SPECTRAL FUNCTIONS OF INVARIANT OPERATORS ON SKEW MULTIPLICITY FREE SPACES

BY MICHAEL WEINGART

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ABSTRACT OF THE DISSERTATION

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by Michael Weingart

Dissertation Director: Friedrich Knop

This thesis extends results on spectral functions of invariant differential operators on multiplicity free spaces to the setting of skew multiplicity free spaces, which are representations of a reductive group whose exterior algebra decomposes into a direct sum of pairwise nonisomorphic irreducibles. We prove in the general skew multiplicity free case that the spectral functions satisfy a vanishing property and a transposition formula which are formally identical to those satisfied by their multiplicity free analogues. We investigate two special cases, the $GL_n\mathbb{C}$ modules $S^2\mathbb{C}^n$ and $\bigwedge^2 \mathbb{C}^n$, for which the spectral functions of invariant operators form a family of supersymmetric functions which can be identified with the factorial Schur Q functions. From this equivalence we deduce several properties of each family, giving the spectral functions a combinatorial interpretation and the factorial Schur Q functions a new representation theoretic one.

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Dedication

To the blessed memory of my parents, who always valued knowledge as an end in itself.

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Chapter 1

Introduction and Overview

Let G be a connected reductive group. A G-module W is said to be *multiplicity free* (MF) if no two irreducible submodules of its symmetric algebra S^*W are isomorphic. A G-module V is said to be *skew multiplicity free* (SMF) if no two irreducible submodules of its exterior algebra $\bigwedge V$ are isomorphic. The primary objects of interest in this study are G-invariant polynomial coefficient differential operators on V where V is an SMF space, and our viewpoint will be that any such invariant operator can be viewed as a complex valued function on the set of highest weights of $\bigwedge V$ as a G-module. The goal is to demonstrate certain general properties of these functions, and to give complete characterizations of them in two special cases.

To justify this viewpoint, let V denote any SMF G-module, and Λ the set of highest weights occurring in $\bigwedge V$. Then we can write the decomposition of $\bigwedge V$ into irreducible G-submodules as

$$\bigwedge V = \bigoplus_{\lambda \in \Lambda} M_\lambda \tag{1.1}$$

Denoting the polynomial coefficient differential operators on V by $\mathcal{PD}(V)$, we consider the natural G-module isomorphisms

$$\mathcal{PD}(V) \cong \mathrm{Cliff}(V \oplus V^*) \cong \bigwedge V \otimes \bigwedge V^* \cong \bigwedge (V \oplus V^*) \tag{1.2}$$

Least familiar among these isomorphisms is $\mathrm{Cliff}(V \oplus V^*) \cong \bigwedge V \otimes \bigwedge V^*$; see [FH, Lemma 20.9] for a proof and discussion. We have

$$\mathcal{PD}(V) \cong \bigwedge V \otimes \bigwedge V^* = \bigoplus_{\lambda,\mu} M_\lambda \otimes M_\mu^* \tag{1.3}$$

Taking G-invariants gives

$$\mathcal{PD}(V)^G \cong \bigoplus_{\lambda,\mu} (M_\lambda \otimes M_\mu^*)^G \tag{1.4}$$

But by Schur's Lemma,

$$\mathcal{PD}(V)^G \cong \bigoplus_{\lambda} (M_\lambda \otimes M_\lambda^*)^G \tag{1.5}$$

Thus for each $\lambda \in \Lambda$ there is a 1-dimensional space $(M_\lambda \otimes M_\lambda^*)^G$ of G-invariant polynomial coefficient differential operators, hence, up to scalars, a unique invariant operator D_λ associated with each $\lambda \in \Lambda$. Moreover, these operators $\{D_\lambda, \lambda \in \Lambda\}$ form a basis for $\mathcal{PD}(V)^G$. The analogous objects in the MF case are, after a suitable normalization, the famous Capelli operators, so that our $\{D_\lambda, \lambda \in \Lambda\}$ may reasonably be termed the *skew Capelli operators.*

Now each such D_λ maps $\bigwedge V$ to itself in a G equivariant way, so that by another application of Schur's lemma, any nonzero image of the restriction of D_λ to a particular M_μ must lie entirely in M_μ, and its action on M_μ must simply be multiplication by a scalar $c_\lambda(\mu)$. In this sense we may interpret D_λ as a complex-valued function $c_\lambda(\cdot)$ on Λ, referred to henceforth as the *spectral function* of D_λ.

The properties of spectral functions in the symmetric case, i.e. for invariant differential operators on MF spaces, have been well investigated by Knop, Sahi, Okounkov, Olshanskii, Benson and Ratcliff, and others. The development of this subject in the skew symmetric case, i.e. that of SMF spaces, follows that of the symmetric case closely. Among the properties which have SMF analogues are the existence of "transposition formulas", in the following sense.

The action of G on V induces an action of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ as differential operators on $\bigwedge V$, and thus of its center $\mathfrak{Z}(\mathfrak{g})$ as invariant differential operators on V. There is an antiautomorphism of $\mathfrak{U}(\mathfrak{g})$, which we call "transposition" by analogy to the work of Knop [K1], whose induced effect on differential operators is to reverse the order of multiplication;

$$(1.6) \qquad x \mapsto x,\ \partial \mapsto -\partial,\ x\partial \mapsto \partial x,\ \partial x \mapsto x\partial$$

Thus for each of our invariant operators D_λ as described above, there exists a transposed operator D^t_λ, hence there also exist the corresponding spectral functions c_λ and c^t_λ, respectively. Chapter 2 derives the following transposition formula, which expresses the value of c^t_λ at a given highest weight as a linear combination of values of several c_μ:

$$(1.7) \qquad c^t_\lambda(\nu) = c_\lambda(\chi - w_0\nu) = \sum_{|\mu|\le|\lambda|} (-1)^{|\lambda|}\frac{d_\lambda}{d_\mu} c_\mu(\lambda) c_\mu(\nu)$$

where w_0 is the longest element of the Weyl group of G, χ is the sum of the weights of V, and d_ν is the superdimension of the irreducible submodule M_ν, whose elements have homogeneous degree $|\nu|$ as tensors. This is a general result on skew multiplicity free spaces which is completely analogous to a corresponding result proved by Knop in the multiplicity free case [K1].

Another property of spectral functions of invariant operators on MF spaces, investigated by several authors beginning with Sahi [S2] and proven in full generality by Knop [K1], is the vanishing condition, $c_\lambda(\mu) = 0$ if $|\mu| \le |\lambda|, \mu \ne \lambda$ and $c_\lambda(\lambda) \ne 0$. This condition, together with simple conditions on symmetry and degree, suffice to determine uniquely a polynomial (with ρ-shifted arguments) which interpolates c_λ at all $\{\mu \in \Lambda\}$. Most remarkably, the spectral functions vanish at many more points than those indicated in the vanishing condition, and indeed satisfy the extra vanishing condition that for all $\lambda, \mu \in \Lambda,\ c_\lambda(\mu) = 0$ if $\lambda \not\subseteq \mu$, in settings in which λ and μ may be interpreted as partitions. Other examples of families of functions satisfying the vanishing and extra vanishing conditions, or suitable analogues thereof, are the shifted MacDonald functions introduced by Okounkov [O2], and even certain non symmetric polynomials introduced by Knop [K5].

Chapter 2 demonstrates that spectral functions of invariant operators on SMF spaces satisfy the same basic vanishing property, $c_\lambda(\mu) = \delta_{\lambda\mu}$ for all $|\mu| \leq |\lambda|$. Since $\bigwedge V$ has only finitely many highest weights for each SMF space V, the interesting cases to investigate are those in which infinite families occur. Howe [H] has classified irreducible SMF spaces for simple groups (possibly augmented by $\mathbb{C}^*$), and apart from the defining representations of the classical groups, the only infinite classes of such spaces are the GL_n modules $S^2\mathbb{C}^n, n \geq 2$ and $\bigwedge^2 \mathbb{C}^n, n \geq 2$.

Chapter 3 investigates the spectral functions of invariant operators on $S^2\mathbb{C}^n$ and $\bigwedge^2 \mathbb{C}^n$ for $n \geq 2$, and proves a characterization theorem analogous to that of Knop in the multiplicity free case:

For each highest weight λ of $\bigwedge V$ there exists a polynomial p_λ on $Span_{\mathbb{C}}\Lambda$ which satisfies the following properties.

CT1a. $p_\lambda(\mu + \rho) = c_\lambda(\mu)$ for every $\mu \in \Lambda_n$.

CT1b. $p_\lambda(\mu + \rho) = 0$ whenever $|\mu| \leq |\lambda|$ unless $\mu = \lambda$, and $p_\lambda(\lambda + \rho) = 1$.

CT2. $p_\lambda \in \mathbb{C}[p_1, p_3, p_5, ...]$, where $p_1, p_3, ..$ are the odd degree power sum polynomials.

CT3. p_λ has degree $\frac{|\lambda|}{2}$.

Furthermore, p_λ is uniquely determined by CT1b, CT2, and CT3. Moreover, c_λ satisfies the extra vanishing condition that $c_\lambda(\mu) = 0$ if $\lambda \not\subseteq \mu$.

A symmetric polynomial $p(z_1, z_2, .., z_n)$ is said to be supersymmetric if for any integers $1 \leq i < j \leq n$, $p(z_1, ..., z_{i-1}, t, z_{i+1}, ..., z_{j-1}, -t, z_{j+1}, ..., z_n)$ does not depend on t. Pragacz (cf. [P]) shows that the odd degree power sum polynomials $p_1, p_3, ...$ generate the algebra of supersymmetric functions. Henceforth we will use the term supersymmetric with this interpretation in mind; we call p supersymmetric if $p \in \mathbb{C}[p_1, p_3, p_5, ...]$. Condition CT2 of the characterization theorem just stated can thus be reformulated; p_λ is supersymmetric.

The factorial Schur Q-functions Q^*_λ are a family of symmetric functions defined by Okounkov and investigated by Ivanov in [I1] and[I2]. Like the classical Schur Q-functions of which they are analogues, the Q^*_λ are supersymmetric and indexed by strict partitions. After replacing each Q^*_λ by a suitably rescaled polynomial function q_λ, the family $\{q_\lambda : \lambda \text{ a strict partition}\}$ are characterized by the same vanishing, supersymmetry, and degree conditions as our spectral polynomials, except that strict partitions are the defining set of evaluation points instead of highest weights of $\bigwedge V$. The argument in chapter 3 associates to each highest weight λ a unique strict partition $\check{\lambda}$, constructs an automorphism ϕ of the algebra of supersymmetric functions such that $\phi(p_\lambda) = q_{\check{\lambda}}$, and concludes that $c_\lambda(\mu) = q_{\check{\lambda}}(\check{\mu})$ for all highest weights λ and μ of $\bigwedge V$.

By virtue of this correspondence between the two families of supersymmetric functions, the spectral functions inherit several important properties of the factorial Schur Q functions, as well as the combinatorial interpretation of $Q^*_\mu(\lambda)$ as a term in a simple formula which counts shifted skew tableaux of shape λ/μ. Conversely, the factorial Schur Q functions acquire a new representation theoretic interpretation as spectral functions of invariant operators, and satisfy a suitable reformulation of the transposition formula.

Chapter 4 employs the Weyl Dimension Formula together with the peculiar combinatorics of the highest weights occurring in $\bigwedge V$, $V = S^2\mathbb{C}^n$ or $\bigwedge^2 \mathbb{C}^n$, to obtain explicit dimension formulas for irreducible submodules of $\bigwedge V$ in terms of the top-row Frobenius coordinates of their highest weights. For each such irreducible M_λ, $\dim M_\lambda$ can be expressed explicitly as a polynomial in n. This makes possible the derivation of an explicit formula for the leading coefficients of the spectral functions, the factorial Schur Q-functions, and the classical Schur Q-functions.

Chapter 5 presents computational examples. Among these are explicitly written operators D_λ, spectral functions, factorial Schur Q-functions, tables of values of these functions, and dimension polynomials.

Chapter 6 sets out several topics for further research which arise from, or are natural extensions of, the current project.

Chapter 2

The Transposition Formula for General Skew Multiplicity Free Spaces

In the following discussion, multiplications are understood to be skew multiplications, i.e. multiplication inside an exterior algebra. Other notational conventions are as follows.

G denotes a connected reductive group. V denotes any skew multiplicity free G-module.
$\mathcal{PD}(V)$ denotes the algebra of polynomial coefficient differential operators on V, and $\mathcal{PD}(V)^G$ the algebra of G-invariant such operators.
For a specific V understood from context, Λ denotes the set of highest weights of $\bigwedge V$ as a G-module, and the elements of Λ are denoted by lower case Greek letters $\lambda, \mu, \nu, ...$
For $\lambda \in \Lambda$, M_λ denotes the irreducible G-module of highest weight λ.
$d_\lambda = (-1)^{|\lambda|}\dim M_\lambda$ denotes the superdimension of M_λ.

Sections 2.1 and 2.2 are largely inspired by Benson and Ratcliff's treatment of the subject in the symmetric case in [BR, 43-57], and closely mimic much of the pattern of their argument.

2.1 Skew Capelli Polynomials and Operators

Assume that V has finite dimension n. To define a pairing $\bigwedge V^* \otimes \bigwedge V \to \mathbb{C}$, let $\{z_i\}$ be a basis of V and $\{\bar{z}_i\}$ a basis of V^*, such that $< \bar{z}_i, z_j >= \delta_{ij}$ and $< z_i, \bar{z}_j >= -\delta_{ij}$. We define ∂_i and $\bar{\partial}_i$ by $\partial_i(z_j) =< \bar{z}_i, z_j >= \delta_{ij}$, and $\bar{\partial}_i(\bar{z}_j) =< z_i, \bar{z}_j >= -\delta_{ij}$.

The G-module isomorphisms $\mathcal{PD}(V) \cong \bigwedge(V \oplus V^*) \cong \bigwedge V \otimes \bigwedge V^*$ permit us to adopt the useful viewpoint that the canonical invariant operator D_λ considered in the introduction can be regarded as a polynomial $\tilde{P}_\lambda$ in skew symmetric variables, which can in turn be viewed as a tensor. We may move freely between these interpretations.

Beginning with the tensor viewpoint, we consider the basis-independent element

$$(2.1.1) \qquad \sum_{i=1}^{|d_\lambda|} v_i \otimes v_i^* \in (M_\lambda \otimes M_\lambda^*)^G \subset (\bigwedge V \otimes \bigwedge V^*)^G$$

where $\{v_i\}$ is any basis for M_λ and $\{v_i^*\}$ is its dual basis, in the sense that $< v_i^*, v_j >= \delta_{ij}$ and $< v_i, v_j^* >= -\delta_{ij}$. We can define the canonical skew invariant

$$(2.1.2) \qquad \tilde{P}_\lambda = \sum_{i=1}^{|d_\lambda|} v_i \otimes v_i^*,$$

and the normalized invariant

$$(2.1.3) \qquad P_\lambda = \frac{1}{d_\lambda} \tilde{P}_\lambda.$$

By the natural isomorphism

$$(2.1.4) \qquad \bigwedge V \otimes \bigwedge V^* \cong \bigwedge (V \oplus V^*)$$

we may regard $\tilde{P}_\lambda$ as a skew-polynomial in the variables $z_1, ..., z_n, \bar{z}_1, ..., \bar{z}_n$, recalling that $z_1, ..., z_n \in V$ and $\bar{z}_1, ..., \bar{z}_n \in V^*$. With this interpretation we write

$$(2.1.5) \qquad \tilde{P}_\lambda(z, \bar{z}) = \sum_{i}^{|d_\lambda|} v_i(z) v_i^*(\bar{z})$$

and refer to $\tilde{P}_\lambda(z, \bar{z})$ as the *skew Capelli polynomial* determined by λ.

Finally, consider the isomorphism of G-modules

$$(2.1.6) \qquad \pi : \bigwedge (V \oplus V^*) \to \mathcal{PD}(V)$$

defined by

$$(2.1.7) \qquad \pi(z_i) = z_i,\ \pi(\bar{z}_i) = \partial_i,\ i = 1, 2, ..., n$$

We define the *skew Capelli operator* determined by λ by

$$(2.1.8) \qquad D_\lambda = \pi(\tilde{P}_\lambda(z, \bar{z}))$$

and write $D_\lambda = \tilde{P}_\lambda(z, \partial)$.

We define a G-invariant form $< \cdot\ ,\ \cdot >$ on $\bigwedge(V \oplus V^*)$ by

$$(2.1.9) \qquad < p, q >= p(\bar{\partial}, \partial)(q(z, \bar{z}))|_{z=\bar{z}=0}$$

The form has the properties

$$(2.1.10) \qquad < \xi z_i, \cdot >=< \xi, \bar{\partial}_i(\cdot) >$$

$$< \xi \bar{z}_i, \cdot >=< \xi, \partial_i(\cdot) >$$

$$< 1, 1 >= 1$$

$$< 1, \xi >= 0 \quad \text{if } \xi \text{ is not a constant.}$$

In the first two statements in (2.1.10) the right side has strictly lower degree than the left, so that the four given statements show that such an inner product on $\bigwedge(V \oplus V^*)$ is unique.

Note also that this form is supersymmetric in the sense that

$$(2.1.11) \qquad < \xi, \eta >= (-1)^{|\xi||\eta|} < \eta, \xi > .$$

Computation of the inner product on concrete elements of $\bigwedge(V \oplus V^*)$ entails replacing the variables of the first argument by suitable differentiation operators.

Examples:

1) $< z_1\bar{z}_2, z_1\bar{z}_2 >=< z_1, \partial_2(z_1\bar{z}_2) >=< z_1, 0 >= 0$

2) $< z_1 z_2, \bar{z}_1\bar{z}_2 >=< z_1, \bar{\partial}_2(\bar{z}_1\bar{z}_2) >=< z_1, -\bar{z}_1\bar{\partial}_2(\bar{z}_2) >=< z_1, \bar{z}_1 >= -1$

Indeed, we can write the following formula for the inner product on straightened monomials:

$$(2.1.12) \qquad < z^a\bar{z}^b, z^c\bar{z}^d >= \delta_{ad}\delta_{bc}(-1)^{\binom{|a|+1}{2}+\binom{|b|}{2}}$$

where $z^a = \prod_{i=1}^{n} z_i^{a_i}$, with each $a_i = 0$ or 1; similarly $\bar{z}^b = \prod_{i=1}^{n} \bar{z}_i^{b_i}$, etc. Thus if ξ and η are monomials then $< \xi, \eta >= 0$ unless ξ equals, up to sign, a permutation of the factors of η.

2.1.1. Proposition. If $M_\lambda \subset \bigwedge V$ and $M_\mu^* \subset \bigwedge V^*$, then the pairing $< \cdot, \cdot >$ restricted to $M_\lambda \times M_\mu$ is zero unless $\lambda = \mu$, in which case it is non-degenerate.

Proof: This follows from the G-invariance of the bilinear form $< \cdot, \cdot >$.

The following computation is central to the overall argument:

2.1.2. Proposition. $< \tilde{P}_\lambda, \tilde{P}_\mu >= d_\lambda \delta_{\lambda\mu}$

Proof: Let $\{v_i\}$ and $\{w_j\}$ be bases for the irreducible submodules M_λ and M_μ, respectively, and let $\{v_i^*\}$ and $\{w_j^*\}$ be the corresponding dual bases. In keeping with our conventions this means precisely that $< v_i^*, v_j >= \delta_{ij}, < v_i, v_j^* >= -\delta_{ij}, < w_i^*, w_j >= \delta_{ij}$, and $< w_i, w_j^* >= -\delta_{ij}$

$$(2.1.13) \qquad < \tilde{P}_\lambda, \tilde{P}_\mu >=< \tilde{P}_\lambda(z,\bar{z}), \tilde{P}_\mu(z,\bar{z}) >$$
$$= \tilde{P}_\lambda(\bar{\partial}, \partial)(\tilde{P}_\mu(z,\bar{z}))$$

$$= \sum_{i,j} v_i(\bar\partial) v_i^*(\partial) w_j(z) w_j^*(\bar z)$$

Since $v_i \in M_\lambda, w_i \in M_\mu$, Proposition 2.1.1 implies that $v_i^*(\partial)w_j(z) =< v_i^*, w_j >= \delta_{\lambda\mu}$. Thus if $\lambda \neq \mu$ we have $< \tilde P_\lambda, \tilde P_\mu >= 0$.

If $\lambda = \mu$, so that the $\{v_i\}$ and $\{w_j\}$ are bases of the same space, we may assume by the basis independence of $\tilde P_\lambda$ that $v_i = w_i$ for each i. Thus we may replace $v_i^*(\partial)w_j(z)$ by $\delta_{\lambda\mu}\delta_{ij}$, and obtain

(2.1.14) $$< \tilde P_\lambda, \tilde P_\mu >= \delta_{\lambda\mu} \sum_i^{|d_\lambda|} v_i(\bar\partial) v_i^*(\bar z)$$

$$= \delta_{\lambda\mu} \sum_i^{|d_\lambda|} (-1)^{|\lambda|}$$

since deg $v_i = |\lambda|$

$$= (-1)^{|\lambda|} |d_\lambda| \delta_{\lambda\mu}$$

$$= d_\lambda \delta_{\lambda\mu}$$

□

The argument in the next section will require the following lemma.

Let $Q = z_1\bar z_1 + ... + z_n \bar z_n$.

2.1.3. Lemma. $\sum_{|\lambda|=k} \tilde P_\lambda = \frac{Q^k}{k!}$

Proof: The left side is the sum of all skew Capelli polynomials of specified degree, which together form a basis of $\bigoplus_{|\lambda|=k} M_\lambda$. Recalling (2.1.5) we can express each skew Capelli polynomial in a basis independent way by

(2.1.15) $$\tilde P_\lambda = \sum_{i=1}^{|d_\lambda|} v_i(z) v_i^*(\bar z)$$

Consider the monomial basis of $\bigoplus_{|\lambda|=k} M_\lambda$, namely $\{z^\alpha : |\alpha| = k\}$, where $\alpha = (\alpha_1, ..., \alpha_n)$ is a multiindex of 0s and 1s, and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. By the basis independence of the sum of skew Capelli polynomials of a given degree, we have that

(2.1.16) $$\sum_{|\lambda|=k} \tilde{P}_\lambda = \sum_{|\alpha|=k} z^\alpha (z^\alpha)^*.$$

$(z^\alpha)^*$, the element of $\bigwedge V^*$ dual to z^α, is determined by formula (2.1.12), which implies that

(2.1.17) $$< \bar{z}^\alpha, z^\alpha >= (-1)^{\binom{|\alpha|}{2}}$$

It follows that

(2.1.18) $$(z^\alpha)^* = (-1)^{\binom{|\alpha|}{2}} \bar{z}^\alpha$$

so that

(2.1.19) $$\begin{aligned} \sum_{|\lambda|=k} \tilde{P}_\lambda &= \sum_{|\alpha|=k} z^\alpha (z^\alpha)^* \\ &= (-1)^{\binom{k}{2}} \sum_{|\alpha|=k} z^\alpha \bar{z}^\alpha \\ &= \frac{(-1)^{\binom{k}{2}}}{k!} \sum_{|\alpha|=k} k! z^\alpha \bar{z}^\alpha \\ &= \frac{(-1)^{\binom{k}{2}}}{k!} \sum_{1 \le i_1 < \ldots < i_k \le n} k!(-1)^{\binom{k}{2}} z_{i_1} \bar{z}_{i_1} \cdots z_{i_k} \bar{z}_{i_k} \\ &= \frac{Q^k}{k!} \end{aligned}$$ □

2.1.4. Corollary. $\sum_{\lambda \in \Lambda} \tilde{P}_\lambda = e^Q.$

2.2 The Transposition Operator

Let $\Delta := \bar{\partial}_1\partial_1 + ... + \bar{\partial}_n\partial_n$. Recall from the beginning of this chapter that by definition, $\partial_i(z_j) =< \bar{z}_i, z_j >= \delta_{ij}$ and $\bar{\partial}_i(\bar{z}_j) =< z_i, \bar{z}_j >= -\delta_{ij}$.

Define an operator $T : \bigwedge(V \oplus V^*) \to \bigwedge(V \oplus V^*)$ by

$$(2.2.1) \qquad (TP)(z,\bar{z}) = (e^{\Delta}P)(z,-\bar{z}) = e^{\Delta}(P(z,-\bar{z})).$$

Define an operator M by

$$(2.2.2) \qquad (MP)(z,\bar{z}) = P(z,-\bar{z}).$$

Then

2.2.1.Proposition. $T = M \circ e^{-\Delta} = e^{\Delta} \circ M = T^{-1}$.

Proof: Observe first that

$$(2.2.3) \qquad \frac{\Delta^k}{k!} = \sum_{1 \leq i_1 < i_2 < ... < i_k \leq n} \bar{\partial}_{i_1}\partial_{i_1}...\bar{\partial}_{i_k}\partial_{i_k}$$

In the expansion of the product $\Delta^k = (\bar{\partial}_1\partial_1 + ... + \bar{\partial}_n\partial_n)^k$, the only terms which do not contain repeated factors and consequently vanish are those of the form $\bar{\partial}_{i_1}\partial_{i_1}...\bar{\partial}_{i_k}\partial_{i_k}$. Furthermore, since there are $k!$ reorderings of each k-tuple of indices $\{i_1, ..., i_k\}$, we have $k!$ copies of each term of the form $\bar{\partial}_{i_1}\partial_{i_1}...\bar{\partial}_{i_k}\partial_{i_k}$ in the expansion of Δ^k.

It suffices to test the equality of $M \circ e^{-\Delta}$ and $e^{\Delta} \circ M$ on elements of the form $z_1\bar{z}_1z_2\bar{z}_2...z_s\bar{z}_s$.

On the one hand,

$$(2.2.4)\quad (-1)^k \frac{\Delta^k}{k!} \circ M(z_1\bar{z}_1 z_2\bar{z}_2 ... z_s\bar{z}_s) = (-1)^k \frac{\Delta^k}{k!}((-1)^s(z_1\bar{z}_1 z_2\bar{z}_2 ... z_s\bar{z}_s))$$
$$= (-1)^{k+s} \sum_{1\le i_1<i_2<...<i_k\le n} \bar{\partial}_{i_1}\partial_{i_1}...\bar{\partial}_{i_k}\partial_{i_k}(z_1\bar{z}_1 z_2\bar{z}_2 ... z_s\bar{z}_s)$$
$$= (-1)^s \sum_{1\le j_1<j_2<...<j_{s-k}\le n} z_{j_1}\bar{z}_{j_1} z_{j_2}\bar{z}_{j_2} ... z_{j_{s-k}}\bar{z}_{j_{s-k}}$$

The second factor of $(-1)^k$, which cancels the first, arises because each $\bar{\partial}_i(\bar{z}_i) = -1$.

On the other hand,

$$(2.2.5)\quad M \circ \frac{\Delta^k}{k!}(z_1\bar{z}_1 z_2\bar{z}_2 \cdots z_s\bar{z}_s) = M((-1)^k \sum_{1\le j_1<j_2<...<j_{s-k}\le n} z_{j_1}\bar{z}_{j_1} z_{j_2}\bar{z}_{j_2} ... z_{j_{s-k}}\bar{z}_{j_{s-k}})$$
$$= (-1)^s \sum_{1\le j_1<j_2<...<j_{s-k}\le n} z_{j_1}\bar{z}_{j_1} z_{j_2}\bar{z}_{j_2} ... z_{j_{s-k}}\bar{z}_{j_{s-k}} \qquad \square$$

2.2.2. Corollary. $T(P_\lambda) = (-1)^{|\lambda|} e^{\Delta}(P_\lambda)$.

Proof: Immediate from the definition and the fact that $\deg_{\bar{z}} P_\lambda = |\lambda|$. $\square$

2.2.3. Lemma. $\{T(P_\lambda) : \lambda \in \Lambda\}$ is a vector space basis for $\bigwedge(V \oplus V^*)^G$.

Proof: $T(P_\lambda) \in \bigwedge(V \oplus V^*)$ is G-invariant and Δ is a G-invariant operator. Moreover,

$$(2.2.6)\quad T(P_\lambda) = (-1)^{|\lambda|} P_\lambda + R_\lambda$$

where $P_\lambda \in \bigwedge(V \oplus V^*)_{|\lambda|,|\lambda|}$ and R_λ is of strictly lower degree, due to the action of e^{Δ}. As the $\{P_\lambda\}$ form a basis for $\bigwedge(V \oplus V^*)^G$, so do the $\{T(P_\lambda)\}$. $\square$

Remark: This differs slightly from argument given in the symmetric case, in which a K-invariant form is used and the statement of the lemma pertains to K-invariants. This difference has no effect on the overall argument.

2.2.4. Definition. The generalized skew binomial coefficients $\begin{bmatrix} \lambda \\ \nu \end{bmatrix}$ are defined for $\lambda, \nu \in \Lambda$ by $T(P_\lambda) = (-1)^{|\lambda|} \sum_{\nu \in \Lambda} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} P_\nu$.

Analogously to the symmetric case, the proof of Lemma 2.2.3 gives the following corollary

2.2.5. Corollary. $\begin{bmatrix} \lambda \\ \lambda \end{bmatrix} = 1$, $\begin{bmatrix} \lambda \\ \nu \end{bmatrix} = 0$ when $|\lambda| \leq |\nu|$ but $\lambda \neq \nu$.

Thus

$$(2.2.7) \qquad T(P_\lambda) = \sum_{|\nu| \leq |\lambda|} (-1)^{|\lambda|} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} P_\nu = (-1)^{|\lambda|} P_\lambda + \sum_{|\nu| < |\lambda|} (-1)^{|\lambda|} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} P_\nu.$$

2.2.6. Proposition. $\sum_{\mu \in \Lambda} (-1)^{|\mu|} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \begin{bmatrix} \mu \\ \nu \end{bmatrix} = (-1)^{|\lambda|} \delta_{\lambda\nu}$.

Proof: $T^2 = 1$ implies that

$$(2.2.8) \qquad P_\lambda = T(T(P_\lambda)) = (-1)^{|\lambda|} \sum_{\nu} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} T(P_\nu)$$

On the other hand,

$$\begin{aligned} (2.2.9) \qquad T(P_\lambda) &= (-1)^{|\lambda|} \sum_{\mu} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} P_\mu \\ &= (-1)^{|\lambda|} \sum_{\mu} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} ((-1)^{|\mu|} \sum_{\nu} \begin{bmatrix} \mu \\ \nu \end{bmatrix} T(P_\nu)) \\ &= (-1)^{|\lambda|} \sum_{\nu} (\sum_{\mu} (-1)^{|\mu|} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \begin{bmatrix} \mu \\ \nu \end{bmatrix}) T(P_\nu) \end{aligned}$$

The proposition follows from the linear independence of the $\{T(P_\lambda) : \lambda \in \Lambda\}$. □

2.2.7. Proposition. For $\lambda \in \Lambda$ and $k \in \mathbb{N}$,

$$\frac{\Delta^k}{k!} P_\lambda = \sum_{|\nu|=|\lambda|-k} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} P_\nu \tag{2.2.10}$$

and

$$(-1)^k \frac{\Delta^k}{k!} T(P_\lambda) = \sum_{|\nu|=|\lambda|-k} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} T(P_\nu) \tag{2.2.11}$$

Proof: We have

$$(-1)^{|\lambda|} \sum_\nu \begin{bmatrix} \lambda \\ \nu \end{bmatrix} P_\nu = T(P_\lambda) = (-1)^{|\lambda|} e^\Delta (P_\lambda) = (-1)^{|\lambda|} \sum_k \frac{\Delta^k}{k!} P_\lambda \tag{2.2.12}$$

For a given k,

$$\frac{\Delta^k}{k!} P_\lambda = \sum_{|\nu|=|\lambda|-k} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} P_\nu \tag{2.2.13}$$

Equating homogeneous components of degree $2(|\lambda| - k)$ on both sides yields (2.2.10).

To prove 2.2.11, apply the operator T to both sides of (2.2.10):

$$e^\Delta \circ M \frac{\Delta^k}{k!} P_\lambda = \sum_{|\nu|=|\lambda|-k} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} T(P_\nu) \tag{2.2.14}$$

$$(-1)^{|\lambda|-k} e^\Delta \frac{\Delta^k}{k!} P_\lambda =$$
$$(-1)^{|\lambda|-k} \frac{\Delta^k}{k!} e^\Delta P_\lambda =$$
$$(-1)^{|\lambda|-k} \frac{\Delta^k}{k!} (-1)^{|\lambda|} T(P_\lambda) =$$

$$(-1)^k \frac{\Delta^k}{k!} T(P_\lambda) = \qquad \square$$

2.2.8. Lemma. Q and Δ are adjoint.

Proof: We compute

(2.2.15) $$< z_j \bar{z}_j \xi, \eta > = < z_j \xi, (-1)^{\deg\, \xi} \partial_j \eta > = < \xi, \bar{\partial}_j \partial_j \eta > .$$

Thus

(2.2.16) $$< Q\xi, \eta > = < \xi Q, \eta > = < \xi, \Delta\eta > . \qquad \square$$

Benson and Ratcliff ([BR, p.54]) cite an unpublished Pieri formula proved by Yan, whose skew analogue is the following:

2.2.9. Theorem. For $\nu \in \Lambda, k \in \mathbb{N}$, $\dfrac{Q^k}{k!} \tilde{P}_\nu = \sum_{|\lambda| = |\nu| + k} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} \tilde{P}_\lambda.$

Proof: The theorem states that

(2.2.17) $$\frac{Q^k}{k!} d_\nu P_\nu = \sum_{|\lambda| = |\nu| + k} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} d_\lambda P_\lambda$$

We have

(2.2.18) $$\frac{Q^k}{k!} P_\nu = \sum_{|\lambda| = |\nu| + k} \frac{< \frac{Q^k}{k!} P_\nu, P_\lambda >}{< P_\lambda, P_\lambda >} P_\lambda$$

since the P_λ of degree $|\nu| + k$ form a basis for invariants of degree $|\nu| + k$

(2.2.19) $$= \sum_{|\lambda| = |\nu| + k} d_\lambda < P_\nu, \frac{\Delta^k}{k!} P_\lambda > P_\lambda$$

by adjointness of Q and Δ (Lemma 2.2.8) and the fact that $d_\lambda =< \tilde{P}_\lambda, \tilde{P}_\lambda >$ (Proposition 2.1.2) , so that $< P_\lambda, P_\lambda >= \frac{1}{d_\lambda}$. Thus we have

$$(2.2.20) \qquad = \sum_{|\lambda|=|\nu|+k} d_\lambda < P_\nu, \sum_{\mu} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} P_\nu > P_\lambda$$

$$= \sum_{|\lambda|=|\nu|+k} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} d_\lambda < P_\nu, P_\nu > P_\lambda$$

by the orthogonality of P_λ and P_ν for $\lambda \neq \nu$, and Proposition 2.2.7.

$$(2.2.21) \qquad = \sum_{|\lambda|=|\nu|+k} \frac{d_\lambda}{d_\nu} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} P_\lambda.$$

Thus

$$(2.2.22) \qquad \frac{Q^k}{k!} d_\nu P_\nu = \sum_{|\lambda|=|\nu|+k} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} d_\lambda P_\lambda$$

$$\text{i.e.} \qquad \frac{Q^k}{k!} \tilde{P}_\nu = \sum_{|\lambda|=|\nu|+k} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} \tilde{P}_\lambda$$

□

2.2.10. Corollary. For $\nu \in \Lambda$, $e^Q \tilde{P}_\nu = \sum_{\lambda \in \Lambda} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} \tilde{P}_\lambda$.

Recall from Chapter 1 that by Schur's Lemma and the skew multiplicity freeness of V, the G-invariant operator $D_\nu = \tilde{P}_\nu(z, \partial)$ acts as a scalar on each irreducible M_λ.

2.2.11. Definition. Let c_ν denote the spectral function of the canonical invariant operator $\tilde{P}_\nu(z, \partial) \in (M_\nu \otimes M_\nu^*)^G$.

Thus for $\nu, \lambda \in \Lambda$, $c_\nu(\lambda) \in \mathbb{C}$ denotes the eigenvalue of D_ν on M_λ.

2.2.12. Proposition. For all $\lambda, \nu \in \Lambda$, $c_\nu(\lambda) = \begin{bmatrix} \lambda \\ \nu \end{bmatrix}$.

Proof: Note that $\tilde{P}_\nu(z,\partial)\tilde{P}_\lambda(z,\bar{z}) = c_\nu(\lambda)\tilde{P}_\lambda(z,\bar{z})$ since $\tilde{P}_\lambda(z,\bar{z}) \in M_\lambda \otimes M^*_\lambda$. We have

$$\tilde{P}_\nu(z,\partial)e^Q = \tilde{P}_\nu(z,\bar{z})e^Q \tag{2.2.23}$$

which holds in the skew case as in the symmetric.

On the one hand,

$$\tilde{P}_\nu(z,\bar{z})e^Q = \tilde{P}_\nu(z,\bar{z})\sum_k \frac{Q^k}{k!} \tag{2.2.24}$$

$$= \sum_k \frac{Q^k}{k!}\tilde{P}_\nu(z,\bar{z})$$

since Q^k is homogeneous of even degree, so that $Q^k\tilde{P}_\lambda = \tilde{P}_\lambda Q^k$ for each k.

$$= \sum_k \sum_{|\lambda|=|\nu|+k} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} \tilde{P}_\lambda(z,\bar{z}) \tag{2.2.25}$$

by Theorem 2.2.9.

$$= \sum_{|\lambda|\geq|\nu|} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} \tilde{P}_\lambda(z,\bar{z}).$$

On the other hand, using Lemma 2.1.3 we find that

$$e^Q = \sum_k \frac{Q^k}{k!} \tag{2.2.26}$$

$$= \sum_k \sum_{|\lambda|=k} \tilde{P}_\lambda(z,\bar{z})$$

$$= \sum_\lambda \tilde{P}_\lambda(z,\bar{z})$$

Thus

$$\tilde{P}_\nu(z,\partial)e^Q = \sum_\lambda \tilde{P}_\nu(z,\partial)\tilde{P}_\lambda(z,\bar{z}) = \sum_\lambda c_\nu(\lambda)\tilde{P}_\lambda(z,\bar{z}) \tag{2.2.27}$$

Having written $\tilde{P}_\nu(z,\partial)e^Q$ in two ways we find that

$$\sum_\lambda \begin{bmatrix} \lambda \\ \nu \end{bmatrix} \tilde{P}_\lambda(z,\bar{z}) = \sum_\lambda c_\nu(\lambda)\tilde{P}_\lambda(z,\bar{z}). \tag{2.2.28}$$

For each λ, equating coefficients of $\tilde{P}_\lambda(z,\bar{z})$ gives $\begin{bmatrix} \lambda \\ \nu \end{bmatrix} = c_\nu(\lambda)$. □

Proposition 2.2.12 and Corollary 2.2.5 imply that the spectral functions satisfy the following vanishing condition:

2.2.13. Corollary. For any $\lambda, \nu \in \Lambda$, $c_\nu(\lambda) = 0$ if $|\lambda| \leq |\nu|$ but $\lambda \neq \nu$; $c_\nu(\nu) = 1$.

2.3 The Transposition Formula

Let the transposition map τ be the antiautomorphism on skew differential operators defined for each $i = 1, ..., \dim V$ by

$$\tau(z_i) = z_i, \ \tau(\partial_i) = -\partial_i. \tag{2.3.1}$$

It then follows that $\tau(z_i\partial_i) = (-1)^{deg(z_i)deg(\partial_i)}(-\partial_i)z_i = \partial_i z_i$, and similarly $\tau(\partial_i z_i) = z_i\partial_i$.

2.3.1. Proposition. The action of $T = e^{\Delta} \circ M = M \circ e^{-\Delta}$ on skew polynomials corresponds to that of τ on skew differential operators in the sense that the following diagram commutes for any $P(z, \bar{z}) \in \bigwedge(V \oplus V^*)$:

$$\begin{array}{ccc} P(z,\bar{z}) & \longrightarrow^{\pi} & P(z,\partial) \\ \downarrow & & \downarrow \\ T(P(z,\bar{z})) & \longrightarrow & \tau(P(z,\partial)) \end{array}$$

Proof: We observe that the following diagrams commute:

$$\begin{array}{ccc} z & \longrightarrow & z \\ \downarrow & & \downarrow \\ z & \longrightarrow & z \end{array} \tag{2.3.2}$$

$$\begin{array}{ccc} \bar{z} & \longrightarrow & \partial \\ \downarrow & & \downarrow \\ -\bar{z} & \longrightarrow & -\partial \end{array} \tag{2.3.3}$$

$$\begin{array}{ccc} z\bar{z} & \longrightarrow & z\partial \\ \downarrow & & \downarrow \\ -z\bar{z}+1 & \longrightarrow & -z\partial+1 = \partial z \end{array} \tag{2.3.4}$$

since $z\partial + \partial z = 1$. To justify that in fact $e^{\Delta} \circ M(z\bar{z}) = -z\bar{z} + 1$, we compute

(2.3.5) $e^{\bar{\partial}\partial} \circ M(z\bar{z}) = e^{\bar{\partial}\partial}(-z\bar{z})$

$= -z\bar{z} + \bar{\partial}\partial(-z\bar{z})$

$= -z\bar{z} + \bar{\partial}(-\bar{z})$

$= -z\bar{z} + 1$

For the induction step, assume that the result holds for polynomials $P(z, \bar{z}) = P(z_1, .., z_{n-1}, \bar{z}_1, ..., \bar{z}_{n-1})$. We must show that it holds for $P(z, \bar{z})Q(z_n, \bar{z}_n)$. It suffices to check this for $Q(z_n, \bar{z}_n) = z_n, \bar{z}_n$, or $z_n\bar{z}_n$. Write

(2.3.6) $\Delta_{n-1} := \bar{\partial}_1\partial_1 + ... + \bar{\partial}_{n-1}\partial_{n-1}$.

Then

(2.3.7) $\Delta = \Delta_n = \Delta_{n-1} + \bar{\partial}_n\partial_n$

and in particular

(2.3.8) $e^{\Delta} = e^{\Delta_{n-1}} e^{\bar{\partial}_n\partial_n}$

Note that Δ_{n-1} and $\bar{\partial}_n\partial_n$ commute, since both expressions have even degree, and therefore $e^{\Delta_{n-1}}$ and $e^{\bar{\partial}_n\partial_n}$ commute as well.

For $Q(z, \bar{z}) = z_n$, observe that $e^{\bar{\partial}_n\partial_n} \circ M(z_n) = z_n$. We have

(2.3.9) $e^{\Delta_{n-1}} e^{\bar{\partial}_n\partial_n} \circ M(P(z, \bar{z})z_n) = e^{\Delta_{n-1}}(P(z, -\bar{z}))e^{\bar{\partial}_n\partial_n}(z_n))$

$= (e^{\Delta_{n-1}}(P(z, -\bar{z})))z_n$

Thus

$$\begin{array}{ccc} P(z,\bar{z})z_n & \longrightarrow & P(z,\partial)z_n \\ \downarrow & & \downarrow \\ (e^{\Delta_{n-1}}(P(z,-\bar{z})))z_n & \longrightarrow & \tau(P(z,\partial))z_n \end{array} \tag{2.3.10}$$

where $e^{\Delta_{n-1}}(P(z,-\bar{z})) \mapsto \tau(P(z,\partial))$ by induction.

For $Q(z,\bar{z}) = \bar{z}_n$, we have $e^{\bar{\partial}_n \partial_n} \circ M(\bar{z}_n) = -\bar{z}_n$, hence

$$e^{\Delta_{n-1}} e^{\bar{\partial}_n \partial_n} \circ M(P(z,\bar{z})(\bar{z}_n)) = e^{\Delta_{n-1}}(P(z,-\bar{z})))(-\bar{z}_n) \tag{2.3.11}$$

Thus we have a commutative diagram

$$\begin{array}{ccc} P(z,\bar{z})\bar{z}_n & \longrightarrow & P(z,\partial)(\partial_n) \\ \downarrow & & \downarrow \\ e^{\Delta_{n-1}}(P(z,-\bar{z}))(-\bar{z}_n) & \longrightarrow & \tau(P(z,\partial))(-\partial_n) \end{array}$$

For $Q(z,\bar{z}) = z_n \bar{z}_n$, we have

$$\begin{aligned} e^{\bar{\partial}_n \partial_n} \circ M(z_n \bar{z}_n) &= e^{\bar{\partial}_n \partial_n}(-z_n \bar{z}_n) \\ &= -z_n \bar{z}_n + \bar{\partial}_n \partial_n(-z_n \bar{z}_n) \\ &= -z_n \bar{z}_n + \bar{\partial}_n(-\bar{z}_n) \\ &= -z_n \bar{z}_n + 1 \end{aligned} \tag{2.3.12}$$

since $\bar{\partial}_n(\bar{z}_n) = -1$, hence

$$e^{\Delta_{n-1}} e^{\bar{\partial}_n \partial_n} \circ M(P(z,\bar{z}) z_n \bar{z}_n) = e^{\Delta_{n-1}}(P(z,-\bar{z}))(-z_n \bar{z}_n + 1) \tag{2.3.13}$$

Thus we have a commutative diagram

$$\begin{array}{ccc} P(z,\bar{z}) z_n \bar{z}_n & \longrightarrow & P(z,\partial)(z_n \partial_n) \\ \downarrow & & \downarrow \\ e^{\Delta_{n-1}}(P(z,-\bar{z}))(-z_n \bar{z}_n + 1) & \longrightarrow & \tau(P(z,\partial))(-z_n \partial_n + 1) = \tau(P(z,\partial))(\partial_n z_n) \end{array}$$

□

Let V be an SMF G-space of dimension n, let χ denote the sum of all weights of V, and let w_0 denote the longest element of the Weyl group of G. Consider the map

$$\bigwedge^i V \otimes \bigwedge^{n-i} V \to \bigwedge^n V \tag{2.3.14}$$

and observe that the module $M_\chi \subseteq \bigwedge^n V$.

If $M_\lambda \subseteq \bigwedge^i V$ then there exists $M_\mu \subseteq \bigwedge^{n-i} V$ such that multiplication is a perfect pairing, $M_\lambda \otimes M_\mu \to M_\chi$. Then necessarily $M_\mu = M_\chi \otimes M_\lambda^*$. Since M_λ^* has highest weight $-w_0\lambda$ it follows that the set of highest weights of $\bigwedge V$ is invariant under $\lambda \mapsto \chi - w_0\lambda$. We can now formulate the following

2.3.2. Proposition. $c_{D^t}(\lambda) = c_D(\chi - w_0\lambda)$.

Proof. The argument closely follows Knop's approach in the symmetric case ([K1, section 2] where the corresponding assertion is that $c_{D^t}(z) = c_D(-z)$).
Let V be any finite dimensional SMF G-space, where G has Lie algebra $\mathfrak{g}$. Let $\mathfrak{Z}(\mathfrak{g})$ denote the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$. The action of G on V induces a homomorphism $\Psi : \mathfrak{U}(\mathfrak{g}) \to \mathcal{PD}(V)$, whose restriction to $\mathfrak{Z}(\mathfrak{g})$ maps to $\mathcal{PD}(V)^G$.

Let e_i be a basis for V consisting of weight vectors such that each e_i has weight χ_i. Let the operator ∂_i be defined by $\partial_i(e_j) = \delta_{ij}$. Write the usual decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. For $\eta \in \mathfrak{h}$ we have $\Psi(\eta) = \sum \chi_i(\eta) e_i \partial_i$. After transposing we have

$$\Psi^t(\eta) = \sum \chi_i(\eta)\partial_i e_i = \sum \chi_i(\eta)(1 - e_i\partial_i) = \chi(\eta) - \Psi(\eta). \tag{2.3.15}$$

If $\eta \in \mathfrak{n}^\pm$ then $\Psi(\eta) = \sum_{i\neq j} a_{ij} e_i \partial_j$. Since e_i and ∂_j anticommute when $i \neq j$, we have

$\Psi^t(\eta) = -\Psi(\eta)$. Since χ is a character of $\mathfrak{g}$ we can define an antiautomorphism τ on $\mathfrak{U}(\mathfrak{g})$ by $\tau(\eta) = -\eta + \chi$ for all $\eta \in \mathfrak{g}$, and thus we have shown that

(2.3.16) $\quad \Psi^t(\xi) = \Psi(\tau(\xi)) \;$ for all $\xi \in \mathfrak{U}(\mathfrak{g})$.

Let $\xi \in \mathfrak{Z}(\mathfrak{g})$ and $D = \Psi(\xi)$. By the Poincare Birkhoff Witt theorem we can write $\xi = \xi_0 + \xi_1$ where $\xi_0 \in \mathfrak{h}$ and $\xi_1 \in \mathfrak{n}^-\mathfrak{U}(\mathfrak{h})\mathfrak{n}^+$. Regarding ξ_0 as a function on $\mathfrak{h}^*$ which takes the value $\xi_0(v)$ at $v \in \mathfrak{h}^*$,

(2.3.17) $\quad \tau(\xi_0(v)) = \xi_0(-v + \chi)$.

Now observe that $\tau(\xi) = \tau(\xi_0)+\tau(\xi_1)$, where $\tau(\xi_0) \in \mathfrak{U}(\mathfrak{h})$ and, since τ is an *anti*automorphism, $\tau(\xi_0) \in \mathfrak{n}^+\mathfrak{U}(\mathfrak{h})\mathfrak{n}^-$. We consider the action upon a lowest weight vector u of M_λ, which has weight $w_0\lambda$, where w_0 denotes the longest element in the Weyl group of G. Since $\mathfrak{n}^-$ annihilates a lowest weight vector, $\Psi(\tau(\xi_1))u = 0$, hence

(2.3.18) $\quad D^t u = \Psi(\tau(\xi_0))u = \tau(\xi_0)(w_0\lambda)u = \xi_0(-w_0\lambda + \chi)u.$

Thus we have

(2.3.19) $\quad c_{D^t}(\lambda) = \xi_0(\chi - w_0\lambda) \;$ for any highest weight λ.

Since λ is the highest weight of M_λ, we can write

(2.3.20) $\quad c_D(\lambda) = \xi_0(\lambda)$.

Since $\chi - w_0\lambda$ is again a weight, we may replace λ by $\chi - w_0\lambda$ to obtain

(2.3.21) $\quad c_D(\chi - w_0\lambda) = \xi_0(\chi - w_0\lambda) = c_{D^t}(\lambda)$. $\square$

This result holds when D is any G-invariant differential operator arising from the center of $\mathfrak{U}(\mathfrak{g})$. On the assumption that D_λ, for $\lambda \in \Lambda$, arises in this way, we obtain at last the desired *transposition formula:*

2.3.3. Theorem. $c_\lambda(\chi - w_0\nu) = \sum_{|\mu|\le|\lambda|} (-1)^{|\lambda|} \frac{d_\lambda}{d_\mu} c_\mu(\lambda) c_\mu(\nu)$ for each $\lambda, \nu \in \Lambda$.

Equivalently, $\begin{bmatrix} \chi - w_0\nu \\ \lambda \end{bmatrix} = \sum_{|\mu|\le|\lambda|} (-1)^{|\lambda|} \frac{d_\lambda}{d_\mu} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \begin{bmatrix} \nu \\ \mu \end{bmatrix}$

Proof: By Definition 2.2.4 we write in terms of unnormalized invariants that

$$(2.3.22) \qquad T(P_\lambda(z, \bar{z})) = \sum_{\mu\in\Lambda} (-1)^{|\lambda|} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} P_\mu(z, \bar{z})$$

In terms of normalized invariants this is written

$$(2.3.23) \qquad \frac{T(\tilde{P}_\lambda(z, \bar{z}))}{d_\lambda} = \sum_{\mu\in\Lambda} (-1)^{|\lambda|} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \frac{\tilde{P}_\mu(z, \bar{z})}{d_\mu}$$

By Proposition 2.2.12 we have

$$(2.3.24) \qquad \frac{T(\tilde{P}_\lambda(z, \bar{z}))}{d_\lambda} = \sum_{\mu\in\Lambda} (-1)^{|\lambda|} c_\mu(\lambda) \frac{\tilde{P}_\mu(z, \bar{z})}{d_\mu}$$

hence

$$(2.3.25) \qquad T(\tilde{P}_\lambda(z, \bar{z})) = \sum_{\mu\in\Lambda} (-1)^{|\lambda|} \frac{d_\lambda}{d_\mu} c_\mu(\lambda) \tilde{P}_\mu(z, \bar{z})$$

Now we apply the map π (see (2.1.6) above) which associates to each polynomial the corresponding differential operator by mapping $z \mapsto z, \bar{z} \mapsto \partial$. Recalling that by Proposition 2.3.1, the action of T corresponds under this map to the transposition operator τ, we have

$$\tau(\tilde{P}_\lambda(z,\partial)) = \sum_{\mu\in\Lambda}(-1)^{|\lambda|}\frac{d_\lambda}{d_\mu}c_\mu(\lambda)\tilde{P}_\mu(z,\partial) \tag{2.3.26}$$

or equivalently,

$$D^t_\lambda = \sum_{\mu\in\Lambda}(-1)^{|\lambda|}\frac{d_\lambda}{d_\mu}c_\mu(\lambda)D_\mu \tag{2.3.27}$$

When each side of this equation acts on an irreducible M_ν, we replace the operator on each side by its spectral function:

$$c^t_\lambda(\nu) = \sum_{\mu\in\Lambda}(-1)^{|\lambda|}\frac{d_\lambda}{d_\mu}c_\mu(\lambda)c_\mu(\nu) \tag{2.3.28}$$

By Proposition 2.3.2 this is

$$c_\lambda(\chi - w_0\nu) = \sum_{\mu\in\Lambda}(-1)^{|\lambda|}\frac{d_\lambda}{d_\mu}c_\mu(\lambda)c_\mu(\nu) \tag{2.3.29}$$ □

Theorem 2.3.3 implies an additional symmetry which the spectral functions satisfy:

2.3.4. Corollary. The expression $\dfrac{(-1)^{|\lambda|}c_\lambda(\chi - w_0\nu)}{d_\lambda}$ is symmetric in λ and ν. □

2.3.5. Corollary. For each $\lambda \in \Lambda$, $c_\lambda(\chi) = (-1)^{|\lambda|}d_\lambda = \dim M_\lambda$.

Proof: By evaluating the transposition formula, Theorem 2.3.3, at $\mu = 0$,

$$c_\lambda(\chi) = c_\lambda(\chi - w_0(0)) = \sum_{|\mu|\le|\lambda|}(-1)^{|\lambda|}\frac{d_\lambda}{d_\mu}c_\mu(\lambda)c_\mu(0). \tag{2.3.30}$$

The vanishing property, Corollary 2.2.13, implies that $c_\mu(0) = 0$ unless $\mu = (0)$, so we have

$$c_\lambda(\chi) = (-1)^0\frac{|d_\lambda|}{d_{(0)}}c_{(0)}(\lambda)c_{(0)}(0). \tag{2.3.31}$$

Since (0) is the highest weight of the trivial representation, $D_{(0)} = 1$, so that $c_{(0)}(\nu) = 1$ for all $\nu \in \Lambda$, and moreover, $d_{(0)} = 1$. Thus (2.3.31) becomes $c_\lambda(\chi) = |d_\lambda|$. □

2.3.6. Corollary. For each nontrivial $\lambda \in \Lambda$, $\sum_{|\mu| \leq |\lambda|} (-1)^{|\mu|} c_\mu(\lambda) = 0$.

Proof: By evaluating the transposition formula at χ,

$$c_\lambda(\chi - w_0(\chi)) = \sum_{|\mu|\leq|\lambda|} (-1)^{|\lambda|} \frac{d_\lambda}{d_\mu} c_\mu(\lambda) c_\mu(\chi) \tag{2.3.32}$$

$$c_\lambda(0) = \sum_{|\mu|\leq|\lambda|} (-1)^{|\lambda|} \frac{d_\lambda}{d_\mu} c_\mu(\lambda) |d_\mu|$$

by Corollary 2.3.5. Since $\lambda \neq 0$ by assumption, $c_\lambda(0) = 0$ by the vanishing property (Corollary 2.2.13), so we have

$$0 = \sum_{|\mu|\leq|\lambda|} (-1)^{|\mu|} |d_\lambda| c_\mu(\lambda) = |d_\lambda| \sum_{|\mu|\leq|\lambda|} (-1)^{|\mu|} c_\mu(\lambda) \tag{2.3.33}$$

$$0 = \sum_{|\mu|\leq|\lambda|} (-1)^{|\mu|} c_\mu(\lambda).$$

Remark: Theorem 2.3.3 and its corollaries are analogous to results obtained by Knop in the general multiplicity free case [cf. K1].

Chapter 3

Spectral functions for $S^2\mathbb{C}^n$ and $\bigwedge^2 \mathbb{C}^n$

In this chapter we restrict our investigation to the two special cases in which $G = GL_n(\mathbb{C})$ and V is the skew multiplicity free space $S^2\mathbb{C}^n$ or $\bigwedge^2 \mathbb{C}^n,\ n \geq 2$. We let Λ_n denote the set of highest weights actually occurring in $\bigwedge V$ as a G-module; the subscript may be omitted when the dependence on n is irrelevant. We will regard these weights concretely as partitions of length at most n, i.e. weakly decreasing n-tuples λ consisting of $\ell(\lambda)$ positive integers followed by $n - \ell(\lambda)$ 0s. It is primarily $\bigwedge V$ whose highest weights are of interest, rather than V, though we continue to denote by χ the sum of all weights of V itself. We also continue to denote by c_λ the spectral function for the action of the skew Capelli operator D_λ on V, so that D_λ acts on the irreducible module M_μ as multiplication by the scalar $c_\lambda(\mu)$.

$p_k = \sum_{i=1}^{n} z_i^k$ denotes the degree k power sum polynomial in n commuting variables, and $\mathbb{C}[p_1, p_3, p_5, ...]$ the algebra of *supersymmetric* polynomials in n variables on V, where n is understood from context.

3.1 Combinatorics of highest weights

There are two useful characterizations of Λ_n when $V = S^2\mathbb{C}^n$. Consider the standard basis $e_1, ..., e_n$ of $\mathbb{C}^n$, so that $\{e_ie_j, 1 \leq i \leq j \leq n\}$ is a basis for $S^2\mathbb{C}^n$. Following Howe's approach in [H, section 4.4], put $e_{ij} = e_ie_j$ and write these basis vectors in a triangular array

$$
\begin{array}{ccccccc}
e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & \cdots & e_{1n} \\
 & e_{22} & e_{23} & e_{24} & e_{25} & \cdots & e_{2n} \\
 & & e_{33} & e_{34} & e_{35} & \cdots & e_{3n} \\
 & & & e_{44} & e_{45} & \cdots & e_{4n} \\
 & & & & \vdots & \vdots & \vdots \\
 & & & & & & e_{nn}
\end{array} \tag{3.1.1}
$$

Now consider subsets of basis vectors with the property that if a given basis element is in the subset, then so are all basis elements above it and/or to the left of it in the array.

$$
\begin{array}{ccccccccc}
e_{11} & e_{12} & e_{13} & \cdots & e_{1r_3} & \cdots & e_{1r_2} & \cdots & e_{1r_1} \\
 & e_{22} & e_{23} & \cdots & e_{2r_3} & \cdots & e_{2r_2} & & \\
 & & e_{33} & \cdots & e_{3r_3} & & & & \\
 & & & \vdots & & & & &
\end{array} \tag{3.1.2}
$$

Howe shows that a highest weight vector of $\bigwedge V$ is obtained by taking the wedge product of all elements of such a subset, and that a highest weight vector for every irreducible submodule can be obtained in this way. Next consider the Young diagram of a highest weight λ defined by such a subset of the triangular array, and observe that the ith row of the array

$$
e_{ii} \quad e_{i,i+1} \quad e_{i,i+2} \quad \cdots \quad e_{i,r_i-1} \quad e_{i,r_i} \tag{3.1.3}
$$

contributes to the Young diagram of λ what Howe defines as an (r_i+1, r_i) hook, namely the Young diagram of $(r_i+1, 1^{r_i-1})$. Note that we must have $r_i \leq n$. Thus the Young diagram of λ is formed by nesting k such $(r+1, r)$ hooks, for a finite number k of different values of r; this property characterizes the elements of Λ_n, which consists of all weights formed by nesting $(r+1, r)$ hooks such that $r \leq n$.

3.1.1. Proposition. The set Λ_n of highest weights occurring in $\bigwedge S^2\mathbb{C}^n$ consists precisely of all weights formed by nesting $(r+1, r)$ hooks such that $r \leq n$.

3.1.2. Corollary. Λ_n embeds in Λ_{n+1} by

$\lambda = (\lambda_1, \lambda_2, ..., \lambda_{\ell(\lambda)}, 0^{n-\ell(\lambda)}) \mapsto (\lambda_1, \lambda_2, ..., \lambda_{\ell(\lambda)}, 0^{n-\ell(\lambda)+1})$ for each $\lambda \in \Lambda_n$

For example, when $n = 3$, the subsets of the triangular array which give rise to highest weight vectors are $\{\}$, $\{e_{11}\}$,$\{e_{11}, e_{12}\}$, $\{e_{11}, e_{12}, e_{22}\}$, $\{e_{11}, e_{12}, e_{13}\}$, $\{e_{11}, e_{12}, e_{13}, e_{22}\}$, $\{e_{11}, e_{12}, e_{13}, e_{22}\}$, $\{e_{11}, e_{12}, e_{13}, e_{22}, e_{23}\}$, and $\{e_{11}, e_{12}, e_{13}, e_{22}, e_{23}, e_{33}\}$. The corresponding highest weights are $(0, 0, 0)$, $(2, 0, 0)$, $(3, 1, 0)$, $(3, 3, 0)$, $(4, 1, 1)$, $(4, 3, 1)$, $(4, 4, 2)$, and $(4, 4, 4)$. Among the Young diagrams of these weights, that of $(2, 0, 0)$ consists of a single $(2, 1)$ hook, that of $(3, 1, 0)$ consists of a single $(3, 2)$ hook, and that of $(4, 1, 1)$ consists of a single $(4, 3)$ hook, while the other nontrivial diagrams consist of nestings of two or three of these.

An equivalent characterization, due to Knop, is based on the form of the Frobenius coordinates of elements of Λ_n. If λ is any partition $(\lambda_1, \lambda_2, ..., \lambda_r)$, then its dual partition $\lambda' = (\lambda'_1, \lambda'_2, ..., \lambda'_k)$ is the partition whose Young diagram is obtained from that of λ by interchanging its rows with its columns. λ is said to have Frobenius coordinates

$$\begin{pmatrix} \alpha_1 & \alpha_2 & ... & \alpha_k \\ \beta_1 & \beta_2 & ... & \beta_k \end{pmatrix},$$

where $\alpha_i = \lambda_i - i$ and $\beta_i = \lambda'_i - i, i = 1, 2, .., k,$ and k is the largest index such that $\lambda_i - i > 0$. This description of a partition λ can be viewed as a decomposition of λ into nested hooks of shapes $(\alpha_1 + 1, 1^{\beta_1}), (\alpha_2 + 1, 1^{\beta_2}), ..., (\alpha_1 + k, 1^{\beta_k})$.

In private conversation, Knop reformulated Howe's characterization of Λ_n by observing that weights which satisfy Howe's nested hook property are precisely those whose Frobenius coordinates are of the form

$$(3.1.4) \quad \begin{pmatrix} \alpha_1 & \alpha_2 & ... & \alpha_k \\ \alpha_1 - 1 & \alpha_2 - 1 & ... & \alpha_k - 1 \end{pmatrix}$$

where $\sum_i \alpha_i \leq n$. Since Frobenius coordinates of this form are determined by their top row alone, whose entries are by definition positive and strictly decreasing, the partitions in Λ_n correspond bijectively to *strict* partitions of integers $\leq n$. The correspondence is indeed bijective, since any strict partition of an integer $\leq n$ can be placed into the top row of Frobenius coordinates of the indicated form and thereby determine a highest weight of $\bigwedge S^2\mathbb{C}^n$.

For each $\lambda \in \Lambda$, denote the associated strict partition $\check{\lambda}$. Observe that each $\check{\lambda}_i$ equals half the number of boxes in the corresponding $(\lambda_i - i + 1, \lambda_i - i)$ hook. It follows that

$$(3.1.5) \qquad |\lambda| = 2|\check{\lambda}|.$$

For example, when $\lambda = (4, 3, 1)$, whose Young diagram is composed of a $(4, 3)$ hook and a $(2, 1)$ hook, we have $\check{\lambda} = (3, 1)$, so that indeed $|(4, 3, 1)|= 8 = 2|(3, 1)|$.

In the case $\bigwedge \bigwedge^2 \mathbb{C}^n$, the same reasoning using the basis $\{e_i \wedge e_j, 1 \leq i < j \leq n\}$ shows that the highest weights are those whose Young diagrams are nested $(r, r + 1)$ hooks and whose Frobenius coordinates are therefore of the form

$$(3.1.6) \qquad \begin{pmatrix} \alpha_1 - 1 & \alpha_2 - 1 & ... & \alpha_k - 1 \\ \alpha_1 & \alpha_2 & ... & \alpha_k \end{pmatrix}.$$

Thus a bijection holds between highest weights of $\bigwedge \bigwedge^2 \mathbb{C}^n$ and strict partitions of integers $\leq n$, by associating to each highest weight its bottom row Frobenius coordinates. Moreover, the characterization by nested hooks shows that λ is a highest weight of $\bigwedge S^2\mathbb{C}^n$ if and only if its dual λ' is a highest weight of $\bigwedge \bigwedge^2 \mathbb{C}^n$, so that λ and λ' correspond to the same strict partition $\check{\lambda}$. This fact will ultimately have the consequence, which is surprising a priori, that our two special cases $\bigwedge S^2\mathbb{C}^n$ and $\bigwedge \bigwedge^2 \mathbb{C}^n$ have essentially the same spectral theory.

This bijective correspondence between highest weights and strict partitions can be realized in another way, which will be crucial to the subsequent discussion. Let

$$(3.1.7) \qquad \bar{\rho} = \left(\frac{n-1}{2}, \frac{n-3}{2}, ..., \frac{-n+1}{2} \right)$$

the half sum of the positive roots, as it were the "true" ρ, and, as in chapter 2, denote by χ the sum of all weights of V. Throughout this chapter we will use

$$(3.1.8) \qquad \rho = \bar{\rho} - \frac{1}{2}\chi$$

When $V = S^2C^n, \chi = (n+1, n+1, ..., n+1)$, so that $\rho = (-1, -2, -3, ..., -n)$ and when $V = \bigwedge^2 C^n, \chi = (n-1, n-1, ..., n-1)$, so that $\rho = (0, -1, -2, ... -n+1)$.

Consider any weakly decreasing m-tuple of integers $(a_1, ..., a_m)$ with at least $a_1 > 0$, and let r be the largest index such that $a_r > 0$. Then let $(a_1, ..., a_m)_+$ denote $(a_1, a_2, ..., a_r)$, i.e. the r-tuple obtained by discarding all 0 or negative entries of $(a_1, ..., a_m)$.

3.1.3. Lemma. Highest weights $\mu \in \Lambda_n$ correspond bijectively to strict partitions of integers $\leq n$, under the map $\mu \mapsto (\mu + \rho)_+$.

Proof: If $V = S^2\mathbb{C}^n$, then the coordinates of $(\mu + \rho)_+ = (\mu_1 - 1, \mu_2 - 2, ..., \mu_k - k)$ are by definition the top-row Frobenius coordinates of μ. Similarly, if $V = \bigwedge^2 \mathbb{C}^n$, and μ has top-row Frobenius coordinates $(\mu_1 - 1, \mu_2 - 2, ..., \mu_k - k)$, then its bottom-row Frobenius coordinates are $(\mu_1, \mu_2 - 1, ..., \mu_k - k + 1) = (\mu + \rho)_+$. $\square$

3.1.4. Lemma. Let $V = S^2\mathbb{C}^n$ and $\rho = (-1, -2, -3, ..., -n)$. For any $\mu \in \Lambda_n$, the unordered set of integers $\{|\mu_1 + \rho_1|, |\mu_2 + \rho_2|, ..., |\mu_k + \rho_k|, ..., |\mu_n + \rho_n|\}$ is precisely the set $\{1, 2, ..., n\}$.

In the statement of the lemma it is understood that if $\ell(\mu) < n$ then $\mu_i = 0$ for each

$i = \ell(\mu) + 1, ..., n$.

A few examples are as follows:

$\mu = (2, 0), \quad \mu + \rho = (1, -2, -3, -4, ...)$

$\mu = (3, 1), \quad \mu + \rho = (2, -1, -3, -4, ...)$

$\mu = (3, 3), \quad \mu + \rho = (2, 1, -3, -4, ...)$

$\mu = (4, 1, 1), \ \mu + \rho = (3, -1, -2, -4, ...)$

$\mu = (4, 3, 1), \ \mu + \rho = (3, 1, -2, -4, ...)$

$\mu = (4, 4, 2), \ \mu + \rho = (3, 2, -1, -4, ...)$

$\mu = (4, 4, 4), \ \mu + \rho = (3, 2, 1, -4, ...)$

A more extensive list is given in Chapter 5, Section 3.

Proof: We prove the result by induction on the number of nested $(r+1, r)$ hooks making up the Young diagram of μ.

If the diagram of μ consists of one hook $(\mu_1, \mu_1 - 1)$, then $\mu + \rho =$
$(\mu_1 - 1, 1 - 2, 1 - 3, ..., 1 - (\mu_1 - 1), -(\mu_1 - 1 + 1), -(\mu_1 - 1 + 2), -(\mu_1 - 1 + 3), ...) =$
$(\mu_1 - 1, -1, -2, ..., 2 - \mu_1, -\mu_1, -\mu_1 - 1, \mu_1 - 2...)$.

The first $\mu_1 - 1$ coordinates are, in absolute value, a cyclic permutation of the integers $1, 2, ..., \mu_1 - 1$. The subsequent coordinates are, in absolute value, $\mu_1, \mu_1 + 1, ...$ Thus the result holds for μ consisting of one hook.

Now suppose that the result holds for weights whose diagrams are composed of k nested $(r_i + 1, r_i)$ hooks. Consider a weight μ composed of $k + 1$ nested hooks. Denote by $\tilde{\mu}$ the partition formed by the first k of these hooks, and write $\nu = \tilde{\mu} + \rho = (\nu_1, \nu_2, ..., \nu_{\mu_1 - 1}, -\mu_1, -\mu_1 - 1, ...)$, so that by induction the integers $|\nu_1|, |\nu_2|, ..., |\nu_{\mu_1 - 1}|$ form a permutation of $1, 2, ..., \mu_1 - 1$. We now consider $\mu + \rho$, whose first k coordinates are identical to those of $\nu = \tilde{\mu} + \rho$.

The coordinates first differ at the $k + 1st$ position, where we add $\mu_{k+1} - k$. They also differ from the $k + 2nd$ to the $\mu_{k+1} - k - 1st$ coordinates, to each of which we add 1;

after this point $\mu + \rho$ agrees with $\tilde{\mu} + \rho$ in all subsequent entries. We can write the entries in which they do differ as
$(\nu_{k+1} + \mu_{k+1} - k, \nu_{k+2} + 1, \nu_{k+3} + 1, ..., \ \nu_{\mu_{k+1}-k-1} + 1)$. It suffices to show that these integers, in absolute value, are a permutation of $(\nu_{k+1}, \nu_{k+2}, ..., \nu_{\mu_{k+1}-k-1})$.

Note the simple but crucial fact that since the hooks are nested, $\tilde{\mu}_{k+1} = \tilde{\mu}_{k+2} = ... = \tilde{\mu}_{\mu_{k+1}-k-1}$, i.e. the length of the $k+1st$ nested hook must be less than that of any previous hook. Thus $\nu_{k+1} = \nu_{k+2} + 1 = ... = \nu_{\mu_{k+1}-k-1} + \mu_{k+1} - k - 2$.

We can be even more specific, and assert that $\nu_{k+1} = -1$. After nesting k hooks, the $k+1st$ row has k boxes. The kth hook cannot be a $(2,1)$ hook, i.e. just a row with 2 boxes, or else we could not nest the $k+1st$ hook. So after nesting k hooks, there are exactly k boxes in the $k+1st$ row, and $\nu_{k+1} = -1$.

Our task of showing that in absolute value,

(3.1.9) $\quad (\nu_{k+1} + \mu_{k+1} - k, \nu_{k+2} + 1, \nu_{k+3} + 1, ..., \nu_{\mu_{k+1}-k-1} + 1)$

is a permutation of

(3.1.10) $\quad (\nu_{k+1}, \nu_{k+2}, ..., \nu_{\mu_{k+1}-k-1})$,

now reduces to showing that in absolute value,

(3.1.11) $\quad (-1 + \mu_{k+1} - k, -1 - 1 + 1, -1 - 2 + 1, ..., -1 - \mu_{k+1} + k + 2 + 1) = (\mu_{k+1} - k - 1, -1, -2, ..., -\mu_{k+1} + k + 2)$

is a permutation of

(3.1.12) $\quad (-1, -1 - 1, ..., -1 - \mu_{k+1} + k + 2) = (-1, -2, -3..., -\mu_{k+1} + k + 1)$.

These do differ by a (cyclic) permutation, and the result follows. $\square$

3.1.5. Lemma. Let $V = \bigwedge^2 \mathbb{C}^n$ and $\rho = (0, -1, -2, ..., -n+1)$.

For any $\mu \in \Lambda_n$, the unordered set of integers $\{|\mu_1 + \rho_1|, |\mu_2 + \rho_2|, ..., |\mu_n + \rho_n|\}$ is precisely the set $\{0, 1, 2, ..., n-1\}$.

Proof: The induction on the number of nested $(r-1, r)$ hooks making up the Young diagram of μ is virtually identical to the proof of Lemma 3.1.4, but uses $\rho = (0, -1, -2, ..., -n+1)$ instead of $(-1, -2, -3, ..., -n)$. □

Examples:

$\mu = (1,1), \quad \mu + \rho = (1, 0, -2, -3, ...)$

$\mu = (2,1,1), \quad \mu + \rho = (2, 0, -1, -3, ...)$

$\mu = (2,2,2), \quad \mu + \rho = (2, 1, 0, -3, ...)$

$\mu = (3,1,1,1), \quad \mu + \rho = (3, 0, -1, -2, ...)$

$\mu = (3,2,2,1), \quad \mu + \rho = (3, 1, 0, -2, ...)$

$\mu = (3,3,2,2), \quad \mu + \rho = (3, 2, 0, -1, ...)$

$\mu = (3,3,3,3), \quad \mu + \rho = (3, 2, 1, 0, ...)$

In each of our two examples of skew multiplicity free spaces V, the number of highest weights λ of $\bigwedge V$ with $|\lambda| = 2d$ equals the number of strict partitions of d. By a celebrated result of Euler, the number of strict partitions of d equals the number of partitions of d into odd parts. Now monomials of degree d in the odd degree power sums $p_1, p_3, p_5, ...$ correspond bijectively to partitions of d into odd parts. For example, the partitions of 6 into odd parts are $(1^6), (1^3, 3), (1, 5)$, and (3^2), and the corresponding monomials are $p_1^6, p_1^3 p_3, p_1 p_5$, and p_3^2. Since monomials of degree $\leq d$ form a basis for the space of supersymmetric polynomials of degree $\leq d$, we have proven the following:

3.1.6. Proposition: For any integers $n \geq d \geq 0$, the number of highest weights $\lambda \in \Lambda_n$ with $|\lambda| \leq 2d$ equals the dimension of the space of supersymmetric polynomials of degree $\leq d$. □

3.2 Factorial Schur Q functions

Okounkov defines [I1, I2] the factorial Schur Q functions, a family of supersymmetric functions which, like the classical Schur Q functions of which they are an analogue, are indexed by strict partitions.

Let $[z \downarrow k] = \prod_{i=1}^{k}(z-i+1),\ k = 0, 1, 2, ...$ For any strict partition λ and $n \geq \ell(\lambda)$, let

$$(3.2.1) \qquad F_\lambda(z_1, ..., z_n) = \prod_{i=1}^{\ell(\lambda)}[z_i \downarrow \lambda_i] \prod_{i \leq \ell(\lambda), i<j \leq n} \frac{z_i + z_j}{z_i - z_j}$$

Then the factorial Schur Q polynomial indexed by λ is defined by

3.2.1. Definition. $Q^*_\lambda = \dfrac{2^{\ell(\lambda)}}{(n-\ell(\lambda))!} \sum_{w \in S_n} F_\lambda(z_{w(1)}, ..., z_{w(n)})$

From this definition it is clear that Q^*_λ has degree $|\lambda|$. Ivanov demonstrates [I2] that Q^*_λ is supersymmetric and vanishes at all strict partitions μ such that $|\mu| \leq |\lambda|$, but not at λ itself. Moreover, Q^*_λ satisfies the extra vanishing condition that $Q^*_\lambda(\mu) = 0$ if λ and μ are strict partitions such that $\lambda \nsubseteq \mu$. Furthermore, for each $n \geq \ell(\lambda)$ we can express Q^*_λ as a polynomial in n variables, but the values taken by Q^*_λ are independent of n.

Ivanov shows that $Q^*_\lambda(\lambda) = H(\lambda)$, where

$$(3.2.2) \qquad H(\lambda) = \prod_{t=1}^{\ell(\lambda)} \lambda_t! \prod_{i<j} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j}.$$

$H(\lambda)$ is indeed defined, since λ is assumed to be a strict partition. We will consider a rescaled version of the factorial Schur Q functions, defined by

$$(3.2.3) \qquad q_\lambda = \frac{1}{H(\lambda)} Q^*_\lambda,$$

so that $q_\lambda(\lambda) = 1$. Thus q_λ has the same properties of supersymmetry, degree, and independence of n as Q^*_λ. The basic vanishing property it satisfies can be expressed as

(3.2.4) $\quad q_\lambda(\mu) = \delta_{\lambda\mu}$ for all strict partitions μ such that $|\mu| \leq |\lambda|$.

Let $V = S^2\mathbb{C}^n$ and $\rho = (-1, -2, ..., -n)$. Let p_k denote the degree k power sum polynomial in n variables, i.e. $p_k = \sum_{i=1}^n z_i^k$.
We define an automorphism ϕ of $\mathbb{C}[p_1, p_3, p_5, ..., p_s]$ in n variables, where s is the largest odd number $\leq n$, by

(3.2.5) $\quad \phi(p_k) = 2p_k + p_k(\rho)$ for each $k = 2j+1, j = 0, 1, 2, ..., \frac{s-1}{2}$

Recall that to each $\lambda \in \Lambda_n$ there is associated a unique strict partition $\check{\lambda}$.

3.2.2. Lemma. $p_k(\lambda+\rho) = 2p_k(\check{\lambda}) + p_k(\rho)$ for every $k = 2j+1, j = 0, 1, 2, ...$ and $\lambda \in \Lambda_n$.

Proof: By Corollary 3.1.5, the coordinates of $\lambda+\rho$ are in absolute value a permutation of $\{1, 2, ..., n\}$; the positive terms are precisely the top row Frobenius coordinates of λ, and are equal to the coordinates $\check{\lambda}_1, ..., \check{\lambda}_{\ell(\check{\lambda})}$ of $\check{\lambda}$. Thus

(3.2.6)
$$p_k(\lambda+\rho) = \check{\lambda}_1^k + ... + \check{\lambda}_{\ell(\check{\lambda})}^k - \sum_{1\leq j\leq n,\ j\neq\check{\lambda}_i} j^k$$
$$= 2(\check{\lambda}_1^k + ... + \check{\lambda}_{\ell(\check{\lambda})}^k) - 1^k - 2^k - ... - n^k$$
$$= 2p_k(\check{\lambda}) + p_k(\rho)$$
□

3.2.3. Proposition. For any $p \in \mathbb{C}[p_1, p_3, p_5, ..., p_s]$ in n variables, $p(\lambda+\rho) = \phi(p)(\check{\lambda})$ for every $\lambda \in \Lambda_n$.

Proof: We can write p as a polynomial in the odd degree power sums of degree $\leq n$, say $p = f(p_1, p_3, ...p_s)$. Then

$$(3.2.7) \qquad \phi(p) = f(2p_1 + p_1(\rho), 2p_3 + p_3(\rho), ..., 2p_s + p_s(\rho))$$

The proposition now follows directly from Lemma 3.2.2. □

3.2.4. Lemma.: If $p \in \mathbb{C}[p_1, p_3, ..., p_s]_{\leq d}$ vanishes at all strict partitions $\check{\lambda}$ such that $|\check{\lambda}| \leq d$ then $p = 0$ identically.

Proof: The rescaled factorial Schur Q functions $\{q_{\check{\mu}} : |\mu| \leq d\}$ form a basis of $\mathbb{C}[p_1, p_3, p_5, ..., p_s]_{\leq d}$, since the vanishing property (3.2.4) which they satisfy guarantees linear independence, and we conclude from Proposition 3.1.6 that they span. It follows that p has a unique expansion $p = a_0 q_{\check{\mu}_0} + a_1 q_{\check{\mu}_1} + ... + a_k q_{\check{\mu}_k}$, where $\check{\mu}_0, ..., \check{\mu}_k$ are all the strict partitions of size $\leq d$, in order of weakly increasing size. But the vanishing property (3.2.4) implies that only $q_{\check{\mu}_0} = q_{(0)}$ does not vanish at (0), hence $a_0 = 0$. Likewise, by induction, each $a_i = 0$, $i = 1, ..., k$, and in fact no such supersymmetric polynomial p of degree > 0 exists. □

3.2.5. Lemma.: For any integers $n \geq d > 0$, if $p \in \mathbb{C}[p_1, p_3, ..., p_s]_{\leq d}$ satisfies $p(\lambda + \rho) = 0$ on the set $\{\lambda \in \Lambda_n : |\lambda| \leq 2d\}$, then $p = 0$ identically.

Proof: Suppose that there exists such a supersymmetric polynomial p of degree $d > 0$. Then by Proposition 3.2.3, $\phi(p)$ vanishes at all strict partitions $\check{\lambda}$ such that $|\check{\lambda}| \leq d$. Since ϕ is an automorphism, the result now follows from Lemma 3.2.4. □

3.3 Characterization Theorem

Let $V = S^2\mathbb{C}^n$, $\rho = (-1, -2, ..., -n)$. The goal of this section is to prove the following Characterization Theorem:

3.3.1. Theorem. For each $\lambda \in \Lambda_n, n \geq \dfrac{|\lambda|}{2}$ there exists a polynomial p_λ which satisfies the following properties.

CT1a. $p_\lambda(\mu + \rho) = c_\lambda(\mu)$ for every $\mu \in \Lambda_n$.

CT1b. $p_\lambda(\mu + \rho) = 0$ whenever $|\mu| \leq |\lambda|$ unless $\mu = \lambda$, and $p_\lambda(\lambda + \rho) = 1$.

CT2. p_λ is supersymmetric.

CT3. p_λ has degree $\dfrac{|\lambda|}{2}$.

Furthermore, p_λ is uniquely determined by CT1b, CT2, and CT3.

Preliminary remark 1: Implicitly, p_λ depends on n, although it will be shown in Section 3.4 that this dependence is much weaker than it appears to be. Indeed, it will be shown that for all n large enough, the values of p_λ on ρ-shifted arguments are independent of n. On the one hand, since G acts on $\bigwedge S^2\mathbb{C}^n$ for each n, we can consider a fixed value of n and study all the functions p_λ. On the other hand we can fix a λ and study the functions p_λ as n increases, in particular for all $n \geq \frac{|\lambda|}{2}$. The statements of the Characterization Theorem should be understood in the latter sense, with λ fixed and for all $n \geq \frac{|\lambda|}{2}$.

Preliminary remark 2: Our strategy is to prove the theorem for a certain class of invariant differential operators, namely those which arise from the center of the universal enveloping algebra of G. A combinatorial argument then shows that in fact all the

G-invariant operators on V belong to this class.

Proof: The action of G on V gives rise to an action of its Lie algebra $\mathfrak{g}$ on V as endomorphisms

$$(3.3.1) \qquad \mathfrak{g} \to End(V)$$

hence of $\mathfrak{U}(\mathfrak{g})$ on V as differential operators

$$(3.3.2) \qquad \mathfrak{U}(\mathfrak{g}) \to \mathcal{PD}(V)$$

and thus, passing to G-invariants, of the center $\mathfrak{Z}(\mathfrak{g})$ of the universal enveloping algebra on V as invariant differential operators

$$(3.3.3) \qquad \Psi : \mathfrak{Z}(\mathfrak{g}) \to \mathcal{PD}(V)^G$$

A priori it is not clear whether Ψ is surjective, but as indicated in the second preliminary remark to this proof, our strategy is to establish CT1, CT2, and CT3 for the class of operators which do arise from $\mathfrak{Z}(\mathfrak{g})$, and leave it to the concluding part of the proof to show that in fact every invariant operator on V arises in this way.

Consider the following diagram, which will be crucial to the proof of each step of the Characterization Theorem.

$$(3.3.4) \qquad \begin{array}{ccc} \mathfrak{Z}(\mathfrak{g}) & \overset{\cong}{\leftarrow} & \mathbb{C}[p_1, p_2, p_3, ...] \\ \Psi \downarrow & & \downarrow \\ \mathcal{PD}(V)^G & \to & Maps(\Lambda_n, \mathbb{C}) \end{array}$$

The map $\mathfrak{Z}(\mathfrak{g}) \leftarrow \mathbb{C}[p_1, p_2, p_3, ...]$ is the canonical isomorphism of symmetric polynomials with $\mathfrak{Z}(\mathfrak{g})$.

The map $\mathcal{PD}(V)^G \rightarrow Maps(\Lambda_n, \mathbb{C})$ is the association of a spectral function c_D to each invariant operator D on V.

The left vertical map Ψ has just been explained. The right vertical map, whose stated properties we wish to verify, assigns to each symmetric polynomial p a function c such that $c(\mu) = p(\mu + \rho)$ for each $\mu \in \Lambda_n$.

Proof of CT1a: Any symmetric polynomial p can be viewed as an element $\xi \in \mathfrak{Z}(\mathfrak{g})$ and then identified with its image $D \in \mathcal{PD}(V)^G$. By the Harish Chandra isomorphism, D acts on $M_\mu \subset \bigwedge V$ by the scalar $c_D(\mu) = p(\mu + \rho)$, which proves that the diagram (3.3.4) is commutative and establishes CT1a for operators arising from $\mathfrak{Z}(\mathfrak{g})$.

Proof of CT1b:

CT1b follows directly from CT1a and Corollary 2.2.13, which establishes the vanishing property for the general case of spectral functions of invariant operators on SMF spaces.

Proof of CT2:

We wish to show that our commutative diagram (3.3.4) can in fact be written

$$
\begin{array}{ccc}
\mathfrak{Z}(\mathfrak{g}) & \leftarrow & \mathbb{C}[p_1, p_3, p_5, ...] \\
\downarrow & & \downarrow \\
\mathcal{PD}(V)^G & \rightarrow & Maps(\Lambda_n, \mathbb{C})
\end{array}
\qquad (3.3.5)
$$

i.e. that the symmetric polynomials, generated by the power sum polynomials, can be replaced by the supersymmetric polynomials, which are generated by the odd degree power sums.

Lemmas 3.1.4 and 3.1.5 have the following immediate consequence:

3.3.2. Corollary. For each even degree power sum polynomial p_{2k} there is a constant N_k such that for every $\mu \in \Lambda_n$, $p_{2k}(\mu + \rho) = N_k$, where $\rho = (-1, -2, -3, ... - n)$. □

Corollary 3.3.2 shows that to each even degree power sum is associated a spectral function which is merely a constant function, or equivalently, that $p_{2k} - N_k$ is in the kernel of the map which assigns spectral functions to symmetric polynomials. It follows that each spectral function has a supersymmetric preimage under this map. This proves CT2, that for each $\lambda \in \Lambda_n$, p_λ can be chosen to be supersymmetric.

Proof of CT3:

In light of the supersymmetry result CT2, we can indeed write

$$\begin{array}{ccc} \mathfrak{Z}(\mathfrak{g}) & \leftarrow & \mathbb{C}[p_1, p_3, p_5, ...] \\ \downarrow & & \downarrow \\ \mathcal{PD}(V)^G & \rightarrow & Maps(\Lambda_n, \mathbb{C}) \end{array} \tag{3.3.6}$$

Consider fixed $d \leq \frac{|\lambda|}{2} \leq n$. Another consequence of the Harish Chandra isomorphism is that if $\xi \in \mathfrak{Z}(\mathfrak{g})$ has degree d then the spectral function c_D of its associated invariant operator $D = \Psi(\xi)$ is interpolated (in the ρ-shifted sense) by a polynomial of degree d. What is not clear a priori is whether D also has degree exactly d; conceivably some cancellation in the highest degree term of D could cause it to have degree less than d. Also unclear a priori is whether $\mathfrak{Z}(\mathfrak{g})$ maps surjectively onto $\mathcal{PD}(V)^G$. The proof will address both of these issues. At this stage the degree conditions implied by the Harish Chandra isomorphism and the fact that $\mathfrak{Z}(\mathfrak{g})$, $\mathcal{PD}(V)^G$, and $\mathbb{C}[p_1, p_3, p_5, ...]$ are all filtered by degree, allow us to restrict our diagram to elements of degree at most d. Thus we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{Z}(\mathfrak{g})_{\leq d} & \leftarrow & \mathbb{C}[p_1, p_3, p_5, ..., p_s]_{\leq d} \\ \downarrow & & \downarrow \\ \mathcal{PD}(V)^G_{\leq d} & \rightarrow & Maps(\Lambda_n, \mathbb{C}) \end{array} \tag{3.3.7}$$

where s is the largest odd integer such that $s \leq d$. We wish to show that there is a bijection between $\mathcal{PD}(V)^G_{\leq d}$ and $\mathbb{C}[p_1, p_3, p_5, ...]_{\leq d}$. First we argue that

$$\begin{array}{c} \mathbb{C}[p_1, p_3, p_5, ..., p_s]_{\leq d} \\ \downarrow \\ Maps(\Lambda_n, \mathbb{C}) \end{array} \tag{3.3.8}$$

is injective for all $n \geq d$. Note first that since $s \leq n$, the odd degree power sums $p_1, p_3, ..., p_s$ are algebraically independent. Furthermore, it follows directly from Lemma 3.2.5 that $\mathbb{C}[p_1, p_3, p_5, ..., p_s]_{\leq d}$ maps injectively into $Maps(\Lambda_n, \mathbb{C})$ if Λ_n contains all highest weights ν with $|\nu| \leq 2d$. This is the case for all $n \geq d$, by Proposition 3.1.1, so that indeed

$$\begin{array}{c} \mathbb{C}[p_1, p_3, p_5, ..., p_s]_{\leq d} \\ \downarrow \\ Maps(\Lambda_n, \mathbb{C}) \end{array} \tag{3.3.9}$$

is injective for all $n \geq d$.

It follows that the composite map

$$\begin{array}{ccc} \mathfrak{Z}(\mathfrak{g})_{\leq d} & \leftarrow & \mathbb{C}[p_1, p_3, p_5, ..., p_s]_{\leq d} \\ \downarrow & & \\ \mathcal{PD}(V)^G_{\leq d} & & \end{array} \tag{3.3.10}$$

is also injective. Now to prove CT3 it suffices to show that this map is also surjective, i.e. that this map preserves degree, or more specifically that the map on the associated filtered algebras preserves degree. This is done by proving that for each d, the dimension of the subspace of degree d modulo degree $d-1$ invariant differential operators equals the dimension of the subspace of degree d modulo degree $d-1$ supersymmetric functions.

$\dim(\mathcal{PD}(V)^G_{\leq d}/\mathcal{PD}(V)^G_{\leq d-1})$ equals the number of highest weights μ with $\frac{|\mu|}{2} = d$, which equals the number of strict partitions of d. On the other hand, $\dim(\mathbb{C}[p_1, p_3, p_5, ...]_{\leq d}/\mathbb{C}[p_1, p_3, p_5, ...]_{\leq d-1})$ equals the number of monomials of degree d in the odd degree power sums; such monomials correspond bijectively to partitions of d into odd parts. We appeal again to Euler's result that the number of strict partitions of d equals the number of partitions of d into odd parts, for all positive integers d. It follows that our injective map $\mathcal{PD}(V)^G_{\leq d} \leftarrow \mathbb{C}[p_1, p_3, p_5, ...]_{\leq d}$ is in fact a bijection, which proves CT3 for those invariant operators on V arising from $\mathfrak{Z}(\mathfrak{g})$. But these are in fact all the invariant operators on V, since the surjectivity of the composite map

$$\begin{array}{ccc} \mathfrak{Z}(\mathfrak{g})_{\leq d} & \leftarrow & \mathbb{C}[p_1, p_3, p_5, ..., p_s]_{\leq d} \\ \downarrow & & \\ \mathcal{PD}(V)^G_{\leq d} & & \end{array} \tag{3.3.11}$$

shows that

$$\begin{array}{c} \mathfrak{Z}(\mathfrak{g})_{\leq d} \\ \downarrow \\ \mathcal{PD}(V)^G_{\leq d} \end{array} \tag{3.3.12}$$

is itself surjective. This completes the proof of CT3.

The spectral polynomials, as a collection of functions, are in fact *characterized* by the supersymmetry, degree, and vanishing properties:

The spectral polynomials $\{p_\mu, |\mu| \leq |\lambda|\}$ form a basis of the space of supersymmetric polynomials of degree $\leq \dfrac{|\lambda|}{2}$; the vanishing property guarantees linear independence, and indexing of polynomials by highest weights guarantees spanning. Now suppose that there exists a supersymmetric polynomial r_λ satisfying the same degree and vanishing properties as p_λ. Then we can write r_λ uniquely as a linear combination of the $p_\mu, |\mu| \leq |\lambda|$. Determining the coordinates of r_λ with respect to this basis is equivalent to solving a matrix system in which both the rows and columns of the matrix (which is square) are indexed by $\{\mu \in \Lambda_n, |\mu| \leq |\lambda|\}$ in order of increasing $|\mu|$, and whose (μ, ν)-entry is $p_\mu(\nu + \rho)$. Since the p_μ themselves satisfy the vanishing condition, the matrix is unit triangular, hence invertible, so that the system has a unique solution. It follows that $r_\lambda = p_\lambda$ identically.

This completes the proof of the Characterization Theorem. □

Remark 1: The uniqueness argument at the end of the proof of the Characterization Theorem is essentially that of Sahi in one of the earliest papers on the subject of spectral functions and the vanishing property [S2, p.573].

Remark 2: The same uniqueness argument shows that the rescaled factorial Schur Q function $q_{\check{\lambda}}$ considered in section 3.2 is uniquely determined by the properties of supersymmetry, having degree $|\check{\lambda}|$, and satisfying the vanishing condition $q_{\check{\lambda}}(\check{\mu}) = \delta_{\check{\lambda}\check{\mu}}$ for all $|\check{\mu}| \leq |\check{\lambda}|$.

We could prove an identical result for the special case $V = \bigwedge^2 \mathbb{C}^n$, but there is an even stronger observation to be made. Recall that μ is a highest weight of $\bigwedge S^2\mathbb{C}^n$ if and only if its dual μ' is a highest weight of $\bigwedge \bigwedge^2 \mathbb{C}^n$, at any rate for all $n \geq \frac{|\mu'|}{2}$. Comparing Lemmas 3.1.4 and 3.1.5 shows that the value of $p_k(\mu + (-1, -2, ..., n))$ in n variables equals the value of $p_k(\mu + (0, -1, -2, ..., n))$ in $n+1$ variables, for any k but in particular for odd k. Thus if we write $p_\lambda = f_\lambda(p_1, p_3, ...)$ in n variables, then $p_{\lambda'} = f_\lambda(p_1, p_3, ...)$ in $n+1$ variables. In other words the corresponding spectral polynomials p_λ and $p_{\lambda'}$ for our two different special cases of skew multiplicity free spaces are identical as expressions in the odd degree power sums, if the number of variables is 1 greater for $p_{\lambda'}$. This is one of the significant surprises of this entire investigation, that the two special cases have the same spectral theory. It had appeared at the beginning of this project that we would obtain two families of supersymmetric polynomials, each obtained by specializing the value of a parameter, by analogy with the Jack polynomials; this turned out not to be the case.

3.4 Equivalence with factorial Schur Q functions, and consequences

Recall from section 3.1 that to each highest weight $\lambda \in \Lambda$ there is associated a unique strict partition $\check{\lambda}$; the coordinates of $\check{\lambda}$ are precisely the top-row Frobenius coordinates of λ, or equivalently, the positive terms of $\lambda + \rho$. Moreover, $|\lambda| = 2|\check{\lambda}|$.

Recall from (3.2.5) above the automorphism ϕ of the algebra of supersymmetric polynomials in n variables, for any n, defined by its action on odd degree power sums:

(3.4.1) $\quad \phi(p_k) = 2p_k + p_k(\rho)$ for each $k = 2j+1, j = 0, 1, 2, ...$

This automorphism carries each spectral polynomial in n variables to its corresponding rescaled factorial Schur Q function:

3.4.1. Proposition. $\phi(p_\lambda) = q_{\check{\lambda}}$ for each $\lambda \in \Lambda_n$.

Proof: Since p_λ is supersymmetric, it can be written as a polynomial in the odd degree power sums, say

(3.4.2) $\quad p_\lambda = f_\lambda(p_1, p_3, ..., p_{s(\lambda)})$,

where $s(\lambda) = \frac{|\lambda|}{2}$ if $\frac{|\lambda|}{2}$ is odd and $s(\lambda) = \frac{|\lambda|}{2} - 1$ if $\frac{|\lambda|}{2}$ is even. Now

(3.4.3) $\quad \phi(p_\lambda) = f_\lambda(2p_1 + p_1(\rho), 2p_3 + p_3(\rho), ..., 2p_{s(\lambda)} + p_{s(\lambda)}(\rho))$

by definition of ϕ. Since $p_\lambda = f_\lambda(p_1, p_3, ..., p_{s(\lambda)})$ vanishes at every highest weight μ with $|\mu| \leq |\lambda|$ except for λ itself, where it takes the value 1, Proposition 3.2.3 implies that $f_\lambda(2p_1 + p_1(\rho), 2p_3 + p_3(\rho), ..., 2p_{s(\lambda)} + p_{s(\lambda)}(\rho))$ vanishes at every strict partition $\check{\mu}$ with $|\check{\mu}| \leq |\check{\lambda}|$ except for $\check{\lambda}$ itself, where it takes the value 1. Moreover, since p_λ has degree $\frac{|\lambda|}{2} = |\check{\lambda}|$ by the Characterization Theorem, so does $\phi(p_\lambda)$. Thus $\phi(p_\lambda)$ satisfies the supersymmetry, vanishing, and degree conditions which uniquely determine the rescaled factorial Schur Q function $q_{\check{\lambda}}$, hence $\phi(p_\lambda) = q_{\check{\lambda}}$. □

3.4.2. Corollary. $p_\lambda(\mu + \rho) = q_{\check{\lambda}}(\check{\mu})$ for each $\lambda, \mu \in \Lambda_n$. Equivalently, $c_\lambda(\mu) = q_{\check{\lambda}}(\check{\mu})$ for each $\lambda, \mu \in \Lambda_n$.

3.4.3. Corollary. The values of c_λ are independent of n for all $n \geq \max(\frac{|\lambda|}{2}, \frac{|\mu|}{2})$.

3.4.4. Corollary. c_λ satisfies the extra vanishing property, i.e. for all $\lambda, \mu \in \Lambda_n, c_\lambda(\mu) = 0$ if $\lambda \not\subseteq \mu$.

3.4.5. Corollary. The expression $f_\lambda(2p_1 + p_1(\rho), 2p_3 + p_3(\rho), ..., 2p_s + p_{s(\lambda)}(\rho))$ considered in statement (3.4.3) in the proof of Proposition 3.4.1, is independent of n.

In addition to the properties indicated in Corollaries 3.3.5 and 3.3.6 which the spectral functions inherit from the $q_{\check{\lambda}}$, they also acquire a combinatorial interpretation. Ivanov proves [I2 p.4200] that if $h_{\check{\lambda}/\check{\mu}}$ denotes the number of shifted skew tableaux of shape $\check{\lambda}/\check{\mu}$, where $h_{\check{\lambda}/\check{\mu}} = 0$ if $\check{\mu} \not\subseteq \check{\lambda}$, then

$$(3.4.4) \qquad h_{\check{\lambda}/\check{\mu}} = 2^{-\ell(\check{\lambda})} h_{\check{\lambda}/(0)} \cdot \frac{Q^*_{\check{\mu}}(\check{\lambda})}{|\check{\lambda}| \downarrow |\check{\mu}|}$$

3.4.6. Corollary. $h_{\check{\lambda}/\check{\mu}} = 2^{-\ell(\check{\lambda})} h_{\check{\lambda}/(0)} H(\check{\mu}) \cdot \dfrac{c_\mu(\lambda)}{|\check{\lambda}| \downarrow |\check{\mu}|}$

In a certain sense this combinatorial interpretation explains the extra vanishing property. It also implies a positivity, rationality, and *non-vanishing* result:

3.4.7. Corollary. $c_\lambda(\mu)$ is a strictly positive rational number for all $\lambda, \mu \in \Lambda_n$ such that $\lambda \subseteq \mu$.

Remark: The shifted Schur functions studied by Okounkov and Olshanskii in [OO1], which up to rescaling are the spectral functions for the multiplicity free space $\mathbb{C}^n \otimes \mathbb{C}^n$, play the same role in the corresponding formula for the number of skew tableaux of a given shape.

Having considered several properties of the $q_{\check{\lambda}}$ inherited by the spectral functions c_λ, we now study a major consequence for the $q_{\check{\lambda}}$ of the equivalence given by Corollary 3.3.4, namely that the $q_{\check{\lambda}}$ satisfy a transposition formula.

Recall from Chapter 2 that transposition of differential operators induces an involution of $\mathbb{C}[p_1, p_3, p_5, ...]$. For each odd degree power sum p_k, denote its image under transposition by p_k^t.

3.4.8. Lemma. $p_k^t(z) = -p_k(z)$ if k is odd.

Proof: The formula $c_{D^t}(\nu) = c_D(\chi - w_0\nu)$ asserted in Proposition 2.3.2 holds for all invariant differential operators which arise from $\mathfrak{Z}(\mathfrak{g})$, and the proof of the Characterization Theorem shows that this accounts for all the invariant operators on $S^2\mathbb{C}^n$. Thus for odd values of k we study the effect of transposition on the power sum symmetric polynomial p_k as follows:

$$(3.4.5) \qquad p_k^t(\nu + \rho) = p_k(\chi - w_0\nu + \rho)$$

Let $z = \nu + \rho$, so that

$$(3.4.6) \qquad p_k^t(z) = p_k(\chi - w_0(z - \rho) + \rho)$$

$$= p_k(\chi - w_0 z + w_0\rho + \rho)$$

Recall that

$$\rho = \bar{\rho} - \frac{1}{2}\chi \tag{3.4.7}$$

Since $w_0\bar{\rho} = -\bar{\rho}$ and $w_0\chi = \chi$, we have

$$\chi + w_0\rho + \rho = \chi + \bar{\rho} - \frac{1}{2}\chi - \bar{\rho} - \frac{1}{2}\chi = 0 \tag{3.4.8}$$

Thus (3.4.6) becomes

$$\begin{aligned} p_k^t(z) &= p_k(-w_0 z) \\ &= p_k(-z) \text{ since } p_k \text{ is symmetric} \\ &= -p_k(z) \text{ since } p_k \text{ has odd degree.} \end{aligned} \tag{3.4.9}$$

□

To produce a version of the transposition formula satisfied by the q_λ, we must calculate the effect of transposition on each odd degree power sum before and after twisting by ϕ. Consider the following diagram

$$\begin{array}{ccc} p_k & \overset{\phi}{\longrightarrow} & 2p_k + p_k(\rho) \\ \downarrow & & \downarrow \tau \\ -p_k & \longrightarrow & -2p_k - p_k(\rho) \end{array} \tag{3.4.10}$$

where the left vertical arrow represents transposition before twisting by ϕ, and the right vertical arrow represents the map, denoted τ, which is transposition after twisting by ϕ.

Thus

$$2\tau(p_k) + p_k(\rho) = -2p_k - p_k(\rho), \tag{3.4.11}$$

hence

$$(3.4.12)\qquad \tau(p_k) = -p_k - p_k(\rho)$$

We can now write a somewhat explicit version of the transposition formula for the rescaled factorial Schur Q polynomials. Since $q_{\check{\lambda}}$ is supersymmetric, it can be written as a polynomial in the odd degree power sums,

$$(3.4.13)\qquad q_{\check{\lambda}} = g_{\check{\lambda}}(p_1, p_3, p_5, ..., p_{s(\check{\lambda})}).$$

Certainly the $q_{\check{\lambda}}$ satisfy the transposition formula in the sense that

$$(3.4.14)\qquad \tau(g_{\check{\lambda}})(p_1, p_3, p_5, ..., p_{s(\check{\lambda})}) = \sum_{|\check{\mu}|\leq|\check{\lambda}|} (-1)^{|\check{\lambda}|}\frac{d_\lambda}{d_\mu} q_{\check{\mu}}(\check{\lambda}) q_{\check{\mu}}$$

which can instead be written

3.4.9. Proposition.

$$g_{\check{\lambda}}(-p_1-p_1(\rho), -p_3-p_3(\rho), -p_5-p_5(\rho), ..., -p_{s(\check{\lambda})}-p_{s(\check{\lambda})}(\rho)) = \sum_{|\check{\mu}|\leq|\check{\lambda}|} (-1)^{|\check{\lambda}|}\frac{d_\lambda}{d_\mu} q_{\check{\mu}}(\check{\lambda}) q_{\check{\mu}}$$

Since this formula holds at all strict partitions, which form a Zariski dense subset of $\mathbb{C}^n$, the formula in fact holds on all of $\mathbb{C}^n$.

For example, when $\lambda = (4, 3, 1)$, so that $\check{\lambda} = (3, 1)$, we have

$$(3.4.15)\qquad q_{(3,1)} = \frac{1}{36}p_1^4 - \frac{1}{36}p_1p_3 - \frac{1}{12}p_1^3 + \frac{1}{12}p_3$$

To obtain the transpose of $q_{(3,1)}$ in 4 variables, we replace p_1 by $-p_1 + 1 + 2 + 3 + 4 = -p_1 + 10$ and p_3 by $-p_3 + 1 + 8 + 27 + 64 = -p_3 + 100$:

$$(3.4.16)\qquad \tau(q_{(3,1)}) = \frac{1}{36}p_1^4 - \frac{1}{36}p_1p_3 - \frac{37}{36}p_1^3 + \frac{7}{36}p_3 + \frac{85}{6}p_1^2 - \frac{250}{3}p_1 + 175$$

Proposition 3.4.9 asserts that its left hand side, given in (3.4.16), should equal its right hand side:

$$(3.4.17)\qquad \sum_{|\breve{\mu}|\leq 4} (-1)^4 \frac{d_{(4,3,1)}}{d_\mu} q_{\breve{\mu}}(3,1) q_{\breve{\mu}}$$

Although the next chapter will derive dimension formulas, we can already compute $d_{(4,3,1)}$ and each d_μ using Corollary 2.3.5, so (3.4.17) becomes

$$(3.4.18)\qquad \frac{175}{1}q_{(0)}(3,1)q_{(0)} - \frac{175}{10}q_{(1)}(3,1)q_{(1)} + \frac{175}{45}q_{(2)}(3,1)q_{(2)} - \frac{175}{50}q_{(2,1)}(3,1)q_{(2,1)} - \frac{175}{70}q_{(3)}(3,1)q_{(3)} + \frac{175}{175}q_{(3,1)}(3,1)q_{(3,1)}$$

Chapter 5 section 3 exhibits the $q_{\breve{\lambda}}$ explicitly up to degree 6:

$q_{(0)} = 1,\ q_{(0)}(3,1) = 1.$

$q_{(1)} = p_1,\ q_{(1)}(3,1) = 4,$

$q_{(2)} = \frac{1}{2}p_1^2 - \frac{1}{2}p_1,\ q_{(2)}(3,1) = 6$

$q_{(2,1)} = \frac{1}{18}p_1^3 - \frac{1}{18}p_3,\ q_{(2,1)}(3,1) = 2$

$q_{(3)} = \frac{1}{9}p_1^3 + \frac{1}{18}p_3 - \frac{1}{2}p_1^2 + \frac{1}{3}p_1,\ q_{(3)}(3,1) = 2$

$q_{(3,1)} = \frac{1}{36}p_1^4 - \frac{1}{36}p_3p_1 - \frac{1}{12}p_1^3 + \frac{1}{12}p_3,\ q_{(3,1)}(3,1) = 1$

Thus we have

$$(3.4.19)\qquad = 175 - \frac{175}{10}\cdot 4p_1 + \frac{175}{45}\cdot 6(\frac{1}{2}p_1^2 - \frac{1}{2}p_1) - \frac{175}{50}\cdot 2(\frac{1}{18}p_1^3 - \frac{1}{18}p_3) - \frac{175}{70}\cdot 2(\frac{1}{9}p_1^3 + \frac{1}{18}p_3 - \frac{1}{2}p_1^2 + \frac{1}{3}p_1) + \frac{1}{36}p_1^4 - \frac{1}{36}p_3p_1 - \frac{1}{12}p_1^3 + \frac{1}{12}p_3$$

After simplifying, this equals (3.4.16), as required.

Chapter 4

Dimension formulas

We continue to restrict our investigation to the SMF spaces $S^2\mathbb{C}^n$ and $\bigwedge^2\mathbb{C}^n$.

Let λ be a highest weight of $\bigwedge V$, and $M_\lambda \subset \bigwedge V$ a submodule of highest weight λ.

Then the following dimension formulas hold:

4.1.1. Proposition. If $V = S^2\mathbb{C}^n$ and $\lambda \in \Lambda_n$ has top-row Frobenius coordinates $\alpha_1, \alpha_2, ..., \alpha_k$, then

$$\dim M_\lambda = \frac{1}{2^k} \prod_{1 \le i < j \le k} \left(\frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j} \right)^2 \prod_{i=1}^{k} \begin{pmatrix} n + \alpha_i \\ \alpha_i \end{pmatrix} \begin{pmatrix} n \\ \alpha_i \end{pmatrix}$$

4.1.2. Proposition. If $V = \bigwedge^2\mathbb{C}^n$ and $\lambda \in \Lambda_n$ has top-row Frobenius coordinates $\alpha_1, \alpha_2, ..., \alpha_k$, then

$$\dim M_\lambda = \frac{1}{2^k} \prod_{1 \le i < j \le k} \left(\frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j} \right)^2 \prod_{i=1}^{k} \begin{pmatrix} n + \alpha_i - 1 \\ \alpha_i \end{pmatrix} \begin{pmatrix} n - 1 \\ \alpha_i \end{pmatrix}$$

Proof of Proposition 4.1.1: As we have seen previously, if $V = S^2C^n, \lambda$ has top-row Frobenius coordinates $\alpha_1, \alpha_2, ..., \alpha_k$, and we use $\rho = (-1, -2, ..., -n)$ then

(4.1.1) $\lambda + \rho = (\alpha_1, \alpha_2, ..., \alpha_k, -1, -2, ..., -\alpha_k + 1, -\hat{\alpha}_k, -\alpha_k - 1, ..., -\alpha_1 + 1, -\hat{\alpha}_1, -\alpha_1 - 1, ..., -n)$.

The proof is an induction on k. When $k = 1$,

(4.1.2) $\lambda + \rho = (\alpha_1, -1, -2, ..., -\alpha_1 + 1, -\hat{\alpha}_1, -\alpha_1 - 1, ..., -n)$.

By the Weyl Dimension Formula,

$$|d_\lambda| = \frac{\prod_{1\le i<j\le n}(\lambda_i + \rho_i - \lambda_j - \rho_j)}{\prod_{1\le i<j\le n}(\rho_i - \rho_j)} \tag{4.1.3}$$

The denominator is simply $(n-1)!(n-2)!..3!2!1!$. The numerator equals

$$\prod_{j=2}^{n}(\alpha_1 - \lambda_i - \rho_i) \prod_{2\le i<j\le n}(\lambda_i + \rho_i - \lambda_j - \rho_j) \tag{4.1.4}$$

$$= \frac{(\alpha_1 + n \downarrow \alpha_1 + 1)}{\alpha_1 + \alpha_1} \frac{(n-1)!(n-2)!...3!2!1!}{(n-\alpha_1)!(\alpha_1 - 1)!}$$

Combining these gives

$$|d_\lambda| = \frac{1}{2}\frac{(n + \alpha_1 \downarrow \alpha_1 + 1)}{\alpha_1!(n-\alpha_1)!} \tag{4.1.5}$$

$$= \frac{1}{2}\frac{(n+\alpha_1)!}{\alpha_1!\alpha_1!(n-\alpha_1)!} \tag{4.1.6}$$

$$= \frac{1}{2}\frac{(n+\alpha_1)!}{\alpha_1!n!}\frac{n!}{\alpha_1!(n-\alpha_1)!} \tag{4.1.7}$$

$$= \frac{1}{2}\left(\frac{\alpha_1}{\alpha_1}\right)^2 \begin{pmatrix} n+\alpha_1 \\ \alpha_1 \end{pmatrix} \begin{pmatrix} n \\ \alpha_1 \end{pmatrix} \tag{4.1.8}$$

Now let λ denote a weight whose Young diagram is composed of k nested hooks, and which has top-row Frobenius coordinates $\alpha_1, \alpha_2, ..., \alpha_k$. Assume by induction that the result holds for the weight μ whose top-row Frobenius coordinates are $\alpha_1, \alpha_2, ..., \alpha_{k-1}$.

By the Weyl Dimension Formula,

$$|d_\lambda| = |d_\mu| \prod_{i=1}^{k-1}(\alpha_i - \alpha_k)\frac{(n + \alpha_k \downarrow \alpha_k + 1)}{\prod_{i=1}^{k-1}(\alpha_k + \alpha_i)} \frac{\prod_{i=1}^{k-1}(\alpha_i - \alpha_k)}{\prod_{i=1}^{k}(\alpha_i + \alpha_k)(\alpha_k - 1)!(n-\alpha_k)!} \tag{4.1.9}$$

since in passing from $\mu+\rho$ to $\lambda+\rho$ the only change in the entries is the replacement of $-\alpha_k$ by α_k, the consequences of which are indicated by this equation.

Simplifying gives

$$(4.1.10)\qquad \frac{|d_\lambda|}{|d_\mu|} = \prod_{i=1}^{k-1}\left(\frac{\alpha_i-\alpha_k}{\alpha_i+\alpha_k}\right)^2 \frac{(n+\alpha_k \downarrow \alpha_k+1)}{2\alpha_k(\alpha_k-1)!(n-\alpha_k)!}$$
$$= \frac{1}{2}\prod_{i=1}^{k-1}\left(\frac{\alpha_i-\alpha_k}{\alpha_i+\alpha_k}\right)^2 \frac{(n+\alpha_k \downarrow \alpha_k+1)}{\alpha_k!(n-\alpha_k)!}$$
$$= \frac{1}{2}\prod_{i=1}^{k-1}\left(\frac{\alpha_i-\alpha_k}{\alpha_i+\alpha_k}\right)^2 \frac{(n+\alpha_k)!}{\alpha_k!\alpha_k!(n-\alpha_k)!}$$
$$= \frac{1}{2}\prod_{i=1}^{k-1}\left(\frac{\alpha_i-\alpha_k}{\alpha_i+\alpha_k}\right)^2 \frac{(n+\alpha_k)!}{\alpha_k!n!}\frac{n!}{\alpha_k!(n-\alpha_k)!}$$
$$= \frac{1}{2}\prod_{i=1}^{k-1}\left(\frac{\alpha_i-\alpha_k}{\alpha_i+\alpha_k}\right)^2 \prod_{i=1}^{k}\begin{pmatrix} n+\alpha_k \\ \alpha_k\end{pmatrix}\begin{pmatrix} n \\ \alpha_k\end{pmatrix}$$

which completes the induction.

Proof of Proposition 4.1.2: For $V=\bigwedge^2 C^n$ use $\rho=(0,-1,-2,...,-n+1)$, so that $\lambda+\rho = (\alpha_1,\alpha_2,...,\alpha_k,0,-1,-2,...,-\alpha_k+1,-\hat{\alpha}_k,-\alpha_k-1,...,-\alpha_1+1,-\hat{\alpha}_1,\ -\alpha_1-1,...,-n+1)$

The same argument holds, with n replaced by $n-1$.

It follows directly from the dimension formulas in Propositions 4.1.1 and 4.1.2 that for each λ, the dimension $|d_\lambda|$ is a polynomial in n. We may reasonably denote this polynomial $|d_\lambda|(n)$, and investigate its properties.

4.1.3. Corollary. If $V=S^2\mathbb{C}^n$ and $\lambda\in\Lambda$ has top-row Frobenius coordinates $\alpha_1,\alpha_2,...,\alpha_k$, then the dimension polynomial $|d_\lambda|(n)$ has degree $|\lambda|$ and leading coefficient

$$\frac{1}{2^k}\prod_{1\le i<j\le k}\left(\frac{\alpha_i-\alpha_j}{\alpha_i+\alpha_j}\right)^2\prod_{t=1}^{k}\left(\frac{1}{\alpha_t!}\right)^2$$

Proof: It follows directly from the dimension formula in Proposition 4.1.1 that the degree of $|d_\lambda|(n)$ equals twice the sum of the Frobenius coordinates, which is precisely $|\lambda|$. The stated leading coefficient follows immediately from the dimension formula as well. □

4.1.4. Lemma. If $\mu, \lambda \in \Lambda$ and $\mu \subset \lambda$, then $\dfrac{d_\lambda}{d_\mu}$ is a polynomial in n of degree $|\lambda| - |\mu|$.

Proof: Since $\mu_i \leq \lambda_i$ holds for corresponding partition numbers of μ and λ, we also have $\check{\mu}_i \leq \check{\lambda}_i$ for corresponding partition numbers of the associated strict partitions $\check{\mu}$ and $\check{\lambda}$, which are precisely the top row Frobenius coordinates of μ andλ, respectively. By the dimension formula in Proposition 4.1.2, d_μ depends on n only through the expression

$$\prod_{i=1}^{k} \binom{n + \check{\mu}_i}{\check{\mu}_i} \binom{n}{\check{\mu}_i} \tag{4.1.11}$$

Since each $\check{\mu}_i \leq \check{\lambda}_i$, every factor in this product of binomial coefficients is also a factor in the corresponding product in the expression for d_λ, so (4.1.11) in its entirety cancels out in the expression for $\dfrac{|d_\lambda|}{|d_\mu|}$. Thus $\dfrac{|d_\lambda|}{|d_\mu|}$, which is a priori only a rational function, is in fact a polynomial in n of degree $2|\check{\lambda}| - 2|\check{\mu}| = |\lambda| - |\mu|$. □

Since p_λ is supersymmetric of degree $\dfrac{|\lambda|}{2}$, it can be written as a polynomial in the odd degree power sums. Although this expression in the odd degree power sums has several monomials of total degree $\dfrac{|\lambda|}{2}$, it is useful to refer to the $p_1^{\frac{|\lambda|}{2}}$ term as the leading term of p_λ, and its coefficient as the leading coefficient of p_λ. With this convention we can state the following:

4.1.5. Proposition. The leading coefficient of p_λ equals the leading coefficient of $|d_\lambda|$.

Proof: The transposition formula asserts that

$$c_\lambda(\chi - w_0\nu) = \sum_{|\mu| \leq |\lambda|} (-1)^{|\lambda|} \frac{d_\lambda}{d_\mu} c_\mu(\lambda) c_\mu(\nu)$$

Consider fixed values of λ and ν but regard n as an indeterminate. By the extra vanishing property (Corollary 3.4.4), the only nonvanishing terms on the right side are those for which $\mu \subseteq \lambda$. By Lemma 4.1.4, each of these nonvanishing terms is a polynomial in n of degree $|\lambda| - |\mu|$, so that the entire right side is a polynomial in n. Thus it is solely the $\mu = (0)$ term in the summation which contributes the leading term of the polynomial, and this leading term is identical to that of $|d_\lambda|$ since the $\mu = (0)$ term simply equals d_λ. The right side is therefore a polynomial in n of degree $|\lambda|$ whose leading coefficient is expressed explicitly in Corollary 4.1.3.

The left side of the transposition formula also depends on n, through the value of $\chi = (n+1, n+1, ..., n+1)$, which has n coordinates. Now $p_1(\chi - w_0\nu)$ has degree 2 in $n, p_3(\chi - w_0\nu)$ has degree 4 in n, ..., $p_k(\chi - w_0\nu)$ has degree $k+1$ in n, so that $p_k^r(\chi - w_0\nu)$ has degree $(k+1)^r$ in n. Thus when the left side is written first as a polynomial in odd degree power sums and then evaluated at $\chi - w_0\nu$, only the leading term $p_1^{|\lambda|/2}$ contributes a term of degree $|\lambda|$ in n.

For example, in degree 6, the highest degree terms in p_λ are $a_1p_1^6, a_2p_1^3p_3, a_3p_3^2$, and $a_4p_1p_5$, where the a_i are constants. Evaluating at $\chi - w_0\nu$, and denoting lower order terms by l.o.t., yields the following:

$$a_1p_1^6(\chi - w_0\nu) = a_1(n^2)^6 + \text{l.o.t.} = a_1n^{12} + \text{l.o.t.}$$
$$a_2p_1^3p_3(\chi - w_0\nu) = a_2(n^2)^3(n^4) + \text{l.o.t.} = a_2n^{10} + \text{l.o.t.}$$
$$a_3p_3^2(\chi - w_0\nu) = a_3(n^4)^2 + \text{l.o.t.} = a_3n^8 + \text{l.o.t.}$$
$$a_4p_1p_5(\chi - w_0\nu) = a_4(n^2)(n^6) + \text{l.o.t.} = a_4n^8 + \text{l.o.t.}$$

Thus the leading coefficient on the left side is precisely the leading coefficient of p_λ. This equals the leading coefficient of the right side, which is the same as that of $|d_\lambda|$. □

4.1.6. Corollary. If $\lambda \in \Lambda$ has top-row Frobenius coordinates $\alpha_1, ..., \alpha_k$, then the rescaled Schur Q-polynomial $q_{\tilde{\lambda}}$ has leading coefficient

$$2^{|\check{\lambda}|-\ell(\check{\lambda})} \prod_{1\le i<j\le \ell(\check{\lambda})} \left(\frac{\alpha_i-\alpha_j}{\alpha_i+\alpha_j}\right)^2 \prod_{t=1}^{\ell(\check{\lambda})} \left(\frac{1}{\alpha_t!}\right)^2$$

Proof: By (3.4.1) and Proposition 3.4.1, the leading coefficient of $q_{\check{\lambda}}$ equals $2^{|\check{\lambda}|}$ times the leading coefficient of p_{λ}. The result now follows from Proposition 4.1.5.

Ivanov's preliminary discussion of the factorial Schur Q-functions shows that each $Q^*_{\check{\lambda}}$ has the same highest degree terms as the corresponding classical Schur Q-function $Q_{\check{\lambda}}$. Recalling (3.2.2) and (3.2.3) we have

4.1.7. Corollary. $Q^*_{\check{\lambda}}$ and $Q_{\check{\lambda}}$ both have leading coefficient

$$2^{|\check{\lambda}|-\ell(\check{\lambda})} \prod_{1\le i<j\le \ell(\check{\lambda})} \frac{\check{\lambda}_i-\check{\lambda}_j}{\check{\lambda}_i+\check{\lambda}_j} \prod_{t=1}^{\ell(\check{\lambda})} \frac{1}{\check{\lambda}_t!} = 2^{|\check{\lambda}|-\ell(\check{\lambda})} \frac{1}{H(\check{\lambda})}$$

Chapter 5

Computed examples

5.1 Skew Capelli operators

We give $\mathbb{C}^3$ the standard basis $\{e_1, e_2, e_3\}$, and $S^2\mathbb{C}^3$ the basis $\{e_1e_1, e_1e_2, e_1e_3, e_2e_2, e_2e_3, e_3e_3\}$, abbreviated by $\{e_{11}, e_{11}, e_{12}, e_{13}, e_{22}, e_{23}, e_{33}\}$; these multiplications are symmetric products, though all subsequent multiplications in this subject will be exterior products. We define ∂_{ij} by $\partial_{ij}(e_{rs}) = \delta_{ir}\delta_{js}$.

$\bigwedge^3 S^2\mathbb{C}^3$ splits into two irreducible GL_3-submodules, $M_{(3,3)}$, and $M_{(4,1,1)}$. These have highest weight vectors $e_{11}e_{12}e_{22}$ and $e_{11}e_{12}e_{13}$, respectively; all multiplications are to be understood in the skew sense.

For each of these two irreducible submodules we find a basis and dual basis for each weight space, multiply corresponding basis and dual basis vectors, and sum. In this way we compute the invariant operators

$$D_{(3,3)} = -e_{11}e_{12}e_{22}\partial_{11}\partial_{12}\partial_{22} - e_{11}e_{13}e_{33}\partial_{11}\partial_{13}\partial_{33} - e_{22}e_{23}e_{33}\partial_{22}\partial_{23}\partial_{33} - \frac{1}{3}e_{11}e_{22}e_{33}\partial_{11}\partial_{22}\partial_{33} - \frac{2}{3}e_{12}e_{13}e_{23}\partial_{12}\partial_{13}\partial_{23} - \frac{2}{3}e_{11}e_{12}e_{23}\partial_{11}\partial_{12}\partial_{23} - \frac{1}{3}e_{11}e_{13}e_{22}\partial_{11}\partial_{13}\partial_{22} - \frac{1}{3}e_{11}e_{12}e_{33}\partial_{11}\partial_{12}\partial_{33} - \frac{2}{3}e_{11}e_{13}e_{23}\partial_{11}\partial_{13}\partial_{23} - \frac{1}{3}e_{11}e_{22}e_{23}\partial_{11}\partial_{22}\partial_{23} - \frac{2}{3}e_{12}e_{13}e_{22}\partial_{12}\partial_{13}\partial_{22} - \frac{1}{3}e_{11}e_{23}e_{33}\partial_{11}\partial_{23}\partial_{33} - \frac{2}{3}e_{12}e_{13}e_{33}\partial_{12}\partial_{13}\partial_{33} - \frac{1}{3}e_{12}e_{22}e_{33}\partial_{12}\partial_{22}\partial_{33} - \frac{2}{3}e_{13}e_{22}e_{23}\partial_{13}\partial_{22}\partial_{23} - \frac{2}{3}e_{12}e_{23}e_{33}\partial_{12}\partial_{23}\partial_{33} - \frac{1}{3}e_{13}e_{22}e_{33}\partial_{13}\partial_{22}\partial_{33}$$

and

$$D_{(4,1,1)} = -e_{11}e_{12}e_{13}\partial_{11}\partial_{12}\partial_{13} - e_{12}e_{22}e_{23}\partial_{12}\partial_{22}\partial_{23} - e_{13}e_{23}e_{33}\partial_{13}\partial_{23}\partial_{33} - \frac{2}{3}e_{11}e_{22}e_{33}\partial_{11}\partial_{22}\partial_{33} - \frac{1}{3}e_{12}e_{13}e_{23}\partial_{12}\partial_{13}\partial_{23} - \frac{1}{3}e_{11}e_{12}e_{23}\partial_{11}\partial_{12}\partial_{23} - \frac{2}{3}e_{11}e_{13}e_{22}\partial_{11}\partial_{13}\partial_{22} - \frac{2}{3}e_{11}e_{12}e_{33}\partial_{11}\partial_{12}\partial_{33} - \frac{1}{3}e_{11}e_{13}e_{23}\partial_{11}\partial_{13}\partial_{23} - \frac{2}{3}e_{11}e_{22}e_{23}\partial_{11}\partial_{22}\partial_{23} - \frac{1}{3}e_{12}e_{13}e_{22}\partial_{12}\partial_{13}\partial_{22} - \frac{2}{3}e_{11}e_{23}e_{33}\partial_{11}\partial_{23}\partial_{33} - \frac{1}{3}e_{12}e_{13}e_{33}\partial_{12}\partial_{13}\partial_{33} - \frac{2}{3}e_{12}e_{22}e_{33}\partial_{12}\partial_{22}\partial_{33} - \frac{1}{3}e_{13}e_{22}e_{23}\partial_{13}\partial_{22}\partial_{23} - \frac{1}{3}e_{12}e_{23}e_{33}\partial_{12}\partial_{23}\partial_{33} - \frac{2}{3}e_{13}e_{22}e_{33}\partial_{13}\partial_{22}\partial_{33}$$

5.2 Spectral polynomials

The spectral polynomials for the operators exhibited in the previous section are

$$p_{(3,3)} = -\frac{1}{36}(z_1^3 + z_2^3 + z_3^3) + \frac{1}{144}(z_1 + z_2 + z_3)^3 + \frac{1}{8}(z_1 + z_2 + z_3)^2 + \frac{3}{4}(z_1 + z_2 + z_3) + \frac{1}{2}$$

and

$$p_{(4,1,1)} = \frac{1}{36}(z_1^3 + z_2^3 + z_3^3) + \frac{1}{72}(z_1 + z_2 + z_3)^3 + \frac{1}{8}(z_1 + z_2 + z_3)^2 + \frac{1}{6}(z_1 + z_2 + z_3) + \frac{1}{2}$$

One can check the correctness of these results as follows.

The highest weights occurring in $\bigwedge S^2\mathbb{C}^3$ are (0,0,0), (2,0,0), (3,1,0), (3,3,0), (4,1,1), (4,3,1), (4,4,2), and (4,4,4). Note that these correspond to strict partitions (0), (1), (2), (2,1), (3), (3,1), (3,2), and (3,2,1), respectively. Setting $\rho = (-1, -2, -3)$ and evaluating $p_{(3,3)}$ at $\lambda + \rho$ for each of these highest weights λ yields

$p_{(3,3)}(0-1, 0-2, 0-3) = p_{(3,3)}(-1, -2, -3) = 0$

$p_{(3,3)}(2-1, 0-2, 0-3) = p_{(3,3)}(1, -2, -3) = 0$

$p_{(3,3)}(3-1, 1-2, 0-3) = p_{(3,3)}(2, -1, -3) = 0$

$p_{(3,3)}(3-1, 3-2, 0-3) = p_{(3,3)}(2, 1, -3) = 1$

$p_{(3,3)}(4-1, 1-2, 1-3) = p_{(3,3)}(3, -1, -2) = 0$

This verifies that $p_{(3,3)}(z_1, z_2, z_3)$ satisfies the vanishing condition. Since the polynomial $p_{(3,3)}(z_1, z_2, z_3)$ is supersymmetric of degree 3, it is a linear combination of $z_1^3 + z_2^3 + z_3^3, (z_1 + z_2 + z_3)^3, (z_1 + z_2 + z_3)^2, z_1 + z_2 + z_3$ and 1. Interpolating the 5 values required by the vanishing condition thus determines the coefficients of the polynomial uniquely. Additional values are as follows:

$p_{(3,3)}(4-1, 3-2, 1-3) = p_{(3,3)}(3, 1, -2) = 2$

$p_{(3,3)}(4-1, 4-2, 2-3) = p_{(3,3)}(3, 2, -1) = 5$

$p_{(3,3)}(4-1, 4-2, 4-3) = p_{(3,3)}(3, 2, 1) = 10$

One obtains the same results by applying the operator $D_{(3,3)}$ as written above, to the corresponding highest weight vectors (h.w.v.). These are

λ	highest weight vector
$(0,0,0)$	1
$(2,0,0)$	e_{11}
$(3,1,0)$	$e_{11}e_{12}$
$(3,3,0)$	$e_{11}e_{12}e_{22}$
$(4,1,1)$	$e_{11}e_{12}e_{13}$
$(4,3,1)$	$e_{11}e_{12}e_{13}e_{22}$
$(4,4,2)$	$e_{11}e_{12}e_{13}e_{22}e_{23}$
$(4,4,4)$	$e_{11}e_{12}e_{13}e_{22}e_{23}e_{33}$

The eigenvalues of $D_{(3,3)}$ on each of these highest weight vectors would appear to be utterly tedious to compute by direct application of the explicitly written operator to the vector. On the contrary, each term $a_I e_I \partial_I$ of $D_{(3,3)}$, where I denotes an index set and a_I a constant, contributes $-a_I$ to the eigenvalue of $D_{(3,3)}$ on the highest weight vector e_J iff $I \subseteq J$. The minus sign is important, and is the result of working in a skew setting; observe that $e_{11}e_{12}e_{22}\partial_{11}\partial_{12}\partial_{22}$ applied to $e_{11}e_{12}e_{22}$ yields $-e_{11}e_{12}e_{22}$ rather than $e_{11}e_{12}e_{22}$, which explains the minus signs appearing in the expressions for $D_{(3,3)}$ and $D_{(4,1,1)}$ above. It is thus not difficult to compute these eigenvalues, which agree with the values of $p_{(3,3)}$ as written above.

One can also check the stated results for $D_{(4,1,1)}$ and $p_{(4,1,1)}$, whose values are

$p_{(4,1,1)}(0-1, 0-2, 0-3) = p_{(4,1,1)}(-1,-2,-3) = 0$

$p_{(4,1,1)}(2-1, 0-2, 0-3) = p_{(4,1,1)}(1,-2,-3) = 0$

$p_{(4,1,1)}(3-1, 1-2, 0-3) = p_{(4,1,1)}(2,-1,-3) = 0$

$p_{(4,1,1)}(3-1, 3-2, 0-3) = p_{(4,1,1)}(2,1,-3) = 0$

$p_{(4,1,1)}(4-1, 1-2, 1-3) = p_{(4,1,1)}(3,-1,-2) = 1$

$p_{(4,1,1)}(4-1, 3-2, 1-3) = p_{(4,1,1)}(3,1,-2) = 2$

$p_{(4,1,1)}(4-1, 4-2, 2-3) = p_{(4,1,1)}(3,2,-1) = 5$

$p_{(4,1,1)}(4-1,4-2,4-3) = p_{(4,1,1)}(3,2,1) = 10$

Making direct use of the supersymmetry, degree, and vanishing conditions we can also compute

$p_{(0)}(z_1, z_2, z_3) = 1$

$p_{(2)}(z_1, z_2, z_3) = \frac{1}{2}(z_1 + z_2 + z_3) + 3$

$p_{(3,1)}(z_1, z_2, z_3) = \frac{1}{8}(z_1 + z_2 + z_3)^2 + \frac{5}{4}(z_1 + z_2 + z_3) + 3$

Now consider the case $n = 4$:

$p_{(0)}(z_1, z_2, z_3, z_4) = 1$

$p_{(2)}(z_1, z_2, z_3, z_4) = \frac{1}{2}p_1 + 5$

$p_{(3,1)}(z_1, z_2, z_3, z_4) = \frac{1}{8}p_1^2 + \frac{9}{4}p_1 + 10$

$p_{(3,3)}(z_1, z_2, z_3, z_4) = \frac{1}{144}p_1^3 - \frac{1}{36}p_3 + \frac{5}{24}p_1^2 + \frac{25}{12}p_1 + \frac{25}{6}$

$p_{(4,1,1)}(z_1, z_2, z_3, z_4) = \frac{1}{72}p_1^3 + \frac{1}{36}p_3 + \frac{7}{24}p_1^2 + \frac{11}{6}p_1 + \frac{35}{6}$

$p_{(4,3,1)}(z_1, z_2, z_3, z_4) = \frac{1}{576}p_1^4 - \frac{1}{144}p_3p_1 + \frac{17}{288}p_1^3 - \frac{1}{36}p_3 + \frac{35}{48}p_1^2 + \frac{25}{8}p_1 + \frac{25}{6}$

$p_{(5,1,1,1)}(z_1, z_2, z_3, z_4) = \frac{1}{1152}p_1^4 + \frac{1}{144}p_3p_1 + \frac{1}{72}p_1^3 + \frac{1}{36}p_3 + \frac{1}{96}p_1^2 + \frac{1}{12}p_1 + \frac{5}{6}$

Observe that the expression for p_λ as a polynomial in the odd degree power sums does indeed depend on n, except for the terms of highest degree.

For $n = 5$:

$p_{(0)}(z_1, z_2, z_3, z_4, z_5) = 1$

$p_{(2)}(z_1, z_2, z_3, z_4, z_5) = \frac{1}{2}p_1 + \frac{15}{2}$

$p_{(3,1)}(z_1,z_2,z_3,z_4,z_5) = \frac{1}{8}p_1^2 + \frac{7}{2}p_1 + \frac{195}{8}$

$p_{(3,3)}(z_1,z_2,z_3,z_4,z_5) = \frac{1}{144}p_1^3 - \frac{1}{36}p_3 + \frac{5}{16}p_1^2 + \frac{75}{16}p_1 + \frac{275}{16}$

$p_{(4,1,1)}(z_1,z_2,z_3,z_4,z_5) = \frac{1}{72}p_1^3 + \frac{1}{36}p_3 + \frac{1}{2}p_1^2 + \frac{139}{24}p_1 + \frac{55}{2}$

$p_{(4,3,1)}(z_1,z_2,z_3,z_4,z_5) = \frac{1}{576}p_1^4 - \frac{1}{144}p_3p_1 + \frac{3}{32}p_1^3 - \frac{1}{16}p_3 + \frac{15}{8}p_1^2 + \frac{475}{32}p_1 + \frac{2475}{64}$

$p_{(5,1,1,1)}(z_1,z_2,z_3,z_4,z_5) = \frac{1}{1152}p_1^4 + \frac{1}{144}p_3p_1 + \frac{1}{32}p_1^3 + \frac{1}{16}p_3 + \frac{67}{192}p_1^2 + \frac{81}{32}p_1 + \frac{1485}{128}$

$p_{(4,4,2)}(z_1,z_2,z_3,z_4,z_5) = \frac{1}{14400}p_1^5 - \frac{1}{1440}p_3p_1^2 + \frac{1}{600}p_5 + \frac{7}{1440}p_1^4 - \frac{7}{360}p_3p_1 + \frac{49}{360}p_1^3 - \frac{199}{1440}p_3 + \frac{7}{4}p_1^2 + \frac{575}{64}p_1 + \frac{441}{32}$

$p_{(5,3,1,1)}(z_1,z_2,z_3,z_4,z_5) = \frac{1}{6400}p_1^5 - \frac{1}{400}p_5 + \frac{11}{1280}p_1^4 + \frac{1}{80}p_3p_1 + \frac{337}{1920}p_1^3 + \frac{17}{120}p_3 + \frac{201}{128}p_1^2 + \frac{2025}{256}p_1 + \frac{5103}{256}$

$p_{(6,1,1,1,1)}(z_1,z_2,z_3,z_4,z_5) = \frac{1}{28800}p_1^5 + \frac{1}{1440}p_3p_1^2 + \frac{1}{1200}p_5 + \frac{1}{1152}p_1^4 + \frac{1}{144}p_3p_1 - \frac{1}{576}p_1^3 - \frac{1}{288}p_3 - \frac{5}{192}p_1^2 + \frac{189}{640}p_1 + \frac{189}{128}$

$p_{(4,4,4)}(z_1,z_2,z_3,z_4,z_5,z_6) = \frac{1}{1036800}p_1^6 - \frac{1}{51840}p_1^3p_3 + \frac{1}{7200}p_1p_5 - \frac{1}{12960}p_3^2 + \frac{7}{57600}p_1^5 - \frac{7}{5760}p_1^2p_3 + \frac{7}{2400}p_5 + \frac{49}{7680}p_1^4 - \frac{49}{1920}p_1p_3 + \frac{49}{288}p_1^3 - \frac{1421}{5760}p_3 + \frac{5831}{2560}p_1^2 + \frac{54047}{3840}p_1 + \frac{62769}{2560}$

$p_{(5,4,2,1)}(z_1,z_2,z_3,z_4,z_5,z_6) = \frac{1}{82944}p_1^6 - \frac{1}{10368}p_1^3p_3 + \frac{1}{5184}p_3^2 + \frac{1}{768}p_1^5 - \frac{5}{1152}p_1^2p_3 - \frac{1}{288}p_5 + \frac{533}{9216}p_1^4 - \frac{67}{1152}p_1p_3 + \frac{2255}{1728}p_1^3 - \frac{85}{3456}p_3 + \frac{47201}{3072}p_1^2 + \frac{217511}{1024}$

$p_{(6,3,1,1,1)}(z_1,z_2,z_3,z_4,z_5,z_6) = \frac{1}{129600}p_1^6 + \frac{1}{12960}p_1^3p_3 - \frac{1}{3600}p_1p_5 - \frac{1}{6480}p_3^2 + \frac{1}{1600}p_1^5 + \frac{7}{1440}p_1^2p_3 - \frac{1}{3600}p_5 + \frac{161}{8640}p_1^4 + \frac{371}{4320}p_1p_3 + \frac{563}{2160}p_1^3 + \frac{1229}{4320}p_3 + \frac{2387}{960}p_1^2 + \frac{1323}{64}p_1 + \frac{24451}{320}$

$p_{(7,1,1,1,1,1)}(z_1, z_2, z_3, z_4, z_5, z_6) = \frac{1}{1036800}p_1^6 + \frac{1}{25920}p_1^3 p_3 + \frac{1}{7200}p_1 p_5 + \frac{1}{25920}p_3^2 + \frac{1}{28800}p_1^5 + \frac{1}{1440}p_1^2 p_3 + \frac{1}{1200}p_5 - \frac{19}{69120}p_1^4 - \frac{19}{8640}p_1 p_3 - \frac{19}{2880}p_1^3 - \frac{19}{1440}p_3 + \frac{199}{4608}p_1^2 + \frac{199}{384}p_1 + \frac{6153}{2560}$

5.3 Factorial Schur Q polynomials

For $V = S^2\mathbb{C}^n$, the correspondence between highest weights λ of $\bigwedge V$ and strict partitions $\check{\lambda}$ is illustrated in the following table. Here $\rho = (-1, -2, -3...)$ so that the positive terms in $\lambda + \rho$ are precisely the top row Frobenius coordinates of λ, which in turn are the partition numbers of the corresponding strict partition $\check{\lambda}$.

λ	$\lambda+\rho$	$\check{\lambda}$
(0)	$(-1,-2,-3,...)$	(0)
(2)	$(1,-2,-3,...)$	(1)
(31)	$(2,-1,-3,-4...)$	(2)
(33)	$(2,1,-3,-4...)$	(21)
(411)	$(3,-1,-2,-4,-5...)$	(3)
(431)	$(3,1,-2,-4,-5...)$	(31)
(5111)	$(4,-1,-2,-3,-5,-6...)$	(4)
(442)	$(3,1,-2,-4,-5...)$	(32)
(5311)	$(4,1,-2,-3,-5,-6...)$	(41)
(61111)	$(5,-1,-2,-3,-4,-6,-7...)$	(5)
(444)	$(3,2,1,-4,-5...)$	(321)
(5421)	$(4,2,-1,-3,-5,-6...)$	(42)
(63111)	$(5,1,-2,-3,-4,-6,-7...)$	(51)
(711111)	$(6,-1,-2,-3,-4,-5,-7,-8...)$	(6)
(5441)	$(4,2,1,-3,-5,-6...)$	(421)
(5522)	$(4,3,-1,-2,-5,-6...)$	(43)
(64211)	$(5,2,-1,-3,-4,-6,-7...)$	(52)
(731111)	$(6,1,-2,-3,-4,-5,-7,-8...)$	(61)
(8111111)	$(7,-1,-2,-3,-4,-5,-6,-8,-9...)$	(7)
(5542)	$(4,3,1,-2,-5,-6...)$	(431)
(64411)	$(5,2,1,-3,-4,-6,-7...)$	(521)
(65221)	$(5,3,-1,-2,-4,-6,-7...)$	(53)
(742111)	$(6,2,-1,-3,-4,-5,-7,-8...)$	(62)
(8311111)	$(7,1,-2,-3,-4,-5,-6,-8,-9...)$	(71)
(91111111)	$(8,-1,-2,-3,-4,-5,-6,-7,-9,-10...)$	(8)

When viewed as partitions, the highest weights of $\bigwedge \bigwedge^2 \mathbb{C}^n$ are precisely the duals of those of $\bigwedge S^2\mathbb{C}^n$, in the sense that their Young diagrams differ by interchanging rows and columns (i.e. by reflection across the diagonal). Equivalently, the bottom row Frobenius coordinates of a highest weight of $\bigwedge \bigwedge^2 \mathbb{C}^n$ are precisely the top row Frobenius coordinates of the corresponding highest weight of $\bigwedge S^2\mathbb{C}^n$. The correspondence between highest weights λ of $\bigwedge \bigwedge^2 \mathbb{C}^n$ and strict partitions $\check{\lambda}$ is illustrated in the following table. Here $\rho = (0, -1, -2, ...)$ so that the positive terms in $\lambda + \rho$ are precisely the bottom row Frobenius coordinates of λ.

λ	$\lambda + \rho$	$\check{\lambda}$
(0)	$(0, -1, -2, -3, ...)$	(0)
(11)	$(1, 0, -2, -3...)$	(1)
(211)	$(2, 0, -1, -3, -4...)$	(2)
(222)	$(2, 1, 0, -3, -4...)$	(21)
(3111)	$(3, 0, -1, -2, -4, -5...)$	(3)
(3221)	$(3, 1, 0, -2, -4, -5...)$	(31)
(41111)	$(4, 0, -1, -2, -3, -5, -6...)$	(4)
(3322)	$(3, 2, 0, -1, -4, -5...)$	(32)
(42211)	$(4, 1, 0, -2, -3, -5, -6...)$	(41)
(511111)	$(5, 0, -1, -2, -3, -4, -6, -7...)$	(5)
(3333)	$(3, 2, 1, 0, -4, -5...)$	(321)
(43221)	$(4, 2, 0, -1, -3, -5, -6...)$	(42)
(522111)	$(5, 1, 0, -2, -3, -4, -6, -7...)$	(51)
(6111111)	$(6, 0, -1, -2, -3, -4, -5, -7, -8...)$	(6)
(43331)	$(4, 2, 1, 0, -3, -5, -6...)$	(421)
(44222)	$(4, 3, 0, -1, -2, -5, -6...)$	(43)
(532211)	$(5, 2, 0, -1, -3, -4, -6, -7...)$	(52)
(6221111)	$(6, 1, 0, -2, -3, -4, -5, -7, -8...)$	(61)
(71111111)	$(7, 0, -1, -2, -3, -4, -5, -6, -8, -9...)$	(7)
(44332)	$(4, 3, 1, 0, -2, -5, -6...)$	(431)
(533311)	$(5, 2, 1, 0, -3, -4, -6, -7...)$	(521)
(542221)	$(5, 3, 0, -1, -2, -4, -6, -7...)$	(53)
(6322111)	$(6, 2, 0, -1, -3, -4, -5, -7, -8...)$	(62)
(72211111)	$(7, 1, 0, -2, -3, -4, -5, -6, -8, -9...)$	(71)
(811111111)	$(8, 0, -1, -2, -3, -4, -5, -6, -7, -9, -10...)$	(8)

The following are examples of the $q_{\check{\lambda}}$, the rescaled factorial Schur Q-polynomials; cf. section 3.2. Observe that the coefficient of each leading term of $q_{\check{\lambda}}$ of total degree k equals 2^k times the corresponding leading term of the spectral polynomial p_λ.

$q_{(0)} = 1$

$q_{(1)} = p_1$

$q_{(2)} = \frac{1}{2}p_1^2 - \frac{1}{2}p_1$

$q_{(2,1)} = \frac{1}{18}p_1^3 - \frac{1}{18}p_3$

$q_{(3)} = \frac{1}{9}p_1^3 + \frac{1}{18}p_3 - \frac{1}{2}p_1^2 + \frac{1}{3}p_1$

$q_{(3,1)} = \frac{1}{36}p_1^4 - \frac{1}{36}p_3p_1 - \frac{1}{12}p_1^3 + \frac{1}{12}p_3$

$q_{(4)} = \frac{1}{72}p_1^4 + \frac{1}{36}p_3p_1 - \frac{1}{6}p_1^3 - \frac{1}{12}p_3 + \frac{11}{24}p_1^2 - \frac{1}{4}p_1$

$q_{(3,2)} = \frac{1}{450}p_1^5 - \frac{1}{180}p_3p_1^2 + \frac{1}{300}p_5 - \frac{1}{180}p_1^4 + \frac{1}{180}p_3p_1 + \frac{1}{180}p_1^3 - \frac{1}{180}p_3$

$q_{(4,1)} = \frac{1}{200}p_1^5 - \frac{1}{200}p_5 - \frac{1}{20}p_1^4 + \frac{1}{20}p_3p_1 + \frac{11}{120}p_1^3 - \frac{11}{120}p_3$

$q_{(5)} = \frac{1}{900}p_1^5 + \frac{1}{180}p_3p_1^2 + \frac{1}{600}p_5 - \frac{1}{36}p_1^4 - \frac{1}{18}p_3p_1 + \frac{7}{36}p_1^3 + \frac{7}{72}p_3 - \frac{5}{12}p_1^2 + \frac{1}{5}p_1$

$q_{(3,2,1)} = \frac{1}{16200}p_1^6 - \frac{1}{3240}p_3p_1^3 - \frac{1}{3240}p_3^2 + \frac{1}{1800}p_5p_1$

$q_{(4,2)} = \frac{1}{1296}p_1^6 - \frac{1}{648}p_3p_1^3 + \frac{1}{1296}p_3^2 - \frac{1}{144}p_1^5 + \frac{1}{72}p_3p_1^2 - \frac{1}{144}p_5 + \frac{1}{72}p_1^4 - \frac{1}{72}p_3p_1 - \frac{5}{432}p_1^3 + \frac{5}{432}p_3$

$$q_{(5,1)} = \frac{1}{2025}p_1^6+\frac{1}{810}p_3p_1^3-\frac{1}{1620}p_3^2-\frac{1}{900}p_5p_1-\frac{1}{90}p_1^5+\frac{1}{90}p_5+\frac{7}{108}p_1^4-\frac{7}{108}p_3p_1-\frac{5}{54}p_1^3+\frac{5}{54}p_3$$

$$q_{(6)} = \frac{1}{16200}p_1^6 + \frac{1}{1620}p_3p_1^3 + \frac{1}{6480}p_3^2 + \frac{1}{1800}p_5p_1 - \frac{1}{360}p_1^5 - \frac{1}{72}p_3p_1^2 - \frac{1}{240}p_5 + \frac{17}{432}p_1^4 + \frac{17}{216}p_3p_1 - \frac{5}{24}p_1^3 - \frac{5}{48}p_3 + \frac{137}{360}p_1^2 - \frac{1}{6}p_1$$

5.4 Values of spectral functions

The highest weights of $\bigwedge S^2\mathbb{C}^3$ are (0), (2), (31), (33), (411), (431), (442), and (444). In the following table of values, the (μ, λ) entry equals $c_\mu(\lambda)$, where each row corresponds to a different λ, and each column to a different μ.

$c_\mu(\lambda)$		μ							
		(0)	(2)	(31)	(33)	(411)	(431)	(442)	(444)
	(0)	1	0	0	0	0	0	0	0
	(2)	1	1	0	0	0	0	0	0
	(31)	1	2	1	0	0	0	0	0
λ	(33)	1	3	3	1	0	0	0	0
	(411)	1	3	3	0	1	0	0	0
	(431)	1	4	6	2	2	1	0	0
	(442)	1	5	10	5	5	5	1	0
	(444)	1	6	15	10	10	15	6	1

Displayed next is the matrix $\{(-1)^{|\lambda|}c_\mu(\lambda)\}$, which is the matrix of the operator T (cf. Definition 2.2.4 and Proposition 2.2.12). Observe that it is indeed self-inverting, as required.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -3 & -3 & -1 & 0 & 0 & 0 & 0 \\ -1 & -3 & -3 & 0 & -1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 2 & 2 & 1 & 0 & 0 \\ 1 & -5 & 10 & -5 & -5 & -5 & -1 & 0 \\ 1 & 6 & 15 & 10 & 10 & 15 & 6 & 1 \end{bmatrix}$$

An expanded table of values of $c_\mu(\lambda)$, but indicating strict partitions rather than their corresponding highest weights, is as follows; observe that both the extra vanishing (Corollary 3.4.4) and non-vanishing (Corollary 3.4.7) properties hold.

	(0)	(1)	(2)	(21)	(3)	(31)	(4)	(32)	(41)	(5)	(321)	(42)	(51)	(6)
(0)	1	0	0	0	0	0	0	0	0	0	0	0	0	0
(1)	1	1	0	0	0	0	0	0	0	0	0	0	0	0
(2)	1	2	1	0	0	0	0	0	0	0	0	0	0	0
(21)	1	3	3	1	0	0	0	0	0	0	0	0	0	0
(3)	1	3	3	0	1	0	0	0	0	0	0	0	0	0
(31)	1	4	6	2	2	1	0	0	0	0	0	0	0	0
(4)	1	4	6	0	4	0	1	0	0	0	0	0	0	0
(32)	1	5	10	5	5	5	0	1	0	0	0	0	0	0
(41)	1	5	10	$\frac{10}{3}$	$\frac{20}{3}$	$\frac{10}{3}$	$\frac{5}{3}$	0	1	0	0	0	0	0
(5)	1	5	10	0	10	0	5	0	0	1	0	0	0	0
(321)	1	6	15	10	10	15	0	6	0	0	1	0	0	0
(42)	1	6	15	8	12	12	3	$\frac{12}{5}$	$\frac{18}{5}$	0	0	1	0	0
(51)	1	6	15	5	15	$\frac{15}{2}$	$\frac{15}{2}$	0	$\frac{9}{2}$	$\frac{3}{2}$	0	0	1	0
(6)	1	6	15	0	20	0	15	0	0	6	0	0	0	1
(421)	1	7	21	15	20	30	5	12	9	0	2	5	0	0
(43)	1	7	21	14	21	28	7	$\frac{42}{5}$	$\frac{63}{5}$	0	0	7	0	0
(52)	1	7	21	$\frac{35}{3}$	$\frac{70}{3}$	$\frac{70}{3}$	$\frac{35}{3}$	$\frac{14}{3}$	14	$\frac{7}{3}$	0	$\frac{35}{9}$	$\frac{28}{9}$	0
(61)	1	7	21	7	28	14	21	0	$\frac{63}{5}$	$\frac{42}{5}$	0	0	$\frac{28}{5}$	$\frac{7}{5}$
(7)	1	7	21	0	35	0	35	0	0	21	0	0	0	7
(431)	1	8	28	$\frac{70}{3}$	$\frac{98}{3}$	$\frac{175}{3}$	$\frac{35}{3}$	28	28	0	$\frac{14}{3}$	$\frac{70}{3}$	0	0
(521)	1	8	28	21	35	$\frac{105}{2}$	$\frac{35}{2}$	21	$\frac{63}{2}$	$\frac{7}{2}$	$\frac{7}{2}$	$\frac{35}{2}$	7	0
(53)	1	8	28	20	36	50	20	16	36	4	0	20	8	0
(62)	1	8	28	16	40	40	30	8	36	12	0	10	16	2
(71)	1	8	28	$\frac{28}{3}$	$\frac{140}{3}$	$\frac{70}{3}$	$\frac{140}{3}$	0	28	28	0	0	$\frac{56}{3}$	$\frac{28}{3}$
(8)	1	8	28	0	56	0	70	0	0	56	0	0	0	28
(432)	1	9	36	35	49	105	21	63	63	0	14	70	0	0
(531)	1	9	36	32	52	96	30	48	72	6	8	60	16	0
(54)	1	9	36	30	54	90	36	36	81	9	0	60	24	0
(621)	1	9	36	28	56	84	42	$\frac{168}{5}$	$\frac{378}{5}$	$\frac{84}{5}$	$\frac{28}{5}$	42	$\frac{168}{5}$	$\frac{14}{5}$
(63)	1	9	36	27	57	81	45	27	81	18	0	45	36	3
(72)	1	9	36	21	63	63	63	$\frac{63}{5}$	$\frac{378}{5}$	$\frac{189}{5}$	0	21	$\frac{252}{5}$	$\frac{63}{3}$
(81)	1	9	36	12	72	36	90	0	54	72	0	0	48	36
(9)	1	9	36	0	84	0	126	0	0	126	0	0	0	84

Corollary 2.3.5, which asserts that $c_\mu(\chi) = |d_\mu|$, is illustrated for n=2, 3, 4, and 5:

	(0)	(1)	(2)	(21)	(3)	(31)	(4)	(32)	(41)	(5)	(321)	(42)	(51)	(6)
(21)	1	3	3	1	0	0	0	0	0	0	0	0	0	0
(321)	1	6	15	10	10	15	0	6	0	0	1	0	0	0
(4321)	1	10	45	50	70	175	35	126	126	0	35	175	0	0
(54321)	1	15	105	175	280	1050	315	1176	1701	126	490	3675	840	0

5.5 Transposition

The following are examples of the the pairing $\lambda \mapsto \chi - w_0\lambda$ discussed in Section 2.3.

In the case $V = S^2\mathbb{C}^n$, the sum χ of all weights satisfies $\chi = (n+1, n+1, ..., n+1)$. In four variables $\chi = (5,5,5,5)$, and the pairing $\lambda \mapsto \chi - w_0\lambda$ is illustrated in the following table.

λ	$w_0\lambda$	$\chi - w_0\lambda$
$(0,0,0,0)$	$(0,0,0,0)$	$(5,5,5,5)$
$(2,0,0,0)$	$(0,0,0,2)$	$(5,5,5,3)$
$(3,1,0,0)$	$(0,0,1,3)$	$(5,5,4,2)$
$(3,3,0,0)$	$(0,0,3,3)$	$(5,5,2,2)$
$(4,1,1,0)$	$(0,1,1,4)$	$(5,4,4,1)$
$(4,3,1,0)$	$(0,1,3,4)$	$(5,4,2,1)$
$(4,4,2,0)$	$(0,2,4,4)$	$(5,3,1,1)$
$(4,4,4,0)$	$(0,4,4,4)$	$(5,1,1,1)$
$(5,1,1,1)$	$(1,1,1,5)$	$(4,4,4,0)$
$(5,3,1,1)$	$(1,1,3,5)$	$(4,4,2,0)$
$(5,4,2,1)$	$(1,2,4,5)$	$(4,3,1,0)$
$(5,4,4,1)$	$(1,4,4,5)$	$(4,1,1,0)$
$(5,5,2,2)$	$(2,2,5,5)$	$(3,3,0,0)$
$(5,5,4,2)$	$(2,4,5,5)$	$(3,1,0,0)$
$(5,5,5,3)$	$(3,5,5,5)$	$(2,0,0,0)$
$(5,5,5,5)$	$(5,5,5,5)$	$(0,0,0,0)$

For $V = \bigwedge^2 \mathbb{C}^n, \chi = (n-1, n-1, ..., n-1)$. For $n = 4$ we have

λ	$w_0\lambda$	$\chi - w_0\lambda$
$(0,0,0,0)$	$(0,0,0,0)$	$(3,3,3,3)$
$(1,1,0,0)$	$(0,0,1,1)$	$(3,3,2,2)$
$(2,1,1,0)$	$(0,1,1,2)$	$(3,2,2,1)$
$(2,2,2,0)$	$(0,2,2,2)$	$(3,1,1,1)$
$(3,1,1,1)$	$(1,1,1,3)$	$(2,2,2,0)$
$(3,2,2,1)$	$(1,2,2,3)$	$(2,1,1,0)$
$(3,3,2,2)$	$(2,2,3,3)$	$(1,1,0,0)$
$(3,3,3,3)$	$(3,3,3,3)$	$(0,0,0,0)$

For $n = 5$ we have

λ	$w_0\lambda$	$\chi - w_0\lambda$
$(0,0,0,0,0)$	$(0,0,0,0,0)$	$(4,4,4,4,4)$
$(1,1,0,0,0)$	$(0,0,0,1,1)$	$(4,4,4,3,3)$
$(2,1,1,0,0)$	$(0,0,1,1,2)$	$(4,4,3,3,2)$
$(2,2,2,0,0)$	$(0,0,2,2,2)$	$(4,4,2,2,2)$
$(3,1,1,1,0)$	$(0,1,1,1,3)$	$(4,3,3,3,1)$
$(3,2,2,1,0)$	$(0,1,2,2,3)$	$(4,3,2,2,1)$
$(3,3,2,2,0)$	$(0,2,2,3,3)$	$(4,2,2,1,1)$
$(3,3,3,3,0)$	$(0,3,3,3,3)$	$(4,1,1,1,1)$
$(4,1,1,1,1)$	$(1,1,1,1,4)$	$(3,3,3,3,0)$
$(4,2,2,1,1)$	$(1,1,2,2,4)$	$(3,3,2,2,0)$
$(4,3,2,2,1)$	$(1,2,2,3,4)$	$(3,2,2,1,0)$
$(4,3,3,3,1)$	$(1,3,3,3,4)$	$(3,1,1,1,0)$
$(4,4,2,2,2)$	$(2,2,2,4,4)$	$(2,2,2,0,0)$
$(4,4,3,3,2)$	$(2,3,3,4,4)$	$(2,1,1,0,0)$
$(4,4,4,3,3)$	$(3,3,4,4,4)$	$(1,1,0,0,0)$
$(4,4,4,4,4)$	$(4,4,4,4,4)$	$(0,0,0,0,0)$

5.6 Dimension Polynomials

This section presents examples of the dimension polynomials discussed in Chapter 4. Observe that for each highest weight λ, the leading coefficient of $|d_\lambda|$ equals the coefficient of the highest power of p_1 in the spectral polynomial p_λ.

We first consider dimensions of submodules of $\bigwedge S^2\mathbb{C}^n$:

$|d_{(0)}|(n) = 1$

$|d_{(2)}|(n) = \frac{1}{2}n^2 + \frac{1}{2}n$

$|d_{(3,1)}|(n) = \frac{1}{8}n^4 + \frac{1}{4}n^3 - \frac{1}{8}n^2 - \frac{1}{4}n$

$|d_{(3,3)}|(n) = \frac{1}{144}n^6 + \frac{1}{48}n^5 + \frac{1}{144}n^4 - \frac{1}{48}n^3 - \frac{1}{72}n^2$

$|d_{(4,1,1)}|(n) = \frac{1}{72}n^6 + \frac{1}{24}n^5 - \frac{5}{72}n^4 - \frac{5}{24}n^3 + \frac{1}{18}n^2 + \frac{1}{6}n$

$|d_{(4,3,1)}|(n) = \frac{1}{576}n^8 + \frac{1}{144}n^7 - \frac{1}{288}n^6 - \frac{5}{144}n^5 - \frac{11}{576}n^4 + \frac{1}{36}n^3 + \frac{1}{48}n^2$

$|d_{(5,1,1,1)}|(n) = \frac{1}{1152}n^8 + \frac{1}{288}n^7 - \frac{7}{576}n^6 - \frac{7}{144}n^5 + \frac{49}{1152}n^4 + \frac{49}{288}n^3 - \frac{1}{32}n^2 - \frac{1}{8}n$

$|d_{(4,4,2)}|(n) = \frac{1}{14400}n^{10} + \frac{1}{2880}n^9 - \frac{1}{480}n^7 - \frac{3}{1600}n^6 + \frac{1}{320}n^5 + \frac{1}{288}n^4 - \frac{1}{720}n^3 - \frac{1}{600}n^2$

$|d_{(5,3,1,1)}|(n) = \frac{1}{6400}n^{10} + \frac{1}{1280}n^9 - \frac{1}{640}n^8 - \frac{7}{640}n^7 - \frac{7}{6400}n^6 + \frac{49}{1280}n^5 + \frac{1}{40}n^4 - \frac{9}{320}n^3 - \frac{9}{400}n^2$

$|d_{(6,1,1,1,1)}|(n) = \frac{1}{28800}n^{10} + \frac{1}{5760}n^9 - \frac{1}{960}n^8 - \frac{1}{192}n^7 + \frac{91}{9600}n^6 + \frac{91}{1920}n^5 - \frac{41}{1440}n^4 - \frac{41}{288}n^3 + \frac{1}{50}n^2 + \frac{1}{10}n$

$|d_{(4,4,4)}|(n) = \frac{1}{1036800}n^{12} + \frac{1}{172800}n^{11} + \frac{1}{207360}n^{10} - \frac{1}{34560}n^9 - \frac{19}{345600}n^8 + \frac{1}{57600}n^7 + \frac{19}{207360}n^6 + \frac{1}{34560}n^5 - \frac{11}{259200}n^4 - \frac{1}{43200}n^3$

$|d_{(5,4,2,1)}|(n) = \frac{1}{82944}n^{12} + \frac{1}{13824}n^{11} - \frac{7}{82944}n^{10} - \frac{5}{4608}n^9 - \frac{19}{27648}n^8 + \frac{7}{1536}n^7 + \frac{419}{82944}n^6 - \frac{85}{13824}n^5 - \frac{161}{20736}n^4 + \frac{1}{384}n^3 + \frac{1}{288}n^2$

$|d_{(6,3,1,1,1)}|(n) = \frac{1}{129600}n^{12} + \frac{1}{21600}n^{11} - \frac{1}{5184}n^{10} - \frac{1}{720}n^9 + \frac{41}{43200}n^8 + \frac{91}{7200}n^7 + \frac{109}{25920}n^6 - \frac{41}{1080}n^5 - \frac{881}{32400}n^4 + \frac{2}{75}n^3 + \frac{1}{45}n^2$

$|d_{(7,1,1,1,1,1)}|(n) = \frac{1}{1036800}n^{12} + \frac{1}{172800}n^{11} - \frac{11}{207360}n^{10} - \frac{11}{34560}n^9 + \frac{341}{345600}n^8 + \frac{341}{57600}n^7 - \frac{1529}{207360}n^6 - \frac{1529}{34560}n^5 + \frac{5269}{259200}n^4 + \frac{5269}{43200}n^3 - \frac{1}{72}n^2 - \frac{1}{12}n$

Chapter 6

Topics for future research

I. Howe's classification of skew multiplicity free spaces is limited to irreducible modules for simple reductive groups. The next step in this area is to classify irreducible SMF spaces for not necessarily simple reductive groups, with the hope that such a classification will yield additional infinite classes of SMF spaces. Each such infinite class presents the problem of characterizing spectral functions for invariant differential operators, and with it the possibility that these functions are, up to a change of variables, some already known and interesting family of symmetric functions. It would also be interesting to classify SMF modules of finite groups, especially those occurring in infinite families.

II. Understanding skew multiplicity freeness in its own right, and not necessarily in connection with symmetric polynomials and invariant operators, is an interesting challenge. Ideally it would be possible to find a geometric characterization of this property, which is not currently known. This would be a significant step toward completing the analogy with multiplicity free spaces, for which there is such a characterization; V is multiplicity free if and only if a Borel subgroup of G has a dense orbit, cf. [K4 Theorem 3.1].

III. Ivanov proves a branching rule for the factorial Schur Q-functions. The equivalence which we have shown between these functions and spectral functions associated with SMF spaces has made it possible to derive properties of one set of functions from the other, in either direction. The next task is to reformulate the branching rule in terms of the spectral functions, and to draw specifically representation theoretic conclusions.

IV. The argument of Chapter 2 proves the vanishing condition for spectral functions in the general SMF case; it remains to be investigated whether an extra vanishing property holds. Furthermore, the transposition formula which holds in the general SMF case requires that the operator D_λ arise from the center of the universal enveloping algebra of G; whether this holds for all invariant operators on SMF spaces is unknown, and certainly does not hold in general for invariant operators on MF spaces.

V. Our study of the spectral functions in the special cases $S^2\mathbb{C}^n$ and $\bigwedge^2 \mathbb{C}^n$ made use of particular choices of ρ. Following the approach of Knop and Sahi, it may be useful to regard ρ as a parameter, and investigate the effects of changing its value, i.e. regarding functions $p_{\lambda,\rho}$ as a deformation of the family p_λ.

VI. At the beginning of this research project it seemed plausible, though difficult to work out, that the explicitly written differential operator D_λ could be interpreted combinatorially in the following sense. D_λ is a sum of terms of the form $a_I e_I \partial_I$. When applied to an element (in particular a highest weight vector) of M_μ, $a_I e_I \partial_I$ contributes $\pm a_I$ or 0 to the value of $c_\lambda(\mu)$, where the contribution is $\pm a_I$ iff $I \subset J$ where J is the index set determined by the highest weight vector of weight μ, and the value of $\pm$ depends only on the residue mod 4 of $\frac{|\lambda|}{2}$. The difficulty lies in the values of a_I. Are the values of a_I, after some appropriate scaling, the solution to a counting problem related to the Young diagrams of λ and μ, and not just through the formula for shifted skew tableaux? This seemed to be the case during the initial work on this project, but it is now quite unclear.

VII. In the special cases considered in Chapter 3, the extra vanishing and non-vanishing properties were proved for spectral functions by means of the equivalence with the factorial Schur Q functions. A direct proof would be desirable, in particular one which makes use of the explicit form of the operator D_λ.

VIII. Difference operators are used in two ways in the symmetric case, both in Knop's proof of the transposition formula and in the Knop-Sahi realization of spectral polynomials as eigenfunctions of such operators. If only to complete the analogy with the symmetric case, or to determine if the cases are not entirely analogous, it would be interesting to construct the appropriate difference operators for the skew case.

IX. One result which holds in the symmetric case whose skew symmetric analogue has yet to be investigated is what Knop [K1] calls the interpolation formula. The analogue, if it exists, would express an arbitrary supersymmetric polynomial as a linear combination of spectral polynomials.

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Vita

Michael Weingart

2007	Ph. D. in Mathematics, Rutgers University
1999	M.A. in Mathematics, Columbia University
1992	B.A. in Classics and Philosophy, Yale University

2007	Teaching assistant, Rutgers University
2005-2006	Instructor, Rutgers University
2004-2005	NSF GK12 Teaching Fellow, Rutgers University
1999-2004	Teaching assistant, Department of Mathematics, Rutgers University
1992-1999	Commodities option trader and analyst, NYAM Options Trading N.V., New York