"SO LET'S PROVE IT!": EMERGENT AND ELABORATED MATHEMATICAL IDEAS AND REASONING IN THE DISCOURSE AND INSCRIPTIONS OF LEARNERS ENGAGED IN A COMBINATORIAL TASK

by

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ABSTRACT OF THE DISSERTATION

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Dissertation Director:

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Through discursive and inscriptive analyses, this research provides an understanding of the mathematical ideas and forms of reasoning that learners build as they collaborate to resolve a combinatorial problem task, set in a non-Euclidean context, "The Taxicab Problem." The analytic method uses data captured on videotape of an extended problem-solving session involving four 12^{th}-grade students and is ethnographical, microanalytical, and epistemological. The microanalytical investigation focuses in detail on the work of the students, and based on their intellectual actions and products, the study suggests possibilities for larger groups of learners. Epistemologically, the investigation contributes understanding of not only the resolutions the learners develop but also of how they build mathematical ideas and forms of reasoning that in some instances they discard and in others use to formulate a resolution of the problem task.

The problem-solving session for this research occurred in May 2000 at the David Brearley High School, in Kenilworth, New Jersey, a diverse working-class, and immigrant community. At the time of this study, the four students were in their twelfth year of involvement in mathematical activities of a longitudinal study of Rutgers University and, as they collaborate to resolve the problem task, display norms of their evolved mathematical microculture.

The study theorizes categories of interlocution whose features and functions reveal discursive practices of mathematics learners and criteria for identifying in discourse mathematical ideas. Other results indicate that students resolve novel mathematical situations as they engage in talk and develop efficacious problem-solving heuristics. They structure and implement their own methods of investigation, co-construct an understanding of the mathematical structure underlining the task, and build mathematical ideas synchronously as individuals and as members of a community of practice. Analyses of their discursive propositions and inscriptions illustrate how students explain and justify their work, evolve heuristic methods and combinatorial algorithms, as well as articulate dynamical links to build isomorphisms. The findings suggest how time influences cognitive development and how significantly more time ought to be spent enabling learners to collaborate, to think deeply about their heuristics, to build mathematical ideas, and ti connect mathematical structures.

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DEDICATION

For Samir, Karma, Jamel, Tajli, and Náela.

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CHAPTER 1: INTRODUCTION

1.1 Nature of the Study

Situated in the discipline of mathematics education, this is a microanalytic, epistemological, and discursive investigation into specific contents of the mathematical awarenesses of learners as they collaborate to resolve a problem task. I use the verb 'resolve' instead of 'solve' to underscore that the focus my inquiry is not with *the* or even *a* solution to the specific problem task given but rather with the mathematical contours and contents of the processes in which the learners engage to satisfy themselves and researchers that they have developed a response to the problem with which they consider both sufficiently satisfying and impervious to challenge, at least for an interval of in their lives, since like most responses to problems, theirs too may be provisional and temporal and depend on context.

The investigation is microanalytic since it focuses in detail on the work of four learners, and based on their intellectual actions and products, the study illustrates possibilities for larger groups of learners. Furthermore, the investigation concerns epistemology in that it is interested in understanding not only the resolutions the learners develop but also, and more fascinatingly and importantly, how they build mathematical ideas and forms of reasoning that in some instances they discard and in others use to formulate a resolution of the problem task. The materials for inquiry are the learners' discourse and inscriptions. With these materials and the analytic tools described in chapter 3, this study aims to understand what the learners' discursive and inscriptive products reveal about their mathematical ideas and forms of reasoning. In these senses, this investigation involves microanalysis, discourse analysis, and epistemology.

1.2 *The Setting and the Participants*

This study is based on a problem-solving session. At the time of the session, however, the participants and the context of the session have a 12-year history. The research session is part of a longitudinal, cross-sectional study, currently in its 15th year, supported by grants MDR-9053597, directed by Robert B. Davis and Carolyn A. Maher, and REC-9814846, directed by Carolyn A. Maher, from the National Science Foundation.[1] The problem-solving session for this research occurred near the end of the school year on 5 May 2000 at the David Brearley High School.[2] Located in Kenilworth, New Jersey, a diverse working-class, immigrant, community, the high school belongs to a school district that during the research session for this study already had a twelve-year partnership with Rutgers University and the Robert B. Davis Institute for Learning. The Rutgers-Kenilworth partnership established itself to create classroom environments in which children would be engaged actively in building mathematical models on an experiential foundation and in which curriculum would be based on students' construction of meaning (see, for instance, Maher,

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[1] Any opinions, findings, and conclusions and recommendations expressed in this dissertation are those of the author and do not necessarily reflect the views of the National Science Foundation.

[2] The longitudinal study is an outgrowth of a three-year teacher development project that began in 1984 in the Harding Elementary School in Kenilworth. See O'Brien (1995) for a ten-year analysis of this teacher-development project. At the same school, the longitudinal study itself was initiated in 1989 in a class of 18 first-graders, one of three such classes. As Maher (2002) relates, "[t]he students in each class remained together for their first three years of elementary school. In grade four, new classes were formed. The study continued with a smaller subset of the original class and several other students who joined" (p. 32). Seven of the original students have been followed for 15 years, including 3 years while they attended their respective universities. The participants of this study are a subset of the seven original students.

1991; Maher & Alston, 1990; Maher, Davis, & Alston, 1993; Maher & Martino, 1996b, 1996c, 2000; Martino & Maher, 1999).

Of significance to this study, over the span of the Rutgers longitudinal study in Kenilworth, researchers (Maher, Martino, & Pantozzi, 1995) have observed shifts in student interactions in problem-solving sessions. Significantly, student interactions have shifted toward growing independence in mathematical thinking and deeper listening to other students. When working collaboratively on a problem, participants of the longitudinal study have developed the mind habit of reflecting on their own ideas and on those of others.

The participants in this investigation are four students—Brian, Jeff, Michael, and Romina—in their senior year of high school. From their entry into first grade, they have participated in mathematical activities of the Rutgers longitudinal study. Over the years, these students have engaged tasks from several strands of mathematics, including algebra, combinatorics, probability, and calculus both in the context of classroom investigations as well as in after school settings (Maher, 2002; Speiser, Walter, & Maher, in press).

1.3 The Problem Task

The problem-solving session for this study was held in a classroom during the late afternoon, after school hours. During the session, which lasted about 1 hour and 40 minutes, the four students collaborate on a culminating task of the research strand on combinatorics that I suggested and helped design—The Taxicab Problem:

> A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the

> intersections indicated on the map. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route.
>
> What is the shortest route from a taxi stand to each of three different destination points? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.

Accompanying this problem statement, the participants have a map, actually, a 6 x 6 rectangular grid (see Figure 1) on which the left, uppermost intersection point represents the taxi stand. The three passengers are positioned at different intersections as blue, red, and green dots, respectively, while their respective taxicab distances from the taxi stand are one unit east and four units south, four units east and three units south, and five units east and five units south.

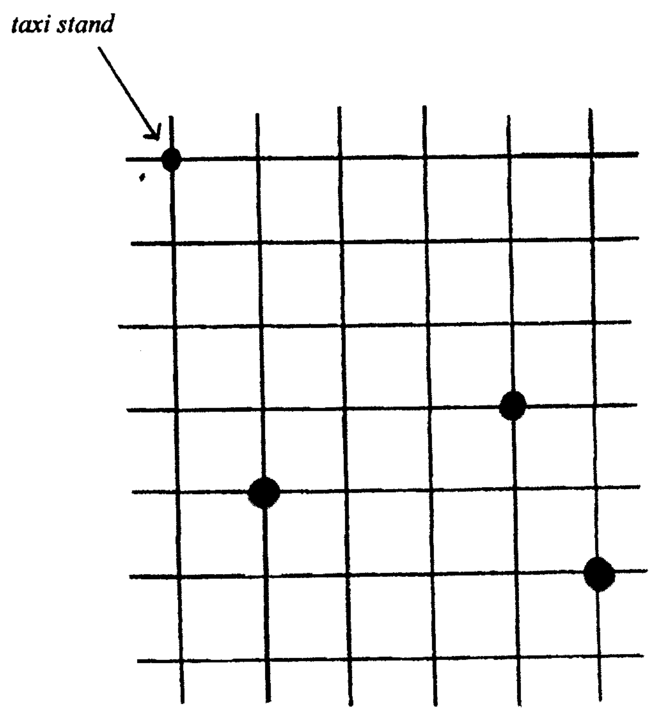

Figure 1. The grid of the Taxicab Problem, one quadrant of the taxicab plane.

This task is a combinatorial problem and set in a non-Euclidean context. See Appendix A for the task as it was actually given to the participants. The problem task has an underlying mathematical structure and contains concepts that resonate with those of other problems on which the students have worked. See Appendix B for the sequence of combinatorial problem tasks on which the participants have worked preceding the task of this study. Working on the present problem task, the participants revisit mathematical ideas they have already built as well as to deepen those ideas and build new ones. The next section details the mathematical environment of the problem task and its curricular significance.

1.4 The Taxicab Problem and Its Significance

The Taxicab Problem lies at the crossroads of combinatorial analysis and non-Euclidean geometry. The particular non-Euclidean geometry in which the problem is embedded has come to be known as taxicab geometry (Golland, 1990; Golland, McGuinness, & Sklar, 1994; Krause, 1973, 1975/1986; Menger, 1952, 1979). In one of his texts on learning and pedagogy, among other topics, Polya (1981) discusses mathematical recursion and, in this context, poses the combinatorial problem of determining "the number of different shortest paths in a network between given endpoints" (p. 68). In relation to this general enumeration problem, the Taxicab Problem is a specialized instantiation of it, situated in the two-dimensional space of Menger's taxicab geometry.

As Menger (1979) recounts, it was Hermann Minkowski (1864-1909), in a collection of his papers (1911/1967), *Gesammelte Abhandlungen,* published posthumously at the beginning of the Twentieth Century, who defined a metric concept "according to which a circle is indeed a square" (p. 217). Menger himself also contributed to this geometry. According to Golland (1990), he initiated the systematic development of abstract distance geometry in 1928 and was the first to use the appellation "taxicab geometry." The name appears in a booklet, *You Will Like Geometry*, a publication accompanying a geometry exhibit that Menger established in 1952 at the Museum of Science and Industry in Chicago, Illinois (Menger, 1952, p. 5). Menger's exhibit attracted the attention of mathematicians as well as non-mathematicians. The taxicab metric and its conceptual implications for the spatial quality of a circle immediately caught the attention of other mathematicians (see, for example, Curtis, 1953).

Besides pedagogical and mathematical purposes, Menger was also interested in the defining metric of taxicab geometry for philosophical and polemical reasons. It is the particular case for $n = 1$ of Minkowski's generalized distance function $\left[|x_1 - x_2|^n + |y_1 - y_2|^n\right]^{1/n}$. Given a fixed point with integral coordinates in the taxicab plane, (x_0, y_0), using the metric of taxicab geometry, the equation—$d_t((x_0, y_0),(x,y)) = |x_0 - x| + |y_0 - y| = 1$— defines the locus of points that lie a unit distance from (x_0, y_0). That is, this taxicab equation of a circle defines a unit square and thus is an example of a square circle. With this idea, Menger (1979) presents contrary evidence to the then prevailing philosophical thought:

> Round squares or square circles have been considered as the pinnacle of absurdity and the quintessence of impossibility by practically all philosophers and philosophical schools in antiquity, in the middle ages and in modern times. In fact, their inconceivability has been one of the few points of universal consensus, even after Minkowski in the early 1900's proved their existence in a new non-Euclidean geometry, then

> considered as ivory tower mathematics. But, years ago, I brought square circles literally to the man in the street. I showed that in a modern city the end points of all taxi rides starting at the same point and having the same length (as proved by fares)—in other words, the points of a circle in the taxicab geometry—lie on a square. (p. 4)

As he further states elsewhere, with the metric of taxicab geometry, mathematicians give "a perfectly meaningful interpretation to the paramount example of what philosophers call *meaningless*" (Golland et al., 1994, p. 28).

The taxicab metric makes meaningful seemingly contradictory categories: square circle and circular squares. However, this is not its only virtue. Krause (1973; 1975/1986) argues that taxicab geometry's accessibility and closeness to Euclidean geometry renders it more pedagogically appropriate than other non-Euclidean geometries. He advocates its educational value for developing the mathematical faculties of secondary school students. Following Krause's lead, other mathematicians and mathematics educators (Borasi, 1981, 1991; Laatsh, 1982; Prevost, 1998; Schattschneider, 1984; Sowell, 1989) have further developed investigations and applications of taxicab geometry. Some of these intersect with areas outside of geometry such as analysis and algebra. The problem used in this study—the Taxicab Problem—among others, involves ideas reflecting Pascal's triangle, binomial coefficients, arithmetical progressions, combinatorial analysis, and distinction between Euclidean and taxicab geometries such as their planar structures, their metrics, and the uniqueness of a path between two distinct points. In textual materials designed to introduce learners to taxicab geometry or to combinatorial analysis, a path problem similar to the one of this study is often an early introductory challenge (Jacobs, 1970/1994; Niven, 1965; Polya, 1981; Prevost, 1998).

1.5 Research Purpose and Guiding Questions

More than a decade ago, Davis (1992a) challenged mathematics education researchers to study the emergence among learners of what lies at the core of mathematics: *mathematical ideas*. He noted that "very little research in mathematics education has focused on the actual ideas in students' minds or on how well teachers are able to identify these ideas, interact with them, and help students improve on them" (p. 732). This study takes up this challenge and posits that such ideas can manifest themselves in learners' discursive interactions and inscriptive products. Through an analysis of these complex interactions and products, this investigation has global as well as local purposes. Globally, it aims to contribute basic scientific understanding of cognitive and discursive behaviors for which mathematical ideas, forms of reasoning, and mathematics learning emerge as by-products of sense making. Locally, by inviting the participants to engage the Taxicab Problem, this investigation endeavors to identify, examine, and illustrate mathematical ideas that the four participants build through their discursive acts.

The problem task is an investigative and cognitively demanding activity. The task admits more than one resolution strategy, each capable of being represented in multiple ways. It requires learners individually and collaboratively to impose meaning and structure on it, make decisions about what to do and how to do it, interpret the reasonableness of their actions, formulate conjectures, verify their conjectures, and justify their procedures and resolutions. In short, the problem task requires learners to reason mathematically.

The analytic foci of this investigation are on talk and inscription. Philosophically, whether revealed in talk or in writing, this investigation views that mathematical ideas, including objects and relations, as constituted in discursive activities, not as preexisting and waiting to be noticed. That is, through communicative acts, mathematical ideas derive their existence and meaning, shape and content.

The participants have been part of the Rutgers longitudinal study and, in the course of it, have worked on problem tasks in which they have built mathematical ideas similar to the underlying mathematical structure of the Taxicab Problem. In the context of this study, this problem task was proposed with the following four central guiding research questions:

1. As the participants engage collaboratively, without assigned roles, to understand and resolve the Taxicab Problem what are the features and functions of learners' discursive practices?
2. As they articulate their emergent understandings of the problem task, what mathematical ideas, associated meanings, and forms of reasoning do they reveal in their discourse?
3. How do participants structure their investigation?
4. How do their conversational exchanges support advances in their problem solving?

1.6 Mathematical Microcultural Norms of the Participants

Over the course of their twelve-year participation in the Rutgers longitudinal research project, engaging mathematical tasks, the four participants of this study along with other members of their cohort have structured a dynamic way of working together. Characteristics of their way of working constitute what can be called their mathematical microculture. Its consists of evolving, situational patterns of behavior that members of the cohort expect of each other and that an observer can notice when cohort members engage problem tasks of the Rutgers research project. Similar to the problem task of this study, these tasks are investigative and cognitively demanding, requiring participants to reason mathematically. For instance, the cohort members demonstrate microcultural norms that include an insistence on sense making, free exchange of ideas, as well as collective and individual justification of ideas.

A window into how these microcultural norms have evolved is to examine characteristics of the teaching interventions of the Rutgers longitudinal study. The following are findings of behavioral characteristics of teacher-researchers when engaging the cohort group in mathematical activities (see, Maher, 1998):

1. Pose tasks, invite learners to participate in building a repertoire of mathematical representations, notations, formalisms, techniques, and lines of reasoning;
2. Assess the ideas that the students have built by observing their activity (such as model building) and listening to their explanations;
3. Require the students to support ideas with suitable justifications and arguments;
4. Encourage the students make public how their ideas emerged so that they develop mathematical tools as well as apply and build upon them;
5. Make standard mathematical tools, notations, representations, and language meaningful and accessible to students;
6. Work to build a classroom culture that encourages the exchange of ideas;
7. Notice and call to the students' attention instances where there are differences and disagreements, and then facilitate conversations among students about these discrepancies;
8. Encourage student-to-student and student-to-teacher efforts to map representations and develop modes of inquiry that might disclose deeper understanding or discrepancies;

9. Facilitate the organization and reorganization of student groups to allow for the timely sharing of information and ideas;
10. Provide multiple opportunities for students to talk about and represent ideas;
11. Keep discussion open and revisit ideas over sustained periods of time;
12. When students have solved a particular problem, challenge them to generalize and extend their solution;
13. Assist students in reflecting on their own learning process so that they can design ways of working on future investigations;
14. Supply students with materials with which to reflect on, reconsider, and make public their ideas; and
15. Treat students and their thinking with interest, dignity, and personal respect.

Remarking on teacher-researchers' expectations, Speiser, Walter, and Maher (in press) note a habit of the mind that evolved into a feature of the microculture:

> From the first grade onward, learners were expected to invent representations that made sense to them, build solutions that they found convincing, and to communicate their findings in an atmosphere of thoughtful and responsive questioning. Over the years, these students became accustomed to extended explorations, in which first impressions often proved to be inadequate.

Extended investigative tasks that require thinking deeply, beyond initial thoughts, are features that contribute to the microculture of the Rutgers longitudinal study. In studies on proofs and representations (Kiczek, 2000; Maher & Martino, 1996c, 2000; Maher & Speiser, 1997; Martino, 1992; Muter, 1999), detail analyses of cohort members changing views of underlying mathematical ideas provide results in the areas of proofmaking and representational systems. Major findings include the following five:

1. Students offered convincing arguments in presenting solutions to their problems. They reasoned using proof by cases and proof by contradiction. The forms of reasoning that evolved included inductive, deductive and recursive.

2. Students expressed forms of reasoning first through informal language and their own notations. The structure of their reasoning broadened, and deepened over time and became increasingly symbolic and generalized.
3. Students' transitions to standard forms of notation occurred naturally. In explaining and justifying solutions to problems, students frequently shifted between personal notational systems and more standard forms to express their ideas.
4. Students developed a deep interest in understanding their own ideas and making sense of them, as well as the ideas of others. The expectation that all ideas would be heard, explored and discussed became the norm.
5. Follow-up interviews revealed that students were able to talk about and reconstruct earlier reasoning about problem tasks. Reconstructed solutions were generally expressed in more formal symbolic notation.

The cohort of the Rutgers longitudinal study, to which the four participants of this study belong, have evolved a microculture, a community of learners that has been shaped by their intellectual actions and the pedagogical behaviors of the teacher-researchers. Consequently, Brian, Jeff, Michael, and Romina, the participants of this study, in the context of the Rutgers longitudinal study have a rich history of problem solving, ways of working on mathematical problems, and internal awareness of researchers expectation that they present convincing arguments for their resolutions.

CHAPTER 2: REVIEW OF LITERATURE

2.1 Introduction

This study introduces itself into disciplinary discussions concerning the psychology and sociology of mathematics education, specifically concerning the influence of discourse on learning. Considerations of discourse are recent to mathematics education whereas discussions of learning have a rather long history and tradition. There are many philosophical traditions that theorize learning in general and about mathematical learning in particular. This review addresses ideas about mathematics learning that underpin the study and then focuses on literature concerned with discourse as vehicle for learning. The aim of this review is arrive at the contribution of this study to discussions of the role of discourse in mathematics learning.

2.2 Learning

Many researchers claim that thought is primary to language and originates in action. Psychologists (for instance, Gattegno, 1973; Gruber & Vonèche, 1977/1995; Piaget, 1973) sustain this position about thinking. Intellectual development or knowing occurs before an individual's acquisition of natural language and, therefore, prior to participation in discourse communities. Piaget notes, "before all language, at the purely sensorimotor level, actions are susceptible to repetition and then to generalization thus building up what could be called assimilation schemes" (1973, pp. 79-80). Elsewhere, he (Piaget, 1960) theorizes about the conditions for the transition of sensorimotor knowledge to reflective knowledge. He argues that essentially intellectual development occurs largely as a result of children's sensory and muscular actions and their assimilation and accommodation of the responses of the world to these actions (1960, pp. 119-155).

Besides sensorimotor factors, Piaget also posits the role of the social on intellectual development. Social development influences intellectual development when "groupings of concrete operations and particularly when those of formal operations are constructed...the problem of the respective roles of social interaction and individual structures in the development of thought arises in all its acuteness" (1960, p. 162). As Tudge and Winterhoff (1993) point out, in the third edition of *The Language and Thought of the Child,* Piaget views that "even during the preoperational stage, children can engage in the sort of discussion likely to lead to development" (p. 69). The influence of social intercourse occurs when individual has the need to understand the thought of others. To communicate thought give rise in an individual to a desire to verify thought. As Piaget states: "The social need to share the thought of others and to communicate our own with success is at the root of our need for verification. Proof is the outcome of argument...[which in turn is] the backbone of verification" (Gruber & Vonèche, 1977/1995, p. 92).

Thinking for and to oneself does not bring about the need for logical thinking or verification of one's thinking. Logical thinking, therefore, is not natural; it is a socialized system of thinking: "logic requires common rules or norms; it is a morality of thinking imposed and sanctioned by others" (Piaget, 1960, p. 163). As quoted in Tudge and Winterhoff (1993), Piaget notes that "[s]ocial life is a necessary condition for the development of logic. We thus believe that social life transforms the individual's very nature" (p. 69).

The relationship between social development and intellectual growth is, however, not one-sided. Piaget acknowledges that the relationship is dialectical. On the one hand, he opines, "[i]t is in fact very difficult to understand how the individual would come to group his operations in any precise manner, and consequently to change his intuitive representations into transitive, reversible, identical and associative operations, without interchange of thought" (1960, p. 164). On the other hand, he states, "every grouping within individuals is a system of operations, and co-operation constitutes the system of operations executed in common" (Piaget, 1960, p. 165-6). Grouping and co-operation are complementary processes, each requiring and influencing the other. Piaget declares, "the most remarkable aspect of the way in which human knowledge is built up...is that it has a collective as well as an individual nature" (as quoted in Tudge & Winterhoff, 1993, p. 69).

Like Piaget, other psychologists theorize the primacy of thought. Gattegno (1973), an early translator of Piaget into English (1949, *Play, Dreams & Imitations* and 1951, *The Child's Conception of Number*[3]), also argues that thought occurs before language. Nevertheless, the acquisition of a language signals the existence of important structures in the mind for doing mathematics. In various writings, Gattegno (1970; 1973; 1988) contends that the acquisition of spoken language evidences that an individual has built mental structures and gained facility in mental processes that are akin to structures and processes of mathematics. Among several findings that he presents are the following three:

[3] While translating from French to English, *The Child's Conception of Number*, Gattegno states that he "had opportunities of getting stuck at a number of points" and "suggested to Piaget that Part II of the published study must be different if he wanted the book published in English with [Gattegno] as the responsible translator. [Piaget] yielded...and the English translation differs...from the original French" (Gattegno, 1983, p. 6).

(a) Children who start school mathematics have already reached a level of mental evolution which is illustrated by, among other things, their considerable mastery of spoken language.

(b) Many activities of children require an organization of mental operations which can only be described as algebraic in nature.

(c) Underlying the notations of mathematics there are verbal components; so the mastery of the spoken language means that it possible to base mathematics on language. (1988, p. 2)

In Gattegno's view, an individual who has acquired a spoken language necessarily possesses algebraic structures in the mind, and by implication, mathematics learning can be based on these already acquired linguistic-mathematic structures.

For Gattegno, learning mathematics has much to do with processes that he collectively terms mathematizing (Gattegno, 1974, 1988). Discussing this construct, Wheeler (1982) suggests that it is "more useful to know how to mathematize than to know a lot of mathematics" and wonders "why do the majority of teachers not encourage their students to 'function like a mathematician'" (p. 45). The suggestion is that to function like a mathematician is to mathematize. For Wheeler, to mathematize is an inherent ability, and he defends this claim indicating that anyone who has learned to speak a language already has shown significant algebraic thinking:

> It has been repeatedly stressed by observers [see, for example, Gattegno (1970; 1973)] that within a very short time of beginning to speak, young children utter grammatically correct phases and sentences that are not copies of any that they have heard others speak. It is clear that [children] could not operate autonomously with the grammar of the language without the capacity to handle classes (nouns, verbs), inclusion and intersection (adjectives, adverbs), relationships (prepositions), transformations (tenses) and substitutions (pronouns)...in fact, the transformational requirements of a confident use of grammar are very complex and still defy complete analysis. But the chief points are issue are that: (a) the child's mastery of grammar can only be adequately described

> in terms of mathematical operations, and (b) this mastery is not derived by imitation. (pp. 45-46)

Wheeler's concern for students functioning as mathematicians and the importance he attributes to mathematizing, parallels Freudenthal's (1991) insistence that

> the learner should reinvent mathematising rather than mathematics; abstracting rather than abstractions; schematising rather than schemes; formalising rather than formulas; algorithmising rather than algorithms; verbalising rather than language. (p. 49)

Understanding proceeds as by-product of functioning attentively in the world. In Piaget's theory, this process of functioning is characterized by his two basic mechanisms to account for development: assimilation and accommodation. He defines these mechanisms as follows:

> Psychological assimilation...is the transformation of the external world in such as way as to render it an integral part of oneself. In the case of intelligence, it is the integration of external objects to the schemata of subjective actions, fusing a preexisting schema and a new object. Any object is then perceived in terms of the actions using it....Accommodation is...a tendency of the organism to compensate for resistance of the object to assimilation by creating a new alternative, or *tertium*, between the application and the nonapplication of a schema to a certain object. (Gruber & Vonèche, 1977/1995, p. 216)

In mathematics, for instance, learners adapt or otherwise modify their understanding to address a new problematic situation when their conception proves inadequate in a situation and, thereby, impedes them from advancing toward a resolution of a problem or when what they consider significant does not coordinate with their previous ideas or resolutions. Learners' growth in understanding occurs as they accommodate their existing web of understanding and assimilate new objects, information, or perspectives.

Investigators in mathematics education have explicated assimilation and accommodation as processes in mathematics learning. In their model for the growth in mathematical understanding, Pirie and Kieren and their collaborators (Kieren & Pirie, 1994; Kieren & Pirie, 1991; Martin, 1999; Pirie & Kieren, 1994; Pirie, Martin, & Kieren, 1996) explicate processes such as "folding back," that echo Piaget's accommodation construct. Davis and Maher and their colleagues (Davis, Maher, & Martino, 1992; Davis & Maher, 1990; Speiser et al., in press; Speiser & Walter, 2000; Steencken, 2001) posit the essential claims of their theory, which implicitly includes the notions of assimilation and accommodation to explain what one does when she or he does mathematics. They state, "to think about a mathematical situation, one must cycle (perhaps many times) through these steps":

1. Build a representation for the input data.
2. From this data representation, carry out memory searches to retrieve or construct a representation of (hopefully) relevant knowledge that can be used in solving the problem or otherwise going further with the task.
3. Construct a mapping between the data representation and the knowledge representation.
4. Check this mapping (and these constructions) to see if they seem to be correct.
5. When the construction and the mapping appear satisfactory, use technical devices (or other information) associated with the knowledge representation in order to solve the problem. (1990, p. 65)

They anchor these points of their theory in the empirical findings of dozens of studies. In one such study, Davis, Maher, and Martino (1992) provide a detail examination and careful analysis of the mathematizing activity of two fifth-grade students resolving a task in the domain of fractions and note that children build knowledge, "using previous experience as assimilation paradigms or 'internal metaphors'" (p. 179). Maher (1998) argues that new data are internalized resulting from learners' well-coordinated actions on objects that may occur in response to exploring a problematic situation. Learners respond by accommodating existing assimilation paradigms. Davis (1984) theorizes that learners construct new ideas from previously learned experiential (not necessarily 'concrete') components by a process he refers to as "assembly" and that through this process the ideas come to be synthesized as metaphor and later as abstract entities. Elsewhere, continuing the assemblage metaphor, Davis (1992b) argues that understanding "occurs when a new idea can be fitted into a larger framework of previously-assembled ideas" (p. 229).

2.3 *Discourse*

In mathematics education, discourse and its influence on mathematical learning has been the subject of numerous studies. In this study, discourse refers to language (natural or symbolic) used to carry out tasks—for example, social or intellectual—of a community. Researchers have been inquiring into influences of discussion between and among peers who bring different perspectives to a task. Mathematics education theorists (Bishop, 1985; Cobb, Wood, & Yackel, 1993; Dörfler, 2000; Seeger, 2002; Sfard, 2000c) claim that mathematical meaning evolves through individual's participation in discursive activity, either verbal or inscriptive interactions. However, attention to the interconnection of discourse and learning in the field of mathematics education is a relatively recent research activity. Searching English-language research journals in mathematics education yielded no literature on discourse before the early 1980s.

In 1984 in a plenary address at a Canadian mathematics education conference, Bishop (1985) discussed what he considered to be the fruitfulness for research in mathematics education of a new theoretical perspective that underscored the social construction of meaning. This new perspective spotlighted what for him are three significant areas. The second of his areas is communication and emphasizes "the process and product of shared meanings" (p. 26). "Meanings and understanding," he wrote, "are about connections one has between ideas" (p. 27). He claimed that

> [c]ommunication in a mathematics classroom is therefore concerned with sharing mathematical meanings and connections. We can only share ideas by exposing them, and 'talk' is clearly a most important vehicle for exposing connections. (p. 27).

In terms of investigations, Bishop suggested that communication in mathematics education proffered rich arenas of research (p. 27).

Parallel with and as a consequence of the emergence of social constructivist and sociocultural perspectives, a stimulus for many studies on discourse has been the claims of the current reform movement in mathematics education that assert discourse as an efficacious vehicle to support students' mathematical development. For instance, the National Council of Teachers of Mathematics (2000) in their document, *Principles and Standards for School Mathematics*, elaborate contentions about the cognitive benefits of communication:

> Communication is an essential part of mathematics and mathematics education. It is a way of sharing ideas and clarifying understanding. Through communication, ideas become objects of reflection, refinement, discussion, and amendment. The communication process also helps build meaning and permanence for ideas and makes them public. When students are challenged to think and reason about mathematics and to communicate the results of their thinking to others orally or in writing, they learn to be clear and convincing....
>
> Students gain insights into their thinking when they present their methods for solving problems, when they justify their reasoning to a classmate or teacher, or when they formulate a question about something that is puzzling to them....
>
> Reflection and communication are intertwined processes in mathematics learning. (pp. 60-61)

The importance of reflection and communication for learning are processes about which mathematics educators agree. As Cobb, Boufi, McClain, and Whitenack (1997) note, consensus on this point within the mathematics education community transcends theoretical differences and include researchers who draw primarily on mathematics as a discipline, on constructivist theory, and on sociocultural perspectives (p. 258).

This consensus notwithstanding, two lines of research broadly speaking can be distinguished. On the one hand, some researchers analyze discourse as a vehicle to investigate relationships between teaching and learning mathematics; essentially teacher-to-student discussions that are teacher lead. On the other hand, other researchers focus their analytic resources to inquire into relationships between student-to-student discursive interactions and consequent mathematical learning; that is, they investigate the mathematical ideas that students build through and constituted by student-student discussions. In the research literature there is a dearth of attention on student-to-student interactions and the mathematical ideas that are built therein.

2.3.1 Teacher-to-student discursive interactions

In the past two decades, the nature of communications and how teachers can encourage and support communicative acts among students that are both productive and mathematical have been the subject of empirical research and theoretical reflection (for example, see Alrø & Skovsmose, 1998; Cobb et al., 1997; Davis, 1996, 1997; Fernandez, 1994; Hiebert & Wearne, 1993; Larson, 1999; Maher, 1998; McNair, 1998; O'Connor, 1998; O'Connor & Michaels, 1996; Phillips

& Crespo, 1996; Pimm, 1987; Seeger, 2002; Sfard, 1994, 2000a, 2000c, 2002; Sfard, Nesher, Streefland, Cobb, & Mason, 1998; Speiser, Walter, & Shull, 2002).

In the context of a five-year case study, Maher and Martino (1996b) present a sequence of classroom and interview episodes to document the mathematical thinking of a learner working on several combinatorial problem tasks and the process by which the learner invents proof by cases. They theorize conditions that encourage student conversation in classrooms.

> These include the following: (a) opportunities for students to work in a variety of social settings, (b) flexibility in the curriculum for students to continue working on a problem or to pursue a new idea, (c) teacher restraint from telling students what to do, and (d) teaching guided by student thinking. These conditions suggest that there be a shift in the teacher's role in helping students, that is, from telling to guiding, and that the classroom be organized in ways that encourage student investigation. (p. 196)

In particular, their findings provide insight into how through conversational exchanges learners may build and invent strategies of proofs. The design of their study provides multiple opportunities for the learner to test, modify, extend, and reflect on her ideas. They raise the question of whether the conditions of the study are integral in the development of student ideas about justification. In a later report, Maher and Martino (2000) argue that "classroom environment," including having sufficient time for exploration and reinvention are "crucial" (p. 269). Exploration and reinvention, as their studies show, can occur through discourse.

Discourse in classrooms, however, is multifaceted. Mercer (1995) distinguishes two types of discourse in schools: educational and educated. The "educational discourse" is the cultural ways in which acceptable talk transpires in the classroom among teachers and learners such as ways of asking and answering questions. In contrast, the "educated discourse" is disciplinary specific and often represents new ways of using language. The phrases 'let x be any nonnegative real number' and 'an ellipse is the locus of a point the sum of whose distances from two fixed points is constant' are examples of educated discourse in mathematics. Mercer argues that teachers

> have to use *educational* discourse to organise, energise and maintain a local mini-community of *educated* discourse....
>
> Teachers are expected to help their students develop ways of talking, writing, and thinking which will enable them to travel on wider intellectual journeys, understanding and being understood by other members of wider communities of educational discourse: but they have to start from where learners are, to use what they already know, and help them go back and forth across bridges from 'everyday discourse' into 'educated discourse'. (pp. 83-84)

Mercer suggests an explicit, mediational role in the apprenticeship of students to the educated discourse of mathematics.

Apprenticing learners to educated discourse can be problematic. Drawing from a quantitative study of secondary school teachers' understanding of their practice in multilingual classrooms in South Africa, Adler (1999) explores the benefits and constraints of "explicit mathematics language teaching." She examines tensions between apprenticing students in the use of the disciplinary language of mathematics or, using Mercer's notion, the educated discourse of mathematics, and the way in which such an apprentice can impede students building mathematical knowledge. From her study, she finds the following:

> For talk to be a resource for mathematics learning it needs to be transparent; learners must be able to see it and use it. They must be able to focus on language per se when necessary, but they must be also be able to render it invisible when they are using it as a means for building mathematical knowledge...There is no resolution of the dilemma...there is only its management through awareness and careful instructional moves when making talk visible in moments of practice. (p. 63)

Adler presents data to illustrate her point about the relative visibility of educated talk. Using a classroom vignette of students working on a trigonometric problem and a teacher's explicit teaching of mathematical language, she notes the importance of talk used for thinking and that inappropriate, untimely attention to apprenticing students into educated discourse can cause students to loose focus on the specific mathematics of the problem on which they are working.

Another perspective on apprenticing learners to educated talk concerns the communicative moves of teachers in socializing and acculturating learners into complex thinking practices in mathematics and science. As a window into this socialization and acculturation process, some researchers characterize for group discussions teacher-student participation frameworks. That is, they examine the particular roles and responsibilities established through language. A case in point is a study by O'Connor and Michaels (1996). They describe and analyze two exemplary teachers' use of what they call revoicing—"a particular kind of reuttering (oral or written) of a student's contribution" (p. 71)—as a discursive move to create a complex participant framework and to foreground propositions in a discussion. They note that revoicing "makes possible an expanded and more contrapuntal set of voices and participant roles in constructing an idea" (p. 97). They contrast framework with the commonly occurring participant framework in teacher-led discussions, what many researchers call the initiation-reply-evaluation (IRE) sequence.[4] Their study

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[4] This IRE discourse sequence can be described more fully: a teacher initiates the sequence by asking a question for presumed known information, followed by a student responding to the question, followed in turn by the teacher evaluating the student's response. As Erickson (1996) observes, the IRE sequence "frames a power-knowledge relationship that is constitutive of the

focuses on the nature of the foregrounded proposition but do suggest that the content of the revoicing move, the foregrounded proposition, is an important issue for study.

Foregrounded actions in classroom discourse is the subject of studies by Cobb and his colleagues' (Cobb et al., 1997; Cobb et al., 1993; Cobb, Yackel, & Wood, 1992). They investigate the ways in which teachers working with children negotiate the multifarious norms of classroom discourse. Cobb et al. (1997) observe a first-grade, whole-class discussions mediated by a teacher to understand "possible relationships between classroom discourse and the mathematical development of students who participate in, and contribute to it" (p. 258). To analyze this relationship, they propose the constructs of "reflective discourse" and "collective reflection."[5] The first construct characterizes the social phenomena of what students and teacher do in action. It becomes objectified and an explicit object of discourse. In this sense, they examine shifts in discourse patterns that support students' reification of their mathematical actions. They hypothesize that learner's participation in reflective discourse "*constitutes conditions for the possibility of mathematical learning*" (p. 264, original emphasis) and emphasize that "it is the students who actually do the learning. Participation in reflective discourse, therefore, can be seen both to enable and

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conventional classroom; that is, it presumes and manifests the belief that 'there is a body of knowledge to be mastered and the teacher has mastery of it. It is that mastery which justifies the teacher's authority in the classroom'" (p. 60, footnote 3).

[5] This construct of 'reflective discourse' is reminiscent of Piaget's (1973) notion of "reflective abstraction," which, in discussing actions and logico-mathematical experience as a necessary preparation for deductive thought, he defines in this way:

> On the one hand, this abstraction 'reflects' (in the same way as a reflector or projector) everything that was on a lower plane (for example, that of action) and projects it to a higher plane, that of thought or mental representation. On the other hand, it is a 'reflective abstraction' in the sense of a reorganization of mental activity, as it reconstructs at a higher level everything that was drawn from the coordination of actions. (p. 81-82)

constrain mathematical development, but not to determine it" (p. 272). Moreover, they address the roles of teachers and symbolization in supporting reflective shifts in classroom discourse. As an example of reflective discourse, they describe first-grade children who generate data on different ways a group of five monkeys can be distributed between two trees. The teacher records the children's suggestions in a table. The discourse pattern shifts when the teacher asks the children how they could be sure that they had accounted for all ways to distribute the five monkeys among the two trees. In the ensuing interactions, the results of their previous activity emerge as explicit objects of discourse.

In this study as well as in the other studies reviewed thus far, the teacher's role is seen as instrumental in triggering on the part of students either reflective discourse or otherwise productive discussions. How and to what extent can such discussions occur among students themselves? There is need for studies that investigate conditions in which the instructor's instrumental role in triggering reflective discourse is minimal and, in turn, students interacting amongst themselves independently catalyze their own reflective abstraction and yield mathematically productive discussions.

2.3.2 Student-to-student discursive interactions

In mathematics education, considerably more analytic attention has been paid to mathematical learning that results from teacher-to-student and teacher-lead interactions than to the mathematical ideas that emerge from the discursive interactions among students in situations in which they structure their own investigation. A question that arises in considering student-to-student interaction is what is meant by student-to-student mathematical discussion.

Pirie and Schwarzenberger (1988) and Pirie (1991) suggest that peer conversations are mathematical discussions when they possess the following four features: purposeful, focus on a mathematical topic, involve genuine pupil contributions, and interactive (Pirie, 1991, p. 143; Pirie & Schwarzenberger, 1988, p. 461). With respect to research into peer discussions of this kind, Pirie notes that it "has received scant attention...but is important as an arena for learning" (p. 143).

Nevertheless, student-to-student interactions do pose a problematic. Discussing such interactions, Yackel, Cobb, Wood, Wheatley, and Merkel (1990) suggest that learners working in small groups engage in two types of problem solving, one having to do with the mathematical task and the other with the social task of working productively. They underscore the obligations of teachers who encourage students to work in small groups: "(1) that they should cooperate to solve the problems and (2) that they should reach a consensus" (p. 19). In such a classroom atmosphere, children as young as second graders are enabled to verbalize their thinking, explain or justify their solutions, seek clarifications, reconceptualize a problem, and extend their conceptual framework to incorporate alternative solutions methods (p. 19). It is a matter of research to ascertain whether to attain these outcomes it is required that teachers encourage students to reach a consensus.

Conversational exchanges about mathematics are of different kinds, some more useful than others. In 1997, at the Third International Conference of History, Philosophy, and Science Teaching, a panel debate was held concerning the efficacy of learning through conversation; that is, whether it is "as good as they say." The authors Sfard, Nesher, Streefland, Cobb, and Mason (1998) state and defended their various positions, which are rather similar to each other. Streefland posits a triad of principles to be observed if discourse is to be effective:

> The participants have to
> (a) be constructive and creative;
> (b) communicate mathematically in a productive manner;
> (c) allow for meta-cognitive shifts. (p. 45)

Following Streefland, Cobb argues that researchers in mathematics education need to investigate the question of the efficacy of classroom discourse empirically to

> look at the conversations that do arise...focus in particular on the nature of the students' participation in those conversations...[and] investigate what students might actually be learning in the course of their participation. (p. 46)

Essentially, Cobb suggests or reflects the interest of mathematics education investigators in two broad research questions concerning connections between students' discursive interactions and mathematics learning: (1) what is the nature of students' participation? and (2) what are students actually learning? Merely participating in a mathematical discussion does not ensure interlocutors' growth in understanding about some piece of mathematics. Cobb et al. (1997) provide evidence that participation in reflective discourse does not guarantee that each participant reflects or reifies prior action as object.

The two broad research questions are manifest in the research agenda of some investigators of mathematics education. For instance, Pirie and Schwarzenberger (1988), in a study design outline a classification scheme that contains three parameters to encode episodes of student-to-student interactions by identifying "the focus of discussion, the kind of language used and they type of statements being made" (p. 466). Each parameter in turn has triadic gradations.

Some researchers employ *a priori* classifications to investigate links between mathematical discussions and mathematical understanding, while others sift their data to discover emergent categories that capture how participants interact with each other in cognitively and mathematically productive ways. For instance, Maher, Martino, and Pantozzi (1995) report on observed shifts in student discursive interactions during problem-solving sessions. Over the span of a longitudinal study, which at the time of the report is in its tenth year, these researchers note the following about the microculture of the focus group of students:

> we have observed changes in student's questions of each other when a group of students grows accustomed to the lack of specific teacher prompts. The teacher/researcher instead models open-ended questions that are based upon the students' thinking, forming a distinct classroom culture. Over time, students begin to question each other in the same manner, probing each other's thinking with questions instead of requesting information or specific answers. The teacher/researcher's role recedes as the students drive the focus of inquiry and use their work as the starting point for additional exploration....Student-to-student questioning moves the activity forward; the students move into a position of mathematical authority since they do not look for the researchers to hand down knowledge. (p. 83)

These researchers observe that important shifts in students' interactive patterns parallel the receding role of teacher-researchers. Significantly, student interactions shift toward growing independence in mathematical thinking and deeper listening to other students. They reflect on their own ideas as well as those of others.

To discover emergent categories, some researchers inquire into relations between interaction characteristics and cognitive behavior of students by investigating dyadic problem solving. Working from the assumption that peer interactions are associated with socio-cognitive conflict and are a means for the co-construction of knowledge, Cobo and Fortuny (2000) use elements of discourse analysis and document interaction types and cognitive influence as pairs of students engage in area-comparison problem solving. In their analysis, they identify a typology of four discursive exchanges and find significant effects on individual development of cognitive and heuristic abilities in the problem-solving process.

2.4 *Intended Contribution of the Study*

In the literature on discourse, investigators have focused considerably more analytic attention to mathematical learning resulting from teacher-to-student and teacher-lead interactions than to mathematical ideas that emerge from discursive interactions among students. In particular, there is a dearth of attention to student peer interactions and therein the mathematical ideas built and the forms of reasoning engaged, especially in environments in which students structure their own problem-solving investigations. That is, a chasm in the research literature on discourse and mathematics learning concerns descriptions and analyses of the nature and function of conversational exchanges, participant frameworks, and the mathematical ideas and reasoning that students develop when they function as mathematicians and engage in mathematical discussions in which they orchestrate their own reflective discourse. It is in this very specific domain that this study contributes.

Toward the aim of analyzing the discursive patterns of learners to understand the mathematical ideas and forms of reasoning that they develop as they engage mathematical situations that are cognitively demanding, the following are the four central guiding questions of this research:

1. As the participants engage collaboratively, without assigned roles, to understand and resolve the Taxicab Problem what are the features and functions of learners' discursive practices?
2. As they articulate their emergent understandings of the problem task, what mathematical ideas, associated meanings, and forms of reasoning do they reveal in their discourse?
3. How do participants structure their investigation?
4. How do their conversational exchanges support advances in their problem solving?

CHAPTER 3: THEORETICAL FRAMEWORK AND ANALYTIC METHOD

3.1 Introduction

Several perspectives undergird this study's theoretical framework and analytic method. Each perspective provides a specific focus for the analytic lens I use to make sense of the data and to arrive at specific and general findings. In what follows, I discuss the notions of critical events, inscriptions, contents of mathematical experience, and interlocution. I end with a discussion of what can be called researcher as a participant interlocutor.

3.2 Theoretical Framework

3.2.1 Maher's notion of critical events

For the past fifteen years, Maher (see, for example, 2002) has lead a longitudinal research program, one that is unique in the field of mathematics education. Observational studies such as this one have played a central methodological role. Indeed, studies that identify, trace, analyze, and theorize about the development of mathematical ideas under particular conditions by individual learners as they work collaboratively in small groups have been the *raison d'être* of the research directed by Maher in which her colleagues and students have participated. From this research program, important analytic tools and theoretical notions have been developed. Gathered data, particularly video recordings, are used to aid inquiry into the moment-by-moment thinking of learners as evidenced in transcripts of their verbal interactions amongst themselves and with teacher-researchers and in records of their inscriptions. In their studies, Maher and her collaborators (Maher, 2002; Maher & Martino, 1996a, 1996b, 1996c, 2000; Maher, Pantozzi, Martino, Steencken, & Deming, 1996; Powell, Francisco, & Maher, 2001; Powell & Maher, 2002, forthcoming; Speiser et al., in press) initiate analyses by identifying particular instances in the mathematical work of learners termed *critical events*. They posit an *event* as a connected sequence of learners' utterances and other actions. An event is called *critical* when it demonstrates a significant or contrasting change from previous understanding, a conceptual leap from earlier understanding, or a cognitive obstacle (Kiczek, 2000; Maher, 2002; Maher & Martino, 1996b; Maher et al., 1996; Steencken, 2001). By connecting sequences of critical events and further analyzing them, for example, by constant comparisons (Glaser & Strauss, 1967), researchers build narratives that initially are amalgams of hypotheses and interpretations and that in turn influence subsequent identification and analyses of critical events. In this sense, critical events and narratives emerge together.

Critical events are contextual. The criticalness of an event is in the eye of the beholding researcher. An event that may be considered critical to one observer may not be considered as such by another observer. Given a particular set of data, events identified as critical depend on specificities of the study. A critical event is a designation that depends on the subject of a researcher's inquiry. Thus, an instance in which learners present a mathematical explanation or argument may be significant for a research question concerned with learners' building mathematical justifications or proofs and, as such, will be identified as a critical event. Similarly, a researcher concerned with the influence of teacher interventions on learners' reflective abstraction or mathematical understanding might deem as critical those events that connect teacher questions and to specific oral, written, or gestic articulation of learners' thinking. Moreover, the relation between critical events and research questions pursued also imply that researchers might identify events as critical that include instances of cognitive obstacles somehow significant to their study. Finally, an event may be deemed critical that is unrelated to the *a priori* theoretical or analytic concern of a researcher but that nevertheless signals his or her attention.

Whether resulting from an *a priori* or *a posteriori* perspective, the identification of an event as critical, therefore, rests on researcher criteria. Following Steencken (2001), who in her study explicates criteria for selecting events as critical, in section 3.2.3, adopting and augmenting categories of mathematical experience suggested by Gattegno (1987), I indicate criteria for this study that I use for determining a critical event.

Importantly, critical events can be affective or cognitive moments in which learners pursue lines of thinking that depart from those privileged in school or academic mathematics. In describing the role of critical events in data analysis, Maher (2002) states the following:

> The analysis begins with the identification of critical events. The mathematical content of each critical event is identified and described, taking into account the context in which the event appears, the identifiable student strategies and/or heuristics employed, earlier evidence for the origin of the idea, and subsequent mathematical developments that follow its emergence. (p. 35)

Critical events are temporally related to other events. Episodes of critical events, whether in expected or unexpected directions, are typically striking, connected to prior events, and prefigure upcoming events. As Maher states, "Each critical event defines a timeline, consisting of a past, a present and a future" and presents the following diagram (2002, p. 35) in Figure 2:

past————————————present——————————future———

(critical event)

Figure 2. Temporal relationship among a critical event and its antecedent and descendent events.

Realizing the temporal position of critical events, therefore, alerts researchers to examine their antecedent events as well as their influence on later understanding as well as to trace the development of ideas manifest in the critical event. Furthermore, when studying, for instance, the development of mathematical ideas or the growth of mathematical understanding, a critical event is associated to a time line and researchers may need to look for other related events in the past and in the future.

Collections of somehow related critical events can be theoretically and analytically important. Besides being temporally related, a critical event may be thematically related to other events. If the related events are critical and lead to growth in understanding, then the set of critical events form what Kiczek (2000) defines as a *pivotal strand* or a *pivotal mathematical strand* (Steencken, 2001). Within a narrative such strands emerge and point to the mathematical ideas and forms of reasoning that learners develop that are key in building a learner's mathematical understanding. Analytically, I would add, it is important to name the pivotal event, thereby indicating what narrative theme to which it belongs. Moreover, when analyzing the discursive interaction of learners, a critical event that qualitatively changes their inquiry trajectory, I coin as a *watershed critical event.* Such an event is often preceded by a series of related critical events that can be collected together as pivotal strand to indicate a discursive thread that begets the watershed critical event. In turn, the watershed initiates a cascade of events some of which may be critical.

3.2.2 Inscriptions

Inscriptions are special instances of the more general semiotic category of signs. A sign is an utterance, gesture, or mark by which a thought, command, or wish is expressed. As Sfard notes, "in semiotics every linguistic expression, as well as every action, thought or feeling, counts as a sign" (Sfard, 2000c, p. 45). A sign expresses something and, therefore, is meaningful and as such communicative, at the very least, to its producer and, perhaps, to others. However, its meaning is not static. A sign's denotation and connotation are subject to modification in the course of its discursive use.

As a discursive entity, a sign is a linguistic unit that has two, connected components. Saussure (1983) proposes that a sign is the unification of the phonic substance that we know as a "word" or *signifier* and the conceptual material that it stands for or *signified*. He conceptualizes the linguistic sign (say, the written formation) as representing both the set of noises (the pronunciation or sound image) one utters for it and the meaning (the concept or idea) one attributes to it (Saussure, 1983, p. 66). Thus, *sign = signifier + signified*.

Saussure observes further that a linguistic sign is arbitrary. A sign being the unity of signifier and signified implies that both components are arbitrary (Saussure, 1983, pp. 67-68). The signifier is arbitrary since there is no inherent link between the formation and pronunciation of a word and what it indexes. A monkey is called *o macaco* in Portuguese and *le singe* in French, and further in English the animal is denoted "monkey" and not "telephone" or anything else.

Just as the signifier is arbitrary so too is the signified. This can be understood in the sense that not every linguistic community chooses to make salient by assigning a formation and a sound image to some aspect of the experiential world, a piece of social or perceptual reality. Consider, for example, the signifieds *cursor, mauve,* and *zero.* They index ideas that not all linguistic communities choose to lexicalize. This is a significant point about Saussure's observation of the arbitrariness signs. I will return to this issue after making a point about the relationship between sign and meaning.

Sfard (2000c) argues that a sign is constitutive rather than representational since meaning is not only presented in the sign but also comes into existence through the it. Specifically, she states that

> mathematical discourse and its objects are *mutually constitutive*: It is the discursive activity, including its continuous production of symbols, that creates the need for mathematical objects; and these are mathematical objects (or rather the object-mediated use of symbols) that, in turn, influence the discourse and push it into new directions. (p. 47, original emphasis).

Mathematical signs—objects, relations, register, talk, and so on—are components of mathematical discourse and are intertwined in constituting mathematical meanings. Signs exit in many different forms, and inscriptions are but one. What are inscriptions? Inscriptions are written signs. They are produced for personal or public consumption and for an admixture of purposes: to discover, investigate, or communicate ideas. As other researchers and mathematics educators (Dörfler, 2000; Lehrer, Schauble, Carpenter, & Penner, 2000; Speiser et al., in press; Speiser et al., 2002) emphasize, building and discussing inscriptions are essential to building and communicating mathematical and scientific concepts. In a discussion of mathematics and science teaching, Lehrer, Schauble, Carpenter, and Penner (2000) illustrate how learners work "in a world of inscriptions, so that, over time, the natural and inscribed worlds become mutually articulated" and the importance of a "shared history of inscription" (p. 357). In mathematics, the invention, application, and modification of appropriate symbols to express and extend ideas are constitutive activities in the history of mathematics (Struik, 1948/1967).

Saussure's theorization about signs and their arbitrariness is applicable to inscriptions. For mathematics education the arbitrariness of signifieds is a more significant point about Saussure's observation concerning the arbitrariness signs. The conceptual material that one lexicalizes with, for example, marks on paper indicates what one sees, one's insight into material reality or the reality of one's mind. In research on mathematics education, participants' inscriptions open windows on their thinking. Their inscriptions present ideas they choose to lexicalize or symbolize. By analyzing inscriptions, researchers can infer participants' thinking. As Speiser, Walter and Maher (in press) underscore, what counts as mathematical in analyzing inscriptions is not the inscription itself, which are "tools or artifacts, but rather how the students have chosen to work" with their inscriptions.

The importance of inscriptions for learners is that they invent or appropriate them. In so doing, they change their relationship to what the inscription signifies and, as such, turn abstract ideas into concrete ones. On the issue of the concrete and abstract nature of objects, Wilensky (1991), arguing for a "revaluation of the concrete," states the following:

> concreteness is not a property of an object, but rather a *property of a person's relationship to an object*...Concepts that were hopelessly abstract at one time can become concrete for us if we get into the "right relationship"

> with time....The more connections we make between an object and other objects, the more concrete it becomes for us...the more ways we have of interacting with it, the more concrete. (p. 198, original emphasis)

As shall be shown in the next section, other researchers echo Wilensky's theoretical position on the concreteness or, by implication, the abstractness of a mathematical object.

3.2.3 Gattegno's contents of mathematical experience

In his penultimate book, the last published before his death, Gattegno (1987), who, among his contributions to mathematics education, developed a pedagogical approach that he employed in his use of Cuisenaire rods, which he popularized worldwide, and geoboards, which he is credited for having invented, provides theoretical justifications for his pedagogy—the subordination of teaching to learning. His book aims to develop a science of education. In the first chapter, he argues for his perspective on how sciences are born from human experiences. In particular, he describes how mathematics emerges from specific kinds of human actions on experience. Along with this idea and Gattegno's construct of the psychological entity that he calls *awareness* (1970; 1974; 1987; 1988), I extract categories of the contents of human experiences upon which mathematics is built. In turn, I adopt and extend these categories for the analytic purpose of crafting criteria for identifying critical events.

In discussing how sciences are born, Gattegno focuses not on the material, economic, or even cultural dimensions of their birth but rather on the psychological, dialogic mechanisms that individuals employ to notice and make sense of their experiential world. His fundamental premise is that "men [and women] can only study the contents of their awarenesses, and these awarenesses go to form the sciences, open to all who share them [the awarenesses]" (1987, p. 3). For example, one can become aware that the length of two or more Cuisenaire rods placed end to end is the same no matter which is positioned first and no matter how one considers grouping them. These two awarenesses not only can become part of the awareness of other observers, based on their own actual or virtual manipulation of Cuisenaire rods, but also a community of observers can formulate and label these specific contents of their awareness as, to use conventional terminology, the commutative and associative properties of addition. Similarly, with awareness of the Euclidean distance between two points, $d_e((x_1,y_1),(x_2,y_2)) = \left[|x_1 - x_2|^2 + |y_1 - y_2|^2\right]^{1/2}$, one can wonder about the implications of a generalization of this notion of distance to the function $\left[|x_1 - x_2|^n + |y_1 - y_2|^n\right]^{1/n}$. Furthermore, one could ask oneself or others, "What if n = 1?" In this example, given certain (mathematical) objects, one can meaningfully enter a dialogic interaction and acquire a new awareness and be curious about implications of it, one of which is that circles have right angles and equal sides. In the first example, mathematical structure is imposed on physical objects of wooden or plastic rods, and in the second case, further mathematical structure is read into a mathematical object of the mind.

Gattegno theorizes how the human capacities to be aware of something and attach importance to it beget the sciences. He argues that the instrument of knowing that allows scientists to be cognizant of the content of their awareness is "a dialogue of one's mind with one's self, which is akin to the action captured in the quotidian phrase, "talking to oneself" (1987, p. 6). This instrument of knowing deployed on different contents of experiences yield different sciences. As Gattegno puts it, "'man can only create the science of the contents of his awareness'" and that "man's dialogues with himself [are] the 'lightings' that generate the various hues which distinguish the various sciences" (p. 10). Different branches of science develop from the repeatable findings or facts that stem from humans' dialogues of minds with themselves specializing on, for instance, their different senses and on specific ways of knowing. He offers examples, one of which is the following:

> Thus, [man] gathered all that struck his eyes, not to talk about what he saw, but to understand why he could trust his sight and the instruments he was using with it. He thus created "the science of optics," which defines the phenomena to be retained for the on-going dialogues: What is light? How is it propagated in the various media? What happened to it if it encounters a surface on which it can bounce or in which it can penetrate? What is color? What makes light white or have color? And many other questions we learn about when we study optics (geometric or physical) in the treatises which sum up the findings of four centuries of work. (p. 11-12)

Optics, from this viewpoint, developed from a specialized focusing of attention on questions related to why one sees what one sees; that is, questions related to particular contents of one's awareness. Gattegno notes that a special watchfulness emerged, concerned with "repeatability," to ensure that "whatever one found, anyone else could find too, confirming the watchfulness at work" (p. 12).

Having established dialogue of one's mind with one's self as human's way to consider moments and their content, Gattegno discusses a special "conquest of the mind at work on itself"—mathematics:

> No one doubts that mathematics stands by itself, is the clearest of the dialogues of the mind with itself. Mathematics is created by mathematicians conversing first with themselves and with one another. Still, because these dialogues could blend with other dialogues which

> refer to perceptions of reality taken to exist outside Man, mathematics were not recognized for what they actually are until recently. Based on the awareness that relations can be perceived as easily as objects, the dynamics linking different kinds of relationships were extracted by the minds of mathematicians and considered per se. (pp. 13-14).

Here Gattegno articulates the particular content of experience from which mathematics develops: dynamics linking different relations. That is, for example, the connection one can perceive between the two processes of (a) raising 2 to consecutively increasing integral powers and (b) the multiplicative process of doubling. In each process, we have a relation (raising to a power and doubling) and objects (2 and the implicit objects ... $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 4, 8...) (Powell, 1995).

Implicit in Gattegno's statement are other contents of experience from which mathematics emerges. From his note that "the awareness that relations can be perceived as easily as objects" we have two other contents of experience, namely, objects and relations among objects. In each process of our example, we have both the objects and the relations among the objects. Thus, from Gattegno's view on the psychological and dialogic development of mathematics, three categories of contents of human experiences upon which the discipline is built can be identified: objects, relations among objects, and dynamics linking different relations.[6]

Formulations of the categories of mathematical experiences of other mathematicians parallel Gattegno's. In November 2002, Keith Devlin, executive director of The Center for the Study of Language and Information at Stanford

[6] Others researcher also focus specifically on how learners' wrestle with mathematical objects. See, for example, Alcock and Simpson (2002), who explore a triad of learners' approaches to reasoning from mathematical definitions in terms of how they handle, among two other constructs, what Alcock and Simpson call *specific objects*, as a framework to understand difficulties that learners experience in the transition from studying school mathematics to real analysis.

University and a consulting professor in Stanford's Department of Mathematics, was interviewed on Fred Goodwin's show, "The Infinite Mind," broadcasted on New York Public Radio (WNYC-FM). This show, titled "One potato, two potato: Numbers and the mind," explores connections between new research on the brain and human mathematical functioning. In the following interview vignette, which occurs on the heels of discussing whether films such as "A Beautiful Mind" are efficacious in relating mathematical ideas to the public, Goodwin, a psychiatrist, asks Devlin to offer advice to math anxious individuals:

> GOODWIN: Once you have math anxiety, I mean, how do people get over it? Do you have any advice for people with it?
>
> DEVLIN: I say to people, `Look, pick up a mathematics textbook at random, open it at a page at random, and ask yourself, how many objects is the--is the author discussing? How many relationships between them are being discussed in this page of mathematics? How complicated are those relationships? How hard is it to understand them? How complex is the web of relationships overall? And how complex is the logic required to follow the mathematics?' And then I say, `Make a note of your answers, switch on a television, watch the first soap opera at random and ask the same questions: How many characters are being discussed? How many relationships? How complicated are those relationships? How complex is the web of relationships, and how complex is the logic of the plot?'
>
> On every count, the soap opera will come out higher than mathematics. In other words, however you measure it, a soap opera episode is more complicated in all aspects than a piece of mathematics. The difference is that the soap opera is about characters that are familiar. They're fictitious, but they're like real people. In mathematics, all of the characters are abstract. They're numbers, triangles, vector fields. The brain did not evolve to deal with abstractions. It finds abstractions incredibly hard to deal with. And that's where the difficulty lies in mathematics. It's not the nature of the--of the reasoning. It's the nature of the objects you're reasoning about. (http://lcmedia.com/mind244.htm)

Several of Devlin's points can be annotated. Pertinent, however, for the theoretical and methodological framework of this study is parallels between his last point and Wilensky's concerning the abstractness or concreteness of an object as well as between Devlin's and Gattegno's categories of the content of mathematical activity. Both Devlin and Wilensky (see quote at the end of section 2.2 of this chapter) maintain that unfamiliarity of mathematical objects as their fountainhead and by implication the basis of the perceived difficulty of mathematics. For each researcher, Devlin and Gattegno, mathematics involves objects, relations among them, and dynamics or a web of links among relations. One the one hand, Devlin posits the difficulty in mathematics is not the kind of reasoning involved but rather the nature of the objects about which one reasons. On the other, Gattegno develops materials manipulable physically or with visual imaginary to enable learners to transcend the extra-ordinary nature of mathematical objects.[7]

In this study, I adopt these categories as criteria for identifying critical events. Nevertheless, I augment these three categories with two others. As the data from this study show, an additional category concerns affectivity and another concerns heuristics.[8] The first category has two components: (1) how one feels while engaged in dialogic contact with the content of one's experience and (2) the flow and intensity of energy that results allowing one to remain or not with a challenge. The second category— problem-solving heuristics—pertains to an awareness of one's method of responding to questions raised in the dialogues

[7] Besides popularizing worldwide the use of the colored rods invented by Georg Cuisenaire, developing a theory and pedagogical practice for them, and writing and publishing teacher and student materials to accompany them, Gattegno (1911-1988) created physical and visual materials for mathematics teaching and learning, including geoboards; prisms and cubes, whose colors are based on the scheme of Cuisenaire rods; animated geometry films, and microcomputer software.

[8] In this study, I use the term heuristics to mean actions that problem solvers perform that serve as means to advance their understanding and resolution of the problem task. This does not mean that an employed heuristic method or strategy will necessarily advance the problem solver toward a solution but only that its intent is to do so. This sense of the term includes the general categories outlined by Pólya (1945/1973, pp. xvi-xvii and 112-114) and others (Brown & Walter, 1983; Engel, 1997; Mason, Burton, & Stacey, 1984; Mason, 1988; Schoenfeld, 1985) and pertains to other actions such as a group of problem solvers decision to assign subtasks to each other to later pool their outcomes to influence their progress on the larger problem at hand.

of the mind with the self and other persons. Therefore, adopting and extending Gattegno's categories, the content of mathematics experiences, whether internal or external to the self, can be *feelings, objects, relations among objects, and dynamics linking different relations,* and *heuristics.*

Gattegno's notion of awareness and the development of mathematics have general implications for mathematics learning and research in mathematics education. Cognitive change or generating knowledge occurs not as a teacher narrates information but rather as learners employ their will to focus their attention to educate their awareness. Learners educate their awareness as they observe what transpires in a situation, as they attend to specific content of their experiences. As a learner remains in contact with a transpiring experience, awareness proceeds from "a dialogue of one's mind with one's self" about the content of that which one experiences. One's will, a part of the active self, commits one to focus one's attention so that the mind observes the content of one's experience and, through dialogue with the self, becomes aware of particularities of one's experience, namely, feelings, objects, relations among objects, and dynamics linking different relations, and heuristics. (Gattegno, 1987; Powell, 1993)

In summary, while investigating the development of mathematical ideas, researchers can observe instances of learners attending to any of these five aspects of their mathematical experience. Learners utter or inscribe either affirmations or interrogatives about these elements of their experience, and as such, researchers can identify or code for ten different types of critical events or moments of awareness. The matrix in Figure 3 contain the type of events that can be identified as critical and, as such, provides criteria for indicating critical events. In the cells are abbreviations that one can use for coding data.

Subject and type of utterance or inscription	Affective	Objects	Relations among objects	Dynamics linking different relations	Heuristics
Affirmations	AA	AO	AR	AD	AH
Interrogatives	IA	IO	IR	ID	IH

Figure 3. Matrix of event types that can be designated as critical.

For this study, the criteria for identifying critical events are based on the event types indicated in this matrix. In their discursive interactions, learners may assert or question their feeling as well as objects, relations, dynamics, or heuristics. I attend to individual learners making statements or posing questions orally or in writing that concerns their ideas about their affectivity as well as the objects, relations among objects, dynamics linking different relations, and heuristics as they collaborate to resolve the Taxicab Problem.

3.2.4 Interlocution

Critical events are identified through close analysis of what learners say and otherwise do. This raises the question of what is the unit upon which analyses are based. The unit of analysis for this study is discursive interaction, the back and forth talk of conversational partners, for which I use the term *interlocution*. I use this analytic unit, a chunk of meaningful conversation, to ascertain the features and functions of the talk among learners collaborate to solve a mathematical problem. It is important that the analytic unit is discursive interaction or interlocution and not just, for instance, some form of articulation, which includes initiating speech such as responding or questioning or not just listening. To analyze listening or some form of articulation the intention of the hearer or speaker must be known or assumed. However, despite the initial intentions, through turns of speech emerge functions of the speech and such functions may even differ from the initial intention of any one of the conversational partners. The function of speech is dialectical and evolves over the course of a conversation. It is influenced by the will of the interlocutors, and at the same time, becomes manifest to the interlocutors. As a researcher, investigating the features and functions of the talk of interlocutors, it is through the back and forth flow of speech that I have evidential clues of the functions of their discursive interactions on their activity of engaging mathematical problems as well as the mathematical ideas they build.

Before detailing the categories of interlocution that I use in analyzing the data of this study, I should explain how the theoretical notion of interlocution arose. In the inductive stage of my quasi-grounded-theory method, a number of codes emerged, and a significant number of these related to the how students articulated their ideas to one another as well as those ideas they chose to pursue. Coincidence with this phase of my analytic work, I read a report by Martin (2001) concerning his investigation of teacher listening in which he employs listening categories proposed by Davis (1996; 1997). Martin elaborates and provides evidence for how the listening patterns of a teacher can occasion opportunities for students "to construct and modify their own images in response to her interventions" (p. 251). It struck me that Davis's theoretical lens could be useful for understanding aspects my data. However, to understand the mathematical ideas students build, it became important for me to have evidence upon which I could monitor how students listen to each other. To gather this evidence, I needed to examine not only the listening of the hearer but also the articulation of the speaker as well as the listener's response to the speaker and so on. This then brought me into analyzing the sets of conversational exchanges that occur around an idea, that is, the interaction of the interlocutors grappling with a mathematical idea. Hence, my inductive analysis turned deductive, nevertheless, informed and refined by further emergent inductive codes from the data.

As described above, the categories of interlocution emerge from the data of this study through a dynamic and dialectical interplay of inductive and deductive coding. What surfaced is grounded in theoretical ideas of Davis (1996; 1997) and the messy realities of open coding of the data but is different from both. Based in philosophical ideas of Levine (1989) concerning the social evolution of human's capacity for listening and a movement toward listening that is an antidote for "alienated meaning" (p. 8), Davis develops a theory of pedagogical listening, according to three different modes of listening, not necessarily mutually exclusive: evaluative, interpretive, and hermeneutic. Evaluative listening occurs when a teacher maintains a "detached, evaluative stance" (Davis, 1996, p. 52). In contrast, a teacher listening interpretively endeavors "to get at what learners are thinking...to open up spaces for re-presentation and revision of ideas—to *access* subjective sense rather than to merely *assess* what has been learned" (1996, p. 52-53, original emphasis). Finally, Davis posits hermeneutic listening as "more negotiatory, engaging, and messy, involving the hearer and the heard in a shared project...an imaginative participation in the formation and the transformation of experience through an ongoing interrogation of the taken-for-granted...the unfolding of possibilities *through collective action*" (1996, p. 53, original emphasis).

Both Martin (2001) and Davis (1996; 1997) inquire into teacher listening and its consequent impact on the growth of student understanding. This study broadens this scope of inquiry as well as applies and extends Davis's category to analyze not just listening but rather discursive practices of learners in conversational exchanges or cognitive interlocution. This theoretical construct contains has four category and guides the inquiry into how learners' discursive exchanges structure their investigation as well as contribute to the mathematical ideas they build and their growth in mathematical understanding. I define the four categories—evaluative, informative, interpretative, and negotiatory—along with their codes as follows:

- *Evaluative* (**E**): an interlocutor maintains a non-participatory and an evaluative stance, judging statements of his or her conversational partner as either right or wrong, good or bad, useful or not.
- *Informative* (**Inf**): an interlocutor requests or announces factual data to satisfy a doubt, a question, or a curiosity (without evidence of judgment).
- *Interpretive* (**Int**): an interlocutor endeavors to tease out what his or her conversational partner is thinking, wanting to say, expressing, and meaning; an interlocutor engages an interlocutor to think aloud as if to discover his or her own thinking.
- *Negotiatory* (**N**): an interlocutor engages and negotiates with his or her conversational partner; the interlocutors are involved in a shared

> project; each participates in the formation and the transformation of experience through an ongoing questioning of the state of affairs that frames their perception and actions.

These are not mutually exclusive categories; a unit of meaningful conversation may have more than one interlocutory feature.

I have explained the criteria that I will use to select critical events. These criteria will allow me to get at the mathematical ideas that participants build. For each critical event, not only will I analyze the mathematical ideas at play but also the nature of participant interlocution. The point will be to understand the features and functions of their conversational practice and to see how it influences their work on solving the Taxicab Problem.

3.3 *Analytic Method*

3.3.1 Data collection

As chapter 1 details, the research session on which this study is based was a problem-solving session involving four participants that occurred on 5 May 2000 at the David Brearley High School in Kenilworth, New Jersey. Three video cameras recorded the session. However, because of the poor quality of the audio portion of the data from one camera, it was not used for this study. In chapters 4 and 5, I make explicit reference to the video data and, for simplicity, use the initials of each videographer to identify their camera view: Lynda Smith (LS) and Sergey Kornienko (SK). I have chosen to base the narrative description on the information that Lynda Smith's video camera captured since its audio portion is most complete. Therefore, the visual and audio images captured by Kornienko's video camera are used to both complement and supplement information derived from that of Smith. The videotapes are digitized, compressed, and stored on five compact disks (three CDs for LS and two CDs for SK) as MPEG1 files.

To elaborate more fully, the data sources for the study consist of the problem task; a video record from the perspective of two cameras of approximately 100 minutes of the activity of the four participants, including roughly 20 minutes in which researchers listened as they presented their resolution and asked questions to follow movements in their discourse toward further justification of their solution; a transcript of the videotapes combined to produce a fuller, more accurate verbatim record of the research session; the participants' inscriptions; and researcher field notes. The participants' inscriptions are scanned and saved as digitized picture documents. The video recordings are digitized, compressed, and stored on five compact disks as MPEG1 files. The transcript is a textual rendering of verbal interactions, specifically, turn exchanges among the participants and between them and researchers and in all consists of 1,619 turns at talk (see Appendix C).

Figure 4 provides information concerning the data collected and used in the analysis of this study.

Category	Content	Data Format	Amount of Data
Video	Two camera views of problem-solving session identified by the initials of the videographers Linda Smith (LS) and Sergey Kornienko (SK).	Videotapes digitized, compressed, and saved on five compact disks (three for LS and two for SK) as MPEG1 files	Approximately 100 minutes of problem session recorded by each camera: LS1, 0:40:41 LS2, 0:39:49 LS3, 0:37:40 SK1, 0:52:10 SK2, 0:56:33
Artifacts	Statement of problem task Participants' inscription	Scanned Microsoft Word picture documents	1 page 23 pages
Transcript	Verbatim rendering of conversation (turn exchange) among participants and between participants and researchers	Microsoft Word text document	88 pages 1,619 turns of talk

Figure 4. Classification of data used in analysis

3.3.2 Data Analysis

The analytic method that I employ involves a sequence of nine interacting, non-linear phases, informed by grounded theory (Charmaz & Mitchell, 2001; Corbin & Strauss, 1990; Creswell, 1998; Pirie, 1998a), ethnography and microanalysis (Erickson, 1992; Goldman-Segall, 1998; Hammersley & Atkinson, 1983; Quantz, 1992), and methods for analyzing video data in mathematics education (Cobb & Whitenack, 1996; Pirie, 2001; Powell et al., 2001). The following are the descriptive heading of the phases of the data analytic method that I employ for this study; accompanying each is a fuller characterization of its activity and rationale:

3.3.2.1 Viewing attentively the video data

To familiarize myself with the content of my video data, I attentively watch and listen to the two videotapes several times and, in this phase, do so without intentionally imposing a specific analytical lens on my viewing with the goal of becoming aware of the problem-solving session in full.

3.3.2.2 Describing consecutive time intervals

I write brief, time-coded descriptions of the video's content. These descriptions are similar to what Pirie (2001) terms "time activity trace." (p. 348) These descriptions could be of two- to three- or even ten-minute intervals. In this phase of work, my descriptions are intended be descriptive and not interpretative or inferential, merely stating what corporal actions and other movements can be seen as well as utterances and other noises can be heard. Pirie (2001) notes that instead of "inferential remarks such as 'He is trying to…' or 'She seems to have…' or even 'A confusing diagram on the board…'" that simple, factual descriptions are best: "'He writes…', "She says…', 'The teacher draws…'" (p. 348). The idea is to map out the video data so that someone else reading the descriptions would have a fairly clear idea of the content of the videotapes. Importantly, descriptions help the researcher become ever more familiar with the data set than one does by attentively watching and listening to the video record. The descriptions also allow the researcher to locate quickly particular vignettes and episodes. My consecutive, time-coded descriptions form the basis of chapter 4, "Detail Description of Data."

3.3.2.3 Identifying critical events

By viewing and describing video data, researchers acquire a comprehensive and in-depth knowledge of the content of the video record. Afterward, researchers move to the next phase of data analysis, which consists of carefully reviewing their videotapes as well as other participant artifacts such as inscriptions and identifying significant moments or, as Maher (2002; Maher & Martino, 1996b, 1996c, 2000) terms them, *critical events*.

As mentioned earlier, what is considered a critical event is related, in part, to research questions pursed. Significant contrasting moments may be events that either confirm or disaffirm research hypotheses; they may be instances of cognitive victories or of cognitive obstacles, correct leaps in logic or erroneous application of logic; they may be any event that is somehow significant to the study's research agenda.

In this study, I identify as critical events instances when participants utter or inscribe either affirmations or interrogatives about their feelings in relation to doing mathematics, mathematical objects, relations among the objects, and dynamics linking different relations, and problem-solving heuristics. Additionally, for each critical event, I indicate the function of the surrounding interlocution, whether evaluative, informative, interpretive, or negotiatory. The critical events that I distinguish compel me to reflect on their antecedents and consequences. Undergirding this analytic phase are the following four guiding research questions:

1. As the participants engage collaboratively, without assigned roles, to understand and resolve the Taxicab Problem what are the features and functions of learners' discursive practices?
2. As they articulate their emergent understandings of the problem task, what mathematical ideas and associated meanings do they reveal in their discourse?
3. How do participants structure their investigation?
4. How do their conversational exchanges support advances in their problem solving?

3.3.2.4 Transcribing the video record

I transcribe the videotapes of the problem-solving session. I realize that it is impossible to render an exact, genuine transcript of verbal and gestic interactions captured on videotape since in the words of Atkinson and Heritage (1984) transcripts are "necessarily selective" (p. 12) and according to Erickson (1992) are "theoretically guided" (p. 219). Nevertheless, it is possible to produce transcripts that are close approximations to being exact and genuine for particular research purposes. Transcripts can be more or less valid representations of interactions and their conventions depending on researchers' analytic purposes (Erickson, 1992, p. 219). I use a modified transcription system based on the one evolved by Jefferson (1984) that has as its purpose to transfer to the page the sound and sequential positioning of talk (see Appendix D). The transcript of the discourse captured by the two cameras is combined and based on conversational turn units. From a hearer's perspective, these are tied sequences of utterances that constitute speakers' turn at talk and at holding the floor. The transcript of the problem-solving session contains in all 1,619 turns at talk of which 1,469 turns or 92 percent are turns of talk of the participants (see Appendix C).

There are essentially four reasons why I think it is important to transcribe my videotapes.[9] First, following analytic procedures within the tradition of grounded theory, I implement an open-coding procedure on my data so that I can see what I might discover above and beyond information that would directly inform my specific, *a priori* guiding research questions and beyond my deductive codes. Second, I am interested in examining whether participants' discursive practices, especially their dialogue, which serve to reveal the mathematical meanings and understandings they construct. Since discursive practices include actions that are not only utterances, I also indicate in the transcripts relevant body movements as well as inscriptions (writings, drawings, sketches, and so on) that participants create. Third, I believe that transcripts, being a permanent record, can reveal important categories not always capable of being discerned by viewing videotapes since, notwithstanding the technology of replay, the visual and aural video images that the viewer's mind eye and ear captures are essentially ephemeral. Instead with transcript data, one can consider more than momentarily the meaning of specific utterances. Fourth, I will transcribe so that later, if and where appropriate in results chapter (5), I provide evidence of findings in the participants own words.

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[9] A debated methodological issue concerning the use of videorecordings is, as Pirie (1998b) phrases it, whether "*the data are the tapes*" or "*the data are the transcripts*" (p. 160, original emphasis). Unlike videos on tape, CD, or DVD, transcripts require no special equipment to access them and, as Maher and Alston (1991) observe, "careful analysis of videotape transcripts of children doing mathematics enables a detailed study of how children deal with mathematical ideas that arise from the problem situation" (p. 71-72). Nevertheless, many things are potentially missed in the movement from videos to transcripts such as subtle nuances in speech and non-verbal behaviors as well as visible influences on patterns of behavior. Consequently, for meaningful analyses of extensive or non-extensive video data, Pirie (1996) advocates transcribing no videos and instead basing analyses exclusively on the tapes. My position, based on experience, is that both transcripts and the original videos are necessary for analysis. However, video data rather than transcript data provides a fuller contextual space in which to perform analysis and interpretation.

Nevertheless, as I code transcripts, I constantly review corresponding vignettes in the video record to perceive subtle nuances in speech and non-verbal behaviors as well as visible influences on patterns of behavior. The importance of transcripts notwithstanding, examining the video record is indispensable to analyze inscriptions since they are built in a layered fashion over time.

3.3.2.5 Coding synchronously the transcripts, videotapes, and inscriptions in an inductive and deductive manner

3.3.2.5.1 Coding transcript

Sequentially, there are two types of coding that I wish to perform on my transcript data, segmenting the transcription of my video data into patterns of information, selecting *in vivo* codes, and imposing *a priori* codes. I will engage in what grounded theorists call line-by-line open coding of the data. In this study, the open coding proceeded turn-by-turn and the need for interlocutory codes emerged from this process of analyzing the data. To do this accurately, I work with both the transcript and the videotapes. This allows me to see whether actions expressed in words are corroborated by bodily actions and to determine whether my inference about a particular event or action is borne out by what occurs before and after it. I attempt to code from the perspective of the participants. Further, to infuse an emic perspective to my coding and resulting analysis, whenever possible and appropriate, I use the exact words of participants in coining a code. However, after the open coding phase, the actual codes I use come largely from the literature.

For the second type of coding, I code from an etic perspective by looking for actions that I interpret within the augmented framework of Gattegno's contents of mathematical experience (Gattegno, 1987) and Maher's analytical sensitivities (Maher, 1988; Speiser et al., in press). Taken together, these perspectives acknowledge and emphasize the dynamic, continuously unfolding phenomenon of the development of mathematical ideas and the interdependence of all participants in an environment. They also recognize that collective or social knowledge and individual understanding co-emerge.

3.3.2.5.2 *Coding inscriptions*

Besides coding transcripts, I identify and code the participants' inscriptions (writings, drawings, sketches, and so on). Theoretically, I content that the mathematics participants come to understand is not an externally created body of knowledge but rather over time is a by-product of discursive practices and focus of attention on patterns and relationships *per se* (Gattegno, 1987, 1988). It is, therefore, important to code participants' artifactual inscriptions since they may provide important insights for understanding participants' actions and accounting for development in their mathematical ideas. The participants' inscriptions are in Appendix E. Furthermore, Appendices F and G contain a table for camera views SK and LS, respectively, of the layered building of participants' inscriptions.

3.3.2.6 Writing analytical memoranda

As I describe, code, and otherwise attend to my data, I will continually write about my emerging and evolving analytic and interpretive ideas, about connections between and among codes, about themes exiting across codes, about larger divisions of categories, and about an emergent central phenomenon.

These memoranda form preliminary hypotheses, jottings about emerging categories and connections between them (Creswell, 1998, p. 241). These memoranda also form an intermediate bridge between my coding of the data and writing first drafts of the results chapter (5).

3.3.2.6.1 Categorizing codes, identifying properties, and dimensionalizing properties within categories

With respect to my guiding research questions and other questions that emerge from coding activities, I raise significant codes to categories, subsume other codes to these categories, identifying them as properties of the categories. In some cases I dimensionalize the categories.

3.3.2.7 Constructing visual representations of codes and categories

I construct matrices containing the names of the participants along one axis and, along a perpendicular axis, their utterances and other actions that correspond to particular categories and associated properties. Such matrices provide a means to visualize data in a reduced, organized form. Also, these matrices help to identify a central phenomena and interrelationships among categories.

3.3.2.8 Constructing a storyline

From among the categories I identify, I engage in axial coding, searching for interconnecting categories. I attempt to select one or two categories of actions that are central in informing my guiding research questions and questions that emerge from coding. These central categories explain and synthesize basic learning processes that I discover within the data. Then, I relate other categories to the central ones and build a narrative in which the data of this study can be analyzed and interpreted. A salient question that emerged concerned the features of the interlocution among the participants and the features became categories for later analysis.

3.3.2.9 Composing a narrative

A narrative, essentially the results and conclusion chapters (5 and 6, respectively) is a composite of narratives or results that emerge from the storyline of critical events. The narrative that I compose embodies what I understand happening in the data along categories of critical events. It is my explanation of what I infer from the discursive and inscriptive actions of the participants. In composing the narrative as with all of the above analytic phases, I sustain a perspective on the work of the participants that focuses on what they do rather than on what they do not do. There is a tradition in mathematics education to document and analyze what learners do not understand and to speculate or through interviews ascertain the root cause of their misunderstanding. Leinhardt, Zaslavsky, and Stein (1990) extensively review the misconceptions literature. Instead of focusing the participants' misconceptions or mathematical deficits, in agreement with Speiser, Walter, and Maher (Speiser et al., in press), I "emphasize the *capabilities* that students demonstrate" and document "how students help each other *overcome* important, perhaps daunting, obstacles that they encounter, and...the kinds of tools and knowledge that they use to do so."

3.4 *Researcher as Participant Interlocutor*

To end this chapter, I mention a theoretical stance that I take as a researcher with respect to the problematic of a researcher's interpretation of participants' interlocutions. This relates directly to analytic method. Sfard (2000b) postulates effective communication: "if it [an utterance] evokes reactions that are in tune with the interlocutors' meta-discursive expectations" and argues that "it is useful to think of an observer [such as a researcher] as a discourse participant, alas not a very active one" (p. 302). For Sfard, meta-discursive expectations are characteristics of responses that permit speakers know that their interlocutors hears them, though not necessarily agree or comply, and these expectations are in contrast to the specific content of an utterance (p. 301). Returning to the issue of interpretation, she remarks that

> there is no reason to assume that the observer's evaluative actions should be essentially different from those of ordinary interlocutors. After all, as conversation participants, people are accustomed to remaining attentive and to monitoring the effectiveness of communication even when only listening to an exchange among their partners. The speaker's utterances evoke interpretations and, thus, certain intentional states in any listener. The listener will have expectations as to the interlocutors' possible responses and will evaluate their utterances as if he or she were the speaker. Thus, the observer's evaluation, like any other's, has the status of a personal interpretation. What makes the observer's evaluation somewhat special is the fact that, being constructed according to certain explicit rules, it may be expected to be to some extent reproducible and thus highly consensual. (p. 302).

As a researcher, present during the problem-solving session as well as even more actively attending during hundreds of hours watching, listening, and interpreting the communicative actions of the participants, I define my analytic role as that of a *participant interlocutor*. From a scientific standpoint, my findings are personal interpretations but at the same time are expected to have the character of being, as Sfard says above, "to some extent reproducible."

As a participant interlocutor, I have to enter the conversations of the other participants. To do so, I must be open and seek to understand their perspective through a stance that can be described, using a notion of Elbow (1973) as Speiser (2002) presents it, as the "believing game" (p. 494). Speiser argues that to understand others' thinking may require suspending doubt, adopting their claim, and then seeking argumentation to support it. As he writes, "listeners begin by accepting and believing, so that lines of reasoning, to support the speakers' claims, emerge *within the listeners*" (p. 495). This is the stance I take as a participant interlocutor listening to the conversational interchanges of the other participants as they engage the Taxicab Problem.

CHAPTER 4: DETAIL DESCRIPTION OF DATA

The data for this study consists of the problem task; a video record of the one hundred minutes and eighteen seconds of the activity of the four participants from the perspective of two video cameras; a transcript of the videotapes combined to produced a fuller, more accurate verbatim record of the research sessions; video recordings of follow up interviews; and field notes. This chapter describes the two main data sources, first the problem and its mathematical significance and then the activity of the four participants working on the problem from the perspective of two video cameras.

4.1 Notation and Nomenclature of the Problem

The next section discusses in detail the content of the videographed session that forms the focal data set of this study. However, before describing the video data, the actions of the four students who worked on the Taxicab Problem and the two researcher-teachers, it is necessary to present lexical items that I will use in the description. Some of these words, phrases, and notational inscriptions are different from what was used by the participants. Others are emic expressions that convey meanings intended by the participants though not elements of the linguistic register of school or academic mathematics. I present the first set of lexicographical items so as to ease communication about the actions of the participants, hopefully without distorting their meanings. The second set of items is present so that the voices of the participants come through in my descriptive narrative of the data.

In the statement of the Taxicab Problem, the taxi driver needs to fetch three different passengers and their different locations, called pick-up points, are indicated respectively by blue, red, and green dots (see Figure 5). Together with the black dot that locates the taxi stand, these dots lie at points of intersection of orthogonal streets, represented by black grid lines (see Figure 5). To describe these and other intersection points, I use terminology and notation that resemble the nomenclature of coordinates in the Cartesian plane as well as representations of binomial coefficients.

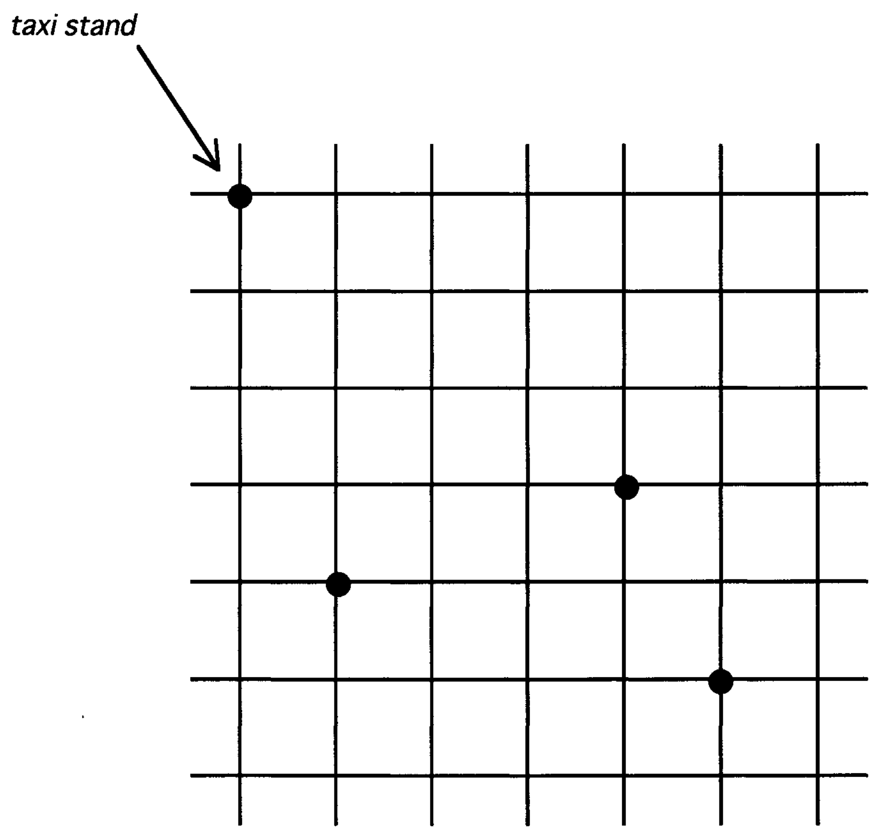

Figure 5. The location of the taxi stand and three pick-up points within the given quadrant of the taxicab plane.

The taxicab driver wishes to travel from the taxi stand through streets that are arranged in an orthogonal grid—a taxicab plane—to pick up passengers at specified corners or intersections. More specifically, for each trip, she wishes to traverse the shortest possible distance, that is, the fewest number of blocks. The taxicab distance between two points of intersection, P and Q, is the length of a shortest path from P to Q composed of horizontal and vertical line segments. To ensure that the taxi driver travels the shortest possible distance, she must not double back and, hence, at each intersection, needs to decide whether to drive either east or south, never west or north.

In the study, the participants describe driving east as "across" or, equivalently, "over" and south as "down." Respectively, for the two directions, one participant uses abbreviations for over (*O*) and down (*D*) as notational inscriptions. Adopting this notation, we can denote driving a block east or south by *O* or *D*, respectively. Then, for instance, a string of *O*'s and *D*'s such as

DOODODO

means, reading from left to right, that the taxi driver drives one block south, then two blocks east, then one block south, then one block east, then one block south, and finally one block east. Following this route by starting from the taxi stand, the intersection at which the taxi driver arrives is the red pick-up point (see Figure 5). The route does not involve doubling back and traverses seven blocks, three down and four across or over. Any efficient or shortest route between the taxi stand and the red pick-up point, therefore, entails driving an admixture of four blocks across and three blocks down, seven blocks in all. As long as one never doubles back by driving west or north, the distance is shortest and independent of the route since at each corner, the driver either goes east or south. To any shortest route from the taxi stand to the red pick-up point there corresponds a string of length seven, four *O*s and three *D*s; and conversely, to any such string of four *O*s and three *D*s there corresponds exactly one seven-block route from the taxi stand to the red pick-up point.

With appropriate modification for generality, the aforementioned is true for any point in the taxicab plane. Consequently, as in Figure 6, each intersection point or corner can be denoted (n,k), where n equals the fewest number of blocks one traverses to reach it or, equivalently, the taxicab distance from the taxi stand to the intersection and k denotes the number of times east or, equivalently, "over" was chosen as the direction to drive.[10] Any route to (n,k) can be written as a list of directions (O and D), with the sum of Os and Ds equals

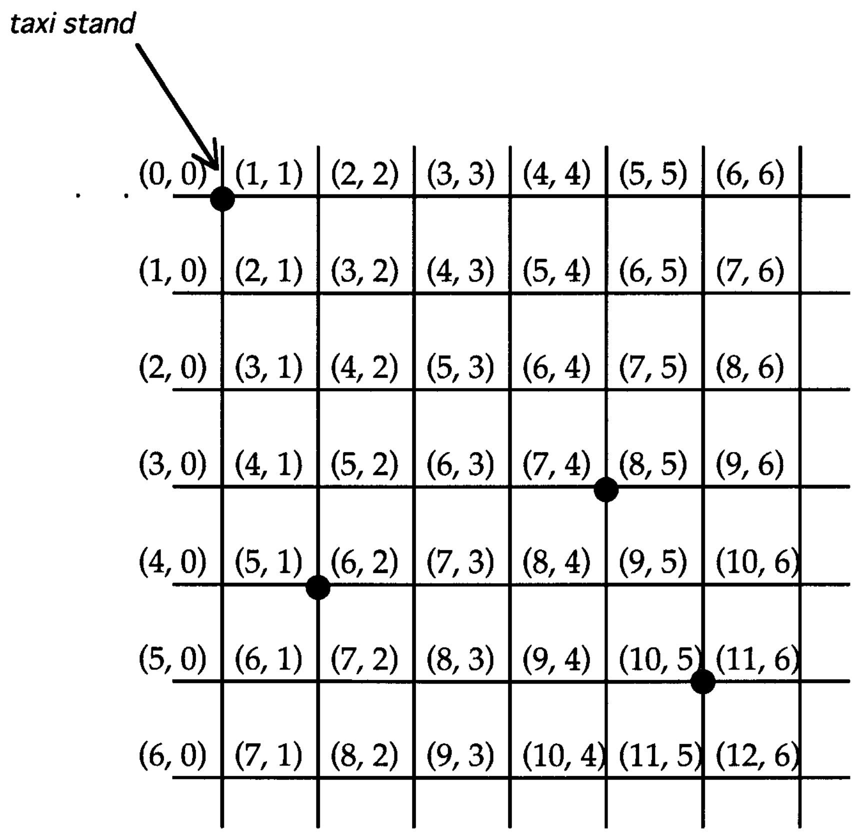

Figure 6. Points of the given quadrant of the taxicab plane denoted (n, k), where n equals the distance from the taxi stand to an intersection point and k denotes the number of blocks "over" that the intersection point lies from the taxi stand.

the route's length or distance. For example, to the intersection (7, 4), the red pick-up point, we have the following route: *DOODODO*. Conversely, any finite

[10] Alternatively, k could denote the number of times south or, equivalently, "down" was chosen as the direction to drive.

sequence of *O*s and *D*s can be identified with an intersection. Hence, *ODODOODODD* corresponds to a route to (10, 5), the green pick-up point.

In the narrative description of the video data as well as in subsequent analyses and discussion of the data, each intersection point on the taxicab plane will be denoted as (n,k). Figure 6 contains the denotation of each point in the portion of the taxicab plane presented in Figure 5. Each ordered pair (n,k) refers to the intersection point or corner to its immediate lower right. The intersection labeled (5, 1) refers to the blue pick-up point. The denotation of this point reveals that the intersection is a distance of five blocks from the taxi stand and a shortest route to it involves driving one block across.

Besides (n,k) naming the intersection points of the given quadrant of the taxicab plane, it will also be used to denote the location of corresponding squares. That is, (n,k) also labels that square whose lower right vertex is the intersection point (n,k). For example, referring to Figure 6, the square that contains the label $(6,3)$, we shall call "square $(6,3)$" and say that the intersection point $(6,3)$ and square $(6,3)$ correspond to each other.

During the research session, the participants often talk about routes to particular intersection points as working in a p-by-q rectangle. Typically, in their reference to a p-by-q rectangle, p represents its horizontal dimension and q its vertical dimension. There are occasions in which the reference is the reverse and, when this occurs, I state so specifically. Otherwise, we shall assume that a p-by-q rectangle means p units across and q units down. In the narrative description, I refer to a p-by-q rectangle as a p-by-q rectangular sub-grid or, simply, a p-by-q

sub-grid since it is a subset of the taxicab grid, where the upper left vertex of the rectangle coincides with the taxi stand.

Finally, the data sources for the narrative description of the research session are two video cameras, each trained on different views of the work of the four participants and the two researcher-teachers such as subset of students and their written work. The description that follows is a composite of visual and audio information recorded by the two cameras. For simplicity, I refer to each camera by the initials of the name of its videographer: Lynda Smith (LS) and Sergey Kornienko (SK). I have chosen to base the narrative description on the information that Lynda Smith's video camera captured since its audio portion is most complete. Therefore, the sight and sound images captured by Sergey Kornienko's video camera are used to both complement and supplement information derived from that of Lynda Smith. Consequently, in the narrative description that follows, when I detail activity not available from Smith's camera view, I indicate it with the following notation to indicate Kornienko's camera, which of two CDs and the inclusive time of the event: [SK_1 or SK_2, hours:minutes:seconds – hours:minutes:seconds].

4.2 *Narrative Description of Video Data*

The research session occurs on 5 May 2000, in the late afternoon, after school. Near a chalkboard in the front of a classroom in their high school, a group of four students (from left to right: Michael, Romina, Jeff, and Brian) sit around a trapezoidal-shaped table (see Figure 7). Atop the table are four black felt-tip makers, sheets of blank paper, and two microphones. Researcher 1 pulls up a chair, sits down between two students (Jeff and Brian) on the right side of the table, thanks the students for coming, distributes the Taxicab Problem (see Appendix A), and asks them to read and see whether they understand it. Afterward, Researcher 1 stands up and, as she backs away from the table, removes her chair. While facing the problem statement, Jeff asks aloud whether one has to stay on the grid lines and whether they represent streets. Researcher 1 responds, "Exactly." Each student has taken a maker. Romina, Brian, Michael, and Jeff discuss that five and seven are respectively the number of blocks it takes to reach the blue, (5,1), and red, (7,4), pick-up points and that different routes to each point have the same length as long as one doesn't go beyond the particular pick-up point. Brian says that they should prove it.

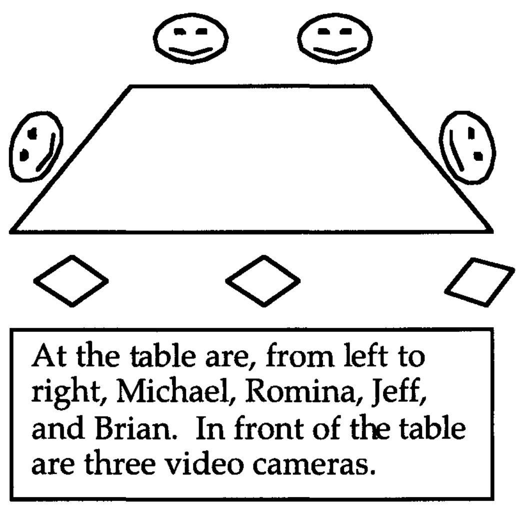

Figure 7. Depiction of the position of the participants and cameras around a trapezoidal-shaped table. The video camera on the right and left are stationary and the middle one is a roving camera.

Researcher 1 walks back over and stands between yet behind Jeff and Brian. She then asks the group to state how they understand the problem. Jeff says that the task is to find the shortest route "from there to here staying on the streets, right?" Researcher 1 adds that it is about finding whether there is more than one shortest route. Both Brian and Romina voice agreement. Researcher 1 goes on to say that if there is more than one, they have to determine how many shortest routes. Jeff inquires with Researcher 1 whether she is asking how many different shortest routes? At about the same time, Brian states that blue has five shortest routes. Researcher 1 says that not only do they have to find the number of shortest routes but also that they will "have to convince us" that they have found all of them. She then walks away from the table.

Jeff asks for colored markers. Romina and Jeff discuss limitations of using markers to keep track of routes that they might draw. Eventually, eight blue, green, and red markers are placed on the table between Romina and Michael. Jeff, Romina, and Brian choose to each work on different pick-up points. Romina says that it is five blocks to the blue point. Brian suggests counting them to be sure. Jeff asks why the length of each route to blue is the same. Romina says that it's a "four by one." Michael agrees and explains that to get the blue point one has to go "four down and right one" since one cannot go backward or diagonally. Romina asks how to devise an area for the red pick-up point. Jeff and Michael tell her that "it's not area." Jeff explains that "it's the perimeter" with the length of each line segment of the grid considered as a unit. Michael states that seven is the length of a shortest route to the red point.

Jeff says that they [the researchers] want to know how many shortest routes there are to the red point. Brian asks whether there are seven possibilities for routes to red. He observes that the length of the shortest route to blue is five and that the number of shortest routes is also five. Jeff says, "aha so." Then he says, "check it out." Romina says that Michael and she will do green, (10,5), and tells Brian and Jeff to do the red. Brian says that he thinks that green is nine. He counts the segments in a shortest route to green and then corrects himself, saying that it's ten. Michael says that there are a lot of routes. Romina says that she is trying to devise a method. Jeff says, "this is hard." Each of the four students point or draw routes with their pen within the grid below the statement of the Taxicab Problem. With a black marker, Romina draws different routes on her grid between the taxi stand and the green pick-up point. Romina says that she has "already lost count," then with her left-hand reaches over to Michael's sheet, and describes a method of counting. With a black marker in his hand and Romina facing in his direction, Michael traces above the grid, several routes between the taxi stand and the green pick-up point and then states "this is a lot." While Michael is talking, Jeff states that he is having difficulty keeping track of what he is doing.

Romina asks whether it is possible to "do towers" [11] on this problem. Overlapping Romina's utterance of her question, while pointing at intersections on his grid, Jeff mentions that one has a choice of going "there or there." Romina and Michael discuss that the length of a shortest route to the green pick-up point is ten. She says ten could be related to the number of blocks in a tower and asks whether the answer to the Tower Problem is two to the *n*. Michael says that the number of shortest routes to the green pick-up point is a lot and that there must be a pattern. Romina asks him whether the number of shortest routes could be equivalent to a block ten high with six different colors. He says that it would be nice if the number of shortest routes from the taxi stand to a pick-up point is half

[11] Romina refers here to a class of problems. She and the other three students have worked on and solved a number of these problems. Appendix B contains the statement of the combinatorial problems on which they have worked. In general, an *n*-Tall Towers Problem can be stated as follows: Your group has two colors of Unifix Cubes to work with. Work together and make as many different towers *n* cubes tall as is possible when selecting from two colors. See if you and your partner can plan a good way to find all the towers *n* cubes tall.

the length of a shortest route to the point. Michael and Romina discuss relationships between number of lines segments from the taxi stand to each point and the number of shortest routes.

Romina asks Jeff and Brian whether they found at least twenty-four routes to the red point. Brian says he has found eight. When Michael says that he has found nine, Brian explains that he is not stumped but is just not working quickly. Brian then asks whether Michael has counted the middle routes.

Michael says let's count routes to the red point. Romina asks him how he is keeping track. Michael responds that he is not sure, just not forgetting. She asks whether they should do like Brian but on the chalkboard. Jeff says that there must be some kind of math. When Brian asks whether there are twenty-four to red, Michael says that he guesses there are twelve but doubts whether he is correct.

Brian is facing Jeff, who with his left elbow, nudges Brian and explains that one can go over or up, while pointing with the tip of his black marker from a point on the lower left side of the grid upward toward the taxi stand. Then, on lower right of his paper, next to the grid, Jeff draws a node with two downward facing branches (binary tree). Romina inquires what he is doing, and Jeff responds that he is not doing anything, just trying to think. He continues this time pointing to another point, $(5,1)$, on his grid and saying that one can go either over or up and points to $(4,0)$ and then to $(4,1)$. With his head cocked downward and close to Brian's, who is facing Jeff's paper, Jeff says that he is not sure what his observation has to doing with anything. Romina tells Jeff that she understands what he is doing. With a black marker in her right hand, she points to Jeff's grid and says that to go to the blue pick-up point from points along the edge of the grid one has two choices. Jeff explains that to the blue point from some intersection points, pointing to the intersection $(1,1)$, one can only go down since the other choice takes you out of your way. He continues to the point $(1,0)$ and says one has two choices but that at the intersection (2,1) one can only go down. Jeff returns to the binary tree he drew at the bottom of his sheet and says that one could "follow all the routes to the endpoint." Brian, pointing with his black marker to Jeff's grid and to his own, explains how he counted fourteen routes to the red pick-up point. Romina says that fourteen does not work with the theory that the number of shortest routes to a pick-up point is half the number of line segments in the rectangle that contains the taxi stand and the pick-up point as vertices. Brian notes that in the rectangle that contains the blue point and the taxi stand as vertices there are thirteen lines altogether, counting the line segments in the middle. Jeff says that prime numbers are not good since there is no way to work with them.

Romina says that she will have to break "it" apart and draw as many possible routes. Brian says, "Yeah." Jeff says, "and have that lead us to something," and that they should do easier ones. He asks for more copies of the problem sheet and for grid paper. Copies of the problem are handed to him. Shortly afterward, paper and transparency versions of a Cuisenaire grid sheet are placed on the table between Romina and Michael.

Romina and Jeff work together tracing routes on the taxicab grid of a problem sheet. Romina asks Jeff to pick a dot and he chooses the grid point $(2,1)$. Using a black marker, Romina counts by outlining without writing on her grid the number of shortest routes to the point from the taxi stand (*SE* and *ES*) and place a 2 in the (2,1) square. She counts the number of shortest routes for two more consecutive points. To the intersection point (3,2), she outlines *EES, ESE,* and *SEE* and writes 3 in the (3,2) square. To the point (4,3), she finds the routes *EEES, EESE, ESEE,* and *SEEE* and then writes 4 in the corresponding square. She says that the array of numbers (2 3 4) looks like the multiplication table.

On her grid, Romina highlights the intersection point (3,1) and without marking the grid outlines three shortest routes between (3,1) and the taxi stand (ESS, SES, and SSE), and writes 3 in the corresponding square. Then she darkens the intersection point (4,2) and then counts and draws routes. After drawing five routes (EESS, ESES, SESE, SEES, and SSEE), she holds her marker on the point (4,2) for several seconds. She and Jeff recount routes from the taxi stand to (4,2), obtaining five (EESS, SESE, SEES, SESE, and SSEE). Jeff then darkens with a blue marker the intersection point (4,2) on his grid and traces five routes (EESS, SSEE, ESSE, ESES, and SEES). He says he cannot remember what he did and that he thinks that he indicated five routes. Romina tells him to do the next intersection point, (5,3), and that they will see whether their results agree.

With a black maker in his right hand, Brian draws shortest routes to the red pick-up point. On the right side of his taxicab grid, he inscribes and numbers staircase-like or step-function-like representations of routes from the taxi stand to the red point. He has represented ten different routes. Alongside one segment of each of the seven representations, he has written a number next to it. In one of these, he has the inscription D2, R1 and, in the other, underneath the letters "D R D," he has the numbers "1 4 2." He draws a line underneath the tenth inscription and numbers the space under the line as "11."

Michael, writing with his left hand, inscribes representations of routes similar to Brian's. His hand right hand and then his left cover up his work. He asks Romina, "What's that?" On her grid, she has written respectively in three different squares of first row the numbers 2, 3, and 4. In the squares (3,1), (4,2), and (5,3), she has written the numbers 3, 5, and 7, respectively. Romina says that the numbers in squares of the grid represent the number of routes for intersection points "diagonally down" from the numbers and tells Michael that she is not sure whether she is counting correctly. Afterward, she darkens the intersection point (4,4) of her grid.

As she points from left to right along her second row of numbers, Romina says they look like prime numbers (see Figure 8). Jeff asks Romina how many routes she found for the intersection point (5,3). She says, "seven." Using his left index and middle fingers, Jeff points to Romina's grid and observes that the sum of the first two numbers in the first row, (2,1) and (3,2), equals the second number in the second row, (4,2), and that the sum of second and third numbers of the first row, (3,2) and (4,3), equals the third number in the second row, (5,3) (see Figure 8). Romina, motioning with a black maker in her right hand, notes that she sees that in the first row the number go 2, 3, 4, 5, 6; as well as down the leftmost column: 2, 3, 4, 5, 6. Along the second row, she says, "3, 5, 7, 9, 11" (see Figure 8).

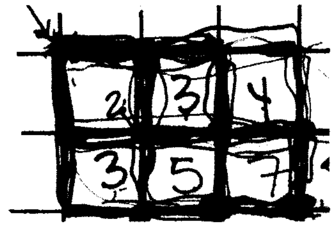

Figure 8. Romina and Jeff's early data on shortest routes in a portion of the taxicab grid from Romina's problem sheet.

She and Jeff discuss how to proceed. They agree to count routes for intersection points associated with the first four squares in the third row of their taxicab grid, namely, $(4,1)$, $(5,2)$, $(6,3)$, and $(7,4)$, as well as the first three squares in the fourth column—$(5,4)$, $(6,4)$, and $(7,4)$. That is, they agree to count routes to the rest of the intersection points within the rectangle that has the red point and the taxi stand as vertices.

Romina counts five routes to the point $(5,4)$, drawing each route in turn: *EEEES, EEESE, EESEE, ESEEE,* and *SEEEE.* She writes 5 in the corresponding square. She then tells Jeff that she is pretty sure about the one she just completed and, while pointing to the corresponding square, that they should count the routes for the point $(6,4)$. He asks whether it is the "four by four" and she responds that it is the "four by two." For this point she traces on her grid nine routes: *EEEESS, EEESES, EESEES, ESEEES, SSEEEE, EEESSE, EESSEE, ESSEEE,* and *SSEEEE.* As she writes "9" in the square, she says, "it's working."

Before she makes this statement, Jeff draws the following seven routes in his "four by two": *SSEEEE, SESEEE, SEESEE, SEEESE, SEEEES, ESSEEE, ESEEES*...[SK_1, 17:59-18:11].

Michael bends his head down, facing his paper and then asks Brian how many routes he has counted for the red pick-up point.

Twelve seconds after Romina announces that "it's working," Jeff points to the nine that she wrote in the square, asks whether she really only counted nine for that point, and invites her to look at how he is counting the routes to that point. At first, as he is talking, she counts the number of shortest routes to the point (4,1). While Romina faces his grid, Jeff counts 11 routes with a blue marker in his left hand: *SSEEEE, SESEEE, SEESEE, SEEESE, SEEEES, ESSEEE, EESSEE, EEESSE, EEEESS, EESEES*, and *EEESES* [SK_1, 18:36-19:07]. Then he says that they're missing some routes. Romina asks which routes are missing. He says, "you're not going like over two, down one, over two, over one." They decide to check their result for the "two by three," the intersection point (5,3). Romina says that she found eight routes. Jeff says that he found more than that and shows Romina how he is counting. Brian says that he is sure that there are more than twenty-two routes to the red pick-up point. Pointing to his grid and drawing routes, he says, "there's only one you can go by going two down. I'm trying to like figure out ways to like cross them out. You know what I'm saying?" Jeff states that there are only three ways of reaching the intersection point where the first move is one unit down (one unit south) and in turn enumerates each route as he draws it. He says, "and then going one down, you can go one, two, three- there's no other ways to go." He draws these nine routes: *SSEEE, SEEES, SESEE, SEESE, SEEES, ESSEE, EESSE, EEESS*, and *ESEES* [SK_1, 20:23 -20:53]. After he draws his last route, Romina tells him that it is a duplication of a previous route. Brian says that there are definitely twenty-three routes to the red pick-up point.

Looking toward Jeff and Brian, Romina says that their current way of working is confusing them. She suggests that they work methodically and determine how many shortest routes there are for each point of the grid, starting from the top of the grid with intersection points $(2,1)$, $(3,2)$, $(4,3)$... in the second row, then the points $(3,1)$, $(4,2)$, $(5,3)$... in the third row, and so on. She observes that for the points in the second row, the number of shortest routes goes up 2, 3, 4, 5 and that it is the same for the intersection points along the second vertical line of the grid. She suggests then that they determine the number of shortest routes to intersection point $(4,2)$ and then to the next point, $(5,3)$. She says that, "If we do that and we see a pattern I'm sure we'll be able to uh..."

Jeff and Romina establish a new method for counting and recording shortest routes to particular intersection points in the taxicab grid. Using a transparency of a Cuisenaire 1-centimeter grid paper, Romina writes in the first three squares of the first row the numbers 2, 3, and 4, respectively, and writes 3 in the square $(3,1)$. Without drawing the routes, she then counts the number of shortest routes to the intersection point $(4,2)$, the lower right vertex of a two-by-two square or sub-grid. Meanwhile, on a new sheet of Cuisenaire 1-centimeter grid paper, starting with the fourth grid line from the top, Jeff draws with a blue marker three horizontal lines completely across the grid, resulting in two rows, each two units in height. Romina announces that she found five and asks Jeff to count the routes, as well. With a blue marker in his left hand, he outlines and counts aloud six routes without drawing them: *SSEE, SESE, SEES, ESSE, EESS,* and *ESES* [SK_1, 22:47-22:58]. Romina tells Jeff that he counted a route twice. Jeff turns to his paper, draws two vertical lines connecting the end points of the three lines that he had drawn, and draws three more vertical lines, creating six two-by-two squares, three in each row. Romina faces his paper. As Jeff draws routes, Romina and he count aloud the routes, six in all. Jeff talks about each of the six routes he draws: *SSEE, SEES, SESE, ESSE, ESES,* and *EESS* [SK_1, 23:12-23:55]. For instance, for the first route— *SSEE*—he says, "that's one. Now that's all the ways you can go by two down." At the end of the count, he observes that each route has its opposite: "There's nothing else to do? Right? Now that would be the opposite of that one. That would be the opposite of that one and that would be the opposite of that one. They're all covered." Romina say that it is good that they re-did their count for the two-by-two sub-grid and writes 6 in the corresponding square, (4,2), on her transparency. Jeff says, "Good. 'Cause at least we're making progress."

For long periods, Michael quietly moves his pen from the taxi stand to the red pick-up point along grid lines, tracing without drawing different routes, and making tick marks on the left side of his taxicab grid.

Romina and Jeff work on three-by-two rectangular sub-grids, where they count the number of shortest routes to the lower right vertex of the sub-grid, that is, the number of shortest routes to the intersection point (5,3). On his 1-centimeter sheet, between the three parallel lines he drew previously to investigate routes to the intersection point (4,2), Jeff draws three vertical lines, resulting in eight three-by-two rectangular sub-grids. Romina crosses out the six routes that Jeff drew in their preceding investigation, saying, "so that we don't have to count those." Jeff uses a blue marker to draw the sub-grids and a red marker to draw the routes. Jeff and Romina find nine routes: *SSEEE, EEESS,*

SEEES, SESEE, SEESE, ESEES, EESES, ESSEE, and *EESSE.* To draw routes beyond the eighth one, Jeff draws three more three-by-two sub-grids and only uses one of them. While counting, Jeff at times uses the idea of counting opposites. At different moments, Romina and he call the routes *SSEEE* and *EEESS* "couples" or "opposites." They find no opposite route for the route "*SEEES.*" Jeff tells Romina that the three routes—*SESEE, SEESE* and *SEEES*—cover "all going through the middle," and she agrees. Jeff then shows Romina the routes in which the first move entails "going to the top" or first moving one unit east. Jeff observes that they have nine routes since one route—*SEES*—"doesn't have a couple." Romina says that perhaps sub-grids with an odd length or width will have an odd number of routes. Jeff says that that would be every other sub-grid. Romina writes 9 in square $(5,3)$, representing the number of shortest routes to the intersection point $(5,3)$, the lower right vertex of a three-by-two rectangular sub-grid. [SK_1, 00:24:10 – 00:26:55]

Jeff asks, "so now where are we at?" Romina writes 5 in square $(5,4)$ and inscribes a 4 in square $(4,1)$, saying it "has to be four." They discuss that 9 should be written in square $(5,2)$, representing the lower right point in a two-by-three rectangular sub-grid, since it is the same as the intersection point they just finished counting (see Figure 9).

2	3	4	5	
3	6	9		
4	9			

Figure 9. Numerical array from which Jeff and Romina perceive a symmetrical relationship in their data.

They decide to work on three-by-three rectangular sub-grids, representing the shortest routes to the intersection point $(6,3)$. On his 1-centimeter paper below their three-by-two rectangular sub-grids, again using a blue marker, Jeff draws two parallel lines and vertical lines, producing twelve three-by-three sub-grids. Jeff tells Romina to draw the routes. As she begins with a red marker, they discuss how to "stick with a pattern," issues of what methodical approach she should take. In two of the three-by-three squares, one under the other, she draws two routes—*EEESSS* and *SSSEEE*—and Jeff says, "opposites" [SK_1, 00:28:14 – 00:28:18]. She then draws routes "going across": *EESESS, EESSES, EESSSE, ESEESS, ESSEES,* and *ESSSEE* [SK_1, 00:28:18 – 00:29:04]. Jeff asks, "And those are all the ways that you can go from the top over?" To which Romina responds, "Yeah. Now going down." She draws routes going down and that are the opposite of ones she has already drawn: *SSEEES, SSEESE, SSESEE, SESSEE,* and *SEESSE.* As Romina draws each route in the second row of three-by-three sub-grids, Jeff asks where is its opposite and then marks with checks and exes those that correspond by opposite to one already drawn. She draws staircase a pattern—*SESESE*—and, with a suggestion from Jeff, its opposite—*ESESES* [SK_1, 00:29:04 – 00:31:26].

After drawing her last route and counting the number of routes she has drawn, Jeff asks whether they are sure there are only 15. With a red marker, he draws routes and checks to see whether they already have it. He draws three routes, but he and Romina notice that they already have been drawn. They decided that maybe there are only 15. Romina writes 15 in square $(6, 3)$ (see Figure 10).

2	3	4	5	
3	5	9		
4	9	15		

Figure 10. Jeff and Romina's data in a taxicab grid after they have indicated the 15 shortest routes that they found in a three-by-three rectangular sub-grid.

Romina asks Michael whether he is working on the red pick-up point. On the left side of his paper, Michael has several rows of staircase-like inscriptions and is moving his black marker along the grid lines from the taxi stand toward the red pick-up point. Romina and Jeff tell him that they are working toward it. On a new sheet of 1-centimeter paper, they then work on determining the number of shortest routes to the intersection point $(6, 4)$, located at the lower right vertex of a four-by-two rectangular sub-grid. With a red marker, Jeff draws three rows of four-by-two rectangular sub-grids, four in each and twelve in all. Using a blue marker, Romina draws the route *EEEESS* in the first sub-grid of the first row. In the first sub-grid of the second row, she draws the route *SSEEEE.* In the three remaining sub-grid of the first row, from left to right, she draws these routes: *SESEEE, SEESEE,* and *SEEESE.* In the second row, starting with the second sub-grid, she draws these routes: *SEEEES, ESSEEE,* and *EESSEE.* In the midst of her drawing the last route, Jeff asks her to "stick with the over ones."

She insists to keep what she's done and then draws the route, *EEESSE*. Romina now says, "over one" and draws *ESEESE* and *ESEEES* [SK_1, 00:35:55 – 00:36:04]. She asks, "can I go over any more?" Jeff says there should be one more. She writes out another route, *ESESEE*, and then crosses it out saying, "it's this one," pointing to the route in the second sub-grid of the first row, which is *SESEEE* [SK_1, 00:36:04 – 00:36:17].

Michael asks Brian how many routes he has found to the red pick-up point. Brian says that he found thirty before but that now that he is writing out the routes he has seventeen. Brian then asks Michael for his count. After a while Michael replies that he thinks he has found thirty-four shortest routes to the red pick-up point. Michael announces that he found thirty different. Brian asks, "So far?" Michael says, "That might be it."

Directly under the three rows, Romina draws another row of four, four-by-two sub-grids. As she begins to draw the new row, Jeff states that there should be one more. With a blue marker, in the first sub-grid of the new row, he draws one more route: *EEESES*. Romina says it's weird that they don't already have that it. Jeff scratches the left side of his head and says that the next one they do they should first "do all the ones that are over one."

While Jeff and Romina discussed the additional route, Brian says that he is writing out the routes for the red point. Brian has written on his paper arrays of numbers. One is head by the letters *O D O D O* and underneath each letter he writes respectively the numbers 2 1 1 2 1. In another row, he begins by writing under the leftmost letter, "2," then hesitates for a few second and places his black marker down on this sheet of paper [SK_1, 00:36:38 – 00:37:02]. He hands Michael his paper, saying that it contains a list of the seventeen routes to red that he has found. Michael asks him how to read it. Brian explains that Ds represent down.

Romina's eyes shift right toward the sheet that Michael is holding up and comments, "That looks nice too, what they're doing." Michael places Brian's paper in the middle of the table, and Brian retrieves it.

Romina writes 12 into square (6, 4) and says that it does not make sense (see Figure 11). She notes that the numbers are all "factors of three."

2	3	4	5	
3	6	9	12	
4	9	15		

Figure 11. Romina and Jeff's data array after they have counted the 12 shortest routes to the intersection point (6, 4).

Pointing to the 9 in square (5,2), Jeff says it does not make sense. Romina responds that it has to be nine and, with a blue marker in her right hand, outlines and counts routes to the intersection point (5,2). After about five routes, she asks Brian to count the routes for a "box two by three" and for him to do his "cool number thing." On the back of the sheet of paper that Brian had handed to Michael, in the lower left, Brian draws one three-by-two rectangular grid (representing the sub-grid for intersection point (5,3)). Brian writes O D and then 3 2 underneath, producing an inscription that resembles an array: $\begin{matrix} O & D \\ 3 & 2 \end{matrix}$. He then writes a second array: $\begin{matrix} D & O \\ 2 & 3 \end{matrix}$, a third array: $\begin{matrix} O & D & O \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{matrix}$, a fourth array:

$\begin{array}{ccc} D & O & D \\ 1 & 3 & 1 \end{array}$, a fifth array: $\begin{array}{cccc} O & D & O & D \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \end{array}$, and a sixth array: $\begin{array}{cccc} D & O & D & O \\ 1 & 2 & 1 & 1 \end{array}$ [SK_1, 00:38:45 – 00:40:52].

Romina asks Michael to count the number of shortest routes to the intersection point $(6,4)$. She and Jeff set out to work on a four-by-three sub-grids, whose dimensions are those of the rectangular sub-grid that has the taxi stand that the red pick-up point as vertices.

Drawing with a blue marker on a blank sheet of 1-centimeter grid paper, Romina produces four rows of four-by-three rectangular sub-grids, each row containing four sub-grids. She says to Jeff that he should draw the routes if he has an organized way of doing it. To his interlocutor, he talks aloud his thinking process, "all the ones you can get by going three down" and with a red maker draws one route: *SSSEEEE*. Then the discusses and draws the one "going two down" and draws these: *SSESEEE, SSEESEE, SSEEESE, SSEEEES*. He asks whether there are other routes first "going two down." He starts to draw another, and realizes that it would be one that he has already drawn. Romina says go "all right, go one down," and Jeff draws these: *SEEEESS, SEEESES, SEESSEE, SESSEEE, SEEESES, SEEE//* [SK_1, 00:40:52 – 00:42:07].

Brian asks a question. Just as Brian begins to speak, Jeff stops drawing his sixth route of the type that goes first "one down." Brian asks Romina and Jeff how many shortest routes they found for a three-by-two sub-grid. Romina says nine. Brian says he found ten. Jeff inquires whether he has a record of them and whether he could do something like what he and Romina have been doing. Brian says that he will write his routes on the chalkboard. To indicate each route, Brian uses the idea of down (D) and over (O) and a step-function-like notation. For each route, he records the number of vertical and the number of horizontal unit segments of which it is composed. Romina asks Michael to determine the number of shortest routes for "three over and two down," representing the sub-grid for the intersection point (5,3).

While Brian writes at the chalkboard, Romina, looking toward Jeff, asks whether they could do something like the towers. She asks whether lines over can be related to the colors and lines down to the number of blocks. She asks for the meaning of 2 and *n* in the expression 2 to the *n* related to the Towers Problem. Jeff answers that *n* is the number of blocks. Michael says that 2 designates the number of colors and that *n* is the number of blocks. With the tip of blue marker that she has in her right hand hovering over the square (2,1), Romina says it does not work of the first one.

Jeff asks Romina to resume what they were doing. Jeff crosses out the partially drawn route that he drew just as Brian had begun to ask to a question and draws two more routes: *SSEESEE* and *SESEESE* [SK_1, 00:44:41 – 00:44:58]. Meanwhile, Romina is facing the chalkboard holding up in her left hand a 1-centimeter sheet that contains marked off sub-grids and routes. Turning back toward the table, she says, "you're right." Jeff begins to draw yet another route and asks Romina to help him [SK_1, 00:44:58 – 00:45:07].

Brian returns to his seat. Romina asks Jeff for a sheet, and after rifling among their papers, he gives her one that has their routes for three-by-two sub-grids. Brian asks Michael whether he sees anything that he is "not getting." Romina reads from her and Jeff's sheet of routes in three-by-two sub-grids. She says that she and Jeff have "down 2 over 3, over 3 down 2." Brian goes to the chalkboard again. He had written his routes in sets. Now, he encloses each set in a rectangle and labels each set x moves, where x is the number of line segments in his representation of routes. Starting on the left of the chalkboard, his groups begin with the labels 2 moves, 4 moves, and 3 moves. (He has other groups but they are not visible on the videotape.) When Brian encloses his first group of routes, which correspond to the first two routes that Romina mentioned, she says, "all right, we got those." Romina continues and notes that on her sheet, there is a route that goes "down 1 over 3." She then says, "except, we don't have one, one, one, one, one, that one....So that nine does equal ten." On her transparency, in two places, she replaces 9 with 10 (see Figure 12).

2	3	4	5	
3	6	10	12	
4	10	15		
5				

Figure 12. Romina and Jeff's taxicab grid after they correct an undercount by one of the number of shortest routes to the intersection point $(5,3)$.

Romina announces, "All right. It's, um, - it's Pascal's triangle." Michael says, "what?" But looking at the array of number on the transparency in front of her, seeing that there is a 12 instead of a 15, she then says, "no it doesn't work out." Also looking at the array of number on the transparency and pointing to particular squares, Jeff notes that the 12 in square $(6,4)$ should be 15 and that in square $(6,3)$ there should be 20 instead of 15.

Romina and Jeff ask Brian to determine the number of shortest routes in a four-by-two sub-grid, that is, to intersection point $(6,4)$. He agrees. They discuss how it's nice to start from nothing, to have no clue, and then to see something familiar. They also mention how easy it is to miss counting routes. Brian writes out routes, using his notation that involves arrays headed by *O*s and *D*s.

Michael has drawn a four-by-two and a four-by-three sub-grid on a sheet of 1-centimeter paper. Holding a black marker moves his hand within the four-by-two grid.

Jeff asks Brian how he does his counting. Brian explains, "the one with two moves, the one with three moves…over three, down two…over, down, over, down." Jeff asks which one Michael is doing, and Michael responds that he's looking for fifteen. Brian asks how many routes are they looking for and, after Jeff says fifteen, he says that he has eight.

Brian has enclosed in six different rectangles his inscriptions for the number of moves and the length of each move. They are arranged on the page as follows: two containing two moves, two containing three moves, two containing four moves, and one containing five moves. Here are the contents within two of his rectangles: $\begin{array}{cc} O & D \\ 4 & 2 \end{array}$ and $\begin{array}{ccccc} O & D & O & D & O \\ 1 & 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \end{array}$. The first rectangle contains his inscription for one route of two moves, where the taxi driver travels 4 blocks "over" or east and two blocks "down" or south. In the second array, he has representations for direction and length of three routes, each of five moves, where the first move is "over" or east. He is writing his fifteenth route, involving five moves, when Jeff asks for his attention.

Brian continues working. Jeff motions to Michael to listen to him [SK_2, 00:00:25]. Jeff explains an idea he has about counting the number of routes for a sub-grid (2-by-4) by decomposing into two parts (two 2-by-2s). Then knowing the number of routes in the left and right parts, adding their number of routes to determine the number of routes in the original sub-grid. Michael joins in thinking about this idea.

While Jeff explains his idea, Brian continues to work on counting routes in a four-by-two sub-grid [SK_2, 00:00:25 – 00:01:04]. After a short while, Brian announces that he found fifteen routes for the four-by-two sub-grid. A few seconds later, Michael says that he too found fifteen and asks what does it mean. Jeff says, "it means that it is the triangle." He also says that in a three-by-three sub-grid it should be twenty shortest routes.

Michael has drawn a three-by-three sub-grid. With a black marker in his right hand, outlines different routes without actually drawing them, and writes down the number of routes he finds. Jeff leaves the room. Michael tells Brian that he wants to verify that in a three-by-three there are twenty routes. At one point, Michael says that he is missing two but that twenty probably is right. Brian asks whether he found the "staircase one."

Romina returns to the table. Brian tells her that for the four-by-two sub-grid, he found fifteen, and she responds that the two-by-four also must be fifteen. Michael returns to counting routes in a three-by-three sub-grid. Romina turns the transparency upside down and re-writes the numbers representing the number of shortest routes that they have empirically found.

2	3	4	5	
3	6	10	15	
4	10			
5	15			

Figure 13. The participants' array of data representing their empirically found number of shortest routes.

Brian says that he will check for the number of routes in a three-by-three sub-grid. Brian draws an array of sixteen dots and then a grid that incorporates the dots, producing a three-by-three sub-grid [SK2, 00:06:53 – 00:06:15]. Romina also works on counting shortest routes for a three-by-three sub-grid. With a green marker but without drawing the outlines of three-by-three sub-grids, she draws the following thirteen routes on a blank sheet of 1-centimeter paper: *EEESSS, EESESS, ESEESS, EESSES, ESSEES, ESSESE, ESESES, SSSEEE, SSESEE, SSEESE, SESSEE, SEESSE, SESESE*. After the last route, she motions as if to count the number of routes she has already drawn and begins to draw a fourteenth route by first moving one unit south, *S*. She hesitates for about nine seconds and announces, "I'm already stuck"[SK2, 00:06:26 – 00:740]. Jeff, who only seconds ago returned to the table, asks her what she is doing. Brian has written four groups of routes, each framed by a rectangle, and is working on a fifth group. Michael announces that he has found twenty.

Jeff asks whether they can "explain why we think...." Michael says that they [the researchers] will ask us, "How do we know...?" Romina writes 20 into square (5,3) of the transparency grid and says let's relate this back to the blocks (towers).

2	3	4	5	
3	6	10	15	
4	10	20		
5	15			

Figure 14. In a portion of the taxicab grid, an augmented array of data the participants found empirically.

Jeff states that the question is "why is it that the Pascal triangle works for this?" Michael asks how do his colleagues know that it's twenty. Jeff says that if they can explain why it's the Pascal triangle up to a certain point that they then do not need to justify why it is twenty. Michael asks Brian whether he can explain how he does his work. Michael and Jeff suggest that if he has a pattern then perhaps it can used to provide explanations. As Brian responds, Romina with a green marker in her hand faces the transparency and moves the maker tracing what seems like routes [SK2, 00:08:52 – 00:09:43]. Brian says that he knows there is a way to reach the pick-up point in two, three, four moves and so on. He confirms Michael's question that he thinks about the routes by the number of moves.

Pointing to the array of number on the transparency, Romina asks what do the numbers 1 2 1 in Pascal's triangle means in relation to towers. She says 2 colors and two to the x. Michael says that two refers to the number of colors. She then asks about the 2 in the array of number on the transparency. Michael explains pointing to the transparency that the 2 stands for the total length it takes to reach the pick-up point (2,1). Brian continues to work on finding routes in a three-by-three sub-grid. With a black marker, Michael points to the line segment from the taxi stand to the square with 15 in it, traversing three blocks east and two blocks south, and says that five represents the total length [SK_2, 00:09:43 – 00:10:01]. Romina says that it's the second row. She decides to re-write the numbers in the array in a triangular form and does so on the right side of the transparency. She writes the first couple of lines of Pascal's triangle.

Brian announces that there are twenty shortest routes in the three-by-three sub-grid [SK2, 00:10:50 – 00:10:52]. Jeff then suggests that they explain and talk through what they have found.

Green ink from Romina's marker stains her sweater. Jeff and Brian advise her on how she might get it out. Jeff tries to rub it out of her sweater. Someone from the River Run staff offers Romina baby wipes. Also, Romina, Jeff, and Brian accept them to clean ink off their hands.

Walking between the chalkboard and the students, Researcher 1 reenters the classroom, carrying a chair. Brian moves himself and his chair to the left [SK_2, 00:14:17 – 00:14:21]. Researcher 1 places her chair at the table between Jeff and Brian, and sits down. Michael says that they cannot justify their answer. Jeff adds that they want talk through things and see where that takes them and states that he needs to leave in five minutes. Researcher 1 says that she really wanted to give the problem of determining the shortest route for all the points on the grid. Romina and Jeff inform her that it is what they did. Brian exclaims that "we've got it, we've got it." Researcher 1 asks whether they liked the problem. Jeff says that doing this kind of "stuff" hurts the brain but otherwise it was fine.

Romina begins to explain that they analyzed from point to point on the grid. Jeff says that they broke the problem down, asking how many ways can one get from the taxi stand to neighboring points. He says that they did a "basic math deal" of making an easier problem. Romina points out the numbers they have in their grid on the transparency and says what they arrived at was Pascal's triangle. Jeff says that if one tilts the grid (rotates it 45 degrees) and put ones along the two sides and a one at the apex, then one has Pascal's triangle. He explains that some of them drew routes, while Brian had a different method of counting. Research 1 asks and Jeff confirms that the numbers in their array were arrived at empirically, by actually counting. Research 1 inquires whether there is any way to justify and to help her see that what they arrived at is Pascal's triangle: What does the second row—1 2 1—mean? After Researcher 1 again says that she does not see Pascal's triangle, with a red marker in his right hand, Jeff points out how their array can be read so as to see that the numbers correspond to rows of Pascal's triangle, indicating as well that the addition rule or, equivalently, the recursive formula—$C_r^n = C_{r-1}^{n-1} + C_r^{n-1}$—works. Researcher 1 responds, "Why does that work, what does the 10 mean?" With her right index finger, she points at an intersection point asks whether there is a way to predict the number of routes for an intersection that you have not found empirically. With a red marker, Jeff marks an intersection point, the one where Researcher 1's finger had been pointing, and says that it would be 35. Michael points out that they cannot justify that statement since they haven't counted. Jeff says that following the pattern is their justification for now.

Researcher 1 says to the students that they noticed a pattern and it fits Pascal's triangle. Michael and Romina begin a side conversation concerning pizzas with one topping, two toppings. Brian asks whether it means that the red is 35 although Michael found 34. Researcher 1 asks the students why do the numbers seem to work and how can they explain them. Romina mentions that she is having difficulty seeing Pascal's triangle. As she states this, she uncaps a green marker and on a blank sheet of paper writes five rows of Pascal's triangle [SK_2, 00:19:41 – 00:19:56]. The following comes into sharp view [SK_2, 00:20:00]:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & \\
 & & & 1 & & 3 & & 3 & & 1 & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Figure 15. Five lines of Pascal's triangle written by Romina.

Romina recaps the marker and now with the cap end points to the 2 in her triangle and says that it comes from when there are two blocks (or Unifix cubes), two blocks with 2 colors. Jeff says one block, then draws a square, and enumerates its two routes. She says, "let's go back to towers." Pointing to the 2 in the second row of the triangle, Romina says that the number 2 comes from when there are two blocks, two colors. Jeff gets up from his chair and departs from the table.

Romina turns toward Michael. Pointing to the 2 in the third row of the triangle, she asks, "How's this go? Just tell me where this comes from." Researcher 1 says farewell to Jeff and invites Brian to sit were Jeff sat. Brian moves into Jeff's seat. Michael with a black marker points back and forth from the numbers in the transparency to the portion of Pascal's triangle that Romina wrote on paper. They discuss their interpretations of the numbers on the transparency and Romina's Pascal's triangle and relate it to the Towers Problem.

With a green marker in her right hand, Romina points with the cap end at the transparency and says that they could name the horizontal blocks As and vertical blocks Bs. Tracing a route from the taxi stand to the intersection point (3,2) and pointing to corresponding 3 in square (3,2) of her transparency grid [SK_2, 00:21:50 – 00:21:52], she says that with two As and one B "there is 3." Now tracing a route from the taxi stand to the intersection point (3,1) and pointing to the 3 in square (3,1)of her transparency grid [SK_2, 00:21:53 – 00:21:55], she points to the 3 and says that it would be two As and one B. She also says that two As and two Bs would correspond to the 6 in square (4,2). Michael pointing with a black marker to the second row of the transparency says that it would be everything with a perimeter of two. Romina says to Michael that they could indicate that an intersection named by eight As and six Bs on the transparency grid and then could find a corresponding number in her Pascal's triangle.

Michael pointing to the transparency grid says that for row six, everything in the row has shortest route of 6 blocks. Pointing at the 2 in her Pascal triangle [SK_2, 00:23:11 – 00:21:14], Romina asks whether the 2 mean one of A color and one of B color. Whereas, now switching to the transparency and pointing to the 2 in its first row, she says, either way one goes its "one across and one down."

Pointing to first 3 in her Pascal triangle, she says that it 2 of A color and one of B color and then pointing to her transparency she says it would be two across and one down, representing two blues and one red. Then pointing to the next 3 says that it would be one blue and two reds, yielding Pascal's triangle.

Michael says that they need some reasoning that they cannot just say that they counted. Michael asks Researcher 1 to say what she is looking for.

Researcher 1 says that Romina seems to know. She asks Romina to state her idea again. Romina, pointing to the 1-2-1 row of her Pascal triangle, says that in relation to towers, it corresponds to two-high towers made from two different colors. Pointing to the 1 on the left, she says that it is a tower made from all one color. Tapping the 2, she says that it's one of each color. The 1 on the right, she states it is a tower made from all other color. She then points to the next row—1 3 3 1—and says that it is towers three high. From left to right, the 1 is all of one color, the three is two of the other color and one of the other. She then points over to the transparency grid, saying that it is basically the same. Pointing to the 2, she says to get to it one goes one across and one down and that there are two different ways to get there. Pointing to the 3 in the first row in square (3,2) and then to the 3 in the second row in square (3,1), she explains that respectively they are two across or down and one down or across. She asks, "do you understand."

Michael says that he understands. Pointing to the third row in Romina's Pascal triangle, he explains that a 3 would represent in towers, color x and two of color y and then pointing to the transparency grid, direction x and two in direction y.

Researcher 1 asks whether Researchers 2 and 3 have any questions to pose. Researcher 3 asks the students how do they know that the number of shortest routes to a particular corner in the taxicab grid corresponds to a number in Pascal's triangle. Romina says that she has not thought about that yet. The researchers say that they will let the students think about it for a couple of minutes.

Brian, Romina, and Michael point to the transparency grid and compute other numbers for squares neighboring the ones they have already filled in based on their empirical findings. Writing with a green marker, Romina appends her Pascal triangle based on these calculations.

							1							
						1		2		1				
					1		3		3		1			
				1		4		6		4		1		
			1		5		10		10		5		1	
		1		6		15		20		15		6		1
	1		7		21		35		35		21		7		1

Figure 16. Romina's seven lines of Pascal's triangle.

Michael says, "we know it follows a pattern," however, he [Researcher 3] wants to why without just saying that it follows a pattern.

Michael and Brian face the area on the table containing the transparency grid and Romina's augmented Pascal triangle. Referring to an intersection five across and two down, point (7,5), Romina says that it means that there are seven blocks. With a marker in her hand, she counts down seven rows in her Pascal triangle and says, "five of one thing and two of another thing." She then crosses out the ones in her triangle, saying that they are not involved. Michael inquires about her meaning for "five and seven." Romina and Brian respond, "five across and two down." She then associates the combinatorial numbers in the seventh row of her Pascal triangle to the idea of "five of one thing and two of another thing," pointing out that, from left to right, the 21 represents two of one color, while the second 21 represents five of one color. During Romina's explanation, Michael follows along with a red marker in his right hand, counting down the rows of Romina's Pascal triangle and moving around the corresponding point on the transparency grid.

Gesticulating with his left hand, Michael asks "why" and leans back into his chair. Romina explains by using as an example the question of the number of shortest routes to the intersection point $(4,2)$, corresponding to the 6 of her transparency grid. Michael sits up and leans forward toward the table, with a red marker in his right hand, he points to the intersection on the transparency grid and then counts down four rows of Romina's Pascal triangle. Romina points out that the distance from the taxi stand to the intersection point is 4 regardless of the shortest route. Since its distance is 4, she says it corresponds to a tower that is 4 blocks tall and to the fourth row of her Pascal triangle. She tells him that to reach the intersection point, one goes two blocks across and two blocks down. Romina says, "its two of one color and two of the another." Then pointing to the fourth row of her Pascal triangle, she explains that, from left to right, the first 4 is "one and three," the of 6 is "two and two," the 4 "three and one," and the 1 is "all one color." Both Michael and Brian express agreement. Romina says, "let's call them in."

Researchers 1 and 2 reenter the classroom. Michael asks Researcher 3 to restate his question. As Researcher 3 begins to restate it, Michael asks him whether its true that he wants to hear a reason why Pascal's triangle works for the problem not just that the numbers looks like it. Pointing to the transparency grid, Romina and Michael agree to give two examples, "the six and the four." That is, they will give examples that involve intersections on the taxicab grid whose number of shortest routes is 6 and 4, respectively. Starting with intersection point (4,3), reachable by four different shortest routes, Romina explains that its distance is 4, it relates to a tower 4-tall, and, therefore, it corresponds to the fourth row of her Pascal triangle. She holds up her left hand with her four fingers extended and her thumb folded into her palm. She also moves her hands, more or less parallel to each other, between the transparency grid and her augmented Pascal triangle. Continuing, she explains that any route will involve 3 moves across and 1 move down. Michael adds that one has to go "three and then one." Romina relates the three moves across and one move down of the transparency grid to three of one color and one of the other of her Pascal triangle. Pointing to her triangle, Romina explains that, in the fourth row, going from left to right, the first 1 represents all of one color but since it means moving across 4 units, she does not use it. The second number in the fourth row, 4, represents one of one color and three of the other, which is the situation she is discussing. Then along the fourth row, the next number represents the number of routes to an intersection involving four moves, two across and two down.

Romina starts to discuss the other example. She explains that to reach the intersection point (4,2) requires 4 moves, two across and two down. Since it requires 4 moves, she says, it means that one still refers to the fourth row of her Pascal triangle. Pointing between the transparency and her Pascal triangle, she shows that having to move two across and two down means that the answer to the number of shortest routes to the intersection point is given by the 6 in her triangular array of numbers.

Michael begins to relate the Taxicab Problem to the Pizza Problem, saying that he sees something but does not know how to say it. Romina, looking toward Michael, suggests that it is same as what she just explained in terms of "the blocks." Michael then relates a topping to driving across and says, "if you're only able to go across one time then you could do it four different ways." Michael points variously with his right and left hands at Romina's Pascal triangle the four different choices one has for moving across and reaching the point that lies at the intersection point (5,2) [SK_2, 00:35:40 – 00:36:54]. He says that the 6 means that one is able to drive 2 blocks across and 2 down, the next 4 means driving 3 across, and the 1 means able to drive nothing across.

While explaining the significance of the last 1 in Romina's Pascal's triangle in relation to the idea of driving across no units, Michael says that the ones do not have meaning in their model.

Romina explains that the ones in Pascal's triangle refer to traversing toward the intersection points on the boarder of the grid since there is only one direction in which to reach those points. In contrast, she points out that reaching intersection points interior to the grid require driving in two different directions.

Michael agrees with Romina. Researcher 3 inquires whether Michael is saying that the number of toppings relates to the number of moves across. Michael goes on to say that all toppings would mean all moves down. Researcher 1 asks whether this means that some are across and some are down.

Researcher 1 asks for how does it work with *A*s and *B*s or *x*s and *y*s and pizza toppings. Pointing to the transparency grid, Romina explains that *x*s correspond to driving across, a topping or a color and *y*s relate to driving down, a topping, or a color.

Researcher 1 asks whether 0s and 1s could be used. Romina says that is Michael's area. Both Brian and Romina encourage Michael to explain. Michael says that it has been awhile since he discussed this. Romina says to him that "one would be every time across" and that "zero would be every time down." Then, with a green maker, Michael writes the following array of binary numbers:

$$\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}$$

[SK2, 00:40:49 – 00:40:53].

He circles the first three in the third row of Romina's Pascal triangle. He explains that this group of numbers represents all that have one 1 and two zeros. He says "next one would" the following array of binary numbers:

$$\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{array}$$

[SK2, 00:41:03 – 00:41:08].

With his left index finger, he points to the second 3 in the third row of Romina's Pascal triangle. Now he assigns 0s as down and the 1s as across. He moves his index finger between the sheet where he has written his array of binary numbers, which is the sheet where Romina wrote her augmented Pascal triangle, and the transparency grid [SK2, 00:41:08 – 00:41:28], explaining that the 1s correspond to going across and that 0s to going down and pointing to each of the 3s in the transparency grid.

Michael asks whether what he has said is good and understood. Researcher 1 asks that Romina and Brian whether they understand. Romina says that she works in towers and that Michael, pointing toward him, works with pizzas and binaries.

Researcher 1 asks whether the student wish to add anything. Brian leans over the table, pulls toward him his work, and with a black marker, moves it down the page as if he were counting. Michael asks if the researchers are convinced. Researcher 1 says that she see where they get their numbers and states that her question is about a general number. With a green mark, Romina distinguishes an intersection on the transparency grid. She draws a green dot at the intersection point (10, 5) [SK2, 00:42:25 – 00:42:31] and says to first determine the number of moves across and then down to the point. For the point she distinguished, she counts across and down and says that it is ten blocks. Gesticulating she says that this tells one that the answer will be on the tenth row of Pascal's triangle and that one has five of one color and five of the other color. Michael says that it will be the middle number in the row since it will contain an odd number of numbers. He places his hand with the palms up and says that there are "ten spots, ten toppings, ten different places to put these numbers."

Researcher 3 asks Romina and Michael to help him understand them. Michael says that the row in Pascal's triangle corresponds to shortest distance. Researcher 1 asks about the *r*th row, and Romina says that it would be *r* moves. Michael says *r* is the shortest distance. Researcher 1 asks whether they have ever seen a problem like this one before, and they respond no.

CHAPTER 5: RESULTS

It takes a mind and the world to create the patterns of mathematics.
Keith Devlin

5.1 Introduction

The problem-solving session lasted for 1 hour, 39 minutes, and 35 seconds. Analysis of the video data reveals that the three researchers do not suggest to the students how to engage the task but rather that through their conversational exchanges, students structure their own investigation. Further analyses of participants' discourse and their inscriptions reveal that they use part of their time to understand and plan how to resolve the problem task; work hard to develop problem-solving strategies and overcome heuristic hurdles; hypothesize and create combinatorial algorithms; build explanations and justifications of their work and solutions; challenge each other to clarify their explanations and justifications as well as accept challenges of the same from researchers; and construct isomorphisms among the Taxicab Problem, the Pizza Problem, and the Towers Problem, using Pascal's triangle as an iconic representation upon which to build their isomorphisms.

This chapter contains this introductory section and seven others. Section 5.2 discusses the structure of the problem-solving sessions in terms of interactions among students and researchers. In sections 5.3 through 5.7, I illustrate results with vignettes from the video data and figures from the inscriptional data. Each vignette presented comes from the turns of speeches that I have coded as a critical event.[12] My analyses are of conversational exchanges and my unit of analysis is turns of speech. As such, I present these

[12] See chapter 3 for a discussion of critical events.

vignettes of critical events as extracts from a numbered transcript, where each number represents a turn of speech. The unit of analysis for some researchers is a line of speech, and thus they number each line of a transcript. Each of the vignettes that I present is numbered and contains information about camera view, compact disk number, and inclusive time codes. When I quote from an exchange that I do not consider part of a critical event, I indicate the turn number from which the quoted speech comes rather than present a portion of the transcript. The entire transcript of the problem-solving session appears in Appendix B.

The results presented in this chapter derive from analyses of the video and inscriptive data. In analyzing these data, I am guided by the following five research questions.

1. As the participants engage collaboratively, without assigned roles, to understand and resolve the Taxicab Problem what are the features and functions of learners' discursive practices?
2. As they articulate their emergent understandings of the problem task, what mathematical ideas and associated meanings do they reveal in their discourse?
3. How do participants structure their investigation?
4. How do their conversational exchanges support advances in their problem solving?

Working within the paradigm of grounded theory, the first question and its results arose during the open-coding phase of analytic work. Interlocution as an analytic category emerged from a dynamic interplay of open-coding, categorizing codes, dimensionalizing categories, and the literature; in particular, my engagement with ideas of Chinn and Anderson (1998), Davis (1996; 1997), Dörfler (2000), Larson (1999), Pirie (1991), Pirie and Schwarzenberger (Pirie &

Schwarzenberger, 1988), Seeger (2002), Sfard (1994; 2000c), and Sfard, Nesher, Streefland, Cobb, and Mason (1998). In turn, the construct of interlocution that emerged from a grounded-theory analysis of the data has become part of this study's theoretical framework. Categories of interlocution are in turn used to analyze further the data and to respond to the other three guiding research questions.

Further responses to first research question are contained in sections 5.3 to 5.7. These sections also inform responses to the second and fourth questions. Results of analyses that point to the third research question are contained in section 5.2, and further details are given in the other sections. In chapter 6, the results are summarized and the light they shed onto the research questions.

5.2 *Structuring the investigation*

In the problem-solving session, students engaged in discursive interactions with each other as well as with researchers. Figure 17 summarizes the major interaction episodes (see Appendix H for detail descriptions of these episodes). Each row of the table represents an episode from the perspective of camera view LS (LS_1, LS_2, or LS_3); indicates the start, end, and elapsed time of the episode; and whether the interval is one in which

1. students interact amongst themselves ($S \Leftrightarrow S$),
2. students initiate interaction with researchers ($S \Leftrightarrow R$), or
3. researchers initiate interaction with students ($Rz \Leftrightarrow S$).

In one episode (LS_1 4), while students engage in discursive interaction with each other, they initiate brief interactions with one of the three researchers. In all, for approximately 77% of the time or 1 hour and 16 minutes, the students spend in discursive interaction with each other, and 92% of this time without the physical presence of the researchers.

Major Interaction Episodes	Start Time	End Time	Elapsed Time	Interaction Type
LS_1 0	00:00	01:05	01:05	Noise. Filming begins. Students enter the classroom and sit at a trapezoidal table in front of three video cameras.
LS_1 1	01:05	01:41	00:36	$R_1 \Leftrightarrow S$
LS_1 2	01:41	03:02	01:21	$S \Leftrightarrow S$
LS_1 3	03:02	03:42	00:40	$R_1 \Leftrightarrow S$
LS_1 4	03:42	40:41	36:59	$S \Leftrightarrow S$, $S \Leftrightarrow R_2$
LS_2 5	00:00	23:59	23:59	$S \Leftrightarrow S$
LS_2 6	23:59	26:57	02:58	$S \Leftrightarrow S$
LS_2 7	26:57	32:18	05:21	$S \Leftrightarrow R_1$
LS_2 8	32:18	36:50	04:32	$S \Leftrightarrow S$
LS_2 9	36:50	39:49	02:59	$S \Leftrightarrow R_1$
LS_3 10	00:00	00:31	00:31	$R_3 \Leftrightarrow S$
LS_3 11	00:31	05:36	05:05	$S \Leftrightarrow S$
LS_3 12	05:36	19:05	13:29	$S \Leftrightarrow R_{3,1}$
Student to student interaction episodes			1:15:59	Total amount of time students spend in discursive interaction without researchers (≈ 77% of total).
Researcher and student interaction episodes			23:36	Total amount of time students spend presenting their ideas and being interviewed by researchers (≈ 23% of total).

Figure 17. Intervals in which students interacted amongst themselves ($S \Leftrightarrow S$), students initiate interaction with researchers ($S \Leftrightarrow R$), and researches initiate interaction with students ($R \Leftrightarrow S$).

Besides developing their own solution to the Taxicab Problem, the four participants in this study—Michael, Romina, Jeff, and Brian—structure their own investigation and, except for the first interaction (LS_1 1), initiate their interactions with the researchers. Neither the task nor the researchers suggest, hint, or otherwise tell the participants how to engage the investigation. The students are given no assignment rolls as is common in many cooperative and collaborative learning models. Of the three researchers who interact with the participants, not one researcher offers them any content information, mathematical or otherwise, or hints of any kind. During the first almost 3.75 minutes of the research session (LS_1 1, LS_1 2, and LS_1 3), one researcher spends a total of 1 minute and 16 seconds with the participants to distribute the problem and later to answer clarifying questions. After this, the students work on their own, both individually and collaboratively, for one hour, three minutes, and fifty-six seconds. Within this nearly 64-minute interval, during episode LS_1 4, another researcher interacts with the students twice and briefly. However, these interactions occur exactly when the students ask for first colored markers and later extra grid paper. A researcher responds by placing the requested materials in the center of the table at which the students are working and utters a few words concerning the nature of the materials. Finally, only when the participants requested (LS_2 7, LS_2 9, and LS_3 12), the researchers listen to the participants report on their work, their ideas, their reasoning, their solution of the Taxicab Problem, and their attempts to relate the features of problem task to isomorphic problems and structures. Within these episodes, two researchers ask the participants questions about their explanations and, for instance in LS_3 10, posed questions to challenge participants' assertions. In reaction to these challenges, the students discuss ideas among themselves and direct themselves to develop responses to the researchers' questions.

As the data show, the students spend a considerable amount of time in the problem-solving session, three-quarters of which they spend in conversational exchanges amongst themselves. The vast majority of this time is purposeful and productive. For only about three of the nearly one hundred minutes do the participants engage in conversation that could be considered off-topic (see episode LS_2 6 in Figure 17 or in Appendix H). This episode notwithstanding, as detailed in subsequent sections of this chapter, during the nearly 1 hour and 16 minutes that the students spend in interlocution, they work to understand the problem task; develop problem-solving strategies and overcome heuristic hurdles; hypothesize and create combinatorial algorithms; build explanations and justifications of work and solutions; challenge each other to clarify their explanations and justifications and accept challenges of the same from researchers; and construct isomorphisms among the Taxicab Problem, the Pizza Problem, and the Towers Problem, using Pascal's triangle as an iconic representation upon which to build isomorphisms.

5.3 *Understanding the problem*

The participants in this study set out to understand the nature of the problem task on which they will work. Their conversational exchanges range among three interlocutory categories: informative, interpretative, and negotiatory. They pose clarifying questions to one of the researchers. To each other, the participants state observations and pose thought-provoking questions about the fundamental objects with which they will work and about relations among these objects.

Fifteen seconds after Researcher 1 distributes the problem task and says, "Why don't I just give you the problem, okay? Um, I'll give you a chance to look at it and see whether you understand the problem," Jeff utters the first question about it.

Vignette 1 [LS_1, 0:02:06 – 0:02:10]

17. JEFF: You have to stay on the lines, right? Those would be streets?
18. RESEARCHER 1: Exactly.
19. JEFF: I agree.

In this informative interchange with Researcher 1, Jeff confirms his interpretation of two issues: allowable movements and object equivalence. He seeks to be sure that the taxi diver must travel only horizontally or vertically, not along non-orthogonal paths, and must travel through the streets, not down alleys ("You have to stay on the lines, right?").[13] He also wants to clarify that the grid lines symbolize streets ("Those would be streets?"). In the taxicab plane, grid lines represent streets and, along with intersection points, are the fundamental objects of its geometry. After Researcher 1 confirms his understanding, Jeff expresses satisfaction ("I agree.")

Following Jeff's student-initiated interchange with a researcher, Romina begins the first student-to-student interaction. It centers on a question about a relation that she notices about which Romina invites her colleagues to comment.

Vignette 2: [LS_1, 0:02:25 – 0:02:59]

20. ROMINA: Isn't it like anyway you go-
21. BRIAN: Pretty much, because look-
22. ROMINA: As long as you don't go like past it. [Facing Brian's direction.]
23. BRIAN: The first one- No.
24. MICHAEL: Well what if you go to the last one-

[13] Jeff's concern reminds me of a comment by Krause (personal communication, 25 October 2002), who has championed the study of taxicab geometry (1973; 1975/1986). In his communication, he notes the following about "alleys" in the taxicab plane: "taxicab geometry means different things to different people. Some people require that an ordered pair have at least one integer coordinate in order to qualify as a point of taxicab geometry. In your initial work with kids, you require both coordinates to be integers. In my book I don't require either coordinate to be an integer. (My cities have alleys all over the place.)" With his question, therefore, Jeff is participating in a wider conversation within the mathematical community about the fundamental structure of a non-Euclidean geometry.

25.	BRIAN:	You can go all the way down and go over and go down three and go over two. [Tracing the routes above the problem sheet with a black marker in his right hand.]
26.	ROMINA:	Isn't it- Don't they all come out to be the same amount of blocks?
27.	BRIAN:	Five.
28.	JEFF:	Five?
29.	ROMINA:	Five? I got seven.
30.	JEFF:	Uh, which one- Yeah, we were both looking at the red one.
31.	BRIAN:	I'm looking at blue. [Michael is tapping his pen on the grid along intersection points.]
32.	JEFF:	Yeah.
33.	ROMINA:	Oh, okay.
34.	JEFF:	All right. I mean pretty much.
35.	ROMINA:	As long as you don't go like past it you're fine. So it's the same thing.
36.	BRIAN:	So, let's prove it.

Romina's interrogative, "isn't it like anyway you go they [the lengths of routes] all come out the same...as long as you don't go past it [the pick-up point]?" suggests that she is aware of a relation among efficient ("as long as you don't go past it") paths or routes between the taxi stand and the red pick-up point. She observes that as long as one does not go beyond the red pick-up point that the numbers of blocks traversed or lengths of routes to red equal each other. Specialized to the red pick-up point, she expresses three awarenesses about relations among objects: (a) an efficient route will be a shortest route, (b) there can exist more than one shortest route, and, her central observation, (c) efficient routes have the same length. These three ideas are important and fundamental for progressing toward a resolution of the problem task.

The conversational exchange that Romina's question precipitates are both interpretive and negotiatory in character. At first, Brian disagrees ("The first one- No, 'cause-") and then, examining routes to the blue pick-up point, attempts to understand Romina's remark ("You can go all the way down and go over and go down three and go over two"). Afterward, Jeff and Romina try to understand Brian's assertion, "five," for the number of blocks traversed by shortest routes between the taxi stand and the red pick-up point. Ultimately, Brian sees that they are speaking about routes to the red point ("Yeah, we were both looking at the red one."). While, they understand that he is referring to the blue pick-up point ("I'm looking at blue."). Taking up Romina's observation for the red pick-up point along with his own for the blue point, Brian suggests, "So, let's prove it."

With this statement, Brian invites the group to think together, to enter into a negotiatory interlocution. His referent for "it" is Romina's statement, "they all come out the same," or, in other words, efficient routes have the same length. His invitation is for the group to justify how they know that there is an equivalency among efficient routes between the taxi stand and a specific pick-up point.

During the first moments after Brian's invitation no one responds to it. Instead, Researcher 1 requests that someone explain his or her understanding of the problem. In an informative exchange with Researcher 1, Jeff says, pointing to the taxi stand and the red pick-up point, that the task is to find the shortest route, "from there to here staying on the streets, right?" (turn 38, see Appendix C). Researcher 1 adds that it is about finding whether there is "more than one shortest route" (turn 39). Both Brian and Romina voice agreement. Researcher 1 emphasizes that if there is more than one, they have to determine how many shortest routes. From the perspective of an interpretative interlocution, Jeff inquires with Researcher 1 whether she is asking "how many different shortest routes?" (turn 46). At about the same time, Brian states that blue has five shortest routes. Researcher 1 reminds the students that not only do they have to find the number of shortest routes but also they "have to convince us [the research team]" (turn 51) that they have found all possible routes.

Even though the participants did not immediately respond to Brian's suggestion that they prove that efficient routes between the taxi stand and particular pick-up points have the same length, about 1 minute and 37 seconds later, Jeff poses a question that places the suggestion back onto the group's agenda.

Vignette 3: [LS_1, 0:04:36 – 0:05:42]

20.	JEFF:	So why- why is it the same every time?
21.	MICHAEL:	You're going left and right.
22.	ROMINA:	Ours is a four by one, right? It's the only way to go.
23.	MICHAEL:	it's the only way you can go. Yeah, it's a four by one, unless you go backwards a couple of times.
24.	ROMINA:	You can't go, well-
25.	MICHAEL:	I know that would be dumb.//
26.	BRIAN:	//[inaudible] the shortest route only if you go forward.
27.	MICHAEL	But the only- You can't go diagonal so you have to go up and down. So if the thing is down this many and//
28.	JEFF:	//Over that many, //it's the same
29.	MICHAEL:	//It's the same-
30.	ROMINA:	//It's the same area
31.	MICHAEL:	No matter how you do it, no matter how you do it it's- you have to- you can't //get around doing that. [Pointing and gesturing around his grid]
32.	ROMINA:	//All right.
33.	MICHAEL:	//You can't get around going four down and right one 'cause -.
34.	JEFF:	All right, yeah. All right.
35.	MICHAEL:	You can't go over there. You can't get around doing that.
36.	JEFF:	Yeah.
37.	ROMINA:	What if I were to go like to the red when I go one, two, three, four- [Pointing at her problem sheet.]
38.	MICHAEL:	But they're not asking for that.
39.	ROMINA:	//Five, //six, seven.
40.	JEFF:	//Five, six, seven. //It's the same thing.
41.	ROMINA:	//Like //how- how am I going to- like //how would I-
42.	JEFF:	//It's the same thing.

43.	MICHAEL:	//It's the same.
44.	ROMINA:	-devise an area for that? Like this- this area up here? [Motioning with her pen on her grid, indicating the area of the rectangular space whose vertices are taxi stand and the red pick-up point.]
45.	BRIAN:	Like plus and [Inaudible].
46.	JEFF:	Well, it's not area.
47.	MICHAEL:	It's not area. It's //just a-
48.	JEFF:	//It's the perimeter. It's like //each one being one.
49.	MICHAEL:	//One, two, three, four, five, six, seven. [Pointing at Romina's paper and counting the length of a route to the red destination point.] [Jeff scratches his head.]
50.	ROMINA:	All right.
51.	MICHAEL:	There's no way you can get around going- [gesturing with his hands]
52.	JEFF:	//Going seven blocks.
53.	ROMINA:	//No, yeah, I understand.
54.	MICHAEL:	Across that many and down that many because you can't go diagonally. Can't- [gesturing with his hands over his problem sheet across to the left and then down]
55.	JEFF:	Yeah.
56.	MICHAEL:	Can't get around it, so- [gesturing with his hands]
57.	JEFF:	I mean, that's the most sensible way I think to say that. Right? And they want to know how many though.

Romina's observation, reiterated by Jeff, and Brain's suggestion become a shared project of the participants. In this sense, their interlocution is negotiatory. Jeff poses his question ("why is it the same every time?") and the others understand his "it" to mean the set of efficient routes to a pick-up point. Michael's immediate response, coming just 4 seconds after Jeff finishes uttering his question, is in contrast to the silence that this question met almost two minutes earlier. The ensuing verbal exchange hints that the issue of the why Romina's observation was true in general remained a concern of the participants and that they are only now prepared to tackle it.

All four students engage Jeff's question. Michael explains that to reach a pick-up point, the shortest distances will always require one to move a fixed number of units down (south) and a fixed number across (east) and observes that within the grid one cannot travel diagonally. Brian reminds the others that only going forward will produce a shortest route. Michael generalizes his awarenesses to all routes. Jeff signals that he is convinced, saying, "I mean, that's the most sensible way I think to say that." In the process of the group's discourse, Jeff and Michael help Romina to see that area is not an operative idea in this task.

Through the conversational exchange, the participants co-construct several new ideas: (a) movement within the given portion of the taxicab plane goes left or right and up or down (turns 73, 77, 83, and 104); (b) diagonal movements are not permissible (turns 77 and 104); (c) the taxi stand and each pick-up point together define a rectangle in which the pair of points are located at opposite ends of a diagonal, and the problem task involves moving along the perimeter but does not a concern for the extent of space that a rectangle occupies (turns 91 to 99); (d) the number of units down plus the number of units across are objects related by addition to produce the length of a shortest path (turn 83); (e) any route to the blue pick-up point will involve four blocks down and one block across (turn 83); and (f) each horizontal and vertical line segment of the grid can be considered as one unit in length (turn 98).

By the end of the vignette, Jeff, who in the form of a question reintroduced Brian's suggestion that they justify the idea that the length of efficient routes from the taxi stand to a pick-up point are equivalent, expresses satisfaction with Brian and Michael's argumentation ("I mean, that's the most sensible way I think to say that."), checks whether the others agree ("Right?"), and reminds his colleagues of the crux of their task by saying, "And they want to know how many though."

Vignettes 1, 2, and 3 are critical events. They present the major occasion in which the participants ferret out the nature of the problem space and build fundamental ideas essential for investigating the problem task. The participants establish what are the basic objects of taxicab geometry (points and line segments or routes); basic awareness of the Taxicab Problem (there can be more than one shortest route to a intersection point in the taxicab plane); and implicitly note a distinguishing feature between Euclidean and taxicab geometries (how distance is measured). This distinction emerges when Michael observes that in the context of the problem task, one cannot travel diagonally, he touches upon the fundamental distinction between the metric of Euclidean geometry and that of taxicab geometry. With the ideas they build illustrated in these critical events, the participants shift their focus to delve further into the problem task and generate considerably more data.

5.4 *Developing problem-solving strategies, overcoming heuristic hurdles*

Knowing to generate data is a heuristic awareness. Generating data and evaluating the feedback that the data give allows for exploration and self-education that can increase the probability of solving a problem. Besides generating data, the participants engage other heuristics that involve knowing to do something as reflected in and different from the heuristics described by Pólya (1945/1973) and others (Brown & Walter, 1983; Mason et al., 1984; Mason, 1988; Schoenfeld, 1985).[14] At the level of non-professional mathematicians, knowing to deploy particular heuristics appropriate for working on mathematical problems

[14] See chapter 3, footnote 3, for a definition of the use of the term heuristics in this study.

include advanced methods useful for, among other settings, international mathematics competitions (Engel, 1997).

The first articulation of a piece of participant-generated datum is by Brian. Responding to Jeff's clarifying question about the task, Researcher 1 emphasizes that if they determine that there is a shortest route, then they must also decide how many are possible. Brian suggests, "let's do blue" (turn 44) and, after six seconds of counting different routes, announces "Blue's got five" (turn 50), and records this datum on his problem sheet. Soon afterward Romina voices agreement ("I have five," turn 54) after also counting routes to the blue pick-up point. This way of working—outlining without drawing as well as counting routes to a pick-up point—is the participants' first heuristic method.

This heuristic quickly raises questions about keeping track of counted routes. Vignette 4 reveals the negotiatory interlocution in which the participants beget two new heuristic methods. Jeff requests colored markers from Researcher 2. However, he, Romina, and Brain discuss whether and how to keep track of routes by drawing with markers. Romina voices concern about being able to distinguish one route from the other. Understanding her point, Jeff suggests seeing what happens in practice. Romina starts to count again the number of shortest routes between the taxi stand and the blue pick-up point but this time draws them. After drawing two routes (*SSSSE* and *ESSSS*) with a blue marker and then expressing concern that the routes will not be distinguishable, she abandons this approach and writes in black on her problem sheet the datum, "Blue: 5." Brian states that they should know how they arrive at the number of shortest routes between the pick-up points and the taxi stand. He suggests that an aspect of their heuristic should entail knowing exactly how to reconstruct their counting scheme for the set of shortest routes to a pick-up point.

Vignette 4: [LS_1, 0:03:45 – 0:04:31]

55. JEFF:	Can we have like a- You have colored like markers? Word! [Responding to Researcher 2's statement that she will give them some markers.]
56. BRAIN:	For what?
57. JEFF:	Because then we can just do each route a different color. To like- [Jeff waves his hand.]
58. ROMINA:	Yeah, but they all kind of go on top of each other.
59. JEFF:	Well, I mean, I don't know. I mean, let's see what it looks like. If it get too ugly then- Which one are you doing?
60. ROMINA:	Which one do you want to do?
61. JEFF:	I'll go to red.
62. ROMINA:	I've got blue.
63. BRIAN:	I did blue.
64. JEFF:	Brian already-
65. ROMINA:	One-
66. BRIAN:	It's just going to look like you're filling //in the boxes.
67. ROMINA:	//Two. Yeah, it is.
68. JEFF:	That's what it's going to end up looking like, right?
69. ROMINA:	Yeah so screw it. There's- Okay, so we know five-
70. JEFF:	Well,- [Romina writes "Blue 5" on her paper to the right of the grid and traces routes with her pen on the grid.]
71. BRIAN:	Just count them and then make sure you know how you got them. You know? [Jeff and Romina count by tapping their pen or marker on the grid. Each of them counts on their own grid.]
72. JEFF:	Yeah.
73. ROMINA:	One, two-

Another heuristic that the participants develop to generate data is to parcel out mini-tasks. Jeff says that he will count the routes from the taxi stand to the red pick-up point (turns 60-64). Romina states that she will do the blue point. Vignette 5 presents another example of parceling out that occurs after Brian conjectures that the number of routes to the red pick-up point equals the length of a shortest route. Jeff indicates interest in the conjecture and urges that the group investigate it. To determine the veracity of Brian's conjecture, Romina parcels out the task of counting routes to pick-up points, saying that she and Michael will count routes to the green point and tells Jeff and Brian to do the same for the red point (turns 121-124).

Vignette 5: [LS_1, 0:05:43 – 0:06:27]

112.	BRIAN:	Are there seven possibilities, though? You know how like blue was five? There's five possibilities but-
113.	JEFF:	Ah, so-
114.	BRIAN:	You know how it's only like five spaces. Like one, two, three, four, five. [Pointing at the grid on his problem sheet.]
115.	ROMINA:	Yeah, so if it goes more.
116.	BRIAN:	Is there seven for blue, I mean red?
117.	JEFF:	Well, check it out.
118.	BRIAN	You've got one- [Pointing at the grid on his problem sheet]
119.	ROMINA:	Here, I'll- //Me and Michael do-
120.	MICHAEL:	//Is that the shortest routes?
121.	ROMINA:	Me and Michael do greens. The green one.
122.	BRIAN:	All right.
123.	MICHAEL:	//Oh, like that's the biggest one. [Pointing at his problem sheet.]
124.	ROMINA:	//And they'll do red.
125.	BRIAN:	Green is nine I think. [Then he begins to check this idea.]
126.	ROMINA:	Well //count how many ways. [They use their pens or markers to count on the grid.]
127.	JEFF:	//All right, we'll look for it.
128.	MICHAEL:	One, two- [counting and pointing at paper]
129.	BRIAN:	Ten. My bad. [Correcting himself on the length of a shortest path to green.]
130.	MICHAEL:	There's a lot.
131.	ROMINA:	Yes I know. I'm trying to devise a- like a-
132.	JEFF:	The- the way to do it?
133.	ROMINA:	Yeah.
134.	JEFF:	This is hard. [Drawing routes in red to the red pick-up point.]

The participants come to realize that parceling out, counting, and drawing routes for particular pick-up points alone are not enough to ensure that they can reconstruct or retrieve how they found the routes. Reconstruction is a heuristic goal that Brain articulated (vignette 4, turn 71). The participants announce their need for a more robust, methodical heuristic for counting. Michaels says there are too many routes to red and Jeff sighs that counting routes to red is hard (vignette 5, turns 130 and 134). Romina states that she trying to devise a heuristic and Jeff says that he understands what she is aiming to do (vignette 5, turns 131-133).

The participants respond to their need for a methodical heuristic in two different ways. One response is to construct combinatorial algorithms that yield manageable, easier approaches to generating data than simply counting and that on the basis of data generated may lead to further conjectures and a resolution of the problem task. I discuss the algorithmic response in section 5.5. A second response is to build isomorphisms, which I treat in section 5.7, that may provide an elegant resolution of the problem task by demonstrating that the task is mathematically akin, in terms of objects and relations among them, to another known and already solved task.

Knowing to construct a combinatorial algorithm and to build an isomorphism are heuristic awarenesses. An example of a heuristic based on the idea of building an isomorphism emerges in the following vignette, where the interaction is both informative and interpretative, and where Romina initiates the interaction:

Vignette 6: [LS_1, 0:07:33 – 0:08:55]

159. ROMINA: Okay, we can't count. Like we need a- can't we- can't we do towers on this.
160. JEFF: That's what I'm saying. Look, all right, you go to here-
161. ROMINA: And they're like blocks.
162. JEFF: All right, you go to here and you got a choice of going there or there. Right? [Indicating a choice of across or down at an intersection point of the grid on his problem sheet.] So then you pick one of those and then you got a choice of there or there. When you get to-

you know what I'm saying? Maybe we can add all those up or something and get like a whole- [Explaining routes on grid paper.] ß

163. ROMINA: All right.
164. MICHAEL: There's a lot.
165. ROMINA: Okay, for ours there's 10 / /
166. MICHAEL: There's more than ten.
167. ROMINA: No. I mean there's 10 blocks. Like 10 lines to get to that thing, right?
168. MICHAEL: Yeah, six by five.
169. ROMINA: So if there's ten, ten could be like the number of blocks we have in the tower.
170. MICHAEL: This is one-
171. ROMINA: How do we do that? Two to the *n*? [Romina moves her pen cap on and off of her pen.]
172. MICHAEL: How- how many? This was five they said? [Pointing to the blue pick-up point on his problem sheet.]
173. ROMINA: Yeah. [Looking back to her problem sheet.]
174. MICHAEL: How much you guys get for the red? Still doing that one?
175. ROMINA: How could-
176. MICHAEL: It's got to be some kind of pattern.
177. ROMINA: Okay, there's 10 lines- 10 lines-
178. MICHAEL: Ten ways of getting there. So you can do. Like you got to-
179. ROMINA: There's ten different lines to get there.
180. MICHAEL: Think of the possibilities of doing this and then doing that. [Pointing at an intersection on his problem sheet grid and gesturing downward and then rightward.]

Romina begins this interlocution after trying to enumerate all shortest routes to the green pick-up point. She announces that counting has become difficult and wonders aloud whether connecting this problem to the Towers Problem[15] might be helpful ("Okay, we can't count. Like we need a- can't we- can't we do towers on this."). Jeff immediately engages Romina and her idea. His take on her comment is the Towers Problem's binomial aspect. Referring to the Taxicab Problem and routes to the red intersection point, he wonders whether the two choices one has at each intersection point could be used additively or otherwise to arrive at the number of shortest routes from the taxi

[15] For a statement of the Towers Problem and related problems see Appendix B.

stand to an intersection point ("So then you pick one of those [intersection points] and then you got a choice of there or there. When you get to- you know what I'm saying? Maybe we can add all those up or something and get like a whole-"). His idea presents a possible combinatorial algorithm.

After listening to Jeff mention his emergent idea, Romina turns toward Michael and thinks aloud about relating block towers to their current problem. She considers whether the length of a shortest route to green, which is ten, could be linked to the height of a block tower and asks Michael to help her recall the meaning of 2^n in the context of the Towers Problem. She would like to move from generating data by counting to building links between both the objects and the relations among them in two systems: Taxicab Problem and Towers Problem. In other words, she attempts a shift from employing the empirical heuristic of counting to the relational heuristic of building an isomorphism. Though Michael appears to hear her, he pursues his idea of a relationship between the length of a shortest route to a pick-up point and the number of all possible shortest routes to that pick-up point.

Together the three vignettes (4, 5, and 6) analyzed thus far in this section form a pivotal strand[16] along with the following critical event:

Vignette 7: [LS_1, 0:14:53 – 0:15:08]

293.	ROMINA:	I think we're going to have to break it apart and draw as many as possible.
294.	BRIAN:	Yeah, //that's what I'm going to do.
295.	JEFF:	//And then have that lead us to something? What if we do- why don't we do easier ones? You know what I'm saying? What if the- the thing- Do you have another one of these papers? [Speaking to Researcher 2.]
296.	ROMINA:	Here, to make it simple, just draw on here.

[16] See chapter 3, section 2.1, for a discussion of pivotal mathematical strand.

The theme of the pivotal strand for which vignettes 4, 5, 6, and 7 are its constitutive critical events can be labeled as 'how to count the number of shortest routes from the taxi stand to the there given points.' Not only is vignette 7 an element of this pivotal strand but also it is a watershed critical event.[17] In it, an agenda for action emerges from the negotiatory interlocution of Romina, Jeff, and Brian. Immediately, Brian and Jeff accept the task implied in Romina's statement ("I think we're going to have to break it apart and draw as many as possible"). Brian agrees to her suggested heuristic and says that he too will shift his approach ("Yeah, that's what I'm going to do"). Responding to Romina, Jeff participates in an unfolding of possibilities by refining her suggestion in his interrogative: "What if we do- why don't we do easier ones?" Romina's statement and Jeff's interrogative establish a new agenda for the group's actions. This plan for collective action represents a milestone in their mathematical investigation, and the participants continue their investigation with a renewed sense of purpose and a new heuristic. From this point onward, they, including Michael, no longer work on the combinatorial problem as given and, instead, pose and work on simpler situations from which to generate data. They choose to work on a more general problem than the given one but sense that it was simpler and that it will, in the words of Jeff, "lead us to something." Reflecting on numerical patterns in their data, they extract insights so as to inform their understanding and resolution of the given problem. Often one's attempt to solve a problem compels one to pose a simpler but structurally related problem, which occurs precisely when one asks, as Jeff does, "what-if" or, as Brown and Walter (1983) insist, "what-if-not" questions.

[17] See chapter 3, section 2.1, for a discussion of watershed critical event.

The problem that the participants pose has an associated new heuristic. Their problem concerns finding all possible shortest routes from the taxi stand to nearby points, at first. In a moment right before the next vignette occurs, responding to Jeff's request for grid paper, Researcher 2 places on the table several paper and transparency copies of Cuisenaire 1-centimeter grids as well as extra problem sheets. In vignette 8, Romina asks Jeff to pick a dot on a new problem sheet, and he chooses the grid point (2,1). To this point from the taxi stand, Romina outlines without drawing on the grid two shortest routes (*SE* and *ES*) and writes 2 in the (2,1) square.

Vignette 8: [LS_1, 0:15:31 – 0:17:15]

306.	JEFF:	All right. So-
307.	ROMINA:	Pick a dot.
308.	JEFF:	Right there.
309.	ROMINA:	One, two.
310.	JEFF:	Two. All right. Here.
	...	
314.	ROMINA:	Okay. So one, two, three- Oh, is this going to be dumb and stuff? One, two, three, four- It looks like a multiplication table.
315.	JEFF:	All right. Uh, one-, two [Inaudible]. [Brian draws his eighth symbol on the right side of the grid and writes "1, 4, 2." On the top of "1, 4, 2" he writes "D, R, D." He also goes back to 7 and writes "D3, R1". He has written a number with each of the first 6 symbols on Brian's paper, too.]
316.	ROMINA:	All right.
317.	JEFF:	Why don't you just- here, use blue. It doesn't matter.
318.	ROMINA:	Yeah. One-
319.	JEFF:	One- //two.
320.	ROMINA:	//Two. Three. //Four.
321.	JEFF:	//Four.
322.	ROMINA:	Five?
323.	JEFF:	Where are you? Wait was that one, two over? The fourth spot? One, two- three- four- five. I don't- I can't remember what I- [Jeff draws routes on a 2 by 2 rectangle.]

324. ROMINA: I think it's five. I think it's five. [Brian draws his ninth symbol for a specific route, with the numbers "2, 4, 1" next to each line on the symbol.]
325. JEFF: I think it is.

Generating data by finding all possible shortest routes from the taxi stand to neighboring points is the current *modus operandi* of the participants. Romina and Jeff work to find all shortest routes to intersection points $(3,2)$, $(4,3)$, $(3,1)$, $(4,2)$, and $(5,3)$, finding respectively 2, 3, 4, 3, 5, and 7 shortest routes. Figure 18 shows a portion of the problem sheet grid on which Romina and Jeff work and how their data are arranged.

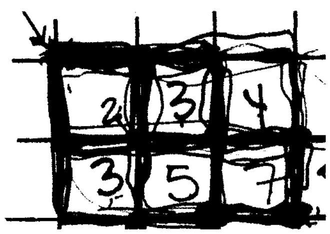

Figure 18. Romina and Jeff's early data on shortest routes in a portion of the taxicab grid from Romina's problem sheet.

Before they continue, Jeff and Romina engage in a negotiatory exchange concerning the intersection point on which they should work next. In vignette 9, Jeff suggests first the point $(9,3)$ and then the red pick-up point, $(7,4)$. Romina disagrees. She argues that "it's going to be too much," presumably meaning that are probably so many routes that given their current counting strategy they could not be confident about the accuracy of the results they would obtain. Instead, she suggests that they continue in the order they have already established and, therefore, determine the number of shortest routes to intersection points $(5,4)$ and $(6,4)$ and afterward routes to points $(4,1)$, $(5,2)$, $(6,3)$, and $(7,4)$. Jeff concurs. Aware that the further an intersection point is away from the taxi stand then the greater the number of shortest routes it is likely to have, Romina keeps herself and Jeff working on easier but incrementally more complex situations:

Vignette 9: [LS_1, 0:18:38 – 0:21:22]

347.	JEFF:	Wait. Why don't we give one of these like to-
...		
349.	JEFF:	To like here. [Pointing to the intersection point $(9,3)$ on the grid of the problem sheet in front of Romina.] No //and then-
350.	ROMINA:	//Because it's going to be too much. Well, //go down and see like when we go down and we do all these and all of these that go out one more and see how much you get. [Pointing to intersection points $(5,4)$, $(6,4)$, $(4,1)$, $(5,2)$, $(6,3)$, and $(7,4)$.]
351.	JEFF:	//For the red one, sorry.
352.	JEFF:	All right.
353.	ROMINA:	One- [Romina starts tracing routes to $(5,4)$ with her pen on the grid.]
354.	BRIAN:	I'm not good at this kind of stuff.
355.	ROMINA:	//one, two, three, four, five.
356.	JEFF:	Where- where you going to?
357.	ROMINA:	Here, this is- this is five. [Writing a 5 in the $(5,4)$ square.] And, go to this one now because- [Pointing at intersection point $(6,4)$.]//I mean that one I'm pretty sure. [Referring to the result obtain for the point $(5,4)$.]
358.	JEFF:	//Was it four by four?
359.	ROMINA:	Uh, four by two.
360.	JEFF:	That's what I meant. I was drawing the right thing. [Jeff draws a four-by-two sub-grid on a sheet of 1-centimeter grid paper and draws routes within the sub-grid.]
361.	ROMINA:	Yeah, it's working. [Romina writes a 9 in the $(6,4)$ square.]
362.	JEFF:	Wait, you only got nine for that!?
363.	ROMINA:	Uh hmm. [Romina writes a 4 in the square in the third row, under the 3 in the second row, after counting routes with a pen on the grid.]
364.	JEFF:	All right, wait a second. Check it out. Um, all right. You go one-
365.	ROMINA:	All right.
366.	JEFF:	Wait- just wait a second.
367.	ROMINA:	No, I know. I'm just- One-

368.	JEFF:	One. Then two. [drawing routes on grid paper]
369.	ROMINA:	Uh hmm.
370.	JEFF:	And then- Uh, three, four, five. [Drawing routes on grid paper.]
371.	ROMINA:	Uh hmm.
372.	JEFF:	Six, seven, eight, nine, ten, eleven- you know what I'm saying? We're missing- [drawing routes on the grid]
373.	ROMINA:	Okay, what am I missing?
374.	JEFF:	You're- we're like
375.	ROMINA:	Did we do that for seven?
376.	JEFF:	Well you're- I don't know. You're not going like over two down one. //Over two over one. [Jeff motions with his pen on the grid.]
377.	ROMINA:	//I'm not doing [inaudible]. I'm not doing that.
378.	JEFF:	So-
379.	ROMINA:	Okay.
380.	JEFF:	You want to go back from the //beginning.
381.	ROMINA:	//Go back to- //go back to seven.
382.	JEFF:	//You got to go- Well, how do you know- we did five right?
383.	ROMINA:	We had to have done five because there was like-
384.	JEFF:	As long as it's right I don't- I don't care. Just as long as it's right. All right, so, which one's the seven one? Two by three?
385.	ROMINA:	I got eight for that, right? [Jeff draws routes on the grid.]
386.	JEFF:	Seven, eight- I got more than that. All right, wait. We got to go through this, and you got to watch.

In vignette 9, Romina and Jeff employ another heuristic, a second form of parceling out. They parcel out the same task to each other. In this case, each one counts routes to intersection point $(6,4)$, Romina does so on a problem sheet and Jeff draws a four-by-two sub-grid on a sheet of Cuisenaire 1-centimeter paper. Romina draws all the routes she finds in the one four-by-two sub-grid defined by the taxi stand and intersection point $(6,4)$. Jeff does the same with his sub-grid. They compare their data. Jeff finds more than nine shortest routes, which is the number that Romina found. The discrepancy between their results and the inconsistency of their methods of counting leads Jeff to articulate to Romina his thinking as he recounts shortest routes to intersection point $(6,4)$:

Vignette 10: [LS_1, 0:21:36 – 0:22:35]

393. JEFF: All right. There's only one you can go by going two down. I'm trying to like figure out ways to like cross them out. You know what I'm saying? And then going one down, you can go one, two, three- There's no other ways to go. [Pointing to his paper and drawing routes.]
394. ROMINA: Mm hmm.
395. JEFF: What about like that? Four?
396. ROMINA: Mm hmm. Mm hmm.
397. JEFF: And then, five, six, seven, eight- [Counting the routes he draws.]
398. ROMINA: You already did that one.

...

400. JEFF: Which one?

...

402. ROMINA: All right guys. This is what we're trying to do. Why don't we try to do this- [Taking a blank piece of 1-centimeter grid paper.]
403. JEFF: All right, what's-
404. ROMINA: We're getting all confused. You see how we're like going to like we're drawing like we're going to here. How many it takes to get to that point and then we're going to here and it's like a- this is just going up like one, two, three- two, three, four, five and then we go down to here and there's the same thing and then like how much we'll get to this point and how much we'll get to that point. [Pointing to intersection points on a blank 1-centimeter grid paper.] Why don't we all try to do that because we're getting confused and we're-
405. JEFF: Yeah.
406. ROMINA: We're doing the same mistakes.
407. JEFF: And it's like real hard. My brain-
408. ROMINA: If we do that and we see a pattern I'm sure we'll be able to- uh

In this vignette, drawing various routes on the same sub-grid does not allow Jeff to distinguish clearly one shortest route from another. This way of working is a heuristic hurdle for Jeff and Romina. Romina argues to the group that they all need to participate in finding the number of shortest routes to intersection points nearby the taxi stand. She expresses confidence that if they see a pattern then they will be able to come up with something.

After vignette 10, another decisive heuristic moment occurs. Romina requests that they start again collecting data on the number of shortest routes from the taxi stand to neighboring intersection points. She takes a transparency of a Cuisenaire 1-centimeter grid paper to record numerical results and writes in the first three squares of the first row the numbers 2, 3, and 4, respectively, and writes 3 in the square (3,1). She and Jeff agree that the number of shortest routes to intersection point (4,2), for which they had found 5, needs to be confirmed. With a blue marker in his left hand, Jeff outlines routes without drawing and counts aloud six routes: *SSEE, SESE, SEES, ESSE, EESS,* and *ESES* [SK_1, 0:22:47-22:58]. Romina tells Jeff, "you're counting one twice" (turn 431, vignette 11). This again illustrates their heuristic hurdle of distinguishing found routes to an intersection point. The routes Jeff outlined are ephemeral. He and Romina cannot reconstruct the routes and check whether any routes in fact were repeated. Their need to overcome this hurdle leads Jeff to draw six two-by-two sub-grids on a blank sheet of 1-centimeter grid paper. Moreover, as he draws each route—*SSEE, SEES, SESE, ESSE, ESES,* and *EESS* [SK_1, 0:23:12-23:55]—six in all, he describes his thinking. For example, he says: "That's one. Now that's all the ways you can go by two down." Vignette 11 contains the negotiatory interlocution between Romina and Jeff.

Vignette 11: [LS_1, 0:24:07 – 0:25:07]

431.	ROMINA:	You're counting one twice.
432.	JEFF:	Six- All right, wait. That's why- here watch.
433.	ROMINA:	Maybe, yeah.
434.	JEFF:	You just go make two by twos. [Drawing three vertical lines on his centimeter paper to create 2-by-2 sub-grids.]
435.	ROMINA:	Mm hmm.
436.	JEFF:	You could go- [Drawing 2-by-2 sub-grids.]

437.	ROMINA:	Yeah, make at least six at the moment.
438.	JEFF:	All right. You can go this way. [Drawing 1 two-down route.]
439.	ROMINA:	Yeah that's one.
440.	JEFF:	That's one. Now that's all the ways you can go- [Drawing a route.]
441.	ROMINA:	Yeah.
442.	JEFF:	-by two down. So then you can go like this...
443.	ROMINA:	Two.
444.	JEFF:	You can go like this. [Drawing 2 one-down routes.]
445.	ROMINA:	Three.
446.	JEFF:	Is there any other ways to go by going down? No.
447.	ROMINA:	Okay.
448.	JEFF:	All right. So then you could- you could go like that. [Drawing 2 one-over routes.]
449.	ROMINA:	Mm hmm.
450.	JEFF:	You could go like that. [Drawing 1 two-over route.]
451.	ROMINA:	Mm hmm. Or you go all the way top to bottom.
452.	JEFF:	There's nothing else to do. Right? Now that would be the opposite of that one. That would be the opposite of that one and that would be the opposite of that one. //They're all covered. [Pointing to pairs of routes on the grid with a pen.]
453.	ROMINA:	//So we got six. Good. Good thing we did that over again.

Vignette 11 not only illustrates the new heuristic of drawing separate sub-grids for each route but also reveals Jeff's algorithmic thinking for counting routes. I discuss further his thinking in this regard in the next section of this chapter. The critical events presented in the three vignettes 4, 10, and 11 constitute a pivotal strand that can be identified as 'building an methodical, retrievable approach to counting shortest routes.' Further, vignette 11 represents a watershed critical event of the strand. This event marks a shift in the manner in which Romina and Jeff proceed to count shortest routes. After this moment, they no longer count routes by tracing them in a single sub-grid. This watershed critical event signals overcoming a heuristic hurdle that they had identified: how to distinguish among routes from the taxi stand to specified intersection points.

Moments after vignette 11, Romina and Jeff investigate other intersection points, drawing individual sub-grids for each shortest route that they find. In contrast, before vignette 11, Brian had already developed an approach to recording his counts of shortest routes that allows him to retrieve the routes that he constructs. Indeed, it was he who had urged, referring to shortest routes, "just count them and then make sure you know how you got them" (turn 71). For most of the first forty-five minutes of the problem-solving session, Brian works on developing his notation for counted shortest routes, offering suggestions, sharing his results with his colleagues, and participating in some discussions. Consequently, analysis of the heuristics that he employs is based on examination of his inscriptions, both the videorecording of his building the inscriptions in successive layers as well as what he says about them and the physical inscriptions themselves.

Brian essentially has two different types of inscriptions and each reveal aspects of his thinking about how to account for shortest routes to particular intersection points in the given portion of the taxicab plane. He creates his inscriptional devices while counting shortest routes from the taxi stand to the red pick-up point. They are represented in Figure 19. On the right side of his problem sheet, he presents a numbered list of step-function-like or staircase-like markings of the routes he finds to the red point. In 7 of 10 different routes that he presents, he places numbers alongside the line segments. He builds his eighth staircase-like presentation and another inscriptional notation in four 4 steps: (a) he draws a short vertical line, then a long horizontal line, and finally another short vertical line, slightly longer than the first; (b) to the right of this one-rung

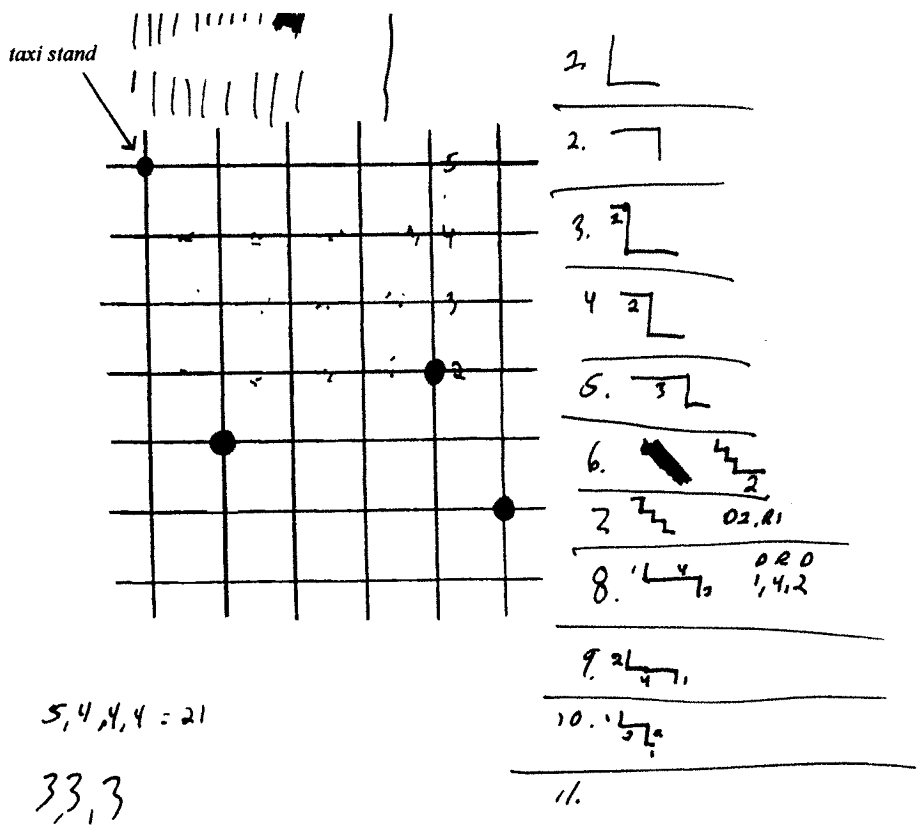

Figure 19. On the right of the taxicab grid, Brian's step-function-like presentations of routes to the red pick-up point.

staircase, he writes the numbers "1, 4, 2"; (c) over these number, he respectively writes D R D; and (d) to the left of the first vertical segment, he writes, "1," underneath the horizontal segment he writes, "4," and to the right of the second vertical segment, he writes, "2" [SK_1, 0:14:17 - 0:15:45].

In Figure 19, Brian has created two notational inscriptions to indicate routes that he finds between the taxi stand and the red pick-up point. The first is the one he creates in steps (a) and (d) and the second in steps (b) and (c). He uses the first notational inscription to indicate his first seven routes to red. He develops the second notation, a variation of the first, as he thinks about his eighth route. The first notational inscription visually depicts the shape of the route the taxi driver takes, while the second is more compact and, like the first, indicates the direction and number of blocks that the motorist traverses. Ultimately, Brian chooses to use the second notational inscription, substituting O for R, indicating 'over' rather than to the 'right,' both of which signify traveling east.

Brian further develops his second inscriptional device into a notation that resembles matrices. Each row presents a route and the columns indicate whether the taxi driver is traveling over (or east) and down (or south). The numerical entry in each matrix cell specifies the number of blocks traversed in the direction indicated by the column in which the number is located.

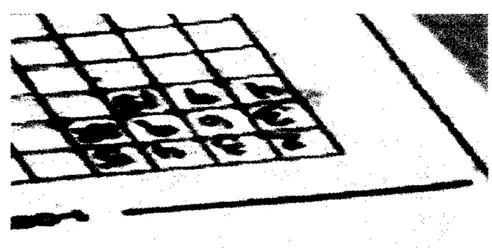

Figure 20. Video image of Romina and Jeff's data array of the number of shortest routes in a taxicab grid after they have counted the 12 shortest routes to the intersection point $(6,4)$.

After nearly forty minutes into the problem-solving session, Romina and Jeff are completing their count for routes to intersection point $(6,4)$. Romina writes 12 into square $(6,4)$ and says that it does not make sense (see Figure 20).

She notes that the numbers are all "factors of three." Jeff says that the result they obtained for (5,2), namely 9, seems wrong. Romina asks Brian to count the routes for "a box two by three" (turns 655 and 657). Brian draws one three-by-two rectangular grid (representing the sub-grid for intersection point (5,3)). Brian writes O D and then 3 2 underneath, producing an inscription that resembles the following matrix array: $\begin{matrix} O & D \\ 3 & 2 \end{matrix}$. He then writes a second array:

$\begin{array}{cc} D & O \\ 2 & 3 \end{array}$, a third array: $\begin{array}{ccc} O & D & O \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{array}$, a fourth array: $\begin{array}{ccc} D & O & D \\ 1 & 3 & 1 \end{array}$, a fifth array:

$\begin{array}{cccc} O & D & O & D \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \end{array}$, and a sixth array: $\begin{array}{cccc} D & O & D & O \\ 1 & 2 & 1 & 1 \end{array}$ [SK_1, 00:38:45 – 00:40:52]. At this stage, he has found eight routes and his inscription allows him to retrieve the routes he has constructed. Aware of this, at one point, Romina engages Brian in order to compare his results with her and Jeff's for the number of shortest routes between the taxi stand and the (5,2) intersection point (turns 707-736).

The above is an example of the heuristic 'parceling out.' Soon afterward, Michael works on counting to the intersection point (6,4) or, equivalently, in a four-by-two sub-grid. Both Brian and Michael participate in further generating data to compare their results with data that Romina and Jeff have obtained (see Figure 20). Their participation indicates the shared project in which the four participants are engaged and in this sense their conversational exchange is negotiatory.

Another heuristic approach that the four participants use is to process their findings with researcher to discover further their ideas. Jeff calls his colleagues attention and suggests this: "All right.... You want to try and explain this and then wherever we get like confused along the way, you know maybe that's how we'll be able to- as we talk through it" (turn 977). As Researcher 1 sits down at the table, Michael says, "we can't justify our answer but we're- we're, uh," and Jeff completes the thought, "We're going to talk through it. And we want to see where that takes us" (turns 1059, 1060, and 1064). Findings related to the emergent and elaborated mathematical ideas that they convey and reasoning that they display are treated below in section 7.

In all, the participants employ 16 different heuristic methods. Figure 29 in section 3.2 of chapter 6 presents the first occurrence of these heuristics. After their first occurrence, the participants elaborate some heuristics to a greater extent than others.

5.5 *Implying, hypothesizing, and articulating combinatorial algorithms*

Central to the participants' task of generating data is devising enumeration procedures. These procedures or algorithms must ensure that the participants account for shortest routes without omission or repetition. Such algorithms can be considered efficient and robust. As participants generate data and based on numerical patterns they perceive in their data, they conjecture the underling mathematical structure of the problem task. Consequently, their data need to be reliable. The challenge of generating reliable data consumes approximately 18% of the time the participants spend in discursive interaction without researchers. To generate reliable data, the participants invent combinatorial algorithms.

The combinatorial algorithms that emerge from participants' activity can be divided into three categories: implied, hypothesized, and articulated. An implied algorithm is a procedure for counting routes about which a participant does not comment but with which he or she generates data. A hypothesized algorithm is one about which a participant discusses but does not explicitly implement, whereas an articulated algorithm is a counting procedure about which a participant discusses and with which a participant generates data.

Implied algorithms can be inferred from an examination of the visual, gestic information in the video record and inspection of the inscriptional data. Hypothesized and articulated algorithms are explicitly described by participants and also evident in the audio information of the video data as well as in the participants' inscriptions.

Both video and inscriptional data reveal implied combinatorial algorithms. An analysis of the order in which participants draw different shortest routes provides a window into their thinking about how to account for all possible routes without omission or repetition. After about five minutes of work on the problem task, while Romina and Michael are counting different shortest routes to the green pick-up point, recognizing that there are a lot of routes, Romina expresses her intention to develop a way of counting:

Vignette 12: [LS_1, 0:06:14 – 0:06:23]

130.	MICHAEL:	There's a lot.
131.	ROMINA:	Yes, I know. I'm trying to devise a- like a-
132.	JEFF:	The- the way to do it?
133.	ROMINA:	Yeah.

Afterward, Romina begins to implement an implied, though partially articulated, algorithm. As she draws routes from the taxi stand to the green pick-up point, (10,5), she enumerates them. The following are the 12 routes plus one unfinished route that she draws: [*EEEEEESSSSS, EEEESSSSSE*],[18] *EEESSSSSEE, EESSSSSEEE, ESSSSSEEEE, SSSSSEEEEE, EEESEESSSS, EESEEESSSS, ESEEEESSSS, EESSEEESSS, ESSEEEESSS, SSEEEEESSS, EEEESSE...* [LS_1, 0:06:31 – 0:06:57]. In the midst of drawing the last route, Romina stops and says,

[18] The routes in square brackets are ones not evident from the video record but ones that I infer that she draws based on inscriptional evidence and the pattern of routes that she is seen to draw from the video data.

"Ha...I already lost count" (turn 138). Nevertheless, from the notational pattern of the routes she draws, emerges an image of her algorithmic thinking about enumerating routes to the green pick-up point. She explains some of her thinking to Michael:

Vignette 13: [LS_1, 0:07:01 – 0:07:11]

140.	ROMINA:	//Well, I'm saying like if you go //all the way over. [Leaning over and pointing with her finger at the grid on Michael's problem sheet.]
...		
142.	ROMINA:	And then //you go all the way// over and leave only one space. [Romina points to Michael's grid and motions with her finger.]
143.	MICHAEL:	//Yeah. One, two, three- Yeah, one, two, three, four, five, six. Six going like that. [Outlining routes on his problem sheet.]

Another type of implied combinatorial algorithm is evident in the inscriptional data attributed to Brian. Nearly 45 minutes into the problem-solving session, Brian announces that he found 10 different shortest routes in a three-by-two sub-grid, representing the intersection point (5,3). Romina and Jeff have found only nine, and in response to Jeff's request, Brian writes his inscriptive data on the chalkboard so that he and his colleagues can compare them with theirs.

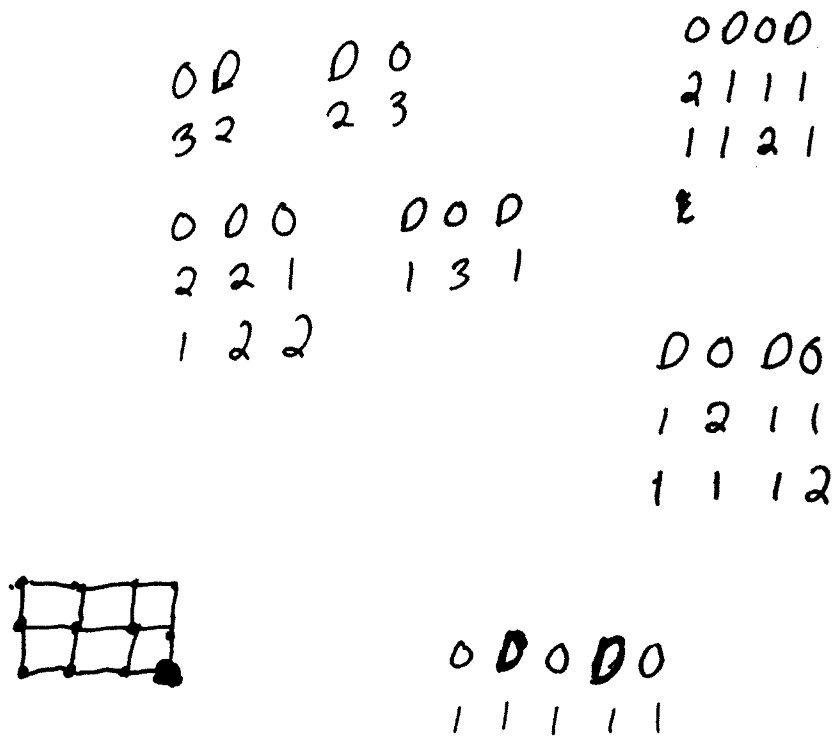

Figure 21. Brian's inscription indicating 10 different shortest routes in a three-by-two sub-grid, representing routes between the taxi stand and the intersection point (5,3).

Figure 21 contains Brian's inscription in which he presents 10 different shortest routes in a three-by-two sub-grid and is the basis for the presentation of routes that he writes on the chalkboard. From Figure 21, it is evident that Brian organizes his procedure for counting routes by the number of line segments of which routes are composed. He firstly considers routes that contain two line segments, first those routes that begin with a horizontal segment or over (*O*) and then those routes that begin with a vertical segment or down (*D*). He finds two different types of two-segment routes. One type begins with a horizontal segment—*O D*—and the other with a vertical segment—*D O*. Then his algorithm entails considering three-segment routes of which he finds two different types. In the first type—*O, D, O*—there are two possible routes; one begins with a horizontal segment two blocks long and with a horizontal one block in length. He finds just one route for the second three-segment type—*D, O, D*; it contains a route that begins with a one-block long vertical segment. Next, his combinatorial algorithm involves considering four-segment routes of which he finds two types, each containing two different routes. Lastly, Brian's counting procedure finds all five-segment routes. This type contains just one route. Traveling along this route resembles a staircase pattern. Brian's implied combinatorial algorithm allows him to retrieve and re-present the routes that he finds. Retrievability is a criterion for which Brian suggested that he and his colleagues strive (turn 71).

Implied combinatorial algorithms such as Brian's though not explicitly articulated are implemented. In contrast, the idea behind a hypothesized algorithm is advanced among the participants but for one reason or another fall short of being implemented. During the problem-solving session, Jeff expresses two hypothesized combinatorial algorithms. The first reveals his awareness of the binomial nature of the Taxicab Problem. The second entails his insight into the decomposition of comparatively large sub-grids into smaller ones whose results are known. Both of these hypothesized algorithms emerge in the context of interpretive interlocutions.

Jeff's discussion of his first hypothesized algorithm emerges a little more than 10 minutes into the problem-solving session. At this point, the participants are working to find the number of shortest routes from the taxi stand to the red pick-up point. Brian, Michael, and Romina all express doubt about the reliability of the data that they have thus far generated for this point. Jeff nudges Brian to engage him as an interlocutor and explains his idea:

Vignette 14: [LS_1, 0:12:17 – 0:13:20]

231.	JEFF:	All right, you- you're here. [Speaking to Brian, Jeff points with his black marker to an intersection point on his problem sheet.]
232.	BRIAN:	Uh hmm.
233.	JEFF:	You get to go over or you can go up. [From (5,1), moving his pen to the left one unit and back and then up one unit [SK_1, 0:11:09- 0:11:12].]
234.	BRIAN:	Mm hmm.
235.	JEFF:	So like here you can go over or up. [On the right side of the problem sheet, drawing a point and from it two lines, producing a binary tree.]
236.	ROMINA:	What are you doing?
237.	JEFF:	I don't know. I'm not doing anything. I'm just trying to think. [Returning to Brian.] And then you get to here. You can either go over or up again. And the same thing. But I don't know what that has to do with anything. My brain is like- just looking at this right now and going like- [Inaudible.] It's just not working. [Jeff waves his hand.]
238.	ROMINA:	But you know, I am- //I understand what you're doing-
239.	BRIAN:	//Just look at the lines and see where you're getting five.
240.	ROMINA:	-but like for this one you know what sucks with this one, is because if you're there you have either one of two choices.
241.	JEFF:	Mm hmm.
242.	ROMINA:	When you get here you have one or two choices, you know, this just doesn't-
243.	JEFF:	Well, yeah, you're here, you can either go there or there. You get here- [Tracing routes on the grid of his problem sheet.]
244.	ROMINA:	Yeah.
245.	JEFF:	-you can go there or there. But if you're here, you're only going to go down. [Pointing at an intersection point on the grid of his problem sheet.]
246.	ROMINA:	Yeah. //That- that- exactly.
247.	JEFF:	//Because you're going out of your way.
248.	ROMINA:	That's exactly what I was doing.
249.	BRIAN:	See this is- //this is what I was thinking of.
250.	JEFF:	//Then you're here and you're only going down or over. Again, this is just down and you can just follow all the routes to the end point- I don't know. [Pointing to the binary tree that he drew on the right side of his problem sheet. (See Figure 22.)]

Jeff tells Brian about his observation concerning the two choices one has traveling from an intersection point to the taxi stand: one can travel one block either over (west) or up (north). Specifically, he points to the intersection point $(5,1)$, on the grid of his problem sheet and says that one can go either over to $(4,0)$ or up to $(4,1)$. He then draws a binary tree to illustrate further his observation (see Figure 22).

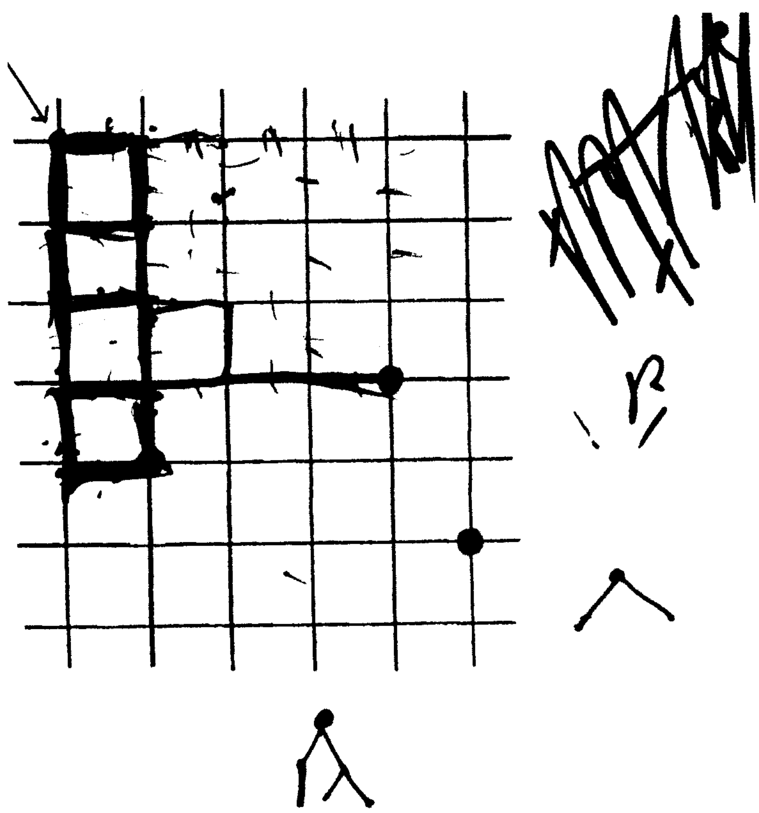

Figure 22. Jeff's inscription on his problem sheet of binary trees presenting his insight into the binomial aspect of the problem task.

Romina points to Jeff's grid and says that to go to the blue pick-up point from points along the edge of the grid one has two choices. Jeff explains that to reach the blue pick-up point from particular intersection points, pointing to the intersection $(1,1)$, one can only go down since the other choice would take one out of the way. Continuing to discuss reaching the blue point, he says that from the point (1,0) one has two choices but that from the intersection (2,1) to one can only go down. Returning to his binary tree, Jeff states that one could "follow all the routes to the endpoint," hypothesizing a procedure for counting routes to a given intersection point such as the blue pick-up point.

Jeff hypothesizes a second combinatorial algorithm. It occurs almost fifty-four minutes into their problem-solving session during a phase in which the participants are investigating the number of shortest routes in a four-by-four sub-grid. Both Michael and Brian are counting routes, and Jeff motions to Brian that he would like his attention. Brian continues to work quietly notating the routes he is counting. Seven seconds after Jeff's request to Brian, Michael agrees to listen. Vignette 15 presents the conversation:

Vignette 15: [LS_2, 0:13:07 – 0:14:11]

843.	JEFF:	All right. All right, what if we even went- let me know when you're done. All right. Because there's an easier way to- All right, Listen to me for one sec.
844.	MICHAEL:	Go ahead.
845.	JEFF:	All right. If- all right. Say in a situation where it's like, uh a two by four.[19] [Drawing a four-by-two sub-grid on 1-centimeter paper.]
846.	MICHAEL:	Uh hum.
847.	JEFF:	All right. If we know that in a four-by-four [really meaning a two-by-two] it's six [shortest routes] then if you figure out all the ways to get to the beginning parts of this, this would all just be six different ways to get from here to here. So you figure out all the ways to get there and you could just add six- you know. [Subdividing the two-by-four sub-grid into two two-by-two sub-grids.]
848.	MICHAEL:	If you have the two, you could find out how many ways it's to get to here and add that where every two is. [Leaning over to Jeff's paper and pointing.]
849.	JEFF:	You know what I'm saying? So like from- from-
	...	

[19] The sub-grid that Jeff draws is what participants have been referring to as a four-by-two sub-grid.

853.	JEFF:	All right. 'Cause from there to there you have six different ways. And then, from there, there's one way. To there there's one way and from there- //
854.	BRIAN:	//Haaa. Tell me when you're done.
855.	JEFF:	Sure. One- two- there's three ways. Um-

In this vignette of an interpretive interlocution, Jeff presents his idea of how to determine the number of shortest routes in a comparatively large sub-grid. His algorithm involves subdividing a sub-grid into smaller sub-grids for which the number of different shortest routes is known. Then he proposes an additive process to calculate the number of shortest routes in the original sub-grid. He offers Michael an example using a four-by-two[20] sub-grid partitioned into 2 two-by-two sub-grids. A four-by-two and a two-by-two sub-grid respectively represent the dimensions of rectangular sub-grids from the taxi stand to the intersection points $(6,4)$ and $(4,2)$. To their satisfaction, the participants have already determined that in a two-by-two sub-grid there are six different shortest routes (see Figure 23). Jeff explains that in the second two-by-two sub-grid from the upper left corner of to its lower right corner there are six different shortest routes. He further explains to tell Michael that, therefore, one only needs to determine how many ways it takes to get from the upper left corner of the first two-by-two sub-grid to reach the second two-by-two sub-grid and then add up the possibilities. In response, Michael contributes his understanding of Jeff's additive algorithm. Jeff spends about another 30 seconds exploring his idea.

[20] Jeff actually says "two-by-four," however; the sub-grid that he draws is what the participants have been calling a four-by-two sub-grid. Here I am use their usual label for the sub-grid.

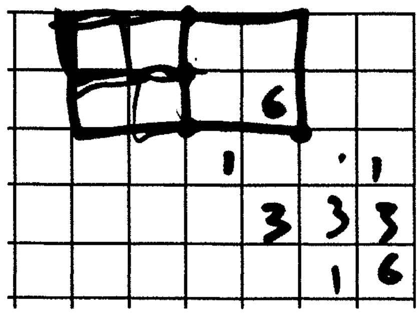

Figure 23. Jeff's inscriptional example of a hypothesized combinatorial algorithm that in part involves partitioning, for example, a four-by-two sub-grid into 2 two-by-two sub-grids.

The two combinatorial algorithms that Jeff hypothesizes are evidenced in the video and inscriptional data. Others participants may have also considered combinatorial algorithms that they did not ultimately implement for one reason or another. However, the video and inscriptional data do not provide clear evidence of other hypothesized combinatorial algorithms. Nevertheless, the two combinatorial algorithms that Jeff hypothesizes form part of the set of mathematical ideas that the participants develop in the problem-solving session.

Beyond hypothesizing combinatorial algorithms but not employing them, the participants do discuss and implement specific counting procedures. These articulated algorithms emerge in the participants' negotiatory conversational exchanges. Vignette 16 is an instance of a negotiatory interlocution about a combinatorial algorithm implemented to determine the number of shortest routes in a two-by-three sub-grid.

Vignette 16: [LS_1, 0:21:14 – 0:22:07]

386.	JEFF:	Seven, eight- I got more than that. All right, wait. We got to go through this, and you got to watch.
...		
391.	JEFF:	Yeah, it's like the hot seat. All right, check it out. One- [On 1-centimeter grid paper, drawing a route in a two-by-three sub-grid.]
392.	ROMINA:	Mm hmm.

393.	JEFF:	All right. There's only one you can go by going two down. I'm trying to like figure out ways to like cross them out. You know what I'm saying? And then going one down, you can go one, two, three- There's no other ways to go. [Drawing more routes on his 1-centimeter grid paper.]
394.	ROMINA:	Mm hmm.
395.	JEFF:	What about like that? Four?
396.	ROMINA:	Mm hmm. Mm hmm.
397.	JEFF:	And then, five, six, seven, eight- [Counting the routes as he draws them.]
398.	ROMINA:	You already did that one.
...		
400.	JEFF:	Which one?

In this vignette, Jeff articulates explicitly his thinking as he implements a procedure for counting routes. He first considers all possible routes obtainable by first traveling two blocks down (south) and finds that there is "only one you can go by going two down." Next, he considers all possible routes that require one first to travel one block down and finds three; and then he reflects on all possible routes that begin with traveling east and finds four. These are the routes Jeff has found: *SSEEE, SEEES, SESEE, SEESE, SEEES, ESSEE, EESSE, EEESS,* and *ESEES* [SK_1, 0:20:23 – 0:20:53]. After Jeff draws the last route, Romina says, "You already did that one." However, since he has drawn all of these routes in a single two-by-three sub-grid, both he and Romina are actually unable determine whether or not he has repeated a route. Indeed, in response to Romina's assertion, Jeff asks, "Which one?"

This vignette and four other critical events about articulated combinatorial algorithms all arise in the context of negotiatory interlocution and form a pivotal strand. The theme of the strand is 'building robust combinatorial algorithms.' Besides attending to issues of omission and repetition, these algorithms involve ideas of controlling for variables and associating a route to its opposite. An example of the latter is the following critical event represented in vignette 17.

Vignette 17: [LS_1, 0:24:11 – 0:25:08]

434.	JEFF:	You just go make two by twos. [Drawing three vertical lines on his centimeter paper to create two-by-two sub-grids.]
435.	ROMINA:	Mm hmm.
436.	JEFF:	You could go- [Drawing two-by-two sub-grids.]
437.	ROMINA:	Yeah, make at least six at the moment.
438.	JEFF:	All right. You can go this way. [Drawing 1 two-down route.]
439.	ROMINA:	Yeah that's one.
440.	JEFF:	That's one. Now that's all the ways you can go- [Drawing a route.]
441.	ROMINA:	Yeah.
442.	JEFF:	-by two down. So then you can go like this…
443.	ROMINA:	Two.
444.	JEFF:	You can go like this. [Drawing 2 one-down routes.]
445.	ROMINA:	Three.
446.	JEFF:	Is there any other ways to go by going down? No.
447.	ROMINA:	Okay.
448.	JEFF:	All right. So then you could- you could go like that? [Drawing 2 one-over routes.]
449.	ROMINA:	Mm hmm.
450.	JEFF:	You could go like that. [Drawing 1 two-over route.]
451.	ROMINA:	Mm hmm. Or you go all the way top to bottom.
452.	JEFF:	There's nothing else to do? Right? Now that would be the opposite of that one. That would be the opposite of that one and that would be the opposite of that one. //They're all covered. [Pointing to pairs of routes on the grid with a pen.]
453.	ROMINA:	//So we got six. Good. Good thing we did that over again. (See Figure 24.)

In this vignette, Romina and Jeff together investigate the number of shortest routes in a two-by-two sub-grid, representing the different shortest routes between the taxi stand and the intersection point $(4,2)$. Jeff uses the heuristic of drawing several two-by-two sub-grids so that each route that they find can be presented in a separate sub-grid. Romina suggests that he draws six such sub-grids. Jeff controls for variables by considering in turn all possible routes two down, one down, one across, and two across (see Figure 24). He speaks as he writes and engages Romina, leaving open the possibility for collective modification of his counting procedure. After drawing the routes—*SSEE, SEES, SESE, ESSE, ESES,* and *EESS* [SK_1, 0:23:12 - 0:23:55]—he notes which route is the opposite of which other route: *SSEE* is the opposite of *EESS, ESSE* is the opposite of *SEES,* and *SESE* is the opposite *ESES.* In so doing, he underscores relationships among the routes and uses the notion of opposite to check whether he has accounted for all possible routes. Romina's participation, concurrence, and subsequent application of the algorithm indicate that they co-construct a method of counting routes.

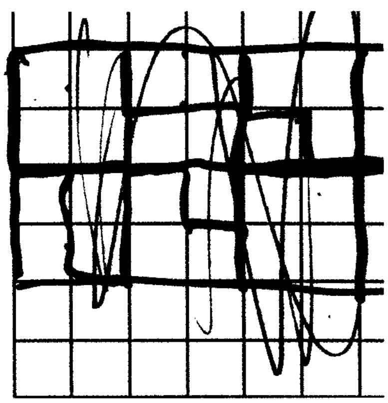

Figure 24. Jeff's inscription for six routes, drawn with a red marker, in two-by-two sub-grids, representing the routes to intersection point $(4,2)$ in the rectangular space between it and the taxi stand of the problem sheet. The routes are crossed out because he and Romina subsequently use the drawn horizontal lines to create several three-by-two sub-grids.

5.6 Building conjectures, counterexamples, and justifications

During the problem-solving session, not only do the participants construct combinatorial algorithms but also they build conjectures and for them provide either counterexamples or justifications. The moments when they do conjecture, present counterexamples, and justify represent critical events in which they identify relations among objects or dynamics linking different relations. As they build their justifications, they involve themselves in negotiatory interlocution. While resolving the problem task, the participants make four conjectures. For two of the conjectures they build justifications and for the other two they present counterexamples. Their four critical events in which they pose conjectures belong to a pivotal strand whose theme is 'the underling mathematical structure of the Taxicab Problem.'

The first conjecture that the participants make and justify concerns the equality of length of efficient routes. Romina observes that efficient routes to a pick-up point traverse the same number of blocks (turns 20-26), where, for Romina, though the term is not hers, efficient means that the route does not go past the pick-up point (turn 22). Moments later, Jeff asks, "why is it the same every time?" (turn 74). The other participants understand his reference to "it" to mean the set of efficient routes to a pick-up point. In turns 75 to 108, Michael with assistance from Jeff explains that to reach a pick-up point, the shortest distances will always require one to move a fixed number of units down (south) and a fixed number across (east) and further observes that within the taxicab grid one cannot travel diagonally. Brian reminds the others that only going forward will produce a shortest route. Michael generalizes his awarenesses to all routes. Finally, in turn 111, Jeff signals satisfaction that the conjecture has been justified.

The participants' next two conjectures propose relationships between pick-up points and the number of shortest routes between them and the taxi stand. The participants later present counterexamples, demonstrating the falsity of these conjectures. The first "false" conjecture posits that the number of shortest routes to a pick-up point equals the length of shortest routes to it. This idea emerges from an observation of Brian. In vignette 18, he notes that the number of shortest routes from the taxi stand to the blue pick-up point is five and speculates that the same relationship might be true of the number of shortest routes to the red point.

Vignette 18 [LS_1, 0:05:42 – 0:05:55]

112.	BRIAN:	Are there seven possibilities, though? You know how like blue was five? There's five possibilities but-
113.	JEFF:	Ah, so-
114.	BRIAN:	You know how it's only like five spaces. Like one, two, three, four, five. [Pointing at the grid on his problem sheet.]
115.	ROMINA:	Yeah, so if it goes more.
116.	BRIAN:	Is there seven for blue, I mean red?
117.	JEFF:	Well, check it out.

Based on their empirical evidence of counting routes between the taxi stand and pick-up points, the participants lay this conjecture to rest. The first hint of documental evidence against it comes from Romina and Michael while they collaborate to find the shortest routes to the green pick-up point. At turns 130 and 131, though the length of shortest routes to the green point is just 10, after spending some time counting routes, they both voice recognition that there are "a lot." Moreover, immediately after affirming that there are many routes, though an indeterminate number of them, Romina says that she needs to devise "like a" and Jeff finishes, "way to do it;" in other words, a combinatorial algorithm (turns 131-133).

The second "false" conjecture that the participants submit again relates the number of shortest routes to a pick-up point to a numerical property of the point and the taxi stand. Moreover, this conjecture has a geometric aspect. Specifically, they surmise that the number of shortest routes between the taxi stand and a pick-up point equals one-half the number of line segments in the rectangular space determined by the two locations. The participants detail their conjecture in vignette 19, taken from three portions of the transcript.

Vignette 19 [LS_1, 0:09:35 – 0:09:43]

190. MICHAEL: There's like- there's ten line- there's ten like lines in here and the answer was five. So I'm waiting for them That's like a half or something.

... [LS_1, 0:11:28 – 0:11:48]

227. ROMINA: //No, there's twenty- No it'd be twelve. Wouldn't it be twelve?
228. MICHAEL: I don't know. How- how much is this?
229. ROMINA: There's ten lines and there's five ways. So if there's //twenty-four lines there would be twelve ways. [Pointing to Michael's problem sheet.]
230. MICHAEL: //but there's one, two, three, four- It's twelve, yeah. We're, we're guessing twelve but that's probably not it. I doubt it. [Counting routes on grid of problem sheet. Romina leaning over her problem sheet and outlining routes.]

... [LS_1, 0:13:57 – 0:14:20]

272. ROMINA: No, I was just saying like if- that wouldn't work with our theory.
273. JEFF: What theory is that?
274. MICHAEL: Divide //it by two.
275. ROMINA: //Divide it by two. It's like a highly- it was like a-
276. JEFF: Was it- like what divided by two? All the- add them all up //[Inaudible]. [Pointing at paper]
277. ROMINA: //Because there's ten lines- ten lines like that are all within this rectangle. [Pointing at paper with pen]
278. JEFF: All right.
279. ROMINA: There's five ways to get to it. So if there are twenty-four lines there would be twelve different lines to get to it. But, it's hard to prove. [Romina points to her grid with a pen.]

This second "false" conjecture materializes after the first one and substitutes it since the first one is earlier demonstrated to be flawed. This new conjecture sways the thinking of Romina and Michael, though they sway between believing it and not. Immediately after Romina states, "it's hard to prove," Brian tosses onto the table a counterexample that carefully examines the number of line segments contained in the rectangular region for which the taxi stand and the blue pick-up point are vertices.

Vignette 20 [LS_1, 0:14:21 – 0:14:50]

280.	BRIAN:	Actually, this whole thing, if you count the middle lines there's thirteen. [Referring rectangular region between the to the blue pick-up point and the taxi stand.]
281.	JEFF:	There is. That's why I- //as soon as I got to thirteen I stopped working because there's none- it's prime.
282.	MICHAEL:	//[Inaudible], right?
283.	ROMINA:	One, two, three, //four, five-
284.	BRIAN:	//It's four on the sides, eight, nine, ten, eleven, twelve, thirteen. [Using his two hands to show routes in the air.]
285.	ROMINA:	//-six, seven, eight, nine, ten, eleven, twelve- There is thirteen.
286.	MICHAEL:	Thirteen what?
287.	JEFF:	Lines //over here.
288.	ROMINA:	//Lines.
289.	JEFF:	That's why I- I threw that out. I wrote- Oh, that's a thirteen but I was like, oh man, prime numbers. [Putting his head in his arms.] No good.
290.	ROMINA:	//thirteen.
291.	JEFF:	There's like no way it could work with a prime number- like you can't even like make something up.

Not only is the number of line segments in the rectangular space defined by the taxi stand and the blue point not 10—as the conjecture asserts it should be—but also and worst, in Jeff's opinion, the number of line segments within it is prime. As the participants' three conjectures indicate, attending to numerical patterns provides grist for the participants' conjecture making mill. Their fourth conjecture arises from noticing a special, known pattern in the data they generate. Throughout the approximately forty-seven minutes of the problem-solving session, the participants consider and discuss other patterns they see in their data. Analyses of the video and inscriptional data reveal an evolution of increased confidence with which the participants ascribe to the data that they generate. They maintain a record of their findings on a transparency copy of a

Cuisenaire 1-centimeter grid. As they obtain a count of the number of shortest routes between the taxi stand and an intersection point, they enter the datum into the corresponding square of the taxicab grid. Figure 25 contains information from their 1-centimeter-grid transparency from two different, adjacent moments in their work.

2	3	4	5	
3	6	9	12	
4	9	15		

A

2	3	4	5	
3	6	10	12	
4	10	15		
5				

B

Figures 25 A and B. Romina and Jeff's taxicab grids. Figure 25A presents their data before correcting the entry for intersection points (5,3) and then (5,2). Figure 25B presents their data after comparing their shortest routes to intersection point (5,3) with Brian's and noticing that they missed one of its shortest routes.

A heuristic of the participants involves parceling out the same task so as to record and compare the resulting data. To generate data, the participants have three inscriptional systems, one used by Romina and Jeff, a second by Michael, and a third by Brian. After about forty minutes of work on the problem task, pointing to the nine in square (5,3) of Figure 25A, Jeff voices concern about the sense it makes. Romina asks Brian to use his algorithm to count the number of shortest routes to intersection point (5,3). The participants wish to coordinate the Romina and Jeff's system with Brian's so as to check the reliability of their implemented combinatorial algorithms. Brian reports that he has found 10 routes for the three-by-three sub-grid, representing the sub-grid for the intersection point (5,3). He writes his routes on the classroom chalkboard, and noticing that she and Jeff neglected to account for a route, Romina says, "we don't have one, one, one, one, one, that one....So that nine does equal ten." Here she starts to count the number of blocks one traverses east and south along a route from the taxi stand to the intersection point (5,3) whose shape resembles a staircase. She notices that this route is one that she and Jeff do not have and as such modifies two entries in Figure 25A for squares (5,3) and (5,2). The new numerical array is given in Figure 25B. After Romina's action results in the new array, then the following conversational exchanges ensues:

Vignette 21 [LS_2, 0:07:09 – 0:07:50]

778.	ROMINA:	All right. It's, um, - it's Pascal's triangle. [Looking at the numerical array of the 1-centimeter-grid transparency.]
779.	MICHAEL:	What is that? Two by three? [Looking and pointing to Brian's inscription on the classroom chalkboard.]
780.	JEFF:	It is?
781.	ROMINA:	Yeah.
782.	JEFF:	Let me see.
783.	ROMINA:	All right. Yeah it is.
784.	MICHAEL:	What?
785.	ROMINA:	It's Pascal's triangle.
786.	MICHAEL:	Two, three-
787.	ROMINA:	No, it's not. It doesn't work out.
788.	JEFF:	See look at- Here, Mike-
789.	ROMINA:	Because twelve that doesn't-
790.	JEFF:	Mike look- just look at it in this thing. You got the 6 and the 4 and the 6 are the 10. That should be a 15- / / that's should be a 20- [Pointing to the 1-centimeter transparency grid that is in front of Romina.]
791.	ROMINA:	/ / But that's not a 15. That is a twelve because he even got the 12.
792.	JEFF:	Well- that should- that should be a 20 right there. [Pointing to the square (6,3) on the transparency that contains the datum 15.]

This negotiatory exchange establishes a new agenda for the participants' collective action. Verifying whether Pascal's triangle indexes the underling mathematical structure of the problem task is now a new mathematical issue around which the participants focus their attention. Romina's observation of the numerical pattern in the data in the grid of Figure 25B, formulated in her statement, "All right. It's, um, - it's Pascal's triangle," proposes to her colleagues the conjecture that pattern in the numerical array of shortest routes from the taxi stand to neighboring intersection points is Pascal's triangle. His colleagues engage the idea and set out to verify the conjecture. In turns immediately after vignette 21, Brian agrees to recalculate, using his algorithmic method of indicating the number of blocks traversed east and traversed south for each shortest route in a four-by-two, representing the intersection point (6,4) of the taxicab grid. Specifically, his combinatorial algorithm consists of considering routes whose geometric shape contains two different segments, then 3 different segments and so on. With this act, the group's objective is to settle the problematic issue of whether the datum for that point should be 12 or 15. As Jeff says to Brian, "you do the four by two, and it should put us, uh, in business" (turn 800).

Vignettes 18 to 21 are critical events that form a pivotal strand. Each critical event illustrates the participants' attention to relations among objects. The objects of their attention are segments, intersection points in the taxicab plane, routes, cardinal numbers, and *n*-by-*m* sub-grids. Among these objects they notice relations such as an efficient route will be a shortest route, there can exist more than one shortest route, and the number of blocks traversed by efficient routes between the taxi stand and intersection points are the same. There is another relation that they notice that is implicit in an action of Romina.

Brian's datum of 10 is for the intersection point $(5,3)$. When Romina modifies the numerical array of Figure 25A, she also places 10 in the square for $(5,2)$. This act evidences her implicit awareness of a symmetrical property of the numerical pattern of shortest routes, which may also contribute to her seeing the pattern in the array of numbers as suggestive of Pascal's triangle. Vignette 21 not only represents the critical event in which the participants give focused attention relating their numerical array to Pascal's triangle but also marks a shift in their investigation and, for this reason, is a watershed critical event. From this event onward, they challenge themselves to justify their new conjecture both with reference to attributes of the problem task itself and with propositions that map the problem task onto other problem tasks on which they have worked in the past.

5.7 *Building and clarifying isomorphisms*

Mapping a problem task or, more generally, a mathematical system onto another in such a way that essential mathematical structures are preserved is a central function of an isomorphism. It is a signifier, in the Saussurian sense,[21] that indexes a signified that exists beyond physical perception and in this sense is an abstraction. An isomorphism is a proposition about dynamical links that in a non-physical manner establishes a one-to-one correspondence between, on the one hand, objects and relations or actions in one system and, on the other hand, objects and relations of another system in such a way that an action on objects of one system maps to an analogous action on the corresponding objects in the

[21] See chapter 3, section 2.2, for a discussion of Saussure's ideas of signifier and signified as components of a sign.

other system. To use the language of the categories of critical events proposed in chapter 3, section 2.3, an isomorphism is a proposition about the dynamics linking different relations and the objects on which they act. What is required to formulate an isomorphism? A prerequisite to articulating or building a proposition that indicates an isomorphism is attending to particular details of objects and relations among the objects within each system to determine whether dynamical links can be formulated between the systems.

Knowing to attend to potential dynamical links between the relations and objects of two systems is a heuristic awareness. It is akin to the heuristic of attending to numerical patterns, which can lead to the formulation of a conjecture. When the participants perceive dynamical links, they may articulate propositions that yield an isomorphism. In section 5.4, when discussing heuristic actions and an interpretative interlocution, an example was analyzed (vignette 6) of the participants' use of the problem-solving strategy of contemplating whether there exist dynamical links between two systems. In fact, vignette 6 was the moment in which the participants initiated their public contemplation of dynamical links between the Towers and Taxicab Problems.

Specifically in that vignette, the participants manifest embryonic thinking about an isomorphism. In the midst of her and Michael's counting of shortest routes to the green pick-up point, Romina announces her perception that the Towers Problem might relate to their current problem task. In the circumstance of her and Michael's work, she asks questions about a state of affairs that frames her perception and presumably Michael's as well as even Jeff and Brian's. She wonders aloud: "can't we do towers on this" (turn 159)..[22] Her public query catalyzes a negotiatory interlocution among her, Michael, and Jeff. Jeff, responding immediately to Romina, says, "that's what I'm saying," (turn 160) and invites her to think with him about the dyadic choice that one has at intersections of the taxicab grid. Furthermore, he wonders whether one can find the number of shortest routes to a pick-up point by adding up the different choices one encounters in route to the point (turn 162).

After Jeff remarks on the similarity in the Towers and Taxicab Problems' binomial aspect, Romina discusses another correspondence or dynamical link between the problems. She speculates, "if there's ten, ten could be like the number of blocks we have in the tower" (turn 169), comparing the length of a shortest route to the green pick-up point to the height of a tower. She lays a propositional foundation for an eventual formulation of an isomorphism by associating object categories between the two systems: numbers of blocks in the taxicab plane to numbers of blocks or Unifix cubes of the Towers Problems or, to say it another way, the length (in blocks) of a shortest route to an intersection point to the height (in blocks) of a tower.

Romina's query concerning the application of towers to the present problem task prompts Michael's engagement with the idea, as well. As if advising his colleagues and himself, he responds in part by saying, "think of the possibilities of doing this and then doing that" (turn 180). While uttering these words, he points at an intersection; from that intersection gestures first

[22] For Romina and other participants in the Rutgers longitudinal research project in Kenilworth, New Jersey, this comment is pregnant with mathematical and heuristic meaning that from their constructed experiences with tasks in the combinatorial and probability strands of the project (Kiczek, 2000; Martino, 1992; Muter, 1999).

downward ("doing this"), returns the to point, and then motions rightward ("doing that"). Similar to Jeff's words and gestures, Michael's actions also acknowledge cognitively and corporally the binomial aspect of the problem task. He, Jeff, and Romina have put into circulation the prospect of as well as insights for building an isomorphism between the Taxicab and Towers Problems. Romina raised the question and both Jeff and Michael provided arguments for the sensibleness of pursuing Romina's idea of associating the two problems.

The prospect and work of building such an isomorphism reemerges several more times in the participants' interlocution. With each reemergence, the participants further elaborate their insights and advance more isomorphic propositions. Eventually the building of isomorphisms dominates their conversational exchanges. Approximately thirty-five minutes after Romina first broached the possibility of relating attributes of the Towers Problem to the problem at hand, the participants reengage with the idea, as illustrated in vignette 22.

Vignette 22 [LS_2, 0:03:45 – 0:04:51]

738.	ROMINA:	Couldn't we just do something like in towers where like lines over are like the color and the lines down are the, um, number of blocks?
739.	JEFF:	All right. And that would?
740.	ROMINA:	Because, okay, lines over- because what is it- the number of blocks to the number of colors?
741.	JEFF:	I don't know what you're- what- what's that?
742.	ROMINA:	Two to the *n*. Two is the amount of blocks or the colors?
743.	MICHAEL:	For what? Like towers on them?
744.	ROMINA:	Yeah.
745.	JEFF:	Colors. *n* is the number of blocks. I think. I don't know. I'm not sure.
746.	MICHAEL:	Well you figure a block has this- you got two- two ten over like this. Or two colors actually. I think it's, uh, the colors and *n* is the blocks.

747. ROMINA: Color two- / /right. [Writing the words "color" and "blocks" on a piece of paper.]
748. JEFF: / /Same thing.
749. ROMINA: We have one color- nah; it doesn't work for the first one.
750. ROMINA: Scratch that idea. [Crossing out the words on her paper.]

In vignette 22, despite Romina's closing imperative, "scratch that idea," new propositional pieces emerge for building an isomorphism between the Taxicab Problem and the Towers Problem. In the vignette's opening question, Romina speculates that between the two problems one can relate "like lines over" to "like the color" and then "the lines down" to the "number of blocks." What is essential here is Romina's apparent awareness that each of the two different directions of travel in the Taxicab Problem needs to be associated with different objects in the Towers Problem. Also in this vignette, through a negotiatory interlocution, she clarifies with Michael and Jeff from the perspective of the Towers Problem the meaning of the entities 2 and n in the expression 2^n ("Two to the n. Two is the amount of blocks or the colors?"). Jeff retrieves, "Colors. n is the number of blocks," but then says, "I think. I don't know. I'm not sure." While Michael adds, referring to the meaning of two and then n, "I think it's, uh, the colors and n is the blocks." In essence, they rebuild and remind themselves of the meaning of the exponential expression (2 to the n) in the environment of the Towers Problem.

On other occasions, the participants rebuild their understanding of the Towers Problem, a requisite action for building an isomorphism between it and the Taxicab Problem. These discussions occur as they generate and verify data on the number of shortest routes between the taxi stand and intersection points in close proximity to it. In Figure 26, the green numerals inside the transparency grid represent the empirical data that the participants have generated.

Romina and her colleagues have recorded the data of shortest routes in the corresponding squares of a transparency of a 1-centimeter grid paper. Given the spatial orientation of their data array (see Figure 26), she says that she

A B

Figure 26. Video (A) and hardcopy (B) of participants' data written on a transparency of a Cuisenaire 1-centimeter grid paper. In green, empirical data of shortest routes between the taxi stand and nearby intersection points. Jeff placed the ones in blue to augment the appearance of the numerical array as Pascal's triangle. From the participant perspective, to the left of Jeff's numbers, Romina wrote in green the numbers 1, 2, and 3 to indicate the rows of the triangular array.

finds it difficult to see Pascal's triangle[23] and rewrites the data (see Figure 27) in an orientation from which, for her, Pascal's triangle is more easily perceived (turn 1168).[24] Earlier (turn 963), as she began to transcribe their data array,

[23] In an earlier remark to Jeff (turn 963), Romina notes that even though they perceive that their data array to be like that of Pascal's triangle, "it's not like in the blocks, is it?" I infer from this that she sees a slight difference between their array of data and the numerical pattern they derived from the problem of block towers.

[24] The orientation of their numerical array resembles the orientation that the seventeenth century French mathematician Blaise Pascal (1623-1662) gave to what he called an arithmetical triangle,

rotated 45 degrees, from the transparency to paper, she remarked to Jeff that their numerical triangle differs from the one of the towers problem. What she could have meant is that from her perspective their numerical array did not appear to contain the row consisting of two 1s, the first row of Pascal's triangle, if one considers to the one above it as the zeroth row (see Figure 27).

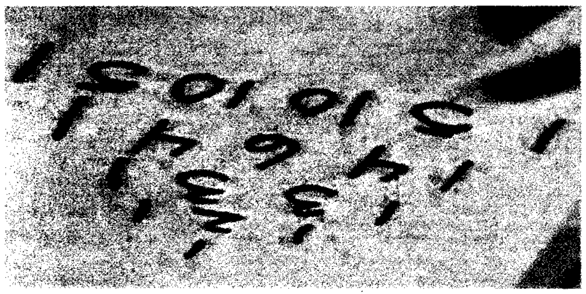

Figure 27. Five lines of Pascal's triangle transcribed onto paper by Romina from data recorded on transparency (see Figure 26).

With this data and with their conviction that Pascal's triangle provides the underling mathematical structure of the Taxicab Problem and that the Towers Problem relates to the present problem task, they use their heuristic of processing their findings with researchers to discover further their ideas and to see *where they lead themselves* (turns 977, 1060, 1062). Following the participants' assertion that the numerical data for shortest routes corresponds to Pascal's triangle,

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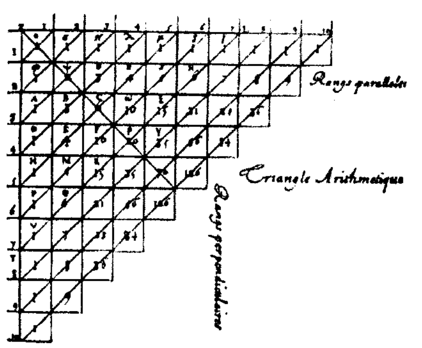

Pascal's original Arithmetical Triangle, from the *Traité du triangle arithmétique* (1665)

given here: . This facsimile is from Pascal's book, *Traité du triangle arithmétique* (*Treatise on the Arithmetical Triangle*) and appears in Edwards (1987, p. ii) as well as Katz (1998, p. 453). The three phrases, from top to bottom, translate as "parallel rows," "arithmetical triangle," and "perpendicular rows." It is conceivable that Pascal conceptualized these arithmetical numbers as entries in an orthogonal grid.

Researcher 1 inquires whether they can explain their data and its relation to the arithmetic triangle of Pascal (turns 1155, 1164, and 1166). In a negotiatory interlocution, Romina and Michael build onto the propositional foundation that the participants have already constructed and further elaborate an isomorphism between the Towers and Taxicab Problems. In vignette 23, Romina suggests labeling a unit of horizontal distance on the taxicab grid *A* and *B* for a unit of vertical distance. She notes that the three shortest routes to the intersection point $(3,2)$ involve two *A*s and one *B*, two unit distances in the horizontal direction and one in the vertical direction. This notational device soon thereafter is used to indicate a correspondence with the Towers Problem. Pointing to a three in the numerical array in Figure 27, she says, "for three that means there's two *A* color and one *B* color, so here it's two across, one down." Here she is using an earlier insight: each of the two different directions of travel in the Taxicab Problem needs to be associated with different objects in the Towers Problem (see vignette 22). This time, she identifies one unit of horizontal distance with one Unifix cube of color *A* and one unit of vertical distance with one Unifix cube of color *B*.

Vignette 23[25] [LS_2, 0:34:18 – 0:36:24]

1196. ROMINA:	That's what it goes one, two, three, four? Because then- okay for this one for the three. If we name all the ones going horizontal- *A*s and ones going down same with *B*. And this would be with two *A*s and one *B* there's three and then there's two *B*s with one *A*, three. [Pointing with a green marker at the intersections points $(3,2)$ and $(3,1)$ on the transparency grid.] And for this one remember like two *A*s two *B*s- / /six. [Now pointing to the intersections point $(4,2)$ and on the transparency grid.]
1197. MICHAEL:	/ /You could say, um-

[25] Approximately, one minute before the time of this vignette, Jeff had to leave the problem-solving session.

1198. ROMINA: Do you understand what I'm saying?

1199. MICHAEL: Like yeah, these are like this row is everything with perimeter two. I mean I half the perimeter, like. [Pointing with his marker to numbers on the transparency grid.]

1200. ROMINA: //Well no I'm saying so to get that-

1201. MICHAEL: //In order to get to that point you have to go over one and down, uh, one or down one and over one. Just like that row. Everything in this row, over two and down two and over one.

1202. ROMINA: Yeah but like I'm just saying like if she were to pick anything like right there we could say it's like eight-eight *A*s and like six *B*s. [Tracing a rectangle on the transparency grid.] You know like- and then we could tell you where you it is in this one. [Pointing to the redrawn Pascal's triangle on the piece of paper.]

1203. MICHAEL: Well you could say all- everything in this row, the shortest route is two. Everything in this row shortest route is three. This one shortest route is four.

1204. ROMINA: Yeah.

1205. MICHAEL: The shortest route is five, six and so on. So that's how you could, you know, name them. This is row six because it has everything in the row has shortest route of six. [Pointing with a marker to diagonals of numbers on the transparency grid.]

1206. ROMINA: No, I understand. I'm just saying like-

1207. MICHAEL: There's a, you know-

1208. ROMINA: To get it-

1209. MICHAEL: //To- to say it like, oh I'll pick this block-

1210. ROMINA: //Because isn't that how- isn't that how we get like these? Like doesn't the two- there's- that I mean, that's one- that means it's one of *A* color, one of *B* color. [Pointing to the 2 on the redrawn triangle on paper.] Here's one- it's either one- either way you go. It's one of across and one down. [Pointing to a number on the transparency grid and motions with her pen to go across and down.] And for three that means there's two *A* color and one *B* color [pointing to the 3 on the redrawn triangle], so here it's two across, one down or the other way [tracing across and down on the transparency grid] you can get three is //two down- [Pointing to the grid.]

1211. MICHAEL: //You mean like one *A* color and two-

1212. ROMINA: Yeah.

1213. MICHAEL: This is one-

1214. ROMINA: Like two blues, one red. Two across, one down or this is two reds, one blue, two down, one across. And that's how we would get the Pascal's triangle. [Pointing to numbers on the redrawn grid and transparency grid.]

Furthering the building of their isomorphism, Michael offers another propositional foundation. Pointing at their data on the transparency grid and referring to its diagonals as rows, Michael notes that each row of the data refers to the number of shortest routes to particular points of a particular length. While pointing to numbers in the linear array—1 4 6 4 1—on their transparency, he observes that each refers to an intersection point whose "shortest route is four." Moreover, he remarks that one could name a diagonal by, for example, "six" since "everything [each intersection point] in the row [diagonal] has shortest route of six." In terms of an isomorphism, Michael's observation points in two different directions (no pun intended): (1) it relates diagonals of information in their data to rows of numbers in Pascal's triangle and (2) it notes that intersection points whose shortest routes have the same length can have different numbers of shortest routes.

Both Michael and Romina's observations provide propositional foundation for the participants' later public articulation of a nuance in the correspondence that they indicate between the Taxicab and Towers Problems. Advancing the correspondence between the length of a shortest route and the height of a tower, they point out in vignette 24 that the number of units of direction *A*, representing direction east or south, is associated to the number of blocks in a tower of color *A*, where *A* represents one of two specific colors.

Vignette 24 [LS_2, 0:37:37 – 0:39:05]

1238. ROMINA: //When we look- whenever we do this we always- we always talk about towers and how this is like a tower of two high with two different colors and there's one- one tower you can make that makes one color and one and one and then all the other color. And- and then for this one it's three high and this is all one color. There's two of one color and one of the other, whatever. And for this it's basically the same thing because this is- let's see. This is- this is two but usually you go one across and one down so there's two different ways to get to that one. And for this one there's going to be two across and one down. Or to go down here it's two down and one across which is basically the same thing and it just goes on. Do you understand? Understand? Was that good? Or, do you want more? [Connecting the data from the grid and the triangle drawings by pointing to the numbers on each back and forth.]

1239. BRIAN: Yeah.

...

1252. MICHAEL: You have, you have a tower of three and you have, you know, two colors. So one- it's either you know- Color *x* and two of color *y*. Well this is direction *x* and two, two directions of *y*, you know-// [Pointing with a marker to the redrawn Pascal's triangle.]

Shortly after the explanation that the participants give in vignette 24, Researcher 3 asks them to clarify how do they know that the number of shortest routes to a particular corner in the taxicab grid corresponds to a number in Pascal's triangle. The participants spend time alone, without the researchers, to develop a response to Researcher 3's question. During this time, they further elaborate their isomorphism between the Towers and Taxicab Problems using Pascal's triangle as an iconic representation upon which to build their isomorphisms. In their negotiatory interlocution, each number in Pascal's triangle represents specific towers of Unifix cubes when there are two colors from which to choose.

Romina, Brian, and Michael augment their array of data on their transparency by assuming the addition rule of Pascal's triangle (see Figure 26). They also add two additional lines to the Pascal's triangle that Romina drew on a separate sheet of paper (see Figure 28).

Figure 28. Romina's augmented version of Pascal's triangle containing seven rows.

The participants focus their attention on their inscriptions in Figures 26 and 28. In the former figure, referring to an intersection five across and two down—intersection point (7,5)—Romina associates the length of its shortest route, which is seven, to a row of her Pascal's triangle by counting down seven rows and saying, "five of one thing and two of another thing" (see vignette 24). She then crosses out the ones in her triangle, saying that they are not involved. Michael inquires about her meaning for "five and two." Both Romina and Brian respond, "five across and two down." She then associates the combinatorial numbers in the seventh row of her Pascal triangle to the idea of "five of one thing and two of another thing," specifying that, left to right from her perspective, the first 21 represents two of one color, while the second 21 represents five of one color.

Vignette 25 [LS_3, 0:01:58 – 0:03:25]

1313. ROMINA: //Okay, five and two- five and two, just add that. That's how many blocks there are. That's seven. So you got to go one, two- no. One, two, three, four, five, six, seven. Gets you down to seven. And five of one thing and two of another thing, so you just- you don't count- we won't count the one because that doesn't involve that. [Pointing between the transparency grid and the redrawn, augmented version of Romina's Pascal's triangle.]

1314. MICHAEL: What do you mean five and two?

1315. ROMINA: What?

1316. MICHAEL:	What are you talking about five and two?
1317. BRIAN:	Five across //and two down.
1318. ROMINA:	//Five across and two down. Like you just count in. It goes- that's with one of one color and that's with two of two- of another color. That's with three, that's with four, that's with five. So it's either the two or the five. Both of them are the same thing. Yeah, we can explain this. Right? If anyone you pick like this one, you know it's one, two, three, four, five, six, seven. You know it's seven and it's going to be one, two, three- six of one color so it's going to be seven. [Pointing to both the redrawn, augmented version of Romina's Pascal's triangle and numbers on the transparency grid.]
1319. MICHAEL:	Are you saying five across- one, two, three, four, five- one, two. [Working with a figure of the first six rows of Romina's Pascal's triangle.]
1320. ROMINA:	So- either way- no, but it's seven blocks. It's five plus two. That's how many blocks you had. For seven blocks you go down. Go one, two, three, four, five, six to the seventh row. And now you know it's five by two so it means there's five of one color, two of another color so if I go to the second one this has to- this is all one color. This is one with one color this is two. So it's either twenty-one or there's three of one color, there's four of one color, and this is five of one color or twenty-one again. [Circling the two 21s on the redrawn, augmented version of Romina's Pascal's triangle.]

Michael asks "why." Romina explains by using as an example the question of the number of shortest routes to the intersection point (4,2), corresponding to the 6 of her transparency grid. She points out that the distance from the taxi stand to the intersection point is 4 regardless of the shortest route, that it corresponds to a tower that is 4 blocks tall, and associates to the fourth row of their Pascal's triangle. She explains further that to reach the intersection point (4,2) one goes two blocks across and two blocks down ("two of one color and two of the another," turn 1342). Then pointing to the fourth row of her Pascal triangle, she indicates that, from left to right, the first 4 is "one and three," the of 6 is "two and two," the 4 "three and one," and the 1 is "all one color."

Both Michael and Brian express agreement, and Romina asks, "who's calling them [the researchers] in" (turn 1347).

The participants construct isomorphisms among in particular the Taxicab Problem and the Towers Problem, using Pascal's triangle as a representation, with which they share a meaningful history, upon which to build their isomorphisms. To summarize, through ten different moments or phases, the participants build their isomorphism. The content of the phases include the following with indication of when from the start of the research session each occurs: (1) there exists a relationship between the Towers and Taxicab Problems, [0:07:37]; (2) Similar to the Towers Problem, the Taxicab Problem has a dyadic choice or binomial aspect, [0:07:39 and 0:08:55]; (3) The length of a shortest route to an intersection point corresponds to the height of a tower, [0:08:15]; (4) Each of the two different directions of travel in the Taxicab Problem needs to be associated with different objects in the Towers Problem, [0:44:26]; (5) Rebuild the meaning of 2 to the *n* in the environment of the Towers Problem, [0:08:26 and 0:44:51]; (6) Identify one unit of horizontal distance with one Unifix cube of color *A* and one unit of vertical distance with one Unifix cube of color *B*, [1:14:59]; (7) A row "diagonal" of their data contains the number of shortest routes for intersection points whose shortest distance from the taxi stand is *n*, [1:16:00]; (8) Intersection points whose shortest routes have the same length can have different numbers of shortest routes, [1:16:37]; (9) A tower 3-high with 2 of one color and 1 of another color, to routes to a point 2 down and 1 across, [1:18:40]; and (10) Intersection point five units east and two south from the taxi stand corresponds to five of one thing and two of another thing and, therefore, go the seventh row of Pascal's triangle and the second and fifth entries of the triangle to find the number of shortest routes from the taxi stand to the intersection point five units east and two south from the taxi stand, [1:22:40]

CHAPTER 6: CONCLUSION

6.1 Introduction

In January 1983, David H. Wheeler (1925-2000), the founding editor of the international journal, *For the Learning of Mathematics*, sent a letter to 60 or so mathematics educators inviting them to engage a daunting task, "to suggest research problems whose solution would make a substantial contribution to mathematics education" (Wheeler, 1984, p. 40). The varied and thought-provoking responses of more than 15 educators were published, some in each of the three issues of the fourth volume of the journal. On Wheeler's mind was the famous example of the 23 problems from various branches of mathematics that David Hilbert (1862-1943) announced in an address delivered to the Second International Congress of Mathematicians in 1900 at Paris (p. 40) and predicted that "from the discussion of which an advancement of science may be expected" (Hilbert, 1900, p. 5).[26] Of all the published responses to Wheeler's challenge, Tall (1984) offered the briefest list of what he considered to be "the central questions": (1) *how do we do mathematics?* and (2) *how do we develop new mathematical ideas?* (p. 25, emphasis added).[27]

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[26] According to Gray (2000), Hermann Minkowski (1864-1909), whose metric concept (order-*p* geometry) provided the theoretical foundation for non-Euclidean, taxicab geometry, was a close friend of Hilbert and urged him to accept the invitation to speak at the Congress: "Most alluring would be the attempt to look into the future, in other words, a characterisation of the problems to which the mathematicians should turn in the future. With this, you might conceivably have people talking about your speech even decades from now" (as quoted in Gray, 2000, p. 1).
[27] It bears noting that Tall's second question is in step with Davis's challenge to investigators in mathematics education to study the emergence of mathematical ideas among learners, to which I refer at the beginning of section 5 of chapter 1.

Hilbert's 23 problems contributed to more than a century of vigorous, fruitful research activity in physics and mathematics.[28] Similarly, considered responses to Tall's two questions require substantial research efforts in different environments over extended periods of time. For researchers in mathematics education to entertain these questions, which I, too, consider are central, we must find ways to observe what learners do as they do mathematics as well as to describe and analyze how they develop their mathematical ideas. These investigative tasks can be accomplished with the observational tool of digital video recordings coupled with analyses of discourse and inscriptions. The descriptive and analytic methods of this study as well as its theoretical perspectives provide a means to glean insights into how the participants do mathematics and how they develop mathematical ideas and forms of reasoning. After mentioning some limitations of the study, I present specific responses to its research questions.

6.2 *Limitations of the Study*

As with other research studies and reports, this study has methodological and narrative limitations. During the open-coding phase of analysis, themes emerged that were emic in nature. Nevertheless, in the case of interlocutory themes, upon categorizing the emergent themes, instead of using *in vivo* codes directly from participants' discourse, I opted to adapt codes already existing in the literature. In contrast, coding for the emergence and elaboration of mathematical ideas and reasoning, I created codes that surfaced from the actions of the participants such as those used to describe their heuristics. However, by

[28] See Grattan-Guinness (2000) for a critical appraisal of "the range of Hilbert's problems against the panoply then evident in mathematics."

adapting an etic perspective in the case of the interlocutory codes, I may have missed making a correspondence between the experience of the participants and the sociocultural and psychological portrayals of learning in the mathematics education literature (Charmaz & Mitchell, 2001). It is, therefore, possible for further research to probe other aspects of the studied phenomena of the participants' discursive practice as they live it, which, in turn, may add supplementary insights for the narrative and methodological outcomes of this study.

In this study, the production channels of participants to which I attended were oral, written, and gestic. In terms of method and narrative, this attention gives analytic privilege to the actions and interactions of externally expressive participants. As such, this foreground of attention to external expressions does not provide analytic scrutiny to the experience and, particularly, the mathematical thinking of less externally expressive participants. Some participants such as Michael appear to have a discursive practice that is largely internal or private. When researching cognition, a researcher can only infer meaning from participants' externally manifested acts. Consequently, by analyzing mainly oral and written expressions, it is difficult and highly speculative to infer the mathematical ideas and reasoning of a participant who is not particularly talkative *and* whose inscriptions are not expansive or elaborated enough to gain a window into their thinking. The narrative portrayal of the development of mathematical ideas and reasoning suffers since the potentially rich and divergent ideas of a participant may be lost. This was the case of Michael.[29]

In the research session of this study, Brian also was not particularly talkative. He, however, did initiate inscriptive presentations of different shortest routes to a pick-up point and in time elaborated a notational scheme that acquired the interest and confidence of his colleagues. Despite his relative quiet, his inscriptions do provide a window into his mathematical and heuristic thinking. From the perspective of ethnographic observations of mathematical thinking, there may be really strong or weak understanding that researchers are not able to notice. Eventually, I suspect, technologies from neuroscience to will allow researchers to monitor neural activities and from them infer the mathematical ideas and reasoning that over time minds build.

6.3 Interpretations: Responding to research questions

Four research questions guided analyses of data in this study. Before summarizing the specific findings for each question, it is worth first summarizing general results from the analyses of the data. The problem-solving session lasted for one and two-thirds hours. During this time, the three researchers do not suggest to the participants how to engage the task. Instead, the participants engage in conversational exchanges through which they shape and direct their own problem-solving investigation. Additional analyses of the participants' discourse and inscriptions reveal that they use part of their time to negotiate their understanding of the problem task; to develop problem-solving

[29] Over the course of the Rutgers longitudinal study, in the case of Michael, it is known that he often possesses wonderful ideas. On this point, see for example Kiczek, Maher, and Speiser (2001).

strategies and overcome heuristic hurdles; hypothesize and create combinatorial algorithms; build explanations and justifications of their work and resolutions; challenge each other to clarify their explanations and justifications as well as accept challenges of the same from researchers; and construct isomorphisms among the Taxicab Problem, the Pizza Problem, and the Towers Problem, using Pascal's triangle as a representation, with which they share a meaningful history, upon which to build their isomorphisms.

Specifically, analyses of the data provide insightful responses to each of the guiding research questions. I discuss each question in turn and respond to each based on data analyses presented in chapter 5.

6.3.1 The first question

The first question concerns the discursive practices of the participants: As the participants engage collaboratively, without assigned roles, to understand and resolve the Taxicab Problem what are the features and functions of learners' discursive practices?

In the data, the participants engage in a variety of discursive interactions directed toward resolving the Taxicab Problem. The features of these interactions coalesce around four categories of interlocution. These interlocutory categories emerge from the data through a dynamic and dialectical interplay of inductive and deductive coding. The following categories surfaced; they are grounded in theoretical ideas of Davis (1996; 1997) and the messy realities of open coding of the data but are different from both:

- *Evaluative*: an interlocutor maintains a non-participatory and an evaluative stance, judging statements of his or her conversational partner as either right or wrong, good or bad, useful or not.
- *Informative*: an interlocutor requests or announces factual data to satisfy a doubt, a question, or a curiosity (without evidence of judgment).
- *Interpretive*: an interlocutor endeavors to tease out what his or her conversational partner is thinking, wanting to say, expressing, and meaning; an interlocutor engages an interlocutor to think aloud as if to discover his or her own thinking.
- *Negotiatory*: an interlocutor engages and negotiates with his or her conversational partner; the interlocutors are involved in a shared project; each participates in the formation and the transformation of experience through an ongoing questioning of the state of affairs that frames their perception and actions.

The data overall and the vignettes presented in chapter 5 in particular contribute to further our understanding of how learners develop mathematical ideas and reasoning through their thoughtful engagement with task situations that they mathematize. Co-constructed mathematical ideas and heuristics emerge from the participants' informative, interpretive, and negotiatory interlocutions. A further outcome of the dominance of these three interlocutory modes is that over time participants build an *esprit de corps*.

Theoretically, this study enlarges Gattegno's constructs of the specific discursive entities around and about which mathematics is built. A reading of Gattegno (1987) reveals that his constructs include objects, relations among objects, and dynamics linking different relations.[30] The results of this study indicate that the category of heuristics—strategies and other actions used to resolve problem tasks—signal behaviors that contribute importantly to building mathematical ideas. Several researchers inquiring into mathematics education mathematicians (Engel, 1997; Mason et al., 1984; Polya, 1945/1973, 1981) recognize heuristics as strategies to be taught so that problem solvers can augment their mathematical "tool kit" and, consequently, increase their ability to resolve mathematical problems. Other researchers (Maher & Martino, 1996b, 1996c; Maher & Speiser, 2001; Silver, Mamona-Downs, Leung, & Kenney, 1996) examine the behaviors that learners develop over time on their own under certain conditions to index learners heuristics. This study contributes to an understanding of the heuristics learners develop and simultaneously to theorizing about constructs associated with ways learners build mathematical ideas.

6.3.2 The second question

The second guiding question concerns participants' emergent understanding and mathematical ideas: As they articulate their emergent understandings of the problem task, what mathematical ideas, associated meanings, and forms of reasoning do they reveal in their discourse?

As participants begin to comprehend the problem task, the theme of their discursive interactions center on fundamental issues about the nature of the problem space and fundamental ideas essential for investigating the problem task. They pose clarifying questions to the researchers concerning allowable

[30] See chapter 3, section 2.3 for a discussion of the theoretical position of Gattegno concerning psychological and dialogical mechanisms involved in constructing mathematics.

movements—orthogonal; clarify object equivalence—the identity between streets and grid lines; establish what are the basic objects of taxicab geometry—points and line segments or routes; note a basic awareness of the Taxicab Problem—there can be more than one shortest route to an intersection point in the taxicab plane; and discuss a distinguishing, spatial feature between Euclidean and taxicab geometries—how distance is measured. Generally speaking, they engage in discourse and, thereby, constitute the objects and relations among the objects of the problem task. With the ideas, the participants shift their focus to delve further into the problem task and generate considerable data.

As the participants engage individually and collaboratively to generate data, they develop problem-solving strategies and overcome heuristics hurdles methods. As an example, Figure 6.3-1 contains a list of the sixteen heuristic methods that the participants employ with indication of the first time they implemented each heuristic, according to time indications for the two compact disks of camera view LS.

	Participants' Heuristics	**First Occurrence**
1.	Counting routes from the taxi stand to a pick-up point while outlining without drawing the routes.	LS_1, 0:02:30
2.	Traveling on the grid lines only east and south.	LS_1, 0:04:41
3.	Parceling out different mini-tasks among group members as well as collecting and recording the data they generate.	LS_1, 0:05:59
4	Counting routes from the taxi stand to a pick-up point while drawing the routes on the same sub-grid.	LS_1, 0:06:15
5.	Attending to dynamical links among objects and relations between two systems.	LS_1, 0:07:31
6.	Attending to numeric patterns in generated data.	LS_1, 0:12:10
7.	Doing easier sub-problems; counting routes from the taxi stand to nearby intersection points while outlining without drawing the routes.	LS_1, 0:14:53

8.	Counting routes from the taxi stand to nearby intersection points while drawing the routes on the same sub-grid.	LS_1, 0:16:12
9.	Parceling out the same mini-task, each counting routes to intersection points nearby the taxi stand, drawing them on the same sub-grid, as well as recording and comparing resulting data.	LS_1, 0:16:50
10.	Planning to count systematically points nearby the taxi stand and anticipating that a numerical pattern will emerge.	LS_1, 0:22:12
11.	Talking aloud how one is finding all shortest routes an intersection point.	LS_1, 0:24:23
12.	Drawing each route between the taxi stand and an intersection point on a separate sub-grid.	LS_1, 0:24:23
13.	Finding opposite routes to each drawn route to ensure that all possible routes are found.	LS_1, 0:25:00
14.	Parceling out mini-tasks to compare the data generated from different combinatorial algorithms.	LS_1, 0:39:08
15.	Building an isomorphism among the Taxicab, Tower, and Pizza Problems, using Pascal's triangle as a representation upon which to build the isomorphisms.	LS_2, 0:21:56
16.	Processing findings with researcher to discover further one's ideas.	LS_2, 0:23:53

Figure 29. The sixteen heuristic methods that the participants employ with indication of the first time they implemented each heuristic, according to time indications for the two compact disks of camera view LS.

As this aspect of data shows, heuristic methods are important ingredients for generating new mathematical ideas. In an article about expert problem solving, Stylianou (2002) observes "Mathematics is as much about axioms and theorems as it is about methods, both highly structured (such as mathematical induction) and less structured strategies or heuristics (such as search for patterns and drawing diagrams)" (p. 303). The conditions of the Rutgers longitudinal project have afforded participants opportunity over time to build and develop facility in their own problem-solving heuristics.

Among other characteristics, the Taxicab Problem is a combinatorial task. It requires one to determine the number of different shortest paths in the given grid between the taxi stand and three distinct intersection points. The participants in this study generalize the problem task, seeking to find the number of different shortest paths between the origin of a grid (the taxi stand) and each intersection point of the grid's network. To accomplish their self-defined goal, among other ideas, they develop enumeration procedures or combinatorial algorithms. Each algorithm that the participants entertain, I identify as one of three types: implied, hypothesized, or articulated. An implied algorithm is a procedure for counting routes about which a participant does not comment but with which he or she generates data. A hypothesized algorithm is one about which a participant discusses but does not explicitly implement, whereas an articulated algorithm is a counting procedure about which a participant discusses and with which a participant generates data. Implied and articulated combinatorial algorithms allow participants to generate data from which to perceive patterns and relations among objects and, in turn, conjecture the undergirding mathematical structure of the problem task. As Davis and Maher (1990) note "Mathematicians *analyze problems* and *create* algorithms, they do not merely memorize algorithms and recall them as needed" (p. 77, original emphasis). Akin to what professional mathematicians do, the participants create their own algorithms and with their algorithms generate data to assist in their resolution of the Taxicab Problem.

Using the data they generate with their various algorithms, the participants build conjectures about the underlying mathematical structure of the problem. For these conjectures, they provide either counterexamples or justifications. In the statement of their conjectures, the participants identify relations among objects or dynamics linking different relations. In their discourse, they essentially pose four conjectures. For two of the conjectures, the participants present counterexamples and for the other two they build justifications. In structuring their justifications, the participants involve themselves in negotiatory interlocution. Together the four critical events in which they pose conjectures belong to a pivotal strand whose theme is 'the underlying mathematical structure of the Taxicab Problem.'

The participants conjecture and employ algorithms and strategies to resolve the problem task to their satisfaction. Through their various heuristic actions, among other consequences, they generate data that they consider reliable. Reflecting on numerical patterns in their data, they conjecture that Pascal's triangle is the underlying mathematical structure of the problem task. How do they justify this conjecture? The data of this study suggest that to justify their conjecture the participants build an isomorphism between the problem task and the Towers Problem. Chiefly, the locus of how they build this isomorphism is through one of their heuristics: *attending to dynamical links among objects and relations between two systems*. The data reveals that early in the problem-solving session by attending to dynamical links three participants—Romina, Jeff, and Michael—articulate awareness of object and relational connections between their current problem task and a former one, the Towers Problem. Later, upon noticing that their array of data resembles Pascal's triangle and conjecturing so, the participants embark on building an isomorphism between the Towers Problem and the Taxicab Problem as an approach to justifying their conjecture since from previous experience they know that Pascal's triangle underlies the mathematical structure of the Towers Problem. In this sense, their reasoning strategy can be interpreted as justifying their conjecture by transitivity (If a ~ b and b ~ c, then a ~ c): (a) Pascal's triangle is equivalent to Towers and (b) Towers is equivalent to Taxicab; therefore implying that (c) Pascal's triangle is equivalent to Taxicab. They know (a) is true and embark on demonstrating (b) to justify and conclude (c). Importantly, the triangular array of the binomial coefficients presents an inscription with deep meaning for the participants and is used to mediate the dynamics or web of relations that they perceive and express between the Taxicab and Towers Problems.

6.3.3 The third question

The third research question inquires into the participants' independence and autonomy as mathematical problem solvers: How do participants structure their investigation?

As discussed in chapter 1, from first grade through the termination of high school, the four participants of this study have been members of a larger cohort of students who have engaged with mathematical activities of the Rutgers longitudinal research project. Over these years, the pedagogical and research practices of the contingent of researchers as well as the social and intellectual practices of the participants have constituted norms of what can be called an evolving mathematical microculture. The conditions of the research session of this study are consistent with the evolved research and pedagogical practices that are constitutive of the mathematical microculture in which the participants and researchers jointly take part. For this study, salient features of this practice are the open-endedness of the problem task, the non-intervention of the researchers in the deliberations of the participants, and, nevertheless, the availability of researchers if and when the participants request an opportunity to present or otherwise process their thinking. Elements of the evolved microcultural norm of researchers, most particularly the questioning patterns and expectations for explanations and justifications have been assimilated by research participants and infused into their ways of collaborating with each other. This point is manifest when Michael comments to his collaborators, "they're going to ask us" (turn 909), and five turns later when Jeff makes explicit Michael's implicit message and says, "why does Pascal's Triangle work for this is the question" (turn 914).

Under these pedagogical conditions and through specifically interpretive and negotiatory interlocution, the participants structure, monitor, and conduct their problem-solving investigation. In all, the students spend approximately 77% of the one and two-thirds hours of the problem-solving session, or 1 hour and 16 minutes, in discursive interaction with each other, and 92% of this time without the researchers being physically present in the room where the participants work. Furthermore, the participants spend about 18% of their time negotiating the construction of combinatorial algorithms, without the presence of the researchers. In interpretive and negotiatory interlocutory modes, the four participants suggest subtasks for each other and heuristic methods as well as identify their need to verify expected outcomes and confirm actual results. In addition, through interpretive and negotiatory interlocution, the participants prompt their need to explain and justify the underlining mathematical structures that they perceive. Negotiatory interlocution is a discursive mode in which participants shape their investigation of the given mathematical task.

6.3.4 The fourth question

The final research question of this study examines more specifically the interlocutory modes and the participants problem solving: How do their conversational exchanges support advances in their problem solving?

Conversational interactions among the participants advance their subsequent individual and collective actions. The data suggest that, among our four interlocutory properties, (1) interpretive and negotiatory interlocution have the potential for advancing the mathematical understanding of individual learners working in a small group, (2) the personal or individual understanding of a learner is intermeshed with the understanding of his or her interlocutors, and (3) the mathematical ideas and understanding of an individual and his or her group emerge in a parallel fashion.

Through their discursive practices, the participants impose meaning and structure, make decisions about what to do and how to do it, and interpret the reasonableness of their actions and solutions. They take intellectual risks as well as respect and value each other's thinking. From the perspective of doing mathematics, they frame and solve problems for themselves, look for patterns, attend to dynamical links between systems, make conjectures, examine constraints, make inferences from their data, abstract, invent, explain, justify and challenge. Finally, interpretive and negotiatory interlocution support reflective abstraction.

6.4 Implications for Research and Pedagogy

Taken together Tall (1984) and Davis (1992a) propose significant challenges. To these challenges, both the limitations and findings of this study raise important questions, pointing to research and pedagogical implications. Specifically, this study contributes to understandings about how learners do mathematics and how learners elaborate mathematical ideas that are new to them. This study has explored these questions through an examination of the discourse and inscriptions in the context of the research and pedagogical conditions that have evolved and have been characteristic of the Rutgers longitudinal study.

6.4.1 Research questions

Attributes of the research design include four students working together, students collaborating without assigned roles, students working without time constraint, the non-intervention of the researchers in the problem-solving process, and observation occurring outside the context of school sessions. Variations of these attributes open perspectives to new research. For instance, open for debate and research are questions such as how within a classroom setting collaborative problem-solving sessions can be adapted or amplified where issues are left unresolved with over several class sessions and what forms of teacher intervention can occur that do not erode student independence and autonomy.

Other questions for further research concern the development of communicative competence for working collaboratively on mathematics. Among other qualities, the discourse of the participants in this study illustrates not only the richness of the mathematical ideas and forms of reasoning that they display but also the sophistication of their skill in leading their own mathematical problem solving and discussions. This skill has developed over a long span of time and under the conditions constituted by the process and content of the Rutgers longitudinal research project and by the affect and intellect of the participants themselves. The participants not only talk about mathematics but also—and this is the important point—make mathematics. This point is evident in the results since this study not only examined features and functions of their discourse but also its mathematical content. A question for further research concerns how the participants have developed over time their skill at engaging in meaningful and productive mathematical discussions. Similarly, what pedagogic actions of the researchers contribute to the skills of participants to conduct meaningful and productive mathematical discussions?

Other questions for further investigation that arise from the study relate to categories of interlocution. These categories enable researchers to track learners' development of mathematical ideas and reasoning, their participation in collaborative efforts in the development of mathematical ideas and reasoning, and the autonomy of learners in their construction of mathematical ideas and reasoning. To what extent are the findings about the participants' discourse patterns and how they develop their mathematical ideas and forms of reasoning parallel learners in other contexts? To what extent can the conditions that contribute to the participants' mathematical development be structured in the context of regular public schools or replicated in out-of-school educational environments? It is also important to learn more about the mathematical ideas of learners who, like Michael in the setting of this study, are intellectually active and whose discourse and inscriptions provide little evidence of their thinking. A finding of this study is that the 'evaluative' interlocutory category played an insignificant role in the discourse of the participants. What factors of the participants' microculture contribute to this finding?

Investigating links between the development of mathematical ideas and interlocution as well as written presentations can reach beyond individual students and groups of students. What are the interlocutory characteristics of professional mathematicians when they collaborate to solve mathematical problems with their peers? In what ways are their discursive practices similar and different from what we observe among learners solving mathematical problems? From the perspective of ethnomathematics (D'Ambrosio, 2001; Gerdes, 1999; Powell & Frankenstein, 1997), what are the characteristics of the interlocution of individual artisans when solving problems collaboratively with their peers?

From particular theoretical perspectives, this study has approached analytically the data through the lenses discourse and inscription to inquire into the development of mathematical ideas and reasoning. It would now be fruitful to analyze the data from other or even multiple theoretical perspectives since understanding (thinking and learning) is too complex of a cognitive process to be described, captured, understood, explained, or accounted for by any single theoretical lens. Moreover, different, complementary, and divergent perspectives may provide more insight on understanding than is possible from any one perspective.

In mathematics education, there is recent precedent for this theoretical and methodological proposal. In Clarke (2001a), he describes a qualitative analytic approach, called Complementary Accounts Methodology. A distinguishing feature of this approach is that a common body of videotape and interview data, which include the retrospective construal of events by participants, is analyzed from multiple theoretical frameworks (Clarke, 2001b). Another example of a multi-perspective analysis of a common data set occurred in 2001 at the twenty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, in Snowbird, Utah, where graduate researchers of the "μ-group," from the University of British Columbia (Pirie, 2001), presented a session of linked papers, each analyzing a set of data from different theoretical perspectives that consider the nature of mathematical understanding. In the case of the data and basic scientific questions of this study, it would be theoretically and methodologically insightful to inquire into the development of mathematical ideas and reasoning through an examination of discourse and inscription from multiple theoretical and analytical frameworks.

6.4.2 Pedagogical implications

From a pedagogical perspective, it would be helpful to know whether the category of interlocution proposed in this study have applicability to learners in other contexts. As has been indicated in section 4 of chapter 5, the participants in this study discuss and negotiate their heuristic methods and in a decisive moment decide to modify and specialize the given problem. Embedded in their negotiation is commentary about their understanding of how best to engage the problem, whether to treat the problem as given or whether to specialize so that they can later generalize on the basis of their specialized results. Indeed, through analyzing by cases and controlling for variables, they specialize the problem and ultimately generalize their findings. In this regard, their cognitive actions challenge a prevailing pedagogical belief that within a mathematical topic problems ought to be presented to students in a sequential form of increased complexity. In this study, however, the participants are given an open-ended problem without its complexity having been removed. Instead, the participants make decisions at each moment about what aspects of the problem to stress and to ignore. Their investigative strategy to find the number of shortest routes between the taxi stand and nearby intersections points leads the participants to notice a pattern that ultimately triggers ideas that in turn guide them toward a resolution. Instead of directing instructional activities to lead students quickly to answer routine or stereotyped problems, this study evidences that pedagogically much more time ought to be given to students to think deeply and with others about their heuristics rather than rush toward answers. Students are far more likely to resolve routine as well as novel, rich mathematical situations as they develop efficacious problem-solving heuristics.

This result of this study concerning the need for time to think deeply and discursively is beginning to receive foreground attention in the literature. A recent, special issue of *Educational Studies in Mathematics* collected several analyses concerning discourse in mathematics classrooms. Commenting on these studies, Seeger (2002) wonders about the possibility of a grand, panoramic theory of learning. In arguing for a comprehensive theory of mathematics education, he suggests that such a theory needs to embrace four metaphors of learning that form the axes "social—individual" and "construction—acquisition," and represents them in a two-by-two grid (p. 289).[31]

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[31] Here Seeger (2002) differs from Sfard's (1998) theorization in which she argues for two metaphors—acquisition and participation or construction— that conceptualizes perspectives on

In addition, Seeger further suggests that "theoretical work has to be balanced by the systematic development of focal problems for practice, theory, and research in mathematics education" (p. 289). He proposes two focal problems for mathematics education, one concerning ecological validity and representation and the other referring to the question of time and change.

Beside epistemological concerns, the question of time and change also concerns methodological issues. Building ideas and understanding are certainly temporal and unbounded. Consequently, there are complex judgments an investigator has to make when inquiring into what learners build, understand, or acquire from a discussion or lesson on a particular issue. When does an investigator examine of what learners say, do, and write? Should these actions be examined in the immediate proximity of the discussion or lesson, in some other, more distinct time, or in some combination of these times? Ball and Lampert (1999) raise somewhat similar questions in a study of teaching practice.

Epistemologically and methodologically, this study contributes to understanding the relation between time and development. It presents many of its results as sequences of vignettes of interactions, selected based on criteria of the study's theoretical perspective and methodological framework. These vignettes range over the problem-solving session of this study and depict particular critical events that culminate in the presentation of important mathematical ideas or evidence particular forms of reasoning. As outcomes of individual and collective constructive actions over the course of the *extended* problem-solving session, the participants build and employ heuristics,

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learning and in which she claims that though complementary they are mutually not amenable to critique.

combinatorial algorithms, and isomorphisms to resolve the Taxicab problem. Among other issues, what the vignettes reveal and make salient is the important relationship between time and development. The results indicate that processes by which the participants build their ideas evidence an epistemological reality: knowledge construction is often a slow process. Mathematical ideas do not develop instantaneously and robustly but rather emerge slowly and in their nascent state are rather fragile.

Ideas dawn and mature over time. To loose fragility, among other things, ideas need to be reflected on deeply, presented publicly, submitted to challenge, available for negotiation, and subject to modification. That is, the essence of developing and understanding mathematical ideas is often a protracted, iterative, and recursive phenomenon, occurring over more time than is usually appreciated or acknowledged in practice in classrooms and in reports in the literature (Pirie & Kieren, 1994; Seeger, 2002). If learners are to develop deep understandings that are less fragile and more durable than is often witnessed by teachers in schools, they need to be offered extended periods of time to wrestle with a problem as well as to debate and negotiate heuristics, to articulate and justify their results, and to have their ideas challenged and then defend or modify their ideas.

On a different but related theoretical note, this study enlarges Gattegno's constructs of the specific discursive entities around and about which mathematics is built. A reading of Gattegno (1987) reveals that his constructs include objects, relations among objects, and dynamics linking different relations.[32] The results of this study indicate that the category of heuristics—strategies and other actions used to resolve problem tasks—signal behaviors that contribute importantly to building mathematical ideas. Several researchers inquiring into mathematics education mathematicians (Engel, 1997; Mason et al., 1984; Polya, 1945/1973, 1981) recognize heuristics as strategies to be taught so that problem solvers can augment their mathematical "tool kit" and, consequently, increase their ability to resolve mathematical problems. Other researchers (Maher & Martino, 1996b, 1996c; Maher & Speiser, 2001; Silver et al., 1996) examine the behaviors that learners develop over time on their own under certain conditions to index learners heuristics. This study contributes to an understanding of the heuristics learners develop and simultaneously to theorizing about constructs associated with ways learners build mathematical ideas.

6.5 Summary

By examining in detail the discourse and inscriptions of our participants as they wrestled with a combinatorial task set in a non-Euclidean environment, called the Taxicab Problem, this study aimed to investigate empirically the features and functions of their discourse patterns as well as the mathematical ideas and forms of reasoning that they build. The study describes their emergent and elaborated mathematical ideas and reasoning, how they structure their mathematical inquiry, their interlocutory patterns, and the ways in which their discourse practices support advance in their problem solving. The results of this

[32] See chapter 3, section 2.3 for a discussion of the theoretical position of Gattegno concerning psychological and dialogical mechanisms involved in constructing mathematics.

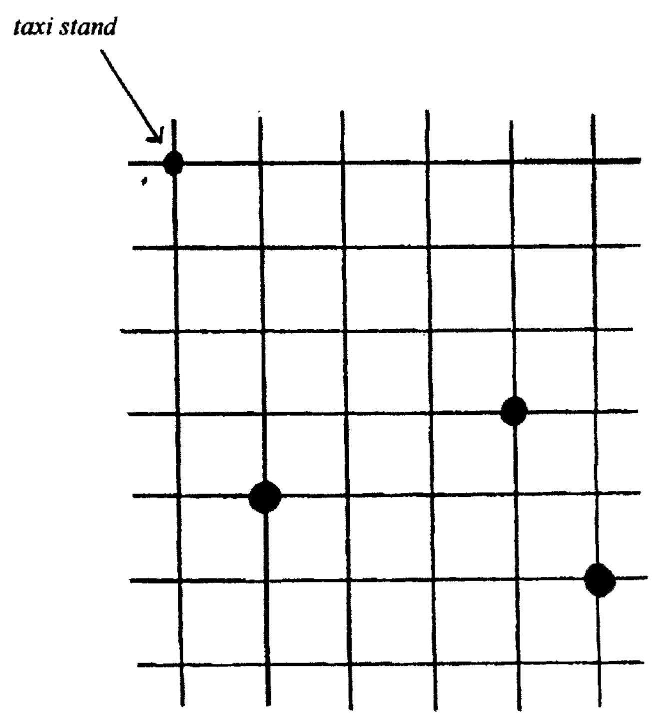
study demonstrate that under its research and pedagogical conditions participants collaborate to create and implement agendas for action, that is, structure and prosecute their own methods of investigation; co-construct an understanding of the mathematical structure underlying the task; and build mathematical ideas synchronously as individuals and as a collective members of a community of practice. Most importantly, as the study focused on participants' *discourse* and *inscriptions,* the results reveal ways that learners engage mathematics, ways in which mathematical ideas and forms of reasoning emerge among learners, and ways that learners elaborate mathematical ideas that are new and challenging to them. Further, this study contributes to the mathematics education literature understanding about how knowledge is built in interaction, discursively co-constructed by learners. In particular, this research illustrates how learners build isomorphisms by identifying their discursive propositions about dynamical links that establish one-to-one correspondences between, on the one hand, objects and relations in one system and, on the other hand, objects and relations of another system in such a way that an action on objects of one system maps to an analogous action on the corresponding objects in the other system. Finally, the results of this study indicate how learners use the mathematical property of transitivity as a form of reasoning to justify their resolution of a mathematical problem.

APPENDIX A: THE PROBLEM TASK

The Taxicab Problem

A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated on the map. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route.

What is the shortest route from the taxi stand to each point? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.



APPENDIX B: COMBINATORICS PROBLEM SEQUENCE

This selection from a strand of mathematical tasks in combinatorics forms part of a larger multi-strand collection of tasks that developed during the first twelve years of the fifteen-year longitudinal, cross-sectional research study lead by Dr. Carolyn A. Maher and funded by, among other sources, the National Science Foundation. These problems preceded the Taxicab Problem, the task of this study, and like it, are deliberately open-ended. How teacher-researchers engage students with them reflect a certain perspective on learning and teaching. Key to this perspective is that knowledge and competence develop most effectively in situations where students, frequently working with others, work on challenging problems, discuss various strategies, argue about conflicting ideas, and regularly present justifications for their solutions to each other and to the entire class. The role of the teacher-researcher, in our research perspective, includes selecting and posing the problems, then questioning, listening, and facilitating classroom discourse, usually without direct procedural instruction.

Shirts and Jeans

Stephen has a white shirt, a blue shirt and a yellow shirt. He has a pair of blue jeans and a pair of white jeans. How many different outfits can he make?

Cups, Bowls and Plates

Pretend that there is a birthday party in your class today. It's your job to set the places with cups, bowls and plates. The cups and bowls are blue or yellow. The plates are blue, yellow or orange.

Is it possible for 10 children at the party each to have a different combination of cup, bowl and plate?

Show how you figured out the answer to the question.

Is it possible for 15 children at the party each to have a different combination of cup, bowl and plate?

Show how you figured out the answer to the question.

Towers 4 -Tall

Your group has two colors of Unifix Cubes. Work together and make as many different towers four cubes tall as is possible when selecting from two colors. See if you and your partner can plan a good way to find all the towers four cubes tall.

Towers 5 -Tall

Your group has two colors of Unifix Cubes. Work together and make as many different towers five cubes tall as is possible when selecting from two colors. See if you and your partner can plan a good way to find all the towers five cubes tall.

Towers 4 -Tall with Three Colors

Your group has three colors of Unifix Cubes. Work together and make as many different towers four cubes tall as is possible when selecting from three colors. See if you and your partner can plan a good way to find all the towers four cubes tall.

Towers 3 -Tall Written Assessment

Please send a letter to a student who is ill and unable to come to school. Describe all of the different towers that you have built that are three cubes tall, when you have two colors available to work with. Why were you sure that you had made every possible tower and had not left any out?

The Pizza Problem with Halves

A local pizza shop fills orders with different choices for each half of a pizza. They offer a cheese pizza with tomato sauce. A customer can then select from the following two toppings: sausage and pepperoni. How many different choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possible choices.

The Four-Topping Pizza Problem

A local pizza shop has asked us to help design a form to keep track of certain pizza choices. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushroom and pepperoni. (No halves!) How many different choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possible choices.

The Four-Topping Pizza with Halves

At customer request, the pizza shop has agreed to offer choices for each half of a pizza. Remember, they offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushroom and pepperoni. There is also a choice of crusts: regular (thin) and Sicilian (thick). How many different choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possible choices.

Towers n -Tall with k Colors

Your group has ***k*** colors of Unifix Cubes. Work together and make as many different towers ***n*** cubes tall as is possible when selecting from ***k*** colors. See if you and your partner can plan a good way to find all the towers ***n*** cubes tall.

Towers 5 -Tall with 2 of One Color

Find all possible towers that are 5 cubes tall, selecting from two colors with exactly two of one color in each tower. Convince us that you have found them all.

Ankur's Problem

Find all possible towers that are 4 cubes tall, selecting from cubes available in *three* different colors, so that the resulting towers contain at least one of each color. Convince us that you have found them all.

Exploring Binomial Expansions

Expand binomial expressions to different powers. Represent and describe what you observe about the results, particularly the coefficients. How would you predict the coefficients for $(a+b)^n$? Prove (or modify) your prediction and explain why it works.

APPENDIX C: TRANSCRIPT OF PROBLEM-SOLVING SESSION

251. RESEARCHER 1: Thank you all, thank you all for coming.

252. BRIAN: These tables aren't [inaudible]

253. MICHAEL: Yeah, I know, I was looking at-

254. RESEARCHER 1: Yeah, well, you know you can move back in. I have a problem for you.

255. JEFF: All right.

256. BRIAN: Yes! [Punching the air with is right fist.]

257. RESEARCHER 1: You're all set.

258. BRIAN: Let's do this.

259. RESEARCHER 1: Okay.

260. JEFF: We're going to do taxicab geometry?

261. RESEARCHER 1: Do you know about it?

262. JEFF: I have no clue.

263. RESEARCHER 1: Did you ever hear of it? I understand that you all love geometry. I was listening to your interviews.

264. JEFF: Awe. [Wiping the left side of his face with his left hand].

265. RESEARCHER 1: I though we would end with a smash of a problem in taxicab geometry. Okay. Why don't I just give you the problem, okay? Um, I'll give you a chance to look at it and see whether you understand the problem. [Leaving the table.]

266. RESEARCHER 1: Why don't I just give you the problem, okay? I'll give you a chance to look at it and see whether you understand the problem

267. JEFF: You have to stay on the lines, right? Those would be streets?

268. RESEARCHER 1: Exactly.

269.	JEFF:	I agree.
270.	ROMINA:	Isn't it like anyway you go-
271.	BRIAN:	Pretty much, because look-
272.	ROMINA:	As long as you don't go like past it. [Facing Brian's direction.]
273.	BRIAN:	The first one- No, 'cause.
274.	MICHAEL:	Well what if you go to the last one-
275.	BRIAN:	You can go all the way down and go over and go down three and go over two. [Tracing the routes above the problem sheet with a black marker in his right hand.]
276.	ROMINA:	Isn't it- Don't they all come out to be the same amount of blocks? [Jeff beginning to draw.]
277.	BRIAN:	Five.
278.	JEFF:	Five?
279.	ROMINA:	Five? I got seven.
280.	JEFF:	Uh, which one- Yeah, we were both looking at the red one.
281.	BRIAN:	I'm looking at blue. [Michael tapping his pen on the grid along intersection points.]
282.	JEFF:	Yeah.
283.	ROMINA:	Oh, okay.
284.	JEFF:	All right. I mean pretty much.
285.	ROMINA:	As long as you don't go like past it you're fine. So it's the same thing.
286.	BRIAN:	So, let's prove it.
287.	RESEARCHER 1:	Okay, does somebody want to tell me what you think you understand the problem to be asking?
288.	JEFF:	Um, what's the shortest route from there to here staying on the streets, right?

289. RESEARCHER 1: Okay, is there more than one shortest route?

290. BRIAN: Yes.

291. ROMINA: Yeah.

292. RESEARCHER 1: In other words, if there is, how many?

293. ROMINA: Ah-

294. BRIAN: Let's do the blue.

295. RESEARCHER 1: Okay?

296. JEFF: All right, how many different shortest routes are there?

297. RESEARCHER 1: Yes.

298. JEFF: Is what you're asking right now? / / All right.

299. RESEARCHER 1: / / Mm hm.

300. BRIAN: Blue's got five.

301. RESEARCHER 1: Okay. And how do you know? You're going to have to convince us. Okay.

302. BRIAN: All right.

303. RESEARCHER 1: If you need us call me or Gina. [Inaudible].

304. ROMINA: I have five.

305. JEFF: Can we have like a- You have colored like markers? Word! [Responding to Researcher 2's statement that she will give them some markers.]

306. BRAIN: For what?

307. JEFF: Because then we can just do each route a different color. To like- [Waving his hand.]

308. ROMINA: Yeah, but they all kind of go on top of each other.

309. JEFF: Well, I mean, I don't know. I mean, let's see what it looks like. If it get too ugly then- Which one are you doing?

310. ROMINA: Which one do you want to do?

311. JEFF: I'll go to red.

312. ROMINA: I've got blue.

313. BRIAN: I did blue.

314. JEFF: Brian already-

315. ROMINA: One-

316. BRIAN: It's just going to look like you're filling //in the boxes.

317. ROMINA: //Two. Yeah, it is.

318. JEFF: That's what it's going to end up looking like, right?

319. ROMINA: Yeah so screw it. There's- Okay, so we know five-

320. JEFF: Well,- [Romina writing "Blue 5" on her paper to the right of the grid and tracing routes with her pen on the grid.]

321. BRIAN: Just count them and then make sure you know how you got them. You know? [Jeff and Romina counting by tapping their pen or marker on the grid. Each of them counts on their own grid.]

322. JEFF: Yeah.

323. ROMINA: One, two-

324. JEFF: So why- why is it the same every time?

325. MICHAEL: You're going left and right.

326. ROMINA: Ours is a four by one, right?

327. MICHAEL: Yeah, it's a four by one, unless you go backwards a couple of times.

328. ROMINA: You can't go, well-

329. MICHAEL: I know that would be dumb.//

330. BRIAN: //[inaudible] the shortest route only if you go forward.

331. MICHAEL But the only- You can't go diagonal so you have to go up and down. So if the thing is down this many and//

332. JEFF: //Over that many, //it's the same

333. MICHAEL: //It's the same-

334. ROMINA: //It's the same area

335. MICHAEL: No matter how you do it, no matter how you do it it's- you have to- you can't //get around doing that. [Pointing and gesturing around his grid]

336. ROMINA: //All right.

337. MICHAEL: //You can't get around going four down and right one 'cause -.

338. JEFF: All right, yeah. All right.

339. MICHAEL: You can't go over there. You can't get around doing that.

340. JEFF: Yeah.

341. ROMINA: What if I were to go like to the red when I go one, two, three, four- [Pointing at her problem sheet.]

342. MICHAEL: But they're not asking for like a //[Inaudible].

343. ROMINA: //Five, //six, seven.

344. JEFF: //Five, six, seven. //It's the same thing.

345. ROMINA: //Like //how- how am I going to- like //how would I-

346. JEFF: //It's the same thing.

347. MICHAEL: //It's the same.

348. ROMINA: -devise an area for that? Like this- this area up here? [Motioning with her pen on her grid, indicating the area of the rectangular space whose vertices are taxi stand and the red pick-up point.]

349. BRIAN: Like plus and [Inaudible].

350. JEFF: Well, it's not area.

351. MICHAEL: It's not area. It's //just a-

352. JEFF: //It's the perimeter. It's like //each one being one.

353. MICHAEL: //One, two, three, four, five, six, seven. [Pointing at Romina's paper and counting the length of a route to the red pick-up point.] [Jeff scratching his head.]

354. ROMINA: All right.

355. MICHAEL: There's no way you can get around going- [Gesturing with his hands]

356. JEFF: //Going seven blocks.

357. ROMINA: //No, yeah, I understand.

358. MICHAEL: Across that many and down that many because you can't go diagonally. Can't- [Gesturing with his hands over his problem sheet across to the left and then down]

359. JEFF: Yeah.

360. MICHAEL: Can't get around it, so- [gesturing with his hands]

361. JEFF: I mean, that's the most sensible way I think to say that. Right? And they want to know how many though.

362. BRIAN: Are there seven possibilities, though? You know how like blue was five? There's five possibilities but-

363. JEFF: Ah, so-

364. BRIAN: You know how it's only like five spaces. Like one, two, three, four, five. [Pointing at the grid on his problem sheet.]

365. ROMINA: Yeah, so if it goes more.

366. BRIAN: Is there seven for blue, I mean red?

367. JEFF: Well, check it out.

368. BRIAN You've got one- [Pointing at the grid on his problem sheet]

369. ROMINA: Here, I'll- //Me and Michael do-

370. MICHAEL: //Is that the shortest routes?

371.	ROMINA:	Me and Michael do greens. The green one.
372.	BRIAN:	All right.
373.	MICHAEL:	//Oh, like that's the biggest one. [Pointing at paper]
374.	ROMINA:	//And they'll do red.
375.	BRIAN:	Green is nine I think. [Then he begins to check this idea.]
376.	ROMINA:	Well //count how many ways. [They use their pens or markers to count on the grid.]
377.	JEFF:	//All right, we'll look for it.
378.	MICHAEL:	One, two- [counting and pointing at paper]
379.	BRIAN:	Ten. My bad. [Correcting himself on the length of a shortest path to green.]
380.	MICHAEL:	There's a lot.
381.	ROMINA:	Yes I know. I'm trying to devise a- like a-
382.	JEFF:	The- the way to do it?
383.	ROMINA:	Yeah.
384.	JEFF:	This is hard. [Romina draws routes on her grid with her pen.]
385.	ROMINA:	Two-
386.	JEFF:	How many was there? For, um, for the blue dot. How many different ways.
387.	BRIAN:	Five.
388.	ROMINA:	Ha...I already lost count. [of the number of shortest routes to the green pick-up point.]
389.	JEFF:	How many //you got for red so far? [Talking to Brian]
390.	ROMINA:	//Well, I'm saying like if you go //all the way over. [Leaning over and pointing with her finger at the grid on Michael's problem sheet.]
391.	BRIAN:	//Two, three- [pointing at paper]

392. ROMINA: And then //you go all the way// over and leave only one space. [Romina points to Michael's grid and motions with her finger.]

393. MICHAEL: //Yeah. One, two, three- Yeah, one, two, three, four, five, six. Six going like that. [Outlining routes on his problem sheet.]

394. BRIAN: One, two, three, //four.

395. JEFF: //You only got five?

396. BRIAN No I'm just.

397. JEFF: Oh, I can't. //I can't keep //track of what I'm doing. [While Romina watches, Michael traces routes with his marker on the grid, without writing.]

398. MICHAEL: //Six this way. //Then you got-

399. JEFF: You know what I'm //saying?

400. MICHAEL: //possibility of doing this. //One, two-

401. ROMINA: //Yeah. How do we get that.

402. MICHAEL: -three, four. Oh, got one. But then you got // Ah, this is a lot

403. ROMINA: //Yeah, you could do this. [Michael counting by tracing with his pen.]

404. MICHAEL: You guys want to do the green? We'll do the blue.

405. JEFF: No that's all right. //We already did the blue.

406. BRIAN: //We already did the blue.

407. ROMINA: Yeah, the blue is fine.

408. BRIAN: We're doing red.

409. ROMINA: Okay, we can't count. Like we need a- can't we- can't we do towers on this.

410. JEFF: That's what I'm saying. Look, all right, you go to here-

411. ROMINA: And they're like blocks.

412.	JEFF:	All right, you go to here and you got a choice of going there or there. Right? [Indicating a choice of across or down at an intersection point of the grid on his problem sheet.] So then you pick one of those and then you got a choice of there or there. When you get to- you know what I'm saying? Maybe we can add all those up or something and get like a whole- [Explaining routes on grid paper.]
413.	ROMINA:	All right.
414.	MICHAEL:	There's a lot.
415.	ROMINA:	Okay, for ours there's ten //
416.	MICHAEL:	There's more than ten.
417.	ROMINA:	No. I mean there's ten blocks. Like ten lines to get to that thing, right?
418.	MICHAEL:	Yeah, six by five.
419.	ROMINA:	So if there's ten, ten could be like the number of blocks we have in the tower.
420.	MICHAEL:	This is one-
421.	ROMINA:	How do we do that? Two to the *n*? [Moving her pen cap on and off of her pen.]
422.	MICHAEL:	How- how many? This was five they said? [Pointing to the blue pick-up point on his problem sheet.]
423.	ROMINA:	Yeah. [Looking back to her problem sheet.]
424.	MICHAEL:	How much you guys get for the red? Still doing that one?
425.	ROMINA:	How could-
426.	MICHAEL:	It's got to be some kind of pattern.
427.	ROMINA:	Okay, there's ten lines- ten lines-
428.	MICHAEL:	Ten ways of getting there. So you can do. Like you got to-
429.	ROMINA:	There's ten different lines to get there.

430.	MICHAEL:	Think of the possibilities of doing this and then doing that. [Pointing at an intersection on his problem sheet grid and gesturing downward and then rightward.]
431.	ROMINA:	Well how many- okay, there's ten. How many lines //end up in the thing?
432.	BRIAN:	//What are you doing man?
433.	JEFF:	I'm just- I'm not, uh, trying to- [Drawing routes on grid paper.]
434.	ROMINA:	Two, //three, four, five, six, seven, eight.
435.	MICHAEL:	//Three, four, five.
436.	JEFF:	-get easier.
437.	MICHAEL:	There's thirty plus- I have thirty. About sixty I think. [Pointing with the pen on the grid.]
438.	MICHAEL:	You might want to-
439.	ROMINA:	So- It couldn't be like a block ten high in six different colors, type deal? That would be- [Counting on the grid with her pen,]
440.	MICHAEL:	There's like- there's ten line- there's ten like lines in here and the answer was five. So I'm waiting for them That's like a half or something.
441.	ROMINA:	So maybe it's thirty? [Counting the number of rows in the array and then draws a symbol. Jeff adds an "L" to the third row, left hand corner box and then adds an "L" to the fourth row left corner box.]
442.	MICHAEL:	It'd be nice if it was.
443.	ROMINA:	How many are there in here. One, two, three, four, twelve, twenty-. You guys got at least twenty-four yet?
444.	JEFF:	Uh, which, wait a sec-
445.	BRIAN:	I'm at eight. What to do you think? What are you guys thinking?
446.	ROMINA:	To get to this one, there could also be five times two but there's ten lines-

447. BRIAN: I've counted it.

448. ROMINA: And there's five ways to go.

449. JEFF: Wait, five?

450. ROMINA: For the blue one.

451. JEFF: There's ten lines?

452. ROMINA: //[Inaudible].

453. MICHAEL: //You got eight for red. I only have nine ways.

454. JEFF: //No but I'm like-

455. MICHAEL: //You have eight?

456. BRIAN: I'm drawing them. I'm not stumped; I'm just like not speeding through it. You know. Did you count the middle lines?

457. MICHAEL: No, I just got eight from the- you know, just- just //[Inaudible].

458. ROMINA: //I didn't- I didn't do it.

459. BRIAN: All right.

460. MICHAEL: I was thinking about that-

461. ROMINA: So ten-

462. MICHAEL: Let's- let's try doing the red one. Try doing the red one.

463. ROMINA: Yeah but, how you going to know when we- how are you keeping track though? [Romina places her hand on her head.]

464. MICHAEL: I don't know. I'm just- see like if I can just not forget. Are you going to like //just write them down?

465. ROMINA: //We can do what Brian's doing. Like we'll just draw a big thing on the board.

466. JEFF: And just go over each way to do it? There's got to- there's got to be

467. ROMINA: / /There's got to be something.

468. JEFF: / /some kind of math- You know what I'm saying? [Placing his hand on his head.]

469. ROMINA: All right.

470. BRIAN: How many do you think for red? Twenty-four?

471. MICHAEL: I was guessing.

472. BRIAN: See that.

473. ROMINA: [Inaudible.]

474. BRIAN: How'd you count that?

475. ROMINA: Or- hold on. There's-

476. MICHAEL: Uh, / /that's not really. It's- it's just a guess.

477. ROMINA: / /No, there's twenty- No it'd be twelve. Wouldn't it be twelve?

478. MICHAEL: I don't know. How- how much is this?

479. ROMINA: There's ten lines and there's five ways. So if there's / /twenty-four lines there would be twelve ways. [Pointing to Michael's problem sheet.]

480. MICHAEL: / /but there's one, two, three, four- It's twelve, yeah. We're, we're guessing twelve but that's probably not it. I doubt it. [Counting routes on grid of problem sheet. Romina leaning over her problem sheet and outlining routes.]

481. JEFF: All right, you- you're here. [Speaking to Brian, he points with his black marker to an intersection point on his problem sheet.]

482. BRIAN: Uh hmm.

483. JEFF: You get to go over or you can go up. [From (5,1), moving his pen to the left one unit and back and then up one unit [SK_1, 0:11:09- 0:11:12].]

484. BRIAN: Mm hmm.

485.	JEFF:	So like here you can go over or up. [On the right side of the problem sheet, drawing a point and from it two lines, producing a binary tree.]
486.	ROMINA:	What are you doing?
487.	JEFF:	I don't know. I'm not doing anything. I'm just trying to think. [Returning to Brian.] And then you get to here. You can either go over or up again. And the same thing. But I don't know what that has to do with anything. My brain is like- just looking at this right now and going like- [Inaudible.] It's just not working. [Jeff waves his hand.]
488.	ROMINA:	But you know, I am- //I understand what you're doing-
489.	BRIAN:	//Just look at the lines and see where you're getting five.
490.	ROMINA:	-but like for this one you know what sucks with this one, is because if you're there you have either one of two choices.
491.	JEFF:	Mm hmm.
492.	ROMINA:	When you get here you have one or two choices, you know, this just doesn't-
493.	JEFF:	Well, yeah, you're here, you can either go there or there. You get here- [Tracing routes on the grid of his problem sheet.]
494.	ROMINA:	Yeah.
495.	JEFF:	-you can go there or there. But if you're here, you're only going to go down. [Pointing at an intersection point on the grid of his problem sheet.]
496.	ROMINA:	Yeah. //That- that- exactly.
497.	JEFF:	//Because you're going out of your way.
498.	ROMINA:	That's exactly what I was doing.
499.	BRIAN:	See this is- //this is what I was thinking of.
500.	JEFF:	//Then you're here and you're only going down or over. Again, this is just down and you can just follow

all the routes to the end point- I don't know. [Pointing to the binary tree that he drew on the right side of his problem sheet.]

501. BRIAN: I don't know. That doesn't sound right. That's one. That's two, three, four, five. That's what I was doing with all of them. That's how I got twenty-four for this one. [Referring to the red pick-up point, pointing at Jeff's paper with his pen.]

502. JEFF: And that's what you thought it was Mike?

503. MICHAEL: //What'd you do?

504. ROMINA: //Yeah.

505. JEFF: Wait, what'd you do? How'd you do it? //That's one-

506. ROMINA: //No, not twenty-four.

507. BRIAN: //That's two. [Brian points to the grid with pen.]

508. JEFF: //That's two, three four //and five. [Brian pointing at Jeff's paper with his pen.]

509. ROMINA: //Twelve. Twelve would work.

510. MICHAEL: But that was not like-

511. JEFF: So then that's one, //that's two- [Pointing to his paper.]

512. MICHAEL: //Good guess.

513. JEFF: And you counted those up for twenty-four?

514. BRIAN: Three, four. [Pointing at paper with pen]

515. JEFF: See, that's what I'm saying.

516. BRIAN: Wait-

517. JEFF: And then the side streets.

518. ROMINA: But then there's more. [Brian counting with his pen on the grid.]

519. JEFF: There's more than fourteen?

520. ROMINA: No, I don't know how many there are.

521. BRIAN: Are you sure you got-

522. ROMINA: No, I was just saying like if- that wouldn't work with our theory.

523. JEFF: What theory is that?

524. MICHAEL: Divide //it by two.

525. ROMINA: //Divide it by two. It's like a highly- it was like a-

526. JEFF: Was it- like what divided by two? All the- add them all up //[Inaudible]. [Pointing at paper]

527. ROMINA: //Because there's ten lines- ten lines like that are all within this rectangle. [Pointing at paper with pen]

528. JEFF: All right.

529. ROMINA: There's five ways to get to it. So if there are twenty-four lines there would be twelve different lines to get to it. But, it's hard to prove. [Pointing to her grid with a pen.]

530. BRIAN: Actually, this whole thing, if you count the middle lines there's thirteen. [Referring rectangular region between the to the blue pick-up point and the taxi stand.]

531. JEFF: There is. That's why I- //as soon as I got to thirteen I stopped working because there's none- it's prime.

532. MICHAEL: //[Inaudible], right?

533. ROMINA: One, two, three, //four, five-

534. BRIAN: //It's four on the sides, eight, nine, ten, eleven, twelve, thirteen. [Brian uses his two hands to show routes in the air.]

535. ROMINA: //-six, seven, eight, nine, ten, eleven, twelve- There is thirteen.

536. MICHAEL: Thirteen what?

537. JEFF: Lines //over here.

538. ROMINA: //Lines.

539.	JEFF:	That's why I- I threw that out. I wrote- Oh, that's a thirteen but I was like, oh man, prime numbers. [Jeff puts his head in his arms.] No good.
540.	ROMINA:	/ / thirteen.
541.	JEFF:	There's like no way it could work with a prime number- like you can't even like make something up.
542.	BRIAN:	All right.
543.	ROMINA:	I think we're going to have to break it apart and draw as many as possible.
544.	BRIAN:	Yeah, / / that's what I'm going to do.
545.	JEFF:	/ / And then have that lead us to something? What if we do- why don't we do easier ones? You know what I'm saying? What if the- the thing- Do you have another one of these papers? [Speaking to Researcher 2.]
546.	ROMINA:	Here, to make it simple, just draw on here.
547.	JEFF:	All right. Well, yeah. We're just going to make a grid.
548.	RESEARCHER 2:	Oh, we got grid paper.
549.	JEFF:	Oh yeah? We could get some grid paper? Or those.
550.	BRIAN:	Whatever.
551.	RESEARCHER 2:	To tell you the truth..[Inaudible].
552.	JEFF:	We're not there yet. We're not-
553.	RESEARCHER 2:	No, I mean like so that you can cover it
554.	JEFF:	Whatever. We're flexible.
555.	RESEARCHER 2:	Okay, here's some more copies if that helps. Okay. And I'll get you
556.	JEFF:	All right. So-
557.	ROMINA:	Pick a dot.
558.	JEFF:	Right there.

559. ROMINA: One, two.

560. JEFF: Two. All right. Here.

561. RESEARCHER 2: We also have more to choose from.

562. JEFF: Jesus.

563. RESEARCHER 2 There's graph paper there. Okay

564. ROMINA: Okay. So one, two, three- Oh, is this going to be dumb and stuff? One, two, three, four- It looks like a multiplication table.

565. JEFF: All right. Uh, one-, two [Inaudible]. [Brian draws his eighth symbol on the right side of the grid and writes "1, 4, 2." On the top of "1, 4, 2" he writes "DRD." He also goes back to 7 and writes "D3, R1". He has written a number with each of the first 6 symbols on Brian's paper, too.]

566. ROMINA: All right.

567. JEFF: Why don't you just- here, use blue. It doesn't matter.

568. ROMINA: Yeah. One-

569. JEFF: One- / /two.

570. ROMINA: / /Two. Three. / /Four.

571. JEFF: / /Four.

572. ROMINA: Five?

573. JEFF: Where are you? Wait was that one, two over? The fourth spot? One, two- three- four- five. I don't- I can't remember what I- [Jeff draws routes on a 2 by 2 rectangle.]

574. ROMINA: I think it's five. I think it's five. [Brian draws his ninth symbol for a specific route, with the numbers "2, 4, 1" next to each line on the symbol.]

575. JEFF: I think it is five.

576. ROMINA: All right. Do the next one. Don't- don't count out loud and we'll see if we get the same thing.

577. ROMINA: [After working silently.] What'd you get?

578. JEFF: Nothing. I've got to start all over again. And what is that? Six?

579. MICHAEL: What 's that?

580. ROMINA: That's...how many I can-

581. JEFF: For each of those points?

582. ROMINA: Yeah. Like the point diagonally down.

583. MICHAEL: Yeah.

584. ROMINA: I'm not sure if I'm right though. I'm not sure if I'm counting right.

585. JEFF: / / One, two, three-

586. ROMINA: I mean this one- this one looks to be like prime numbers- I know this one going up- [Romina points to an intersection on the rectangle with her pen.]

587. JEFF: How many did you get?

588. ROMINA: Hol'- I think- Seven.

589. JEFF: All right. Well, the only thing I'm seeing right now with this right, is those together with that and those together with that.

590. ROMINA: / /Well I-

591. JEFF: / /So hopefully-

592. ROMINA: I'm going two, three, four, five, six. Two, three, four, five, six. Five- three, five, / /seven, nine.

593. JEFF: / /Seven, nine.

594. ROMINA: Eleven and then we're going to go up again?

595. JEFF: Well go for it. Yeah.

596. ROMINA: Here go- go and we'll have to [Inaudible]. [Brian writes tally marks on the top of his grid. Brian crosses out two of the tallies on his paper.]

597. JEFF: Wait. Why don't we give one of these like to-

598. MICHAEL: Brian, how many did you get, get so far? [Romina and Jeff wrote a number in each of the squares in a three by two rectangle that they drew on their grid. The top row contains the numbers 2, 3 and 4 and the bottom row 3, 5 and 7.]

599. JEFF: To like here. [Pointing to the intersection point $(9,3)$ on the grid of the problem sheet in front of Romina.] No / / and then-

600. ROMINA: / / / Because it's going to be too much. Well, / / go down and see like when we go down and we do all these and all of these that go out one more and see how much you get. [Pointing to intersection points $(5,4)$, $(6,4)$, $(4,1)$, $(5,2)$, $(6,3)$, and $(7,4)$.]

601. JEFF: / / For the red one, sorry.

602. JEFF: All right.

603. ROMINA: One- [Romina starts tracing routes to $(5,4)$ with her pen on the grid.]

604. BRIAN: I'm not good at this kind of stuff.

605. ROMINA: / / one, two, three, four, five.

606. JEFF: Where- where you going to?

607. ROMINA: Here, this is- this is five. [Writing a 5 in the $(5,4)$ square.] And, go to this one now because- [Pointing at intersection point $(6,4)$.] / / I mean that one I'm pretty sure. [Referring to the result obtain for the point $(5,4)$.]

608. JEFF: / / Was it four by four?

609. ROMINA: Uh, four by two.

610. JEFF: That's what I meant. I was drawing the right thing. [Jeff draws a four-by-two sub-grid on a sheet of 1-centimeter grid paper and draws routes within the sub-grid.]

611. ROMINA: Yeah, it's working. [Romina writes a 9 in the $(6,4)$ square.]

612. JEFF: Wait, you only got nine for that!?

613. ROMINA: Uh hmm. [Romina writes a 4 in the square in the third row, under the 3 in the second row, after counting routes with a pen on the grid.]

614. JEFF: All right, wait a second. Check it out. Um, all right. You go one-

615. ROMINA: All right.

616. JEFF: Wait- just wait a second.

617. ROMINA: No, I know. I'm just- One-

618. JEFF: One. Then two. [Drawing routes on grid paper.]

619. ROMINA: Uh hmm.

620. JEFF: And then- Uh, three, four, five. [Drawing routes on grid paper.]

621. ROMINA: Uh hmm.

622. JEFF: Six, seven, eight, nine, ten, eleven- you know what I'm saying? We're missing- [drawing routes on the grid]

623. ROMINA: Okay, what am I missing?

624. JEFF: You're- we're like

625. ROMINA: Did we do that for seven?

626. JEFF: Well you're- I don't know. You're not going like over two down one. //Over two over one. [Jeff motions with his pen on the grid.]

627. ROMINA: //I'm not doing [inaudible]. I'm not doing that.

628. JEFF: So-

629. ROMINA: Okay.

630. JEFF: You want to go back from the //beginning.

631. ROMINA: //Go back to- //go back to seven.

632. JEFF: //You got to go- Well, how do you know- we did five right?

633. ROMINA: We had to have done five because there was like-

634. JEFF: As long as it's right I don't- I don't care. Just as long as it's right. All right, so, which one's the seven one? Two by three?

635. ROMINA: I got eight for that, right? [Jeff draws routes on the grid.]

636. JEFF: Seven, eight- I got more than that. All right, wait. We got to go through this, and you got to watch.

637. BRIAN: I got at least twenty-two for red. I assure you of that.

638. JEFF: Assure you?

639. BRIAN: It's not raining no more. I'm sweating.

640. ROMINA: Yeah.

641. JEFF: Yeah, it's like the hot seat. All right, check it out. One- [On 1-centimeter grid paper, drawing a route in a two-by-three sub-grid.]

642. ROMINA: Mm hmm.

643. JEFF: All right. There's only one you can go by going two down. I'm trying to like figure out ways to like cross them out. You know what I'm saying? And then going one down, you can go one, two, three- There's no other ways to go. [Drawing more routes on his 1-centimeter grid paper.]

644. ROMINA: Mm hmm.

645. JEFF: What about like that? Four?

646. ROMINA: Mm hmm. Mm hmm.

647. JEFF: And then, five, six, seven, eight- [Counting the routes as he draws them.]

648. ROMINA: You already did that one.

649. BRAIN: / / I don't remember if I did that.

650. JEFF: Which one?

651. BRIAN: / /There's definitely twenty-three.

652. ROMINA: All right guys. This is what we're trying to do. Why don't we try to do this- [Taking a blank piece of 1-centimeter grid paper.]

653. JEFF: All right, what's-

654. ROMINA: We're getting all confused. You see how we're like going to like we're drawing like we're going to here. How many it takes to get to that point and then we're going to here and it's like a- this is just going up like one, two, three- two, three, four, five and then we go down to here and there's the same thing and then like how much we'll get to this point and how much we'll get to that point. [Pointing to intersection points on a blank 1-centimeter grid paper.] Why don't we all try to do that because we're getting confused and we're-

655. JEFF: Yeah.

656. ROMINA: We're doing the same mistakes.

657. JEFF: And it's like real hard. My brain-

658. ROMINA: If we do that and we see a pattern I'm sure we'll be able to- uh

659. JEFF: Hey, you know what we could even do, we could, uh- where are those transparencies? We could exploit the fact that we have those. You know what I'm saying? Like- [Michael silently writes.]

660. ROMINA: Well I was going to go over like to see how far we've gone. That's good. Oh, that's not the same side. [Romina takes a transparency with a grid on it.]

661. JEFF: No, but even- I mean you could say, all right, um, on-on the- you could do, um, a hundred six squares here. You could do- [Pointing at paper.]

662. ROMINA: Yeah.

663. JEFF: You know what I'm saying? And then just-

664. ROMINA: So- we definitely know this is two, right?

665. JEFF: Here, knock yourself out.

666. ROMINA: We definitely know that's two, right? Now give me a blank. [Romina writes 2, 3 and 4 in the first row of the grid and a 3 directly below the 2.]

667. JEFF: Well go to- which one are we having the most trouble with right now?

668. ROMINA: Well, I'm just going to write numbers. Two-

669. JEFF: Yeah, but I'm saying like before we get involved in all this, let's find out like how many there are and- [Michael and Brian silently write.]

670. ROMINA: Okay, let's make sure that's- Let's make sure how much that is. I'm going to go with that's- [Jeff draws rows of two by twos while Romina rewrites her numbers in the squares, only writing the top 3 numbers and the number in the second row, first position on the left.]

671. ROMINA: I think that's five.

672. JEFF: That is five? [On centimeter grid paper, drawing three horizontal lines across the page, creating two sets of parallel lines 2 centimeters away from each other.]

673. ROMINA: Mm hmm. Oh well, you do it too.

674. JEFF: Oh.

675. ROMINA: I mean-

676. JEFF: Which- by what by what?

677. ROMINA: Two by two.

678. JEFF: Two by two?

679. ROMINA: Let's get that done.

680. JEFF: One. All right. One, two, three, four, five- [Using a transparency of a centimeter grid paper, traces in the air shortest routes for a 2 by 2 square.]

681. ROMINA: You're counting one twice.

682. JEFF: Six- All right, wait. That's why- here watch.

683. ROMINA: Maybe, yeah.

684. JEFF: You just go make two by twos. [Drawing three vertical lines on his centimeter paper to create two-by-two sub-grids.]

685. ROMINA: Mm hmm.

686. JEFF: You could go- [Drawing two-by-two sub-grids.]

687. ROMINA: Yeah, make at least six at the moment.

688. JEFF: All right. You can go this way. [Drawing 1 two-down route.]

689. ROMINA: Yeah that's one.

690. JEFF: That's one. Now that's all the ways you can go- [Drawing a route.]

691. ROMINA: Yeah.

692. JEFF: -by two down. So then you can go like this...

693. ROMINA: Two.

694. JEFF: You can go like this. [Drawing 2 one-down routes.]

695. ROMINA: Three.

696. JEFF: Is there any other ways to go by going down? No.

697. ROMINA: Okay.

698. JEFF: All right. So then you could- you could go like that? [Drawing 2 one-over routes.]

699. ROMINA: Mm hmm.

700. JEFF: You could go like that. [Drawing 1 two-over route.]

701. ROMINA: Mm hmm. Or you go all the way top to bottom.

702. JEFF: There's nothing else to do? Right? Now that would be the opposite of that one. That would be the opposite of that one and that would be the opposite of that one.

		//They're all covered. [Pointing to pairs of routes on the grid with a pen.]
703.	ROMINA:	//So we got six. Good. Good thing we did that over again.
704.	JEFF:	All right, well. Yeah, good because at least we're- you know, //we're- we're making progress. [Romina writing 6 on her transparency of centimeter paper in the square that represents a two-by-two grid.]
705.	ROMINA:	//Yeah, all right. And go-
706.	JEFF:	All right. //The three-
707.	ROMINA:	//Three and two. [Jeff draws three vertical lines, creating four three-by-two rectangles.]
708.	JEFF:	The greatest MC in the world. [Singing.] Look at that. Beautiful. [Drawing three-by-two rectangles on the grid and crossing out the others]
709.	ROMINA:	Tell me you know how to count those. All right. [Jeff crossing out the 6 different 2 by 2s he just drew shortest routes on.]
710.	JEFF:	All right. We can go like this, and that's the only way- [Drawing the one 2-down, 3-across route.]
711.	ROMINA:	Right.
712.	JEFF:	-to do that.
713.	ROMINA:	You want-, you want to do them in couples?
714.	JEFF:	Now the opposite of that is that right there. So that's- that covers those two. [Underneath the previous route, drawing the one three-over route.] Now, the other way- now we've got to go one down like that. [Using a red marker to draw a route one-down, three-over.] And the couple of that would be- //I'm not-
715.	ROMINA:	//[Inaudible].
716.	JEFF:	-not exactly sure so wait.
717.	ROMINA:	We can't go in couples I mean.
718.	JEFF:	Yeah well-

719.	ROMINA:	All right, I'm going to open the windows.
720.	JEFF:	Ah yeah? [Draws 2 more one-down routes in his 3-by-2 grids.]
721.	Cameraperson:	What's making noise here? [Inaudible]?
722.	ROMINA:	[Inaudible].
723.	Cameraperson:	I understand.
724.	JEFF:	All right.
725.	ROMINA:	What'd you get?
726.	JEFF:	I don't know. I'm waiting for you, man.
727.	ROMINA:	All right. One, two-
728.	JEFF:	All right. That's that- [With his pen, pointing at the different three-by-two routes on the grid in which the first move is one down.]
729.	ROMINA:	Mm hmm.
730.	JEFF:	And that's that. And then, you know, that's going one over. It's going two over. It's going //three over.
731.	ROMINA:	//Three over.
732.	JEFF:	That covers all going through the middle.
733.	ROMINA:	Mm hmm.
734.	JEFF:	Correct?
735.	ROMINA:	Yes.
736.	JEFF:	All right. So now we've got to start going to the top. You can go one over, down, over. You can go one over or two over, down. You could also go one over, down two and over. You could also go- [Drawing the route.]
737.	ROMINA:	We've got eight so far, right?
738.	JEFF:	Could also go, um, two over, two down and over. [Draws the route.]

739. ROMINA: Mm hmm.

740. JEFF: Anything else? That's one, two, three, four, five, six, seven, eight, nine. Oh, it's nine because that one doesn't have a couple.

741. ROMINA: Yeah, //okay.

742. JEFF: //Those are couples, uh, this one and that one are couples, uh- [Pointing to routes and matching them with marker.]

743. ROMINA: //The one going-

744. JEFF: //These two are couples. [Pointing with marker.]

745. ROMINA: The one going all the way across in the middle is never going to have a //couple.

746. JEFF: //Never going to have a couple.

747. ROMINA: Because-

748. JEFF: That's- //so that will always be odd.

749. ROMINA: //All right, so you can't [Inaudible].

750. JEFF: So every other one will be odd because there will be one going fully across the middle. Right? That's why that's nine.

751. ROMINA: Well that can't be odd because it's-

752. JEFF: Hey- 'cause that- that won't-

753. ROMINA: Maybe any one with an odd length or width.

754. JEFF: Which would be every other one.

755. ROMINA: Yeah.

756. JEFF: All right. So now where are we at? [Romina writing a 5 next to the 4 and I notice she has a 6 and 9 next to the 3 in the second row. She also writes a 4 under the 3 in the second row.]

757. ROMINA: This is five. Okay, do you want to go down this- has to be four [Inaudible]. This should be nine too. [Pointing to the square below the 6.]

758.	JEFF:	Right. Because that is the- that is the same as that. [Jeff rotates the grid that Romina is writing numbers on.] Exactly. So that should be nine too. And- all right, do you want to go three by three? [Pointing to the routes and matching them.]
759.	ROMINA:	Yeah.
760.	JEFF:	You write it.
761.	ROMINA:	Um,-
762.	JEFF:	Cut a long one.
763.	ROMINA:	[Inaudible].
764.	JEFF:	It's you?
765.	ROMINA:	No. I thought it was you.
766.	JEFF:	Are you serious?
767.	ROMINA:	No, I didn't know.
768.	JEFF:	Oh man. It's the greatest MC in the world. [Singing.] [Brian working silently. He has a symbol for 10 now.]
769.	JEFF:	You're going to draw the lines on this one because it's- [Jeff draws rows of 3 by 3 rectangles.]
770.	ROMINA:	So I can mess it up?
771.	JEFF:	Yup. Because I can't handle it no more.
772.	BRIAN:	Oh man.
773.	ROMINA:	He's getting a little like kidish.
774.	BRIAN:	What are you starting?
775.	JEFF:	None of that.
776.	ROMINA:	One-
777.	BRIAN:	Isn't your head like-
778.	ROMINA:	Two, right? [Drawing a route in the first and second three by three on the grid.]

779.	JEFF:	Opposites. Uh, wait, wait, wait. Stick with a pattern. Do all the ones like going down first. You know what I'm saying? [Pointing to paper in front of Romina.]
780.	ROMINA:	I was going to do all the ones going across first. [Pointing to paper]
781.	JEFF:	All right. Then do that. But, [Inaudible] we're just doing all the ones that are going like one across.
782.	ROMINA:	Like I'm going to do two to one. Instead of one to-
783.	JEFF:	All right. Just don't blow it.
784.	ROMINA:	One.
785.	JEFF:	Where you going with that?
786.	ROMINA:	Uh, well, you know what //I meant.
787.	JEFF:	//Go for the whole deal now.
788.	ROMINA:	Should I- on the next one should I go all the way down? [Drawing routes]
789.	JEFF:	Yeah. That's another two over piece.
790.	ROMINA:	Okay. I'm going to go one- one over, one two over, one three over. All those? [Drawing routes]
791.	JEFF:	Yeah. And those are all the ways that you can go from the top over?
792.	ROMINA:	Yeah. Now going down.
793.	JEFF:	Now- now before you even do that, can't you just move these all the other way?
794.	ROMINA:	Yeah I know but- shouldn't we draw them just to make sure though?
795.	JEFF:	Yeah. Well let's do the opposites then like the same way we did the other thing.
796.	ROMINA:	Okay, so now we're going to go //two down. [Pointing to paper]
797.	JEFF:	//So, two down over-

798.	ROMINA:	Over- / / All the way?
799.	JEFF:	/ / All the way.
800.	ROMINA:	Down.
801.	JEFF:	Right? Where's that there? That's- that's that right there. One over- [Jeff marks off two of the 3 by 3s to show couples.]
802.	ROMINA:	All right.
803.	JEFF:	So you could either just like, uh- [Drawing routes.]
804.	ROMINA:	Down- Here? [Drawing routes.]
805.	JEFF:	You can do whatever you want and we'll just- all right- [Marking off two more 3 by 3s with routes drawn.]
806.	ROMINA:	Down over-
807.	JEFF:	Uh, where do you see- All right. Um, where's that one the other way? [He marks off 2 more three by threes with routes drawn.]
808.	ROMINA:	Hold on not yet- I'm All messed up now. Okay, so now I'm going to go down one- [drawing routes]
809.	JEFF:	Mm hmm.
810.	ROMINA:	Over down across. [Drawing a horizontal line of their grid paper] Down one- down one over two. Okay.
811.	JEFF:	All right, wait, I 've got a question. Where's the other one of this? That's the other one of that? But, that's already the other one of that. So there's more ways that you can have two boxes open- Yeah, it's getting- it's getting heavy. All right. Well, just continue. [pointing at paper]
812.	ROMINA:	Down one. Down one over two. I already went down two. I could do- I could do one of these little babies. [drawing a "staircase" route]
813.	JEFF:	What? You don't know? What about the opposite of that. Of that one. Got that?

814.	ROMINA:	I already did the ones that are in bold. [Drawing another "staircase" route]
815.	JEFF:	How many are there?
816.	ROMINA:	One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen - It doesn't work.
817.	JEFF:	That's all of them?
818.	ROMINA:	Well, no it's working. It's going. It's going.
819.	JEFF:	[Inaudible]. [Counting routes in the air with his marker silently.]
820.	JEFF:	Hmm.
821.	ROMINA:	Hmm. [Jeff points to the numbers on Romina's grid.]
822.	JEFF:	Uh, but are we sure there's only fifteen? Is my question. Coming out [singing, inaudible] What about like, um, down one- Where's all the down ones? All starting here?
823.	ROMINA:	Second row.
824.	JEFF:	Uh, down one over two- Where's that? You got that?
825.	ROMINA:	Down one- down one's are over here.
826.	JEFF:	Down one, over two. Hmm- [Drawing a route then crosses it out.]
827.	ROMINA:	You just draw, I'll find it. [Inaudible] yeah-
828.	JEFF:	That one's already there. Huh? Come here. Maybe there is only fifteen. Or- I wish we knew like-
829.	ROMINA:	Three by three? Is that what we're doing?
830.	JEFF:	I guess. Put it in for now. [Romina writing a 15 in the square next to the 9 so the 3rd row reads 4, 9, 15.]
831.	ROMINA:	What are you working on? //The red?
832.	MICHAEL:	//Uh, yeah. Did you do the red ones?
833.	JEFF:	We're working on, uh-

834.	ROMINA:	Oh, we're getting there.
835.	JEFF:	We're / /just working on stuff.
836.	ROMINA:	/ /Where's red? Three by four? That's our next one.
837.	JEFF:	Nah, [inaudible]. So, if we could get to there it would be, uh, big time you know what I'm saying?
838.	ROMINA:	It's this one.
839.	JEFF:	This one right here?
840.	ROMINA:	Mm hmm.
841.	JEFF:	Three by-
842.	ROMINA:	Yeah.
843.	JEFF:	All right, well we're on- what are we on? Two by four? [Brian writing rows of numbers 0, 1, 2 or 3 silently. Michael has routes drawn all over his paper. He continues working.]
844.	ROMINA:	Mm hmm. That's not going to be enough.
845.	JEFF:	That's all right. We can make more.
846.	ROMINA:	Now I'm really starting to hate doing this.
847.	JEFF:	Oh yeah?
848.	ROMINA:	One.
849.	JEFF:	Yep.
850.	ROMINA:	Two. All right.
851.	JEFF:	Go all the way across.
852.	ROMINA:	Two down? [Romina draws routes.]
853.	JEFF:	We already did all the two downs. [Pointing to paper]
854.	ROMINA:	What if we go two down and across. / /Did that.
855.	JEFF:	/ /We already did that. Um, what about- All right that's all of / /them.

856. ROMINA: / /Over one. [pointing to paper]

857. JEFF: All right.

858. ROMINA: Over one. [Romina draws routes for each four by two rectangle.]

859. JEFF: Uh um. Wait, stay with- go over ones. Do all the over ones.

860. ROMINA: Oh I was going- I was doing this / /one. [Pointing to paper] [She continues drawing routes.]

861. JEFF: / /All right then. Well, whatever you want to do. You're out of control.

862. ROMINA: Now, I want to go over one.

863. MICHAEL: What'd you get so far for the red one? What are you up to / / now?

864. ROMINA: / / Can I go anymore?

865. BRIAN: Before I got at least like thirty.

866. MICHAEL: How many? Thirty?

867. BRIAN: Right now I'm actually writing them out, I got like seventeen.

868. JEFF: Over.

869. ROMINA: No that's this one.

870. BRIAN: What do you have?

871. MICHAEL: I have to count them up.

872. JEFF: Oh man. Wait, there should be one more.

873. ROMINA: Yeah, I'm drawing-

874. JEFF: I'm saying - I think there should be one more.

875. MICHAEL: I got thirty-four.

876. BRIAN: So far?

877. MICHAEL: That might be it. [Counting routes on Brian's grid. Romina counting with a pen over grid in air.]

878. JEFF: Um- No, over to there, down there, like that. [Drawing routes]

879. ROMINA: [Inaudible]. [Brian working with the rows of numbers 0,1,2 or 3 and adding more rows.]

880. JEFF: You got that?

881. BRIAN: Man, I'm giving up on it.

882. ROMINA: Hum, that's kind of weird.

883. BRIAN: Mike here's the list. So far I've got some of the things [Inaudible].

884. MICHAEL: How many have you got there, thirty?

885. BRIAN: No that's only like //[Inaudible].

886. JEFF: //I think- I think we should like [inaudible] on like the next one we do, I think we should just like //do all ones over one. [Motioning across with his pen.]

887. BRIAN: //D's is like down- //down one.

888. JEFF: //All those, you know what I'm saying?

889. BRIAN: Like the order.

890. JEFF: What else? Is there anything else?

891. ROMINA: No.

892. JEFF: That should be it.

893. ROMINA: That looks nice too, what they're doing.

894. JEFF: What?

895. ROMINA: Brian, see that looks like a much- when you do like the-

896. BRIAN: Opposites and all that?

897. ROMINA: All right, this is- start counting- see, uh,- Which one's this?

898. JEFF: Well it's- well it should //be twelve.

899. BRIAN: //Oh man.

900. ROMINA: Twelve-

901. JEFF: You got the [Inaudible] right now.

902. BRIAN: All right //now this-

903. ROMINA: //That makes no sense. Oh well they're all – oh yeah they're all factors of three.

904. JEFF: Yeah, now this is the one that's- that's making it tough.

905. ROMINA: But this one has to be nine. One, two, three- Here Brian, do a box-

906. JEFF: Well- well you know what //that is?

907. ROMINA: //Two by three.

908. JEFF: That's plus five, plus six, maybe it's plus seven there? That's plus one, plus one, plus one. That's all plus threes. Or that's- I don't know what that is. I don't know. But that's plus- You know what I'm saying? Plus five, plus six? Plus fifteen plus seven um- twenty-two? [Placing his finger over each number on Romina's grid. Romina wrote a "12" next to the "9" in the second row of her grid.]

909. ROMINA: Mm hm.

910. JEFF: Is that an option for what that is?

911. ROMINA: How many did you guys get by the- the-

912. JEFF: To the red.

913. ROMINA: Three by- or four by three? To the red?

914. MICHAEL: I got thirty-four.

915. BRIAN: [Inaudible]. So what am I doing with the box two by three?

916. ROMINA: How many- how many to the one at this one. [Pointing to Brian's grid.]

917.	BRIAN:	From here to here?
918.	ROMINA:	Yeah. Do your down- do that cool number thing.
919.	JEFF:	Cool number thing. [Brian beginning writing rows of 2 numbers, then 3 numbers in a row.]
920.	ROMINA:	We have to have some of these wrong.
921.	JEFF:	Well just- I don't know, is this just twelve? I mean I'm saying we found this one like in a second. That's it? We quit after we found that one? [Pointing to paper]
922.	ROMINA:	Only because it's not like long enough to be going like zigzagging through. [Romina zigzagging with a pen in the air.]
923.	MICHAEL:	Which one, that point?
924.	ROMINA:	Mm hmm.
925.	MICHAEL:	What's that's two by-//.
926.	ROMINA:	He's doing two by three, now you do four by two.
927.	MICHAEL:	Let me finish the two by four.
928.	JEFF:	Here Mike, you got all this man.
929.	BRIAN:	Ohaa.
930.	JEFF:	Mike, because then you could- you see how we're doing it? Like you could just do, you know, on all different-
931.	ROMINA:	Because we're going to be working on the one that you just did now. What is this? Four by two?
932.	JEFF:	[Inaudible].
933.	BRIAN:	Do you have like a formula that you're wanting to see if it works with this one?
934.	ROMINA:	No, we're just guessing.
935.	JEFF:	What are you doing?
936.	ROMINA:	Nothing, I was just going to put these under here so-

937. JEFF: All right, just- [inaudible]

938. ROMINA: Sure I'll do it. Are you doing four by three?

939. JEFF: It's big money.

940. JEFF: Now you're cooking with oil. Good. She's really good.

941. ROMINA: [Inaudible]. [Brian continuing by writing rows of 4 numbers 0, 1, 2 or 3. Romina makes rows of 4 by 3s on the grid.]

942. JEFF: Oh yeah?

943. ROMINA: Here, and we'll show- we'll even show you our patterns.

944. JEFF: Well wait, let him do his first.

945. ROMINA: Yeah, then-

946. JEFF: Because you're going to- it'll- gets in your [Inaudible] brain.

947. ROMINA: All right. Here why- if you have an organized way why don't you do it?

948. JEFF: All right- All the ones that you can go three down and get- Right, come over here. There's that and that's all the ones you can go three down and get. Right?

949. ROMINA: Mm hmm.

950. JEFF: All right. So going two down, if you go two down then you could either go, um,-

951. ROMINA: Over one down? [Michael counts with the pen on grid silently.]

952. JEFF: Over one down and over. If you go two down, you can go over two, down and over. You go two down, you can go over three down and over. You go two down, you can go over four and down. Um, is there any other place you can go if you go two down? No. What about- yes you can. You can go two down, over two, uh- No you can't. I was going to say and then down and over but we //already got that. [drawing routes on the 4 by 3 rectangles]

953. ROMINA: //You just messed up the box. [Romina cross out the box, 4x3 sub-grid, in which Jeff drew an incorrect route.]

954. JEFF: You're out of control.

955. ROMINA: All right. Now go one down?

956. JEFF: You go one down. You could go all the way over. You go one down you can go almost all the way over. You go one down, you can go two over. You go one down, you can go one over. Now you can go one down- [Drawing routes on the 4 by 3 rectangles.]

957. BRIAN: //How many did you think was going to be for this one?

958. ROMINA: Nine.

959. JEFF: What'd you get?

960. BRIAN: Ten.

961. JEFF: Do you know what they are?

962. BRIAN: Yeah.

963. JEFF: Can you do them- can you do it //on something like this?

964. ROMINA: //Here- Where's- where's ours?

965. JEFF: Which one is-

966. MICHAEL: I got twelve for the one you've just got-

967. JEFF: For the one we got twelve for? //All right.

968. ROMINA: //Here. Those are the ones we have for that one.

969. MICHAEL: They probably the same thing

970. ROMINA: Yes.

971. JEFF: What are you looking for [inaudible]

972. ROMINA: Did- we did that one.

973. JEFF: All right.

974. BRIAN: What did- do you know what your twelve are?

975. ROMINA: Nine.

976. JEFF: One, two-

977. MICHAEL: Me?

978. BRIAN: //Him.

979. ROMINA: //Oh.

980. MICHAEL: I haven't- don't have them written down but I know-

981. ROMINA: These are our twelve. [Handing Brian a sheet containing her and Jeff's work counting routs on a 3x2 sub-grid.]

982. BRIAN: All right, let me do it on the board for you.

983. ROMINA: Mike, do three over and two down.

984. MICHAEL: Huh?

985. ROMINA: three over and two down.

986. BRIAN: Writing up on the board. [Jeff drawing a four by 3, draws in routes, then crosses it out with his pen in the air. Brian draws his symbols or taxonomy of routes on the board with a number next to each edge.]

987. JEFF: One, two- Uh, that's [Inaudible].

988. ROMINA: Couldn't we just do something like in towers where like lines over are like the color and the lines down are the, um, number of blocks?

989. JEFF: All right. And that would?

990. ROMINA: Because, okay, lines over- because what is it- the number of blocks to the number of colors?

991. JEFF: I don't know what you're- what- what's that?

992. ROMINA: Two to the n. Two is the amount of blocks or the colors?

993. MICHAEL: For what? Like towers on them?

994. ROMINA: Yeah.

995. JEFF: Colors. *n* is the number of blocks. I think. I don't know. I'm not sure.

996. MICHAEL: Well you figure a block has this- you got two- two ten over like this. Or two colors actually. I think it's, uh, the colors and *n* is the blocks.

997. ROMINA: Color two- / /right. [Writing the words "color" and "blocks" on a piece of paper.]

998. JEFF: / /Same thing.

999. ROMINA: All right, here we have one color- nah; it doesn't work for the first one.

1000. ROMINA: Scratch that idea. [Crossing out the words on her paper.]

1001. JEFF: Well- why- you know, what happened to- to what we were doing?

1002. ROMINA: No, I know. Just keep on going. [Jeff, Brian and Michael working silently.]

1003. JEFF: All right.

1004. ROMINA: You're right [inaudible] three by two.

1005. JEFF: Can you help me out?

1006. ROMINA: What- what [Inaudible] / /by two of this sheet?

1007. BRIAN: / /That's what I got so far.

1008. ROMINA: / /You need one [Inaudible]?

1009. BRIAN: / /That's how far right there. It's on the board. / /The board.

1010. ROMINA: / /I know, I'm looking for- [Jeff continuing to draw routes.]

1011. BRIAN: Mike do you see anything that I'm not getting?

1012. ROMINA: / /Three by three.

1013. MICHAEL: //Which one you doing?

1014. BRIAN: Two by three.

1015. ROMINA: Three by two. All right, here. This is what we got.

1016. JEFF: It's really hot in here.

1017. ROMINA: All right, we got down two over three. Over three, down two. [Brian drawing routes on the chalkboard while Romina reads off her possibilities.]

1018. BRIAN: //Okay.

1019. ROMINA: //That's one of those? The first one.

1020. BRIAN: [Inaudible].

1021. ROMINA: All right we got those. [Brian continues writing on the chalkboard.] Got a down one over three.

1022. ROMINA: Except they don't have one, one, one, one, one, that one.

1023. JEFF: That's one we don't have?

1024. ROMINA: We don't have his last one over there. Check. I think that was the only one. So that nine does equal ten. [Brian writing, "start over" on the chalkboard and the word "Moves" up top.]

1025. JEFF: I don't see uh- Um-

1026. ROMINA: Because we don't have that one?

1027. JEFF: No, we don't have that one. [Inaudible]. [Romina erases the 9s and writes in 10s. She also writes a 5 under the 4.]

1028. ROMINA: All right. It's, um, - it's Pascal's triangle. [Looking at the numerical array of the 1-centimeter-grid transparency.]

1029. MICHAEL: What is that? Two by three? [Looking and pointing to Brian's inscription on the classroom chalkboard.]

1030. JEFF: It is?

1031. ROMINA: Yeah.

1032. JEFF:	Let me see.
1033. ROMINA:	All right. Yeah it is.
1034. MICHAEL:	What?
1035. ROMINA:	It's Pascal's triangle.
1036. MICHAEL:	Two, three-
1037. ROMINA:	No, it's not. It doesn't work out.
1038. JEFF:	See look at- Here, Mike-
1039. ROMINA:	Because twelve that doesn't-
1040. JEFF:	Mike look- just look at it in this thing. You got the 6 and the 4 and the 6 are the 10. That should be a 15- //that's should be a 20- [Pointing to the 1-centimeter transparency grid that is in front of Romina.]
1041. ROMINA:	//But that's not a 15. That is a twelve because he even got the 12.
1042. JEFF:	Well- that should- that should be a 20 right there. [Pointing to the square (6,3) on the transparency that contains the datum 15.]
1043. ROMINA:	[Inaudible].
1044. MICHAEL:	Up to here is been a one, one, one, one and-
1045. JEFF:	Huh.
1046. BRIAN:	So what's wrong?
1047. MICHAEL:	It should be six- fifteen.
1048. ROMINA:	Do- do a four by two.
1049. MICHAEL:	Yeah.
1050. JEFF:	You do the four by two, and it should put us, uh, in business.
1051. BRIAN:	All right.

1052. ROMINA:	And then- because we'll compare it to all-
1053. JEFF:	If this comes through it just-
1054. ROMINA:	If it's Pascal's triangle it'll just give us problems.
1055. JEFF:	No but it- it's just nice how- you start- like when you start from nothing. You know what I'm saying? Like we have no clue what we're doing. [Putting his hand on his forehead and then waves his hand in the air by his forehead.]
1056. ROMINA:	But he even got twelve when he did it.
1057. MICHAEL:	I might be missing two.
1058. JEFF:	It could be- it's not hard to miss three, right? [Jeff waves his hand in the air.]
1059. MICHAEL:	Two.
1060. JEFF:	Three.
1061. ROMINA:	So for the next one Jeff we missed five?
1062. JEFF:	It's very easy. I mean, //there's a lot of things going on.
1063. MICHAEL:	//That's kind of a lot.
1064. JEFF:	We like blew like a lot of these. You know what I'm saying? [Waving his hand in the air and puts it back on his forehead.]
1065. ROMINA:	Yeah. I think we, uh, got a few wrong. So what.
1066. JEFF:	That's what I'm saying. So why- like it wouldn't be totally out of control. [Removing his hand from of his forehead and waving his hand toward the grid.]
1067. BRIAN:	Oh.
1068. ROMINA:	Do- do it the other way. Just turn it around. That'll make our life- that- because that's we did. It's the same thing but- [Brian writing rows of numbers silently, this time adding a 4 too.]
1069. BRIAN:	Is that the air that just turned on?

1070. JEFF:	Yeah. But, it don't work though.
1071. ROMINA:	I'll be right back. [Leaving the room.]
1072. JEFF:	So how do you do your deal? I don't know how to do your deal.
1073. BRIAN:	It's nothing, the ones with two moves, the ones with three moves so I just go like three moves over- starting over first. over three down two boom, boom boom, boom boom. Then, then I go to over down, over down. This row gets eliminated pretty much. [Jeff nods his head at Brian.]
1074. JEFF:	All right. But you're not going to get there. I hear you. [Jeff shakes his head "no" and then him and Brian work silently.]
1075. JEFF:	It'd be so much easier if some of us were lefties. [Brian already wrote his rows of 2 numbers, 3 numbers and now is writing rows of 4 numbers choosing from 0, 1, 2, 3 and 4]
1076. BRIAN:	Why?
1077. JEFF:	You'd just block like, uh, you try to see what someone does and it's just like- I mean like what is Mike looking at when I'm writing right now. //You know what I'm saying?
1078. MICHAEL:	//Oh yeah. [Michael is drawing routes in the air on top of the 4 by 2 rectangle drawn on his grid.]
1079. JEFF:	It's like, what- Which one are you doing, man?
1080. MICHAEL:	We're looking for fifteen for this one, right? [Brian works silently.]
1081. JEFF:	Didn't you get that?
1082. MICHAEL:	Hmmm. Don't know, yet. [Jeff works silently.]
1083. BRIAN:	What number are we looking for on this one?
1084. JEFF:	Fifteen. How many you got?
1085. BRIAN:	Eight. [Jeff is drawing routes on his 4 by 2 rectangles.]

1086. JEFF: When it rained he went home and as soon as it stopped he came back out. Annoying bastard. I can't take it no more. [Inaudible singing]. [He is referring to the driver of an ice cream truck the noise that the truck makes.]

1087. MICHAEL: What is that?

1088. BRIAN: Zero.

1089. MICHAEL: I got twelve so far.

1090. JEFF: Yeah, twelve's like the number that we got stuck on last time.

1091. MICHAEL: I think I got it.

1092. BRIAN: [inaudible] it's fifteen.

1093. JEFF: All right. All right, what if we even went- let me know when you're done. All right. Because there's an easier way to [Inaudible]. Listen to me for one sec.

1094. MICHAEL: Go ahead.

1095. JEFF: All right. If- all right. Say in a situation where it's like, uh a two by four. [Drawing a two-by-four sub-grid on 1-centimeter paper.]

1096. MICHAEL: Uh hum.

1097. JEFF: All right. If we know that in a four-by-four [really meaning a two-by-two] it's six [shortest routes] if you figure out all the ways to get to the beginning parts of this, this would all just be six different ways to get from here to here. So you figure out all the ways to get there and you could just add six- you know. [Subdividing the two-by-four sub-grid into two two-by-two sub-grids.]

1098. MICHAEL: If you have the two, you could find out how many ways it's to get to here and add that where every two is. [Leaning over to Jeff's paper and pointing.]

1099. JEFF: You know what I'm saying? So like from- from-

1100. BRIAN: I got fifteen.

1101. JEFF: You did?

1102. BRIAN: Yeah.

1103. JEFF:	All right. 'Cause from there to there you have six different ways. And then from there, there's one way. To there there's one way and from there- / /
1104. BRIAN:	/ / Haaa. Tell me when you're done.
1105. JEFF:	Sure. one- two- there's three ways. Um-
1106. MICHAEL:	I got fifteen also.
1107. BRIAN:	Yeah Mike. [Inaudible]. [Leaving the room.]
1108. MICHAEL:	So what does that mean?
1109. JEFF:	It means that it is the triangle. Right here? [Pointing to paper]
1110. MICHAEL:	Mm hm.
1111. JEFF:	You have fifteen there?
1112. MICHAEL:	I got fifteen.
1113. JEFF:	That's good. Yeah, because then- Yeah. This- then in a three by three it should be twenty. That'll be, uh,- [Pointing to paper and writing a 6 in the lower right hand square.]
1114. MICHAEL:	Is nine blocks for that one? [Pointing to intersection point (6,3) on the transparency]
1115. JEFF:	In the nine block it should be twenty. [Jeff writes the numbers 1, 3, and 6 in squares vertically with two 3s to the left of the other 3.] [inaudible]
1116. MICHAEL:	Where'd they go?
1117. JEFF:	What?
1118. MICHAEL:	Where'd they go?
1119. [Brian returns.]	
1120. JEFF:	Well, Brian's just standing there. I don't know. I need to go to the bathroom though. I'm like going to pass out. [Jeff leaves.]
1121. BRIAN:	So what are we doing now.
1122. MICHAEL:	No idea.

1123. BRIAN: Did you figure out the five by five?

1124. MICHAEL: Five by five? I'm doing three by three right now.

1125. BRIAN: It's the green. If we already know what it is then we have to figure out-

1126. MICHAEL: I just want to make sure that's twenty first, so- [Counting routes with his pen on his grid.]

1127. MICHAEL: I'm missing two. That's probably right though.

1128. BRIAN: Did you get the, uh, staircase one?

1129. MICHAEL: Which one? For the three by three?

1130. BRIAN: Yeah. [Romina returning.] All right, I got twelve, fifteen, right.

1131. ROMINA: Oh, you guys went and wrote on this didn't you?

1132. MICHAEL: I didn't do it.

1133. ROMINA: All right.

1134. BRIAN: Did Jeff tell you?

1135. ROMINA: What?

1136. BRIAN: I got fifteen for this one-

1137. ROMINA: For which one?

1138. MICHAEL: //For-

1139. BRIAN: //Four by two.

1140. ROMINA: So you did get fifteen? So now it's working? And then the two by four has to be fifteen too. Now if we do three by three and that's twenty, then we're done.

1141. BRIAN: That's what he's doing?

1142. ROMINA: What?

1143. BRIAN: He said he was off by two. [Romina erasing the numbers on the grid transparency then takes a new

	transparency with a grid on it.] You can just get another one.
1144. ROMINA:	I'll just turn this around. [Referring to the transparency of a centimeter grid. She then writes 2 and 3 in the squares of the first row of the transparency.]
1145. BRIAN:	It's only a couple of numbers.
1146. ROMINA:	Did it again. You got twelve for this one? Fifteen, I mean? [Rewriting the numbers on the grid and adds a 15 to the right of the 10 and under the 10.]
1147. BRIAN:	Yep. Now, which one are you expecting to be twenty? Three by three?
1148. BRIAN:	I guess I'll do it. Check it out.
1149. ROMINA:	I don't think- it's here- he has- He was just doing three by three wasn't he? [Looking through her papers.]
1150. BRIAN:	Yeah. It's no big deal.
1151. ROMINA:	I'm already stuck. [Brian drawing a three-by-three sub-grid on his paper. Romina draws in shortest routes for the "imaginary" three-by-threes on her grid. Romina's pen stops when drawing a route.]
1152. JEFF:	You shouldn't be. Where you going?
1153. ROMINA:	Three by three. [Showing the paper to Jeff.]
1154. JEFF:	You said F making the- the boxes.
1155. MICHAEL:	Yeah, I got twenty for that one.
1156. JEFF:	For three by three?
1157. MICHAEL:	Yeah.
1158. JEFF:	All right well then- I mean can't we explain why we think- well- all right. [Waving his hand.]
1159. MICHAEL:	/ /They're going to ask us-
1160. JEFF:	/ /All right then the next question is why- / /why-
1161. ROMINA:	/ /Now-

1162. MICHAEL: / /How do you know-

1163. ROMINA: / /Just relate this back to the / /blocks. [Pointing to the 1-centimer grid on the transparency with his marker.]

1164. JEFF: / /Wait- Why is this- why does the Pascal's triangle work for this is the question.

1165. ROMINA: / /Exactly. Relate it to the blocks.

1166. MICHAEL: / /Just think first how do you know it's twenty? You know, how do you know it's not nothing else?

1167. JEFF: Well F that. If we could explain-

1168. ROMINA: Stop saying that.

1169. JEFF: Why- why this is the Pascal's triangle up to here [Pointing to the numerical array on the transparency 1-centimer grid.], we don't need to explain how we're positive this is twenty. / /You know what I'm saying?

1170. ROMINA: How does it go- this is-

1171. JEFF: One-

1172. MICHAEL: It should be ones on all the sides. [Jeff writing ones on the outside lines of their numeral array on the transparency 1-centimer grid.]

1173. ROMINA: Yeah right. So- [Writing out Pascal's triangle.]

1174. JEFF: That's six-

1175. ROMINA: This is just one, two, three. / /So-

1176. JEFF: / /What's that?

1177. ROMINA: With one- / /there's only one possibility.

1178. MICHAEL: / /All right, how- / /how is he getting them?

1179. ROMINA: / /Two-

1180. MICHAEL: How are you getting yours? Maybe the way you're doing will give us-

1181. JEFF: Has some kind of- Yeah, we can work something out.

1182. BRIAN:	[Inaudible].
1183. MICHAEL:	Do you just like- are you guessing or do you have some kind of pattern?
1184. BRIAN:	I'm just- doing it man. I'm just- you know-
1185. MICHAEL:	Ah- [Romina pointing to the numbers on her transparency with a marker.]
1186. BRIAN:	I know there's a way to make two and get there in two moves. I know there's a way to make it in three moves. Four moves.
1187. MICHAEL:	So you're going by the moves, right?
1188. BRIAN:	Yeah.
1189. JEFF:	Don't use that one.
1190. ROMINA:	Hold on. For the Pascal's triangle-
1191. MICHAEL:	Yeah.
1192. JEFF:	You're making thumbprints again.
1193. ROMINA:	The one, //two, one-
1194. JEFF:	//Bringing it back to eighty-six.
1195. ROMINA:	-that's with what? With?
1196. MICHAEL:	Um-
1197. ROMINA:	Two colors- It's, it's two to the *x* like that? [Pointing to the second diagonal "row" of the array of numbers on the 1-centimeter-square transparency, containing the numerals 1, 2, 1.]
1198. MICHAEL:	Yeah it's two.
1199. ROMINA:	So it's two colors-
1200. MICHAEL:	Think of it as zero, one, two- you only have two colors of choices - zero, one, two. Three-
1201. ROMINA:	Huh
1202. MICHAEL:	Three toppings on a pizza.

1203. ROMINA: Yeah, like- so then how could this- this is two what? Two? Two different ways- like- [Pointing to the top numbers on the transparency with her marker.]

1204. MICHAEL: Two- Uh- it's the total. One, two, three- That's, that's the total length that you can get, have to get there- to get there. [Pointing at numbers on transparency with marker.]

1205. ROMINA: Yeah, okay.

1206. MICHAEL: You know?

1207. ROMINA: So for this one, the total length is three.

1208. MICHAEL: But then this one is one, two, three, four, five and you get ten. You know? [Pointing at the 6 on the transparency grid]

1209. ROMINA: But you're in the second row. [Pointing at triangle]

1210. MICHAEL: Yeah. [Romina taps her marker on the table.] Right. This is one, two, three- four, five, six and you get twenty. [Pointing at the 20 on the transparency grid.]

1211. ROMINA: All right.

1212. BRIAN: All right.

1213. ROMINA: I'm going to write it this way because I'm having a- I don't know about you people but- How does this go? It's not like in the blocks, is it?

1214. JEFF: What? For the thing?

1215. ROMINA: Yeah.

1216. JEFF: Yeah, it'll fit. Why- why don't you start like in the middle like here.

1217. ROMINA: Yeah.

1218. JEFF: Or why don't you use a different transparency?

1219. ROMINA: Well I just want it like- I'm just doing it so I can see it.

1220. MICHAEL: Why don't you do it like //that that way we can see it. [Pointing to the transparency grid with his marker.]

1221. JEFF:	//Why do you keep- //you're starting all the way over on the side every time.
1222. BRIAN:	//All right. There's twenty
1223. MICHAEL:	Start like this. It's easier to figure out like a two by two box. Over here [Inaudible].
1224. ROMINA:	No, I know. It's just- it's just so I can see it so that's one block, two block, three block, ok.
1225. JEFF:	All right.
1226. MICHAEL:	That would be seven- twenty-one- thirty-five-
1227. JEFF:	All right. You want to, um- You want to try and explain this and then wherever we get like confused along the way, you know maybe that's how we'll be able to- as we talk through it we could even- Oh sorry. I tried to stop it from hitting your leg. I don't even see it.
1228. MICHAEL:	It's a wet erase marker it will come off with water.
1229. BRIAN:	Oh man.
1230. JEFF:	At least you don't have grease all over your pants.
1231. ROMINA:	Yeah but this is like my favorite-
1232. JEFF:	Oh I hate these pens.
1233. ROMINA:	Would you lick it because my fingers have blue on them? Lick your fingers Jeff.
1234. JEFF:	It'll be all right. It's going to be there for a little while, you'll have to deal with it. All right, can we try to explain this?
1235. MICHAEL:	To who?
1236. JEFF:	Anyone who wants to hear it.
1237. JEFF:	Here's some water for it.
1238. BRIAN:	Jeff-
1239. JEFF:	All right. Uh-

1240.	MICHAEL:	Put the caps on so they don't roll.
1241.	BRIAN:	We've got like five minutes.
1242.	BRIAN:	You're just going to spread it all over the place.
1243.	JEFF:	Well don't get mad at me. Relax. I'm trying to work this out here. What do you think? Should //I continue.
1244.	BRIAN:	//You ain't working nothing that way.
1245.	JEFF:	What about a tissue that we could dab- we could-
1246.	BRIAN:	What about Romina, when you go home-
1247.	JEFF:	Yeah, you put a little stain stick on it-
1248.	BRIAN:	A little Shout. //It'll Shout it out.
1249.	JEFF:	Shout it out.
1250.	ROMINA:	I don't have those things.
1251.	JEFF:	You could go to the //store and pick them up.
1252.	BRIAN:	//Go to the store. You could buy little Shout wipes.
1253.	JEFF:	Yeah, they're real cheap. You could clean it up and you'll never have to worry about it.
1254.	ROMINA:	I'm very upset right now.
1255.	BRIAN:	How many they got?
1256.	JEFF:	Do you see this?
1257.	UNKNOWN:	Romina do you want a baby wipe?
1258.	ROMINA:	Yeah please. We'll try this.
1259.	JEFF:	Can I get one of those just for my hands?
1260.	UNKNOWN:	Yes.
1261.	ROMINA:	Yeah, my hands are-
1262.	UNKNOWN:	Anybody else? Baby wipe Brian?

1263. BRIAN: Nah, I'm clean man.

1264. UNKNOWN: Mike?

1265. MICHAEL: No I'm good.

1266. JEFF: Let me see your hands.

1267. BRIAN: I ain't working with the markers.

1268. JEFF: Oh that's-

1269. ROMINA: So when I asked someone to lick my shirt you were obviously not going to.

1270. BRIAN: Well Romina, I'm going to come over and lick your shirt. That's what I'm going to do.

1271. ROMINA: Lick your fingers. I didn't mean lick my shirt.

1272. BRIAN: And you see what it did? It spread it all over your shirt.

1273. JEFF: Why don't you go for that on your shirt?

1274. ROMINA: Because I'm going to try to-

1275. BRIAN: I love the smell of baby wipes dude.

1276. JEFF: They do smell good.

1277. ROMINA: Oh. It's just getting worse.

1278. BRIAN: Now it's going to be a wet stain.

1279. JEFF: Ah-

1280. BRIAN: Romina has it- Romina if it didn't-

1281. MICHAEL: You just better leave it.

1282. JEFF: If it- Just stop.

1283. BRIAN: You're making it worse.

1284. ROMINA: It's already bad.

1285. BRIAN: You're going to ruin it beyond repair.

1286. BRIAN:	Is she busy? She can't come and visit us?
1287. UNKNOWN:	She's just all the way down the hall.
1288. BRIAN:	All right.
1289. JEFF:	How- / /how are those kids doing?
1290. ROMINA:	/ /Yeah you know they're probably done with the assignment.
1291. JEFF:	Are they- / /what are they doing?
1292. ROMINA:	/ /They're on problem five.
1293. MICHAEL:	This takes a long time to figure this out.
1294. BRIAN:	You know probably- if we just think about it what do we work on every single / /time.
1295. JEFF:	/ /Yeah I know, but we go to-
1296. MICHAEL:	We got to explain it.
1297. JEFF:	You got to figure it out. You know what I'm saying? You got to go through it.
1298. BRIAN:	She's going to look at these, and she's like I have no idea what you're doing.
1299. JEFF:	I'm out- I've got to leave / /in five minutes.
1300. ROMINA:	/ /No she's going to go like this. You're still on this?
1301. JEFF:	She won't say that.
1302. MICHAEL:	No they got- they got a different problem than us.
1303. JEFF:	They have the same- do they have the same problem down- down there?
1304. ROMINA:	Are they done?
1305. RESEARCHER 1:	Um, they're working on a different problem.
1306. JEFF:	They have a different problem?
1307. BRIAN:	All right.

1308. ROMINA: Like they didn't get //this one to work on?

1309. MICHAEL: //We- we can't justify our answer but we're- we're, uh-

1310. JEFF: We're going to talk through it.

1311. ROMINA: //[Inaudible].

1312. JEFF: //And we want to see where that takes us. I'm going to have to leave in five minutes though.

1313. RESEARCHER 1: Oh so, you've got to talk fast.

1314. JEFF: So we're going to talk fast. All right.

1315. RESEARCHER 1: Okay, the problem I really wanted to give you was for all the points on the grid.

1316. ROMINA: Oh good. //We did that.

1317. JEFF: //Oh yeah. That's what we did.

1318. ROMINA: Why don't we do points-

1319. BRIAN: We got it. We got it.

1320. JEFF: You know-

1321. ROMINA: Points up to-

1322. RESEARCHER 1: All right. So tell me, tell me.

1323. BRIAN: Pens are flying now.

1324. RESEARCHER 1: Yeah. Did you like the problem?

1325. ROMINA: No. Nah, it was okay.

1326. JEFF: It's just-, doing all this kind of stuff really hurts your brain, but other than that //it was all right.

1327. ROMINA: //It- your eyes. All right. What we did is we- we took it-

1328. JEFF: We broke it down.

1329. ROMINA: Yeah, we just went from point to point on the thing.

1330. JEFF: Yeah. Like we even- we'll just say we started making the box like that. How many different ways can you get from this point to this point? You know, make an easier problem. Like the basic math- deal. [Romina draws in points with her marker and points to the numbers on the transparency grid.]

1331. ROMINA: So we did like up to this point there's two. Up to this point there's three, four, six, three- So that- those are our numbers. Those are up to the points like down and diagonal. And what we got is Pascal's triangle. [Jeff points to the numbers on the transparency grid with his marker.]

1332. JEFF: Yeah. We started, you know, and then as we started, you know, like it takes two to get to there. Three to- you know, to get there as Romina just went through and did. And then as we started filling it out we noticed that if you tilt it like that [Rotating the transparency.] and throw ones on the outside and a one on top, I mean you're looking at Pascal's triangle. And so we stopped at this point [Jeff points to a point on the transparency grid with his marker.] because I mean making, you know, like thirty plus different things like this it gets- it just gets confusing you know. [Drawing a curve on his paper.]

1333. RESEARCHER 1: Hm.

1334. JEFF: And so Brian had a- Brian was get- like doing, you know, we were- some of us were drawing out all the ways. [Jeff begins to draw on his paper.] Brian had another method of finding out the ways to do it. //You know. //And we just- [Jeff waves his hand to Brian.]

1335. ROMINA: //And then we just compared //them. And like //whatever he didn't have-

1336. JEFF: //-brought it all together and then that's kind of what we're looking at right now.

1337. RESEARCHER 1: So you found those numbers, all of them, by counting? [Referring to the numbers on the taxi grid]

1338. ROMINA: Yeah. //The ones we have written. Yeah.

1339. JEFF: //Well up- up to here. [Jeff points to a number on the transparency grid.] Right. What is written we counted through them. [Making a circular motion with his hand.]

1340. RESEARCHER 1: Okay. So is there anyway you can justify if I were to say to pick- you said these are like rows, like so this one, two, one would be what row? These points here //of the triangle? [Pointing to 3 vertices on the grid.]

1341. JEFF: //What? Um, I'm not-

1342. RESEARCHER 1: You put ones on the side I noticed.

1343. JEFF: Yeah.

1344. RESEARCHER 1: So if you were to look at //one, two, one-

1345. MICHAEL: //Do you mean like this row? [Pointing to triangle]

1346. RESEARCHER 1: Well pick any row.

1347. JEFF: All right. All right. We'll say one, two, one because that's an easy place to start from.

1348. RESEARCHER 1: Right.

1349. JEFF: What's the question though?

1350. RESEARCHER 1: Right. So-

1351. BRIAN: //One, two, three, one-

1352. RESEARCHER 1: //-that's the second row.

1353. JEFF: //Yeah.

1354. MICHAEL: //I mean I guess we're saying-

1355. JEFF: That's the second- yeah. [Pointing to triangle]

1356. MICHAEL: Things with, uh, one- one block. [Pointing to the transparency grid with pen.]

1357. RESEARCHER 1: Okay.

1358. MICHAEL: Two blocks, three blocks, four blocks.

1359. JEFF: And then this would be five blocks then- [Pointing to triangle]

1360. MICHAEL: Not four. That doesn't make sense.

1361. RESEARCHER 1: How would that be?

1362. MICHAEL: Six, you could, hum, things that- I don't know.

1363. JEFF: See I'm still not exactly sure what you're asking.

1364. MICHAEL: //Yeah, I don't know.

1365. ROMINA: //Yeah.

1366. RESEARCHER 1: I didn't ask anything yet.

1367. JEFF: I'm all-

1368. RESEARCHER 1: I was- I was saying that you-

1369. JEFF: What are you trying to-

1370. RESEARCHER 1: you're showing me that's Pascal's triangle but I don't' see it. Help me see it.

1371. JEFF: You don't see it?

1372. RESEARCHER 1: Right. Can you //show it to me?

1373. JEFF: //All right, well here. The one, one two one, one three three one, one four six four one- [Pointing to transparency grid with marker]

1374. RESEARCHER 1: Okay let- show me again where's the four?

1375. JEFF: All right. We're on- all right. The- that one right there- [Pointing to grid]

1376. RESEARCHER 1: Mm hm.

1377. JEFF: -that we added in and this two is the three. The two in that one there is a three and there's two ones on the outside- [Placing his finger on each number as he speaks.]

1378. RESEARCHER 1: Mm hum.

1379. JEFF: So you get one three three one. And then one- the one and the three for the four. Three and the three for the six. The three and the one for the other four and then the other one on the end and then continuing through the four and the one together is the five. The four and the six is the ten. Six and the four is the other ten. Four and the one is the five. //Do you see it? [Pointing with his finger and marker to the numbers on the transparency grid.]

1380. RESEARCHER 1: //Okay. So can you explain to me, for instance, why that works? Why- where this ten comes from when you know- you're just saying well there's a pattern here because you found them, but is there a way where you haven't' found them that makes sense to predict the number of paths of one you haven't found?

1381. JEFF: Well like to here, I mean we //would say-

1382. RESEARCHER 1: //You understand my question?

1383. JEFF: Well like to here we would say it was thirty-five. [Pointing to a square on the transparency grid with marker]

1384. RESEARCHER 1: Right. How would you- how- where //would the thirty-five come from?

1385. MICHAEL: //You can't justify it because- You can justify these because you can say you counted. You can't justify that because you can't say you counted.

1386. JEFF: Yeah because we didn't count it. We're saying we're following the pattern- [Waving his hand.]

1387. RESEARCHER 1: Right.

1388. JEFF: That's- that is our justification as of now.

1389. ROMINA: [Inaudible].

1390. RESEARCHER 1: Right.

1391. JEFF: That we're just following //the pattern.

1392. RESEARCHER 1: Do you understand my next question Jeff? What I'm asking?

1393. JEFF: //Yeah.

1394.	ROMINA:	/ /What if three- what if Pascal-
1395.	RESEARCHER 1:	Because so you notice this pattern and the pattern fits Pascal's triangle.
1396.	BRIAN:	So does that mean there's / /thirty-five for the red one? [Romina and Mike are counting. Mike writes something.]
1397.	MICHAEL:	/ /Only these are zeros. This is like one topping- you know on the pizza? [With Jeff looking at the transparency grid. Jeff pointing to a number on the grid.]
1398.	ROMINA:	Yeah, one topping, two toppings.
1399.	BRAIN:	Remember how- Mike you had thirty-four for the red one, right?
1400.	MICHAEL:	Um- Yeah I think that was the problem.
1401.	BRIAN:	It's thirty-five.
1402.	JEFF:	Yeah, it's thirty-five.
1403.	MICHAEL:	Oh, I probably missed one.
1404.	JEFF:	Good, uh, deduction.
1405.	RESEARCHER 1:	So you counted thirty-four by brute force / /and you're saying that by this pattern, um, you would feel more comfortable with the pattern in saying thirty-five.
1406.	JEFF:	/ /Yeah.
1407.	BRIAN:	But-
1408.	RESEARCHER 1:	Right?
1409.	ROMINA:	Did you actually get thirty-five?
1410.	MICHAEL:	I got / /thirty-four.
1411.	BRIAN:	/ /He got thirty-four but you know he's been off by like one cause you know. Yeah, it could- it could of / /been one.
1412.	ROMINA:	/ /Natural tendencies? Um,-

1413. MICHAEL: Stop that.

1414. RESEARCHER 1: Okay. So why is- why is that-

1415. ROMINA: All right. [With Jeff studying the transparency grid.]

1416. RESEARCHER 1: Why do you think that- Why do those number seem to work? How could you explain those numbers? That's- that's really- isn't that interesting?

1417. JEFF: Yeah. It- it hurts though. It really does.

1418. ROMINA: Yeah, I'm having trouble seeing Pascal's triangle. [Rewriting the triangle the way she is used to seeing it.]

1419. RESEARCHER 1: It's hard to see the other way, isn't it?

1420. ROMINA: All right. So for this one the two comes from when there's- [Pointing to numbers in the triangle with her marker]

1421. JEFF: One block.

1422. ROMINA: One-

1423. JEFF: Block.

1424. ROMINA: Is that-

1425. JEFF: One block.

1426. ROMINA: Isn't that two blocks?

1427. JEFF: One, two.

1428. ROMINA: No. Um, let's go back to towers. The two comes from- this is one block. This is two blocks with two colors. [Continuing to point to numbers in the triangle with her marker.]

1429. JEFF: I have to leave. I'm kind of out.

1430. ROMINA: Hold on. How's this go? Just tell me where this comes from.

1431. MICHAEL: What happened?

1432. ROMINA: Okay. This is with- with just one block?

1433. MICHAEL:	This is nothing.
1434. ROMINA:	This is nothing? This is one block?
1435. MICHAEL:	This is like- yeah, //one. All right.
1436. ROMINA:	//One block, two- this one tells how many blocks.
1437. MICHAEL:	One block. Two blocks. [Pointing to the 1 and 2 in the triangle Romina redrew.] Not two blocks but like- [He points to the numbers on the transparency grid.] [Inaudible.]
1438. ROMINA:	One block, two blocks, three blocks- Oh no, this is zero block, one block, two block?
1439. MICHAEL:	For one block you get two. Right? Or two blocks-
1440. ROMINA:	All right.
1441. MICHAEL:	Three- three- three blocks. One- So you can't really say it because there's three for three and then you get four here. You can't really- I don't think you can use that. That- that row thing. [Pointing to the numbers on the transparency grid.]
1442. ROMINA:	All right. Yeah. I know. I'm just trying to- because for like-
1443. MICHAEL:	There's got to be some type of, you know, way. If I could see-
1444. ROMINA:	Can't you just go one, two, three, four?
1445. MICHAEL:	Uh hum [nodding his head in agreement]
1446. ROMINA:	That's what it goes one, two, three, four? Because then- okay for this one for the three. If we name all the ones going horizontal- *A*s and ones going down same with *B*. And this would be with two *A*s and one *B* there's three and then there's two *B*s with one *A*, three. [Pointing with a green marker at the intersections points $(3,2)$ and $(3,1)$ on the transparency grid.] And for this one remember like two *A*s two *B*s- //six. [Now pointing to the intersections point $(4,2)$ and on the transparency grid.]

1447. MICHAEL: //You could say, um-

1448. ROMINA: Do you understand what I'm saying?

1449. MICHAEL: Like yeah, these are like this row is everything with perimeter two. I mean I half the perimeter, like. [Pointing with his marker to numbers on the transparency grid.]

1450. ROMINA: //Well no I'm saying so to get that-

1451. MICHAEL: //In order to get to that point you have to go over one and down, uh, one or down one and over one. [Pointing to the intersection point $(2,1)$.] Just like that row. Everything in this row, over two and down two and over one.

1452. ROMINA: Yeah but like I'm just saying like if she were to pick anything like right there we could say it's like eight- eight *A*s and like six *B*s. [Tracing a rectangle on the transparency grid.] You know like- and then we could tell you where you it is in this one. [Pointing to the redrawn Pascal's triangle on the piece of paper.]

1453. MICHAEL: Well you could say all- everything in this row, the shortest route is two. Everything in this row shortest route is three. This one shortest route is four. [Pointing to a diagonal of numbers—1 4 6 4 1—in the transparency grid.]

1454. ROMINA: Yeah.

1455. MICHAEL: The shortest route is five, six and so on. So that's how you could, you know, name them. This is row six because it has everything in the row has shortest route of six. [Pointing with a marker to diagonals of numbers on the transparency grid.]

1456. ROMINA: No, I understand. I'm just saying like-

1457. MICHAEL: There's a, you know-

1458. ROMINA: To get it-

1459. MICHAEL: //To- to say it like, oh I'll pick this block-

1460. ROMINA: //Because isn't that how- isn't that how we get like these? Like doesn't the two- there's- that I mean, that's one- that means it's one of *A* color, one of *B* color. [Pointing to the 2 on the redrawn triangle on paper.] Here's one- it's either one- either way you go. It's one of across and one down. [Pointing to a number on the transparency grid and motions with her pen to go across and down.] And for three that means there's two *A* color and one *B* color [pointing to the 3 on the redrawn triangle], so here it's two across, one down or the other way [tracing across and down on the transparency grid] you can get three is //two down- [Pointing to the grid.]

1461. MICHAEL: //You mean like one *A* color and two-

1462. ROMINA: Yeah.

1463. MICHAEL: This is one-

1464. ROMINA: Like two blues, one red. Two across, one down or this is two reds, one blue, two down, one across. And that's how we would get the Pascal's triangle. [Pointing to numbers on the redrawn grid and transparency grid.]

1465. MICHAEL: But there's like- you know, there's got to be a way that we could just say, all right this one's three. //So five down this has to be this because of some kind of-

1466. ROMINA: //I know, I'm just saying-

1467. ROMINA: So if it were-

1468. MICHAEL: Pattern- I mean like, you know, reasoning. You can't just say I counted them.

1469. ROMINA: I know. I'm just saying so like- and then that could relate back to this but that is this, so it's believable- and for-

1470. MICHAEL: So what- what are you looking for right now?

1471. ROMINA: Yeah like-

1472. RESEARCHER 1: I think Romina knows what I'm looking for. I think she's said it very articulately. That if I were to pick any point right on-

1473. MICHAEL: Mm hmm.

1474. RESEARCHER 1: If I were to make a larger grid- right Brian? I think he //knows what I'm looking for.

1475. BRIAN: //Yeah.

1476. RESEARCHER 1: She's looking for a way to come up with a particular pattern that she's identifying that. I think I'm hearing you say- you're trying to look at blocks-

1477. ROMINA: Mm hmm.

1478. RESEARCHER 1: Colors?

1479. ROMINA: Yeah.

1480. RESEARCHER 1: And then you're doing *A*s and *B*s.

1481. ROMINA: Mm hmm.

1482. RESEARCHER 1: That's what I'm hearing you say? And you were trying to say maybe that could get you to some general point. Why don't you try saying that again? I- I thought I followed you but I'm not so sure that Brian and Michael followed what you said.

1483. ROMINA: Like why-

1484. RESEARCHER 1: So say it again. What you were-

1485. ROMINA: Like why this and this are related?

1486. RESEARCHER 1: Yeah.

1487. ROMINA: Well-

1488. RESEARCHER 1: Throw out your idea //again for them so we can hear it.

1489. ROMINA: //When we look- whenever we do this we always- we always talk about towers and how this is like a tower of two high with two different colors and there's one- one tower you can make that makes one color and one and one and then all the other color. And- and then for this one it's three high and this is all one color. There's two of one color and one of the other, whatever. And for this it's basically the same thing because this is- let's see. This is- this is two but usually you go one across and one down so there's two different ways to get to that one. And for this one there's going to be two across and one down. Or to go down here it's two down and one across which is basically the same thing and it just goes on. Do you understand? Understand? Was that good? Or, do you want more? [Connecting the data from the grid and the triangle drawings by pointing to the numbers on each back and forth.]

1490. BRIAN: Yeah.

1491. ROMINA: Or do you want more?

1492. RESEARCHER 1: I don't know. I don't know if Michael-

1493. BRIAN: Mike do you understand?

1494. MICHAEL: Yeah, I understand what you're talking about.

1495. ROMINA: Yeah. Yeah.

1496. MICHAEL: This would be, um, one- we'll think of it as pizza because that's the thing I like but-

1497. ROMINA: Think of towers.

1498. MICHAEL: Or towers. I mean this will be a tower of three-

1499. RESEARCHER 1: Think of it as pizzas.

1500. MICHAEL: A pizza. A pizza with, um, three possible choices for toppings and- I like the tower.

1501. ROMINA: Yeah, the tower is easier.

1502. MICHAEL: You have, you have a tower of three and you have, you know, two colors. So one- it's either you know- Color x and two of color y. Well this is direction x and two, two directions of y, you know-// [Pointing with a marker to the redrawn Pascal's triangle.]

1503. ROMINA: //Yeah.

1504. MICHAEL: //of y. So that makes- that makes sense.

1505. ROMINA: So for like the three, it would be two x, one y or two y, one x// [Referring to the taxi grid.]

1506. MICHAEL: //Yeah, I got that.

1507. ROMINA: And this would be-

1508. RESEARCHER 1: Okay. Well- where I'm still having a little trouble is, um, - Okay, so you're talking about these blocks, right?

1509. ROMINA: Mm hmm.

1510. RESEARCHER 1: So what are you labeling them? These blocks? [Referring to blocks on the taxi grid.] Which is the *A* and which is the *B* and why is it / /okay to call them *A*s and *B*s?

1511. ROMINA: / /We'll do it- how about x and y?

1512. RESEARCHER 1: Sure.

1513. ROMINA: *x* will be the ones that go horizontal. [Motioning across with her marker on transparency grid.]

1514. RESEARCHER 1: Okay.

1515. ROMINA: And y will be the ones that go over there, basic graphing skills. [Moving her marker down.]

1516. RESEARCHER 1: Does that make any sense Brian?

1517. BRIAN: Yeah.

1518. RESEARCHER 1: Brian, do you think so?

1519. BRIAN:

1520. I think so. Yeah. I'm- I'm hanging out. I'm doing good now. You know what I'm saying. Oh, I was like what is that? A research paper.

1521. RESEARCHER 1: Researcher 3, Researcher 2, do you have any questions?

1522. RESEARCHER 3: Well I mean I have a very simple question. That is, it's still not clear to me how- how they know that the- to get to any particular corner corresponds to one of the numbers in Pascal's triangle.

1523. ROMINA: You see I haven't done that either yet.

1524. RESEARCHER 1: Okay, why don't you think about that for a couple of minutes?

1525. ROMINA: All right let's say- [Drawing on Michael's representation of Pascal's triangle]

1526. RESEARCHER 1: Let me just leave you be while you think.

1527. ROMINA: What would that be anyway?

1528. BRIAN: We'll say thirty-five there. [Romina writing 35 on the transparency grid.]

1529. ROMINA: You know, why don't we do this one?

1530. BRIAN: Thirty there.

1531. ROMINA: This is thirty?

1532. MICHAEL: No, no that's uh-//

1533. ROMINA: No.

1534. BRIAN: No //twenty-one.

1535. ROMINA: //This is twenty?

1536. BRIAN: Twenty-one.

1537. ROMINA: No, you know, why don't we do it this way.

1538. MICHAEL: That should be twenty-one.

1539. BRIAN: That one right there should be twenty-one.

1540. ROMINA: One, six- [Drawing on triangle.]

1541. BRIAN: And that should be a six. Fifteen plus six, twenty-one. And twenty- [Pointing to a number on triangle]

1542. ROMINA: Like that? Is that one of them? One //six-

1543. MICHAEL: //No. The next one. The next one.

1544. ROMINA: All right so that's one, seven- [Writing more rows of the triangle on paper.]

1545. MICHAEL: Twenty-one.

1546. ROMINA: Okay, I' not just- I, I'm doing-

1547. BRIAN: Thirty-five.

1548. ROMINA: And one- //seven.

1549. BRIAN: //Seven.

1550. MICHAEL: Like we know it is that.

1551. ROMINA: Okay- //So this-

1552. MICHAEL: Without- //without just saying //oh it follows the pattern.

1553. BRIAN: //[Inaudible] why.

1554. MICHAEL: He wants to know why. Yeah.

1555. ROMINA: So this one is- is that thirty-five again? Or no, this one's thirty-five. [Writing the numbers in on the transparency grid.]

1556. BRIAN: This one's thirty-five.

1557. ROMINA: This one's thirty-five so then this one is?

1558. BRIAN: Twenty-one.

1559. ROMINA: Twenty-one. So let's see. One, two, three, four, five- one, two- I don't know. I see how it would go. [She draws lines in between the numbers in the 7th row.]

1560. MICHAEL: I- I know- we know it follows a pattern but he wants to know-

1561. ROMINA: Okay. Five-

1562. MICHAEL: Without saying oh it just follows a pattern. //Why is it-

1563. ROMINA: //Okay, five and two- five and two, just add that. That's how many blocks there are. That's seven. So you got to go one, two- no. One, two, three, four, five, six, seven. Gets you down to seven. And five of one thing and two of another thing, so you just- you don't count- we won't count the one because that doesn't involve that. [Pointing between the transparency grid and the redrawn, augmented version of Romina's Pascal's triangle.]

1564. MICHAEL: What do you mean five and seven?

1565. ROMINA: What?

1566. MICHAEL: What are you talking about five and seven?

1567. BRIAN: Five across //and two down.

1568. ROMINA: //Five across and two down. Like you just count in. It goes- that's with one of one color and that's with two of two- of another color. That's with three, that's with four, that's with five. So it's either the two or the five. Both of them are the same thing. Yeah, we can explain this. Right? If anyone you pick like this one, you know it's one, two, three, four, five, six, seven. You know it's seven and it's going to be one, two, three- six of one color so it's going to be seven. [Pointing to both the redrawn, augmented triangle and numbers on the transparency grid.]

1569. MICHAEL: Are you saying five across- one, two, three, four, five- one, two. [Working with a figure of the first six rows of Pascal's triangle.]

1570. ROMINA: So- either way- no, but it's seven blocks. It's five plus two. That's how many blocks you had. For seven blocks you go down. Go one, two, three, four, five, six to the seventh row. And now you know it's five by two so it means there's five of one color, two of another color so if I go to the second one this has to- this is all one color. This is one with one color this is two. So it's either twenty-one or there's three of one color, there's four of one color, and this is five of one color or twenty-one again. [Circling the two 21s on the redrawn, augmented version of Romina's Pascal's triangle.]

1571. MICHAEL: But suppose you were to say not colors but like //ups and downs, you know-

1572. ROMINA: //Or like that- this is //with two- two-

1573. MICHAEL: //But why- you know, why is it thirty-five? If you go- Or why is it- let's go- go a little easier. Why is it, you know, four if- of, um-

1574. ROMINA: All right. Four, right? Four is- all right, why don't we do six? Six is a little harder. Six is one two- the one with six. There's one, two, three, four- you know there's four- [Pointing to triangle.]

1575. MICHAEL: It's two and two. All right. Two, four-

1576. ROMINA: This one.

1577. MICHAEL: One, two, three, four.

1578. ROMINA: It's because it's four blocks. No matter how you go there you had to take four spaces. And any direction you take has to be four spaces, right? So that means it's four- it's four blocks high. So you go to the fourth one. So you know it's in here. [Circling the 4th row of the triangle.] And it's- to get here it's two across and two down. So whatever, like you know- Do you understand? Whatever route you take you'll end up two across two down. So it means there's-

1579. MICHAEL: Two across and two down that would be this one because this would be one across and two down and this is two across and two down and this is- Wait, two down- two down and one across. One across and two down and this is two across and one down. [Pointing to redrawn triangle.]

1580. ROMINA: No, this is three across one down.

1581. MICHAEL: Oh whatever. Three.

1582. ROMINA: And this is //three down-

1583. MICHAEL: //No it's imposs-. //It doesn't make sense.

1584. ROMINA: //Three across.

1585. MICHAEL: Three across would be at- you'd be in- you'd be somewhere else.

1586. ROMINA: No you won't. Three across, one down is still in that row.

1587. MICHAEL: Yeah but you- you're doing this- this square right here, right? Two and two.

1588. ROMINA: I'm doing the six, right? You want me to do the six?

1589. MICHAEL: Yeah. That square right there. [Pointing at taxi grid transparency]

1590. ROMINA: That's still a four.

1591. MICHAEL: Mm hum.

1592. ROMINA: That's two across two down. That's four so you're in the four blocks. And then it's this- to get to here the only way to get to here is somewhere you got to go two across and two down. So there's two of one color and two of another. This is all one color. This is one and three. Two and two. Three and one.// [Pointing to grid and redrawn triangle]

1593. MICHAEL: //All right. Yeah - That makes sense//

1594. ROMINA: //All one color. And the- the four is still three and one but then it's three across and one down so it means it's three of one color and one of the other color. [Pointing to triangle]

1595. MICHAEL: That- that's a pretty good explanation.

1596. BRIAN: It's cool.

1597. ROMINA: Who's calling them in?

1598. BRIAN: Don't call them in yet. Let's hang out. I'm going to go home //I'm going to weigh a hundred and ten pounds.

1599. ROMINA: //Does it look better?

1600. MICHAEL: Yeah.

1601. BRIAN: You didn't have to get them.

1602. RESEARCHER 1: Oh.

1603. ROMINA: We're ready for his question.

1604. BRIAN: Romina's got something good.

1605. RESEARCHER 1: Okay, ready for your question.

1606. ROMINA: Come on down.

1607. RESEARCHER 1: RESEARCHER 3.

1608. BRIAN: RESEARCHER 3.

1609. ROMINA: He's our summer buddy.

1610. MICHAEL: All right. Ask- ask your question again so we know what we're-

1611. ROMINA: Exactly what you're saying.

1612. RESEARCHER 3: Uh, my question was you said that you found Pascal's triangle here and um, it wasn't clear to me that if you go, let's take-

1613. MICHAEL: You want a like reason why- how it relates?

1614. RESEARCHER 3: Yeah.

1615. ROMINA: Okay.

1616. MICHAEL: Not because it looks like it? You want to know why.

1617. ROMINA: Now we just picked any point. Let's say we picked this point. No matter how you get to this point-

1618. MICHAEL: Do the six one. The six one-

1619. ROMINA: Well we'll do the six and the four.

1620. MICHAEL: All right.

1621. ROMINA: Okay, to this point you know you need to take at least- you have to take four moves. That's the shortest amount of moves because just like a simple one, two, three, four. So that means it's- let's say you we're relating back to this four moves equals four blocks. So I'd have to go down to the four block area. So that's one, two, three, four. [Pointing to the fourth row of her Pascal's triangle.] And now here you're going three across and one down. Or- so- [Illustrating the moves on the taxi grid and pointing to the numbers on the grid and redrawn triangle.]

1622. MICHAEL: There's no possible way you could-

1623. ROMINA: //Do anything else.

1624. MICHAEL: //You have to- no matter how or which way you go you have to go three and then one.

1625. ROMINA: Right. In any move you're going one down and three across no matter- in any direction you take. So the three across and one down, that relates to three colors and then-

1626. MICHAEL: Of one-

1627. ROMINA: Three of one color and one of another. So you go and you look in here. Say- Okay, here's with all one color. This is with one of one color-

1628. MICHAEL: That's- that's nothing.

1629. ROMINA: No that's all one color but we're not using that because you can't all go all in the same direction. That's all one color. That's with one of one color and three of the other. So that's four and that's what we have and if you go down to here this is two and two and this is three and one which is the same thing. So there's your other four. And if you go to the sixth, the only way you can get there again is by four moves. It goes one- one, two, three, four. So you're in the four block again but this time you have to take, no matter what you do, you go two across and two down anyway you do it. So that would be two and two which is your six but you're still in like the four block area. [Relating the taxi grid to Pascal's triangle.]

1630. MICHAEL: Like you know what the uhm- let me write this down. Like when you write the Pascal's triangle, this is really like- like- all right, let's say-

1631. ROMINA: [Inaudible].

1632. MICHAEL: Let's say it's like, uh- I don't know how to say it- like, um, like a pizza or something. All right, you have choice of four toppings.

1633. RESEARCHER 3: Okay.

1634. MICHAEL: All right. This one is the pizza with nothing. So you'll only- there's only one possibility without any toppings on the pizza. [Pointing at the triangle.]

1635. RESEARCHER 3: Uh hum.

1636. MICHAEL: Now if you have one choice of topping you get four. I- I see it but I don't know how to like say it. [Waving both hands.]

1637. RESEARCHER 3: Maybe you can help me see how you're relating the number of toppings and the number of //blocks.

1638. MICHAEL: //To this?

1639. RESEARCHER 3: Yeah. To the- get- getting to any- to a particular corner.

1640. MICHAEL: I like see something and I- if I say it'll- it'll make it a lot clearer but I just don't- don't know how to say it.

1641. RESEARCHER 3: Why don't you just try saying it?

1642. MICHAEL: All right. Well- I'm trying to think of like a- a way//

1643. ROMINA: //Mike, if we were to use pizza could you explain this 'cause I don't know how to do this, okay, that means you have four toppings- [Pointing with Michael to the 4th row of the triangle.]

1644. MICHAEL: This is, um,- Yeah, four toppings.

1645. ROMINA: //Plain. [Pointing to the 1st number in the 4th row.]

1646. MICHAEL: //You have one topping, you're going to make //four different kinds of pizzas.

1647. ROMINA: //One topping. //Two toppings. [Pointing to the 2nd # in the 4th row]

1648. MICHAEL: //Two toppings. [Pointing to the 3rd #]

1649. ROMINA: //Three toppings. [Pointing to the 4th number]

1650. MICHAEL: //You can make six.

1651. ROMINA: Four toppings. [Pointing to the 5th number]

1652. MICHAEL: Yeah.

1653. ROMINA: All right. So, you can do that. Just do-

1654. MICHAEL: Don't know where to go from there though. Like how to relate toppings to that.

1655. ROMINA: Just the same way I just did with the blocks. It's the same thing.

1656. MICHAEL: All right, think of a topping as like, um, being able to go across so if you're only able to go across one time then you could do it four different ways and-

1657. ROMINA: That's one topping.

1658. MICHAEL: Here. You could do this- This- this one right here. Go across this time and go down this time and go down this time and that time. The rest is all going down. The rest of your moves are all going down. [Tracing moves on grid]

1659. RESEARCHER 3: So you're say one topping-

1660. MICHAEL: Yeah. Yeah, one topping would be like you're only able to walk across or go across or drive across actually it's a taxi, one time- one block.

1661. RESEARCHER 3: Okay.

1662. MICHAEL: Now the six would mean you're able to drive two blocks across and two down. Um, four would be you're able to drive three across and the last- and this one right here is you're able to drive- wait four, um, you're able to drive four across which- I mean, drive four down- no, nothing across. I'm trying- I'm trying to say- I can't really say-

1663. BRIAN: Good job.

1664. MICHAEL: Yeah, this would mean you would drive nothing across. It wouldn't even get you to that- wouldn't even get you there. So, that's why, you know, the ones don't really count in our- in our like model. Like- [motioning with fingers in air and pointing to redrawn triangle and grid triangle]

1665. ROMINA: The ones- the ones //would be if you just could-

1666. MICHAEL: //The only thing-

1667. ROMINA: -if you're going just to this point because it's only- you're only going in one direction. Like you can't get to any of the inside points because you have to use two directions.

1668. MICHAEL: Yeah. So on the odd do you see like four-

1669. RESEARCHER 3: What I understood you say- you're saying is that the number of toppings related to-

1670. MICHAEL: To the number of times you go across.

1671. RESEARCHER 3: Okay. So that one that you have at the corner there-

1672. MICHAEL: This one right there? [Pointing to a number in the redrawn triangle]

1673. RESEARCHER 3: Uh hum. How many toppings is that one?

1674. MICHAEL: That's all the toppings. But you really- you can't get there by going all- you know- um-

1675. ROMINA: Those would be like the across- toppings.

1676. MICHAEL: Yeah. This one actually- this would be, uh, all toppings, which would really mean all down.

1677. RESEARCHER 1: So are you telling me that some of those are across and some of those are down?

1678. MICHAEL: Yeah, like how I was saying it.

1679. ROMINA: This one would be two across- [Pointing at 4 in triangle]

1680. MICHAEL: No, no. This would be one across and-[pointing at 4 in triangle]

1681. ROMINA: One across, yeah.

1682. MICHAEL: -and three down. All right? That's- [Pointing at one by three in grid]

1683. ROMINA: No-

1684. MICHAEL: No, one across and three down. [Pointing at grid]

1685. ROMINA: Yeah, that one-

1686. MICHAEL: All right, this one you go two across and two down and three across and one- and one down. [Pointing at grid]

1687. RESEARCHER 1: So how does that work with the A's and the B's and the toppings? So I see what you mean by across and down but now if I'm thinking of *A*s and *B*s or x's and y's, right. Would you say that just one more time? I know that you've said it.

1688. MICHAEL: I- I said it?

1689. RESEARCHER 1: No. Somehow it came out of the conversation. Somebody said it.

1690. BRIAN: Romina was bringing it up.

1691.	ROMINA:	Um, I'm sorry. What am I trying?
1692.	BRIAN:	x's and y's like-
1693.	RESEARCHER 1:	I think it was Romina who did it, yes. She used x's and y's for across and downs.
1694.	ROMINA:	Okay, so if we're doing the same one with, um, with no- no x's then you'd have to go four down- four y's down and that would be this one. [Motioning across and down on grid] But you're not going to get there. Whatever. But if you're trying to get there it's one x and then you go three y's. So that's your four. If you're trying to get to this one over here it's two x's, two y's then three x's, one y and they all- they all equal four but they all have different amounts of x's and y's and that's how we get this. Yes? No? [Referring to the drawing of Pascal's triangle.]
1695.	RESEARCHER 3:	And the x's and y's- What does s correspond to again?
1696.	ROMINA:	//x is across.
1697.	BRIAN:	//Going across. And y is //down.
1698.	ROMINA:	//Or a topping or a color. All the same thing. And all our y's are down, toppings, color.
1699.	RESEARCHER 1:	Could you use zeros and ones?
1700.	ROMINA:	Sure.
1701.	RESEARCHER 1:	How does that work
1702.	ROMINA:	That's his area.
1703.	MICHAEL:	I don't believe it.
1704.	BRIAN:	Come on Mike.
1705.	RESEARCHER 1	Is that Michael's area?
1706.	ROMINA:	Come on Mike. Zero, one.
1707.	BRIAN:	//Break out the binary.
1708.	RESEARCHER 1:	//Does that work with zeros and ones?

1709. MICHAEL:	Uh man, I haven't seen that in a while. Uh, I really- don't remember.
1710. ROMINA:	Well just- the same thing-
1711. MICHAEL:	Oh like-
1712. ROMINA:	One would be every time across-
1713. MICHAEL:	Yeah, one-
1714. ROMINA:	Zero would be every time down.
1715. MICHAEL:	Just- All right, this- right there. This group is, you know, everything that has one, one and two zeros. [Writing binary codes: 100, 010, and 001.]
1716. RESEARCHER 1:	Uh hum.
1717. MICHAEL:	That's that. The next one would be- [Writing binary codes: 110,011 and 101] two ones and one zero. That's this. And I guess the one you could call going across and two down. Across and two down. Twice and down. You know you go two ones- [Pointing to a two by one on the grid.]
1718. RESEARCHER 1:	//Mm hm.
1719. MICHAEL:	//or two across' and one down there's a zero. That's a- is that good?
1720. RESEARCHER 1:	I don't know. Is that another way?
1721. MICHAEL:	Do you- like do you see how you can relate the zeros //across and down.
1722. BRIAN:	//The same thing.
1723. RESEARCHER 1:	Brian- //Brian thinks-
1724. MICHAEL:	The one moving across and the zero would mean down.
1725. RESEARCHER 1:	Romina?
1726. ROMINA:	Yeah, see I can't work like that. I work in, um, towers.
1727. RESEARCHER 1:	You're working in towers.

1728. ROMINA: He works in pizzas and binary.

1729. RESEARCHER 1: Brian are you- work both ways Brian?

1730. BRIAN: No. No I'm totally not a binary kid. I don't-

1731. ROMINA: We- see me and Brian were absent when we did binaries in like sixth grade.

1732. BRIAN: I missed a week.

1733. ROMINA: We obviously weren't there.

1734. BRIAN: What class was that?

1735. MICHAEL: Seventh grade.

1736. ROMINA: Seventh grade. We weren't there.

1737. BRIAN: I wasn't in that class all year man.

1738. ROMINA: I was in surgery.

1739. BRIAN: I was playing basketball all year in that class.

1740. RESEARCHER 1: Wow. That's really neat. Do you have anything else to add?

1741. BRIAN: Um, no. I mean I'm-

1742. MICHAEL: I mean- I mean did that convince you?

1743. RESEARCHER 3: Well sort of.

1744. RESEARCHER 1: Well I see- I see how you get the numbers. I see how you get those numbers.

1745. MICHAEL: How you figure-

1746. RESEARCHER 1: I guess my- my question still is suppose once we get just a general number there, um-

1747. ROMINA: Okay, that-

1748. RESEARCHER 1: How would you talk about some general numbers?

1749. ROMINA: All right. We'll just pick this one. [Drawing the intersection point $(10,5)$.]

1750. RESEARCHER 1: Um hum.

1751. MICHAEL: We've proved to you that you understand why it relates to the Pascal's triangle.

1752. ROMINA: Yeah.

1753. RESEARCHER 1: Oh yeah.

1754. MICHAEL: So you give us a general number, we look at the triangle.

1755. ROMINA: You pick a general number=

1756. MICHAEL: That's basically-

1757. ROMINA: To get the simplest way you're going to go all your overs and all your downs at one time so that'll tell you this is going to be one, two, three, four, five- five across so one and / / then one, two, three, four, five and five down. [Counting with marker on grid.]

1758. MICHAEL: / / And five down.

1759. ROMINA: So you know there's going to be a total of ten blocks.

1760. RESEARCHER 3: Mm hm.

1761. ROMINA: And then- so you've got your ten block row and then you're going to know it's five of one color and five of the other color.

1762. MICHAEL: There's going to be one right / / in the middle.

1763. ROMINA: / / There's going to be a number- yeah, it's going to be the one right in the middle. It's going to- Well I don't know. I don't know what it's going to be but-

1764. MICHAEL: The one that like-

1765. ROMINA: The one right in the middle of everything.

1766. MICHAEL: I don't- That's- that's-

1767. RESEARCHER 3: Which- which row?

1768. MICHAEL: -that's way up there. That's-

1769. ROMINA: It's going to be the tenth row because you took ten moves to get there. So you're going to go down to the tenth row.

1770. MICHAEL: Yeah, it's going to be the tenth row because you have-

1771. ROMINA: And the tenth row that has five of one color and five of the other color, that's your number and that's how many ways you can get to that point.

1772. MICHAEL: Which that one will be in the middle.

1773. ROMINA: Uh hum.

1774. MICHAEL: Because just the way it's set up. That one will end up in the middle.

1775. ROMINA: Plus it's like an even //the square.

1776. MICHAEL: //One- yeah, it's- no it's an odd number. That's why it's in the middle.

1777. ROMINA: Yeah it's a square.

1778. MICHAEL: Even numbers- there is no-

1779. RESEARCHER 3: How do you know it's the tenth row?

1780. RESEARCHER 1: Yeah.

1781. ROMINA: Because it took us five moves to get- uh, ten moves to get there.

1782. MICHAEL: Because you have ten spots. Ten toppings and-

1783. ROMINA: Because you know //you can always-

1784. MICHAEL: //Ten different places to put these numbers.

1785. ROMINA: Yeah.

1786. MICHAEL: Which is ten.

1787. ROMINA: And you know- and this ten there's- there's only ten moves you can take because this is like the simplest way. You go all the way across and all the way down.

1788. RESEARCHER 3: Mm hmm.

1789. ROMINA: And that's going to be like the simplest way and that's going to mean that's the shortest way to get there. Like-

1790. RESEARCHER 3: Maybe help me understand that by running us through-

1791. ROMINA: Okay //like-

1792. RESEARCHER 3: //-each story //from the first row-

1793. ROMINA: //-this one?

1794. RESEARCHER 1: Yeah, what's the first row.

1795. RESEARCHER 3: -of Pascal's triangle.

1796. ROMINA: This one, there's only two moves you can get to this one. You go over one down one. //Two moves.

1797. MICHAEL: //You mean like the first row that would be-

1798. ROMINA: To the second row because there's two high in block terms. And for this one it's two across and one down-

1799. MICHAEL: I mean, like I said before, the rows correspond to the //shortest distance.

1800. ROMINA: //Yeah.

1801. MICHAEL: I mean the //shortest route.

1802. ROMINA: //Yeah. So this is //three moves.

1803. MICHAEL: //Everything in this row, two. [Pointing to triangle]

1804. ROMINA: Third row.

1805. MICHAEL: And this one three. So that's how-

1806. RESEARCHER 3: Say it again please.

1807. ROMINA: Okay, this one. There's three moves.

1808. BRIAN: One, two, three.

1809. ROMINA: And this is the third row. [Pointing at row 3]

1810. RESEARCHER 3: So the-

1811. ROMINA: This one's four moves, fourth row. [Pointing at row 4]

1812. MICHAEL: If the shortest route is ten, then it- then it's //in the tenth row.

1813. ROMINA: //Tenth row.

1814. RESEARCHER 3: I'm still a little confused.

1815. MICHAEL: All right. If you pick any point on- [Pointing to the grid.]

1816. RESEARCHER 3: Start- start from the very first row please.

1817. MICHAEL: The first- the first one.

1818. ROMINA: The first-

1819. MICHAEL: All right.

1820. ROMINA: No moves. There's only- you're stationary there. That's one. Just one. [Pointing at first row of her Pascal's triangle]

1821. RESEARCHER 3: So it's the top row of-

1822. ROMINA: Yeah, that's just //your Pascal's.

1823. RESEARCHER 3: //-Pascal's triangle?

1824. ROMINA: Yeah. You go down to here. There- You're going to go over one, down one. There's only //two moves. [Pointing to the grid]

1825. RESEARCHER 3: //Two.

1826. ROMINA: That's the simplest way you can go.

1827. RESEARCHER 3: Uh hum.

1828. ROMINA: So that's Pascal's like second row, two blocks, two toppings, whatever you want to say. [Pointing to the redrawn triangle.]

1829. RESEARCHER 3: Uh hum.

1830. ROMINA:	And this one, you go over two and down one so that's a total of three moves. The simplest moves so that's the third row and you can go-
1831. RESEARCHER 3:	So it's the second going over two blocks-
1832. ROMINA:	Yeah.
1833. RESEARCHER 3:	-and it's which row of Pascal's triangle?
1834. MICHAEL:	//That's in the third row.
1835. ROMINA:	//The third row. [Pointing to the third row in the triangle.]
1836. MICHAEL:	//Because it takes three to get there.
1837. ROMINA:	//Because you have two and one. And you're going over two over one. You're doing three complete moves. And that move just happens to be two and one. [Inaudible]. [Gesturing across and down on the grid.]
1838. RESEARCHER 3:	Uh hum.
1839. ROMINA:	And- and this one here you're making- you're going over three and down one so that's a total of four moves. That's the fourth row.
1840. RESEARCHER 1:	So, what about the r^{th} row?
1841. MICHAEL:	Would be-
1842. ROMINA:	The r^{th} row would be- r moves
1843. MICHAEL:	Yeah, *r* moves *r* shortest distance. Whatever-
1844. ROMINA:	Yeah.
1845. RESEARCHER 3:	Uh hum.
1846. MICHAEL:	*r* half the perimeter whichever, you know-
1847. RESEARCHER 3:	Okay.
1848. RESEARCHER 1:	Are you convinced?
1849. RESEARCHER 3:	Yeah.

1850. RESEARCHER 1: It's really very interesting. Interesting problem. Did you ever do anything like this before?

1851. MICHAEL: No, no I've never seen it before in my life.

1852. ROMINA: We just discovered Pascal's triangle.

1853. BRIAN: Didn't we have to- didn't we have to do something in Pantozzi's class with the subway?

1854. RESEARCHER 1: What's that?

1855. ROMINA: Yeah but we didn't do it though.

1856. BRIAN: Uh, no. Something- I don't know, somewhere like-

1857. ROMINA: We can-

1858. BRIAN: If a person is let off at like this subway station and they want to go to this building what's the shortest way to go or something?

1859. MICHAEL: No it was like- no it was a bunch of subway stops-

1860. ROMINA: Yeah.

1861. MICHAEL: And there's some subway stop is three blocks away from this building- something- //which stop should he get off at?

1862. BRIAN: //Something like this.

1863. MICHAEL: In order to get there. And //then-

1864. BRIAN: //It wasn't exact. It wasn't exact so we're not going to get into it.

1865. RESEARCHER 1: So some of the same kind of-

1866. MICHAEL: Yeah.

1867. RESEARCHER 1: -reasoning you used.

1868. MICHAEL: Yeah. That was last year though.

1869. RESEARCHER 1: You- you are wonderful for staying and working this hard. I just have one general question to ask you. You're going to be the last ones here [Inaudible] for coming and staying so long.

APPENDIX D: TRANSCRIPTION CONVENTIONS

The following table contains transcription convention used in the transcript of the research session presented in Appendix C.

Transcription Symbol	Meaning
Numeral.	Ordinal position of a speaker's turn at talk.
NAME:	The name of the speaker of the preceding utterances.
[ ]	Transcriber's descriptive commentary appears between square brackets.
//	The moment at which overlapping speech begins.
-	Indicates an interruption in a speaker's utterance.

APPENDIX E: PARTICIPANTS' INSCRIPTIONS

This appendix contains twenty-five pages, twenty-three of which comprise the inscriptional work of the four participants, Michael, Romina, Jeff, and Brian. They had access to extra copies of the problem task (see Appendix A), blank paper, paper and transparency copies of Cuisenaire 1-centimeter grids, as well as black, red, blue, and green makers. The participants worked both individually and collaboratively and freely exchanged ideas. For most inscriptions, therefore, though one person scribed the work, the presented thinking has a collective authorship. The table below lists for each inscriptional page the name of the participant or participants who scribed it and the page in this appendix where it appears. Next to the participants' names in parenthesis is an ordinal number, *n*, representing the n^{th} inscription scribed by those participants. Finally, the inscriptions are listed in the order of they appear in the combined video record.

Inscriptional Works		
	Scribe(s)	Page
1	Brian (1)	286
2	Romina (1)	287
3	Jeff (1)	288
4	Jeff (2)	289
5	Michael (1)	290
6	Romina (2)	291
7	Jeff (3)	292
8	Jeff (4)	293
9	Jeff and Romina (1)	294
10	Romina (2) - transparency	295
11	Jeff and Romina (2)	296
12	Brian (2)	297
13	Brian (3)	298
14	Michael (2)	299
15	Michael (3)	300
16	Jeff and Romina (3)	301
17	Brian (4)	302
18	Jeff and Romina (4)	303

19	Michael, Romina, Jeff, and Brian (1)	304
20	Brian (5)	305
21	Jeff and Romina (5)	306
22	Romina (3)	307
23	Michael, Romina, Jeff, and Brian (2)	308

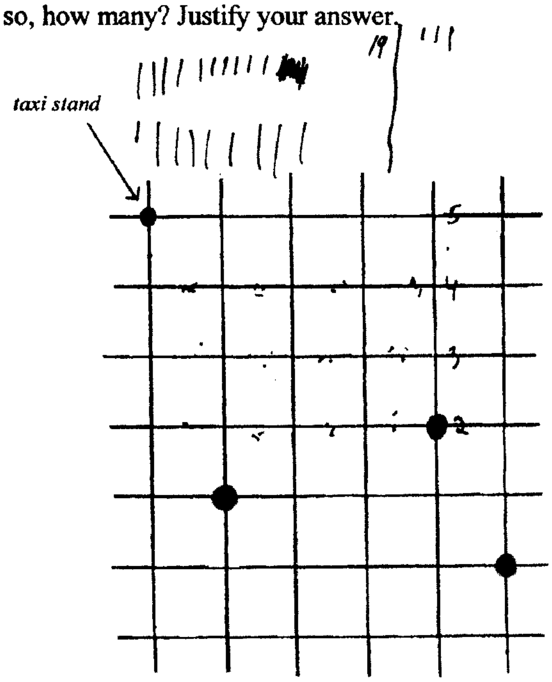
Brian

The Taxicab Problem

A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated on the map. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route.

What is the shortest route from the taxi stand to each point? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.

22



Blue - 5

2.
2.
3.
4
5.
6.
7. 02, R1
8. ORD 1,4,2
9.
10.
11.

27

5, 4, 4, 4 : 21

33, 3

13 21

Brian (1)

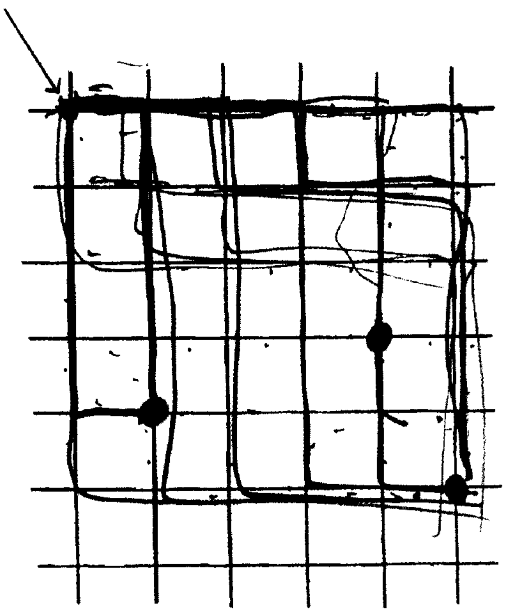
The Taxicab Problem

• Romina

A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated on the map. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route.

What is the shortest route from the taxi stand to each point? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.

taxi stand



Blue: 5

10

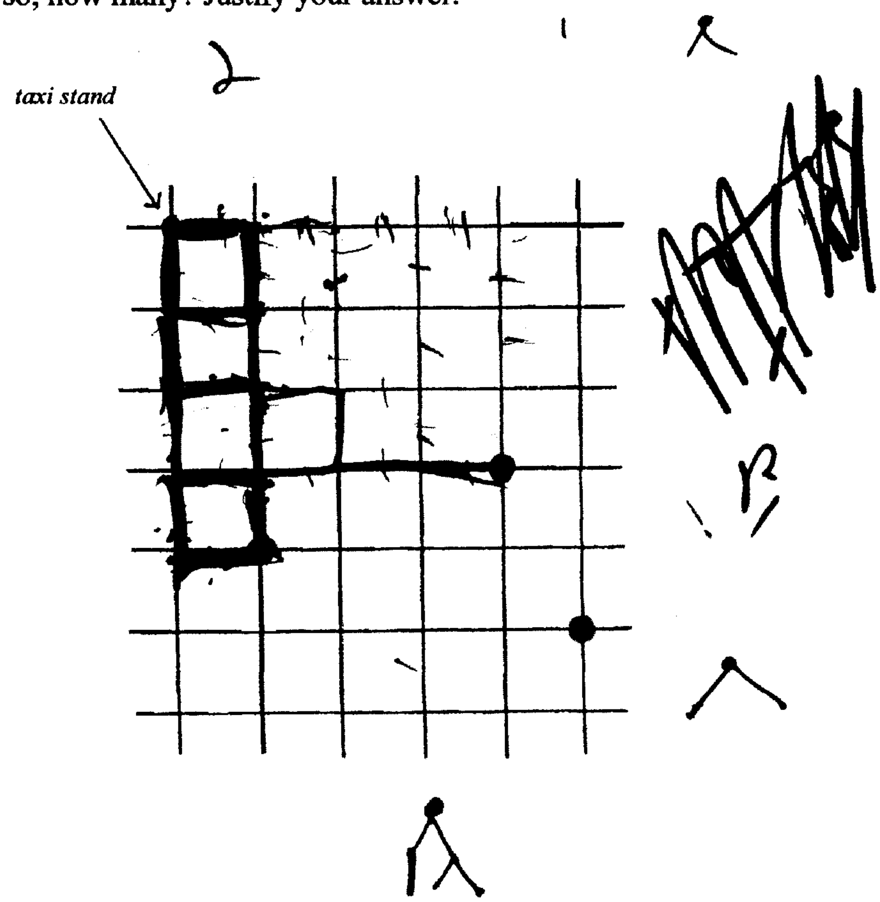
30

Romina (1)

The Taxicab Problem

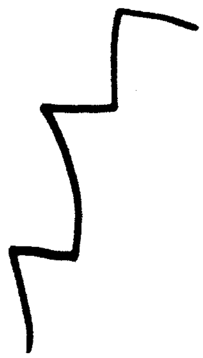
A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated on the map. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route.

What is the shortest route from the taxi stand to each point? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.



Jeff (1)

Jeff (2)



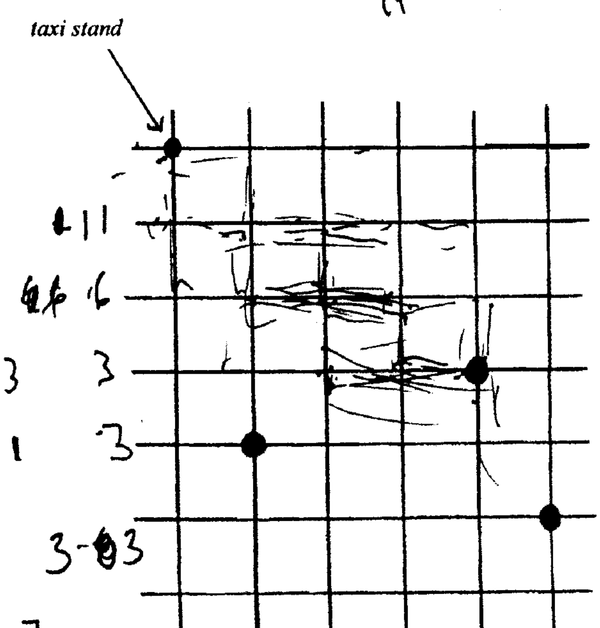
The Taxicab Problem

A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated on the map. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route.

What is the shortest route from the taxi stand to each point? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.

Mike

14 20



12
3
4
5 11
24

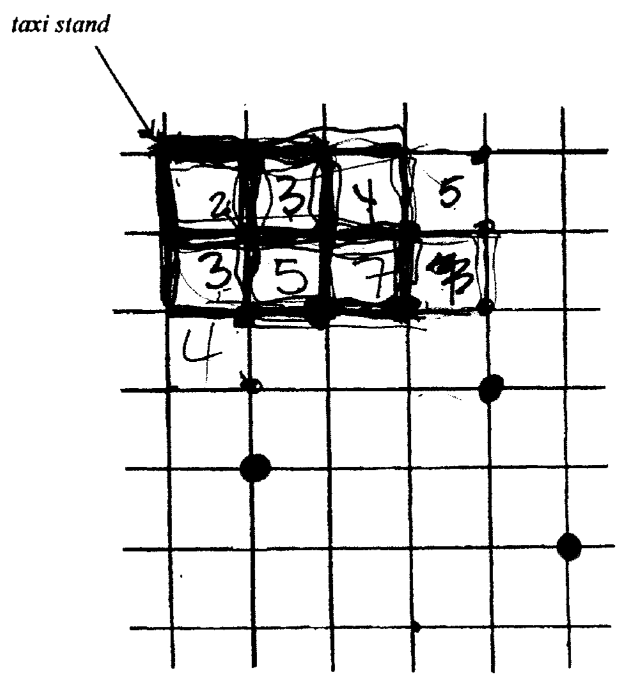
Michael (1)

Romina

The Taxicab Problem

A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated on the map. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route.

What is the shortest route from the taxi stand to each point? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.

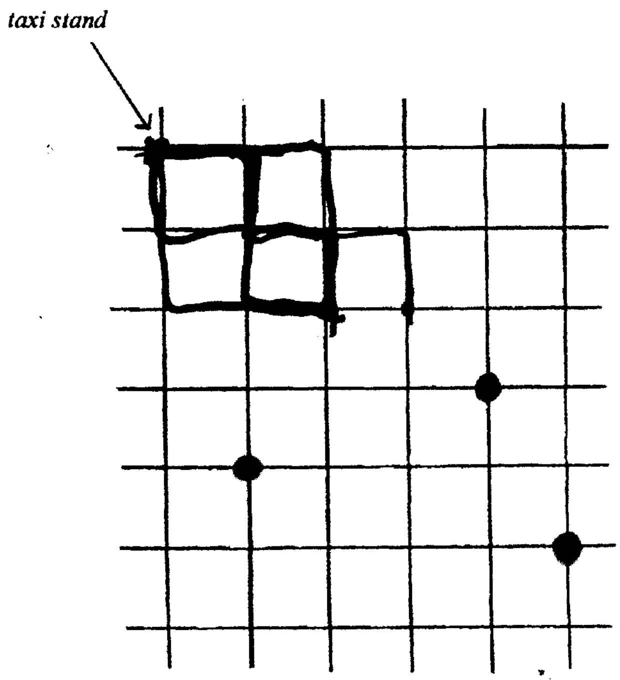


Romina (2)

The Taxicab Problem

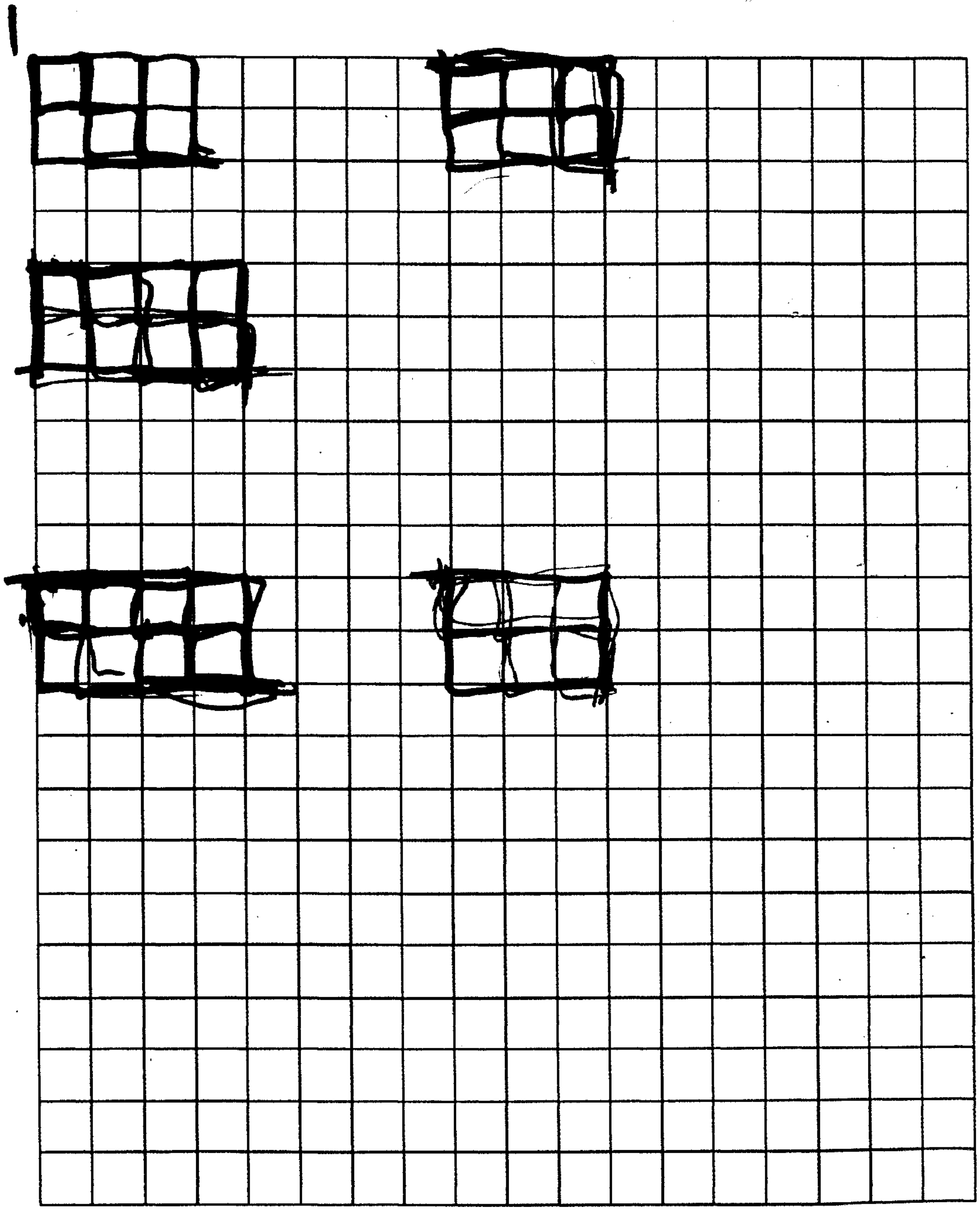
A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated on the map. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route.

What is the shortest route from the taxi stand to each point? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.



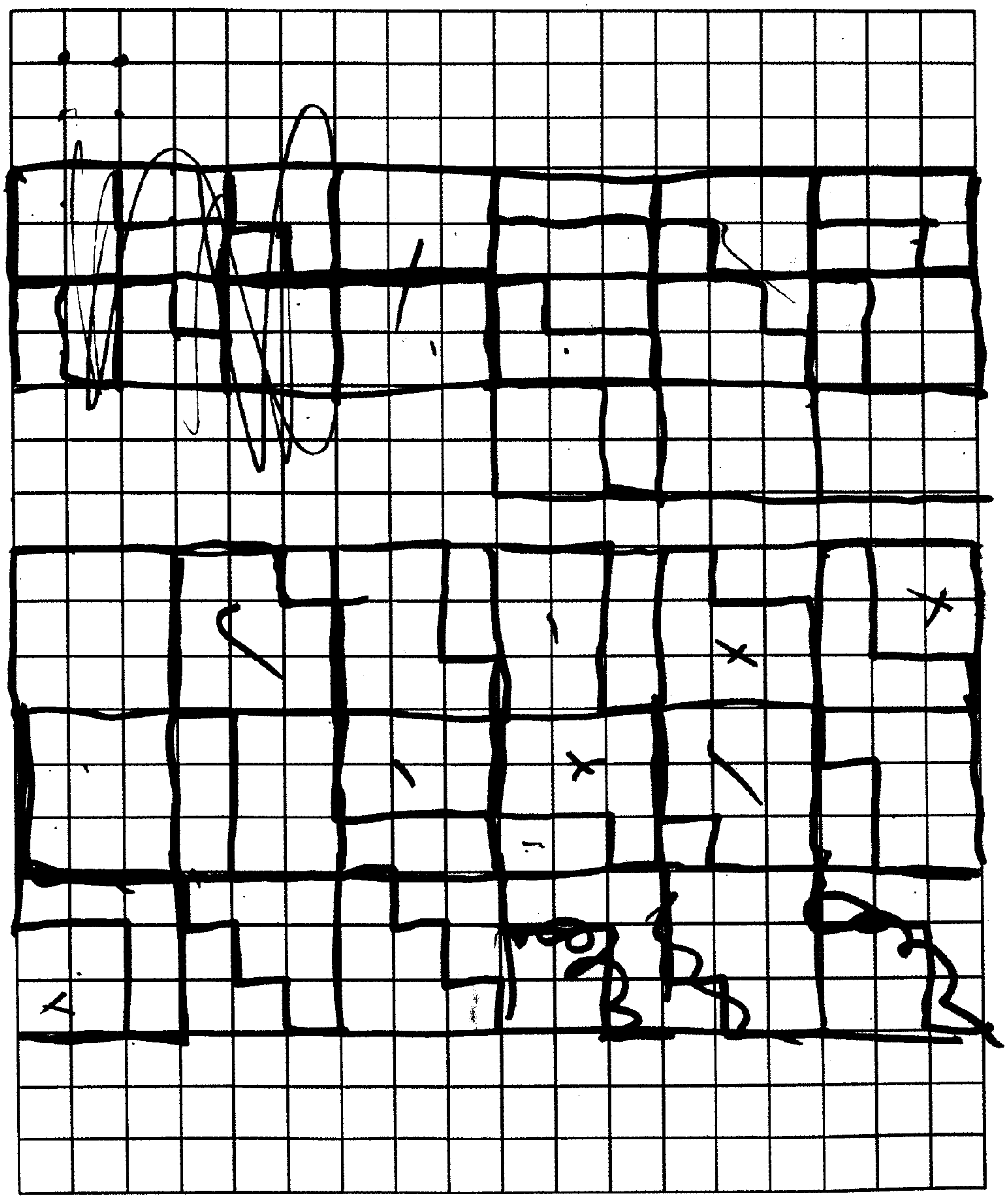
Jeff (3)

1-CENTIMETER GRID PAPER



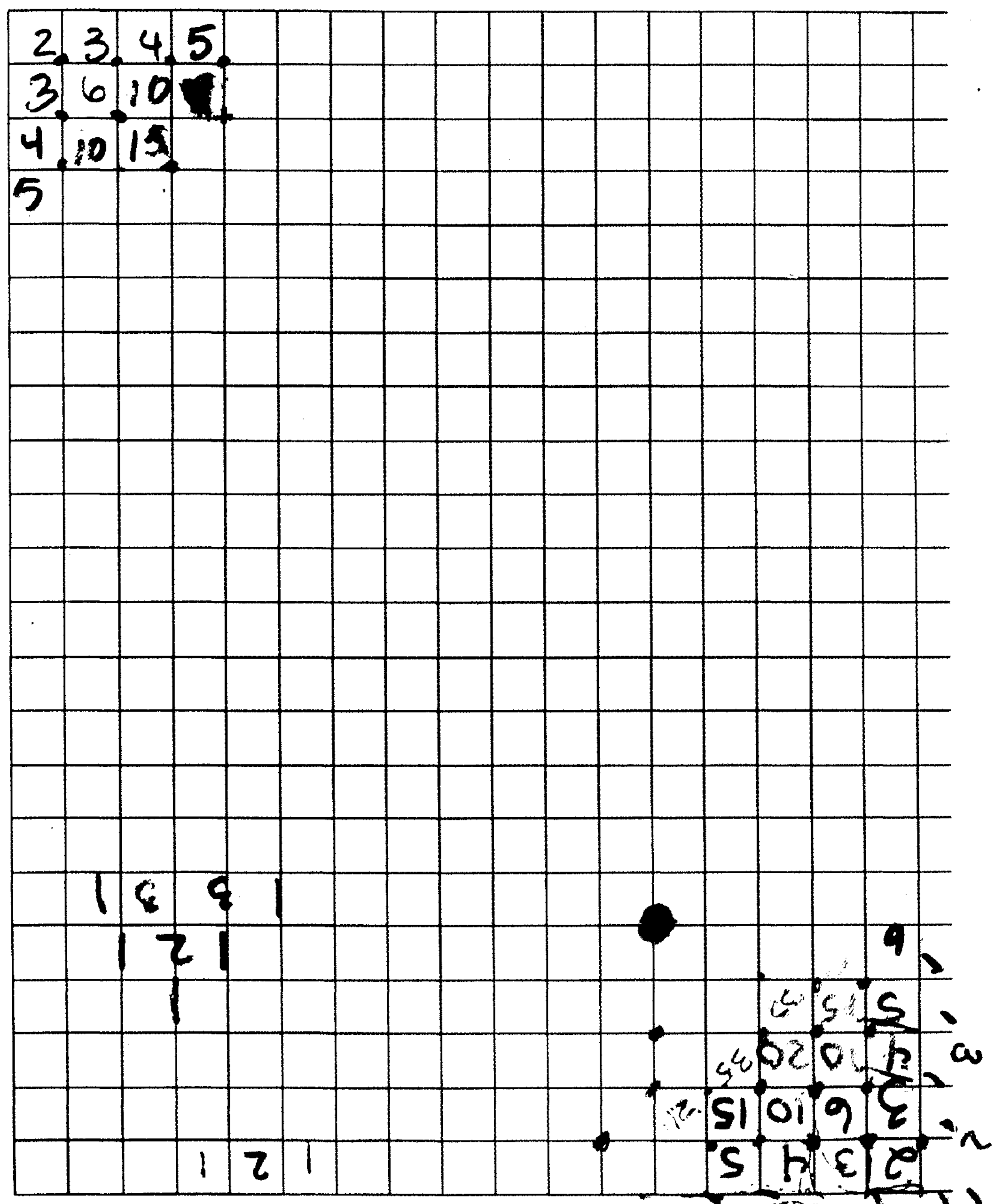
Jeff (4)

1-CENTIMETER GRID PAPER



Jeff and Romina (1)

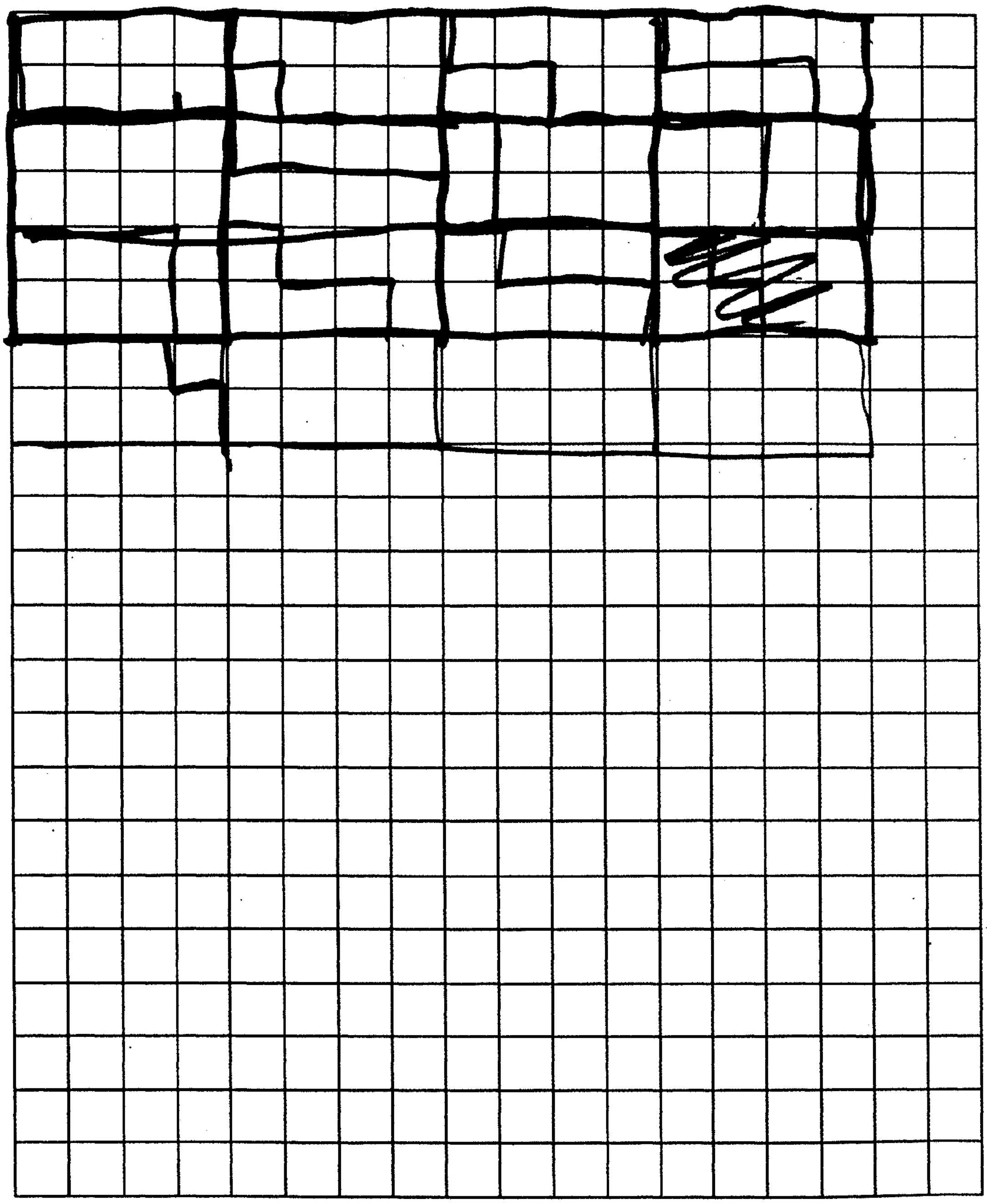
1-CENTIMETER GRID PAPER



Romina (2) — transparency

Romina 4x2

1-CENTIMETER GRID PAPER



Jeff and Romina (2)

Brian (2)

Brian

0 0
3 2

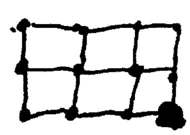
0 0
2 3

0 0 0 0
2 1 1 1
1 1 2 1
2

0 0 0
2 2 1
1 2 2

0 0 0
1 3 1

0 0 0 0
1 2 1 1
1 1 1 2



0 0 0 0 0
1 1 1 1 1

0 0	0 0 0	0 0 0 0
4,3	3,3,1	1 4 2
3,4	2,3,2	2 4 1
	1,3,3	

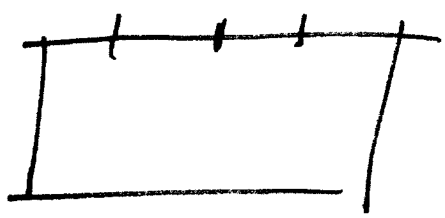
0 0 0 0
3 1 1 2
3 2 1 1
2 1 2 2
2 2 2 1
1 1 3 2
1 2 3 1

0 0 0 0
1 3 2 1
1 2 2 2
1 1 2 3

0 0 0 0 0
2 1 1 2 1
2

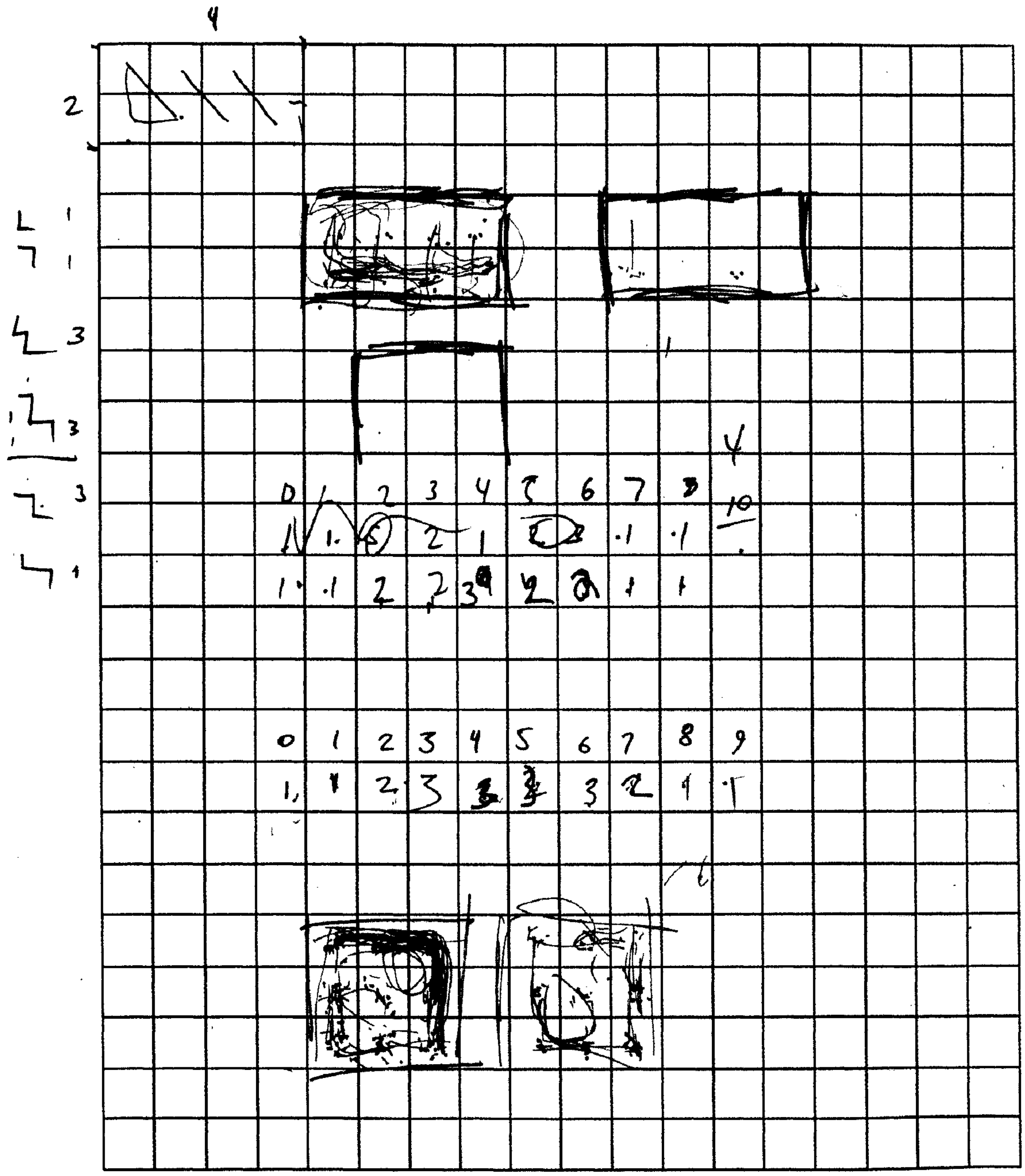
Brian (3)

Michael (2)



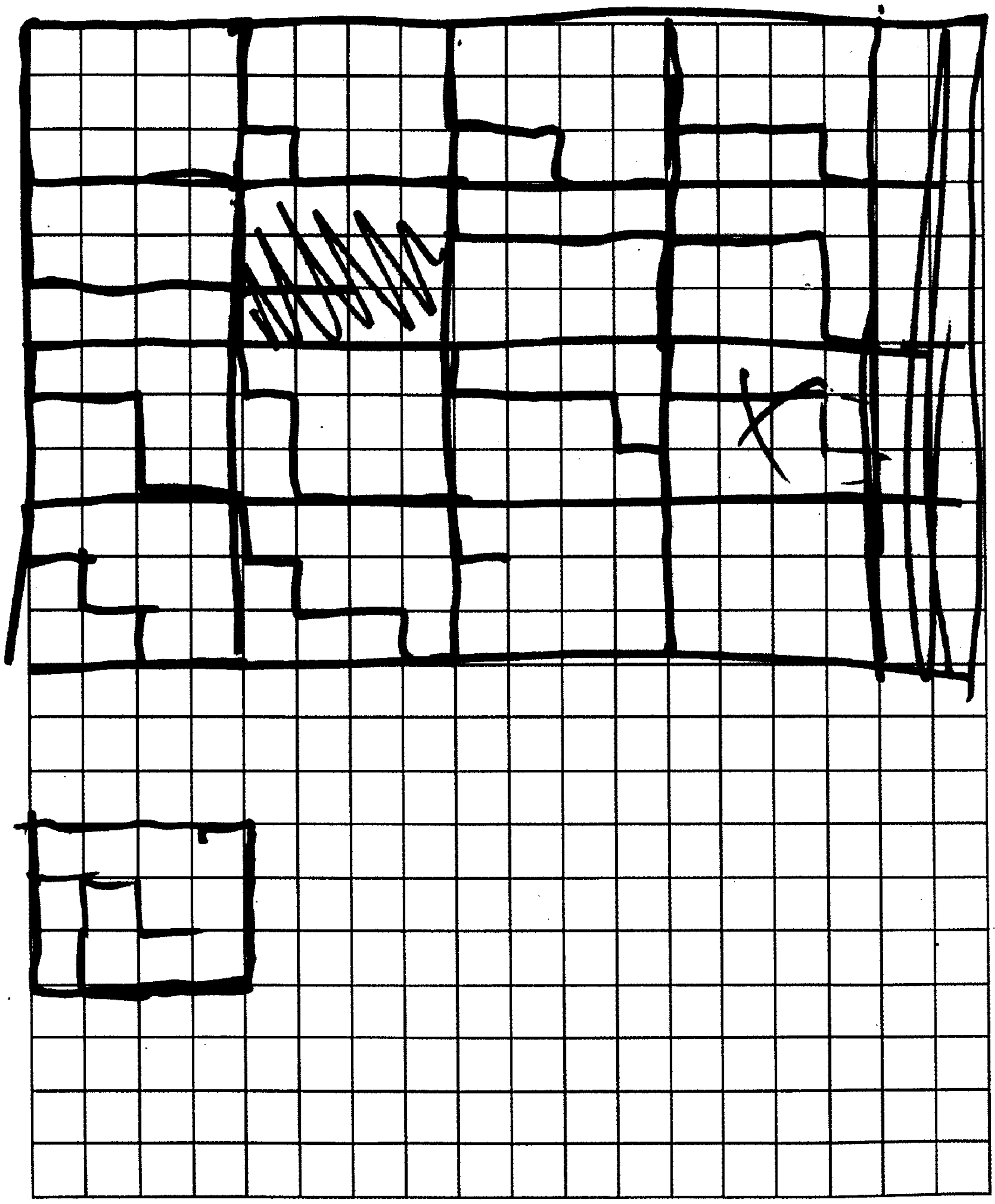
Michael

1-CENTIMETER GRID PAPER

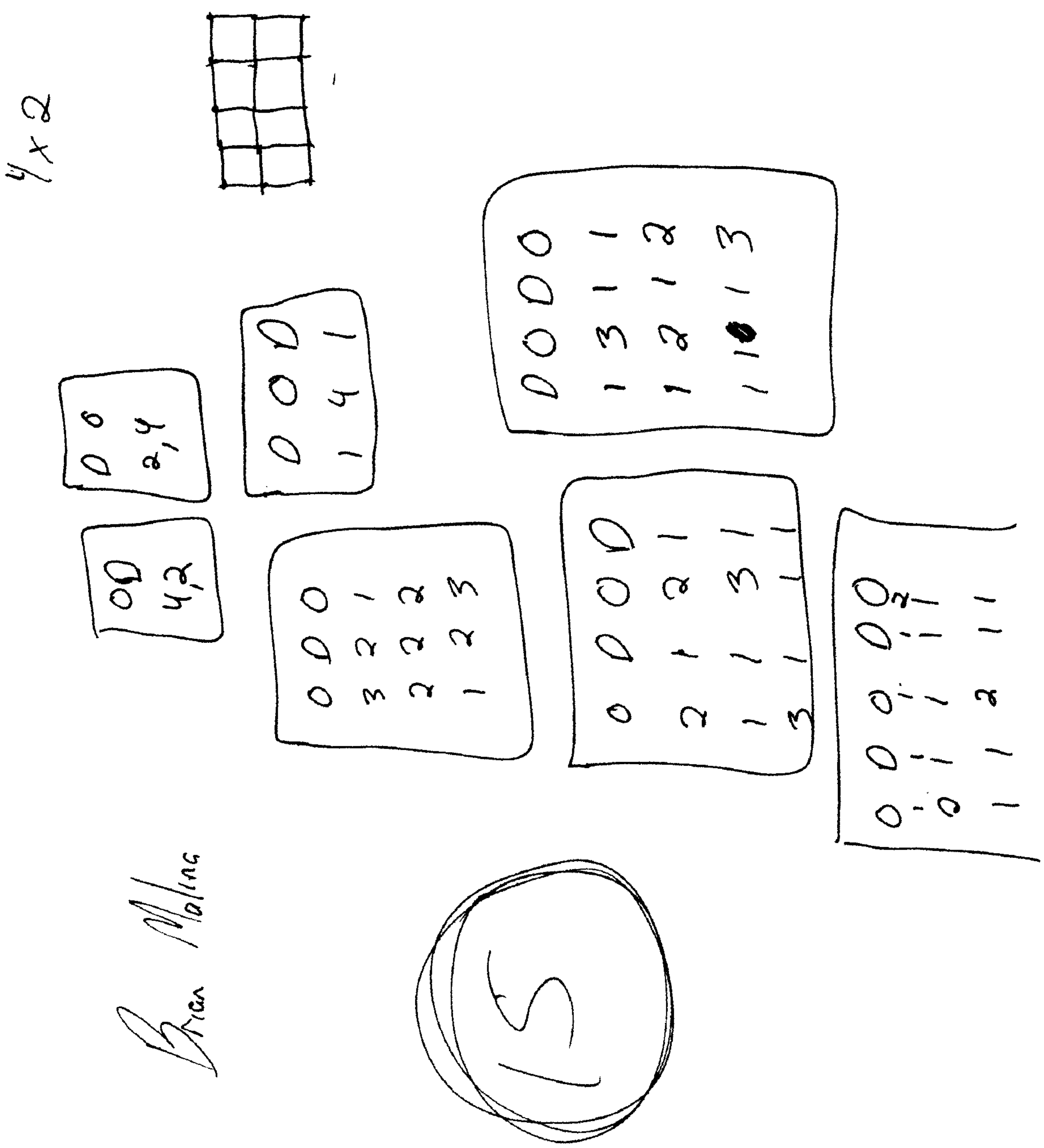


Michael (3)

GRID PAPER



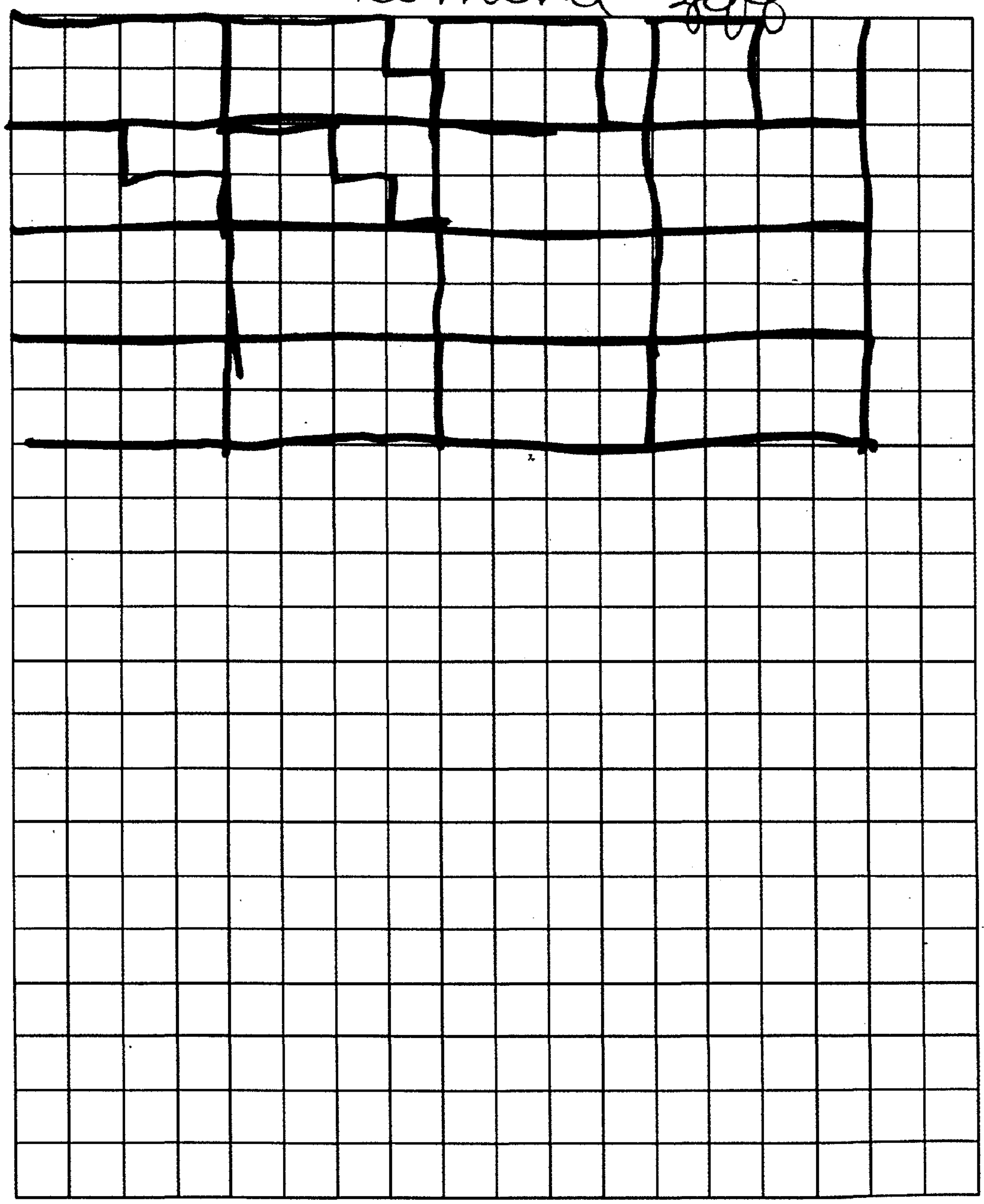
Jeff and Romina (3)



Brian (4)

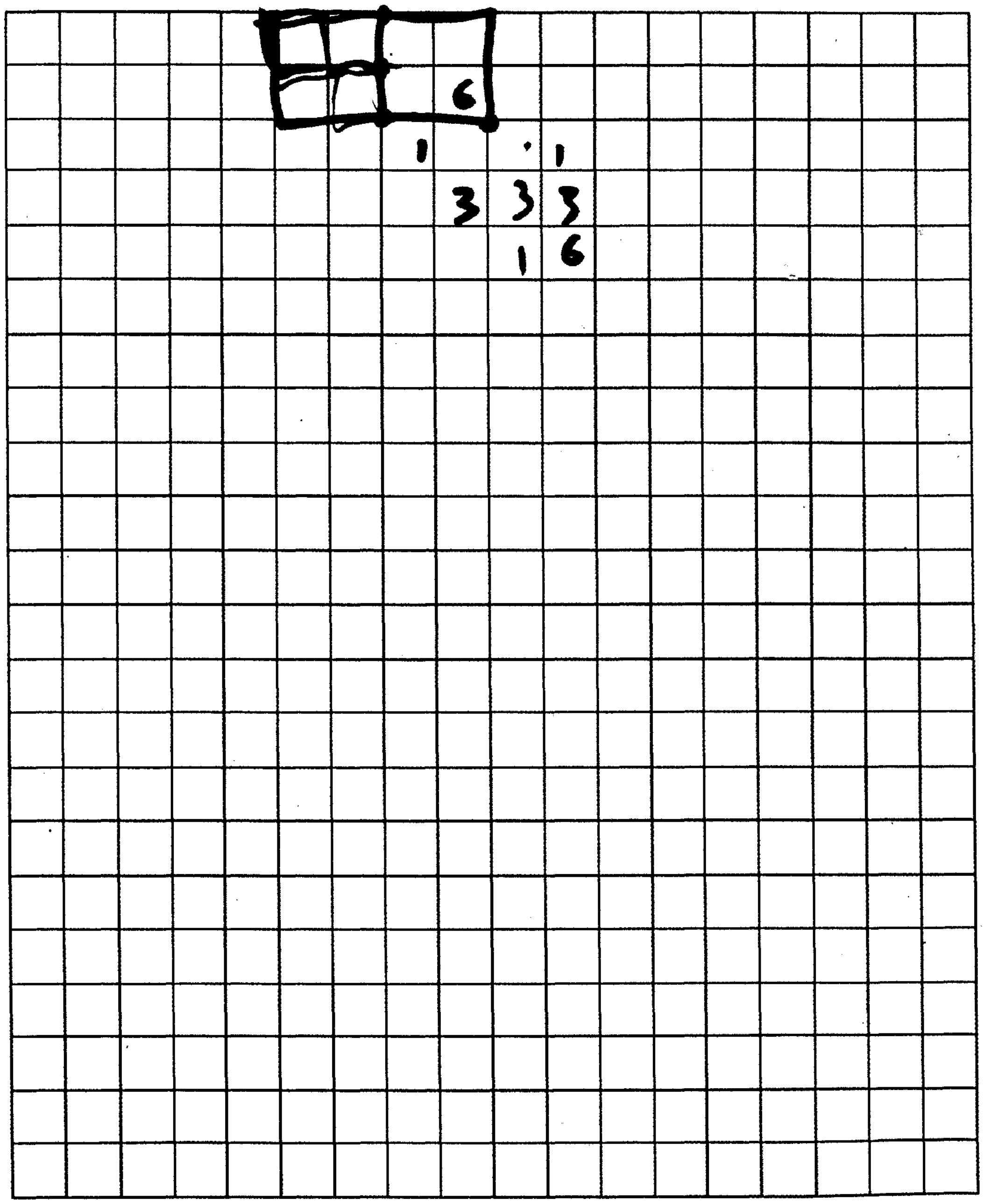
1-CENTIMETER GRID PAPER

Romina? Jeff

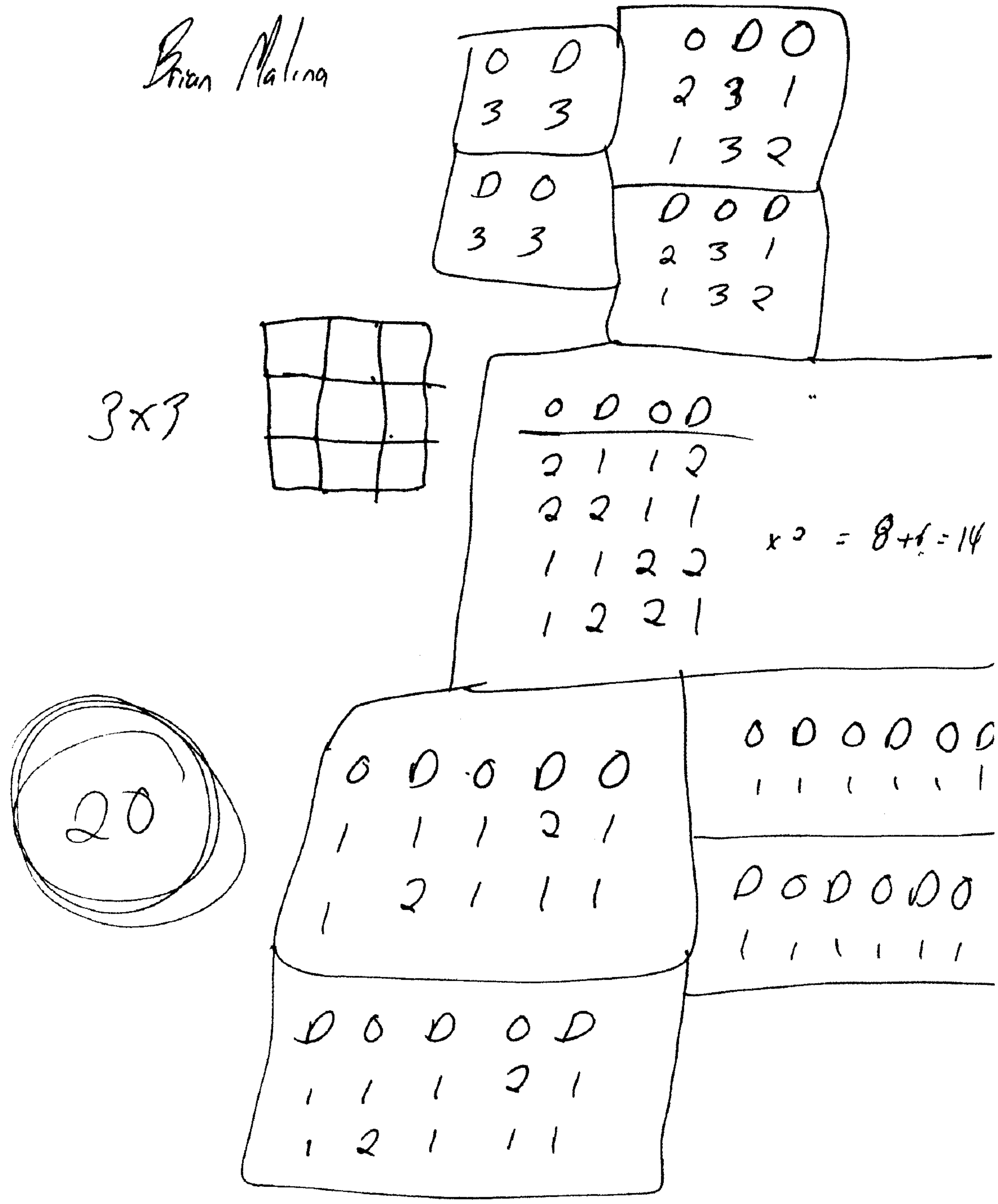


Jeff and Romina (4)

1-CENTIMETER GRID PAPER

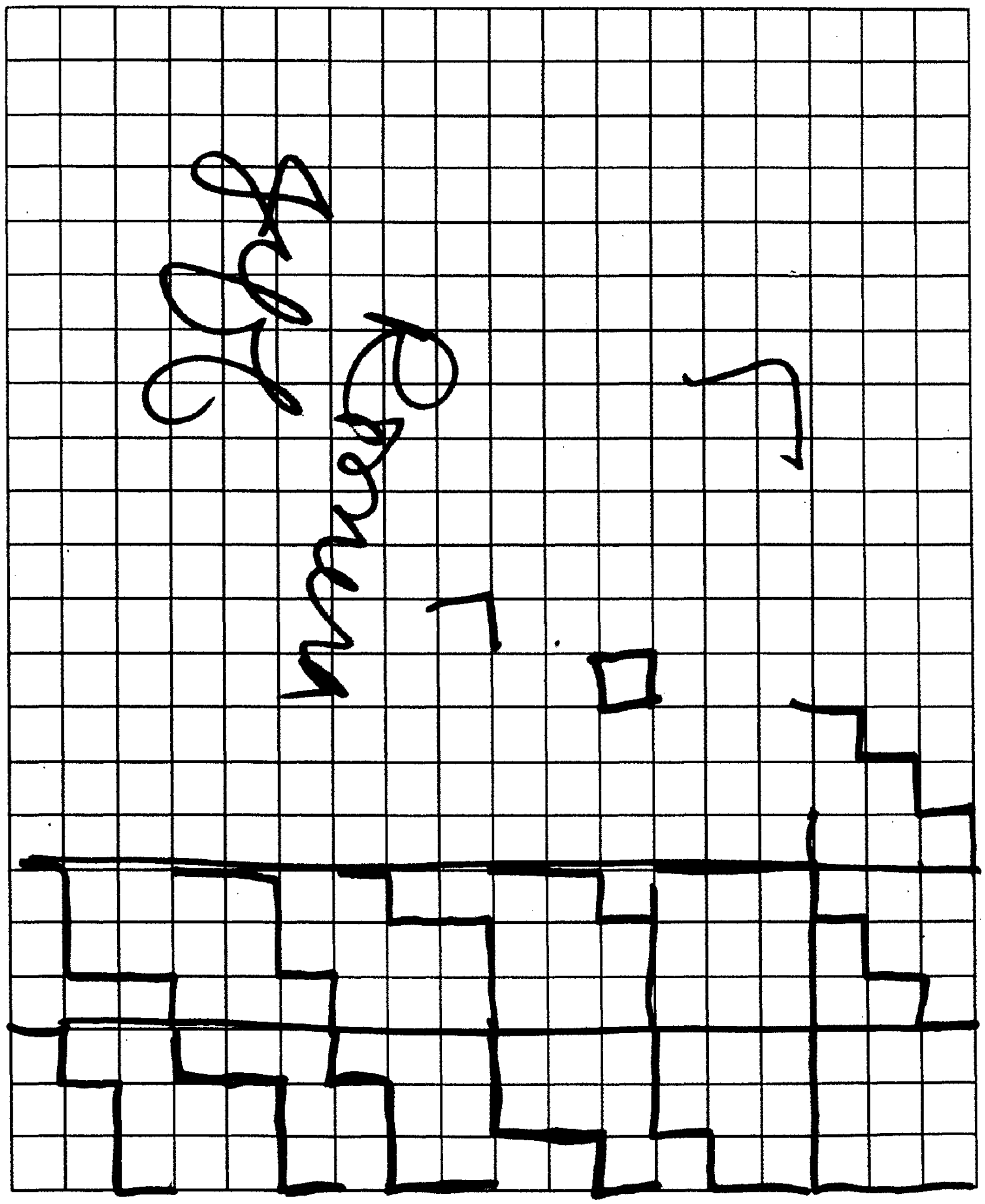


Michael, Romina, Jeff, and Brian (1)



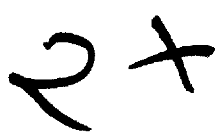
Brian (5)

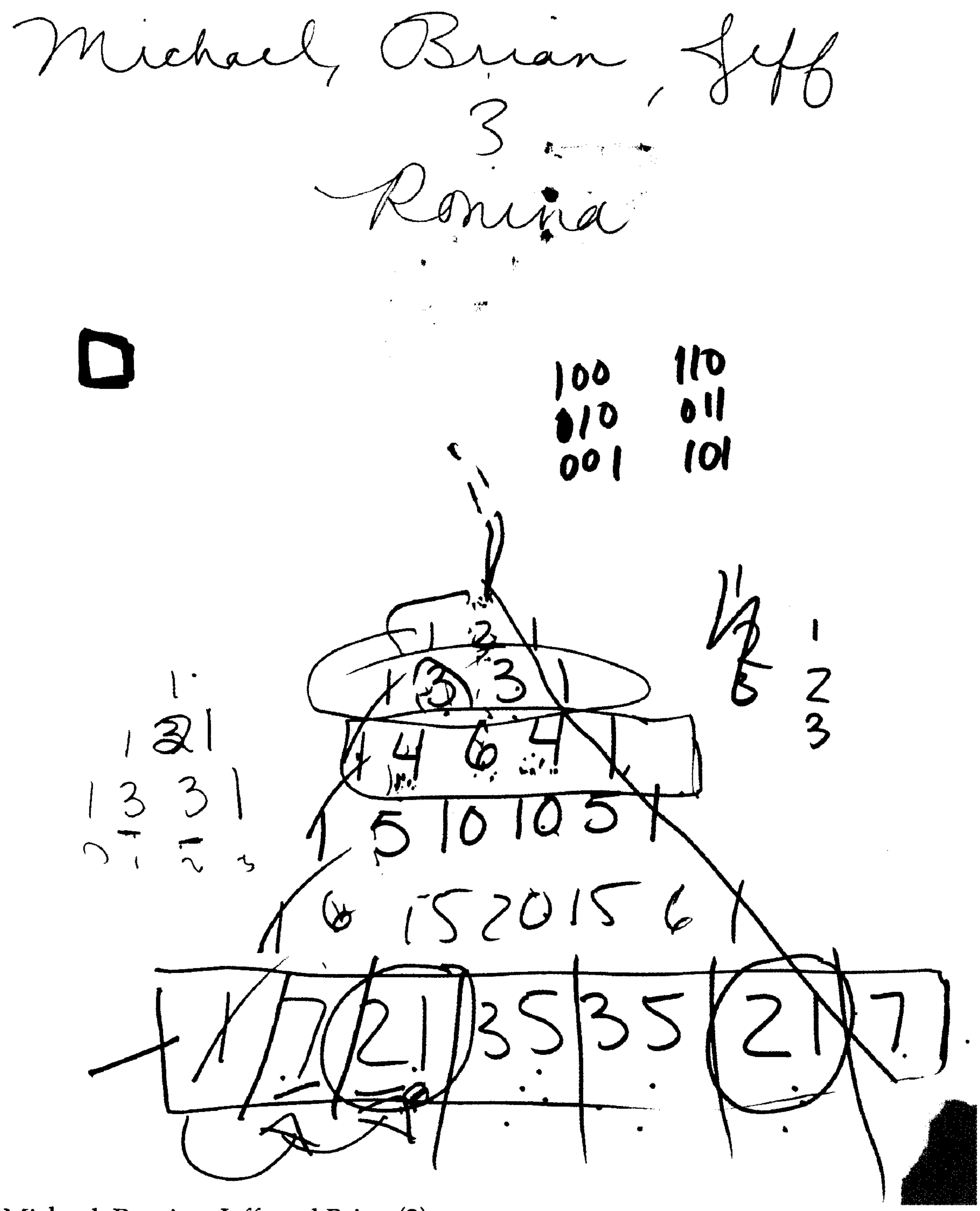
1-CENTIMETER GRID PAPER



Jeff and Romina (5)

Romina (3)





Michael, Romina, Jeff, and Brian (2)

APPENDIX F: LAYERED BUILDING OF INSCRIPTIONS (SK)

Video Data Base: SK_1 and SK_2

Inscription Name	Camera View	Time Interval	Comments
1) Taxicab problem	SK_1	0:0:49- 0:03:08	Participants appear to be reading and thinking about the problem. They ask questions, discuss the problem, and request color markers.
2) Brian (1)	SK_1	0:02:36- 0:02:40	Brian wrote "Blue – 5" on the left side of his grid.
3) Jeff (1)	SK_1	0:03:09- 0:03:35	Jeff draws a red line from the taxi stand to the red point.
4) Romina (1)	SK_1	0:03:37- 0:03:45 0:04:01- 0:04:50	First view of Romina's paper shows a rectangle in blue from the taxi stand to the blue point. Also she wrote "Blue: 5" with a black pen on the right side of her grid. All participants exchanged ideas about the problem.
5) Brian (1) / Jeff (1) / Romina (1) / Michael (1)	SK_1	0:04:54- 0:05:03	The participants start to count the number of routes to each pick-up point, each using their own method of keeping track.
6) Brian (1)	SK_1	0:05:03- 0:05:12 0:05:24- 0:05:51	Brian keeps track of the number of routes by numbering the possible options, which he draws as like staircase-like routes.
7) Romina (1)	SK_1	0:05:13- 0:05:24	Romina draws more routes using a black pen from the taxi stand to the green point.
8) Brian (1)	SK_1	0:05:30- 0:05:49	Continuing with his method, Brian draws two more staircase-like route and numbers them 4 and 5, respectively.
9) Jeff (1)	SK_1	0:05:50- 0:05:53	Jeff traces another route from the taxi stand to the red pick-up point.
10) Michael (1)	SK_1	0:05:55- 0:06:27	With a covered black marker, Michael traces different routes between the taxi stand and the green pick-up point. Also, he and Romina discuss how to keep track of the different routes they find.
11) Jeff (1)	SK_1	0:06:31- 0:06:57	Jeff gives Romina some suggestions on how to identify the routes. He points out that one has two choices at each intersection.
12) Jeff (1)	SK_1	0:06:57- 0:07:12	Jeff works some more on his sheet. He wrote "2" as he moves from the taxi

			stand to the point (2,1).
13) Brian (1)	SK_1	0:07:14-0:07:53	Brian uses his inscriptive method to check and record a sixth possible route.
14) Jeff (1)	SK_1	0:07:54-0:08:22	With a blue marker, Jeff traces and records routes starting from the taxi stand to the blue point (5,1).
15) Brian (1)	SK_1	0:08:23-0:08:36	Brian visually traces route from the taxi stand to the red point (6,4).
16) Jeff (1)	SK_1	0:08:39-0:09:24	Jeff keeps track of the number of routes on the right side of his grid and records them in blue. It seems that he deliberately choose the color of the pen to match the color of the different points.
17) Romina (1) / Michael (1)	SK_1	0:09:40-0:09:55	Looking at Michael work and her own paper, Romina realizes they had a problem keeping track of their results and proposes to use the board.
18) Jeff (1)	SK_1	0:09:58-0:10:18	Jeff shows sign of frustration. He scratches the work he has done on the right side of his grid when he was keeping track of the different routes he had found.
19) Romina (1) / Michael (1)	SK_1	0:10:18-0:10:49	Romina overlooks Mike's paper and points to him a connection between the number of lines and the number of possibilities.
20) Jeff (1)	SK_1	0:11:00-0:11:42	Jeff explains to Brian his way of thinking about the problem while going over and over the previous routes he traced.
21) Jeff (1)	SK_1	0:11:43-0:12:47	Romina, Jeff, Mike and Brian share their ideas about ways of solving the problem. All participants direct their attention to Jeff's paper.
22) Romina (1) / Jeff (1)	SK_1	0:12:47-0:13:10	Pointing at her own work, Romina proposed that there is a theory of dividing by 2 behind the problem, but it's hard to prove.
23) Romina (1)	SK_1	0:13:11-0:13:39	Brian directs the attention of the group to Romina's paper and explains how he got thirteen possible routes to a specific point.
24) Jeff (1)	SK_1	0:13:39-0:14:16	Romina suggests that they divide the problem into parts. Jeff agrees and Romina starts making a grid on a blank

			sheet. At this point Carolyn gives them some grid paper.
25) Brian (1)	SK_1	0:14:18-0:15:48	Brian not only represents his possibilities by case-like routes on which he puts numbers, but also he uses a code, "D R D" with numbers under them, representing movement: down, right, down.
26) Brian (1)	SK_1	0:14:40-0:15:23	Brian draws he's eighth stair-like route and labels it: D1, R4, D2 (down 1, right 4, and down 2).
27) Brian (1)	SK_1	0:15:24-0:16:06	Brian draws he's ninth and tenth stair-like routes and labeled them respectively 2 down, 4 over, 1 down; and 1 down, 2 over, 2 down, 1 over.
28) Jeff (3)	SK_1	0:15:27-0:15:33	While Brian is working on his grid, we get a first glance at Jeff (3).
29) Romina (1) / Jeff (3)	SK_1	0:15:47-0:16:20	Jeff and Romina try to divide the routes into blocs. Jeff counts 5 different routes so far.
30) Romina (1) / Jeff (1)	SK_1	0:15:48-0:16:22	Romina and Jeff are still working in counting the number of routes. Jeff however finds it hard to remember the results he gets when he counts every route.
31) Jeff (4)	SK_1	0:16:23-0:16:44	Introduction of Jeff (4). He uses a grid paper to draw 3 X2 and 4x2 squares in which he traces different routes from the left, up-most bloc to the far-right, bottom square.
32) Brian (1)	SK_1	0:16:48-0:16:44	Brian uses tallies to keep track of the number of routes he found. He writes them on top of his grid.
33) Romina (2)	SK_1	0:17:32-0:17:56	Jeff and Romina work together on Romina (2) trying to reach the point (6,4).
34) Jeff (4)	SK_1	0:17:57-0:18:11	As Romina explores more routes trying to reach the point ((5,4), Jeff draws a 4x2 square on his sheet to work toward the same point as Romina.
35) Romina (2)	SK_1	0:18:16-0:18:33	Romina works on point (3,1) and found 4 possible routes.
36) Jeff (4)	SK_1	0:18:36-0:19:10	Jeff and Romina work on the second 4x2 grid on Jeff's paper. Together, they count the number of routes and find more than 11 routes. They realize that they had missed some routes

			previously.
37) Jeff (4)	SK_1	0:20:32-0:20:57	Jeff draws the last 3x2 grid on Jeff (4) in an attempt to get the right number of routes.
38) Jeff and Romina (1)	SK_1	0:21:03-0:21:30	Romina marks the four points on top left of the grid in order to explain to Jeff that breaking the route into blocks will be easier and less prone to mistakes.
39) Romina trans	SK_1	0:21:45-0:22:31	Romina started using the transparency. She wrote 2 in the first square of the grid.
40) Jeff and Romina (1) / Romina trans	SK_1	0:22:33-0:23:59	Jeff started drawing 2x2 sub-grids that he then uses individually to trace routes. He finds 6 different paths. Romina record that number in the transparency.
41) Jeff and Romina (1)	SK_1	0:24:12-0:27:16	Jeff draws 3x2 sub-grids. He traces different path in each one. He makes sure that he draws opposite routes in a pair. He notices that every other path will either pair or odd.
42) Jeff and Romina (1)	SK_1	0:28:14-0:33:33	Using the same sheet, Romina draws a 3x3 sub-grid. Jeff overlooks as Romina traces the different paths. They find 15 different paths.
43) Brian (2)	SK_1	0:34:04-0:35:26	Brian writes 3 series of 4 columns of numbers under the heading: ODOO (Over, Down, Over, Over).
44) Jeff and Romina (2)	SK_1	0:35:29-0:36:35	Jeff and Romina draws 4x2 sub-grid. Romina traces different paths in each one. He makes sure that he drew opposite routes in a pair.
45) Brian (3)	SK_1	0:38:37-0:38:44	Romina puts a dot on 3x2 grid on Brian's paper while asking him to check the number of routes he has gotten for that table using his own method.
46) Brian (3)	SK_1	0:38:44-0:39:30	Brian works on the 3x2 table. He writes down all possible routes using his notation of *O*s and *D*s.
47) Jeff and Romina (3)	SK_1	0:39:44-0:39:55	Firs view of Jeff and Romina (3). Romina draws 4x3 grids.
48) Brian (3)	SK_1	0:39:56-0:40:56	Brian writes 4 columns of numbers under the heading: *O D O O* (Over, Down, Over, Over).
49) Jeff and Romina (3)	SK_1	0:41:00-0:45:12	Jeff and Romina work together on the 4x3 grid. With a red marker Jeff traces

			the possible paths while Romina looks and comments.
50) Jeff and Romina (1)	SK_1	0:45:32-0:46:25	Romina compares their results on "Jeff and Romina (1)" with Brian. The latter draws on the board the different routes.
51) Romina trans	SK_1	0:46:37-0:47:23	Romina observes that they are dealing with Pascal triangle. She writes the number 10 in square (4, 2) and (4, 3). She also writes the number 5 in square (4,1) and confirms, "that's Pascal triangle".
52) Brian (4)	SK_1	0:47:31-0:50:38	The group asks Brian to work on the 4x2 grid to confirm that the problem is based upon Pascal triangle.
53) Michael (3)	SK_1	0:50:40-0:51:33	Michael works on a 4x2 sub-grid. He mentally draws the routes and keeps track of his findings by writing down the numbers.
54) Jeff and Romina (4)	SK_1	0:51:37-0:52:08	Like Brian and Mike, Jeff works on a 4x2 sub-grid. He uses a green color marker to make the grid and a red one to trace the route.
55) Michael, Romina, Jeff, and Brian (1)	SK_2	0:0:30-0:01:10	Jeff tries to find an easier way to come up with the number of routes using a 4x2 sub-grid.
56) Romina trans	SK_2	0:01:49-0:52:08	Jeff uses the transparency to show Michael how the numbers they have been finding relates to Pascal's triangle. Based on his assumption, he declares, "the ninth block should be twenty".
57) Romina trans	SK_2	0:05:07-0:05:40	Romina turns the transparency bottom-up and begin to write back the numbers based on Pascal's triangle.
58) Brian (5)	SK_2	0:05:55-0:06:21	Brian start working on a 3x3 sub-grid using a black marker. He writes on the right his findings using his system of "OD" with corresponding numbers underneath
59) Jeff and Romina (5)	SK_2	0:06:25-0:07:46	Romina works on a 3x3 grid in the same time as Brian. She uses a green marker to draw different routes. She traces 6 routes in the first row and on the second draws their opposite.
60) Michael (3)	SK_2	0:07:52-0:08:23	Michael declares he has gotten 20 routes for the 3x3 sub-grid while Romina draws squares around her own 3x3. As Romina add 20 to their transparency,

			Jeff wonders if they could find a way to relate their finding to Pascal triangle.
61) Romina trans	SK_2	0:08:25-0:09:17	The participants focus their attention on the Pascal triangle. Jeff and Romina attempts to number the rows and columns of the triangle as 1, 2, 3…
62) Romina (3)	SK_2	0:09:18-0:10:40	Romina wrote "2^X" in an attempt to associate 2 colors with the Pascal triangle.
63) Romina trans	SK_2	0:10:42-0:11:18	Romina tries to make a smaller Pascal triangle in order to have a better understanding of the principle behind it. She wrote respectively in a triangle shape, 1/1,2,1-/1,3,3,1.
64) Jeff+Romina-5	SK_2	0:15:03-0:15:10	Jeff draws a small box in red to show Caroline how they started working on the problem.
65) Romina trans	SK_2	0:15:10-0:16:06	Romina and Jeff use the transparency to explain how they discover that they were dealing with Pascal triangle. Jeff observes that as they tilted the transparency and put 1's on top and on the outside of the transparency grid, they were in presence of the Pascal triangle.
66) Romina trans	SK_2	0:17:23-0:19:39	The participants declare finding a pattern that fits Pascal triangle. Also they become convince that the results they obtain using this pattern are more accurate than when they count the routes. Looking at their triangle and using his finger, Jeff points out to Caroline how they obtain the pattern by adding two numbers in the previous row.
67) Michael, Romina, Jeff, and Brian (2)	SK_2	0:19:40-0:20:41	Romina draws the triangle on a different sheet of paper. Along with the others, she tries to explain the pattern that they found.
68) Romina trans, Michael, Romina, Jeff, and Brian (2)	SK_2	0:20:42-0:26:01	They brainstorm, and use representation such as tower, color, (a, b), and (x, y), and even pizza toppings to provide a logic explanation for the pattern they identify.
69) Romina trans, Michael, Romina, Jeff, and Brian (2)	SK_2	0:26:02:-0:27:36	Mike tries using a pizza-topping imagery to achieve a similar goal. The toppings represent the rows of numbers in their image (on Brian + Jeff + Mike +

			Romina). Mike defines topping as the number of ways to go across.
70) Romina trans, Michael, Romina, Jeff, and Brian (2)	SK_2	0:27:40-0:32:35	Romina and Brian work in completing the Pascal triangle. They added 35 and 21 at the bottom of the triangle. When they were all done, the bottom of the triangle (7th row) reads from the left to the right, 7, 21, 35, 35, 21, and 7.
71) Michael, Romina, Jeff, and Brian (2)	SK_2	0:34:00-0:34:50	Pointing at the triangle, Romina explains the relation to Pascal triangle using different color code and blocks.
72) Romina trans, Michael, Romina, Jeff, and Brian (2)	SK_2	0:34:52-0:39:08	Once again Mike uses a pizza-topping imagery to explain the relation with Pascal triangle. He refers to the triangle as the sum of all the toppings of a pizza.
73) Romina trans, Michael, Romina, Jeff, and Brian (2)	SK_2	0:39:33-0:40:24	Romina explains the relation between (a, b), (x, y) and the toppings. Romina uses both triangles that they drew and identifies the x's as across, colors, or toppings; and the y's as down, toppings, and colors.
74) Romina trans, Michael, Romina, Jeff, and Brian (2)	SK_2	0:40:46-0:41:36	Mike uses a series of binary numbers to explain the Pascal triangle. He wrote on top of the triangle 2 groups of 1's and 0's. 1's meaning across, and 0's down.
75) Romina trans, Michael, Romina, Jeff, and Brian (2)	SK_2	0:39:33-0:43:53	Romina marks point (10,5) indicating a number and try to justify how their finding can explain what the shortest route will be. In this case, along with Mike, Romina determines that there is 10 ways of getting there by adding the number of rows to the number of columns.
76) Romina trans, Michael, Romina, Jeff, and Brian (2)	SK_2	0:44:40-0:46:00	The participants start from the very beginning of the triangle to explain their conclusion. Romina uses both triangles with a covered green pen to show how they have gotten the numbers in the triangle They explained that the shortest route is the number of row.

APPENDIX G: LAYERED BUILDING OF INSCRIPTIONS (LS)

Video Data Base, LS_1, LS_2, and LS_3

Inscription Name	Camera View	Time Interval	Comments
1) Taxicab problem	LS_1	0:01:30-0:04:09	Researcher 1 distributes the problem task. The participants discuss it and request color markers.
2) Romina (1)	LS_1	0:04:09-0:04:51	Romina draws a route with a red marker and writes "Blue 5" with a black pen. Participants comment her findings.
3) Jeff (2)	LS_1	0:04:56-0:04:58	Jeff draws in red the shape of the typical route for the problem on Brian's. [Note paper length difference between Jeff's own paper]
4) Jeff (1)	LS_1	0:05:02-0:05:07	First view of Jeff's problem sheet. He traces a red line from the taxi stand to the red pick-up point.
5) Brian (1)	LS_1	0:05:06-0:06:04	First view of Brian's problem sheet. On the left side he writes "Blue-5", referring to the number of shortest routes from the taxi stand to the blue pick-up point.
6) Jeff (1)	LS_1	0:06:16-0:06:26	Using a red marker, Jeff draws more routes showing other ways to reach the red point.
7) Romina (1)	LS_1	0:06:31-0:06:56	Using a black marker, Romina draws more routes from the taxi stand to the green point.
8) Romina (1)	LS_1	0:09:40-0:10:30	Period during which Romina wrote "10" and "30" on her sheet.
9) Romina (1)/ Michael (1) / Jeff (1)	LS_1	0:10:52-0:11:15	Romina remarks that they had problem keeping track of their findings and later proposed that they write them on the board. Mike says he is trying to remember what he had found. Jeff thinks there must be some type of math formula for solving the problem. He shows sign of frustration and draws graffiti in blue over some of his work on the right side of his grid.
10) Jeff (1)	LS_1	0:12:19-0:13:44	Jeff brainstorms with Brian and Romina while he repeatedly retraces possible routes with a black marker on the grid and represents these possible routes at the bottom and to the right of his grid.

11) Brian (1)	LS_1	0:13:46-0:13:54	Second view of Brian's problem sheet. He uses staircase-like shape to represent the possible paths and numbers them.
12) Romina (2)	LS_1	0:15:38-0:16:51	Romina draws routes, numbers the squares, and counts the different possibilities while Jeff watches and comments.
13) Jeff (3)	LS_1	0:16:53-0:17:11	With a blue marker, Jeff draws a 2x2 square. Within this square he draws all possible paths from the taxi stand.
14) Brian (1)	LS_1	0:17:17-0:17:26	Brian works progresses to include more record of staircase-like representation of different routes. He has at this point has found ten.
15) Michael (1)	LS_1	0:17:34-0:17:57	Mike's very first work representing staircase-like with numbers on the right side of grid. Also, he seems to add numbers.
16) Romina (2)	LS_1	0:17:58-0:19:48	Both Romina and Jeff work on new strategies on Romina's paper. As they focus on Romina's 2x3 square, Jeff points to the square (8, 3) as a suggestion for the next move. Instead, Romina proposes that they investigate a 3x3 sub-grid, which she illustrates by going around it slightly with her black marker.
17) Jeff (4)	LS_1	0:19:51-0:20:19	Jeff works on the first Cuisenaire 1-centimeter grid paper, where he draws routes in 3x2 and 4x2 sub-grids.
18) Romina (2)	LS_1	0:20:19-0:22:44	Comparing his work to Romina's, Jeff observes that she was missing a route in her 3x3 grid. With his pen he slightly goes over the points that Romina missed. Starting from the taxi stand, he notices that she did not draw the route over 2, down 1, over 2 and down 1.
19) Jeff + Romina (1)	LS_1	0:22:15-0:22:34	Romina marks in black ink 4 dots to propose that they find how many ways there are to get from the stand to each point on the grid corresponding to a point on the taxicab grid. She predicts that by doing so they will see a pattern. As a follow up, Jeff proposes the use of a transparency.
20) Romina-trans.	LS_1	0:22:47-0:23:42	Romina begins using the transparency to predict what the next number will be based on the results they have already

			obtained. She writes with a blue marker the numbers: 2, 3, 4, and 5.
21) Jeff + Romina (1)	LS_1	0:23:42-0:24:10	In a set of blue 3x3 sub-grids, Jeff traces routes with a blue marker.
22) Michael (1)	LS_1	0:24:15-0:27:48	Mike traces routes and records numbers
23) Romina-trans.	LS_1	0:27:58-0:28:37	Romina works more on the transparency. She writes the following numbers in the second and third rows respectively: 3, 6, 10 and 4, 10, 15.
24) Jeff + Romina (1)	LS_1	0:28:41-0:29:11	Using a blue marker, Jeff traces the second set of 3x3 sub-grids, and then gives the grid to Romina.
25) Jeff + Romina (1)	LS_1	0:29:21-0:34:22	Jeff asks Romina to "stick with a pattern." Romina uses a red pen to trace routes.
26) Romina-trans.	LS_1	0:34:22-0:34:38	Romina records their findings onto the transparency. She added a third row of numbers: 4, 9, and 15.
27) Jeff and Romina (2)	LS_1	0:34:38-0:34:44	Jeff takes a blank sheet and positions it between Romina and himself.
28) Michael (1)	LS_1	0:34:44-0:35:09	Mike is still working alone on Mike2. Romina asked him whether he was still working on getting to the red point…He said he was
29) Jeff and Romina (3)	LS_1	0:35:12-0:35:40	Jeff started to draw a 2x4 table using a red marker.
30) Michael (1)	LS_1	0:35:41-0:35:50	Mike is still working on the same sheet. We can see more stair-like shapes with more numbers next to them and, at the bottom right of the grid, an addition of 12, 3, 4, 5 for a total of 24.
31) Jeff and Romina (2)	LS_1	0:35:58-0:38:30	Jeff passes the sheet with the 2x4 sub-grids he has just worked on to Romina so she can look for all possible routes within each one. After tracing 11 possible routes, they observe that they need one more 2x4 sub-grids
32) Michael (1)/Brian (2)	LS_1	0:37:10-0:37:45 0:38:16-0:38:30	Mike and Brian exchange information about their work. Brian tells Mike that he found 30 for one and 17 for the other.
33) Brian (2)	LS_1	0:38:30-0:38:41	Romina comments on Brian's work. She thinks that he's using a "nice" method.
34) Jeff and Romina (2) /	LS_1	0:38:42-0:39:31	Jeff and Romina review their routes in their 2x4 sub-grids and try to make sense

Romina-trans.			of their transparency. They count the number of routes they have found in their 2x4 sub-grids and write the number in the transparency. This time Romina writes 12 as the fourth number in the second row.
35) Brian (3)/Jeff and Romina (2) / Michael (3)	LS_1	0:39:31-0:40:41	All 4 members of the group started sharing information. Romina inquires how many possibilities Mike and Brian found for the 4x3 sub-grids. They both say 34. Brian draws a 3x2 grid. Romina showed him the point at the bottom right towards which he needs to work.
36) Michael (2)	LS_1	0:40:20-0:40:26	Jeff and Romina propose that Michael works on a 4x2 sub-grids while Brian works on a 2x3. Mike turns over his problem sheet and draw a square which he marks at four locations as a way to make a 4x2 sub-grids
37) Michael (3)	LS_1	0:40:26-0:40:41	Jeff handed over a Cuisenaire to Michael to replace the problem sheet he was using.
38) Brian (3)	LS_2	0:0:15-0:0:33	While working on the 3x2 sub-grids he draws, Brian inquires if Jeff and Romina had come up with a formula. Above his grid, Brian displays 3 columns of a series of numbers under the letters O and D, meaning over and down.
39) Jeff and Romina (3)	LS_2	0:0:39-0:01:03	Jeff and Romina divide in blue a new grid into a 3x5 sub-grid. They will use "Jeff and Romina (2)" to show their pattern to Brian after Brian has come up with his own.
40) Jeff and Romina (3)	LS_2	0:01:09-0:02:35	Romina invites Jeff to use his method for finding the shortest routes.
41) Jeff and Romina (3) / Brian (3)	LS_2	0:02:36-0:02:52	Brian compares his findings with Romina and Jeff.
42) Jeff and Romina (3) / Brian (3) /	LS_2	0:02:52-0:03:19	Mike's result of 12 matches Romina and Jeff's.
43) [Blackboard]	LS_2	0:03:20-0:05:35	Brian proposed to write his findings on the chalkboard for a 3x2 sub-grid.
44) Jeff + Romina (1)	LS_2	0:05:50-0:07:29	Romina and Jeff compare their results with what Brian writes on the chalkboard.

45) Romina-trans.	LS_2	0:07:35-0:08:38	Romina, Jeff, Mike, and Brian discuss the logic behind the numbers they came up with. Jeff propose, 15; and Romina and Brian, 12.
46) Brian (4)	LS_2	0:08:50-0:09:00	Brian works on "Brian (4)" to show different combination.
47) Michael (3)	LS_2	0:09:05-0:09:36	Mike traces and records routes.
48) Brian (4)	LS_2	0:09:43-0:10:26	Jeff and Brian work together. Jeff looks over as Brian explains to him his method
49) Michael (3)	LS_2	0:10:28-0:11:00	Both Mike and Jeff are looking for number 15.
50) Jeff and Romina (4)	LS_2	0:11:16-0:11:29	Jeff uses a green marker to draw twelve 2x4 squares.
51) Brian (4)	LS_2	0:11:30-0:13:02	Brian finds only 12 possibilities instead of 15 as was predicted. On his sheet, Brian has 4 boxes of numbers.
52) Michael, Romina, Jeff and Brian (1)	LS_2	0:13:03-0:14:52	Jeff thinks there is an easier way. He proposes to divide the 4x2 into two 2 x2. He gets input from the others, and records his findings.
53) Michael (3)	LS_2	0:14:53-0:17:12	Mike draws a fourth square (3x3) on his sheet after writing down the numbers 0 to 9. He works inside the box and writes down what he found.
54) Brian (4)	LS_2	0:17:20-0:17:34	Brian announces to Romina that he found 15 routes in a 4x2 sub-grid. Romina replies that implicitly (2x4) has to be the same. Furthermore, she suggested that (3x3) must have 20 routes.
55) Romina-trans.	LS_2	0:17:44-0:18:16	Romina turns the transparency upside down and writes the number of routes they have agreed upon
56) Brian (5)	LS_2	0:18:17-0:18:40	Brian proposes to look for the number of routes in a 3x3 table. As he starts writing, Romina follows along.
57) Jeff and Romina (5)	LS_2	0:18:42-0:19:49	Romina turns the grid upside down and draws different routes in each of her thirteen 3x3 sub-grids.
58) Brian (5) / Jeff and Romina (5)	LS_2	0:19:51-0:20:31	Brian writes more numbers in his 3x3 tables while Romina shares her work with Jeff.
59) Michael (3) / Jeff and Romina (4)	LS_2	0:20:32-0:21:50	Mike told Jeff and Romina that he had gotten 25 for the 3 x3. The group started discussing the relation to Pascal's

			triangle.
60) Romina (3) / Jeff and Romina (4)	LS_2	0:21:50-0:24:00	Romina explained the Pascal's triangle relation. Mike added more details. Jeff proposed that they spell out their idea and talk about it so they can understand it. They ask to see the researchers.
61) Romina (3)	LS_2	0:22:03-0:22:06	Romina wrote " 2^x " on this sheet in an attempt to explain two colors as being 2 to the x.
62) Romina-trans. / Jeff and Romina (4)	LS_2	0:27:12-0:28:05	The group shared they feelings about the problem with researcher 1. Told her that they had considered all points on the grid, and informed her about the steps they took such as breaking the problem down. This and other considerations, Romina pointed out, leaded them to the Pascal's triangle.
63) Romina-trans. / Jeff and Romina (4)	LS_2	0:28:05-0:32:08	Jeff explained in details how they came to make the connection to Pascal's triangle. However, participants could not explain how the pattern would help them predict the number of paths for any location.
64) Romina-trans. / Michael, Romina, Jeff and Brian (2)	LS_2	0:32:08-0:39:49	Researcher 1 asked the participants at this point to justify the pattern found. They brainstormed, and used representation such as tower, color, (a, b), and (x, y), and even pizza to put logic on the pattern they had found.
65) Romina-trans. / Michael, Romina, Jeff and Brian (2)	LS_3	0:0:04-0:05:23	Researcher 2 wants to know how do they relate each location to a number from Pascal's triangle. Romina, Brian and Mike discuss ways to explain the pattern.
66) Romina-trans. / Michael, Romina, Jeff and Brian (2)	LS_3	0:05:56-0:07:37	Romina tries to explain a relationship between Pascal's triangle and the Towers problem.
67) Romina-trans. / Michael, Romina, Jeff and Brian (2)	LS_3	0:07:38-0:11:51	Brian tried using a pizza-topping image to achieve a similar goal. The toppings representing the rows of numbers in their image in Brian + Jeff + Mike + Romina. Mike defines topping as the number of ways to go across.

68) Romina-trans. / Michael, Romina, Jeff and Brian (2)	LS_3	0:11:52-0:13:10	Researcher 1 asks Romina to explain the relation between (a, b), (x, y) and the toppings. Romina identify the x's as across, colors, or toppings; The y's as down, toppings, and colors.
69) Michael, Romina, Jeff and Brian (2)	LS_3	0:13:11-0:14:40	To answer Researcher 1 question concerning the use of binary, Brian wrote in green the series of one's and zero's.
70) Romina-trans. / Michael, Romina, Jeff and Brian (2)	LS_3	0:15:18-0:18:28	Romina marks a point (5 down, 5 across) indicating a number and try to justify how their finding can explain what the number will be.
71) Romina-trans. / Michael, Romina, Jeff and Brian (2)	LS_3	0:18:29-0:19:00	Researcher 1 asks the participants how they would explain the Rth row? Romina thinks it's their moves, and Brian observes it will be the shortest distance.

APPENDIX H: STUDENT-RESEARCHER INTERACTION EPISODES

The following details information summarized in Table 1. Each row of the table represents an episode from the perspective of camera view LS (LS_1, LS_2, or LS_3); indicates the start, end, and elapsed time of the episode; and whether the interval is one in which students interact amongst themselves ($S \Leftrightarrow S$), in which interaction with researchers is initiated by students ($S \Leftrightarrow R_z$), or in which interaction with students is initiated by researchers ($R_z \Leftrightarrow S$).

Major Interaction Episodes	Time			Interaction Type
	Start	End	Elapsed	
LS_1 0	00:00	01:05	01:05	Noise. Camera begins to roll, students enter the classroom, sit at a trapezoidal table in front of three video cameras.
LS_1 1	01:05	01:41	00:36	$R_1 \Leftrightarrow S$: Researcher 1 welcomes the students, distributes the Taxicab Problem (see Appendix A), asks students to read it to see whether they understand, and leaves the table but remains nearby.
LS_1 2	01:41	03:02	01:21	$S \Leftrightarrow S$: Students read the problem. Jeff asks question of Researcher 1. The four students ask each other questions and otherwise discuss features of the task.
LS_1 3	03:02	03:42	00:40	$R_1 \Leftrightarrow S$: Researcher 1 asks the students for their understanding of the problem. They ask a few clarifying questions. Afterward, Researcher 1 leaves the room.
LS_1 4	03:42	40:41	36:59	$S \Leftrightarrow S$, $S \Leftrightarrow R_2$: In various combinations and at various times, Michael, Romina, Jeff, and Brian work individually and collaboratively. Based on student request, Researcher 2 places several differently colored markers on the table and extra grid paper.
LS_2 5	00:00	23:59	23:59	$S \Leftrightarrow S$: In various combinations and at various times, Michael, Romina, Jeff, and Brian work individually and collaboratively. Jeff suggests that they

				explain and talk through what they have found. A cameraperson or someone from the research staff informs Researcher 1 that the students are ready to present their work.
LS_2 6	23:59	26:57	02:58	S ⇔ S: Green ink from Romina's marker stains her sweater. Jeff and Brian advise her on how she might get it out. Someone from the River Run staff offers Romina baby wipes. Jeff and Brian also accept baby wipes to clean ink off their hands.
LS_2 7	26:57	32:18	05:21	S ⇔ R_1: Students request that Researcher 1 listen to their report on what they have found and asks questions to understand what they mean. Researcher 1 asks the students how could they explain why Pascal's triangle emerges in the Taxicab Problem.
LS_2 8	32:18	36:50	04:32	S ⇔ S: The students discuss approaches to answering the question. Jeff leaves in the middle of this session. Michael asks Researcher 1 to explain again the question.
LS_2 9	36:50	39:49	02:59	S ⇔ R_1: In response to Michael's question, Researcher 1 asks Romina to explain her idea. Michael then attempts to explain how he understands. Researcher 1 asks the students to further explain the relationship they see between the Taxicab Problem and the Pizza Problem.
LS_3 10	00:00	00:31	00:31	R_3 ⇔ S: Researcher 3 asks the students to explain why the number of shortest routes to any intersection point corresponds to a number in Pascal's triangle. Researchers 1 and 3 then leave the table and room to let the students think through their explanation.
LS_3 11	00:31	05:36	05:05	S ⇔ S: The students collaborate to negotiate their justification of how the number of shortest routes to any intersection point corresponds to a number in Pascal's triangle.
LS_3 12	05:36	19:05	13:29	S ⇔ $R_{3,1}$: The students suggest that they are now prepared to responded to Researcher 3's question. In response to

				Michael's request, Researcher 3 reiterates his question. The students then present their explanation of the relationship between the number of shortest routes to any intersection point corresponds to a number in Pascal's triangle. In their response they relate the Taxicab Problem to the Pizza and Towers Problems.
Student to student interaction episodes			1:15:59	Total amount of time students spend in discursive interaction without researchers (≈ 77% of total).
Researcher and student interaction episodes			23:36	Total amount of time students spend presenting their ideas and being interviewed by researchers (≈ 23% of total).

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EDUCATION

9/72-5/76	Hampshire College, Amherst, Massachusetts Bachelor of Arts – Mathematics
9/76-5/79	University of Michigan, Ann Arbor, Michigan Master of Science – Mathematics (12/77)
9/99-5/03	Rutgers, the State University of New Jersey, New Brunswick Doctor of Philosophy – Education (Mathematics)

EMPLOYMENT

9/79-12/80	City University of New York Queens College, Flushing, New York 113567 Mathematics Department, Adjunct Lecturer
9/79-6/81	Educational Solutions, Inc. 95 University Place, New York, New York 10001 Research Associate in Mathematics to Caleb Gattegno
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9/80-5/84	District Council 37 Campus, New York, New York 10017
9/85-1/86	Harlem Campus, New York, New York School of New Resources (adult education) Mathematics Department, Adjunct Lecturer Courses taught: Research and Statistics, Computer Programming (BASIC and Logo), and Teaching Mathematics in Elementary Schools.
7/81-present	Rutgers University, Newark, New Jersey 07102 Faculty of Arts and Sciences Academic Foundations Department Associate Professor (7/87-present) Assistant Professor (7/81-6/87) Chair (7/90-12/92, 7/94-1/97) Acting Chair (9/89-6/90) Developmental Mathematics Coordinator (9/82-9/91) Courses taught: from developmental mathematics to the calculus and mathematics pedagogy (K-8). Faculty Research Scientist, Robert B. Davis Institute for Learning, Graduate School of Education (8/00-present) Member, Center for African Studies (8/02-present)

3/93-6/93	University of the Western Cape, Belville, South Africa Mathematics Department Visiting Scholar
7/93-12/93	Instituto Superior Pedagógico, Maputo, Mozambique Departamento de Matemática Professor Auxiliar

PUBLICATIONS

Articles

Fernández, E., & Powell, A. B. (2000). Remembering Anneli Lax (1922 - 1999). *Journal of Mathematical Behavior, 19*(1), 1-7.

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