TRACING THE GROWTH IN UNDERSTANDING OF FRACTION IDEAS: A FOURTH GRADE CASE STUDY

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ABSTRACT OF THE DISSERTATION

Tracing the Growth in Understanding of Fraction Ideas: A Fourth Grade Case Study

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The case study of a class of twenty-five fourth graders was designed to trace the growth and development in thinking about fraction ideas prior to the inception of rules and algorithms within the school curriculum. The project was an outgrowth of a long-term teacher development collaboration between Rutgers University and the Conover Road School in Colts Neck, NJ.

Twenty-five children in Mrs. Phillips' class met with a team of Rutgers researchers fifty one and one-half hour sessions during the school year. This study reports on the first seven of the twenty-five sessions focusing on fraction activities.

During all sessions, children were invited to explore activities working with partners or in small groups. They discussed their solutions and built models to illustrate their findings. Children explained and supported their ideas, first to other small groups and then to the entire class. The fourth-graders built a mathematical community in which ideas were presented, explored and debated.

Four pivotal mathematical strands developed in the children's thinking. These strands included a growing understanding of 1) fraction as operator and fraction as number, (2) attention to the naming of the unit or the construction of an assimilation paradigm, (3) fraction comparisons, and (4) equivalence.

The work of the children is offered as a powerful existence proof of the mathematical understanding that learners can develop before exposure to the rules and definitions presented in formal curricula.

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A STATEMENT OF THE PROBLEM

The difficulty many students encounter in attaining clear meaning and understanding of fraction ideas presented itself to me while I was the instructor of a mathematics education course for elementary education majors (1995-1998). I posed the following question to the class: How might you expect your students to think about the following question: 'Which is larger one half or one third?' Some members of the class quickly responded that they felt children would respond with "one half". One student explained that there was no answer to the question because I hadn't given any indication of a half of "what" or a third of "what" in stating the problem (1996). This response heightened my awareness of assumptions that we make implicitly that may not be explicit to the thinking of others. The difficulties in building an understanding of fraction ideas can follow the learner through adulthood.

Streefland (1993) points to this lack of understanding as a worldwide dilemma. The problem has been documented by mathematicians, mathematics educators and psychologists. In a 1991 book, Streefland refers to the underestimation of the complex nature of fractions, coupled with the mechanical approach to the teaching of vague, yet rigid rules, as the rationale behind students' difficulties. One such difficulty is reconciling the notion of fraction as operator with the notion of fraction as number. Noticing how children think of whole numbers illustrates this concept.

Young children begin explorations into number by recognizing specific sets of objects. In recognizing 'two' objects they may refer to two toys, two hands, two shoes. This specificity can be referred to as working at an operator sense of number. Over time children begin to abstract the 'twoness' from individual sets and sees 'twoness' across a variety of sets. Once children begin to internalize this concept, they begin to 'see' number in an abstract sense.

The arbitrary labels of word and symbol attached to 'twoness' vary from culture to culture - for example, zwei, ni, duo, / /, II, and our Hindu-Arabic 2. However, the meaning of “2” is not arbitrary. Toddlers reciting numbers as they would the alphabet may have attached little meaning to the recitation or may give different labels as they attempt to imitate adult conversation [1, 2, 4, 8 may seem as logical as 1, 2, 3, 4]; hence it is an arbitrary listing of labels. For a very small child, it is in the operator sense of number that meaning begins to be attached to each label.

Movement from the operator sense of number to the abstract notion of number does not imply the negation or elimination of the operator sense. A child can imagine the objects the operator sense demands, while talking of twoness. The operator sense is viewed as implicit and absorbed in the abstract notion of number sense. This movement does imply that <u>both</u> are essential in the understanding of number.

A similar development of a child's mathematical progress is hoped for in the development of ideas relating to fractions, namely achievement of both the operator and number sense of fraction. A child begins with an operator sense in which fraction ideas are specific to particular objects. He wants 1/2 of a cookie, 1/2 of the box of crayons, (Alston, Davis, Maher, Martino, 1994).

> Fractions, for example, are typically thought of as operators for an extended period of time, often years, before they come to be thought of as numbers. For example, a fourth or fifth grader, asked to explain what "fractions" are, will typically answer in terms of "one half of" something, or "one fourth of" something.
>
> (Davis and Maher, 1993, p. 13)

With this understanding in mind, it was natural to ask, what conditions might be employed so that children’s learning of fraction ideas might become a mathematical building experience for them?

The Colts Neck Study

Researchers at Rutgers University have long been studying the development of mathematical ideas in children. Davis and Maher conducted a three and one-half year study in three school districts: the urban district of New Brunswick, the blue collar district of Kenilworth, and the suburban district of Colts Neck, all located in New Jersey. Maher and colleagues traced the development of the mathematical idea of proof making with a focus group of students for twelve years[1].

In working with the fifth-graders in Kenilworth, Davis and Maher noticed the difficulties the children were experiencing during investigations focusing on fraction ideas. The children were struggling with finding solutions to fraction tasks by trying to utilize the algorithms given to them as part of their regular class curriculum. As a result of the events in Kenilworth, another research project was designed and implemented to explicitly explore the development of fraction ideas in children before the onset of imposed rules (For example, see Maher, Martino, & Davis, 1994). The project was implemented as an extension of an already established collaboration with the Conover Road School in Colts Neck, New Jersey. In particular, the research subjects were twenty five heterogeneously grouped children in a fourth grade class. The classroom teacher, Mrs. Joan Phillips, worked with Rutgers researchers to support this intervention, to communicate with parents, and to participate in the planning of grouping within the classroom.

Fourth grade was selected because it is typically the year prior to the introduction of formal algorithms concerning the operations with fractions; this was the case in the Conover Road School. These nine and ten year old children, as part of their prior primary education in mathematics, had

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[1] See Appendix A for a selected list of citations focusing on the longitudinal study.

been offered a strong experience in which they had explored ideas associated with fraction as operator.

Classroom conditions were established in which students were invited to work together and conduct thoughtful investigations with appropriate materials. Davis and Maher (1990, p. 77) emphasize the importance of giving students the opportunity to work similar to how a mathematician works, the opportunity of studying and solving problems, analyzing results and inventing their own rules and procedures. Their studies show that children can generate the same inventive thinking that mathematicians do.

This study focuses on the first seven of the one and one-half hour sessions, videotaped using two to three cameras. Triangulation is possible from analysis of the videotapes, student journals and researcher notes. The archival data set make possible a detailed study of the development of fraction ideas in children prior to the formal school teaching of fractions. As a member of the data collection team, this researcher videotaped many of the sessions and participated in the debriefing' sessions following each meeting with the children.

According to Yin (1994), this undertaking is thus defined as a case study of these fourth graders: coping with a distinctive and complex situation, relying on multiple sources of evidence, and based on prior development of a theoretical perspective (p. 13). Additionally the study is basically exploratory in nature, with a goal of "developing pertinent hypotheses and propositions for further inquiry" (p. 5).

With these conditions in place, the following questions were posed:

1. What fraction ideas do children build?
2. What representations do children use in expressing these ideas?
3. How do mathematical ideas travel within a classroom?

CHAPTER II

REVIEW OF THE LITERATURE

The word fraction comes from the Latin, *frangere*, "to break apart" (Wheat, 1937, p. 84). This Latin word was translated from the Arabic, *kasra*, meaning broken (Guedj, 1997, p. 81). The following is offered as an example of recent children's literature relying on the broken unit, or part-of-the-whole, relationship that fractions often elicit.

> "Are there any precious stones [in the mine]?" asked Milo excitedly. "…I'll say there are. Look here." The Mathemagician reached into one of the carts and pulled out a small object, which…sparkled brightly. "But that's a five," objected Milo, for that was certainly what it was. …So that's where they come from," said Milo looking in awe at the glittering collection of numbers. He returned them to the Dodecahedron… but as he did, one dropped to the floor with a smash and broke in two… "Oh don't worry about that", said the Mathemagician as he scooped up the pieces, "We use the broken ones for fractions." (Norton Jester, The phantom toothbrush, 1961, quoted by Guedj, 1997, p. 163)

A Brief History of Broken Numbers

Representations for fractions developed in the Bronze Age when man's advancing culture required him to measure and weigh his possessions and those of others as he began to barter and trade. Struik (1987, p. 12) cites the lack of evidence of use of fractions in the Stone Age; people survived by simply hunting and gathering. Boyer (1989, p. 5) notes that as fractions developed, a custom of selecting increasingly smaller units was sometimes employed.

Systems developed for fraction use were patterned after the systems for whole numbers. Menninger (1992, p. 208-213) cites an ancient system of finger gestures where fractions follow rules governing whole numbers[1]. Such historic knowledge about mathematics and its corresponding systems is acquired from surviving written recordings (drawings, symbols, and texts).

Weil (1984, p. 5) comments that the origins of many mathematical ideas are nearly impossible to trace, since they may predate the written recordings of surviving documentation. Evidence found in such documents implies that early and important contributions to the history of fractions were made by Egyptian and Babylonian peoples, but the Ancient Orient, notably Hindu and Chinese cultures, and later-day Greek and Roman influences, were also essential to the development of fractions as a decimal notation system.

Each indigenous trading group developed its own fraction system, based on a collective of cultural, scientific and sometimes religious beliefs. The practical needs of a civilization conducting the functional aspects of daily life drove the development of fractions and their notational systems. Thus the development of fractional number systems parallels the development of mercantile communities.

Surviving Egyptian mathematical texts contained problems of a practical or commercial nature – i.e., computing the capacity of a granary, recording recipes for bread and beer, calculating the number of bricks needed for building. Identification and study of Egyptian mathematics is primarily based on the Rhind or Ahmes Papyrus (1650 BC)[2]. Boyer (1989, p. 14) counts eighty-four

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[1] Since antiquity, finger gestures for whole numbers, have been recorded in drawings. These gestures were extended to include a system for fractions. For example, merchants in the areas of the Red Sea, Arabia, and East Africa are still observed using a silent finger language to signify fractions. If a merchant is observed soundlessly stroking his middle finger from the middle joint knuckle toward the tip, he is saying –1/2, if he strokes the index finger toward the knuckle he means + 1/2.

[2] Four lesser writings of some importance include the Moscow Papyrus, the Kahun Papyrus, the Berlin Papyrus, and the Leather Roll (Bunt, Jones, & Bedient, 1988, p. 5-6)

widely assorted mathematical problems, plus two tables, as comprising the surviving Rhind Papyrus, as recorded by the scribe, Ahmes.

Egyptians used a decimal or base ten system for natural numbers (Bunt, Jones & Bedient, 1988, pp. 43). Boyer (1989, p. 10) notes that no standard place-value system existed in Egypt during this period since numerals were recorded sometimes with the smallest on the left, or sometimes vertically, or sometimes in a reverse orientation.

Egyptian fractions were generally represented as unit fractions. Wheat (1937, p. 85) conjectures that the Egyptian view of unitizing fractional quantities was reasonable. If one should consume one fourth of a pie, what would you name the uneaten remains? Wheat claims that the answer is unclear as 3/4, since it is not three of anything, it is one part of something. Egyptian fractions, in the period recorded in various papyrus, would represent the remains as 1/2 + 1/4. Wheat views this unitizing as looking at the <u>size</u> of the part remaining, not at the <u>number</u> of parts remaining; he sees the Egyptian recording of any fractional quantity as a sum of unlike units written one at a time as a logical conclusion.

Logical as this procedure might appear; it is quite a cumbersome process. Using modern notation for the Egyptian system, the fractional quantity 7/8 would be expressed as 1/2 + 1/4 + 1/8 (Adkins, 1963, p.224), or 11/25 as 1/3+ 1/50 + 1/300 (Bunt, et al., 1988, p. 25), or 14 and 28/97 as 14 + 1/97 + 1/56 + 1/679 + 1/776 + 1/194 + 1/338 (Struik, 1987, p. 25).

Additionally, the Rhind Papyrus shows Egyptians represented division by the fraction form 2/n. For example, 2/101 is represented as 1/101 + 1/202 + 1/303 + 1/606. Boyer (1989, p. 14) notes that this 2/n table was followed by a table of n/10. The first six problems of this Papyrus obliged the scribe Ahmes to record the division of 1, 2, 6, 7, 8, or 9 loaves of bread among 10 men. The n/10 table was used for all 6 problems. Ahmes recorded 10 men as receiving 2/3 + 1/10 + 1/30 of a loaf of 8 loaves of bread were to be shared (8/10 or 4/5 of a loaf of bread). Similarly, Ahmes recorded, with considerable detail that 10 men would receive 2/3 + 1/5 + 1/10 + 1/30 of a loaf, (1, or a whole loaf). Struik (1987, p. 23) notes that these elaborate and cumbersome calculations lasted into the Middle Ages, although experimentation with similar unit fraction methods continued much longer[3].

Ratios and proportional reasoning were evident in a manipulation of numbers, similar to the "Rule of Three" in ratios which is credited to the Greeks (Boyer, 1989, p. 15; Weil, 1984, p. 5) and Chinese (Ronan, 1981, p.13). The "Rule of Three" is a method of finding a fourth number from three given numbers where the ratio between two of them is the same as that between the third and the unknown fourth number. For example, given the relationship of 2 to 4, the relationship of 8 to a fourth number is only defined by the number 16 (2:4 :: 8:16 or 2/4 = 8/16).

Babylonian mathematical history was recorded on clay tablets, with surviving tablets dating from 1900 B. C. to 1600 B. C. (Bunt et al. 1988, pp. 42-43). Numbers recorded on clay tablets (1800 B. C. - 1200 B. C.) show symbols for simple fractions, not all as unit fractions (Struik, 1987, p. 15). Boyer (p. 28) labels the surviving Babylonian cuneiform tablets "table texts", since they comprise tables for multiplication, reciprocals, squares, cubes, square and cube roots – all written in cuneiform sexagesimals[4].

Boyer (1989, pp. 27-28) regards the Babylonian system for fractions as much superior to that of the Egyptians. The mathematicians of Mesopotamia took the place-value or positioning system of their numerals and extended it to their ordering of fractions. Unlike the Egyptian pre-

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[3] J.J. Sylvester, 1814-1897, recorded his own method of compiling unlike unit fractions. His method extracted the largest unit fraction, and calculated the remainder. Repetition of this process continued until a unique unit fraction remained. 2/35 can be represented as 1/21 + 1/105 <u>or</u> 1/30 + 1/42 and 1/20 + 1/140 (Bunt, et al.,t,1976, p. 17-18).

[4] In modern notation, 2 and 1/2 would equal 2 and 30/60. In a sexagesimal system, this would be written 2;30. The semicolon here is used to separate the integer value from the fractional value; a comma would follow if there were more sexagesimal fractional parts involved. For example, 2/27 would read as 0;4,26,40 (Bunt et al., p. 45-46). This would be seen as $4/60^1+26/60^2 + 40/60^3$.

occupation with unit fractions, Babylonians employed an incomplete sexagesimal system; they had no symbol for 0 and repeated two symbols for units of ten and units of one (Bunt et al p. 44).

Menninger (1992, p. 164) gives a reasonable explanation for the adoption of a sexagesimal system: Babylonians used common fractions of measure, such as 1/2, 1/3, and 2/3. As their culture evolved, they found it necessary to express fractional parts of the larger measure as whole numbers of the smaller. For example, in describing 1/3 mina in shekels, two separate groups of measure – halves and thirds were employed. The least common multiple is 6; yet this would have made the measures too close to each other; the distinction between the large and the small measures would be minimized. Thus the number base of 60 was chosen as a multiple of ten: 1 mina = 60 shekels and 1/3 mina = 20 shekels. The sexagesimal system continued in existence long after the Babylonian civilization ended (Bunt et al., 1988, p. 45, p. 237-238).

Multiple systems were sometimes in place within a single civilization. The tradition of using successively small units rather than fractional parts of a larger unit prevailed in the Ancient Orient, although the *Chou Pei Suan Ching* (the Arithmetical Classic of the Gnomon and the Circular Paths of Heaven) cites problems involving numbers such as 247 and 933/1460; these numbers were written without any symbolic notation. Ronan (1981, p. 9) notes that the *Chui Chang Suan Shu* of the Later Han period represented more advanced mathematical knowledge, although still of a practical nature. The text includes 246 problems concerning "land surveying, engineering, the fair distribution of taxation, and other subjects, all of which bring various mathematical operations into play". In addition to a facile use of decimal fractions, this text introduced the "Rule of Three". Emperor Chhin Shih Huang Ti (221 B. C.) unified measurement for a formerly ambiguous decimal system for fractional measurements focusing on parts of the human body - for example, finger, hands, forearms.[5]

The *Nine Chapters on the Mathematical Art* (206 – 220 A. D.) and the *Zhou bei* form some of the basis of what is known about Chinese mathematics. Struik (1987, p. 32) notes that the material contained in the *Nine Chapters* may be much older. As far back as the second millenium B. C., Chinese scribes wrote numbers using a decimal system with place value for both whole numbers and fractions. Mathematics in China remained traditionally one of technical use; practical examinations required candidates to correctly cite memorized texts. Indian mathematical texts were written in metric stanzas to help in memorization. Struik (1987, p. 33) regards the mathematics of Chinese, Indian, Egyptian, Babylonian and other cultures as basically unchanged until the Greek development of mathematics as a science.

In ancient Greece, the word "number" was used to signify only whole numbers. Boyer (1989, p. 53) reports fractions were thought of only in terms of a ratio between whole numbers. Bunt, et al. (1988, p.47) believe that the sexagesimal system used by Babylonians was acquired by the Greeks for use in astronomy. Bunt et al. (1988, p. 68) and Boyer (1989, p. 60) agree that the Greeks also used the Egyptian system of unit fractions. The Greeks developed a system in which the use of the 'fraction bar' was introduced, although the denominator was written <u>above</u> and the numerator was written <u>below</u> the fraction bar.

Klein (1992, p. 37-60) discusses the Greek view of the unit in the 4th century B. C., a time when mathematics was the subject of philosophers. Klein quotes Plato (Republic 525 E) as saying that expert mathematicians would laugh at the idea of subdividing the unit. Plato's view is that by

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[5] This ruler assigned 6 *chhih* as the length of a double pace. His advisors used the following to note smaller measures, in decimal notation: 1 *cchih* = 10 *tshun*, 1 *tshun* = 10 fen, 1 *fen* = 10 *li*, 1 *li* = 10 *fa*, 1 *fa* = 10 *hao*. Thus, Lui Hui in the third century A. D., in his commentary on the *Chui Chang*, expresses a diameter of 1.355 feet as 1 *chhih, 3 tshun, 5 fen, 5 li* (Ronan, 1981, p. 37)

taking increasingly small units, one would multiply them, and not divide any unit into parts. These units would be referred to as Plato's "counting units". Klein continues that the Greeks viewed the unit as "permanently the same and irreducible basic element which is met in all counting – and thus in every number" (p. 50).

Weil (1984, p. 5) reports that mathematical ideas often present themselves simultaneously through perhaps different settings, under varying conditions and often long before a realization of the ideas of others are presented. Through the survival of ancient documents, notational systems and symbols are noticed "traveling" across the bounds of civilizations and cultures as mercantile communities moved between countries and civilizations. For example, Struik (26-27) notes the evident that trade and conquest saw the flow of Babylonian mathematics into both Hindu and Greek traditions. He continues (p. 73-74) by observing that Chinese mathematics was not an isolated activity, there existed considerable trade and cultural relations with other Asian countries as well as Europe.

Novel systems and symbols for fractions confronted people, as trade, commerce and travel increased; the systems were examined, modified, accepted or rejected. Boyer (1989, p. 244) cites the adoption of the Chinese decimal system by Arabic countries. Barnett (1998) attributes three properties of our current-day numeric system to the Hindu-Arabic tradition: (1) place value; (2) the use of zero as a placeholder; and (3) the use of base ten. Barnett comments that Hindu-Arabic numbers appeared in Western Europe and were recorded in the writings of Gebert d'Aurillac (Pope Sylvester II) in 999. Prior to this time, the Roman Empire moved from sexagesimal fraction notations to duodecimal, and then to a decimal system (Ronan, 1981, p. 32). The Romans introduced the symbolic notation of the fraction bar with the numerator written above the denominator, although using Roman numerals[6] until the time of Sylvester II. Boyer (1989, p. 254-255) notices that Fibonacci (ca. 1180-1250) in his book, *Liber abaci*[7], used three different notational systems for recording fractions: common, sexagesimal and unit. According to Barnett (1998), Fibonacci's book made use of algorithms refined by Hindu-Arabic techniques. Italian mathematicians in this period were called abacists, or *maestri d'abbaco*, and supported themselves by teaching arithmetic computation (p. 70). By 1579, the French mathematician, Viete, urged the use of decimal fractions instead of the still accredited sexagesimal system. In 1585, Simon Stevin of Bruges impatiently pleaded for adoption of the decimal system for fractions. Boyer (1989, p. 316-317) cites Stevin's full and detailed explanations for winning over both common people and mathematicians.

Historic documentation provides evidence that different cultures evolved with distinct fraction systems and the notations that accompanied their recordings. The systems continued in use, sometimes long after the civilization credited with their development had ceased to exist. As countries expanded trade and commercial venues and explored each other's cultures, codification of a fraction system evolved. As a helpful tool to inform the citizenry, textbooks were written and students began to practice the rules.

Barnett (1998) regards the quest for algorithms as "the driving force of mathematical development" (p. 76). Kamii and Dominck (1998) argue that although these rules were convention in times long ago, today they are harmful for two reasons: (1) children relinquish their own ideas, and (2) they disconnect content from place value concepts (p.135). In reporting on a classroom observation project on fifth graders' fractions ideas, Huinker (1998), cautions that a premature introduction of algorithms is damaging to students, because the nature of mathematics is distorted

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[6] For Example, 1/3 would be written I/III in this system. (Boyer)
[7] Title translated: "The book of Calculations" (Barnett, 1998, p. 70)

by perplexing rules (p. 160). The nature of the mathematics focusing on rational numbers presents a view of many complicated elements amassed under one appellation, “fraction”. Behr, Harel, Post and Lesh (1993) cited the difficulty that children have in attaining clear meaning and understanding of fraction ideas.

Complex Issues

According to Carraher (1996), a fraction as simply a number of the form *a/b* (where *a, b* are integers and *b* is not zero), is incomplete. He argues for a broader meaning; that is, fractions are also a meaningful representation of relationships, relationships that are not always physical, obviously countable objects.

In 1983, Freudenthal described fractions as “the phenomological source of the rational number – a source that never dries up” (p. 134). He noted that the pedagogy of fractions takes a unified course, in which students are “so advanced as to be satisfied with one approach from reality” (p. 134). This notion is in contrast to the multiple views students acquire of natural numbers and in total disregard for the multiple uses of fractions. Freudenthal maintained that the singular approach in the teaching of fractions is incorrect, and this failure in didactical reasoning causes many people to never reach a complete awareness of fraction. He offered a list of the uses of fractions while cautioning the reader against accepting his catalogue as complete; Freudenthal wrote that he did not wish his list to be interpreted as an oversimplification of a complex organization. In his collection, Freudenthal used the following classifications for fractions: (1) as a “fracturer” (folding in two, whole and part relationships - with definite or indefinite wholes); (2) as comparers (copper is half as heavy as gold), (3) as ratio (5 out of 6 people); (4) as transformer (stretching a rubber band 2 1/2 times); and (5) as measurer (a segment on a number line). He maintained that the operator sense is visible in all aspects.

Kieren (1994) attributes four subconstructs to the meaning of fractions: (1) as quotients – sometimes referred to as partitioning (10 pies shared by 7 people); (2) as measures (1/2 yard); (3) as operators (2/3 of the population); and (4) as ratios (3 out of 4 dogs). The 1994 work of Behr, Wachsmuth, Post and Lesh extended Kieren's model to include a fifth subconstruct – the part-whole relationship. Pitkethley & Hunting (1994) cite the work of Behr et al as confirming that these five subconstructs have "stood the test of time" (p. 6). Kieren (1994) feels this fifth category is really a notion related to his operator subconstruct. Within Kieren's four subconstructs, a learner's everyday language employs many meanings for fraction.

Witherspoon (1993) lists a learner's uses of a common unit fraction. For example, one half might represent: (1) a subdivided unit of a continuous quantity (a pizza); or (2) a discrete set of objects (a subset of team members); (3) a ratio, such as 1 out of 2 patients in a study; (4) a division, such as the amount of food shared in a group; and (5) a member of a set of rational numbers, such as one half existing on a number line between 0 and 1. Freudenthal (1986) cautioned that learning a new idea with so many different associated meanings presses the student to sort and appropriately attach a proper interpretation in each instance before involving any arithmetic approach to the situation.

Compounding the complex ideas associated with fraction, the traditional way in which children learn about fractions detaches their learning from their real-world experiences and settles them into a memorized and often vague, rule-driven school experience. Freudenthal (1983) offered, as an example, a report on an interview he conducted with a ten-year-old girl who viewed the following problem, "Eight bottles of beer, three persons and each of them gets his fair share". He commented that the girl immediately used a long division algorithm to solve the problem and complained that "it" (the quotient) did not terminate. Freudenthal intervened by reminding her that the people did share the beer. He reported she reacted as though she had "awakened from a dream – suddenly she noticed more things between heaven and earth than are dreamt of in the arithmetic lessons she had had so far" (p. 161). Freudenthal's story indicates the thoughtfulness of a young student's correct thinking about the concept of a repeating non-terminating decimal while trying to solve a real-world problem in which such decimals seem non-existent.

Maher, Davis, and Alston (1991) questioned the school approach of moving too quickly from concrete modeling to abstract symbolization. They presented the four-year case study of one student, Brian, whose mathematical ideas concerning fractions were traced from 4th through 8th grade. Observations of Brian were recorded on videotape as he explored tasks in classroom sessions and interviews. In grade 5, Brian successfully drew rectangular and hexagonal models and built concrete models (using pattern blocks) to represent his mathematical thinking. In a grade 6 interview, prior to introduction of a classroom unit on fractions, Brian successfully used drawings and concrete models (again, using pattern blocks[8]) for new tasks. He told the interviewer that he was "not that sure about numbers, but if I just did it with stuff like these (manipulatives) I could figure it out" (p. 179). After the unit on fractions was completed, another interview was conducted with Brian. He used Cuisenaire rods[9] to complete a model of his solution to the task: "Which is larger 2/3 or 3/4?" Brian then displayed his answer numerically. During an interview in May of 7th grade, Brian was presented with a task using a recipe for chili for 24 servings and reducing the amounts of the ingredients to make chili for 8 servings. He drew a model of four rectangles to represent 4 lbs. of beef, the meat required for 24 servings of chili, and cleverly subdivided the units and responded the solution was to use 1 and 1/3 lb. of beef. During an interview in September of 8th grade, Brian chose to solve the same recipe task numerically. He used subtraction and division to find the amount of beef and said, "Cause 24 [his gestures and intonation make clear his

[8] Brian attempted to build a solution using Cuisenaire rods and then switched to pattern blocks

meaning…and 8 people are going. I divided by 3. I take away 3 from the 4 and get 1 (one pound)". In analyzing his problem solving techniques, Maher (et al) observed, among other remarks, that Brian's understanding of powerful relationships grew[10], although as he sought solutions using only numeric approaches in 7th and 8th grade, some nonsensical statements resulted[11].

The inconsistencies associated with an incomplete understanding of symbolic notation use in algorithms and real-world relationships can follow students into adult life, causing discomfort and frustration. Lave's (1995) writing supports this view and she offers the following example:

A group of dieters, at their regular meeting, was asked to fix a serving of cottage cheese, supposing that the amount allotted for the meal was three-quarters of the two-thirds cup the program allowed. Lave writes (p.165):

> The problem solver in this example began the task muttering that he had taken a calculus course in college (an acknowledgment of the discrepancy between school math prescriptions for practice and his present circumstance). Then after a pause he suddenly announced that he had "got it!" From then on he appeared certain he was correct, even before carrying out the procedure.

Lave reports further regarding the operation with fraction the dieter used in order to complete his task

> He filled a measuring cup two-thirds full of cottage cheese, dumped it out onto a cutting board, patted it into a circle, marked a cross on it, scooped away one quadrant, and served the rest... At no time did the Weight Watcher check his procedure against a paper and pencil algorithm, which would have produced 3/4 cup (sic) x 2/3 cup = 1/2 cup.

The subject returned to his understanding of fraction as operator, physically maneuvering two thirds of the cup of cottage cheese. One might ask why he could not recall the appropriate algorithm. Davis and Maher (1993, p. 13) suggest that for most Americans, mathematics is a

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[9] Brian had worked on this problem during a classroom session. Using Cuisenaire rods, he built a model to represent his solution and showed that 3/4 was bigger than 2/3 by 1/12.

[10] For example, he noticed that 1/n is larger than 1/(n+1) because you would be sharing with fewer other people. (p. 209)

[11] For example, he stated that " twice 1/3 must be 1/6"

memorization of facts and rules which are practiced without meaning. When placed in problem-solving situations, many people struggle to recall which rule to use and how the rule might be applied, and often retrieve an incorrect one.

Traditional textbooks involving fractions are often 'solved' by applying certain rules. These rules, or algorithms, often model solutions that hold little if any meaning for the students using them. Freudenthal (1983) offers an explanation for why some students (like Lave's dieter who 'mastered' calculus) 'succeed' in spite of this traditional approach:

> Pupils with a knack for digesting algorithms learn to operate on fractions anyhow, pupils who are less or not at all gifted in this specific way learn it by trail and error or not at all. After one or two years of fractions, some pupils master the algorithms though they have no idea what fractions mean and what you can do with them; others do not even know the names of the particular fractions. The phenomenological poverty of this approach seems to me largely responsible for this didactic failure. (p. 144-145)

According to Streefland (1991), the traditional school approach to the study of arithmetic ideas concerning fractions has become one of perfecting obscure techniques. Techniques which are often irretrievable for the learner. Procedural steps learned without understanding, become no more than the singsong recitation of counting one observes in young children - the response is there, but without an underlying understanding.

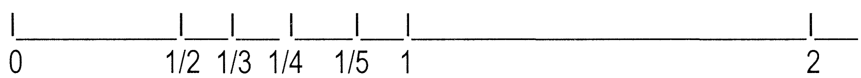
A disconnect between real-world understanding of fraction as operator and sense-making of fraction as number has occurred during the imposition of meaningless rules. The work of many researchers supports the view that the operator sense dominates discussion of the meaning learner's attribute to fraction[12], while the algorithms concerning fractions are derived from the concept of fraction as number. For example, a student might be asked to find the appropriate answer to a problem such as "½ + ¼ = ?". Many children, trying to find solutions to these problems, resort to mindless symbolic manipulation of school taught algorithms. A large segment of school

[12] See for example, Dienes; Kieren; Behr, et al; Freudenthal; R.B. Davis; G. Davis; G. Davis & R. Hunting.

curriculum in grades five and six is normally devoted to the study of rules used in computation involving fractions. Algorithms concerning the operations involved with fractions are revisited, reviewed and re-taught throughout the ensuing elementary grades.

Additionally, the rules governing fractions are derived from whole numbers (positive integers). Carraher (1996) cites an inherent danger and ensuing difficulty if one assumes the results involving operations with fractions will always follow the patterns of those operations with whole numbers.

Russell (1920/1993) highlighted the importance of the idea of ordering in the development of mathematical thought. A common assessment task involving the ordering of fractions is to request the placement of simple unit fractions [1/2, 1/3, 1/4, 1/5...] between the interval 0 and 1 on a number line. When taught meaningless rules, it is not extraordinary for someone to offer the following as his determination of their positions; this is almost expected.



This arrangement might suggest the placement of these unit fractions as determined by a focus on the denominator and a treatment of the fraction as a whole number [placed consecutively higher from left to right, although this may be only one explanation. The following research suggests another.

Reporting on an interview with Brian (a sixth grader), Davis, Maher and Alston (1991) noted Brian had correctly placed 1/2 between 0 and 1 on the number line. When asked about the placement of 1/4, Brian hesitated, pointed to the whole number 4 on the number line, and then settled on correctly placing it between 0 and 1/2. Davis et al commented his actions suggested that Brian was focusing on the *size* of the rational numbers he was placing (p. 250). However, when

Brian was asked to place 2/3 on the number line, he placed 2/3 approximately half way between 1 and 2 on the line. With a question about the placement of 2/4, Davis et al based Brian's reasoning not on the *size* of the fractions, but on their *classification.* (classifying 1/4 first as near to 4, but then positioning 1/4 between 0 and 1; similarly placing 2/3 as between 1 and 2; and 2/4 as between 1 and 2/3). When asked, Brian explained his method: first he concentrated on the <u>numerator</u> and then placed the fraction somewhere to the left of the number in the numerator. He has built an inappropriate, although consistent, rule for solving number line tasks. During subsequent questioning in the interview, Brian's reliance on his numeric method was deemed to be inconsistent with his understanding of fractions as physical partitions of a unit.

Maher and Alston (1989) offered a contrast to Brian's dependence on his numeric method in their earlier reporting of a case study of one girl's understanding of fraction ideas. Ling Chen, a student in a "talented and gifted" program had just completed the fifth grade when the interview occurred. She professed an easy use of school-taught algorithms or concrete model building to represent her solution to a given task. During the interview, she was asked the following: "Jane has 1/3 of a candy bar. She gave half of what she has to Mike. How much of the candy bar does she give to Mike?" Using pattern blocks, Ling Chen came to the solution that Mike was given 1/6 of the candy bar, because as she said, "1/2 of 1/3 is 1/6" (p. 245-6). She was asked to solve the problem with numbers. She wrote the following:

$$1/3 \div 1/2 = 1/3 \text{ X } 2/1 = 2/3$$

$$1/2 \div 1/3 = 1/2 \times 3/1 = 3/2$$

$$1/3 \div 1/2 = 1/3 \times 1/2 = 1/6$$

Ling Chen was asked which of her three answers she believed; she responded, "still 1/6". Maher and Alton commented, "She cannot close the gap between meaningless ritual and tangible reality."

(p. 248) Her reliance on concrete images of fractions enabled her to uphold her correct response when faced with numeric manipulations that offer conflicting solutions.

The complex issues of reality and numeric understandings of fraction are often further confounded when students try to transfer their knowledge of operations with whole numbers to those of fractions. It is natural for students to conjecture, based on their knowledge of multiplication of the natural numbers, that multiplication 'makes bigger', or that division 'makes smaller. For example, placing 6 apples in each of 4 compartments (4 x 6) will yield 24 apples, 6 x 4 = 24; and 36 yards of ribbon partitioned into 3 pieces results in each piece being 12 yards in length, 36 ÷ 3 = 12. So it is not surprising that this reasoning carries over into an extended domain, where it is no longer valid. For example, 6 x 1/2= 3, and 36 ÷ 1/2 = 72.

Freudenthal (1983) reminded us that there are increasingly more complex issues of fractions; he likened his presentation of fractions in their "full phenomenological wealth" as an ocean in which he could possibly drown (p. 134). The immensity of fraction concepts are then often presented to students in a unified, but rote and meaningless way, as a set of definitions and rules with a difficult time to be had in making the connections between reality and ritual.

Attention has been given to alternative approaches to the development of meaningful understanding of fraction ideas. A few decades ago, Dienes (1967) recommended an operational approach to the learning of fraction ideas in which a fraction would be represented in two ways: as a “state of affairs”, or as a “command”. His “state of affairs” mode patterns the sense of fraction as operator; that is, a fraction as part of some unit or set. Dienes' “command” mode begins with the issue of an order of operation. For example, Dienes gave as an example, two thirds of a set as a “state of affairs’. The instruction to, “take two thirds of a set” suggests a multi-step “command”. This involves (1) splitting the set into three subsets (a division) and (2) taking two of the subsets (a multiplication). The multi-step “command” for 3/5 + 3/7, although completely valid, becomes quite cumbersome.

Dienes (1967) suggested that children begin their introduction to fractions by working with a set of everyday objects, such as stones, leaves, etc. He noted that the management of everyday objects could become possibly overwhelming and suggested the use of Cuisenaire rods, among other tools, for children's exploration of fraction ideas (p. 1). Freudenthal (1983) suggested the use of objects which concentrate on the <u>length</u> and <u>area</u> as the most natural means for visualizing magnitudes in exploring ideas concerning fractions (p. 152). R. B. Davis (1980) expressed his preference for the use of Cuisenaire rods as a tool in working with fractions (p. xiv). Davis' view echoed the opinion of Piaget (1935/1965), who viewed the rods as an improvement over the colorless design tools used by Miles Audemars and Lafendel at the Maison de Petits in Geneva (p. 704).

R. B. Davis used the rods in two ways: (1) To foster an understanding of the operator sense of fractions (one rod as half as long as another rod) while asking for a justification (Can you convince me?) and (2) To allow children to experience fraction as number (If I call the green rod one, what number name shall I give to the red rod?) (pp. xiv-xvii).

Dienes (1967) viewed Cuisenaire rods as useful abstractions of reality, while representative of mathematical objects. In 1963, he highlighted three types of “mathematical play” as essential to a learner's growth of understanding: (1) “exploratory-manipulative” play, such as when a child becomes aware of the properties of some objects; (2) “representational” play, when the objects begin to stand for something they are not and imagination is introduced by the child; and (3) a “rule-bound” play, when rules are developed – or imposed- and then used. (p. 20-32.)

Preferring what Piaget (1972/1995, p. 726) called the “Platonic model” of learning, traditional educators sometimes avoid the use of mathematical play, in which perhaps teachers feel the subject matter exists as independent of the representation of mathematical objects. Piaget (1972/1995,p. 727) asserts that an aversion to play may be understandable, given a view that "empirical experiences will harm the deductive and rational mind, which characterizes their discipline". He denounces this concern by citing psychological studies to the contrary and offers assurances to educators and mathematicians alike.

Mathematical play occurs when a learner works alone or with others. Piaget (1924/1995, p. 92-93) remarked that the social discourse in which students involve themselves is both meaningful and essential (p. 93). In 1964, he clarified his position on this issue while discussing a fundamental need for reflection by the learner:

> ...A reflection is more than an internal deliberation, in other words, a discussion with him/herself or with another child or adult or with real contradictions or external contradictions. We can say also that reflection is the social behavior of discussion, but made more internal. They are the general rules that come before, like if you reflect internally on yourself the behaviors that you achieve because you are with others. Or the social discussion is an external reflection. Really this problem, as well as all analogous questions, is the same as what comes first, the egg or the hen. Among human behaviors, we have social and individual perspectives. (1964, p. 61, unofficially translated by R. Mazoreis)

According to Piaget (1972/1995, p. 727), a child's mathematical actions should include two types of experiences [13]: Physical experiences working on objects and logico-mathematical experiences gathering in information from the actions carried out by the child. Each kind of experience might be, as Piaget suggested, as personal as it is social. Piaget assured, however that these are not equivalent experiences, but essential and qualitatively different experiences in the child's development, 'Traditional' school mathematics tends to preclude both at the personal level and at the social level.

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[13] These experiences are viewed as the antithesis of the Platonic model and more closely related to the format of Dienes' stages of play.

Streefland's work in the Netherlands supported this view of necessary experiences by children. In 1991, he reported on a study using children in two groups; one as a rule-imposed 'traditional' control group, a second as a 'realistic' experimental group (p.117). Streefland noted the children in the 'realistic' group invented their own 'clever calculations'. The children who were allowed to develop their own rules demonstrated a greater understanding of mathematical concepts than their peers who were taught traditional algorithms did. In developing a prototype for a course on fractions, Streefland's control group, which was taught using a "mechanistic" approach, lagged dramatically behind the "realistic" group who worked on their own clever calculations. In fact, the algorithmic orientation of the control group with the tendency to apply rules was not advantaged with respect to the experimental group.

Other international studies have observed the work of fifth graders such as the study by Koyama (1997) in Japan. The focus of this project was to explore students' ideas concerning fractions. The fifth-graders were asked, for example, to compare 4/5, 3/5, and 3/4. Koyama reported that the children used previous knowledge, such as converting to decimal notation, subtracting each fraction from the number one and ordering the unit fractions remaining, drawing line-segment or thin rectangular pictures to examine the measurements in terms of length, and using known rules to establish equivalent relationships. This was research conducted to establish understanding of children's procedural and non-procedural knowledge, prior to the introduction of formal procedures for the reduction of fractions. An interesting variety of strategies were evidenced in the children.

In 1998, Huinker (p. 170-181) conducted an observational study of a four-week instruction unit based on a problem-solving approach to fifth graders' building of algorithms involving the addition and subtraction of fractions. The project, in a large urban school district, was a first attempt by a classroom teacher as part of an "instructional team" to put the textbook aside and try a different approach. Children were given word problems and encouraged to find solutions in ways that were sense-making to them. Huinker records that students employed different strategies in finding solutions to their tasks. Two algorithms for addition of fractions emerged: "making a whole" (seeing if two fractions combine to more than one 'whole'); and "renaming" or "trading" (using equivalent names for pieces of the same size). Two algorithms emerged for the subtraction of simple fractions, two more for the subtraction of mixed numbers. During the last two weeks of the unit, fraction tasks involving multiplication and division were introduced. Students received a pre-test and a post-test in which significant gains in scores were recorded, although the team of two teachers had hoped for higher scores.

In 1995, Watanabe presented the ideas of unit fractions by American fifth graders through his individual interviews of the children. He stated in his findings that the students in his study, with multiple years of fraction instruction, had "little quantitative sense" of fraction (p. 394). Watanabe proposes problem-solving experiences involving proportional relationships as a basis for building an understanding of fractions.

Several studies involving computer microworlds have been cited in exploring children's understanding of fraction. Hunting, Davis and Bigelow designed and implemented a software program in 1991 called *Copycat* that is used with a graphing device called Super-Paint. Their studies focus on an iterative model of fraction schemes as well as the building of part-whole relationships. Within children's explorations using *Copycat,* seven and eight year olds were asked to predict the outcomes of using a 1/2 or 1/3 machine, as well as deciding how the 'machine' was working internally and how it might be recording the results (p. 76-88).

Steffe and Wiegel (1994, 117-132) speak of essential cognitive play on the part of the learner through the medium of a microworld called *Toys*. *Toys* focuses on whole number counting strategies involved with five motifs. Following *Toys*, *Sticks* was introduced as a microworld environment for studying iterative patterns of line segments. Steffe and Tzur (1994, p.99-115) report on interviewing two ten-year-old children on partitioning and iterative understanding of fraction ideas through the use of this software.

Within all mediums of exploration, learners need to function and focus on the mathematics under investigation. Watson, Campbell and Collis (1993) propose three modes of functioning intellectually when presented with a new mathematical problem: sensorimotor, ikonic (imagery, reality, "aha" experiences, diagrams), and concrete symbolic. They state their results as 'tentative' but argue that

> Movement along the concrete symbolic route, without complementary ikonic mode development will both limit a deeper understanding of the meaning of concrete symbolic manipulation, and also limit options in complex problem solving. (p. 60)

They suggest more research investigating how children build ideas through concrete experiences.

A review of the literature confirms that research has been done to trace the learning of fractions in young children and to point to differences in student learning. However, what is lacking are studies about the development of fraction ideas in young children prior to the teaching of formal algorithms in the context of a student-centered classroom community in which certain conditions that foster students thinking are put into place. This study addresses that need.

Theoretical Perspective

The research is based on the view that attention to the ways in which children explore, discover / invent, and then discuss, through their language, their writings and their actions, the mathematics that offers insights into their understanding. The retention of procedural information presented as rote learning is, at best, a difficult task. Repetition of an appropriate procedure to a given task may appear to signal understanding, but the action does not determine if understanding has taken place. According to Piaget (1972/1995, p. 731) the learner must reinvent/invent newly experienced concepts for understanding to develop.

This necessity for invention will require a change in educational environments. R. B. Davis (1997) acknowledged the current agreement on the need for change in school mathematics. He viewed the debate as one of alternative learning environments: one in which children are told or shown mathematics, or one in which children "build up mathematical ideas themselves, in their own minds" (p. 87).

The theoretical framework in which this study is set proscribes the later model -- the experiential learning environment in which children are provided with opportunities to develop cognitive skills and build mathematical understanding. Certain classroom conditions must be in place for children to prosper in such an environment.

Maher (1996) includes conditions that develop a culture in which students are expected to support and represent their ideas, and discuss the ideas of others – conditions that nurture the exchange of ideas. In 1998, Maher augments her index of classroom conditions to include the child's need for adequate time both to think deeply about questions and to build explanations and justifications for solutions. This essential time factor should occur both within and outside of the classroom (p. 105).

An essential question in the research is, "How do children build mathematical knowledge?" According to Davis (1984), a learner builds mental representational structures that are framed within his/her prior experiences. Through these structures, the learner builds a collection of assimilation paradigms. Davis and Maher (1993) describe assimilation paradigms, using Piagetian language, as a set of information processing activities in which the learner sees a new experience as 'just like' or 'similar to' some recalled earlier experience. New experiences create data for the learner to process.

This study assumes the proposed steps fundamental to the processes of human thought as defined by Davis and Maher (1990, p. 65): A mental data representation is built by the learner who uses a memory search to retrieve relevant knowledge. A mapping between the data representation and the retrieved knowledge representation is constructed and checked for correctness. When constructions and representational mapping are sufficient, any technical aids or other information associated with the knowledge representation is employed. This does not imply a single linear progression; these steps may be repeated and cycled through many times in this process[14]. In 1998, Davis and Maher, re-emphasized that additional building blocks for constructing knowledge representations come from ideas that one has already built as a result of previous experience.

A question arises concerning how a learner might meet the challenge of building knowledge when previous experiences prove insufficient. One view is that thoughtful problem solving may facilitate the process. Maher, Martino, & Alston, 1993, suggest that the development of a new idea may come about in the process of tackling new problems. They indicate that the learner may be challenged to reorganize and extend existing knowledge when new ideas are presented. Maher et al write: "The process of tackling the problem may trigger the construction of a more adequate representation and provide the incentive to reorganize or extend their available existing knowledge" (p. 13).

Maher (1998a) indicates that data are internalized through a learner's well-coordinated actions on objects that may occur in response to explorations of a problematic situation. The

[14] See also Davis, Maher, Martino (1992).

learner responds by drawing upon existing mental representations that are already built. As the learner enters new data, these representations are either validated, modified or rejected on the basis of the fit with existing knowledge structures.

In 1991, G. Davis offered the view that the disquiet associated with the learner's need for modification, rejection, or re-evaluation of his/her ideas is essential. He defines disequilibrium as part of the process of moving from a prevailing mental construction of things," the internal relational net", to the building of a new relational scheme (p. 229). He views this disequilibrium as essential in the breaching of established realities before the building of newer ones can begin. Davis cautions that within this disequilibrium reflection is fundamental; actual experiences cannot lead to new relational comprehension without reflections by the learner. Both internal and external reflections as described by Piaget (p. 22, this study) are central to studying the development of children's mathematical thinking.

Within the processes of building mathematical knowledge representations, advances and setbacks, conflicts and confirmations are expected in the real sense of 'doing mathematics'.

> Given the view of Lakatos (1976), mathematics does not proceed in a Vauban-like manner, making step-by-step sure advances in a pre-determined direction, but like the daring exploits of the cavalryman of the advance guard, the forays into new territory may be flawed. Mathematical thinking, as opposed to the reflected organization of mathematical thought, is a creative activity that brings with its the possibility of human error. Indeed the very possibility of error is what makes the major advances such moments of human success (Ervynck,1991, p. 52).

Fundamental to the development of mathematical ideas is the learner's ability to function as a community member, to be positive about one's ability to explore and develop ideas, to explain and justify opinions, to challenge and modify personal ideas as well as those of others. According to Tymocko, "Mathematicians, even ideal mathematicians, are able to do mathematics and to know mathematics only by participating in a mathematical community" (quoted by Hanna, 1991).

Hunting, Pitethley & Pepper (1991) caution that the terminology associated with mathematics holds different meaning for adults (the teacher) and children.[15] Within a mathematical community, the representations offered by all members (teacher included) form a framework for negotiation and analysis which can lead to accepted meaning by the community and subsequent understanding by individual members. The representations of language and their precise meanings become invaluable data.

Bakhtin (1934-35) speaks of *heteroglossia* as the condition necessary for meaning to be understood. Holquist (1981, p. 428) calls this condition "the primacy of context over text". Probing into the meaning of the children's mathematics is essential. A. A. Leontev (1981) writes supporting the views of social interactions that are fundamental to meaning. Confirming Voloshinov's work from almost a half a century ago, Leontev quotes, "Meaning is not in a word and not in the speaker's soul and not in the hearer's soul. Meaning is the effect of the interaction of the speaker with the listener on the material of the given sound complex.... Only social interaction involving speech gives the word the color of its meaning." (p. 254)

Pitkethly and Hunting (1994, p. 7), point to the development of meaning for the language and symbols of fractions as notably relevant. Within their study, they argue for student involvement in an action-based context where the student will be enabled in the use of these mechanisms for developing an understanding of iterable fraction units and how to combine and reconfigure the units. They conjecture that as the child grows in his/her use of language and symbols, the growing complexity and sophistication of his/her experience will meet the challenge of new concepts concerning fractions.

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[15] They cite examples of words used such as "group, same, and different" in a study conducted with seven-year-olds (p. 109).

The interpretations of language and the importance of meaning described by these authors is expanded to include the importance of all forms of representations offered by the children in this study: language, both verbal and non-verbal, drawings, writings, and models. According to Maher (1998), children's ideas are not always written or verbal, but often communicated by actions on objects used to build models. With a focus on the representations children might offer, some areas of observation become critical in focusing on the mathematical thinking of learners.

Writing about the role of mathematics curricula, Schoenfeld (1994) cites five such areas requiring great attention (p. 58): (1) Content, classically conceived (as a rich, deeply connected collection of ideas); (2) Problem solving strategies or heuristics; (3) Control, which is concerned with how well or how effectively people use mathematical resources at their disposal; (4) Beliefs; and (5) One's ability to function as a member of a mathematical community. With the role of the mathematical community as a focal point, the portrayal of one member – the teacher, becomes critical.

According to Maher and Martino (1999), children often begin by building personal knowledge representations and then become interested in the ideas of others. The non-traditional role of the teacher[16] is fundamental in the development and execution of the process of orchestrating the exploration and communication of children's ideas. The teacher becomes an active participant who "attends to children's cognitive development and encourages discourse in the classroom community" (Maher 1998). Ball (1993,p.159) notes that this community member must chose and build models, narratives, representations, and activities that promote children's mathematical development. If a classroom is to be truly student-centered, the role of the teacher is one of almost continual evaluation and re-evaluation of plan and direction which encourages

[16] Teacher/researcher in this study.

student responsibility for the acquisition of new ideas. In 1996, Maher presented a list of characterizations of a constructivist teacher as one who: (1) provides experiences from which a learner can build a powerful bank of mental images to draw upon for new constructions; (2) assesses and estimates student learning by observing and listening to them; (3) encourages student justification and explanation of their ideas; (4) makes effort to build a classroom culture conducive to an exchange of ideas; (5) makes students aware of differences and disagreements over mathematical issues; (6) organizes and reorganizes student groups; (7) encourages dialogue between students and between teacher and student; (8) provide opportunities for student to express ideas and representations; (9) sustains discussion and revisiting of ideas; and (10) pursues opportunities for students to build generalizations and extensions.

The accommodation of these theoretical approaches will become the framework of this study.

CHAPTER III

DESIGN OF THE STUDY

The Colts Neck Project was designed as an enrichment strand[1] in mathematics for a fourth grade class. A problem-solving approach was the focus of the activities. The sessions began with investigations involving the idea of fraction as operator and fraction as number. All sessions were led by a team of teacher/researchers from Rutgers University; the team will be referred to in this study as T/R 1 and T/R 2[2]. During all sessions, the classroom teacher, referred to as CT in this study, was present as an observer, interacting occasionally with the children while they worked in their groups. The teacher/researchers worked together with the classroom teacher to support this intervention, to communicate with parents, and to participate in the planning of grouping within the classroom. The tracking of the children's seating assignments was important[3]. The teacher/researchers and the classroom teacher observed the relationships developing between the children and adjustments in the pairings and grouping were discussed and implemented.

The classroom sessions were organized so that the twenty-five children worked together with partners, in small groups, and in whole class discussions[4]. Twenty-five of the fifty sessions focused on explorations with fractions. The self-contained nature of this classroom provided time for extended math lessons (usually 60 - 90 minutes in length). All sessions were videotaped using two or three cameras. At least two of the cameras were continually manned, one facing the

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[1] The issue of a grade in mathematics was the responsibility of the classroom teacher.
[2] T/R 2, a former middle and high school teacher, is currently a fourth grade teacher at the Conover Road School.
[3] Room charts were designed to follow the placement of the children (See Appendix B for sample room charts).

children from the front of the room, the second facing the children from the side of the room. The third camera, usually unmanned, captured the front of the class and recorded the activities of the teacher/researcher and the presentations by the children on the overhead projector.

In addition to the teacher/researchers and classroom teacher, graduate students were sometimes present in the class as field researchers in the collection of data. A few times during the sessions, the school principal visited the class and interacted with the children. Data include the videotape library, children's written work and researchers' notes.

The teacher/researcher began each session by asking students to investigate a new activity or to talk about what they had been doing in the previous session. The teacher/researchers walked around the room, observing, asking questions when appropriate, challenging children with task extensions as necessary, and orchestrating the children's presentations to their classmates. Children would be encouraged to discuss, explain and challenge their classmates. Each child, or pair of children, was invited to build a model(s) of their solution, and then compare their model and ideas about that model with others. Students prepared their solutions for class sharing and discussion as appropriate.

The design was student-centered and teacher-researchers would base decisions about follow-up activities on their best estimates of the children's progress. Classes were conducted without 'closure'; it was the children who offered solutions, outcomes and opinions as issues arose.

Children were offered the use of Cuisenaire rods (see Appendix C) as a tool for building models to represent their ideas. Transparent Cuisenaire rods for the overhead projector were available for the children to use in their presentations to the class. They were also encouraged to write about their findings, sometimes during class, sometimes as an after-session suggested assignment.

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[4] When all children were present, eleven partnered groups and one small group of three was implemented

Methodology

This study will center on the first seven sessions, which are viewed (1) as a catalogue of the activities, focusing on the mathematical implications for the study of fractions (Commonalties and differences in the structure of the sessions will be noted); and (2) as an identification of significant episodes in the progress of the children. These sessions can be viewed as a developing framework in which the children continued their explorations throughout the subsequent sessions in the project.

In studying the development of mathematical ideas by these children, the significant episodes will be viewed as "critical events". These events were originally defined as "conceptual leaps" - moments of mathematical insights (Maher, C. A., Martino, A. M., 1996, Maher, C. A., Pantozzi, R., Martino, A. M., Steencken, E. P., Deming, L. S., 1996). A developing definition of "critical events" expands to include serious mathematical misconceptions.

The mathematical content of each critical event will be identified and described within the following criteria:

1. What is the context in which the event appears?
2. What identifiable strategies and/or heuristics do the children employ?
3. What circumstances appear on prior videotape as evidence for the evolution of each event?
4. What, if any, subsequent mathematical developments follow the emergence of the critical event?

Seeking answers to these questions provides the necessary "trace" – the data tracking the development of children's thinking (Maher et al., 1996). The conduct of these children focuses on their interactions as significant in the development of ideas. As each event unfolds, the reaction of other students is described and analyzed. These events are seen as stories in the children's development of ideas. Coding of all data surrounding each critical event focused on five major strands, or categories[5]. The selection of these categories does not imply a distinct partitioning of the critical events. Rather, these classifications are woven together and help to fashion the fabric unfolding as the stories of these children. The trace of critical events comes together as a collection of pivotal mathematical strands. Kiczek (2000) defines a mathematical strand as a collection of critical events that become key in the building of mathematical understanding. This definition is expanded to include a "collection" of pivotal strands, which is as intertwined in the telling of the children's mathematical stories as the identification of critical events.

The possibility of changes throughout data coding and analysis was expected; the presence of easily accessible data (videotapes, written work, and observer notes) insured the validity of any modifications or changes in the process of coding. Chosen "critical events", prior evidence of their foundations, and subsequent tracing of the reactions to their emergence within the community, were transcribed and analyzed. Independent researchers verified transcription (with videotape included) and analysis of data. Graduate students working as field researchers collected data and analyzed their findings. Their work was used as a measure of reliability in this study.

The operation of three cameras in the classroom produced multiple videotapes of each session. The data from all camera views were correlated and then transcribed before coding was completed.

[5] Additional coding includes a systematic listing of the activities presented by the teacher/researchers as well as those tasks offered by the children. Time working on tasks without obvious interaction is also noted

Data Coding

The following coding scheme is an adaptation of the original plan proposed for this study. Analysis of the transcriptions caused necessary adjustments in the subcategories of the codings. The modifciations were made to present a finer grained picture of the children's thinking and the roles of other community members.

Note: **Shading** any code denotes a student's actions as answering a different question.

OP denotes work done at the overhead projector

1. **[R]** Student representations

- the language of the students

  [lv] verbal

    [t] precise mathematical statements

    [f] first-stage mathematical statements

  [ln] non-verbal

    [g] gestures describing mathematical ideas

    [e] emotional responses to the ideas of others

  - **[m]** the models they build

  - **[w]** their written work

    [t] mathematical statements

    [f] first-stage mathematical statements

    [d] pictures, graphs, drawings of models built in class

2. **[I]** Mathematical ideas developed and/or expressed by the children

   Subcategories include

   - **[f]** first evidence of a new idea – a student's discovery
   - **[r]** restatement of existing ideas
   - **[e]** extensions to existing ideas

3. **[M]** Mathematical reasoning

   - **[r]** acknowledgement of relationships between mathematical objects
   - **[p]** recognition of patterns
   - **[e]** explanations / justifications of ideas
   - **[c]** connecting present knowledge to earlier knowledge/experiences

4. **[C]** Community structure and the actions/reactions

   - **[i]** individual

     [s] self-correction of ideas

     [r] reaffirmation of ideas

     [e] extension of own ideas

     [q] questioning teacher

   - **[p]** partners

     [qi] questioning each other's ideas

     [qc] questioning for clarification

     [b] building a common idea

[c] correcting each other's reasoning

[d] disagreement

[i] working independently

- **[g]** small group

  [q] questioning each other's ideas

  [b] building a common idea

  [c] correcting each other's reasoning

  [d] disagreement

  [i] working independently

  [t] response to T/R's direct questioning

- **[c]** whole class

  [q] questioning each other's ideas

  [b] building a common idea

  [c] correcting each other's reasoning

  [d] disagreement

  [i] working independently

  [t] multiple voices answering teacher

5. **[T]** The role of the teacher/researcher

- **[f]** as facilitator, the teacher might be viewed as

  [i] sharing or giving information

  [p] presenting a task /activity/problem to an individual or small group of students or the whole class

[n] introducing notation

[s] summarizing student ideas

[l] Enculturating students into precise mathematical language

- **[o]** As observer, the teacher may withdraw or never enter into discourse with students(s) as a direct attempt to not interrupt students' activities

- **[q]** As questioner, a teacher may

  [al] lead students to a certain path of reasoning

  [ar] nudge them in a certain direction – the notion of "appropriate rigging"[6]

  [u] check for student understanding: 'convince me', restating of task or questions

  [c] check for class understanding: 'convince me', restating tasks or questions

  [e] emphasize significant statements, to shift attention to/from certain ideas, to give clues to fruitful avenues of pursuit

- **[m]** As mediator, the teacher may

  [ci] praise (or reject) individual's participation

  [cw] praise (or reject) whole-class participation

  [f] keep students focused on the task

  [d] encourage and support debates and discussions

---

[6] 'Appropriate rigging' is a teacher's strategy for stimulating student thinking. Examples include the entire spectrum from 'telling' to 'not telling', specific questions, interventions and decisions.

[e] encourage less vocal students to voice their opinions

[r] encourage student reflection

6. **[A]** The activities/ tasks of each session, tasks will be categorized by their solution(s):

    - **[s]** single solution represented by only one model
    - **[sm]** single solution, represented by multiple models
    - **[m]** multiple solutions
    - **[n]** no solution as task is stated

7. **[TW]** Student time working. The segments of videotape not transcribed are labeled and the duration of time spent is noted. For example, a non-transcribed segment of video showing students working on an activity might be recorded as [TW] 2 min.

The codes were used to identify and trace the development of children's mathematical thinking in this study. Samples of coded transcription are included in Appendices D, F and H. Coding charts were developed from the transcription and examples are given in Appendices E, G and I. The charts illustrate the activities and resulting interactions of all community members and include notation of the critical events as each occurred.

CHAPTER IV

RESULTS

The fourth-grade problem-solving intervention established specific classroom conditions. These conditions afforded opportunities for the children to explore, explain and discuss their ideas. The research questions posed in the study of the first seven sessions include the following: (1) What fraction ideas would the children build? (2) What representations would the children use in expressing these ideas? (3) How do mathematical ideas travel?

The first three sessions were designed to invite the children to make observations about the physical attributes of the rods and to explore relationships between the rods. The first comparison problem was introduced in Session 3. Activities leading to discussions concerning equivalence in fractions began in Session 4.

Results have been organized in the following manner;

1. Each session's narrative begins with a listing of the activities for the day.
2. Each session includes illustrations of the children's models whenever appropriate
3. Appendices are noted and used to include samples of: room charts (Appendix B), samples of coded transcription (Appendices D, F, H), samples of coding charts (Appendices E, G, I), and copies of the children's written work (Appendices J, K, L, M)
4. The teacher/researchers are identified as T/R 1 and T/R 2. The classroom teacher is identified as CT. TP is a Rutgers researcher visiting the classroom.

5. A discussion of the children's written work is included at the end of the seven sessions, in a separate section.

6. A set of Cuisenaire rods contains 10 colored wooden rods in increments of one centimeter (See Appendix C for illustration of Staircase model of Cuisenaire Rods.)

   White – W – 1 cm

   Red – R – 2 cm

   Light Green – LG – 3 cm

   Purple – P – 4 cm

   Yellow – Y – 5 cm

   Dark Green – DG – 6 cm

   Black – BK – 7cm

   Brown – BR – 8 cm

   Blue – B – 9 cm

   Orange – O – 10 cm

7. Children built models to represent their solutions, sometimes by constructing "trains" of Cuisenaire rods. "Trains" can be formed by placing rods together, either by using the same colored rods or of a mix of different colored rods. See Figure 2, page 57, for an example of a set of trains.

Session 1

The Activities

Presented by teacher/researcher (T/R 1):

1. I claim the light green rod is half as long as the dark green rod. What do you think? What would you do to convince me?
2. What number name would we give to the light green rod if I called the dark green rod one?
3. Someone told me that the red rod is half as long as the yellow rod. What do you think?
4. Someone told me that the purple rod is half as long as the black rod. What do you think?
5. Someone told me that the red rod is one third as long as the dark green rod. What do you think?
6. If I call the dark green rod one, what number name would I give to the red rod?
7. Someone told me that the light green rod is one third as long as the blue rod. What do you think?
8. If I call the blue rod one, what number name would I give to light green?
9. What number name would I have to give the dark green rod if I wanted red to be one?
10. If I call the brown rod one, what number name would I give to red?
11. If I call the red rod one, what number name will I give to brown?
12. I want to call the white rod one half. What rod will I call one?

T/R1 encouraged students to pose and solve their own tasks. The following were created by the children and T/R 1 and T/R 2:

1. If the red rod is considered one fifth, what would the orange rod be? [Art]
2. If light green is one whole, what is blue? [Betty]

3. If blue is one, what is light green? [Matt]

4. If white is one, what is orange? [Julie and Kristin]

5. If orange is one, what is white? [T/R 1]

6. If purple is one half, what is one? [Maria]

7. If light green was one half, what would be a whole? [Ed]

8. If white is considered one fifth, what would one be? [Art]

9. If I call purple two, what would one look like? [T/R 2]

10. If white is three, what is six? [Ed]

11. If I call the white rod one, what (rod) would you call seven? [Maria]

12. If red is one third, what (rod) would be one? [Sami]

13. Find a rod whose number name is one sixth. [T/R 1]

14. If I want green to be six, what would white be? [T/R 1]

Session 1

Narration

Children were introduced to Cuisenaire rods through free-play, giving them time to explore differences in size and color within the set of rods. The idea of permanent color names and variable number names for the rods was introduced through explorations such as the following:

1. I claim the light green rod is half as long as the dark green rod. What do you think? What would you do to convince me?
2. Someone told me that the light green rod is one third as long as the blue rod? What do you think?
3. If I call the red rod one, what number name will I give to the brown rod?

Art, Ed, and Dan responded to the challenge of finding a number name for the dark green rod, when the red rod was given the number name one.

T/R 1: Ah, let's see, who wants to give me an answer? ...Okay, Art?

Art: Three wholes.

T/R 1: Do you want to tell me why you think so? Are you all hearing what Art says?

Ed: [Art's partner] I know. I know why.

Art: Okay, if the red is considered one [He points to the red rod in Ed's model] then the green one is a lot bigger. So it would have to be, it would take three whole ones to make another green so it should be considered three wholes.

Ed: [Continuing] Well, I think, well, if you, if you say that this would be one [He holds up towards the teacher one red rod]. This is one, and it takes three of the one, the one wholes to equal up one of these [He points to a dark green rod on his desk]. And it that's one whole, umm, one whole plus one whole plus one whole would equal three wholes. So the green would have to be three wholes.

T/R 1: Does that make any sense? Do you understand what Ed is saying? What do you think, Dan?

Dan: I thought the same thing. If red is one, green would have to be two more wholes.

Later in the same session, the children were asked to pose problems of their own. Ed and Art, who worked as partners, represented one fifth with a red rod, calling the orange rod one. Ed put five red rods next to an orange rod. Ed began to search for the rod that would be one third as long as the orange rod and Art offered a solution.

Ed: Yes, I got it. [He puts two purple rods next to the orange rod.]

Art: No, those won't make it.

Ed: What makes thirds?

Art: Thirds, thirds out of a, thirds out of this? [He points to an orange rod.] Probably the greens.

Ed: Light green,

Art: Light green would make thirds out of the orange. [He places 3 light green rods next to the orange rod.]

Ed: Yeah.

Art realized that the light green rod was not one third as long as the orange rod. The boys continued to search. Art noticed the length of the orange rod was not 12 cm.

Art: There's got to be one.

Ed: No, but what makes it [thirds]?

Art: Nothing can divide twelve into thirds except

Ed: Red.

Art: No. [He counts on the five red rods next to the orange rod] Two, four, six eight, ten. Ten divided into thirds. No, ten can't be divided into thirds.

Ed: But nine can.

Art: Nine can, but there is no nine rod. Oh, yeah there is.

Ed: Eleven, this is twelve though. [Art holds up the orange rod.]

Art: No, it isn't, look [Art counts on the five red rods next to the orange rod] Two, four, six, eight, ten. The orange rod is ten.

Ed offered a 9 cm blue rod for their model of "one", and a 3 cm light green rod as the solution to the task he posed.

Ed: Okay, ten. So that's ten, this must be nine. [He holds up a blue rod.] And this divided into thirds must be

Art: It takes

Ed: Light green

Art: It takes green to divided the nine into thirds.

Ed: Blue

Art: No, we are doing this one. I'm doing this one, the one I made up [the problem of finding a rod that is one fifth the length of the orange rod].

Ed: [Simultaneously] I'm doing this one [the three light green rods with one blue as a model]. Yeah.

Session 2

Activities

1. If I call the blue rod one, what rod will I call one half?
2. If I call the yellow rod one half, what rod will I call one?
3. Can you design a rod that is half as long as the blue rod?
4. If we call the orange rod "two", what can we say about yellow?
5. If we call the orange rod "six", what number name can we give to yellow?
6. If we call an orange and light green train "one", can you find a rod that has the number name "one half"?

Session 2

Narration

T/R 1 began by asking the children to review what they remembered about the previous session. Ed offered:

> Well, let's see [He picks up a 9 cm blue rod]. If we said that the blue rod would be one whole, um, we'd figure out what, we'd take all the blocks and try and figure out what would be one half of it.

The children were invited to find a solution. Five times children offered a simpler problem with a solution: finding a rod that is one third as long as the blue rod. Dan offered his explanation, employing a strategy focusing on upper and lower bounds:

Dan:	I don't think that you can do that because if you put two yellows that'd be too big, but then if you put two purples that's uh, that's uh, that'd be too short and
T/R 1:	What about something between purple and yellow?
Dan:	I don't think there is anything.
T/R 1:	Why not? [Dan pauses] Show us what you have there, Dan. Why do you think there isn't any? Cause I think you built it to show us. Can you show us your yellow and your purple? ...Dan, why don't you come up here and explain your reasoning. What's your reasoning? Let's listen to what Dan has to say
Dan:	[He comes to the overhead and put a blue rod onto it. He places a yellow rod and a purple rod, end to end, adding one white rod – to equal the length of the blue rod.] All right. You see usually, um, they are only one, with the shorter one, only one block apart. Like that and so these, but then if you have for the blues, like if you have two yellows, it would be too tall and if you have two purples. [He puts two yellow rods, end to end, next to the blue rod and then one purple rod next to one of the yellow rods.]
T/R 1:	[T/R 1 hands him a purple rod.]Do you need another purple? Here

Dan used the rod staircase to justify his solution of no rod existing between the yellow and purple rods (see Figure 1).

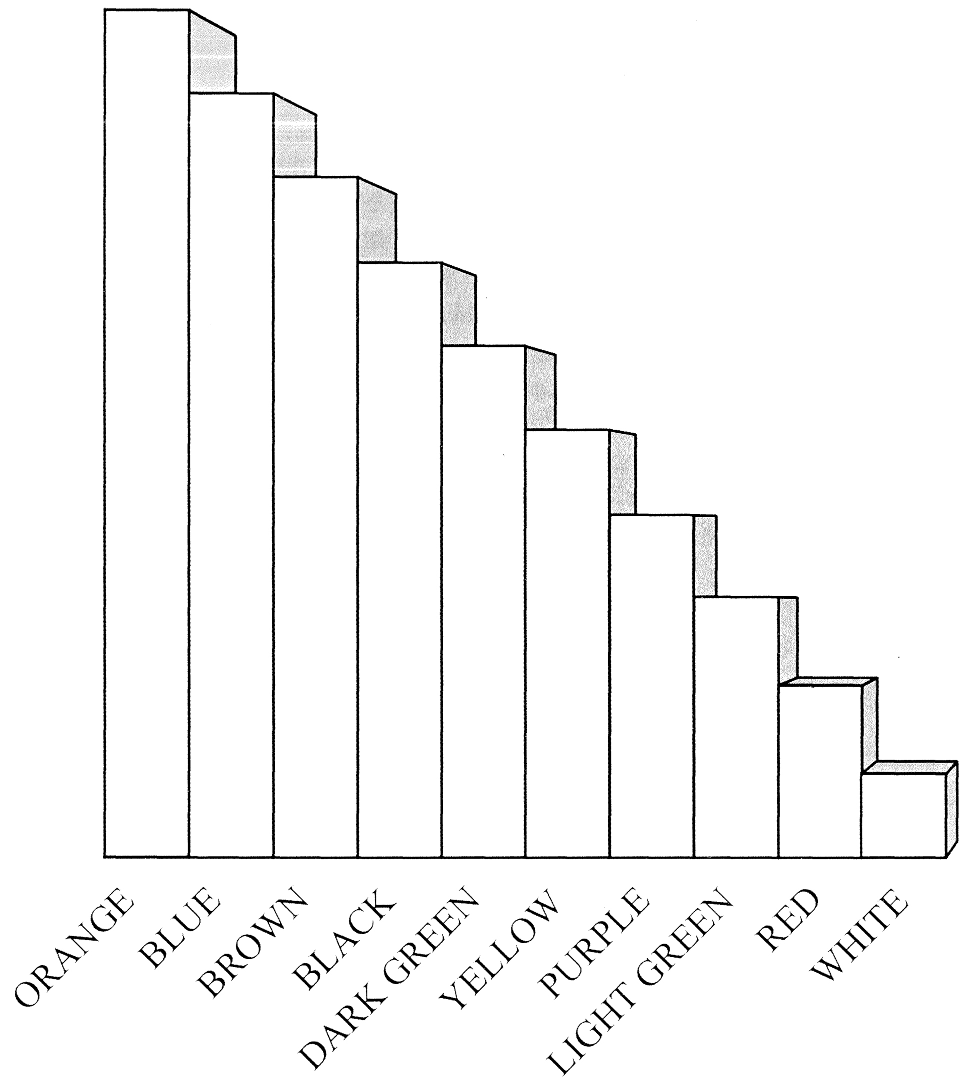


Figure 1. A Staircase Arrangement of Cuisenaire Rods.

Dan: That'd be too short and then there's, there's really nothing in between 'cause if you do [He builds a 'staircase' of rods, beginning with the longest, orange rod, then places blue, etc. until he reaches the shortest rod, the white one.] And then here [between the yellow and the purple rods], there's nothing in between, right here, so there's no way that you can do it.

T/R 1: [to the class] Are you convinced?

Ed satisfied one of the two conditions necessary for the definition of "halfness".

Ed: I think you could do it, but they're… See, I figure if you take a yellow and a purple it's equal [to the length of the blue rod]. They're not exactly the same, but they're both halves. Because the purple would be half of this even though the yellow is bigger because if you put the purple on the bottom and the yellow on top it's equal, so they're both halves, but only one's bigger than the other. So it equals up to the same thing. If this would be one whole [the blue rod], you could take the yellow to be and you could call it one half [holding a yellow rod next to the blue]. But if you took another yellow it would be too big. But if you took a purple with the yellow, and put it on top of yellow, it equaled to the blue. So, the purple would be a half and the yellow would be a half, except that the yellow would just be one bigger than the other would. Or maybe you could call this three quarters [holding the yellow rod] and you could call this one quarter [holding the purple rod]. And, but it would still equal up to the whole.

T/R 1: What do you think, Dan?

Dan satisfied the second condition.

Dan: I didn't think of that. [Ed chuckles. Dan places a yellow and a purple rod end to end, next to a blue rod.] Cause I was thinking that you would need the same.

T/R 1 asked the class if someone could summarize the issue under discussion. Art responded and offered another task using the blue rod as "one".

Art: You can't, if you're div, you can't divide that into halves, because you'd have to use rods that are of different sizes, but you could divide it into thirds using rods that are the same size which, which is the light green rods.

Dan offered an explanation as he sorted the rods in his staircase into "odd" and "even" subsets.

Dan: [at OHP, pointing to the rods on the OHP] I think that some of these that you can't do like this would be odd. [He moves the white rod to one side.] This could be even. [He begins a new group with the red rod.] This would be odd. [He moves the light green rod next to the white rod.] Be even. [He moves the purple rod next to the red rod. Continuing in this manner, he

> moves the yellow, black and blue rods next to the white and light green rods. He moves the dark green, brown and orange rods next to the red and purple rods.] This, be, you see, then when you get up to here, blue would be odd, but like with brown, you could take these two [He places two purple rods next to the brown rod.] and put them together and that would be even. Take the orange, put two yellow, with the orange and that would be even [He does this as he is speaking].

The children concluded that no solution existed for finding a rod that is half as long as the blue rod within the set of existing rods. They were then challenged to design a new set of rods which contained a rod half as long as the blue rod. Ed and Art, working as partners, began an exploration with Ed explaining:

> You can't divide it into halves. "Cause I put this up here and there are nine of these and one, two, three, four, five. One, two, three, four [pause] four, one two, there four five. One, two, three, four. One, two, three, four. One, two, three, four, five [He is counting the two groups of white rods next to the blue rod].

Art suggested a new activity and Ed joined the search.

Art: All right um, what I'm going to do right now is make out of everything, I'm going to halve or third every color, I can third every color. I can halve every color.

Ed: Except blue.

Art: You can third.

Ed: You can third. You can third.

Art: And ninth.

Ed: And ninth.

Art: Now black. [Art chooses to explore his activity focusing on the black rod – 7 cm in length. The class reconvenes before he can continue the exploration.]

A lovely classroom discussion ensued in which the children considered possible designs and shared their ideas with their classmates. Some of the children's comments on the issue of building new sets include:

Mario: If you're going to make a new rod, then you'd have to make a whole new set because there'd have to be a half of that rod, too.

Bob: No matter what, there'll always be something... No matter what there'll always be something that won't be equal to something, like... If you cut these little ones in half, then there wouldn't be something for the little ones to make a half out of them.

Dan: Well, what I told you. I thought that, uh, to cut it in half, too, but then I realized that, uh, that you would have to make a whole set ...and make a half for every one.

Maria: Well, you could just, if you do that then you'd have to cut the ones that are separate, the little blocks into halves, all of them, so then you could make it equal.

Julie: Um, it, I agree with Mario. 'Cause if you do that, um, it changes the whole pattern 'cause this has a set-in pattern to it and the whole thing would change.

The activity led to the generalization that it would be impossible to ever be finished designing new sets of rods, since there would always be a rod for which one half as long would not exist, that is, the smallest rod in any set of rods.

T/R 2 posed a new task: "If we call the orange rod "two", what can we say about yellow?"

Maria: You used all the yellow [She goes to back of room. Her partner Katy raises her hand. She has built a model of two yellow rods under the orange rod. Sami raises her hand. Maria returns to her desk] Oh! She called orange two. One half? Two? This [yellow rod] would be one.

Benny: You put two yellows together and they're the same size as the orange. This [orange rod], is considered two. These two [yellow rods] are considered like an orange, each would be one.

Ed used proportional reasoning in his explanation.

Ed: I have another name. You can call it another name. Do you have to call the orange two? If you could call it one, then two yellows would be a half.

If you would consider the orange two, then call those [yellow rods] one. If you can call it [orange] one, then you call it [yellow] one half.

Benny: [at overhead]. There might be other ways. ...You can do thirds, or like that... [voice trailing off]

T/R 2 posed another task: "I'm going to change the name for orange to six. What number name will I give for yellow?" Katy responded.

Katy: Five. Look here [pointing to Benny's model] before you said that [the orange rod] would equal two, and then Benny said that [yellow rod] would equal one. So now you're saying that [orange rod] equals six, so I figured that [yellow rod] equals five now.

T/R 2: That's interesting. So you're saying when I call the orange two, yellows are each one. So if I call the orange six now, yellow is five. What do you think about that? Did you all here Katy's argument here?

SS: No.

T/R 2: She's saying that when we called this one [the orange rod], that the number name for each yellow is one. If we called the orange six now, we call that [the yellow rod] five. [Katy sits down.]
[Maria and others shake their heads negatively.]

Some people are shaking their heads and I want to know why. Art?

Art offered a different solution using a strategy employing multiplication.

Art: [Goes to the overhead] You said that the orange rod was six. And before you said it was two and this [yellow rod] was one. So now if you're calling this [orange rod] six, half of six is three.

T/R 2: Okay. We have another argument? What do you all think about Art's calling this [yellow rod] three when this [orange rod] is six? Maria?

Maria: Yes.

T/R2: Julie?

Julie: I agree with Art. Half of six is three so

T/R 1: I'm curious. Katy, how did you think of five? Help me to understand. Before when orange was two, yellow was one. So now orange is six and you said yellow is five. That's where I am confused. If this [yellow rod] is five and this [yellow rod] is five, this [orange rod] is six?

Katy explained how she found her solution by using information about the yellow rods in the previous activity.

Katy: I made a mistake from some before. I figured it out now. I forgot that adding one and one is two, five and five isn't six, so

T/R 1: What would the orange rod be called if the yellow rod was called five?

Katy: Ten.

T/R 1: You'd have to call orange ten. Do you agree with that? What a class! It's hard to stump this class.

T/R 1 posed a new problem: "Suppose we made a train, take Erik's idea from earlier, call it orange and [light] green together, call an orange and [light] green train together one. I'm curious; can you find a rod that has the number name one half?" The children offered a variety of models for their solution to finding a number name for one half of the 13 cm train. Benny with his partner Eileen chose to construct a new rod that would be as long as one dark green and half of a white rod. T/R 2 questioned his solution.

T/R 2: There was no rod that was one half [of the train]?

Benny: No, because ten and three is thirteen and thirteen is an odd number.

T/R 2: What does that have to do with it?

Benny: With thirteen you can't split thirteen in half equally. Except take a twelve [2 green rods] and split one rod [white] in half. Like what we did last time.

Benny then offered another model in which one half of the train would be represented by two light green rods and one half of a white rod. He incorporated Jon's earlier model of three light green rods in building his second model.

Benny: You could probably do it another way. That's what Jon did and I saw it probably with these [light green rods]. Maybe it would work, it would probably work. When he was using the blue with the nine, he was using these others [light green rods], so I thought

[He places four light (LG) green rods,12cm, under the train of orange and light green,13 cm.]

No, no. Oh yeah you could do this like we just did. [He places one white (W) rod between the light green rods. His train is LG-LG-W-LG-LG]

Liza and Jen built a model similar to Benny's first model.

Jen: We had to invent a new rod. So first we thought half would be dark green. We had to put that [two green rods]. That didn't work, we need a white.

T/R 2: So what would one half of this orange and light green train be? Can you show me?

Jen: [Stacking one green (G) rod on top of the other: G then G-W on bottom] Well

T/R 2: So what do you think, Liza? Do you know what I'm asking her? I want to be able to see the one half in my head.

Jen: This [holding up a train of green and white rods] would be half.

T/R 2: Okay, that [green rod] and the white?

Jen: Well, it's sort of in thirds, but if you, if you like say if this [orange and light green train] is one, then this [green-green-white train (G-G-W)] would be two. And you have to like pretend that this [G-G-W] was one whole right here.

T/R 2: What do you think, Liza?

Liza: I think that one of these greens and half of this one [white] would be half.

T/R 2: Okay, so

Jen: Yeah, half of the white. So half of the white and this green and half of the white and that green would be the halves.

Maria and Sami, as partners, constructed a model similar to Brian's first model. Maria focused on the 13 cm length of the train. Sami focused on the color of the rods.

Maria: It's thirteen. So we have a seven? What's seven?

Sami: Green doesn't work.

Maria: We need six.

Sami: Blue.

Maria: Not a six, see watch. Ten, nine eight, less than this [holding a brown rod].

Sami: It won't work.

Maria: Oh yeah, it will. I'll prove it, watch. [She puts Y-W-Y under the train of O-LG.] What's highest after seven?

Sami: Dark green doesn't work.

Maria: Who said it doesn't. Yes it does, remember halves. [She changed her model from Y-W-Y to G-W-G.]

Sami: Yes, I do. [Both girls raise their hands.]

Maria: Oh, oh, it works! [T/R 2 joins them.]

Maria: Yeah, I took two greens and a white. Then you have six and a half and six and a half and these. You would have it.

T/R 1 called the class together for a discussion of their ideas and solutions. Alex and Matt came to the overhead and built a model using one dark green rod and half of one white rod as their model for one half the length of the train. T/R 1 asked Maria if she followed the boys' explanation. Maria responded using addition and noticing that the length of the 13 cm train was "odd".

Maria: Well, because you have seven, seven and six. This number block, seven, two of the Well, take two dark greens and a white. And they're no blocks with halves, uneven, odd numbers and you need halves.

Benny: Last time with Mrs. Maher, like the block of gold. One you fit in the middle, split in half like we did last time.

T/R 2: Benny had another model of all light green rods and one white in the middle. He split the white rod in the middle.

Benny built 4 models the same length as the orange and light green train. In each of the first three models, he created a new train one half as long as the orange and green train by splitting the white rod into halves. (For example in third train the train one half as long as the original train would be three red rods and one half of a white rod in length.) In his fourth model, Benny split the yellow rod into halves to make his new train, purple and one half of the yellow rod, one half as long as the orange and light green train. (See Figure 2.)

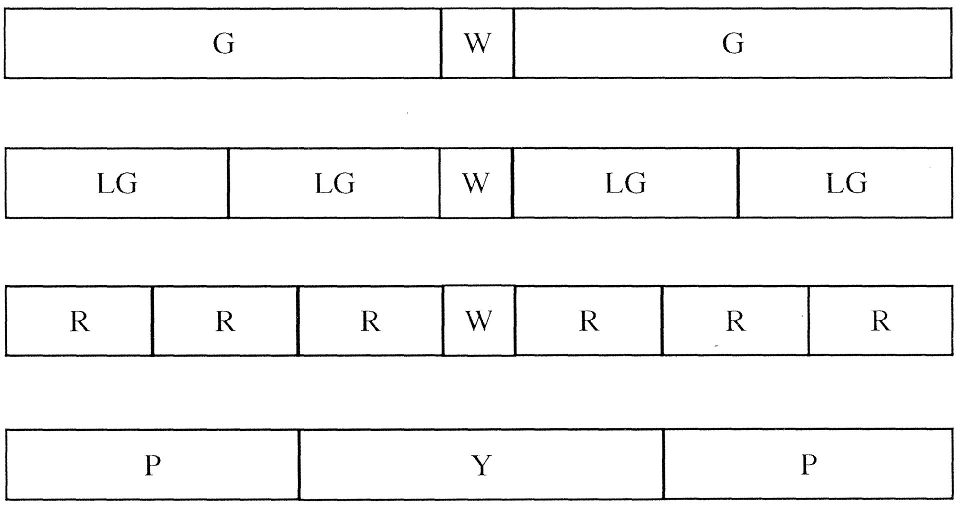


Figure 2. Benny's models of 13 cm trains.

Sami and Maria went to the overhead. Maria offered her view of the orange and light green train, focusing on its 13 cm length. She constructed the following train of the same length. See Figure 3.

Y	LG	Y

Figure 3. Maria's model of a 13 cm train.

Maria: Thirteen. Yellow is I think yellow is about five long, and green in the middle [Counting cm in the train] Ten [two yellow rods], eleven, twelve, thirteen [for the light green rod], thirteen yellows.

T/R 2: You were thinking of the whole length of the train as being thirteen of what?

Maria: Thirteen

T/R 2: Thirteen blues, thirteen oranges, thirteen what?

Maria: Thirteen yellows.

T/R 2: Thirteen yellows?

Maria: Turn light green into yellows.

T/R 2: I don't understand.

Maria seemed to be answering a different question. She explained how the repainting of half the light green rod into a yellow rod (1 and 1/2 cm long) and attaching this new yellow rod to the existing yellow rod (5 cm long) would produce a train one half (6 and 1/2 cm long) as long as the original 13 cm train.

Maria: Well, if you cut that [light green rod] in the middle. Paint light green of each piece yellow and you're making it thirteen and it will be equal to the train.

T/R 2: Do you understand my question? I don't understand when she's saying thirteen for the train of orange and green. I don't understand where she's getting the number thirteen from. Why thirteen?

Ed suggested an explanation for Maria's ideas and offered another solution in which he partitioned the length of the train into two evenly distributed subsets of rods plus a remainder (see Figure 4). T/R 2 asked questions while Ed redistributed the members of his two subsets

Ed:	If you take one of the orange rods and take all these little things [white rods] and put them up to it, it will equal ten. And if you do the same with the light green rod, you have three. And if you have ten and three you have thirteen.
T/R 2:	Oh! So if you line up the white rods along the train of orange and light green and you have thirteen.
Ed:	I have another solution. [He goes to the overhead and puts two light green rods under the orange and light green train. He adds seven white rods to the right of the light green rods.]. I figured you could take two [light green rods] and put them there. After that I took clear ones [white rods]; I put down seven of them. I figured you have this, put a match. Light green, add clear. I took all the little ones and I figured that I have three four [He motions that he is adding one W to the LG, one W to the other LG, etc.], and then four, five, five, six, six, seven.
T/R 2:	Put seven on each of them? So there'd be seven and seven?
Ed:	Yeah, well, not seven and seven, seven and six. It's an odd number, it wouldn't be seven when

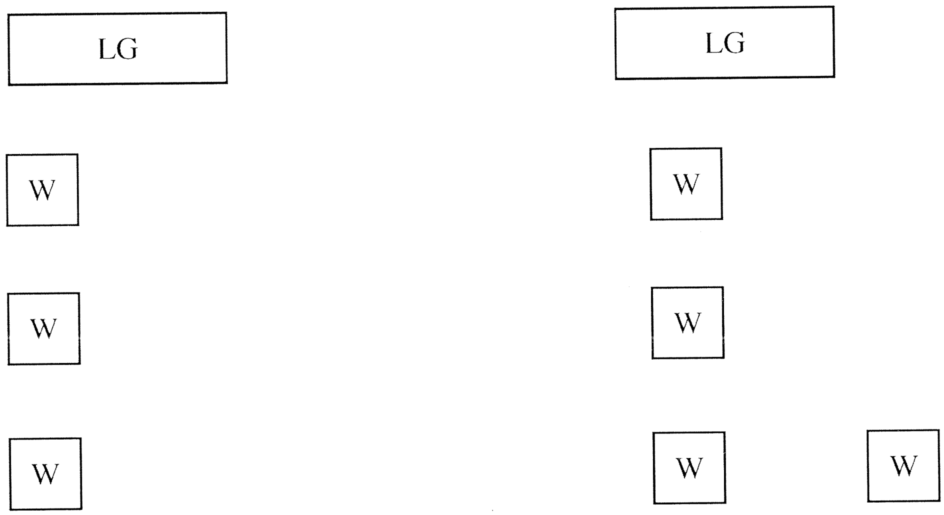


Figure 4. Ed's first arrangement using two equal subsets plus a remainder.

Ed paused and T/R 2 questioned him further.

T/R 2: What happens to this guy? [She points to the white rod to the far right] How can I be fair in making my two halves the same size? What could I do?

Ed: What you could do is, you could take this [white]. You could take those three whites and replace them with a light green. [He removes three white rods and places a light green in his model.]

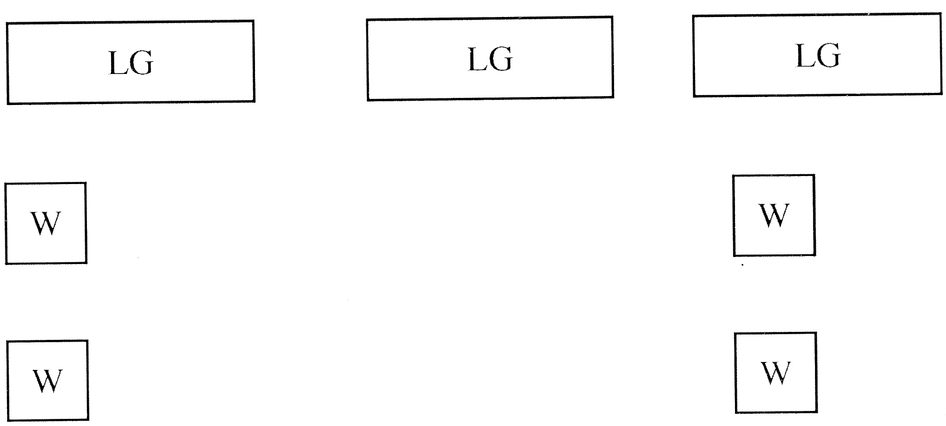


Figure 5. Ed's second arrangement using two equal subsets plus a remainder.

Ed: And then it goes

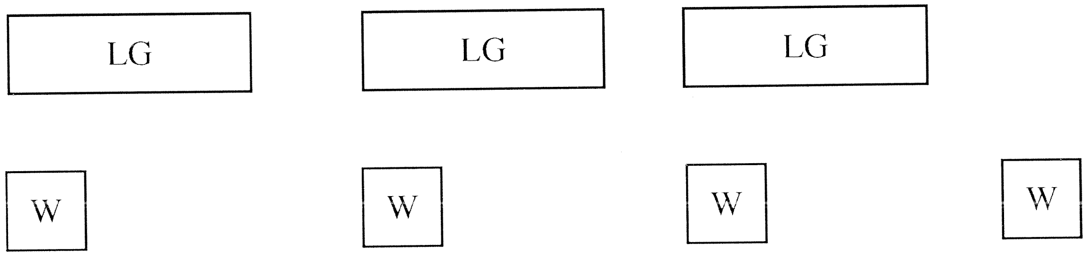


Figure 6. Ed's third arrangement using three equal subsets plus a remainder.

T/R 2: Then what about this guy? [She points to the remaining W on the far right.]

Ed: Oh, what this guy would do

[T/R 2 rearranged the rods to appear as they do in Figure 4.]

T/R 2: We ran into the same problem, didn't we? Would you agree that if we went back to this model where we had these [She rearranges the rods]. Would you agree that maybe I could take this one [white rod] and saw it in half, if I had a saw?

And we were divvying them up [two halves of sawed white rod]

Ed: Yeah.

T/R 2: And then what could I do with it?

Ed: Then you could put it here and here [pointing to the two columns of rods]

Class ended as T/R 2 suggested that the children, "Write about what we worked on the past few days." (See Appendices J, K L and M for samples of the children's writings.)

Session 3

Activities

1. If purple is called 1/2, what number name shall we give to brown?
2. If purple is called 1, what number name shall we give to brown?
3. If orange is called 2, what number name shall we give to yellow?
4. *A train of yellow and light green is called 2, what number name shall we give to red?
5. *A train of yellow and light green is called 1, what number name shall we give to red?
6. Which is bigger, 1/2 or 1/3? And by how much?

*Tasks 4 and 5 were given simultaneously.

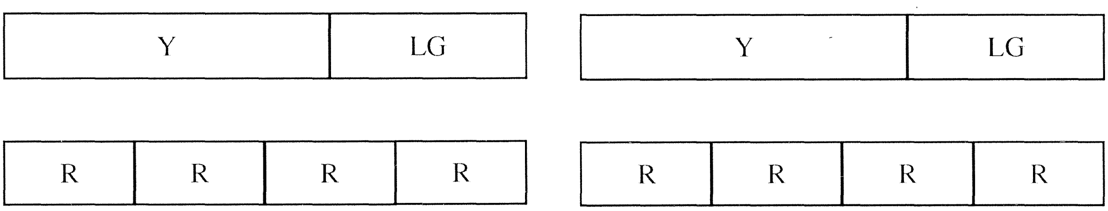
Session 3

Narration

The notion of making 'trains' of rods, that is, making new rods by placing existing rods end to end, was introduced in the previous session. The following questions were posed, almost simultaneously:

> If a train of yellow and light green rods has the number name "two", what number name will we give the red rod?
> If a train of yellow and light green rods has the number name "one", what number name will we give the red rod?

All of the children placed four red rods under their trains of yellow and light green rods. Sami and Anne presented their model using overhead rods and built two identical models with different number names (see Figure 7).



The train of yellow and light green rods is 2.
What is the number name for the red rod?

The train of yellow and light green rods is 1.
What is the number name for the red rod?

<u>Figure 7.</u> Models of trains by Sami and Anne

The school principal, who frequently visited classes and interacted with the students while they worked, questioned some of them about their models

Anne: Oh, first we put the red rods up to the yellow and light green rods and then we said if the yellow and light greens rods, and then we said if the yellow and the green was two, what would we call the red rod? And we thought that we would call it one and one fourth. And if it [the train] was one, we would call it one fourth.

Principal: Okay, so if it was one, you said you would call it one fourth, and if it was two, what did you say?

Anne: It would be one and one fourth.

T/R 1 asked the children if they agreed with Anne's solution. Some indicated that they did not. T/R 1 asked Anne and Sami what they would have to do to convince the class, first requesting that they illustrate the train whose length was 'one'. Using this train the girls explained they named the red rod one fourth because there were four red rods equal to the length of the train. The class agreed with the girls' name for the red rod when the train was called "one"; they disagreed with the name for the red rod when the train was called "two". Bob and Jackie joined Sami and Anne at the overhead and pointed out that the red rod should be called "one half". Dan, also, joined the group and explained why the red rod should be given the number name "one-half":

> Okay, so this is two [the train given the name 'two'], and this would be a half [a red rod] because if you put another one and another one that'd be two. [He aligned four red rods on the overhead.] And if you take away these [two red rods] that would be one and took away that [another red rod], leaving one red rod, that would be a half.

T/R 1 asked the class what they found so confusing about this problem. Ed explained.

> I think the confusion is, they think, that they think, they have the temptation of calling, since there are four red blocks, they think they are gonna call it one fourth 'cause they forgot that the yellow and the [light] green are two. ...Because, see, if you have one, there'd be two halves, but if you have two it's two halves plus two halves which would be four halves. Therefore, you would have to call one of the reds one half.

T/R 1 began with a story about two friends, Tom and Amy, each of whom had been given half of a chocolate bar. She reported that Tom was happy with the size of his piece but Amy complained that her amount was unfair. T/R 1 posed to the class how that might have happened.

Matt replied: "You probably gave Tom a bigger half than Amy." T/R 1 asked if that made sense. Maria responded with an example using yellow and light green rods.

> Well, see this was one (the train of a yellow and a light green rod), and then you gave this much to Tom (pointing to the yellow rod which was given the name "five eighths") and this much to Amy (pointing to the light green rod which was given the name "three eighths"). That wouldn't be a fair cut."

Dan said that he agreed with Maria and pointed out that a half should be "even". T/R 1 continued with the candy bar metaphor. She displayed a large and a small candy bar and said that she gave half of the large one to Tom and half of the small one to Amy. The children giggled and agreed that it was unfair to talk about halves with different "wholes".

Next, a comparison problem involving unit fractions was posed to the class, "Which is bigger, one half or one third?" The teacher invited the students to comment on the meaning of the problem and Mario responded.

> Well, normally, one half is bigger than one third, but if you got a bigger size of candy bar or pizza, and if you get one third of that, then that'd be more than one half of a little pizza.

The children worked with their partners. Some children observed the half was bigger than a third. T/R 1 asked Julie and Bob, "By how much?"

Julie:	By an inch.
Bob:	No, by red.
T/R 1:	By red? What number name would you give the red then?
Bob:	A fourth.
T/R 1:	Remember what you called one.
Julie:	One fourth.
T/R 1:	What number name, prove to me that red is a quarter. [Julie moves closer to Bob to see what he is doing.] This is red, that's a half [the dark green rod]. Sure it's a quarter? Change your mind?

Julie: Yeah.

T/R 1 asked Julie if she could explain that and left the partners to continue their discussion.

Bob: Maybe… Okay, so what would these be? [Julie counts the red rods.] They're sixths.

Dan and Maria were observed building balance beams as they explored possible models. The class ended and as the children began to ready themselves to leave, T/R 1 asked the children to assemble around Dan and listen as he explained what he had built. The whole class gathered as Dan rebuilt and explained his solution to the task, using a balance beam. (See Figure 8.)

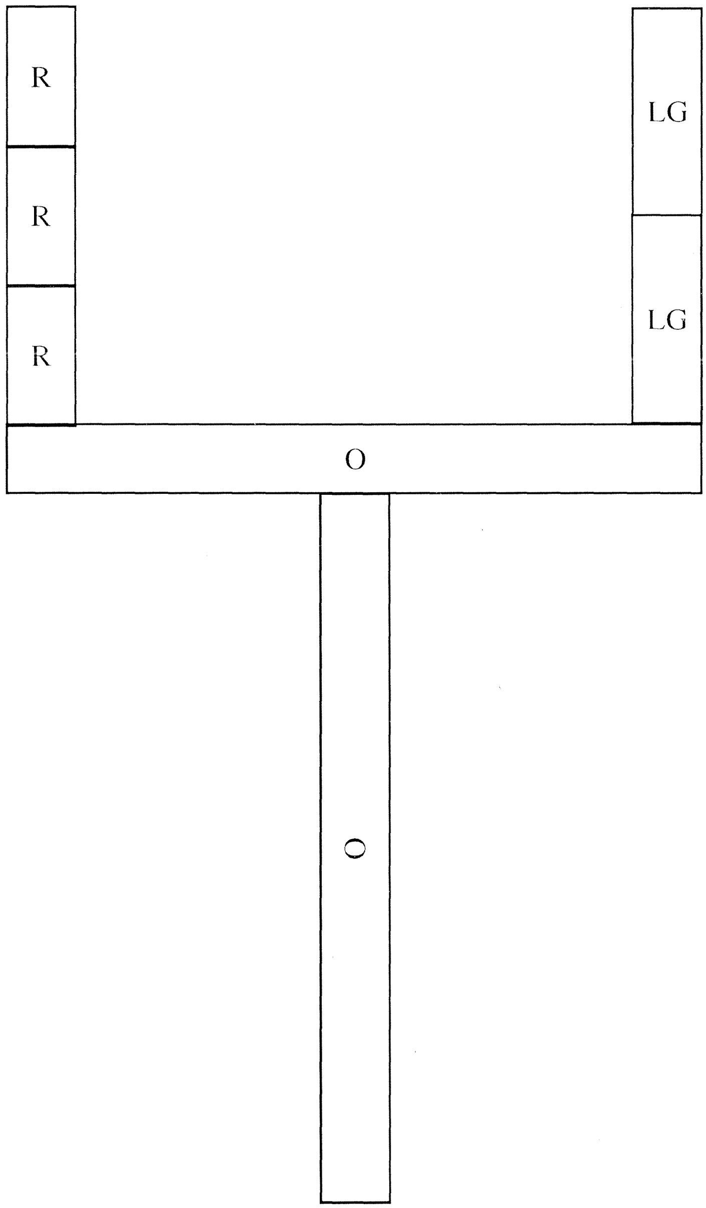


Figure 8. Dan's Balance Beam Model

Dan: All right, I made a balance and the whole thing is dark green and the light green is a half and the reds are the thirds, but then what I'm doing is, um, I'm making a balance so when I take off that [one light green rod] and then the two reds, then I think it will fall to the side and show a half is bigger.

Dan placed two light green rods on one side and the three red rods on the other. When he removes the one light green rod and the two red – leaving one light green rod (a half) and one red rod (a third) – his structure fell to the side of the light green rod (signifying that side of the balance is heavier - bigger).

T/R 1 probed for an answer to the question, "And by how much?"

T/R 1: It did fall to that side. Your prediction was right. Okay, now the question I'm going to ask you, when you work on this balance what would you have to put there to stop it from falling? What other rod could you have put on the left side so that it wouldn't fall when you took that off? Do you understand my question? What did you take off?

Dan: I took off the two reds and a light green.

T/R 1: Okay, now if you don't want it to collapse, right? You said it fell to the right the way you had it built, okay?

Dan: Um.

T/R 1: And the red rods were on the right side? Is that correct and the greens were on the other side, or was it the other way?

Dan: Well, the reds were on the left side.

T/R 1: On the left side. So you took the two reds from the left side and the green from the right side. Okay, what would you have had to put on that other side so it wouldn't tip? Once you took the two reds and the green off? Do you understand my question?

Dan: Um, let's see

T/R 1 asked Dan to predict which rod would have created the new balance between one half and one third, which rod added to the red rod would equal to the length (and weight) of the light green rod.

Dan: Um, maybe a little white? [He placed a light green rod next to a red rod and added a white rod to the red rod.]

T/R 1: A little white? Okay, we could try that experiment on Monday, right? That's a good guess. Why did you guess that? I think you went looking for something specific. Why were you looking for that one?

Dan: Well, 'cause when I went like this I just saw there was one space in between and I knew that there white is that space.

T/R 1: Okay, what number name can you give to white?

Dan did not answer and T/R 1 suggested that this was something for the class to think about as the class was dismissed.

Session 4

Activities

1. If I call the orange rod one, what number name will I give to the white rod?
2. If I call the orange rod one, what number name will I give to the red rod?
3. If I call the orange rod one, what number name will I give to two white rods?
4. If I call the orange rod ten, what number name will I give to the white rod?
5. If I call the orange rod fifty, what number name will I give to the yellow rod?
6. If I call the orange rod fifty, what number name will I give to the white rod?
7. Which is bigger, one half or one third? And by how much?

Session 4

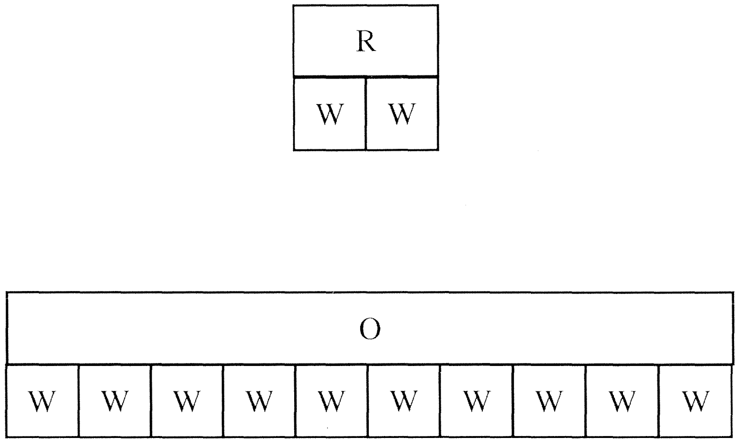
Narration

The children were asked to investigate specific relationships between the rods and assign appropriate number names to the rods. T/R 2 asked: "If I call the orange rod one, what number name will I give to the white rod?" The children named the white rod "one tenth". T/R 2 continued: "If I call the orange rod one, what number name will I give to the red rod?" The children named the red rod "one fifth". She then asked: "If I call the orange rod one, what number name will I give to two white rods?" Two boys, Matt and Alex, answered, "one fifth". They went to the overhead to present their solution, which showed the length of the two white rods as equal to the length of the red rod. They gave the red rod the number name one fifth.

T/R 2 then asked if there were other answers. In response, Maria, Betty, Sami, and Dan presented their solution. The students placed ten white rods under the orange rod and Maria and Betty explained:

Maria: I think it's two tenths. Take the red away and put this [white rod] up to the orange. When we did it before, we said that orange measures ten whites. If you put the whites up it would have ten. Two of ten is two tenths.

Betty: Since ten of these [white rods] equal one orange, then if you took two of these it would be two tenths because one equals one tenth and you just count one more and then you have two tenths.



Alex and Matt's Model

Maria, Betty, Sami and Dan's Model

<u>Figure 9.</u> Models of 2/10 and 1/5

T/R 2 asked the class how there could be two answers, two tenths and one fifth. Bob held up two white rods in one hand and one red rod in his other and explained:

Bob: Even if two white cubes equal up to one red cube, it's still not like imagining that this was another red cube so I think it's two tenths because it actually is two tenths.

T/R 2: Because you can see two there?

Bob: Yeah, I also see one fifth but what you're seeing right here is two tenths, not a fifth.

T/R 2 asked the class if it were possible for the two white rods to have both number names. Maria repeated that the two white rods should be called two tenths, and said: "There's only two of them. They're not joined together. If you wanted to join them together you should use a red."

For the remaining ten minutes of the session, the children were asked to think about the problem from the last session: "Which is bigger one half or one third? And by how much?" Art and

Ed, Betty and Sami experimented with balance beam models, similar to the one constructed by Dan in Session 3 (see Figure 8, p. 67). The children again agreed that one half was larger than one third. The following conversation occurred between the school principal with Art and Ed.

Art built two different length models; the first with two yellow rods next to an orange rod (10 cm), and a second with three light green rods next to the blue rod (9 cm). Art modeled one half to be bigger than one third by using two different length units. The principal questioned his use of these models to find a solution to the task.

Prin:	Well, how can you convince me that one half is greater than one third?
Art:	There's the half (the yellow rod). That (the light green rod) would be one third of the blue rod, but you can't divide the orange into thirds.
Prin:	Are you allowed to use two different rods? Can you use the blue rod to get one third and the orange rod to get one half? What do you think, Ed?

Ed explained his reason for not using models with two different units. A discussion ensued between the boys over the use of unlike models for the unit "one".

Ed:	You can't compare. If you use orange for halves, you can't use a blue rod for thirds because the blue rod is smaller than the orange.
Art:	Nevertheless, if you could make a new rod to make the orange into thirds, it still wouldn't be as big as the yellow.
Prin:	Art wants to use the orange rod for one half and the blue rod
Ed:	And then compare it, but you can't compare a larger rod with a smaller rod.
Art:	I know, but even if you made thirds with the orange, they wouldn't be as big as the half.
Ed:	So that is the answer right there. You don't have to use the blue rod to compare with the orange, and you're saying we do.
Art:	You'd have to make a new rod.

Ed questioned the possibility of finding a rod to represent the unit “one” which could be divided by two and three. He then explained how he found such a unit, the dark green rod (6 cm).

Ed: Exactly, I don’t know if you can divide a single rod into one half and one thirds

Art: You can divide any rod you want into one half.

Ed: I know, no, you can’t. You can’t divide any rod you want into halves...wait. I just got a rod that you can divide into both things [the dark green rod].

[To the principal] I used dark green and then I experimented. I studied the rod, and I said, maybe you could use light greens, so I put then up against it [the dark green rod] so you could use them as one half. Then I studied it again. I go, maybe the ones could do it, but then I looked at it and I go, no. So then I thought maybe one larger and then I go, “Oh”, the red rods! So I put red rods up against it [the dark green rod] and divided it into thirds.

Prin: So which is bigger?

Ed offered his reason for why one half was larger than one third.

Ed: One half. Because if you have one whole, and you wanted to divide it into halves, the halves have to be so big that you can only divide it into two parts. And if you want to into thirds, they have to be big enough to divide into three parts. So if you only wanted two parts, the whole has to be big enough to divide into two equal parts. So if you have two parts, two is less than three, but if you divide into two parts, they have to be bigger than one third.

Prin: Interesting. Art, do you follow what he is saying?

Art: Yeah I have it here.

Ed refined his explanation in repeating his ideas to Art and the principal.

Ed: The number three is bigger than two but if you cut something into two parts, they, technically, if you count by numbers, the smaller number is the larger [rod]. Using Cuisenaire rods, if you’re cutting into halves and thirds, the smaller the number, the larger the half, the larger the piece.

Prin: Does that always work? What do you think, Art?

(As the class ended, a break in the tape caused further documentation to be unavailable, from any of the three camera views.)

Session 5

Activities

1. Does 1/5 = 2/10?
2. Number names for 1/2 of a 3-by-4 scored candy bar.
3. Which is bigger, 1/2 or 1/3? And by how much?

Session 5

Narration

T/R 1 wrote the following problem on the overhead projector: "Does 1/5 = 2/10?" Maria came to the overhead and built a model using an orange rod to represent one. She placed two yellow rods beneath the orange rod and explained that the yellow rod would have number name one-half. She then placed five red rods underneath two yellow rods and put two white rods together. Maria explained that two white rods would be named "tenths" while the red rod would be named "one fifth".

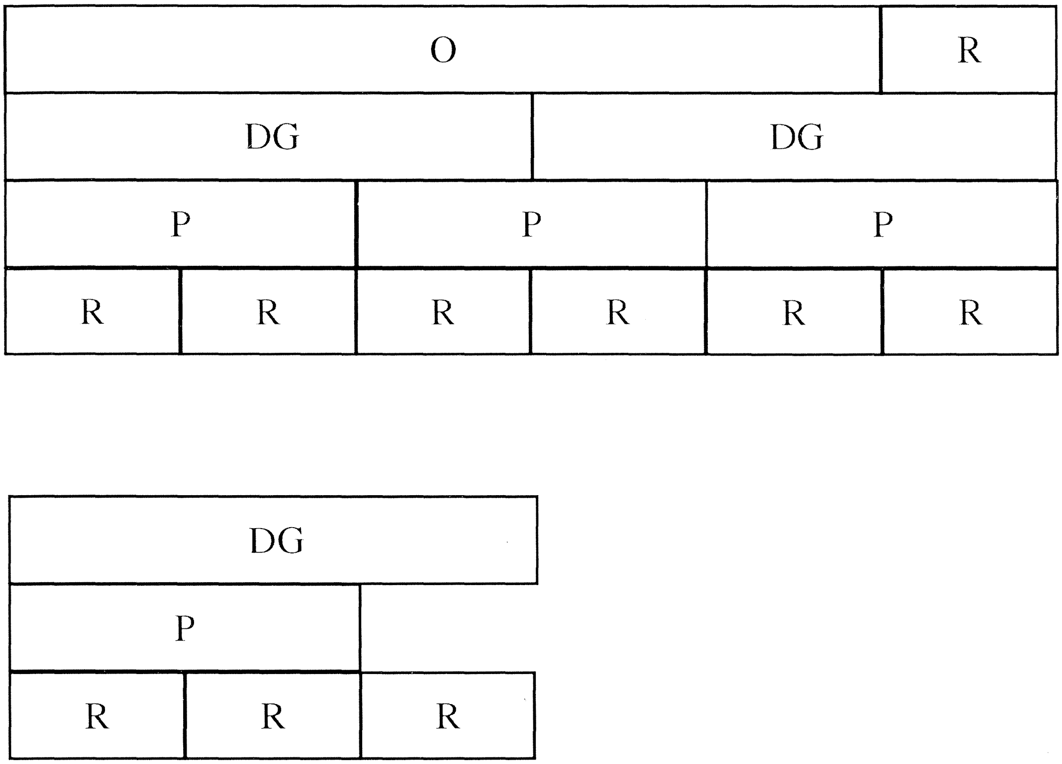
Maria:	...And if you take one of them [She moves one red rod above the two white rods] it is equal to two tenths.
T/R 1:	So what is your conclusion if I ask you the question, is two tenths equal to one fifth?
Maria:	Yes.

Eric repeated Maria's explanation to the rest of the class; Bob followed with a similar explanation. The class agreed that the same lengths could be measured with different units, tenths and fifths.

T/R 1 then produced an overhead transparency of a candy bar [scored in a 3-by-4 grid] and asked the children to give number names for one-half of the bar. Jackie answered: "...six twelfths, because there are twelve pieces in all, and she got six, and six makes a half." Dina offered: "Two fourths...Because if she got a half, then the top two rows, um, is a half, and then that's two fourths." Bob added: "...three sixths. Yeah, because there are six of them there, there,

there. I found groups of sixths." T/R 1 recorded the children's comments on the overhead projector. She wrote: 1/5 = 2/10, 1/2 = 6/12, 2/4 = 3/6.

T/R 1 then posed the earlier problem: "Which is bigger one half or one third, and by how much?" Jen and Liza volunteered to give their solution at the overhead. They used a train of orange and red rods to represent their unit. They explained that 1/2 was bigger than 1/3. When the T/R 1 asked by how much, and the girls responded that it was a **red bigger.** When asked to give the red rod a number name, Anne, who joined the other two girls, responded, "It's a third bigger, I think." Jen and Liza agreed, and Jen explained: "I think it's one third bigger, too, because if you put the red to the [dark] green, you need three and if you put the purple one to it also, and then it takes one third of them." Jen measured the length of the red rod in comparison to the length of the dark green rod, which was one half in her model (see Figure 10).



Jen's model comparing 1/2 and 1/3

Jen's model illustrating "And by how much?"

<u>Figure 10.</u> Jen's model: "Which is bigger 1/2 or 1/3? And by how much?"

Jen: I think it's one third bigger too because if you put the red to the green [dark] you need three and if you put the purple one to it also and then it takes one third of them.

Kristin: Yes. If you have, um, a red, if you hold, um if you have.. We used these and we went and then we like held reds up and we showed that um, that um, one half is bigger by, because this part is smaller, and this is supposed to be one, one third so that's how we did it.

T/R 1 repeated the girls' solution, that one half is bigger than one third by one third. The girls concurred.

Kristin: Yeah, you can put three of these, three reds up to one green and then it would take one, one third of the red to um, like um, to go there, like.

T/R 1 asked how many agreed, weren't sure, or disagreed and why. Some children raised their hands indicating that they weren't sure or disagreed. Bob offered a different number name for the red rod and went to the overhead projector to explain his reasoning, using the girls' model.

Bob: Well, when they said one third is bigger than one half by one third. I think they said, is that what they said? Well, I don't really agree, well if you split, if you split one of the thirds In half which would make, which would make a sixth. I think it's a sixth bigger. Like, well, should I go up there? [He goes to the overhead.]

Well, see for um, when they said it was one half bigger, if you split a third in half it'd make a sixth, like one, two, three, four, five six. Like pretending they were, like pretending they were split in half. If you split one of these in half and you have three of them up there they'd make six and anyway, and when you split them in half right in the middle over there it's kind of like this, it's kind of like this, there was this was, that was the one third [points to a purple rod] and that was the one half [points to the dark green rod] or the bottom so it's just like, and the red I'm pretending is like, is like, is a half of one of the purples and you see when I split it in half it's, it's one sixth and, and it equals, and it equals up to a green.

T/R 1 repeated Bob's explanation that red rod was named one sixth. Bob explained that one half of one third would be one sixth.

Bob: Well, I mean a red, I'm considering red one sixth because two of these [red rods] equals, see there two, there two halves of one purple and the purple is a third and the half of one third is a sixth.

T/R 1 asked the class if the answer could be both a sixth and a third. Joyce offered a second model in which the dark green rod was named "one" (see Figure 11).

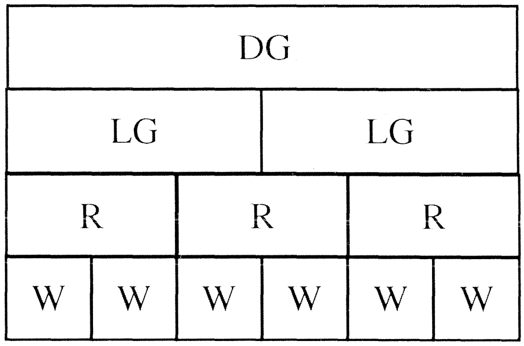


Figure 11. Joyce's Model: "Which is bigger 1/2 or 1/3? And by how much?"

Joyce: Well, maybe if we call this dark green one and the red one third and the light green one half, and we thought the, we thought one third was bigger by one of these white ones.

Jen: Oh, I think they are making a different size candy bar

T/R 1 asked if different sized candy bars were allowed.

Jen: No, because it's not fair because, say you give someone half of this one [12cm] and then one half of that one [6cm] and this is bigger than that one [takes a light green and dark green rod in hand].

T/R 1 questioned the girls; asking if dark green was still "one", as it was when they started the problem. They agreed. She then asked, "which is bigger, one half or one third? And by how much?"

Joyce: Okay, we think that a half is bigger than a third. Yeah, and we think light green is a half [of the 6 cm model]. Okay, this is a half [light green] and the red is a third.

T/R 1 asked Joyce to explain why she named the light green rod "one half"

Joyce: Because if you put these all together they equal up to the one and we think the light green which is a half is bigger than the red by, by one which is this white one. Actually, I used this to tell that the light green is one white bigger.

Ed offered an explanation of the girls' solution.

Ed: I think they mean that they want to call this, the dark green one, one whole, and they want to call this, yeah, like you line all the whites up to it which I think should be six and they want to call it one sixth. I think that's what they're trying to say but they just, they're just not saying it. I think they just, they want to call it one sixth.

The girls agreed they meant to say one sixth. T/R 1 questioned them, "How much bigger is one half than one third?"

Girls: Um, one, one sixth.

T/R 1: Which?

Girls: One sixth, sure, yeah.

The class ended as T/R 1 asked the children to write about the different arguments that were presented during the session. (See Appendices J, K, L and M for samples of the children's written work.)

Session 6

Activities

1. Which is bigger, 1/2 or 1/3? And by how much?
2. Which is bigger, 1/2 or 1/4? And by how much?

Session 6

Narration

The session began with a discussion of the magnitude of the difference between one half and one third. Some children expressed that they could provide a convincing argument that the difference must be one sixth. Jen offered a model using a train of an orange and red rod to represent the number "one" and showed the difference to be "one sixth". Art's model made use of a dark green rod to represent "one" and also showed the difference to be "one sixth". Jen challenged Art's model,

Like remember, you [T/R 1] said that it can be only one size candy bar and that's it.

Joyce and Kristin, then Art agreed that it was appropriate to build models with different lengths for the unit, "one", as long as the comparison stayed within each unit (one third and one half would be compared within one model or another, not across models).

Joyce: There can be candy bars of different sizes.

Art: You just can't switch the candy bar.

T/R 1 reminded the class, "trading candy bars of different sizes is unfair, but to make models with different sizes is okay, because what you call "one" changes."

T/R 1 then asked, "What rod has the number name one sixth if we call the orange and red [train] one?" Sami answered, "red". T/R 1 then asked the children to name the white rod in this model. Various answers were suggested: "one sixth" [Jon], "one twelfth" [Bob], and "one tenth" [Liza]. After a lively discussion, Jon changed his mind and went to the overhead. He placed twelve white rods under the train and explained that each white rod would be called "one twelfth". The children concurred. T/R 1 prompted the class:

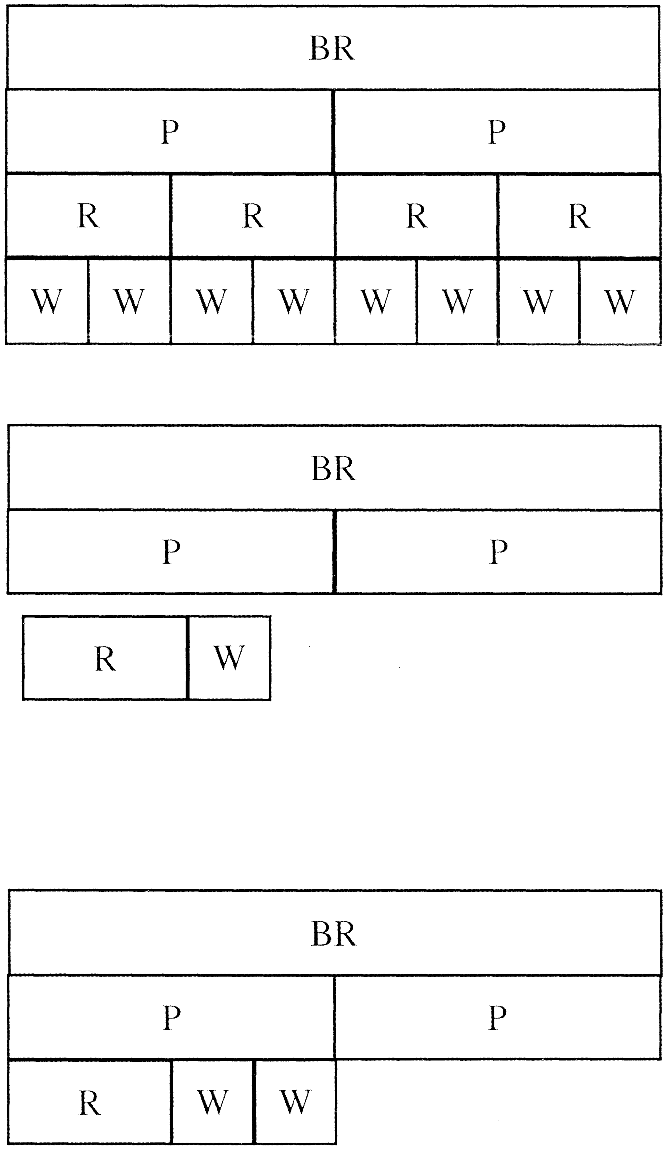
> Everybody agrees we have two different ways of showing that one half is bigger than one third by one sixth? Who can, and who cannot write about two different ways and why?

A second problem was posed: "Which is bigger one half or one fourth? And by how much?" The children built a variety of models.

T/R 1: Who has more models One? Two? Five? More than five?

She suggested the class listen to different solutions, building what was being done at the overhead with the rods at their desks. Graham, Brian and Michael came to the overhead and shared their model with an orange and red train named "one", a dark green rod named "one half", and a light green rod named "one fourth". They demonstrated that one half was bigger than one fourth by one fourth.

Jen, Alyce, and Jon went to the overhead to present a different solution. They built a model that used a brown rod as "one", and two purple rods, each as "one half". They placed 8 white rods under the purple rods and called the white rods "eighths". They measured the difference between the purple rod [1/2] and two white rods [1/4] as one white rod. They stated that one half was bigger than one fourth by one eighth. Maria came to the overhead and showed the group that they still had room in their model for one more white rod, which would make the difference two eighths, not one eighth (see Figure 12).



Jen Alyce and Jon's Model

Maria's Model

Figure 12. Models: "Which is bigger, !/2 or !/4? And by how much?"

Maria explained that the difference was two eighths <u>or</u> one "quarter" (fourth):

> Yes. Okay. This is an eighth [one white rod]. It's not one eighth cause there's still a space. We're calling that an eighth [the white rod]. If you take another one it, um, it could be bigger by two eighths and, or it could be bigger by one quarter. One quarter or two eighths. It's the only way it could be one half.

The children concurred.

Session 7

Activities

1. Which is bigger, 1/2 or 2/3? And by how much?

Session 7

Narration

The problem, “Which is bigger two thirds or one half? And by how much?”, was investigated by the children. Some quickly produced a model to show their solution; they were then asked if they could build more than one model. Maria was seated at her desk when T/R 2 approached and asked her to explain the models[1] (see Figure 13).

<table>
<tr><td colspan="3">LG</td><td colspan="3">LG</td></tr>
<tr><td colspan="2">R</td><td colspan="2">R</td><td colspan="2">R</td></tr>
<tr><td>W</td><td>W</td><td>W</td><td>W</td><td>W</td><td>W</td></tr>
<tr><td colspan="6">DG</td></tr>
</table>

<table>
<tr><td colspan="10">O</td><td colspan="2">R</td></tr>
<tr><td colspan="6">DG</td><td colspan="6">DG</td></tr>
<tr><td colspan="4">P</td><td colspan="4">P</td><td colspan="4">P</td></tr>
<tr><td>W</td><td>W</td><td>W</td><td>W</td><td>W</td><td>W</td><td>W</td><td>W</td><td>W</td><td>W</td><td>W</td><td>W</td></tr>
<tr><td colspan="2">R</td><td colspan="2">R</td><td colspan="2">R</td><td colspan="2">R</td><td colspan="2">R</td><td colspan="2">R</td></tr>
</table>

Maria’s Model 1

Maria’s Model 2

Figure 13. Maria's models: "Which is bigger 1/2 or 2/3? And by how much?"

Maria built the two models illustrated in Figure 13, in the order in which they are shown; she chose to explain her models to T/R 2 in the reverse order. She chose two white rods to measure the length between two thirds and one half in Model 2 and named the difference "two twelfths".

> And then you take the two thirds [she removes two purple rods and one dark green rod from Model 2], and you take two twelfths [the white rods] and then you put it up to two thirds and it is bigger than it by two tens... two twelfths.

Maria then explained the difference, using Model 1. She named the difference "one-sixth". T/R 2 asked Maria if there were another number name for the difference between two thirds and one half in Model 2. Maria had named the difference "two twelfths" and now offered another number name, "one sixth".

> Um, yeah, well, maybe... [Maria places 6 red rods below the train of 12 white rods, and placed a red rod above 2 of the train of 12 white rods.] ...Yeah, and maybe since two of these little white ones equals up to one of these. [She put 1 red rod on top of 2 white rods in the train, showing that a red rod is the same length as a train of 2 white rods.] Or it's one fifth, oh, I mean one sixth.

Shortly after this exchange, T/R 1 asked the class to share their ideas with one another. Jen, Eileen, and Joyce presented their solution to the entire class at the overhead and displayed the model that they built. Their model used the same rods for representing "one" as Maria's Model 2 (See Figure 13). They chose two dark green rods to represent halves, three purple rods for thirds, and six red rods for sixths. One red rod was used to indicate the difference between one half and two thirds. They named the red rod "one sixth". T/R 1 asked if there were any other solutions. Maria responded eagerly and joined the girls. She placed twelve white rods under the six red rods

[1] See Maher(1995), Steencken and Maher (1998), Steencken and Maher (in press).

displayed on the overhead. Maria chose two white rods to measure the difference. She stated that two thirds was larger than one half by "two twelfths". Mario strongly objected.

> No. ...No, they can't do that. Because the, the two thirds are bigger than a half by a red. So they can't use those whites to show it.

The T/R 1 asked Mario about the model he built. Mario explained and T/R 1 questioned further.

T/R 1:	And you got two thirds to be bigger than one half?
Mario:	[Politely impatient] Yes.
T/R 1:	By how much?
Mario:	[Deliberately] By one sixth.
Maria:	Or, or two twelfths.
Mario:	[Shaking his head in dissent] No. [Mutterings in the classroom of 'no']

Maria supported her idea and provoked a lively debate among her classmates. Ed offered his understanding of Maria's solution and added an opinion for his calling the difference 'combined'.

> Yeah, but see just the two whites together. That's right, it would be two twelfths. But you have to combine them. You can't call them, you can call them separately, but you can also call them combined and if you combine them it would be a, a, one sixth.

The T/R 1 questioned further.

T/R 1:	So, so what is Maria saying here?
SS:	There's two answers.
T/R 1:	Are there two answers?
Mario:	[Simultaneously with Ed] No, they're the same answer.
Ed:	No, they're the exact same thing, except she, she took the red and divided it into half, she divided it into halves, into half and called, and called each half one twelfth. They're the exact same answer except they're just in two parts.

T/R 1: So we're talking about the length of the red rod, the length of the red rod is the same as the length of the two white rods? Is that true? Do you all agree to that?

SS: Yeah. Yes.

Ed: And she's calling a white rod one twelfth and the other white rod one twelfth and the red is really one sixth. Well, when she calls them two twelfths, the two twelfths are actually just two white rods put together to equal a red, so it should be really, it's really one sixth...

T/R 1 joined the four girls at the overhead projector. She began to record the spoken words of the children in a formal notation (see Figure 14).

R 1/6

2W 2/12

1R = 2W

1/12 + 1/12 = 2/12

1/2 of 1/6 = 1/12

1/6 = 2/12

Figure 14. Children's Recorded Statements.

T/R 1: Let me write this down. This, what you are saying here is so important, here. Let me see if I can write this down. You're saying that you're calling the red, you're giving red the number name, right? The length of the red, right? We'll give it the number name, what did you say?

SS: One sixth.

T/R 1: [She writes R 1/6] One sixth. And two whites, can I write two 'w ' for two whites?

SS: Yeah.

T/R 1: And you're calling two whites

SS: Two twelfths.

T/R 1: [She writes 2W 2/12] But what Eric just told me, right?, is something about red and white.

Ed: Yeah. A red, one red equals, one red rod up here, one red equals two of the white ones.

T/R 1: [She writes 1R = 2W] So we're talking about the length of the red rod, the length of the red rod is the same as the length of the two white rods?

Is that true? Do you all agree to that?

SS: Yeah. Yes.

Ed returned to Maria's solution of two twelfths and expressed his decision to name the difference in lengths as "one sixth". T/R 1 continued, throughout the conversation, to record the mathematical statements of the children.

Ed: And [Maria] she's calling a white rod one twelfth and the other white rod one twelfth and the red is really one sixth. Well, when she calls them two twelfths, the two twelfths are actually just two white rods put together to equal a red, so it should be really, it's really one sixth. Because two whites, two whites

T/R 1: She says one white is one twelfth [She writes 1/12] and then if you put it together with another one twelfth [She writes +1/12], she's saying you get two twelfths [She writes = 2/12].

Ed: And it's one sixth, it's one sixth.

T/R 1: And you're saying if you have, if you take one half that's all right? If you're taking one half of one sixth, you're saying you get one twelfth [She writes 1/2 of 1/6 = 1/12]
You're saying that. That's the two things I'm hearing. And you're saying that one sixth, the length of one sixth is the same as the length of two twelfths. [She writes 1/6= 2/12] Is that what you are saying?

Ed: Yeah.

T/R 1: All those things, are they true?

Ed agreed the statements were true, but reaffirmed his opinion on naming the difference between one half and two thirds as "one sixth".

Ed: Yeah. But I don't really think you could call, call them two twelfths because two twelfths equal exactly to the same size as one sixth. Well, if you want to you could call them, I guess. But I think it would be easier just to call them one sixth cause you wouldn't want to exactly call them one twelfth and another twelfth. I'd just call them one sixth. Therefore I think you just really call them one sixth.

Bob: Well, maybe you can call them

Ed: Well you can call them, if you want to, but

T/R 1 prompted the children to clarify the issue of different number names for the same length rod.

T/R 1: Well, we have different number names for these rods

Mario: They're not different

Bob: There's just half of one, there's just half of one.

T/R 1: So you're saying that one half of the one sixth is another way of saying one twelfth.

Bob: They're just two answers.

T/R 1: Well, you're saying if you took a twelfth, a rod that has length one twelfth, and another rod that has length one twelfth and put them together, right? That rod would have length two twelfths. Isn't that what you said?

Mario restated Ed's view.

Mario: What Ed said is that two whites equal one red, so it would be the exact same thing.

T/R 1: Or a rod that has length one sixth, that would be the red rod in this problem, would also have length two twelfths. Is that what you said when you talk about the length of these rods? So are all of these [pointing to the recorded notations on the overhead projector] true statements?

SS: Yeah. Yes.

T/R 1 invited the class to think further about these problems and to write about their solutions using several models to support their reasoning. They were also asked to write about what was alike and what was different about these models. (See Appendices J, K, L, and M for samples of the children's written work.)

Children's Written Work

The teacher/researchers, T/R 1 and T/R 2, requested that the children record their ideas and submit their writings so that the Rutgers researchers could read about their ideas. During the course of these seven sessions, T/R 1 and T/R 2 suggested writings by the children on four separate occasions: (1) A request for the children to write about what they were working on – during and after Session 2; (2) "Which is bigger one half or one third? And by how much?" – after Session 5; (3) "Which is bigger one half or one fourth? And by how much?" – after Session 6; and (4) "Which is bigger one half or two thirds? And by how much?" -- after Session 7. In requests 3 and 4, the children were also asked to write about similarities and differences in their models.

The first requested writing, open-ended in nature, was suggested at the end of the second session. Twenty one children responded. In this first writing task, the children all used text to describe their feelings and to write about their ideas; no drawings were included.

Eleven children wrote about their emotional reactions to the intervention; nine commented positively that the project was "fun", or "fun, but hard". Jon commented, "They has some really puzzling questions. At first I didn't get it, but then I got it and it was easy as pie. Mostly they tried to stump us." Sami complained that she was extremely unhappy with her partner. Eileen noted that she would have to learn fractions, because "if someone is a lawer (sic) or accountant you have to know your math". Mario commented that "I never thought the rods could teach me so much about fractions"; he continued by saying how he saw fractions everywhere, "my bookshelf that's divided into thirds, my dresser is divided into fourths".

Three of the children made reference to the use of the overhead projector. For example, Liza stated, "We got a chance to work on the overhead projector to explain about what we were doing. Rutgers asked us questions and we would have to explain our answer with the rods."

Seven of the children posed illustrations of the kinds of tasks they were exploring; some included solutions. For example:

1. If a red block was one whole, what would be the orange block? [Alex]
2. If white is one, how many is orange? Ten [Joyce]
3. Dr. Mar (sic) would say, "If I call the white rod one, what would I call the orange rod?" If we should have eleven white rods, your fraction answer would be 1/11 [Anne]
4. "They taught us that if you call a rod 1 whole you can figure out what 1 half is and call it 1 half. You can also call something 6 or 10 and figure out the half of it and call it half. For instance if you have something that you call it 6 1 half would be three." [Ed]

Ten of the children referred to a unit, either by writing about what they did in class, or by posing a problem task of the kind they explored in class. Six of these students used the language of "whole"; for example, Dan wrote, "If I was going to call white 1 whole then what would red be? 2 whole". Five children used "one"; all of these students posed questioned patterning the questions that were presented during class. For example, Maria asked, If blue was considered 1 what color name would 1/3 be? Light Green". One student used both the language of whole and the language of one; Bob noted, "...we talked about halfes (sic) and thirds and wholes and how many there are in one."

Twenty one children responded to the second writing task, "Which is bigger one half or one third? And by how much?" All students recorded, with models and text, that one half was bigger than one third. One student, Cassie, whose first submission stated that one half is bigger than one third by one sixth, drew an appropriate model of the task in a second submission and then wrote, "I think there about the same. Because the red is the same as the dark green and light green."

Seventeen children chose to draw models using rods or similarly drawn rectangles; two used pie charts or circles, and two drew rectangular candy bars, scored into 6 parallel sections. Of the children who used the rods, 2 wrote descriptions of their models by referring to candy bars; one child, Liza, drew rod models and offered an explanation in terms of pies: "From the chart (2 rod models) you can see that 1/2 is bigger. If you divide a pie in half the pieces will be bigger than if you divide it into thirds." In describing their solutions, the two boys referred to ideas concerning area, associated with the circles they had drawn:

Matt: "I think 1/2 is bigger than 1/3 because 1/2 has more space. ...1/2 takes up more room"

Jon: "As you see the thirds circle there are three big shapes, but the halfs (sic) circle has two even bigger shapes."

Eight children suggested looking at and comparing their models to notice why one half is bigger than one third. Dina pointed out, "See red is smaller"; Dan suggested, "put red to white". Maria said, "...put one 1/3 piece on top of one 1/2 piece, the 1/2 piece is bigger." Five children reasoned that the number of pieces (in the denominator) determined the size of the pieces; for example, Art stated, "...The smaller the number that you divide the rod, the bigger the pieces will be". Three other children referred to the size of the pieces; for example, Bob commented, "The 1/2 is bigger because all you need is 2 of them to make 1."

Four children offered solutions to the second part of the task, "And by how much?' Cassie said, "…you can see that the red rod (1/3 in her model) would need a little white rod to be the same (as the light green rod, labeled '1/2' in her model)." Sami drew a model and responded to the second question by drawing an arrow, labeled "Why 1/2 is larger", to her model with the difference of a white rod labeled '1/6'. Betty presented two models; she wrote, " Answer: 1/6" on the first and drew an arrow to a white rod labeled "1/6" on the second.

Two children drew models with mislabeled rods. Dan correctly labeled his "whole as 4 cm in his model (the purple rod), but referred to the "1" as the yellow rod (5 cm) in his description. Joyce was the fourth child to respond to the second question on the task; she wrote that 1/2 is larger than 1/3 by 1/4 "is a purple rod. If y 2 – (sic) red from dark green, it will equal purple".

The third writing request prompted the children to record their ideas about the following task, "Which is bigger one half or one fourth? And by how much?" During class they were also asked to confer with other children and write a response to the following: "Is it possible to get a different answer with a different model?" Eighteen students responded; two other children, who did not respond to this task, in addition to one of the group of eighteen, who did respond, submitted writings on the previous task.

The three submissions were all second responses to the task of the comparing of one half and one third; all drew rod models and wrote that one half was bigger than one third by one sixth. One of the children, Jen, extended the task of comparing one half and one third to also comparing one half to two thirds. She stated, "1/2 is larger than 1/3 but it is smaller than 2/3 by one red. Also red represents 1/6."

In the task comparing one half and one fourth, eighteen children responded that one half was bigger than one fourth. Fourteen children drew rod models to illustrate their findings. Two girls, Sami and Betty, chose to represent their solutions with pie charts. Two children wrote their explanations, explaining the rods without illustration.

In answering, “And by how much?”, five children wrote “1/4” and three children wrote “a quarter”. Eileen explained, “1/4 is half as long as 1/2”. Anne drew two models that illustrated the difference as one fourth and two eighths; she labeled the difference as “1/8”. Two children used equivalent fractions in their ideas; Maria used two eighths and one fourth, and George wrote “2/4 makes 1/2”.

The question, “Is it possible to get a different answer with a different model?”, produced various responses from the children. Six children answered yes to the question, five answered no, seven children did not respond. George said different answers as possible “because thier (sic) might be a different model that we never did. Matt suggested, “If someone else builds a model a different way you could have different answers.”

Mario stated, “I agree with the model below (one he had drawn), because it shows what I am trying to explain that 1/2 is bigger than 1/4 by 1/4. This is true for any model.” Ed disagreed that different answers are possible “because no matter what rods you use you will usually get the same answer.”

Six children reported that they had conferred with classmates, other than their partners. Katy comments that she agreed with the “other team’s” answer of 1/4”because if I take one 1/2 & 1/4 out of my model it clearly shows that I nead (sic) one more 1/4 to make 1/2.”

The fourth writing task suggested children record their ideas about the following problem, “Which is bigger one half or two thirds? And by how much?” They were asked to answer the following: “What is the same and what is different about your models?” Twenty three children responded, reporting that two thirds is bigger than one half and illustrating their work with rod models. Bob's writings were in reply to the previous task, Which is bigger, one half or one fourth?" He wrote that he agreed with Dan, because Dan's model "has 1 whole, 2/2, 4/4 like it should be." Bob also commented, "it is possible to get different answer with different models. For example, odd # (model) cannot be divided in 1/2. Even # (model) can be divided in 1/2 (shown with illustrations)."

With three exceptions, the children drew different length models to support their solution. Jen and Benny illustrated their answers, each with one model; Katy illustrated her work with two models of the same length (12 cm). Alex drew rod models, but wrote his explanation in terms of pies, "If you had two pies. You divide one into 3rds and the other into halfs (sic). You take two 3rds out of the pie and 1/2 out of the other. Then you'll see your answer."

Thirteen of the children responded that 2/3 is bigger than 1/2 by one sixth; Maria and Liza added that the comparison was also bigger by two twelfths. Mario noted, "It takes 6/6 to equal a whole and 1/6 is always half of 1/3. It takes three 1/6 to equal 1/2, but you need 4 1/6 to equal 2/3. That proves that 2/3 is bigger also 2/12 is just like saying 1/6."

In responding to the question of what was the same about their multiple models, eleven children responded by noting they have the same answer; Anne called this commonality the "same value". Some children focused on other features of their models: George commented that the models "equaled up to the same amount", Dina said the models "all have white rods"; Alyce noticed that all of the models had "ones, and 1/2's and 1/3's"; Joyce noticed "halfs (sic) and thirds". Ed noted, "Each whole was able to be divided into halfs (sic) thirds and sixths." Dave commented, "they all have the same answers. Most of the models have 1/2 and all of the models we did we had fractions."

Seventeen of the children responded to the difference in their models as being differences of size; Maria noted, "Some have the same colors but different measurements." Ed added, "The third halfs (sic) and sixths were different sizes." Mario noticed, "There is nothing different with my models except the size of my whole, 1/2, 1/3, and 1/6."

CHAPTER V

FINDINGS

The research questions posed in the study concern the fraction ideas that children build before the introduction of definitions and rules, how children represent their thinking, and how mathematical ideas travel in a classroom where discourse is freely encouraged. Findings from the study are classified in relation to the meaningfulness in light of other relevant studies, to the author's theoretical perspective[1] and to the pivotal strands of mathematics explored and developed by the children.[2]

Relevant Studies

The works of R. B. Davis, Dienes, Freudenthal, Maher, Streefland and others (Chapter II)[3] emphasized the need for a different, non-traditional approach to the learning of fractions. Some studies were designed to explore children's thinking about fractions through interviews (see Steffe and Tzur, Watanabe, G. Davis and Hunting) while others focused on classroom research.
For example:

(1) Streefland's study was based on a comparison of children's thinking in differently structured classroom settings, with a goal of recording student achievement on standardized tests through the use of specific texts.

1 See Steencken (1999), Steencken and Maher (1998), Steencken and Maher (2000), Maher, Speiser & Steencken (1999).
[2] Unless otherwise noted, all results of the Colts Neck Project are documented in greater detail in Chapter IV.
[3] Unless otherwise noted, all literature and studies cited are discussed in greater detail in Chapter II.

(2) Huinker's study focused on a four-week project with two fifth-grade teachers in which students explored ways in which to solve fractions-as-number tasks involving four operations.

Reported studies, such as Huinker's, did not cover the length of the Colts Neck Project with a teacher/researcher conducting the investigations in a classroom setting. Streeefland's study paralleled the Colts Neck Project in its focus on children's mathematical creativity; however two important differences combined to make the studies dissimilar. Unlike the Streefland study, the Colts Neck Project was not designed (1) to center on a comparison of groups receiving dissimilar approaches to learning, and (2) to emphasize a formal assessment component.

The Colts Neck Project was designed to establish specific and essential classroom conditions focusing on a student-centered approach to learning. Conditions included:

(1) Expectations that the children are responsible for presenting their ideas and listening to those of others. For example, teacher questioning regularly included the following comments: "How would you convince me?"; "Why don't you talk to your partner?"; and "What do the rest of you think?".

(2) Encouragement to support ideas and justifications, to discuss differences and disagreements over mathematical issues. For example, in Session 4 the teacher solicited the children's comments by asking, "Can we have two different answers?".

(3) Necessary time for children to visit and revisit tasks and ideas. The list of activities presented during a session and the revisiting of the same and similar tasks offered time for children to discuss and revisit their ideas.

(4) Opportunities for children to build generalizations and extensions. For example, the activity of designing a new rod in Session 2 led to generalizations about the set of rods.

Watson, Campbell, and Collis called for more concrete experiences enabling students to build mathematical knowledge. The Colts Neck Project began with a focus on children's explorations of fractions through the use of specific mathematical objects, namely Cuisenaire Rods. Dienes stressed the need for mathematical play; some examples noted and analyzed in the sessions include, but are not limited to the following:

(1) "exploratory-manipulative" play, such as when a child becomes aware of the properties of some objects. Examples: the free play of Session 1, where children were encouraged to spend some time in examining the rods and noting their properties; and the children's attention to the operator sense of fraction in the rods such as exploration of whether the light green is half as long as dark green which occurred in Session 1.

(2) "representational" play, when the objects begin to stand for something they are not and imagination is introduced by the child. Example: the activity, "If I call the orange rod "two", what number name will I give to the yellow rod?" under exploration in Session 2.

(3) "rule-bound" play, when rules are developed – or imposed -- and then used. Example: the children imposed rules in the designing of new sets of rods in Session 2

Piaget cited two essential types of experiences for children to build understanding: physical experiences working on objects (examples include those cited in the discussion above on Dienes' mathematical play) and logico-mathematical experiences. Logico-mathematical experiences occurred when the children used the mathematics their rod models demonstrated. For example, the children began building an understanding of equivalence, in Sessions 4 through 7.

A Theoretical Perspective

R. B. Davis discussed the learner's building new knowledge through the frameworks of previous structures. For example, some children wrote about the explorations with rods, using pie chart representations or, as in the case of one child, drew rod models and wrote her explanation in terms of pie charts.

Davis and Maher referred to the Piagetian view of assimilation paradigms built by the learner during the processing of new data information. Some children built an assimilation paradigm of the candy bar discussed in class and used the reference to develop meaningful attention to the unit under investigation in the activities, as expressed in Sessions 3 and 5.

Maher discussed the input of new data and the process of its internalization by the learner who validates, modifies, or rejects new data on the basis of fit within existing mental representations. For example, in Session 7, Mario stated that two thirds was bigger than one half by one sixth, and not by two twelfths. Mario's writings after the session showed a modification in his thinking: "That proves that 2/3 is bigger also 2/12 is just like saying 1/6".

Within this experience, as well as others, G. Davis' disequilibrium was observed. For example, in the above-mentioned session, Mario emphatically proclaimed "You can't do that" when Maria presented two twelfths as well as one sixth for the difference between the lengths of one half and two thirds. Previously in Session 4, disequilibrium was observed in the explanation of Maria and then Benny. Maria stated that the naming of two rods as one fifth wasn't appropriate because two units were being named, not one. G. Davis stressed the need for reflection during the disequilibrium faced by the learner in order for mathematical knowledge to be built. Whether reflection occurred on an internal or external basis, as described by Piaget, subsequent ideas presented by the children on several occasions marked obvious modifications in their thinking. For example, as noted above, Mario's written work after Session 7, showed an expanding understanding of equivalence; and Maria's and Benny's discussions in class during Session 5 demonstrated their acceptance of one fifth as equal to two tenths.

Learners need time to reinvent in the tradition of Piaget. The essential use of blocks of time was designed into the study with sessions ranging from 60 to 90 minutes in length. Additionally the children were given time to explore and revisit activities under investigation. Art first used two different units for comparing one half and one third in Session 4, using two different units named "one" in the same model. In Session 5 he commented to Joyce, "You just can't switch the candy bar.", signifying his attention to and subsequent use of the same unit in building his model. The revisiting of tasks and exploration of similar tasks (see list of activities in each session) helped to contribute to children's reinvention of ideas.

Pivotal Strands

The findings of this study are organized through a trace of the critical events that highlighted the children's work. In turn, these traces became a collection of "pivotal mathematical strands"[4] in the development of understanding of fraction for the children. Pivotal mathematical strands in this study have been identified as (1) fraction as operator and fraction as number, (2) attention to the naming of the unit or the construction of an assimilation paradigm - the candy bar, (3) fraction comparisons, and (4) equivalence. These strands are not distinct, but actually build upon each other, beginning with the central idea of the sense of fraction as operator and fraction as number. Throughout these strands, some findings arise concerning issues of the role of communication and community.

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[4] See Chapter III for detailed discussion.

The representation of fractional ideas became more precisely expressed, first in natural language, then in the production of a physical model, and finally in notation as the sessions progressed (See samples of the children's written work in Appendices J, K, L, and M). Children talked about "wholes" to express units or parts of units at the beginning of the study. In Session 1, the reference was made to "whole", "three wholes", two more wholes". Gradually the children began to use the more precise language introduced by the teacher. In subsequent sessions, children assigned the number name "one", as well as other specific number names. The children used some of the operations involved in fractions through explanations given in their natural language. For example, in Session 3 the children were exploring number names for a red rod, when a train of a yellow and light green rod was named "one" and named "two". For the train named "two", the children suggested two names for the red rod – "one half" and "one and one fourth". Ed explained the confusion over the two answers and then used addition in a very natural way to justify his solution of one half:

> I think the confusion is, they think, that they think, they have the temptation of calling, since there are four red blocks, they think they are gonna call it one fourth 'cause they forgot that the yellow and the [light] green are two.
>
> Because, see, if you have one there'd be two halves, but if you have two its two halves plus two halves which would be four halves. Therefore, you would have to call one of the reds one half.

Given the opportunity to restate ideas, the children's language became more refined. For example, in Session 4, the school principal asked Ed and Art to explain their models in finding a solution to the comparison task: Which is bigger one half or one third?" Both boys agreed that one half was bigger and Ed offered his explanation:

> One half. Because if you have one whole, and you wanted to divide it into halves, the halves have to be so big that you can only divide it into two parts. And if you want to into

> thirds, they have to be big enough to divide into three parts. So if you only wanted two parts, the whole has to be big enough to divide into two equal parts. So if you have two parts, two is less than three, but if you divide into two parts, they have to be bigger than one third.

The principal asked Art if he followed what Ed was saying. Art responded, "Yeah I have it here." Ed repeated his explanation:

> The number three is bigger than two but if you cut something into two parts, they, technically, if you count by numbers, the smaller (denominator) number is the larger (fraction). Using Cuisenaire rods, if you're cutting into halves and thirds, the smaller the number, the larger the half, the larger the piece.

Children began to develop a common meaning for "different" in terms of "different" models. Realistically, two duplicate models can be labeled distinct and "different" models; for example two trains each built with one orange and one red rod are both 12 cm in length, but can be viewed as distinct and different, since they are not using exactly the same rods. Another understanding of distinct and "different" models develops when comparing two trains of the same length that are not duplicates of one another; for example the aforementioned train of one orange and one red rod compared with a train of one blue rod and one light green rod, both of which are 12 cm in length. The children in this study began to settle on the term "different" model to designate yet another meaning. In the sessions, the children noted one model as "different" from a second model when the length of the first was different from the length of the second, with both models used as examples in exploring a specific mathematical relationship. For example, children built models of different lengths to explain their solutions to comparison problems such as the comparison of 1/2 and 2/3. In this activity, a model could be 6 cm long and another could be 12 cm long; the children referred to these as "different models", while a second 6 cm model would not be referred to as "different". The generalization of 2/3 being always bigger than 1/2 through the use of "different" models was an essential idea under investigation. The children were asked to write about their understanding of what was the same and what was different in these models (See Appendices J, K, L, and M) for samples of the children's writing). The children's subsequent work in the classroom built upon a common understanding of this meaning of "different".

The children employed multiple strategies and techniques in exploring problem tasks while working alone and/or with their partners or classmates, as well as during classroom discussions and presentations. Session 2 examples include:

(1) Dan's use of upper and lower bounds in explaining why there can be no rod half as long as the blue in the given set of rods (class presentation);

(2) Ed's physical search for a rod half as long as the blue rod demonstrates an exhaustive inspection as a proof-by-cases (individual exploration);

(3) Benny's construction of an appropriate model by building on the earlier work of another student, Jon (exploration building on work of another classmate);

(4) Ed's sorting and resorting of the rods into equal subsets of a 13 cm train, while trying to find a rod or train that would be half as long as the original orange and light green train (class discussion).

The teacher/researchers "positioned" the children in the class, often asking them to check on other partnered classmates' models and ideas as part of the idea of "appropriately rigging" (Chapter III, p. 39). For example, T/R 1 called the entire class back to Dan's desk so that they could join in on his explanation of the balance beam (Session 3). When Dan could not give a number name to the white rod in his balance beam model, the teacher suggested that this was something for the class to think about as the class was dismissed. In the following session, other children began experimenting with balance beams; Art and Ed, Betty and Sami, for example. Classmates were given time to name the white rod as part of the next session's activities; Dan decided the white rod would be named "one sixth".

Children sometimes answered a different question than the one that was being asked. For example, in responding to the request for her written understanding of the task, "Which is bigger one half or one third? And by how much?" Joyce drew four versions of the same model of dark green (6 cm) as her "one" first with halves, then with thirds, then with one half removed and lastly, with two thirds removed. Joyce wrote, "1/2 is larger than 1/3 by how much? by 1/4 is a purple. If y 2 – red from dark green, it will equal purple." She has separated 2 red rods leaving one red rod (one third) in her model. She measured the length of the "2 red" rods appropriately as the length of a purple rod, which is 4 cm. Joyce was answering a question about the naming of the difference between one and one third, not the difference between one half and one third.

Classmates also came to assist one another in presenting models and justifying their solutions. For example in Session 6, Jen, Alyce, and Jon came to the front of the classroom to share their solution with the rest of their class by building a model using overhead rods. They had built an appropriate model for the problem that called for a comparison of one fourth and one half. They placed eight white rods onto the overhead to measure the length of their unit, a brown rod (See Session 6, Figure 7, p. 78). In using the white rods to measure the difference between one half (a purple rod) and one fourth (a red rod) they measured the difference with one white rod, one eighth. Maria came to the overhead and demonstrated that there was room for one additional white rod to measure the length of the difference, thus the solution was two eighths.

Children freely expressed their thinking in the classroom[5], allowing mathematical ideas to travel within the community of learners. Not unlike the communication between peoples, which caused the Ancients (and others succeeding them) to listen, examine, modified, and accept or reject the mathematical ideas of others in the building of mercantile communities, similar reactions

[5] Children communicated verbally and non-verbally, as well as with drawings, constructions and written exchanges.

by the children illustrated the fourth-graders' thinking and building of fraction ideas. One such example followed the seventh session in which Mario challenged Maria's notion that two twelfths, as well as one sixth, could represent the difference between one half and two thirds (see Session 7, Figure 13). He insisted that the difference should be one sixth and a lively classroom discussion occurred amongst the children. In a writing assignment he submitted the next day, Mario wrote:

> I think that 2/3 is bigger than 1/2 by 1/6. It takes 6/6 to equal a whole and 1/6 is always half of 1/3. It takes three 1/6 to equal 1/2, but you need 4 1/6 to equal 2/3. That proves that 2/3 is bigger also 2/12 is just like saying 1/6."

In his writing, Mario employed the use of the symbolic notation formally introduced in Session 7. The introduction of writing mathematical statements in the session followed the patterning of the statements and ideas of the children. These patterns of notation are evidenced in the labeling of drawings and notations in written text submitted the children (see Appendices J, K, L and M).

<u>Fraction as Operator, Moving to Fraction as Number</u>

The children began modeling solutions to the fraction as operator activities such as finding a rod half as long as some given rod (Session 1). They found solutions to tasks where the number names of the rods were requested in relation to a given rod named "one" (Session 1 and subsequent sessions). It was at this time that they began incorporating the idea of fraction as number.

Some of the subsequent activities were designed to introduce proportional relationships between tasks. Sometimes these tasks were presented sequentially; sometimes they were presented simultaneously. At times the children themselves posed their own proportional explorations, as Ed did in Session 3. Given the task of finding a number name for the yellow rod when the orange rod was named "two", Ed built upon the fact that that if he could call the orange rod "one", the yellow rod would be "one half"; so if he called the orange rod "two", the yellow rod would be "one".

The children often used different methods to find solutions and often used each other's ideas to assess and/or modify their own thinking. For example, an extended task in proportionality was next posed in Session 3: Finding a number name for the yellow rod when the orange rod was named "six". Katy used the name for the yellow rod from the previous problem and employed the strategy of thinking about "one plus what number equals six", and named the yellow rod "five". Art used a multiplication strategy to find his solution of "one half of six equals three" and named the yellow rod "three". After hearing Art's response, Katy changed her solution, agreed the yellow rod should be named "three", and explained that her solution could not be "five", since five plus five was not equal to six.

Another set of proportional tasks presented simultaneously to the class was given in Session 3: (1) A train of yellow and a light green rod was named "two", what number name would be given to the red rod? (2) A train of a yellow and a light green rod was named "one", what number name would be given to the red rod? In Task 1, the red rod was one fourth as long as the train, and had the number name "one half". In Task 2, the red rod was one fourth as long as the train, and had the number name "one fourth". Two different answers were presented for Task 1, one and one fourth and one half. Dan offered his solution of one half, using the both operator sense of the red rod and the number sense of the naming of the red rod. Other children concurred with Dan. The teacher/researcher's questioning of the children, asking what the difficulty of these problems might have been, elicited an important response from Ed. He explained the confusion when the red rod is viewed as 1/4 the length of the train (the sense of fraction as operator) and the red rod is named one half if the train is called 'two" (the sense of fraction as number).

The children demonstrated the need for time to explore working with fraction ideas and to build a lasting understanding of their discoveries. Evidence of the need for continued exploration presented itself, beginning with Session 1. For every given Cuisenaire rod, the given set does not contain a rod that will be one half as long or one third as long as the given rod. The children's explorations of this property of the rods was demonstrated in activities during Session 1: "Someone told me that the red rod is half as long as the yellow rod. What do you think?"; and "Someone told me the purple rod is half as long as the black rod" In the subsequent discussion by Ed and Art, the boys demonstrated their need to continue explorations. For example, finding a rod that is one third as long as the orange rod (Ed and Art to each other in Session 1), finding a rod that is one half as long as the blue rod (Ed posed to class in Session 2). Another student, Dan, presented a convincing argument to the claim that no rod exists that is half as long as the blue rod. Later in the same session, Art (in a conversation with his partner Ed), hypothesized that he can "half <u>or</u> third" every color rod, "half <u>and</u> third" every color rod. Ed pointed out that they could not halve the blue rod, he stated that halving the blue rod would result in "Four and a half. You can't make a rod that's four and a half." Art agreed, "So you can't divide into anything." Ed countered, "Except thirds." The boys then begin a search through the rods, choosing to look for a rod half as long as the black rod, another example of a rod for which no rod half as long exists within the given set of rods – the black rod is 7cm in length. The thoughtful work of the boys highlights the need for further exploration when first recognition of an idea presents itself. The identification of an idea does not always demonstrate lasting understanding of the meaning behind such recognition.

The children built staircase models with Cuisenaire rods (see Appendix C). In the beginning they were useful as an ordering system for viewing all of the rods in ascending/descending order. In subsequent sessions, mathematical arguments about relationships between given rods were supported with the staircase arrangement. For example, in Session 2, Dan modeled the staircase and presented an argument using the upper and lower bounds of the rods. He also used them to discuss the odd and even lengths of the rods by noticing the centimeter lengths of each rod. Other children used the property of odd and even throughout early discussions of the rods.[6]

Children negotiated the meaning of mathematical terms. For example, Ed's suggested an activity in Session 2, finding a rod that is half as long as the blue rod, created the conditions that led to a class discussion on the two conditions necessary for "halfness": (1) two pieces equal to the length of the unit one, and (2) two pieces be equal. The classmates also built generalities as they worked with the rods. For example, the class was asked to design a new set of rods that would include a rod half as long as the blue rod. After several solutions were offered, Mario, and then other members of the class, generalized that the process of new sets of rods would repeat itself infinitely.

In Session 2, Ed offered a purple rod and a yellow rod as "not exactly the same, but they're both halves" (of the blue rod). In the Session 1, he observed two relationships focusing on the blue rod: the light green rod was one third as long as the blue rod, and the purple rod was not one half as long as the blue rod. The videotape showed Ed, while assuring his classmates that there had to be a rod half as long as blue, sorting through the rods, establishing a proof by cases for the non-existence of any such rod. He methodically chose two of each rod and held them up against the blue rod, measuring the appropriateness of their lengths[7]

Tasks under investigation focused on the lengths of the rods; surprisingly, some of the children's investigations focused on another attribute as well. A few children built some sort of

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[6] See Chapter IV, Session 2 for an example given by Maria.

[7] Although the use of 3 cameras and researcher field notes was invaluable in recording Erik's ideas, it was not always the case that all data was captured. For example, in Session 4, the school principal discussed Caitlin's ideas concerning proportionality of her models. This was never found recorded on any camera view or field notes.

balance models with the rods, focusing on the balance between the length and weight of the rods. One student, Dan chose to build a model and present his explanation using a balance beam made from the rods. (See Session 3, Figure 3.)

Children posed their own activities, as they were asked to do in Sessions 1 and 2. Sometimes children posed or extended activities without prompting. Previously mentioned, Ed posed a proportional task to the one given by the teacher. Children also extended tasks with their partners. For example, in Session 4, the class is asked to find a number name for the yellow rod, if the orange rod was called fifty. Dan told his partner Maria that the yellow rod would be named "twenty five". He then added, "If this was one hundred (orange rod), those would be fifty (yellow rods), two of these would equal one hundred."

An Assimilation Paradigm: The Candy Bar

During explorations in Session 1, children gave various names to the lengths of the rods. The teacher/researcher asked how the red rod and the light green rod could be given the same name – one third. Mario answered, "Cause there is a different size whole". With Mario's statement the issue of "attention to the unit", to what they would call the "whole", and later "one" was brought into the classroom forum. This idea was reinforced with the image of a shared candy bar. The candy bar was introduced in two different ways to the children: (1) In comparing candy bars of different sizes (Session 3), the children were exploring the unit, and (2) In naming the shared parts of half of a single candy bar (Session 5), the children were additionally looking at equivalent names for one half of the unit.

Children built different models for their solutions, with notable conversations resulting. In Session 5 when children built two models for the same task, Jen remarked, "Oh I think they are making a different sized candy bar". In Session 6, Jen built a 12 cm model and Art built a 6 cm model. She challenged Art's model, saying that he had used a different sized candy bar. In doing so, Jen had once again employed the candy bar as an assimilation paradigm. Joyce interjected saying: "There can be candy bars of different sizes;" Art added, "You just can't switch the candy bar." In the composition Jen submitted after the seventh session, she was asked to comment on different models used in the task, "Which is bigger, one half or two thirds? And by how much?" She wrote, " I think 2/3rds are bigger by 1/6. They both are different models and they have different sized rods. They both have the same answer."

In Session 4, Art had built a model with two different sized units (p. 67-69). His partner, Ed, explained to Art and the principal how his own thinking helped him to use the same unit, although neither boy referred to the candy bar at that time.

<u>Comparing Common Fractions</u>

Comparison activities were structured to begin by asking a question, first in comparing unit fractions, then to comparing non-unit fractions with unit fractions (in the seven sessions reported here). The question was asked, "Which is bigger"? Children were requested to build models of their solutions and explain their answers. Initially the question, "And by how much?", was not asked until the children decided which was bigger; this custom was soon replaced by asking both questions at the same time.

Before the children began to work on the first unit fraction comparison problem in Session 3, the teacher/researcher asked for the "common understanding" of the problem. Mario explained the importance of focusing on a unit for the comparison, the need to focus on one half and one third of the <u>same</u> sized unit. The children explored this task as part of the activities in Sessions 3, 4, and 5. The ideas were revisited and became more clearly articulated by children over these sessions. Each visiting of the activity brought a building of ideas which would follow through this class of tasks- the comparison of fractions. In Session 4, the children were asked to find by how much one half was larger than one third. Some children offered that one half was bigger than one third by the naming of a particular rod's color – for example in a 6 cm model the red rod represented the difference.

The naming of this rod in relation to the models they built set the stage for how they would develop names for rods in subsequent sessions. In offering names for this rod in Session 5, two solutions were given – one third and one sixth. The inappropriate response of one third was triggered by Jen's placing the red rod in relation to the rod she had named one half and calling it "one" – she had switched candy bars (Session 5). Some children thought both answers were appropriate, some did not. Ed used a sense-making approach to the solution, how could one half be bigger than one third by one third that would mean that adding one third and one third to two thirds would equal one half and that was not possible. The teacher ended the session with a request that the children think about the two answers that were posed[8]. In Session 6, Jen presented a 12 cm model and Art presented a 6 cm model in which one half was bigger than one third by one sixth. There was no discussion of the solution of one third, there was the previously mentioned discussion of the different sized models and then the children began exploring another unit fraction activity. In the next activity, the comparison of 1/2 and 1/4, there was general consensus that the answers of both 1/4 and 2/8 were valid.

In the next comparison activity in Session 7, the children explored a non-unit and unit fraction comparison, 2/3 and 1/2, for the first time in the sessions. Again they were asked which was bigger and by how much. The children were asked to build more than one model to support

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[8] This was one of many examples of "appropriate rigging" (See Chapter III, p. 39).

their solutions. In the building of these multiple models for a given comparison problem, children began to see an important generalization about the solutions they found. Mario commented in his writing task, "I agree with the model below (one he had drawn), because it shows what I am trying to explain that 1/2 is bigger than 1/4 by 1/4. This is true for any model."

Equivalence

The pivotal strand of equivalence was first explored in Session 4. The children were asked to find a number name for two white rods, when the orange rod was named one. Some children declared the name of two white rods as one fifth, some stated the name as two tenths. The teacher asked the class if it were possible for the two white rods to have both number names. Maria repeated that the two white rods should be called two tenths, and said: "There's only two of them. They're not joined together. If you wanted to join them together you should use a red." Maria and three of her classmates stated that they viewed the rods as discrete objects and in doing so, could not count them as anything except two of the objects called tenths.

In Session 5, the teacher presented on the overhead the following problem: *Does 1/5 = 2/10?* Maria built a model to explain her solution, one fifth is equal to the length of two tenths. Eric repeated Maria's explanation to the rest of the class and Bob followed with a similar explanation. The class concurred. Then the children were asked to give number names to one-half of a scored candy bar. Various answers were suggested and explained: "six twelfths", "two fourths", and "three sixths". The teacher wrote on the overhead "1/5 = 2/10 and ½ = 6/12 and 2/4 = 3/6".

In Session 6, in the next comparison activity, Maria offers the solution of 1/4 and 2/8 as the same for comparing the difference between 1/4 and 1/2. The class concurred.

In Session 7, another comparison task was presented, "Which is bigger, two thirds or one half, and by how much?" When Maria suggested two answers, 1/6 and 2/12, Mario commented that the solution of 2/12 was not acceptable. A heated classroom discussion evolved over the two answers; some children thought they were different solutions, some saw them as the same. As the conversation continued, the teacher recorded their language symbolically (Session 7). Ed saw the two answers as the same, stating that you could name the difference with either response, but it "would be easier" to use 1/6. At the end of class she asked the children to think more about the problems and to write about their solutions using several models to support heir reasoning. They were also asked to write about what was alike and what was different about their models (See Appendices J, K, L, and M).

The findings include evidence of the children's problem solving and model building strategies. They explored particular fraction ideas concerning an attention to the unit, equivalence, and the comparison of fractions – all building upon explorations involving fraction as operator and fraction as number. Children used the models built in all seven sessions to illustrate their solutions and to share their thinking with classmates and the teacher. Children drew pictures of their models coding and labeling the images, oftentimes on overhead transparencies. They reasoned from the written models in presenting their explanations and justifications to the class. As a result, invaluable and often lively conversations emerged as ideas and opinions moved between members of the class. The children's understanding grew in a non-linear, non-uniform progression as they explored, conjectured, and began to challenge each other's ideas. This flow of ideas allowed the children to build important meanings as they develop their own understanding of fraction concepts.

In later sessions the children moved to more complicated comparison problems, representing fractions on a number line, working with the operations of addition, subtraction, multiplication and division of fractions. They used the tool of the Cuisenaire rods less often, but referred to many of their ideas in terms of the rods. As the children explored these ideas, they moved into working with positive and negative numbers.

An interesting finding, not in the research design of the study, was noted:

As recorded in Chapter II, the development of fraction systems followed the building of mercantile communities, first nationally and then internationally. In earlier times, mathematical ideas were discussed evaluated, modified and adopted by community members. An interesting finding is that the children's development of mathematical understanding of fraction ideas. Striking similarities to that of the ancients' thinking of fractions was observed in that of the children in two ways. One for example, can be found in Benny. In his explanation of naming two white rods one fifth or two tenths in Session 3, he voiced the same argument that Wheat attributes to the reasoning of the Egyptians. As Benny viewed the two pieces, each representing one tenth, the Egyptians, according to Wheat's example, would have similarly viewed 3/4 as 1/4 + 1/4 + 1/4 or 1/4 + 1/2.

The second instance can be found in the discussions focusing on the size and ultimate naming of the unit, of what the children were calling the "whole" and later "one". The children's conversations played a significant role in the building of their understanding. This "attention to the unit" formed a necessary part of the children's negotiations of meaning attached to an important mathematical concept. The same attention was evidently a topic of discussion in ancient times as Plato tried to settle the argument from his, as well as other Greek mathematicians', perspectives on the size and naming of the unit.

Implications

Davis, Freudenthal, Maher, Streefland, and others have cited the disconnection between children's concrete mathematical experiences involving fractions and the rules and techniques surrounding the operations associated with fractions. The findings of this study document evidence of children thoughtfully building a sense of fraction as number in natural ways, bereft of imposed definitions and rigid rules. The research team had set specific classroom conditions, creating an environment in which the children became active participants in their own learning. The team had no set expectations of what these children might do and how far their thinking would take them, or what effect the absence of grading might have on participation by the students. The children never ceased to surprise and amaze all of those connected with the project with both their enthusiasm for participating in the sessions and the quality and depth of their thinking.

The fourth graders eagerly investigated activities and presented powerful explanations and justifications for their findings. The research team challenged the thinking of the children, to investigate more deeply, to think more profoundly and to refine their ideas; the children met and/or exceeded such challenges. In doing so, the children consistently met the Process Standards outlined in 2000 in the National Council of Teachers of Mathematics (NCTM) Principles and Standards. The process standards include "problem solving, reasoning and proof, communication, connections and representations" for pre-kindergarten through grade 12 students (p. 402).

Problem Solving

The children became "problem solvers", demonstrating the ability to build new knowledge, to apply appropriate strategies, and to monitor and reflect on the processes of their own thinking, and that of others.

Reasoning and Proof

The children demonstrated their reasoning and development of ideas leading to proof by recognizing and presenting conjectures, explanations and justifications for mathematical ideas. They listened, reflected and reacted to the ideas of others, evaluating and developing their own mathematical arguments to support views and opinions

Communication

The children communicated freely and thoughtfully. They did so through verbal language as well as non-verbal gestures, models and written work. They developed meaning for mathematical terms and use of precise mathematical language.

Connections

The children connected previous understanding of fraction as operator and moved to a sense of fraction as number in natural ways. They interconnected representations and mathematical operations associated with fractions building important mathematical ideas such as equivalence and the ability to make accurate comparisons between fractions.

Representation

The children created and used models to organize, record, and communicate their findings. The study documented the children's work with models as well as their verbal and non-verbal language, drawings and written work (see Chapter III), recorded both on paper and overhead transparencies. In their labeled drawings and written statements, the children adopted symbolic notations for their ideas. The children responded, not only the ideas presented by their representations, but also to those of other class members.

The achievement of these students is significant in light of the goals of NCTM. NCTM's Principles and Standards offers achievement goals for children in grades 3-5: "all children in grades 3-5 should develop and use strategies to estimate computations involving fractions and decimals in situations relevant to students' experiences" and "use visual models, benchmarks and equivalent forms to and subtract commonly used fractions and decimals seems attainable". Additionally, the achievement goals in grades 6-8 indicate that "all students should understand the meaning and effects of arithmetic operations with fractions, decimals and integers" (p. 391-402). Classroom conditions established in this fourth grade provided ample opportunities for children to develop and exceed NCTM's standards. The success of the children in the study further indicates that the goals of NCTM are reasonable and attainable. In fact, this success may be regarded as an existence proof for NCTM's objectives.

The data show the powerful thinking of young children about fraction ideas. Also, they indicate the appropriateness of the investigations as problem-solving tasks out of which children can examine their ideas and those of others, and offer explanations and justifications for their thinking. The classmates developed abilities to present and share their ideas as well as to question and challenge each other. Indeed, they worked as a community to explore, to build and to revisit fundamental and important mathematics. The findings found in this study have broader implications for the teaching of other mathematical concepts. Future studies may provide more insight into the appropriateness of these and similar activities for young children before their exposure to rigid language in the labeling of mathematical ideas as well as to particular rules and procedures to solve certain classes of problems.

The rich data from the Colts Neck Project continues as an ongoing research opportunity under investigation. Data from the next sessions suggests the development of powerful mathematical ideas by the children. The work that began this study will continue by this researcher and others who wish to follow this development as well as parallel studies in other settings.

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APPENDIX A

Selected List of Citations

from the

Twelve Year Study

Selected List of Citations
from the
The Twelve Year Study

Davis, R.B. & Maher, C. A. (1990). What do we do when we do mathematics. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), Constructivist Views on the Teaching and Learning of Mathematics: Journal for Research in Mathematics Education Monograph No. 4 (pp. 65-78). National Council of Teachers of Mathematics. Virginia:Reston.

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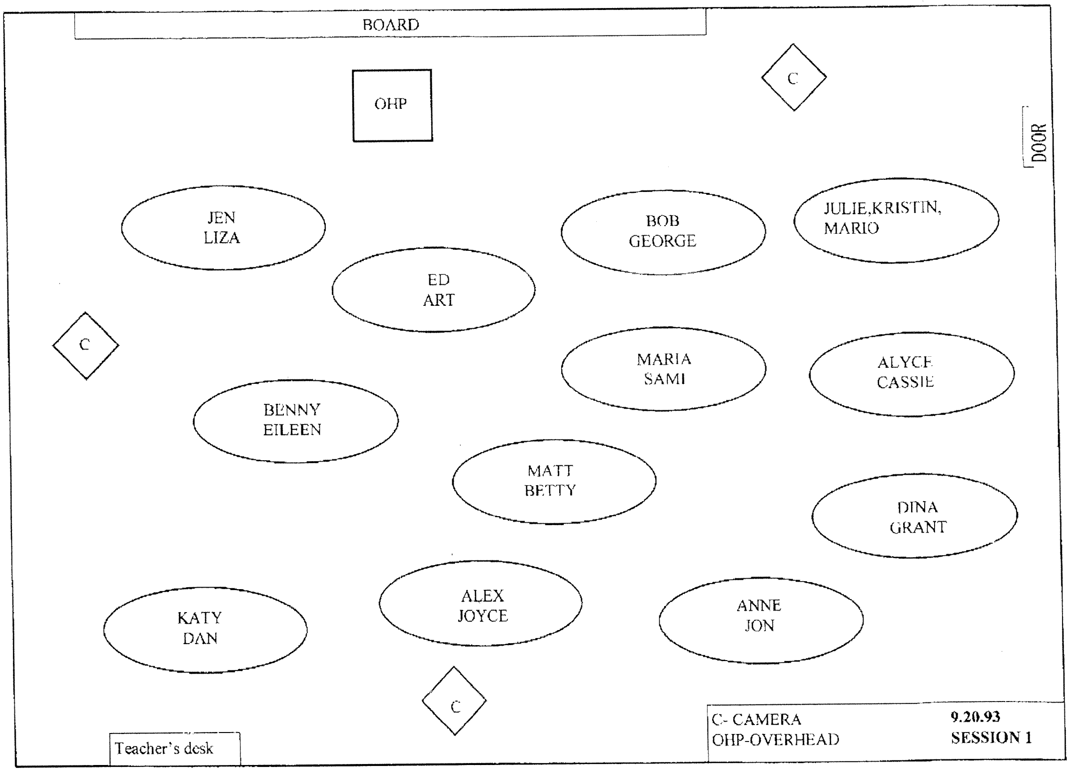
APPENDIX B

Sample Room Charts

Sessions 1, 2, 3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ALYCE
CASSIE
BENNY
EILEEN
MATT
BETTY
DINA
GRANT
ALEX
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DAN
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Teacher's desk
C- CAMERA
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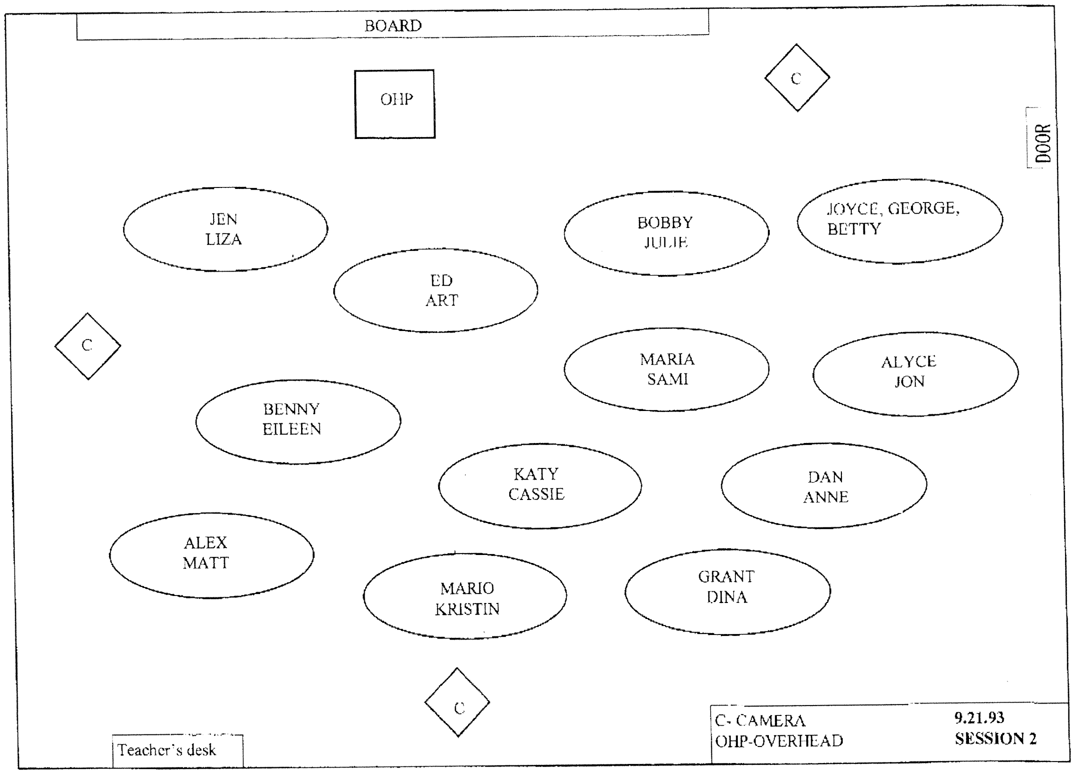
SESSION 1

Session 1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ALYCE
JON
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EILEEN
KATY
CASSIE
DAN
ANNE
ALEX
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DINA
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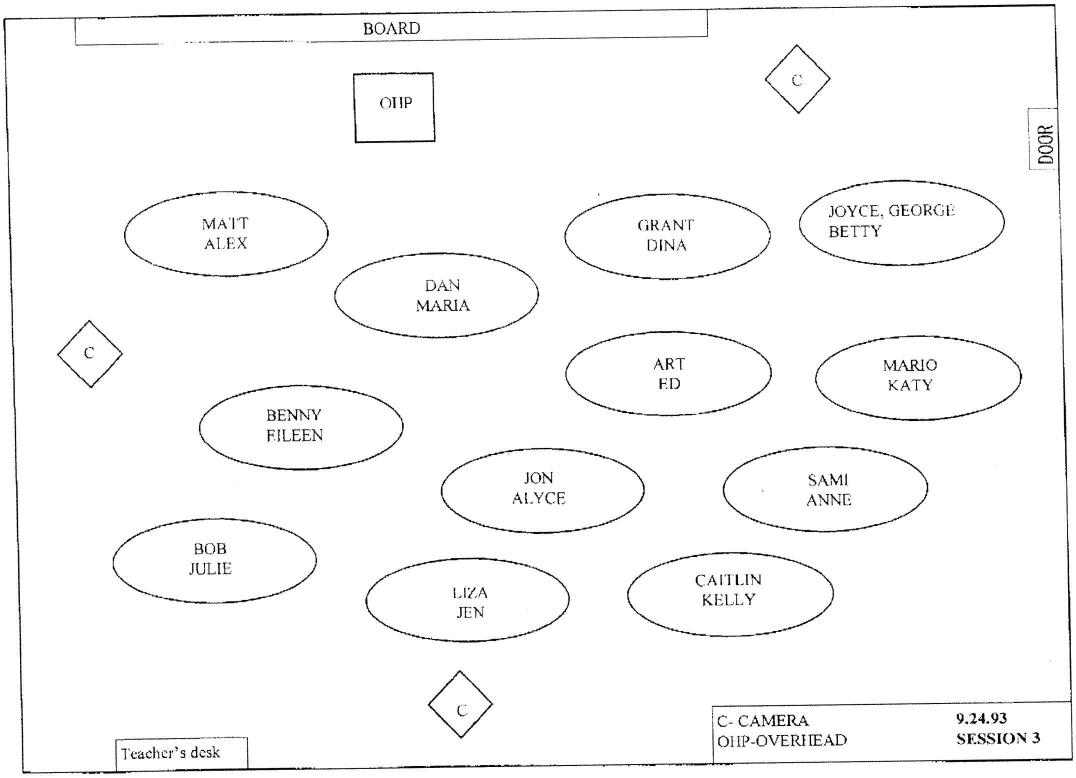
SESSION 2

Session 2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KATY
BENNY
EILEEN
JON
ALYCE
SAMI
ANNE
BOB
JULIE
LIZA
JEN
CAITLIN
KELLY
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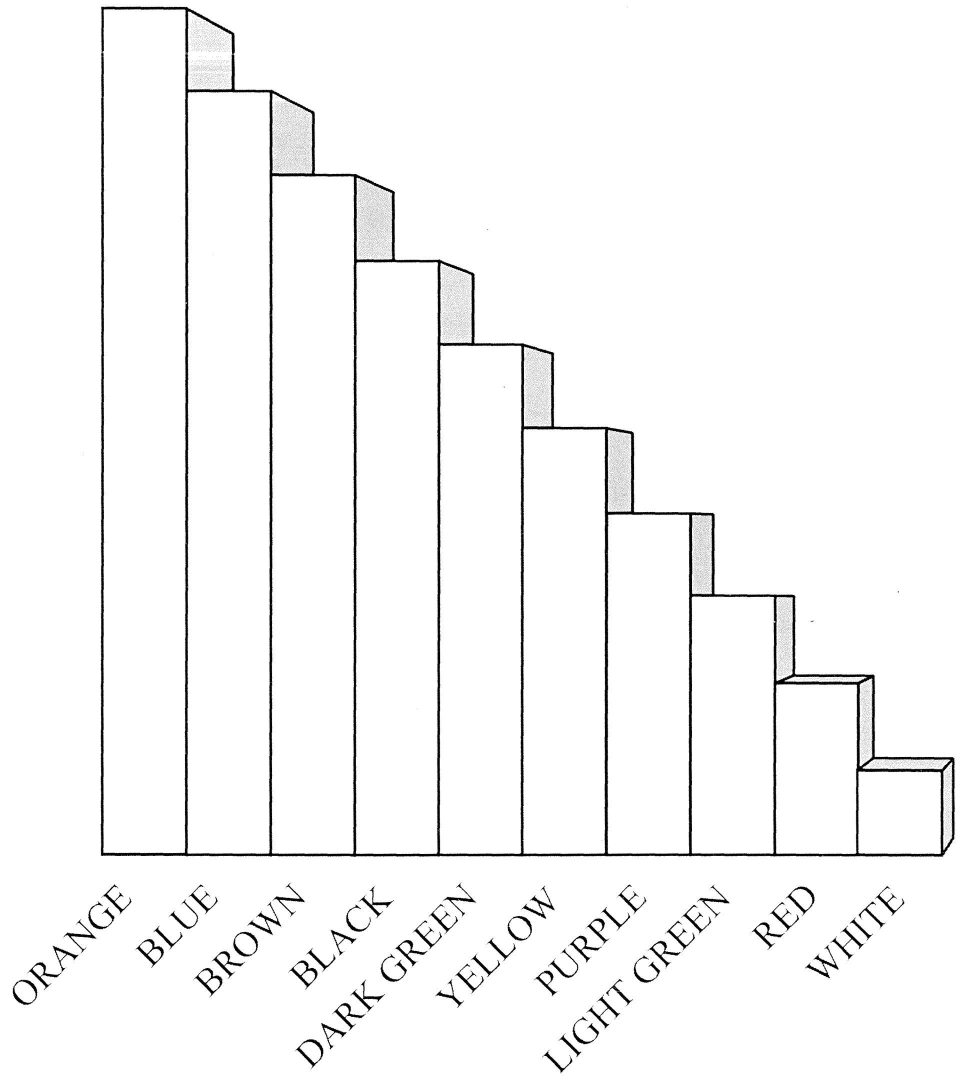
SESSION 3

Session 3

APPENDIX C

Staircase Model

Of Cuisenaire Rods



Staircase Model of Cuisenaire Rods

APPENDIX D

Coded Transcription

Session 1

Session 1
Side, front cameras - OHP camera unmanned
T/R1 leads
T/R 2 and CT present

Small group: Julie, Kristin, Mario
Partners: Ed and Art, Sami and Maria, Betty and Matt, Alyce and Cassie, Dina and George, Anne and Jon, Alex and Joyce, Bob and George, Jen and Liza, Benny and Eileen, Katy and Dan.

Activities:

1. I claim the light green rod is half as long as the dark green rod. What do you think? What would you do to convince me?
2. What number name would we give to light green rod if I called the dark green rod one?
3. Someone told me that the red rod is half as long as the yellow rod. What do you think?
4. Someone told me that the purple rod is half as long as the black rod. What do you think?
5. Someone told me that the red rod is one third as long as the dark green rod. What do you think?
6. If I call the dark green rod one, what number name would I give to the red rod?
7. Someone told me that the light green rod is one third as long as the blue rod. What do you think?
8. If I call the blue rod one, what number name would I give to light green?
9. What number name would I have to give the dark green rod if I wanted red to be one?
10. If I call the brown rod one, what number name would I give to red?
11. If I call the red rod one, what number name will I give to brown?
12. I want to call the white rod one half. What rod will I call one?

T/R1 encouraged students to pose and solve their own tasks. The following were created by the children and T/R 1 and T/R 2:

1. If the red rod is considered one fifth, what would the orange rod be? [Art]
2. If light green is one whole, what is blue? [Betty]
3. If blue is one, what is light green? [Matt]
4. If white is one, what is orange? [Julie and Kristin]
5. If orange is one, what is white? [T/R 1]
6. If purple is one half, what is one? [Maria]
7. If light green was one half, what would be a whole? [Ed]
8. If white is considered one fifth, what would one be? [Art]
9. If I call purple two, what would one look like? [T/R 2]
10. If white is three, what is six? [Ed]
11. If I call the white rod one, what (rod) would you call seven? [Maria]
12. If red is one third, what (rod) would be one? [Sami]
13. Find a rod whose number name is one sixth. [T/R 1]
14. If I want green to be six, what would white be?

The session began with T/R 2 asking children what activities they remembered from past Rutgers visits with them. [About half of the class had been visited while they were third graders.] Bob remembered Pizza problems, Ed - the Hat Problem, Maria – Towers, Betty – estimating beans in a jar. The children were asked if they knew anything about the rods that were on their desks. Some children said they had used rods before for math - in multiplication, building trains, staircases. T/R 1 began:

Time	Code	Speaker	Text
10:00	**ASs**	T/R1:	[Introducing the task, speaking slowly, repeating question several times] I claim that the light green rod is half as long as the dark green rod. What do you think? How many think that's true? What would you do to convince me? What would you do to convince me that's true? Do you want to think about that for a minute with your partner? Alex has already decided how to convince me. I think Cassie also has decided. And Bob and George. [Approx. 1 min. given to class as children raise their hands when ready.] Okay, Eileen, you've never done this before.
	Rm	Eileen:	It's true. [She puts two light green rods next to the dark green rod.]
			Take two light green rods and put them together next to dark green.
12:00	**ASs**	T/R1:	What number name would we give the light green rod if I called the dark green rod one? What number name would we give the light green rod? Talk to your partner and see if you agree.
	Rlv:	Maria:	[to Sami] One half.
		T/R1:	Someone told me... Joyce?
	Mr	Joyce:	Umm, yeah. Light green is half of dark green.
		T/R1:	Joyce thinks so. How many of you agree with Joyce? [Most of the students raise their hands.] What would you do to convince me? You want to come up here and convince me, Joyce, on the overhead? [Joyce goes to the overhead.] This would be half of the green rod? You
	Tmd		all think about that for a minute. If I called the dark green rod one, what number name would I give the light green rod? Why don't you talk to your partner and see if you agree. [T/R 1 walks around the room, talks to children, attempting to see who agrees.]
		T/R1:	Art and Ed, do you agree?
			[Both boys mutter in agreement.]

Time	Code	Speaker	Text
			Did you talk? [Boys nod in agreement.] Whisper.
	Rlv	Art:	[Whispering] One half.
	Tqc	T/R1:	[Addressing whole class] Is there anyone who disagrees? So if I call the dark green rod one, I would call the light green rod one half? Okay, that's interesting.
14:00	**ASs**		Someone told me, someone told me that the red rod is half as long as the yellow rod. What do you think?
		Ed:	Which red rods?
		Art:	These little ones. [He is holding up red rods to show Ed.]
		T/R1:	Someone told me that the red rod is half as long as the yellow rod. What do you think?
	Rlv	Maria:	No.
	Rlv	Sami:	No.
	ASn	Art:	[To Ed and to himself] No. Look. [He holds up the red rods to show Ed.]
	ASn	Ed:	Nope.
		Art:	There is enough to fit more here. There is enough to fit another light green in here.
		T/R1:	Dina, what do you think?
		Dina:	No.
		T/R1:	Dina thinks no. What can you do to convince me that that's not true? Can you speak nice and loud?
	Mr	Dan:	Two red rods don't fit. You need to put more.
15:00	**ASn**	T/R1:	Someone told me that the purple rod
		Art:	[He hold up the purple rod] This one
		T/R1:	Is half as long as the black rod.
	Rlv	Maria:	[with Sami and T/R2] No.

Time	Code	Speaker	Text
	Rm		Look. Two purples are too large.
	ASn	Ed:	No.
		T/R1:	What do you think?
		Ed:	No.
		[T/R 1 approaches the two boys.]	
		Art:	Nope.
		T/R1:	[To Ed] The black rod.
		Ed:	Oh, the black rod. [He puts back the blue rod, which he has used by mistake; he takes out a black rod.]
	CP,**Mr**	Art:	It would take another light green to make a whole and that's not half. [He is holding up the black rod with the purple rod in one hand and with his other hand, he takes the light green and puts it together with the purple rod to show that the train of purple and light green is equal in length to the black rod.]
	CP	Ed:	Yeah, it is, look. [Ed puts two purple rods in a train next to the black rod.]
	CP	Art:	That is not as long as the black, it would take another light green one.
	CP	Ed:	Oh.
		T/R1:	[To the class] What do you think? Dan?
	Rm,Iv	Dan: No, two purples are too large.	
17:00	**ASs**	T/R1:	Can you find the dark green? Are you ready for this one? Someone told me, that the red rod is one third as long as the dark green rod. What do you think?
	RIv	Ed:	Yep.
	RIv	Art:	Yep.
		Ed:	Umm.

Time	Code	Speaker	Text
		Art:	Umm. Cause two of these makes
	Tmd	T/R1:	Discuss it with your partner.
		Ed:	Yeah, I think so.
		Art:	Umm, cause if you did it like this
	Mr		[Alex has the dark green rod on the table and is putting three red rods; Ed points to the red rod in his staircase]
		Jackie:	That the red rod is one third as large, as long as the dark green.
		T/R1:	What do you think, guys? Mario?
	Rm	Mic:	You put the red rods, um, like below the green rod.
		T/R1:	How many did that? Okay, what a smart class. You sure everyone didn't work with this last year? What a smart class. That's lovely, thank you, Joyce. [Joyce has put the dark green rod with three red rods on the overhead.] [On the overhead there is a dark green rod with three red rods below it. T/R 1 takes one red rod away.
	ASs	T/R1:	If I had to give another name, a number name for the red rod, if I called the dark green rod one, what would I call the red rod? What number name would I give to it? If I called the dark green rod one, what number name would I give to the red rod? [Most of the students have their hands raised.] How many of you think you know a number name, for the red rod?
	Tmd		Would you talk to your partner a little bit and see what you think?
	Ct	SS:	We already did.
		T/R1:	You did already? [Most hands are still raised.]
		T/R1:	Betty, you want to tell us?
	Rlv	Betty:	One third.
	Rm	Sami:	[to T/R2] There are three of them. [She points to the red rods under the dark green rod.] So there is a third.

Time	Code	Speaker	Text
	Tqc	T/R1:	How many think one third? [All hands go up.] You all agree. Can you tell me why you would give it the number name one third?
	Mr	Betty:	Because if you put three on them it makes one whole.
		T/R1:	Okay, so if it makes three, it would make one whole. Do you agree with that?
		Betty:	Umm.
20:00	**ASs**	T/R1:	So we give it the number name one third. Okay, very good. Someone told me that light green is one third as long as blue. What do you think? Someone told me that light green is one third as long as blue. What do you think? What do you think, Jen?
		Jen:	It is.
	Tqs	T/R1:	Jen thinks it is. And how would you convince me, Jen?
	Mr	Jess:	Because you could, if you have, you have three of these you could put it up to the blue, and it's one whole.
	ASs	T/R1:	So if I call the blue rod one, what number name would I give to light green, everybody?
	Ct	SS:	One third.
	Tqc	T/R1:	I would give it the number name one third. Now notice, if I called the dark green rod one what number name would I give to the red?
	Ct	SS:	One third.
	Tqal	T/R1:	So are the number names always the same?
	Rlv	SS:	[Tentatively] No.
	Tqal	T/R1:	Are the color names always the same?
	Ct	SS:	No.
	Tqal	T/R1:	Does this have another name other than blue? Am I ever going to call this something other than blue?
		SS:	No, oh yeah.

Time	Code	Speaker	Text
		T/R1:	What am I going to call it?
	Ct	Ed:	Dark blue.
		T/R1:	Well, dark blue. [The question is again raised.]
	Rlv	Ed:	Color names, no.
	Tqc	T/R1:	Ed thinks that the color names don't change. Right, do we agree that the color names are always the same?
		SS:	Umm.
	Tqal	T/R1:	Do the number names change?
	Ct	SS:	Yes.
	Tqc	T/R1:	Okay, tell me why the number names change. Give me an example of why the number names change. Ed?
	Rlv	Ed:	You could think, you could say that, you could take an orange block and a blue block and that would be, that would be three thirds of it.
	Tqs	T/R1:	Is it?
	Rlv	Ed:	Wait. No, wait, hold on. Would, no, I mean, if you ta, if you take a blue rod and you could call it one whole and you would take an, a diff, a smaller rod and
	Tqs	T/R1:	Which one?
	ASs	Ed:	Um, well, oh, you take a light green, and that'll be a third of it.
	Tqc	T/R1:	Okay, so I called the light green one third and I called the red one third. Why could the light green be a third and why could the red be a third? How is that possible for both of these to be a third?
		Ed:	[Raising his hand] Oh, oh, I know.
		T/R1:	Mario?
	CE, Me	Mic:	Cause there is a different size whole.

Time	Code	Speaker	Text
		T/R1:	Because there is a different size whole. There is a different number name for what I call one, okay?
24:00			Look at color names, see what you find. Then we'll do some problems with number names. [Many students build staircases.]
25:00			How many think there are ten (colors)? Many of you have built similar models, Kristin? Tell us what you did.
	Rm, Iv	Kristin:	[Describing the staircase that she built] I started with orange, then blue, then, brown, black, green, yellow, pinkish
		T/R1:	We call that one purple.
		Kristin:	Purple, light green red, and white.
28:00	**Tqc**	T/R1:	Okay, so we are starting again. Dark green has the number name one. Three red equal green, why can't we call red one? Some students think so. Mario?
	RIv,	Mich:	Because two whites equal a red.
		T/R1:	But you said if dark green is called one, red is called one third, you said so.
		Mich:	Oh, yeah.
	Tqc	T/R1:	Can I call red one?
	Mr	Ed:	No. You're comparing red to dark green.
	Mr	Mich:	Red can't be one. Green is bigger than red.
	ASs	T/R1:	If I call red one, what number name shall I give dark green? What number name would I have to give [dark] green if I wanted red to be one? [Pause as students work on this task.]
	Tmd		You want to talk to your partner for a minute?
	CP	Ed:	[To Art] three wholes?
	CP	Art:	Can't give it a name because it can't be put into, into two. Because look, [Art points to the dark green rod, which is part of a staircase on his desk.] I don't know.
	Mr	Ed:	But the dark green is bigger.

Time	Code	Speaker	Text
		T/R1:	Okay, suppose I want the red to be one, what number name would I have to give dark green for red to be one?
	Mr	Ed:	[To Art] Three wh, three wholes. Because if this is a one [He holds up a red rod]
		T/R1:	[She comes to their desks.] Hi, Ed. What Ed?
	Mr	Ed:	[Still to Art] It would be three wholes. Because if this is a one [He is holding up a red rod.]
		T/R1:	Right.
		Ed:	It takes three to figure this out. [He takes three red rods and puts the next to the dark green rod.] And if all the reds are one.
		T/R1:	So dark green would have to be three.
		Ed:	And there are three, that'll be three wholes.
	Tqs	T/R1:	Do you agree, Art?
		Art:	[Nodding] Umm.
	Tqs	T/R1:	Does that make sense, Art?
		Art:	Umm, yes, I guess so.
		T/R1:	okay.
	CPb	Art:	Three wholes [Art nods to Ed.]
		T/R1:	Okay, are you ready to argue that, Ed?
		Ed:	Yeah.
30:00		T/R1:	Okay. [to the class] Some students say three, some say one third. [Bob?] There are two different answers. I would like to hear both answers. Ah, let's see, who wants to give me an answer? [Ed and Art raise their hands.]
	Tqe		I heard, I heard a couple of different answers. Only this table wants to give me an answer? heard a different one. Okay, Art?
	Rlv	Art:	Three wholes.

Time	Code	Speaker	Text
		T/R1:	Nice and loud, Art.
		Art:	Three wholes.
	Tqc	T/R1:	Do you want to tell me why you think so? Are you all hearing what Art says?
		Ed:	I know. I know why.
		Art:	Okay, because the, the
		T/R1:	Ah, you need to talk much louder.
	Mr	Art:	Okay, if the red is considered one [He points to the red rod in Ed's model] then the green one is a lot bigger. So it would have to be, it would take three whole ones to make another green so it should be considered three wholes.
	Me	Ed:	[Continuing] Well, I think, well, if you, if you say that this would be one [He holds up towards the teacher one red rod]. This is one, and it takes three of the one, the one wholes to equal up one of these [He points to a dark green rod on his desk]. And it that's one whole, umm, one whole plus one whole plus one whole would equal three wholes. So the green would have to be three wholes.
		T/R1:	Does that make any sense? Do you understand what Ed is saying? What do you think, Dan?
	Me	Dave:	I thought the same thing. If red is one, green would have to be two more wholes.
33:00	**ASs**	T/R1:	If I call the brown rod one, what number name would I give to red? Dina?
	Rlv	Dan:	One fourth.
	Tqs	T/R1:	What would you do to convince me?
	Mr	Dan:	Put four red onto the brown.
35:00	**ASs**	T/R1:	Now I want to call the red rod one, what name would I give to brown? Julie?
	Rlv	Julie:	Four.

Time	Code	Speaker	Text
	Tqs	T/R1:	What would you do to convince me?
	Mr	Julie:	you line them up. They add up to four wholes
	ASs	T/R1:	I want to call the white rod one half. What rod will I call one?
	Tmd	Talk to your partner. Liza?	
	Rlv	Liza:	Red.
	Tmd	T/R1:	How many agree? What would you do to convince me?
37:00	**Tfp**	T/R1:	When your partners have made up a problem for me and the rest of the class. When you think you have one, be careful how you are going to ask it. Practice how you are going to ask the problem and then raise your hand.
	CP,Rm	Ed:	[Ed puts five red rods next to an orange rod.] Ha, ha, ha, hm.
		Art:	No, that's one fifth.
		Ed:	I know
	CP	Art:	Oh, yeah. [He starts to set up Ed's problem.] We are out of reds. Oh, well.
	Tmd	T/R1:	Try to get a hard one and try to stump us. [Art raises his hand.]
		Ed:	Yes, I got it. [He puts two purple rods next to the orange rod.]
		Art:	No, those won't make it.
	CP	Ed:	What makes thirds?
	Mr	Art:	Thirds, thirds out of a, thirds out of this? [He is pointing to an orange rod.] Probably the greens.
		Ed:	Light green,
		Art:	Light green would make thirds out of the orange. [Art puts light green rods next to the orange rod.]
		Ed:	Yeah.
	CP	Art:	No, it wouldn't.

Time	Code	Speaker	Text
	CP	Ed:	Yeah, it would.
	CP	Art:	No, it doesn't. Try it.
		Ed:	Then what does?
		Art:	I know what makes thirds.
		Ed:	What?
	Mr	Art:	There's got to be one.
	Tmf	T/R1:	[T/R 1 approaches their desks.] Oh, this is an interesting one. [She points to the orange rod with five red rods next to it.]
	CPqc	Ed:	[To T/R 1] Which one makes thirds? What makes
	Tqs	T/R1:	[To Art] This would be an interesting problem, what would you ask me here, Art?
		Art:	If, if the red rod was considered one fifth, what
	Tfl	T/R1:	Or if the orange rod is considered, if the red rod is one fifth, what would the orange rod be?
		Art:	Umm.
	Tmci	T/R1:	Good problem, that is a good one to ask. Okay, good problem.
	CPqi	Ed:	No, but what makes it?
	Mr	Art:	Nothing can divide twelve into thirds except
	Mr	Ed:	Red.
	Mr	Art:	No. [He counts on the five red rods next to the orange rod] Two, four, six eight, ten. Ten divided into thirds. No, ten can't be divided into thirds.
	Mc	Ed:	But nine can.
	Mr	Art:	Nine can, but there is no nine rod. Oh, yeah there is.
	Rm	Ed:	Eleven, this is twelve though. [Art holds up the orange rod.]

Time	Code	Speaker	Text
	Rm	Art:	No, it isn't, look [Art counts on the five red rods next to the orange rod] Two, four, six, eight, ten. The orange rod is ten.
	Mr	Ed:	Okay, ten. So that's ten, this must be nine. [He holds up a blue rod.] And this divided into thirds must be
		Art:	It takes
	Mr	Ed:	Light green
		Art:	It takes green to divided the nine into thirds.
	Rlv	Ed:	Blue [the "nine" Art is referring to]
		Art:	No, we are doing this one. I'm doing this one, the one I made up.
		Ed:	[Simultaneously] I'm doing this one [the three light green and blue model]. Yeah.

Julie, Kristin and Mario build the same model of ten white rods lined up next to an orange rod. No question is heard.

	ASm	Maria:	[to Sami] How many colors would it take to make a blue rod?
	Rm	Sami:	Everyone will have different answers.
	ASm	Maria:	How many different ways can you have [T/R1 appraoches]
	Tqs	T/R1:	We want to talk about number names. That's really a hard problem.
	Tmd	T/R1:	[to class] Okay, I'm ready to hear some questions.
			[Art is at the overhead with T/R 1; he puts five red rods next to the orange rod.]
40:00	**ASs**	Art:	If one red rod was considered one fifth, what would a whole be considered?
	Tqc	T/R1:	Okay, do you understand the question? One more time, ask the question. That's really a hard question.
		Art:	If the red rod would be one fifth, what would one, what would one be? [Art gestures to George to respond.]

Time	Code	Speaker	Text
		Ed:	Me?
		T/R1:	If the red rod was one fifth, what would we call one?
	Rlv	George:	The orange rod.
		Art:	Umm.
		T/R1:	Nice and loud, Ed.
		Ed:	No, he called on George.
		George:	The orange rod.
		T/R1:	Oh, George.
		George:	The orange rod.
	Tqs	T/R1:	Can you prove it?
	Mr	George:	Five red ones make up an orange rod.

Next to present – Betty and Matt are partners

Time	Code	Speaker	Text
	ASs	Betty:	If light green is one whole, what is blue?
	Tfl	T/R1:	If light green is one, what number name shall we give to blue?
	Rlv	Ed:	Three wholes.
	Tfl	T/R1:	Just say "three".
	ASs	Matt [If blue is one, what is light green?]	
	Rlv	Julie:	One third.
	Mr		Put three light greens up to the blue. Each is one third.

Julie and Kristin as small group with Mario.

Time	Code	Speaker	Text
	ASs	Julie and Kristin [If white is one what is orange?]	
	Rlv	Ed:	Ten wholes.
	Mr		Because you need ten of them (white rods) for orange.
	ASs	T/R1:	If orange is one, what is white?

Time	Code	Speaker	Text
		Joyce:	One tenth.
	ASs	Maria:	If purple is one half, what is one?
	Rlv	Alyce:	Brown.
	Mr		I took one purple and, and you need two purples to equal up to the whole.
48:00	Tfp	T/R1:	In the last five minute today, try to stump your partner.
		Ed:	If light green was one half, what would be a whole?
		T/R1:	What would be one?
		Art:	Dark green.
		[T/R2 is sitting with Art and Ed.]	
	ASs	Art:	If the white one was considered one fifth, what would be considered one? [He holds up a white rod.]
		Ed:	What?
		Art:	If this was one fifth
		T/R2:	It's a good one.
		Art:	What would be one?
	Rlv	Ed:	Yellow.
		Art:	Right.
		T/R2:	Are you going to let him get off that easily?
		Art:	He knows it anyway. [Ed starts to put five rods next to the yellow rod.]
		T/R2:	Just in case you couldn't remember it in your head, you should always go back and prove it.
	Mr	Ed:	Umm, and also I did it, I just counted up five [Ed counts up one the staircase on his desk.]

Time	Code	Speaker	Text
		T/R2:	Oh.
	Mr	Ed:	And I know that that's half of [He points to the orange rod], and I know that yellow is half of orange, which is ten.
		T/R2:	Clever. So you're using [She accidentally knocks Ed's staircase], Oh, I'm sorry. [She straightens them]
		Ed:	That's okay.
	Tmci	T/R2:	You're using the staircase then to help you, so you don't have to do all that. That's very clever.
Break in tape.			
		T/R2:	I have one for you.
		Art:	Okay.
	ASs	T/R2:	Okay, let me see. No, that's too easy, let's see. [She takes a purple rod.] Okay. If I call this two, what would one look like? Which rod would one be?
		Ed:	That's two.
		T/R2:	That's two.
	CP,Rlv	Ed:	Then one would be red.
		Art:	Umm.
		T/R2:	Okay, why?
	CP,Me	Ed:	Well, you see if that's two [He points to the purple rod in his staircase]
		T/R2:	Umm.
	CP,Me	Art:	Umm. And half of two is one. [Art puts a red rod next to the purple rod that T/R 2 has selected.] And you take another one.
		Ed:	[He puts another red rod next to Art's red rod.] I have one for you, Art.

Time	Code	Speaker	Text
		T/R2:	Clever. Okay, go ahead. [She leaves them.]
	ASs	Ed:	If this is three [He holds up a white rod], what is six? If this is three what is six?
		Art:	If that little thing is three, what is six?
		Ed:	Yeah.
	Mr	Art:	This? [He holds up a light green rod.]
		Ed:	This [He is shaking his head 'no' and holds up a white rod again.]
		Art:	No, if that, if that, if that
		Ed:	This is three, what is six?
		Art:	If that was considered a
	Cir	Ed:	Hold on, I've got to check. [He checks Art's answer, while covEileeng his model with one hand.]
		Art:	Three of something
	CPRlv	Ed:	Oh, wups, you were right. Sorry, sorry.
		Art:	All right now
		Ed:	I was thinking it was that [He points to his staircase.]
		Art:	Let me bump you off with one.
		Ed:	Like you can.

Maria and Sami as partners.

ASs	Maria:	[to Sami] Call the white one. What would you call seven?
Rlv	Sami:	Brown.
	Maria:	Nope.
	Sami:	[She places a black rod on her desk, and counts white rods as she places them on top of the black rod] One, two… Black.

Time	Code	Speaker	Text
		Maria;	Yup.
		Sami:	If red is one third, what would one be? [T/R1 joins the girls]
	ASm	T/R1:	I want to find a rod whose number name is one sixth. Can you find it?
	Rlv	Maria:	Green, dark green?
	Tqs	T/R1:	If dark green is one sixth, what is one?
	Rm	Maria:	[She places six white rods under a dark green rod.]
	Tqs	T/R1:	Green is one sixth?
	Rlv	Maria:	No.
	Tqs	T/R1:	Which rod has the number name one sixth?
	Rlv	Maria:	White.
	Tqs	T/R1:	Then what is the number name for dark green?
	Rlv	Sami:	One.
	Rlv	Maria:	Six.
	Tqs	T/R1:	One or six?
	Rlv	Maria:	Six, oh! It's one.
	Tqs	T/R1:	If I wanted green to be six, what would white be?
	Rlv	Maria:	One.
	Tmr	T/R1:	You have to be very careful.
55:00	End of class.		

APPENDIX E

Coding Chart*

Session 1

*See Chapter III for coding notation

Sequence of events	Tea	Ss	Eil	Joy	Art	Dina	Ed	Alex	Bett	Sam	Jen	Mar	Dan	Jul	Liza	Gra	Matt	Kris	Mari	Aly
A: I claim LG is half as long as DG	ASs																			
T: Can you convince me?	qc																			
R: True… Take 2 LG, put them next to DG			m																	
A: # name LG if DG called 1	ASs																			
T: Talk to your partner	md																			
R: One half [to Sarah]																			lv	
M: Yeah.. LG half of DG [OHP]				r																
T: Alan, Erik? Do you agree?	qs																			
R: One half					lv															
A: someone told me R half as long as Y	ASn																			
CP, R: No [to Sarah]																			lv	
CP, R: No [to Meredith]]										lv										
R: no [shows Erik 2 reds]					m															
R: no 2 reds don't fit						m														
A: P half as long a BLK	ASn																			
CP, M: No. 2P are too large [to Sarah and T/R2]																			r	
M: no [using B]							r													

CP, R: no [using BLK]					m															
CP, R: no [using BLK]							m													
M: No. 2P are too large													r							
A: someone told me R is 1/3 as long as G	ASs																			
T: discuss w/ partner	md																			
CP, R: yeah							lv													
CP, R: do it like this					m															
R: 3R placed by G								m												
R: R 1/3 as large, as long as G				lv, m																
R: put reds below green												m								
A: If G called 1, # name for R?	ASs																			
T: Talk to your partner	md																			
R: We already did		lv																		
R: R=1/3									lv											
M: There are 3R, so R=1/3 [toT/R2]										r										
M: 3 make 1 whole									r											
A: Someone told me LG is 1/3 as long as B	ASs																	7		

R: it is											lv									
T: how could you convince me?	qs																			
M: 3 put up to B, B is one whole											r									
A: B called 1, # name for LG? Everybody?	ASs																			
R: One third		lv																		
A: If G called 1, # name for R?	ASs																			
R: One third		lv																		
T: # names always same?	qal																			
R: no		lv																		
T: color names always same?	qal																			
R: no		lv																		
T: another name for B?	qc																			
R: Dark B							lv													
T: Well... color names always same?	qal																			
R: No, oh yeah		lv																		
R: color names, no							lv													
T: Do you agree?	qc																			

R: Yeah		lv																		
T: Do # names change?	qc																			
R: Yes		lv																		
T: Give an example	qc																			
A: B=1 whole, LG = 1/3							lv													
A: Well… possible for LG=1/3, R= 1/3?	qe																			
CE, M: Different sized whole												e								
T: Different # name for what is 1	fl																			
T: Look at color names, see what you find	fp																			
R: [many build staircases]		m																		
R: I started with orange …[end with] white																		lv		
T: G - # name 1, 3R=G, why can't R be called 1?	ASs																			
T: Discuss w/ partner	md																			
R: Because 2W=R												lv								
T: But if DG=1, R= 1/3, you said so	qe																			
R: Oh, yeah.												lv								
T: Can I call R 1?	qc																			

M: No, you're comparing R to G							r													
M: R not 1, G bigger than R												r								
T: If I call R 1, # name for G?	ASs																			
T: Talk to your partner	md																			
CP, R: Three wholes?							lv													
CP, R: no name, not into two					lv															
M: DG > R							r													
R: [To T/R1] Three wholes…							lv													
M: …because it takes three							r													
T: Do you agree Alan?	qs																			
R: Yes, I guess so					lv															
T: 2 answers. Some say 3, some say 1/3.	qc																			
R: Three wholes					lv															
M: R considered 1, G>R, 3 whole R make G					r															
M: 1whole+ 1 whole+ 1 whole, so G is 3 wholes							e													
M: If R is 1, G is 2 more wholes													e							
A: If I call BR 1, # name for R ?	ASs																			

R: One fourth						lv															
T: What would you do to convince me?	qs																				
M: Put 4R onto BR						r															
A: I call R1, # name for BR?	ASs																				
R: Four														lv							
T: What would you do to convince me?	qs																				
M: Line them up. They add up to 4 wholes.														r							
A: Call W one half, what rod is 1?	ASs																				
T: Talk to your partner	md																				
R: Red															lv						
A: Make up problems, practice with partners	fp																				
CP, R: 5R next to O							m														
CP, R: That's 1/5					lv																
CP, R: 2P next to O, what makes 1/3?							m														
CP, R: Probably LG					lv																
CP, R: There has to be 1/3					lv																
T: [pointing to 5R and O] What would you ask me?	qs																				

A: If R= considered1/5, what would O be?		ASs					
T: Good one	mci						
CP: What makes it? [Divide O into thirds]			qp				
CP: Nothing divides 12 into thirds, except...		b					
CP, R: red			lv				
CP, M: [counting Rs next to O]		r					
CP: M: But 9 can			e				
CP, R: O is 12		lv					
CP, R: no, O is 10			lv				
CP, A: B=9, B/3=?			ASs				
CP, R: LG= 1/3			lv				
R: 10 W next to O [no question heard]		m			m	m	
A: [to Sarah] How many colors to make B?							ASm
M: Everyone will have different answers				r			
A: How many different ways… [T/R1 approaches]							ASm
T: We want…# names. That's a really hard problem.	qs						
T: [to class] Ready to hear questions	md						

A: If 1 R considered1/5,
what would whole be? ASs
R:
Orange lv
T:
Can you prove it? qs
M:
5 R make up O r
A: If LG is 1 whole,
what is B? [with Mark] ASs
T: If LG is 1, what # name
shall we give to B? fl
R:
Three wholes lv
T:
Just say three fl
A: If B is 1,
what is LG? [with Beth] ASs
M: One third. Put 3 LG up to
B, each 1/3. r
A: If W is 1, what is O?
[with Michael as group] ASs ASs
M: Ten wholes. Because
need 10 [W] for O r
A: [to class]
If O is 1, what is W? ASs
R:
One tenth lv
A: If P is 1/2,
what is 1? ASs
R:
BR is 1 lv

T: Try to stump your partner md

CP, A: If LG was 1/2,			
what would be a whole?			ASs
CP, T:			
What would be one?	fl		
CP, R:			
DG		lv	
CP,A: If W considered 1/5,			
what considered 1?		ASs	
CP, R:			
Y			lv
CP, T: {T/R2]			
You let him off easily?	qs		
CP, R:			
He knows it		lv	
CP, M:			
I just counted up 5			e
CP, M: I know y is 1/2 of O,			
O is 10 [points to staircase]			e
T: Using staircase,			
very helpful	mci		
CP,A [T/R2]: If P is 2, what			
would I look like?	ASs		
CP, R:			
1 would be R			lv
CP, Why?	qs		
CP, M: [using staircase]			
1/2 of 2 is 1		e	
CP, M:			
Take another 1, 1+1		e	
CP, A: [T/R2 leaves]			
If W is 3, what is 6?			ASs
CP, R: This?			
[Holds up LG]		m	

CP; No… hold on,				
I have to check		ir		
CP,R;				
You were right		lv		
CP,A: [to Sarah] Call W 1,				
what would you call 7?				ASs
CP,R:				
Brown			lv	
CP,R:				
Nope				lv
CP, M: [places BLK, puts W's				
on top] One, two… Black			r	
CP,A: If R is 1/3, what would				
be 1? [T/R1 joins]			ASs	
A: Find a rod whose # name is				
1/6	Asm			
CP,R:				
Dark green?				lv
T:				
If G is 1/6, what is 1?	qs			
CP, M: [places 6 W under G]				
No.				r
T:				
Which rod has # name 1/6?	qs			
CP, R:				
W				lv
T:				
What is # name for DG?	qs			
CP,R:				
One			lv	
CP,R:				
Six				lv
T:				
One or six?	qs			

CP,R:		
Six, oh! It's one.		lv
T: If I wanted green to be 6,		
what would W be?	qs	
CP,R:		
One		lv
End of class		

APPENDIX F

Coded Transcription

Session 2

Session 2
Side, front cameras - OHP camera unmanned
T/R1 leads
T/R 2 and CT present

Small group: Joyce, George, Betty
Partners: Ed and Art, Bob and Julie, Maria and Sami, Alyce and Jon, Dan and Anne, Grant and Dina, Mario and Kristin, Alex and Matt, Benny and Eileen, Katy and Cassie, Jen and Liz

Activities

1. If I call the blue rod one, what rod will I call one half?

2. If I call the yellow rod one half, what rod will I call one?

3. Can you design a rod that is half as long as the blue rod?

4. If we call the orange rod "two", what can we say about yellow?

5. If we call the orange rod "six", what number name can we give to yellow?

6. If we call an orange and light green train "one", can you find a rod that has the number name "one half"?

The second class begins with T/R 1 asking students to tell Tom Purdy what they had done the previous day.

Time	Code	Speaker	Text
	recall	T/R 1:	What went on here yesterday? ...Somebody want to help him out? Thank you, Jen.
		Jen:	Um. We did activities with rods and we um had to see like which rods were bigger and we had to...um, we did math problems with them.
		T/R 1:	Okay. Somebody want to sum up a little bit more? Mario?
		Mario:	Um. We, well, we, what we did is was, we called one one and them we had to decide the littler one, what it would be called, one thirds, one fourth, or a half of the, the bigger block.
		T/R 1:	You know, I don't want to embarrass Mr. Purdy, but you have to go very slow for him. He often needs an example.

Time	Code	Speaker	Text
		Tom:	I need to see it.
		T/R 1:	He needs to see it. He just… that's the way he learns. Can you help him a little better then, Mario? Maybe make up and example for him or somebody? We really need to help him out…Ed, you want to help while Mario is thinking of something else?
CE	Asn	Ed:	Well, let's [He picks up a blue rod.] If we said that the blue rod would be one whole, um, we'd figure out what, we'd take all the blocks and try and figure out what would be one half of it. [He holds the purple rod next to the blue rod.] And let's, and I figured that the purple block would be half of it. So, well, no, not exactly, but…
		T/R 1:	Mr. Purdy goes through the same thing, Ed.
		Ed:	But if we call this one whole [holding up the blue rod], we'd figure out which block one would be one half of it
		Tom:	Uh huh.
		Ed:	And which block would equal up the two blocks of… these two blocks of it, that would equal up to one of these we'd call that one half of the whole block. So, that's basically what we did.
		T/R 1:	You're not going to help solve it for him? [to Tom]
		Tom:	I was going to say, did you find it? Or
		Ed:	Oh, oh.
		Tom:	[inaudible]
		Ed:	Well, yeah we did, but
		Tom:	You're making me believe maybe you can't do it.
	CE	Ed:	No. We did find it. I just can't remember which one it was. [He holds up two dark green rods, end to end, next to the blue rod and discards them when he sees that two dark green rods are not equal in length to the blue rod.] I don't think there is one. [He measures two yellow rods, end to end, to the blue rod.]
		T/R 1:	Maybe some of you can help Ed out.

Time	Code	Speaker	Text
		Ed:	I think it was the dark green.
		Tom:	You're saying the blue one is one [Children are working with the rods.]
		S:	try the yellow.
	1/3	Art:	the little green one was the thirds. The yellow was the half. No. The yellow is the halves of the orange one.
		Tom:	I think you picked a good one.
		T/R 1:	Ed, Ed. Suppose I wanted, suppose I wanted to call the yellow one "one half". Suppose I wanted to do that.
		Ed:	Found it!
		T/R 1:	But suppose I wanted to call the yellow rod, I wanted to give it a number name one half. Can you tell me what I would have to call one?
		S:	Oh, oh!
		T/R 1:	I think you need to get your rods and build it for me.
		S:	Oh.
		T/R 1:	If I wanted to call the yellow one half, can you show me
		Art:	Easy
		T/R 1:	What would I have to call one?
		Art:	It's orange. It's easy. See? The orange one. [Ed is working with his rods.]
		T/R 1:	Bob, you want to tell Mr. Purdy?
		Bob:	Well, these two blocks equal up to this one whole. [He shows two yellow rods beside the orange rod.]
		Tom:	Those two blocks equal up to one whole. So how much is each one? Each one of the yellows?

Time	Code	Speaker	Text
		Bob:	One half.
		T/R 1:	So you are going to call the yellow one half? I'm still worried about Ed's problem. Ed wants to call this one 'one' [She holds up a blue rod] and Ed is trying to call something one half. Did you want to help Ed out?
	**	Ed:	I don't think there is one.
	1/3	Art:	A little green makes a third out of that. Look I can do it
		T/R 1:	If you call the dark blue one "one"
		Art:	One, two, three.
		T/R 1:	Dan, what do you think? What does Dan think? I can't hear you, Dan. Hold on.
		Dan:	I don't think that you can do
		T/R 1:	Why, Dan? Slowly and loud.
CE	**	Dan:	I don't think that you can do that because if you put two yellows that'd be too big, but then if you put two purples that's uh, that's uh, that'd be too short and
		T/R 1:	What about something between purple and yellow?
		Dan:	I don't think there is anything.
		T/R 1:	Why not? [Dan pauses.] Show us what you have there, Dan. Why do you think there isn't any? "Cause I think you built it to show us. Can you show us your yellow and your purple?
		Dan:	Well, I was thinking. 'Cause there's usually, the tall one... [inaudible]
		T/R 1:	Dan, why don't you come up here and explain your reasoning. Dan doesn't think it's possible because Mr. Purdy said, "Maybe it's not possible." So let's, let's see. Let's help him out a little. Here's the two yellows and here's the two purples. What's, what's your reasoning? Let's listen to what Dan has to say.

Time	Code	Speaker	Text
	**	Dan:	[He comes to the overhead and puts a blue rod onto it. He places a yellow rod and a purple rod, end to end, with one white rod.] All right. You see usually, um, they are only one, with the shorter one, only one block apart. Like that and so these, but then if you have for the blues, like if you have two yellows, it would be too tall and if you have two purples [He puts two yellow rods, end to end, next to the blue rod and then one purple rod next to one of the yellow rods.]
		T/R 1:	Do you need another purple? Here
		Dan:	That'd be too short and then there's, there's really nothing in between 'cause if you do [He builds a 'staircase' of rods, beginning with the longest, orange rod, then places blue, etc. until he reaches the shortest rod, the white one.] And then here [between the yellow and the purple rods], there's nothing in between, right here, so there's no way that you can do it.
		T/R 1:	are you all convinced? Jen? Jen has a question for you, Dan.
	1/3	Jen:	But if you put three greens to it you could
		Dan:	Yeah, but Ed said, Ed wants the half. [inaudible]...'cause I figured that out, too.
	**	Ed:	I think you could do it, but they're... See, I figure if you take a yellow and a purple it's equal [to the length of the blue rod]. They're not exactly the same, but they're both halves. Because the purple would be half of this even though the yellow is bigger because if you put the purple on the bottom and the yellow on top it's equal, so they're both halves, but only one's bigger than the other. So it equals up to the same thing.
		T/R 1:	Did you all hear what Ed said? Ed, do you want to say that one more time? How many heard what Ed said? How many would like to hear it a second time? Ok, Ed, would you say that one more time to Dan and the rest of us?
		Ed:	If this would be one whole [the blue rod], you could take the yellow to be and you could call it one half [holding a yellow rod

Time	Code	Speaker	Text
		Ed:	next to the blue]. But if you took another yellow it would be too big. But if you took a purple with the yellow, and put it on top of yellow, it equaled to the blue. So, the purple would be a half and the yellow would be a half, except that the yellow would just be one bigger than the other. Or maybe you could call this three quarters [holding the yellow rod] and you could call this one quarter [holding the purple rod]. And, but it would still equal up to the whole.
		T/R 1:	What do you think, Dan?
	**	Dan:	I didn't think of that. [Ed chuckles. Dan places a yellow and a purple rod end to end, next to a blue rod.] Cause I was thinking that you would need the same.
		T/R 1:	You think you would need the same?
		Dan:	Yeah, but that might
		Ed:	You don't really.
	**	T/R 1:	You don't need the same? In other words, I could call this a half [the yellow rod] and this a half [the purple rod]. Suppose this is a brick of gold and we're going to share it, Ed. And I'm going to take the yellow half and you get the purple half. Fair?
		Ed:	Yeah.
		T/R 1:	Do the rest of you agree? Do you like that? [Chorus of no's] Betty? No. Betty doesn't like that. Katy? Does it matter? Ed doesn't care. Do you care?
		Katy:	Yes, cause the pink, the purple is smaller than the yellow and the person who got the purple wouldn't have as much.
	**	Ed:	Yeah, but you could call this three quarters and this one quarter and it would still equal up to the whole. Then it, just wouldn't be halves, it would be quarters. But it would still look like you're dividing it into halves, but you're dividing into quarters.
		T/R 1:	What do you think Benny?
	1/3	Benny:	Well, you could, you could use say, if there, if there was three people – you could at least split it into thirds.

Time	Code	Speaker	Text
		T/R 1:	Is that, is that the question?
		Dan:	Well, no. It's not. You see we're trying to do it in halves.
		T/R 1:	[To Benny] We're trying to work on halves.
		Benny:	Oh.
		T/R 1:	Okay. Art.
	**	Art:	When you're dividing thinks into halves, both halves have to be equal – in order to be considered a half. [inaudible] this isn't a half. Those two aren't both even halves.
		T/R 1:	Ed?
		Ed:	Yeah?
		T/R 1:	What do you think of that?
		Ed:	Well
		T/R 1:	Can you divide things in halves and have them different sizes? I think that's what Jen is asking Art and Dan.
	**	Ed:	Well, see. This isn't exactly dividing in halves. But I'm still using two blocks, but not… I'm dividing it in half still using two blocks, but one block is bigger than the other block. So it's like using three quarters and one quarter, but you're only using two blocks so it's almost like dividing it in half.
		T/R 1:	Alex? What do you think about that, Alex?
		Alex:	Well if he's saying, he's saying that he wants a half, but if he puts that, a purple and a yellow, he won't have half. He would have three quarters and one quarter. And he wants a half.
		T/R 1:	It seems to me we have some differences here, don't we? Um. How many of you agree with Ed? [no hands are raised, children giggle] How many of you disagree with Ed? [all hands are raised, more giggling]. Hm, okay, what's the issue, do you think, here in the disagreement? Can someone summarize the issue? Art, do you want to try again?

Time	Code	Speaker	Text
	1/3	Art:	Um. You can't, if you're div, you can't divide that into halves, because you'd have to use rods that are of different sizes, but you could divide it into thirds using rods that are the same size which, which is the light green rods.
		Ed:	But I didn't want thirds.
		T/R 1:	[inaudible] can be very helpful to Mr. Purdy. Because I think, go ahead, Dan. What do you think?
CE		Dan:	[at OHP, pointing to the rods on the OHP] I think that some of these that you can't do like this would be odd. [Dan moves the white rod to one side.] this could be even. [Dan begins a new group with the red rod.] This would be odd. [He moves the light green rod next to the white rod.] Be even. [He moves the purple rod next to the red rod. Continuing in this manner, he moves the yellow, black and blue rods next to the white and light green rods. He moves the dark green, brown and orange rods next to the red and purple rods.] This, be, you see, then when you get up to here, blue would be odd, but like with brown, you could take these two [He places two purple rods next to the brown rod.] and put them together and that would be even. Take the orange, put two yellow, with the orange and that would be even {He does this as he is speaking].
		T/R 1:	Okay, let me see, I think that we have. Maybe, Ed, the way we can resolve this is, I don't think I'm hearing you say, Ed, that you want to call the yellow one half and the purple one half. I don't, I don't hear you say that. You're not saying that, are you?
		Ed:	No [agreeing that Dan is not saying that].
		T/R 1:	You're saying that you agree with the rest of the class that if you call something one half of something
		Ed:	Yeah
		T/R 1:	They have to be the same size.
		Ed:	Yeah, yeah.
		T/R 1:	Right?
		Ed:	Yeah.

Time	Code	Speaker	Text
		T/R 1:	You are in essence answering a different question, maybe?
		Ed:	Yeah.
		T/R 1:	Where you were saying, "Well, if I call this one, there are other rods that make up one and maybe they're not the same size." I think you're very generous, Ed. Not as generous as Betty and Katy. And if we're talking about bricks of gold, letting me have the larger one if we're sharing one half. I really appreciate your generosity. I know Mr. Purdy wouldn't be so generous. Is that right, Mr. Purdy?
		Tom:	That's right.
	**	T/R 1:	That's right. But I do appreciate your generosity, so we'll have to talk later about some, some sharing. Um. We could go into business together, Ed. But I think that what we're saying from this is the point that Dan is making and Art and some of you have expressed very nicely, tht if we are calling a rod one half, okay, if we call rod one half, of, let's say, of a rod that we called one, was given a number rod one, there are two conditions that have to be satisfied. Can you tell me what those conditions are? And I think one more time as a summary because you're saying that purple could not be considered one half because one of the conditions isn't met, right? I mean, they're both the same size.
		Dan:	Um, hm. But they don't, um, if you put like that [He puts two purple rods together.], they don't, uh, they're not as big as the blue.
		T/R 1:	Do you agree? Do you all see the second condition that's not met? See the space in here? Or if you can put them like this, see the space? And I think that Dan has made another very powerful, interesting argument that I'd like you to think about. He claims that that's missing, right? And that there couldn't be another rod in between to do it, right? That's interesting, now, you know, suppose you had to manufacture there rods and make another color. Okay? Here we have a purple rod that's too small, right? To qualify to be a half. Do you agree the purple's too small? And here we have a yellow one, right? That's too big, right? To qualify, do you see that? If you were designing a new set of rods and you wanted to cal the blue rod one, okay? Can you tell me what the new rod would look like so that you would be able to call it a half? ...Do you understand my question? We have rods here with ten, we have ten colors, don't we? You told me that yesterday.

Time	Code	Speaker	Text
		SS:	Yeah.
	ASm	T/R 1:	right? And you told me that if I wanted to call blue one in terms of the box you have, right? You can't find a rod that you could give a number name one half. Isn't that what you all told me? [mumbles of agreement] That's a problem. Because, um, there's another school that wants to have rods where they want to call blue one and have another rod that they can give a number name one half. Okay? Now can you tell me what the design of that rod might begin to look like? Why don't you talk to your neighbor and think about that problem? Do you understand my problem?
			[inaudible]
			We know it can't be purple and we know it can't be yellow. [T/R 1 and Dan confer at the OHP. Their conversation cannot be heard.]
		Ed:	It can't be anything 'cause you can't divide nine equally. You see if this is
		Art:	If you could
		Ed:	No you can't. This is ten.
		Art:	If you could make a rod.
		Ed:	If this is ten [the orange rod], then this [the blue rod] is nine. It's impossible to divide this evenly.
		Art:	Different rods. You might be able to, like if you divide a blue rod in half you could that that like and make a new color and that would equal up to halves. Which would mean it would be like [noise]
		Ed:	It's impossible. You can't divide it in half. You can't divide it in half, Art.
		Art:	Right, you could divide it in half id you had [inaudible] parts. You could divide it in half, but having equal parts, but you couldn't have equal numbers.
		Ed:	[inaudible]
		Art:	If you cut this [the blue rod] down the middle, it would be four and a half, [inaudible] the same length.

Time	Code	Speaker	Text
		Ed:	Four and a half. You can't make a rod that's four and a half.
		Art:	Um, hm. So you can't divide into anything.
		Ed:	Except thirds.
		Art:	Except thirds. Or, or singles.
		Ed:	You can't divide it into halves. "Cause I put this up here and there are nine of these and one, two, three, four, five. One, two, three, four [pause] four, one two, there four five. One, two, three, four. One, two, three, four. One, two, three, four, five [He is counting the two groups of white rods next to the blue rod].
		Art:	Over here you have thirds.
		Ed:	You can divide it into thirds, but you can't divide it into halves.
		Art:	You can divide it into thirds. You can divide it into ninths.
		Ed:	But you can't divide it into halves.
		Art:	You can't divide it into anything else but thirds and ninths.
		Ed:	Exactly, you're right.
		Art:	Just thirds or ninths. That's all you can do. That's productive reasoning.
		Ed:	What?
		Art:	Productive reasoning. So there can be only thirds and ninths. And they are singular rods. And you can't divide it into halves.
		Ed:	Exactly. It's impossible to divide it in halves.
		Art:	That can't be done.
		Ed:	It's impossible, Art. You can't divide it into halves.
		Art:	It's been proven.
		Ed:	Exactly. [noise]

Time	Code	Speaker	Text
		Art:	What I'm going to do right now is make out of everything, I'm going to halve or third every color, I can third every color. I can halve every color.
		Ed:	Except blue.
		Art:	[inaudible]
		Ed:	You can third. You can third.
		Art:	And ninth.
		Ed:	and ninth.
		Art:	Now black.

Rm Bob and Julie built staircases independently.

Sami and Maria sat quietly; T/R 2 approached the partners.

	Tqs	T/R 2:	Do you understand the problem? What are you being asked to do?
		Sami:	[to Maria] You don't know.
		Maria:	Yes, I do.
		Sami:	Well, you do it.
		Maria:	She's asking us to find a rod that will make blue. Find one that will fit.
	Tqs	T/R 2:	If I we could go and build a new rod, how would you describe it?
	Mr	Maria:	This [orange] is ten, this [blue] is nine. Cut the blue in half. It has to be four and a half. And you would have to take one, say, this is four, this is four [purple rods]. You'd have to have half on each four.
	Tqs	T/R 2:	What do you think, Sami?
			[Sami nods yes and T/R 2 leaves.]
		Maria:	All you need is

Time	Code	Speaker	Text
21:00	Tcq	T/R 1:	Okay, now I'd like us, if you don't mind, if we can stop for a minute and I'm going to ask Betty, George, and Jackie to come up and pose their solutions. I heard a few of your solutions. I know Dan has a solution I heard already, up front. I'd like to hear some other possible solutions. You can clear off there [the OHP] what you don't need.
		Betty:	I got it, right here.
		Jackie:	[Places purple then white then purple rods in a line on OHP] We thought that to make a new rod we would make, um, we would cut this white one in half and attach it
		T/R 1:	Could you speak nice and loud? "Cause I'm a student back here and I can't hear you. Do you want to try and talk really loud?
		Jackie:	We thought of to cut the white one in half and add it to one rod [purple] and then add it to the other rod [purple]. And
		George:	Call it light pink.
		Jackie:	And we thought the color would be light pink.
		George:	And the smallest one would be a half 'cause it was the white one.
		T/R 1:	Did you all hear what they said? No, they, Katy didn't hear you, dear.
		Jackie:	We thought to cut the white one
		T/R 1:	[inaudible] nice and loud, I know you can [inaudible]
		Jackie:	We could cut the white one in half and add it to the purple rod and add one one half to one purple rod and the other to the other one and we thought that we could call the color light pink.
		T/R 1:	And you said something else, what would your smallest rod be?
		Jackie:	Oh, yeah. Our smallest rod would be half of the white one.
		T/R 1:	What are you going to call that? [some giggling] You're the designers. What are you going, it's not going to be white, what do you think? You want to help them out? We have other consultants to this design. Why don't you call on someone for help and consulting? George? Betty?

Time	Code	Speaker	Text
		Jackie:	[inaudible]
		George:	We cut the clear one [the OHP version of the white rod] in half to like make this. Then you'd, then you would have to cut like a reg, a regular one in half to be your smallest one
			[pause]
		T/R 1:	I see some hands up. Why don't you see if…?
CE	**	Mario:	If you're going to make a new rod, then you'd have to make a whole new set because there'd have to be a half of that rod, too.
			[pause]
		T/R 1:	What do you think, George?
			[some giggling]
			What do the rest of you think? Do you think there would have to be a whole new set? There are some other people who have opinions. Why don't you go, who's going, why don't you take it, Jackie? You call on people, okay?
		Jackie:	Um, Benny.
		Benny:	No matter what, there'll always be something [inaudible]
		T/R 1:	Nice and loud, Benny, I can't hear you.
		Benny:	No matter what there'll always be something that won't be equal to something, like
		T/R 1:	Can you say a little more about that, Benny? Nice and loud.
		Benny:	If you cut these little ones in half, then there wouldn't be something for the little ones to make a half out of them.
			[laughter]
		T/R 1:	Did you all hear what Benny said? That's, Benny, I, we didn't hear back here. Katy and I are trying hard. Can you turn around and say it nice and loud?

Time	Code	Speaker	Text
		Benny:	If you cut one of these in half then there wouldn't be a half of the litt, of the one that you, of the halves of these.
		T/R 1:	What do you think about that? Dan, you had your hand up. Do you have a different point?
		Dan:	Well, what I told you. I thought that, uh, to cut it in half, too, but then I realized that, uh, that you would have to make a whole set.
		T/R 1:	Yeah.
		Dan:	And make a half for every one.
		T/R 1:	Okay, Sot that's what we heard, um, Mario tell us. Maria has something to say to the group.
		Maria:	Well, you could just, if you do that then you'd have to cut the ones that are separate, the little blocks into halves, all of them, so then you could make it equal.
		T/R 1:	What do you think, Julie?
		Julie:	Um, it, I agree with Mario. 'Cause if you do that, um, it changes the whole pattern 'cause this has a set in pattern to it and the whole thing would change.
		T/R 1:	It's an interesting question, isn't it? It's an interesting question. So in other words, when you designed a solution, you're telling me, for the problem where you're making now a pink rod, is that what you're calling it?
		Jackie:	Yes.
		T/R 1:	You're creating a pink rod. And as I understand it, the pink rod is made up of purple and half of white. Is that what you said? Um. You solved the problem of having a rod that you can all one half when you call the blue rod one, right? But then, as some of you pointed out, then your smallest rod is then, with this new design, your smallest rod is
		Jackie:	Half
		T/R 1:	Half of the
		Jackie:	White

Time	Code	Speaker	Text
		T/R 1:	White rod, right? And what are you going to call that? Let's give that a name. Let's give that a name. Can you give that a name?
		Jackie:	[inaudible]
		T/R 1:	It's not white any more. It's half of white. What color name shall we give it?
		Jackie:	Light blue.
		T/R 1:	Pardon? Light blue? Okay, so your smallest rod is going to be light blue. But I heard some other people say, like Benny in particular, and some others, Maria, that, okay, you've solved that problem, but you could expect new problems. Yeah. That's interesting. Well, okay. That's something to think about. You did a really nice job. Did anybody have another way to make the argument? Jon? [Jon goes to OHP]
		Jon:	Well, I thought that if you had a blue rod as one, you could take light green, imagine there are two others here. Then you could split the middle one in half and you could call that a light blue rod.
		T/R 1:	Is that okay? That's another way, huh? Does anybody have another one? ...Do you think there's still another way? [pause] Do you think there are other ways than this?
		Maria:	Yes.
		T/R 1:	Do you think there are other ways than this? That's really so? Mr. Purdy, did that help you?
		TP:	Yes.
	Tmcw	T/R 1:	That's great. That's been very helpful. What do you think, Dr. Martino? ...Do you have more problems?

T/R 2 posed a new problem:

28:00	**ASs**	T/R 2:	Let's try something a little different now. If we call the orange rod "two", what can we say about yellow? Think about it. Do you want to talk to your partner?

Ed and Art discuss, but conversation inaudible.

CT asked Dina to explain. She said "one". CT noticed that Dina's partner, Grant, was frowning.

Time	Code	Speaker	Text
	Tqs	CT:	He's [Grant] not convinced; he's a little shaky. Can you explain to him why you'd call it one?
	Rm	Dina:	Because there's two [yellow rods next to an orange rod]. Then these [yellow rods] should be one. Because see you needed two of them. We call that [orange rod] two. We call these one [yellow rods] because this [orange rod] is two.
	Mc	Grant:	When the orange is one, we went like a half down
	Tqs	CT:	[To Grant] Is this [orange rod] still one?
		Greg:	No, it's two.
	Tqs	CT:	So then would the yellow rod be called one?
	Rlv	Greg:	Yeah, it's one.

Sami and Maria worked as partners.

	Mr	Maria:	You used all the yellow [she goes to back of room].
			[Katyberly raises her hand. She has built a model of two yellow rods under the orange rod. Sami raises her hand. Maria returns to her desk]
			Oh! She called orange two. One half? Two? This [yellow rod] would be one.

Class called together by T/R 2

	Tqc	T/R 2:	Let's see. When we call the orange rod two, what number name do we give to yellow? Benny, what do you think?
	Rlv	Benny:	One.
		T/R 2:	Why one? Come up. Come up and show us. [Benny goes to overhead.]
	Rm	Benny:	You put two yellows together and they're the same size as the orange. This [orange rod], is considered two. These two [yellow rods] are considered like an orange, each would be one.
	Tqs	T/R 2:	Okay, so each of these [yellow rods] is considered two?
	Mr	Benny:	No. Together they equal the same as that [orange].
	Tqs	T/R 2:	So the number name you're giving to yellow is what?
	Rlv	Benny:	One.

Time	Code	Speaker	Text
	Tqc	T/R 2:	Okay, one. What do you think about this? Who agrees with that number name for yellow? Does anybody disagree? Ed?
	Mr	Ed:	I have another name. You can call it another name. Do you have to call the orange two? If you could call it one, then two yellows would be a half. If you would consider the orange two, then call those [yellow rods] one. If you can call it [orange] one, then you call it [yellow] one half.
	Rlv	Benny:	[at overhead]. There might be other ways. ...You can do thirds, or like that... [voice trailing off]
32:00	**ASs**	T/R 2:	I'm going to change the name for orange to six. What number name will I give for yellow?
	Rlv	Katy:	Five.
		T/R 2:	Five. That's interesting. Can you come up and tell us about that? [Katy goes to the overhead.]
	Mr	Katy:	Look here[pointing to Benny's model] before you said that [the orange rod] would equal two, and then Benny said that [yellow rod] would equal one. So now you're saying that [orange rod] equals six, so I figured that [yellow rod] equals five now.
	Tqc	T/R 2:	That's interesting. So you're saying when I call the orange two, yellow's are each one. So if I call the orange six now, yellow is five. What do you think about that? Did you all here Katy's argument here?
		SS:	No.
		T/R 2:	She's saying that when we called this one [orange], that the number name for each yellow is one. If we called the orange six now, we call that [yellow] five. [Katy sits down.] [Maria and others shake their heads negatively.] Some people are shaking their heads and I want to know why. Art?
	Mr	Art:	[Goes to the overhead] You said that the orange rod was six. And before you said it was two and this [yellow rod] was one. o now if you're calling this [orange rod] six, half of six is three.
	Tqc	T/R 2:	Okay. We have another argument? What do you all think about Art's calling this [yellow rod] three when this [orange rod] is six? Maria?
	Rlv	Maria:	Yes.
		T/R2:	Jacqueline?

Time	Code	Speaker	Text
	Mr	Jacq:	I agree with Art. Half of six is three so
		T/R 1:	I'm curious. Katy, how did you think of five? Help me to understand. Before when orange was two, yellow was one. So now orange is six and you said yellow is five. That's where I am confused. If this [yellow rod] is five and this [yellow rod] is five, this [orange rod] is six?
	Me	Katy:	I made a mistake from some before. I figured it out now. I forgot that adding one and one is two, five and five isn't six, so
	Tqs	T/R 1:	What would the orange rod be called if the yellow rod was called five?
	Rlv	Katy:	Ten.
	Tmcw	T/R 1:	You'd have to call orange ten. Do you agree with that? What a class! It's hard to stump this class.
		T/R 2:	Okay, let's try another one. Okay if we call [long pause]...
		T/R 1:	...Suppose we made a train, take Ed's idea from earlier, call it orange and [light] green together, call an orange and [light] green train together one. I'm curious; can you find a rod that has the number name one half?

Ed and Art as partners:

		Art:	Ed, look at the biggest I can find.
		Ed:	I'm trying to figure it out.
	Mr	Art:	There's no way to call something half [in this train].
		Ed:	How do you know?

Benny and Eileen as partners. Benny raised his hand and T/R 2 joined them. Benny built the following model: O-LG train with G-W-G train directly beneath.

	Mr	Benny:	You split this [white rod] in half and then put one half on one side [green rod] and then put the other half on the other side [of the other green rod].
		T/R 2:	There was no rod that was one half [of the train]?
	Mr	Benny:	No, because ten and three is thirteen and thirteen is an odd number.
	Tqs		What does that have to do with it?

Time	Code	Speaker	Text
	Mr,c	Benny:	With thirteen you can't split thirteen in half equally. Except take a twelve [2 green rods] and split one rod [white] in half. Like what we did last time.
		T/R 2:	Oh, that's interesting.
		Benny:	You have to change the color.
		T/R 2:	Okay, so we are going to invent a new rod. What do you think, Eileen? Do you agree?
			[Eileen nods affirmatively.]
	Ie, Mc	Benny:	You could probably do it another way. That's what James did and I saw it probably with these [light green rods]. Maybe it would work, it would probably work. When he was using the blue with the nine, he was using these others [light green rods], so I thought
	Rm		[He places four light green rods (12cm) under the train of orange and light green (13 cm).]
	Mr		No, no. Oh yeah you could do this like we just did.
			[He places one white rod between the light green rods. His train is LG-LG-W-LG-LG]
			Yeah, I think so, yeah.
		T/R 2:	Okay, so show me where one half would be. One half of that [orange and light green] train.
		Benny:	Well, right there [he points to the white rod]. Down the center there.
	Tmci	T/R2:	Nice thinking, Benny.

T/R 2 left Benny and Eileen and joined Liza and Jen. T/R 2 questioned the girls about Jen's model [a train of G-G-W beneath the train of O-LG]

	Rm	Jen:	...for one half. We had to invent a new rod. So first we thought half would be dark green. We had to put that [two green rods]. That didn't work, we need a white.
	Tqs	T/R 2:	So what would one half of this orange and light green train be? Can you show me?
		Jen:	[Stacking one green rod on top of the other: G then G-W on bottom]. Well

	T/R 2:	So what do you think, Liza? Do you know what I'm asking her? I want to be able to see the one half in my head.
Rm	Jen:	This [holding up a train of green and white rods] would be half.
	T/R 2:	Okay, that [green rod] and the white?
Mr	Jen:	Well, it's sort of in thirds, but if you, if you like say if this [orange and light green train] is one, then this [green-green-white train] would be two. And you have to like pretend that this [G-G-W train] was one whole right here.
	T/R 2:	What do you think, Liza?
Mr	Liza:	I think that one of these greens and half of this one [white] would be half.
	T/R 2:	Okay, so
Rlv	Jen:	Yeah, half of the white.
	T/R 2:	Okay, so if I imagine that I had a saw, a small saw and I could cut that [white rod] in half and take a green and the half of white
Mr	Jen:	So half of the white and this green and half of the white and that green would be the halves.

Maria and Sami as partners:

Mr	Maria:	It's thirteen. So we have a seven? What's seven?
Rm	Sami:	Green doesn't work.
Mr	Maria:	We need six.
Rm	Sami:	Blue.
Rm	Maria:	Green, no.
Rm	Sami:	No purple's right here [pointing to her staircase].
Rm	Maria:	Not a six, see watch. Ten, nine eight, less than this [holding a brown rod].
Rm	Sami:	It won't work.
Rm	Maria:	Oh yeah, it will. I'll prove it, watch. [She puts Y-W-Y under the train of O-LG.] What's highest after seven?

Time	Code	Speaker	Text
	Rm	Sami:	Dark green doesn't work.
	Rm	Maria:	Who said it doesn't. Yes it does, remember halves. [She changed her model from Y-W-Y to G-W-G.]
	Rm	Sami:	Yes, I do. [Both girls raise their hands.]
40:00	**Rm**	Maria:	Oh, oh, it works! [T/R 2 joins them.]
	Tqs	T/R 2:	Can I join you? Have you come up with an idea?
	Mr	Maria:	Yeah, I took two greens and a white. Then you have six and a half and six and a half and these. You would have it.
	Tmci	T/R 2:	Okay, so I see what you're doing.
	Tqc		[To the class] I want to hear what you're doing. I want to hear from some people. Show me one half of the orange and [light] green train. Andrew? Did you work with Mark? Do you both want to come up here? The rest of you watch and se what they do.
	Rm	And:	We took two dark green and a white. If we split white in half, then it would be half.
		Maria:	I have a solution.
		T/R 2:	Do you follow that? Dos this work inventing new rods again? Why do you think it works?
	Mr	Maria:	Well, because you have seven, seven and six. This number block, seven, two of the Well, take two dark greens and a white. And they're no blocks with halves, uneven, odd numbers and you need halves.
	Ir	Benny:	Last time with Mrs. Maher, like the block of gold. One you fit in the middle, split in half like we did last time.
	Tqe	T/R 2:	Benny had another model of all light green rods and one white in the middle. He split the white rod in the middle. [Benny built 4 models for the orange and light green train: G-W-G, LG-LG-W-LG-LG, R-R-R-W-R-R-R, P-Y-P]

Sami and Maria went to the overhead.

45:00	**Rm**	Maria:	[She builds a train of Y-LG-Y.] You can add one and a half to yellow on each side.

Time	Code	Speaker	Text
	Tqc	T/R 2:	Where did she get one and a half from? How did this become one and a half? What piece of it [the train] becomes one and a half? I don't understand.
	Mr, RIng	George:	She split it in the middle, one and a half on each side. [He holds up the light green rod and shows cutting through the middle of it.]
	Tqc	T/R 2:	Okay, all right. So if I cut that [Light green rod] down the middle, I see, okay. Well, see if we're calling this light green three, what are you calling the train of light green and orange?
	Rm	Maria:	Thirteen. Yellow is I think yellow is about five long, and green in the middle[Counting cm in the train] Ten [two yellow rods], eleven, twelve, thirteen [for the light green rod], thirteen yellows.
		T/R 2:	You were thinking of the whole length of the train as being thirteen of what?
		Maria:	Thirteen
		T/R 2:	Thirteen blues, thirteen oranges, thirteen what?
		Maria:	Thirteen yellows.
		T/R 2:	Thirteen yellows?
		Maria:	Turn light green into yellows.
		T/R 2:	I don't understand.
	Rm	Maria:	Well, if you cut that [light green rod] in the middle. Paint light green of each piece yellow and you're making it thirteen and it will be equal to the train.
	Tqc	T/R 2:	Do you understand my question? I don't understand when she's saying thirteen for the train of orange and green. I don't understand where she's getting the number thirteen from. Why thirteen?
	Rm	Ed:	If you take one of the orange rods and take all these little things [white rods] and put them up to it, it will equal ten. And if you do the same with the light green rod, you have three. And if you have ten and three you have thirteen.
		T/R 2:	Oh! So if you line up the white rods along the train of orange and light green and you have thirteen.
	Rm	Ed:	I have another solution. [He goes to the overhead and puts two light green rods under the orange and light green train. He adds seven white rods to the right of the light green rods.].

Time	Code	Speaker	Text
48:00		Ed:	I figured you could take two [light green rods] and put them there. After that I took clear ones [white rods]; I put down seven of them. I figured you have this, put a match. Light green, add clear. I took all the little ones and I figured that I have three four [He motions that he is adding one W to the LG , one W to the other LG, etc.], and then four, five, five, six, six, seven.
	Tqs	T/R 2:	Put seven on each of them? So there'd be seven and seven?
	Me	Ed:	Yeah, well, not seven and seven, seven and six. It's an odd number, it wouldn't be seven when

Model arrangement:

LG	LG	
W	W	
W	W	
W	W	W

Time	Code	Speaker	Text
	Tqs	T/R 2:	What happens to this guy? [pointing to the white rod to the far right] How can I be fair in making my two halves the same size? What could I do?
	Rm	Ed:	What you could do is, you could take this [white]. You could take those three whites and replace them with a light green. [He moves three whites and places a light green in his model.]

Model arrangement:

LG	LG	LG
W	W	
W	W	

And then it goes

Model arrangement:

LG	LG	LG	
W	W	W	W

Time	Code	Speaker	Text
	Tqs	T/R 2:	Then what about this guy? [She points to the remaining W on the far right.]
		Ed:	Oh, what this guy would do
	Tqe	T/R 2:	We ran into the same problem, didn't we? Would you agree that if we went back to this model where we had these [She rearranges the rods]. Would you agree that maybe I could take this one [white rod] and saw it in half, if I had a saw?

Time	Code	Speaker	Text
			Model arrangement:

LG	LG	
W	W	
W	W	
W	W	W

Time	Code	Speaker	Text
			And we were divvying them up [two halves of sawed white rod]
		Ed:	Yeah.
	Tqs	T/R 2:	And then what could I do with it?
	Mr	Ed:	Then you could put it here and here [pointing to the two columns of rods]
	Tqc	T/R 2:	We are out of time and we have to clean up. [Benny, directly in front of T/R2, raises his hand and shows his model of P-Y-P. As children are putting rods into boxes, the background noise makes hearing Benny difficult.]
	Mr	Benny:	...the yellow one in the middle. You can cut it in half to make halves.
	Tfp		That's really nice. ...Write about what we worked on the past few days.

54:00 End of class.

APPENDIX G

Coding Chart*

Session 2

*See Chapter III for coding notation

Sequence of events	Tea	Ed	Art	Ben	Dan	Jen	Katy	Alex	Bett	Joy	Geo	Mar	Mari	Jul	Jon	Din	Gra	Liza	Eil	Bob	Sam
T: recall yesterday	fp																				
CE A: find rod 1/2 as long as blue		ASn																			
R: There isn't any		m																			
R: we did find it. (cases) I don't think there is one		m																			
M: reasoning 1/3 of blue			r																		
A: Y = 1/2, O = ?	ASs																				
R: O = 1		m	m																		
A: find rod 1/2 as long as blue	ASn																				
M: I don't think there is one		r																			
M: reasoning 1/3 of blue			r																		
CE I: upper and lower bounds					f																
M: Y too big, P too small					r																
T: rod between Y and P?	qs																				
R: OHP staircase, M: no rod					m,r																
M: reasoning 1/3 of blue						r															
R: 1/2 not 1/3 of blue					lv																
M: 2Y > B, Y+P=B, 3/4 + 1/4		r																			

I: need the same (for halves)					e																
M: don't need same, 2 pieces		r																			
T: Brick of gold	qs																				
M: 2 pieces		r																			
T: Do you agree?	qc																				
M: P < Y							r														
M: 3/4 & 1/4= whole																					
M: reasoning 1/3 of blue				r																	
M: doing halves					r																
I: halves have to be equal			e																		
T: halves w/ different sizes?	qc																				
M: 2, almost divide in half		r																			
M: won't have half, quarters								r													
T: Summarize issue?	qc																				
M: reasoning 1/3 of blue			r																		
CE I: odd & even rods					f																
R: sort staircase					m																

T: They have to be same size?	qs																				
R: Yeah		lv																			
T: 2 conditions	fs																				
M: 2 P<B					r																
A: design new rods	ASm																				
CP, M: can divide 9 equally		r																			
CP, M: could make new rod			r																		
CP,M: O = 10, B = 9 can't divide B		r																			
CP, M: make a new color			r																		
CP, M: impossible to divide		r																			
CP, M: 1/2 of 9 = 4 1/2			r																		
CP, M: can't only thirds		r																			
CP, R: thirds or singles			lvf																		
CP, M: thirds not halves		r																			
CP, R: thirds and ninths			lvt																		
CP, M: halves, impossible		r																			
CP, A: halve or third every color			ASn																		

CP,A: Halve & third every color			ASn																		
CP, M: except blue		r																			
CP,R: building staircases														m						m	
T2: [to Mere & Sarah] Do you understand the problem?	qs																				
T2: [to Mere & Sarah] Describe new rod	qs																				
CP, M: O=10, B=9, cut B in half - 4 1/2; 1/2B = P+1/2W													r								
CP, T: what do you think, Sarah?	qs																				
CP, R: [Sarah nods affirmatively]																					Inv
T: New rod 1/2 of B	qc																				
CG, M: OHP pink= 4 1/2, R: m									r,m	r,m	r,m										
CG, M: smallest= 1/2 of W										r											
T: name the 1/2 of white	qs																				
CG, R: clear											lv										
CE I: always need to make another set												f									
M: always something not equal				r																	
M: always new set					r																
M: cut separate ones													r								

M: change pattern of set														r							
T: name the 1/2 of white	qc																				
R: light blue										lv											
T: another way?	qc																				
M:3LG, split middle LG into lt blue															r						
A [T2]: O=2, Y=?	ASs																				
T: [CT] explain to your partner [to Danielle]	qs																				
CP, M: Y=1																r					
CP, M: Y= 1/2 down from O, 1																	r				
R: 2Y train under on O							m, nvl														
R: O=2, Y=1													m, lv								
M: OHP Y = 1, Y+Y = O				r																	
M: OHP O=2, Y=1, O=1, Y= 1/2		r																			
A [T2]: O=6, Y=?	ASs																				
M: Y=5							r														
M: Y= 3			r																		
M: Y= 3						r															

T: why 5? If Y=5, O=6?	qs																				
M:Y=3, Y+Y=O, 5+5=10, not6						r															
A: O+LG=1, 1/2=?	ASm																				
CP, M [T2]: 1/2 = DG+ 1/2 W				r																	
T [T2]: There was no rod half as long as O-LG?	qs																				
M: no O+LG=10+3=13, 1/2 of 13 not equally				r																	
CP, R: 1/2 = DG+ 1/2 W																		Ing			
Cp, I, M: do it another way, like James [using LG]				e,c																	
R: 4LG then 2LG-W-2LG				m																	
CP,M: 1/2=DG+W						r															
CP, M: 1/2 = DG+ 1/2 W																	r				
CP, M: 1/2 = DG+ 1/2 W						r															
CPR: Y-W-Y test, then G-W-G												m									
T [T2]: can I join you?	qs																				
CP,M: 6 1/2 and 6 1/2 do it												r									
CP, M: OHP 1/2 = DG+ 1/2 W [with Mark]							r														
T[T2]: BrG's model LG-LG-W-LG-LG, "he split W"	qe																				

I: 7+ 6=13, uneven, odd													r								
I: last time, gold, split one in middle				r																	
R: Brian: G-W-G, 2LG-W- 2LG, 3R-W-3R, P-Y-P				m																	
CP, R: OHP [with Sarah] Y-LG-Y train													m								
CP, M: add 1 1/2 to Y on each side													r								
T [T2]: where did she get 1 1/2 from?	qc																				
M: Split it [LG] in the middle											e										
T [T2]: I don't understand; 13, 13, why 13?	qc																				
R: turn LG into yellows, get 13													lv								
M: # of W in train, O+LG = 10+3= 13		e																			
R: 7+6, 2LG+7W		m																			
M: LG+3W=1/2, W left		e																			
M: 3 LG+2W, W left		e																			
T [T2]: LG+3W, use saw on W	qe																				
M: LG+3 1/2W = !/2		r																			
M: P-Y-P, P+1/2Y = 1/2																					r
T[T2]:write about what we worked on	fp																				

APPENDIX H

Coded Transcription

Session 3

Session 3
Side, front cameras - OHP camera unmanned
T/R1,T/R 2, CT
Dr. L., principal, arrives during session

Small group: Joyce, Betty, George
Partners: Katy and Mario, Anne and Sami, Cassie and Kristin, Liza and Jen, Jon and Alyce, Ed and Art, Maria and Dan, Dina and Grant, Matt and Alex, Julie and Bob, Benny and Eileen

Activities:

1. If purple is called ½, what number name shall we give to brown?
2. If purple is called 1, what number name shall we give to brown
3. If orange is called 2, what number name shall we give to yellow?
4. *A train of yellow and light green is called 2, what number name shall we give to red?
5. *A train of yellow and light green is called 1, what number name shall we give to red?
6. Which is larger, ½ or 1/3 and by how much?

*Tasks 4 and 5 were given almost simultaneously.

T/R 1 talks about students going on a trip to look for fossils and then they discuss 'math bowling'. Math bowling is described by the students as a game where there are teams with three players on each team. Scores on the team are averaged. T/R 1 asked multiple questions about averages. Alyce says that you divide by the number of people on your team [others had said to divide by two to find the average]. T/R 1 asks why they need to take and average; the children do not respond. She leaves the topic with questions about when and how the team average gets better, worse or stays the same.

Time	Code	Speaker	Text
0:5:50	**ASs**	T/R 1:	Maybe Anne can tell us. Anne, if purple has the number name one half, what number name would be given to dark brown?
	Rm		[Dan and Maria build models that are erect - one dark brown rod next to two purple rods.]
	Rlv	Mere:	One half.
		T/R 1:	We have a consultation. Sami [Anne's partner], you can help Anne decide. You can discuss it with her.
TW ~1.5 min.			
			Okay, your consultation is over. Anne?
	Rlv	Anne:	One half.

Time	Code	Speaker	Text
	Tqs	T/R 1:	IF we give the purple the number name one half, we're going to give the brown the number name one half?
	Rlv	Anne:	No, I mean, um, the dark brown would be one.
		T/R 1:	The dark brown would be one. How many of you agree with that?
			[All visible hands are raised. Anne is smiling.]
	Tqs		And why is that, Anne? Can you tell me how you would convince me?
	Mr	Anne:	'Cause I put the purple rods up against the brown rod and I got two purple rods.
	Tqc	T/R 1:	How many of you agree with that?
			[All visible hands are raised.]
	ASs		What if I gave the purple rod the number name one? What number name would I give the brown rod? Liza?
	Rlv	Liza:	Two.
	Tqc	T/R 1:	How many of you agree with Liza?
			[All visible hands are raised.]
			Can you tell us why, Liza?
			[T/R 1 restarts Liza twice, because Liza is not speaking loud enough.]
	Rm	Liza:	If you put two of these [purple rods] together and each of these is one, then one, two. And that [the brown rod] would be called two.
		T/R 1:	Do you all agree with that? How many of you feel pretty good at doing that?
			[Hands go up.]
			Okay, that's neat. All right.

Time	Code	Speaker	Text
9:00	**ASs**		[At overhead] And you remember if we give the orange rod the number name two, can you tell me what number name we give to yellow? Do you remember how to do that?
		Mere:	Oh, oh [raising her hand].
		T/R 1:	Julie?
	Rm	Julie:	[At her desk] If you take two yellows up against the orange rod, they match up. And if this [the yellow rod] is half of it. Well, if the orange is two, this [yellow rod] is one.
		T/R 1:	Do you all agree? Thanks. Well, that's fantastic. Suppose I have a train now...
			[Some conversation between T/R 1 and T/R 2 takes place about whether the next problem had already been examined.]
10:30	**ASs**		If I make a train of one yellow and one light green, one yellow and one light green, and I call this train two, what number name would I give to red? If I call yellow and light green two, what number name would I give to red? Remember now, you must convince me why. When you seem to have an answer, why don't you discuss it with your partner and see if your partner agrees with you.
11:00	**RIne**		[Dan and Maria's hands go up.]
		T/R 1:	If you think you have done the problem and you're waiting for someone to come around,
			[T/R 1, T/R 2, and Dr. L. circulate among the children, asking questions about their answers.]
11:30	**ASs**		I'd like you to make yellow and light green one and tell me the number name for red. Remember the problem, if yellow and light green is two what number name would I give to red? I want you to do both problems...
			[Bob and Julie build two physically identical models.]
		Joyce:	One fourth.
		T/R 1:	Let's see if you fall into a trap. [She talks with Dan and Maria.]

Time	Code	Speaker	Text
			How many of you changed your mind from what you first thought?
			[Some hands go up.]
			Some of you changed your thinking. Uh, huh! Okay, if you're ready to discuss the answers, raise your hands. Remember there are two problems. If yellow and green are two, what number name for red? If yellow and green are one, what number name for red?
			[Only some hands are raised. Some children are talking to their partners, more time is given.]
14:30	**RIv**	BrG:	I think it'd be fourths.
			[Eileen's comment is inaudible.]
			One eighth. [Pause.] I mean, yeah, I mean this is like a half and this would be… one [cannot see his model].
			[Both BrG and Eileen are not speaking; they seem to be deep in thought.]
			Which one? This is one [BrG holds up yellow rod as camera leaves them.]
	RIv	Julie:	You would call it one fourth [She indicates the red rod]. This is two, wait, yeah, this is two and this is another yellow and this is one. This is one fourth [she puts four red rods under the yellow and light green train]. And this is, one half [She puts four red rods under the yellow and green train that was called 'two']. One fourth, one half.
	RIv	Bob:	[thoughtfully] one fourth and one half.
		Camera captures T/R 1 as she leaves Jen. T/R 1comments that Jen seems confused and needs to think some more about this.	
17:36		T/R 1:	okay… Can someone explain to Dr. L… Dr. L., can you call one someone?
		Dr. L:	Art!
		Art:	We made a train and if this is considered two, what would the red rod be?

Time	Code	Speaker	Text
		Dr. L.:	Uh, huh! Boy, that's some problem.
		Art:	And the other one was if this [train] would be one, what would the red rod be?
		Dr. L.:	Okay, was anyone able to figure it out?
			[Some hands are raised.]
			Anne, what did you come up with?
			[Sami and Anne go to the OHP. Anne builds a model of the train Y - LG]
19:00	**Rlv**	Anne:	Oh, first we put the red rods [she places four red rods underneath the train] up to the yellow and light green rods and then we said if the yellow and the green was two, what would we call the red rods? And we thought that we would call it one and one fourth. And if it [the train] was one, we would call it one fourth.
		Dr. L:	Okay, so if it was one you said you'd call it one fourth and if it was two, what did you say?
		Anne:	It would be one and one fourth.
		Dr. L.:	One and one fourth, I don't know if I understand that. [To the class] Do you all agree with that? Did you come up with the same names for that? No?
	Tqc	T/R 1:	How many of you agree that if we call the yellow and the green one two
		Anne:	Two, it would be
		T/R 1:	The red would be one and one fourth. How many of you agree?
			[No hands are raised.]
	Tqs		You're not having people agree with that, so you are going ton have to convince them, Sami and Anne. What would you do to convince the class that it would have the number name one and one fourth? But before we ask you to convince them, I'm curious about the other. If you call the yellow and light green together one, what would you call the red?

Time	Code	Speaker	Text
		Anne:	One fourth.
	Tqc	T/R 1:	How many of you agree with that? [Hands are raised.] So we have some people agreeing with that. They agree with your second solution, but not your first. So let's hear some arguments...
	Mr	Anne:	Well, because see, the yellow and the green was the same size as the brown, so if we put the reds up against, no, wait, no. See, because there's, if there was one, if it was brown you would normally call it one. And if we put the reds up against it we would all call it one fourth, so we thought if we call the yellow and the green one, it would be the same thing as the brown.
	Tqc	T/R 1:	How many of you agree with that argument for calling the red one fourth when the yellow and the green [train] are one? How many of you agree with the argument that Anne just gave us? [T/R 1 restates Anne's argument.] Ed?
		Ed:	I agree.
		T/R 1:	You agree?
	Mr	Ed:	Yeah, because see if the brown and the yellow and green are equal and they both are called one, and four reds to equal up to one, therefore they'd have to be fourths.
	Tqc	T/R 1:	Would you raise your hands if you agree? Up high, so I can see them. Some hands are down; does that mean you disagree or you are not sure? Bob?
	Mr	Bob:	We disagree. We thought the two, we called that one half.
		T/R 1:	Okay, we're talking about when we call it [the train] one. You're talking about the other... Do you agree that when we call it one, red is one fourth?
		Bob:	Yes.
		T/R 1:	Okay, now the second part, all of you hear... I like the brown rod up there to show you there's another way to call it [the train] one.

Time	Code	Speaker	Text
			That's very nice. Some of you didn't do this. It's something Sami and Anne introduced that I think is very nice.
			But let's hear the other argument. You think you get one and one fourth when they called the brown rod two?
			[Anne and Sami are quiet. They seem unsure.]
			You're not sure you have an argument? Do you want to pull back and listen to someone else? Bob, do you want to come up here… And Julie?
			[Bob and Julie come to the overhead.]
	Mr	Bob:	We think the two [he moves the train of yellow and green] would be called a half.
		T/R 1:	The two of what, Bob?
		Bob:	?
	Mr	Julie:	When this [the train] is two, this [the red rod] would be called a half.
		T/R 1:	You're saying the red would become a half?
		Julie:	Yeah.
	Tqs	T/R 1:	That's an interesting idea. So when yellow and green become two, how can you convince us? How many of you agree? A few of you agree. Now you have to help convince the rest of us. Can you convince us?
		Julie:	Not really.
		T/R 1:	Bob?
		Bob:	Julie thought it was. So she should be able to explain it.
		Julie:	Well, this is called two [the train] and all of these would be one half [the red rods].
			[She sighs and strums on the overhead projector.]

Time	Code	Speaker	Text
	Tqs	T/R 1:	Dan, do you want to help them out? You also thought one half, you and Maria. [Dan goes to the overhead projector.]
26:06	**Mr**	Dan:	Um, so if this is called two [the yellow and light green train] and then this would be two too [the brown rod]. So then this would be one [indicating the two red rods]. But then if we take this away [one red rod] it would be one half over there [the red that is remaining] and put another one half that would be one and another would make up to be two [realigns the four red rods to equal the length of the yellow and light green train].
		T/R 1:	Did you all follow what, what Dan said? Dan, you're going to have to do it again. I think some people had a little trouble following it. All right. Mario, did you follow it?
		Mario:	Yeah.
		Dan:	All right, so...
		T/R 1:	You can help say it another way. It might help other people follow it so give Dan another chance and then maybe Mario can help him out, and Maria.
		Dan:	Okay, so this is two [the yellow and light green train], and this would be a half because if you put another one and another one that'd be two [He aligns four red rods]. And if you take away these [two red rods] that would be one and took away that [He takes away another red rod], leaving one red rod], that would be a half.
		T/R 1:	How many of you understand? How many of you followed what Dan said? Raise your hand. Is that what you were thinking, Julie?
		Julie:	Yeah. I just couldn't get the words out.
		T/R 1:	You couldn't get the words out. Do you want to try it again?
		Julie:	No.
		T/R 1:	Who wants to give it a try? Bob?

Time	Code	Speaker	Text
		Bob:	Well
		T/R 1:	I liked Anne's trick of finding out what one was in the other problem. Remember Anne and Sami came up with the brown rod. Do you remember that?
	Mr	Bob:	If you take these two things [two red rods], that would be one half.
		Julie:	One.
		Bob:	One. And if you take one away then it could be a half [the red rod].
		T/R 1:	That's sort of what I heard Dan say, but you were beginning to say something else...
			[T/R 1 goes to the overhead and moves two of the reed rods.]
			The temptation I noticed many of you used, you [Bob] wanted to call these two reds a half and these two reds a half. And you saw Julie shaking her head...
			As I walked around I saw lots of people doing that...
			Is it okay to call this [two red rods] a half and this[the other two red rods] a half sometimes?
	Mr	Julie:	Well if this, if you call both of these one half, then this [the train of yellow and light green] would be one.
		T/R 1:	...What do you think is so confusing here?
31:06	**Mr**	Ed:	I think the confusion is, they think, that they think, they have the temptation of calling, since there are four red blocks, they think they are gonna call it one fourth 'cause they forgot that the yellow and the [light] green are two.
	Tqs	T/R 1:	What are they thinking that the yellow and the green are when they do that?
	Mr	Ed:	One.
		T/R 1:	They are thinking that the yellow and the green are one when they do that?

Time	Code	Speaker	Text
	Mr	Ed:	Because, see, if you have one there'd be two halves, but if you have two its two halves plus two halves which would be four halves. There fore, you would have to call one of the reds one half.
		T/R 1:	How many of you understand?
			[Hands are raised.]
	Asm		When asked how many had fallen into the trap of calling two reds a half, many students raised their hands. T/R 1 poses a new question to the class. She speaks of Tom and Alyce liking chocolate and how she gave each of them a half of a candy bar. Tom was happy and Alyce felt it was unfair.
		T/R 1:	How could that be? Matt?
	Rlv	Matt:	You probably gave Tom a bigger half than Alyce.
	Tqc	T/R 1:	Can a half be a bigger half? You told me when I called this one [the red rod], right, two reds make a half and two other reds make another half... Maria, does that make sense to be a bigger half?
33:25	**Rlv**	Mere:	Mn, mmn [negative response].
			Well, see this was one [indicates a yellow and light green train] and then you gave this much to Tom [yellow rod] and this much to Alyce [the light green rod]. That wouldn't be a fair cut.
		T/R 1:	I agree with that but I wouldn't call that a half. Why wouldn't I, if I called this one I wouldn't call green a half and I wouldn't call yellow a half. If I did, Dr. L. wouldn't let me come back. She'd day stay out of that class, what are you teaching these students? Would I have called it a half? Dan?
	Rlv	Dan:	No, because it wasn't even.
	Tqs	T/R 1:	What do you mean by that, Dan?
	Mr	Dan:	Well, um, a half should be even so that the other side is the same as it is. So the yellow is bigger than the green and the half should be the same size.
	ASs	T/R 1:	If I called the brown rod the candy bar, can you find a rod that would be a half of the candy bar? Remember how the brown rod

Time	Code	Speaker	Text
			is the same size as the yellow and light green train. I want to use Sami and Anne's trick and I'll call yellow and green dark brown…
	Rm	Bob:	Two purple equal the brown rod.
		Dan:	Two purples equal up to the brown rod.
		T/R 1:	Remember I did not violate the condition that halves be equal when I gave the pieces of candy to Tom and Alyce. What could I have done to make Alyce so annoyed with me? Anne, what do you think?
	Rm		Anne shows T/R 1 the model that Maria had offered. She says that Alyce got the light green half and Tom got the yellow half and that's why Tom's was bigger.
	Tqc	T/R 1:	I didn't do that. I really made the candy bars have halves the same size. I don't know why Alyce was so angry. What else could I have done that would make her feel badly? Do you want to know what I did? How many of you want to know? Tell me what I did and if I'm right or wrong. [She holds up a large candy bar.] I gave Tom Purdy half of this candy bar, right down the middle… I gave him half. And [She holds up a small candy bar.] right down the middle, right into two equal pieces [Children are giggling.] Why should she [Alyce] be annoyed with me? Has anyone pulled that on you? You wouldn't do that with your younger brother or sister…
		SS:	Yes!
		T/R 1:	Why was Alyce so angry? I didn't trick her that way. Cassie?
	Mr	Cassie:	They're not the same size.
		T/R 1:	That's right! What does that have to do with what we are doing, if anything? Bob?

Time	Code	Speaker	Text
		Bob:	We're working with halves.
		T/R 1:	that's true, we're working with halves.
	Rm	Anne:	Because, see, we took these two together [She indicates a yellow and light green train] and you called it two and it would be like one candy bar. And the other candy bar and well, if you put the reds on top, um, I think, someone, they said that if it was a half, if you put two reds on top of the green and it isn't 'cause the two reds is bigger than one light green and it can't be half and just like the chocolate bar couldn't be a half.
		T/R 1:	That's very interesting.
			[T/R 1 uses the large candy bar to find one half of it; uses the smaller candy bar to find one half of it]
	Tqc		...I'm mixing up my ones. Is that allowed when you're comparing things? To mix up my ones? 'Cause then I could say to you, is it fair to compare different sizes?
		SS:	No.
	Tqc	T/R 1:	When we ask the question, which is bigger one half or one third, what are we assuming? What's the common understanding about that?
46:00	**Mr**	Mario:	Well, normally one half is bigger than one third, but if you got a bigger size of candy bar or pizza, and if you get one third of that, then that'd be more than one half of a little pizza.
		T/R 1:	Okay. We don't want to fall into that trap, can we have an agreement in this class? And maybe you want to think about that for the rest of your life in mathematics. When we compare fractions, it is the same thing. If I ask you which is bigger one half or one third, we mean of the same object. We're not allowed to switch.
	ASs		One last question. Which is bigger one half or one third? What do you think is bigger, one half or one third... You can think of this candy bar... [The candy bar is scored in a three by four grid pattern.]

Time	Code	Speaker	Text
	Rlv	Julie:	Cut it in half the long way.
		T/R 1:	Why not the other way?
	Mr	George:	'Cause there's three of them [three sections across the bar]. You get six parts.
		T/R 1 asks what about a third of the candy bar.	
	Mr	Dina:	You get four wedges out of twelve. One half is six out of twelve; one third is four out of twelve.
		T/R 1:	Who gets more? The person with one half or one third?
		Alyce:	The person with one half.
			[All children agree.]
		T/R 1:	Is it possible that I was talking about different sized candy bars? Can you imagine that one third is larger than one half? One third of big, one half of small?
		Alex:	I want one third
		T/R 1 discusses children sharing, the dishonesty of switching sizes of candy	bars or pizza pie.
		T/R 1:	So what's the question you should always be asking yourself when you compare fractions? Maria?
	Mr	Mere:	Which one's bigger.
	Tqc	T/R 1:	Which? Which thing is bigger?
46:45	**ASm**	T/R 1:	One last problem... Make me a model to show me which is bigger one half or one third. And I want you to tell me how much bigger and be able to convince me.
	Rm		[Dan and Maria are busy constructing bArtce beams with rods for their model.]
TW: ~ 7 min.			
53:38		T/R 1:	What do you have there?

Time	Code	Speaker	Text
	Rm	Bob:	If you take two dark greens and you make each one a half and you make these [purple] a third, they'd be equal.
		T/R 1:	So which is bigger?
		Bob:	They're equal these colors [indicates the length of the whole – which train is he using?]
		T/R 1:	What number name is this [a dark green rod]?
		Bob:	A half.
		T/R 1:	What number name is this [purple]?
		Bob:	A third.
		T/R 1:	Which is bigger a half or a third?
	Mr	Bob:	The half.
		T/R 1:	The half is bigger
		Julie:	Oh yeah
	Tqs	T/R 1:	Right, by how much?
	Mr	Julie:	By an inch.
	Mr	Bob:	No, by red.
	Tqs	T/R 1:	By red? What number name would you give the red then?
	Mr	Bob:	A fourth.
		T/R 1:	Remember what you called one.
	Mr	Julie:	One fourth.
	Tqs	T/R 1:	What number name, prove to me that red is a quarter.
			[Julie moves closer to Bob to see what he is doing.]
			This is red, that's a half [the dark green rod]. Sure it's a quarter? Change your mind?

Time	Code	Speaker	Text
		Julie:	Yeah.
		T/R 1:	Okay, great, can you explain that?
		Bob:	Maybe…
			[T/R 1 walks away as the students continue]
			Okay, so what would these be?
			[Julie counts the red rods.]
	Mr		They're sixths.
	Rm		Liza and Jen are invited to go to the overhead and share their solution. They use a train of three purples and a train of two dark greens as their model. When they are asked which is larger, one half or one third, they reply the dark green was bigger.
	Tqs	T/R 1:	How much bigger?
	Mr	Jen:	One red bigger.
	Tqc	T/R 1:	What number name would you give to the red rod, and why?
			Why don't you think about this over the weekend?
			If the orange and red train were 'one', what would be the number name for the red rod?
57:46	Mr	Art:	One sixth. We know already that, that, three reds would make a dark green and if there are two dark greens to make an orange and the red rod then it would take six red rods to make the orange and the red rod.
			The class is getting ready to leave and Dan is setting up a bArtce. T/R 1 calls T/R 2 to listen to Dan. The whole class begins to gather around Dan.
	Rm	Dan:	All right, I made a bArtce and the whole thing is dark green and the light green is a half and the reds are the thirds, but then what I'm doing is, um, I'm making a bArtce so when I take off that [one light green rod] and then the two reds, then I think it will fall to the side and show a half is bigger.

Time	Code	Speaker	Text
	Mr		[He places two light green rods on one side and the three red rods on the other. When he removes the one light green rod and the two red – leaving one light green rod (a half) and one red rod (a third) – his structure falls to the side of the light green rod (signifying that side of the bArtce is heavier - bigger)].
	Tqs	T/R 1:	It did fall to that side. Your prediction was right. Okay, now the question I'm going to ask you, when you work on this bArtce what would you have to put there to stop it from falling? What other rod could you have put on the left side so that it wouldn't fall when you took that off? Do you understand my question? What did you take off?
		Dan:	I took off the two reds and a light green.
		T/R 1:	Okay, now if you don't want it to collapse, right? You said it fell to the right the way you had it built, okay?
		Dan:	Um.
		T/R 1:	And the red rods were on the right side? Is that correct and the greens were on the other side, or was it the other way?
		Dan:	Well, the reds were on the left side.
		T/R 1:	On the left side. So you took the two reds from the left side and the green from the right side. Okay, what would you have had to put on that other side so it wouldn't tip? Once you took the two reds and the green off? Do you understand my question?
		Dan:	Um, let's see
		T/R 1:	What would you have guessed it should have been?
	Mr	Dan:	Um, maybe a little white?
			[He has a light green rod next to a red rod and equals the length of both by adding a white rod to the red rod.]
	Tqs	T/R 1:	A little white? Okay, we could try that experiment on Monday, right? That's a good guess. Why did you guess that? I think you went looking for something specific. Why were you looking for that one?

Time	Code	Speaker	Text
		Dan:	Well, 'cause when I went like this I just saw there was one space in between and I knew there white is that space.
		T/R 1:	Okay, what number name give to white?
			Dan did not answer and T/R 1 suggested that this was something for the class to think about.

End of tape.

APPENDIX I

Coding Chart*

Session 3

*See Chapter III for coding notation

Sequence of events	Tea	Ann	Sami	Mari	Dan	Liza	Jul	Ben	Bob	Ed	Geo	Mar	Matt	Cas	Joy	Gra	Aly	Jen	Art	Dina
A: P=1/2, Br =?	ASs																			
R: erect models , I Br, 2 P				m	m															
R: erect models "1/2"				lv																
R: "1/2"		lv																		
?: P =1/2, Br =1/2?	qs																			
R: Br=1		lv																		
Convince me	qs																			
M: P to Br, 2P =Br		r																		
?:How many agree?	qc																			
A: P=1, Br =?	ASs																			
R: 2, 1+1=2						m														
?:How many agree?	qc																			
A: O=2, Y=?	ASs																			
R: O=2, Y=1							m													
A: Train Y+LG=2, R=? (10:30)	ASs																			
R: hands raised				lne	lne															
A: Train Y+LG=1, R=? (11:30)	ASs																			
R: "1/4", "1/8"								lv												

R: "1/4"(1), "1/2"(2)							lv													
R: "1/4"(1), "1/2"(2)									lv											
R: "1/4"(1), "1 1/4"(2)		lv																		
?: how many agree?	qc																			
?: convince us Sarah & Audra	qs																			
M: Y+G=Br, R=1/4		r																		
?: how many agree?	qc																			
M: Y+LG=1, 4R=1, R=1/4										r										
?: how many agree?	qc																			
M: Br=2, R=1/2							r		r											
?: Can you convince us?	qs																			
R: not really							lv													
?: David, can you help?	qs																			
M: Y+LG=1, 2R=1, R=1/2					r															
M: 2R=1/2 of Y+LG									r											
R: 2R=1							lv													
M: 2R=1, R=1/2									r											
M: If 2R=1/2, then Y+LG=1							r													

?: What is so confusing?	qc																			
M: 4R -> R=1/4, but Y+LG=2										r										
?: If R=1/4, Y+LG=?	qs																			
M: then Y+LG=1										r										
M: 2/2+2/2=4/2, R=1/2										r										
A: Candy bar w/ Amy & Tom	Asm																			
R: Bigger half to Tom													lv							
?:Can halves be bigger?	qc																			
R:no, Y-->T, LG->A, no fair cut				lv																
R: no it's not even					lv															
?: what do you mean?	qs																			
M: even, same size					r															
A: Br=cbar, find 1/2 cbar	ASs																			
R: 2P=Br									m											
R: LG->Amy, Y->Tom		m																		
?:halves same, show lg & sm	qc																			
M: not the same size bars														r						
R: LG not 1/2 of train		m																		

?:Fair to use dfferent sizes?	qc																			
A: Which is bigger 1/2 or 1/3?	ASs																			
..what are we assuming?	qc																			
M: depends size of bar/pizza											r									
A: Which is bigger 1/2 or 1/3?																				
…3x4 candy bar shown	qc																			
R: cut 1/2 the long way														lv						
M: 3 sections, you get 6										r										
?: what aout 1/3?	qc																			
M: 1/3=4of12, 1/2=6of12																				r
?:who got more 1/2 or 1/3?	qc																			
R: person with 1/2																lv				
?:What should you ask?	qc																			
R: Which one is bigger				lv																
?Which thing is bigger?	qc																			
A: 1/2or1/3 how much bigger?	ASm																			
R: balance beams				m	m															
CP,R: G=1/2, P=1/3							m		m											

CP, M: 1/2 > 1/3									r											
?: By how much?	qs																			
CP, M: by one inch							r													
CP,M: No, by a Red									r											
?: what number name for R	qc																			
CP,M: R=1/4									r											
CP,M: R=1/4							r													
?: G = 1/2, R= 1/4?	qs																			
M: they're (R) sixths!									r											
CP, R: G=1/2, P=1/3						m												m		
?: How much bigger??	qs																			
R: 1/2>1/3 by 1 R																		lv		
?: What # name for R?																				
M: 1/6, 6R = O+R																			r	
R: Balance beam, 2LG, 3R					m															
M: take off LG, fall to side of R					r															
R: take off 2 R, 1LG					m															
?:what needed for balance?	qs																			

R: a little white					m															
?: What #name for W?	qs																			
M; one space					r															
End of class																				

APPENDIX J

Samples of Children's Work

First Submission

Working with fractions with Rutgers

Today Rutgers came into our classroom. We did fractions using quizzinaire rods. For example:

If blue was considered 1 what color name would $\frac{1}{3}$ be? [redacted] Light Green

Another example:

If orange was considered 1 what color name would $\frac{1}{5}$ be? [redacted] Red

E. D.

Wroking with Fractions with Rutgers 9/20-9/21

When we worked with Rutgers with fractions They taught us that if you call a rod 1 whole you can figure out what 1 half is and call it 1 half. You can also call something 6 or 10 and figure out the half of it and call it half. For inctance if you have something that you call it 6 1 half would be three. We also used thirds and wholes.

Working with fractions with Rutgers

M. a

9/20/93

I enjoyed making fractions with rods. It was fun when we got to make fractions for the class to solve. I liked the rods because they were a new way to do fractions. I also learnd that not every rod that you put down could be made. I never thought that sticks could be divided by smaller sticks. I never thought that rods could teach so much about fractions. Now just about everywhere I see fractions. I'm really looking forward to do more. Some things that I see now are: are my book shelf that's divided into thirds, and my dresser is divided into fourths. Thank you I reall learnd alot.

Working with fractions with Rutgers 9/20 9/21

B F

When Rutgers came we work on fractions and we talked about halfes and thirds and wholes and how many there are in one. I liked it, it was fun.

Working with fractions with Rutger

We were working with fractions using the rods. We thought of many different fractions. The rods could be an easy way to find out fractions. We used certain rods to make other rods. The partner we worked with made up fractions for us to figure out. We got to show our solutions on the overhead projector.

A. L.

Working with fractions with Rutgers 9-20 + 9-21

O : I

First Dr. Mar gave us a problem with the cuisinaire rods. If she gave us a problem using orange ro and the white rod, she'd say, if I called the white rod one what would I call the orange rod? After that we would put some white rods up against the orange rod, to see how many rods it would take to add up to the orange rod. Then, if we should have 11 white rods, your fraction answer would be $\frac{1}{11}$.

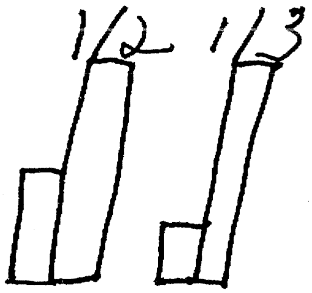
APPENDIX K

Samples of Children's Work

Second Submission

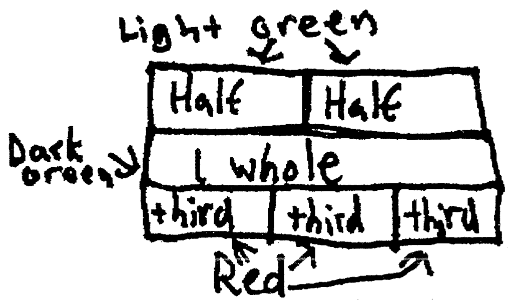
G m 4.Rh Sept 27 1993

What is bigger 1/3 1/2



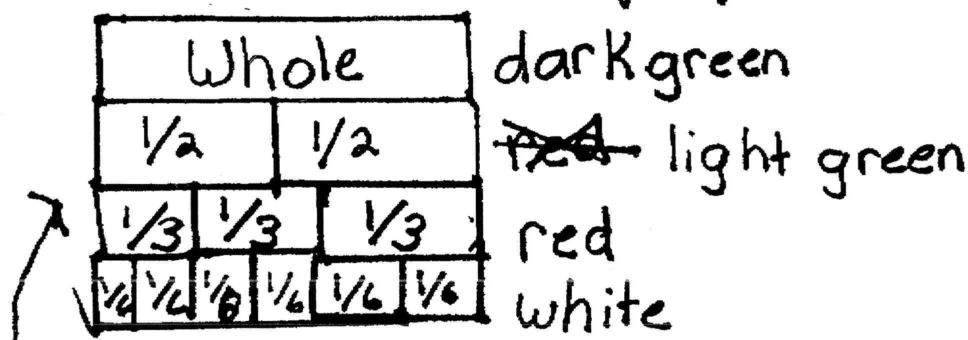
1/2 is bigger because if get 1 half and tak away one.

E 4-Ph 9-27-93



I think 1/2 is larger than 1/3 because if you have a whole and cut into halfs you have two equal parts, third you have three. If you cut something in half it is parts and two may be a larger number but in fractions 1/2 is larger than 1/3. it takes a larger piece to be a half becouse it is two and three you could make smaller, and thirds are three parts.

B R 9/27/93



Which is larger 1/2 or 1/3?
why? 1/2 is larger.

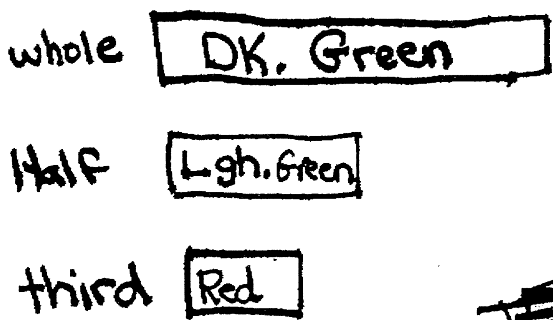
J Math

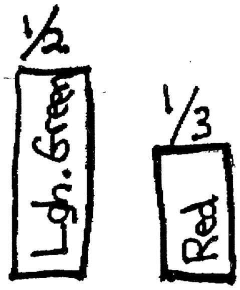
Sept. 27, 1993
4-Ph

Which is larger 1/2 or 1/3? Why?

I know 1/2 is larger because if you compair the light green to the red the green would be bigger.

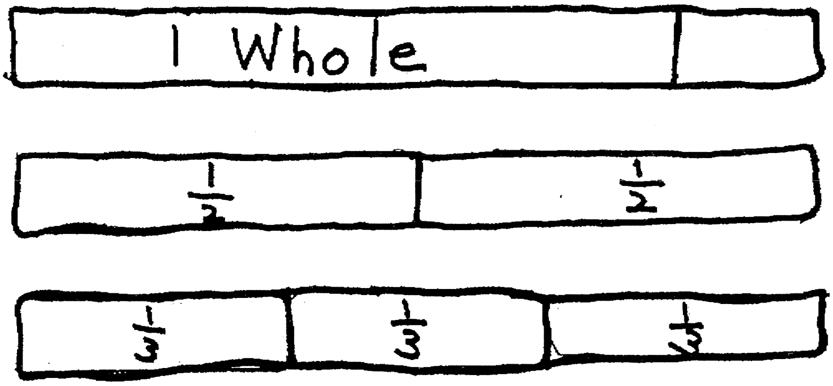
Key
Lgh. = light
DK = Dark





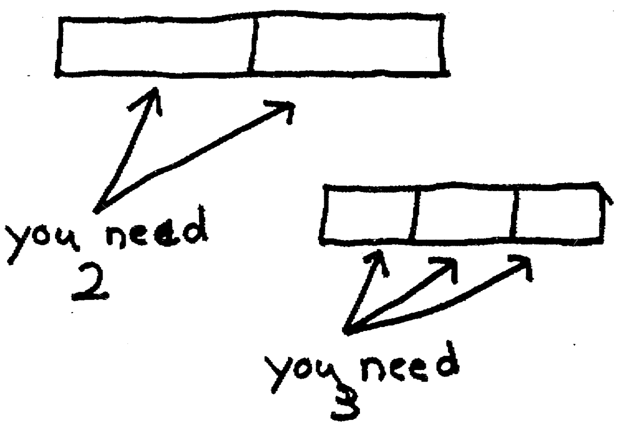
B F 9/27/93 Rutgers

I made a train



The $\frac{1}{2}$ is bigger:

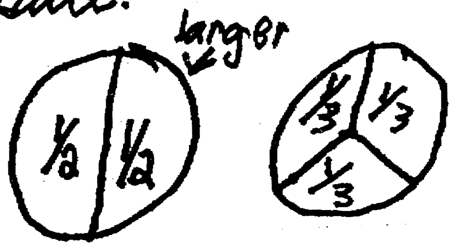
because all you need is 2 of them to make 1.



Which is larger 1/2 or 1/3? why?

M M 4 Ph.

I think 1/2 is bigger than 1/3 because 1/2 has more space.



As you can see in the picture 1/2 takes up more room than 1/3 so 1/2 is your answer.

APPENDIX L

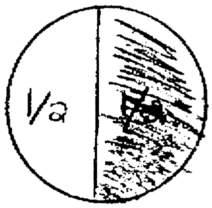
Samples of Children's Work

Third Submission

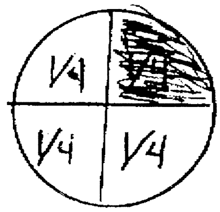
B a

Is 1/2 bigger 1/4 if so by how much

① halves

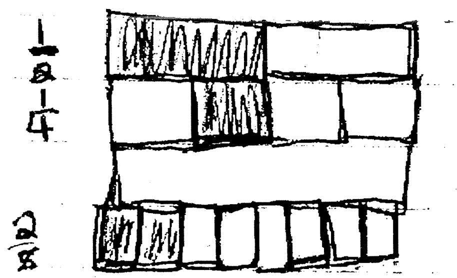


② fourths



I think that 1/2 is bigger then 1/4. Because if you look at the picture 1/2 looks like one quarter more then 1/4.

M. F.

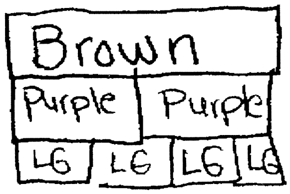


$\frac{1}{2}$ is bigger than $\frac{1}{4}$ by $\frac{2}{8}$ or $\frac{1}{4}$.

Yes.

J G

In Math we had to figure out all different problems. One example of a problem goes like this.....



Just for know you have to pretend that the blocks are even.

The brown would be a whole the halve would be purple and the fourth would be light green.

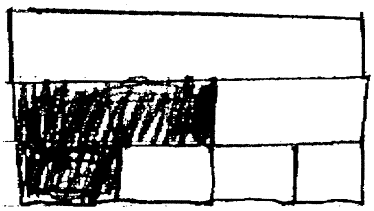
It is not possible to get different answers with different models.

M. M.

① $\frac{1}{2}$ is larger than $\frac{1}{4}$ by $\frac{1}{4}$ because if you measure $\frac{1}{4}$ it is smaller by $\frac{1}{4}$.

$\frac{1}{2}$

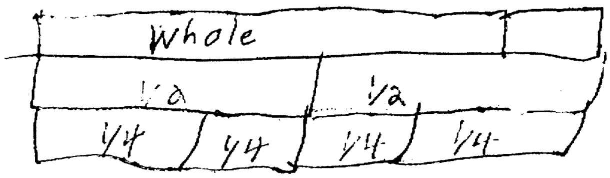
$\frac{1}{4}$



② Yes it is true you can get different answers with different models. If someone else builds a model a different way you could have differen anwers.

M A 10/3/93

I think half is bigger than 1/4 by 1/4 because it ~~them~~ takes two 1/4 to equal a half and I think ~~that~~ they will always get 1/4, because it takes 4 1/4 to equal a whole and it will always take 2 halfs to equal one whole. I agree with the model below because it shows what I am trying to explain that 1/2 is bigger than 1/4 by 1/4. This is true for any model.



E

I compared my model with D 's model. My model was two orange rods. I figured out that two yellow rods equal one organe and four yellow rods equal 2 organge rods. D 's model was one purple rod and the red rods as the halls and the white rods as the fourths. I dissagree that you can get different answers, because no matter what rods you use you will usally get the same answer.

<table>
<tr><td colspan="4">1 whole</td></tr>
<tr><td colspan="2">1 half</td><td colspan="2">1 half</td></tr>
<tr><td>1 fourth</td><td>1 fourth</td><td>1 fourth</td><td>1 fourth</td></tr>
</table>

<table>
<tr><td colspan="4">1 whole</td></tr>
<tr><td colspan="2">$\frac{1}{2}$</td><td colspan="2">$\frac{1}{2}$</td></tr>
<tr><td>$\frac{1}{4}$</td><td>$\frac{1}{4}$</td><td>$\frac{1}{4}$</td><td>$\frac{1}{4}$</td></tr>
</table>

APPENDIX M

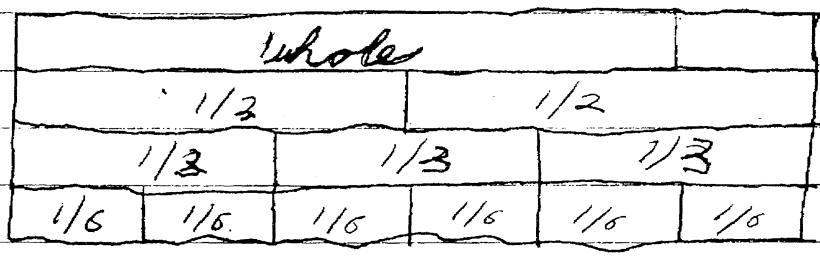
Samples of Children's Work

Fourth Submission

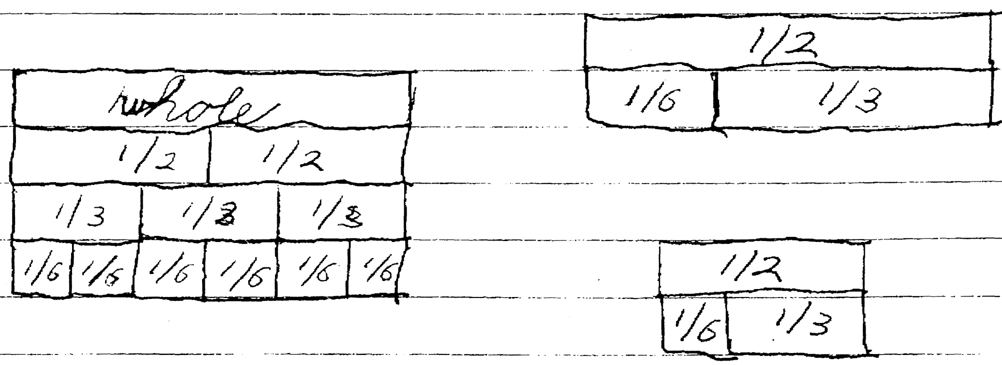
2/3 is larger then 1/2. By how much? A. P.

I agree that 1/2 is less then 2/3 because if you had two pies. You divide one into 3rds and the other into halfs. You take two 3rds out of the pie and 1/2 out of the other. Then you'll see your answer.

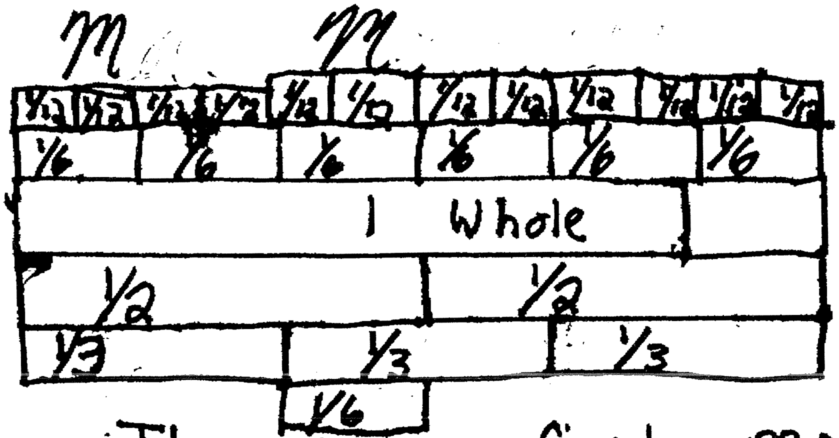
1



2

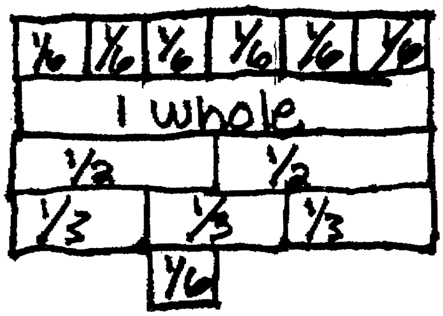


model



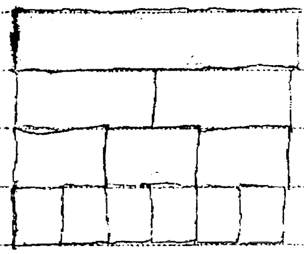
This is my first model
it shows that 1/2 is
smaller than 2/3 by 1/6.
because there is six red blocks.

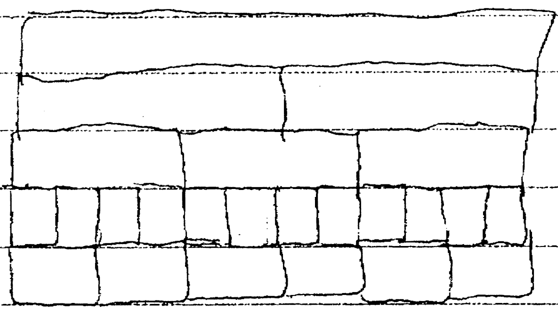
#2
model



This is my second model
it also shows that 1/2 is
smaller than 2/3 by 1/6.

M. F.





$\frac{2}{3}$ is bigger than $\frac{1}{2}$ by $\frac{1}{6}$ or $\frac{2}{12}$. I know this because if you take a $\frac{1}{2}$ piece and $\frac{2}{3}$ piece of my second model and you put the $\frac{1}{2}$ piece next to the $\frac{2}{3}$ pieces the $\frac{2}{3}$ pieces are bigger. Then you take a $\frac{1}{6}$ piece or a $\frac{2}{12}$ piece and you put either of the pieces next to the $\frac{1}{2}$ piece. It is equal to $\frac{2}{3}$rds.

Same: Different
Answer, Some are the same colors but different measurements.

1

6

I think $\frac{1}{2}$ is samler than $\frac{2}{3}$ because if you divide somethi into thirds there is no half-way point but in halfs there is a half-way point. If there is no half-way point it is either larger or smaller. If you were working with $\frac{1}{3}$ it would be smaller but we are working with $\frac{2}{3}$ so $\frac{2}{3}$ would be larger by one sixth.

Diagram #1

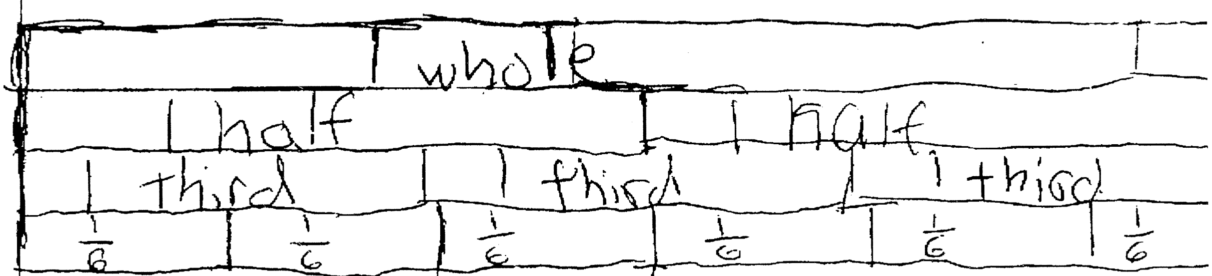
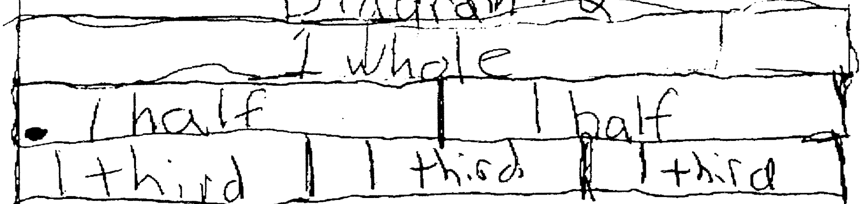


Diagram #2



The second question was that the same thing about my models was that I got the same answer each time and each whole was able to be divided into halfs, thirds and sixths. The differences in my modles were that the thirds, halfs and sixths were different sizes.

M . A. Fractions

I think that 2/3 is bigger than 1/2 by 1/6. It takes 6/6 to equal a whole and 1/6 is always half of 1/3. It takes three 1/6 to equal 1/2, but you need 4 1/6 to ~~eqal~~ equal 2/3. That proves that 2/3 is bigger also 2/12 is just like saying 1/6. There is nothing different with my ~~Model~~ models except the size of my whole, 1/2, 1/3, and 1/6. The things they have in comon is 2/3 is bigger by 1/6 in both.