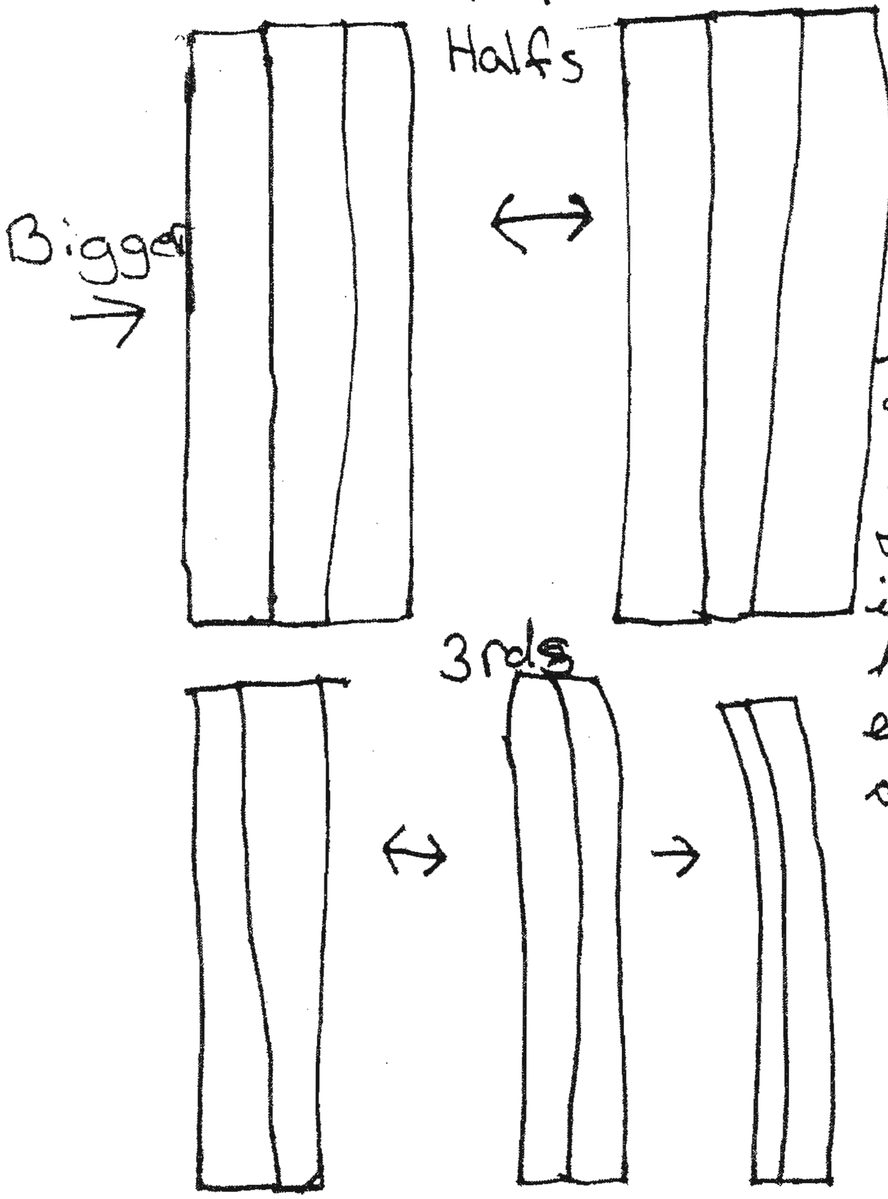
I used the dark green rod for a candy bar. Then I used light green for a half. Then I used the red rod for the thirds.



Wich is Larger?

Audra
4 Ph.

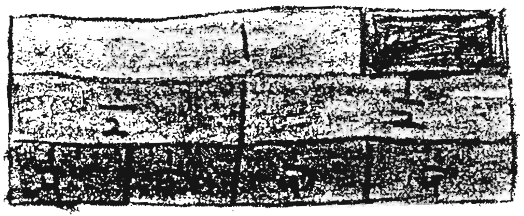
Sept 27, 1993



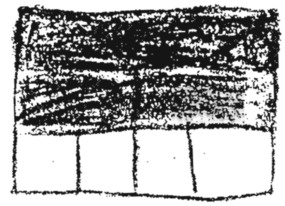
If you have a ~~cochoclate~~ chocolate bar with 6 peices in it, and you divided it in half, you'd ~~had~~ have 3 peices in each half. If you divided it in 3rds you'd have 2 peices in each third. So one half is bigger.

Brian Fractions

Brian



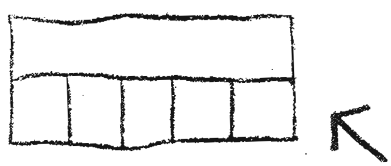
Dave

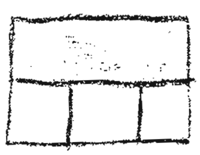


1 $\frac{1}{2}$ $\frac{1}{4}$

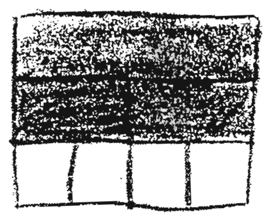
① I agree with Dave because Dave's model has 1 whole, $\frac{2}{2}$, $\frac{4}{4}$ like it should be.

② Yes. It is possible to get a different answer with different models. For example:





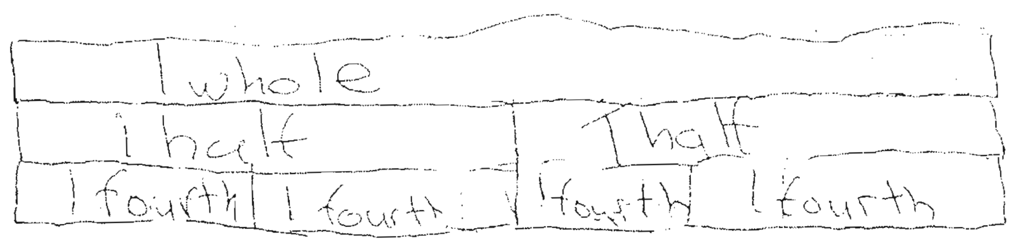
Odd # cannot be divided in $\frac{1}{2}$.

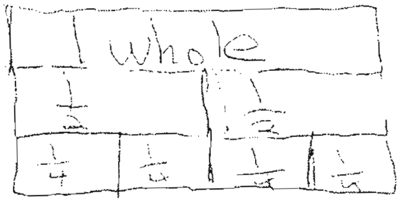


Even # can be divded in $\frac{1}{2}$.

Erik

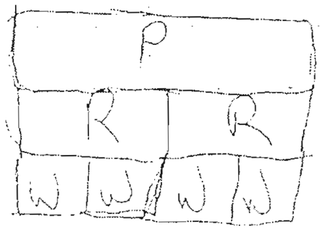
I compard my model with David's model. My model was two orange rods. I figured out that two yellow rods equal one orgone and four yellow rods equal 2 organge rods. David's model was one purple rod and the red rods as the halfs and the white rods as the fourths. I dissagree that you can get different answers, because no matter what rods you use you will usally get the same answer





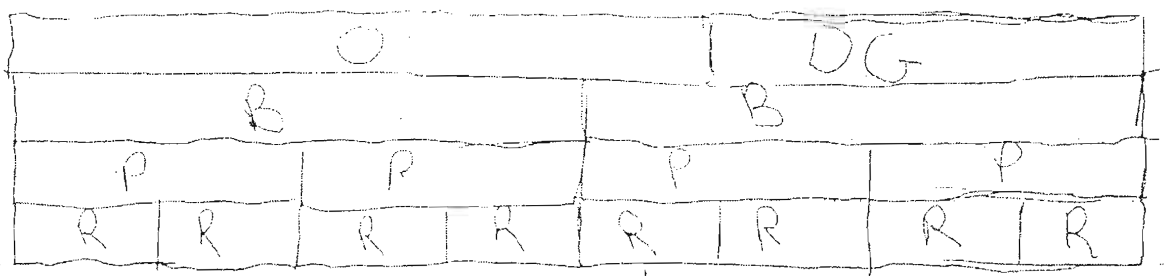
Audra

Mine



agree

Other's

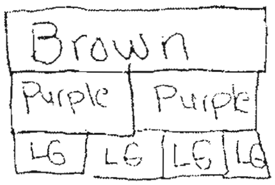


8-Biger

It is possable to get different answers.

Jessica

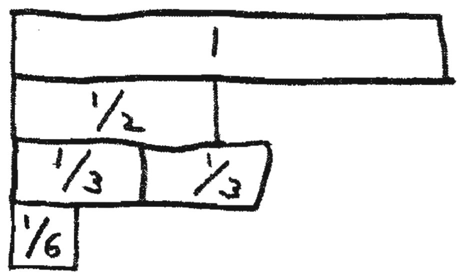
In Math we had to figure out all different problems. One example of a problem goes like this......



Just for know you have to pretend that the blocks are even.
The brown would be a whole the halve would be purple and the fourth would be light green.
It is not possible to get different answers with different models.

Brian Fractions

1. Which is more $\frac{1}{2}$ or $\frac{1}{3}$?

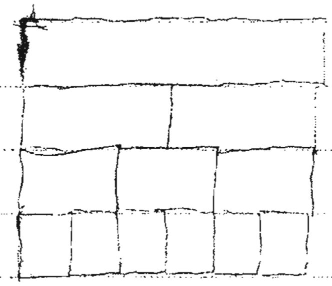


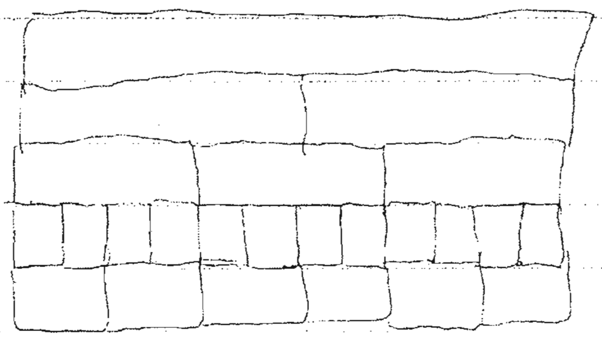
1. $\frac{2}{3}$ are larger than $\frac{1}{2}$ because it takes $\frac{3}{6}$ to equal $\frac{1}{2}$ but it needs $\frac{4}{6}$ to equal $\frac{2}{3}$ and so it makes $\frac{1}{6}$ extra.

2. What is the same about your models? They all have fractions.

3. What is different about your models? They have different sizes.

Meredith

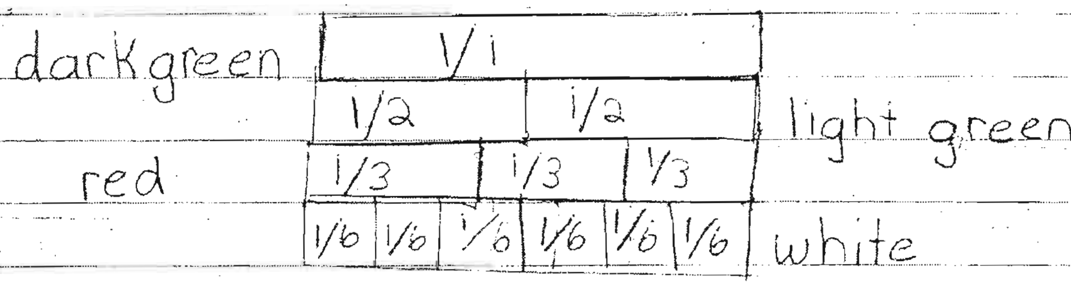




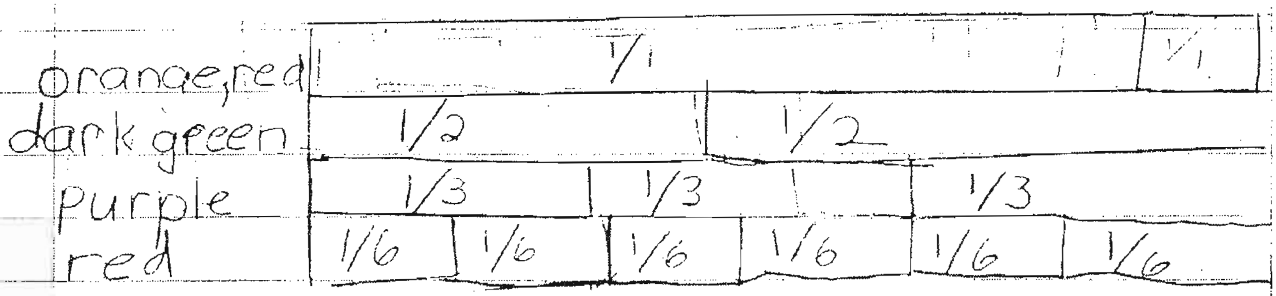
$\frac{2}{3}$ is bigger than $\frac{1}{2}$ by $\frac{1}{6}$ or $\frac{2}{12}$. I know this because if you take a $\frac{1}{2}$ piece and $\frac{2}{3}$ piece of my second model and you put the $\frac{1}{2}$ piece next to the $\frac{2}{3}$ pieces the $\frac{2}{3}$ pieces are bigger. Then you take a $\frac{1}{6}$ piece or a $\frac{2}{12}$ piece and you put either of the pieces next to the $\frac{1}{2}$ piece. It is equal to $\frac{2}{3}$ rds.

Beth

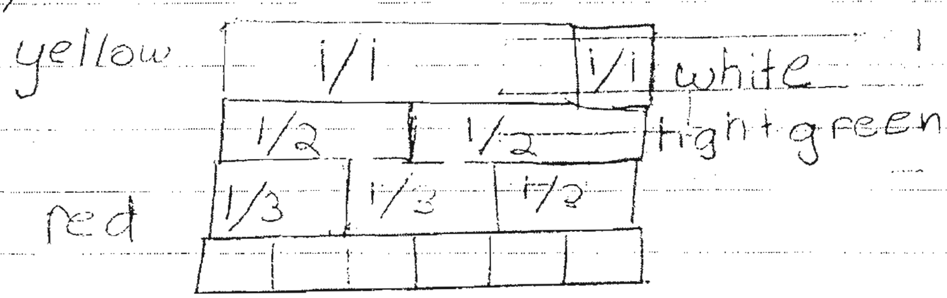
Is 2/3 larger then 1/2, if so by how much?



2/3 is bigger in this picture. By 1/6



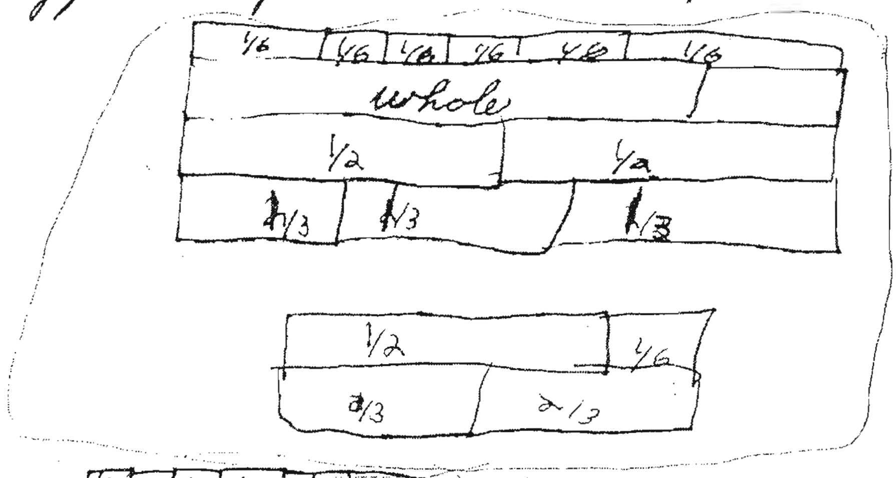
2/3 is bigger in this picture. By 1/6.

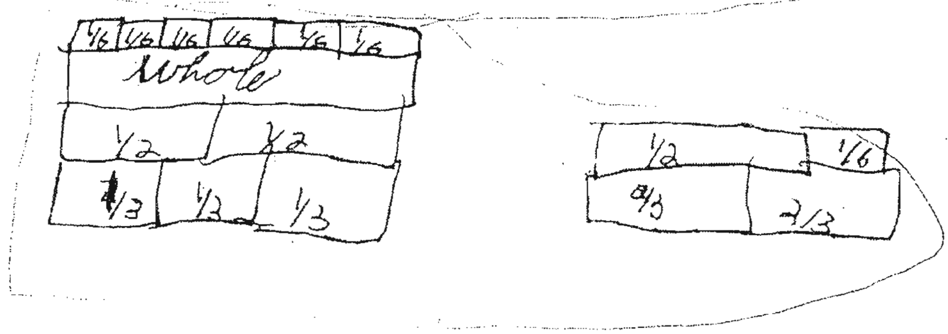


2/3 is bigger in this picture. By 1/6.

Michael   Fractions

I think that 2/3 is bigger than 1/2 by 1/6. It takes 6/6 to equal a whole and 1/6 is always half of 1/3. It takes three 1/6 to equal 1/2, but you need 4 1/6 to ~~eqal~~ equal 2/3. That proves that 2/3 is bigger also 2/12 is just like saying 1/6. There is nothing different with my ~~Model~~ models except the size of my whole, 1/2, 1/3, and 1/6. The things they have in comon is 2/3 is bigger by 1/6 in both.





Erik

I think $\frac{1}{2}$ is samler than $\frac{2}{3}$ because if you divide somethin into thirds there is no half-way point but in halfs there is a half-way point. If there is no half-way point it is either larger or smaller. If you were working with $\frac{1}{3}$ it would be smaller but we are working with $\frac{2}{3}$ so $\frac{2}{3}$ would be larger by one sixth.

Diagram #1

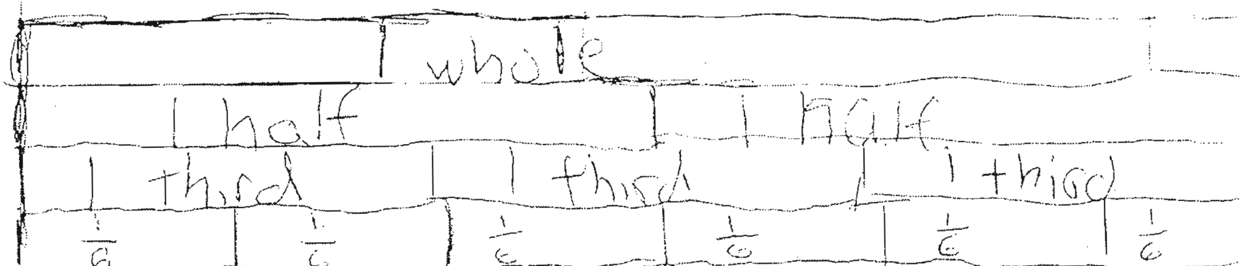
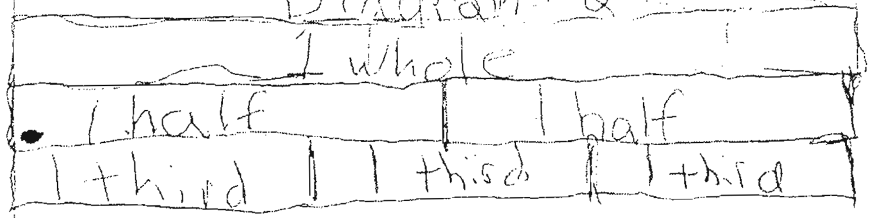


Diagram #2



Brian Fractions

(1) Which is larger 3/4 or 1/2
by how muck?

B&R	Whole			
O	1/2		1/2	
Y	1/4	1/4	1/4	1/4

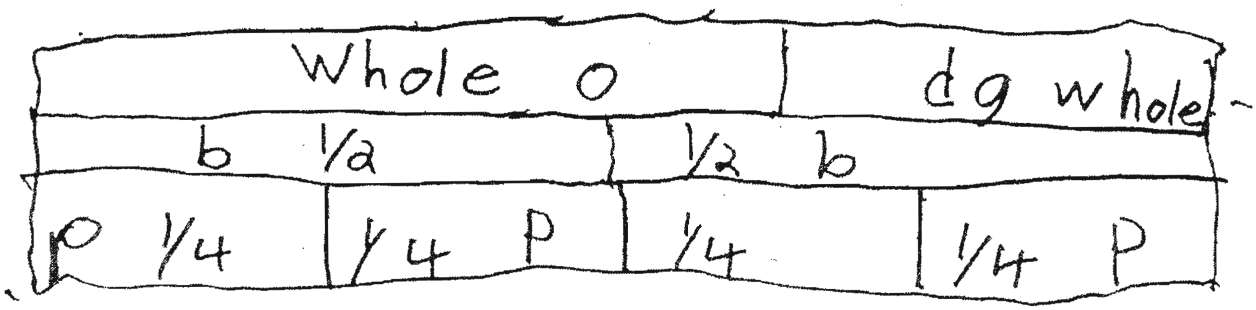
Whole				P
1/2		1/2		R
1/4	1/4	1/4	1/4	W

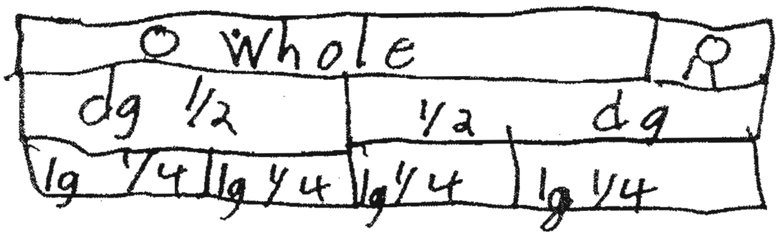
(2) 3/4 is larger than 1/42
by 1/4 because, it takes
2/4 to equal 1/2 but the
question is 3/4 and so
there is 1/4 left.

Michael Fractions

2Q Which is larger 3/4 or 1/2?
By how much?

A 3/4 is bigger than 1/2 by 1/4, because it takes 2/4 to equal a half and but the question is 3/4 so thier is 1/4 left.





Laura

Which is larger 3/4 or 1/2?

1 whole - Brown			
1/2 - Purple		1/2 - Purple	
1/4 - red	1/4 - red	1/4 - red	1/4 - red

This model shows that 3/4 is larger than 1/2 by 1/4

1 whole - orange + Red			
1/2 - dark green		1/2 - dark green	
1/4 - green	1/4 - green	1/4 - green	1/4 - green

This model shows the same as the first model. It is bigger by 1/4.

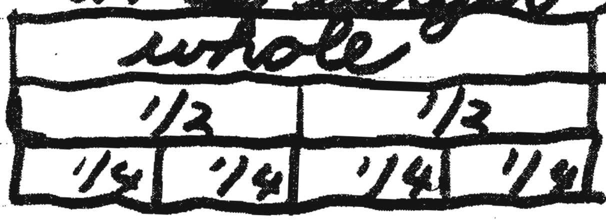
1 whole - Purple			
1/2 - red		1/2 - red	
1/4	1/4	1/4	1/4

All the models show the same thing. They all show that 3/4 is bigger than 1/2 by 1/4. It doesn't matter if the models are different sizes you'll always get the same answer.

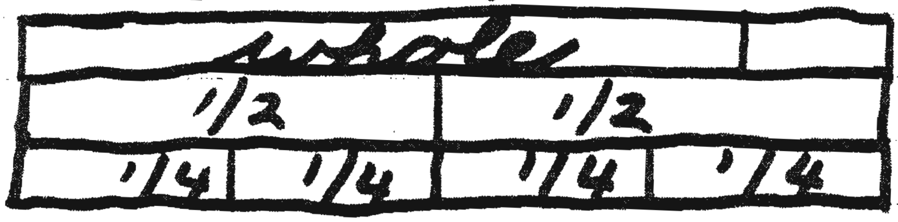
Andrew 10/6/93

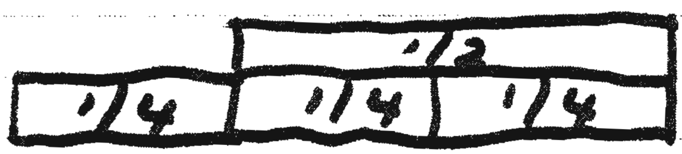
Sheet #1

① My first ~~model~~ modle looks like this. I think 3/4 is larger then 1/2 by 1/4



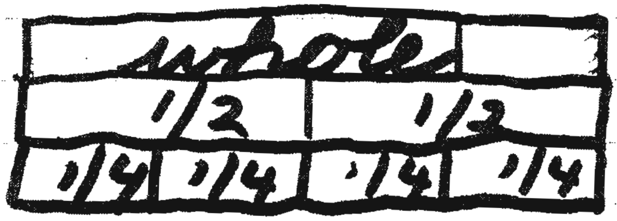
② The second modle is almost the same only biger.

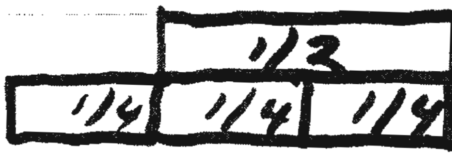




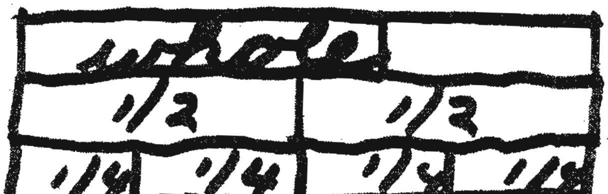
Andrew 10/6/93

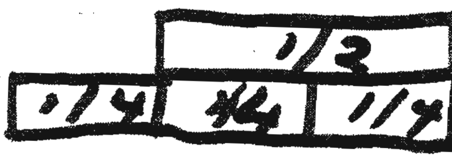
③ Sheet #2
The 3rd modles has the same thing of the 1st and 2nd.





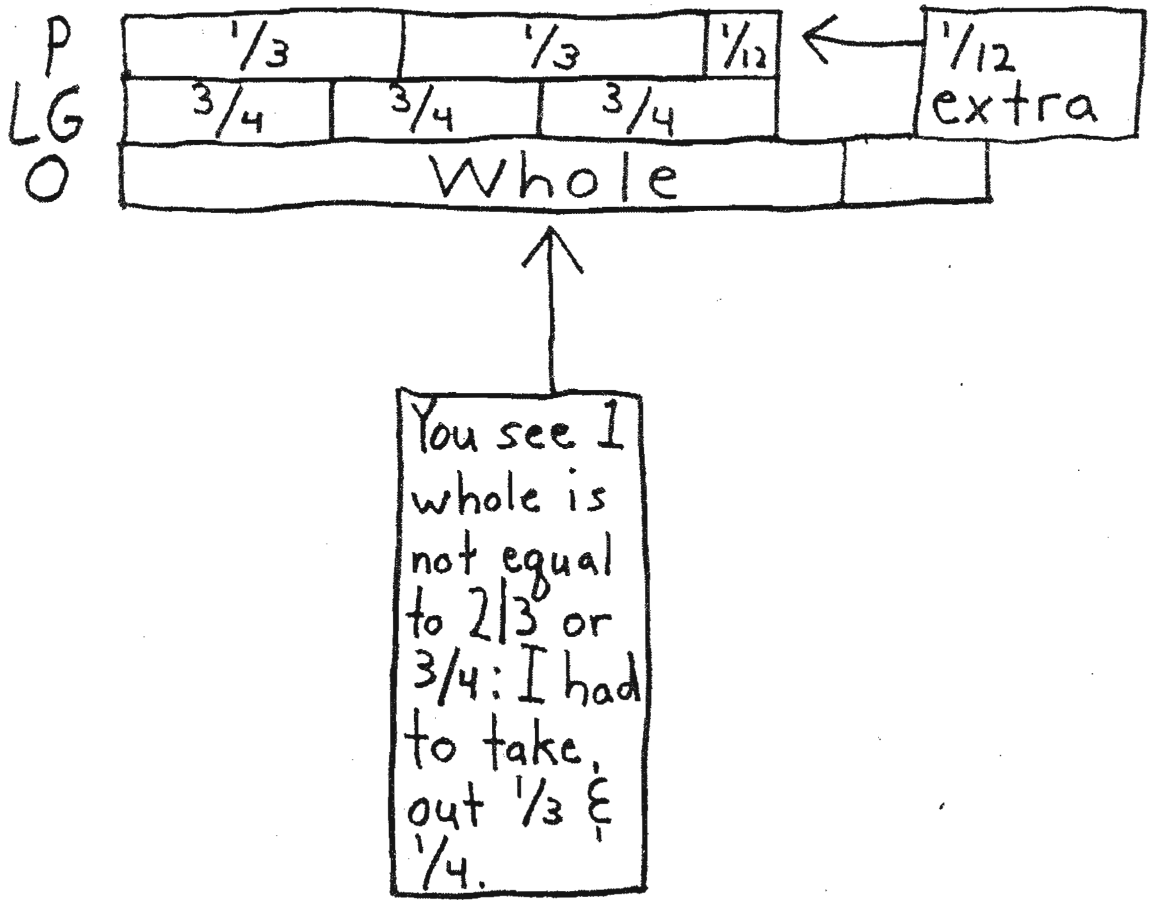
④ The final modle is 1 whole, 1/2 and 1/4.





Brian Fractions

① Which is larger 2/3 or 3/4 by how much?



Brian Fractions

1) 3/4 are larger than 2/3 by 1/12 because if you put 12 cubes and put it against 1, 12 cubes would equal 1. And so if you put in 1/12 it would make the 3/4 and 2/3 equal.

if you slide in 1/12 it would make 3/4 = to 2/3

Erin

Which is larger 3/4 or ½?

3/4 is larger. by 1/4

	Orange 1 Whole										Red	
	D.Green $\frac{1}{2}$						D.Green $\frac{1}{2}$					
	L.Green $\frac{3}{4}$			L.Green $\frac{3}{4}$			L.Green $\frac{3}{4}$			L.Green $\frac{3}{4}$		
3/12	ω	ω	ω	ω	ω	ω	ω	ω	ω	ω	ω	ω

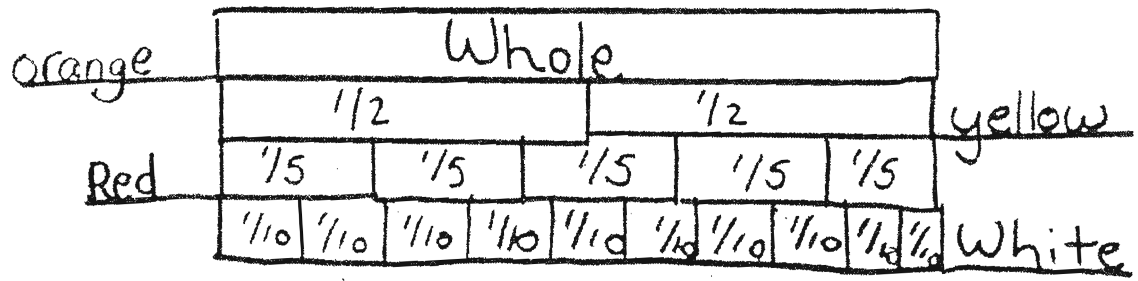
	D.Green 1 Whole					
	L.Green $\frac{1}{2}$			L.Green $\frac{1}{2}$		
3/4	ω	ω	ω	ω	ω	ω

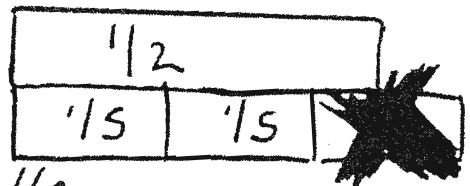
Brown 1 whole			
Purple $\frac{1}{2}$		Purple $\frac{1}{2}$	
Red $\frac{3}{4}$	Red $\frac{3}{4}$	Red $\frac{3}{4}$	Red $\frac{3}{4}$

	Purple 1 whole			
	Red $\frac{1}{2}$		Red $\frac{1}{2}$	
3/4	ω	ω	ω	ω

Sarah Which is larger 1/2 or 2/5?

By how much? By 1/10





1/2 ~~2/5~~ is larger then 2/5 ~~1/2~~.

Orange × 4 = 1 whole
Purple × 10 = 10ths
orange × 2 = 1 half
red × 20 = 20ths
Brown × 5 = 5ths
White × 40 = 40ths

Model # 4

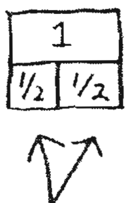
Comparing 2/5 and 1/2

1. model with 1 orange
2. 2 orange
3. 1 brown 1 re

Conover Road School Grade 4

12/2/93

Name Brian

1 

$\boxed{1/2} + \boxed{1/2} = \boxed{1}$

2 whites

$1 \div 1/2 = 2$

How many whites are in red? 2

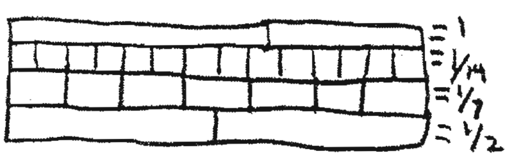
How many 1/2's are in 1? 2

2 

$1 \div 1/8 = 8$

3 

$1 \div 1/15 = 15$

4 

$1 \div 1/4 = 4$ or

$1 \div 1/7 = 7$

Alan
Math

Dec. 14, 1993
4 Ph

1. $2\cancel{7}$
$\times\ 3$

2. 20
$\times 3$
60

3. $\cancel{2}7$
$\times 3$
21

4. $21 + 60 = 81$

I did the problem this way because it would be easier to understand.

In step 1, you take the 7 off the 27 and you get 20.

In step 2, you do this. You multiply 20 three times and you get 60.

In step 3, you now only have the 7 left so, you multiply 7 × 3 and get 21.

In step 4, you add your 2 answers, 60 and 21 and get 81.

Kimberly	Dec. 14 '93
math Elbows	4-Phillips

~~The~~ The problem I worked ~~on~~ on was $27 \times 3 = 81$. Here is the way I do ~~the~~ the problem.

$$\begin{array}{r} {}^{2}27 \\ \times\ 3 \\ \hline 81 \end{array}$$

~~I take~~ I ~~do~~ do $\begin{array}{r} 7 \\ \times 3 \\ \hline 21 \end{array}$

I write the one ~~two~~ down the one $\begin{array}{r} {}^{2}27 \\ \times\ 3 \\ \hline 81 \end{array}$ and carry the two. I do $\begin{array}{r} 3 \\ \times 2 \\ \hline 6 \end{array}$ and add the two

that I carried and ~~I~~ add it ~~to~~ two the six and get eight. That ~~I~~ is how I get my answer.

In the problem the 27 is really ~~2~~ 27 meters and the 3 is really $\frac{1}{3}$. They are used like a regular 27 and 3. That is the best I can expeain it. I am still a little confused with my own problem.

Beth
Math bows

December 14, 1993
4- Phillips

I made 3 bows out of 1 meter. I think that works because a 3rd means one out of 3 pieces. Out of three meters you can make 9 bows since multiplying 3X3 equals 9. I can make 27 bows out of 9 meters, since multiplying 3X9 equals 27. Out of 27 meters I can make 81 bows since multiplying 3X27 equals 81. I can make 243 bows out of 81 meters since multiplying 3X81 equals 243.

Why can you carry?

I think you can carry numbers, but if the number is going in the 10's clowmn it has to have a ten value. Same with the 100's 1,000's ect.

847,289/1/9/9/643

Michael 12/13/93
Math Bows

I timesd every thing by 3. For eremple $3 \times 3 = 9 \times 3 = 27 \times 3 = 81$. I did this because there are 3 1/3 in a meter and you take the 3 and times it by ~~whatever~~ ~~however~~ whatever number of meters you have so I kept doing that and got alot of answers. I think this works because it takes 3 1/3 to equal a whole (or meter) and then you have a certain amount of meters and you times that by 3 because it it takes 3 1/3 to equal a meter.

Audra Dec. 14, 1993
4 Ph.

Math Bows

Jakki and I started the problems by multiplying. For example, 3 meters of ribbon and we have to divide them by 3rds, would be 9. Because, there are 3 meters and there are 3 3rds 3X3=9. When the numbers got to big for the calculator we had to add because the numbers were too big for us to multiply. Our last number was 2854499941!

Laura | Dec. 14, 1993
Math | 4-Phillips

1) I think out of 1 meter you can make 3 bows. I think this because there are 3 ⅓ in 1 meter.

2) I think out of 3 meter you can make 9 bows. I think this because there are 9 ⅓'s in 3 meters.

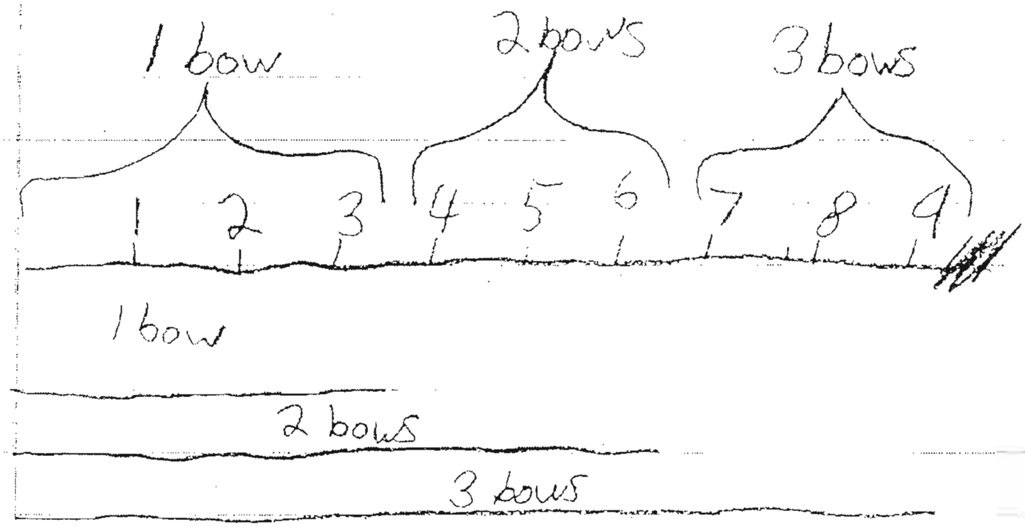
3) I think out of 9 meters you can make 27 bows. I think this because there are 27 ⅓'s in 9 meters.

4) I think out of 27 meters you can make 81 bows. I think this because there are 81 ⅓'s in 27 meters

5) I think out of 81 meters you can make 243 bows. I think this because there are 243 ⅓'s in 81 meters

Dave Dec 14, 1993
Mathematics

Eric, Brian C., Caitlin, Erin and I worked on the 9 meter problem. Each bow was 3 meters so 9-3=6, 1 bow, 6-3=3, 2 bows, and 3-3=0, 3 bows. We also could have just said 3X3=9 but thats how you can prove it. No can prove it!

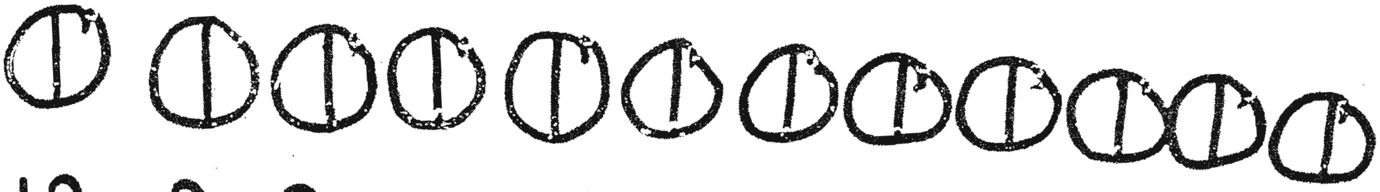


Erik 12/14/93
Math 4-Phillips

I n math I measured the 9 meter ribbon and if each bow is 3 meters long you can make 3 bows. If each bow is $\frac{1}{3}$ of a meter you can make 27 bows. If you have 9 meter ribbon and divide it into 3 parts each part would be 3 meters. If you divided 9 meter ribbon into 27 parts each part would be $\frac{1}{3}$ of 1 meter.

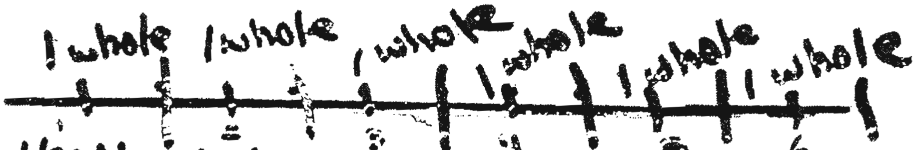
12/15/93

Picture this as 12 meters



12x2=24

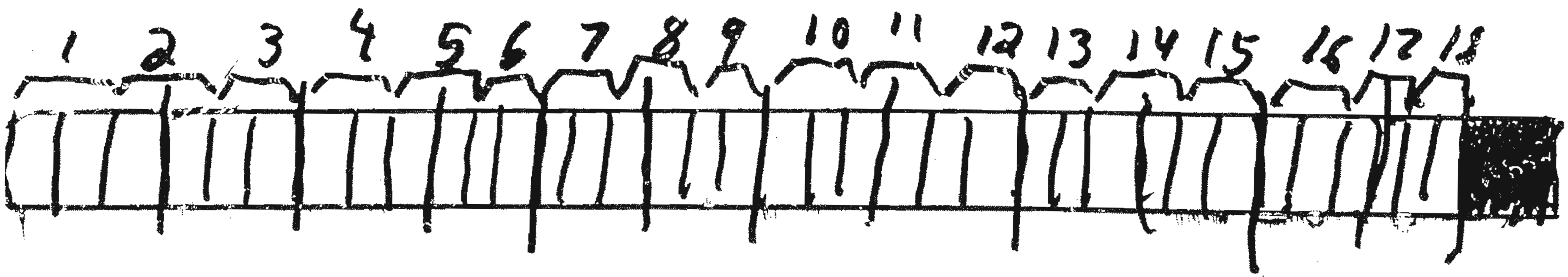
Picture this a 12 meter



1/2+1/2=1 whole

6 1 wholes

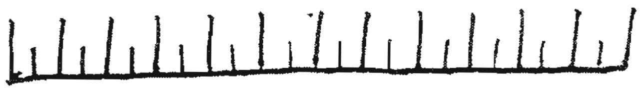
I added 1/2 12 times

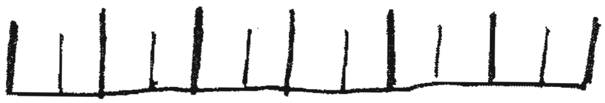


18

Alan
Kim

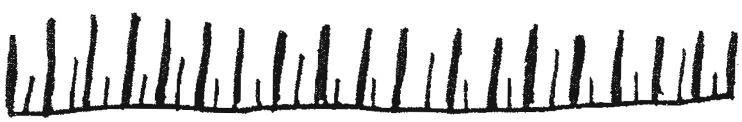
Brian Dec. 15

1.  24

2.  6

3.  36

4.  2

5.  18

12 m = 

1 bow =

Appendix D

Written Problem Handouts

Name_________________________________

Fraction Problem Sheet 1
December 2, 1993

For each problem draw your model and write a division number sentence to describe your model.

1. If we give the red the number name 1, what number name would we give to white?

   How many whites are in red?

   Write a number sentence to describe this relationship.

2. If we give the brown the number name 1, what number name would we give to white? What number name would we give to purple?

3. If we give the orange and yellow train the number name 1, what number name would we give to white? What number name would we give to light green? What number name would we give to yellow?

4. If we give the blue and yellow train the number name 1, what number name would we give to white? What number name would we give to red? What number name would we give to black?

HOLIDAY BOWS

We have just received a shipment of ribbon to be used for bows for the holidays.

(1) Red ribbon comes packaged in 6 meter lengths;
(2) Gold ribbon comes packaged in 3 meter lengths;
(3) Blue ribbon comes packaged in 2 meter lengths; and
(4) White ribbon comes packaged in 1 meter lengths.

Bows for different sizes of gift boxes require pieces of ribbon that are different lengths.

Your job is to find out how many bows of particular lengths can be made from the packaged lengths for each color ribbon.

Please fill out the charts below with your solutions.

I. White Ribbon	Ribbon Length of Bow	Number of Bows
1 meters	1/2 meter	
1 meters	1/3 meter	
1 m	1/4 meter	
1 m	1/5 meter	

II. Blue Ribbon	Ribbon Length of Bow	Number of Bows
2 meters	1/2 meter	
2 meters	1/3 meter	
2 m	1/4 meter	
2 m	1/5 meter	
2 m	2/3 meter	

III. Gold Ribbon	Ribbon Length of Bow	Number of Bows
3 meters	1/2 meter	
3 meters	1/3 meter	
3 m	1/4 meter	
3 m	1/5 meter	
3 m	2/3 meter	
3 m	3/4 meter	

IV. Red Ribbon	Ribbon Length of Bow	Number of Bows
6 meters	1/2 meter	
6 meters	1/3 meter	
6 m	1/4 meter	
6 m	1/5 meter	
6 m	2/3 meter	
6 m	3/4 meter	

For each color and each bow length, write an explanation or make a drawing to show what you did.

I. White Ribbon

1/2 m bow

1/3 m bow

1/4 m bow

1/5 m bow

II. <u>Blue Ribbon</u>
1/2 m bow

1/3 m bow

1/4 m bow

1/5 m bow

2/3 m bow

III. <u>Gold Ribbon</u>
1/2 m bow

1/3 m bow

1/4 m bow

1/5 m bow

2/3 m bow

3/4 m bow

IV. Red Ribbon

1/2 m bow

1/3 m bow

1/4 m bow

1/5 m bow

2/3 m bow

3/4 m bow