TO SYMBOLS FROM MEANING:

STUDENTS' INVESTIGATIONS IN COUNTING

by

ELIZABETH B. UPTEGROVE

A dissertation submitted to

The Graduate School of Education

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements

for the degree

Doctor of Education

Graduate School of Education

approved by

____

Carolyn A. Maher, Chair

____

Lara J. Alcock

____

Steven M. Barnhart

____

Arthur B. Powell

New Brunswick, New Jersey

January 2005

ABSTRACT OF THE DISSERTATION

To Symbols From Meaning: Students' Long-term Investigations in Counting

By ELIZABETH B. UPTEGROVE

Dissertation Chairperson Carolyn A. Maher, Ed. D.

This research provides an analysis of how a cohort group of five students learned standard notation for combinatorics over an 18-month period. The students were among the participants in a long-term study of the development of mathematical ideas and reasoning. Over the years, the students worked on open-ended and challenging mathematical problems from a combinatorics strand, such as building towers of different heights from different colored cubes, counting pizzas with different toppings, and counting taxicab routes.

The group was videotaped doing mathematics during their sophomore and junior years of high school, when they revisited combinatorics tasks that they had worked on during middle and elementary school. During these reinvestigations, they were introduced to the standard notation for combinatorics and to Pascal's Triangle and they explored the addition rule for Pascal's Triangle (Pascal's Identity). Three years later,

three of the students were videotaped during task-based interviews in which they again revisited the combinatorics problems and the notation.

Analysis of their work shows that the students used the combinatorial tasks with which they were already familiar to give meaning to the standard notation and to entries of Pascal's Triangle. They used the understanding of the combinatorics problems that they developed and refined over the years, including their recognition of the isomorphic relationships among the pizza, towers, and taxicab problems (structurally similar problem with different surface features), to build Pascal's Identity. A major contributing factor for the representation of Pascal's Identity in standard notation was their retrieval of earlier images of pizzas and towers that had meaning for them. Other important factors for their success included working together in collaborative investigations and having sufficient time for revisiting and rethinking ideas related to the problems. Follow-up interviews, in which individual students were asked again about Pascal's Triangle, provided evidence that students, independently, were able to rebuild or reconstruct what they had built together earlier as a group.

This research provides evidence of the power of giving meaning to symbols. It suggests that students who build meaning first and then develop the symbolic vocabulary can acquire lasting understanding.

ACKNOWLEDGEMENTS

Brian Malina, Robert Sigley, and Kirstin Uptegrove provided assistance with CDs, tapes, transcripts, and other administrative and technical needs, for which I am grateful.

My fellow graduate students provided encouragement and sympathetic ears when they were most needed.

I thank my committee members Lara J. Alcock, Arthur B. Powell, and Steven M. Barnhart, who provided much helpful advice and generous support.

I am grateful to my advisor Carolyn A. Maher, who has provided encouragement, advice, and support in more ways than I am able to enumerate.

Dirck Uptegrove has been accommodating, understanding, and tolerant, as always.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1: STATEMENT OF THE PROBLEM

1.1 Introduction

This study is an examination of the work of a group of high school students who learned conventional mathematical notation for combinatorics by building on personal representations that they had developed over many years of investigating various combinatorics problems related to Pascal's Triangle.

In this section, I discuss the reasons why learning standard notation is important and the rationale for building that learning on students' personal notation; I note some shortcomings of the current recommendations regarding introducing standard notation. In the next section, I discuss how the current study can contribute to the research on the learning of notation.

It is hardly disputable that it is important that students learn standard notation. Conventional mathematical notation can provide a common language for communicating mathematically; it can also help students extract the important features of a mathematical problem, by highlighting what is significant as opposed to what is superficial (Lehrer, Schauble, Carpenter, and Penner, 2000). The fact that standard notations apply to broad groups of problems is also important; Smith (2003) notes "representations that are general – that aid the solution of classes of problems – are essential to mathematics.

Children must connect their idiosyncratic representations with mathematical ones if they are to progress very far within the discipline" (p. 263). However, it is not easy to specify the best way to for students to learn conventional notation. In the past, it was often recommended that students learn the notation first and then build up the ideas from the notation. For example, in 1982, Gary Perlman, writing in *Mathematics Teacher* magazine, advised teachers to teach notation first: "When students begin learning an area of mathematics, a large proportion of effort should be devoted to teaching notation" (p. 466). Many textbooks, particularly at higher levels, continue today to teach notation first. And it has been argued that there are occasions in which children do learn notational symbols before understanding the meaning. For example, Pimm (1990) notes that young children learn to say the alphabet, count, and recite before they understand the meaning of the words associated with those activities.

Nevertheless, it is widely recommended today that students should construct their own representations first; when standard notation is later introduced, students then learn about it by building on the representations that they have already made. For example, the National Council of Teachers of Mathematics (NCTM) recommends in *Principles and Standards for School Mathematics* (2000) that students "create and use representations to organize, record, and communicate mathematical ideas" (p. 402). These representations might include tables, scale models, discrete models (pictures representing quantities), and diagrams (pp. 280-281). Then students can use their own personal representations to help them learn standard notation:

> Teachers should introduce students to conventional mathematical representations and help them use those representations effectively by building on the students' personal and idiosyncratic representations when necessary (pp. 362-3).

However, students' personal representations, although they have the advantage of being very familiar to and well-understood by the students who build them, can have shortcomings. They are not necessarily useful for communicating with a wider group, they might not focus on the most important aspects of the problem, and they might not lend themselves to easy generalization. Skemp (1987) describes how recognition of the possible shortcomings of their own notations can help encourage students to adopt the standard notation. Skemp notes:

> These ways of expression may often be lengthy, unclear, and differ between individuals. By experience of these disadvantages, and by discussion, children may gradually be led to the use of established mathematical symbolism in such a way that they experience its convenience and power for communicating and manipulating mathematical ideas (p. 183).

The NCTM recommends that teachers should decide when it is appropriate to introduce standard notation to students:

> Teachers need to use sound professional judgment when deciding when and how to help students move toward conventional representation. Although conventional representational forms have many advantages, introducing representations before students are able to use them meaningfully can be counterproductive (p. 284).

But the NCTM does not provide much information as to how to guide students from notations that are personal and idiosyncratic, but not incorrect, toward conventional notation, nor how to determine whether students are ready or not ready to learn conventional notation. Examining specific instances where students did use personal representations in the process of learning standard notation could therefore provide a useful window into the process by which students might make sense of standard notation.

1.2 The Current Study

Until recently, combinatorics has not been widely taught in the United States before college. But with the NCTM now recommending that discrete mathematics be taught throughout the pre-college curriculum (p. 31), combinatorics concepts are being addressed at the elementary and middle school level. Therefore, the issues related to teaching and learning combinatorics notation merit individual study.

In this research, I examine one case where students learned the notation associated with combinatorics after they had already built personal representations. These representations derived from their work on two particular combinatorics problems over many years. (These were the towers problems and the pizza problems, which will be described and discussed further in Chapter 2.) The students related the answers to those problems to the numbers found in Pascal's Triangle and used these relationships to help them make sense of the standard notation. In this study, I investigate what strategies these students used in order to make sense of the combinatorics tasks, the numbers in Pascal's Triangle, and the standard combinatorial notation. By detailing the experience of these students, I seek to provide information about possible ways that competency in the use of standard notation might be achieved.

The questions that guide this analysis are:

1. How did the students represent their ideas about the combinatorics tasks?
2. What connections did the students make among their personal representations, the combinatorics tasks, and the standard notation?
3. What strategies did the students use to make sense of Pascal's Triangle? What strategies did the students use to make sense of the standard notation?
4. Is there evidence that individual students remembered or could reconstruct their ideas about Pascal's Triangle and the standard notation up to three years later?

1.3 Organization of This Document

The data for this work comes from videotaped problem solving sessions that were part of a long-term longitudinal study of children's mathematical thinking directed by Carolyn A. Maher. In the following two chapters, I place the current research in the context of that study, as well as the context of the literature on representations and sense-making in mathematics. In Chapter 4, I describe my methods and criteria for data selection, collection, and analysis. In Chapter 5, I describe the problem-solving sessions in detail and summarize the results. In Chapter 6, I discuss connections the students made among their own representations, the standard notation, and Pascal's Triangle. In particular, I show how their organizational strategies contributed to their ability to make sense of the standard notation and of the relationships between the combinatorics problems, Pascal's Triangle, and Pascal's Identity. I also discuss the evidence that the students' understanding of Pascal's Triangle and related combinatorics problems is long-lasting, and I offer observations about implications of this study.

CHAPTER 2: BACKGROUND

2.1 Introduction

This study is based on data from a Rutgers University longitudinal study of children's mathematical thinking. In this chapter, I briefly describe that longitudinal study and I note how my study connects to it. I also explain the combinatorics problems which the students in my study investigated, and I describe previous work of the students on those problems.

2.2 The Longitudinal Study

Ankur, Brian, Jeff, Michael, and Romina, the students in my study, were among the original participants in the longitudinal study. This longitudinal study has followed a group of students from the Kenilworth, NJ public schools from first grade through college. (See, for example, Kiczek, Maher, and Speiser, 2001; Maher, 1998; Maher and Martino, 1999; Maher, 2002; Speiser, Walter, and Maher, 2003; and Uptegrove, 2003). The group originally consisted of 18 first-graders; other students joined the study in later years. Seven students, including the five students in the present study, have continued in the longitudinal study for fifteen years. At first, researchers met with students in the classroom approximately four times a year, for two to three days at a time, in sessions lasting from one to three hours. When the students were in the last three years of high school, the meetings (again one to three hours long) took place after school also about four times a year. As college undergraduates, some of these students continued to meet informally with researchers for one- to three-hour sessions.

Over the years of this longitudinal study, students were presented with open-ended challenging problems, many dealing with combinatorics. It was not uncommon for students to revisit problems during later sessions, over periods of months to years. The purpose of the longitudinal study was not to teach the students particular topics in mathematics. Rather, a goal was to establish a culture where the correctness of an answer came from the sense-making of the students, rather than from the authority of the researcher. Students' justifications for their reasoning were triggered by researcher questions about what was convincing, what made sense, and how students developed their answers. Moreover, students were not presented with the standard notations of combinatorics; they used whatever representations they and their classmates created.

2.3 The Current Study

Because students in the longitudinal study were not given the language of mathematical exposition at the beginning, data from the longitudinal study provide a unique opportunity for my research. Only after students generated their own representations and used them to make sense of the combinatorics problems were they provided with the standard notation.

Ankur, Brian, Jeff, Michael, and Romina, all of whom participated in the longitudinal study since first grade, worked together on generating Pascal's Identity in a session in their junior year of high school. At that time, they were already familiar with the rule for generating successive rows of Pascal's Triangle from existing rows, they had already demonstrated the ability to explain entries of Pascal's Triangle in terms of the towers and pizza problems, and they had also explained the addition rule in terms of manipulations on towers and pizzas. They put everything together at that session, writing the addition rule in standard notation and in general form. In order to trace how this event came about, I selected for review earlier problem-solving sessions in which they investigated and generalized the relationships among different combinatorics problems and Pascal's Triangle. I also included a problem-solving session from one year later and interviews with three of the students (Ankur, Michael, and Romina) during their second year of college, for the purpose of seeing whether they could still generate and explain Pascal's Identity and whether they still used the same strategies for understanding it. A complete list of the sessions is given in Chapter 4.

2.4 Combinatorics Problems

A major strand of activities in the longitudinal study was a set of counting problems that the students worked on from second grade through high school. Problems in this strand that are relevant to this study are pizza problems, towers problems, and the taxicab problem. (Refer to Appendix A for a complete list and full descriptions of all the combinatorics problems.)

The four-tall towers problem was first given to these students when they were about nine years old (grade 3, 1990) in the following form:

> Your group has two colors of Unifix cubes. Work together and make as many different towers four cubes tall as is possible when selecting from two colors. See if you and your partner can plan a good way to find all the towers four cubes tall.

They revisited this problem and its variations (such as the five-tall towers problem) in grades four, five, and ten.

These students first encountered the following five-topping pizza problem in grade five (1993). It is:

> A local pizza shop has asked us to help design a form to keep track of certain pizza choices. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many different choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possible choices.

The students worked on variations of this problem in grade five, and they revisited the five-topping problem in grade 10 (1997).

Four of the students (Brian, Jeff, Michael, and Romina) worked on the taxicab problem at the end of their senior year of high school (2000). The taxicab problem is:

> A taxi driver is given a specific territory of a town, shown in Figure 1. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated on the map. To pass the time, she considers all the possible routes she

could have taken to each pick-up point and wonders if she could have chosen a shorter route. What is the shortest route from a taxi stand to each of three different destination points? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.

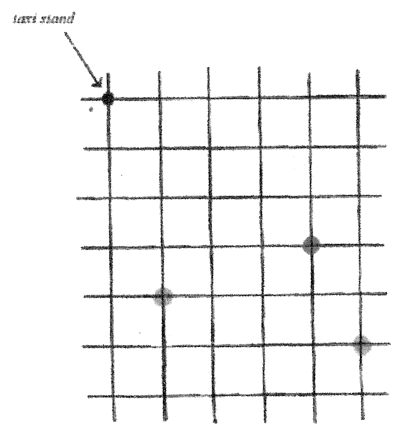


Figure 1. Taxicab problem grid

These problems are isomorphic. In general form, they are:

1. Towers: How many towers n-cubes tall can you build, selecting from blue and red Unifix cubes? Answer: For each cube in the tower, there are two choices (blue or red). Hence, for a tower n cubes tall, the number of possible towers is 2^n.
2. Pizzas: How many possible pizzas can you make when selecting from n different toppings? Answer: There are two choices for each topping, on or off the pizza. So there are 2^n possible pizzas.

3. Taxicab: How many ways can you move n blocks away from the origin on a rectangular grid, when you are allowed moves in two directions (to the right and down)? Answer: At each intersection, there are two choices, right or down. So for a move of n blocks, there are 2^n possible routes.

These problems are also related to Pascal's Triangle (Figure 2).

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\ldots & & & & & & & & & &
\end{array}
$$

Figure 2. Pascal's Triangle, with arrows indicating instance of Pascal's Identity

If the 1 at the top of the triangle is called row 0, and if the first number in each row is called the 0^{th} entry, then the r^{th} entry of the n^{th} row of Pascal's Triangle gives the number of combinations of r objects selected from n objects. There are several standard notations to indicate this number. Five are given in Figure 3.

$$ {}_{n}C_{r} \quad C_{(n,r)} \quad \begin{vmatrix} n \\ r \end{vmatrix} \quad C^{n}_{r} \quad C(n,r) $$

Figure 3. Five standard notations for combinations

Pascal's Triangle can also be written in combinatorics notation (using any of the notations described above). Figure 4 shows one way to do this.

$$\begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 \\ 0 \end{vmatrix} \qquad \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

$$\begin{vmatrix} 2 \\ 0 \end{vmatrix} \qquad \begin{vmatrix} 2 \\ 1 \end{vmatrix} \qquad \begin{vmatrix} 2 \\ 2 \end{vmatrix}$$

$$\begin{vmatrix} 3 \\ 0 \end{vmatrix} \qquad \begin{vmatrix} 3 \\ 1 \end{vmatrix} \qquad \begin{vmatrix} 3 \\ 2 \end{vmatrix} \qquad \begin{vmatrix} 3 \\ 3 \end{vmatrix}$$

$$\begin{vmatrix} 4 \\ 0 \end{vmatrix} \qquad \begin{vmatrix} 4 \\ 1 \end{vmatrix} \qquad \begin{vmatrix} 4 \\ 2 \end{vmatrix} \qquad \begin{vmatrix} 4 \\ 3 \end{vmatrix} \qquad \begin{vmatrix} 4 \\ 4 \end{vmatrix}$$

Figure 4. Pascal's Triangle in combinatorics notation

The relationship between the numbers in Pascal's Triangle and the binomial coefficients is shown in Figure 5.

$$(a+b)^n = \sum_{r=0}^{n} C(n,r)a^{n-r}b^r$$

Figure 5. Coefficients of the expansion of $(a+b)^n$ are found in Pascal's Triangle

This is the context in which Pascal's Triangle has long been taught in schools, as a useful mnemonic device for remembering the binomial coefficients. But Pascal's Triangle is related to other problems in combinatorics and number theory, including the combinatorics problems discussed above. $C(n,r)$ also gives:

1. The answers to the pizza problem, organized according to number of toppings: $C(n,r)$ is the number of pizzas with exactly r toppings when there are n toppings to choose from.
2. The answer to the towers problem, organized according to the number of red cubes: $C(n,r)$ is the number of towers that are n cubes tall containing exactly r red cubes, when there are red and blue cubes to select from.
3. The answer to the taxicab problem, organized according to the number of moves to the right: $C(n,r)$ is the number of ways that it is possible to move from the origin to a point n blocks away, making exactly r moves to the right, when the only allowed moves are in two directions (to the right and down).

The students in this study recognized these relationships, and they used the connections they made among these problems and between these problems and Pascal's Triangle to help them understand Pascal's Identity, as will be discussed in Chapter 5.

2.5 Early Student Representations

Student work on the towers problem was reported by Maher, Alston, and Martino (1993); they noted that students first used random methods, building new towers and then attempting to match them to towers already built. They then moved to locally organized methods, for example building "opposites" (exchanging the two colors) and "cousins" (building upside-down versions). Eventually, they found that organizing the towers by cases (according to how many red blocks, for example) and recognizing patterns helped them justify their assertion that all towers had been found (Maher and Martino, 1996, and Maher and Martino, 2000).

Another report on the longitudinal study (Bellisio, 1999) examined the notation used by elementary school students in open-ended investigations involving algebraic reasoning by fifth-graders. Included among others were the five students from this study. She reported that the notation used by these five students (and others) in order to solve the four-topping pizza problem included drawings of pizzas, symbols (such as lines drawn on the pizzas to represent sausage), and letters (such as *S* for a sausage pizza). They made partially organized lists, but they did not present a convincing justification that they had found all possible pizzas (although they did in fact find all sixteen). One early classification scheme was "whole" vs. "mixed." They categorized pizzas as whole or mixed according to whether the pizza had one topping or more than one topping. As an example, Figure 6 (Bellisio, 1999, p. 141) shows a list dictated by Ankur and written by Romina.

Whole
Pepper = P
Sausage = S
Mushroom = M
Pepperoni = PE
Plain = Pl
Mixed

P/S	~~/P~~	M/PE
P/M	S/M	M/Pl
P/PE	S/PE	
P/Pl	S/Pl	

Figure 6. Romina and Ankur's first enumeration of four-topping pizzas (from Bellisio)

This shows an early instance of the use of codes in student work, and it illustrates an appreciation of the value of having an organizational scheme. How they built on these codes and the idea of an organizational scheme will be discussed in Chapter 5.

CHAPTER 3: LITERATURE REVIEW AND THEORETICAL FRAMEWORK

3.1 Representations

> All of us have an intuitive idea of what it means to represent a situation; we do it all the time when we teach or do mathematics. … And we *think* about things using "private" representations and mental images that are often difficult to describe. But what do we mean, precisely, by "representation," and what does it mean to represent something? These turn out to be hard philosophical questions that get at the very nature of mathematical thinking.
>
> *A. Cuoco (2001, p. x)*

3.1.1 Internal and External Representations

Since one focus of this paper is student use of representations, it is important to be explicit about the term *representation*. Janvier (1987) says, "A representation can be considered as a combination of three components: symbols (written), real objects, and mental images" (p. 68). In a similar vein, Goldin's (1998) proposed model for what he calls a *representational system* (p. 143) includes internal representations and external representations, corresponding respectively to the mental images and the written symbols listed by Janvier. A researcher might focus on either the observable symbols or the mental images. For example, Davis (1992) focuses on the mental images; he says, "the key pieces of mental representations are usually *not* about written notations … but rather about the things denoted by the symbols" (p. 227). Others prefer to concentrate only on what can be observed. Lesh, Post, and Behr (1987), for example, define representations "in a naïve and restricted sense as external (and therefore observable) embodiments of students' internal conceptualizations" (p. 33). I concentrate here on what Goldin would call external representations (such as spoken and written words, gestures, and symbols) that students use to communicate about mathematical ideas. The reason is that I am reporting on what I observe, not on what I infer. Further, even inferences of internal representations must be made on the basis of what is seen. As Goldin notes, internal representations "cannot be directly observed; they can only be inferred from observable behavior" (p. 145).

3.1.2 Examining the Role of Students' Representations

Skemp (1987) notes that representations play various roles in mathematics; symbols function as a tool for communicating, recording knowledge, and classifying ideas, among other things (p. 46). Smith (2003) emphasizes the personal nature of representations; the same representation can be interpreted in different ways by different individuals (p. 266).

According to Goldin (1998), one important reason to study students' use of representations is that observing the way students use representations is a way to evaluate how well they understand the mathematics. Sfard (2000), too, notes that "to be able to use mathematical names and symbols with proficiency" is an indicator of mathematical understanding (p. 91). Therefore, my examination of students' use of combinatorics representations can help shed light on how well they understand combinatorial mathematics.

Another reason to look at students' representations is that problems that students have with representations are often seen as an obstacle to mathematical achievement. For example, Kaput (1987) considers problems with representations to be a common source of student problems with mathematics; he says that there are "well-known, deep, and continuing difficulties experienced by students in translating between different representations of mathematical ideas, and between common experience and mathematics" (p. 19).

Goldin and Shteingold (2001) contend that apparent difficulties in student mathematical understanding are really problems with representations:

> … we suggest here that the apparent limitations in some children's understandings are *not intrinsic*. Rather, they are a result of internal systems of representation that are only partially developed, leaving long-term cognitive obstacles and associated affective obstacles (p. 3).

Davis and Maher (1990) also consider representations crucial to solving mathematical problems: "*there is no reliable way to go from a problem statement to a solution procedure unless you get a correct representation of the problem*" (p. 75).

Therefore, examining both the problems that the students in my study have with their representations and how (and if) those problems are resolved can shed light on the how students might overcome obstacles to understanding that imperfect representations can present.

Goldin and Shteingold (2001) note that making connections between representations is important. Cuoco (2001) agrees; he notes that "representations don't just match things; they preserve *structure*" (p. x). Similarly, Davis and Maher (1997) indicate that an important part of the process of making connections between mathematical problems involves mapping corresponding relational structures.

In summary, it is important to look at students' use of representations because this is one way to evaluate their mathematical understanding. Indicators of mathematical competency include mastery of the conventions of mathematical discourse (standard notation, for example), the ability to move between different representations (personal representations and standard notation, for example) with ease, and the ability to connect problems which may be represented differently but which are structurally the same. In addition, problems with mathematical understanding may be related to problems with representations.

3.1.3 Mathematical Representations and Their Meanings

Representations are not necessarily meaningful in themselves; they can take on different meanings depending on context. According to Goldin and Shteingold (2001):

> A mathematical representation cannot be understood in isolation. A specific formula or equation, a concrete arrangement of base-ten blocks, or a particular graph in Cartesian coordinates makes sense only as part of a wider *system* within which meanings and conventions have been established (p. 1).

The meaning of the representation therefore is not inherently part of the representation; it comes from the situation in which it is being used. Moreover, the meaning of the representation is not fixed; it depends on the context of the discussion. Sfard notes that representations have no meanings in and of themselves; they take on meaning as students create them, work with them, and talk about them: "Whatever path we chose to usher young people into the world of mathematics, we may be better off if we think of this world as symbolized into being rather than merely represented with symbols" (1994, p. 95). Sfard gives the name *reification* to the process through discourse an individual uses when bringing mathematical objects into being.

Although students in this study were presented with mathematical symbols (the standard notation for combinatorics) that already had meaning to the mathematical community, I will show how the meaning for these students emerged from their discourse about shared combinatorics tasks and their relationship to both students' personal and students' shared representations.

3.1.4 Research on Student Representations

There has been considerable research addressing issues of students' use of personal representations and standard notation in areas of mathematics other than combinatorics. For example, Kamii, Kirkland, and Lewis (2001) looked at young children's early use of representations when learning numeration and arithmetic. Students' learning of representations for functions has been studied by Bridger and Bridger (2001), Coulombe and Berenson (2001), Bellisio (1999), and others. The use of representations in algebra has been studied by Friedlander and Tabach (2001) and others.

The research on combinatorics notation is more sparse. Tarlow (2004) reported on the work of a different cohort of students from the longitudinal study, who retrieved and modified earlier personal representations when working on combinatorics problems. Warner (2005) described how middle school students used personal representations and standard notation in investigations of some isomorphic problems in combinatorics. Glass (2001) reported on college students' work on combinatorics problems, including the pizza and towers problems.

Other reports on students' work on combinatorics include three from the longitudinal study that will be discussed further in Chapter 5, because they report on work that is relevant to the present study. They are Muter (1999), Kiczek (2001), and Powell (2003).

3.4 Mathematical Understanding

Since I discuss students' use of mathematical notation as a way to evaluate their understanding, it is appropriate to investigate the meaning of mathematical understanding. Piaget provided some early perspectives on the concept of knowledge and understanding in general. He proposed that acquiring knowledge is not a matter of receiving information; rather, a learner acquires knowledge through interaction. Learning is "an operation that constructs its objects" (Furth, p. 7). According to Piaget, learning is a two-step process involving accommodation (transforming the learner's knowledge structures to admit new information) and assimilation (fitting new information to the learner's knowledge structures). More recently, constructivist and similar theories of learning have built on this foundation; mathematical meaning is said to be constructed by the student, not conveyed by the teacher. (For example, refer to Goldin, 1990, Davis and Maher, 1990, and von Glaserfeld, 1990.) In this study, I will argue that the meaning of the standard notation could be said to be constructed by the students rather than imparted by the researcher.

In 1976, Skemp proposed distinguishing between instrumental understanding and relational understanding. Instrumental understanding is knowing what to do but not knowing why to do it, and relational understanding is knowing both what to do and why. In explaining why he believed relational understanding to be more desirable, Skemp compared a person with a "set of fixed plans" for finding his way around town to a person with a mental map of the town. The person with fixed plans has instrumental understanding

> *What has to be done next is determined purely by the local situation.* There is no awareness of the overall relationship between successive stages, and the final goal. ... the learner is dependent on outside guidance for learning each new 'way to get there' (2002, p. 14).

Skemp contrasted this with relational understanding: "In contrast, learning relational mathematics consists of building up a conceptual structure (schema) from which its possessor can (in principle) produce an unlimited number of plans for getting from any starting point within his schema to any finishing point" (pp. 14-15). Subsequent writers have expressed the idea differently, but there is general agreement that relational understanding is desirable. Further, understanding is often viewed as a process than a destination. Lamon (2001) calls understanding "a moving target" (p. 155). Pirie and Kieren (1994) list eight different levels of mathematical understanding, from primitive knowing (knowledge brought to the task at hand) up to what they call inventising (extending knowledge through making and testing conjectures). According to Davis and Maher (1997), a student's mathematical understanding is not static; it grows through experience and reflection.

Skemp and others have provided definitions of what it means to understand mathematics. Skemp said, echoing Piaget, "*To understand something means to assimilate it into an appropriate schema*" (1987, p. 29). G. Davis and Tall (2002) expand on the notion of a schema: It is "a connected collection of hierarchical relations" (p. 132).

In a similar vein, R. Davis (1992) says, "'Understanding' occurs when a new idea can be fitted into a larger framework of previously-assembled ideas" (p. 299).

But it is not necessarily easy for one individual to determine the level of understanding possessed by another individual. Current theories of mathematical learning suggest that observing learners' mathematical communication is a fundamental way to determine their understanding. Dörfler (2000) is a proponent of this view: "The only available and observable indicator that a subject has grasped the meaning – whatever this is – of a linguistic or symbolic entity is that the subject has a thorough command of its social use" (p. 101). Sfard (2000) too emphasizes the central role of communication. In fact for Sfard, mathematical knowledge is not just observed in discourse, it is *created* through discourse. Sfard also notes that importance of forming new ideas in demonstrating mathematical understanding. She says, "the most natural way to assess one's understanding of a mathematical idea is to estimate the ease with which one reasons and discovers new facts about it" (p. 49). Greer and Harel (1998) cite specific reasoning that demonstrates mathematical understanding: recognizing structural relationships (isomorphisms) and distinguishing them from surface features.

Student communication about mathematics includes spoken words and other sounds, written words and other written inscriptions such as figures and graphs, and gestures. Detailed examination of students' communication is therefore a useful way to evaluate their understanding. Media that give researchers the opportunity to engage in such detailed examination are important components of a comprehensive analysis.

Videotape data (of classroom sessions, individual interviews, and small-group problem-solving sessions) and associated student written work are examples of such media (Powell, Francisco, and Maher, 2003).

In summary, although it is not necessarily easy to evaluate the level of understanding possessed by another person, the ability to communicate mathematically is seen as essential. Therefore, I will use the students' communications about the standard notation and the meaning of the mathematical tasks as a way to judge whether understanding of the notation and the underlying mathematics is present.

3.5 Theoretical Framework

The work of Davis and Maher provides the theoretical framework for this study. According to Davis and Maher (1996), mathematics is "a *way of thinking* that involves *mental representations* of problem situations and of relevant knowledge, that involves dealing with these mental representations" (p. 73). Further, they note (1990) that children learning mathematics are capable of engaging in real and challenging mathematical activities; like mathematicians, learners can "*analyze problems* and *create* algorithms," not just "memorize algorithms and recall them as needed" (p. 77). But in order to do this, learners must be provided with the appropriate tools in an environment that supports mathematical thinking. According to Martino and Maher (1999), the environment should be one that provides learners with sufficient time for exploration of mathematical ideas and with the opportunity to express their own ideas. The tools consist of repertoires of representations based on appropriate mathematical experiences (Davis and Maher, 1997). Learners use these representations as "powerful tools to think with" (p. 114) which they can use to make meaning of new mathematical ideas.

According to Davis (1984), learners of mathematics recall existing knowledge and representations in order to work on a new problem. If an existing representation is not adequate to solve the problem, the learner may need to construct a better representation and to reorganize existing knowledge. Those who guide children's mathematical development, therefore, should present mathematical tasks that tax learners' existing representations.

According to Maher, Martino, and Alston (1993), the tasks in the longitudinal study which supplies the data for this research were carefully selected with this aim in mind. Tasks were "chosen to challenge children to reorganize or to extend available existing knowledge" (p. 13). Maher, Martino, and Alston also emphasize the importance of asking learners to explain and justify their answers. It is in discussion and explication of new ideas that learners often extend, reorganize, and generalize both their knowledge and their knowledge representations.

Davis and Maher (1997) also acknowledge the importance of using appropriate notations. Although mathematics is concerned with abstract ideas, it also "involves, in an equally important way, *notations* that enable us to deal with numbers and algorithms and polynomials and functions and sophisticated geometries" (p. 97). They maintain that an important characteristic of notation is that it be based on "*extensive personal experience*" and that it deal "*in an accurate and relevant way with important parts of real mathematics*" (p. 97)

Maher, Martino, and Alston (1993) note that learners naturally invent notation, which may be modified or replaced if it is unsuitable to the task. This is another way that learners extend or generalize their mathematical knowledge. "The process of coordinating notation with a mental representation could also result in a modification, reorganization, and/or refinement of the original idea" (p. 14). In this study, I note instances where student-invented notation assisted students in these ways, helping them to reorganize and refine their understanding of some combinatorics problems and providing a tool for later generalization.

In summary, this study provided the conditions that Davis and Maher maintained could be used to build new mathematical ideas through the use of powerful personal representations. The students were presented with challenging tasks, they had the opportunity to explore and express their ideas, and their old representations were not necessarily suitable for exploring the new questions. Therefore this provides an opportunity to test Davis and Maher's theory that under such conditions, students can analyze problems, create algorithms, and build new mathematical ideas. My assertion, supported by the evidence in Chapter 5, is that the students in this study did demonstrate the ability to analyze problems, create algorithms, and build new mathematical ideas. They analyzed the combinatorics problems by observing and describing their isomorphic relationship. They created algorithms by building Pascal's Identity, the generating rule for Pascal's Triangle in standard notation. They built their own mathematical ideas by explaining the workings of Pascal's Identity and describing how it related to the combinatorics problems they had analyzed.

CHAPTER 4: METHODOLOGY

4.1 Introduction

In order to investigate the students' use and understanding of personal representations, standard notation and the combinatorics tasks, my plan was to explore the relationships that I found among:

1. Personal representations of the student participants,
2. Standard mathematical notations supplied to the student participants,
3. Links that students made between personal representations and standard notations, and
4. Combinatorial tasks used by the students to describe properties of Pascal's Triangle and Pascal's Identity.

The intention was to explain how the interactions of the various representational systems and choice of tasks contributed to or undermined the students' ability to make sense of standard combinatorial notation and of the mathematics of Pascal's Triangle.

4.2 The Videotapes

I began my analysis with the session of May 12, 1999 (called the "Night Session"), because it was in that session that the group of students first made use of the standard notation to write Pascal's Identity. But because I was also interested in where their ideas came from and whether their ideas were long-lasting, I also worked backwards and forwards from that date. I looked for earlier and later sessions in which they investigated Pascal's Triangle and Pascal's Identity and I arranged interviews with some of the students, who were now in college.

I searched videotapes made between December 1997 (when, after a hiatus of almost two years, students again began meeting with researchers) and May 2000 (the end of their senior year in high school) for sessions that included my student participants. During that time, the students worked mostly on problems in combinatorics and probability; I selected all tapes related to their work on combinatorics problems. In order to locate the tapes, I reviewed logs of all videotapes made during that time. In cases where the log was unclear, I viewed the videotapes or read the transcripts. I selected for primary analysis five sessions during which the students worked on combinatorics problems related to Pascal's Triangle: December 12, 1997; December 19, 1997; January 9, 1998; February 6, 1998; and March 6, 1998.

1. December 12, 1997: In this session, the group of five students worked on the five-topping pizza problem, explored the use of binary notation as a problem-solving tool, found a general solution to the *n*-topping pizza problem, and began an investigation of the relationship between the towers and pizza problems.

2. December 19, 1997: In this session, the students continued to investigate the relationship between the towers and pizza problems, and they proposed a solution to the m-color n-tall towers problems. They also explored the binomial expansion.

3. January 9, 1998: In this session, the students first worked on some variations of towers problems in which they made use of binary notation. Then they were introduced to the various notations for combinations and subsequently challenged to make sense of the numbers in Pascal's Triangle in terms of the binomial expansion.

4. February 6, 1998: In this session, in an interview with a visiting researcher, three of the students explored the relationships among the numbers in Pascal's Triangle and their relationships to the towers problem, the pizza problem, and the binomial expansion.

5. March 6, 1998: In this session, four members of the group re-worked some of the February 6 investigations for the benefit of Michael, who had been absent. The researcher helped them recall the standard combinatorial notation, and they linked it to the answers to some towers problems.

I located five other tapes not directly linked to the group's work on combinatorics problems but where that work was discussed: A session on June 12, 1998 when the primary focus was on probability but when Pascal's Triangle and the binomial expansion were mentioned; two interviews with Michael (December 14, 1998 and January 29, 1999), when Michael explained instances of Pascal's Identity in terms of the pizza problem; and two tapes when students worked on the World Series problem (January 22 and 29, 1999), when the relationships among Pascal's Triangle, binary notation, and the pizza problem were discussed.

The ten tapes described above comprise all the tapes I was able to find in which Pascal's Triangle was discussed to any extent by these students before the night session. It is thus intended to be a comprehensive selection including all known videotaped work of these students on this task before the night session.

In assessing durability of student understanding, I used four sessions after the night session. There was only one problem-solving session after the night session in which the students in this group worked on a combinatorics problem related to Pascal's Triangle; this was the taxicab session on May 5, 2000 analyzed in detail by Powell (2003). I refer to Powell's results in discussing the durability of their knowledge. I also requested interviews with all the students. Three students were available; Michael was interviewed in April 2002, and Romina and Ankur were interviewed, separately, in July 2002. The purpose of these interviews was to ascertain whether the students were still able to explain Pascal's Triangle, Pascal's Identity, and the standard combinatorial notation. Besides providing an opportunity to assess durability of student knowledge, these interviews also afforded the opportunity for observing the relationship between the strategies students used in the original construction and the strategies they use to reconstruct that work.

4.3 Analysis Plan

My analysis followed the recommendations of Powell, Francisco, and Maher (2003). They recommend an eight-step analysis (viewing the videotapes, describing consecutive time intervals, identifying critical events, transcribing, coding, writing analytic memoranda, constructing a storyline, and composing a narrative). My analysis plan, adapted from their recommendations, follows.

1. Viewing the videotapes and describing consecutive time intervals. Powell, Francisco, and Maher recommend that researchers review the videotape several times before beginning analysis, in order to become familiar with the session but without a preconceived notion of where the viewing will lead. Writing time-coded descriptions enables a researcher to gain more familiarity with the data, and using time codes provides for easy access to interesting episodes. I chose to work with three-minute intervals; I found that this provided a balance between keeping the number of intervals in a typical one-hour tape from becoming unwieldy and having descriptions that were too long. The summaries were prepared using vPrism (described below) and stored as Word documents. (The summaries are attached in Appendix E.) In these summaries, and in subsequent transcriptions, descriptions, and analyses,

I used codes for researcher names; Appendix F gives the names of the researchers associated with the codes.

2. Identifying and coding critical events. Davis, Maher, and Martino (1992) describe critical events as students' moments of insight, interesting speculations, or conceptual leaps. Powell, Francisco, and Maher further note that critical events are whatever events are significant for the research agenda. My research relates to representations, combinatorics problems, and Pascal's Triangle, and so that is how I selected critical events. I coded for events related to students' ideas and insights that could be used to make sense of Pascal's Triangle, Pascal's Identity, and standard notation. These included student-developed representations, references to standard notations, and use of combinatorics problems. I included codes for references to Pascal's Triangle itself and student and researcher participation. This gave me three different types of codes for each event; the use of multiple codes allowed me to show connections made by students or researchers. For example, a code of S (student), P (pizza problem), and PI (Pascal's Identity) indicates that a student used the pizza problem in a discussion of Pascal's Identity. Being able to show that students linked tasks and/or notations was important to my analysis because I consider identifying appropriate connections as evidence of sense-making. Although the fact that a student discussed pizzas and Pascal's Identity together does not in itself constitute definite evidence of sense-

making, the presence of codes indicating linkages identified an event that warranted further examination for sense-making. Figure 7 shows the list of critical event codes.

Code Type	**Code**	**Description**
Participant Identification	S	Student
	R	Researcher
Combinatorics Problem	P	Pizza problem
	T	Towers problem
	X	Taxicab problem
	C	Binomial coefficients
	O	Other problem
Representations	B	Binary notation
	N	Standard Notation
	D	Other representation (e.g. diagram, letters)
Pascal	PT	Pascal's Triangle
	PI	Pascal's Identity

Figure 7. Table of critical event codes

Since it is not always possible to identify which events are critical at an early stage of analysis, I coded events that I called candidate critical events, those that seemed likely to be seen as critical during later analysis. As analysis progressed and in the light of later events, I refined the critical events list. For example, an early candidate critical event was a brief mention by Ankur of factorial notation on December 12, 1997; it was listed with the brief description "discussion of factorial notation." It was unclear from the immediate context whether this would be relevant to the later analysis, but factorial notation came up on three later occasions (including twice during the night session), and so that candidate critical event became part of the timeline for factorial notation. (See below for a discussion of timelines.) If factorial notation had not come up at all in later discussions, that candidate critical event would not have become part of the analysis.

3. Transcribing. Powell, Francisco, and Maher recommended transcribing critical events, but I sometimes had full-session or partial-session transcripts to work from. Partial transcripts of the December 12, February 6, and March 6 sessions had already been prepared and verified by other graduate students, Powell (2003) included a transcript of the May 5 session, and I had already prepared a full transcript of the night session (verified by another graduate student). For the other tapes, I transcribed critical events, and those transcriptions were verified by other students. In all cases, transcripts were stored both as Word documents and in vPrism (a computer program developed by LessonLab, Inc. that allows simultaneous entry of codes, transcriptions, and side notes. Since vPrism works with digitized media, it also provides for stable time coding. (With magnetic media stored on videotapes, times can change from one viewing to the next, due to stretching of the tape; this does not happen with digitized videos.) Appendices G, H, and I contain critical event transcripts of sessions prior to the night session, Appendix J contains a full transcript of the night session, and Appendix K

contains critical event transcripts of the taxicab session and of the three interviews.

4. Writing analytic memoranda. I organized the critical events into episodes. An episode is characterized by a common theme (for example, students working on determining the general pizza problem answer), and it can include several critical events along with associated discussions. I compiled a list of all episodes, showing dates, times, participants, and brief descriptions. (Refer to Appendix C.) From that I created timelines for the episodes of the night session. Generating the timeline enabled me to see how events from before the night session contributed to the episodes of the night session. For example, the time line showed the contributing events to Michael's night session explanation of row 3 of Pascal's Triangle (written in standard notation) in terms of pizzas. It started with Michael's introduction of binary notation in December 1997 and Researcher 1's introduction of standard notation in January 1998. The night session explanation itself had its roots in the explanation first proposed in February 1998 and reiterated in March 1998. Michael repeated his explanations in December 1998 and recalled it during the work on the World Series problem in January 1999. A timeline for this episode is shown in Figure 8. All timelines are shown in Appendix D.

12/12/97 Michael introduces binary notation.	**01/09/98** R1 introduces standard notation.	**02/06/98** Ankur, Jeff, and Romina explain specific instances of Pascal's Identity.	**03/06/98** Michael explains how 3+3=6 in terms of towers.	**12/14/98** Michael explains specific cases of Pascal's Identity in terms of pizzas.
01/22/99 Michael explains Pascal's Identity in terms of pizzas to Ankur, Brian, and Romina.	**01/29/99** Michael re-explains Pascal's Identity in terms of pizzas.	**05/12/99** Michael uses pizzas (and binary notation) to explain one instance of Pascal's Identity	**05/05/99** The students discuss Pascal's Triangle during the taxicab problem work.	**2002 Interviews** April: Michael explains the addition rule in terms of pizzas. July: Romina explains the addition rule in terms of pizzas.

Figure 8. Example time line for links between pizza and Pascal's Triangle

5. Constructing a storyline and formulating interpretations. First I constructed a storyline of the night session itself, describing the events that led up to the students' generation of Pascal's Identity. Then I went back to the earlier dates and showed how earlier critical events contributed to making the work of the night session possible, and then I connected what happened in the night session to the later session and to the interviews three years later. A major component of interpretation involved analysis of sense-making evidence. In looking for evidence of sense-making, I considered student explanations of the combinatorics problems, Pascal's Triangle, and standard notation. To me, evidence of sense-making involved not only making mathematically correct statements but also

providing justifications of why the statements were correct. Specifically, sense-making evidence includes: a) giving correct problem solutions with understandable justifications; b) providing correct explanations of the relationships among the problems, the representations, and Pascal's Triangle; or c) providing convincing explanations of Pascal's Identity. For example, one critical event from March 6, 1998 involved making connections between towers built with white and blue cubes and pizza problems. (Refer to Appendix C and Chapter 5 for more details.) Ankur said, "the colors don't represent anything." Michael then held up a white cube and said "no topping" and a blue cube and said "topping." Michael demonstrated sense-making because he gave a brief but correct explanation of the relationship: one color represents a topping placed on the pizza and the other color represents a topping available but not placed on the pizza. Ankur, however, did not demonstrate sense-making. Note that this is not the same as saying that Ankur demonstrated lack of understanding, just that evidence for sense-making is lacking. Interpretation also involved showing how the students' work on these problems fit into the theories of Maher and Davis (and others). For example, Maher and Davis noted that repertoires of representations, built from previous mathematical experiences, enable students to deal with new mathematical ideas. The discussion of factorial notation in Chapter 6

> illustrates students' building of repertoires of representations for thinking about this notation. The first time the students mentioned factorial notation, they had some questions about what kinds of problems called for its use. A few months later, they were quick to recognize an occasion where it was appropriate. A year later, they described at least three different situations where factorials were used.

Starting with basic non-analytic descriptions enabled me to become very familiar with exactly what the students were doing in each session. Preparing transcripts provide for further detailed familiarization with the data. This familiarity aided my later work on interpretation and analysis, making it easier for me to find events in earlier sessions that gave rise to student understanding in later sessions and to find connections between the different tasks the students addressed and the different notations they used.

CHAPTER 5: RESULTS

5.1 Introduction

The session of May 12, 1999 (the night session) is the primary focus of this analysis, because it was at this session that the students provided the formalization of Pascal's Identity in standard notation. But the events of the night session were founded in events that happened over the previous 18 months. Therefore, Section 5.2 summarizes ten sessions (eight problem-solving sessions and two interviews) whose work led to the night session, providing descriptions of events that led up to the critical events of the night session. Section 5.3 is an analysis of the night session episodes, including discussion of relevant contributing events from the earlier sessions.

In Section 5.4, I discuss four sessions following the night session. The first is the session from May 5, 2000, in which four of the students worked on the taxicab problem. The others were interviews conducted in 2002 with the three students who were available then. One of the interviews was conducted by Researcher 1; I conducted the others. In examining these sessions, I was primarily interested in assessing whether the students could recall the connections that they had made between Pascal's Triangle and the combinatorics tasks. In the case of the taxicab session, I was also interested in examining how the students made connections between the taxicab problem and the other combinatorics problems and between the taxicab problem and Pascal's Triangle.

In Section 5.5, I summarize the findings. Note: Although there are five students in the group under study, not all five participated in every session, and not all five participated in every discourse. In the following discussion, I will use the student's name if only one student's work is relevant. If two or more students are involved, I will talk about "the group" or "the students." Partial transcripts are included in this chapter. Refer to Appendices G, H, and I for critical events transcripts of the sessions before the night session, Appendix J for a transcript of the night session, Appendix K for a critical events transcript of the taxicab session, and Appendix L for critical event transcripts of the interviews. Appendix F lists the researchers involved in these interviews.

5.2 Summaries of Earlier Combinatorics Sessions

The ten sessions discussed here include five earlier sessions in which the foundations for the students' work with Pascal's Triangle were laid and five later sessions in which that work was revisited. In the five earlier sessions, spanning December 1997 through March 1998, these students were reintroduced to the towers and pizza problems. They found general solutions to those problems; they found a way to organize their solution lists to prove that all solutions were present; they recognized that those problems were related to each other, to the binomial coefficients, and to Pascal's Triangle; and they used those problems to form preliminary ideas about the meaning of Pascal's Identity.

After March 1998, they moved on to other investigations, but they revisited Pascal's Triangle and the pizza and towers problems on occasion. The five occasions where Pascal's Triangle were revisited include a session in June 1998 where the primary focus was probability but where Researcher 1 brought up the subject of Pascal's Triangle, two interviews with Michael (in December 1998 and January 1999), where he explained instances of Pascal's Identity in terms of pizza problems, and two World Series sessions in January 1999 where the students discussed Pascal's Triangle and Pascal's Identity. (Although Michael was the key player in the two World Series sessions, the insights he brought forth then were later expressed by other member of the group.) During these later sessions, students' discussions about Pascal's Triangle, Pascal's Identity, and the combinatorics problems became increasingly general.

5.2.1 December 12, 1997

In the session of December 12, 1997, early in the sophomore year, the group worked on the four-topping pizza problem, which they had last encountered in fifth grade (1993). As described by Muter (1999), Ankur, Brian, Jeff, and Romina, working in groups of two but discussing their work among themselves, settled on a numerical code (1, 2, 3, and 4 for the four toppings) and prepared an organized enumeration of all the possibilities. These coding and organizational schemes were not significantly different from those used in the fifth-grade sessions described by Bellisio (1999). But Michael developed a binary coding scheme that formed the basis for some later investigations into other combinatorics problems and into the relationships between those problems and Pascal's Triangle. Michael proposed that pizzas be represented by binary numbers; a four-topping pizza would be represented by a four-digit binary number, with a 1 in the k^{th} digit representing the presence of the k^{th} topping and a 0 representing the absence of the k^{th} topping. For example, all one-topping pizzas are represented by all four-digit binary numbers with exactly one 1: 0001, 0010, 0100, and 1000. Michael also described how to expand the solution: to add another topping, add another binary digit. In the following episode, Michael explained his thoughts to the other four students. At this point, the group had already agreed that the number of possible four-topping pizzas was 16 (which they thought of as 15 pizzas with toppings and one plain pizza). The other four students claimed there were 30 possible five-topping pizzas.

00:40:39	ROMINA:	Five is wrong?
00:40:41	MICHAEL:	I think it's thirty-two, with that cheese. And without the cheese, it'd be thirty-one.
00:40:51	ANKUR:	Mike, tell us the one that we're missing, then.
00:41:35	MICHAEL:	I'll tell you, I'll tell you why. The 1's would mean a topping; 0 means no topping. So if you had a four-topping pizza, you have four different places. In the binary system you'd have, the first one would just be 1. The second one would be that. That's the next number up. You remember what that was? That was like one. This was like two, and this was like three. [Michael writes 1, 10, and 11.]
00:41:25	JEFF:	I don't remember what you're talking about.
00:41:26	MICHAEL:	You don't remember the binary-
00:41:27	JEFF:	No, I understand, I know exactly what you're talking about. ... It was computers and this and that. I don't know how to add them or-
00:41:31	MICHAEL:	Well, I don't know. I think, I have this thing in my head. It works out in my head is all I want to say. [Michael draws four columns on his paper, labeled 8, 4, 2, and 1.] So you

got four toppings. Four different- this is like four. [Michael indicate the four different columns.] There's four places of the binary system. It all equals up to 15. [Michael writes 15.] That's the answer for the four-topping. Five toppings would be-

00:41:56 JEFF: No, I, I don't understand what you're trying to do. What are you doing? …

00:42:03 MICHAEL: For just one topping pizzas, you've got like four different toppings. It would be like, one. [Michael writes 0001, 0010, 0100, and 1000.] That would be four, right?

00:42:24 JEFF: Uh-huh.

00:42:24 MICHAEL: All right, two toppings, you could have like- [Michael writes 11.] That. [Michael writes 101.]

00:42:28 JEFF: Uh-huh.

00:42:28 MICHAEL: That. And all those little combinations like that. You know what I'm saying? And for three, you have combinations of like this. That. And four, you'd have that. [Michael writes 1010, 1110, 1011, and 1111.] All right? And it would, like, you would, you would have as many different combinations as, like, ah, I can't say it.

00:42:43 ROMINA: So, is, is the, the 1. Is that your topping?

00:42:56 MICHAEL: Yeah.

00:42:58 JEFF: Each 1 is a topping. The 0's are no toppings, the1's are toppings.

00:43:00 ROMINA: That's all you had to say.

00:43:01 MICHAEL: So you go from this number [0001], which in the binary system is 1, to this number [1111], which is 15. And that's how many toppings you have. There's 15 different combinations of 1's and 0's if you have four different places.

00:43:16 JEFF: All right.

00:43:17 MICHAEL: I, I don't know how to explain that, but it, it works out in my head. I have weird things going on in my head. And if you have an extra topping, you just add an extra place, and that would be 16, and that would be 31. [Michael writes the 16.]

00:43:30 JEFF: And then you'd add the cheese?

00:43:31 MICHAEL: Yeah. Plus the cheese is 32.

00:43:33 JEFF: So which one-

00:43:33 MICHAEL: You add another one, it would be 32 plus 31.

This episode also illustrates some common themes of these students' interactions. There were assumptions about shared knowledge that permitted an abbreviated communication style. But when Michael's initial explanation was unclear, Jeff and Romina called for clarification, a practice that became common in these combinatorics sessions. Another notable aspect of this discussion is that Michael was able to convince the group that his solution was correct without ever addressing Ankur's request. ("Mike, tell us the one we're missing, then.") He used the structure of his solution to convince the others that the solution was complete, even though he did not enumerate all the answers.

The group then made a connection between the two representations (numerical codes and binary notation). They explained how to link the numerical codes (which stood for topping names) to the binary solution: Use the topping names as column headers for the list of binary digits. (Refer to Figure 9 for their table.)

8	**4**	**2**	**1**	
O	**M**	**P**	**S**	
1	0	0	0	← onion pizza
0	1	0	0	← mushroom pizza

Figure 9. Students' table linking topping codes and binary notation [annotation added]

They could thus explain that the difference between 1000 and 0100 is, as Jeff said, "the difference between an onion pizza and a pepperoni pizza."

00:51:53	R1:	So what am I supposed to think about when I think about those numbers one, two, three, and four? What do they represent?
00:51:59	MICHAEL:	Well, the different toppings.
00:52:01	R1:	The different toppings. So when I read that 0100, what? They're the different toppings?
00:52:09	MICHAEL:	That would be one combination of- that would be one pizza.
00:52:11	R1:	Which one?
00:52:12	JEFF:	Each, any set of four numbers is a setup of one pizza.
00:52:16	R1:	Is it a particular pizza? 0100? ... What's the difference between 1000 and 0100?
00:52:26	JEFF:	Well, that would be like, that would be the difference between an onion pizza and like a pepperoni pizza. 'Cause you could give eight, you could make, like, the eight up there be onions. [Jeff indicates the leftmost column, labeled 8. Refer to Figure 9.] ...
00:52:35	MICHAEL:	Every time you see a 1 in there, that's an, that pizza has an onion in it. [Michael indicates the leftmost column.] Let's give it onions, uh, what is it? Mushrooms. [Michael labels the second column.]
00:52:41	JEFF:	Yeah. Pepperoni [third column].
00:52:45	ROMINA:	"*pp*." [Romina is suggesting a code for pepperoni.]
00:52:45	R1:	Sausage [rightmost column]?
00:52:45	MICHAEL:	Sausage, OK? Every time you see a 1 there [in the leftmost column], you know that pizza has onions. Every time you see a 1 in this, in this place right there [the next column] you know it has mush, uh, mushrooms. And a 1 in this place [third column], you know it has a pepper- no, pepperoni. And a one in this [rightmost column] is a sausage. That's what each of those 1's mean.

The explanation in which they discussed the connection between two different notational systems was echoed many times in later sessions, when the students discussed equivalences among the pizza and towers representations, the binomial coefficients, and the binary notation.

The question of linking the pizza and towers problem first came up in this December 12 session. When the group was asked if the pizza problem reminded them of anything, Brian responded, "Everything we do is like the towers problem." But when the group was asked to reflect further on this, they determined, as Ankur put it, that "It's similar but it's not exactly alike." The main obstacle to their belief in an isomorphic relationship was the contention (voiced by Ankur but generally agreed upon) that the problems could not be identical because a red-yellow tower is different from a yellow-red tower, but a peppers-pepperoni pizza is the same as a pepperoni-peppers pizza. This argument is significant because it re-emerged later (discussed below) even after the group had apparently concluded (following Michael's explanation) that the problems did have the same structure. It appears that they were convinced at the time by Michael's argument, but in his absence they reverted to the argument with the stronger association.

Following their claim that the pizza and towers problems were not the same, Researcher 1 asked the group to find the answers to the 2-topping and 3-topping pizza problems and the 2-tall and 3-tall towers problems. Even though the group determined that the pizza problem and the towers problem had the same solution in those cases (4 and 8, respectively), they did not identify an isomorphic relationship to the researchers.

5.2.2 Problem Session December 19, 1997

In the December 19 session, the group resumed the discussion of a possible relationship between pizza and towers problems. As noted by Muter (1999), Michael found another use for binary notation, this time using it to represent the two colors of the towers problem. He specifically noted that instead of relating binary digits to the presence or absence of pizza toppings, he could relate them to colors of cubes: "Zero is blue and one is red." In fact, with the help of the binary notation, Michael contended at an early point in the session that the towers and pizza problems were really the same, and he also pointed out that height of the tower could be related to the number of pizza toppings. This explanation was accepted, and, as reported by Muter (1999), the group determined that 2^n gave the general solution for both the *n*-topping pizza problem and the *n*-tall towers problem, noting that *n* is the height of the tower and the number of pizza toppings. Figure 10 shows Michael's table of solutions for both the two-topping pizza problem and the two-tall towers problem. The column headers 1 and 2 represent the two pizza topping choices and the two levels of the tower. The 1 and 0 represent topping/no topping and blue cube/red cube.

2	1
1	0
0	1
1	1
0	0

Figure 10. Michael's listing of two-tall towers and two-topping pizzas

Another concept briefly explored in this session was the use of factorial notation. As the group explored an expansion of the towers problem (to four colors), Ankur suggested that factorials would give the answer; he said, "Isn't that the factorial thing?" Michael noted that factorial use was not appropriate for the four-color towers problem because when selecting color of cubes to place on towers, colors could be reused. This is noted here because there were other discussions of factorials later on and in the night session; in subsequent sessions, the students' repertoire of representations for discussing factorials became larger.

During the factorial discussion, Researcher 1 asked the group why multiplication worked. Their response was similar to the one they later gave during the night session, and, like many episodes during the night session and elsewhere, it illustrated how their explanations became clearer each time they revisited problems.

00:58:32	R1:	OK. Have you explored why multiplying works? …
00:58:55	ROMINA:	Each time you get a different thing, you ... so you multiply.
00:59:05	MICHAEL:	I think that thing we did a lot in sixth grade that would work for this. Cause you would like, it would be like the, the, 1 2 3, 1 2 3, and it would be another one, because you got three different colors. [Michael writes the numbers 1 2 3 on the board three times.] And you'd have all these lines here, you know. [Michael draws lines under the numbers.] And you would multiply these lines here by that. I don't know.
00:59:23	R1:	What do you think, Ankur? What is he saying, Ankur? 'Cause you're agreeing there, your facial expression.
00:59:30	ANKUR:	He's saying that with each one of those top two, after he connects the lines with the top to the middle-
00:59:39	MICHAEL:	'Cause I have, like, this line could go three different ways. [Refer to Figure 11.]
00:59:42	ANKUR:	And each line from there can also go another three different ways. So it's times another three ways.

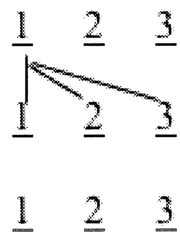


Figure 11. Michael's diagram explaining why you multiply

Michael seemed to find it difficult to explain at first (saying "I don't know" and then halting his explanation), but when Ankur added a few words, Michael resumed and Ankur finished up the explanation. This account (in substance and in style) was echoed in the night session, when these students were also asked "Why do you multiply?" At that time, the students also worked their way to a cogent explanation after some initial hesitation, using similar notations and representation.

At the end of this session, the binomial expansion was introduced by Researcher 1. The students were asked to explore the meaning of $(a+b)^n$ for n equal to 2 and 3. Although the students did not mention any relationship between this expansion and the combinatorial problems at this time, this is noted as the first time the binomial expansion came into the combinatorics discussions.

5.2.3 Problem Session January 9, 1998

In this session, the students continued their re-investigation of the five-topping pizza problem that they had begun the previous month. They were asked to revisit a problem from fourth grade, that of finding the number of five-tall towers with exactly two red cubes when selecting from red and yellow cubes. Ankur and Michael used binary notation and organized the cases by controlling for variables; they found and justified the solution in about two minutes. Figure 12 shows their work, consisting of two lists by cases. First, they listed towers according to the proximity of the two red cubes (represented by 1's); this list is shown at the top of Figure 12. Then they used a strategy of controlling for variables: first, they listed all possible towers with the red cube in the topmost position; then they listed all possible towers with the highest red cube moving down through the rest of the tower positions; since the towers are five cubes tall, there are four such cases. This list is shown at the bottom of Figure 12. As can be seen in Figure 12, the number of towers for each case is four, three, two, and one, summing to ten.

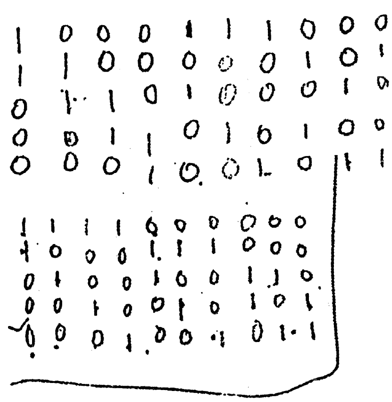


Figure 12. Ankur and Michael's complete listing of five-tall towers with two red cubes

problem from fourth grade, that of finding the number of five-tall towers with exactly two red cubes when selecting from red and yellow cubes. Ankur and Michael used binary notation and organized the cases by controlling for variables; they found and justified the solution in about two minutes. Figure 12 shows their work, consisting of two lists by cases. First, they listed towers according to the proximity of the two red cubes (represented by 1's); this list is shown at the top of Figure 12. Then they used a strategy of controlling for variables: first, they listed all possible towers with the red cube in the topmost position; then they listed all possible towers with the highest red cube moving down through the rest of the tower positions; since the towers are five cubes tall, there are four such cases. This list is shown at the bottom of Figure 12. As can be seen in Figure 12, the number of towers for each case is four, three, two, and one, summing to ten.

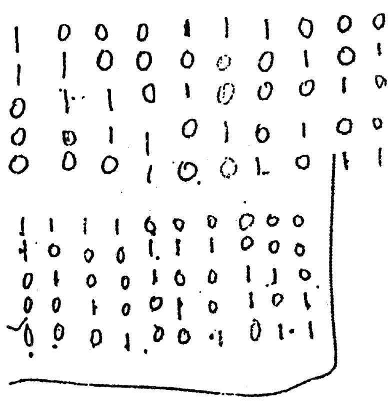


Figure 12. Ankur and Michael's complete listing of five-tall towers with two red cubes

Later, the students recalled from the previous session's work that the n-topping pizza problem and n-tall towers problem had the same solution of 2^n. I mention this, even though the discussion was brief, because it shows that they did not forget the common answers to the two problems, although, as will be seen in later sessions, not all of the students had a firm grasp of the isomorphism.

At the end of this session, Researcher 1 introduced some concepts and notations of combinatorics. She stated that asking how many five-tall towers have exactly two red cubes is the same as asking how many combinations there are when selecting two of five objects. She showed four different ways to write this, as shown in Figure 13.

$$ {}_5C_2 \quad C_{(5,2)} \quad \binom{5}{2} \quad C_2^5 $$

Figure 13. Notation introduced by Researcher 1 for selecting two of five objects

Researcher 1 concluded with a discussion of the binomial expansion and Pascal's Triangle. (Refer to Appendix H for the full transcript.) She wrote the expansion of $(a+b)$ to powers 0 through 3, drew Pascal's Triangle, and then asked the students to think about the relationship. Refer to Figure 14 for Researcher 1's drawings.

$$
\begin{array}{lll}
(a+b)^0 = & 1 & 1 \\
(a+b)^1 = & 1a+1b & 1 \quad 1 \\
(a+b)^2 = & 1a^2+2ab+1b^2 & 1 \quad 2 \quad 1 \\
(a+b)^3 = & 1a^3+3a^2b+3ab^2+1b^3 & 1 \quad 3 \quad 3 \quad 1 \\
 & & 1 \quad 4 \quad 6 \quad 4 \quad 1
\end{array}
$$

Figure 14. Researcher 1's listing of the binomial expansion and Pascal's Triangle

In the following excerpt, she asked the students to think about how their work fit into a larger picture.

01:44:56	R1:	The question is, what's the relationship here? How could you model it? How could you show this relationship? And why does it work? That's the question. So that's sort of the direction. Are you interested in knowing that? I think you have the bits and pieces to put it together.
01:45:15	ANKUR:	Some of the pieces are really small.
01:45:18	R1:	They're bigger than you think. You've been working on this for a long time.
01:45:22	ROMINA:	Is this what we did today though?
01:45:24	R1:	You've been dealing with some of this today. So think about it.
01:45:27	ANKUR:	So are all of the things we learned for the past eight years sort of combined into one thing?
01:45:31	BRIAN:	Imagine that.

Immediately following that discussion, Researcher 1 asked the students to make concrete the numbers in Pascal's Triangle, by thinking about them in a "very real way" (linking them to towers problems).

01:46:16	R1:	… when you first came in here today, you produced that number ten. [She refers to the first 10 in row 5 of Pascal's Triangle – 1 5 10 10 5 1.] Right?

01:46:22	ANKUR:	Yes.
01:46:23	R1:	And what problem were you solving?
01:46:25	ANKUR:	Two were red and three something else. ... Three of another color.
01:46:30	R1:	OK. So you can think of that ten in a very real way, if you want to, right?
01:46:36	ANKUR:	Yeah.
01:46:37	R1:	Can you think of those other numbers in a real way? Does that help?
01:46:46	ANKUR:	The 1 is, in 1 4 6 4 1, the 1 represents all red. The other 1 represents all yellow I guess ...
01:46:58	R1:	All red and all yellow for what?
01:47:04	ANKUR:	Of four high.
01:47:05	R1:	So this is four high. [R1 points to row 4 of Pascal's Triangle.] And these are all red. [R1 points to the first 1 in that row.]

This marks the first time the towers problem was explicitly linked with Pascal's Triangle. Row 4 of Pascal's Triangle (counting the first row as row 0) was connected to the 4-tall towers problem.

Just before the session ended, Researcher 1 asked the students to think about the meaning behind the addition rule for Pascal's Triangle in the specific case of how the 6 in Row 4 was generated from the two 3's in row 3. Although no answer was attempted at this session, it is noted here as the first time they were asked to think about Pascal's Identity.

5.2.4 Problem Session February 6, 1998

In this session, three students (Ankur, Jeff, and Romina), in a first meeting with Researcher 5, explored the relationships among the pizza problem, the towers problem, and Pascal's Triangle, and (for the first time) they discussed Pascal's Identity in terms of operations on physical objects (adding cubes to towers).

When Researcher 5 asked about their recent work, they mentioned the binomial expansion, towers, and pizza problems. Researcher 5 asked how the problems were related and why the same numbers kept coming up in the answers. The students did not mention an isomorphic relationship between the towers and pizza problems. Instead, as they had done back in December (before they had found that the n-topping pizza problem and n-tall towers problems had the same answer of 2^n), they maintained that the problems were different. Romina, recalling Ankur's earlier reasoning, said that red-blue was different from blue-red but sausage-pepperoni was the same as pepperoni-sausage. Ankur added that a five-topping pizza problem is like a five-color towers problem. Jeff agreed, saying that a tower could have two of the same color but that a pizza could not have pepperoni-pepperoni. Although all three had participated in the earlier, correct, discussion of the relationship between the towers and pizza problems, what they recalled later was their own original ideas, not Michael's idea.

When the group could not recall the notation for combinations, Researcher 1 again wrote the notations shown in Figure 13 and stated how the notation was related to the five-tall towers problem. She went on to show the expansion of $(a+b)^2$ and to ask an explicit question: What are the relationships, if any, among $(a+b)^5$, the five-tall towers problem, and the five-topping pizza problem? This is significant in terms of the students' later work, as it represents the first time the students were asked to think about a three-way link, between the binomial expansion and the two combinatorics problems. Researcher 1 also asked the students to relate those three problems to row 5 of Pascal's Triangle. The students were able to make the connection, evoking and expanding Ankur's explanation from the previous month of how the four-tall towers problem could be found in row 4 of Pascal's Triangle. (This also anticipated their night session explanation of entries in Pascal's Triangle in terms of pizzas.) In the following excerpt, the students linked the binomial expansion to the towers problem.

00:55:19	R1:	What, what are the *a*'s and the *b*'s here?
00:55:21	ROMINA:	Colors. …
00:55:47	ANKUR:	*a* and *b* is red and- red and blue. …
00:55:59	R1:	What do you mean by red and blue?
00:56:01	ANKUR:	*a* is red and *b* is blue.
00:56:02	ROMINA:	*b* is blue.
00:56:04	ANKUR:	That's [red-blue tower] *a b*. So *b a* would be a blue red.
00:56:09	R1:	So how, if you have them in front of you, how would they look different?
00:56:12	ANKUR:	Red and blue, red's on top.
00:56:13	JEFF:	Red's on top.
00:56:14	ANKUR:	And blue's on the bottom. Blue's on top and red's on the bottom.

In the following episode, Ankur explained to the others how to find the answers to the five-tall towers problem in row 5 of Pascal's Triangle, and then Jeff and Romina located the pizza answers in row 6. (Row 6 of Pascal's Triangle, referred to in this segment, is 1 6 15 20 15 6 1.)

01:05:12	ANKUR:	This [1] is no red.
01:05:13	JEFF:	Yeah.

01:05:14	ANKUR:	So there's one with no red. There's six with one red. … There's fifteen with two reds. Twenty with three reds. Six with five reds.
01:05:25	JEFF:	And one with no-
01:05:25	ANKUR:	And one with no-
01:05:27	ROMINA:	No.
01:05:27	ANKUR:	No. Six reds.
01:05:28	JEFF:	One with six reds. … All right. Now. What does that have to do with pizza?
01:05:38	ANKUR:	Just relate the tower problem to the pizza problem.
01:05:39	JEFF:	Well, we're saying that this [1] is a pizza with just plain.
01:05:42	ROMINA:	Yeah. That'll be the plain pizza.
01:05:44	JEFF:	Plain. This [6] is with all your six toppings.
01:05:46	ROMINA:	That's with one topping.
01:05:49	ANKUR:	You can't exactly relate these numbers to the pizza problem.
01:05:51	JEFF:	Well, we'll try really quick.
01:05:52	ROMINA:	Yeah. You can. 'Cause this [1] is plain, just plain pizza.
01:05:54	ANKUR:	And what will the other 1 represent?
01:05:56	ROMINA:	With everything on it.
01:05:57	ANKUR:	OK.
01:05:58	JEFF:	So this is plain.
01:05:59	ANKUR:	OK. Six with-
01:06:01	JEFF:	With one of each. Fifteen is with-
01:06:07	ROMINA:	Two toppings.
01:06:07	JEFF:	Just two toppings out of your six. Twenty is with three toppings. Fifteen is with the four toppings. Six is with the five toppings.
01:06:14	ROMINA:	Five toppings.
01:06:14	JEFF:	And the other one is-
01:06:15	ROMINA:	And the one is with all of them.
01:06:16	JEFF:	Like the supreme.
01:06:19	ROMINA:	Is that good?
01:06:20	ANKUR:	Cool. We're on fire today.

Thus Ankur, Jeff, and Romina used two combinatorics problems they knew in order to explain the numbers in Pascal's Triangle. This was the first time they connected the pizza problem to Pascal's Triangle.

After explaining the link between specific numbers in Pascal's Triangle and the pizza and towers problems, the students described an instance of the addition rule in terms of towers problems. They explained the instance of Pascal's Identity shown in Figure 15 below. They described the two 10's in row 5 and of the 20 in row 6 as counting classes of five-tall towers, and they described the process by which the 20 (counting a different class of six-tall towers) could be generated from the two 10's. The transcript below gives portions of their discussion. (Refer to Appendix H for the complete transcript.)

01:15:04	R5:	What are those tens counting? And what do the, what does the twenty count?
01:05:06	JEFF:	The tens show-
01:05:09	ANKUR:	The tens show two of one color.
01:05:12	ROMINA:	And three of another.
01:05:12	JEFF:	and the other one's three of-
01:05:14	ANKUR:	One color and two of another color. ... That's why it's ten and ten. But then you, at the top of each one, you can put either-
01:05:27	JEFF:	Yeah. You could either put a red or like blue.
...		
01:16:02	R5:	The first ten in that row of five high has two reds and three blues? We're counting reds?
01:16:10	ANKUR:	Yes.
01:16:12	R5:	And the second ten has-
01:16:13	ANKUR:	Three reds and-
01:16:15	R5:	-three reds and two blues. Now coming down here, the twenty is supposed to count the ones that have three of each.
01:16:20	ANKUR:	Three red. Three reds and three blues.
01:16:23	JEFF:	Right.
01:16:23	R5:	So how do the two tens add to give the twenty?
01:16:26	ANKUR:	Because in these ten, where there's three reds and two blues, you want to make it three reds and three blues. So you put a blue on top of each one.

This is the first instance where a connection was made between Pascal's Identity and a specific concrete combinatorics problem.

$$\begin{array}{ccccccccccccc} & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ 1 & & 6 & & 15 & \searrow & 20 & \swarrow & 15 & & 6 & & 1 \end{array}$$

Figure 15. One instance of Pascal's Identity which students linked to towers problems

5.2.5 Problem Session March 6, 1998

Ankur, Jeff, Michael, and Romina attended this session. Much of the session was devoted to attempts by Ankur, Jeff, and Romina to explain to Michael specific instances of Pascal's Identity in terms of towers and the binomial expansion. In the February session, as described above, the students had successfully explained the addition rule shown in Figure 15 in terms of towers. But this time, when asked to explain the similar case shown in Figure 16 they provided a different explanation.

$$\begin{array}{ccccccccc} & 1 & & 3 & & 3 & & 1 & \\ 1 & \searrow & 4 & \swarrow & 6 & & 4 & & 1 \end{array}$$

Figure 16. Another instance of Pascal's Identity discussed in terms of towers problems

They correctly mapped the 1 and 3 into towers (1 = white-white-white and 3 = blue-white-white, white-blue-white, and white-white-blue, as shown in Figure 17).

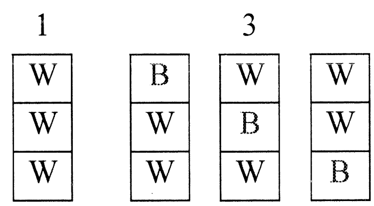


Figure 17. Students map specific towers to 1 and 3 in row 3 of Pascal's Triangle

But they tried to create the 4 by breaking apart the tower representing 1 and distributing its cubes among the other three towers. After Researcher 1 questioned this method, Michael gave a different explanation for how to create the 4.

00:46:19	JEFF:	We've got this [the white-white-white tower, representing 1]. And we're saying how this goes together. [Jeff has assembled the three towers each with one blue cube to represent the 3. Refer to Figure 17.] We're saying- [Jeff starts to dismantle the white-white-white cube.]
00:46:25	R1:	No. No. Don't take that apart. Because-
00:46:27	JEFF:	Well, that's why I made this. So I could.
00:46:30	ANKUR:	We made another one so we can take that one apart. … And show you.
00:46:34	R1:	You mean, you mean, you mean you get the four by taking something apart?
00:46:36	ANKUR:	You're not taking it apart.
00:46:37	ROMINA:	You're not taking it apart, you're just seeing how they go together. …
00:46:54	MICHAEL:	You don't really have to take it apart to show this, 'cause look. Each one, the reason why they combine, each one of these four blocks [towers] is going to have something added to them to equal the same thing.

00:47:01 ANKUR: Yeah.
00:47:02 MICHAEL: These blocks [towers] are going to have, they're going to have a white block added to them. [Michael indicates the three 3-tall towers with one blue cube.]
00:47:05 ANKUR: They're going to have a *b* added to them.
00:47:08 MICHAEL: And this one's [the white-white-white] going to have a, a blue added to it.
00:47:09 ANKUR: An *a* added to it.
00:47:10 MICHAEL: And they're going to equal the same thing. That's why you're going to have the four. [Refer to Figure 18 for a diagram of Michael and Ankur's suggestions.]

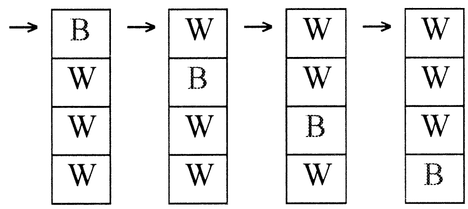


Figure 18. Michael and Ankur illustrate 1+3=4 by adding cubes to towers

The other students accepted Michael's explanation and apparently comprehended the process, as evidenced by their later work in the night session and subsequent interviews. In addition, Ankur reiterated a link noted in the February session, observing that the *a*'s and *b*'s in the binomial expansion could be connected to the blue and white cubes, respectively, in the towers. The same connection was made a year later during the night session.

Next, Researcher 1 asked the students to relate the towers problem to binary notation and the pizza problem; she said, "If you had to make up a pizza problem to model this row [row 2 of Pascal's Triangle], what's the pizza problem?" Ankur reiterated the distinction that had been made in two previous sessions, that a peppers and pepperoni pizza is the same as a pepperoni and peppers pizza. The group noted that the n^{th} row of Pascal's Triangle could be linked to the n-topping pizza problem, but they did not propose an explanation about how to use the numbers in the n^{th} row to enumerate pizzas. Researcher 1 asked for clarification. Michael stepped in with his own explanation, which the group accepted. This episode began at 00:56:34; the entire discussion about relating pizzas to towers and Pascal's Triangle, of which this was the concluding section, started about seven minutes earlier. Refer to Appendix H for the full transcript. Figure 19 shows the four two-tall towers the group made.

00:56:34	R1:	Now wait. Now I'm lost again. What, what, what was this? … [R1 indicates the single white and blue cubes representing row 2 of Pascal's Triangle.]
00:56:39	ANKUR:	The colors don't, don't look at the colors.
00:56:40	MICHAEL:	No. No. No.
00:56:40	ANKUR:	Just look at this [Pascal's Triangle]. … But the colors don't specifically represent anything.
00:56:51	ROMINA:	Yeah.
00:56:52	MICHAEL:	Yes. It does.
00:56:52	ANKUR:	No, it don't.
00:56:53	MICHAEL:	Topping. [Michael points to the blue cube.] Or no topping. [Michael points to the white cube]. Just say like that. And if you look at it like this, you know.
00:56:58	ANKUR:	So all of the whites are no topping?
00:56:59	MICHAEL:	Yeah. [Michael takes the white-white-white tower.] Then this is a plain pizza with a choice. If you had a choice of three toppings.
00:57:08	JEFF:	All right.
00:57:08	ANKUR:	OK.
00:57:08	ROMINA:	OK.
00:57:10	MICHAEL:	This [the blue-white-blue tower] would be a pizza-

00:57:11 ROMINA: Oh. With the one. Ooh.
00:57:12 MICHAEL: -with two different toppings, without the other, third topping.
00:57:14 ROMINA: That's what I was asking.
00:57:14 ANKUR: OK.
00:57:18 JEFF: … Well, yeah. Well, if you're just saying that this [the white-white-white tower] is the pizza with three no toppings, it's plain.
00:57:22 ROMINA: It's just a plain pizza.
00:57:23 ANKUR: All right. All right. so that's [blue-blue tower] two toppings.
00:57:25 ROMINA: Yeah.
00:57:25 JEFF: Yeah. All right. So.
00:57:26 MICHAEL: That’s [white-white tower] … a choice of two, but you want it plain.
00:57:27 ANKUR: You have a choice of two toppings.
00:57:28 JEFF: Yeah, so this is, this [blue-blue tower] is choice of two using two. This [blue-white tower] is choice of two using one.
00:57:32 ANKUR: Two using one.
00:57:33 JEFF: This [white-blue tower] is choice of two using the other one.
00:57:34 ANKUR: That's using the other one. And that's [white-white tower] using nothing.
00:57:35 ROMINA: Yeah.
00:57:35 R1: And that's all the possibilities?
00:57:37 ANKUR: Yes.
00:57:37 ROMINA: Yeah.
00:57:37 R1: You like that?
00:57:37 ANKUR: Those are the only possibilities.
00:57:38 ROMINA: Oh, wow!

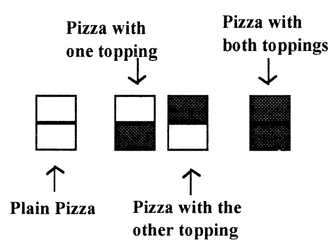


Figure 19. The students’ link between two-tall towers and two-topping pizzas

The other three members of the group immediately accepted and built upon Michael's brief remarks. All he had to say was "topping" and "no topping," and all three of the others began to relate specific individual towers to specific pizzas. This represented the fifth discussion of the pizza problem in four months, and at least three members of the group apparently began this discussion without a clear idea of what was the essential feature of the problem (topping vs. no topping), as opposed to a surface feature (the fact that the toppings could be selected in any order). But it appears that this discussion helped them finally to make sense of the isomorphic relationship, because the pizza problem was the one that the group selected during the night session to explain Pascal's Identity. (This is discussed in Section 5.3.4.)

After the pizza question, Researcher 1 presented a new towers problem: In general form, it is: When building n-tall towers and selecting from n colors, how many towers are there containing exactly n different colors?

00:59:22	R1:	Now I'm asking you the problem where you're selecting from three colors, the towers are three tall, how many are there with one of each color? And now you're selecting four tall, now you're selecting from four colors, how many are there with one of each color. And you're making your towers five tall-
00:59:52	ANKUR:	Five of each color.
00:59:53	R1:	Do you understand my question?
00:59:53	JEFF:	Yeah. Five of-
00:59:54	R1:	And so forth. Um. n tall. Right? Seven tall. Twelve tall. Fifteen tall.
01:00:00	ANKUR:	n tall. n of each color.

01:01:01	R1:	You can start with, you know one tall. Right? One tall, one of each color.
01:01:03	ANKUR:	One is one.
01:01:11	R1:	Do you understand my problem?
01:01:11	ROMINA:	I think so.
01:01:13	ANKUR:	Two is two. … One is one. Two is two.
01:01:21	JEFF:	So we're saying then two tall would be two colors.
01:01:24	ANKUR:	Two is either blue and white or white and blue.
01:01:25	JEFF:	Yeah. [A long discussion ensures, about the 3-tall towers case. Ultimately, the group decides that there are six three-tall towers containing exactly three colors. Then they start to work on the case of n=4. Refer to Figure 21 for the table they create.]
01:02:05	ANKUR:	It's, it's-
01:02:06	MICHAEL:	Factorial.
01:02:08	JEFF:	Wait. So-
01:02:10	ANKUR:	This one's factorial. Because look. Can you see this?
01:02:16	ROMINA:	Yeah. It is.
01:02:17	ANKUR:	'Cause you have four spaces. [Ankur draws four dashes.] You have to use all four colors. For the first one, you have four choices. For the second one, you only have three because you used one here. Two, one. …
01:02:33	ROMINA:	Twenty-five, is that what you're saying?
01:02:34	ANKUR:	Twenty-four. And then times five would be the next one.
01:02:35	JEFF:	Wait. Wait. Wait a minute. Wait a second. Are we doing-
01:02:36	ROMINA:	Oh, no. I'm adding wrong. Sorry.
01:02:37	JEFF:	Aren't you supposed to- wouldn't you be adding them?
01:02:39	ROMINA:	No.
01:02:40	JEFF:	No. You'd multiply.
01:02:40	ROMINA:	Factorial is multiplying. … And then five would be- …
01:02:49	ANKUR:	Twenty-four times five. …
01:02:53	JEFF:	Five. One twenty? …
01:03:04	R1:	So, n tall with choices of n different colors?
01:03:07	ANKUR:	It would be n factorial.
01:03:08	ROMINA:	n factorial. …
01:03:12	MICHAEL:	What's the sign for that?
01:03:13	ROMINA:	It's an exclamation point.

On December 12 (described in Section 5.2.1), the group had discussed a situation where factorial was not the appropriate operation. (After a color was selected, it was still available.) Here, the students found a situation where factorial was appropriate. (After a color was selected, it was not available.) They would make use of these insights when they explained the meaning of the factorial notation during the night session. (This is described in Section 5.3.2.)

At the end of this session, Researcher 1 introduced the idea of writing Pascal's Triangle in the notation of combinatorics and showed the group how to write a general row of Pascal's Triangle, including a general entry in that row.

01:08:30	R1:	Remember the other notation I showed you? Let's start with this row [row 2 of Pascal's Triangle]. You see that 1 2 1? Right. That means you're building two high. So that 1 means maybe you're selecting no blues. Right? [R1 writes 2 choose 0.] That was the 1. And this one means that you're selecting-
01:08:50	ANKUR:	One.
01:08:51	JEFF:	One blue out of two choices.
01:08:53	ANKUR:	One blue, two-
01:08:53	R1:	And?
01:08:54	JEFF:	And one red.
01:08:54	R1:	Right. [R1 writes 2 choose 1.]
01:08:56	JEFF:	'Cause you picked the one blue.
01:08:59	R1:	And then this is two two. [R1 writes 2 choose 2.] Right?
01:09:00	JEFF:	That means you picked two blues.
01:09:01	R1:	OK. And these [2 choose 0 and 2 choose 2] turn out to be ones. And then we could have gone to three high. Right? Okay?
01:09:07	ANKUR:	Three one. Three two. Three three. [R1 writes row 3. Refer to Figure 20.]
01:09:09	R1:	Exactly.
01:09:10	JEFF:	It would be three one. Three two. Yeah. And so on.

01:09:12	R1:	Remember that notation? And you know what that number is. You're going to be playing with those ideas later at some point. Um. And so if we kept doing this, right? OK. To the, let's say, n^{th} row. That would be n.
01:09:27	ANKUR:	Over zero.
01:09:28	R1:	Uh-huh..
01:09:28	JEFF:	*m* one. [Jeff is misreading *n* as *m*.] *m* two.
01:09:40	ANKUR:	Until the bottom number equals *n*.
01:09:41	MICHAEL:	The bottom number equals *n*.
01:09:42	ROMINA:	Yeah.
01:09:45	R1:	OK. So let's take a general term, an *r* term in there, right? OK. [R1 writes row *n* of Pascal's Triangle. Refer to Figure 20.] … One of the things I want you to tell me, if this is *r*, right? The r^{th} term, the general term, right? … How can you tell me, in general, where that r^{th} term comes from?

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{n}{0} \quad \binom{n}{1} \quad \binom{n}{2} \vdots \binom{n}{r} \quad \binom{n}{n}$$

Figure 20. Rows of Pascal's Triangle written in standard notation by Researcher 1

The form shown in Figure 20 is exactly what Researcher 1 asked the students to generate in the night session the following year. So these four students had seen the general row before the night session, although the evidence suggests that they did not remember it. (During the night session, they took some time to figure out how to write the general form, and they used different variables – *N* and *X* instead of *n* and *r*.)

Although Researcher 1 asked them to think about a general rule for producing the r^{th} entry of the n^{th} row, it was not discussed in these sessions until the night session.

5.2.6 June 12, 1998

Ankur, Brian, Jeff, and Romina participated in this session with Researcher 1. The primary purpose of the session was to introduce conditional probability. But it is mentioned briefly here because Researcher 1 provided the students with a reminder of the relationships among Pascal's Triangle, the binomial expansion, and the towers problems. Researcher 1 also asked the students in this session to think about the meaning of the addition rule in terms of towers and binomial coefficients. Even though probability problems became the primary focus of these students' problem-solving sessions, the idea of thinking about Pascal's Triangle and explaining Pascal's Identity in terms of familiar problems had not been forgotten. A transcript of this segment is included in Appendix H.

5.2.7 Interview December 14, 1998

On December 14, 1998, Michael met with Researcher 1 primarily to discuss probability. But at the end of the session, they spent some time discussing the relationship between the pizza problem and Pascal's Triangle. Michael had already written the eight 3-digit binary numbers (000, 001, 010, 011, 100, 101, 110, and 111) and explained that they represented an exhaustive list of all possible 3-topping pizzas. When Researcher 1 asked him to explore the distribution of pizzas with zero through three toppings, Michael brought up Pascal's Triangle and explained that the four numbers in row 3 (1 3 3 1) represented zero, one, two, and three toppings respectively. This part of the session is described in Kiczek, Maher, and Speiser (2001). Later in the session, Michael provided an explanation of one instance of Pascal's Identity (1+3=4) in terms of binary representations of pizza toppings.

00:39:20	R1:	Now this is my question to you. OK? You know one of the characteristics of this triangle … let's go here. This 1 and this 3 equals this 4. [Refer to Figure 21.] Now could you explain to me with your pizzas. … Here you're making it with three toppings. [Researcher 1 points to row 3 of Pascal's Triangle.] Right? … And these have none and these have one. [Researcher 1 points to the first 1 and 3 of row 3.] Right? Now you're adding an extra topping. Right? OK. And this 4 [the first 4 in row 4], right? Tells you how many you're going to have, right? With exactly one topping when you can select from four toppings. Right? So how does that work? … You know which pizza this is, don't you? [Researcher 1 points to the first 1 in row 3.]
00:40:32	MICHAEL:	Yeah. That is, with that [000].
00:40:36	R1:	OK, you know that, right? And you know which pizzas these are, you told me that. [Researcher 1 points to the first 3 in row 3.]
00:40:39	MICHAEL:	Yeah, I know what those are. …
00:40:42	R1:	How, how does that addition of that other topping end up with that? Would you explain that to me?
00:40:48	MICHAEL:	Let me write down those three pizzas.
00:40:52	R1:	Sure.
00:41:06	MICHAEL:	… Let's just say these are pizzas again. [Michael writes 100, 010, and 001.] In those four, you're going to have another, I guess, either a 1 or a 0 added on to it. On, I guess on this side [the left side]. And this one I guess has to be zero. [Michael adds 0's to the left of the three binary numbers, giving 0100, 0010, and 0001.] … But that, that pizza with nothing on it, could have a 1 now. [Michael

writes 1000. Refer to Figure 21.] … That's my explanation.

00:41:40 R1: OK, so you're not adding any toppings because they already have a topping.

00:41:44 MICHAEL: But the one … without anything, has one.

$$
\begin{array}{ccccccccc}
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Three Topping Pizzas		Four-Topping Pizzas
Add 0 to the front of the three	001 → 0001	This gives all possible
1-topping pizzas:	010 → 0010	1-topping pizzas when
	100 → 0100	there are 4 toppings
Add 1 to the front of this one:	000 → 1000	to choose from.
(the pizza with no toppings)		

Figure 21. Michael explains instance of Pascal's Identity with pizzas and binary notation

In February and March, the students had explained instances of Pascal's Identity using towers. But this was the first time that pizzas and binary notation figured in the explanation. Moreover, the particular toppings did not figure in this explanation at all.

Michael followed up this discussion with an email, in which he gave further examples of the meaning of the numbers in Pascal's Triangle in terms of pizzas and of the addition rule in terms of adding toppings to pizzas (Kiczek, Maher, and Speiser, 2001).

Michael reiterated these explanations of instances of Pascal's Identity, using either the pizza problem or binary notation, on three subsequent occasions: during the group's work on the World Series problem on January 22; in a discussion of the World Series problem with Researcher 5 during the second World Series session on January 29; and in a conversation with Researcher 10 about the email described above on January 29. These discussions are described below.

5.2.8 World Series Session January 22, 1999

Pascal's Triangle came up at the end of the January 22, 1999 World Series session. Researcher 1 asked Michael to recall how he had previously use pizzas to explain the growth of Pascal's Triangle. In the following segment, Michael referred to the addition shown in Figure 22.

01:00:14	MICHAEL:	These, these [the first 3 from row 3] with the mushrooms. If they get anchovies, they're gonna be three different, three different pizzas with two toppings. They're gonna go here [toward the 6]. If they don't, they're still going to be three different pizzas that have one topping [following the arrow to the 4]. So, you understand? Like, um, when you add a, when you add a topping or whatever, uh, this many is gonna, is gonna like, if you think of it as steps, like one, two- this one's going to, like, move up a step 'cause it's-
01:00:48	BRIAN:	Get a topping.
01:00:48	MICHAEL:	-either gonna get one or not, all right? So there's twice as many possibilities now. And three of them will have, and three then will not have, that extra one.
01:00:59	R1:	Does that make sense?
01:01:00	ROMINA:	Yeah, we understood this.
01:01:01	BRIAN:	You got something, you move up one. ... If you don't, you join up with the one of those over there.
01:01:07	R1:	Could you say that again, Brian? That was good. One more time. If you've got something- ...

01:01:13 BRIAN: If you get something, you move up [to the right], and if you don't, then you just join the other one [add with the number to the left].

Figure 22. Michael indicates addition from row 3 to row 4 on Pascal's Triangle

In this segment, Michael explained the doubling aspect of the numbers in Pascal's Triangle. (Each pizza in one row represents two pizzas in the following row – one with the new topping and one without.) This segment also illustrates the interactions among these students. Although Michael (who was asked to explain) provided the initial explanation, Romina indicated that she understood, and Brian provided a summary of the doubling rule.

5.2.9 Interview January 29, 1999

Just before he joined the others to discuss the World Series problem again, Michael met with Researcher 10. Researcher 10 asked Michael to explain his email message describing how the pizza problem can be used to describe Pascal's Identity. Refer to Kiczek, Maher, & Speiser (2001) for further information about Michael's email. Michael began by explaining what the numbers in Pascal's Triangle meant in terms of pizzas. In the following segment, he explained rows 0 and 1.

00:00:35	R10:	I don't understand this 1 up here [at the top of Pascal's Triangle]. Where does it come from?
00:00:37	MICHAEL:	That's, you don't have- All right, you go to a pizza place. "I want a pizza." You don't have any toppings to choose from. You can only make one pizza. A pizza without anything. …
00:00:55	MICHAEL:	Now, when the pizza guys says, "all right, you can put a mushroom on that pizza." … You can either have a plain one or a mushroom one. That's it. You can't have anything else.
00:01:06	R10:	OK. So this [first 1 in row 1] is without mushrooms and this [second 1] is with mushrooms.

In the next segment, Researcher 10 asked Michael to explain the addition rule he had indicated in his email. (Refer to Figure 23.)

00:02:25	R10:	… you underlined the 1 here. And you underlined the 2 here. And somehow you got to the 3 here.
00:02:35	MICHAEL:	Yeah. All right.
00:02:36	R10:	I don't completely understand that.
00:02:37	MICHAEL:	This 1, um, has no toppings on it, all right?
00:02:43	R10:	This 1 at the end?
00:02:44	MICHAEL:	Has no toppings on it.
00:02:44	R10:	OK. OK.
00:02:45	MICHAEL:	When that pizza man says, "All right, yeah. I give you, I'll give you, uh, another topping to choose from." All right. That plain pizza you just made, you could either put that mushroom on it, or you can't.
00:02:56	R10:	Oh, OK.
00:02:57	MICHAEL:	Let's say you already made that one pizza. He goes, "Oh. I just got some mushrooms." You could either put it on there, or don't. So then you could make two different pizzas with that.
00:03:04	R10:	OK.
00:03:05	MICHAEL:	And the one without it is that 1, and the one with the mushroom that you just put on, is part of that 3.

1

1 1

1 **2** 1

1 **3** 3 1

1 4 6 4 1

1 5 10 10 5 1

Figure 23. Michael's email diagram of the addition rule

This was the third time that Michael explained Pascal's Identity in terms of making pizzas, but this explanation was the most detailed and concrete. He included specifics about buying pizzas from a "pizza man" and specifics about how new toppings become available. In other sessions that month, Michael used less concrete imagery. It appears that he included that kind of information in order to make things clearer for Researcher 10, with whom he had not previously discussed these problems, and who indicated some difficulty in understanding.

5.2.10 World Series Session January 29, 1999

During the follow-up World Series session on January 29, 1999, binary notation was used to explain the numbers in Pascal's Triangle. Michael did almost all of the talking in this segment, but Jeff and Ankur contributed answers at one point, indicating that they followed the discussion. In the following brief segment, Michael described the meaning of row 2 of Pascal's Triangle to Researcher 5.

00:12:43 MICHAEL: I want to explain something. You're familiar with Pascal's Triangle, right?

00:12:49 R5: Yeah. Yeah.

00:12:50	MICHAEL:	I had- there's a way to, um, connect that and binary numbers together. You are familiar with that?
00:12:56	R5:	Uh-huh.
00:12:57	MICHAEL:	All right. So I used that to find a solution to that problem. Right? Let me just-
00:12:59	R5:	How does it go?
00:12:59	MICHAEL:	All right. Well, the connection would be, like, this is a two number. [Michael points to row 2 of Pascal's Triangle – 1 2 1.] Like the binary number with two numbers in two places.
00:13:27	R5:	Yes.
00:13:28	MICHAEL:	The first one is the amount, the total number, that only have, that have no 1's in it. The second one would be with one. And the third place would be the amount that had two, right?
00:13:39	R5:	So you're looking at the row 1 2 1. ...
00:13:44	MICHAEL:	This column has 1 [the first 1 in 1 2 1] with just whole 0's in it. [Michael writes 00.] This one [the 2 in 1 2 1] has- there's two in that ...
00:13:50	R5:	What are the two?
00:13:51	MICHAEL:	This one and that one. [Michael writes 10 and 01.]
00:13:52	R5:	Uh-huh.
00:13:52	MICHAEL:	All right. And the last 1 [the last 1 in 1 2 1], all 1's. [Michael writes 11.]

He followed up with an explanation of the addition rule involving adding 0's and 1's to the binary numbers, similar to the explanation he had offered in the interview the previous December. But this time, Michael did not mention pizza toppings, relying only on the nature of the binary numbers in his explanation. This is the first time that the addition rule was discussed without the involvement of an anchoring concrete problem.

5.2.11 Summary of Early Sessions

Between December 1997 and January 1999, these students first found general solutions to the pizza and towers problems, using letter and number codes and binary notation to enumerate the pizzas and towers. They organized their lists of solutions, organizing the pizza problem solutions according to number of toppings and the towers problems solutions according to the number of cubes of one color. This organized list not only provided a way to show that all cases were present, but it also provided the means to associate those cases with the numbers in Pascal's Triangle. In discussions with Researcher 1 and other researchers, these students described the isomorphic relationship between the pizza and towers problems, using words and written inscriptions and sometimes referring to concrete materials (as when Michael held up a blue cube and a white cube and said "topping" and "no topping.") Led by Michael, who explored the growth of Pascal's Triangle primarily in terms of pizza problems and binary notation, these students used combinatorics problems to make sense of Pascal's Identity. Although they were introduced to the standard notation during this time, these students did not themselves use that notation in their own explanations of Pascal's Triangle and Pascal's Identity.

5.3 Night Session May 12, 1999

In the night session, the students continued the investigations of Pascal's Triangle that they had begun in the sessions described in the previous sections. By the end of the night session, they had written Pascal's Identity in standard notation and provided a sound explanation of its meaning. They did this by looking at general forms of the towers and pizza problems, referring back to how they had previously explained specific instances of Pascal's Identity in terms of towers and pizzas, and making use of the binary notation introduced by Michael in 1997. The following episodes show the progression from a discussion of the meaning of factorial and combinatorial notation through an explanation of specific instances of Pascal's Identity to the final generalization. For each episode, I include a timeline showing links between current episode to past and future episodes.

5.3.1 Episode 1: Thinking About "10 choose 2"

At the beginning of the session, the first three students to arrive (Jeff, Michael, and Romina) talked about that day's classwork, which had been to find the coefficients of the expansion of $(a+b)^n$. Jeff brought up what they called "choose" notation, the notation to denote combinations, using the symbols from their calculators, ${}_nC_r$.

In this episode, when Jeff was trying to explain how to find the coefficients of the expansion of $(a+b)^{10}$, Romina spontaneously introduced the towers problem with the words "ten high" and "two reds." Jeff and Michael elaborated that this meant building towers ten cubes tall, selecting from two colors, and counting how many there are containing exactly two cubes of one of the colors (red). Jeff went on to associate *a* with one of the colors and *b* with the other.

00:03:11	JEFF:	… If we were looking for like *a* plus *b* to the tenth say, um, obviously … it was a one *a* to the tenth and then ten … ten *a* to the ninth *b* to the first, right? … [The next coefficient] was forty-five but we were working on how to figure it out when we were doing it. We knew it was the choose thing, whatever that means. … What was it? Ten choose two? … Like, uh, was it N-C-R? [Jeff is referring to buttons on his calculator.] Actually that's supposed to be lower case. Two, is that how you do it? [Jeff writes 10 NCr 2.] Right? … And that equals forty-five and that's the answer. … We're not really sure how all this works but it's like … If you have ten different. What is it? Ten different things …
00:04:25	ROMINA:	Ten high. Ten high.
00:04:26	JEFF:	Ten high. How many.
00:04:26	ROMINA:	How many would have two reds, only two reds.
00:04:27	JEFF:	How many would have two, two reds.
00:04:29	R1:	One more time.
00:04:31	JEFF:	… If you have towers with ten high and two colors.
00:04:32	MICHAEL:	How many different places can you put two reds in there?
00:04:35	JEFF:	And like *a* would be one color and *b* would be blue, um, *b* would be the other color. Then how many would you have, *a* being two in the whole thing? And that would be forty-five and that's, that's what this number would be. [Jeff points to his paper, which says $1a^{10} \quad 10a^{9}b^{1} \quad \underline{\ \ }45$.]

When they began to talk about towers, their style of speaking was personal and abbreviated. (All Romina had to say was "ten high," and Jeff immediately began to talk about the towers problem, also in abbreviated form: "How many would have two, two red.") But when asked to clarify ("one more time"), they expanded the explanation, not only providing a fuller description of the towers problem, but also describing how it related to the binomial expansion. This is a theme seen frequently in their work: an initial discussion among the students in a brief telegraphic style, using shared representations, followed by an expanded exposition for the benefit of the researchers asking questions.

As shown in Figure 24, the link made by the students between the towers problem and the binomial coefficients can be traced back to three earlier sessions: January 9, 1998; February 6, 1998; and March 6, 1998. In January, Researcher 1 drew Pascal's Triangle, referred to the binomial expansion, and encouraged the students to think about the numbers in Pascal's Triangle in a concrete way. In February, in response to a question ("What are the *a*'s and *b*'s here?"), Ankur explained how the *a*'s in the expansion of $(a+b)^n$ could be linked to one of the colors in a tower and the *b*'s to the other color. He reiterated this in March, in another context (an exploration of Pascal's Identity in terms of towers).

01/09/98	**02/06/98**	**03/06/98**	**05/12/99**
The students work on problem with towers: How many 5-tall towers are there with exactly two red cubes? R1 introduces standard notation.	They discuss how the binomial expansion is related to towers and Pascal's Triangle.	They link towers to coefficients "*a* will be blue and *b* will be red."	They explain 10 choose 2 using 10-tall towers with 2 red towers. R1 asks the group to consider towers. Jeff explains 5 choose 2 in terms of towers 5 tall with 2 of one color.

Figure 24. Timeline for thinking about "10 choose 2"

5.3.2 Episode 2: Making Sense of Factorial Notation

Earlier in the session, Michael had provided the formula for expanding "choose" notation; refer to the left side of Figure 25. (He said another student had given it to him.) Jeff had written the formula shown on the right side of Figure 25, and the three students were exploring the meaning of ${}_3C_2$ in terms of that formula. When Ankur arrived, the three students were asked to explain their work. Michael suggested that they use a familiar problem which he called "people on a line." Jeff, Michael, and Romina used this problem in order to discuss why 3! gives the number of ways to arrange three different items, and then Ankur provided the explanation requested by Researcher 1, about "why you multiply," by using a different problem (arranging three colors – red, white, and blue).

$$\binom{n}{x} = \frac{n!}{(n-x)!x!} \qquad \binom{3}{2} = \frac{3!}{1!\cdot 2!}$$

Figure 25. Michael's formula for *n* choose *x* and Jeff's formula for 3 choose 2

00:14:51	MICHAEL:	Like, you should use the explanation like she used. Like the people on the line. ...
00:15:00	JEFF:	All right, say we're doing us three right here.
00:15:01	MICHAEL:	Yeah, on the line. ...
00:15:04	ROMINA:	There's three different people to fill in the first spot.
00:15:07	JEFF:	Yeah. Then there's, after, then once one goes there, there's only two people left to fill in this spot. And then-
00:15:10	ROMINA:	So you multiply three and two.

00:15:12	JEFF:	Three times two and then once, once someone goes in the other, there's only one person left. And they get the last spot, so that's times the one.
00:15:18	ROMINA:	And that's everyone.
00:15:19	JEFF:	That make more sense?
00:15:27	R1:	… so why are you multiplying? … Why aren't you adding? [Researcher 1 leaves in order to give the students time to prepare an answer. Ankur brings up another problem, that of arranging three different colors, and the group agrees to discuss this new representation.]
00:16:18	JEFF:	… All right, how you saying this?
00:16:22	ANKUR:	The red, white and blue, right? You take, if red goes over here, that means you only have, with red there could go either go white and blue. Like it's each one of those three goes with two more. You know what I mean? There's three things here and then there's two things here. …
00:16:40	JEFF:	All right, yeah.
00:16:41	ANKUR:	Each one of those, those three goes with //two other.
00:16:42	JEFF:	//Those three things go with-
00:16:43	ROMINA:	//OK, like with our line thing.
00:16:44	ANKUR:	//So it's three times two.
00:16:45	JEFF:	All right.
00:16:45	ROMINA:	Like our line thing.
00:16:47	MICHAEL:	Or you could say like you have two more colors to add on. So you could do, you could make these into two different combinations. …
00:16:57	MICHAEL:	So that's two.
00:16:53	JEFF:	Yeah. That's- Yeah, that's why. All right.
00:16:54	MICHAEL:	That's like times. That's why you multiply. … [They call Researcher 1 back.]
00:17:16	JEFF:	All right, say you have three colors, red, white, and blue.
00:17:23	ANKUR:	… One of those colors goes in the first spot.
00:17:24	JEFF:	So, say you have your three spots. [Jeff draws three dashes in a column on the board.] Say red goes in the first one, all right? [Jeff writes R on the first dash.] Then you could do- …
00:17:31	ANKUR:	One, one color goes in the first spot, so there's two colors left. So there's three different colors that can go in the first spot and each of those colors can go with two other colors. [Refer to Figure 26.]

R W B
B W
W R B
B W
B R W
W R

Figure 26. Jeff's diagram for why you multiply

Discussions of factorial notation are listed in the timeline shown in Figure 27. Twice in December 1997, Ankur suggested using factorials to work on the towers problem and Michael disagreed. In March 1998, the group agreed that *n* factorial gave the number of *n*-tall towers that could be build when selecting from *n* colors and when each color had to be present. In the current episode, the students moved easily between representations of problems using factorials, using themselves to represent "people on a line," and then using codes to represent color choices and dashes to represent positions to be filled.

12/12/97	**12/19/97**	**01/09/98**	**03/06/98**	**05/12/99**
Ankur asks if they use factorial for 4-tall towers with two different colors. Michael says no.	Ankur asks if factorial is appropriate for towers with more than 2 colors. Michael says no.	A brief mention of having learned factorial notation in school.	Problem involving factorials: *n*-tall towers selecting from *n* colors having at least 1 of each color.	Making sense of factorial notation: People on a line and arranging three colors.

Figure 27. Timeline for making sense of factorial notation

5.3.3 Episode 3: Further Exploration of Factorials and Combinations

After discussing 3!, the students moved on to combinatorics notation. For about ten minutes, they used various representations in thinking about the purpose of the denominator in the "choose" equation (Figure 25). First Michael brought back the "people on a line" problem (interpreting ${}_5C_2$ as arranging five people on a line but only being concerned about the positions of two of the people). He said the purpose of dividing by $(n\text{-}x)!$ (in this case by 3!) was to eliminate repeats.

00:19:35	MICHAEL:	Those three, it … doesn't matter where they are. That's why you want, you want, you know, eliminate them. Because you only, you're only worrying about the two. How many different combinations that you could put those two in. It's five choose two. You're only worried about like the, the two. … Five people on the line, you want to know how many different places you could put those, those two people. Now there's going to be, there's going to be a lot of, a lot of repeats because you're also going to count those other three people where they're going to be and you're not worried about those other three people. So that's, that's why you would divide, to get rid of the, to get rid of them.
00:20:12	JEFF:	To subtract them.
00:20:15	MICHAEL:	No, it's divide. Why divide that, *n* minus *x*?

But Michael said he was not able to explain the other division (by $x!$). Jeff began to work out a specific example (${}_5C_2$); as he wrote, the meaning of the division seemed to emerge for the others.

00:22:18	ANKUR:	I get it. I get it. I get it. I get it.
00:22:21	MICHAEL:	I, I got it now. All right, then the last number would be-
00:22:26	JEFF:	Because this just gives you the number.
00:22:28	MICHAEL:	You have- Yeah. Yeah.

00:22:29	JEFF:	You're going to multiply by the number.
00:22:29	MICHAEL:	Those, those, you want to get rid of those. The, all the combinations that the three are moved around and those, those two aren't.
00:22:33	JEFF:	Yeah, they-
00:22:35	MICHAEL:	But then those two themselves will be repeat-
00:22:36	JEFF:	Yeah-
00:22:37	MICHAEL:	-will be mixed up.
00:22:38	JEFF:	You're repeating that's what you-
00:22:39	MICHAEL:	That's why you want to get rid of that, too.
00:22:40	JEFF:	Exactly. And then, so that would be just two.
00:22:42	MICHAEL:	Yeah.

Although the students expressed confidence, their explanation was not entirely comprehensible. The researchers asked for clarification, and the following explanation was offered. (Note that they often used the word "combinations" in the sense of arrangements or permutations; they did not use it in the mathematical sense.)

00:23:15	MICHAEL:	All right. The top thing, the *n* to the, the *n* to the factorial was going to give you how many?
00:23:21	ROMINA:	That's all the combinations.
00:23:22	MICHAEL:	That's every single combination. … Right? Now you're, you're only worried about them, those two people in that line. So there's going to be some instances where those two people are going to be in the same place and those three will be, you know, will be switch, you know, changing. So that's, that would be the, the three factorial. You want to, you want to get rid of that. You want to get rid of them. …
00:23:43	ANKUR:	All right, so you have the five minus two, is that what you're explaining on there? …
00:23:46	MICHAEL:	All right. But you don't think like when those two people are going to be in these two spots-
00:23:52	JEFF:	And everyone else is changing.
00:23:54	MICHAEL:	-not those other three.
00:23:54	JEFF:	And those make no difference because all we're worried about are where those two people are.
00:23:56	ROMINA:	Oh like when, oh, oh, okay, okay, okay.

00:23:58	MICHAEL:	All right, those two people are going to be moving around and it- you know, they're like the two people staying in the same place. So that's why you get rid of that. ... But then those two people themselves could switch places too. You know what I'm saying?
00:24:08	ANKUR:	Um-huh. So you got to get rid of those, too. That's why you do the *x*. ...
00:24:19	MICHAEL:	Because you're not worried about every, each person.
00:24:23	ROMINA:	Just the two. Yeah, we have, I got it. I'm good.

Although Romina and Ankur said they understood Jeff and Michael's explanation, the researchers pressed for further clarification. First, Ankur explained.

00:24:40	ANKUR:	The top number is five factorial, that's the total number of possibilities for five people. And then the five minus two comes in where you're not worried about everyone, you're just worried about two people at a time. So we need to subtract the five minus two. Those get, that gives you, and you do factorial, that gives you all the possibilities of just two people, right?
00:25:05	ROMINA:	Three people.
00:25:06	JEFF:	No, three extras.
00:25:07	MICHAEL:	No. The three that you don't, you're not worried about.
00:25:08	JEFF:	That's going to eliminate everyone except the two people you're worried about.
00:25:12	ANKUR:	OK. Everyone except the two people you're worried about. And then the *x* factorial eliminates ... when the two people are switched back and forth when you have the same ones over again.

Then, in response to one more request for explanation, Michael contributed the following.

00:25:43	R1:	Michael, start from the beginning very slow.
00:25:45	MICHAEL:	All right. You have five people. [Michael goes to the board.] I'm going to write it nice and clear... [Michael writes an equation on the board; refer to Figure 28.] All

		right. You got five people in a line. You agree with me that's how many different combinations you can put those five people. [Michael indicates 5!]
00:25:10	R3:	That part I understand. …
00:25:14	MICHAEL:	Now and you're only worried, you want to know how many different places you can put those two people. All right? So in all the combinations you're going to have, they're going to be repeated a lot. A lot. When you have like, the two people in a certain place and you know, those three. If the three are, are like this. [Michael points to the board and moves his hands back and forth.] And then one of them switches, that's another combination. And you get a lot of repeats like that.
00:25:37	R3:	Oh, I see. OK.
00:25:39	MICHAEL:	So by eliminating that, you eliminate the combinations that repeat by the three people moving around. Then let's say you just have those two people in, in any given combination. If this guy switches the place with this guy they're different combinations, but in this we're not worried about where they are. … That's why we get rid of the, the two factorial to, to, um, eliminate the amount like as many times as you could, as many combinations as you could put those two people. Right? Like the three would, would be to eliminate the combinations you could put those three people that you're not worried about. Then the two, they would repeat because those people, they move around. They move around in the line also. And then when you're done with all that, you just get, um, you get how many places you can just put that two. Like you're not worried if, like you don't care who they are. You don't care like if this guy has a switch with this guy.

$$\frac{5!}{3!\times 2!}$$

Figure 28. Diagram for Michael's discussion of five choose two

Finally, the researcher asked the group to explain ${}_5C_2$ in terms of towers.

00:27:41	R1:	OK. I don't want to say those are people. I want to think of the tower now. Isn't that what Jeff said? And now I'm thinking of towers that are five tall? And we're talking of those that have two reds? Explain it to me with that.
00:27:50	JEFF:	All right. Say we're doing, we're doing towers that were five tall. Towers of five tall with two different colors in it. Then that's the total amount of possibilities is the five factorial that you could have. All right, in, with, with five high combinations. So that's where, that's the five factorial on top. Then the three factorial on the bottom would be five different, five different spots minus the two spots that you're concerned about, leaving you with the three other spots ... that you don't care about. That's going to eliminate all of them.
00:28:29	ROMINA:	That's like, if you say like the reds. Let's say reds are our two colors that they stay in the same place, and like ... the two stay in the same place and then the other three are just switching while they're staying in the same place.
00:28:39	JEFF:	Yeah, they're staying in the same spot.
00:28:40	ROMINA:	But we're not concerned with those. ...
00:28:43	MICHAEL:	It's going to repeat like six times.
00:28:44	JEFF:	Yeah. So that's where the three factorial comes from, and you're multiplying that by the two factorial. Those are what you're-
00:28:50	ROMINA:	That's to say like the first place and the third place and then they just switch. ...
00:28:54	MICHAEL:	They just don't have a name on them so they're the same thing.

It is noteworthy that all four students participated in these discussions and that all four had no difficulty in moving between different representations of ${}_5C_2$ (people on a line, red cubes in a tower) that did not present obvious similarities. Michael's final remark (that all red blocks are "the same thing" because they are unnamed) is also worth noting; it gives evidence that he recognized the kind of counting that was appropriate for this task.

5.3.4 Episode 4: Using Pizzas to Explain One Instance of Pascal's Identity

When the researcher asked the group how their explanation of "choose" notation fit in with their work in class that day, Michael drew a few rows of Pascal's Triangle on the board and explained that a sample row could be expressed in "choose" notation: the row 1 3 3 1 could be called "3 choose 0" through "3 choose 3." Researcher 1 asked them to talk about the addition rule in that notation. Romina suggested that Michael use the pizza problem to explain. Michael selected a specific row, which Jeff related to binomial coefficients. In this segment, Michael listed all possible pizzas, using binary notation, and then he used the pizza problem to explain what addition meant.

00:32:21	ROMINA:	You started talking about toppings. …
00:32:27	MICHAEL:	… Let's go to this one. This would be like three different places, I guess. [Michael indicates row 3, which is 1 3 3 1.] …
00:32:41	JEFF:	That would be *a* plus *b* to the third.
00:32:42	MICHAEL:	All right, let's say you have like, here's a number, all right? [Michael writes 000.] Zero means no toppings. … One would be a topping. So first category is everything with no toppings. [Michael points to the first 1 in row 3.] And that's – you can't make – that's that's your number for that one. [Michael points to 000.] That's like, like binary numbers or something. Next would be- [Michael writes 001, 010, then 101.] There's all the, the ones that have one topping.
00:33:12	JEFF:	Right, you got to write that 0 at the end. You messed up. Last one should be a hundred, not a hundred and one.
00:33:15	MICHAEL:	I knew that. [Michael changes 101 to 100.] There's all the ones that have one topping. … There's your 3 choose 1 and there's three different combinations you could put that. … But, um, when you have a new – when you add another

place, another topping. [Michael draws three dashes next to the numbers already there. Refer to Figure 29.]

00:33:34 JEFF: That could be one or the other, one or the other, one or the other.

00:33:36 MICHAEL: So, it could be one or the other. It could be a zero or one, a zero or one, zero or one. [Michael writes 0 and 1 above each dash; refer to Figure 29.] … So all these threes would either move up a step onto the next category and, uh, have two toppings. [Michael points to the 6 in row 4, which is 1 4 6 4 1.] Or they might stay behind and still only have one if they have the zero. [Michael points to the 4 in row 4.] So three [Michael draws lines from the first 3 in row 3 to the 4 and 6 in row 4.] … get a topping, go to this one [Michael points to 6.] and three won't, will stay. [Michael points to 4.] And obviously this guy's going to get a topping. [Michael draws a line from the first 1 in row 3 to the 4 in row 4.] That's why you add this one. [Michael points to the first 1 in row 3.] … So now this guy's going to have- without toppings. You're going to add a topping on to him. That's going to be one topping. These three [Michael points to the first 3 in row 3.] with one topping won't get one so, you know, you can put them in the same category as this one.

00:34:14 JEFF: That's their four? Yeah.

00:34:17 MICHAEL: That's four. … And you know, the three that had two toppings won't get any. [Michael draws a line from the second 3 in row 3 to the 6 in row 4. Refer to Figure 29.] And you could put them in together with the ones that did get something. That's why you would add. Keep on adding.

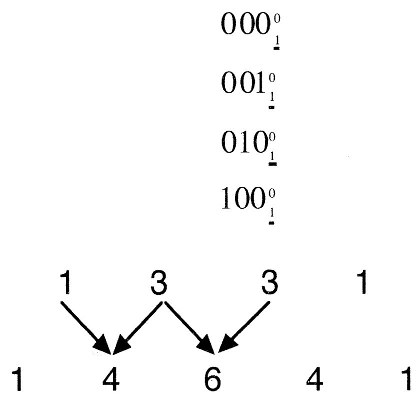


Figure 29. Binary listing of 3 choose 0 and 3 choose 1; Pascal's Identity examples

Although Michael dominated the discussion involving binary notation, it seems clear that Jeff was following, since he was able to correct Michael's coding. In explaining the addition rule, Michael moved easily from talking about "choose notation," to pizza problems, to binary notation, and back to pizza problems again.

The Timeline in Figure 30 shows earlier episodes leading up to this discussion of Pascal's Identity. Michael had taken the lead in discussions of binary notation and the pizza problems, and that is reflected in this table. Michael's binary diagram in this session was similar to the diagrams he drew in December 1998 and January 1999, when he explained the link between Pascal's Triangle and the pizza problems to Researcher 1, Researcher 5, and Researcher 10 in four separate sessions. This table also shows that the others had participated in discussions of Pascal's Triangle in terms of both towers and pizzas and they had explained instances of Pascal's Identity in terms of towers.

12/12/97 Michael introduces binary notation.	**01/09/98** R1 introduces standard notation. The binomial expansion is discussed.	**02/06/98** Ankur, Jeff, and Romina discuss Pascal's Triangle in terms of pizzas and towers. They use towers to explain instances of Pascal's Identity.	**03/06/98** Michael explains how 3+3=6 in terms of towers. They discuss a relationship between binomial expansion and towers.
12/14/98 Michael explains specific cases of Pascal's Identity in terms of pizzas to R1.	**01/22/99** Michael discusses pizzas, and Pascal's Identity with Ankur, Brian, and Romina.	**01/29/99** Michael explains Pascal's Identity in terms of pizzas to R10. He discusses binary with R5.	**05/12/99** Michael and Jeff use pizzas (and binary notation) to explain one instance of Pascal's Identity.

Figure 30. Timeline for using pizzas to explain one instance of Pascal's Identity

Jeff's parenthetical note (that row 3 is $(a+b)^3$) shows that he knew of a relationship between Pascal's Triangle and binomial coefficients. A connection between binomial coefficients and Pascal's Triangle was first discussed by this group on January 9, 1998 and reiterated on March 6, 1998 and June 12, 1998, but this represents the first time a connection was extended to include the pizza problem.

5.3.5 Episode 5: Pascal's Triangle and "Choose" Notation

Following the explanation of the specific incidence of Pascal's Identity, the researcher asked the students to consider Pascal's Triangle as written in "choose" notation. Michael had already named row 3 in terms of "choose" notation. Researcher 1 rewrote it, as shown in Figure 31.

$$1 \quad 3 \quad 3 \quad 1 \qquad \binom{3}{0} \; \binom{3}{1} \; \binom{3}{2} \; \binom{3}{3}$$

Figure 31. Researcher 1 shows two ways to write row 3 of Pascal's Triangle

Researcher 1 then asked the students to write more rows of Pascal's Triangle in this notation, including a general row (row *n*). Michael wrote rows 0, 1, 2, and 4.

The students had watched Researcher 1 write row *n* a year earlier, but they had not worked with the notation themselves. Now they struggled to write a general row, and ultimately the researcher told them how to finish it up (by writing "*n* choose *n*"). But they made a connection between Pascal's Triangle and the binomial expansion, and they went on to relate the n^{th} row to a tower *n*-blocks high in attempting to write the general row.

00:36:38	MICHAEL:	So it would be like *n* over, not two. ... *n* choose.
00:36:46	ROMINA:	Well, and *n*, make *n* like your height or something. ... *n* equals height.
00:36:52	ANKUR:	Well write the *X*. Write *a* plus *b* to the whatever it is next to it. You know what I mean? ...
00:37:07	JEFF:	Yeah well, *a* plus *b* to the second, so it would be if, or *a* plus *b* to the n^{th} .... [The students all talk at once for a few seconds.]
00:37:28	MICHAEL:	Well if you had an *n* it would be, uh.
00:37:30	ANKUR:	To the height of the tower which is *n*, right?
00:37:32	MICHAEL:	We'd have a bunch of *n*'s. ... There'd be *n* plus one *n*'s going this way.
00:37:37	JEFF:	Yeah. ... It would be *n* over zero.

00:37:38	MICHAEL:	All right? So if n was three you'd have four n's going this way. … And the bottom numbers would be just going from zero to … Zero to n. [Michael writes the first four entries of the n^{th} row; see Figure 32.]
00:37:51	JEFF:	… To n. Whatever n equals. … So it would be n over zero to the n^{th}. …
00:38:08	MICHAEL:	Zero, what are you talking about?
00:38:09	ANKUR:	What are you talking about zero to the n?
00:38:11	MICHAEL:	Zero minus n? That would be negative.
00:38:13	JEFF:	No not minus, like that's two, whatever. n is n over zero, n over one. … Not divided by. I was using bad, uh, bad looking things there.
00:38:34	MICHAEL:	Each of those would be a number.
00:38:35	JEFF:	Yeah, it's what, zero to n.
00:38:37	ANKUR:	And n represents the height of the tower?
00:38:39	ROMINA:	Height of the tower, yeah.
00:38:49	MICHAEL:	… It's like height of the tower with two colors. You have two numbers.
00:38:59	JEFF:	Yeah. How do you, how are you, can you write that to get this?
00:39:04	ROMINA:	Like that's what I meant. Like I didn't mean factorial. I meant like when we used four first and like three first. I don't know how to write that, though.
00:39:10	R1:	So you go zero, one, two, three, dot, dot, dot, up to n. Can we get one in the middle there, like n choose r? [Michael rewrites row n, including the last entry, as shown in Figure 32.]

$$\binom{n}{0}\ \binom{n}{1}\ \binom{n}{2}\ \binom{n}{3}\cdots\binom{n}{n}$$

Figure 32. Michael's version of row n of Pascal's Triangle

Although they had just finished connecting Pascal's Triangle to the pizza problem, they returned to the towers problem when building the general row. They also again explicitly called forth the association with binomial coefficients, when Ankur told

Michael to write $a+b$ to the "whatever it is" power next to each row of Pascal's Triangle, and Jeff added "$a+b$ to the n^{th}".

Michael's final remark was also important: he explicitly noted that Pascal's Triangle relates to an n-tall tower built from *two* colors. Michael had first brought this up on December 19, 1997, when the group explored m-color towers problems.

The students did not immediately respond to the request for a general entry ("n choose r"), but after Jeff rewrote rows 0 through 4 of Pascal's Triangle in "choose" notation, followed by row n, the researcher again asked for a general entry. Jeff used N choose X. Refer to Figure 33.

$$\binom{N}{0}\cdots\binom{N}{X}\cdots\binom{N}{N}$$

Figure 33. Jeff's version of row N of Pascal's Triangle with a general entry

The timeline for this episode is given in Figure 34. It shows that Researcher 1 had actually written a general row for the students back on March 6, 1998, when she asked them to think about the meaning of the general addition rule. But the generalization of the addition rule was not discussed again until this night session. As noted earlier, it seems likely that the students approached the problem anew at the night session, since their notation was slightly different (N choose X instead of n choose r) and also because they used pizzas when generalizing the addition rule. (Row *n* was brought up previously in the context of towers problems.)

01/09/98 R1 draws Pascal's Triangle, calls attention to binomial coefficients, and asks why we keep finding those numbers (answers to towers problem) in there.	**02/06/98** R1 introduces standard notation.	**03/06/98** R1 writes Pascal's Triangle in "choose" notation, including row *n* and a general (r^{th}) entry in row *n*. The students explain to her what various entries mean in terms of towers.
01/22/99 Michael explains Pascal's Identity in terms of pizzas to Ankur, Brian, and Romina.	**05/12/99** They work on writing Pascal's Triangle in "choose" notation.	**2002 Interviews** Michael uses standard notation. He explains it in terms of pizzas. Romina explains the addition rule in terms of towers. Ankur explains how to get from C(3,1) and C(3,2) to C(4,2) in terms of towers.

Figure 34. Timeline for Pascal's Triangle and "choose" notation

5.3.6 Episode 6: Specific Instances of Pascal's Identity

After Jeff redrew Pascal's Triangle, Researcher 1 asked the students to express an instance of the addition rule using "choose" notation. Michael stated that ${}_4C_2 + {}_4C_3 = {}_5C_3$. When Jeff asked for an explanation, Michael used a general form of the pizza problem to justify his reasoning.

00:43:02 R1: Show me that 3 plus 3 is 6. Which ones would it be? …
00:43:15 MICHAEL: This one and that one. [Michael points from 3 choose 1 and 3 choose 2 to 4 choose 2.]

00:43:16	JEFF:	OK, is that it? [Jeff draws arrows as indicated by Michael; refer to Figure 35.] Is that all, that's all you want? ...
00:43:25	R1:	OK, so you're saying 3 choose 1 plus 3 choose2 equals 4 choose 2. Right? ... OK. So what's 4 choose 2 plus 4 choose 3?
00:43:35	JEFF:	... 4 choose 2 plus 4 choose 3? That would be, that would be 5-
00:43:41	ANKUR:	5 choose. ...
00:43:43	MICHAEL:	5 choose 3.
00:43:50	JEFF:	... Why is he 5 choose 3?
00:43:53	ANKUR:	... Because it's, it's always the one on the right.
00:43:55	MICHAEL:	Because, see, this guy gets another topping, I guess. ... Because he's moving up, this bottom number's going to change. ...
00:44:09	R1:	Explain that one more time, Michael, please.
00:44:11	MICHAEL:	[Michael indicates 4 choose 2.] Um, wherever this guy goes, wherever this guy goes, he's going to get another topping because he's moving this way. [Michael points to the right.] ... So this bottom one's going to change too. [Michael points to 4 choose 3.] This guy's not going anywhere. The bottom number stays the same. ...
00:44:	ROMINA:	I'm with you.

$$\binom{3}{0}\binom{3}{1}\binom{3}{2}\binom{3}{3}$$

$$\binom{4}{0}\binom{4}{1}\binom{4}{2}\binom{4}{3}\binom{4}{4}$$

Figure 35. Jeff's arrows showing 3+3=6 in combinatorics notation

Figure 36 shows the timeline for this episode and for the following episode. The question about what the addition rule meant was raised during three previous problem-solving sessions. On January 9, 1998, Researcher 1 first brought up the question, although the students did not discuss an answer then. On February 6, three of the students (Ankur, Jeff, and Romina) explained an instance of the addition rule to Researcher 5 in terms of towers (adding a red block to one group of towers and adding a blue block to another group, in order to preserve the number of red blocks). On March 6, Michael provided a similar explanation using towers, emphasizing that the goal was to add something to one group of towers with one property (add a blue block to the group with fewer blue blocks) and something else to another group of towers with a different property (add a white to the group with more blue blocks) in order to create a new group of towers, all with the same property (one block higher with the same number of blue blocks). In December 1998 and January 1999, Michael provided a comparable explanation in terms of pizzas: adding a topping to the pizzas with fewer toppings and not adding a topping to the pizzas with more toppings. It is this pizza explanation that is repeated in this episode.

It is also worth noting Ankur's response to the question of why ${}_4C_2 + {}_4C_3$ yields ${}_5C_3$. He said, "It's always the one on the right," meaning that the 3 in ${}_5C_3$ came from the 3 in ${}_4C_3$, indicating perhaps that he noticed a pattern that would come into play later when the group produced the generalization.

01/09/98 R1 shows standard notation.	**02/06/98** Ankur, Jeff, and Romina explain specific instances of Pascal's Identity in terms of towers.	**03/06/98** They discuss how 3+3=6 in terms of towers.	**12/14/98** Michael explains Pascal's Identity in terms of pizzas.
01/22/99 Michael discusses pizzas and Pascal's Identity in terms of pizzas with Ankur, Brian, and Romina.	**01/29/99** Michael explains Pascal's Identity to R10 in terms of pizzas. Michael explains Pascal's Identity to R5 in terms of pizzas, using binary notation.	**05/12/99** Jeff and Michael show 3+3=6 in "choose" notation.	**2002 Interviews** Michael discusses the addition rule in terms of pizzas. Romina explains the addition rule in terms of towers. Ankur discusses the addition rule in terms of towers.

Figure 36. Timeline for Pascal's Identity (specific instance and general form)

5.3.7 Episode 7: Pascal's Identity

Brian arrived after the group had been working for almost an hour; in explaining to Brian what they had been doing, Jeff rewrote row *N* of Pascal's Triangle with additional entries and added row *N*-1. Jeff's work is shown in Figure 37; the dialog follows.

00:45:15 JEFF: We got um, *N* choose 0, and over here we have *N* choose *X*, and then over here we have *N* choose *N*. All right? Then this right here would be – Oh, we're explaining the general addition, the addition rule using this type of – to fill out the triangle. Using chooses to fill out the triangle and this here would be *N* choose *X* plus 1 and then *N* choose *X* plus 2 and so on to whatever *N* equals. … And this here would be *X* minus 1 and then … That'd be *X* minus 2 and so on each way. Right? So it'd be that. ... And the row above this would be N minus 1 right. Yeah. ... Um, choose 0. This again would be N minus 1 choose X and then N minus 1, N minus 1.

$$\binom{N-1}{0} \cdots \binom{N-1}{X} \cdots \binom{N-1}{N-1}$$

$$\binom{N}{0} \cdots \binom{N}{X-2} \binom{N}{X-1} \binom{N}{X} \binom{N}{X+1} \binom{N}{X+2} \cdots \binom{N}{N}$$

Figure 37. Jeff writes rows N-1 and N of Pascal's Triangle

In contrast to the earlier episode, in which he had struggled over writing and naming the entries, this time Jeff completed the task without hesitation and in about half the time (not only adding entries to row N but also adding the row above row N).

Researcher 1 asked the group to write the general equation and explain it to Brian. The general equation is shown in Figure 38.

00:49:41	R1:	Can you write it- can you write it as an equation? Just like you wrote three plus three equals six.
00:49:48	ANKUR:	N plus, just that plus that. [Ankur points to the board.]
00:49:51	MICHAEL:	... N choose X.
00:49:52	JEFF:	N choose X, um, plus N choose X plus one. [Jeff writes on the board as he speaks. Refer to Figure 38.]
00:49:57	MICHAEL:	Equals that. ...
00:50:00	JEFF:	Plus one equals that right there. [Jeff points to N+1 choose X+1.] ... Then, well, that's, that's because this would be gaining an X and going into the X plus 1. [Jeff points to N choose X.]
00:50:14	MICHAEL:	Yeah.
00:50:15	JEFF:	And this would be losing an X. [Jeff points to N choose X+1.]

00:50:16	MICHAEL:	No, no, not losing, not getting anything.
00:50:16	ANKUR:	Staying the same. …
00:50:22	MICHAEL:	… And the top numbers have changed because you have more.
00:50:24	JEFF:	Because you're adding; you have more things. …
00:50:28	R1:	Say it so Brian can follow it because he wasn't here for the earlier pizza discussion. …
00:50:35	JEFF:	What, what we're doing is the next line of the triangle. Remember how today in class the other triangle was one, two.
00:50:40	BRIAN:	Yeah.
00:50:41	JEFF:	Three, that whole row there. Well that's the increase in *N* and then the *X* plus one. … Say we're doing pizzas.
00:50:50	BRIAN:	All right.
00:50:51	JEFF:	If you add another topping onto it.
00:50:53	ROMINA:	You know how we get the triangle and how we go 1 2 1 and add those two together?
00:50:56	BRIAN:	Yeah.
00:51:02	JEFF:	… We were explaining why you add.
00:51:03	BRIAN:	All right, keep going.
00:51:04	JEFF:	… Because when you add another topping like onto it, this one – say the toppings were 1 and 0. [Jeff points to *N* choose *X*+1.] … If it gets a topping, that's why it goes up to the *X* plus 1. [Jeff points to *N*+1 choose *X*+1.]. And since it doesn't get anything it'll stay the same. And in this one, it's staying the same, right? [Jeff points to *N* choose *X*+1 and looks at Michael, who nods.] … And that's why it's going there. … You're going to there. Make sense?
00:51:28	BRIAN:	Yes. It actually does.
00:51:30	JEFF:	So, so that would be the general addition rule in this case.

$$\binom{N}{X} + \binom{N}{X+1} = \binom{N+1}{X+1}$$

Figure 38. Pascal's Identity in students' notation

These students showed that Pascal's Identity can be explained in terms of a general pizza problem as follows: The next row of Pascal's Triangle can be generated through actions on the pizzas representing the numbers in the current row: add the new topping (representing the new row) to one group of pizzas (those that have fewer toppings) and do not add the new topping to another group of pizzas (those that already have the requisite number of toppings).

Although this was the first explanation of Pascal's Identity accompanied by general notation, the students had described instances of Pascal's Identity in earlier sessions, as listed in the timeline shown in Figure 36. In February 1998, Ankur, Jeff, and Romina had provided an explanation using towers, and the group had reprised the explanation in March 1998. (Add a blue cube to the towers in one group – the ones with fewer blue cubes – and a white cube to the towers in another group – the ones with more blue cubes – in order to create a new group with the desired property – a given number of blue cubes.) Michael had used pizza problems to explain instances of Pascal's Identity in the December 1998 and January 1999 interviews.

Two years later, in individual interviews (discussed more fully in the following sections), Michael, Romina, and Ankur were asked to recall this work. Although none recalled the night session, all three were able when prompted to write the formula, and all three were able to provide a cogent explanation of the addition rule.

5.4 Later Sessions

5.4.1 Taxicab Session

The session of May 5, 2000, the last meeting before the end of high school, was attended by Brian, Jeff, Michael, and Romina. The taxicab problem (described move fully in Chapter 2 and Appendix A) is to find the number of routes from the origin to a destination *n* blocks away (*r* blocks to the right and *n-r* blocks down), when the only allowed moves are to the right and down. This session is fully analyzed by Powell (2003); a summary of critical events is presented here. Refer to Appendix K for a critical events transcript.

Twice in the beginning of the session (first after about seven minutes and then about 37 minutes later), Romina suggested that the group think about using towers in their work on the taxicab problem. But the suggestion was not followed up by the others. However, after 47 minutes of work, Romina announced that Pascal's Triangle was involved. Investigating the relationships – between Pascal's Triangle and the taxicab problem and between the taxicab problem and the other combinatorics problems – consumed most of the remaining time (almost an hour) of the taxicab session.

In the following segment, Romina explained to Brian and Michael how she linked the taxicab problem to the towers problem by giving some specific examples from row 4 of Pascal's Triangle.

01:24:56 ROMINA: That's two across, two down. That's four, so you're in the four blocks [the fourth row of Pascal's Triangle]. … So there's two of one color and two of another. … And the,

		the four is ... three across and one down, so it means it's three of one color [and] one of the other color.
01:25:19	MICHAEL:	That, that's a pretty good explanation.
01:25:22	BRIAN:	It's cool.

Later, Romina and Brian discussed the relationship again with Researcher 12. They had previously talked about linking towers and taxicab solutions to x and y. Although they had not explicitly stated what the x and y stood for, it is possible that they were referring to the binomial expansion.

01:33:02	R12:	And the x's and y's. What does the x correspond to again?
01:33:06	ROMINA:	x is across.
01:33:07	BRIAN:	Going across. And y is down.
01:33:09	ROMINA:	Or a topping. Or a color.
01:33:11	R12:	Uh-huh.
01:33:12	ROMINA:	All the same thing. And all our y's are down, topping, color.

Pascal's Identity and standard notation were not discussed in the taxicab session. But links were made among all the combinatorics problems and between the problems and Pascal's Triangle. Although Michael and Jeff had taken the lead in the night session, it was Romina who first proposed the relationship between the taxicab problem and Pascal's Triangle in this session. Brian, the latecomer to the night session, also contributed to these taxicab discussions. So, although not all aspects of the night session work were addressed in the taxicab session, the taxicab session did provide important information about students' abilities to make connections among the combinatorics tasks and between the tasks and Pascal's Triangle, one year after they had formalized the relationships.

5.4.2 Interview with Michael

Over the years in which he worked on the combinatorics problems, Michael progressed from drawings and codes through his personal (binary) representation system to the standard combinatorics notation. From the time he introduced his ideas about binary notation to his fellow students to his most recent interview over five years later, he demonstrated the ability to make sense of the problems and of the notation, both through the use of his chosen notation and through the use of the combinatorics tasks. The timelines show Michael taking the lead in devising representations, finding connections, and making sense of the tasks. He recognized structural similarities between problems; he moved between different representations with ease; and he extended, generalized, and reorganized his knowledge when he discussed it with others.

In the interview of April 3, 2002, Michael was asked to recall or reconstruct the work from the night session. Appendix K contains a transcript of this interview.

Researcher 1 asked Michael to recall how the group used the pizza problem to think about Pascal's Triangle. A paper showing Pascal's Triangle in combinatorial notation (Figure 38) was in view. This segment shows that Michael had no specific memory of the evening in question, and he did not quite remember the details of the pizza problem.

00:02:17 R1: And I'm thinking now at the night session that was the night before your prom. And you brought a lot to that session, … which went back from some interviews we had done with you earlier, where you had begun to think about Pascal's Triangle. And you had to think about how one could look at that triangle, from the lens of some of the problems you had worked on, like towers and pizzas.

00:03:12 MICHAEL: OK, I think I remember that. … I do remember what you were talking about the Pascal, like how I looked at it, sort of like, each thing is like relating to probability. Is that what you're talking about? Like what she had on her sheet right there? With the choose? [Michael points to the paper shown in Figure 38.] 'Cause I don't remember the specific night.

00:03:33 R1: It was looking at how the triangle grew. [Michael nods.] And that was the question. How can you talk about how that triangle grows? And you had used the example of pizzas to think about the movement from one row of the triangle to the other.

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

Figure 38. Pascal's Triangle in combinatorics notation (interview with Michael)

Michael recalled that the relationship between Pascal's Triangle and the pizza problem had something to do with relating the rows of the triangle to pizza toppings, and he attempted to rediscover that relationship by working with a specific row. After struggling to recall the exact problem, Michael asked for clarification, and Researcher 1 posed the two-topping pizza problem.

00:04:40	MICHAEL:	I'm going to draw a small little triangle. [Michael draws rows 1 and 2 of Pascal's Triangle.] I do remember, I kind of related each row ... with a certain pizza topping situation. [Michael looks off into space.] … I'll just pick a row, three for example. 1, 3, 3, and 1. So, I don't know if that was, um, two toppings, or...?
00:05:20	R1:	Well, think about it.
00:05:21	MICHAEL:	Yeah, I'm trying to, you know, get it back. OK, I think this row, was like, you have two toppings to choose, and, like different, uh, no, that's not gonna... hold on a second. No, it would have to be, it would have to be more than that. … Can you refresh the pizza thing? Was it you have a certain amount of pizzas? Or a certain amount of toppings?
00:06:23	R1:	A certain amount of toppings. … So the question is, if you can select from two toppings, how many different pizzas can you make?

At first, Michael struggled to recall the previous work. But after some false starts, he reconstructed the connection between the pizza problem and Pascal's Triangle.

00:06:52	MICHAEL:	OK. I think I found …where I was supposed to be. Um, like you said, the two toppings, um. [There is a pause.] I just. [Michael pauses and then shakes his head.]
00:07:04	R1:	…Feel free to pick the two toppings. Make them up.
00:07:10	R7:	Mushroom.
00:07:11	MICHAEL:	OK. If you had no toppings, that would be one pizza.
00:07:15	R1:	OK. So where is that on the triangle?

00:07:17 MICHAEL: Well, I'm going to just draw it. … And then we'll find it. … If you're … using just one topping, you can make two possible pizzas with that. And then if you have all the toppings, that's one. Right. And then automatically I see that relates to this row. [Michael points to row 2 of his Pascal's Triangle (1 2 1); R1 nods.] And I'm pretty sure it would go down, this is like a third topping and a fourth topping. [Michael points to rows 3 and 4 of his Pascal's Triangle.] Now I think the way I thought about it is, like, the row on the outside would be your plain pizza. And there's only one way to make a plain pizza. And … the next one over would be how many pizzas you could make using only one topping, and then so on until you get to the last row which is all your toppings. And, once again, you can only make one pizza out of that. … If you go to the next row, it would like follow. … So you see like a physical connection, we can see the numbers ... Now why do they?

Note: When he said "row on the outside," Michael was referring to the leftmost number in a given row of the triangle, and when he said "last row," he meant the rightmost number in a row. He had offered an almost identical description to Researcher 10 in January 1999. (Refer to the transcript in Appendix I.)

Michael rejected the suggestion by the interviewers that he use specific toppings to aid his recollection and that he immediately locate the numbers he was deriving in Pascal's Triangle. Further, Michael, unprompted, indicated that the numerical relationship needed to be explained. (He said, "Now why do they?")

Researcher 1 then suggested that Michael use the notation shown in Figure 38 for a specific case: "Why don't you use that notation explicitly from one row to the next?"

00:08:47 MICHAEL: Well, we'll take the one-two-one row. It's basically written, um, in that choose. I don't know what it's called. …

00:08:53	R1:	Choose is fine. …
00:08:57	MICHAEL:	It would be, um, two choose zero? If I'm writing it correctly. Do I have to put the brackets around it?
00:09:04	R1:	That's the way we do it.
00:09:06	MICHAEL:	OK. And it would be 2 choose 0, 2 choose 1, and then 2 choose 2. [Michael writes row 2.] … And the next row would be the same, which is with three. And you have another, you know, you can go all, as far as, up to three. [Michael writes row 3.] So it's going to be four on the bottom. Now to connect the two rows together. See if I can remember that. … I kind of understand why each row is made of the other two combined.
00:10:11	R1:	Why is that?
00:10:12	MICHAEL:	Well, you're starting off with this, you know, this group of pizzas that has no toppings. [Michael points to 2 choose 0.] And this group of pizzas that has one. [Michael points to 2 choose 1.] So when you, when you go up, you have the choice of adding one more. … To that one that had nothing, you could either not give it that extra topping. … Or you can. So for those pieces that you do give that extra topping, it moves to the right. And for the others it moves to the left. [Michael draws arrows as shown in Figure 39.] And that's kind of why it doubles. … The one that doesn't get the topping, you know, still stays in the one topping category. So then they combine.

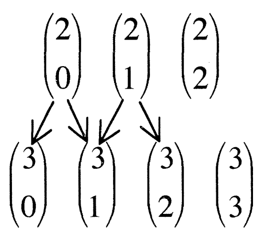


Figure 39. Michael's illustration of the addition rule

Michael was unsure about what to call the notation, but he was nevertheless able to explain how it related to the pizza problem. Michael went on to describe in more detail what happens to the two pizzas represented by 2 choose 1.

00:11:21	MICHAEL:	For those two, you can make four more pizzas by adding another topping on it. You can keep, you know, have those two the same way they were. Or just add, you know, the topping on those two. So half of them go to the left. Which would keep it in the one topping category. And half would go to the right, which would...
00:11:41	R7:	Go to the next one.
00:11:42	MICHAEL:	Yeah.

Michael used the pizza problem to explain the doubling pattern of Pascal's Triangle (the sum of the numbers in row *n* is twice the sum of the numbers in row *n*-1). This echoed the explanation he gave to Researcher 10 in January 1999.

About two minutes later, Michael was asked to write a general equation. This segment includes the discussion as Michael wrote the equation.

00:11:45	R1:	And at that session [the night session] … I asked them to write an equation to show, for instance, how that might happen from one row to the next. So can you just do that, write. …
00:12:10	MICHAEL:	Like a general equation?
00:12:11	R1:	Well, that was what I was going for ultimately. …
00:12:20	MICHAEL:	To give an amount for any spot in the row.
00:12:22	R1:	Right. …
00:12:24	MICHAEL:	All right, so I guess we'll give, you know, the row a name. … Call that *r*. And I guess the spot in the row, like, you know, zero topping, one topping. Call that, *n* sounds fine. Just to like pick, you know, one spot, and then see what... OK? [There is a pause; then Michael writes the left part of the equation shown in Figure 40.] I'm just going to like

work this out in my head and see if it actually works. [Michael adds the right part of the equation.]

This equation is equivalent to the textbook version, usually written in a slightly different form. (See Figure 40.)

$$\binom{r}{n} + \binom{r}{n+1} = \binom{r+1}{n+1}$$

Figure 40. Michael's equation for Pascal's Identity

In order to solve the problem of finding a general equation, Michael did not rely on symbol manipulation. He linked the numbers to a problem task that made sense to him, and then he expressed the relationships in that task in symbolic form. This could explain why his notation differed both from standard notation and from what his group had used three years earlier. Michael was not remembering the form; he was reconstructing the substance.

The interviewers observed that Michael's group had often solved a general problem when they were asked to investigate a specific problem. (They were referring specifically to the taxicab problem, which the group had worked on in May 2000. Refer to Appendix A for details about the taxicab problem.) Michael was asked to talk about why they did that.

00:16:15	R7:	Why did you go to the general question?
00:16:16	MICHAEL:	I mean, I don't remember that specifically. ... I haven't taken math courses in a while, but usually ... I make

everything into a problem first, OK? For me, it's easier to understand if you just take one, you know, specific case. ... You see the r's and the n's and you can't combine them, so you know, I don't know if you understand. ... Let's say something was a 2 and a 3 and I combine them into a 5. It doesn't mean anything, I mean when it's a number. But, like, I feel comfortable ... using a general form, putting in equations. That's the way I feel comfortable with. Dealing with problems like that.

00:17:06 R1: ... Am I hearing you say that when you see the general, you also see the particular ones in there?

00:17:14 MICHAEL: Yes.

It is interesting that, in his second year of college, Michael reported that he had not "taken math courses in a while," but he was nevertheless comfortable and secure (and accurate) when working on a question about combinatorics he had not seen in some time. Further, he observed not only that using a specific case helped him to make sense of the general form ("For me, it's easier to understand if you just take one ... specific case"), but also that using the general form helped explain specific cases. In the sessions leading up to the night session, Michael had observed the general form several times. On December 12, 1997, Michael had explained his binary coding in a general way (1 = topping and 0 = no topping). On December 19, 1997, he did something similar with towers (1 = red and 0 = blue). On March 6, 1998, he made the link between pizzas and towers (white = no topping and blue = topping). But he also brought forth specific instances when called for. In March, he identified the white-white-white tower as the pizza with three toppings available, none of which were selected. During the night session, he again equated that particular plain pizza with a specific entity, this time 3 choose 0.

5.4.3 Interview with Romina

Romina had apparently not taken a leading role in many of the critical events of the night session and earlier. Many times, it was Michael who expressed an insight (sometimes during discussions or arguments with Ankur), and in the night session, Michael and Jeff dominated. But Romina took a leading role in the taxicab session. And this interview suggests that, three years after the night session, Romina was able to state the combinatorics problems related to Pascal's Triangle and to regenerate Pascal's Identity.

In this interview, which took place in July 2002, Researcher 8 asked Romina to recall or reconstruct the night session work on Pascal's Triangle and Pascal's Identity. In the first episode, Romina was shown two versions of Pascal's Triangle (see Figure 41) and asked to talk about the triangle and the addition rule in terms of pizzas and towers.

00:02:00	R8:	Michael … would explain things by talking about, you're going from a number of pizza toppings to a different number of pizza toppings, and how the addition rule works there. And ... you seemed to think of towers as the primary way to do Pascal's Triangle. … I sort of refreshed your memory a little bit. Do you remember anything from, from how you guys worked on this or how, you know, the addition rule would apply here?
00:02:24	ROMINA:	… I think this is [Romina points to Pascal's Triangle A in Figure 41.] how many toppings. Like the top number, like the 1 choose 1 or 1 choose 0 would be how many toppings. Or I mean if we were talking about towers … [Romina points to row 2 of Pascal's Triangle A.] This would be with two high with zero reds, one red, two reds. And it just keeps going like three high, zero reds, one red, two, then

three reds. So it would be like three high and like out of those, you choose how many blocks of each color.

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1
\end{array}
$$

Figure 41. Pascal's Triangle A (left) and Pascal's Triangle B (right)

Romina recalled that the triangle related to numbers of pizza toppings, although she did not make explicit exactly what the relationship was. She did, however, give a clear explanation of how the entries of Pascal's Triangle were linked to the towers problem.

Next, Romina was asked to explain the addition rule in terms of either towers or pizzas. She chose towers. See Figure 42 for the lines Romina drew.

00:03:21	R8:	Now how would the addition rule work in terms of towers? I mean, I know the addition rule, one plus two equals three, and so on. Does it make sense to talk about in terms of either towers or pizzas?
00:03:45	ROMINA:	… This was Michael's area of expertise. … OK, I think, the way it goes, it's like this 2. [Romina points to 2 choose 0 in Pascal's Triangle A.] … You could be either direction with it. You're either gonna add another red, say, so it goes to

		two red, or you're gonna add another blue, so it stays one red.
00:04:27	R8:	… Why don't you just write on it? Show me the branches again.
00:04:30	ROMINA:	Like this, this one [Romina points to 2 choose 1.] goes off of that to either of those. [Romina draws lines from 2 choose 1 to 3 choose 1 and 3 choose 2.]
00:04:34	R8:	And when it goes out to the left...
00:04:38	ROMINA:	This is, we're adding a blue, say, and this one [Romina points to3 choose 2.] we're adding another red, so this changes to two.
00:04:44	R8:	OK. And then, so this one [points to 2 choose 0] also goes to that [3 choose 1].
00:04:47	ROMINA:	Uh-huh. Cause this one, yeah, you have the same. [Romina draws lines from 2 choose 0 to 3 choose 0 and 3 choose 1. Refer to Figure 42.] This one you could either add a red...
00:04:55	R8:	OK. Or not add a red.
00:04:56	ROMINA:	Or you add another blue. Yeah.

Romina referred to specific rows of Pascal's Triangle, but her explanation of the addition rule was a general one: moving to the left means adding a blue block to a tower and moving to the right means adding a red block. This echoes Ankur's remark in the night session (when he said, regarding the variables in the general equation, that the total number of pizza toppings is "always the one on the right").

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}$$

Figure 42. Romina draws lines to illustrate the addition rule

Researcher 8 wrote general rows of Pascal's Triangle, using the notation that Michael had chosen three months earlier, with the variables reversed from the standard textbook notation and different from what the group had used in the night session. (Refer to Figure 43.)

$$\begin{array}{l}
\binom{R}{0}\binom{R}{1}\cdots\binom{R}{N}\cdots\binom{R}{R} \\
\binom{R+1}{0} \qquad\qquad \binom{R+1}{R+1}
\end{array}$$

Figure 43. Rows *R* and *R*+1 of Pascal's Triangle drawn by Researcher 8

Researcher 8 asked Romina to write an equation in "choose" format. The intention was to ask Romina to use specific numbers. But Romina wrote the general form, as shown in Figure 44. The discussion follows.

00:10:44 R8: So she's [Researcher 1] saying, instead of writing one plus two equals three [R8 writes 1+2=3.], you're going to write

		this [R8 points to 1.] in choose notation and this [R8 points to 2.] in choose notation and this [R8 points to 3.] in choose notation. 'Cause this three isn't really a three, it's a three choose one. And this one isn't really a one, it's a two choose zero. …
00:11:20	ROMINA:	So you mean, *N* choose *R*, like that? [Although Romina says *N* choose *R*, she writes *R* choose *N*.] Plus ... *R* choose *N* plus one [Romina writes *R* choose *N*+1] equals *R* plus one over...I'm not even sure if it would be *N* plus one. [Romina writes *R*+1 choose *N*+1. Refer to Figure 44.]

$$\binom{R}{N}+\binom{R}{N+1}=\binom{R+1}{N+1}$$

Figure 44. Romina's version of Pascal's Identity

Although she wrote a correct equation, Romina indicated some uncertainty about exactly what the terms meant. Researcher 8 asked for an explanation.

00:12:57	R8:	So tell me what these letters would be here. [R8 points to the equation in Figure 44.]
00:13:01	ROMINA:	The *R* would be how high. [Romina points to *R* choose *N*.]
00:13:04	R8:	All right.
00:13:30	ROMINA:	… The *R* plus one, obviously, you know because you added a block to it, so … this became one higher. … And the choose, … you had to have gone up, because you added a color. … That's where, like, your *N* plus one comes from, I think.

Romina was clear about the size of the set; she said *R* increased to *R*+1 because "you added a block" to the tower. But she was less clear about the selected subset. She explained *N*+1 by saying, "you added a color." But in her earlier explanation of the specific case, she had more correctly described the process as adding a block of the selected color (red).

Romina had indicated a preference for using the towers problem to talk about Pascal's Triangle, although she also demonstrated some familiarity with the pizza problem. Researcher 8 asked her if she could link the addition rule for Pascal's Triangle to the pizza problem. Their discussion follows.

00:14:31	R8:	Can you look at this in terms of pizzas too? …
00:14:40	ROMINA:	… That's not like my preference. … You have two toppings to pick from. And then, what he [Michael] did with this one [Romina indicates row 2 of Pascal's Triangle; see Figure 38.] is either you could ... now, you could add a third toppings to your pizza. [Romina indicates row 3 of Pascal's Triangle.] Like you have three options, you could either not add anything to the pizza. [Romina indicates the line from 2 choose 0 to 3 choose 0.] Or you could just add one more topping. [Romina indicates the line from 2 choose 0 to 3 choose 1.]
00:15:09	R8:	All right. So when you said "add one more topping," or "not add one more topping," ... Can you relate that to red and blue?
00:15:18	ROMINA:	You either, it's either, like, you add one more red block, or you just keep it consistent and add, just add another blue. So blue would be like nothing, like not an ingredient, and red would be an ingredient. … Like, his binary [Michael's binary notation], it does the same thing. A 0 would be blue or no topping. And a red one, which would be a 1, would be a topping.
00:15:50	R8:	So if I pointed to any one of these, you would be able to say, like say this one. [R8 points to 3 choose 3.] … In terms of pizzas.
00:16:01	ROMINA:	Three toppings, and he put all three toppings on the pizza.

Romina had said she preferred to think about Pascal's Triangle in terms of towers, but she had no difficulty in switching to the pizza representation. Further, Romina easily explained the isomorphism between pizzas and towers when asked, and she added an unsolicited explanation of how it all related to binary notation.

5.4.4 Interview with Ankur

During Ankur's interview on July 31, 2002, he and Researcher 8 watched a video of portions of the night session. Although he did not remember the night session, he was able to recall the meaning of Pascal's Triangle in terms of towers, and he explained Pascal's Identity in terms of towers. This was similar to the explanations Ankur had provided to Researcher 5 in February 1998 and to Researcher 1 in March 1998.

In the first episode, Ankur discussed the meaning of the numbers in Pascal's Triangle after he saw Pascal's Triangle written in combinatorics notation.

00:11:26	R8:	… Looking at the chooses in a different way, looking at them in terms of the towers or something.
00:11:43	ANKUR:	I remember that, with the towers. Now it's, now it's recalled. … I'm trying to think. I remember something. We, we used like these numbers to represent like the various towers, I think. Some- something towards- of that nature.
00:11:55	R8:	OK.
00:11:55	ANKUR:	Something like with what- Say there was like a tower of three tall with red and blue, like. Like, if the first one represented only one red, or no reds. One red, two reds.
00:12:04	R8:	OK. … All right, so 3 tall would be-
00:12:11	ANKUR:	This was no reds, one red, two reds, three reds. And those numbers would match here. [Ankur indicates row 3 of Pascal's Triangle – 1 3 3 1.] … There would be one with

no red, three with one red, three with two red and one with three reds.

This was similar to the explanation he offered in February 1998. Later in this interview, Ankur wrote a general row of Pascal's Triangle and then wrote Pascal's Identity. (Refer to Figure 45.) Researcher 8 asked Ankur to explain the meaning of Pascal's Identity in terms of pizzas, but he expressed a preference for the towers problem and used towers to explain both a specific instance of Pascal's Identity and the general equation.

00:40:46	ANKUR:	I don't remember the pizza problem really. I remember the towers though.
00:40:49	R8:	So, OK, so this was towers again. And this [row 3 of Pascal's Triangle] was three-tall towers.
00:40:57	ANKUR:	With no red, one red, two red, three red.
00:40:59	R8:	So how would this [row *N*] be in terms of towers? This would be-
00:41:02	ANKUR:	A tower *N* tall ... with none of one color. ... And then one of one color.
00:41:11	R8:	So it's gonna be zero reds, one red, all the way up to-
00:41:13	ANKUR:	Up to *N*.
00:41:14	R8:	All reds.
00:41:14	ANKUR:	All reds.
00:41:16	R8:	... Does it make sense to explain this addition rule in terms of adding, in terms of going to the next row of towers?
00:41:28	ANKUR:	I mean, you could do it the same way. It's-
00:41:31	R8:	All right, so this one is a tower *N* tall.
00:41:34	ANKUR:	With *X* amount of reds. And then, you add it with the tower the same height with one more red. And you get that a tower one taller. ... [They decide to consider the case of 3-tall towers.]
00:42:51	R8:	This [3 choose 1] is a 3-tall tower with one red. So it's got- it's 3 tall, so it's got one red and two blues. And then you changed that into a 4-tall tower with two reds. So how are you gonna do that?

00:43:18	ANKUR:	With two reds? You just add a red.
00:43:20	R8:	You have to put a red on it. OK. But this one's [3 choose 2] a 3-tall tower that already has two reds. And it goes to a 4-tall tower that has two reds.
00:43:27	ANKUR:	So you got to add a blue.
00:43:28	R8:	So you're gonna add a blue to that one. OK. So this one combines with this one in the sense that-
00:43:46	ANKUR:	You add … a red to all three of those and a blue to all three of those and that's how you get-
00:43:51	R8:	Yeah. OK. That makes sense.
00:43:52	ANKUR:	That's why you have the extra- that's why the bottom number's *X* plus 1. Because these you added the same color.
00:44:04	R8:	… All right, here's an *N*-tall with *X* reds [*N* choose *X*]. And how are you gonna get down there [to *N*+1 choose *X*+1]?
00:44:12	ANKUR:	You're gonna add a red. …
00:44:15	R8:	And you go from there [*N* choose *X*+1] to there [*N*+1 choose *X*+1]?
00:44:17	ANKUR:	By adding the other color.

$$\binom{N}{X} + \binom{N}{X+1} = \binom{N+1}{X+1}$$

Figure 45. Ankur writes Pascal's Identity

This explanation recalled Ankur's earlier explanation from February 1998, when he also talked about adding one color to one group of towers and the other color to the other group. But this time, when Researcher 8 prompted him to continue, Ankur went further and explained general notation in terms of operating on a general towers problem.

5.5 Summary of Results

The members of this group made use of well-understood problems in order to make sense of factorial and combinatorial notation. They first explored these problems during elementary school. In high school, when the researchers asked them to take the exploration deeper, they investigated general properties of the problems, examined the underlying structures, and looked for relationships among the problems and between the problems and Pascal's Triangle.

When they were asked to provide an explanation of a combinatorial concept, they referred not to a formula or memorized definition, but to a specific representational situation which had meaning for them. First they introduced the relationship between towers problems and the binomial expansion in order to think about combinatorics notation. Then they used their own familiar representations (diagrams and well-known combinatorics problems) in order to give meaning to factorial notation. They explained specific instances of Pascal's Identity, written in standard notation that was new to them, by referring to the pizza and towers problems and using binary numbers to represent pizzas. They went on to give a general form of Pascal's Identity, written in standard notation, which they explained by again referring to the pizza problem.

Although these students expressed their ideas using familiar problems and familiar notation, such as the pizza and towers problems and Michael's binary notation, they also adapted to the new notation, by using the familiar to make sense of the unfamiliar. For example, when Michael explained how row 3 of Pascal's Triangle generates row 4 (episode 4 of the night session), he started by representing individual pizzas as binary numbers. This enabled him to give a general addition rule that encapsulated the doubling pattern of Pascal's Triangle: Adding the place for a new digit to the binary number corresponds to making a new pizza topping available. Adding a 1 to the binary number in that place corresponds to adding that new topping to the pizza and adding a 0 corresponds to not adding that new topping. Later, when Jeff wrote the general equation, he referred to its variables in terms of a more familiar representation, pizzas. (One goes from N to N to $N+1$ by making a new topping available. One changes X to $X+1$ by placing the new topping on those pizzas, and one retains $X+1$ by keeping the same number of toppings on the pizzas.) In repeated discussions of these problems over about 18 months, the students extended, reorganized, and generalized their knowledge about these isomorphic problems, Pascal's Triangle, and Pascal's Identity.

One year later, four of the students made sense of the taxicab problem by relating it to this previous work. Romina first noted the connection to Pascal's Triangle, and then she and the others described how the taxicab problem could be linked to the pizza and towers problems.

In interviews three years later, three students demonstrated that they were still able to discuss and explain these concepts. Michael regenerated an explanation of Pascal's Identity in terms of pizzas in terms similar to those he had used in December 1998, and he wrote the equation using idiosyncratic notation. Romina recall three ways to explain the numbers in Pascal's Triangle (towers, pizzas, and binary notation) and explained the relationships among those representations. Ankur and Romina both explained the general form of Pascal's Identity in terms of towers, even though the equation had originally been explained in terms of pizzas, thus demonstrating the ability to move between different representations.

CHAPTER 6: CONCLUSIONS

6.1 Introduction

In their second year of high school, Ankur, Brian, Jeff, Michael, and Romina began a reinvestigation of the pizza problem and the towers problem, two combinatorics problems from their middle school years. Exploring previously-unexamined complexities of these problems was a mathematically challenging task of the sort recommended by Davis and Maher (1990) to foster their ability to engage in real mathematics (developing their own mathematical theories, for example). Conditions recommended by Maher and Martino (1999) were in place: they had ample time for exploration of mathematical ideas and the opportunity to express their own ideas. The students' existing representations were taxed by new questions about how to relate these problems to each other, to Pascal's Triangle, and to the binomial coefficients. Hence, according to Davis (1984), there was a need to reorganize existing knowledge and to construct tools for dealing with these new ideas. These tools would consist of repertoires of representations based on appropriate mathematical experiences according to Davis and Maher (1997).

As noted in Chapter 3, Davis and Maher (1990) stated that children learning mathematics are capable of engaging in real and challenging mathematical activities, as mathematicians do, provided that the right conditions are in place. Given that all the conditions mentioned by Davis and Maher were in place, one can ask whether the students did engage in challenging mathematical activities like mathematicians. The evidence suggests that they did. This evidence is given in sections 6.2 through 6.5, which summarize the answers to my research questions. In section 6.2, I discuss how the students represented their ideas. In section 6.3, I discuss the connections the students made among their personal representations, the combinatorics tasks, and the standard notation. In section 6.4 and 6.5, I discuss the strategies the students used to make sense of the standard notation and of Pascal's Identity. In section 6.6, I discuss the evidence for individual students' ability to recall or reconstruct their ideas about Pascal's Triangle and Pascal's Identity. In Section 6.7, I discuss implications of this study.

6.2 Representing Their Ideas

When they first started working on the pizza and towers problems, the students built towers and drew pictures of pizzas. While still in middle school, they began to use symbolic notation. (For example, they used letter and number codes to stand for the objects they were investigating.) Besides continuing the use of codes during high school, the students also found increasing use for the standard notations of mathematical discourse. For instance, on March 6, 1998, they worked on a new towers problem. (This question was: How many n-tall towers can be built with exactly one of each color when there are n colors to choose from?) After noticing the pattern of the answers for low values of n (1, 2, and 3) and then examining the structure of the problem, the students decided that factorial notation was appropriate. By the time of the night session, when they were asked to explain the meaning of the factorial notation, they had a repertoire of representations to choose from (people on a line, a selection of colors, and towers problems), based on this and other previous mathematical experiences.

In December 1997, Michael introduced the idea of using binary numbers to represent first pizza toppings and later towers. Binary notation was more powerful than the codes the students had been using because it was easily extended (adding a cube to the tower or a topping choice to a pizza corresponded to adding a binary digit) and it was applicable to both problems, thus making it easier for the students to identify the similar structures of the two problems. (The other students had been using different codes for the two problems – number codes to represent pizza toppings and letters to enumerate towers.) Although Michael was the driving force behind the use of binary notation (the students often continued to refer to it as "Mike's binary"), the others showed evidence of understanding its meaning, even if it was not their first choice of notation. Ankur used it when he and Michael worked on a towers problem in January 1998. (How many 5-tall towers are there with exactly two red blocks when you are selecting from red and yellow blocks?) Ankur and Romina used it when exploring the link between towers and pizzas in March 1998. (Refer to the transcript in Appendix H, at 00:51:56.) In the night session, when Michael proposed an exhaustive list of three-topping pizzas, Jeff corrected a mistake in Michael's notation. (Refer to the transcript in Appendix J, at 00:33:12.) Romina explained in the interview of July 22, 2002 how binary notation could be linked to towers and pizza problems. Using binary notation helped the students focus on the isomorphic structural aspects of the combinatorial problems (the duality of the choices) rather than the surface features (the different pizza toppings for example).

Binary notation was also an easily-generalizable notation in three ways: 1) adding an extra digit corresponded to making a new pizza topping available and to adding another block to the tower; 2) adding a 1 corresponded to adding that newly available topping and to adding a block of the designated color to the tower; and 3) adding a 0 corresponded to not adding the new topping and adding a block of the other color. This idea that it was not necessary to know the current number (or names) of pizza toppings or the current height of the tower in order to describe what happened next was important in the students' generation of the general equation for Pascal's Identity. (Refer to Figure 46.)

$$\binom{N}{X} + \binom{N}{X+1} = \binom{N+1}{X+1}$$

Figure 46. Pascal's Identity as written by students in the night session

6.3 Making Connections

As noted in Chapter 3, recognizing isomorphisms can be considered a mark of mathematical competence (Greer and Harel, 1998). During the course of discussions over four months (December 1997 through March 1998) these students first noted that the pizza and towers problems had the same answer in specific cases. Then they linked specific answers to pizza and towers problems to entries in Pascal's Triangle. In March, they described the links among binomial coefficients, pizza toppings, and towers. (Blue block = a = topping on the pizza; white block = b = topping off the pizza.) During the night session, they built on their knowledge of these links. When they were discussing binomial coefficients in the beginning of the session, they turned to towers problems to explain "choose" notation. Later, when making sense of some explicit entries in Pascal's Triangle written in that "choose" notation (rows 3 and 4), they turned to pizza problems. But when dealing with the general row (row n), they used towers and binomial coefficients (n-tall towers and $(a+b)^n$.) As noted in Chapter 3, being able to map corresponding mathematical structures among these three representations was a mark of mathematical competency (Davis and Maher, 1997). Further, being able to discuss Pascal's Triangle and Pascal's Identity in any of these ways shows a command of the "social use" described by Dörfler (2000) as a measure of understanding.

6.4 Strategies for Making Sense of Standard Notation

These students were shown the standard notation of combinatorics in these problem-solving sessions three times before the night session: in January 1998, February 1998, and March 1998. In January, Researcher 1 informed them that they had been working on problems in combinatorics (counting ways to select two items from a group of items) and showed them four ways to write the number of ways to select two objects from a group of five. A month later, when the students were explaining their work to Researcher 5, Researcher 1 reminded them of this notation. Finally, in March 1998, Researcher 1 wrote Pascal's Triangle in the notation of combinatorics, including a general row *n*. The students themselves had not used that notation in these sessions, although there was some evidence that they had encountered it in school by the time of the night session. (For example, Jeff referred to a calculator function for finding the coefficients of the binomial expansion. He said, "We knew it was the choose thing, whatever that means.") Romina's response to the question about explaining "the choose thing" was to refer to towers. So it seems clear that by the night session, at least some members of the group recalled that there was an association between the standard notation for combinatorics and at least one of the combinatorics tasks.

The students brought up Pascal's Triangle when discussing "choose" notation. Michael wrote Pascal's Triangle in the traditional way (with numbers, not "choose" notation), but he referred to entries in terms of "choose," and he used binary notation when explaining them in terms of selecting pizza toppings. After Researcher 1 prompted the students to rewrite the triangle in that notation, they wrote rows 0 through 4 without discussing the relationship to any tasks. But when they wrote row n, they made reference to the towers problem (Romina said, "make n your height.") and the binomial expansion. (Ankur said, "write a plus b to the whatever it is" next to row n, and Jeff said row n was associated with "a plus b to the n^{th}.") These repertoires of representations were tools for making sense of the generalized form of the notation, according to Davis and Maher (1990).

6.5 Making Sense of Pascal's Identity

6.5.1 Davis's View of Understanding

According to Davis (1992), students demonstrate understanding by fitting new ideas into frameworks of older ideas. Ankur, Brian, Jeff, Michael, and Romina have been seen to accomplish this throughout the years of this study. They began their exploration into the meaning of Pascal's Identity in January 1998 when Researcher 1 showed them how their answer to that day's combinatorics problem was found in Pascal's Triangle. (The entry 10, representing all five-tall towers with exactly two red cubes, was found in row 5.) Ankur then noted that the 1's on the outside of row 4 represented, respectively all red and all yellow four-tall towers and Brian identified the other numbers in the row as "mixed." This can be seen as a preliminary effort to fit the new ideas about the meaning of the numbers in Pascal's Triangle into an existing framework of ideas about towers problems.

The following month, Ankur, Jeff, and Romina made further connections between rows of Pascal's Triangle and the pizza and towers problems. They explained to Researcher 5 how towers problems could be used to explain instances of Pascal's Identity. (One produces the twenty 6-tall towers with exactly three red blocks by adding a blue block to the ten 5-tall towers that already have three red blocks and adding a red block to the ten 5-tall towers that have only two red blocks.) They followed up in March (when Michael joined them) by relating two tower colors – this time blue and white – with the binomial expansion, equating blue with a and white with b. These are all instances of fitting new ideas (about Pascal's Triangle, about Pascal's Identity, about the relationships among the combinatorics problems) into previously-assembled ideas (about the answers to the towers and pizza problems and about how to multiply binomials and determine binomial coefficients).

In interviews almost a year later, Michael described how to explain Pascal's Identity in terms of pizza problems, and he shared that description with the other four students during their work on the World Series problem. During the night session, the group recalled this work, explaining specific instances of Pascal's Identity in terms of the pizza problems. (The pizzas in row 4 with exactly two toppings come from the pizzas in row 3 that have one topping – they get the new topping – and the pizzas in row 3 that have two toppings – they do not get the new topping.) These are instances of fitting new ideas about how Pascal's Identity can be explained in terms of pizzas into an existing framework of ideas about relating pizza problems to Pascal's Triangle.

During the night session, the students first wrote an equation for a specific instance of Pascal's Identity, as shown in Figure 47. When asked for a general equation, Ankur noted the form of that equation. He said, "it's always the one on the right." (The bottom number on the left side of the equation came from the bottom number of the rightmost addend as seen in Pascal's Triangle.) Michael followed up by noting that the first addend represented pizzas that had a topping added, and the second represented pizzas that did not have a topping added. They made use of this observation when they wrote the general equation. Here again, they fit the new idea (the general form of the equation) into the framework of a previously-assembled idea (making sense of one instance of Pascal's Identity).

$$\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$$

Figure 47. Instance of Pascal's Identity discussed by Ankur

6.5.2 Other Lenses

Dörfler (2000) suggests that students show that they understand a concept by demonstrating the ability to discuss it (having "a thorough command of its social use," p. 101). Similarly, Sfard (2000) suggests that students demonstrate understanding by showing that they are proficient in using mathematical names and symbols. During this study, these students engaged in articulate discussions about the relationships among binary notation, pizza problems, towers problems, coefficients of the binomial expansion, Pascal's Triangle, Pascal's Identity, and standard notation. This can be said to demonstrate a proficiency in the "social use" of these concepts. Skemp (2002) might say that they were able to accomplish this by building up conceptual schemas which enabled them to be aware of the overall structure of Pascal's Triangle. Skemp might also note that they possessed relational understanding. Because they were able both to generate the numbers in Pascal's Triangle and to explain where the numbers came from and why the addition made sense, they knew both what to do and why.

6.6 Individual Students' Recall or Reconstruction of Ideas about Pascal's Triangle

In this study, I focused primarily on the work of the cohort group of students rather than on individual students. But the sessions following the night session provided an opportunity to examine the work of individual students. In the taxicab session one year later, Romina (who had not taken a leading role in the night session) was the first to articulate the relationship between the taxicab problem and Pascal's Triangle. Brian (who missed most of the night session) contributed to Romina's explanations to the researchers. All three participants in the individual interviews three years after the night session were able, when asked, to discuss Pascal's Triangle in terms of one or more of the combinatorics problems; they could also reconstruct Pascal's Identity and explain it in terms of at least one of those problems. This suggests that, although the work of the night session was a group effort, individual members of the group retained an ability to make sense of the group's work.

6.7 Other Findings

This research might also contribute information about the way in which students build on personal experience when learning new notation. I found that the students' use of a good organizational strategy was a key element that helped them to form connections among the towers problem, the pizza problem, and Pascal's Triangle. I also found that they made extensive use of their personal representations at the beginning of the process. But once those connections were formed, the students' use of personal representations lessened as they began to make general statements about Pascal's Triangle and Pascal's Identity. Finally, although they were able to articulate general information about Pascal's Triangle and Pascal's Identity, they did not represent the generalizations symbolically until the night session. These observations are discussed briefly below.

6.7.1 Organization

The students had organized the towers by a number of cubes of a particular color as early as the fourth grade (Martino, 1992); they had organized pizza answers by number of toppings starting in the fifth grade (Bellisio, 1999). This particular organization may have been driven by researcher questions about particular groups of towers. (For example, a question asked first in fourth grade and revisited in tenth grade was: When building five-tall towers and selecting from two colors, how many towers have exactly two red cubes?) But it was also an easy way to demonstrate that all towers or pizzas had been found. Because this organization for n-tall towers and n-topping pizzas corresponds to row n of Pascal's Triangle, using this organization seems to have helped students to observe the connections between the problems and Pascal's Triangle.

6.7.2 Using Personal Representations

During the time that these students were learning about Pascal's Triangle and Pascal's Identity, they made extensive use of their personal representations of the problem situations. They using letter codes or binary notation to describe towers, for example, and the use of codes that they had created (along with the organizational strategy described above) seemed to help them enumerate all possibilities. On one occasion (March 6, 1998, when they were investigating Pascal's Identity), they built towers, and this seemed to help them understand the mechanism by which Pascal's Triangle was built. But after they were aware of the association between Pascal's Triangle and the combinatorics problems, their reliance on personal notations diminished. For example, when the group discussed the pizza problem with Researcher 1 on January 22, 1999, they told her that specific numbers were related to specific pizzas (and they described those pizzas), but they did not draw them or enumerate them using any symbols.

6.7.3 Generalizing

After they had made the association between Pascal's Triangle and the combinatorics problems, the students demonstrated an ability to describe any selected entries in Pascal's Triangle in terms of the combinatorics problems. For example, on February 6, 1998, they described the numbers in row 6 as representing six-tall towers with zero through six red cubes respectively. They also verbally described some general entries. For example, Michael stated on December 14, 1998 that the leftmost number in each row of Pascal's Triangle referred to plain pizzas and the rightmost number in each row referred to "all topping" pizzas. The students also talked about Pascal's Identity in a general way. For example, during the session on January 22, 1999, they said that adding a topping meant moving to the right on Pascal's Triangle (adding to the number on the right to produce the number below and to the right), and not adding a topping meant moving to the left (adding to the number on the left to produce the number below and to the left). The fact that they could explain any instance suggested that they had an idea of the general rule; but without the standard notation, they could express their general ideas most easily by referring to specific examples. By the time of the night session, these students seemed to be at a point where they required a general notation in order to be able to make general statements about what they knew about Pascal's Triangle and Pascal's Identity.

6.8 Implications

The findings described above suggest that one way that teachers can follow the recommendation by the NCTM (2000) to "use sound professional judgment when deciding when and how to help students move toward conventional representation" (p. 284) is to help students develop a powerful organization, one that lends itself to a mapping onto formal notation.

This research also provides an example of a cohort of students who were not given the standard notation immediately but who nevertheless made sense of it and used it appropriately. This demonstrates that it is possible to understand standard notation even if it is introduced long after the initial concept is introduced. Further, the use of certain early personal representations by these students decreased. Perhaps the earlier personal representations became part of a repertoire consisting of more elaborate representations that became more useful as a tool for communicating deeper understanding. This suggests that students who begin work on a mathematical concept by relying on personal representations can build on them as their knowledge grows.

In summary, this research suggests that students who build meaning first and then develop the symbolic vocabulary can acquire a deep and durable understanding.

APPENDIX A: COMBINATORICS PROBLEMS

Listed here are the combinatorics problems the students encountered from elementary school through high school, along with brief discussions of solutions.

1. Shirts and Jeans (May 1990, Grade 2; October, 1990, Grade 3) – Stephen has a white shirt, a blue shirt, and a yellow shirt. He has a pair of blue jeans and a pair of white jeans. How many different outfits can he make?

   He can make six different outfits; each of the two pairs of jeans can be matched with each of the three shirts. The outfits are: blue jeans/white shirt, blue jeans/blue shirt, blue jeans/yellow shirt, white jeans/white shirt, white jeans/blue shirt, and white jeans/yellow shirt.

2. Towers 4-tall (October 1990, Grade 3; December 1992, Grade 5) – Your group has two colors of Unifix cubes. Work together and make as many different towers four cubes tall as is possible when selecting from two colors. See if you and your partner can plan a good way to find all the towers four cubes tall.

   At each position in the tower, there are two color choices. Therefore, there are $2\times2\times2\times2=16$ possible towers that are four cubes tall. This can be generalized to an n-tall tower with two colors to choose from; there are $2\times2\times2\ldots\times2=2^n$ possible towers that are n cubes tall, when there are two colors to choose from. This can also be generalized to an n-tall tower with m colors to choose from; there are $m \times m \times m \ldots \times m = m^n$ possible towers that are n cubes tall with m colors to choose from. In the following discussions, I will call the first generalization (the n-tall tower with two colors) the towers problem, and I will call the second generalization (the n-tall tower with m colors) the generalized tower problem.

3. Cups, Bowls, and Plates (April 1991, Grade 3) – Pretend that there is a birthday party in your class today. It's your job to set the places with cups, bowls, and plates. The cups and bowls are blue or yellow. The plates are blue, yellow, or orange. Is it possible for 10 children at the party each to have a different combination of cup, bowl, and plate? Show how you figured out the answer to this question.

   Each of the two cup choices can be matched with each of the two bowl choices, and each cup-bowl pair can be matched with any of the three different plate choices. Therefore, there $2 \times 2 \times 3 = 12$ are possibilities. So it is possible for 10 children at the party each to have a different combination of cup, bowl, and plate.

4. Relay Race (October 1991, Grade 4) – This Saturday there will be a 500-meter relay race at the high school. Each team that participates in the race must have a different uniform (a uniform consists of a solid colored shirt and a solid colored pair of shorts). The colors available for shirts are yellow, orange, blue, or red. The colors for shorts are brown, green, purple, or white. How many different relay teams can participate in

the race?

There are four choices for shirts and four choices for shorts, so there are $4 \times 4 = 16$ ways to make uniforms. Sixteen different relay teams can participate.

5. Towers 5-Tall (February 1992, Grade 4; December 1997, Grade 10) – Your group has two colors of Unifix cubes. Work together and make as many different towers five cubes tall as is possible when selecting from two colors. See if you and your partner can plan a good way to find all the towers five cubes tall.

   There are $2^5=32$ towers five cubes tall.

6. Towers 4-Tall with Three Colors (February 1992, Grade 4) – Your group has three colors of Unifix cubes. Work together and make as many different towers four cubes tall as is possible when selecting from three colors. See if you and your partner can plan a good way to find all the towers four cubes tall.

   Since there are three choices for each of four positions, there are $3^4=81$ possible towers that are four cubes tall when selecting from three colors.

7. Guess My Tower (February 1993, Grade 5) – You have been invited to participate in a TV Quiz Show and the opportunity to win a vacation to Disney World. The game is played by choosing one of four possibilities for winning and then picking a tower out of a covered box. If the tower you pick matches your choice, you win. You are told

that the box contains all possible towers that are three tall that can be built when you select from cubes of two colors, red, and yellow. You are given the following possibilities for a winning tower:

All cubes are exactly the same color.
There is only one red cube.
Exactly two cubes are red.
At least two cubes are yellow.

Which choice would you make and why would this choice be better than any of the others?

In order to decide which is the best choice, we need to find the probability of each choice. The total number of 3-tall towers is 8. The probabilities are:

All cubes are exactly the same color: There are two ways (all red or all yellow). The probability is $2 \div 8 = 0.25$.

There is only one red cube: There are three ways; the red cube can be on the top, in the middle, or on the bottom. The probability is $3 \div 8 = 0.375$.
Exactly two cubes are red: This is the same as saying exactly one cube is yellow. The probability is the same as for exactly one red cube: $3 \div 8 = 0.375$.

At least two cubes are yellow: This is equivalent to saying that either exactly two cubes are yellow or exactly three cubes are yellow. As discussed above, the probability that exactly two cubes are yellow (the same as the probability that exactly two cubes are red) is 0.375. Since there is one way for exactly three cubes to be yellow, that probability is $1 \div 8 = 0.125$. The probability of either event is therefore $0.375 + 0.125 = 0.5$. (We can add because the two events are mutually exclusive.)

"At least two cubes are yellow" is the most likely event.

Assuming you won, you can play again for the Grand Prize which means you can take a friend to Disney World. But now your box has all possible towers that are four tall (built by selecting from the two colors yellow and red). You are to select from the same four possibilities for a winning tower. Which choice would you make this time and why would this choice be better than any of the others?

The total number of 4-tall towers is $2^4=16$. The probabilities are:

All cubes are exactly the same color: There are two ways (all red or all yellow). The probability is $2\div16=0.125$.

There is only one red cube: There are four ways; the red cube can be on the top, second from the top, second from the bottom, or on the bottom. The probability is $4\div16=0.25$.

Exactly two cubes are red: The number of ways to accomplish this is $C(4,2)=6$. The probability is therefore $6\div16=0.375$.

At least two cubes are yellow: This means that exactly two cubes are yellow, exactly three cubes are yellow, or exactly four cubes are yellow. As discussed above, the probability that exactly two cubes are yellow (the same as the probability that exactly two cubes are red) is $6\div16=0.375$. The probability that exactly three cubes are yellow is the same as the probability that one cube is red: $4\div16=0.25$. Since there is one way for exactly four cubes to be yellow, that probability is $1\div16=0.0625$. The probability of any one of the three events is therefore $0.375+0.25+0.0625=0.6875$.

"At least two cubes are yellow" is the most likely event.

8. The Pizza Problem with Halves (March 1993, Grade 5) – A local pizza shop has asked us to help them design a form to keep track of certain pizza sales. Their

standard "plain" pizza contains cheese. On this cheese pizza, one or two toppings could be added to either half of the plain pizza or the whole pie. How many choices do customers have if they could choose from two different toppings (sausage and pepperoni) that could be placed on either the whole pizza or half of a cheese pizza? List all possibilities. Show your plan for determining these choices. Convince us that you have accounted for all possibilities and that there could be no more.

With two topping choices, there are four possibilities for the first half pizza, because each topping can be either on or off that half of the pizza. The four choices are: plain (sausage off, pepperoni off), sausage (sausage on, pepperoni off), pepperoni (sausage off, pepperoni on), and sausage/pepperoni (sausage on, pepperoni on). Consider each of the four possibilities in turn.

> Case 1: Plain. There are four possibilities for the other half of the pizza, the four listed above (plain, sausage, pepperoni, and sausage/pepperoni).
>
> Case 2: Sausage. There are three possibilities for the other half of the pizza: sausage, pepperoni, and sausage/pepperoni. (We omit plain, because we already accounted for the plain-sausage pizza in Case 1.)
>
> Case 3: Pepperoni. There are two possibilities remaining for the other half of the pizza: pepperoni and sausage/pepperoni. (Plain and sausage are already accounted for.)
>
> Case 4: Sausage/pepperoni. There is only one possibility left for the other half of the pizza; that is sausage/pepperoni.

There are 4+3+2+1=10 possible pizzas with halves.

9. The Four-topping Pizza Problem (April 1993, Grade 5) – A local pizza shop has

asked us to help design a form to keep track of certain pizza choices. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many different choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possible choices.

There are $2\times2\times2\times2=16$ possible pizzas.

10. Another Pizza Problem (April 1993, Grade 5) – The pizza shop was so pleased with your help on the first problem that they have asked us to continue our work. Remember that they offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. The pizza shop now wants to offer a choice of crusts: regular (thin) or Sicilian (thick). How many choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possible choices.

    Each of the 16 4-topping pizzas has two choices of crust, so there are 32 pizzas.

11. A 5-Topping Pizza Problem (December 1992, Grade 5; December 1997, Grade 10) – Consider the pizza problem, focusing on the number of pizza combinations that can be made when selecting from among five different toppings.

    There are $2^5=32$ different pizzas.

12. A Final Pizza Problem (April 1993, Grade 5) – At customer request, the pizza shop has agreed to fill orders with different choices for each half of a pizza. Remember that they offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushroom, and pepperoni. There is a choice of crusts: regular (thin) and Sicilian (thick). How many different choices for pizza does a customer have? List all the possible choices. Find a way to convince each other than you have accounted for all possible choices.

    The first half of the pizza can have $2^4 = 16$ possible topping configurations, as described above. Consider each of those configurations in turn. Following the procedure described above for the 2-topping half-pizza problem, we find that there are 16+15+14+...+3+2+1 possible pizzas; this sum is given by$16\times17\div2$. Since each pizza can have a thick or thin crust, we multiply by 2. The number of possible pizzas is$16\times17\div2\times2=272$.

13. Counting I and Counting II (March 1994, Grade 6) – How many different two-digit numbers can be made from the digits 1, 2, 3, and 4? Each of four cards is labeled with a different numeral: 1, 2, 3, and 4. How many different two-digit numbers can be made by choosing any two of them?

    Counting I: Assuming that you are not permitted to reuse digits, there are four choices for the first digit and three for the second digit, giving 12 two-digit numbers. (They are 12, 13, 14, 21, 23, 24, 31, 32, 34, 41, 42, and 43.)

Counting II: There are four choices for the first digit and four choices for the second digit. This makes 16 different two-digit numbers. (They are 11, 12, 13, 14, 21, 22, 23, 24, 31, 32, 33, 34, 41, 42, 43, and 44.)

14. Ankur's Problem (January 1998, Grade 10) – Find all possible towers that are 4 cubes tall, selecting from cubes available in three different colors, so that the resulting towers contain at least one of each color. Convince us that you have found them all.

Suppose the colors are red, blue, and green. We are counting the towers in three cases: 1) those with two red cubes, one blue cube and one green cube, 2) those with one red cube, two blue cubes, and one green cube, and 3) those with one red cube, one blue cube, and one green cube. The following equation gives the number of ways of selecting m groups of objects of size r_1 through r_m:

$$\binom{n}{r_1, r_2, \ldots, r_m} = \frac{n!}{r_1! \cdot r_2! \cdot \ldots \cdot r_m!}, \text{ where } \sum r_i = n$$

So the number of 4-tall towers containing exactly two red cubes, one blue cube, and two green cubes is:

$$\binom{4}{2,1,1} = \frac{4!}{2! \cdot 1! \cdot 1!} = 12$$

Similarly for the other two cases:

$$\binom{4}{1,2,1} = \binom{4}{1,1,2} = 12$$

Hence the number of towers with the required condition is 12+12+12=36.

15. The World Series Problem (January 1999, Grade 11) – In a World Series two teams play each other in at least four and at most seven games. The first team to win four games is the winner of the World Series. Assuming that the teams are equally matched, what is the probability that a World Series will be won: a) in four games? b) in five games? c) in six games? d) in seven games?

The number of ways for a team to win the series (four games) in n games is the number of ways it can win three times in n-1 games (and then win the last game). This is given by $C(n\text{-}1,3)$. The probability of any given set of outcomes for n games is $1 \div 2^n$ (since there are two equally likely outcomes for each game). So the probability that one team wins the series in n games is given by $C(n\text{-}1,3) \div 2^n$, and the probability of a win for either team is double that: $C(n\text{-}1,3) \div 2^{n-1}$. The probabilities are:

a) $C(4\text{-}1,3) \div 2^{4-1} = C(3,3) \div 2^3 = 1 \div 8 = 0.125$.
b) $C(5\text{-}1,3) \div 2^{5-1} = C(4,3) \div 2^4 = 4 \div 16 = 0.25$.
c) $C(6\text{-}1,3) \div 2^{6-1} = C(5,3) \div 2^5 = 10 \div 32 = 0.3125$.
d) $C(7\text{-}1,3) \div 2^{7-1} = C(6,3) \div 2^6 = 20 \div 64 = 0.3125$.

16. The Problem of Points (February 1999, Grade 11) – Pascal and Fermat are sitting in a café in Paris and decide to play a game of flipping a coin. If the coin comes up heads, Fermat gets a point. If it comes up tails, Pascal gets a point. The first to get ten points wins. They each ante up fifty francs, making the total pot worth one

15. The World Series Problem (January 1999, Grade 11) – In a World Series two teams play each other in at least four and at most seven games. The first team to win four games is the winner of the World Series. Assuming that the teams are equally matched, what is the probability that a World Series will be won: a) in four games? b) in five games? c) in six games? d) in seven games?

    The number of ways for a team to win the series (four games) in n games is the number of ways it can win three times in n-1 games (and then win the last game). This is given by $C(n\text{-}1,3)$. The probability of any given set of outcomes for n games is $1 \div 2^n$ (since there are two equally likely outcomes for each game). So the probability that one team wins the series in n games is given by $C(n\text{-}1,3) \div 2^n$, and the probability of a win for either team is double that: $C(n\text{-}1,3) \div 2^{n\text{-}1}$. The probabilities are:

    a) $C(4\text{-}1,3) \div 2^{4\text{-}1} = C(3,3) \div 2^3 = 1 \div 8 = 0.125$.
    b) $C(5\text{-}1,3) \div 2^{5\text{-}1} = C(4,3) \div 2^4 = 4 \div 16 = 0.25$.
    c) $C(6\text{-}1,3) \div 2^{6\text{-}1} = C(5,3) \div 2^5 = 10 \div 32 = 0.3125$.
    d) $C(7\text{-}1,3) \div 2^{7\text{-}1} = C(6,3) \div 2^6 = 20 \div 64 = 0.3125$.

16. The Problem of Points (February 1999, Grade 11) – Pascal and Fermat are sitting in a café in Paris and decide to play a game of flipping a coin. If the coin comes up heads, Fermat gets a point. If it comes up tails, Pascal gets a point. The first to get ten points wins. They each ante up fifty francs, making the total pot worth one

hundred francs. They are, of course, playing "winner takes all." But then a strange thing happens. Fermat is winning, 8 points to 7, when he receives an urgent message that his child is sick and he must rush to his home in Toulouse. They carriage man who delivered the message offers to take him, but only if they leave immediately. Of course, Pascal understands, but later, in correspondence, the problem arises: how should the 100 francs be divided?

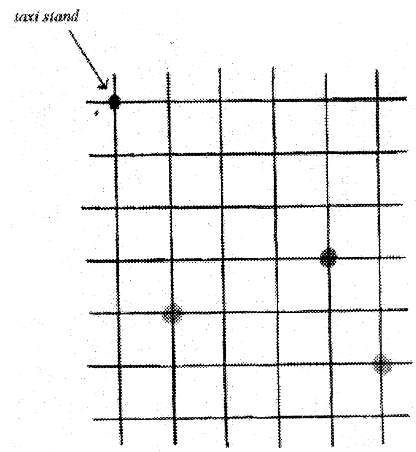
We can list all the circumstances where Fermat gets two points before Pascal gets three points. He can do this in two flips, three flips, or four flips. (The game cannot proceed past four flips. As soon as both players get to nine points, the next flip will produce a winner. It takes three flips for this to happen.)

a) Two flips: Fermat wins both. Probability $=1\div2^2=1\div4$.
b) Three flips: Fermat wins one of the first two and the last one.
Probability $= C(2,1)\div2^3 = 1\div4$.
c) Four flips: Fermat wins one of the first three and the last one:
Probability $= C(3,1)\div2^4 = 3\div16$

Probability of any of these events $= 1\div4 + 1\div4 + 3\div16 = 11\div16$. Therefore Fermat should get $100 \times 11\div16$ Francs ≈ 69 Francs and Pascal should get 31 Francs.

17. The Taxicab Problem (May 2002, Grade 12) – A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated on the map. To pass the time, she considers all the

possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route. What is the shortest route from a taxi stand to each of three different destination points? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.



I use Powell's (2003) notation to denote coordinates on the taxicab grid. (n,r) indicates a point n blocks away from the taxi stand and r blocks to the right. So the blue dot is at (5,1), the red dot is at (7,4), and the green dot is at (10,6). Taking the shortest route means going in two directions only (down and to the right). Finding the number of shortest paths from the taxi stand (0,0) to any point (n,r) involves the number of ways to select r segments of one kind of movement in a path that includes two kinds of movements; i.e. the number of shortest paths to (n,r) is $C(n,r)$. For the specific cases given above, the shortest paths are:

Blue: $C(5,1) = 5$.
Red: $C(7,4) = 35$.

Green: $C(10,6) = 210$.

APPENDIX B: A BRIEF HISTORY OF PASCAL'S TRIANGLE

Introduction

This is Pascal's Triangle as it is usually written today:

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \leftarrow \text{Row } 0 \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
\ldots & & & & & & & & & & & & & &
\end{array}
$$

The triangle goes on indefinitely. If we call the 1 at the top of the triangle row 0, then we can name the r^{th} number of the n^{th} row; call this *C(n,r)*. An additive rule for generating a number in one row from the previous row is:

$$C(n,r) = C(n-1,r-1) + C(n-1,r)$$

A multiplicative rule for generating one number in a row from the previous number in the same row is:

$$C(n,r) = \frac{n-r+1}{r} C(n,r-1)$$

Although the triangle is named for 17^{th}-century French mathematician Blaise Pascal, Pascal was not the creator of the triangle; he was not the first to write about the triangle; and he was not the first to relate it to patterns of numbers. Arrays of numbers (arranged in a triangular pattern or in some other form) have been used since ancient times to investigate various numbers patterns. Three patterns of particular interest in relation to Pascal's Triangle throughout history have been called figurate numbers, combinatorial numbers, and binomial coefficients. Today this is an artificial distinction. (All three sets of numbers are in fact the same.) But it is useful to maintain the distinction when considering the history of Pascal's Triangle because the triangle evolved in response to three separate kinds of questions, which mathematicians only later realized involved the same numbers.

Figurate Numbers

Figurate numbers are progressions of numbers created by an inductive rule:

1. Some initial progression is specified. For Pascal's Triangle, this is 1, 1, 1, …
2. To create successive progressions, for m greater than 1 and any natural number n, find the n^{th} number of the m^{th} progression by adding the first n numbers of the $(m-1)^{st}$ progression.

The first four progressions of figurate numbers starting with 1, 1, … are therefore:

1. 1, 1, 1, 1, …
2. 1, 2, 3, 4, … These are the natural numbers.
3. 1, 3, 6, 10, … These were named triangular numbers by the Greeks, due to the fact that the numbers can be represented by triangular arrays of dots.
4. 1, 4, 10, 20, … The Greeks named these tetrahedral numbers. They are sometimes called pyramidal numbers.

Greek mathematicians such as the Pythagoreans were particularly interested in figurate numbers because of their belief that certain number patterns possessed mystical properties.

Combinatorial Numbers

Combinatorial numbers arise in certain kinds of selection problems called combinations. If we start with a set of n objects, then the purpose of a problem in combinations is to find out how many ways to select a subset consisting of r of them. (r is between 0 and n.) For early mathematicians, combinatorial problems arose in connection with practical and philosophical issues. For example, the Greek Porphyrus investigated the number of ways to make pairs of five Aristotelian voices (genus, species, proprium, differentia, and accidens). Indian mathematicians investigated the number of ways to combine subsets of the six tastes (sweet, pungent, astringent, sour, salty, and bitter).

Consider finding the number of pairs of the five Aristotelian voices. Note that there are four ways to combine the first voice with the other voices, three ways to combine the second voice with the other voices, two ways to combine the third voice with the other voices, and one way to combine the fourth and fifth voices; $4+3+2+1 = 10$. This generalizes to n objects: there are $(n-1) + (n-2) + \ldots + 1$ ways to make pairs of n objects. Since the sum of the first $(n-1)$ numbers is given by $n\times(n-1)\div 2$, this formula also gives the number of combinations of two objects selected from n objects. This formula was known to Greeks in the third century.

For the general case, selecting r objects, consider that there are n choices for the first selection, $n-1$ choices for the second selection, and so on to $n-r+1$ choices for the r^{th} selection. This can be written as:

$$\frac{n!}{(n-r)!}$$

But this formula considers a group of objects selected in different orders as different groups, and order should not be considered when selecting combinations. (For example, in selecting two of the five voices, genus/species is the same as species/genus.) So it is necessary to divide out the selections that have been over-counted. There are $r!$ ways to arrange the selected r items in r slots; therefore divide by $r!$, giving the formula for the number of combinations of r objects selected from n objects as:

$$\frac{n!}{r!(n-r)!}.$$

The number of combinations of r objects selected from n objects, often called *n choose r*, can be written today in several ways, including the notations shown in Figure B1.

$$\binom{n}{r} \qquad {}_nC_r \qquad C_r^n \qquad C(n,r) \qquad c(n,r)$$

Figure B1. Five modern notations for combinations

Pascal's Triangle can be written in the notation of combinations. One way is:

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
\cdots & & & & & & & & & & \\
\binom{n}{0} & & \cdots & & & \binom{n}{r} & & \cdots & & & \binom{n}{n}
\end{array}
$$

So its addition rule can be understood in terms of combinations as well:

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1} \text{ or } C(n,r)=C(n-1,r)+C(n-1,r-1)$$

This rule can be proved algebraically, using the equivalent factorial notation, and it can also be proved in terms of the meaning of combinations as follows. Consider a set of n objects of which a subset of r objects is to be chosen. (The problem is to find $C(n,r)$.) Any subset will or will not contain the n^{th} object, so $C(n,r)$ is the sum of the numbers of combinations for those two possibilities. If the n^{th} object is in the subset, then we are selecting the other r–1 objects from n–1 objects; this is $C(n-1,r-1)$. If the n^{th} object is not in the subset, then we are selecting r objects from n–1 objects; this is $C(n-1,r)$. Hence $C(n,r)=C(n-1,r)+C(n-1,r-1)$.

Binomial Coefficients

Binomial coefficients arise in connection with the binomial expansion formula $(a+b)^n$. The following can be shown by induction:

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

The coefficient of $a^{n-r}b^r$ is given by:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

This number is the r^{th} entry in the n^{th} row of Pascal's Triangle. For example, the expansion of $(a+b)^3$ corresponds to the third row of Pascal's Triangle:

$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

The fact that a triangular array of numbers can be used to generate the coefficients of the binomial expansion was known to Persian mathematicians by about 1100. Chinese mathematicians knew of this rule by the 11th century; by 1300 a Chinese mathematician (Chu Shih-chieh) published a triangle for determining binomial coefficients remarkably similar to the one used today (Edwards, 2002).

Pascal's *Traite du Triangle Arithmétique*

By the seventeenth century, Western mathematicians knew a great deal about figurate numbers, combinatorial numbers, and binomial coefficients. Figurate numbers had been related to combinatorial numbers and to binomial coefficients. The arithmetical triangle had been published in various forms. Additive and multiplicative rules for generating the arithmetical triangle were well known. Furthermore, a compendium of current knowledge about combinatorics (the *Harmonicorum universelle*) had been written by French mathematician Father Marin Mersenne in 1636 (Edwards, 2002). At the time, Blaise Pascal, age 14, was a student at an informal school run by Father Mersenne, and so it is known that Pascal was familiar with Father Mersenne's work.

As described above, Pascal did not create the triangle, nor was he the first to relate it to patterns of numbers. But his landmark work (*Traite du Triangle Arithmétique*, 1654), provides justification for giving his name to the triangle. Edwards (2002) explains the contributions of the treatise:

> … it was this work that brought together all the different aspects of the numbers for the first time. In it Pascal developed the properties of the numbers as a piece of pure mathematics (often using mathematical induction in his proofs) and then, in a series of appendices, showed how these properties were relevant to the study of the figurate numbers, to the theory of combinations, to the expansion of binomial expressions, and to the solution of an important problem in the theory of probability (p. xiii).

The triangle that appeared in Pascal's book (a copy of the triangle that Mersenne had previously published) contained squares determined by parallel horizontal and vertical lines, so it looked like today's familiar Pascal's triangle lying on its side. Pascal (1654/1952) describes how to generate this triangle by an inductive additive rule (pp. 447-8):

1. Assign any number to the cell in the top left corner; Pascal calls this number the generator. In the standard version of Pascal's Triangle, the generator is 1.

2. The number for any other cell is the sum of the numbers in the cell to the left and the cell above the given cell. Pascal calls these the perpendicular and parallel cells.

Pascal goes on to describe nineteen properties of the triangle that he derives from the definitions. For example, in what he calls the First Consequence, he proves that the first horizontal row (which he calls the first parallel row) and the first vertical column (which he calls the first perpendicular row) must have the same value as the generating cell. Here is his reasoning:

> For by definition each cell of the triangle is equal to the sum of the immediately preceding perpendicular and parallel cells. But the cells of the first parallel row have no preceding perpendicular cells, and those of the first perpendicular row have no preceding parallel cells; therefore they are all equal to each other and consequently to the generating number (p. 448).

In later sections of the treatise, Pascal shows that the triangle contains the three sets of numbers previously discussed (figurate, combinatorial, and binomial) and gives proofs of various relationships between the numbers. For example, he derives an instance of the addition rule for the triangle in terms of combinations. In modern terminology, he shows that $C(4,2) = C(3,1) + C(3,2)$. He goes on to assert without proof the general version: $C(n,r) = C(n-1,r-1) + C(n,r-1)$.

The information in these sections of the treatise is not new, although Pascal's organization and rigor can be considered fresh. But the section related to the Problem of

Points is original work that had a great impact on the fledgling mathematical science of probability. Edwards (2002) describes the problem:

> Briefly, this problem concerns the division of the stakes between two players when a game has to be left unfinished. For example, suppose the game consists of tossing a penny, player A to count heads as points in his favour, and player B to count tails, the winner to be the first to score an agreed number of points. How should the stakes be split if the game is interrupted when A still needs a points to win, and B still needs b? It is supposed that the coin is a fair one, and that at the outset each player contributed an equal stake (p. 58).

Note that if Player A needs a points to win and Player B needs b points, then at most $(a+b-1)$ further coin tosses are required to determine the winner.

Pascal saw that the entries of the n^{th} row of the triangle give all the ways for heads to come up in n coin tosses. In other words, the r^{th} number from the left in row n gives the number of ways to get r heads when a coin is tossed n times. Therefore, row $a+b-1$ (which has $a+b$ entries) gives all possible results (i.e. possible number of heads tossed, from 0 to $a+b-1$) for all remaining $a+b-1$ games. The r^{th} number in that row, called $C(a+b-1,r)$ in the combinatorial notation introduced previously, represents the number of ways to toss the coins $(a+b-1)$ times and get r heads. So the number of ways A can win is obtained by adding all the ways to get a or more heads; this is given by the sum $C(a+b-1,a) + \ldots + C(a+b-1,a+b-1)$. The number of ways B can win is similarly obtained by adding all the ways to get fewer than a heads; this is given by the sum $C(a+b-1,0) + \ldots + C(a+b-1,a-1)$. Convert the counts of ways to win to probabilities by dividing by the total number of possible games; this is 2^{a+b-1}.

Pascal (1654/1952) gives a numerical example involving one player (Player A) who is two points away from a win and another (Player B) who is four points away from a win (p. 464). At most five (4+2–1) coin tosses are required. So consider row 5 of the triangle. Its six numbers are 1, 5, 10, 10, 5, and 1. In combinatorial notation, these are $C(5,0)$, $C(5,1)$, $C(5,2)$, $C(5,3)$, $C(5,4)$, and $C(5,5)$. Player A wins in the four cases involving at least two heads: $C(5,2)$ through $C(5,5)$. Player B wins in the other two cases: $C(5,0)$ and $C(5,1)$. The probability that A will win is therefore $(10+10+5+1)\div 32 = 13\div 16$ and the probability that B will win is therefore $(1+5)\div 32 = 3\div 16$.

Pascal's use of the triangle and the associated mathematics of combinatorics and the binomial distribution was regarded as a major contribution to the new mathematical science of probability. Pascal knew this; he "regarded his taming of the vagaries of chance as a major contribution in itself" (Edwards, p. 79).

Notations for Combinatorics

As shown in Figure B1, there are currently at least six standard notations to indicate the combination of *n* objects taken *r* at a time. Pascal himself did not use any of these notations. Pascal (1654/1952) used no notation at all; he just described his results in words. The mathematical notation he did use included symbols for addition, multiplication, division, and ratios. For example, he wrote B:E::3:4 to indicate that $B\div E = 3\div 4$ (p. 452).

Mathematicians began to use notation for factorials and combinatorics shortly after Pascal's time (Cajori, 1928a). During the 18th century, it was often (but not always) standard for multiplication to take precedence over addition and subtraction, as is standard today. But this standard did not apply to the multiplication associated with factorials. So it was common to use $n \cdot n-1 \cdot n-2$ or $n \times \; n-1 \times n-2$ to indicate $n(n-1)(n-2)$. However, some mathematicians used parentheses or vinculums (bars over the symbols) to indicate grouping.

Figure B2 gives some notations that were used in the past to indicate combinations (Cajori, 1928b).

Date	**Mathematician**	**Notation**	**Equivalent**
1708	Leibniz	$\dfrac{n \cap n-1 \cap n-1, etc., n-k+1}{1 \cap 2 \cap 3, etc., \cap k}$	$C(n,k)$
1747	Jones	$n' = n \cdot \dfrac{n-1}{2}$	$C(n,2)$
1806	Euler	$\left(\dfrac{n}{\rho}\right)$	$C(n,\rho)$
1827	von Ettingshauser	$\left(\begin{matrix} m \\ \rho \end{matrix}\right)$	$C(m,\rho)$
1834	Schillbach	$(r, n! a_\delta)$	$C(n,r)$

Figure B2. Some historical notations for combinations

By the early 20th century, parentheses were in use; Cajori (1928a) used *(n,r)*. At present, as noted previously, there are at least five standard notations in use. Clearly, there has never been and there is still no single standard notation for combinations. This is not uncommon in mathematics. For example, there are several ways to express derivatives; the choice of which terminology to use is dependent on the needs of the particular context. And notation in new fields of mathematics is expected to be in flux. For example, Askey (1996), in describing the application of q-series and q-polynomials to new fields in mathematics, notes that their notation has not yet been standardized (p. 18).

The choice of notation can be an assistance to mathematical advances or a hindrance. For example, Seife (2000) notes that 18th-century British mathematicians, clinging to Newton's notation for differential calculus, were unable to keep up with other European mathematicians who used Leibniz's notation:

> … the English stuck to Newton's fluxion notation rather than adopting Leibniz's superior differential notation… English mathematicians fell far behind their Continental counterparts when it came to developing calculus (p. 123).

In the case of combinatorics, there have been suggestions for improved notations. A generalized notation is given by Putz (1986) for more than one partition. It is:

$$\binom{n}{r_1, r_2, \ldots r_k} = \frac{n!}{r_1! r_2! \ldots r_k!}, \text{ where } \sum r_i = n$$

Hilton and Pederson (1999) propose something similar for the binomial form:

$$\binom{n}{r\ s}, \text{ with } n = r+s$$

This makes the symmetry explicit. For example, in this notation (p. 171), Pascal's Identity is:

$$\binom{n}{r\ \ s} = \binom{n-1}{r-1\ \ s} + \binom{n-1}{r\ \ s-1}$$

Hilton and Pederson propose various extensions to Pascal's Triangle (for example Pascal's Hexagon, a form that comes out of a generalization that allows n, r, and s to be nonpositive, and Pascal's Tetrahedron, a three-dimensional generalization that applies to trinomial coefficients).

Since the time of Pascal and before, the numbers in Pascal's Triangle have been looked at as solutions to questions about enumeration, philosophy, number theory, and gambling. The various notations have included written descriptions such as that used by Pascal himself, notations that emphasize the operations (e.g. that of Leibniz), notations that can be considered mnemonic (such as C for combinations), notations intended to emphasize certain features or regularities (such as that proposed by Hilton and Pederson), and notations that promote generalization and extensions (Putz). Using binary notation and thinking about the relationships in terms of pizza toppings and towers problems fit into the history of thought about Pascal's Triangle.

APPENDIX C: TABLE OF CRITICAL EVENTS

Code Type	**Code**	**Description**
Participant Identification	S	Student
	R	Researcher
Combinatorics Problem	P	Pizza problem
	T	Towers problem
	X	Taxicab problem
	C	Binomial coefficients
	O	Other problem
Representations	B	Binary notation
	N	Standard Notation
	D	Other representation (e.g. diagram, letters)
Pascal	PT	Pascal's Triangle
	PI	Pascal's Identity
Who	A	Ankur
	B	Brian
	J	Jeff
	M	Michael
	R	Romina
	R*n*	Researcher *n*

Start	**End**	**Codes**	**Who**	**Description**
December 12, 1997				
00:41:35	00:53:07	S, R, P, B, D	A, B, J, M, R, R1	Michael introduces binary notation; the students explain how binary notation can be related to the code the other four had been using.
01:00:48	01:07:53	S, R, B, P, T	A, B, J, M, R, R1	"Everything we ever do always is like the tower problem." They investigate this relationship. Michael uses binary notation to investigate towers.
01:02:14	01:02:37	S, R, N, T	A, B, J, M, R, R1	Discussion of factorial notation. When is it appropriate to use factorials?
December 19, 1997				
00:19:20	00:19:35	T, B	M	Michael explains the link between binary and towers: 0 is blue and 1 is red.

Start	End	Codes	Who	Description
00:34:20	00:39:03	S, R, P, T, B	A, B, J, M, R	Michael says the pizza problem is the same as the tower problem with 2 colors. They discuss.
00:39:43	00:40:19	S, N	A, M	They discuss whether factorial is appropriate for towers with more than two colors.
00:58:32	00:59:49	S, R, N	A, B, J, M, R, R1	They address the question, "Why do you multiply?"
January 9, 1998				
01:35:00	01:39:44	R, N, PT, T	R1, A, B, J, M, R	R1 introduces standard notation for combinatorics and illustrates with 5-tall towers problem.
01:40:31	01:40:39	N	A, B, R	A brief mention about having learned factorial notation in school.
01:40:40	01:44:02	R, C, T, PT	R1	R1 draws Pascal's Triangle, calls attention to binomial coefficients, and asks why we keep finding those numbers (answers to towers problem) in there.
February 6, 1998				
00:50:41	00:53:35	S, R, T, N	A, J, R, R1	R1 reminds them of the notation for combinations and talks about how it is related to the towers problems.
00:53:35	00:56:16	S, R, T, C, PT	A, J, R, R1	They discuss how the binomial expansion is related to towers and Pascal's Triangle.
01:04:36	01:06:19	S, P, T, PT	A, J, R	The students work out among themselves what row 6 means in terms of towers and pizzas.
01:10:38	01:11:22	S, R, T, P, PT	A, J, R	They explain to R5 how row 6 of Pascal's Triangle is related to towers and pizzas.
01:13:21	01:16:58	S, R, T, PI	A, J, R, R1, R5	Ankur, Jeff, and Romina explain specific instances of Pascal's Identity.
March 6, 1998				
00:05:26	00:06:42	S, R, PT, C	A, J, M, R, R1	R1 asks the students to review how the binomial coefficients are found in Pascal's Triangle.

Start	End	Codes	Who	Description
00:33:49	00:34:28	S, C, T, PT	A, J, M, R	They link towers to coefficients: "*a* will be blue and *b* will be red."
00:46:56	00:48:20	S, R, T, PI	A, J, M, R, R1	Michael explains how 3+3=6 in terms of towers.
00:56:50	00:57:37	S, R, T, P, PT	A, J, M, R, R1	Ankur (in trying to explain pizzas in terms of towers) says, "the colors don't represent anything." Michael says they do; white is no topping and blue is topping. Then they explain all the 2-tall towers in terms of pizzas.
00:59:22	01:03:08	S, R, T, N	A, J, M, R, R1	Problem involving factorials: n-tall towers selecting from n colors having at least 1 of each color.
01:08:14	01:10:24	S, R, T, N	A, J, M, R, R1	R1 writes Pascal's Triangle in "choose" notation, including row n and a general (r^{th}) entry in row n. The students explain to her what various entries mean in terms of towers.
June 14, 1998				
00:06:55	00:09:28	R, C, PT	A, B, J, R, R1	R1 reminds students to think about the relationship between Pascal's Triangle and the binomial coefficients.
00:09:33	00:13:18	R, C, T, PT	A, B, J, R, R1	R1 asks students to explain Pascal's Identity in terms of towers and binomial coefficients.
December 14, 1998				
00:32:00	00:35:49	R, S, B, P, PT	M, R1	In an interview with R1, Michael explains the relationship between binary numbers, pizza problems, and Pascal's Triangle.
00:39:20	00:43:43	R, S, B, P, PT	M, R1	In the interview, Michael explains Pascal's Identity in terms of pizza problems. (He follows up with a more detailed email explanation.)
January 22, 1999				
00:57:03	01:00:59	S, P, PT	A, B, M, R	Michael explains Pascal's Identity in terms of pizzas to Ankur, Brian, and Romina.
January 29, 1999 (interview)				

Start	End	Codes	Who	Description
00:00:35	00:02:04	S, R, P, PT	M, R10	In an interview with R10, Michael explains the relationship between Pascal's Triangle and pizza problems.
00:02:25	00:05:41	S, R, P, PT	M, R10	In the same interview, Michael explains instances of Pascal's Identity in terms of pizzas.
00:07:43	00:08:16	S, R, P, PT	M, R10	Michael goes through the Pascal's Identity explanation one more time.
January 29, 1999 (problem-solving session)				
00:12:50	00:18:42	S, R, B, P, PT	A, J, M, R, R5	Michael explains Pascal's Identity to R5 in terms of pizzas, using binary notation.
May 12, 1999				
00:04:20	00:04:55	S, T, N	J, M, R	They explain 10 choose 2 using 10-tall towers with 2 red towers.
00:27:41	00:29:04	S, R, N, T, O	J, M, R, R1	R1 asks the group to consider towers. Jeff explains 5 choose 2 in terms of towers 5 tall with 2 of one color.
00:14:27	00:17:57	S, R, N, O	A, J, M, R, R1	Making sense of factorial notation.
00:22:18	00:24:24	S, R, N, O	A, J, M, R, R1	Further exploration into factorial notation.
00:32:21	00:34:28	S, R, N, B, P, PI	J, M, R, R1	Michael uses pizzas (and binary notation) to explain one instance of Pascal's Identity.
00:36:38	00:39:22	S, R, C, T, N, PT	A, J, M, R, R1	They work on writing Pascal's Triangle in "choose" notation.
00:43:02	00:44:34	S, R, N, PI	J, M, R1	Jeff and Michael show 3+3=6 in "choose" notation.
00:45:15	00:46:32	S, N, PT	A, J, M, R	Jeff writes rows *N* and *N*-1 in "choose" notation.
00:49:41	00:51:34	S, R, PT, PI	A, J, M, R, B	They write the general equation.

Start	End	Codes	Who	Description
May 5, 2000				
00:07:07	00:09:09	S,T,X	M, R	Romina asks if they can use towers to work on the taxicab problem. She and Michael discuss the solution to the towers problem. (Romina asks, "is it two to the *n*?")
00:44:01	00:45:07	S, T, X	J, M, R	Romina brings up towers again, and again they discuss two to the *n*. Jeff says *n* is the height of the tower. But Romina's answers do not fit with what she expects and she again abandons the idea. (She says, "scratch that idea.")
00:47:25	00:49:07	S, X, PT	B, J, M, R	Romina says, "it's Pascal's Triangle." They discuss whether their findings fit Pascal's Triangle. (They decide to work out some more answers.)
00:54:42	00:58:38	S, X, PT	B, J, M, R	After getting more answers, they agree that Pascal's Triangle is involved.
01:00:53	01:01:06	S, T, X, PT	J, M, R	They discuss how to prove that there is a relationship between the taxicab problem and Pascal's Triangle. Romina suggests using towers.
01:02:13	01:04:05	S, T, P, X, PT	J, M, R	In the discussion about proving the relationship, Romina brings up powers of 2; Michael brings up pizzas.
01:08:07	01:10:58	S, R, X, PT	R1, J, R	Romina and Jeff tell R1 that the problem is related to Pascal's Triangle.
01:16:00	01:19:57	S, T, X, PT	B, M, R	Romina links colors of towers to across and down in the taxicab problem. Brian and Michael say they understand.
01:22:28	01:25:22	S, T, X, PT	B, M, R	Romina reiterates the explanation in discussions with Brian and Michael.
01:26:00	01:33:15	S, R, T, X, PT	R1, R12, B, M, R	They explain to R12 the relationships among the taxicab problem, the towers problem, and Pascal's Triangle.
01:33:17	01:34:31	S, R, X, B	R1, B, M, R	R1 asks about binary notation. Romina, Michael, and Brian respond.
April 3, 2002				

Start	End	Codes	Who	Description
00:07:11	00:08:30	S, P, PT	M, R1	Michael links pizzas to Pascal's Triangle (explaining that the leftmost number in each row represents the plain pizza and the rightmost represents the "all topping" pizza). Michael explains row 2 of Pascal's Triangle in terms of pizzas.
00:07:32	00:09:20	S, R, P, N	M, R1	Michael uses standard notation. He explains it in terms of pizzas.
00:09:20	00:13:31	S, R, P, N, PT, PI	M, R1	Michael uses pizzas to explain the addition rule and then writes the addition rule in standard notation.
00:07:11	00:08:30	S, P, PT	M, R1	Michael links pizzas to Pascal's Triangle (explaining that the leftmost number in each row represents the plain pizza and the rightmost represents the "all topping" pizza). Michael explains row 2 of Pascal's Triangle in terms of pizzas.
July 22, 2002				
00:02:24	00:02:59	S, P, T, N, PT	R	Romina relates the top number in choose notation to the number of pizza toppings. Then she relates the top number to the height of the tower and the bottom number to the number of red cubes.
00:03:25	00:04:58	S, T, PT, PI	R, R8	Romina explains the addition rule in terms of adding red or blue blocks to towers.
00:06:04	00:06:19	S, R, T, N	R, R8	Romina restates the relation between the notation and the towers.
00:11:19	00:11:37	S, R, N	R, R8	Romina writes the addition rule in standard notation.
00:12:58	00:14:03	S, R, T, N, PT, PI	R, R8	Romina explains the notation in the addition rule in terms of towers.
00:14:31	00:15:06	S, R, P, N, PT, PI	R, R8	Romina explains the addition rule in terms of pizzas.
00:15:19	00:16:04	S, R, T, P	R, R8	Romina explains the relationships among towers, pizzas, and binary notation. Then she explains $C(3,3)$ in terms of pizzas.

Start	End	Codes	Who	Description
00:24:40	00:25:49	S, R, X, T, PI	R, R8	Romina explains the addition rule in terms of towers.
July 31, 2002				
00:11:26	00:12:26	S, R, T, PT	A, R8	Ankur explains row 3 of Pascal's Triangle in terms of towers.
00:40:46	00:44:23	S, T, PT, PI	A, R8	Ankur explains how to get from C(3,1) and C(3,2) to C(4,2) in terms of towers and then generalizes.

APPENDIX D: TIMELINES

Episode 1: Thinking About "10 choose 2"

12/97	**01/09/98**	**02/06/98**	**03/06/98**	**06/12/98**
	The students work on problem with towers: How many 5-tall towers are there with exactly two red cubes? R1 introduces standard notation.	They discuss how the binomial expansion is related to towers and Pascal's Triangle.	They link towers to coefficients "*a* will be blue and *b* will be red."	

12/14/98	**01/22/99**	**01/29/99**	**05/12/99**	**2002 Interviews**
			They explain 10 choose 2 using 10-tall towers with 2 red towers. R1 asks the group to consider towers. Jeff explains 5 choose 2 in terms of towers 5 tall with 2 of one color.	Michael explains row 2 of Pascal's Triangle. Romina relates the top number in choose notation to the number of pizza toppings. Then she relates the top number to the height of the tower and the bottom number to the number of red cubes. Ankur explains row 3 in terms of towers.

Episodes 2 and 3: Factorial Notation and Combinations

12/97	**01/09/98**	**02/06/98**	**03/06/98**	**06/12/98**
Discussion of factorial notation at both December sessions. They address the question, “Why do you multiply?”	A brief mention of having learned factorial notation in school.		Problem involving factorials: n-tall towers selecting from n colors having at least 1 of each color.	
12/14/98	**01/22/99**	**01/29/99**	**05/12/99**	**2002 Interviews**
			Making sense of factorial notation. Further work on factorials.	

Episode 4: Using Pizzas to Explain One Instance of Pascal's Identity

12/97	**01/09/98**	**02/06/98**	**03/06/98**	**06/12/98**
Michael introduces binary notation.	R1 introduces standard notation.	Ankur, Jeff, and Romina explain specific instances of Pascal's Identity.	They discuss how 3+3=6 in terms of towers.	
12/14/98	**01/22/99**	**01/29/99**	**05/12/99**	**2002 Interviews**
Michael explains specific cases of Pascal's Identity in terms of pizzas to R1.	Michael discusses pizzas and Pascal's Identity with Ankur, Brian, and Romina.	Michael explains Pascal's Identity in terms of pizzas to R10. Michael discusses binary notation with R5.	Michael and Jeff use pizzas (and binary notation) to explain one instance of Pascal's Identity.	Michael explains the addition rule in terms of pizzas. Romina explains the addition rule in terms of pizzas.

Episode 5: Pascal's Triangle and "Choose" Notation

12/97	**01/09/98**	**02/06/98**	**03/06/98**	**06/12/98**
	R1 draws Pascal's Triangle, calls attention to binomial coefficients, and asks why we keep finding those numbers (answers to towers problem) in there.	R1 introduces standard notation.	R1 writes Pascal's Triangle in "choose" notation, including row *n* and a general (r^{th}) entry in row *n*. The students explain to her what various entries mean in terms of towers.	
12/14/98	**01/22/99**	**01/29/99**	**05/12/99**	**2002 Interviews**
	Michael discusses pizzas and Pascal's Identity in terms of pizzas with Ankur, Brian, and Romina.		They work on writing Pascal's Triangle in "choose" notation.	Michael uses standard notation. He explains it in terms of pizzas. Romina explains the addition rule in terms of towers. Ankur explains how to get from C(3,1) and C(3,2) to C(4,2) in terms of towers.

Episodes 6 and 7: Pascal's Identity

12/97	01/09/98	02/06/98	03/06/98	06/12/98
	R1 shows standard notation.	Ankur, Jeff, and Romina explain specific instances of Pascal's Identity.	They discuss how 3+3=6 in terms of towers.	R1 asks students to explain Pascal's Identity in terms of towers and binomial expansion.
12/14/98	**01/22/99**	**01/29/99**	**05/12/99**	**2002 Interviews**
Michael explains Pascal's Identity in terms of pizza problems.	Michael discusses pizzas and Pascal's Identity in terms of pizzas with Ankur, Brian, and Romina.	Michael explains Pascal's Identity to R10 in terms of pizzas. Michael explains Pascal's Identity to R5 in terms of pizzas, using binary notation.	Jeff and Michael show 3+3=6 in "choose" notation. They write the general equation.	Michael writes the addition rule in standard notation. Romina writes the addition rule in standard notation. Romina explains the notation in terms of towers and pizzas. Ankur discusses the addition rule in terms of towers.

Summary: Recognizing Isomorphisms

12/97	01/09/98	02/06/98	03/06/98	06/12/98
They investigate the question: How are towers and pizzas. Michael explains the link between binary and towers: 0 is blue and 1 is red. Michael says the pizza problem is the same as the tower problem with 2 colors.	R1 draws Pascal's Triangle, calls attention to binomial expansion, and asks why we keep finding those numbers (answers to towers problem) in there.	R1 reminds them of the standard notation and talks about how it is related to the towers problems. They discuss links among binomial, towers, and Pascal's Triangle. Link: row 6 of Pascal's Triangle to towers and pizzas.	R1 asks the students to review how the binomial expansion are found in Pascal's Triangle. Michael links towers and pizzas: white is no topping and blue is topping. Link: Pascal's Triangle, pizzas, and towers.	R1 reminds students to think about the link between Pascal's Triangle and the binomial expansion.
12/14/98	**01/22/99**	**01/29/99**	**5/5/00**	**2002 Interviews**
In an interview with R1, Michael explains the link between binary numbers, pizza problems, and Pascal's Triangle.		In an interview with R10, Michael explains the link between Pascal's Triangle and pizza problems.	They describe how the taxicab problem is related to towers, pizza, and binary notation.	Romina explains the relationships among towers, pizzas, and binary notation.

APPENDIX E: SUMMARIES

Summary of December 12, 1997

Minutes	**Summary**
0-3	Romina, Jeff, Michael, and Ankur introduce themselves. Researcher 1 asks them to fill out some permission forms.
3-6	They talk about school. The other math students don't want to know why; they just want the answers.
6-9	Researcher 1 asks about their previous math classes. They say they didn't like geometry. Brian says that he likes this school better because the teacher actually teaches. Pizza is distributed. Michael asks for a clarification of the pizza problem. Researcher 1 gives the 5-topping pizza problem, and then she introduces Researcher 9.
9-12	They eat and chat. They work on components of the 5-topping problem: how many pizzas with one topping, two toppings, etc. Michael says, speaking of the two-topping cases, that there are five choices for the first topping and four for the second. That's 5 times 4. They continue working on the problem. Jeff and Romina work on the problem together, Brian and Ankur work together, and Michael works alone.
12-15	Romina says they have 60 for 3 toppings. Ankur tells Brian to write numbers for the toppings. Brian says for one topping, he has 5; he says for 2 toppings it would be 20. Ankur says each of the 5 can be with each of the 4 others. Now they have to convince Researcher 1. Brian draws lines to indicate how to select toppings. Ankur says 2 with 1 is the same as 1 with 2. So there are actually 10 pizzas with two of the five toppings. Meanwhile Jeff and Romina are talking about the same thing; they also get 10. (They are using letter codes, e.g. S and P.)
15-18	Ankur and Brian work on the 3-topping case (the number of pizzas with exactly three toppings when selecting from five toppings). Brian starts to make a list: 123, 124, 125, etc. He and Ankur get 9. Ankur says factorial is not the right operation to use because onions and peppers is the same as peppers and onions. Jeff and Romina say they got 10 for the 3-topping case. Ankur asks them to read them off, and he asks them to change from letters to numbers. Romina describes her codes. Ankur tells her they "just wrote it out like this." She looks at his paper.
18-21	Romina says she thinks he's right. She shows Jeff and then hands the paper back. Romina asks how many they got for the four-topping

Minutes	**Summary**
	case. Ankur and Brian say 3. Researcher 1 tells Ankur to think about explaining it to a younger kid. The camera switches over to Jeff and Romina. Michael asks how many it is with 4 (the four-topping case); he says he got 5. Jeff looks at Michael's work, and then he asks Ankur to read off his results for 4 toppings. Ankur does so, and they find that Jeff and Romina were missing 1245. Jeff enumerates the list.
21-24	Romina says, "so there's 30?" Romina asks if they should try the 4-topping case. Michael and Jeff look at Michael's paper. He got 10 for the 3-topping case; they find that Brian and Ankur missed one pizza with three toppings. Michael is seen to be writing 0's and 1's. Ankur says there are five pizzas with exactly 4 toppings. Researcher 1 asks them to convince each other before explaining it to her. She asks them to work on the 4-topping problem if they finish early.
24-27	They start to work on the 4-topping problem. Ankur suggests how to enumerate. Romina says for 2 she has 4. Ankur says they got 6; so did Jeff. Ankur says there are 3 with 4 toppings. Then they look at the three-topping case. Romina says, "7, right?" Jeff asks Ankur what they got for three toppings (when selecting from four). Ankur lists the four pizzas that have exactly three toppings.
27-30	Ankur says they got 7 for the three-topping problem. Jeff asks if it was 15 for 4. Ankur says yes. They enumerate the three answers for the two-topping problem. Jeff says, "for 6 there's going to be 60." For 9, there will be some large number. Romina asks if they have a formula. Researcher 1 says suppose there were 5 choices, would they have been able to order "this pizza" (meaning a plain pizza). Ankur asks if they should add the plain. Romina says that would be like 6 toppings. Ankur disagrees. Researcher 1 asks if they want to think about that interpretation. Jeff says you can't have peppers and cheese. "Cheese is already on. It comes standard."
30-33	Jeff argues that since Researcher 1 asked how many combinations there are, the plain can not be considered a combination of five toppings. Researcher 1 says it depends on how you interpret the language; she says it's a legitimate choice to say you want none of the toppings. Romina says it's not how many toppings; it's how many different pies are possible. Romina says that with five toppings, there are 31 different pies. So with 6, says Jeff, there are 61. Researcher 1 says she's not convinced. Jeff says to start with the easy case, two toppings. Jeff goes to the board. He calls the toppings 1 and 2.

Minutes	Summary
33-36	Jeff enumerates the pizzas 12, 1, 2, and plain: 4 pizzas. He asks if she's convinced. Researcher 1 consults with Researcher 4 and others. They're all convinced. Jeff moves on to 3 toppings. He lists 123, 12, 13, 1, 2, 3, 23, plus plain. This is 8. He does 4 toppings: 1234, 123, 124, 134, 234. Then Researcher 1 asks for the projection; they say 15+1=16. He writes 3+1=4 and 7+1=8 for the other 2 cases. He finishes enumerating pizzas.
36-39	He counts the pizzas: 15. He enumerates the possibilities for 4, 3, 2, and 1 toppings. Jeff reiterates the four-topping case: they got 15 pizzas with toppings plus a plain, giving 16. Jeff says the projection for 5 was 30 + 1. Jeff says the pattern works well if you add the cheese. He and Ankur say there might be something wrong with 5 because it breaks the pattern.
39-42	Jeff says they'll look at 5 again. Researcher 1 offers cookies. There are side conversations. Michael says it's 31 plus the cheese. Michael says he used the binary system: 1's mean the topping is present and 0's mean no topping. You use four places in the binary system to represent four toppings; this is 15.
42-45	Michael says for just one topping pizzas, you would have 0001, 0010, 0100, 1000. He enumerates two topping pizzas. Romina asks if the 1 is a topping. Jeff says yes, and 0's are no toppings. Michael says you go from 1 in binary to 15 in binary and those numbers correspond to the 15 pizzas. "It works out." If you have an extra topping, you have an extra binary digit, giving 32possibilities. Romina asks if he worked it all out. Michael shows how to write 1 through 5 in binary. Eventually you get to 1111, which is 15; there are 15 possible combinations of 1's and 0's using four binary digits. They discuss their 8th grade teachers. (Michael learned binary notation in 8th grade.)
45-48	Michael is asked to go to the board, but he refuses. Researcher 1 says Romina will help him. When Researcher 1 asks him to explain binary notation, Michael tells her that he knows she must know the binary system, because he learned it in 7th grade. Researcher 1 says she learns from them every week. Michael explains that the binary system is a different way of writing numbers. Instead of having 1 through 10, you have 0 and 1. The first place is 1, then 2, 4, and 8. Romina asks if the places double. Michael says yes. He shows how to write 1, 2, 3, 4, and 5 in binary notation.
48-51	Romina asks if 1000 is 16. Michael says that it is. He says it's 2 to "whatever" power. Researcher 1 says 2 to a power is a good idea.

Minutes	**Summary**
	Researcher 1 asks how to think of 1; Michael says 1 is 2^0. Researcher 1 says she likes these ideas. Michael says if you have 4 different toppings, all the numbers between 1 and 1111 are the different combinations of pizzas. "If you have a 5-topping pizza, you add the next place." And so on to 32, 64, etc.
51-54	Researcher 1 says before they had 1, 2, 3 and 4, but now they're using 1's and 0's. Michael puts 1, 2, 3, and 4 as column headers. Researcher 1 asks what she should think about when she thinks about those numbers. What pizza is 0100? What is the difference between 1000 and 0100? Jeff says, "it would be the difference between an onion pizza and a pepperoni pizza." He says you could label the columns with pizza toppings. Michael says that each 1 in the onion column means that the pizza has onions. Researcher 1 asks how many combinations there are for 10 toppings.
54-57	They calculate and get 512. There is some discussion about whether or not to add 1. Researcher 9 asks what the answer would be for *n* toppings. Jeff asks if it's *n* to a power. Ankur says it's 2^n. Michael says it's *n*-1. Researcher 1 asks them to talk about it. They discuss.
57-60	Ankur says, "32 is to the 6, right? … So it's 2^{n+1}." After some discussion, they settle on an answer of 2^n.
60-63	They argue about who thought the answer was 2^n. Researcher 1 asks if it reminds them of anything else they did. Brian says, "everything we do is like the towers problem." Researcher 1 asks what he means. Brian says, "instead of building a tower, you're building a pizza." Researcher 1 asks how it's like the towers problem. Romina says instead of building it, they're writing it. Researcher 1 asks them to explain the tower problem to Researcher 9. Romina says for each topping you have a different color cube. Ankur says it's five high with two different colors. Brian explains the 2-tall, 3-tall, and 4-tall towers problems. Brian says instead of having 1, 2, 3, and 4 toppings, you just have cubes.
63-66	Brian says, you can use the same one twice. Researcher 1 asks him to show this. Ankur says with towers you can have red on top and yellow on the bottom, and yellow on top and red on the bottom. But with pizzas, you can't have peppers and pepperoni and then pepperoni and peppers. Researcher 1 asks if the problems are therefore not alike. Ankur says they're similar but not exactly alike. Researcher 1 suggests talking about towers 2-tall. Ankur says red on top, yellow on the bottom, yellow on top and red on the bottom, all red, and all yellow. Michael says that's four, the same as the pizzas.

Minutes	Summary
	Michael says 2 is a bad number to test with, so Researcher 1 says to try 3. Jeff says with pizzas you could have different numbers of toppings, but with towers, you can't have different heights. But Researcher 1 says she wants to return to towers. So Jeff says let's pick 3. Ankur enumerates towers.
66-69	Michael says he's using binary again. "'Cause it might work for this" [towers]. The rest start to enumerate using letters (R and Y). Brian lists them and says there are 8 possible 3-tall towers. Ankur predicts 16 for 4 tall. He says, "it's all the same." Researcher 1 says this is a good place to stop. She asks if they can come after school next Friday. They agree.
69-72	Side conversations. Researcher 1 asks if they are on email, because she'd like to keep in touch. Michael asks Researcher 1 what she does at Rutgers. She tells him that she's a professor.
72-75	Researcher 1 talks about the importance of students' learning to do thoughtful mathematics. Researcher 9 asks Michael where binary numbers are used.
75-78	Michael explains how computers use binary numbers. On and off are 1 and 0. Researcher 1 suggests meeting a couple times a month.
78-81	Side discussions. Researcher 1 offers her email address.
81-84	They write email addresses. Researcher 1 talks about a problem they did in the 6^{th} grade that Michael solved in an interesting way. The problem is to determine how many 2-digit numbers can be made selecting from the digits 1 through 4.
84-87	Researcher 1 asks them to write their initial thinking about solving that problem, which they all solved in the 6th grade. Ankur says 16. Michael says there are 4 numbers with a 1 in the 10s place, and the same goes for the other places. Michael shows his work.
87-90	Researcher 3 shows Michael's 6^{th}-grade solution. Michael wrote 1 2 3 4 and then underneath he wrote 1 2 3 4. He drew lines from the first row to the second row (10 of them). Then he doubled 10, because each pair can be used twice. (For example, 2 and 3 gives 32 as well as 23.)
90-93	Researcher 3 says Michael's final step was to subtract duplicates (11, 22, 33, and 44). 20-4 gives 16. Michael said it was a hard way. Researcher 3 says it was beautiful. Side conversations.
93-96	They take orders for sandwiches. Tape ends.

Summary of December 19, 1997

Minutes	**Summary**
0-3	We see the students sitting at a table and eating. Left to right, they are Ankur, Romina, Jeff, Michael, and Brian. Researcher 1 asks them, while they are waiting for sandwiches, to say what they had been doing in the other room with their teacher. Romina says they were using Sketchpad. Jeff says they started with $y=x$, a straight line, then $y=x^2$, a parabola, then $y=x^3$, and they got "that thing" (he gestures). Romina draws the curve. Jeff says they went to x^4, which was a flatter parabola, and then x^5. Researcher 1 asks them to explain. Romina goes to the board and draws. Jeff says their teacher had the SketchPad set up so that there were dots on the side. Each dot controlled something, like coefficients of the terms in the polynomial.
3-6	Romina says this is related to their work on functions. They keep finding differences until they get a linear function. She shows an example. If the first set of difference is linear, the function is a parabola; if the next set of differences is linear, it's x^3. Romina says x^3 is a graph that looks "like a squiggly S.".
6-9	Jeff says x^4 is a parabola that hugs the x-axis more because it's a fraction times a fraction. Romina says the even ones are parabolas and the odd ones are like the S curves. Researcher 1 asks if Brian is studying this in his class. He says no. Jeff says that their teacher knows they have already had something like this. Researcher 1 says Researcher 6 and Researcher 2 cut the tape that they watched (of themselves as fourth graders working on a towers problem). Romina says they were doing the same thing then as now, but they know algebra now.
9-12	Researcher 1 reminds them where they left off last time. Last time, Brian said something about towers. Researcher 1 asks what they figured out last time. Michael says it was how many pizzas can be made with a certain number of toppings. Researcher 1 asks how many choices there are with 20 toppings. Michael says 2 to the n-1 or n+1 "or something like that." Jeff and Ankur say it's 2^n, so for 20 it's 2^{20}. Researcher 1 asks if they believe it. They calculate 2^5 and agree that it's 2 to the number of toppings. Researcher 1 says that's connected to the binary numbers, and she asks if they can all talk about those connections. They nod. Researcher 1 asks if that's connected to building towers. If no, why not, if yes, in what way?
12-15	Romina says, "didn't we say it was different?" Researcher 1 says

Minutes	Summary
	they can talk about it while they eat. Ankur says the result is the same but other things are different. Green peppers and onions are the same as onions and green peppers, but blue-red and red-blue are different. But "something cancels out so the answer is still the same." Researcher 1 asks them to convince her with something very explicit as to whether the problems are the same or different. "Just don't give me a general argument... Imagine I'm a sixth grader." Sandwiches are distributed.
15-18	Researcher 1 says she'd like to see them prepare a presentation to themselves as sixth graders as to how the problems are alike or different and in what way. Jeff asks what can you do with pizzas that you can't do with towers? Brian says, "nothing." Michael says with three different color towers, his chart would be 0, 1, or 2, instead of just 0 or 1. Brian says, "everything we've done in our life has something to do with towers. … Maybe not directly, but it will somehow link back to it in like two years."
18-21	Brian says with the pizza problem, you added things to a basic pizza, but there is no one basic tower. There are two basic towers: all blue and all red. Ankur says with a pizza, you can have just pepperoni, but with the towers you can't have just one red in a tower three high. Michael says he can still use binary notation with two colors; 0 could be blue and 1 could be red.
21-24	Side conversations.
24-27	Side conversations.
27-30	Side conversations.
30-33	Researcher 1 asks about the problem of building towers n tall. She asks how many towers there are 5-tall when selecting from two colors. At first, Romina says 32, but then she says she thought they were talking about pizzas. Brian says there would be 32 towers. Jeff says they looked at this problem last week. Researcher 1 says they'll have to look at it again and be able to convince her. Michael asks how many different colors. Researcher 1 says to start with two colors.
33-36	Jeff asks, "and how many do we get for 2-topping pizzas?" Romina says 2^2, which is 4. Jeff and Romina start to enumerate as Michael explains to Brian. Michael says, "you can see the pattern." Michael says it's 2^n. "That's with two colors. If you have 3, there's a problem." Michael says the towers problem is the same as the pizza if the only variable is height. Jeff says, "if we start to change color, what would that be like changing on the pizza?" Ankur says,

Minutes	Summary
	"wouldn't that be like changing the 2?" Jeff says, "changing the height would be like changing the number of toppings. Changing the color would be like changing... what?" Jeff says changing the height of the tower from 2 to 3 is like changing the number of possible pizza toppings from 2 to 3. Michael says changing colors (from 2 to 3) is like a half a topping ("like chopped up"). A 0 would be no topping, and a 1 would be a chopped up topping, and a 2 would be a whole topping. Romina says height is supposed to be the variable. Jeff says they're just trying to explain what it would be like to change the colors. Michael says towers can be related with the pizza. Jeff says with pizza, there's only one variable – it's on or not on.
36-39	Jeff says the color doesn't coincide with pizza. Michael reiterates: If there are two different colors, it would be the same as the pizza. We see that Brian is writing. Jeff and Ankur ask him what he's doing. Brian says he's figuring out what it would be for three colors. He says it's 100 times more. Ankur asks if he has them all. Brian says no; he is working on 3-tall towers choosing from 3 colors. Romina asks why they don't do the height one first. Jeff replies that they can pretty much explain that one. Jeff says he'll use what he has and then Mike will come in with his binary presentation. Romina says that's just one case. Jeff says they'll go to 3 high. They write. Romina tells Jeff she got 8, which is what they were supposed to get. Jeff continues to work. Michael tells Jeff there's going to be a pizza with all 3 toppings. Jeff adds the plain one.
39-42	Michael says for 3 colors, there are 9 possibilities, but he's not 100% sure he has them all. He says for each place, there are 3 colors; 3 times 3 is 9. He adds that for four colors, it would be 16. Ankur says to Romina, "isn't that the factorial thing?" Romina nods. Ankur repeats, "that's the factorial thing." Michael says it's not factorial, because factorial is used in cases where once you use something, it's not available again. Brian interrupts to ask if they're doing 3 high with 2 colors. Brian says, "why wouldn't that just be 3^n?" The reason is that with 2 colors, the answer is 2^n. Jeff says it would be 2^3. Michael says with 2 colors, 2 high, the answer would be 4. With 3 high, it would be 2 times 2 times 2 = 8. Jeff agrees that the answer is 8. Michael explains that factorial is used in problems such as arranging the letters in a name. "But if you can use each letter as many times as you want, that's different." Ankur agrees. Michael says for 4 colors, it would be 4 times 4. Jeff asks if it would be 4 times 4 times 4 for 3 high. Michael says yes. Ankur says you're

Minutes	Summary
	making the first number the number of colors.
42-45	Ankur says with 2^n, the 2 represents the number of colors, and if you do 3 colors, that's 3 to the second or third. Ankur says the big number is the number of colors and the second number is how tall. They discuss what to call the numbers. Ankur says the exponent is how high. Michael asks if everyone understands. Everyone says yes. Brian has a proposal. He says it's 2 *n* if there's 2 different toppings, 3 *n* if there's 3 different toppings, 4 *n* if there's 4. Brian says *n* stands for toppings. Jeff tells Brian he has to write it as 3^n.
45-48	Brian says *n* equals the toppings. "If you have one topping, you put 1 for *n*, you have one pizza." But Jeff says there are two possible pizzas when there is one topping to choose from. Brian says, "you don't count the cheese, you add 1." Michael and Ankur disagree. Jeff says if *n* is 1, 2^1 is 2. Michael says that with 2^n, 2 is the number of toppings. Ankur says no, *n* is the number of toppings. Ankur says the 2 just worked out. Brian says the 2 has to stand for something. Michael says 2 is either 0 or a 1; "you could have a topping or not." Jeff says the first number is the number of colors and the *n* is the height. Ankur says with the pizza problem, the exponent would be the number of toppings. Michael tries to figure out how to equate three-color towers to pizza problems; he says you could have a square mushroom or a triangular mushroom, but it would still be a mushroom. Ankur says you would use 0, 1, and 2 if there were three choices about toppings.
48-51	Michael says Brian related his work to the pizzas. Romina asks which is height. Jeff says *n* is height, and *n* is toppings in the pizza problem. Jeff says there are 16 possibilities for 4 colors 2 high. Michael says they learned that from Mr. Poe. Ankur says, "he put out those places and then he multiplied them." Brian asks if it's correct that there are 16 towers if there are 4 different colors. Researcher 1 suggests that Romina go to the board. Romina writes 2^n. Jeff asks her to explain the 2^n. She says the 2 represents topping or no topping. Jeff says they didn't understand last time why there was a 2.
51-54	Jeff says the 2 means on or off, and Romina says *n* is the number of toppings. Researcher 1 says she understands, but she asks them to be more explicit: show how there are 8 choices when there are 3 toppings. Michael goes to the board. He writes "mushrooms" and "pepperoni" on top of two columns. He says 1 stands for a topping and 0 stands for no topping. He writes 00 for a pizza with no

Minutes	Summary
	toppings, 01 and 10 for one-topping pizzas, and 11 for a pizza with both toppings. Researcher 1 asks about 3 toppings. Michael adds another column.
54-57	Michael says they can relate pizzas to the tower problem. He says with two colors, the towers and pizza problems would be exactly the same. Instead of toppings, the columns would be the height of the tower. Michael says 0 is red and 1 is blue. Jeff said they were having trouble figuring out the relationship when there are three colors for towers. Michael says they figured out today for the first time that 2 means toppings or no toppings.
57-60	In discussing two-tall towers selecting from 3 colors, Michael says they would now have 0, 1, and 2. Jeff says there would be 9 possibilities: 3 times 3. Michael explains that there are 3 possibilities for the first color and 3 for the second. Researcher 1 asks why multiplying works. They laugh. Michael draws lines to indicate the possibilities. Ankur is asked to explain. He says there are 3 possibilities for each of 3 possibilities. Brian agrees; he says he was the one who first brought it up.
60-63	Researcher 1 says she has to give a talk to the MAA, and she'd like them to write up what they've done today. She wants to show their thinking to the MAA.
63-66	They talk about mentioning this program in their college applications. Researcher 1 says they've done a mapping of problems that are equivalent in structure. Some of her graduate students can't do that. They've been doing this since second grade; today they carried these ideas another step. There may be bits of this that some of them don't understand as well as they might. Writing helps them understand.
66-69	Researcher 1 says they should do their own write-ups first and then they might take part in an interview. "Part of the learning is explaining it to someone else."
69-72	Researcher 1 writes $a+b$ and $(a+b)^2$. She asks what it is. They say it's $a^2 + 2ab + b^2$ She asks what $(a+b)^2$ means. Jeff says you could be trying to find the area. Jeff draws a square with a side of $a+b$. Michael draws the lines to show a^2, b^2, ab, and ba.
72-75	Michael talks about learning binary notation. Jeff recalls some games they used to play in math class: the elevator game and the mail game. Researcher 1 returns. Jeff says Michael remembered how to extend the lines (on the drawing of $(a+b)^2$) to show regions, which Jeff shows. Researcher 1 asks if ab and ba are the same

Minutes	Summary
	regions, and Jeff shows how the regions are the same, but rotated. Jeff says it's as if one is 5 by 3 and the other is 3 by 5. Researcher 1 asks them if that reminds them of anything they've seen before. Researcher 1 asks them to look at $(a+b)^3$. Ankur says that's the 3rd dimension.
75-78	Jeff asks for blocks. Researcher 1 offers them Unifix cubes. Researcher 1 asks them to build $(a+b)^2$ too. Some of them start building.
78-81	Jeff builds and Michael draws a diagram.
81-84	Michael says he understands what Jeff did. Jeff calls Researcher 1 over. She asks Romina to explain. Romina explains the diagram for $(a+b)^2$. Researcher 1 asks how that works for cubed. Jeff starts to work on that. Researcher 1 asks what all the terms of $(a+b)^3$ would be. Michael continues with his diagram as the others work on building a model with Unifix cubes.
84-87	They continue to build. Jeff asks Ankur to write out what *ab* times *ab* times *ab* is. He then corrects *ab* to $a+b$. Ankur says he will just multiply $a^2+2ab+b^2$ times $a+b$.
87-90	Ankur shows his work to Jeff. He writes $a^3 + 3a^2b + 3a^2b + b^3$. Jeff starts to relate it to the three-dimensional model that he is building. Romina points out the a^3. They discuss their model.
90-93	They get additional colors of Unifix cubes. They continue to build.
93-96	They continue building.
96-99	The camera moves over to Brian, who is writing. Ankur asks Brian what he's doing. He says he's making a diagram.
99-102	Jeff and Romina continue trying to put the Unifix cube groups together. Brian is building something else using all yellow cubes.
102-105	Jeff has to leave. Romina and Brian discuss Brian's work. Brian says he picked apart the cube and did each wall separately. He turned it so that ab^2 would match. Researcher 1 says she's not convinced about how the *ab*'s work.
105-108	Researcher 1 says there are four regions and she sees that two of them collapse. She talks about where the $3ab^2$ and $3a^2b$ came from, before the numbers collapsed. Michael said it was one ab^2 and $2ab^2$; they are alike so you can add them together. Researcher 1 asks if they can show a model. Ankur shows what $a+b$ times $a+b$ equals. Researcher 1 says they build it up from four regions; she wants to see the regions. She asks Michael to talk about the regions he has drawn. Michael describes them.
108-111	Michael continues to explain by referring to the cube they have built.

Minutes	Summary
	Romina describes a^2b and b^2a to Michael. Researcher 1 asks if they know about zero exponents. Michael replies, "it's always 1." Researcher 1 says $a+b$ to the first power is $a+b$. She describes $(a+b)^2$ and $(a+b)^3$.
111-114	Researcher 1 asks them about $(a+b)^4$. She asks them to guess the first and last terms. They say a^4 and b^4. Researcher 1 asks if there's a way to figure it out exactly; she asks them to think about what it might be before they do it exactly. Michael says there will be 12's in there. Brian continues to work with cubes. Michael writes it out and shows his work to Researcher 1. Ankur suggests he can combine things. Researcher 1 says she will leave that for them to talk about. She asks them to think about $(a+b)^8$.
114-117	Researcher 1 asks them to think about making a model. She asks Brian to talk about what he was doing. He says he drew a cube, and then he took each face and tried to construct each part. He says it's too confusing, but Romina understood.
117-120	After a side conversation, they discuss Brian's cube.
120-end	Researcher 1 thanks everyone. They clean up. The tape ends.

Summary of January 9, 1998

Minutes	Summary
0-3	As the tape starts, we hear Researcher 1 asking if they understand the question. She says, "You have to convince me that you found them." [She is talking about the work they did in fourth grade, when they found that when you are building 4-tall towers selecting from two colors, there are 10 towers with exactly two red blocks.] The camera focuses on Michael and Ankur. Ankur says, "She said only two red and three yellows. She didn't say three red and two yellows." Michael asks if it could be two yellows and three red. Researcher 1 says she'll accept the answer to either question. Ankur says the answers are 10 and 20. Researcher 1 asks how they would convince her that they found them all. They show their paper on which they enumerated the possibilities using a 1 to indicate red and a 0 to indicate yellow. Michael says, first you put a one on the top, and then you put a one in the second space, etc. They enumerate their other cases. Ankur asks if Researcher 1 is convinced. She says she is. Michael asks if the other group (Brian, Jeff, and Romina) is doing the same thing.

Minutes	Summary
3-6	Romina says, "You guys proved it already?" Michael answers yeah. Researcher 1 tells Michael his notation is very powerful. Michael says he can use it [binary notation] for any problem she gives them. We hear Jeff and Brian talk about working on the problem by finding opposites. Researcher 1 asks Ankur and Michael to generate a problem. Ankur poses the problem of finding how many 4-tall towers there are when selecting from 3 colors, when there must be one of each color in each tower. Michael and Ankur start to discuss the problem as Researcher 1 moves over to Jeff, Brian, and Romina. Brian is saying there are 20. Romina asks Michael and Ankur why the answer is 10. They don't answer. Romina asks, "Are they on a different problem already?"
6-9	We watch Ankur and Michael work on Ankur's problem. Jeff asks, "Can we say we got it and then go to another one?" Brian asks how they can get the next one if they didn't get the first one. Researcher 1 tells them they have to convince her that they have them; they can't just say they told me there are 10. Romina says they didn't write out everything. Brian says it's going to be the same thing: The *y*'s could be the *r*'s and the *r*'s could be the *y*'s. Having two *r*'s and three *y*'s is the same as having three *r*'s and two *y*'s. Jeff says, "What's the problem?" Romina and Brian laugh. Researcher 1 says you are building towers 5 tall, with two colors to select from; how many of these towers will have exactly two reds and three yellows? How do you know you found them all? She says, "That was the problem that I saw on the tape when you were in the 4th grade." We hear Michael and Ankur talking as Romina's group talks quietly.
9-12	Jeff, Romina, and Brian discuss binary notation. They talk about finding the opposites of some patterns. Brian says as long as they're 5 high and there are three of one color and two of another, there's going to be 10. Romina asks why. Brian builds some towers.
12-15	Brian says the color doesn't matter; it's the height that matters. Researcher 1 is speaking to Michael and Ankur. They are talking about 4-tall towers selecting from three colors. Ankur says 3 to the 4^{th}. Michael says 4 to the 3^{rd}. Ankur says, "Remember when we have two colors and it was five high, it was 2 to the 5^{th}, right?" Michael says OK. Ankur says, "The *n* represents the height of the thing." Michael says, "Or the amount of toppings on the pizza." Ankur agrees. Michael agrees with 3 to the 4^{th}. He says, 9, 27, ... 71. Ankur corrects 71 to 81. The camera switches over to Romina, who is saying, "I don't know the binary system." We hear Ankur and

Minutes	**Summary**
	Michael discussing Ankur's problem.
15-18	Romina asks Jeff what he's doing. He says that he's thinking. Meanwhile we hear Ankur and Michael at the other side of the table, working on Ankur's problem. Jeff says, "Five high with zero of the other color, five high with one of the other color, five high with two of the other color. That's 10." He asks Romina if she understands. Brian says there are two towers that have no opposites.
18-21	Brian shows his towers: a group of five with one white on top and another white in each of the four other positions. Brian says there is 4 in this group, plus the opposites, plus the 2 that have no opposites; that's 10. Romina says there has to be an equation. Jeff asks Brian to explain what he did. Brian says there are 4 original positions that have opposites, that's 8, plus the 2 that don't have opposites. He says, "If you try to make the opposite of that, you'll have three whites." Romina asks Jeff to tell what he did. Jeff explains to Brian. Brian says, "That's a nice little thing." Jeff calls Researcher 1 over.
21-24	We hear Michael and Ankur talking, but the camera is on Romina and Brian. Romina and Brian are discussing the enumeration of 5-tall towers. Researcher 1 returns. Jeff says, "If you have one color in it, one color times the 5 different spots it could go in is 5. Then they have the opposite colors. So it would be 10. There are 10 with 2 of one color, flipped over would give you 20. And then all reds and all yellows is 2, giving 32." Researcher 1 says they answered more than she asked. Jeff responds that they tried to find some kind of connection. Researcher 1 says it connects to what she wanted to talk about today. Researcher 1 says Michael and Ankur should put their problem on hold, but Michael says no. Researcher 1 asks Ankur to tell the problem to the others. Ankur says you have four high, three colors, you have to use at least one of each color in each tower. Jeff asks what the answer is.
24-27	Michael says they have an answer, but it's not working. Ankur says the answer is 72. Romina says to use 1s 0s, and Xs to enumerate the possibilities. Romina and Jeff start writing; Brian builds towers. They discuss how to enumerate the towers.
27-30	They discuss Ankur's new answer (54). They discuss ways to write 0s, 1s, and Xs.
30-33	Brian says take the number of spaces times the number of colors and multiply that by 4. Brian says you could move that blank into 4 different spots. Romina says there might be doubles, but Brian says there can't be that many doubles. Brian says to Michael, "36, I can

Minutes	Summary
	beat that." Ankur says, "We definitely have it now." Michael says, "Don't be too sure." Ankur says, "I am sure."
33-36	Brian asks Romina about her notation. Romina asks if they cancelled out 54 as a possibility. Ankur says yes; the answer is less than 54. Someone asks about a formula. Michael says there's no formula.
36-75	They work on Ankur's problem.
75-78	Researcher 1 tells the group they will get together with a mathematician from Utah, Researcher 5, who has seen a lot of their tapes and used them in his classes; he will be there on Friday, February 6. Jeff says if he's playing, he'll have to leave at 5. Jeff leaves. Researcher 1 thanks him. Michael asks Romina to explain the 36 (answer to Ankur's problem) again because he wasn't paying attention before. Romina goes to the board. Michael asks what 1 0 and X are. Romina says they're three different colors. She says the 4th color has to the same as one that's already there. Michael agrees.
78-81	Romina draws boxes on the board. She starts to explain her solution. Michael tells them to talk among themselves while he finishes something. Ankur says he doesn't want to write on the board; Romina volunteers to do it.
81-84	Romina goes to the board. Romina writes columns as Ankur says 11 10, 2220, 3330, and then 1101, 2202, and then "do the same thing but the zeroes are in the second." Romina continues to write columns with the 0's in new positions. Ankur says 1=red, 2=blue, 3=yellow, and 0=any one of the 3.
84-87	Ankur explains his reasoning for his answer of 27. [This is a partial answer to Ankur's problem; he is listing the towers that do not have at least one of each color.]
87-90	Romina writes 11, 22, 11, 33 and more as Ankur dictates. He says 1221, 1331, 2112, and asks if they see where this is leading: he notes that these are 18 more towers that don't have one of each color; 27 plus 18 = 45 that don't have 3 of each color. (These are 2 of one color and 2 of the other color.) Researcher 1 asks if they have all the possibilities and Ankur says yes. Romina says it's an extended version of what her group did. Researcher 1 asks Michael what he thinks. He says he believes it. Researcher 1 asks where his notion of variable got him in trouble. Ankur says, "We made a mistake in the first thing we did." He asks Michael to explain.
90-93	Michael goes to the board to explain where he and Ankur had a problem; there were some doubles that they did not find right away. He says Romina's group's explanation was better.

Minutes	**Summary**
93-96	Researcher 1 asks if that same reasoning would work if you were doing towers five tall. Ankur says he really doesn't care. Romina asks if anyone else besides this group knows about towers. Researcher 1 says yes; a lot of people are using these tasks throughout the country and other parts of the world. She says this is called discrete mathematics; it is used for networking problems, for example. In discrete math, there aren't always simple formulas; sometimes you have to think through the problem. Ankur says, "we start with simple formulas and then we review it and make it a little better until we finally get it." Researcher 1 says that these are called combinatorics problems.
96-99	Researcher 1 says when she asks about towers five tall, exactly how many are two red, she is asking how many combinations you can pick of two of a particular kind with a total of five. In this case there are red and not red. Researcher 1 writes ${}_5C_2$ and asks if the students have ever seen it. She says this is called the combination or the selection of 2 of a particular kind from a set of 5 of them. She shows some other ways of writing "5 choose 2." These are found in college or high school math books. Researcher 1 asks how many towers have exactly three red cubes. Ankur says 10. Researcher 1 asks why.
99-102	Ankur says that in the one they already solved (exactly two red), there are three that are not red. Researcher 1 asks more questions about the symmetrical nature of the answers. She writes numbers from Pascal's Triangle on the board and asks them to relate those numbers to the towers problem answers. Researcher 1 asks why a 5-tall tower with four reds is the same as a tower with one red. Ankur says because it has exactly one block that is not red. Researcher 1 asks how many 5-tall towers there are. They say 32. Brian says, "That's the first time you gave us the answer." Researcher 1 says, "I didn't give the answer; I just summarized what you did." Researcher 1 says, there was one time in the 4th grade; they went through a whole argument of this particular case. Ankur says, "When we learned factorial, this was like every problem." Researcher 1 says it's important to look at how you think about it. These students are not just moving symbols around, the way a lot of people do. They have to be convinced; they have to make sense of this themselves. Researcher 1 says she has something else for them to think about. She starts writing on the board powers of $(a+b)$. Romina says they could probably work it out.

Minutes	Summary
102-105	Researcher 1 writes $(a+b)^2$ is $1a^2 + 2ab + 1b^2$. She writes $(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$. She asks how she can do this so fast. She writes $(a+b)^1$ as Ankur says, "We never learned this." She continues with $(a+b)^0 = 1$. She says $(a+b)^1$ is one *a* and one *b*. She writes 1 1. She says $(a+b)^2$ is $1a^2 + 2ab + 1b^2$. She writes 1 2 1. Researcher 1 pauses and Romina says 1 3 3 1 and adds that they have seen this before. Ankur says, "You're adding what's in between." Ankur gives the next row, 1 4 6 4 1. Romina says they did this in Mr. Poe's class. For the next row, Ankur says, "1 5 10 ... that thing." Researcher 1 says that's something for them to think about: Why does that work? What does that have to do with anything? There is a pause. Researcher 1 says the question is, what does it all mean? She says they know what $(a+b)^5$ means, and they have a way of thinking about 2^5. Researcher 1 says that this particular relationship is called Pascal's Triangle. Romina and Ankur say they remember learning this. Researcher 1 says you can have a first-grader generate Pascal's Triangle. They know how to do simple addition. But the question is to explain the relationship.
105-108	Researcher 1 tells them that they have been working on this for a long time. Romina asks if this is what they have been doing today. Researcher 1 says that's some of it. Ankur says it's all of the things that they learned for the past eight years combined into one thing. Researcher 1 goes to the board and tells them that what did they did is a piece of Pascal's Triangle. She circles the first number 10 from 1 5 10 10 5 1 and what it meant. Ankur and Brian respond that it's two reds. Researcher 1 tells them to think of where the 10 came from – it's not just a number; it has to mean something. Ankur says the 10 came from the 6 and the 4. Researcher 1 asks what problem they were solving when they produced the number 10 earlier. Ankur says a tower with 2 red and 3 of another color. Ankur says that in 1 4 6 4 1, the 1 represents all red. The other 1 represents all yellow, for 4-tall towers. Researcher 1 says, "where did the 4 come from, where did the 6 come from?" She draws lines from the two 3's in the row above to the 4 and the 6.
108-111	Romina says this would be good training for an actuary; she discusses what an actuary does. Researcher 1 asks them to write up the solution to Ankur's problem.
111-114	Researcher 1 says they can give Ankur's problem to their teachers. Brian responds, "My teacher won't know what I'm talking about." Ankur says, "I bet you half the teachers in this school won't be able

Minutes	Summary
	to solve it." Ankur restates his problem, and they discuss the importance of proper wording.
114-117	Researcher 1 says the problem is, "You haven't told how many colors to select from. Does it matter?" Romina suggests saying, "How many towers can you build 4 high with a choice of 3 colors, and having all 3 colors in each tower?" Researcher 1 says that is different from what Ankur was saying; careful language is important. Brian says he never thought about how one word could change the whole thing.
117-120	Researcher 1 asks them to write up Ankur's problem. She thanks them and tells them they're doing beautiful mathematics.

Summary of February 6, 1998

Minutes	Summary
0-3	Researcher 1 introduces Researcher 5 to Jeff, Romina, Ankur, and Michael. She tells the students that they are to help Researcher 5 understand what they have been doing. Researcher 1 leaves. Researcher 5 asks what they have been doing. They respond, "Ankur's problem." Romina says that the problem is to make 4-tall towers selecting from three colors, and using all the colors; one color will be used twice. Jeff gives an example: red, white, blue, and then red. "How many towers can you make?" Researcher 5 repeats the question and asks, "How do you know?"
3-6	Romina draws a diagram. Jeff says, "1 represents a color, 2 represents a color, 3 represents a color." Researcher 5 says if there are two 1's, it means the first two blocks in the tower are the same color. Researcher 5 adds, "it's one of three possible same colors." Romina says yes; she adds that there are six different possible positions. Researcher 5 asks how she knows. Jeff says, "you can see there's no other place to put two 1's besides what's on the paper." Researcher 5 asks how to see that. Ankur responds by enumerating all the possibilities with the 1 in the first position. Then he enumerates the possibilities with 1 in the second position and 1 in the third position. Researcher 5 asks if this is like anything they know from somewhere else. Jeff responds, "Rutgers?" He says they've been doing things like this for a long time. Romina says they organized the 2-color towers. Researcher 5 asks if they used that organization right away in this problem. Jeff says that when you do

	something a long time, "you have a certain way of thinking." When you see a new problem, "you try to do it the way you know how to do it." Romina adds, "it's the easiest way to organize it." Romina continues with the problem.
6-9	They describe Romina's solution to Ankur's problem. Researcher 5 says he's sure there are not more than 36. He asks if they are sure they haven't counted anything twice. The students say yes.
9-12	Ankur says if there's not any duplicate in the first case, then there won't be any duplicates in the other cases either. They discuss variations of Ankur's problem.
12-15	Researcher 5 asks if they have thought about towers 5-high in 3 colors, and all the colors have to appear in each tower. Romina draws a diagram and they discuss the solution.
15-18	Michael says he has to go to his job. He leaves. The others continue to discuss the new problem.
18-21	They discuss how to organize the answer to ensure there are no duplicates.
21-24	They continue to work on the variation of Ankur's problem.
24-27	Romina asks, "could you explain the problem again?" Researcher 5 says, "we're kind of negotiating the problem." They continue to enumerate possibilities using codes of 1's, 0's, and X's.
27-30	They continue to work on the variation of Ankur's problem.
30-33	They continue to work on the variation of Ankur's problem.
33-36	They continue to work on the variation of Ankur's problem.
36-39	They come up with an answer of 90. Researcher 5 says he's convinced. He asks about connections to other math they've done. Ankur recalls that they did something with binomials, something like x+1 times x+1. Jeff says they made squares. Researcher 5 asks what happens when you square x+1. Ankur says it looks like x^2+2x+1. Researcher 5 asks what they found, and Romina says, "nothing." Researcher 5 expresses disbelief.
39-42	Researcher 5 asks them to explore x+1 times x+1. Jeff draws a square with side x+1, but Romina says they used *a* and *b* the last time they did it. Romina says they moved to towers with Ankur's problem, and Jeff says they did pizza problems with Mike's binary notation.
42-45	Researcher 5 asks how all these things can come up at once (finding areas, binary, towers, pizza). Romina says, "combinations." Ankur says that a pizza problem with five possible toppings is like a towers problem with possible colors. Jeff talks about towers of various heights. For pizzas, Jeff says the answer is 2 to the number of

toppings. Romina agrees, but Ankur says, "plus 1" (for the plain pizza). Researcher 5 asks if the number of possible pizzas with 3 toppings is 2 times 2 times 2. Jeff says it is, and for 2 toppings it's 2 times 2. Researcher 5 says he's seeing the same number coming up. Is that an accident? Romina contrasts towers and pizzas. She says in the tower problems, you have a solid height; for example it could only be three high.

45-48 Romina says with towers, you have the colors red blue and blue red. But with the pizzas you can have any height, because sausage and pepperoni and pepperoni and sausage are the same thing. Jeff says that if you have 3 choices, you can have just pepperoni on a pizza, but you can't have just one cube when you're doing 3-tall towers. You can repeat colors on a tower, but on a pizza you can't have pepperoni peppers pepperoni. Researcher 5 asks if they mean that there are more colors, but some of the combinations have collapsed. Jeff agrees; you can't have peppers and peppers. Researcher 5 says he sees the same numbers coming up, powers of 2. Is that an accident? Romina repeats, "combinations." Ankur says, "1+5+5+10+1, remember that?" Jeff says they came across this when they saw that they were powers. Jeff said they try to find an easy way out and try to find connections. Researcher 3 calls Researcher 1 back. Researcher 5 says they're talking about pizzas and towers and rectangular diagrams and cubes. Is it possible that all these things connect together? The students say yes, and they are showing how powers of 2 are related. Romina says there's a way to write combinations with a *C*. Researcher 1 says she will repeat her previous explanation about the notation.

48-51 Researcher 5 asks if it is an accident that you get powers of 2. Researcher 1 says she will review notation for Jeff's sake. First she talks about the video from fourth grade, where they convinced her that there were 10 five-tall towers with exactly two reds. Researcher 1 says that last time, she told the students, "there's a lot of ways to write this; it's called combinations."

51-54 Researcher 1 shows different forms of combinatorial notation. She says that when they were 4th graders, they were able to convince her there were exactly 10 towers with exactly 2 red. Romina asks how it's connected. Researcher 1 writes $(a+b)^2$.

54-57 Ankur says 1 a^2 plus $2ab$ plus b^2. Researcher 1 says 1 b^2; it's good to remember that there's a 1 in front of the b^2. She asks them to think about those numbers and letters. Ankur says it's the same problem as the tower; it's two high with two colors. Researcher 1 asks what the *a*'s and *b*'s mean. Researcher 5 asks, "what about *ba*?" Ankur says red is *a* and *b* is blue, so *ba* would be blue-red. Researcher 1 asks how they would be different. Ankur says one would be red on top, blue on the bottom, and the other would be blue on top, red on the bottom. Researcher 5 says there are two lines of reasoning about pizzas and towers. Ankur says a pizza with five toppings is a tower five tall, and you can have a pizza with just one topping, but you can't have a five-tall with just a red cube.

57-60 Jeff says the pizza is like a tower with five different colors with no restriction on height. Ankur says the order of the toppings doesn't matter; pepperoni and sausage is the same as sausage and pepperoni. Researcher 5 says the order matters with towers. "But how come we keep getting the same numbers? … They seem to be about games that have different looking rules." Ankur says the basic rule is the same. Researcher 5 asks what the basic rule is. Researcher 1 says they will leave and let them think about that. Romina asks Researcher 1 to restate the problem. Researcher 1 says they can relate 5-tall towers to row 5 of Pascal's Triangle. She asks them to fill in the missing part of $(a+b)^5$ and then think of what this means in terms of towers and pizzas. Is there any connection or relationship? Researcher 1 and Researcher 5 leave.

60-63 They work out $(a+b)$ to the 3rd, 4th, and 5th powers. Ankur writes the numbers in rows 4 and 5 in Pascal's Triangle.

63-66 Ankur tells Romina how to add to get the numbers in Pascal's Triangle. Ankur says if you add up the numbers in the 5th row of Pascal's Triangle, you get the 6-tall towers. Romina asks, "what's the question?" Ankur says to relate to the pizza problem. Ankur explains 1 6 15 20 15 6 1 row in terms of towers (no reds, 1 red, 2 reds, up to 6 reds). Jeff asks how this relates to pizzas.

66-69 Jeff points to the 1 and says, "this is the plain pizza." They relate the numbers in turn to one topping, two toppings, up to "the supreme." Researcher 1 and Researcher 5 return. Ankur says they found out the answer, but there is a discussion for a moment as they clarify whether row 6 is related to $(a+b)^5$ or $(a+b)^6$.

69-72 Ankur says $(a+b)^6$ is the same as 1 red, 2 reds, etc. Jeff says that relating the triangle to the pizza problem helps them understand "why we never needed the plus 1." Romina says zero toppings is zero red. Romina says the rest of the numbers go with one topping, two toppings, etc. Researcher 1 asks how did it become 2 to the 5th? Ankur says that's how it breaks down. Researcher 1 and Researcher 5 say they follow.

72-75 Researcher 5 asks about the 6 (the number of towers 6 blocks high with 1 red block). How does 6=1+5 work out? The 1 is the 5-tall towers with 0 reds, the 5 is the 5-tall towers with 1 red, and the 6 is a 6-tall tower with 1 red. Ankur says the 6 represents one red in the six different positions in the tower (at the bottom, at the second from the bottom, etc.). He says you can add a red to the top of the 1 (the 5-tall tower with no reds), and it will still have one red. Researcher 5 says the 1 counts 5-tall towers that have no reds. Ankur agrees. Researcher 5 says the 5 counts the 5-tall towers with 1 red. Ankur says you get the 6 by putting a block on top of each tower. Researcher 5 asks what color; Ankur replies, "not red." Ankur reiterates: put a not red on top of the five with one red, and then put a red on top of the one that has no reds. Researcher 5 says you have two sets of towers and you treat them slightly differently. Ankur says yes. Researcher 1 asks Romina to explain for 10+10=20.

75-78 Romina says you're adding another block so you're doubling the number of towers. Researcher 5 asks what the 10's count and what the 20 counts. Ankur says 10's show two of one color and three of another; Jeff adds that the other 10 is three of one color and two of another – so they're opposite. Jeff says that's why they're the same number. Ankur and Jeff say on the top of each one, you could put a red or a blue. Romina says the new group will all have three reds. Researcher 5 asks, "the first 10 has two reds and red blues, if you're counting reds?" Ankur says yes. Researcher 5 adds that the second 10 has three reds and two blues, and the 20 counts the ones that have three of each. Ankur says you add a red to the ones that has two reds and three blues and a blue on top of the others. Researcher 5 asks Romina if that's what she's saying too. She agrees. Researcher 1 says, "how far can you go with block towers?" Ankur says, "pretty far."

78-81 Researcher 1 asks if Researcher 5 can visit again. They agree to schedule another visit. Researcher 1 says they did well, and she's impressed.

Summary of March 6, 1998

Minutes	**Summary**
0-3	As we see the test pattern, we hear a discussion of how the cameras and students will be positioned. We see Ankur, Romina, and Jeff sitting together at the left and Michael sitting at the end of the table.

Minutes	Summary
	Michael asks what they did last time [February 6, 1998]. Romina says they just explained what they did the week before [actually, it was January 9, 1998.] and then they explained it to Jeff. Jeff says they did the thing when you multiply *a b* squared. He says you get "1 2 2 1." Michael says, "Didn't we do that the week before?" Jeff says they talked about it again. Researcher 1 asks what he means by "multiply *a b* squared." Romina says it's $a+b$; it's the "whole pyramid thing." Jeff starts to write but hands the paper to Romina. Ankur dictates 1 *a* squared, 2 *ab*, 1 *b* squared. Romina writes.
3-6	Jeff writes a few rows of Pascal's Triangle. Michael says he remembers. Jeff explains to Michael how to generate rows of the triangle. Researcher 1 asks Jeff to write four rows of the triangle. Jeff does so. Researcher 1 asks Michael if he remembers what that triangle is called. He doesn't. Researcher 1 asks if anyone remembers. Ankur says it's Pascal's Triangle. Researcher 1 adds a 1 to the top of the triangle Jeff drew. Researcher 1 asks if they all understand what $a+b$ has to do with this. Michael says the coefficients of $a+b$ squared are in the second row of the triangle. Researcher 1 says $a+b$ is called a binomial because there are two terms; what they've done is expanded the binomial. She says when you expand $a+b$ to the third, which is $a+b$ times $a+b$ times $a+b$, you do all that multiplication.
6-9	Researcher 1 says these numbers (row 3 of Pascal's Triangle) are the coefficients of $a+b$ to the third. Researcher 1 asks if they have thought about the *a*'s and the *b*'s. Jeff says they're just the variables. She points to row 2 of Pascal's Triangle. She says this is 1; Jeff says *a* squared. She points to the 2; Jeff says two *ab*. Researcher 1 says that was from *ab* and *ba*. She points to the last 1; Jeff says that's *b* squared. She points to the third row. Romina says *a* cubed, then 3 *a* squared maybe? Michael says it's *a* squared *b*. Ankur says 3 *a* squared *b*, then 3 *b* squared *a*. Researcher 1 asks about the fifth row. Ankur says the first one would be *a* to the fifth. Researcher 1 says they're expanding $a+b$ to the fifth. Ankur says 5 is *b* to the fourth and *a* to the first. Then he says *a* to the third and *b* to the second. Researcher 1 suggests that they write this down. Jeff writes as Ankur says 1 *a* to the fifth, 5 *a* to the first *b* to the fourth, 10 *a* to the second *b* to the third, 10 *a* to the third *b* to the second, 5 *a* to the fourth *b* to the first and one *b* to the fifth. Researcher 1 asks if they all concur. Jeff says yes. Researcher 1 says it's interesting how he went from *a* to the fifth to *a* to the first. Researcher 1 asks what that row would

Minutes	Summary
	look like if you did it the other way, and are they equivalent? Jeff rewrites in order of descending powers of *a*.
9-12	Ankur says they are equivalent. Jeff finishes writing up to 1 *b* to the fifth. Researcher 1 asks how they got the numbers so fast. Jeff shows how he added. Researcher 1 says that one of the questions last time was, "have you explored why this addition works?" Researcher 1 asks them to talk this over. She says, "the issue is, this 5 means that you have 5 with 4 factors of *a* and 1 of *b*. And this 10 means you have 10 with 3 factors of *a* and 2 factors of *b*." She adds, "but when you combine these two, what are you going to get?"
12-15	Jeff asks, "*a*'s were the whats?" Romina and Ankur say *a* to the fifth. Ankur says to start with the smaller ones; they'll be easier to do than going from *a* to the fifth to *a* to the sixth. Romina writes the first row as *a* and the second row as $a + b$. Jeff says, after mentioning a few other possibilities, that 1 1 is just *a* and *b*. Jeff, Ankur, and Romina say the next one is 1 *a* squared, 2 *a b*. Ankur says to write it out. Romina writes as she and Ankur speak: 3 *a* squared *b* 1 plus 3 *a b* squared plus *b* cubed. Jeff says "if you put an $a+b$ over here," [to the side of the triangle] you can see you are multiplying $a+b$ times what was already there. Ankur says that what they have to explain is how you add two numbers to get the third number. Jeff says the reason it goes to 3 is that you're adding an extra *a*. Ankur asks why it wouldn't be *a* to the third. Jeff says there's only two *a*'s to multiply, so you can't get an *a* to the third. "It's impossible." Romina says, "we're talking about why this jumps to this, right?"
15-18	Ankur says they're looking at why 1 plus 2 equals 3 and what that means. Romina says it's like a pattern. There is a pause. Romina tells Michael he's supposed to be doing this too. He says he's thinking. Michael says there's got to be some kind of relationship. There is a pause. Ankur asks if Michael knows the answer. Michael says he doesn't know the question. Ankur says it's about adding 1+2=3. Why does that work? How are they related? There is another pause. Ankur tells Romina to write the next one, the 1 4 6 4 1. She does so.
18-21	Jeff says you had the 3 *a* squared *b* and 3 *b* squared *a*. Ankur says to write what's underneath that. Romina writes 4 *a* squared *b* plus 6 *a* squared *b* squared. They talk about what the right numbers are. Ankur says, "you're adding the numbers and keeping the subscripts."
21-24	Romina asks how does it get the 6? Ankur says 3+3. Romina says

Minutes	Summary
	they need to explain it a different way. Romina asks Michael what he's writing. He doesn't say. Jeff asks Researcher 1 to come over. Ankur says they wrote it out. Jeff asks if *a* squared plus *a* squared equals *a* to the fourth. Romina says no. Jeff asks if it's 2 *a*. He says when you multiply, you add [the exponents]; what do you do when you add them? Ankur says it's 2 *a* squared. Jeff asks how they explain keeping the thing [the exponent]. Ankur says when you're adding, you just keep it. Ankur asks Researcher 1, "When you add *a* squared plus a squared, does that equal 2 *a* squared?" Jeff says, "but it wouldn't equal 3." Jeff says 3 *a* is different from 3 *a* squared. Jeff asks if 2*a* times *a* would be 3 *a* squared? Romina says she doesn't think so. Ankur says it's 2 *a* squared.
24-27	Romina says, "it's 2 *a* squared, isn't it?" Jeff says, "but it's a 3;" he says that makes no sense. Romina says you don't just add; you have to multiply things and then add. Jeff says 10 times 10 is a hundred and 3 times 25 is 75. [He's multiplying coefficients.] Ankur says that's not right. Michael interrupts, "you want to know why that's a 3, right?" Jeff says yes. Michael says you're really putting parentheses and multiplying it out. He says the reason you get 3 is because you're multiplying one thing by *a* and another thing by *b*. "That's where the third one comes from." Jeff says they were trying to figure out why 2 and 1 gives 3 and why adding 3 and 3 gives 6. Researcher 1 says she wants to clarify; she says, "this particular 1 means 1 of something." Jeff says that's the a^2. Researcher 1 says, "this 1 a^3 and this 3 a^2b, when they go to 4, it's 4 what?" Ankur says 4 a^3b.
27-30	Ankur says this is a^2b and when you combine them, you get 4 a^3b. Researcher 1 asks why that works. Ankur says you're adding them. Romina says you're multiplying. Researcher 1 asks, are you adding or multiplying? Ankur says he meant multiplying. Researcher 1 says that she and Jeff and Michael are not following; she asks Jeff to summarize. Jeff says1 means a^3 and 3 means 3 a^2b. If you multiply another *a* into that, then that would equal the same as the 4. Michael says he knows that. Jeff works it out and gets 4 a^3b. Researcher 1 asks, going back to the original triangle, how you would explain 10+10=20. Ankur says the same way; "you just write out what they mean."
30-33	Researcher 1 says theoretically you should be able to predict the terms and where they came from. They agree. She says the particular terms they used as an example were a^3 and $3a^2b$; the

Minutes	Summary
	numbers in front of those terms have a particular meaning in terms of those terms. She asks if they can see where the meaning comes from. She said they squared $a+b$. They combined the *ab* and the *ba*. Researcher 1 says she remembers from last time that Ankur made a relationship about these *a*'s and *b*'s to building towers. She says Michael did not hear that. Ankur says that's when the other guy [Researcher 5] was here. Researcher 1 asks them to explain 3+3=6 to Michael in terms of towers. She leaves.
33-36	They get out the Unifix cubes. Jeff says *a* would be blue and *b* would be white. Ankur tells Jeff to start with 1 2 1. Jeff says 1 a^2 as he puts down a tower with two blue cubes. Then he puts down a tower with two white cubes. Ankur says to put down the two things in his right hand. [They are two identical 2-tall towers (white on top, blue on the bottom).] Jeff says this would be 1 2 1 1. Ankur says yes. Michael says, "wouldn't that be $2a$?" Michael indicates the 2-tall tower with two blue cubes. Romina says that's a^2. They have a discussion about whether individual cubes can represent powers of *a*. Jeff and Romina continue to build, with Romina and Ankur directing. Ankur says it's not really 3 a^2; it's 3 with two whites. Ankur says to make them all the same or she'll say something. [They have three identical 3-tall towers for the 2-white/1 blue case.] Romina puts the three towers in the middle. Romina says they multiply by another *a*. Ankur says they are putting a white on the top row.
36-39	There is a discussion about how to add cubes to show movement to the next row of the triangle. Researcher 1 returns and asks Michael to explain. Michael says that the group of 2-tall towers is 1 2 1. The all blue is *a* squared, the all white is *b* squared, and the two white-blue towers are $2ab$. Researcher 1 says she's confused about the two *ab* towers. She says she didn't hear Jeff say it was an *ab* and an *ab*; Jeff it was an *ab* and a *ba*. Romina and Ankur say it's the same thing. Researcher 1 says [after she has changed the white-blue to blue-white] that she can see these are all possible towers two cubes tall. She says, "I was taking Jeff literally. I think Jeff made an important point earlier."
39-42	Michael says we multiply $a+b$. He says it's like putting a blue on top of each tower. Jeff shows the new towers. Researcher 1 points out that again not all 3-tall towers are represented. [Again, they have made identical towers to represent 3 *a* squared *b*.] Jeff rearranges. Researcher 1 says, "can you tell me in tower language why 4+6=10?" There is a pause. Jeff starts to show 3+1=4. Researcher 1 says some

Minutes	Summary
	of the 3-tall towers look like duplicates.
42-45	Researcher 1 says, "it's hard for me to keep track because they look exactly the same." Jeff says, "we're going from these [three tall] to these [four tall]." Researcher 1 asks if they could say what the set would look like and why, without building it. Researcher 1 says, "we're here and we're here" [indicating the relevant 3-tall towers]. She says "this one [group of 3-tall towers] is what?" Ankur says 3 *a* to the second *b*. Jeff says no, it's *b* to the second *a* to the first. Ankur agrees. Jeff says the last one is *b* to the third. Researcher 1 asks Michael if he agrees. Michael says yeah, but Researcher 1 says she thinks Michael's not convinced. Researcher 1 asks what the next set look like, without building it. Ankur starts talking about the four 3-tall towers [3 with one blue and 1 with all whites].
45-48	Ankur says each one of the whites goes over here [to the three towers with one blue each]. Jeff says "and then you have one left over to get the 1 *b*." Jeff builds new towers and then starts to take them apart. Researcher 1 says, "you mean you get the four by taking something apart?" Romina says you're not taking it apart, you're just seeing how it goes together. Michael says each of the four towers has "something added to equal the same thing." Michael says the three towers with one blue block are going to have a white block added and the all-white tower will have a blue, and "they'll equal the same thing." Michael says, regarding how 3+3=6, one group of three will have a blue block added and the other will have a white block added. Researcher 1 says the set is going to grow to four tall, and it's going to keep two blue blocks. Researcher 1 asks the others what they think. Romina says she follows it. Ankur says you're not really taking the towers apart. Michael says you're not actually combining them. You're using the *a* and the *b* that you're multiplying by.
48-51	Researcher 1 says 1 3 3 1 is *a* plus *b* quantity cubed. She says this is 8, the number of towers you can make three tall selecting from two colors. She asks how many there are in the next row. Ankur says 16. Jeff says each one will turn into two. Ankur says each one is going to have a blue distributed to it. Jeff adds, "and a white." Researcher 1 asked if they remember the pizza problem, when Michael wrote out his binary numbers. "That was very nice," using binary notation to represent the solution to the pizza problem.
51-54	Researcher 1 asks what that has to do, if anything, with this. Ankur says it's like 111, you could add a 1 or a 0 to it. Jeff and Romina agree: instead of blue and white, it's 1 and 0. Researcher 1 asks how

Minutes	**Summary**
	you can explain that with towers, if at all. Ankur says you substitute blues for the 1's and whites for the 0's. Researcher 1 asks what pizza problem is 3-tall towers? Ankur says 3 toppings. Jeff asks what would *a* squared *b* be? Romina says it's got to be two different toppings. Ankur enumerates pizzas: plain, peppers, pepperoni, peppers, and pepperoni. He says peppers and pepperoni is the same as pepperoni and peppers. It's not clear what entries in Pascal's Triangle he is referring to.
54-57	Researcher 1 says she's not following. Ankur says the 1 is just a plain pizza at the top of Pascal's Triangle. Romina asks under what circumstances Pascal wrote his triangle. Jeff asks why people multiply *a* and *b* together in the first place. Researcher 1 says they will have more intuition when they figure out the pizza problem. Researcher 1 asks Ankur to repeat. Ankur says the top of the triangle is a 1; that's a plain pizza with no toppings, and there's only one possibility. Romina and Jeff say the 1 at the top is *a*. Researcher 1 says she thought they said that the first 1 in row 1 [1 1] 1 *a* and the second 1 was 1 *b*. Ankur says the 1 at the top of the triangle is just 1, not *a*. Researcher 1 says, "let's put that aside." In explaining row 2 of Pascal's Triangle, Ankur says the colors don't represent anything. Michael says they do; he says blue stands for topping and white stands for no topping. Ankur says, "so all the whites are no topping?" Michael says yes; the three-tall tower with all white cubes is a plain pizza with three choices. Michael says, "this [blue-white-blue] is a pizza with two different toppings without the third topping." Ankur says OK. Jeff says, that the two-tall tower with two blue cubes is "choice of 2 using 2." Researcher 1 asks if they have all possibilities. They all answer yes.
57-60	Researcher 1 asks if they have any questions. Romina asks what Pascal was thinking about. Researcher 1 says she doesn't know; his name was Blaise Pascal. There is some discussion among the students of the Pascal programming language. Researcher 1 says she will bring them reading material about Pascal next time. Researcher 1 poses a new problem. In general form, it is: when you are selecting from *n* colors and building *n*-tall towers, how many towers are there with exactly *n* different colors? Ankur says 1 is 1, 2 is 2, and 3 is 3.
60-63	Jeff enumerates possibilities for three-tall towers, using red, white, and blue. Ankur says the answer is 6. Jeff continues to write possibilities. Ankur says 1 is 1, 2 is 2, and Romina adds that 3 is 6. Ankur says it's factorial. Ankur says to work out the case for four-

Minutes	Summary
	tall towers by drawing four dashes. There are four choices for the first space; Jeff adds that the remaining choices are 3, 2, and 1. They calculate that the answer for 4 is 24 and for 5 it's 120. Ankur says it's *n* factorial. Jeff says you write factorial with an exclamation point. Researcher 1 asks if they're all convinced. Romina says yes. Researcher 1 asks if things are coming together.
63-66	Jeff says they were really quiet in the beginning, but when it started to come together, they got vocal. Researcher 1 says one of the questions that Researcher 5 would ask is, how do you know that you're not getting duplicates? Jeff says once you find the formula, you take it for granted that the formula is correct. Researcher 1 says you shouldn't do that. To build a formula you have to show that it's correct. Ankur says they went from 1 to 4 with no duplicates, why would there be duplicates at 5? Researcher 1 tells them, "Give an argument in general why there are no duplicates." They discuss the two- and three-color cases. They talk about scheduling the next meeting (March 20).
66-69	Researcher 1 asks if they know about the mathematician Fermat. They don't. She says these are classic problems that they have been doing; they establish the work they will do later in Algebra II. The students indicate that Algebra II is moving very slowly. Jeff says, "it's kind of frustrating." Michael says the regular work "is so easy I don't even bother looking at it." Researcher 1 says she's going to challenge them with a classical famous problem; she suggests they think about it using towers or binomial coefficients. She says they should make sure they have a strong understanding of why the addition rule works. Researcher 1 writes Pascal's Triangle in "choose" notation, including row *n*.
69-72	Researcher 1 says, "let's take a general term, an *r* term in there." She writes *n* choose *r*. She says she wants them to tell her, in general, where the r^{th} term comes from. She says you could call the row n+1 and the previous row *n*, or you could make it row *n* and the previous row n-1. Researcher 1 says she wants them to tell her in general how it grows. Researcher 1 and Ankur talk about building the next group of towers from three-tall towers. Researcher 1 asks them to come up with some kind of generalization, thinking about how it works with the towers. Researcher 1 offers them Unifix cubes; Romina takes them. Researcher 1 asks if they'll have a chance to think about that over the next few weeks. Jeff says they will. Researcher 1 says they're worked hard and done some nice mathematics.

Minutes	Summary
72-75	Researcher 1 asks when they started playing with the cubes. Romina says fourth grade. Researcher 1 says, "some of you were building towers in grade 3." The students leave and Researcher 1 and graduates students converse.

Summary of June 12, 1998

Minutes	Summary
0-3	Jeff, Ankur, Romina, and Brian are seated at a table with Researcher 1. Researcher 1 is talking about building and arranging towers. She talks about the "Gang of Four" tape (from fourth grade). "We focused on the thinking. ... Stephanie had an organization that was a proof by cases." She mentions Jeff's question, "Why do you have to have a pattern?"
3-6	Researcher 1 says, "we're interested in how these patterns are represented by you." She says they invented their own notation for solving problems; the representations were windows into how they were thinking about the problem. She reminds them about how Michael represented the pizza problem in terms of 0's and 1's. Romina recalls how Michael and Ankur used the binary numbers to immediately answer the question: How many 5-tall towers are there with exactly two red cubes. Researcher 1 tells the students that they have tools that they call up as needed for solving problems. That's what doing mathematics is all about.
6-9	Researcher 1 tells the group that they've been spending time on combinatorics problems, relating to the binomial theorem. She suggests a review. She writes $a+b$. The first power is $1a + 1b$. She writes the second and third powers: $1a^2+2ab+1b^2$ and $1a^3+3a^2b+3ab^2+1b^3$. She asks if it looks familiar. They nod. Researcher 1 tells them that they don't have to multiply it out; they can follow the pattern.
9-12	They work out $a+b$ to the fourth: $1a^4+4a^3b+6a^2b^2+4ab^3+1b^4$. Researcher 1 says what's really important to think about is why the addition rule works. She tells them that Stephanie used block towers to explain the addition rule, and she asks them to think about why $3a^2b$ and $3\ ab^2$ gives the 6. She asks, "If this were a tower, what would the tower look like?" She says, "We talk about this [a row of Pascal's Triangle] as a class of towers." She asks the students if there is a difference between the two classes of towers [the class for row 3

Minutes	Summary
	and the class for row 4]. She asks them to think about what the two 3's mean in terms of towers. Jeff and Ankur start to discuss it in terms of light and dark cubes. Researcher 1 asks where the 6 comes from. Ankur says you have 3 with two *a*'s and a *b*. Jeff says there's 3 different combinations with two *a*'s and a *b*.
12-15	Researcher 1 asks them to explain, and be very particular. Ankur says you add an *a* to the $3ab^2$ and a *b* to the $3a^2b$, and that gives $6a^2b^2$. Researcher 1 asks if the two 3's are different; she asks them to imagine what they look like. Ankur says they are two dark and one light. Researcher 1 asks if they can see how the tower images can help. She says what they are doing is called the binomial expansion and the numbers in Pascal's Triangle are the coefficients. She says you can think of these as a distribution of towers. She asks if they do probability in school. Jeff says they did something with flipping pennies. They discuss.
15-18	Researcher 1 tells the students that there are two approaches to probability. "One is with a model that is supposed to be theoretically perfect." For example, you can assume a perfectly balanced coin, although in practice a coin is never perfectly balanced. Researcher 1 discusses the meaning of "in the long run." She discusses using a sample space; the sample space for flipping a coin is H or T. Theoretically, you could assign numbers to probabilities. They must be between 0 and 1 and they must add up to 1. She asks what to assign to the probabilities of H and T. Jeff suggests 0.5.
18-21	Researcher 1 says that all the events in a sample space are called sample points, and the numbers assigned to them are called probabilities. This approach is called *a priori*; before you do anything, you invent a model. She asks about the sample space for flipping two coins. The probability of each event is 1/4.
21-24	Researcher 1 tells the students that they always have to check that the probabilities are between 0 and 1 and they must add up to 1. She reminds them that when they were in the fifth grade, they had some probability problems with towers. She says the sample space is towers of a certain height, for example two. She asks them to imagine towers two-tall selecting from two colors. Jeff says it's the same as tossing two coins. Researcher 1 asks them to think of a problem with towers that might assign numbers to probabilities. Jeff suggests, what is the probability of getting a tower of all one color? Jeff starts to enumerate probabilities (1/3 is mentioned), and Researcher 1 asks him to clarify.

Minutes	Summary
24-27	Jeff says there are 3 possibilities in the sample space. Jeff says that one of each is the same thing. Researcher 1 says she would argue that it depends on how you ask the question. Jeff says it doesn't matter because the two "one of each" cases look the same when you turn one upside down. Researcher 1 asks about the probability of getting a blue cube on the bottom. Jeff says in that case, the sample space would have four things. But if you're just looking at pulling all reds or all blues, it doesn't matter. Researcher asks them to find the probability of pulling out a tower with all of one color.
27-30	Jeff says the probability of pulling each of the towers is 1/4; he says the probability is 1/2 for all one color. Researcher 1 says there are four points in the simple sample space; when you combine them, it's not simple any more. She says you can create problems around those probabilities. Each of the points in the sample space is separate, not overlapping. In order to find probabilities of certain collections of towers, you need to build a probability space. She assumes they could do that for $a+b$ to the fifth. She asks them to suppose she wants the probability that a five-tall tower had at least three red cubes. They did that in fifth grade. Jeff starts to write the expansion of $(a+b)^5$. Romina and Ankur discuss.
30-33	They finish writing $(a+b)^5$, and then they start to count the cases that satisfy the condition. Ankur says it's out of 32. Jeff and Romina count the terms that satisfy the condition.
33-36	They calculate. Jeff says it's 16 out of 32. They call Researcher 1. Jeff says the problem was how many had three a's in $(a+b)^5$. Researcher 1 asks if they can relate that to towers. Ankur says it's the probability of getting at least three of a specific color (red). Jeff says they converted the towers to $(a+b)^5$.
36-39	They put the entries into the sample space. Jeff explains the binomial expansion in terms of colors. a is red and b is blue. The second term is four reds and a blue, etc. Jeff says 1 out of 32 is all red. You have 1 out of 16 chance of pulling all of one color (2/32). Researcher 1 introduces the concept of conditional probability. Suppose she puts a condition on the probability of an event.
39-42	If you put a condition, are you dealing with all the points? No. You're taking a subset of the original sample space. They discuss conditional probabilities with regard to coin tossing. Another problem: suppose a person flips one coin and then a second coin. Is that the same as flipping them both at the same time?
42-45	They discuss whether the probability changes. They confirm that the

Minutes	Summary
	probability that there are at least three red cubes is 1/2. Researcher 1 asks them to consider a new problem: "Now suppose you are given that the bottom cube is red." What is the probability of at least three reds?
45-48	The students count the cases which satisfy the condition. Jeff notes that not all 10 have the bottom cube being red. Ankur says to look at four high with two reds, and the third red would come from moving to the next row. You would put the red on the bottom. They discuss. Jeff asks about the 10. Was that with three reds in each? They discuss what towers should count.
48-51	Jeff says it's like the last one they did, which wasn't even part of the probability. "If you flip a tail, it doesn't count." Romina says you have to count certain cases. There is a pause. It appears that Jeff is writing while the others wait. Jeff counts cases. Jeff and Ankur discuss, but their words are inaudible.
51-54	Jeff calls Researcher 1 over. Ankur says the answer is 11 out of 32. Looking at cases in terms of the numbers in row 5 of Pascal's Triangle, Jeff says the first one definitely fits because the one on the bottom has to be red. (It's the all-red tower.) The next case is a^4b. Four of those five towers will have one on the bottom. The other tower in that group doesn't fit the criteria but it's still part of the 32.
54-57	In the next group of 10 (a^3b^2) , only six fall into the category of red on the bottom. Ankur says he changed it to a simpler problem. He looked at four-tall towers. That's $(a+b)^4$. He counted all of the 4-tall towers with two reds. Then adding a red on the bottom (in order to get to five-tall towers) makes the third red. Jeff says the answer is 11 out of 32. But Ankur says it's 11 out of the 16 4-tall towers which have two reds, and 11 out of 32 with 1 red at the bottom. Researcher 1 says she's confused; she asks what's the relation between the two problems. Jeff says the one problem leads into the other. Researcher 1 says there has to be a red on the bottom. Ankur says, "I know, I'm making a simpler problem first."
57-60	Researcher 1 says she is confused about whether there are 32 sample points with red on the bottom. Ankur says there are 32 towers. She says she wasn't asking the probability with a bag containing all the towers. Jeff says that the bag doesn't even contain those other towers. Researcher 1 restates the problem. They are building towers five-tall when selecting from red and blue. What is the probability that the tower has at least three red cubes, given that the bottom cube is red?

Minutes	Summary
60-63	Ankur says, "Out of the entire bag." Jeff says you weeded out the ones without red on the bottom; they're in a different bag now. Researcher 1 asks if that makes sense. Jeff says it makes a lot of sense. Ankur says it's the same problem as four-tall towers having two reds. Ankur says they got the same answer, but they were answering different questions. Researcher 1 asks Brian how he thought about Ankur's problem. Brian talks a little about five-tall towers with 3 reds. Researcher 1 asks for help understanding Ankur's problem. Ankur and Jeff say it's the same problem.
63-66	Jeff says when Ankur cut the problem in half, 2 out of 4 is the same as 3 out of 5. Researcher 1 says that conditional probability reduces the sample space. She asks them to make up a conditional probability problem related to towers.
66-69	Researcher 1 says they can figure out towers problems right away. For example, she asks what's the probability of getting a tower with all the same color cubes with 7-tall towers? They say 2 out of 2 to the seventh. Jeff says, about 5-tall towers, excluding the ones that are all one color, how many have at least one blue? Researcher 1 asks for Researcher 6's work on the tower problem. Researcher 1 shows this to the group. Researcher 6 wrote out all the towers. Jeff says he started to do that.
69-72	Researcher 1 introduces two new problems. One of her colleagues used these problems. A fair coin is flipped four times, each time landing heads up. What is the most likely outcome if the coin is flipped a fifth time? It could be either way. Jeff says the probability of 000 in a lottery is the same as 456. Romina says what about the probability of three heads in a row.
742-75	Ankur brings up the odds of having four boys and then a girl, as opposed to the odds of five girls in a row. Researcher 1 asks what's the probability of a head the first time? Jeff says 1 out of 2. Researcher 1 asks about the second time. Does the first toss have anything to do with it? Jeff says no. Researcher 1 says a lot of people have problems with that. If you have a string of 20 heads, what is the probability of a string of 20 heads? One over 2 to the 20. What's the probability of the next one being heads or tails? They discuss the probability of three girls in a row. Jeff says if you have one kid, the probability of the next child being a girl is still 1/2. Researcher 1 asks the next problem. What is the probability of four heads in a row? Jeff says 1/2 times 1/2 times 1/2 times 1/2.
75-78	They discuss the probability of HHHHT and HHHHH. Researcher 1

Minutes	Summary
	asks about the probability of having four heads out of 5 tosses. Ankur says 5 out of 32. Which of the following is the most likely result in five tosses of a fair coin? HHHTT, THHTH, THTTT, and another one. Is one more likely, or are all four equally likely? Ankur asks if this is the exact order. Researcher 1 says yes.
78-81	Researcher asks if they are imagining five-tall towers. They discuss scheduling. Researcher 1 says they will try to do a summer institute next summer. Jeff asks what they write on the college applications about this program. Researcher 1 says Rutgers Math. They discuss doing this next year.
81-84	They discuss jobs and other non-math topics.

Summary of December 14, 1998

Minutes	Summary
0-3	As the tape starts, Michael is saying, "that was a while ago." He goes on to say that he starts by figuring out the probabilities of having various children. [It becomes apparent that he is talking about the boy-girl probability problem: There are two children in the family. If you knock on the door and a girl answers, what is the probability that the other child in the family is a girl?] Michael lists the various possible combinations of two-child families: *bb*, *bg*, *gb*, and *gg*. He says he can eliminate the *bb* case, since a girl answers the door. In the three cases that are left, only one has two girls; therefore, the probability that the other child is a girl is 2 out of 3. Researcher 1 suggests looking at a different version of the problem: One where there are three children in the family.
3-6	Researcher 1 asks, if a girl answers the door, what is the probability that at least one of the other children is a girl? Michael says a 3-D diagram would be required. He things for a while and then states that he thinks it would be 50-50. Researcher 1 says that's his first intuition; she asks how he would justify it. Michael starts to say there are four possibilities. Researcher 1 notes that he found four possibilities in the previous problem. Are there still only four possibilities? Michael thinks some more and says there are eight.
6-9	Michael draws three dashes and says there are two possibilities for each of the three slots. Researcher 1 asks him to list all the possibilities. He says you can write either a *b* or a *g* in front of each of the four possibilities he already listed. He lists the eight

Minutes	Summary
	possibilities: *bbb*, *bbg*, *bgb*, *bgg*, *gbb*, *gbg*, *ggb*, and *ggg*. He says there are three cases with two boys and one girl, one case with all girls, one case with all boys and three cases with two girls and a boy. Researcher 1 asks, "what's the probability that at least one of them has a girl?"
9-12	Michael draws houses for each case. He omits the three-boy case. He counts and says that the probability is 4 out of 7, because in four out of the seven remaining cases (after the *bbb* case was omitted), the girl has a sister. Researcher 1 proposes a different problem: Suppose there are 10 children in the family, and a girl answers the door. What is the probability that they are all girls? Michael says it's 1 over 2 to the 10^{th}, since there are 10 places. Researcher 1 asks if there couldn't be something like boy-boy-boy-boy in those spots.
12-15	Michael says he multiplies one-half times one-half, etc. because there is a 50% chance of a girl each time. Researcher 1 asks if that's true if she says that a girl answers the door. Michael says that in that case, it would be 2 to the 9^{th} because "one of those spots was revealed to us." Researcher 1 says his thinking is very good; what he is doing is called conditional probability. The condition that she gave reduced the sample space to fewer outcomes. Researcher 1 gives another problem. It's an *n*-children family. If a girl answers the door, what is the probability of an all-boy family?
15-18	Michael asks for clarification. Researcher 1 says, "The whole family has all boys." Michael says 1 over 2 to the *n*-1. Researcher 1 asks, in a two-child family, if a girl answers the door, what is the probability of two boys? Michael says one-fourth. They discuss what the question means, and then Michael says it's impossible, so the probability is zero for both the two-child and *n*-child families. Michael says that the probability that the family has all girls (the question he thought he was answering at first) is 1 over 2 to the *n*-1. Researcher 1 suggests a coin-tossing problem. She flips a coin 20 times, and the first coin comes up heads.
18-21	Researcher 1 asks for the probability that the rest of the coins are tails. Michael says to plug the numbers into "there" (indicating the paper on which he wrote the answers to the boy-girl questions). Researcher 1 asks if it's the same as the boy-girl problem. Michael says yes; it should be 1 over 2 to the 19^{th}. Researcher 1 asks if Michael remembers doing probability problems before. Michael says that he probably did, but he doesn't have a good memory. Researcher 1 asks, "supposing you were building towers, selecting

Minutes	**Summary**
	from two colors. Does that have anything to do with this?" She asks Michael to make up a tower problem. Michael says, "You have a classroom who are building 5-tall towers."
21-24	Michael says one student builds a tower. "What's the chance that another kid would make the exact same tower?" Researcher 1 asks how to solve it. Michael says there's a 1 over 2 to the 5th chance of building that same tower. Michael notes that it can be difficult to tell the orientation of a tower because it looks the same upside-down as rightside-up. Researcher 1 asks if that would be a problem with pizza problems. Michael says he remembers using binary to solve pizza problems. Researcher 1 asks how they solved the 5-topping pizza problem.
24-27	Michael says he expressed the different pizzas using binary notation. Michael says pizzas are different from towers in that upside-down is definitely different from rightside-up. If you place peppers and mushrooms in the third and fourth spaces, when you turn it around, it's not mushrooms and peppers anymore. "It's sausage and whatever." There is a discussion about whether two towers should be considered the same if one is an upside-down version of the other.
27-30	They continue the discussion of upside-down vs. rightside-up towers.
30-33	After further discussion of upside-down towers, Researcher 1 asks Michael what constraint he would place so there would be no problem. Michael says, "That each block has a top and bottom to it." Researcher 1 asks if the towers problem is like the pizza problem with that constraint. Michael says yes. Researcher 1 asks him to explain. He says that with binary notation, all the numbers between 000 and 111 include all possibilities. Researcher 1 and Michael confirm that 000 stands for the pizza with no toppings and 111 stands for the pizza with all three toppings. Researcher 1 asks how many pizzas there are with 0, 1, 2, and 3 toppings. Michael asks if she's heard of Pascal's Triangle. "That has a lot to do with it." Researcher 1 says, "Show me."
33-36	Michael starts to write Pascal's Triangle. [He omits row 1; refer to Figure E1.] Michael says that Pascal's Triangle tells exactly which pizzas have which toppings. Indicating row 2 [121], he says that one pizza would have no toppings, one pizza would have both, and two of them would have only one. For three toppings [1331], he says one pizza would have no toppings, one would have all three toppings, and there are six left. Researcher 1 asks how he would distinguish between the two 3's. There is a pause. Researcher 1 reminds

Minutes	Summary
	Michael that there are eight different pizzas when there are three toppings to select from. Michael says the first number 1 is for no toppings and the last 1 is for 2 toppings. Researcher 1 asks Michael what the 111 is. He says it's three toppings. He goes on to say that the first three is for one topping, and the second is for two toppings. Michael elaborates that 1 3 3 1 is 0, 1, 2, and 3 toppings. Researcher 1 asks Michael to identify the one-topping cases in the list of binary numbers. Michael marks off 001, 010, and 100. Researcher 1 asks him to identify the 2-topping cases. Michael marks 011, 101, and 110. Researcher 1 asks Michael to talk about row 4.

1
1 2 1
1 3 3 1
1 4 6 4 1

Figure E1. Michael's rendition of Pascal's Triangle

39-42	Researcher 1 asks Michael where 16 is. He says, "You can't do 16 with 4." [The four-digit binary numbers go from 0 to 15.] But he adds that there are 16 possibilities. Researcher 1 says one of the characteristics of the triangle is that, for example, 1+3=4. Researcher 1 asks Michael to explain this with pizzas. She says the 1 is the pizzas with no toppings and the 3 is pizzas with one topping. She says, "now you're adding an extra topping, right? … And the 4 tells you how many you will have with 1 topping when you can select from 4 toppings." Researcher 1 asks Michael to explain how that works. Michael writes the three 1-topping pizzas [100, 010, 001]. He says that you add a 0 to the left of those three numbers and a 1 to the left of the 000 representing the pizza with no toppings. Michael says that you're not adding any toppings to the ones that already have one, and you're adding a topping to the one that doesn't have any. Researcher 1 asks Michael to do another one; show how 3+3=6.
42-45	Researcher 1 notes that the first 3 is the 1-topping pizzas and the second 3 is the 2-topping pizzas. Michael says that the ones that didn't get any added went to the 4 and the ones that did went to the 6. Researcher 1 asks Michael to write up this explanation of why 1+3=4 and email it to her. She says no one has ever shown this explanation of Pascal's Triangle with pizzas and binary numbers before.
45-48	They exchange email addresses and discuss future get-togethers.

Summary of January 29, 1999

Minutes	**Summary**
0-3	The tape begins with a test pattern as we hear Michael and Researcher 10 speaking. Michael is saying, "if you have a choice of 0 toppings, there's only one pizza you can make." If you have one topping, you can have the pizza with the topping or without it. The test pattern goes away as Researcher 10 says, "so this is 1, 2, and 1?" Researcher 10 asks, "so this [row 2] is with two toppings?" Michael says it should be, but he might have to correct himself at the end. Researcher 10 asks where the 1 at the top of Pascal's Triangle comes from. Michael responds that that's when you don't have any toppings to choose from. "You can only make one pizza, a pizza without anything." Michael says everything in "this column" [the leftmost number in each row of Pascal's Triangle] represents a plain pizza. Then he explains that the next row represents one choice of toppings, for example mushrooms; you can have a pizza with mushrooms or without. Researcher 10 says this [the first 1] is without mushrooms and this [the second 1] is with mushrooms. Michael agrees. Michael says for row 2, if the choices are mushrooms and peppers, "you could still order that cheese pizza." [He indicates the first 1.] Or you could order one of the two pizzas with exactly one topping: mushrooms or peppers, or you could have it with both toppings. Researcher 10 asks about the next row: 1331. Michael says that row corresponds to three toppings. Researcher 10 suggests mushrooms, peppers, and sausage. Michael says the 1 on the left is no toppings, the next 3 is three different pizzas with one topping each, the next 3 is pizzas with two toppings, and the last 1 is all three. Researcher 10 says she understands. Researcher 10 asks how he got from the 1 and 2 in row 2 to the 3 in row 3. Michael says "this 1 has no toppings on it." He says when another topping becomes available, "that plain pizza you just made, you could either put that mushroom on it, or you can't."
3-6	Michael says, "suppose the pizza man just got some mushrooms in. … The one without the mushrooms is that 1 [the first 1 in row 3]. And the one with it is part of the 3." Researcher 10 asks him to repeat his explanation. Michael says that if "they just got some mushrooms," you could either put mushrooms on the pizza or not. If you put the mushrooms on the plain pizza, it will now have one topping and be in the one-topping category for row 3. If you don't

Minutes	Summary
	put the mushrooms on it, it will stay in the same category [no toppings]. Researcher 10 asks him to explain where the second 3 in row 3 comes from. Michael says if you add the mushrooms to the two pizzas that have one topping, they become part of that 3 (which represents pizzas with two toppings), and if you don't add mushrooms to the one pizza that has two toppings, it will stay in the two-topping category. Michael says the pizza that got a new topping "moved up in the world of pizzas." If it gets a topping, it goes right; if it doesn't, it goes left. Researcher 10 asks if it "happens here too" (pointing to the first 1 from 1331). Michael says yes. Researcher 10 says she doesn't understand the 6 in the middle of row 4 (14641); she asks Michael to explain.
6-9	Michael explains how the 6 came from adding groups of pizzas. Researcher 10 asks Michael to talk her through row 4. Michael says the first 1 came from the 1 in the previous row that didn't get a topping; the 4 came from 1 pizza that got the extra topping and 3 that didn't. The 6 came from 3 that got a topping and 3 that didn't. The 4 came from 3 that got a topping and 1 that didn't. The 1 came from the 1 that got an extra topping. Researcher 10 says that it makes sense. It helps to have someone talk through it. Researcher 10 asks Michael if there are any other problems besides pizzas that you can think of in "that Pascal way" for going from one row to the next. Michael says he didn't really use pizzas in this one, he used binary notation. He talks about how there are two choices when adding a binary digit to an existing binary number.
9-12	Researcher 10 says it's like a tree. Michael agrees. "It branches out. … Each one of those combinations becomes two." Researcher 10 says she was wondering if there are other problems that seem to have the doubling rule could be described using Pascal's Triangle. Michael mentions the World Series problem. Researcher 10 asks if you could use Pascal to think about the World Series problem. Michael says he personally used it. He indicates row 3. He says the 1 at the end represents Team A winning all 3 games. Researcher 10 says 1331 is 3 games. Michael says only 1 of those 8 represents 3 wins by Team A. Researcher 10 asks him to explain the first 1. Michael says Team B won all 3. Researcher 10 writes BBB for the first 1 and AAA for the last 1. She asks what the first 3 is, and Michael says, "A only won once." He says ABB, BAB, and BBA, which Researcher 10 writes. He says the second 3 is "A won twice." She writes AAB, ABA, BAA.

Minutes	**Summary**
12-15	Michael says the probability of winning 3 games in a row is 1/8, which he asks Researcher 10 to write down. He adds that there's a half chance of winning the next game; that gives 1/16. Researcher 10 says, "so that was really the next row you're interested in." He says you have to take into account the probability of winning 3. They continue discussing probability. Then they discuss scheduling future meetings, as Michael gets ready to leave.

Summary of May 12, 1999

Minutes	**Summary**
0-5	Romina, Jeff, and Michael are sitting at a table talking with Researcher 1. Researcher 1 asks them what they did in class that day. Romina asks Jeff about what they were doing; she says it was something with e that was almost like Pascal's triangle. Romina asks for a calculator, and Researcher 3 distributes calculators to the students. They do some work on calculators and Romina shows the display to Researcher 1. Romina says, "we did a hundred and we took ten percent of it," and then got 1 2 1, 1 3 3 1. Then Jeff says that in class today, they were looking at *e*. Michael says they looked at how *e* connects with *ln*. Romina adds that they looked at $(a+b)^{n.}$ Jeff says that they were using "the choose thing" to find the coefficients of the terms in $(a+b)^{10}$, for example. He says they use the ${}_nC_r$ button on the calculator to find 45; that is 10 ${}_nC_r$ 2. They mention ten-high towers; Jeff says, "If you have towers with ten high and two colors," and Michael adds, "How many different places can you put two reds in there." Jeff adds that the answer is 45. Jeff asks if the two from 10 ${}_nC_r$ 2 is from the two colors, and Michael says no.
5-10	Jeff asks if it is implied that there are only two colors. Romina says she doesn't know, but "I'll go with the yeah." Michael says that the problem is to pick two things out of ten; "how many different places could you put them?" Researcher 1 asks what the answer would be for "eight red ones or eight A's." Michael answers 45. Researcher 1 asks Michael how he did that so fast, and he shows the button on the calculator. Jeff asks why that is the case, and Romina says that "it's almost like switching colors." Researcher 1 asks how they would get the answer without using the calculator. Jeff starts to draw a tower. Michael says that Bob Sigley had a formula. Romina tells Jeff to use a three-tall tower. Jeff draws a three-tall tower and writes X's to

Minutes	Summary
	show places to put two colors. Romina makes a list consisting of BBB, BBR, BRB, and RBB]. She says, "that's how we figure them out when we have to write them out." Jeff says there's a formula, but he does not know how it goes. Researcher 1 asks Michael to go to the board and show what is on his paper. Michael starts to write the equation on the board. Ankur arrives.
10-15	Michael, Jeff, and Romina are talking about "the choose thing." Jeff says that if you're doing three, "that would be three times two times one." Romina says x is the "number we want to get, the choose number." Then they talk about taking away the repeats. They use their calculators. Michael tells Jeff to change some of what he entered into the calculator. Jeff's calculator shows 6/((3-2)!*2!). The next line displays 3. Michael pushes some buttons on Jeff's calculator and says the answer is 3. (The calculator shows 3 ${}_{n}C_{r}$ 2, and the next line has a 3.) Michael says that it works. Jeff says that Michael can explain it to Ankur when Ankur finishes eating. Ankur sits down and then there is a break in the video. When it resumes, Jeff and Romina are sitting down. Jeff goes to the board. He says "this was going to cover all the total possibilities of your tower." He says, for towers of three, three factorial "will cover all of the different combinations that you could put three in with two colors." He asks, "All right? Yes?" Researcher 1 tells him to go through it, and she will ask questions when he is all done. Michael tells Jeff to use "people on the line."
15-20	Jeff says, "say we're doing us three right here." Romina says there are three "people to fill in the first one." Then Jeff says that leaves two people for the next spot. Romina says you multiply three and two. Jeff says there is "only one person left and they get the last spot so that's times the one." Researcher 1 asks why they multiply; Romina says that they don't like that question. Researcher 1 asks them to figure it out, and she leaves the table. Ankur mentions red, white, and blue. Romina says they are only doing two colors, but she says, "to explain it, maybe you want to do three different colors." Ankur says with red in the first position, there is a choice of white and blue. He says that "each of those three goes with two other. So it's three times two." Romina calls Researcher 1 and Jeff tells Researcher 1, "I think we're good with this." Jeff writes R, W, and B on the board and asks Ankur to continue. Ankur says, "One of those colors goes in the first spot." Then he says there are two colors left. "So there's three different colors that can go in the first spot and each

Minutes	Summary
	of those colors can go with two other colors." Researcher 1 asks what this has to do with the towers and the $(a+b)^n$ problem. Michael responds that they just answered why they multiply; Romina adds that "we're not there yet." Jeff writes $3!\div(n-x)!$ on the board. Michael says that the n is the number you are choosing from, and you subtract x. Jeff asks, "How come the x is there?" Michael says you subtract the x from n, and Jeff adds that "the factorial will eliminate all the other ones that you don't want." Michael says, with regard to 5 choose 2, that with five people on the line, you "want to know how many different places you could put those two people." Michael says that you divide to get rid of the repeats, and also because "you're not worried about those other three people." Jeff mentions subtraction, and Michael says, "no, it's divide." Jeff, Romina, and Michael talk some more, and Jeff says they still don't know where "the final x comes from." Researcher 1 tells the students to work out a piece of the problem; she says Ankur's not convinced. Michael says 5! gives "all the combinations they put everybody in." Michael says the reason for the $n-x$ is that "there's going to be a lot of times where those two are going to be in the same spot as the other three are going to be ..." Jeff asks Researcher 1, "Is this all good?" She replies, "I'm waiting for the whole thing." Ankur says that he gets it. Michael says that the 3 factorial gets rid of when those three move around, and the 2 factorial gets rid of when those two move around. Jeff says that's 120 divided by 12: 10. Michael does it on the calculator and agrees. Michael asks if Researcher 3 and Researcher 1 get it, and Researcher 3 says no. Michael says the numerator represents all the combinations. Michael, Jeff, Romina, and Ankur talk about the 3 factorial, which they say represents the three people moving around, and the 2 factorial, which they say represents the two people moving around. Researcher 1 asks Ankur to explain. Ankur says that the top number (5!) is the total number of possibilities for five people, and the 5 minus 2 is because "you're just worried about two people at a time."
20-25	Jeff says that the 3 factorial eliminates "everyone except the two people you're worried about." Ankur says it's "when the two people are switched back and forth." Researcher 3 responds, "It's getting better" and asks for an example. Researcher 1 asks Michael to start from the beginning very slow. Michael goes to the board and writes $5!\div3!\times2!$ He starts by saying that five people are in a line. He points to 5! and says you agree that's how many combinations you

Minutes	Summary
	could put those people in. Researcher 3 says that she understands. Michael then says that when the other two people are in a certain spot, the other three move, causing repeats that we have to eliminate. The two is for the two people switching back and forth. Researcher 1 says she wants to think of them as towers, not people (towers five tall with two reds). Jeff says that if you are doing towers five tall with two different colors, "the total amount of possibilities is the five factorial." He says the "the three factorial on the bottom would be five different spots minus the two spots that you're concerned about leaving you with the three other spots that you don't care about." Romina adds that, if the two colors are red, "the two stay in the same place and then the other three are just switching..." Michael says that repeats six times; Jeff adds, "that's where the three factorial comes from, and you're multiplying that by the two factorial." Michael says they [the two red blocks] don't have a name, so "they're the same thing." Researcher 1 says she follows this, and she asks, "why were we doing this?" Romina replies, "you wanted us to explain the choose thing, which goes back to Pascal's triangle and ... $(a+b)^n$." Researcher 1 asks what it has to do with what they did in class today. Jeff says they were looking at $a+b$. Michael goes to the board and writes rows 0 through 3 of Pascal's Triangle.
25-30	Michael says that the last row (1331) starts with 3 choose 1; Romina says that's when "it's all one color. Michael then says it's 3 choose 0. He says the rest of the numbers (331) are 3 choose 1, 3 choose 2, and 3 choose 3. Researcher 1 asks Michael to write some more rows of Pascal's triangle in "the choose way," and show the addition rule with this new notation. Michael goes back to the board and writes the next row of the triangle. Researcher 1 asks Michael to explain why you add to get new rows of Pascal's triangle. Michael says the addition rule is if you add another topping, the three will go either to the six or to the four depending on whether they got a topping or not. He says, "the three that had two toppings won't get any, and you could put them in together with the ones that did get something. That's why you would add."
30-35	Researcher 1 writes the 1331 row of Pascal's triangle in choose notation. She says that she wants them to write Pascal's triangle that way and then give the general rule. Michael adds the fourth row. Researcher 1 refers to towers and says Ankur can explain. Romina says n is height, and Jeff refers to each row of the triangle as $(a+b)$ to a power. He says the n^{th} row would be $(a+b)^n$. Michael says the tops

Minutes	Summary
	would be n and the bottoms would go from 0 to n. Michael asks if n refers to the height of the tower, and Romina says it does. Michael says n is the height of the tower with two colors. Researcher 1 asks if they can put n choose r in the middle. Jeff says you can put it wherever, as long as r does not exceed n.
40-45	Researcher 1 asks them to rewrite their work nicely because she wants to ask a question. Jeff rewrites Pascal's triangle in choose notation. He asks how far to go. Researcher 1 says to go to the n^{th} one. Jeff writes row N. Researcher 1 asks him to put N choose N at the end and N choose X in the middle. Jeff does this and then sits down. Researcher 1 asks Jeff to go to the board and show her the addition rule from the fourth row to the fifth row. She asks him to show 3 plus 3 is 6, and how to draw the arrows. Jeff draws the arrows, and Researcher 1 says that 3 choose 1 plus 3 choose 2 equals 4 choose 2. She then asks what 4 choose 2 plus 4 choose 3 equals. Michael answers, "5 choose 3." Researcher 1 says, "I don't know if Romina's convinced." Jeff asks, "Why is he 5 choose 3?" Michael goes to the board and says, "because this guys gets another topping...because he's moving up, this bottom number's going to change." Researcher 1 asks him to explain it again. Michael says as you move to the right (he gestures to the right), you get another topping.
45-50	Jeff writes row N of Pascal's triangle. Then he writes some entries of row N-1. Researcher 1 asks Jeff to explain to Brian. Jeff says the work that he has written on the board is Pascal's Triangle using "the choose situation." Jeff uses the third row as an example. He tells Brian that 3 choose 0 is 1, 3 choose 1 is 3, 3 choose 2 is 3, and 3 choose 3 is 1. Romina shows it to him on paper. Brian asks what the exclamation mark is, and they all say "factorial." Romina points to Brian's paper and says, "that's all the combinations." Jeff asks where Researcher 1 wants them to go with this, and she responds, "I want you to show me how the addition rule works in general." Together Jeff and Michael say n+1 over x+1. Researcher 1 asks them to write an equation. Jeff writes the equation shown in Figure E2.

Minutes	Summary

$$\binom{N}{X} + \binom{N}{X+1} = \binom{N+1}{X+1}$$

Figure E2. Pascal's Identity in students' notation

50-55 Jeff says the top would be gaining an x while the bottom loses an x. Michael and Ankur correct him: the bottom stays the same. Michael says the top number changes because you add a topping. Researcher 1 asks Jeff to explain it for Brian, since he was not there for the earlier pizza discussion. Jeff talks about adding another topping, and Romina says, "you know how we get the triangle and how we go 1 2 1 and add those two together." Jeff says they are explaining why you add. He says if it gets a topping, it goes to x+1, and if it doesn't, it stays the same. Researcher 1 asks them to write the addition rule using factorials. Michael goes to the board and writes, with Jeff and Ankur giving him some of the terms. Michael sits. Researcher 1 asks if all the students agree. Michael goes back to the board and writes the general equation under what he wrote before, and Jeff says they are the same. Researcher 1 asks if they checked their arithmetic. Ankur tells Michael to change the denominator of the last term.

55-60 Michael makes a correction and then he sits down. Researcher 1 remarks on how frightening the formula is and says, "I wonder if there's any way of simplifying it." Researcher 1 says that the term that Michael just wrote could be a little simpler. Jeff starts writing and then asks if you can cancel factorials. Romina asks if you can cancel the (n+1). Jeff asks if that would leave you with "factorial divided by factorial." Michael goes back to the board and simplifies. The final version is shown in Figure E3. They conclude with a discussion about simplifying factorials.

$$\left(\frac{n!}{(n-x)!x!}\right) + \left(\frac{n!}{(n-x+1)!(x+1)!}\right) = \left(\frac{(n+1)!}{(n-x)!(x+1)!}\right)$$

Figure E3. Factorial version of the addition rule

Summary of April 3, 2002

Minutes	Summary
0-3	Researcher 1 tells Michael she's interested in talking about the work of the night session, where the group looked at Pascal's Triangle from the lens of the pizza and towers problems. She tells Michael that it was the night before the prom.
3-6	Michael says he does not remember the specific night. Researcher 1 tells him that they used the example of pizzas to look at how the triangle grows. Michael says he does remember that. He takes a pen and paper and starts to draw Pascal's Triangle. He refers to row 2. After a pause, he asks Researcher 1 to refresh "the pizza thing."
6-9	Researcher 1 tells Michael that the question for row 2 is, if you can select from two toppings, how many pizzas can you make? Michael hesitates again, and Researcher 1 suggests he specify the toppings. Researcher 7 suggests mushrooms. Michael says the two-topping pizza relates to row 2 (1 2 1) because there are two one-topping pizzas and one pizza with both toppings. He says row 3 would go with three toppings and row 4 would go with four toppings. He says, "the row on the outside would be your plain pizza." (He means the leftmost number in each row.) Then the next number in each row would be the one-topping pizzas, and the last number in each row would be the one pizza with all the toppings. "And, once again, you can only make one pizza out of that." Michael asks if he can look at the paper with Pascal's Triangle in choose notation. Researcher 1 suggests he use that notation himself.
9-12	Michael writes row 2 of Pascal's Triangle: 2 choose 0, 2 choose 1, and 2 choose 2, and the he adds row 3. He explains how the addition works in the case of 2 choose 0 plus 2 choose 1: "You're starting off with this group of pizzas that has no toppings. And this group of pizzas that has one." For each set of pizzas, you can add a topping or not add a topping. Adding a topping means moving to the right and not adding a topping means moving to the left. He says that's why the number doubles each time. Michael explains the addition rule in a few more cases. Researcher 1 says that she asked the participants in the night session to write a general equation. Michael asks if she wants him to write a general equation. Researcher 1 responds, "that was what I was going for ultimately."
12-15	Michael says he'll name the row *r*, and he'll name the position in the row *n*. Michael write the general equation as shown in Figure E4. Researcher 1 remarks that it's nice that he reversed the notation

Minutes	Summary
	"because it's evidence that you've rebuilt it and not memorized the formula." Researcher 7 relates an anecdote about Richard Feynman. Someone told Feynman that he thought something through and then wrote it down. Feynman said no; "the writing is the thinking." Researcher 7 asks Michael to what extent the writing is part of the thinking.

$$\binom{r}{n} + \binom{r}{n+1} = \binom{r+1}{n+1}$$

Figure E4. Michael's version of Pascal's Identity

15-18	Michael says the writing is definitely part of the thinking. He says he didn't have it all in his head when he started writing. "You only have so much in your head," and "when it's on paper, you can make sense of it." Researcher 1 says when they worked on the taxicab problem, they didn't just answer the question; they gave a general solution. She asks Michael to talk about that. Michael says he doesn't remember that problem and he hasn't taken math courses in a while. He adds that it's easier to understand when you start with a specific case. But he feels comfortable putting things in the form of equations. Researcher 1 asks, "am I hearing you say that when you see the general, you also see the particular ones in there?" Michael says yes. Researcher 7 says, "what I think I'm hearing you say is, the specific numbers can sometimes obscure patterns?" Michael agrees. He says you won't get the pattern if it's just numbers. Researcher 1 asks why he thinks they proposed a general solution to the taxicab problem.
18-21	Michael says, "it's more convincing to propose a general solution" because if you give one solution, you don't know if it will work in another case. And if you had to answer the question for another case, you would have to "redo the whole process with that other number." But if you have a general solution, once you verify that it's right, it will always work. Researcher 1 asks Michael to talk about the role of being convincing. Michael says, "If someone's not fully convinced, then, it's like you failed." The point is to convey your meaning so that others understand it. Researcher 7 tells Michael he saw a tape of Michael working on the Tower of Hanoi problem. They discuss Michael's solution. Michael notes that it's the explanation of the process that makes the answer convincing.

Summary of July 22, 2002

Minutes	**Summary**
0-3	Researcher 8 tells Romina that she is interested in the session of May 12, 1999. Researcher 8 says she wants to talk about how they created "the formal notation for Pascal's Triangle, the addition rule." Researcher 8 says the group wrote it in "choose notation," which Romina says she remembers. (Romina says, "three choose two, yeah.") Researcher 8 says that Michael used pizzas to talk about Pascal's Triangle, but Romina seemed to prefer towers. Researcher 8 asks Romina if she can remember "how you did the addition rule." Romina says the top number (in choose notation) would be the number of toppings. Then she says if you were doing towers, the bottom number would be how tall the tower is, and the top number would be the number of reds.
3-6	Researcher 8 asks Romina how the addition rule would work in terms of towers or pizzas. Romina points to row 2 of Pascal's Triangle and says, "you're either going to add another … red, say, so it goes to two red, or you're going to add another blue, so it stays one red." Researcher 8 asks Romina where the numbers of the bottom (of the choose notation) come from.
6-9	Romina says the number on the top is "how many high it is," and the number on the bottom (for the example of a 3-tall tower) is "How many reds are in each of these towers three high." Researcher 8 asks about Michael's general notation. She writes the r^{th} row of Pascal's Triangle, including *r* choose *n* and asks Romina if she remembers. Romina says, "kind of, but I have to use *r* and *n*?" Researcher 8 says, "…whatever letters that you guys picked." Researcher 8 writes the next row (the r+1 row) in choose notation. Romina asks where the *n* would come in.
9-12	Researcher 8 asks Romina to write 1+2=3 in choose notation. Romina says she isn't sure, and Researcher 8 says the 1 is two choose zero, and the two is two choose one. Romina writes the general equation. Researcher 8 says she was really asking Romina to write a specific equation, not the general one, but she didn't do a good job of explaining.
12-15	Researcher 8 asks if Romina remembered or reconstructed the notation. Romina says she reconstructed it. She says she remembers Researcher 1 asking them to do something like that. "But I don't remember what I did." Researcher 8 asks Romina to explain the letters. Romina says the *r* is how high. She says when you add a

Minutes	Summary
	block, it becomes one higher. She says when you add a color, "that's where your $n+1$ comes from, I think." Referring to the 0 and 1 entries, she says, "when you add no color and one color, you have one color." Researcher 8 asks if Romina remembers how Michael did it with pizzas. Romina points to the second row of the triangle and says, "if you have a possibility of two toppings, in this one [2 choose 0], you don't have any toppings. And this one [2 choose 1], you have one topping."
15-18	Referring to the first entries in the third row, Romina says, "when you have a third topping, you could either not add anything to the pizza, or you could just add one more topping." Researcher 8 asks Romina to relate adding a topping or not adding a topping to red and blue. Romina says, "blue would be like nothing, not an ingredient, and red would be an ingredient." Romina says binary is the same thing: 0 is blue or no topping and red is 1 or a topping. Researcher 8 points to an entry (3 choose 3) in Pascal's Triangle and asks Romina to describe it in terms of pizzas. Romina says it's "three toppings, and he puts all three toppings on the pizza." Researcher 8 points to 3 choose 2 and says it's "three toppings, and he puts two of them on the pizza." Romina agrees. Romina says order doesn't matter with toppings, because "it's not like you put pepperoni, then sausage. It's you have pepperoni and sausage." Romina says pizzas are easier than towers because order doesn't matter. Researcher 8 says Romina was telling Researcher 12 that this has something to do with the taxicab problem. Researcher 8 says she is not sure how the addition rule works with the taxicab problem. Romina says she is not sure either. Romina says the top number is the number of blocks to get from one point to another, but she isn't sure about the bottom number. She draws a grid for an example taxicab problem (moving over 4 and down 2 for a total of 6).
18-21	Researcher 8 notes there's no row 6 on the copy of Pascal's Triangle they're using. Romina draws another grid (moving over 2 and down 2 for a total of 4). She points to row 3 of Pascal's Triangle. Researcher 8 asks, "It wouldn't be the next one?" Romina agrees that she means row 4. Researcher 8 says for the grid Romina has drawn, "no matter how you do it, you have to go over 2 and down 2." Romina agrees. Researcher 8 asks if she can relate this to a place on Pascal's Triangle, because "I also heard you say [on the taxicab session tape] that right means red." There is a pause. Romina draws some lines. She says, "the way I'm thinking about it, it's zeroes if

Minutes	Summary
	you could just go in one direction. … But you always have to go down one." [She is looking at a grid where the movement is over 3 and down 1.] Researcher 8 says that it might work with what she drew. Researcher 8 says that in the grid Romina already drew, going over four would be the zero. Romina agrees and says that's why it goes "like that." [She draws a diagonal.] Researcher 8 says that on the tape [of the taxicab session], the Pascal's Triangle "looked kind of sideways." Romina asks for a copy of the taxicab problem, but they don't find one so Romina draws her own grid.
21-24	Romina writes numbers in the grid: a 2 in the top left box, and two 3s in the boxes to the right and below that box. Romina points to the points at the top of the box and the two points to its right and below and says, "this would be 111." Then to get to the number below, she says, "to get to this one [a point two steps away from the starting point – one to the right and one down], you either take one red one blue or one blue one red." She moves the pen over one path and then the other as she speaks. Researcher 8 asks Romina to show the red and a blue, blue and a red point from the grid on the Pascal's Triangle. Researcher 8 says that's a 2-tall towers, and she says she thinks it's the 2 from the 121 row and asks Romina if she agrees. Romina agrees but she says she doesn't understand the 1's. Researcher 8 says that maybe it's the 1's from the diagonals. Romina says she was looking at the diagonals shifted up. Researcher 8 asks if there's a way to explain the addition rule that way. Romina points to the 2 from the 1 2 1 set on the grid and says you are adding to go to the right.
24-27	Romina draws a new grid. She indicates one point and says it is two blues. She calls another point a blue and a red. She says, "to get to this three, we were either here [the two blues] or here [the blue and the red]. … We either added another blue or added a red." Researcher 8 asks again. Romina says, "we either got to this three from an all blue tower or a tower that had one blue and one red. And then we just added one, so we either added another blue or we added a red." Researcher 8 says "or else we went from two downs and added a right, or we went from a right and a down and we added a down."

Summary of July 31, 2002

Minutes	Summary
0-3	Ankur talks about his experiences in college. He started at Rutgers but he is now attending Kean, majoring in physical therapy and economics.
3-6	Researcher 8 talks about the night group; it was the night before the Junior Prom: Brian was late and his tux didn't fit.
6-9	Researcher 8 tells Ankur that he called their explanation of what they had been doing "the Reader's Digest Version." Researcher 8 shows Ankur some diagrams of Pascal's Triangle and talks about what they did in the night session with it. Ankur says it looks a little familiar. Researcher 8 tells Ankur that they had their favorite metaphors: pizzas, towers, and binary notation. "If any of that is coming back to you, I'd be interested in hearing how ... can you tell me how you figured this out?" Ankur asks to see a video. There is a break while they access the video. We return to see Researcher 8 and Ankur looking at a computer screen. Ankur asks if Michael was interviewed. Researcher 8 says yes; she was sick the day Michael was interviewed, so Researcher 1 interviewed him. Michael used his binary notation. Ankur says Michael brought that into everything.
9-12	They continue to look at the computer screen. We hear Michael's voice talking about mushrooms and peppers. Researcher 8 says Michael is talking about his binary notation. We hear Researcher 1 say, "I want to think of them as chooses." Researcher 8 asks if it would help if Ankur can see a transcript. Researcher 8 sets up the computer to show a partial transcript. We hear Researcher 1 again say, "I want to think of them as chooses..." Researcher 1 says, "let's take this row 1 3 3 1." Ankur says, "I have a bad memory." Researcher 1 says, "I'd like you to write ... what would a general row look like?" Ankur says he remembers the towers. "We used like these numbers to represent the towers."
12-15	Ankur says if there was a tower with reds and blues, the first one (first entry in row 3) would represent no reds. Then the others would be one red, two reds, and three reds. Ankur says he doesn't remember the videotape, but he remembers what he thought. They watch more of the videotape. Jeff is talking; Researcher 8 says he's picking up on the notation there. We hear Romina's voice as well. Michael says the height of the towers is *n*. Researcher 8 tells Ankur that Researcher 1 asked to see the n^{th} row; the group is struggling with that. Ankur says, "this number [the top number in choose

Minutes	Summary
	notation] is the height of the tower; this number [the bottom in choose notation] is the one pertaining to the particular color in the tower." Researcher 8 asks if anything is coming back about the n^{th} row. Researcher 8 asks Ankur to write the n^{th} row in choose notation. Ankur says the top would be n – that would be the height of the tower. Ankur asks if Researcher 8 wants the whole row. Researcher 8 says to put a random position in the row. "Could you start writing the row? What would the first number be in the row?" Ankur says the first number would be zero. Researcher 8 asks Ankur to write that.
15-18	Ankur says, "it would be one" as Researcher 8 says, "and the second one would be n 1." Ankur says, "all the way to n over n." Ankur writes the n^{th} row. Researcher 8 starts the videotape again. She says, "they are trying to figure out what to put in the middle" of the row. They watch Michael and Jeff talking about the n^{th} row. On the videotape, Michael says something about zero to the n. Ankur says he doesn't understand what Michael means now, and in the video he says the same thing. Researcher 8 says Michael misspoke there. Researcher 8 notes that Jeff wrote division instead of choose notation, like Ankur. We hear Jeff say, "it goes from zero to n." Then we hear Researcher 1 tell Jeff to remove the division symbol. Then she asks them to put something in the middle of the n^{th} row.
18-21	Researcher 8 asks Ankur if he gets what it is that Researcher 1 is asking. Ankur says he's not really sure; he says n choose r. Researcher 8 says the group decided not to use r. Researcher 8 says, "we have n for the n^{th} row; we want a general way to write some random entry between 0 and n in that row. That's what she's trying to get everybody to do here." Ankur asks if it's n minus something. Researcher 8 says they talked about that a little bit but then they decided on something else. They listen as Jeff says n choose r, as long as r doesn't exceed n. Researcher 1, referring to the n^{th} row, says "something in here could be an n choose r... or an n choose x." Researcher 1 asks them to rewrite the triangle nicely. As Jeff is doing that, Researcher 8 tells Ankur that Researcher 1 is saying, "do this, do that."
21-24	Researcher 8 says that seemed unusual. Ankur asks why. Researcher 8 says it's because she usually asked more general questions. Ankur says she probably asked them to write on the board so that the camera could see what they were doing. They return to watching the video; Jeff is writing the 5^{th} row. Researcher 1 tells

Minutes	Summary
	him to write the n^{th} row. Researcher 8 tells Ankur that they wrote n choose x in the middle to represent kind of a middle one. On the tape, Researcher 1 asks Jeff to show an addition rule of Pascal's Triangle from the 4th row to the 5th. She says 3+3=6. Researcher 8 says, "She wants to see 3+3 there" [on the triangle]. Researcher 8 says "you are saying that 3 choose 1 plus 3 choose 2 equals 4 choose 2, basically." Ankur agrees.
24-27	We see Researcher 1 on the tape say 3 choose 1 plus 3 choose 2 is four choose 2. "So what's 4 choose 2 plus 4 choose 3?" As Jeff prepares to answer, Ankur says, "it's the next row." Researcher 8 asks him to write the next row "so we can work on that." Ankur does so. Researcher 8 repeats Researcher 1's question: "What is 4 choose 2 plus 4 choose 3?" Ankur looks at what he wrote and says, "Did I write that right?" Researcher 8 says it looks good; she repeats the question. Ankur says, "5 choose 3." Researcher 8 asks him to draw the arrows. Ankur says, "I think I see where this is leading" as the tape continues to play and Michael says 5 choose 3. Researcher 8 asks if he remembers or if he now sees where it is leading. Ankur says he can probably remember what he thought. Ankur says, "if you gave me these two, that would lead to 6 choose 3." Researcher 8 asks him to squeeze the 6 choose 3 under the 5^{th} row so they can put the arrows in. They continue to watch the video; on the video Ankur says, "it's always the one on the right." Ankur says that's what he was thinking right now. Researcher 8 says, "tell me what that means, that it's always the one on the right." Ankur says for example 2 choose 0 plus 2 choose 1 leads to 3 choose 1. "The one on the right refers to this 1" (he points to the 1 in 2 choose 1). Researcher 8 says you go to the next level, which means the 2 becomes a 3, and it takes the bottom number from the number on the right. Ankur agrees. Ankur says, "whatever the reasoning is behind it, which I probably couldn't tell you right now." Researcher 8 says it sounds like you could do any two of them "even if they weren't on the triangle."
27-30	Ankur says, "that's what led us to write the formula." Researcher 8 says Michael is explaining why this works in terms of pizza toppings. On the tape, Michael says, "this guy is going to get another topping... this guy is not going anywhere..." Ankur laughs. He says he was reading ahead in the transcript to where Brian showed up and had to eat. Ankur laughs as Brian talks about his tux. There is a break in the tape, and then we watch Jeff talking about writing Pascal's Triangle in terms of chooses. Ankur writes what Jeff is saying (n

Minutes	Summary
	choose x, etc.) on their copy of the triangle. As Jeff writes the row above the n^{th} row, Researcher 8 asks Ankur to write that row.
30-33	Researcher 8 says that's like the n-1 row. Ankur writes the n-1 row. Ankur asks how to write the x entry. He asks, "n-1 choose x?" Researcher 8 tells Ankur she is not sure where n-1 choose x would go, because the rows are offset a little. She asks Ankur if the n-1 choose x will be to the right of n choose x. Ankur says "over there" (to the right). Researcher 8 agrees. Ankur also writes n-1 choose x-1. They watch Jeff write the last entry in the n-1 row. Jeff asks Researcher 1 where to go from here. Jeff says the third row is 3 choose 0, 3 choose 1, 3 choose 2, and 3 choose 3.
33-36	Jeff is telling Brian how they wrote Pascal's Triangle in choose notation. He tells Brian that n! means n factorial. Romina explains to Brian where the factorials come in. Researcher 8 asks, can you see that that (the factorials) is n choose x?" On the tape, Researcher 1 says, "I want you to show me how the addition rule works in general." Researcher 8 says to Ankur, "I think you were the first one to figure out what she was asking you to do there. Do you remember?" Ankur laughs. He says, "isn't that what we were just talking about?" Researcher 8 says, "she's asking you to do it with the n-1 row and the n row, but I'm not sure that's how they did it." Ankur said, "add this row with this row and show what would come after that." Researcher 8 asks if Ankur could show that now or if he wants to see more of the video. Ankur says, "Add this and this?" as he points to n choose x and n choose x+1. Researcher 8 says yes, and Ankur starts to write. "I guess this would be n plus 1 and that would be x+1." Researcher 8 says, "you were saying the bottom number always comes from the right." Ankur agrees. Researcher 8 says the next thing she asks for is writing 3+3=6 without the arrows.
36-30	Researcher 8 says you could also write 3 choose 1 plus 3 choose 2 equals 4 choose 2. Researcher 8 asks Ankur to write a regular equation without arrows for n choose x, etc. Ankur says, "I understand what you're saying, but I don't understand what you're looking for." Researcher 8 says "an equation with a plus sign and an equals in it." She takes the pen and writes 3+3=6, "only now I'm changing..." So she writes 3 choose 1 plus 3 choose 2 equals 4 choose 2. Ankur writes n choose x plus n choose x+1 = n+1 choose x+1. Researcher 8 says, "she was pushing to see an actual equation there." They watch the video as Jeff and Michael write the equation. Ankur watches himself tell Jeff what to write. On the tape,

Minutes	Summary
	Researcher 1 asks Brian to have them explain it to him.
39-42	On the tape, they explain to Brian why you add. Researcher 8 tells Ankur that they're explaining it in terms of pizza toppings. Ankur says that's what Michael was talking about earlier. Jeff is saying, "if it gets a topping, it goes to the x+1..." Researcher 8 says she's not sure she followed all the explanation about pizza toppings. Ankur says he's not sure either. Ankur says he remembers the towers. Researcher 8 points to row 3 of Pascal's Triangle and says this is three-tall towers. Ankur says, "no red, one red, two red, three red." Researcher 8 asks what row n would be in terms of towers. Ankur says a tower n tall with none of one color. Researcher 8 says it goes all the way up to n reds. Researcher 8 asks if it makes sense to explain the addition rule in terms of going to the next row. Ankur says you could do it the same way. Researcher 8 says n choose x is a tower n tall, and Ankur says it has x reds. Ankur says you add it with a tower the same height with one more red, and you get a tower one taller. Researcher 8 asks how you get to the one taller one from here. Ankur says it's n+1.
42-45	Researcher 8 says she's trying to visualize how that works with towers, how the towers look. Ankur says they're both the same height, and one has one more red. Researcher 8 says the next row is one taller. The n choose x towers become one taller so you add a cube to them, and the same for the n choose x+1 ones. Ankur agrees. Researcher 8 asks why it's like that. Ankur says he couldn't say right now. Researcher 8 suggests that they look at it in terms of the three-tall tower with one red. Researcher 8 asks how you change that into a 4-tall tower with two reds. Ankur says you put a red on it. Then Researcher 8 asks about the 3-tall tower that already has two reds that goes to a 4-tall tower that has two reds. Ankur says you add a blue. Ankur says you add all blues to this one (the one on the right) and all reds to that one (the ones on the left). Researcher 8 says you have to add one, because you're going from 3 to 4. Ankur says you add a red to all three of those towers. That's why the bottom number is x+1, because these you added the same color. Researcher 8 asks him to repeat. Ankur says you go from n choose x to n+1 choose x+1 by adding a red and from n choose x+1 to n+1 choose x+1 by adding the other color. Researcher 8 says that makes a lot of sense. She says that was the main thing she wanted to explore, how they went from the special case to the general case. She intended to ask what he remembered about Michael's explanation. He didn't remember

Minutes	Summary
	that, but he told her another way to look at it.
45-end	The discussion ends.

APPENDIX F: TRANSCRIPT INFORMATION

List of Researchers

Code	Name
Researcher 1 (R1)	C. Maher
Researcher 2 (R2)	E. Steencken
Researcher 3 (R3)	R. Kiczek
Researcher 4 (R4)	R. Pantozzi
Researcher 5 (R5)	R. Speiser
Researcher 6 (R6)	E. Muter
Researcher 7 (R7)	G. Davis
Researcher 8 (R8)	E. Uptegrove
Researcher 9 (R9)	E. Mayansky
Researcher 10 (R10)	A. Alston
Researcher 11 (R11)	L. Alcock
Researcher 12 (R12)	A. Powell

Transcript Conventions

Symbol	Description	Usage
–	dash	an interrupted or incomplete utterance
…	ellipses	omitted words
[]	brackets	parenthetical notes, e.g. pauses or gestures
//	slashes	simultaneous speech

APPENDIX G: TRANSCRIPTS OF 1997 SESSIONS

December 12, 1997

Time	Speaker	Transcript
00:00:00	–	During the first 40 minutes, the students work on the three-, four-, and five-topping pizza problems. They settle on respective answers of 7, 15, and 30 (excluding the plain pizzas). The students begin to question the answer for the five-topping pizza because it does not fit the pattern.
00:40:39	Romina:	Five is wrong?
00:40:41	Michael:	I think it's 32. And without that cheese, it would be 31. I'm thinking. I'll show you what. [Michael starts to write. Jeff looks over his shoulder and Michael moves away.]
00:40:48	Jeff:	No, no, you go.
00:40:50	Michael:	All right.
00:40:51	Ankur:	Mike, tell us the one that we're missing, then.
00:40:53	Romina:	We have 30.
00:40:55	Michael:	Ah, OK. So, you know like the binary system a little while ago? The ones and zeroes? Binary. Right?
00:41:04	Jeff:	Uh-huh.
00:41:04	Michael:	The ones would mean toppings. So if you had a four-topping pizza, you'd have four different places, all right, in the binary system. So you'd have- the first one would be just 1, and that would be- that, that's the next number up. You remember what that was? That was like 1 is sort of like 2. And this [11] was 3.
00:41:25	Jeff:	I don't remember what you're talking about.
00:41:26	Michael:	You don't remember the binary-
00:41:27	Jeff:	No, I understand, I know exactly what you're talking about. It's the thing we looked at in Mr. Poe's class. It was computer and this and that.
00:41:30	Michael:	Yeah.
00:41:31	Jeff:	Yeah. I don't, I don't know how to add 'em or-
00:41:35	Michael:	Well, I don't know. I think, I have this thing in my head. It works out in my head, is all I want to say.
00:41:42	Jeff:	Maybe for a- [Inaudible.]
00:41:42	Michael:	[Michael writes on the blackboard.] So you got four toppings. Four different- this is like four. [Michael indicate the four different columns.] There's four places of the binary system. It all equals up to 15. [Michael writes 15.] That's the answer for the four-topping. Five toppings would be-

Time	Speaker	Transcript
00:41:56	Jeff:	No, I, I don't understand what you're trying to do.
00:41:58	Michael:	OK. OK. Um.
00:42:02	Jeff:	What are you doing?
00:42:03	Michael:	For, for just- [Pause. Michael takes another sheet of paper.] For just one topping pizzas, you've got like four different toppings. It would be like, one. [Michael writes.] That would be four, right?
00:42:24	Jeff:	Uh-huh.
00:42:24	Michael:	All right, two toppings, you could have like- [Michael writes.] That.
00:42:28	Jeff:	Uh-huh.
00:42:28	Michael:	That. [Michael writes combinations of two 1's and two 0's.] And all those little combinations like that. You know what I'm saying? And for three, you have combinations of like this, that. [Michael writes groups of three 1's and a 0.] And four, you'd have that. [Michael writes 1111.] All right? And it would, like, you would, you would have as many different combinations as, like- I can't say it.
00:42:53	Romina:	So, is, is the, the 1. Is that your topping?
00:42:56	Michael:	Yeah.
00:42:56	Jeff:	Each 1 is a topping. The 0s are no toppings. The 1s are toppings.
00:43:00	Romina:	That's all you had to say.
00:43:01	Michael:	So you go from this number [0001], which in the binary system is 1, to this number [1111], which is 15. And that's how many toppings you have. There's 15 different combinations of 1's and 0's if you have four different places.
00:43:16	Jeff:	All right.
00:43:17	Michael:	I, I don't know how to explain that, but it works out in my head. I have weird things going on in my head. And if you have an extra topping, you just add an extra place, and that would be 16, and that would be 31. [Michael writes.]
00:43:30	Jeff:	And then you'd add the cheese?
00:43:31	Michael:	Yeah. Plus the cheese is 32.
00:43:33	Jeff:	So which one-
00:43:33	Michael:	You add another one; it would be 32 plus 31.
00:43:37	Jeff:	Which would be-
00:43:38	Romina:	How does that go with 15?
00:43:39	Michael:	Because, all right. You have. You know how to write binary?
00:43:44	Romina:	Did you work it all out? Or-
00:43:45	Michael:	You write- this is how you write binary, I don't know how to say. This is 1, this is 2, this is 3, this is 4, this is 5. [Michael writes 1,

Time	Speaker	Transcript
		10, 11, 100, and 101.] And those are the different combinations. You know? Of, of, uh. And eventually, you will come up to this [1111]. That's it for four different, four different-
00:44:07	Jeff:	Yeah, you could, you see.
00:44:07	Romina:	Yeah.
00:44:10	Jeff:	He's just leaving out like these [zeroes in front of the binary numbers], right?
00:44:11	Michael:	Yeah, I'm not writing them.
00:44:11	Jeff:	He's not writing them. Which is, which, which confused me at first. 'Cause now it looks a lot.
00:44:16	Michael:	And then eventually you come up to that number, which- you remember the binary. This, the first place was 1, the second was 2, and that was 8.
00:44:24	Jeff:	Uh-huh.
00:44:25	Michael:	Remember that? And it all leads up to 15. 'Cause that's how many different combinations of 1's and 0's you could have with four places.
00:44:34	Jeff:	Seventh grade?
00:44:36	Michael:	I didn't have Mrs. Toy's class.
00:44:37	Jeff:	Oh, [Inaudible.] class.
00:44:37	Michael:	Yeah.
00:44:36	Jeff:	Eighth grade.
00:44:37	Brian:	You had Miss Toy.
00:44:38	Jeff:	We had her seventh. We had Poe sixth grade.
00:44:39	Romina:	Ah. Seventh. I wasn't there.
00:44:41	Brian:	Well, we were there.
00:44:42	Jeff:	Um.
00:44:43	Romina:	I'm just making sure it's not the wrong way.
00:44:44	Ankur:	So actually [Inaudible.] too.
00:44:47	Romina:	Hm?
00:44:48	Jeff:	[Aside.] When did we graduate from grammar school?
00:44:53	Michael:	That's how I came up with it.
00:44:54	Romina:	[Side comment about grammar school.]
00:44:59	Jeff:	All right, you want to try and explain that?
00:45:00	Michael:	What?
00:45:01	Jeff:	See what you think about it?
00:45:03	Michael:	They got it on tape.
00:45:04	Jeff:	They want you up to the board.
00:45:06	Michael:	I'm not going to the board.
00:45:08	Jeff:	Yeah, go up to the board.

Time	Speaker	Transcript
00:45:08	Michael:	Like-
00:45:09	Jeff:	I went there.
00:45:09	Michael:	I can't even explain it to you.
00:45:12	Jeff:	'Cause then she, she'll-
00:45:13	R1:	Romina will help you. Right, Romina?
00:45:15	Jeff:	Sure.
00:45:15	Romina:	OK.
00:45:16	R1:	Romina will help you at the board. So if Romina can follow it, right?
00:45:20	Jeff:	Yeah, so you'd be the perfect, she's like the, she's like the little kid.
00:45:23	Michael:	I don't know if anyone knows what I'm, what I'm thinking.
00:45:24	R1:	You're just asking for the questions.
00:45:26	Jeff:	Now. She'll continue to do that.
00:45:27	R1:	So why don't
00:45:28	Michael:	I don't even know if anyone knows what I'm thinking.
00:45:29	R1:	Why don't you come here and show us? …
00:46:43	Michael:	Well, you know. Somebody [Inaudible.]. You know what the binary system is?
00:46:47	R1:	Yeah. I know what the binary system is. But tell me what you think it is.
00:46:49	Michael:	Now this is how I interpret it-
00:46:50	R1:	What do you think of when you think of it?
00:46:51	Michael:	-Into the pizza problem. You know what it is? You understand it?
00:46:52	R1:	Tell me what you think the binary system is.
00:46:53	Brian:	I don't know what it is.
00:46:54	Michael:	[To Brian.] Shut up. [To Researcher 1.] You know what it is.
00:46:56	R1:	What is it?
00:46:57	Romina:	Is this just replacing the [Inaudible.]
00:46:58	Brian:	Yeah, what is, what is it?
00:46:58	Michael:	You know.
00:46:59	Ankur:	Mike.
00:46:59	Michael:	It's just like -
00:46:59	Ankur:	You want me to help you?
00:47:00	Michael:	It's like a different way of writing numbers.
00:47:01	R1:	Yeah. Ankur will help.
00:47:02	Michael:	a different way to look-
00:47:03	R1:	Ankur. Do it together. Go ahead, Ankur.
00:47:04	Ankur:	OK. I'll dictate and you write.

Time	Speaker	Transcript
00:47:06	Michael:	OK. I'll write everything you say.
00:47:08	Ankur:	You start explaining it and I'll put in my two cents.
00:47:11	Michael:	I got to explain it?
00:47:12	Ankur:	Start explaining it and I'll help you.
00:47:13	Michael:	It's just a different way of writing numbers. Instead of having place, as places as one through ten, it's, or one through nine, one through, I don't know, we're just having one, one up through one, zero and one. So, the first place would be one. The second would be two. The third would be four. And this one would be, well this is, if only you have four places. That would cover all numbers.
00:47:35	R6:	The places as place value?
00:47:36	Michael:	Place value. Yeah.
00:47:37	Romina:	Like do they double?
00:47:39	Michael:	Yeah. It goes by double. Like if you want, if you have to express the number one, it's just this [Michael writes 1 on the board]. The number three, two would be expressed by this [10]. Eh? Number three would be expressed by this [11]. Four [100]. Five [101].
00:47:57	Jeff:	And then you'd skip to eight, because there'd be-
00:47:58	Michael:	And then that [111] would be, uh, six, no, uh-
00:48:03	Jeff:	Seven.
00:48:03	Romina:	Seven.
00:48:04	Michael:	Five. Yeah. Seven. You, you understand? Right?
00:48:06	Romina:	Michael, if I wanted to get like, now if I had a one zero zero zero zero, would that be sixteen?
00:48:14	Michael:	One. How many zeros?
00:48:15	Romina:	Four zeros. [Actually, there are four digits, but three zeroes.]
00:48:17	Jeff:	Yeah, exactly. Yeah.
00:48:17	Michael:	Yeah. Sixteen. That-
00:48:18	Jeff:	That would be-
00:48:19	Romina:	It doubles each time. Is that it?
00:48:20	Jeff:	Yeah. There's one line in the sixteen, there'd be a sixteen column.
00:48:22	Michael:	Yeah. Sixteen. It just doub- It's just-
00:48:23	Romina:	Two to the six-
00:48:25	Michael:	-Whatever power times, like one-
00:48:26	Romina:	Like it's double.
00:48:29	Michael:	-Times.
00:48:31	Jeff:	Mike, times two. Look at it.
00:48:32	Michael:	Times two. Yeah.
00:48:32	Jeff:	One times two is two. Two times two is four.

Time	Speaker	Transcript
00:48:35	Romina:	That's all I wanted to know.
00:48:35	Jeff:	Four times two is eight.
00:48:37	R1:	OK. But, but when you said two to the power, um, let's not throw that idea away. It's an interesting idea. Can, can you think about that as two to different powers?
00:48:44	Michael:	Mm. Yeah. I think it's right, I guess.
00:48:47	R1:	How?
00:48:47	Michael:	Two to the first. [Michael points to the two.] Two to-
00:48:48	Ankur:	To the third.
00:48:48	Michael:	To the second. Two to the third.
00:48:49	Ankur:	Except for the one.
00:48:50	Michael:	Two to the fifth is what?
00:48:51	R1:	So how can you think of the one?
00:48:52	Michael:	Two to the zero.
00:48:53	R1:	Two to the zero.
00:48:54	Michael:	What is two to the fifth?
00:48:54	R1:	So you can think of it as two to different powers.
00:48:56	Brian:	Thirty-two.
00:48:57	R1:	You can think of it as two to the zero. Which kind of tells you why they defined two to the zero as one.
00:49:02	Michael:	Three to the two.
00:49:02	R1:	It kind of makes it look pretty.
00:49:03	Brian:	Thirty-two.
00:49:04	R1:	OK. So you, I like your idea of two to the different powers, and I like your idea of the doubling, and I like your examples. Your examples are very clear.
00:49:12	Michael:	OK, so if you have four toppings, this will show you how many different, um, pizzas, no matter, pizzas with one topping, pizzas with two toppings or three. 'Cause you, you, if you go by the number system, the lowest number you can get is this. [Michael points to the 1.] The highest you can get is this. [Michael writes 1111.] And if you add, if you- the value of this number is 15. And that's the number of pizzas. 'Cause you will have one, which is-
00:49:41	R1:	Wait. Wait. Where are you getting fifteen?
00:49:42	Michael:	'Cause then, um-
00:49:45	Jeff:	Eight plus four.
00:49:45	Michael:	Add these numbers up. You add them up. You, you understand?
00:49:49	R6:	So you're saying it's one one, one two, one four, and one eight.
00:49:53	Romina:	Yeah.

Time	Speaker	Transcript
00:49:54	Michael:	What?
00:49:55	Romina:	Yeah. Yeah.
00:49:56	Michael:	It's, yeah. All right. Yeah. Um. That's it. You can go from the low, from the lowest number, just this [writes 1] and the highest would be that [1111] and all the numbers in between are the different combinations of pizzas. You can get like, um, you know, something like this. [1011] It will all, the highest number you can get is this [1111], which is fifteen. There's fifteen different combinations for four toppings.
00:50:19	Jeff:	'Cause you went through all the other combinations.
00:50:21	Michael:	And if you have a five topping pizza-
00:50:22	Jeff:	To get all ones across.
00:50:22	Michael:	-You add the next place, which is a sixteen. And there's uh, thirty-one. This is without the plain. You just add one for the plain pizza. The next place would a thirty, thirty, thirty-two. The next, then that. And the place after that would be like a sixty-four. And then all we got to do is add, add, like do the binary system, find the value of that.
00:50:43	Michael:	You will, you will get the number of pizzas.
00:50:44	Romina:	I got it, Mike.
00:50:45	Jeff:	Going from just the one line to all four lines, it has to go through each combination to give you every number in there.
00:50:50	Michael:	Yeah, it has to-
00:50:50	Jeff:	So that's how-
00:50:51	Michael:	-Go through-
00:50:51	Romina:	Yeah, I understand that.
00:50:52	Michael:	-Every single ones and zeros combinations possible.
00:50:54	Jeff:	To get, to move up every number. It's got to go to a different one, so it-
00:50:58	R1:	I still have one thing that I'm still confused about. I, I believe I follow this. This is really very helpful. But before you had one, two, three, and four. And now you're just doing it all with ones and zeros. How come? How can you do this?
00:51:12	Romina:	It's almost like-
00:51:12	Ankur:	The ones and zeros are representing-
00:51:14	Jeff:	We're saying that-
00:51:15	R1:	What are they representing?
00:51:16	Jeff:	That one represents a topping, any topping.
00:51:18	Michael:	No. 'Cause, OK. I'll tell you what-
00:51:20	Jeff:	Go for it.

Time	Speaker	Transcript
00:51:20	Michael:	Let's say you got, uh, like ones, two, three, and four.
00:51:24	Ankur:	You got a four, huh?
00:51:24	Michael:	Ones, twos, threes, and fours. There, instead of having these numbers on top, they had like one, two, threes, and fours and a pizza with just a one would be that [1000], you know, in my interpretation would be that. a pizza with just a two in, in mine would be this [0100]. So they just, they just had like um-
00:51:43	Jeff:	a different set-up.
00:51:43	Michael:	So as many toppings as you have for mine, it's like one, two, four and eight, you just give it four places. But theirs, they gave as numbers. I don't know. And they just matched up the numbers until they couldn't find any more.
00:51:53	R1:	So what am I supposed to think about when I think about those numbers one, two, three, and four? What do they represent?
00:51:58	Jeff:	Well, well.
00:51:59	Michael:	Well, the different toppings.
00:52:01	R1:	The different toppings. So, so when I- when I read that zero one zero zero what? They're the different toppings?
00:52:09	Michael:	That would be one combination of, that would be one pizza.
00:52:11	R1:	Which one?
00:52:12	Jeff:	Each, any set of four numbers-
00:52:14	R1:	Any of them?
00:52:15	Jeff:	-Is a set up of one pizza.
00:52:15	Michael:	Each.
00:52:16	R1:	Is it a particular pizza?
00:52:17	Michael:	This would be-
00:52:17	R1:	Zero one zero zero?
00:52:18	Michael:	This is a pizza. That's a different pizza.
00:52:19	Jeff:	Well, you're not, you weren't asking for the different, like the actual combinations.
00:52:21	R1:	But I'm asking it now. Like-
00:52:23	Michael:	You want that.
00:52:23	R1:	What's the difference between one zero zero zero and zero one zero zero?
00:52:26	Jeff:	Well, that would be like - that would be the difference between an onion pizza and like a pepperoni pizza. 'Cause you could give eight, you could make, like, the eight up there be onions. You could- you could-
00:52:34	Michael:	Yeah.
00:52:34	Jeff:	Write, label the-

Time	Speaker	Transcript
00:52:35	Michael:	OK, every-
00:52:35	Jeff:	The other things on top.
00:52:35	Michael:	Every time you see a 1 in there, that's an, that pizza has an onion in it. [Michael indicates the left column.]
00:52:39	Jeff:	In the first column.
00:52:41	Michael:	Let's give it onions, uh, what is it? Mushrooms.
00:52:42	Jeff:	Yeah. Pepperoni.
00:52:43	Michael:	Pepperonis.
00:52:44	Jeff:	Pepperoni.
00:52:45	Romina:	"*pp*."
00:52:45	R1:	Sausage?
00:52:45	Michael:	Sausage- OK? Every time you see a 1 there [in the *o* column, you know that pizza has a - onions. Every time you see a 1 in this in this place right there [the *m* column], you know it has mush, uh, mushrooms. And a 1 in this place [the *pp* column], you know it has a pepper- no, pepperoni. And a 1 in this [the *s* column] is a sausage. That's what each of those 1s mean. Those- those are- that's how I expressed my- this is my different combinations, right there
00:53:07	R1:	OK. So what do you think of this way to do it?
00:53:09	Jeff:	I'm impressed.
00:53:09	Brian:	Yeah.
00:53:11	R1:	Sup- suppose you have ten toppings to select from?
00:53:14	Jeff:	Well, then you'd go one, two, you'd go-
00:53:16	R1:	You'd go like that? So, so how many, how many, how many-
00:53:19	Michael:	You want me to give you a number?
00:53:20	R1:	-Choices do you have? How many choices do you have?
00:53:21	Michael:	Eh? Let's see. You got-
00:53:22	Jeff:	Well, you go-
00:53:25	Michael:	One, two, three, four. [Michael is counting places as he writes multiples of two place values.]
00:53:27	Romina:	Eight.
00:53:28	Michael:	Five. Six. Seven. Eight. What's that?
00:53:33	Jeff:	One over, one over.
00:53:34	Michael:	Give me numbers. More.
00:53:34	Jeff:	One twenty-eight. One twenty-eight.
00:53:35	Michael:	That would be, uh, one, two, three, four, five, six, seven, eight. Two more.
00:53:41	Ankur:	Yeah. Write eight. Two fifty-six.
00:53:41	Jeff:	Two fifty-six.

Time	Speaker	Transcript
00:53:43	Michael:	Two more. Two fifty-six. Eight. Nine.
00:53:48	Ankur:	Five twelve.
00:53:49	Michael:	That would be five-
00:53:50	Ankur:	Twelve.
00:53:50	Jeff:	Twelve.
00:53:52	Michael:	Six. Two fifty-six.
00:53:52	Jeff:	Five twelve.
00:53:52	Ankur:	Five twelve.
00:53:54	Michael:	Two fifty-six.
00:53:55	Jeff:	Yes. Five twelve. [Michael writes 512.]
00:53:55	Michael:	OK. And then you just add those numbers up. And then it would be plus one.
00:54:00	Jeff:	And then it would be 513.
00:54:01	Michael:	Yeah, plus one.
00:54:01	Jeff:	With the plain pizza if you add ten. That's kind of crazy though, thinking about that.
00:54:04	Michael:	No. It's not five thirteen. It's five twelve plus this.
00:54:07	Ankur:	It's five twelve plus
00:54:08	Brian:	All them, plus that
00:54:08	Michael:	That plus that plus that plus that
00:54:09	Jeff:	Oh. Plus that.
00:54:10	Michael:	Plus that, plus that one with the cheese.
00:54:12	Ankur:	Yeah.
00:54:12	Michael:	With the plain pizza if you add ten.
00:54:13	Jeff:	That's nuts.
00:54:14	Michael:	Cheese would be zero zero.
00:54:14	Jeff:	Zero zero.
00:54:15	Michael:	All zeros.
00:54:15	Jeff:	All zeros.
00:54:15	Michael:	Not - there'd be no one in a cheese pizza.
00:54:17	Romina:	I understand that.
00:54:21	Michael:	Uh huh.
00:54:21	Jeff:	That's a lot of toppings.
00:54:22	R9:	And if you had n toppings?
00:54:24	R1:	If you had n pizzas?
00:54:26	R9:	n toppings.
00:54:27	R1:	n toppings.
00:54:27	Michael:	n toppings, it'll be-
00:54:28	Jeff:	n toppings?
00:54:29	Michael:	n times, uh, plus-

Time	Speaker	Transcript
00:54:31	Jeff:	*n* to the power, right?
00:54:33	Michael:	No. It's uh-
00:54:34	Ankur:	It's something.
00:54:35	R1:	Why don't you talk about that? If you had *n* toppings. And I'm not saying that works for what you gave us- [Inaudible.] How many choices would you have?
00:54:40	Ankur:	Look, Mike.
00:54:41	R1:	I like R9's question.
00:54:42	Ankur:	If you had two to the zero, it's-
00:54:43	Michael:	'Cause when we had four, when we had four toppings we had-
00:54:46	Jeff:	If you have two to the second-
00:54:47	Ankur:	If you have one topping-
00:54:48	Michael:	Fifteen. So then-
00:54:49	Ankur:	-It's two to the first.
00:54:49	Jeff:	Uh-huh.
00:54:50	Ankur:	And two toppings, two to the second. It's two to the n^{th} power.
00:54:52	Jeff:	So if you had three, it would be two-
00:54:55	Michael:	Two to the *n* minus one power.
00:54:57	Jeff:	You're saying *n*-
00:54:58	Michael:	All right?
00:54:59	Jeff:	*n* minus one?
00:55:00	Michael:	No, it's one. It's not.
00:55:01	Ankur:	No.
00:55:03	Jeff:	'Cause then you're saying you have zero combinations for two to the first.
00:55:05	Michael:	Two, uh, first, two to the-
00:55:07	Jeff:	Ach!
00:55:08	Michael:	-First is two. Zero is the first power. This is the second power. And this is the third power, right? [Michael points to each column in turn.] Two to the third power?
00:55:14	Jeff:	Uh-huh.
00:55:14	Ankur:	Uh-huh.
00:55:14	Michael:	And you have to- two to the *n* minus one? Has to be plus two to the- What the hell am I doing? [There is laughter as Michael erases the board.]
00:55:24	Jeff:	Hey, Romina.
00:55:28	R1:	Why don't you sit and talk about that for a little bit and then tell us what you're-
00:55:30	Michael:	You want, you want us to give you like *n*, *n* amount of toppings?
00:55:33	R1:	No. I want you to, I want you to- R9 asks a very interesting

Time	Speaker	Transcript
		question.
00:55:35	Ankur:	It's two to the five and then it's-
00:55:36	R1:	That you told me how many-
00:55:37	Ankur:	-Plus two to the fourth, plus two to the third.
00:55:39	R1:	-Pizzas there would be if you have - what was it?
00:55:42	Brian:	[To Ankur.] It's x plus 1.
00:55:43	R1:	If you had five toppings?
00:55:44	Ankur:	Two to the fifth, plus two to the fourth.
00:55:46	R1:	No. Just for five toppings. Only five toppings.
00:55:49	Ankur:	Two to the, uh-
00:55:50	R1:	Why are you adding them? I'm, I'm not too sure I understand that addition.
00:55:53	Brian:	'Cause you know how they're add-
00:55:53	Ankur:	Because that's the way, it is, because you see like-
00:55:55	Jeff:	We just explained that.
00:55:56	Ankur:	-For those four toppings how we added eight, four, two and one?
00:55:59	Michael:	And then you get the-
00:56:01	Jeff:	Fifteen.
00:56:02	R1:	You mean you're getting five- What are you getting when you add them? What question I- the question I asked you, if you could select from these five toppings.
00:56:09	Michael:	Uh-huh.
00:56:10	Ankur:	Yes.
00:56:10	R1:	How many pizzas can you make? What was the answer to that?
00:56:13	Michael:	Uh, what's-
00:56:15	Jeff:	Thirty-two? Plus one.
00:56:16	Michael:	Thirty-one-
00:56:17	Brian:	Thirty-two.
00:56:18	Jeff:	No.
00:56:18	Michael:	-Plus cheese.
00:56:19	Jeff:	Thirty-two plus cheese
00:56:20	Brian:	Thirty-one-
00:56:20	Romina:	It's like-
00:56:21	Michael:	Thirty-one plus one is the thirty-two. That's with cheese. It's thirty-two. That's the answer.
00:56:25	R1:	Well, there seems to be some people who aren't convinced about that, Mike, so could you-
00:56:25	Michael:	OK. Without cheese, it's-
00:56:30	Jeff:	[Inaudible.] My numbers.
00:56:31	Michael:	Without cheese, it's what?

Time	Speaker	Transcript
00:56:32	Jeff:	It's thirty-one.
00:56:34	Michael:	With cheese, it's thirty-two.
00:56:35	Ankur:	Thirty-two.
00:56:35	R1:	OK so if you're, if you're finding pizzas, when you can pick from five toppings, how many are there?
00:56:40	Michael:	Thirty-two.
00:56:43	Brian:	Yeah.
00:56:43	R1:	Thirty-two. Are you all in agreement on that?
00:56:44	Romina:	Yeah.
00:56:44	Brian:	We all agree on that.
00:56:45	R1:	You believe that.
00:56:45	Michael:	If you, if you want to count the cheese as a topping, as a, as a combination-
00:56:48	R1:	As a choice.
00:56:48	Michael:	-as a combination, as a choice-
00:56:50	Brian:	It's not a topping; it's a choice.
00:56:50	Michael:	-as a choice for a pizza, then it's, then it's thirty-two.
00:56:53	R1:	It is a, it is a real choice in my world.
00:56:55	Michael:	OK.
00:56:55	R1:	Because I order that.
00:56:56	Brian:	That's right.
00:56:57	Michael:	Then it's thirty-two.
00:56:58	R1:	OK.
00:56:59	Ankur:	So thirty-two is two to the [pause] sixth?
00:57:03	Michael:	What?
00:57:03	Ankur:	Thirty-two is two to the sixth, right?
00:57:06	Michael:	Thirty-two is not two to the sixth.
00:57:08	Ankur:	What is it then?
00:57:09	Michael:	Hold on, hold on.
00:57:10	Jeff:	Yeah. It would be. Yeah.
00:57:11	Michael:	Thirty-two.
00:57:11	Ankur:	So it's two to the n plus one power.
00:57:12	Jeff:	Wait.
00:57:14	R1:	Are you all convinced that thirty-two is two to the sixth?
00:57:17	Ankur:	'Cause if it was five toppings.
00:57:20	Jeff:	Uh-huh. Two to the n plus one, we're saying?
00:57:21	Ankur:	Yeah.
00:57:23	Jeff:	So if it was- Say we had two toppings-
00:57:25	Brian:	It's not $n+1$.
00:57:26	Jeff:	-It would be two to the second.

Time	Speaker	Transcript
00:57:28	Ankur:	If there's two toppings.
00:57:30	Jeff:	Uh-huh.
00:57:31	Ankur:	It's two to the third power. Because it's two toppings plus one.
00:57:34	Michael:	I don't think so.
00:57:36	Ankur:	It's two to the third power.
00:57:38	Jeff:	That could be it. You think that would go?
00:57:40	Brian:	Yeah. Right.
00:57:41	Jeff:	With two toppings, we would have four there.
00:57:43	Michael:	With two toppings, we would have three pizzas, right?
00:57:46	Romina:	With two toppings, you would have how many?
00:57:49	Michael:	Three pizzas.
00:57:50	Ankur:	Two to the third is four, isn't it?
00:57:51	Romina:	No, we have four with two toppings, because we're adding the, uh-
00:57:54	Jeff:	It would be two to the second.
00:57:54	Brian:	Yeah.
00:57:55	Michael:	One plus two.
00:57:56	Brian:	It's two to the number of toppings.
00:57:57	Jeff:	And then for three toppings, it would be the number of toppings.
00:57:59	Brian:	Two to the third.
00:58:00	Ankur:	Plus one.
00:58:01	Jeff:	No. That would be two to the fourth.
00:58:02	Brian:	No.
00:58:03	Jeff:	It would be two to the third minus one. Two times-
00:58:05	Brian:	If we had three toppings.
00:58:06	Jeff:	No. It would be two to the-
00:58:07	Brian:	If we had three toppings.
00:58:08	Jeff:	All right. Two to the second is two times two, which is four. Two to the third is two times two times two, which is eight. Two to the fourth is two times two-
00:58:15	Brian:	So it's [Inaudible.].
00:58:16	Jeff:	-Times two times two is sixteen. Two to the fifth is two-
00:58:19	Brian:	Multiply again and then that's it.
00:58:20	Jeff:	Times two times two, which is thirty-two.
00:58:21	Brian:	So it's not plus or minus anything.
00:58:22	Michael:	If you have two toppings. You know, two to the sixth, it's-
00:58:24	Ankur:	There's thirty-two.
00:58:25	Romina:	Yeah, there's-
00:58:25	Michael:	It's three.
00:58:27	Ankur:	Two to the fifth is sixteen.

Time	Speaker	Transcript
00:58:28	Michael:	Give me four. Well, that's with the cheese.
00:58:30	Romina:	No, but with the cheese one, with the cheese one.
00:58:31	Michael:	OK.
00:58:31	Jeff:	Yeah, but the cheese one makes it, makes them go where-
00:58:35	Ankur:	You need the cheese one.
00:58:36	Jeff:	Yeah, makes it go right-
00:58:37	Brian:	Yeah.
00:58:37	Jeff:	-Into what we're doing. It gives, it, it's going to be an odd number if we don't add the cheese to it. The cheese makes it-
00:58:43	Romina:	I'm just saying, 'cause Mike is saying, OK, you know how we put the one under the two and the one under the one and we add them together, it equals three.
00:58:48	Jeff:	Uh-huh.
00:58:49	Brian:	Yeah.
00:58:49	Romina:	That's why he's saying that.
00:58:49	Jeff:	With the- Yeah, I'm saying-
00:58:51	Michael:	Yeah, yeah. That's, and then you could have two zeros, that would be another combina- and that's another choice.
00:58:56	Jeff:	Yeah, that'd be four.
00:58:57	Michael:	And that's how it this works.
00:58:57	Jeff:	And that'd be two to the second. With two toppings, two to the second would be four in total - a total of four toppings.
00:59:01	Michael:	And so with cheese-
00:59:01	Ankur:	And so it would be two to the *n* plus one?
00:59:03	Jeff:	You're not listening to me and that's really bothering me, 'cause we just
00:59:06	Brian:	That's why I'm-
00:59:07	Jeff:	We're saying it over and over again. And you're just not listening. Why aren't you not agreeing, or not telling me what you- you're just like mumbling and not telling me what's up.
00:59:18	Romina:	Isn't it just two *n* then?
00:59:20	Brian:	Yeah.
00:59:20	Jeff:	Exactly. That's what I've been saying for the past half hour!
00:59:22	Romina:	I though you guys were arguing two *n* plus one.
00:59:23	Michael:	I-
00:59:25	Jeff:	Then you sat there and called me stupid. That's the wrong thing and I've been trying to say that.
00:59:29	R1:	Yeah, but you have to convince each other and not just us.
00:59:31	Jeff:	I've been trying. No one listens to me.
00:59:32	Romina:	I think it's two *n*.

Time	Speaker	Transcript
00:59:33	R1:	Well. Now how can you convince-
00:59:34	Jeff:	Two *n*. That's what we said.
00:59:35	Michael:	That's it.
00:59:36	Jeff:	Exactly.
00:59:37	R1:	I don't think Ankur-
00:59:38	Michael:	Case closed.
00:59:39	Jeff:	Me- I'm-
00:59:39	Michael:	What's the problem?
00:59:40	Ankur:	I'm convinced.
00:59:41	Michael:	Shut up.
00:59:41	Jeff:	Ankur came up with the answer with me and Brian.
00:59:43	Michael:	Ankur, give us, give us an example.
00:59:44	R1:	Ankur said *n* plus one.
00:59:45	Jeff:	But that was a couple of minutes ago.
00:59:46	R1:	Ankur, do you believe, two the *n* plus one or two to the *n*?
00:59:48	Ankur:	No. Two to the *n*.
00:59:48	Jeff:	Let's record this. You could rewind the tape and you'll see Ankur go, "Oh, no. It is two *n*."
00:59:54	R1:	OK. You're all convinced.
00:59:56	Michael:	Two *n*. I'm convinced.
00:59:57	Romina:	I think it's two to the *n*.
00:59:58	R1:	You do, too, Romina? I heard you.
00:59:59	Romina:	I did, yeah. I just didn't want to start an argument.
01:00:00	Michael:	All, all right. It's still the binary thing, you know.
01:00:02	Jeff:	But you had the same answer we did.
01:00:04	Romina:	No, but I, that's why I didn't say, 'cause I thought it was two *n* the whole time.
01:00:06	Ankur:	What do you think?
01:00:07	Romina:	And I just didn't want to look dumb.
01:00:08	Jeff:	And Ankur was going two minus one.
01:00:08	Ankur:	Two plus one.
01:00:09	Romina:	And you guys are saying two plus *n* minus one.
01:00:11	Brian:	Well, that's 'cause I was thinking- 'cause you're adding the cheese.
01:00:13	Jeff:	Yeah, but-
01:00:14	Brian:	Jeff brought the whole thing out. If you didn't add the cheese, it would be odd.
01:00:17	Jeff:	It would be an odd number.
01:00:17	Brian:	And then you just get two squared.
01:00:20	Romina:	OK.
01:00:20	R1:	You all worked very, very hard. I guess you're doing junior level

Time	Speaker	Transcript
		math now.
01:00:25	Romina:	Yeah.
01:00:26	Michael:	I don't know what we're doing.
01:00:27	Romina:	Yeah. We're doing junior level.
01:00:28	Ankur:	We're with like the seniors.
01:00:28	Jeff:	We're trying to do it.
01:00:29	R1:	You really work very very hard.
01:00:31	Michael:	I don't know.
01:00:31	R1:	We really- I, I appreciate your coming. You, you guys are hard workers, good thinkers.
01:00:37	Romina:	We're actually- [laughter]
01:00:39	R1:	Does it, does it remind you of anything else that we've done before?
01:00:43	Romina:	Um.
01:00:44	Michael:	I remember about this- I was like-
01:00:46	Romina:	I remember doing this before.
01:00:47	Michael:	-In eighth grade, we were doing something. Or seven.
01:00:48	Brian:	Everything we ever do always is like the tower problem.
01:00:49	Michael:	I always come back to that prob-
01:00:51	R1:	All right. What, what, what problem?
01:00:52	Jeff:	Do you want something to drink? [To Romina.]
01:00:53	Brian:	Everything, the tower problem, with the little Unifix cubes.
01:00:56	R1:	Yeah.
01:00:56	Romina:	We'd always-
01:00:56	Brian:	Everything is like that 'cause like-
01:00:58	Michael:	That's like the same thing.
01:00:58	Brian:	Except like different colors.
01:00:59	R1:	What do you mean? What do you mean?
01:01:00	Michael:	Because instead of different combina- mm – wait.
01:01:03	Romina:	It's the same thing, only-
01:01:03	Michael:	[Inaudible.] Different combinations.
01:01:04	Brian:	Instead of building a tower, you're building a pizza.
01:01:06	Michael:	Yellow, black, or-
01:01:07	Brian:	You got different color blocks.
01:01:07	Michael:	White block, or what whatever colors combinations you want to use.
01:01:11	R1:	I want to know about that. Tell me.
01:01:12	Michael:	Ah. See now, you had to do that.
01:01:13	R1:	That's great. Tell me. How is this like the tower problem or unlike it?

Time	Speaker	Transcript
01:01:16	Romina:	Because instead of, instead of building it, we're like writing it.
01:01:19	Brian:	You're building, you're building a pizza, you're building a tower.
01:01:22	R1:	What's the tower problem again? Because I don't think R9 knows what it is.
01:01:24	Ankur:	It's the same.
01:01:24	Brian:	Yeah.
01:01:25	Romina:	It's, it's not a problem.
01:01:25	R1:	She wasn't here when you did that.
01:01:26	Ankur:	It's-
01:01:26	Brian:	It's like a- you have different colors cubes.
01:01:29	Michael:	Different color combina- you would have a tower with like-
01:01:31	Romina:	You'd have like for each topping-
01:01:32	Ankur:	Building a tower.
01:01:32	Romina:	-You'd have a different color.
01:01:33	Ankur:	One person tell her.
01:01:34	R1:	Pardon?
01:01:34	Romina:	All right. For each, OK, for each topping you'd have a different color with the little cubes. And you'd build 'em into towers showing like one tower would be one pizza. Another-
01:01:41	R1:	Now, wait. The tower problem is-
01:01:43	Ankur:	Is like five high with two different colors.
01:01:45	R1:	Five high with two different colors? That's not what I heard Romina say. Romina said you have a different color for each toppings. And you're saying five high, two different colors. So there are two different ideas here and I'd like to hear both of them. Tell us your idea. Either one of you go first.
01:01:56	Romina:	I always disagree!
01:01:57	Ankur:	I'm sorry, I didn't hear what you said.
01:01:58	Brian:	No. Ankur is actually right. Like-
01:02:00	Romina:	I'm just kidding. I'm kidding.
01:02:01	Jeff:	Yeah, that was-
01:02:01	Brian:	Like, we started off with, like-
01:02:02	Michael:	Yeah.
01:02:03	Romina:	Uh-huh.
01:02:04	Brian:	-Like different heights of towers.
01:02:05	Romina:	And we move the thingies around
01:02:05	Brian:	Like-
01:02:06	Ankur:	And we had two different colors.
01:02:07	Brian:	Like the tower had to be two blocks high, two different colors, how many different combinations that can be.

Time	Speaker	Transcript
01:02:10	Michael:	Yeah.
01:02:10	Jeff:	And it was always two.
01:02:11	Brian:	And it was like three high, two different colors.
01:02:12	Jeff:	With two different colors. Four high.
01:02:13	Brian:	Then four would be two different colors.
01:02:14	R1:	Well, how many how do you fit- how do you reason how many were there, let's say four high with two different colors to select from?
01:02:19	Brian:	Just write it out. Just do the same thing Mike did.
01:02:22	Ankur:	Now that was factorial, wasn't it?
01:02:23	Romina:	Yeah. I think that was.
01:02:24	Jeff:	No.
01:02:24	Michael:	That - something was factorial, something to do-
01:02:26	Jeff:	Yeah, something to do-
01:02:27	Romina:	'Cause I remember Mr. Poe told us, eighth grade.
01:02:28	Jeff:	Well, we found out and we were like-
01:02:29	Romina:	And we got so mad. [Laughter.]
01:02:30	Michael:	Factorial would be like finding out how many different number combin- like a lottery number. That would work for factorial, wouldn't it?
01:02:37	R1:	OK. Let's hear Brian's idea. So Brian, you were saying-
01:02:41	Brian:	I don't have an idea.
01:02:42	R1:	You were saying about these towers, uh.
01:02:45	Brian:	Well, instead of having like [Pause.] I don't know. I'm trying to explain it, but I can't. Like instead of having one, two, three and four, representing like, toppings, you can just have like one and two as cubes and like you just can't have the same combination. Like you can use the same one twice, but you just can't have the same combination. They're kind of like this, but not really. You know what I'm talking about?
01:03:08	R1:	Do you want to show me? I have some towers with me. Just happen to have 'em in my bag.
01:03:12	Brian:	[Brian throws his hands to his face and then in the air.] If you want. If you want.
01:03:14	Ankur:	Wait. Wait. Wait. Wait.
01:03:15	Jeff:	I really want to play with the cubes.
01:03:16	Brian:	I know.
01:03:16	R1:	You're telling me about the towers.
01:03:19	Ankur:	I just want to say something.
01:03:21	R1:	We have them. We have them.

Time	Speaker	Transcript
01:03:21	Jeff:	I really want-
01:03:21	Romina:	No.
01:03:22	Jeff:	I haven't, haven't played with those cubes.
01:03:23	R1:	I carry them with me everywhere.
01:03:26	Brian:	I always wanted to build them. And the teachers would never let us.
01:03:29	R1:	Next time.
01:03:29	Ankur:	Like they, the towers-
01:03:29	Brian:	They should-
01:03:30	Jeff:	They used to come in those big boxes. And I, they were never in order. It always used to bother me. They always used to be all different colors and stuff.
01:03:35	R1:	Um.
01:03:36	Ankur:	With the towers, you can have like red on top and yellow on the bottom and then yellow on top and red on the bottom.
01:03:42	R1:	OK.
01:03:42	Ankur:	But with the pizzas you can't have- [Inaudible.]
01:03:42	Michael:	But when you put them, the tower on-
01:03:45	R1:	That's two possibilities.
01:03:45	Ankur:	But with the pizzas you can't have peppers and pepperoni and then pepperoni and peppers. [Pause.] //Understand that?
01:03:50	R1:	//So you're saying it's not like it? [Romina nods.]
01:03:51	Michael:	It's similar.
01:03:51	Jeff:	We did. We just-
01:03:53	Ankur:	It's similar, but it's not exactly alike.
01:03:56	R1:	I'm confused.
01:03:57	Michael:	Because you would have-
01:03:58	R1:	So let's go back. Let's go back. Help me here now.
01:04:00	Romina:	OK. It's-
01:04:00	R1:	Let's talk about towers two tall. Two tall.
01:04:01	Michael:	Like with the towers-
01:04:02	Jeff:	You talk about-
01:04:02	Ankur:	Red and yellow and yellow and red.
01:04:05	Jeff:	Everyone's talking at one time. No one's listening.
01:04:07	Jeff:	Slow down.
01:04:07	Brian:	Ankur, go.
01:04:09	Ankur:	OK.
01:04:09	R1:	OK.
01:04:09	Ankur:	You can have red on top and yellow on the bottom. You can have yellow on top and red on the bottom. You can have all yellow or

Time	Speaker	Transcript
		all red.
01:04:16	Michael:	That's four. The same thing with the pizzas.
01:04:17	Romina:	It's not only that-
01:04:18	Michael:	'Cause you got that extra cheese, or whatever.
01:04:20	Brian:	Yeah. You're going to only have four.
01:04:22	Ankur:	The only difference is you can't// have peppers and-
01:04:23	Michael:	//That's only with two, that's only number two.
01:04:24	Ankur:	-Pepperoni and pepperoni and peppers.
01:04:25	Michael:	But I think with three it would be-
01:04:26	Romina:	And you can't have pepper and pepper and pepperoni and pepperoni.
01:04:27	Jeff:	Yeah, but with the blocks you can't have no choice.
01:04:28	Ankur:	Yeah.
01:04:30	Jeff:	You can't say, all right.
01:04:31	Michael:	Yeah, so it's-
01:04:33	Ankur:	So you can't have, like, one toppings, right?
01:04:35	R1:	Well, I'm confused.
01:04:37	Michael:	Well, two is, two is a bad number anyway. Because two is also like-
01:04:40	Ankur:	a bad number.
01:04:41	Jeff:	Yeah, two is a bad number.
01:04:42	Michael:	Just pick a, just pick a different-
01:04:43	Jeff:	Pick three.
01:04:43	Michael:	It's a bad number to pick, because-
01:04:46	Jeff:	The only thing different is that you can have
01:04:48	R1:	Pick three. Tell me for three.
01:04:49	Jeff:	Like you could, you could have a just, a just mushroom pizza with one thing on it. You couldn't have like we didn't count that one Unifix cube.
01:04:55	Michael:	Yeah.
01:04:56	Jeff:	You know, you understand what I'm saying?
01:04:59	Ankur:	No.
01:04:59	R1:	No.
01:05:00	Romina:	See-
01:05:00	Jeff:	All right. No. You want to go? Are you going to say the same thing I am?
01:05:03	Romina:	No. Go ahead.
01:05:04	Jeff:	Well, we're saying that, all right, with pizzas you could have either one toppings, two toppings, or three toppings. With the blocks you didn't, we couldn't do it. We could have three high,

Time	Speaker	Transcript
		two high or one high. It was like three high, how many combinations could you get for three high? Not, no, we didn't count the two or the one high.
01:05:16	Ankur:	You understand that?
01:05:16	Jeff:	Which would be equivalent to like if we had mushrooms, peppers and onions, we could have a mushroom.
01:05:22	Ankur:	Mushroom. Mushroom and pepperoni.
01:05:23	Jeff:	Yeah. You know? Do you understand what I'm saying?
01:05:25	R1:	I understand what you're saying.
01:05:26	Brian:	You couldn't just have one by itself.
01:05:27	Jeff:	Yeah.
01:05:28	R1:	But I want to go back to the towers again. So. I'm a little confused. Uh. 'Cause Mike said and Ankur said that if you're making them two high, you said there are how many towers you could make two high?
01:05:41	Ankur:	Four, but-
01:05:41	Michael:	Four. But two's a bad number. You can't-
01:05:42	R1:	'Cause, that's what I, I heard Mike say that, that two's a bad number. So then, uh, Jeff said, "let's pick three." So you're making towers three high.
01:05:51	Ankur:	All red. All yellow and then red, red, yellow.
01:05:55	Jeff:	Yel-
01:05:56	Ankur:	Red, yellow, red. Yellow, //red, red.
01:05:58	Jeff:	//Red, red.
01:05:59	Ankur:	Yellow, red, yellow.
01:06:01	Jeff:	Yellow, yellow, yellow.
01:06:02	R1:	Are you all, are you all believing him? Or are you checking him out?
01:06:04	Romina:	Yeah.
01:06:04	Brian:	Yeah.
01:06:04	Romina:	That's all there is.
01:06:05	Ankur:	If you want to write it, it would be easier.
01:06:06	Brian:	I'm checking in my head.
01:06:06	Jeff:	Yellow, yellow, yellow.
01:06:07	R1:	If you're picking from two colors.
01:06:08	Jeff:	Yellow, yellow, yellow. Is that is?
01:06:11	Ankur:	Uh-huh.
01:06:12	Brian:	Yellow, yellow, yellow.
01:06:12	R6:	But three toppings.
01:06:13	R1:	None. [To R6.]

Time	Speaker	Transcript
01:06:16	Jeff:	What about -?
01:06:17	Ankur:	Try it.
01:06:18	R1:	Three tall.
01:06:19	Jeff:	How many you got over there, Mike?
01:06:20	Michael:	I don't know. I'm just writing my binary again. [Michael chuckles.] 'Cause it might work for this!
01:06:25	Jeff:	Ah. Yeah, but let's try it. Let's-
01:06:27	Ankur:	All right.
01:06:28	Romina:	You want me to, uh-
01:06:29	Ankur:	Write.
01:06:29	Romina:	I'm usually the one that does the neatest writing.
01:06:30	Ankur:	Just write. [Brian does the writing.] RRR.
01:06:32	Michael:	I don't care.
01:06:33	Jeff:	RRY. RYY.
01:06:37	Brian:	R what?
01:06:38	Romina:	Hold on.
01:06:39	Romina:	//RRY.
01:06:40	Jeff:	//RRY.
01:06:40	Brian:	//RRY. Y. RRR.
01:06:41	Romina:	Y.
01:06:41	Brian:	Y.
01:06:46	Ankur:	RYR
01:06:49	Jeff:	//YRR.
01:06:49	Romina:	// YRR.
01:06:52	Ankur:	YRR.
01:06:53	Jeff:	YRY.
01:06:58	Romina:	YYR.
01:06:59	Ankur:	Do you have YYR?
01:07:01	Jeff:	Yes?
01:07:01	Romina:	No.
01:07:02	Jeff:	No. Oh, wait. But you could do that.
01:07:03	Ankur:	That's different.
01:07:03	Jeff:	Is that, is that how we're doing this?
01:07:05	Ankur:	Yeah.
01:07:05	Jeff:	Oh, well. YYR. And then we can have, uh, Y, we have Y, yeah we have YRR.
01:07:11	Ankur:	OK. That's it.
01:07:12	Brian:	And then the opposite of that. The opposite of that is that.
01:07:16	Jeff:	That and that.
01:07:17	Romina:	Remember that?

Time	Speaker	Transcript
01:07:18	Michael:	How much did you get?
01:07:18	Ankur:	Opposite of that-
01:07:19	Romina:	We used to pair 'em up and look up-
01:07:20	Ankur:	Do we have YY? Yeah, it's the last one.
01:07:21	Romina:	Fourth grade.
01:07:22	Ankur:	The last one.
01:07:23	Jeff:	Yeah.
01:07:24	Brian:	That, that.
01:07:25	Michael:	How much you get?
01:07:26	Jeff:	Two, four, six, eight.
01:07:28	Romina:	Eight.
01:07:29	Brian:	Look at that.
01:07:30	Ankur:	It's the same.
01:07:30	Brian:	Hmm.
01:07:31	Michael:	It would be like the same thing. It's just that, 'cause look at the numbers.
01:07:35	Ankur:	It's a little different, but-
01:07:35	Michael:	The one's that change when// you flip 'em up.
01:07:36	Ankur:	//It's the same thing, but-
01:07:38	Brian:	a little bit different.
01:07:39	Michael:	Names are different, same thing, but the answers are different.
01:07:41	Ankur:	The answers are different, but-
01:07:42	R1:	So what do you predict four tall?
01:07:44	Ankur:	Eight. No. Four.
01:07:45	Jeff:	No. Wait. Sixteen.
01:07:46	Ankur:	Sixteen.
01:07:46	Jeff:	Sixteen. What do you mean, the answers are different? Then I don't- aren't the answers the same?
01:07:48	Ankur:	The answers are the same, I said.
01:07:50	R1:	Oh, my! I'm really-
01:07:51	Jeff:	Yeah.
01:07:52	R1:	You've got a lot of explaining to do.
01:07:53	Ankur:	It's all the same.
01:07:53	Brian:	Next time, next time we'll do that. [Brian chuckles.]
01:07:55	R1:	OK. That's, that might be a good place to stop. Some things to think about. You know you have really some very interesting powerful ideas on the table. You want to come back?

December 19, 1997

Time	**Speaker**	**Transcript**
00:00:00	–	During the first 10 minutes, the students talk about their work in class that day: graphing functions. Then R1 asks, with regard to last week's work on pizza problems, how many choices there are with 20 toppings to choose from.
00:10:58	Michael:	Two to the *n* or two to the *n* minus 1?
00:10:59	Jeff:	It was, no, it was-
00:11:01	Romina:	It would be *n*.
00:11:04	Ankur:	2 to the-
00:11:04	Jeff:	2 to the *n*.
00:11:05	Ankur:	*n*. …
00:11:50	R1:	My next question is, is that related to building towers? Yes or no. If not, why not? If yes, in what way?
00:12:00	–	[Insert here from latest transcript.]
00:18:44	Brian:	There is no basic tower. It might be all blue or all red.
00:19:02	Michael:	Like, how many colors is she asking?
00:19:05	Romina:	It would be the same thing no matter what. …
00:19:20	Michael:	'Cause I could still use the binary. ... Instead of making the number of colors ... you just use the number of blocks. 0 is blue and one is red.
00:19:35	–	[Skip about 19 minutes.]
00:34:20	Jeff:	What if we had, so we- If you just change, if the only variable we're changing is height, we could- it would be the same-
00:34:26	Michael:	It would be the same as the pizza.
00:34:29	Jeff:	If we tried to change //color.
00:34:30	Michael:	//color.
00:34:30	Jeff:	What would that be like changing on the pizza, though? 'Cause like you know-
00:34:33	–	[Skip about five minutes.]
00:39:16	Michael:	I'm thinking of something now.
00:39:16	Jeff:	Are you sure you got it?
00:39:17	Michael:	I'm not. I'm not that sure. But this is what I'm thinking. You have two different places to put colors. The first place you have a possibility of three numbers. The second place you have a possibility of three numbers. If you times 'em, it comes, it's nine.
00:39:32	Jeff:	Uh-huh.
00:39:33	Michael:	That will give you the number of combinations you could have. And if you have three, if you have four colors, it would be sixteen.

Time	Speaker	Transcript
00:39:42	Romina:	[Inaudible.] –Towers.
00:39:41	Jeff:	Three is [Inaudible.]
00:39:43	Ankur:	That's the factorial thing.
00:39:45	Michael:	If you have three colors, three colors-
00:39:46	Ankur:	That's the factorial thing.
00:39:48	Romina:	We got it, too. 'Cause Jeff said so.
00:39:47	Michael:	That is.
00:39:51	Jeff:	You're not going to tell us?
00:39:52	Romina:	No. [All students talking at once.]
00:39:55	Ankur:	All right.
00:39:55	Michael:	No, it's not.
00:39:56	Ankur:	Well, like-
00:39:57	Michael:	It's not.
00:39:57	Romina:	Then what is?
00:39:58	Ankur:	Not exactly.
00:39:59	Michael:	'Cause, no, 'cause-
00:39:59	Ankur:	I know what you're talking about.
00:40:00	Jeff:	Wouldn't that be four times three times times 1?
00:40:01	Romina:	I know, but I'm-
00:40:02	Michael:	But that's, that's just, that's only if you have, um, like a set of numbers, and, let's say you have these letters. How many different combinations. I could have, I could have-
00:40:11	Jeff:	Yeah, you could have-
00:40:12	Michael:	The first one I could have five, but since, two-
00:40:13	Jeff:	Five. No, four.
00:40:14	Michael:	Four, four, I mean four. There's four letters in his name. The second one, you only have three, because I already took the N.
00:40:19	Ankur:	OK.
00:40:20	Michael:	You know?
00:40:20	Brian:	Are we doing three high with two colors, right?
00:40:23	Jeff:	We're just talking right now.
00:40:24	Michael:	We're just talking.
00:40:24	Brian:	But I was-
00:40:25	Michael:	We're not doing anything, we're just talking.
00:40:26	Jeff:	I was doing three high with two colors, with
00:40:31	Brian:	That would just be three to the n. 'Cause when there's one color- [Students discuss selecting from four colors for a minute and a half.]
00:41:55	Ankur:	All you're doing is you're making the first number the number of colors. Like the 2 to the *n*, the 2 represents the number of colors.

Time	Speaker	Transcript
		... 3 times 3 times 3, that's 3 to the third. ... The first number is the number of colors, and the ... coefficient? ...
00:42:21	Michael:	-is the number of blocks.
00:42:23	Ankur:	How high.
00:42:23	Michael:	The number of colors would be- what's that? What's that number? what is that?
00:42:29	Ankur:	The big one?
00:42:30	Michael:	The small one was called the-
00:42:31	Ankur:	The coefficient.
00:42:32	Michael:	What was the big one called?
00:42:33	Ankur:	The big number.
00:42:34	Michael:	The big number. Yeah. Uh.
00:42:35	Ankur:	The original number.
00:42:36	Michael:	I don't know.
00:42:37	Ankur:	I don't know where- the base number maybe. ...
00:42:43	Michael:	So the number, the number of colors would be like the big number. And the number of blocks
00:42:50	Ankur:	Height. How-
00:42:51	Michael:	Height
00:42:52	Ankur:	High would be the coefficient. Or the exponent.
00:42:54	Michael:	Just say exponent.
00:42:56	Ankur:	Exponent.
00:42:59	Michael:	I guess that's it. I don't know. That's what I think.
00:43:29	Brian:	[Brian introduces his ideas about 2 to the *n*.]
00:45:22	Michael:	When we got, when we got to the equation 2 to the *n*, we understood, we understood what the *n* was, but we-
00:45:31	Jeff:	2 was just a number.
00:45:31	Michael:	2 was the number of toppings, though.
00:45:33	Ankur:	No it wasn't. *n* was the toppings.
00:45:34	Michael:	No, no, no, not toppings. The number- what? What the-
00:45:37	Ankur:	2 just worked out.
00:45:38	Michael:	No, it, it does. It has a-
00:45:40	Ankur:	Two.
00:45:41	Brian:	It has to stand for something.
00:45:41	Michael:	It stands for something.
00:45:42	Jeff:	And-
00:45:43	Michael:	*n* was the number of toppings, and 2-
00:45:46	Jeff:	-was the other thing, and that's probably the same thing.
00:45:46	Michael:	The 2 is that you can either have an oh or a 1. You can have a topping or not.

Time	Speaker	Transcript
00:45:50	Ankur:	Yeah. Topping or not.
00:45:51	–	[Skip about five minutes.]
00:50:07	Jeff:	All right, we're talking about how, how, well-
00:50:13	R1:	Romina, you start.
00:50:14	Jeff:	Yeah, you start.
00:50:15	Romina:	Well, what are we going by? The towers?
00:50:17	Jeff:	We're talking about how-
00:50:18	Ankur:	How the-
00:50:19	Jeff:	How they relate to each other.
00:50:20	Romina:	What are we talking about? ... [Romina writes on the board x to the n and 2 to the n.] That's all I want to know.
00:50:28	Jeff:	Start out, start out with the 2 n, explain that, and then we'll work down to the x n.
00:50:34	Ankur:	All right.
00:50:34	Jeff:	All right?
00:50:36	Romina:	OK. For the pizza problem, the 2 n, the 2 represents either topping or no topping. Right?
00:50:43	Michael:	There's two different possibilities for each.
00:50:44	Jeff:	So that's when, that's what that 2 stands for. There's no- that's why.
00:50:46	Michael:	You can't have-
00:50:47	Jeff:	That's why, that's why there's two. We didn't know, I don't think we explained that last time, why it was two.
00:51:01	Jeff:	Topping or no topping, and that's what the 2 is. Now the n, Romina.
00:51:04	Romina:	Is toppings.
00:51:07	Jeff:	The number of toppings. [R1 asks them to be explicit.]
00:52:32	Michael:	[Michael goes to the board.] Make up two toppings, mushrooms and peppers. Let's say we had two toppings. OK. If you want, if you had, like pizzas. 1 would stand for, you have a topping, 0 would stand for, you don't have a topping. [Michael writes 01.] Mushroom pizza. [Michael writes 10.] Pepperoni pizza. [Michael writes 11.] Mushroom and pepperoni and you got a nothing, plain. [Michael writes 00.] And that-
00:53:02	Jeff:	and that would be 2 to the 2.
00:53:02	Michael:	That's everything you could have. [They explain what this shows there are 4.]
00:54:01	R1:	I see. I see. [They explain that you add another column for 3 toppings.]
00:54:15	Michael:	Now if you want to link it to the tower problem. If you had two

Time	Speaker	Transcript
		colors, it would look exactly the same.
00:54:25	R1:	Explain to me how.
00:54:29	Michael:	This would be, instead of this being the toppings [Michael erases column headings.], this would be-
00:54:32	Jeff:	Height.
00:54:33	Ankur:	Height.
00:54:35	Michael:	This would be the, the height. You have height 1, height 2, height 3. [Michael writes new column headings.]
00:54:42	Ankur:	Under one, it's either-
00:54:43	Michael:	If you had only, uh-
00:54:45	Jeff:	There's the, the height is going across the top. The number of colors are like in the grid.
00:54:51	Michael:	There's only two. You understand, you can only have blue or red. ... [Michael fills in the grid for a 2-tall tower.] And it's looking the same like it did before. The same exact combination with the pizza. That does look familiar?
00:55:15	Jeff:	So it's the same.
00:55:16	Michael:	... mushroom and pepper looks the same. You see?
00:55:19	Brian:	... Mike [Brian asks a question about using neither red nor blue.]
00:55:33	Michael:	... 0 is red.
00:55:34	Brian:	... 1 is blue.
00:55:36	R1:	Romina, does that make any sense?
00:55:36	Romina:	Yeah, I understand. …
00:55:45	Michael:	You understand how if you have a tower with two colors, it's the same as the pizza problem.
00:55:48	Jeff:	[They discuss building towers with a choice of three colors.] All right, well, if the toppings on the pizza is the *n*, We're trying to bridge the gap between the two.
00:56:25	Michael:	'Cause we never knew what 2 meant.
00:56:25	Jeff:	We never knew what it meant.
00:56:27	Michael:	We didn't know what 2 meant. Now I, we figured out today what the 2 means. The 2 means-
00:56:32	Jeff:	You could either have toppings or no toppings.
00:56:34	Michael:	The, the, the, the amount of digits you could put in this one space. [Michael points to one of the binary columns on the blackboard.] Like with two. A 0 or a 1. But when you go in to towers, you know, it's different colors, you-
00:56:43	Jeff:	You could put three in there.
00:56:44	Michael:	Let's say you have three colors. I want to make that oh, 1, and a 2. That's three different colors.

Time	Speaker	Transcript
00:56:50	Jeff:	That, and that would change the *x* with the 3-
00:56:52	Michael:	That would change this number to a 3. [Michael points to the 2 of 2 to the n.
00:56:55	Jeff:	And then-
00:56:55	Michael:	So then you would have like things that look like a 2 and a 1. [Michael rewrites some of the numbers in the columns.] An oh and a 2. You get what, what, the thing's gonna look? It's crazy but like, you see, like, a 1 and a 2. You wouldn't be using that oh color because it's only a 2 tower.
00:57:12	Jeff:	But you could have a 1. You could, you could have 0 1.
00:57:15	Michael:	Yeah.
00:57:15	Jeff:	You could have 0- Erase those things. You could have 0 0. 0 1. 0 two.
00:57:21	Ankur:	Write them all out.
00:57:21	Michael:	I don't want to write them out.
00:57:23	Jeff:	Yeah, 'cause that would be
00:57:25	Brian:	A lot.
00:57:26	Jeff:	No, it'd be
00:57:26	Romina:	?
00:57:27	Michael:	Can I take it
00:57:28	Brian:	No, when we're going to the
00:57:30	Jeff:	You're saying for the first round, for the-
00:57:31	Michael:	Yeah.
00:57:32	Jeff:	-preliminary thing, it would be nine, 'cause you could-
00:57:33	Ankur:	Three.
00:57:34	Jeff:	Three possible, three digits in each block, and two, two different columns.
00:57:38	Michael:	But see, when I did this, I didn't use the, the binary system because it-
00:57:41	Jeff:	Because you couldn't do the 2 in there.
00:57:43	Michael:	You know, it's too-
00:57:44	Jeff:	It doesn't work.
00:57:45	Michael:	As long as it- [Inaudible.] You have two possible, possible- you have two places, two blocks. [Michael writes two dashes on the board.] I'll leave, I'll put something like that. You can have, let's see, how many, three different colors?
00:57:57	Ankur:	We have three possible-
00:57:58	Michael:	There's three different colors for a, for a first space. I'm going to put a three down.
00:58:03	Ankur:	And then for the second space, you can also, it also has three

Time	Speaker	Transcript
		possibilities.
00:58:05	Jeff:	That has three colors. And then the *n* would show the number of spaces you have, and the three represents how many you could put in each, what, the number you put in each space. So say there's 3 to the fourth power, there'd be four lines for you to put in, that's //3 times 3 times 3.
00:58:21	Michael:	//Three times three times three. Because you have three different possibilities for each-
00:58:23	Jeff:	Three for each different space.
00:58:25	Michael:	-spot.
00:58:28	Ankur:	If you multiply, you'll get the total number of possibilities.
00:58:29	Jeff:	... works in x to the n-th.
00:58:31	R1:	Have
00:58:31	Jeff:	'Cause it's to the power.
00:58:32	R1:	OK. Have you explored why multiplying works? [Students laugh.]
00:58:41	Michael:	Uh. [Students discuss for a minute.]
00:58:54	Brian:	You go boom, boom, boom.
00:58:55	Romina:	Each, each thing is going to give you- [Inaudible.] Each time you get a different thing, you, so you multiply.
00:59:05	Michael:	I think that thing we did a lot in sixth grade that would work for this. Cause you would like, it would be like the, the, 1 2 3.
00:59:13	Brian:	1 2 3.
00:59:14	Michael:	The 1 2 3 and it would be another one, because you got three different colors. [Michael writes the numbers 1 2 3 on the board in three rows.] And you'd have all these lines here, you know. [Michael draws lines under the numbers.] And you would multiply these lines here by that. I don't know.
00:59:22	Jeff:	It would be 3 times 3 times 3.
00:59:23	R1:	What do you think, Ankur?
00:59:24	Ankur:	Yeah. That-
00:59:25	Jeff:	He's-
00:59:26	R1:	What is he saying, Ankur? 'Cause you're agreeing there, your facial expression.
00:59:30	Ankur:	He's saying that with each one of those top two, after he connects the lines with the top to the middle-
00:59:39	Michael:	'Cause I have, like, this line could go three different ways.
00:59:42	Ankur:	And each line from there can also go another three different ways. So it's times another three ways.
00:59:49	R1:	What do you think? Jeff? Romina?

Time	Speaker	Transcript
00:59:51	Romina:	Yeah.
00:59:52	Jeff:	I agree with what he's saying.
00:59:52	Brian:	Totally. Totally.
00:59:53	R1:	Are you convinced?
00:59:54	Brian:	I was one of the ones that brought it up. [Ankur laughs.] I got to get credit for something.

APPENDIX H: TRANSCRIPTS OF 1998 SESSIONS

January 9, 1998

Time	Speaker	Transcript
00:00:00	R1:	Do you understand the question? Now first of all, maybe there aren't ten. Maybe there are twelve. There have to be ten because they found ten, right? [We see a test pattern for 10 seconds. Then we see Michael, Ankur, Jeff, Romina, and Brian sitting around a table.] Unless they-
00:00:10	R6:	Do you have ten there?
00:00:12	Ankur:	Yeah.
00:00:13	R1:	Now you have to convince me that you have found them all. There couldn't be any others. [The question is, when building five-tall towers and selecting from two colors, how many towers have exactly two red cubes?] So why don't you think about that for a minute. That wasn't really what I was going to ask you to do today, but it's certainly a good way to start, given, uh, where we should be going. [Ankur and Michael are starting to talk in the background.]
00:00:29	Michael:	If the 0 is like yellow and these are like red. [Michael is speaking to Ankur.]
00:00:32	R1:	You can talk to each other or you can think yourself.
00:00:34	Ankur:	[To Michael.] Then you mean if like this is yellow and that's red? So she said-
00:00:37	Michael:	So it's like twenty.
00:00:38	Ankur:	No. She said with only two red and three yellow. She didn't say three red and two yellow. You know what I mean?
00:00:44	Michael:	Uh. What's her name, Dr. R?
00:00:46	Ankur:	Trust me. Trust me.
00:00:48	Michael:	Dr. R? Uh.
00:00:49	Ankur:	Researcher 1.
00:00:49	Michael:	Researcher 1? Uh, if we had like this, let's say like the ones are like red. And those are like yellow.
00:00:58	R1:	OK.
00:00:59	Michael:	Do we like, could this be like-
00:01:01	Ankur:	He wants to know if we could make this-
00:01:01	Michael:	Two yellows and three reds be a different ones?
00:01:06	R1:	Well, what do you think?
00:01:06	Michael:	You said two- I don't know if you wanted two-

Time	Speaker	Transcript
00:01:07	Ankur:	You wanted two red and three yellow and how many you could make with that.
00:01:10	R1:	Uh-huh.
00:01:11	Michael:	//And can it be two yellow and three red?
00:01:11	Ankur:	//So it's ten.
00:01:12	Michael:	You just said two-
00:01:12	Ankur:	No, she just said two red and three yellows.
00:01:13	Michael:	No, she said, she said how many different combinations are two of the same-
00:01:13	Ankur:	She didn't say two of one and three of another.
00:01:15	R1:	I did say both, so you can answer either question. I'll accept your answer to either question.
00:01:21	Michael:	Ten and ten.
00:01:21	Ankur:	Ten and twenty.
00:01:22	Michael:	Or twenty. Whichever.
00:01:23	R1:	So how would you convince me that you found them all?
00:01:25	Michael:	I just did like-
00:01:26	Ankur:	It goes with the first number. It's a 1 there. And then he puts a 1 here, and then the rest [pause] are 0's.
00:01:32	Michael:	See this? That's a, uh, the first top of the tower [Michael gestures up with his left hand.] with a color.
00:01:35	Ankur:	Red.
00:01:36	Michael:	You would have this one, and you can also have this one. [Michael draws.] All right? Then you'd go for the second tower. I mean the second space from the top.
00:01:45	Ankur:	Is red.
00:01:46	Michael:	Always that color.
00:01:48	R1:	Uh-huh.
00:01:50	Michael:	You have- you couldn't have a [Inaudible.] one up here. Because you're either there- It sounds like I'm telling you that problem with the lines. Um- And then it-
00:01:58	Ankur:	And then with the third one-
00:01:59	Michael:	And the same goes for them. And then that one. So that's it.
00:02:02	R1:	Uh-huh.
00:02:03	Michael:	I could probably make in little lines. Probably, you know, with the one, two, three or something like that. [Michael draws a tree diagram.]
00:02:07	R1:	Uh-huh. That's another way you could probably do it. But this works, you said. This works. OK. [Refer to Figure H1 for Michael and Ankur's work.]

Time	Speaker	Transcript

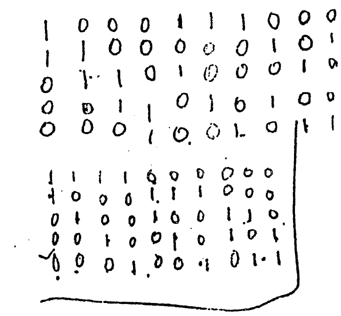


Figure H1. Michael and Ankur's list of 10 towers with exactly 2 reds

00:02:11	Ankur:	Are you convinced?
00:02:12	R1:	Yeah, I'm convinced. Um, so you might be curious to know what you did when you were in the fourth grade.
00:02:20	Ankur:	Probably the same answer?
00:02:21	R1:	Do you think you dealt with 1's and 0's? You got the same answer.
00:02:24	Michael:	Well, we probably had something like red.
00:02:27	Ankur:	We probably, had like, yeah.
00:02:28	Michael:	Ah. Red and blue.
00:02:30	Ankur:	Did we have towers, though?
00:02:30	R1:	Yes.
00:02:31	Ankur:	Then we probably built them.
00:02:32	R1:	You did. You did build them. But after you built them, you, you pulled out-
00:02:37	Michael:	We wasted our time building towers.
00:02:38	R1:	You built them, and you pulled out ten of them, but how do I know you had them all? You had to organize them in a particular way [Ankur nods.] to convince me you had them.
00:02:44	Ankur:	We organized them like this.
00:02:47	R1:	Very much so. But you did them with the towers rather than with the numbers. That's exactly right. So, uh, OK [Inaudible.] patiently watching-
00:02:58	Michael:	Are they doing the same thing?
00:02:59	R1:	Uh-huh.
00:03:01	Romina:	You guys proved it already?
00:03:02	Ankur:	Yeah. [Romina laughs.]
00:03:05	Brian:	Don't ask.
00:03:05	R6:	No, no, no.

Time	Speaker	Transcript
00:03:06	R1:	No, no. We're going to wait to hear what you do after- and you're going to hear what they did.
00:03:11	Brian:	Just tell us now.
00:03:12	R1:	Take your time.
00:03:15	–	[Students finish this problem and work on Ankur's problem for 1 hour and 16 minutes.]
01:19:48	Ankur:	I don't like writing.
01:19:49	Romina:	I'd write for you, but I-
01:19:50	R1:	It's easier for me to understand.
01:19:51	Ankur:	I can't-
01:19:52	R1:	Maybe someone will write for you.
01:19:54	Romina:	I'll write for you if you tell me what to write. I'll write on the board.
01:19:57	R1:	Yes. Romina gave you- [Inaudible.] What do you think? Do you accept?
01:20:03	Ankur:	Decline.
01:20:04	Romina:	But you, you're gonna be aggravated. You're gonna want to do it yourself anyway. So why don't you just do it yourself?
01:20:09	Brian:	You lazy bum.
01:20:12	Ankur:	That's right. [Ankur laughs.]
01:20:14	R1:	Come on.
01:20:16	Ankur:	I got to wait for him. [Ankur gestures toward Michael, who has been writing and who has just raised his hand.]
01:20:17	R1:	Yeah, we'll wait.
01:20:17	Michael:	No. You don't have to wait for me. You just
01:20:18	Ankur:	You have to do this part.
01:20:19	Michael:	Write, write this exactly. One, look, see how it looks on the paper? [Michael shows paper to Ankur.] Write it up there. But neater. [Ankur laughs.] Write it up there. Do that.
01:20:29	Ankur:	Did you, did you write them out?
01:20:30	Michael:	Don't worry about it. Look at all those numbers I have there.
01:20:32	Ankur:	That's what you got to write up it.
01:20:33	Michael:	I'm not gonna write that up.
01:20:35	–	[They discuss Ankur's problem for 13 minutes.]
01:33:10	Ankur:	36.
01:33:12	Ankur:	36 and 45.
01:33:14	R1:	So suppose you were doing towers five tall. Would that same reasoning work?
01:33:20	Ankur:	I really don't care. [Romina laughs.] No one's going to ask me that on the street.

Time	Speaker	Transcript
01:33:26	Romina:	Does anyone else other than us know about towers?
01:33:30	Ankur:	Like scientists who study towers.
01:33:34	Romina:	I guess they know about patterns.
01:33:36	R1:	Well, we- Yeah. A lot of people are using this, this task throughout the country now and in different parts of the world actually. Um, but there's something that, you know, I really. This is really very nice work you've done here. The kind of work you're doing, you might say, who cares. Um. This is a branch of what's called discrete mathematics. And this is widely used in lots of problems like-
01:34:02	Ankur:	Like this?
01:34:05	R1:	Oh, yes. This is, um, like, people who work in networking problems, or people who do, they're very very classical problems that people are working on and solving. It's this kind of reasoning. And the problem is there aren't always simple formulas. You said a couple of times, Romina, when you do it by math. Sometimes the way you have to think through the problem, you do exactly what you've done. And see if it makes sense. Do you know what I'm saying? Sometimes...
01:34:32	Romina:	Yes, like we just assume.
01:34:33	Ankur:	We start with simple formulas.
01:34:35	Romina:	Yeah.
01:34:35	Ankur:	And we get an answer.
01:34:36	R1:	That can get you in big trouble because it may be perfectly sensible.
01:34:38	Ankur:	And we review it, and we make it a little better. And then we make it even a little better. And finally we get this.
01:34:46	R1:	But you got to be careful. Because if someone just wanted to know an answer. And you are questioning each other some, right? And you're really very honest. If you don't understand it, you tell each other. Do you think everyone is like that? [Romina shakes her head.] Right. They say, sure, sure, whatever you say.
01:35:02	Romina:	Our math class-
01:35:02	Michael:	We even do that sometimes.
01:35:04	Romina:	I know. When I don't want to hear it, I'll go-
01:35:05	R1:	When you're tired and you don't want to think about it, right? You're sort of not engaged.
01:35:08	Romina:	That happens in our math class every day. We try to explain. "Yeah, yeah, yeah, we get it."
01:35:12	R1:	Do you get it?

Time	Speaker	Transcript
01:35:13	Romina:	Oh, no. When we try to explain to other people. I'm, I'm only like that when I don't want to deal with it any more.
01:35:18	R1:	When you're tired.
01:35:19	Romina:	Yeah.
01:35:19	R1:	You've had enough. That's exactly right. What's that?
01:35:23	Ankur:	Huh?
01:35:26	R1:	Um. What were you going to say?
01:35:29	Brian:	No, I was thinking about something. [The camera has been focused on Brian, who has been laughing.] Something that happened a long time ago, like in sixth grade. [Inaudible.]
01:35:37	R1:	[She holds up a tower.] These particular problems are in the field of math that people talk about as combinatorics. And these problems are very important for a whole branch of study in probability and statistics, as well as discrete math. So it, it's a field, um, that you'll be seeing. Have you ever heard- You use the word combinations, right? And in a sense, when I asked you if you're building towers five tall, exactly how many are two red, all right, I'm asking you how many combinations, right? When you pick two of a particular kind. Right? From a total of five. When, in this case, there were two kinds, right? Red and not red, if you like. Um, and some of you may, may look in books, um. When you solved that problem, remember how many you got, when we didn't care what color? Let's say red. If you would have picked five, then you want exactly two to be red.
01:36:37	Michael:	Ten?
01:36:37	R1:	You found ten. So, some books will write something like this. Have you ever seen this? There are different ways of writing it. Some will write it like this. If you have any older brothers or whatever, books. This is called a combination. Where the selection of two of a particular kind from a set with five of them. It's also sometimes written like this. Like a large parentheses? Where you're selecting two from five. Is there another way to do it that you can think of? [To Researcher 6.]
01:37:11	R6:	Um. Above and below the C.
01:37:14	R1:	Ah yes. [Refer to Figure H2 for the notations.] Like this, right?

$$ {}_5C_2 \qquad C_{(5,2)} \qquad \binom{5}{2} \qquad C_2^5 $$

Figure H2. Researcher 1's list of notation for combinations

Time	Speaker	Transcript
01:37:15	R6:	Yeah.
01:37:16	R1:	That's another way. This is, you can look in college books, and you can look in high school books. This is, I don't know if you do this, this is probably- If you get it, it would be algebra 2 late, or third-
01:37:28	R6:	Pre-calculus.
01:37:29	R1:	Or pre-calculus would be third-year math.
01:37:32	R6:	You could do it in statistics classes.
01:37:33	R1:	Well, you use this in a lot of courses, in a lot of other mathematics courses.
01:37:37	R6:	I took a whole course in it in graduate school.
01:37:40	R1:	You can take a whole semester of graduate study in this particular area. So the ideas that you're dealing- you want to know who wants to know, a lot of people think this is a pretty important foundation. Um, so, when this particular group here, they were working on an interesting problem because they were looking at- [She erases the board.] I don't know what notation you prefer. I like the large parentheses. [She writes on the board.] OK. They were looking, um- [She writes.]
01:38:17	Romina:	So were we wrong, or-?
01:38:19	Ankur:	Equals-
01:38:19	R1:	You're going to tell me. I'm not going to tell you. I think you're going to be able to tell yourself. [She continues to write.]
01:38:31	Brian:	[Inaudible.]
01:38:34	R1:	OK, you, you were looking at a version of this particular problem. Um, for this one you found 10, right?
01:38:41	Romina:	Uh-huh.
01:38:42	R1:	OK. I'm going to ask you a question. If exactly two are red, right? How many are there if exactly three are red?
01:38:50	Ankur:	Ten.
01:38:52	R1:	Why?
01:38:52	Ankur:	Because, if in, in that one, there were three not red.
01:39:00	R1:	OK, so you all buy that?
01:39:03	Romina:	Yeah. That's what we-
01:39:04	R1:	You think of that as not red. OK, exactly one red?
01:39:04	Ankur:	Five.
01:39:05	Romina:	Five.
01:39:06	R1:	Where are they? Can you imagine it?
01:39:09	Ankur:	One, two, three.
01:39:10	Romina:	Yeah.

Time	Speaker	Transcript
01:39:11	R1:	Exactly no red?
01:39:13	Ankur:	One.
01:39:13	Romina:	One.
01:39:14	R1:	Which one is that?
01:39:15	Romina:	That's all the other one.
01:39:17	R1:	One. [She continues to write.] OK, so go on. Which is this?
01:39:21	Ankur:	The same as-
01:39:23	Romina:	Isn't it the same as-
01:39:24	Ankur:	The same as five and one.
01:39:25	Romina:	Five and one.
01:39:26	R1:	This is exactly four red. Right?
01:39:29	Ankur:	There's five.
01:39:30	Romina:	So- five.
01:39:31	R1:	So why is it the same as exactly-
01:39:32	Ankur:	Because it's like exactly one not red.
01:39:33	Romina:	[Inaudible.]
01:39:35	R1:	OK.
01:39:35	Brian:	And there's one at the end.
01:39:38	R1:	How many is that?
01:39:39	Ankur:	20, 32.
01:39:40	Romina:	32.
01:39:40	R1:	Oh. Isn't that interesting. Which you told me was two to the fifth. [Researcher 1 finishes writing. Refer to Figure H3.]

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$1 + 5 + 10 + 10 + 5 + 1 = 32$$

Figure H3. Counting five-tall towers using combinatorics notation

Time	Speaker	Transcript
01:39:44	Brian:	That's the first time we ever got an answer.
01:39:46	R1:	I didn't give you an answer; I just summarized what you did.
01:39:47	Ankur:	You told us something.
01:39:48	Brian:	Ever.
01:39:49	R1:	All I did was- that's really not true. Actually, Researcher 6-
01:39:52	Ankur:	There was one time.
01:39:53	R1:	Researcher 6- [To Brian.] Maybe you weren't in the class, though.
01:39:54	R6:	No, he was.

Time	Speaker	Transcript
01:39:55	R1:	He was. She saw you. In the fourth grade, we went through a whole argument of this particular case, and it's on view. Researcher 6 wants to bring it in to show you. We actually did do that. You probably fell asleep by then. [To Brian.] It was about an hour and 30 minutes in the class. I'm only teasing. So you see, I want to ask you to think about for next time. Does this work for four, for three?
01:40:18	Brian:	Of course.
01:40:18	R1:	And six?
01:40:19	Romina:	So wouldn't we just be like taking-
01:40:20	Brian:	Oh, man. [Brian leans back and puts his hands over his face.]
01:40:21	R1:	So I'm going to ask you another question.
01:40:23	Ankur:	Oh my God.
01:40:24	R1:	You like this? Is this cool?
01:40:27	Brian:	It's amazing how we, all we do is-
01:40:28	Ankur:	We thought-
01:40:29	R1:	I told you, you were doing college math in the fourth grade.
01:40:31	Ankur:	We thought factorial was good.
01:40:31	R1:	You didn't know that.
01:40:32	Romina:	I told my parents that we're doing college things. We just don't know it.
01:40:34	Ankur:	When we learned factorial, we're like, this is like every problem.
01:40:38	Brian:	Yeah.
01:40:39	Romina:	Yeah.
01:40:40	R1:	Look at, look at how you think about it. That's the important thing. So you're, you're not just saying I'm just moving these symbols around. A lot of people do that, and they don't know what they're doing. I'm sure you must see that. Always ask why. If you're never sure, don't let anyone try to talk you into it. That's why I'm saying. Move off to the side, exactly what you do, Mike. Hide your paper. Go and think about it yourself. Don't let- You have to be convinced, right? Each of you. And you're right. Because you're all doing, you know. You really have to make sense of this yourself. So, that, that's my, my sort of lecture for today. But I have something else for you to think about, right? Um. [She writes on the board, but she is not on camera.] You told me last week what this was, right?
01:41:27	Brian:	That, that. [Inaudible.]
01:41:30	R1:	Right? [She continues to write.] I really believe you all know what even this is. But why don't you think about that?

Time	**Speaker**	**Transcript**
01:41:40	Romina:	We could probably work that out.
01:41:41	R1:	I think you know. I think you worked it out. But-
01:41:43	Romina:	We have?
01:41:44	R1:	I'm going to leave you to think about that. But I ...
01:41:46	Romina:	In one of these?
01:41:47	R1:	Pardon?
01:41:47	Romina:	In one of these sessions?
01:41:48	R1:	Think about and see if we'll talk about it next time.
01:41:52	Ankur:	What it looks like?
01:41:53	R1:	Yeah. What, what does this look like? If *a* plus *b* quantity squared.
01:41:56	Ankur:	It's like *a* plus *b* times *a* plus *b*.
01:41:58	R1:	Last time you told me that you had 1 *a* squared.
01:42:01	Brian:	[Inaudible.]
01:42:02	R1:	Plus 2 *ab* plus 1 *b* squared. Remember that?
01:42:05	Romina:	Yes.
01:42:06	R1:	Isn't that right?
01:42:07	Brian:	Yeah.
01:42:08	R1:	OK? And *a* plus *b* quantity cubed, you told me you had 1 *a* cubed. And then you told me you had 3 *a* squared *b*. [She writes on the board.] How am I doing this so fast? Plus 3 *a b* squared. Plus-
01:42:21	Ankur:	How are you doing this so fast?
01:42:22	R1:	*b* cubed. [She puts down the chalk. The camera focuses on the blackboard.] And I have a terrible memory. Right?
01:42:28	Ankur:	'Cause there is something.
01:42:29	R1:	You're right. There is. And *a* plus *b* to the first. [She writes on the board.]
01:42:34	Ankur:	We never learned this.
01:42:35	Romina:	I know.
01:42:36	R1:	Is 1 *a* plus 1 *b*. Or *a* plus *b*, right?
01:42:39	Ankur:	Can you do *a* plus *b* to the fifth like that?
01:42:40	R1:	And *a* plus *b* to the 0, right? Is merely 1. [Refer to Figure H4 for what has been written so far.] Now I'll show you something. [She erases.] I'm going to erase, OK? *a* plus *b* to the 0 is 1, right? You understand the *a*'s and *b*'s there? [She writes 1 on the board.] Right? *a* plus *b* to the first power, right? Is 1 *a* and 1 b, right? [She writes 1 1.] Let's remember that, right?

Time	Speaker	Transcript

$$
\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= 1a + 1b \\
(a+b)^2 &= 1a^2 + 2ab + 1b^2 \\
(a+b)^3 &= 1a^3 + 3a^2b + 3ab^2 + 1b^3
\end{aligned}
$$

Figure H4. Binomial expansion (powers 0 through 3)

01:43:14	Ankur:	OK.
01:43:15	R1:	*a* plus *b* to the second power is?
01:43:18	Romina:	1 *a*.
01:43:19	Ankur:	1 *a* squared.
01:43:24	R1:	[She writes.] Right?
01:43:27	Brian:	Ah.
01:43:31	Romina:	1 3 3 1.
01:43:32	Michael:	I've seen that before.
01:43:35	Romina:	Yeah, we're, we have seen this before.
01:43:37	Ankur:	She making it, she's just adding.
01:43:39	R1:	What am I doing?
01:43:40	Ankur:	You're adding what's in-
01:43:42	Romina:	1 4 4.
01:43:44	Ankur:	1 4.
01:43:44	Romina:	4.
01:43:45	Ankur:	6 4 1.
01:43:48	R1:	Oh. What do you think the next row is going to be?
01:43:49	Romina:	We did this in Mr. Poe's class.
01:43:51	Ankur:	1 5 10 10 5 1. That thing. [Refer to Figure H5.]

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1
\end{array}
$$

Figure H5. Pascal's Triangle (rows 0 through 4)

01:43:56	R1:	Well, that's something for you all to think about. Why does that work? What does that have to do with anything?

Time	Speaker	Transcript
01:44:02	Ankur:	That's the same thing as one of those things.
01:44:06	R1:	Can you say more, Ankur?
01:44:07	Ankur:	You know what I'm saying.
01:44:10	R1:	No. [She laughs.]
01:44:18	Romina:	[To Ankur.] Where did we do this? Do you remember doing that? And then we had it.
01:44:20	Brian:	I remember doing it. We did that in Poe.
01:44:21	Romina:	We added. We did, added. Didn't we add something?
01:44:23	R1:	The question is, I'm asking you the question. What does it all mean? You know what *a* plus *b* to the fifth means. You have a way of thinking of 2 to the fifth, don't you? What that could mean?
01:44:34	Ankur:	1 5.
01:44:35	R1:	There are lots of ways to think about that.
01:44:33	Ankur:	10 10 5 1.
01:44:42	R1:	This particular relationship, by the way, is called Pascal's Triangle.
01:44:46	Ankur:	Yeah.
01:44:46	Romina:	[She nods.] Yeah, we did learn.
01:44:47	Ankur:	We did this.
01:44:48	R1:	But anyone could do this. You can have a first-grader generate Pascal's Triangle. Right?
01:44:52	Ankur:	We learned about Pascal's Triangle.
01:44:53	R1:	They know how to do simple addition.
01:44:55	Michael:	We just forget.
01:44:56	R1:	The question is, what's the relationship here? How could you model it? Right? How could you show this relationship? And why does it work? That's the question. So that's sort of the direction. Are you interested in knowing that? I think you have the bits and pieces to put it together.
01:45:15	Ankur:	Some of the pieces are really small.
01:45:18	R1:	They're bigger than you think. You've been working on this for a long time.
01:45:22	Romina:	Is this what we did today though?
01:45:24	R1:	You've been dealing with some of this today.
01:45:27	Ankur:	So are all of the things we learned for the past eight years-
01:45:30	R1:	So think about it.
01:45:30	Ankur:	-sort of combined into one thing?
01:45:31	Brian:	Imagine that.
01:45:32	R1:	That's what I'm hoping. Now look here. What did you do here?

Time	**Speaker**	**Transcript**
		[She circles the first 10 from Figure H3; refer to Figure H6.] That's a piece of Pascal's Triangle.

$$1 + 5 + 10 + 10 + 5 + 1 = 32$$

Figure H6. Calling attention to the meaning of an entry in Pascal's Triangle

01:45:37	Romina:	That was one two three four.
01:45:41	Ankur:	That's the next step in that.
01:45:42	R1:	Well, what was, what did you do? What did this mean [She points to Figure H6.] when we did that? Where did this ten come from?
01:45:47	Ankur:	Two red.
01:45:48	Brian:	Two red and three ...
01:45:49	Ankur:	Three.
01:45:54	Romina:	[Inaudible.] [Brian leans back and stares at the board.] Three [Inaudible.] four. [Inaudible.]
01:46:02	Ankur:	I have no idea.
01:46:04	Romina:	Yeah, we see the two and the three.
01:46:06	R1:	Well, think of where the ten came from. It's not just a number.
01:46:08	Ankur:	The ten came from-
01:46:09	R1:	It has to mean something.
01:46:10	Ankur:	The ten came from the- [Pause.]
01:46:12	R1:	When we first came here today-
01:46:15	Ankur:	From the six and the four, which came-
01:46:16	R1:	No. But I'm saying when you first came in here today, you produced that number ten. Right?
01:46:22	Ankur:	Yes.
01:46:23	R1:	And what problem were you solving?
01:46:24	Romina:	Two.
01:46:25	Ankur:	Two were red and three-
01:46:26	Romina:	Yeah. Two red.
01:46:26	Ankur:	And three something else.
01:46:28	Romina:	Three something else.
01:46:29	Ankur:	Three of another color.
01:46:30	R1:	OK. So you can think of that ten in a very real way, if you want to, right?
01:46:36	Ankur:	Yeah.
01:46:37	R1:	Can you think of those other numbers in a real way? Does that help?

Time	Speaker	Transcript
01:46:40	Ankur:	That's the problem.
01:46:42	Romina:	If you- The two and the three are up there. Like when we're getting ten.
01:46:46	Ankur:	The 1 is, in 1 4 6 4 1, the 1 represents all red. The other 1 represents all yellow I guess, and the 4 is probably-
01:46:57	Brian:	Mixed.
01:46:58	R1:	All red and all yellow for what?
01:47:00	Ankur:	For the one.
01:47:01	R1:	For what, though?
01:47:02	Ankur:	All red.
01:47:03	Romina:	All red.
01:47:04	Ankur:	Of four high.
01:47:04	R1:	Four high.
01:47:05	Ankur:	Four high.
01:47:05	R1:	So this is four high. [She points to the fourth row of Pascal's Triangle.] And these are all red. [She points to the first 1 in that row.]
01:47:08	Ankur:	Four red. And that's four high and-
01:47:09	Brian:	[Inaudible.]
01:47:10	Ankur:	One red.
01:47:11	R1:	Four red.
01:47:12	Ankur:	Yeah, four red, four yellow.
01:47:13	Romina:	There'd be one yellow.
01:47:14	Ankur:	And then that's one yellow, one red, and then ... [There is a pause.]
01:47:22	Brian:	[Inaudible.]
01:47:22	Romina:	[Inaudible.]
01:47:23	R1:	So the question is, if that's true, then why does this work? [She points to lines drawn from the 1 and 3 of row 3 of Pascal's Triangle to the 4 in row 4. She draws lines from the two 3s in row 3 to the 6 in row 4. See Note 2.] You know, where did the four come from? Where did the six come from? [She puts down the chalk.] But that's next time. It's 5:15.
01:47:33	Brian:	[Inaudible.]
01:47:34	Romina:	Is that where the six came from? [Inaudible.]
01:47:35	Brian:	[Inaudible.]
01:47:36	Ankur:	The six comes from ...
01:47:37	R1:	So, so.
01:47:38	Ankur:	Like you have four and you add either a red or a yellow to it.
01:47:44	R1:	Can you show us?

Time	Speaker	Transcript
01:47:45	Romina:	The six.
01:47:46	Ankur:	Not really.
01:47:47	R1:	Are you tired?
01:47:47	Romina:	The six would be everything else, wouldn't it?
01:47:49	Brian:	[Inaudible. He tosses some cubes towards Ankur.] Knock yourself out.
01:47:55	Ankur:	Too far away. I'm kidding.
01:47:57	R1:	You've been working very hard. I know you're tired. But you've done a lot of mathematics. If, if you want to stop and do this next time, and think about it, you can.
01:48:06	Brian:	Yeah.
01:48:07	Romina:	This would be really good if you're going into some statistical math.
01:48:10	R1:	What do you think, Romina?
01:48:12	Romina:	What?
01:48:13	R1:	What did you say?
01:48:14	Romina:	This will, this will help if you went to, I was thinking of going into actuary, what I was telling them about before.
01:48:17	Ankur:	Actuary.
01:48:18	R1:	This is exactly the foundation for that.
01:48:20	Romina:	Yeah. I didn't know that before I thought about being that.
01:48:22	Ankur:	You want to do this for the rest of your life?
01:48:23	R1:	It's hard work, Romina.
01:48:24	Romina:	I didn't know that.
01:48:26	Ankur:	Do you want to do this for the rest of your life?
01:48:28	Romina:	Uh, I don't know.
01:48:29	Brian:	I'd just [Inaudible.].
01:48:30	Ankur:	Seven or eight years. Why stop now?
01:48:34	Romina:	Maybe I'll just-
01:48:35	R1:	Probably she's done it ten years.
01:48:37	Ankur:	Ten years.
01:48:39	Romina:	'Cause someone told me about what an actuary does. And I was like, oh, I find that interesting. Little did I know that I did it for a really long time.
01:48:48	Michael:	[Inaudible.]
01:48:50	R1:	What do you think, Mike? What are you thinking?
01:48:52	Michael:	I'm asking what an actuary does.
01:48:53	Romina:	Yeah, I just don't have all the ...
01:48:54	Brian:	It's an insurance company person thing. I don't know.
01:48:56	Romina:	Actuary? Oh, here we go again.

Time	Speaker	Transcript
01:48:58	Ankur:	Explain it.
01:49:00	Romina:	I've explained it.
01:49:00	Ankur:	It's your profession.
01:49:01	Romina:	OK. You figure out, like, statistics for insurance companies so they know, like, who to insure and who not to insure. Sort of.
01:49:09	Brian:	They better insure me. [Ankur and Romina laugh.]
01:49:12	R1:	OK, so, um, you know, I really just hate to let you ...
01:49:18	Michael:	It's dopes like you that make guys have more insurance.
01:49:21	Romina:	What?
01:49:22	Brian:	It's wrong.
01:49:23	R1:	Or less.
01:49:24	Brian:	We never have less insurance.
01:49:25	R1:	Oh, for the young men, yes.
01:49:27	R6:	Do you guys have, are you on the honor roll?
01:49:30	Romina:	We're gonna get ...
01:49:31	R6:	When you go for your car insurance, make sure you send them a copy of the ... a statement that you were on the honor roll.
01:49:35	Ankur:	Yeah, send them a copy of your ... [A discussion of car insurance follows.]
01:50:26	R1:	OK. Now are you going to write up your solution to Ankur's problem?
01:50:30	Brian:	[Side conversation.] One month, 21 days.
01:50:31	R1:	This is Ankur's problem. You're going to write Ankur's problem.
01:50:33	Ankur:	Ankur's problem.
01:50:34	R1:	The statement of Ankur's problem and the solution to it.
01:50:36	Romina:	Hold on a second. This?
01:50:37	R1:	You have two parts to it.
01:50:38	Romina:	This?
01:50:39	R1:	That's part one.
01:50:40	Ankur:	And the other is part two.
01:50:41	R1:	And the other is part two. So you can get on this together and write it up?
01:50:43	Romina:	So I'll do part one.
01:50:43	Ankur:	Can't you just look at the tape and exactly what we did on the board?
01:50:46	R1:	No, because I'd like to see your writing. I want to give you credit.
01:50:50	Brian:	Yeah, Ankur.
01:50:52	R1:	So I need your write-up.
01:50:53	Ankur:	This credit.
01:50:54	Romina:	We could always give you this picture. You couldn't get it from

Time	Speaker	Transcript
		the picture?
01:50:56	R1:	I, I think we, we can get it but we want you to write it up.
01:50:59	Romina:	I'm just doing part one. They're doing part two. Are we combining it?
01:51:00	Michael:	How about we give them this and let them try to figure out what we were doing.
01:51:03	R1:	Let's say ...
01:51:04	Romina:	I know. Here you go.
01:51:04	Ankur:	From ours they won't be able to ..
01:51:06	Michael:	You know, just look at it.
01:51:07	R1:	Write up, write up the statement of Ankur's problem first. You might want to give it to your teachers. Say, "This is the problem."
01:51:15	Ankur:	They won't be able to solve it.
01:51:16	Brian:	My teacher won't know what I'm talking about.
01:51:18	Ankur:	I bet you half the teachers in the school won't be able to solve it.
01:51:21	R1:	You really think so? There's a study for you.
01:51:21	Ankur:	Honestly.
01:51:22	R1:	You can gather some data.
01:51:23	Romina:	Ankur's problem.
01:51:24	Ankur:	What's the point?
01:51:25	Romina:	OK. What's the problem? [They finish up with a discussion of Ankur's problem and proper wording of problems.]

February 6, 1998

Time	Speaker	Transcript
00:00:00	–	For the first 37 minutes, the students (Ankur, Jeff, and Romina) discuss with Researcher 5 what they have done recently: finding areas, multiplying binomials, binary notation, towers problems, and pizza problems.
00:37:55	R5:	Did you do other things with these, uh, I don't know what to call them, these blocks coloring problems? Did you connect it with other mathematics that you know?
00:38:04	Jeff:	I don't know.
00:38:05	Ankur:	Well, she [Researcher 1] did.
00:38:05	Romina:	She did, yeah.
00:38:06	Ankur:	Didn't she do something with like binomials?
00:38:07	R5:	How did that go?

Time	Speaker	Transcript
00:38:08	Jeff:	Yeah. We, uh, it was like-
00:38:11	Romina:	Like, yeah.
00:38:12	Ankur:	x+1 times x+1, or something like that.
00:38:16	Jeff:	Yeah. Like if we made these, we made it like-
00:38:18	Romina:	And then, then you add another one in-
00:38:20	Jeff:	We were working with squares.
00:38:21	Ankur:	x+1 quantity squared.
00:38:22	Jeff:	Yeah. I don't really remember what the problem was, though.
00:38:24	R5:	What happens when you square x+1?
00:38:27	Jeff:	It's-
00:38:28	Ankur:	It looks like x squared plus-
00:38:31	Romina:	Yeah. that we were supposed to figure out-
00:38:32	Ankur:	-2x plus 1.
00:38:34	Jeff:	Yeah. Yeah. Exactly.
00:38:37	Romina:	This is what she was doing. x, like it was plus 1.
00:38:42	R5:	Suppose you just multiply that out by hand. Yeah.
00:38:44	Romina:	[Romina does the multiplication on her paper.] Then you just kept going on, what would happen.
00:38:51	R5:	What did you find?
00:38:52	Romina:	Nothing.
00:38:54	R5:	Nothing!
00:38:55	Ankur:	Well, there was something.
00:38:56	R5:	Something has to be happening here.
00:38:57	Romina:	No, yeah, she explained to us, like-
00:38:59	R5:	There's got to be something.
00:39:00	Romina:	Yeah, it was, it was a while ago, so.
00:39:02	Jeff:	It was, yeah.
00:39:04	Romina:	[To Jeff.] Did she use like a C?
00:39:04	R5:	A month ago, that's a lot of water under the bridge.
00:39:07	Romina:	She, she wrote something like this.
00:39:09	Jeff:	[Inaudible.]
00:39:10	Romina:	Like this, and this, and this, like. Remember?
00:39:12	Jeff:	Yeah, we, it just, I was trying to think about that problem we did where x became a part of-
00:39:17	Ankur:	She did a whole [Inaudible.] up there.
00:39:18	Romina:	Yeah. She-
00:40:07	Ankur:	She showed us how it related.
00:39:21	Jeff:	Because we started out doing, I don't remember we were doing, but it came-
00:39:25	Romina:	But that's how we started.

Time	Speaker	Transcript
00:39:26	Jeff:	Well. And then it came to the fact that- it was like, we had, I don't, it was, talking about, into a certain- we were working with a square in a certain way. And we were looking at this being, like, this length being x, knowing that this side's is like 10. We know that that was like x, and that was like 5.
00:39:48	R5:	Ah.
00:39:49	Jeff:	And then this was like 5, and that was x, and that, that was like 1, like this. And, I don't- and then we tried to, three dimen- we tried to make it three-dimensional.
00:39:55	Romina:	Three-dimensional.
00:39:57	Jeff:	And I just can't remember how we were doing it.
00:39:59	Romina:	Because we never, because we multiplied this but after this we got a little-
00:40:04	Ankur:	We were trying to show, like-
00:40:05	Romina:	Yeah.
00:40:08	R5:	How about if we start with what, the part you're saying that with x+1 that you have a feeling that you sort of have a grip on that. So why don't we explore that one first and then see how much of that we could carry forward into the next case? Would that, does that make sense?
00:40:25	Jeff:	Yeah. All right.
00:40:25	Ankur:	Yes.
00:40:25	Jeff:	This is what, this is what we do.
00:40:26	R5:	Yeah.
00:40:27	Jeff:	We're looking at something like this. [Jeff draws a square.] We're saying all right, that's we're, this doesn't dissect it perfectly. [He draws lines to divide the square into four parts.] We'll just, this is like-
00:40:35	R5:	OK. I'll buy that.
00:40:36	Jeff:	X one.
00:40:37	R5:	OK.
00:40:37	Jeff:	So this would, this side would equal x+1.
00:40:39	R5:	OK.
00:40:40	Jeff:	All right?
00:40:40	R5:	Uh-huh.
00:40:40	Jeff:	The same thing with this side, x and 1. And this is $x + 1$.
00:40:44	R5:	OK.
00:40:45	Jeff:	And then we looked at x times x, the, the area of this square here would be x squared.
00:40:50	Romina:	You know. We used a and b.

Time	Speaker	Transcript
00:40:51	Ankur:	It's the same.
00:40:52	Jeff:	But it's the same thing.
00:40:54	Romina:	Well, OK. I'm just saying because that would equal this.
00:40:57	Ankur:	What? Well, it is a square. It's got a-
00:41:01	Jeff:	Yeah. And then we looked at x times 1, which x-
00:41:04	Ankur:	x.
00:41:05	Jeff:	1 times 1 is 1. And x times 1 is x. And that equals// x squared plus 2x plus 1, which-
00:41:12	Ankur:	//x squared plus 2x plus 1.
00:41:14	Jeff:	-gives you, which makes sense, because it's the answer to this.
00:41:17	R5:	OK.
00:41:18	Jeff:	What does this have to do with blocks?
00:41:21	Ankur:	Remember she showed us something?
00:41:23	Jeff:	I know this-
00:41:23	Romina:	Yeah.
00:41:24	Jeff:	We were working with it and then she said all right, well, what if we wanted to make this a cube and make it-
00:41:29	Ankur:	Times another x+1.
00:41:30	Jeff:	-make it x+1 high. And then we did, we started to work on that, but I don't, I don't know why we were even there. I can't remember.
00:41:37	Romina:	And then we moved to the towers, 'cause Ankur came up with the problem.
00:41:39	Jeff:	Yeah, but we weren't even doing towers when we were doing this, right? What were we doing?
00:41:44	Ankur:	We were just, no, we-
00:41:46	Jeff:	We were doing pizza problems.
00:41:47	Ankur:	We went to the towers and then-
00:41:49	Jeff:	And then Mike used his-
00:41:50	Ankur:	She was about to give us a problem. Mike used his binary.
00:41:53	Jeff:	Mike used the binary system. And then we went from pizzas to this.
00:41:57	R5:	This is really wonderful. You're multiplying polynomials. You're finding areas. You're saying that there's a connection to the towers. Um. You're mentioning the pizza problem.
00:42:07	Jeff:	Which was kind of like-
00:42:10	R5:	How could all these things come up at once?
00:42:11	Jeff:	'Cause it's all the same thing, you're just-
00:42:12	Ankur:	'Cause they're basically the same thing.
00:42:13	Romina:	It's the, yeah-

Time	Speaker	Transcript
00:42:13	Ankur:	-behind them all.
00:42:13	Romina:	It's different combinations.
00:42:14	R5:	Could you explain that a little?
00:42:15	Jeff:	All right. Like this was like, the pizza problem was like, you got a, you have a pie, and you could have like five toppings on it, say. Right? And that's how it's the same thing, like we worked, like-
00:42:29	Ankur:	It could be-
00:42:30	Jeff:	One and like, this one he, like, say there could be-
00:42:35	Ankur:	The only difference is, it's- there's five possible toppings, which is like five possible colors.
00:42:41	Jeff:	Yeah. It's like five possible colors in a five high thing.
00:42:43	R5:	I'm not sure I followed that.
00:42:44	Jeff:	No.
00:42:44	Ankur:	No.
00:42:48	Jeff:	It's like having, you have five colors and it's saying you can put 'em anywhere you want, but-
00:42:53	Ankur:	No, no, no.
00:42:53	Jeff:	There's no restriction on height.
00:42:55	Ankur:	We had a pizza pie and there's five-
00:42:56	Jeff:	Yeah, but the height only is-
00:42:57	Ankur:	You have a choice of five toppings but can only use two of them on the pie at once. Was that it?
00:43:02	Romina:	No. It was, you have five pie, toppings, and you can-
00:43:04	Ankur:	And you get the pizza-
00:43:05	Romina:	Then we, remember, we figured out that, yeah, because it did the height, that height wasn't a factor, like you had just-
00:43:09	Jeff:	Yeah, 'cause height didn't-
00:43:10	Romina:	Pepperoni and-
00:43:12	Jeff:	Yeah. Like you're saying that you can have, if you have five colors but it doesn't make a difference how high. You could have red.
00:43:16	Romina:	But it does make a difference, 'cause-
00:43:17	Jeff:	You could have a red-blue, you have a red-blue-green. It doesn't make a, you're not saying-
00:43:19	Romina:	You see, you couldn't have-
00:43:20	Jeff:	-that they have to be-
00:43:20	Romina:	A red and a blue or a blue and a red. 'Cause that would be the same thing.
00:43:23	Jeff:	Yeah, so it-

Time	Speaker	Transcript
00:43:24	Romina:	So when you take out some restrictions-
00:43:25	Jeff:	You take away-
00:43:26	Romina:	Here. You put some in there.
00:43:26	Ankur:	You take away some restrictions.
00:43:27	Jeff:	You take away the height factor.
00:43:27	Ankur:	But you also add in some restrictions.
00:43:28	Jeff:	So, like-
00:43:29	R5:	Oh. I see.
00:43:29	Romina:	That's what we were saying before.
00:43:31	Jeff:	So, like that-
00:43:32	Ankur:	It just turned out to be the same-
00:43:32	Jeff:	-kind of deal.
00:43:32	Ankur:	-answer.
00:43:33	Jeff:	'Cause this is the same thing right here, if two high?
00:43:35	Ankur:	But wasn't it plus one, because of like a plain?
00:43:38	Romina:	Yeah.
00:43:39	Jeff:	And then, and at the end-
00:43:39	Romina:	At the end, yeah.
00:43:40	Jeff:	You add one. It was two, it was two times two times gives you the answer.
00:43:44	Ankur:	It was, uh.
00:43:44	Jeff:	It was like two to the number of toppings.
00:43:46	Romina:	Two. Yeah.
00:43:47	Jeff:	Was it? It was.
00:43:48	Ankur:	Plus one.
00:43:49	Jeff:	It was like if you could have, yeah, two to the number-
00:43:50	R5:	Are you saying the number of possible pizzas where you can choose among, say-
00:43:54	Jeff:	Five.
00:43:54	R5:	-three toppings.
00:43:54	Ankur:	Yes.
00:43:55	Jeff:	Well, say it's two.
00:43:55	R5:	-is two times two times two?
00:43:57	Jeff:	Yeah. You're looking at, like, say, you have one pizza. Or you have two toppings, right? That would be 2 to the, 2 to the second. [Jeff looks at Ankur and Romina.] With two toppings. Two times two is four. And that's one topping, the other topping.
00:44:10	R5:	Uh-huh.
00:44:11	Jeff:	The two toppings together, and the plain. And that would be just

Time	Speaker	Transcript
		like an *x* would be plain. And that would be four. Those are the only possibilities that you can get out of those.
00:44:19	R5:	That's true.
00:44:20	Jeff:	And then, say you had three toppings. You could have 1, 2, 3, you could have 1 and 2, you could have 1 and 3, you could have 2 and 3, and you could have 1 and 2 and 3. That would be one, two, three, four, five, six, seven, and *x* being the plain. Eight, 2 times 2 times 2.
00:44:39	Romina:	And then that's how we know- [Inaudible.]
00:44:43	Jeff:	And then we moved on from there to-
00:44:44	R5:	I see the same numbers coming up.
00:44:46	Romina:	Yeah.
00:44:46	R5:	But I'm not sure I'm seeing-
00:44:48	Ankur:	The only difference is-
00:44:49	R5:	Is it an accident that the same numbers are coming up?
00:44:51	Ankur:	The number can't be rearranged in a different order.
00:44:52	Romina:	Yeah, like in this, in the tower problems.
00:44:55	Jeff:	Like in towers- [Inaudible.]
00:44:56	Romina:	That, you have a solid height, like it can only be three high.
00:44:59	R5:	Yeah.
00:44:59	Romina:	But you can put the colors red-blue and blue-red, you know, on this one, and that's it. And in this one, you can have any height you want, 'cause it-
00:45:04	Jeff:	Yeah, it's like-
00:45:05	Romina:	You can't have, like pepperoni and sausage and then sausage and pepperoni.
00:45:10	Jeff:	Yeah.
00:45:10	Romina:	It's the same thing.
00:45:10	R5:	[Researcher 5 nods.] It's the same thing.
00:45:11	Jeff:	Yeah, 'cause it's saying that if you have one that's three high, you can't just put in a red and say that's it, like we can do with this and say we just want, like, pepperoni. You could do that on a pizza, but, but you can't do that in three high. On the other hand, you can go 1 2 1, but that, but on a pizza, you can't have pepperoni, peppers, and then pepperoni. You can't do it. So as you take away the height in the problem, you're also, you're saying that you can't double up like that.
00:45:37	R5:	[Researcher 5 nods.] OK, so there are more colors, but-
00:45:41	Ankur:	There are other restraints.
00:45:42	R5:	Some of the combinations sort of collapse.

Time	Speaker	Transcript
00:45:45	Jeff:	Yeah, you can't have peppers and peppers and peppers and peppers.
00:45:48	R5:	Uh-huh. So far, what I'm seeing is the same numbers coming up. [Jeff nods.] On the towers of two, right? But is that an accident?
00:46:00	Jeff:	Obviously not.
00:46:01	Romina:	No, because she found- she, she told us that it was like combinations, and she used combinations where she put like- Remember how she put-
00:46:09	Ankur:	Remember that tower she did, where it was like //1+5+10+10+5+1?
00:46:13	Jeff:	//Weren't we-
00:46:14	Ankur:	Remember that?
00:46:14	Romina:	Yeah. It was like binary.
00:46:14	Ankur:	Like 5 represented, like, the possible-
00:46:19	Jeff:	'Cause we were, we came across this when you and Mike were flying through your stuff. And we were stuck, and we were just looking for like connections on-
00:46:25	Romina:	Yeah.
00:46:26	Jeff:	-just what we do and we saw how, that they were like, powers or whatever. And they just kind of worked that out and just kind of found some other stuff. 'Cause that's the easiest way to do some stuff, is just to try to find an easy way out and you just find the stupid connections.
00:46:38	Romina:	Yeah.
00:46:39	Jeff:	And then it just seems that it worked out.
00:46:41	R5:	Is she in the other room?
00:46:44	R6	Would you like us to get her? Researcher 1? We have a question for you.
00:46:51	Ankur:	Does she watch it over there?
00:46:53	R6	[Inaudible off-camera conversations.]
00:47:00	R1:	[Researcher 1 sits down.] Oh, we're up to that. OK, that's great.
00:47:04	R5:	They really got me confused. They're talking about pizzas. They're talking about towers.
00:47:08	R1:	Uh-huh.
00:47:09	R5:	They're talking about rectangular diagrams and cubes.
00:47:14	R1:	Uh-huh.
00:47:15	R5:	They're built up x's and 1's. And Romina was saying a's and b's.
00:47:20	R1:	Uh-huh.
00:47:21	R5:	And I'm asking them: is it possible that all these different things

Time	Speaker	Transcript
		connect together? They're telling me yes.
00:47:27	Romina:	And you- 'cause she-
00:47:28	Ankur:	You showed us-
00:47:29	R5:	And they're showing me-
00:47:30	Romina:	Yeah. You wrapped it all up for us.
00:47:30	R5:	They're showing me how powers of two are coming up.
00:47:30	Romina:	It was like a *C* and there was like one and something. Wasn't it? or something like that?
00:47:34	R5:	Yeah.
00:47:34	Ankur:	There's a way to write it.
00:47:35	Romina:	Yeah.
00:47:35	Ankur:	You showed us everything.
00:47:37	Jeff:	I left early for, for the big explanation.
00:47:39	R1:	You can't blame Jeff. [Laughter.] It's time for me to leave, Jeff, too. Why don't we both go and see how Romina and Ankur do with this? Right? Fair enough?
00:47:50	R5:	Shall I step out of the room while you guys prepare? [Inaudible.]
00:47:52	R1:	No. No. No. No, I'll, I'll be very happy to repeat what I said then. 'Cause I think it's not gonna help answer the question. Um.
00:47:59	Jeff:	What is, what is the question? Just how it all ties in together?
00:48:04	Ankur:	Isn't that the answer to the question?
00:48:04	R5:	Was it an accident that you just keep getting powers of two or is there some reason?
00:48:07	Romina:	[Inaudible.]
00:48:08	R1:	Let me just go back and review for Jeff. So that there isn't really much of a hole, but it's a question of notation. You remember when you first came in and, um, I told you we saw a videotape of some of you earlier when you were in fourth grade and you were building towers five tall when you could select from two colors? And I was looking at a little clip of tape, um, after you had all built your towers. You weren't in that class, Romina, but you two [Ankur and Jeff] were. Um. Many of you found- do you remember how many you found, five all towers when you can select-
00:48:45	Ankur:	Thirty-two.
00:48:46	R1:	-from two colors? You found thirty-two. You really couldn't find any more, but you really didn't come up with an argument why you thought you had them all. And one of the questions I

Time	Speaker	Transcript
		asked you is, well, how, how many do you have that have exactly two reds?
00:49:01	Ankur:	And then I broke it down and it was something like with all, all red was one and then all blue was one on the other side. And then there was like five with one color and four of the other. And five with one-
00:49:16	Romina:	You remember this?
00:49:17	Ankur:	-of the other color and- now I remember it.
00:49:20	Jeff:	It just, it just makes me- [Inaudible.]
00:49:21	Ankur:	You know what I'm saying.
00:49:22	Jeff:	Yeah, it just-
00:49:23	Ankur:	It just all adds up. It's one plus five plus ten plus ten plus five plus one.
00:49:27	R5:	You found ten there. Is there a connection?
00:49:30	R1:	OK. So that- [Inaudible.]
00:49:34	Ankur:	This is the-
00:49:35	R1:	[Inaudible.] Just to go back to Jeff.
00:49:37	Ankur:	[Inaudible.] I-
00:49:37	R1:	Hold that idea. But the question was-
00:49:39	Romina:	[Inaudible.]
00:49:39	Ankur:	OK. That helps a lot.
00:49:40	R1:	But I only asked you in the fourth grade, I only asked you in the fourth grade, um, how many did you find that had exactly two reds. And so what all of you did is you looked at your 32 and you picked them out and you said and you counted and what did you find?
00:49:58	Jeff:	[Inaudible.]
00:49:59	Ankur:	Out of two red? Ten.
00:50:00	R1:	Exactly two red, you found ten. I said, but wait a minute. How do you know you found 'em all?
00:50:04	Jeff:	I think we picked them out.
00:50:06	Ankur:	Yeah.
00:50:08	R1:	But maybe there are more. Can you convince me that you have all possible five towers tall with exactly two reds when you can select from two colors? And when I told you that day is when you came in that fourth grade class you did convince me. And I was asking you if you remembered how. And so you were working with Michael. And you showed me. And then you and Brian and Romina were working and you showed me, but you showed me differently.

Time	Speaker	Transcript
00:50:38	Jeff:	Yeah. That was the deal. 'Cause they went through and that's what we were saying to him.
00:50:40	R1:	But all-
00:50:41	Jeff:	That they went through it really quickly.
00:50:41	R1:	But all I told you, lest my graduate student friends and my dear friend Researcher 5 will blame me of telling you too much, just for the record, is all I told them was, when I asked you how many two reds can you find from towers five tall, we can write that a lot of ways. It's called a combination.
00:51:01	Romina:	Yeah. That was it.
00:51:01	Ankur:	Yeah. That was it.
00:51:02	R1:	That you're selecting-
00:51:03	Romina:	[Inaudible.]
00:51:04	R1:	-two of a particular kind from five. And we can write it also this way. [She writes $C(5,2)$.] And we can-
00:51:09	Jeff:	What does the *C* represent?
00:51:10	R1:	Selection. Or we use the word combination. That we're finding combinations. that we're selecting from this set of all possible ones of a particular characteristic. In this case, reds. And this tells you how many of them, from a set and this is the-
00:51:27	Jeff:	Oh, the height.
00:51:27	R1:	That are five tall.
00:51:28	Jeff:	The restriction.
00:51:29	R1:	OK. So I was only then saying that you- if you've seen this in maybe in another book in your school, you've seen this notation. and what was- we said some others. I'm not-
00:51:39	Ankur:	there were like five or six.
00:51:41	R1:	What was another notation? We also said you could-
00:51:42	Ankur:	There was something underneath.
00:51:43	R1:	-put a *C* and they do sometimes a five, two. [Refer to Figure H7 for the different notations written by Researcher 1.]

$$\binom{5}{2} \quad C_2^5 \quad C_{(5,2)}$$

Figure H7. Researcher 1 lists three notations for "5 choose 2"

00:51:45	Romina:	Like over and under.
00:51:47	R1:	Right. And, and so you were solving problems like this. that would be an example of a problem like that. And so you told me the answer to that was ten, right?

Time	Speaker	Transcript
00:51:58	Jeff:	Yeah.
00:51:58	R1:	And I could've asked you, what if I asked you this problem?
00:52:00	Jeff:	And we said-
00:52:01	R1:	Do you know what I'd be asking you?
00:52:02	Jeff:	And we said it was five.
00:52:03	R1:	You said it was five, right.
00:52:04	Jeff:	And then if you did five-
00:52:06	Ankur:	Five and three.
00:52:06	Jeff:	Five and three.
00:52:07	Jeff:	We said it was-
00:52:08	Ankur:	Fifteen.
00:52:08	Jeff:	Fifteen.
00:52:08	R1:	OK. so, um, and that's all I, I sort of did and when, when you told me this, you said one. What did you say, five, ten?
00:52:16	Ankur:	Ten.
00:52:16	R1:	Five, one. Which you told me, right? I said we could've written this differently. we could've written this one, I, I chose this notation as five. [Researcher 1 begins to rewrite row 5 of Pascal's Triangle in "choose" notation.]
00:52:26	Ankur:	Five, one.
00:52:27	R1:	And you're selecting, you're selecting how many?
00:52:27	Ankur:	One.
00:52:29	Romina:	One.
00:52:30	R1:	There's one, one way to select-
00:52:31	Romina:	Zero?
00:52:32	R1:	-exactly one red towers five tall?
00:52:34	Jeff:	There's none.
00:52:35	Ankur:	Yeah.
00:52:36	Jeff:	Wait. Oh. You're saying that there's-
00:52:38	Ankur:	There's only- you can only have one tower all red.
00:52:38	Jeff:	We're saying that you don't have to use every color once.
00:52:41	R1:	OK. So. So if it's all red.
00:52:43	Ankur:	It's zero.
00:52:44	R1:	It's picking none of the other color, right? And there are five towers where you are picking them with exactly one red.
00:52:50	Ankur:	One red.
00:52:50	R1:	Right? And there are ten, you told me, when you're picking-
00:52:53	Jeff:	Two reds.
00:52:54	R1:	Right? What did I do wrong? Zero, one-
00:52:56	Ankur:	Wait. It's ten. One, five, ten, ten, five, one.

Time	Speaker	Transcript
00:52:59	R1:	Oh. Good. Thank you. [Researcher completes row 5. Refer to Figure H8.] So this is three, right? And this is five, and this is [Inaudible.]. Now, I still didn't go back to the initial question. When you were fourth graders, you, you were able to convince me that you found all possible five tall towers with exactly two reds. And you convinced me that, that- you all convinced , you convinced me of a lot more than that, but that was one question. So, can I leave now? Or. Oh! The other thing I said, for your benefit. What else did we talk about that day, Romina? Do you remember?

$$\begin{matrix} 1 & 5 & 10 & 10 & 5 & 1 \\ \binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5} \end{matrix}$$

Figure H8. Row 5 of Pascal's Triangle written by Researcher 1

00:53:35	Romina:	And then how-
00:53:36	Ankur:	You did-
00:53:36	Romina:	How this related back to this-
00:53:37	Ankur:	You did something with-
00:53:39	R1:	Yeah.
00:53:39	R5:	Yeah.
00:53:40	Ankur:	-with that [Inaudible.] like-
00:53:41	R5:	So that's really where-
00:53:42	Ankur:	x plus 1 times that x plus quantity-
00:53:43	R5:	-this discussion started?
00:53:43	Romina:	Yeah.
00:53:44	Ankur:	About five times.
00:53:45	R1:	Well, I asked the question.
00:53:45	Ankur:	You showed up something.
00:53:46	R1:	Is this connected to, well, can you show me, let's take a plus b quantity squared. You know what that is, right?
00:53:54	Romina:	Yeah.
00:53:55	Jeff:	Yeah. We were just, we were just doing that.
00:53:56	Romina:	a squared.
00:53:57	Jeff:	a squared plus-
00:53:58	R1:	That's one a squared, isn't it?
00:53:58	Romina:	Yeah.

Time	Speaker	Transcript
00:53:59	Jeff:	Well, yeah.
00:53:59	Ankur:	One *a* squared plus two *a b*.
00:54:00	Romina:	Yeah.
00:54:00	Ankur:	Plus *b* squared.
00:54:01	R1:	I'm gonna write 1 *b* squared.
00:54:03	Jeff:	You do that? You put the 1 before it?
00:54:06	R1:	I'm just doing that now. I don't, no, we don't usually do it as a convention. For now it would be good-
00:54:10	Ankur:	It would be easier to see something.
00:54:10	R1:	-for us to remember it, right? It would be good for us to remember that we mean that, right? And so for now, I want, I want you to think about those numbers a little bit, right? And I want you to think about those letters a little bit. So you might just think about that. *a* plus *b* to the zero. Put, think about the numbers, too.
00:54:30	Ankur:	Those number kind of related something to this.
00:54:31	Romina:	Yeah, 'cause then-
00:54:33	Ankur:	They relate to this.
00:54:33	Romina:	Wouldn't this be, wouldn't this be like-
00:54:34	Ankur:	One of-
00:54:35	Romina:	Here would be, would be, we'd have a 2 here or something. Wouldn't it be?
00:54:39	R1:	OK. So you're saying that we have-
00:54:41	Ankur:	Wouldn't it just be *a* plus *b*?
00:54:42	R1:	-two here.
00:54:42	Ankur:	-quantity second is gonna be this one of them.
00:54:45	Romina:	Uh. *C* 2 of 1.
00:54:46	R1:	So what's this one?
00:54:47	Ankur:	That's-
00:54:49	Romina:	That's, that would, that's like this, right?
00:54:51	Ankur:	That's the same problem as the tower.
00:54:53	Romina:	That shows us that there's it's all one color.
00:54:55	Jeff:	Yeah.
00:54:57	R1:	So tell me more.
00:54:59	Jeff:	It could be like anything over, it's just the same thing here.
00:55:02	Romina:	Yeah.
00:55:02	Jeff:	Like your- it could be that.
00:55:03	Romina:	It's, it's all one color and none of the other color.
00:55:05	R1:	It's 5 select 0.
00:55:06	Romina:	It's 2 0.

Time	Speaker	Transcript
00:55:06	Jeff:	2 0.
00:55:07	Ankur:	Well, that's the same problem-
00:55:08	Jeff:	Well, yeah.
00:55:08	Ankur:	-as having a tower.
00:55:08	Jeff:	It's to the second.
00:55:09	Romina:	Yeah.
00:55:10	Ankur:	Two high with two colors.
00:55:11	Jeff:	And that would be this one would be 2 1, right?
00:55:13	Ankur:	I think, yes.
00:55:16	Jeff:	That would be 2 1.
00:55:16	Romina:	That would be 2 2.
00:55:18	Jeff:	And then that would be 2 2.
00:55:19	R1:	What, what are the *a*'s and the *b*'s here?
00:55:21	Romina:	Colors.
00:55:22	Ankur:	The *a*'s.
00:55:23	Romina:	No?
00:55:24	R1:	What does the *a* squared mean and the *ab* mean and the *b* squared? I know what this means. This means you have one of some things.
00:55:30	Ankur:	Well.
00:55:31	R1:	This means you have two of something and this means you have one of something.
00:55:32	R5:	There's something missing.
00:55:34	R1:	Yeah. What's-
00:55:35	R5:	What about *b a*?
00:55:36	R1:	Yeah, what about *b a*?
00:55:37	Ankur:	*b a* is the same thing as *a b*.
00:55:39	Romina:	As *a b*.
00:55:39	Ankur:	But there's two of 'em.
00:55:40	Romina:	Two of 'em.
00:55:40	Jeff:	That makes, that's why it's two *a b*.
00:55:42	R5:	Tell me more.
00:55:43	R1:	Yeah.
00:55:43	Jeff:	'c-cause *a*-
00:55:45	Ankur:	*a* and *b*-
00:55:46	Jeff:	*a* times *b* is-
00:55:47	Ankur:	*a* and *b* is red and yel-
00:55:47	Jeff:	-is the same thing as-
00:55:48	Ankur:	Red and blue-
00:55:49	Jeff:	*b* times *a*.

Time	Speaker	Transcript
00:55:49	Ankur:	Yeah.
00:55:50	Romina:	And if you actually work it out, you have-
00:55:50	Jeff:	And there's one of each.
00:55:51	Ankur:	And the same thing with red and blue and blue and red.
00:55:52	Romina:	You have one *a* squared.
00:55:53	Jeff:	and so it becomes two *a b*.
00:55:53	R5:	Ooh!
00:55:54	R1:	So say that again, Ankur.
00:55:56	Ankur:	Like red and blue are the same thing.
00:55:59	R1:	What do you mean by red and blue?
00:56:00	Jeff:	Like the colors are like this.
00:56:01	Ankur:	*a* is red and *b* is blue.
00:56:03	Romina:	*b* is blue.
00:56:04	Ankur:	That's a *b*. So *b a* would be a blue red.
00:56:09	R1:	So how if you have them in front of you, how would they look different?
00:56:12	Ankur:	Red and blue, red's on top.
00:56:13	Jeff:	Red's on top.
00:56:14	Ankur:	And blue's on the bottom.
00:56:14	R1:	Oh.
00:56:15	Jeff:	And then-
00:56:15	Ankur:	Blue's on top and-
00:56:16	Jeff:	Blue's on top.
00:56:16	Ankur:	-and red's on the bottom.
00:56:16	R1:	Oh. OK.
00:56:17	R5:	Wait a second. I'm confused. This is a part, you weren't here. Um. There were two lines of reasoning that were coming up in looking at this bunch of related problems. One was if you had a pizza where the person wanted pepperoni and mushrooms, it was the same as a pizza that had mushrooms and pepperoni.
00:56:33	Jeff:	And pepperoni.
00:56:34	R5:	OK?
00:56:34	Ankur:	Yeah.
00:56:35	R5:	On the other hand, if you had a tower that's say two blocks high, if it had red-
00:56:40	Jeff:	That's what we were saying.
00:56:40	R5:	-on the bottom and blue on the top, that was different-
00:56:42	Ankur:	Yes.
00:56:42	R5:	-from the blue on the bottom and the red on that top and that we were watching carefully which of these that we were calling the

Time	Speaker	Transcript
		same and which we were calling different.
00:56:48	Ankur:	The difference in the pizza problem is you could have five toppings-
00:56:49	R5:	Right.
00:56:50	Ankur:	-which would be a tower five tall, but-
00:56:52	Jeff:	But you could have-
00:56:53	Ankur:	If you have a tower five tall-
00:56:54	Jeff:	[Inaudible.] one tall.
00:56:55	Ankur:	-you can't have just a red.
00:56:56	Jeff:	It's like you're saying that if you have in the pizza-
00:56:59	Ankur:	Just having pepperoni.
00:57:00	Jeff:	[Inaudible.] You're saying that you could, you have five different colors, no restriction on height.
00:57:03	Ankur:	But there's no- yeah.
00:57:04	Jeff:	None at all. You could have one, just, as many combinations as you can make, period. But if you have five colors, five blocks in front of you, as many combinations as you can make with five blocks.
00:57:13	Ankur:	And the order of the topping doesn't matter.
00:57:15	Jeff:	Yeah.
00:57:16	Ankur:	So pepperoni and sausage-
00:57:17	Romina:	[Inaudible.]
00:57:18	Ankur:	-would be the same thing as sausage and pepperoni.
00:57:19	R5:	So in some of these cases, the order in which it comes in matters, like bottom to top on the tower.
00:57:25	Ankur:	Yeah.
00:57:26	Jeff:	So, like, so like the difference-
00:57:27	Romina:	At the end, it's all the same.
00:57:27	R5:	In other cases-
00:57:28	Jeff:	The difference- [Inaudible.]
00:57:29	R5:	In those other cases, like the- [Inaudible.]
00:57:29	Ankur:	But then the restrictions put on the pizza problem-
00:57:30	Romina:	[Inaudible.]
00:57:32	R5:	But then how come we keep getting the same numbers?
00:57:35	Ankur:	Because the numbers are all related.
00:57:37	Romina:	Yeah.
00:57:39	R5:	But why are they related? 'Cause they seem to be about games that have different looking rules, don't they?
00:57:43	Romina:	But, see like-
00:57:44	Ankur:	But the basic rule behind them all is the same.

Time	Speaker	Transcript
00:57:46	Jeff:	Yeah, the-
00:57:48	R5:	Could you explain that?
00:57:50	R1:	Yeah. That's interesting. What is that basic rule behind them all?
00:57:54	Ankur:	Didn't we-
00:57:54	R5:	What is the basic rule?
00:57:56	R1:	That was the question that we were all supposed to be thinking about. What was that basic rule behind them all? If- is it the same? And if it is, in what way? And if it's not, in what way is it different? That's sort of the question where we are today.
00:58:08	R5:	Yeah.
00:58:08	R1:	That's a good question.
00:58:10	R5:	Yeah. We're hitting the very same question.
00:58:11	R1:	A very good question. Mm. [Pause.]
00:58:18	Romina:	[Inaudible.]
00:58:21	R5:	See, here's a place where-
00:58:22	R1:	Well, maybe what we should do is we need a break.
00:58:24	R5:	OK.
00:58:26	R1:	So we'll go out and let you think about that for a minute. Do you understand the problem?
00:58:30	R5:	Yeah. I could use another drink.
00:58:31	Jeff:	Could you, could you-
00:58:32	Romina:	Just restate it one more time, please.
00:58:34	Jeff:	[Inaudible.] too.
00:58:35	R1:	OK. We can think of building towers five tall. And so you could explain to me how this represents the different categories of five tall towers, right? So if you could do that, then I would expect that you could do this problem: *a* plus *b* to the fifth, right? [Researcher 1 writes $(a+b)^5$.]
00:59:02	Jeff:	Yeah.
00:59:02	Ankur:	Yeah.
00:59:03	R1:	Right? I'd like you to fill in the missing parts here, the *a*'s and the *b*'s. I'd like you to do that. You understand what I'm saying?
00:59:13	Jeff:	Well, our idea was-
00:59:14	R1:	When I put the *a* and the *b* here-
00:59:15	Ankur:	It's like that-
00:59:15	R1:	I said one *a* squared. I'll help you. This is-
00:59:18	Ankur:	'Cause-
00:59:18	R1:	1 *a* to the fifth and this is 1 *b* to the fifth.
00:59:21	Ankur:	That was obvious.

Time	Speaker	Transcript
00:59:22	R1:	I'll be a good sport about that. I just thought I'd give you a hand. Just like I helped you here, right?
00:59:26	Ankur:	It'll be easy, because-
00:59:28	R1:	OK, I want you to do that, OK?
00:59:29	Ankur:	OK.
00:59:29	R1:	All right. And you can do that with each other. Uh. And I'm, and then I want you to think of what each of these mean in terms of towers, right? Now I want you to imagine pizzas, selecting from five toppings. Selecting from five toppings making that and see if there's any connection or relationship and if so, how? That's the question. So you have two, two parts. And we're gonna go get a drink. We'll leave you to work on it. Take some fresh paper. I have great confidence in all of you.
01:00:00	R5:	Me, too. [They leave.]
01:00:03	Ankur:	Do you know how to do it?
01:00:04	Jeff:	[Inaudible.]
01:00:05	Romina:	[Inaudible.]
01:00:07	Jeff:	Wait, if we're [Inaudible.], wait.
01:00:08	Ankur:	I just wanted to show-
01:00:09	Jeff:	What does, what does it look like?
01:00:10	Ankur:	I just was gonna show you what *a b* quantity to the three looks like.
01:00:14	Jeff:	Yeah. That's what I want to do right now.
01:00:15	Ankur:	It looks like-
01:00:16	Jeff:	Wait, that's-
01:00:17	Ankur:	It's just-
01:00:18	Jeff:	*a b* times-
01:00:19	Ankur:	[Inaudible.]
01:00:20	Jeff:	I'd really like to see it really quick.
01:00:22	Ankur:	I'll show you what it looks like.
01:00:24	Jeff:	Now you can go. Go ahead. Talk.
01:00:25	Ankur:	It looks like one- one some-
01:00:28	Jeff:	One *a* to the third.
01:00:29	Ankur:	I'm not- I'm just gonna tell you the numbers.
01:00:30	Jeff:	All right. One.
01:00:31	Ankur:	One three three one.
01:00:33	Jeff:	So that would be one *a* to the third, 3 *b*, one, 3 *a*-
01:00:40	Ankur:	No. Multiply.
01:00:44	Romina:	Here. You want to-
01:00:44	Jeff:	No. No. Wait.

Time	Speaker	Transcript
01:00:45	Romina:	You want to try it?
01:00:46	Jeff:	I messed up. Wait. It would be, it would be 1 *a* to the third.
01:00:48	Ankur:	Multiply this, Jeff. One-
01:00:50	Jeff:	That would be *x* squared plus
01:00:57	Ankur:	Multiply *x* squared plus 2- what was that thing? What's *a b* to the second?
01:01:04	Romina:	2 *a*. No.
01:01:05	Jeff:	No.
01:01:06	Romina:	*x* squared plus-
01:01:08	Ankur:	*x* plus 1 to the second, just multiply that.
01:01:11	Romina:	It's *x* squared plus 2*x* plus 1. You want me to multiply that by-
01:01:16	Ankur:	Multiply that by another *x* plus 1. [Pause.]
01:01:25	Romina:	[Romina is talking to herself.] *x* to the third plus 1, 1 *x* squared plus 2*x* squared plus 1.
01:01:46	Jeff:	So that, that would be-
01:01:47	Ankur:	Combine them.
01:01:48	Jeff:	-be *x* squared plus 3, there's 3, 2, oh, there's 3 *x* squareds.
01:01:55	Ankur:	1 3 3 1.
01:01:56	Romina:	Then 1 3.
01:01:57	Jeff:	Wait, wait, wait, wait.
01:01:59	Ankur:	The next one's gonna be 1.
01:02:02	Romina:	It's gonna be *x* to the fourth plus 4 *x*.
01:02:06	Ankur:	And the numbers are gonna look like this. [Ankur pulls a paper toward him.]
01:02:09	Romina:	4 *x* to the squared. Is that right? Plus 4*x* plus 1?
01:02:14	Ankur:	Well, you-
01:02:15	Jeff:	One one.
01:02:16	Ankur:	It'd look like 1 4 6 4 1.
01:02:20	Jeff:	Yeah.
01:02:21	Ankur:	1 4 6 4 1.
01:02:22	Romina:	I hated that. I, OK, so then this would be 6.
01:02:30	Ankur:	1 4 6 4 1. That's *a* to the what?
01:02:31	Jeff:	To the-
01:02:32	Ankur:	That's to the fourth?
01:02:33	Romina:	*a* to the fourth.
01:02:35	Ankur:	So the next number's like 1 5 10 10. [Ankur has been drawing Pascal's Triangle. Refer to Figure H9 for the final version showing rows 3 through 6; the annotation was added.]

Time **Speaker** **Transcript**

1 3 3 1 ← Row 3
1 4 6 4 1 ← Row 4
1 5 10 10 5 1 ← Row 5
1 6 15 20 15 6 1 ← Row 6
1 7 21 35 35 21 7 1 ← Row 7

Figure H9. Ankur's drawing of Pascal's Triangle (rows 3 through 7); row numbers added

01:02:37	Jeff:	10 10 5 1.
01:02:39	Ankur:	5 1. The answer to the-
01:02:40	Jeff:	To the problem.
01:02:41	Ankur:	Five tall two-
01:02:42	Romina:	I hated that stuff.
01:02:43	Jeff:	What did you hate?
01:02:45	Romina:	Like, I remember Mr. Poe doing that thing. [Inaudible.]
01:02:48	Jeff:	When do you do that?
01:02:49	Ankur:	We never did that.
01:02:49	Romina:	Yes, we did.
01:02:49	Ankur:	No, we didn't.
01:02:50	Jeff:	We never did that.
01:02:51	Ankur:	We never did that.
01:02:52	Romina:	Yes, he did. Remember that.
01:02:54	Ankur:	Never did that.
01:02:55	Romina:	We had these stupid pyramids.
01:02:55	Ankur:	We never did that.
01:02:55	Romina:	Yes, we did.
01:02:56	Jeff:	Oh! When we, I, I, remember, and he caught me cheating on that test. 'Cause there was like, it was that maze and-
01:03:03	Romina:	I didn't know.
01:03:04	Jeff:	How many different way you could go.
01:03:04	Romina:	Yeah. I didn't know how to do that.
01:03:06	Jeff:	And I had no clue and-
01:03:07	Ankur:	Was this in eight grade or sixth grade?
01:03:08	Romina:	I don't know.
01:03:09	Jeff:	I just remembered, do you remember how you take a triangle? [Jeff starts to draw.]
01:03:12	Romina:	And that's how-
01:03:13	Jeff:	And you break it up.
01:03:14	Romina:	How do you do the next one?
01:03:15	Jeff:	I never learned. Oh!

Time	Speaker	Transcript
01:03:16	Ankur:	You just add 'em.
01:03:17	Romina:	Yeah, Ankur.
01:03:17	Ankur:	[Ankur points to row 6.] 1 6 15 20 15 6 1.
01:03:23	Romina:	Now, how do you add 'em, you go, how do you do it?
01:03:25	Ankur:	And then add a 1 at the end.
01:03:26	Romina:	And how do you do it? That?
01:03:27	Jeff:	[Inaudible.]
01:03:28	Ankur:	You put a 1at the beginning. Then 1 plus 5.
01:03:29	Romina:	Yeah.
01:03:30	Ankur:	5 plus 10. 10 plus 10.// 5 plus 1.
01:03:32	Jeff:	//10. 5. Plus 1. 1.
01:03:33	Ankur:	And then 1 at the end.
01:03:35	Romina:	OK. So-
01:03:36	Jeff:	You don't- you sure you don't remember it? It was, yeah, all right. You have a triangle, right?
01:03:39	Ankur:	And then when you add all these up-
01:03:39	Romina:	Yeah. [Inaudible.]
01:03:40	Jeff:	And when you come out, you have two ways to go out of everything, like you had two ways to go, and then from there you'd have and it was j- and it made this big pyramid?
01:03:47	Ankur:	I honestly don't remember that at all.
01:03:48	Jeff:	I totally, I just, I never, that's what I-
01:03:50	Ankur:	Oh! I know what you're talking about: the maze!
01:03:51	Romina:	Yeah, didn't-
01:03:52	Jeff:	Yeah, it was the maze.
01:03:52	Romina:	Didn't we talk about that the last time we did this?
01:03:54	Ankur:	I don't think we talked about this, these numbers.
01:03:56	Romina:	Yeah. That's what she was doing up there. Oh, I guess we didn't say that.
01:03:59	Ankur:	I just remember the maze.
01:03:59	Jeff:	Yeah. I just, I remember-
01:04:01	Romina:	Remember? That's what-
01:04:01	Jeff:	I never knew how to do it.
01:04:02	Romina:	That's when she started doing it, I was like, "Oh, I don't know how to do this."
01:04:04	Ankur:	And if you add these up, it will be the answer to the-
01:04:07	Jeff:	[Inaudible.]
01:04:08	Ankur:	-six tall two colors.
01:04:09	Jeff:	So it would be 1 7 7, right? 1 7 7-
01:04:14	Ankur:	1 7 7. No. It's-

Time	**Speaker**	**Transcript**
01:04:16	Jeff:	No?
01:04:16	Ankur:	That's a 15. 1 7 21-
01:04:19	Jeff:	Wait. Wait.
01:04:20	Romina:	Oh.
01:04:20	Jeff:	One.
01:04:21	Romina:	Wouldn't it be 7?
01:04:22	Jeff:	7.
01:04:23	Ankur:	21.
01:04:23	Romina:	21. Yeah.
01:04:24	Jeff:	21.
01:04:25	Romina:	35.
01:04:26	Ankur:	35. 35. 21.
01:04:27	Jeff:	35. 21.
01:04:30	Ankur:	7 1. [He writes row 7 of Pascal's Triangle, as shown in Figure H9.]
01:04:32	Jeff:	7 1. Boy. All you have to do is get the-
01:04:36	Ankur:	And then when you add these up, like every single number here-
01:04:38	Romina:	That's how we [Inaudible.].
01:04:39	Ankur:	It will be the answer to the six tall-
01:04:40	Romina:	And this is all with one color.
01:04:42	Ankur:	-with two colors.
01:04:43	Romina:	That's like all red.
01:04:43	Ankur:	You understand that?
01:04:43	Romina:	Red. All the blue, all the-
01:04:44	Jeff:	Mm. Yeah.
01:04:45	Ankur:	And this is- Yeah.
01:04:46	Jeff:	Well, you're really-
01:04:47	Ankur:	Well, no. This is no red.
01:04:48	Romina:	What's-
01:04:49	Ankur:	One red. Two reds.
01:04:49	Romina:	What's the question-
01:04:50	Ankur:	Three reds.
01:04:51	Romina:	-we're supposed to be answering?
01:04:51	Ankur:	Four red, five red, six red.
01:04:52	Jeff:	I don't know, but we should make sure we know what's up.
01:04:56	Romina:	Yeah. What, what was the question we were supposed to answer?
01:04:57	Ankur:	How does it relate to the pizza problem now?
01:04:58	Jeff:	Wait. You don't have to tell them to come back in now.
01:05:00	R6:	Oh. OK.

Time	Speaker	Transcript
01:05:01	Jeff:	Wait one more minute.
01:05:04	R6:	OK.
01:05:04	Jeff:	Um.
01:05:05	Ankur:	Because it's-
01:05:05	Jeff:	All right. Say again-
01:05:08	Ankur:	The answer would-
01:05:09	Jeff:	-what you just said about how this equals one red. Say that again. What you were-
01:05:12	Ankur:	This is no red.
01:05:13	Jeff:	Yeah.
01:05:14	Ankur:	So there's 1 with no red. There's 6 with one red.
01:05:17	Jeff:	Uh-huh.
01:05:17	Ankur:	There's-
01:05:19	Jeff:	15.
01:05:20	Ankur:	-15 with two reds.
01:05:20	Romina:	Two reds.
01:05:21	Jeff:	Two reds.
01:05:21	Jeff:	20 with three reds. 15 with
01:05:23	Ankur:	20 with three reds. 15 with four.
01:05:24	Romina:	Four.
01:05:24	Jeff:	Four reds. 6 with five reds.
01:05:24	Ankur:	6 with five reds.
01:05:24	Jeff:	And 1 with no-
01:05:25	Ankur:	And 1 with no-
01:05:27	Romina:	No.
01:05:27	Ankur:	No. Six reds.
01:05:28	Jeff:	1 with six reds.
01:05:29	Ankur:	Six reds.
01:05:30	Jeff:	All right. We're saying. Yeah. All right. Now. How, what does that have to do with pizza?
01:05:35	Ankur:	Pizza problem.
01:05:36	Romina:	Pizza. That-
01:05:37	Jeff:	This is-
01:05:38	Ankur:	Just relate the tower problem to the pizza problem.
01:05:39	Jeff:	Well, we're saying that this is-
01:05:40	Romina:	One.
01:05:41	Jeff:	-a pizza with just plain.
01:05:42	Romina:	Yeah. That'll be the plain pizza.
01:05:44	Jeff:	Plain. This is with all your six toppings.
01:05:46	Romina:	That's with one topping.

Time	Speaker	Transcript
01:05:47	Jeff:	This is-
01:05:48	Ankur:	No, you can't-
01:05:48	Jeff:	Yeah. Yeah, that's, this is-
01:05:49	Ankur:	You can't exactly relate these numbers to the pizza problem.
01:05:51	Jeff:	Well, we'll try really quick.
01:05:52	Romina:	Yeah. You can. 'Cause-
01:05:52	Jeff:	'Cause you're saying that this-
01:05:53	Romina:	-this is, has-
01:05:53	Jeff:	-is plain.
01:05:53	Romina:	This plain, just plain pizza.
01:05:54	Ankur:	And what will the other one represent?
01:05:56	Jeff:	But-
01:05:56	Romina:	With everything on it.
01:05:57	Jeff:	Everything.
01:05:57	Ankur:	OK.
01:05:58	Jeff:	So this is plain.
01:05:59	Ankur:	OK. 6 with-
01:06:01	Jeff:	With one of each. 15 is with-
01:06:03	Romina:	Two.
01:06:04	Jeff:	Two. All right. Like, say, uh, two.
01:06:07	Romina:	Two toppings.
01:06:07	Jeff:	Just two toppings out of your six. 20 is with three toppings. 15 is with the four toppings. 6 is with the five toppings.
01:06:14	Romina:	Five toppings.
01:06:14	Jeff:	And the other 1 is-
01:06:15	Romina:	And the 1 is with all of them.
01:06:16	Jeff:	Like the supreme.
01:06:19	Romina:	Is that good?
01:06:20	Ankur:	Cool.
01:06:22	Jeff:	We're ready.
01:06:23	R6:	You ready?
01:06:24	Jeff:	Yes.
01:06:24	R6:	They're ready.
01:06:25	Ankur:	We're on fire today.
01:06:27	Jeff:	It's 'cause Brian and Mike aren't here.
01:06:30	R1:	Oh?
01:06:32	Ankur:	No Mike.
01:06:33	Romina:	[Inaudible.] Brian-
01:06:35	R1:	Are you ready to talk to us?
01:06:36	R5:	Are you thirsty? Would you like anything?

Time	Speaker	Transcript
01:06:37	R1:	Anybody want a cool drink?
01:06:38	Jeff:	Can I get a Coke, please?
01:06:39	Ankur:	[Inaudible.] please.
01:06:40	R1:	I know, I know you have to leave, but can you stay two more minutes?
01:06:42	Jeff:	[Partially inaudible. Much background noise.] Yeah, I want to stay just to see-
01:06:46	R1:	You have something to tell us?
01:06:47	Ankur:	You can just look at the videotape. It will tell you everything.
01:06:49	R1:	[Partially inaudible. Beverages are distributed and opened.] We absolutely positively want you to tell us.
01:06:57	R5:	Classic OK? [He hands a Coke to Ankur.]
01:06:58	Ankur:	Thank you.
01:06:59	R5:	All right.
01:07:00	R1:	So now use the board?
01:07:01	Romina:	No.
01:07:02	Jeff:	It would be easier-
01:07:03	Romina:	OK.
01:07:04	Ankur:	It's in front of you.
01:07:05	Romina:	Well, we first discovered, we, we-
01:07:06	Ankur:	We related, we found out-
01:07:07	Romina:	Yeah.
01:07:08	Ankur:	This is the answer to-
01:07:09	R1:	Wait, here, can you-
01:07:10	Jeff:	All right.
01:07:11	Ankur:	Just write it over.
01:07:13	Romina:	I guess I'm, I'm secretary. OK. The-
01:07:17	Ankur:	x. x. It's-
01:07:21	Romina:	This one.
01:07:21	Ankur:	a b quantity squared to the fifth.
01:07:23	Romina:	Yeah. [Romina writes.] a plus b.
01:07:26	Ankur:	a plus b.
01:07:27	Romina:	OK. Would be this.
01:07:29	Ankur:	One.
01:07:31	Romina:	Six.
01:07:32	Ankur:	Those are the numbers in front of-
01:07:35	Jeff:	Yeah.
01:07:35	Ankur:	What everything else is gonna be. [Pause.]
01:07:40	Jeff:	Now you know where those numbers come from, right?
01:07:42	R1:	What number is this up here?

Time	Speaker	Transcript
01:07:43	Romina:	Five.
01:07:43	Jeff:	Five.
01:07:44	Ankur:	Five.
01:07:44	Romina:	I wrote squared. Sorry.
01:07:45	R1:	You want this to be to the fifth?
01:07:47	Jeff:	Yeah. To the fifth. *a b* times *a* times *a b*, so, so, do you, do you understand what it is?
01:07:52	Ankur:	*a* plus *b* times-
01:07:53	Jeff:	*a* plus *b* times *a* plus *b*.
01:07:54	Romina:	And this is, this is how many different toppings there'd be.
01:07:56	Jeff:	Wait. Wait.
01:07:57	R1:	What would it be to the fourth?
01:07:58	Jeff:	Wait. Wait. Wait. Stop. Stop. Stop. It would be-
01:07:59	Ankur:	To the fourth.
01:07:59	Romina:	Fourth. It would be- [There is a discussion about whether they mean $a + b$ to the fifth or to the sixth power.]
01:10:04	R1:	So that's *a* plus *b* to what power?
01:10:05	Jeff:	To the sixth.
01:10:06	R1:	OK, I thought you said that was to the fifth.
01:10:06	Ankur:	No, that's uh-
01:10:11	R1:	One of you said it was the fifth.
01:10:12	Romina:	Oh, you guys, remember, where's, where's the other sheet? That's where we, that's when I was trying to figure out how to do it, that's when we went on to this thing.
01:10:18	R1:	Oh, so it's not the fifth. It's the sixth.
01:10:19	Ankur:	It's the next one.
01:10:20	Jeff:	It's-
01:10:22	R1:	OK. Did you understand why I asked you those other questions?
01:10:26	Ankur:	It was to the-
01:10:26	R1:	Well, because you told me this was the fifth, and I was confused.
01:10:27	Jeff:	It's the sixth.
01:10:29	Romina:	OK.
01:10:32	R1:	So, go ahead, I'm sorry.
01:10:34	Ankur:	These numbers relate to the power problem because-
01:10:36	Romina:	If you add them all up, that's how many powers.
01:10:38	Ankur:	Like, *a* plus *b* to the sixth is the same thing as having towers six call with two colors.
01:10:45	Jeff:	You [Inaudible.] to us. You understand because you told us.
01:10:47	Ankur:	No reds, one red, two reds, three reds, four reds, five reds, six reds. [Ankur points to the entries in Pascal's Triangle as he

Time	Speaker	Transcript
		speaks.]
01:10:52	Jeff:	And then, the pizza problem, this is with-
01:10:57	Romina:	Nothing, just a plain cheese one.
01:11:00	Jeff:	This is the plain one. And this also understands why we never needed the plus one.
01:11:01	Ankur:	Plus one.
01:11:02	Jeff:	We were never sure-
01:11:03	Ankur:	This is the plain, this is the one with all the toppings.
01:11:05	Jeff:	This is like
01:11:06	Ankur:	The only one with all the toppings.
01:11:07	Romina:	The zero toppings, that's zero red.
01:11:09	Jeff:	Yeah. And this is with-
01:11:11	Ankur:	There's five toppings, one of each.
01:11:12	Jeff:	This is one of each, this is two of each-
01:11:15	R5:	Uh-huh.
01:11:16	Jeff:	Three of each.
01:11:17	Romina:	Four.
01:11:18	Jeff:	Four of each. And this is all five.
01:11:20	Romina:	And this would be one topping, two toppings, three toppings, four toppings, and all toppings.
01:11:22	R1:	How does it become two to the fifth?
01:11:25	Jeff:	Because it's-
01:11:28	Ankur:	That's how it breaks down. Because when you add all these up-
01:11:31	Jeff:	That's what it is.
01:11:32	Ankur:	They give you the total amount of toppings.
01:11:34	Jeff:	That will give you 32, which is-
01:11:38	Ankur:	-five toppings and a lot of pizza.
01:11:39	R1:	I think I follow that. Do you follow that?
01:11:41	R5:	I think I do.
01:11:42	R1:	Yes.
01:11:45	R5:	So, I have another question. How did you get that last line that goes 1 6 15?
01:11:51	Romina:	One plus five.
01:11:52	R5:	Oh, you
01:11:54	Ankur:	You start with the one because you take-
01:11:56	R5:	I believe the one.
01:11:57	Ankur:	All right.
01:11:59	Jeff:	Then you just add these two. When you were first explaining this, I had no clue what was going on.
01:12:01	Romina:	Yeah.

Time	Speaker	Transcript
01:12:02	Ankur:	Did you know how I was doing it?
01:12:03	Jeff:	And-
01:12:04	Romina:	Yeah.
01:12:05	Jeff:	As long as it took me just to- you were just spitting out numbers like telling me how to get *a* plus *b* to the third. I didn't understand it. But then once you just sit back for a second. You-
01:12:15	Ankur:	You see how it-
01:12:16	Jeff:	How it all fits in. And I wish I had know this. I would have gotten a good grade on that test.
01:12:22	R5:	I've got a question. See, I'm looking at that six and I'm thinking from what you said before. That's like the number of towers six blocks high with one red. 'Cause there are six places where you can put the red. These are kinds of things you told me before. Now how does that six come up with the one plus the five? Where you're talking about shorter towers.
01:12:41	R1:	Five tall towers.
01:12:42	R5:	Yeah.
01:12:43	R1:	Right? These are the five tall towers with-
01:12:47	Jeff:	I really thought you were going to explain it to him. And I was like-
01:12:49	Ankur:	I know. That's how I thought.
01:12:51	R1:	These are the five-tall towers with how many reds? This one represents?
01:12:54	Ankur:	Zero.
01:12:54	Romina:	Zero.
01:12:55	R1:	Zero reds. And these are the five-tall towers with-
01:12:56	Ankur:	One red.
01:12:57	R1:	One red.
01:12:58	Ankur:	In each place.
01:12:59	R1:	And this represents a six-tall tower with-
01:13:00	Ankur:	That's-
01:13:02	R1:	[Inaudible.]
01:13:04	Ankur:	You can put one. Right now there's-
01:13:05	Romina:	Extras.
01:13:06	Ankur:	One at the bottom, one at the second from the bottom, one at the third, one at the fourth, and one at the fifth from the bottom. And this is just all, there's no red in it.
01:13:13	R1:	Uh-huh.
01:13:14	Ankur:	Now if you put a red at the top, then there's going to be six.

Time	Speaker	Transcript
01:13:19	R1:	Uh-huh.
01:13:20	Ankur:	And there's still going to be one red of each.
01:13:20	R1:	Uh-huh.
01:13:22	Ankur:	That's how 5 plus 1 is the six.
01:13:25	R5:	I'm not sure I follow. The one counts towers that are five blocks tall that have no reds, right?
01:13:30	Ankur:	Yes.
01:13:32	R5:	And the five counts the towers that are five blocks tall that have one red.
01:13:37	Ankur:	Yes.
01:13:38	R5:	OK. So I'm imagining all those towers. And I think I can sort of see them.
01:13:42	Ankur:	If you can picture it, there's one up here.
01:13:43	R5:	I can picture it.
01:13:44	Ankur:	One down and then on the diagonal.
01:13:45	R5:	Right.
01:13:46	Ankur:	And then when you add-
01:13:47	R5:	OK, how do you get the six possible-
01:13:48	Ankur:	You put-
01:13:49	R5:	-Below?
01:13:49	Ankur:	You put a block on top of each one of the towers. [Researcher 5 nods.] And-
01:13:53	R5:	What color is that block?
01:13:55	Ankur:	It's gonna be not red.
01:13:57	R1:	Not red for both of them?
01:13:58	Ankur:	Right now, there's five with one red each. [Researcher 5 nods.] And the first one has no reds.
01:14:03	R5:	Right.
01:14:04	R1:	And this represents- tell me what these towers look like again.
01:14:06	Ankur:	OK, they're gonna be.
01:14:08	Jeff:	Six with one red each.
01:14:09	Ankur:	Six with one red each.
01:14:10	R1:	OK, so they have to end up with one red each.
01:14:12	Jeff:	Yeah.
01:14:14	Ankur:	A block on top of each. The first one- you can't put any blocks on top of this, because it's only one red.
01:14:19	R5:	Oh, I see, you're trying to preserve the one red.
01:14:20	Jeff:	Yeah.
01:14:22	Ankur:	And this one has no red. So if you put one red at the top-
01:14:24	R5:	Oh, I see.

Time	Speaker	Transcript
01:14:25	Ankur:	It won't look like any of these towers.
01:14:27	R5:	Oh, I see, so you have two groups of towers and you treat them slightly differently, so that-
01:14:29	Ankur:	Yes.
01:14:30	R5:	I see! OK.
01:14:33	R1:	Do you know what they're talking about, Romina?
01:14:34	Romina:	Oh, yeah.
01:14:34	R1:	Tell me.
01:14:35	Romina:	They're talking about how we have, the difference here is, we have, these are five high so when we're putting on one more, you're like-
01:14:45	R1:	Does that always work? Can you explain that for, let's say, the twenty? These ten plus ten being twenty? Can you explain that to me with towers?
01:14:52	Jeff:	You're saying that you have the three reds, right? Yeah. [Pause.]
01:15:01	Romina:	[Romina writes.] You're adding another one, so you're gonna, you're just kind of doubling it?
01:15:04	R5:	What are those tens counting?
01:15:06	Ankur:	The tens look like-
01:15:07	R5:	And what do the, what does the twenty count?
01:15:09	Jeff:	The tens show-
01:15:10	Ankur:	The tens show two of one color.
01:15:12	Romina:	And three of another.
01:15:12	Ankur:	And three of another color.
01:15:13	Jeff:	Another color, and the other one's three of-
01:15:14	Ankur:	One color and two of another color.
01:15:16	Jeff:	Two of another color.
01:15:16	Romina:	Two of another color.
01:15:17	Ankur:	So they're the same.
01:15:17	Jeff:	But opposites like, kind of like that.
01:15:18	Ankur:	You understand that?
01:15:19	R1:	Uh-huh.
01:15:20	Jeff:	And that's why there're, that's why they're, they're the same number.
01:15:22	Ankur:	That's why it's 10 and 10. But they you, at the top of each one, you can put either-
01:15:27	Jeff:	A, yeah. You could either put a red or like-
01:15:28	Ankur:	Red.
01:15:29	Jeff:	-a blue.

Time	Speaker	Transcript
01:15:30	Ankur:	Or blue.
01:15:31	Jeff:	Showing then why you add them together giving you the 20 to go there.
01:15:34	Ankur:	Like these will have four reds and these already have, each one has three reds.
01:15:42	Jeff:	And they're just adding the fourth one on top.
01:15:43	Ankur:	And if you just put a red or a non-red on top.
01:15:46	R1:	These will have four reds?
01:15:47	Romina:	These will have three and that will be kind of splitting them in half and-
01:15:49	Ankur:	One, two, three, four.
01:15:50	R5:	Three reds and three blues?
01:15:50	Romina:	The other color it has to be.
01:15:51	Ankur:	These will have four reds, because- no, these will have three reds.
01:15:54	Jeff:	Yeah. Yeah. And then the fourth one makes it twenty 'cause there are in three- Yeah, three in like-
01:16:02	R5:	Wait a minute. The first 10 in that row of five high has two reds and three blues? We're counting reds?
01:16:10	Ankur:	Yes.
01:16:12	R5:	And the second 10 has-
01:16:13	Ankur:	Three reds and-
01:16:15	R5:	-Three reds and two blues. Now coming down here, the 20 is supposed to count the ones that have three of each.
01:16:20	Ankur:	Three red. Three reds and three blues.
01:16:23	Jeff:	Right.
01:16:23	R5:	So how do the two 10's add to give the 20?
01:16:26	Ankur:	Because in these 10, where there's three reds and two blues-
01:16:29	R5:	Yeah.
01:16:30	Ankur:	You want to make it three reds and three blues.
01:16:31	R5:	Uh-huh.
01:16:31	Ankur:	So you put a blue on top of each one.
01:16:32	R5:	Ah!
01:16:32	Ankur:	And these, when there's, when there's two- [Pause.]
01:16:38	Romina:	Red.
01:16:38	Ankur:	-Blues and three reds, I think. You put, no these are three red and two blues, and you put a red on top, no, a blue on top of each one.
01:16:47	R1:	To preserve the three reds.
01:16:48	Ankur:	Yeah.

Time	Speaker	Transcript
01:16:49	R5:	OK.
01:16:50	R1:	Does that make sense?
01:16:51	Romina:	Yeah.
01:16:51	R5:	Is that what you're seeing too?
01:16:53	Jeff:	Yes.
01:16:54	R5:	Yeah.
01:16:55	R1:	Wow! How far can you go with block towers?
01:16:57	Ankur:	I think pretty far. I think we reached the limit.
01:17:00	R1:	You think we reached the limit.
01:17:01	R5:	You think you reached the limit for today.
01:17:02	Ankur:	Yes.

March 6, 1998

Time	Speaker	Transcript
00:00:00	--	Michael asks what they did the last time (February 6, 1998), when he was absent. Jeff draws Pascal's Triangle and Ankur talks about the binomial expansion.
00:05:05	R1:	You all remember that. Now I don't remember if Mike was here for some other conversation we had... You were here, Mike, when this particular triangle ... There's a special name for the triangle. ...
00:05:26	Ankur:	Pascal's Triangle.
00:05:27	R1:	Pascal's Triangle. Very good, Ankur. Pascal's. ... It usually has a little 1 up here, also. [She draws a 1 at the top of the triangle, which Jeff has drawn starting with 1 1. Refer to Figure H10.] Now you understand about this? About what *a*+*b* has to do with this?

```
        1        ← Row 0
      1   1      ← Row 1
    1   2   1    ← Row 2
  1   3   3   1  ← Row 3
1   4   6   4   1 ← Row 4
```

Figure H10. Jeff's drawing of Pascal's Triangle (annotation added)

Time	Speaker	Transcript
00:05:49	Jeff:	Yeah.
00:05:50	R1:	You were here? Can you help me, Michael, then? Just to be sure. What does the *a*+*b*, what do that- what do these have to do

Time	Speaker	Transcript
		with these?
00:05:59	Michael:	Um, for *ab*, *a+b* squared, it would be 1 *a* squared plus 2*ab*, and you would take that. What are these called? Coefficients or something like that?
00:06:09	R1:	Uh-huh.
00:06:09	Jeff:	Yeah.
00:06:10	Michael:	That would be the number that goes there.
00:06:13	R1:	So the coefficients-
00:06:14	Michael:	Yeah, that's what it-
00:06:16	R1:	OK. So here we have- you know what this is called? *a+b*, do you know what that, that's called, what the technical- It doesn't really matter it's called the binomial because there are two, bi. And so you expanded your binomial. OK. And you have *a+b* to that quantity cubed, right? When you expand that, which is *a+b* times //*a+b* times *a+b*.
00:06:40	Jeff:	//a+b times a+b.
00:06:40	R1:	You do all that multiplication. These numbers are the coefficients. OK. Now what about the *a*'s and the *b*'s?
00:06:51	Jeff:	What about them?
00:06:51	R1:	Did you think about those, what- Have you thought about the *a*'s and the *b*'s at all? What that has to do with anything?
00:07:01	Jeff:	I mean, they're just the variables that come to the problem. ... That we should be working on.
00:07:05	R1:	OK. So you told me that this [the first 1 in row 2] was 1.
00:07:12	Jeff:	Yeah. *a* squared.
00:07:13	R1:	*a* squared. And this [the 2 in row 2] was-
00:07:15	Jeff:	Two *ab*.
00:07:16	R1:	Two *ab*, which you got from *ab*-
00:07:18	Jeff:	Time-
00:07:18	R1:	-and *ba*.
00:07:19	Jeff:	Yeah.
00:07:20	R1:	That's, that's what Jeff was showing, the origin of that 2 of them, right?
00:07:23	Jeff:	Uh-huh.
00:07:23	R1:	And this [last 1 in row 2] is?
00:07:25	Jeff:	*b* squared.
00:07:26	Romina:	*b* squared.
00:07:26	R1:	*b* squared. So this [first 1 in row 3] is?
00:07:28	Romina:	*a*-
00:07:28	Ankur:	*a* cubed.

Time	Speaker	Transcript
00:07:28	Romina:	-*a* cubed. 3 [as Researcher 1 points to the 3]. 3 *a* squared maybe?
00:07:35	Michael:	I like *a* squared *b* or something.
00:07:36	Ankur:	3.
00:07:37	Romina:	Yeah. Like // *a* squared *b*.
00:07:38	Ankur:	*a* squared *b*.
00:07:38	Michael:	And then that's like *ab* squared.
00:07:40	Ankur:	Three *b* squared *a*.
00:07:40	Jeff:	That would be the same thing.
00:07:41	Romina:	Yeah. It would start-
00:07:42	Jeff:	No. Yeah, it would be - ...
00:07:44	R1:	OK, so what would it be in this row, can you tell me? [Researcher 1 asks them to consider why the addition rule works. The students discuss this among themselves for about 18 minutes.]
00:25:29	Michael:	You want to know why that [the 1 and 2 from row 2 of Pascal's Triangle] turns to like a 3, right?
00:25:32	Jeff:	Yeah. We're trying to figure out why these two- [There is the sound of shuffling papers.]
00:25:38	Michael:	I was just looking at something. Pretend the first thing was like *a* squared, two. [Michael writes $a^2 + 2ab + b^2$.] What you're really doing is like putting a parenthesis. [Michael puts parentheses around what he just wrote and adds $a+b$. It now says $(a+b)\ (a^2 + 2ab + b^2)$.] You know, you're putting, just going-
00:25:47	Jeff:	Yeah. And multiplying that-
00:25:48	Michael:	Yeah. And the reason-
00:25:49	Jeff:	-times that.
00:25:49	Michael:	-why you get three is 'cause you're doing this by *a*. Right? [Michael circles the 2*ab* in Figure H10.] You're doing two *a* squared plus *b*. [Michael writes $2a^2\ b$.]
00:25:56	Ankur:	Uh-huh.
00:25:57	Michael:	And then you, you, that *b*, it's timesed right there, too. You see?
00:25:59	Jeff:	Uh-huh.
00:26:00	Ankur:	Yeah, you add a *b* to that.
00:26:01	Romina:	Yeah. That's-
00:26:01	Michael:	And that's how you get 3, you know? That's where the third one comes from.
00:26:05	Jeff:	Does it look good?
00:26:06	Romina:	[Inaudible.] -do it. Yeah. 'Cause we just did like the final.
00:26:08	Michael:	Like, you were trying to figure out why it turns to 3?

Time	Speaker	Transcript
00:26:10	Jeff:	We were trying to figure out why you add, if, how you add these two [the 1 and the 2], why this works? Why adding 2 and 1-
00:26:15	Ankur:	2-
00:26:16	Jeff:	-gets 3.
00:26:17	Ankur:	-will give you 3.
00:26:18	Jeff:	And why adding 3and 3 will give you the 6. And the 6 and 4 will give you the 10.
00:26:26	Ankur:	[Inaudible.]
00:26:27	Michael:	What?
00:26:28	Jeff:	All right. I'm just thinking 'cause that was- [Pause.]
00:26:31	Michael:	I don't know.
00:26:34	R1:	But if I can clarify what I think I hear Mike said, um, why, let's say, 3 and 1 gives you the 4. [She points to their diagram of Pascal's Triangle.] This particular 1 [the first 1 in row 3] means one of something in this particular row. Isn't that right?
00:26:50	Jeff:	Yeah.
00:26:50	Ankur:	That one means-
00:26:50	Romina:	Uh-huh.
00:26:50	Jeff:	That would, that one adds-
00:26:52	R1:	That's *a*.
00:26:52	Jeff:	-to this *a* squared that comes into the thing.
00:26:54	Romina:	Yeah. And see-
00:26:54	Ankur:	That's-
00:26:54	Jeff:	That could turn into this.
00:26:56	Romina:	This one turns into this.
00:26:57	Jeff:	Yeah.
00:26:58	Romina:	Like this, this one [the first 1 in row 3] turns into this [the first 4 in row 4] because you're a-
00:27:02	Jeff:	It equals-
00:27:03	Romina:	Because you're adding, like it, it goes, it goes up 1. So this is the one that's by itself. Thank you.
00:27:05	R1:	So, so this [first 1 in row 2] 1 *a* cubed.
00:27:10	Ankur:	Plus-
00:27:11	R1:	and this 3 *a* squared [the first 3 in row 3].
00:27:12	Ankur:	*a* squared *b*. Uh-huh.
00:27:13	R1:	Right? When they go to the 4, right? There were 4 of what?
00:27:17	Ankur:	There's 4-
00:27:18	Romina:	4 *a* squared.
00:27:20	Ankur:	*a*-
00:27:20	Romina:	No.

Time	Speaker	Transcript
00:27:21	Ankur:	-to the third.
00:27:21	Romina:	*a* to the third.
00:27:21	Ankur:	*b* to the f- No. 4 *a* cubed *b*.
00:27:24	Romina:	Yeah.
00:27:27	R1:	OK. Do you all agree with that?
00:27:28	Romina:	Yeah.
00:27:28	Ankur:	Yeah. That's, that's because this is *a*- this is *a a*, this is *a* cubed. [Ankur points to the first 1 in row 3.]
00:27:37	R1:	Uh-huh.
00:27:38	Ankur:	And this is *a a* to the s- *a* squared b. [Ankur points to the first 3 in row 3.]
00:27:43	R1:	Uh-huh.
00:27:44	Ankur:	And when you combine them, you get 4 *a* cubed *b* to the- 4 *a* cubed *b*.
00:27:52	R1:	Why does that work? [Pause.]
00:27:56	Ankur:	'Cause you're adding them.
00:27:59	R1:	Are you? [There is a discussion about adding vs. multiplying.]
00:28:32	Jeff:	Just say it one more time.
00:28:33	R1:	Thank you, Jeff.
00:28:35	Romina:	OK. It's-
00:28:36	R1:	Here.
00:28:37	Ankur:	Write- write- write what this means [the first 1 in row 3], what this means [the first 3 in row 3], and what that means [the first 4 in row 4].
00:28:41	Michael:	Stop using that pen 'cause it's making dots on everything.
00:28:42	Romina:	Yeah. Anything else, Ankur?
00:28:43	Ankur:	You got something against dots, Mike?
00:28:44	Jeff:	Yeah. And that's, and that's why they all equal that's why they're saying-
00:28:47	Romina:	And then you-
00:28:48	Jeff:	They come together to get 4.
00:28:49	Romina:	Yeah. That's why you like kind of you multiply that.
00:28:51	Jeff:	I could've said that.
00:28:51	Romina:	Yeah. See?
00:28:52	Jeff:	Yeah.
00:28:53	Romina:	They know exactly what we were saying.
00:28:54	R1:	OK.
00:28:55	Jeff:	I'm really hot.
00:28:56	R1:	It is warm in here. Um. OK. Can you, can you say it, summarize it for us? Romina, please?

Time	Speaker	Transcript
00:29:04	Ankur:	Say what you said to him, to Mike.
00:29:05	R1:	Yes.
00:29:07	Jeff:	[Inaudible.] All right. If you write out what 1, what 1 means here, it would be *a* squared. Right? [Jeff writes a^2.] Write out what 3 means, which would be-
00:29:12	Michael:	*a*.
00:29:13	Jeff:	3.
00:29:14	Ankur:	No, wait, Jeff. this is *a* to the, *a* cubed.
00:29:15	Romina:	*a* cubed.
00:29:16	Jeff:	*a* cubed. Oh, well.
00:29:17	Michael:	3 *a* squared *b*. [Inaudible.]
00:29:18	Ankur:	It's the same thing, but-
00:29:18	Jeff:	*a* cubed. 3 *a* squared *b*.
00:29:20	Romina:	Uh-huh.
00:29:21	Jeff:	Right? If you add the, add.
00:29:23	Romina:	Multiply.
00:29:24	Ankur:	Multiply.
00:29:25	Romina:	Another *a*, like your *a* plus *b*.
00:29:27	Michael:	Into both of them, right?
00:29:29	Ankur:	Yeah.
00:29:30	Jeff:	Yeah, into that, right? You're multiplying that into that?
00:29:33	Romina:	Yeah.
00:29:34	Jeff:	You're multiplying $a + b$. then that would equal this, that right there. [Inaudible.]. It would be $a + b$, right? [Jeff writes $a^3 \cdot 3a^2b \cdot (a+b)$.]
00:29:49	Romina:	Uh-huh.
00:29:49	Jeff:	And that would equal the same as what this 4 would, what the 4 here, it would be 4, it would be-
00:29:50	Michael:	Yeah. I know what it would be.
00:29:50	Jeff:	Yeah. That's what we were trying to say. [Inaudible.]
00:29:52	R1:	Can you go through it for me, please?
00:29:54	Jeff:	Um. What was it, 4? What is this, to the fourth? So it would be 4 *a* to the -
00:30:02	Ankur:	*a* cubed.
00:30:05	Jeff:	*a* cubed *b* to the first-
00:30:06	Ankur:	*a* cubed *b*.
00:30:07	Jeff:	*a* cubed *b*? [Jeff writes $4a^3b$.]
00:30:08	Romina:	[Inaudible.] Where's the paper underneath? Here.
00:30:17	R1:	OK. So. Um. Let's go back to the original triangle for a moment. Um. So if you had to explain this 10 plus 5 equals 15

Time	Speaker	Transcript
		or this 10 plus 10 equals 20.
00:30:35	Ankur:	You'd just do it the same way.
00:30:36	Jeff:	Yeah.
00:30:36	Romina:	Yeah.
00:30:37	R1:	What's the same way?
00:30:37	Jeff:	You could just say that's-
00:30:38	Ankur:	Write out what the 10 means.
00:30:39	Jeff:	The quantity of this [Jeff points to the 10.] times the quantity, well, the quantity of these. [Jeff indicates both 10s.]
00:30:43	Ankur:	Times the *a* plus *b*.
00:30:45	Jeff:	Times the *a* plus *b* would give you-
00:30:45	Ankur:	Would give you the-
00:30:47	Jeff:	-quantity of this.
00:30:48	R1:	Uh-huh. OK. I, I follow that.
00:30:52	Jeff:	That's cool.
00:30:53	R1:	You like that?
00:30:54	Romina:	It's cool.
00:30:55	R1:	OK. So then theoretically, um, you should be able to predict, um, what those terms are and where they came from, right?
00:31:07	Ankur:	[Inaudible.]
00:31:08	R1:	You should be able to go backwards even and tell me where they came from. In any, in any of these terms on the triangle. Is that right? OK. Now the particular terms you chose to, to give as an example here, the *a* cubed and the 3 *a* squared *b*, right? Um. Where you have one of the *a* cubed terms and 3 of the *a* squared *b* and you talked about where the 3 of the *a* squared *b* came from. I heard you saying that, Romina, earlier. Um. [Pause.] The numbers in front of those terms then have a particular meaning with respect to those terms. Can, can you all see where, where that meaning comes from? [Pause.] I mean, have you thought about like where it all comes from. You went through the $a + b$ when you square it, right? The quantity $a + b$ squared. And right in the very beginning, Jeff was saying *a* squared plus *ab* plus *ba* plus *b* squared. And then when you combine the *ab* and *ba*, he said that's a 2 *ab*, right? And so on and so forth. So, so you understand that the, where that particular term comes from each time, right? It's a tedious exercise, isn't it? But, but you can figure it out.
00:32:26	Jeff:	Yeah. Yeah.
00:32:27	R1:	OK. Now what I do remember from last time, um, I think it was

Time	Speaker	Transcript
		Ankur, but I could be wrong. We could always go back and look at the tape and to see who introduced this. Um. You were making a relationship about these *a*'s and *b*'s to building towers. Can, can you-
00:32:49	Jeff:	That was that-
00:32:49	R1:	-remember that, Ankur?
00:32:50	Ankur:	That's- yeah.
00:32:50	R1:	Can you, can you all for a minute- I don't think Michael heard that conversation.
00:32:54	Jeff:	That was last week?
00:32:55	R1:	Well.
00:32:56	Jeff:	I thought it was two weeks ago.
00:32:57	Ankur:	It was, it was when that guy was here.
00:32:58	Romina:	Yeah. It was last week. Well, the last time.
00:32:59	R1:	So, so, Michael-
00:33:00	Ankur:	Last time.
00:33:00	R1:	-wasn't here for that. Do you understand the question, Michael?
00:33:01	Michael:	No, I [Inaudible.] -
00:33:02	Jeff:	Like, how do they-
00:33:03	Ankur:	How do these numbers relate to the towers?
00:33:05	Michael:	Yeah.
00:33:06	Romina:	Yeah.
00:33:07	Ankur:	Two colors. And-
00:33:08	R1:	So, so, in other words, if I asked you for example to show me how in the towers, uh, 3 plus 3 equals 6 and how that works, can you show me that?
00:33:22	Jeff:	Yes.
00:33:22	R1:	OK. I'm going to leave you to show that to Michael, so that Michael can show me when you're done. Fair enough, Michael? So you can't let them off the hook.
00:33:30	Ankur:	You can just show her right now.
00:33:31	Jeff:	We could do it. Yeah. Right now.
00:33:32	R1:	But I want Michael to be able to show me.
00:33:34	Michael:	Get some cubes, you guys.
00:33:39	Jeff:	Pass 'em down.
00:33:40	Ankur:	Pass what down?
00:33:41	Romine	The cubes.
00:33:42	Jeff:	The blocks. My little cousin just bought them.
00:33:45	Ankur:	Josh?
00:33:46	Jeff:	I was all excited about it.

Time	Speaker	Transcript
00:33:48	Ankur:	[Inaudible.]
00:33:49	Jeff:	[Inaudible.] OK. We'll start out, *a* will be blue and *b* will be white. [Jeff sneezes.]
00:34:04	Romina:	Bless you.
00:34:05	Michael:	[Inaudible.]
00:34:05	Ankur:	[Inaudible.]
00:34:06	Jeff:	Thank you.
00:34:07	Ankur:	Make *b* squared.
00:34:07	Jeff:	This is, this would be one right here. That, this is 1 *a b*, which is the first thing right there.
00:34:12	Ankur:	Don't even bother-
00:34:12	Michael:	It's the bottom one, right?
00:34:13	Ankur:	-with the first two. Start with 1 2 1.
00:34:14	Jeff:	All right.
00:34:15	Ankur:	'Cause those really don't show you anything.
00:34:16	Jeff:	So this would be-
00:34:17	Ankur:	Make-
00:34:17	Jeff:	-one-
00:34:18	Romina:	*a* squared.
00:34:18	Jeff:	-*a* squared.
00:34:19	Ankur:	Put two. *b a*.
00:34:20	Jeff:	*b* squared.
00:34:21	Ankur:	Two of those.
00:34:22	Jeff:	And then it would be two, what is that? This. Right here. That's it?
00:34:28	Ankur:	Right. Two of those things in your right hand.
00:34:31	Jeff:	So this would be 1 2 1 1, right?
00:34:35	Ankur:	Yeah.
00:34:35	Jeff:	Correct?
00:34:36	Michael:	And that would be-
00:34:36	Ankur:	And then make the row underneath it.
00:34:37	Michael:	Wouldn't that be 2*a*?
00:34:38	Jeff:	Where?
00:34:41	Ankur:	That's *a* squared.
00:34:42	Romina:	That's *a* squared.
00:34:43	Jeff:	It's *a* squared.
00:34:44	Michael:	That's 2*a*.
00:34:45	Romina:	Mike. It's *a* squared. Isn't that OK?
00:34:46	Michael:	It's *a* and *a*.
00:34:47	Romina:	Well, how would you like us to make *a* squared?

Time	Speaker	Transcript
00:34:48	Ankur:	*a* and *a*, *a* squared.
00:34:49	Jeff:	Yeah, how do, how do you want us to make a [Inaudible.]? Do you want us to [Inaudible.]?
00:34:51	Michael:	Because you can't, you can't make a block-
00:34:52	Ankur:	Look.
00:34:52	Michael:	-be *a* like *a* and *a*, because-
00:34:54	Ankur:	Look, Mike. Mike, I'll show you. It's a 3-dimensional thing.
00:34:56	Romina:	Well, do you know what we mean?
00:34:57	Ankur:	See this? [Inaudible.]
00:34:58	Michael:	Yeah. I know what you mean, but I'm saying-
00:34:59	Ankur:	Watch. Wait. It's-
00:35:00	Jeff:	How do you want to build it? You show us how.
00:35:01	Michael:	'Cause if this, if this thing a part- [Inaudible.].
00:35:05	Romina:	So then just deal with it.
00:35:07	Michael:	Then this would be a squ- [Inaudible.]
00:35:08	Jeff:	Yeah. We're saying that this like-
00:35:09	Michael:	Like this face of it.
00:35:10	Jeff:	Yeah, like that would be an *a*.
00:35:11	Romina:	All right. Let's just, that's an *a*.
00:35:12	Jeff:	This is an a. And actually, if this *a* squared, there would be- [Inaudible.]
00:35:16	Michael:	This would be *a* squared. The face of it.
00:35:17	Jeff:	Yeah, if the face was an *a*-
00:35:18	Ankur:	Look, this one means-
00:35:19	Michael:	No. No. No. That would, that would, that would be *a* cubed. The face would be *a* squared. And this little side here-
00:35:23	Jeff:	Yeah. A line would be an *a*, like the edge, is that what-
00:35:26	Michael:	Yeah.
00:35:26	Jeff:	-would be *a*?
00:35:27	Romina:	Can we just imagine here?
00:35:29	Jeff:	-*a* squared and this would be- [Inaudible.]
00:35:30	Michael:	OK. [Inaudible.] Yeah, *a* squared.
00:35:32	Romina:	Thank you. All right. *a* cubed. And three-
00:35:37	Ankur:	[Inaudible.] Three.
00:35:38	Romina:	That's-
00:35:40	Ankur:	Then you're gonna have-
00:35:41	Romina:	Give me-
00:35:44	Ankur:	-three of those.
00:35:45	Romina:	How do you want those?
00:35:48	Ankur:	No. Wait. Those two are different.

Time	Speaker	Transcript
00:35:48	Michael:	No. They're the same.
00:35:50	Ankur:	No. They're the same. Never mind.
00:35:51	Michael:	Doesn't-
00:35:52	Ankur:	Yeah, I know.
00:35:53	Michael:	In actuality, it's like not really three *a* squared.
00:35:54	Ankur:	Yeah.
00:35:55	Michael:	It's like-
00:35:56	Romina:	OK.
00:35:57	Ankur:	-three with two whites and [Michael starts to arrange the towers.] make 'em the other way just in case.
00:36:04	Romina:	Yeah, just make 'em-
00:36:04	Ankur:	Like the same.
00:36:05	Romina:	-all the same. Right.
00:36:06	Ankur:	She'll say something. [Michael continues to build towers.]
00:36:08	Michael:	What are they like that?
00:36:16	Jeff:	Uh-huh. Where, where, where-
00:36:17	Ankur:	Then you got to have-
00:36:19	Jeff:	Was this *a* cubed?
00:36:20	Romina:	Am I missing something?
00:36:21	Ankur:	No.
00:36:22	Jeff:	Where's the paper with all of 'em written?
00:36:24	Ankur:	1 3 3 1.
00:36:25	Romina:	Here's, put 'em in the middle so we can see the-
00:36:28	Ankur:	And we got to show how-
00:36:29	Jeff:	So this is just the going to-
00:36:31	Romina:	And then we multiply it by another one of these. So-
00:36:33	Jeff:	Yeah. So you start out, yeah, like you-
00:36:36	Ankur:	No. It's just like this. See these two on the top row, the top right? You're just adding-
00:36:40	Jeff:	Yeah. Just look at it.
00:36:41	Ankur:	You're putting a white.
00:36:42	Romina:	Yeah.
00:36:43	Jeff:	Yeah. It's like taking this and just going-
00:36:44	Ankur:	Exactly.
00:36:44	Michael:	Put one over there.
00:36:45	Jeff:	Right there.
00:36:46	Michael:	One over there.
00:36:46	Jeff:	One there, one there.
00:36:47	Michael:	One there. Not one there.
00:36:48	Jeff:	Well, no. Well-

Time	Speaker	Transcript
00:36:49	Michael:	Uh-huh.
00:36:51	Jeff:	That would go with these to make-
00:36:52	Romina:	Yeah. 'Cause-
00:36:52	Jeff:	-the third one there.
00:36:53	Romina:	Yeah.
00:36:53	Jeff:	And then when you do this, it would be-
00:36:54	Romina:	That's gonna be-
00:36:55	Jeff:	-that one would lead to there. Or this one would bring it to there. That would go there.
00:36:58	Ankur:	It's like adding a blue-
00:36:59	Romina:	Then that one would go there.
00:37:00	Ankur:	-on top of 'em all.
00:37:01	Romina:	And then you, joining them together.
00:37:02	Jeff:	Exactly. Get that blue over there. We can explain this.
00:37:04	Romina:	OK.
00:37:05	Jeff:	Where, where's the blue that was on there?
00:37:05	Ankur:	It's like adding a blue one on top of all 'em and then-
00:37:09	Jeff:	Yeah, first you-
00:37:10	Ankur:	-a white on top of all 'em.
00:37:10	Jeff:	Yeah. And then-
00:37:10	Romina:	Yeah. 'Cause you have to-
00:37:11	Ankur:	And then that's what you get there.
00:37:12	Romina:	[Inaudible.]
00:37:13	Jeff:	Which we could show- [Inaudible.]
00:37:14	Ankur:	Easily.
00:37:16	Jeff:	[Inaudible.]. All right. Um. [Inaudible.] Do you want to come over for a second?
00:37:24	Romina:	Here. You know what. I'm- [Inaudible.] I'll make another one of those things.
00:37:30	Ankur:	You don't have to make another one.
00:37:31	Romina:	Yeah. 'Cause that way we can show her how we did-
00:37:32	Ankur:	We can show- [Inaudible.]
00:37:33	Jeff:	I can show her.
00:37:34	Ankur:	We can just show her.
00:37:36	Romina:	OK.
00:37:37	Ankur:	[Inaudible.]
00:37:38	Romina:	I was gonna say then we could like when we add something, then we could make it like you know and show her how it gets to this.
00:37:41	Ankur:	I know what- [Inaudible.]
00:37:43	Romina:	Do you want to do that?

Time	Speaker	Transcript
00:37:44	Jeff:	Well, I just, that's why I have four of these. [Inaudible.]
00:37:46	Romina:	Yeah. Well, then you need another one for these.
00:37:48	Jeff:	Well, then we'll take it off.
00:37:49	Ankur:	Well, then take it off.
00:37:50	Romina:	I was just- [Inaudible.]
00:37:51	Jeff:	It's a little extra work. All right. Here we go. All right, we're saying-
00:37:53	R1:	So, Michael.
00:37:54	Michael:	Yeah.
00:37:55	R1:	Are you gonna show me?
00:37:56	Jeff:	Oh. You're gonna show? All right. I'll-
00:37:58	Michael:	Um.
00:37:58	Jeff:	You could, you gonna do what we did while we had all these-
00:38:01	Michael:	Yeah, that.
00:38:01	Romina:	Yeah. Would you like us to-
00:38:02	Michael:	Give me one, one of those.
00:38:03	Jeff:	Start out with one of these. Yeah.
00:38:04	Michael:	Like this would be the, um-
00:38:06	Jeff:	We're saying-
00:38:07	Michael:	1 2 1. this is *a* squared. See this-
00:38:09	Jeff:	We're saying that this is *a*-
00:38:10	Michael:	This blue is an *a* and the white is a *b*. It would be *a* squared. And this would be 2*a*, no 2 *ab*, right?
00:38:18	Jeff:	Yeah.
00:38:19	Michael:	Can you see that?
00:38:20	R1:	I'm confused.
00:38:22	Romina:	Told ya.
00:38:23	Michael:	Un, this is *b*.
00:38:25	R1:	No. No. Can I tell you what I'm confused about?
00:38:27	Michael:	Yeah.
00:38:28	R1:	OK. I'm confused about- these are all 2-tall towers.
00:38:34	Michael:	That's what I was saying, too.
00:38:35	Jeff:	Yeah. You see exactly-
00:38:37	Ankur:	You can't do it.
00:38:37	Romina:	You can't-
00:38:37	Ankur:	You can't show *a* squared.
00:38:38	Jeff:	'Cause *a*, we're saying that *a*-
00:38:39	Michael:	With a 3-dimensional object.
00:38:40	Jeff:	Yeah, like a would be-
00:38:41	Michael:	-would be just be the line where the, the-

Time	**Speaker**	**Transcript**
00:38:43	Ankur:	-this. Right? This.
00:38:44	Michael:	*a* squared would be-
00:38:45	Jeff:	*a* squared would be the face.
00:38:45	Romina:	The face.
00:38:45	Michael:	-face and *a* to the third the block.
00:38:46	Jeff:	*a* cubed would be the block.
00:38:47	Ankur:	When we mean *a* squared, we mean a tower with two *a*'s.
00:38:50	Michael:	Just pretend.
00:38:50	Romina:	Yeah.
00:38:51	R1:	Right. No. I don't have trouble with this. I can see that this [the 2-tall tower with two blue cubes] is *a* squared and I could see that this [the 2-tall tower with two white cubes] is *b* squared.
00:38:56	Ankur:	And the 2*ab* is-
00:38:57	Romina:	This is just-
00:38:58	R1:	And I could see that this is-
00:38:58	Ankur:	There's 1 *a* and *b*.
00:39:00	Romina:	And maybe I-
00:39:00	Ankur:	And there's two of 'em.
00:39:01	Romina:	I guess we can do it like this.
00:39:02	R1:	But there isn't an *a*-
00:39:03	Ankur:	That's it.
00:39:04	R1:	*ab*.
00:39:04	Ankur:	There's *a*-
00:39:04	R1:	-and an *ab*. That's not what I heard Jeff say before.
00:39:06	Ankur:	There's an *a* in each one, and a *b* in each one.
00:39:08	R1:	I heard Jeff say there was an *ab*-
00:39:09	Jeff:	And a *b*, uh, a *ba*. You can do this.
00:39:12	R1:	And a *ba*. That's what I heard Jeff say.
00:39:13	Ankur:	It's the same.
00:39:14	Romina:	Well, it's the same thing.
00:39:14	Ankur:	It's the same thing.
00:39:14	R1:	OK.
00:39:15	Jeff:	*a* times *b* equals *b* times *a*.
00:39:15	R1:	Oh. OK. But where I get confused is, I could see-
00:39:17	Jeff:	Yeah, that-
00:39:17	R1:	-that these are all possible towers two tall.
00:39:19	Jeff:	Uh-huh.
00:39:20	R1:	And I can see that-
00:39:22	Michael:	Well, then, then somebody fix all of 'em.
00:39:22	R1:	-these are one with *a* and one with *b*. But you see where I, I

Time	Speaker	Transcript
		could get confused?
00:39:26	Jeff:	Yeah, I understand. Yeah. I was gonna do that before.
00:39:27	R1:	I was just taking Jeff literally. I think Jeff made an important point earlier. That they, they really are different.
00:39:32	Ankur:	Switch them around.
00:39:33	Jeff:	Relax.
00:39:34	Michael:	I'm gonna switch 'em.
00:39:35	Jeff:	All right. So anyway-
00:39:36	R1:	I just, I just get confused easily. My students will tell you that I'm very easily confused. You have to go slow with me. I like that *ab*, *ba*. It helps.
00:39:46	Jeff:	That's the way it is.
00:39:46	Michael:	OK. Then we, you like multiply-
00:39:49	Jeff:	You multiply it by-
00:39:50	Michael:	-*a* plus *b*. It would be like first- I don't know which one is first.
00:39:52	Jeff:	It goes, it goes- [Inaudible.]
00:39:54	Romina:	*a*.
00:39:54	Michael:	*a*, *a* is first. It would be like putting one of these [blue cubes] on top of each tower.
00:39:58	R1:	OK.
00:39:58	Jeff:	Which we can show for the-
00:39:59	Michael:	Well, don't put them on yet.
00:40:00	Jeff:	Well, I'm not, like, you don't have them to squeeze them down. Just throw them on top to show-
00:40:03	Michael:	And that's-
00:40:04	Jeff:	And that will show-
00:40:04	Michael:	They're in there.
00:40:05	Jeff:	And that shows you-
00:40:06	Michael:	They're in.
00:40:08	Jeff:	That's this one right here, right?
00:40:09	R1:	Uh-huh.
00:40:10	Jeff:	That is-
00:40:11	Ankur:	Two of 'em are over there.
00:40:12	Jeff:	So that would equal that. Two of those go there, correct? And one of those go over there. Right? [Jeff arranges the 3-tall towers as he speaks. See Figure H11.]

Time	Speaker	Transcript

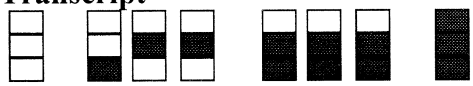


Figure H11. Jeff's arrangement of three-tall towers representing 1 3 3 1

00:40:18	R1:	Uh-huh.
00:40:19	Jeff:	//Now you take-
00:40:19	Michael:	//Then you're also-
00:40:20	Jeff:	Take these off. 'Cause now you're, 'cause-
00:40:22	Michael:	Now you-
00:40:22	Jeff:	Now you're multiplying it by-
00:40:23	Ankur:	Now you're distributing the *b*.
00:40:24	Michael:	But you still have those, you know.
00:40:24	Jeff:	These are still-
00:40:25	Michael:	And you just stick a bunch of whites on there.
00:40:27	Jeff:	And then-
00:40:27	Michael:	'Cause you're, you're like-
00:40:29	Ankur:	-distributing the blue.
00:40:30	Jeff:	And that would represent that one right there. These two that you just made would represent these two here. That would represent that. And that's how you break it down. That's how-
00:40:40	R1:	Except that, again, these two-
00:40:41	Michael:	Yeah, they're-
00:40:42	Romina:	Yeah. Well-
00:40:43	R1:	I'm supposed to imagine.
00:40:44	Jeff:	Yeah.
00:40:44	Romina:	Yeah.
00:40:44	Jeff:	We, we're, we're-
00:40:44	Michael:	Pretend. [Inaudible.]
00:40:45	R1:	I'm supposed to imagine-
00:40:46	Ankur:	[Inaudible.]
00:40:47	R1:	I'm supposed to imagine that-
00:40:48	Jeff:	Yes.
00:40:49	R1:	OK. And, and similarly. OK. That's very interesting. OK. So in general then if, um, if you're moving from towers four tall to five tall, without building it, then can you tell me in tower language why 4 plus 6 is 10?
00:41:12	Ankur:	Um. [Pause.]
00:41:32	Jeff:	All right. How about we tell you why 1 plus 2 is-
00:41:33	Ankur:	'Cause it'll be-
00:41:34	R1:	That's fine. You could-

Time	**Speaker**	**Transcript**
00:41:35	Ankur:	It'll be smaller.
00:41:36	R1:	That'll work too.
00:41:36	Ankur:	That's what I was just gonna say.
00:41:37	Jeff:	All right. Here we go.
00:41:38	R1:	You like that better.
00:41:39	Jeff:	All right. [Pause.] Here. All right. Look. We got-
00:41:42	Ankur:	Without towers.
00:41:42	Jeff:	We got this. Without towers?
00:41:43	Ankur:	Yeah.
00:41:44	Romina:	Yeah. On the paper.
00:41:45	Jeff:	I thought we were supposed-
00:41:46	Romina:	Tower-wise.
00:41:47	R1:	Any way.
00:41:48	Jeff:	Well, we can show-
00:41:48	R1:	Just tell me in tower language what-
00:41:49	Ankur:	We can show towers.
00:41:49	Jeff:	-with towers. All right. All right. Check it. We're gonna add, if we added an *a*, we got what we did before, right? To all of these? [Jeff places a blue block on top of four of the 3-tall towers.] Um. Put the *a*'s across. *b*'s, wait, I don't have a *b*. [Jeff continues to place blue blocks.] Is that, is that- Wait. Wait. The *b* equals the- [Jeff hesitates as he places a blue block on top of the blue-white-white tower; refer to Figure H12.]

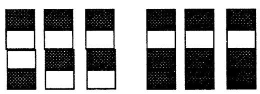


Figure H12. Jeff's first attempt to show the addition rule with towers

00:42:18	Romina:	You're getting- [Inaudible.]
00:42:19	Jeff:	Yeah. And then- [Aside to Romina.] It doesn't really matter- and then that would be-
00:42:25	Ankur:	2 plus 1, just make the 2.
00:42:26	Jeff:	Well, the 2-
00:42:27	Ankur:	Which is-
00:42:28	Jeff:	The 2 was what?
00:42:29	Ankur:	That. [Ankur points in the general direction of a group of towers.]
00:42:31	Jeff:	We're, we're past that. Now we're going from 3 to 4.
00:42:33	Ankur:	So we're doing this 3 plus 1.

Time	**Speaker**	**Transcript**
00:42:34	Jeff:	So this 3 plus 1-
00:42:35	Ankur:	Equals 4.
00:42:36	Jeff:	Yeah.
00:42:36	Ankur:	All right. So make the 3.
00:42:37	Jeff:	The 3 are-
00:42:39	R1:	I get confused 'cause these look like duplicates to me. [Researcher 1 points to the two white-blue-white towers.]
00:42:41	Jeff:	All right.
00:42:41	R1:	And do you really mean them to be duplicates 'cause-
00:42:44	Romina:	No.
00:42:44	Ankur:	No. We just mean-
00:42:44	R1:	Why don't, why don't we make them-
00:42:45	Ankur:	You know what we mean.
00:42:46	R1:	But it's hard for me to keep track, because they look exactly the same. [The students rearrange the three towers with one blue cube each, so they are all different, as shown in Figure H13.]

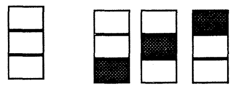


Figure H13. Towers representing b^3 and $3ab^2$ (a is blue and b is white)

00:42:49	Jeff:	Put the white on the bottom. All right. We got- All right. We're going from these [the three 3-tall towers with one blue cube], from these to the next set, right?
00:43:04	Ankur:	You can just show how this [the three towers with one blue, as shown on the right side of Figure H13] plus this [the tower with all white cubes, as shown on the left side of Figure H13] equals the 4. You don't even need those over there.
00:43:08	Jeff:	All right. You do it then.
00:43:09	R1:	OK. So can you explain to me without building it, what this set would look like without building it and why.
00:43:16	Jeff:	All right.
00:43:17	R1:	That's my question.
00:43:17	Jeff:	It would be-
00:43:18	R1:	Do you understand my question?
00:43:18	Jeff:	Yes. It would be *a b* squared. 'Cause you'd, *b* squared pl- it would be a 4-
00:43:23	Ankur:	4.
00:43:24	Jeff:	No. It would be *a*, wait, what, what, what is this? This would be

Time	Speaker	Transcript
		a to the fourth?
00:43:28	Romina:	Yeah. Are you going from-
00:43:29	Jeff:	*a b* to the fourth.
00:43:29	Romina:	You're going from this one to this one?
00:43:30	Jeff:	We're going-
00:43:31	Ankur:	We're going 3 plus 1 equals 4.
00:43:32	Jeff:	We're going from the third to the fourth. Yeah.
00:43:33	Romina:	Yeah. OK. Then that one-
00:43:33	Jeff:	All right. So that would be 3, 3, *a*, *a*, *b* to the third.
00:43:42	Romina:	Here, why don't you use this one.
00:43:43	Jeff:	And then *a* to the fourth. Yeah. It's be 3 *a b* to the third.
00:43:46	Ankur:	*b*.
00:43:48	Jeff:	*b*. Yeah. 3. Well, we did. Well, yeah. Right here.
00:43:50	Ankur:	Three *a*.
00:43:51	Jeff:	*a* to the third *b*. Actually it would be 3, wait.
00:43:55	Ankur:	No. It would be 4. 'Cause we're doing the 3 to the 1 and then 4.
00:43:56	Romina:	Yeah. We're going-
00:43:57	Ankur:	It would be four *a* to the third *b*. [Ankur points to row 4 in Pascal's Triangle.]
00:44:02	R1:	OK. Now we're here. Right? We're here and we're here. [Researcher 1 indicates the three towers with one blue block as shown on the right side of Figure H13 and the one tower with all white blocks as shown on the left side of Figure H13.]
00:44:06	Ankur:	Got it. It's-
00:44:07	R1:	OK. So let's just-
00:44:07	Ankur:	It's 4 *a* to the third *b*.
00:44:08	R1:	So what is what-
00:44:09	Ankur:	Because-
00:44:09	R1:	What, OK, this one is what? [Researcher 1 indicates the towers with one blue block, as shown in on the right side of Figure H13.]
00:44:11	Ankur:	That's this. The 3. [Ankur points to the first 3 in row 3 of Pascal's Triangle.]
00:44:13	Romina:	That's the-
00:44:14	R1:	Which? 3 what?
00:44:15	Ankur:	Two uh-
00:44:16	Romina:	Two-
00:44:16	Jeff:	That's 3.
00:44:17	Ankur:	3 *a* to the second *b*.
00:44:18	Jeff:	No it's *b* to the second, *a* to the first.

Time	Speaker	Transcript
00:44:19	Romina:	Yeah. It's *b* to the second.
00:44:19	Ankur:	*b* to the second *a*.
00:44:20	Jeff:	And this is *b* to the third. [Jeff indicates the all-white tower.]
00:44:24	R1:	Do you agree on that? Michael, do you agree with that?
00:44:28	Michael:	Yeah.
00:44:28	R1:	I think Michael's-
00:44:30	Romina:	He's not-
00:44:31	R1:	I think Michael's, um-
00:44:33	Ankur:	You don't know that this is *b* to the third?
00:44:35	Michael:	Yeah. I understand.
00:44:36	Ankur:	And that's-
00:44:36	Michael:	That's *b* to the third.
00:44:37	Ankur:	*b* to the second *a*?
00:44:37	Michael:	I'm not stupid. I know.
00:44:38	Ankur:	*b* to the second *a*, *b* to the second *a*. OK.
00:44:40	Michael:	I know. I knew, like from the first time they did it.
00:44:45	R1:	OK. So I'm asking you, without building it.
00:44:47	Michael:	Yeah.
00:44:47	R1:	What does the next set look like? They're gonna, you told they're gonna be, I believe you, there are gonna be four of them, right?
00:44:52	Ankur:	It would be-
00:44:53	R1:	But, but, but-
00:44:54	Ankur:	I'll-
00:44:54	R1:	4 of what?
00:44:55	Ankur:	I'll, I'll tell you.
00:44:56	R1:	And how did they come from here? What in here made them become those four? [Researcher 1 points to the towers shown in Figure H13.]
00:44:59	Ankur:	What in here? [Ankur points to the same towers.]
00:45:01	R1:	Well, you started with 3.
00:45:03	Ankur:	[Ankur laughs.] I'll tell you. All right.
00:45:04	R1:	Three *a b* squared.
00:45:05	Ankur:	You have this [the three towers with one blue block each], right?
00:45:07	R1:	Uh-huh.
00:45:07	Ankur:	And you have this [the one all-white tower]. And you're combining 'em to make four of something, right?
00:45:11	R1:	Uh-huh.
00:45:12	Ankur:	And the four of something have, they're gonna be four tall, right?
00:45:16	R1:	Uh-huh.
00:45:17	Ankur:	So right now you have this [the three one-blue block towers] and

Time	Speaker	Transcript
		this [the one all-white 3-tall tower]. They combine so- this joins with this. There's no more blue over here.
00:45:25	R1:	Uh-huh.
00:45:26	Ankur:	So there's still gonna be only one *a*. Do you understand that?
00:45:30	R1:	Uh-huh.
00:45:31	Ankur:	So it's gonna be *a* to the first or just *a*. And since there's two whites over here and three whites over here, each one of the whites goes to each one over here.
00:45:43	R1:	Are you following that, Romina?
00:45:46	Ankur:	'Cause then there would be three.
00:45:48	Romina:	I, I missed from where he went to this turns into this.
00:45:51	R1:	Uh-huh.
00:45:52	Ankur:	'Cause this [the 3-tall all-white tower] is gonna go to this [the 3-tall one-blue towers]. There's gonna be 3 *b*'s and 1 *a*.
00:45:55	Jeff:	And then 3 *b*'s and 1 *a*.
00:45:57	Romina:	And then this- OK. Yeah. Oh. Sorry.
00:45:58	Ankur:	See?
00:45:59	Jeff:	And then you'd have you'd have one left over and get the one *b*.
00:46:00	Romina:	Yeah. The one.
00:46:01	Ankur:	Yeah.
00:46:02	Jeff:	Yeah. So it would be like, all right. That's all right.
00:46:03	Romina:	Yeah.
00:46:03	R1:	No. No. Leave this alone.
00:46:03	Jeff:	All right. All right.
00:46:04	R1:	I'd rather, I'd rather you'd build it again.
00:46:05	Jeff:	We don't want-
00:46:06	R1:	I'd rather you'd build it again.
00:46:08	Jeff:	By- two-
00:46:09	Romina:	Don't flip out, all right?
00:46:12	Jeff:	Who's flipping out here?
00:46:13	Ankur:	You.
00:46:14	Jeff:	Me?
00:46:15	Ankur:	No. You're gonna make-
00:46:19	Jeff:	[Jeff is building towers.] We've got this. And we're saying how this goes together. [He has built another set of towers as shown in Figure H13.]
00:46:24	R1:	Uh-huh.
00:46:24	Jeff:	We're saying-
00:46:25	Ankur:	That's-
00:46:25	R1:	No. No. Don't take that apart. [Jeff has started to dismantle one

Time	Speaker	Transcript
		of the towers.] Because-
00:46:27	Jeff:	Well, that's why I made this.
00:46:28	Ankur:	We're not-
00:46:29	Jeff:	So I could. But-
00:46:30	Ankur:	We made another one so we can take that one apart.
00:46:31	Jeff:	Yeah. So we could, so we could-
00:46:32	Ankur:	And show you.
00:46:33	Jeff:	-take it apart.
00:46:33	Ankur:	You told us not to take that one apart.
00:46:34	R1:	You mean, you mean, you mean you get the 4 by taking something apart?
00:46:37	Romina:	Well, you don't-
00:46:38	Ankur:	You're not taking it apart.
00:46:38	Romina:	You're not taking it apart, you're just seeing-
00:46:39	Ankur:	No, this is what you're ac-
00:46:39	Romina:	-how they go together.
00:46:40	Ankur:	You're actually-
00:46:40	Michael:	It's not actually-
00:46:43	Ankur:	You're not actually taking it apart. You're taking-
00:46:43	Jeff:	You're taking
00:46:44	Ankur:	-another.
00:46:45	Jeff:	You're making like an *a*, an *a*.
00:46:47	Romina:	Yeah.
00:46:48	Ankur:	You're taking another-
00:46:49	Romina:	Since you're adding something to it to all of 'em, you're just-
00:46:51	Michael:	It's like-
00:46:52	Romina:	Kind of like joins this group.
00:46:53	R1:	All right -
00:46:54	Michael:	You don't really have to take it apart to show this, 'cause look.
00:46:55	Ankur:	You don't have to take it apart, you just-
00:46:56	Michael:	Each one, the reason why they combine, each one of these four blocks is gonna have something added to them to equal the same thing.
00:47:01	Ankur:	Yeah.
00:47:02	Michael:	These blocks are gonna have, they're gonna have a white. [Michael indicates the three towers with one blue block.]
00:47:05	Ankur:	They're gonna have a *b* added to them.
00:47:08	Michael:	-a white block added to 'em and this one's gonna have a, a blue-
00:47:09	Ankur:	An *a* added to it.
00:47:10	Michael:	-added to it and they're gonna equal the same thing.

Time	**Speaker**	**Transcript**
00:47:12	Ankur:	So there's-
00:47:13	Michael:	That's why you're gonna have the 4.
00:47:14	Ankur:	There's 3 and 4 of them.
00:47:14	Michael:	Whatever. And these blocks goes for the same. They're gonna have one block, I don't know which color, but they're gonna be added to 'em to equal the same thing. They're gonna have these, all three are gonna have a blue and these are all gonna have a white. They're all gonna equal 6 of something that are all the same. And the same goes on this side. You know?
00:47:31	R1:	OK. So what you're telling me is that this new set, right?
00:47:35	Michael:	Is gonna have, is gonna be-
00:47:36	R1:	Because, go ahead, say it again.
00:47:37	Ankur:	There's gonna be-
00:47:38	R1:	This new set here.
00:47:38	Michael:	This new set is gonna be like-
00:47:40	R1:	How many will have how many blues?
00:47:41	Michael:	6.
00:47:42	R1:	How many blues will it have in it?
00:47:43	Michael:	It will have, I mean like, how many-
00:47:46	Ankur:	Well.
00:47:46	Michael:	Actual-
00:47:46	Ankur:	What do you mean, how many blues?
00:47:47	Michael:	Blue like blocks?
00:47:48	R1:	Blue blocks. This new set, four tall, will have how many blue blocks?
00:47:51	Ankur:	One, two, three, four, five, six, seven, eight.
00:47:53	Michael:	It will have two wh-
00:47:53	Ankur:	Nine, ten, eleven, twelve.
00:47:54	Michael:	No. Two blue blocks and two whites.
00:47:55	Ankur:	There's two, there's gonna be two in each one.
00:47:57	Michael:	Two in each.
00:47:57	R1:	OK. So. so the question is that if this set is gonna grow to be four tall. [Researcher 1 indicates the three towers containing two blue blocks.]
00:48:02	Michael:	Four tall.
00:48:03	R1:	Right? The way it's gonna keep exactly two blue blocks-
00:48:07	Michael:	Is by adding a white.
00:48:07	Ankur:	By adding a white.
00:48:08	R1:	-is by adding a white.
00:48:08	Michael:	And this is by adding a blue. [Michael indicates the three towers

Time	Speaker	Transcript
		containing two white blocks. Refer to Figure H14.]

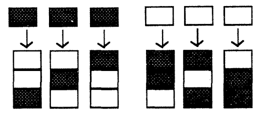


Figure H14. Michael explains 3+3=6 in terms of adding cubes to towers

00:48:10	Ankur:	By adding a blue.
00:48:10	R1:	But this set to grow two tall will grow by adding-
00:48:12	Michael:	But it's adding like something to it to equal the same thing.
00:48:14	R1:	Uh-huh.
00:48:15	Michael:	That's why they're being combined. 'Cause they're all like you know
00:48:17	R1:	What do you think about that, Romina?
00:48:18	Romina:	I follow it. Like I understand like what he's talking about.
00:48:20	Michael:	It's a lot easier than-
00:48:21	Romina:	Yeah.
00:48:22	Michael:	-taking it and you know.
00:48:22	Jeff:	Apart.
00:48:23	Ankur:	'Cause you're not really taking them apart.
00:48:23	Michael:	You're not, you're not actually combining them.
00:48:24	R1:	Uh-huh.
00:48:25	Michael:	You're just, you're putting this *a* and the *b* that you're gonna multiply into it.
00:48:29	R1:	Uh-huh.
00:48:25	Michael:	And-
00:48:30	Jeff:	And putting it in.
00:48:31	Michael:	And they like, they equal the same.
00:48:32	R1:	Uh-huh.
00:48:32	Michael:	So you could like, ought to categorize it as you know.
00:48:35	R1:	Uh-huh.
00:48:35	Michael:	It would be four of this.
00:48:36	R1:	Uh-huh.
00:48:36	Michael:	You know, you're not actually like combining 'em.
00:48:39	R1:	OK. So, so that's, that's really very nice. So, so when you have, let's say, this one here. 1 3 3 1, right? And this is the *a* plus *b* quantity cubed, right? Is that right?
00:48:51	Jeff:	Uh-huh.
00:48:52	R1:	So you have, um, 1.

Time	Speaker	Transcript
00:48:53	Ankur:	Yeah.
00:48:55	Jeff:	3 3 1.
00:48:56	R1:	Is that right?
00:48:57	Jeff:	Yeah.
00:48:58	Ankur:	Uh-huh.
00:48:58	Jeff:	And that would go into the 1 4 6 4 1 .
00:49:02	R1:	So, so, so that's, that's eight. Right? That's eight.
00:49:05	Jeff:	Yes.
00:49:05	R1:	Possible three cube towers.
00:49:08	Jeff:	Towers.
00:49:09	R1:	That you can select from two colors, right?
00:49:10	Jeff:	Uh-huh.
00:49:11	R1:	And then if you do this next row, there would be altogether how many?
00:49:15	Jeff:	Uh.
00:49:16	Ankur:	Sixteen.
00:49:17	Jeff:	Yeah.
00:49:18	Ankur:	'Cause each one's gonna have-
00:49:19	Jeff:	Gonna turn into, um, three, no two. Right?
00:49:24	Ankur:	Uh-huh.
00:49:25	Jeff:	Yeah, 'cause that would be-
00:49:25	Ankur:	Each one would have a blue.
00:49:28	Jeff:	Eight and sixteen is two.
00:49:29	Ankur:	Or a-
00:49:29	Jeff:	Yeah, every one would add, take a blue and a white.
00:49:30	Ankur:	Each one's gonna get a blue distributed to it and-
00:49:31	Jeff:	And, and a white.
00:49:32	Ankur:	-a white.
00:49:33	Jeff:	Like a blue put on, taken off. A white put on, taken off and that would be [Inaudible.] more.
00:49:36	Ankur:	No. Just a blue taken on and then the same tower with-
00:49:37	Jeff:	The white.
00:49:38	Ankur:	Yeah.
00:49:39	Jeff:	Exa- that's what I was trying to say.
00:49:40	Ankur:	Yeah.
00:49:41	Jeff:	It sounded real funny.
00:49:42	R1:	Fair enough where we are?
00:49:43	Jeff:	Uh-huh.
00:49:43	R1:	So. OK. The last question that we really didn't quite explore but we touched on that I'd like to pose to you one more time. Um.

Time	Speaker	Transcript
		You show me how you can make the relationship between these block towers and build them up to how you can take your binomials, right? And expand then, right? You can take *a* plus *b*, *a* plus *b* quantity squared, *a* plus *b* quantity cubed. That's pretty cool, isn't it? And you could show me how you can get those coefficients. It'll save you a time on a college board SAT test to know what the term is and the coefficient when they ask you that question, if you think about this. And if you really think about this model, you can save yourself a lot of multiplication time if you can make sense of these patterns.
00:50:27	Michael:	If they ask you what's *a* plus *b* to the seventh.
00:50:30	R1:	Or, you know, what's, you know, right. How many terms are there with three *a*'s and three *b*'s or something like that.
00:50:36	Ankur:	They don't get that advanced on SAT's.
00:50:38	Jeff:	Yeah, like that kind of-
00:50:39	Romina:	Yeah.
00:50:39	R1:	Well, maybe the advanced math test.
00:50:41	Jeff:	Yeah, but like the, the, they, they don't really like-
00:50:45	Romina:	They weren't - they were-
00:50:45	Jeff:	-basic math.
00:50:46	Romina:	Just long.
00:50:46	R1:	They were long. OK. Well, maybe there's some reason some day, you know, you'll like to able to do this fast. And maybe there isn't. That's a possibility. But this is my next question. Um. Remember when you did the pizza problem?
00:50:57	Jeff:	Yeah.
00:50:59	Romina:	Yeah.
00:50:59	Michael:	Pizza.
00:51:00	R1:	Right. And you, you s- I remember when, when Michael, uh, did it immediately.
00:51:04	Jeff:	Are we talking about the pizza problem we did this year?
00:51:05	R1:	The pizza problem we did this year. And I asked you, you how many, you know, pizzas you can make when you can select from four toppings, right? The four toppings and remember, Michael, what you did? You, your solution?
00:51:16	Michael:	Remember it? Yeah, I remember it.
00:51:18	R1:	Remember what you did? Just-
00:51:20	Michael:	What do you mean by that?
00:51:21	R1:	How you solved it?
00:51:22	Michael:	Yeah. With the binary.

Time	Speaker	Transcript
00:51:23	R1:	With the binary numbers. Remember how he did that? And he wrote out all his binary numbers, right?
00:51:27	Romina:	Yeah.
00:51:28	R1:	OK. That was very nice. Now, and then he got, he said and all of you said later, in fact Romina used that binary expression-
00:51:35	Michael:	We expanded it to do the-
00:51:36	R1:	-to do her proof.
00:51:36	Michael:	With the-
00:51:37	R1:	Right. And so you, so you were able to show that binary way of representing the solution to the pizza problem, right? My question is, um, what does that have to do if anything-
00:51:47	Michael:	With this?
00:51:48	R1:	-with this? Is that a fair question?
00:51:50	Romina:	Like you can kind of-
00:51:52	Ankur:	It's the same thing.
00:51:53	Romina:	You can see like the 1's.
00:51:53	R1:	Well, why don't you talk to each other and decide and then tell me? OK?
00:51:56	Ankur:	It's the same thing. Like it's, it's like 1 1 1.
00:52:00	R1:	Or is it just chance? You know.
00:52:02	Ankur:	You can add a 1 or a 0 to it.
00:52:03	Jeff:	Yeah.
00:52:04	Romina:	Yeah.
00:52:04	Jeff:	Um.
00:52:05	Ankur:	Call her back.
00:52:07	Romina:	Yeah. We're done.
00:52:08	Jeff:	I can explain.
00:52:11	Ankur:	Isn't that it really, basically?
00:52:12	Michael:	You know what she says.
00:52:13	Jeff:	Basically.
00:52:13	Ankur:	Instead of blue and white-
00:52:14	Romina:	Yeah like-
00:52:14	Ankur:	-1 and 0. And at each one you can add a 1.
00:52:16	Michael:	Yeah, but you know what? She's asking for more.
00:52:18	Romina:	Right?
00:52:19	Ankur:	So when she asks for more-
00:52:20	Romina:	And then, but the only thing we do is we do-
00:52:21	Michael:	What the hell is that [the symbols on Romina's paper]? Equals?
00:52:22	Romina:	No. It can go to this or to this. [Romina has written 111; she draws arrows as shown in Figure H15.]

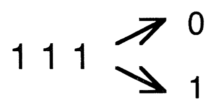


Figure H15. Romina shows how to go from 3-digit to 4-digit binary numbers

Time	Speaker	Transcript
00:52:25	Michael:	Oh.
00:52:26	Jeff:	That's, that's gonna-
00:52:29	Romina:	And then we just put, we go through all the ones first and then we add all the ones on the bottom.
00:52:32	R1:	My question still is how can you explain that to me with what you built with towers, if at all?
00:52:37	Romina:	OK. 'Cause the binary, we used 1's and 0's, right?
00:52:40	R1:	Let's talk about what we see here. What we have here.
00:52:43	Ankur:	And here we used blue and white.
00:52:43	Romina:	Yeah, so it's like-
00:52:44	Ankur:	So you just substitute blues-
00:52:45	Romina:	-the same thing.
00:52:46	Ankur:	-for the 1's and whites for the 0's.
00:52:48	Romina:	Like that would be-
00:52:48	R1:	Tell, I'm, I'm very, I'm very concrete. I mean I, I need to understand. What pizza problem would we be talking about here and how would that be the same as that? [Researcher 1 indicates all the 2-tall towers.]
00:53:01	Romina:	Like with-
00:53:02	R1:	If you had to make up a pizza problem to model this row, what's the pizza problem?
00:53:06	Jeff:	Well that would, that would be-
00:53:08	Romina:	One. No, it can't be.
00:53:09	Jeff:	Like without, um-
00:53:10	Ankur:	Three toppings.
00:53:11	Jeff:	No. There's only, how could there be three toppings?
00:53:13	Ankur:	Two toppings.
00:53:14	Romina:	Yeah, but see, on this one. It has-
00:53:15	Jeff:	Well, wait. We're looking at this. We're saying that-
00:53:17	Romina:	Yeah.
00:53:17	Jeff:	Yeah, but, like-
00:53:18	Romina:	When it moves on to this ones, would it be-

Time	Speaker	Transcript
00:53:19	Jeff:	All right. *a*, what would *a* squared be, though? Like what and what?
00:53:22	Ankur:	We figured this out before. [Pause.]
00:53:28	Romina:	Well, so, that would be two different toppings, right?
00:53:30	Ankur:	Wait a minute. *a* squared.
00:53:32	Jeff:	Well, *a* squared can't, *a* equals, *a* is a-
00:53:35	Ankur:	All right, look.
00:53:36	Jeff:	*a* is the same thing regardless.
00:53:37	Ankur:	This is a plain pizza. That's it. This is no toppings.
00:53:39	Jeff:	Wait. Where, where is that? What's that on?
00:53:40	Ankur:	The one. [Ankur has placed a single white cube on the table.]
00:53:41	Romina:	The 1 all the way at the top.
00:53:42	Jeff:	All right. All right. We got plain.
00:53:44	Ankur:	This is-
00:53:45	Jeff:	Uh. Peppers.
00:53:46	Michael:	Yeah, but like, like a pizza, just-
00:53:47	Ankur:	One topping.
00:53:47	Romina:	One topping.
00:53:48	Jeff:	One topping. Yeah that's one topping.
00:53:48	Ankur:	Peppers. It's either peppers or-
00:53:50	Romina:	No peppers.
00:53:50	Ankur:	-without peppers.
00:53:51	Jeff:	Yeah.
00:53:51	Ankur:	And this is two toppings. Peppers, um- [Ankur points to $a+b^2$, which is apparently meant to be $(a+b)^2$.]
00:53:56	Romina:	Yeah, that could be like-
00:53:57	Ankur:	Pepperoni.
00:53:58	Romina:	Plain.
00:53:58	Jeff:	Plain.
00:53:59	Romina:	With.
00:53:59	Jeff:	With stuff.
00:54:00	Romina:	And all. [Romina has pointed in turn to the 1, 2, and 1 in row 2 of Pascal's Triangle.] But you see that double? [Romina points to the 1 1 in row 1 of Pascal's Triangle.]
00:54:02	Ankur:	Wait. Wait. Wait. Wait.
00:54:02	Jeff:	It's like- [Inaudible.]
00:54:04	Ankur:	It's like peppers, pepperoni.
00:54:05	Jeff:	Yeah. Both.
00:54:06	Ankur:	Peppers and pepperoni.
00:54:07	Romina:	Yeah.

Time	Speaker	Transcript
00:54:08	Ankur:	And plain. [Ankur has also pointed in turn to the 1, 2, and 1.]
00:54:09	Jeff:	Wait. That's five though. I mean.
00:54:10	Ankur:	No it isn't.
00:54:10	Romina:	Yeah.
00:54:10	Ankur:	It's still four, because peppers and pepperoni is the same thing as pepperoni and peppers.
00:54:14	Romina:	Yeah, but see-
00:54:14	Jeff:	I understand what you're saying.
00:54:15	Romina:	-here it's different. It's, it's-
00:54:16	Ankur:	Yeah, but the number is still gonna be the same.
00:54:18	Romina:	The number is gonna be the same, but, OK.
00:54:19	Jeff:	Yeah, cause that since they're the same thing.
00:54:19	Romina:	Yeah.
00:54:20	Jeff:	That's how you balance out your um-
00:54:22	Ankur:	Extra.
00:54:23	Jeff:	Yeah. What about this right here, though? Like this right here? [Jeff indicates the three towers shown on the right side of Figure H13.] This would be, uh, mushrooms, pep-
00:54:26	Ankur:	That's three toppings.
00:54:27	Jeff:	Yeah, but say that would be mushrooms, peppers, and like, uh, something else? Wouldn't that-
00:54:31	Ankur:	No. Wait. Let's just do it like this.
00:54:32	Jeff:	Oh. No. Wait. No. that would only be two. That would be like double-
00:54:34	Romina:	How can-
00:54:34	Ankur:	Mushrooms-
00:54:36	Jeff:	Uh-huh.
00:54:36	Ankur:	Pepperoni.
00:54:37	Jeff:	Yeah.
00:54:41	Ankur:	Mushroom. Mushrooms and peppers.
00:54:42	Jeff:	Yeah. Oh. Mu- all right.
00:54:43	Ankur:	'Cause you can use two and you don't have to use all three.
00:54:45	Jeff:	I hear you.
00:54:46	Ankur:	So that's how it makes out. [Pause.]
00:54:50	R1:	I'm not following.
00:54:51	Ankur:	'Cause.
00:54:54	R1:	Help me understand.
00:54:55	Ankur:	This one is just a plain pizza. At the top of Pascal's Triangle. [Ankur points to the 1 in row 0 of Pascal's Triangle as shown in Figure H10.] That's 1.

Time	**Speaker**	**Transcript**
00:55:00	Jeff:	How did you know the name of that?
00:55:02	Romina:	Yeah. How did you remember that?
00:55:03	Ankur:	I just remembered it.
00:55:05	Jeff:	I can just imagine. This guy must have been really excited when he figured this out. Like when you're going through it like this.
00:55:11	Romina:	Well.
00:55:12	Jeff:	There's nothing more exciting than when you're watching something and you see, like you're trying to figure out like shortcuts and stuff, and you just watch it all come together.
00:55:15	Romina:	Well, what was he doing? Was he actually doing this?
00:55:17	Jeff:	Was he looking-
00:55:18	Romina:	Like, was he actually multiplying a and b together?
00:55:20	Jeff:	Why would people multiply a and b together in the first place? To like, normally, why would he be hanging around doing that?
00:55:25	Romina:	This is like a long time ago so it must have been like-
00:55:27	Jeff:	Yeah.
00:55:28	Romina:	When they were discovering all this stuff.
00:55:29	Jeff:	Like I don't, I don't-
00:55:29	R1:	Well, you know, maybe when we figure out the pizza problem, we'll have more intuition.
00:55:33	Jeff:	We'll worry about that after we-
00:55:35	R1:	After we figure out the pizza problem maybe.
00:55:36	Michael:	Didn't we do it?
00:55:36	R1:	You'll have the answer to that question yourself. But what were you saying, Ankur? And I'm not sure-
00:55:40	Ankur:	The top of the-
00:55:41	R1:	I really am not sure I'm following.
00:55:42	Ankur:	The top of this-
00:55:43	R1:	There's so many ideas. They're not coming together for me.
00:55:44	Ankur:	The top of the triangle is a 1, right? And the 1 can be a plain pizza, with no toppings. And that way there's only one possibility, right?
00:55:54	R1:	I don't know. You, you all decide.
00:55:56	Ankur:	We all agree.
00:55:57	Romina:	Yeah.
00:55:57	Jeff:	Exactly. That it is.
00:55:58	Romina:	It's a. It's just like the one thing.
00:55:59	Jeff:	It's a. You can only have a.
00:56:00	Romina:	Pizza or no pizza.
00:56:01	Ankur:	a or a.

Time	Speaker	Transcript
00:56:02	R1:	Is that, is that what that 1 means? 1 *a*?
00:56:04	Jeff:	Yeah. We're a-
00:56:04	Romina:	Yeah.
00:56:05	R1:	'Cause I'm confused. 'Cause I thought-
00:56:06	Jeff:	No, that would be *a b*.
00:56:07	R1:	I thought you told me that this [a white cube] was *a*, 1 *a*, and this [a blue cube] was 1 *b*. that's what you told me before.
00:56:14	Jeff:	Isn't, all right, wait, isn't the 1-
00:56:16	Romina:	This has to, what's the 1? What is it?
00:56:19	Ankur:	I don't know what the 1 is.
00:56:20	Romina:	The 1 is just 1.
00:56:21	Ankur:	The 1 is just 1.
00:56:22	Jeff:	Yeah.
00:56:22	Romina:	It's not an *a* or a *b*.
00:56:23	Jeff:	Yeah.
00:56:23	Romina:	It's nothing. It's just 1.
00:56:23	Ankur:	It doesn't represent anything in the *a* plus *b*.
00:56:24	Jeff:	No a b there.
00:56:25	R1:	OK. Well. Let's put that aside for now. Why don't we start with-
00:56:28	Jeff:	We'll start with 2.
00:56:29	Ankur:	All right.
00:56:30	Romina:	You go get-
00:56:31	Ankur:	It's 2. It's one topping.
00:56:31	R1:	Now.
00:56:31	Ankur:	Either pepperoni or without pepperoni.
00:56:34	R1:	Now wait.
00:56:34	Ankur:	It's two possibilities.
00:56:34	R1:	Now wait. Now I'm lost again. What, what, what was this? Did we move from here to here? [From row 1 to row 2.]
00:56:38	Michael:	This, this was-
00:56:39	Ankur:	The colors don't, don't look at the colors.
00:56:40	Michael:	No. No. No.
00:56:40	Ankur:	Just look at this.
00:56:41	Michael:	This would be- you, you-
00:56:41	R1:	I'm, I'm trying to make sense.
00:56:42	Michael:	You don't look at the towers like-
00:56:43	R1:	I'm trying to make sense of it with the towe-
00:56:44	Michael:	-lying down like this. 'Cause it's different.
00:56:47	Ankur:	All right. Then lie them down.

Time	Speaker	Transcript
00:56:48	Michael:	Because then, then you have like, this, this.
00:56:50	Ankur:	But the colors don't specifically represent anything.
00:56:51	Romina:	Yeah.
00:56:52	Michael:	Yes. It does.
00:56:52	Ankur:	No, it don't.
00:56:53	Michael:	Topping. [Michael points to the blue cube.] Or no topping. [Michael points to the white cube.] Just say like that. And if you-
00:56:56	Ankur:	OK.
00:56:57	Michael:	Look at it like this, you know.
00:56:58	Ankur:	So all of the whites are no topping?
00:56:59	Michael:	Yeah, but then-
00:57:01	Romina:	So like what is like-
00:57:02	Michael:	Yeah.
00:57:02	Romina:	This is a whole no topping?
00:57:04	Michael:	Then this is a plain pizza. [Michael takes the three-white-cube tower.]
00:57:05	Romina:	This is a two-topping.
00:57:06	Michael:	With a choice. If you-
00:57:07	Ankur:	Oh. OK. OK.
00:57:07	Michael:	-had a choice of three toppings.
00:57:08	Jeff:	All right.
00:57:08	Ankur:	OK.
00:57:08	Romina:	OK.
00:57:10	Michael:	This [the blue-white-blue tower] would be a pizza.
00:57:11	Romina:	Oh. With the one. Ooh.
00:57:12	Michael:	With two different toppings, without the other, third topping.
00:57:14	Romina:	That's what I was asking.
00:57:14	Ankur:	OK.
00:57:15	Michael:	It's not like a pizza with-
00:57:16	Ankur:	OK.
00:57:17	Michael:	Well.
00:57:18	Jeff:	Two, two toppings. Well, yeah. Well, if you're just saying that this is the pizza with three no toppings, it's plain.
00:57:22	Romina:	It's just a plain pizza.
00:57:23	Ankur:	All right. All right. So that's two toppings.
00:57:25	Romina:	Yeah.
00:57:26	Michael:	That's-
00:57:26	Jeff:	Yeah. All right. So.
00:57:26	Michael:	A choice of two, but-

Time	Speaker	Transcript
00:57:27	Jeff:	Yeah.
00:57:27	Ankur:	You have a choice of two toppings.
00:57:28	Michael:	You want it plain.
00:57:28	Jeff:	Yeah, so this is, this [blue-blue tower] is choice of two using two. This [blue-white tower] is choice of two using one.
00:57:32	Ankur:	Two using one.
00:57:33	Jeff:	This [white-blue tower] is choice-
00:57:33	Ankur:	That's using the other one.
00:57:34	Jeff:	-of two using the other one.
00:57:34	Ankur:	And that's [white-white tower] using nothing. [Refer to Figure H16.]

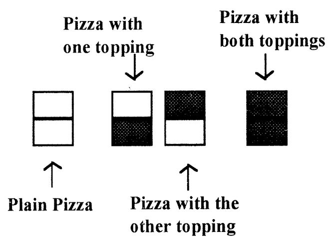


Figure H16. Connecting two-tall towers to two-topping pizzas

00:57:35	Romina:	Yeah.
00:57:35	Jeff:	And that's-
00:57:35	R1:	And that's all the possibilities?
00:57:36	Jeff:	-the choice of two.
00:57:37	Ankur:	Yes.
00:57:37	Romina:	Yeah.
00:57:37	R1:	You like that?
00:57:37	Ankur:	Those are the only possibilities.
00:57:38	Romina:	Oh. Wow.
00:57:40	R1:	OK. Is that interesting? You've worked so hard. I have one last question.
00:57:44	Jeff:	All right.
00:57:45	R1:	Unless you have questions for me? OK.
00:57:47	Romina:	What was he thinking?
00:57:48	Jeff:	That's what he-
00:57:49	R1:	Well, what do you think? That's, I don't know what he was thinking, I can't speak to that, but-
00:57:52	Romina:	Well, how long ago was this?
00:57:54	Jeff:	I'd say, yeah, like how old is this?

Time	Speaker	Transcript
00:57:55	Michael:	That's like when they like first started with discovering these-
00:57:56	Romina:	Like that's like when all the Greeks.
00:57:57	Jeff:	Isn't there a computer program, like isn't that like Pascal, isn't that a deal?
00:58:00	R1:	Blaise Pascal.
00:58:01	Michael:	What kind of program?
00:58:03	R1:	Blaise Pascal.
00:58:04	Michael:	What does it do?
00:58:05	R1:	No, it's not a program. It's a, a-
00:58:06	Michael:	Oh.
00:58:07	R1:	It's a, it's a-
00:58:08	Jeff:	Isn't that a way to write in-
00:58:09	R1:	No. It is. That's absolutely right. But, but this was a person, a-
00:58:12	Jeff:	Yeah.
00:58:13	R1:	-mathematician who thought about these-
00:58:14	Michael:	A computer language.
00:58:15	R1:	-things. Who thought about-
00:58:16	Jeff:	Yeah. It's like C++.
00:58:17	R1:	-these math ideas. Pascal is, is a-
00:58:19	Romina:	This the only thing he came up with?
00:58:20	R1:	-computer language. No, he did a lot of stuff. You want some more stuff to read on Blaise Pascal, we'll bring a few next time.
00:58:25	Romina:	Sure.
00:58:25	R1:	OK.
00:58:26	Michael:	I'm just waiting-
00:58:27	R1:	That, that'll be our job to bring you stuff to read. My next question-
00:58:32	Jeff:	Yeah.
00:58:32	R1:	Um. Suppose I wanted to talk about, um, a *c* in this? You know, we talked about *a* plus *b* quantity cubed.
00:58:44	Ankur:	*a* plus *b* plus *c* quantity squared?
00:58:45	Jeff:	Yeah.
00:58:45	R1:	Yeah.
00:58:46	Ankur:	A trinomial.
00:58:47	Jeff:	That would add another- [Pause.]
00:58:50	Ankur:	Color.
00:58:51	R1:	OK. I don't, I don't know if that works the same way. I'm gonna leave you to think about that, but my, my question is, um-
00:58:57	Jeff:	Are we gonna be doing this now? Are we gonna do that now or are we gonna-

Time	Speaker	Transcript
00:58:59	R1:	No, but we're going to take a piece of it to look at now. And the little piece is this. Remember Ankur gave us this very interesting problem, um, a few weeks ago. And, and that problem was, I believe, towers were being made that were four tall and we were selecting from plastic cubes of three colors. And in Ankur's problems there had to be-
00:59:21	Ankur:	One of every, each color.
00:59:22	R1:	-one of at least each color, right? Now I'm asking you the problem where you're selecting from three colors, the towers are three tall, how many are there with one of each color? And now you're selecting four tall, now you're selecting from four colors, how many are there with one of each color. And you're making your towers five tall-
00:59:52	Ankur:	Five of each color.
00:59:53	R1:	Do you understand my question?
00:59:53	Jeff:	Yeah. Five of-
00:59:54	R1:	And so forth. Um. *n* tall. Right? Seven tall. Twelve tall. Fifteen tall.
01:00:01	Ankur:	*n* tall. *n* of each color.
01:00:03	R1:	You can start with, you know one tall. Right?
01:00:06	Ankur:	One of each.
01:00:07	R1:	One tall, one of each color.
01:00:10	Ankur:	One is one.
01:00:11	R1:	Do you understand my problem?
01:00:11	Romina:	I think so.
01:00:13	Ankur:	Two is two.
01:00:14	Michael:	[Inaudible.] each color [Inaudible.] ever-
01:00:15	Jeff:	That would be two tall.
01:00:16	Romina:	Come on, when there's two tall-
01:00:17	Michael:	It's not a function.
01:00:18	Ankur:	One is one. Two is two.
01:00:21	Jeff:	So we're saying then two tall would be two colors.
01:00:24	Ankur:	Two is either blue and white or white and blue.
01:00:25	Jeff:	Yeah.
01:00:26	Ankur:	Three is either red, white, and blue.
01:00:30	Jeff:	And blue. Blue, white, red. Or can't we- no, we need to have, we need to use all three.
01:00:33	Ankur:	You have, yeah, so-
01:00:34	Jeff:	Or, or we can, all right-
01:00:35	Ankur:	Three is three. Four would be-

Time	Speaker	Transcript
01:00:36	Jeff:	I have a question. Is this, um, would red, blue, white [Jeff writes RBW.], is that different from this? [Jeff writes WBR.]
01:00:42	Ankur:	Yeah.
01:00:43	Jeff:	So there would be a lot more than three for three.
01:00:45	Ankur:	No. It'd be three.
01:00:46	Romina:	Three. How, what other possibilities can you get other than-
01:00:48	Ankur:	Try it.
01:00:48	Romina:	Now.
01:00:49	Ankur:	Seriously. Try it.
01:00:50	Jeff:	All right. All right.
01:00:50	R1:	Do you want another color?
01:00:51	Jeff:	No. Wait. Red, blue, white. Wait. [Jeff writes RBW.]
01:00:53	Ankur:	It, it'd be six.
01:00:54	Jeff:	White, blue, and red. [Jeff writes WBR.]
01:00:55	Ankur:	It would be six.
01:00:56	Romina:	Oh. Yeah. OK.
01:00:57	Ankur:	It would be six.
01:00:57	Jeff:	Red, blue.
01:00:58	Romina:	All right.
01:00:58	Jeff:	Everybody slow down. [Jeff writes RBW again.] And then like red. [Jeff writes RWB.] How come I can only get three?
01:01:03	Ankur:	Six.
01:01:05	Jeff:	Agh.
01:01:06	Ankur:	One is one. Two is two.
01:01:07	Jeff:	Yeah. Two is-
01:01:08	Ankur:	Two.
01:01:09	Romina:	Two.
01:01:10	Jeff:	Two.
01:01:10	Ankur:	Write that.
01:01:10	Jeff:	How about-
01:01:11	Romina:	Three is-
01:01:11	Ankur:	Three is six.
01:01:12	Jeff:	So it would be one.
01:01:13	Romina:	Here.
01:01:13	Ankur:	One, two, six.
01:01:15	Jeff:	One times one color. [Jeff writes 1*1.] They use the, how many, wait, how many-
01:01:19	Romina:	One is one, right? Two is two.
01:01:21	Jeff:	Wait. How many, how many are there for three? Wait.
01:01:22	Ankur:	This is like-

Time	Speaker	Transcript
01:01:23	Jeff:	Wait. Wait. Wait.
01:01:23	Romina:	Oh. I thought it was six. I thought you figured that out.
01:01:24	Jeff:	Wait. Wait. Wait. Wait. Wait. We didn't finish yet. Wait. Wait. Wait. Wait.
01:01:26	Ankur:	This is like the function. You put in like-
01:01:27	Jeff:	Yeah. Wait.
01:01:28	Ankur:	So-
01:01:28	Jeff:	Wait. How, how many are left here? Wait.
01:01:30	Ankur:	Four.
01:01:31	Jeff:	Wait.
01:01:32	Romina:	Well, let's see. Is there six?
01:01:33	Jeff:	Wait.
01:01:33	Ankur:	There is six.
01:01:34	Jeff:	Yeah. There's, uh, two, for- Yeah!
01:01:35	Ankur:	Like, you put a red here.
01:01:36	Romina:	Like the-
01:01:37	Ankur:	Then 1 1 or-
01:01:38	Jeff:	Yeah. It was like 0, like remember you did all this?
01:01:40	Ankur:	Yeah.
01:01:40	Romina:	Yeah.
01:01:40	Jeff:	And then it would be red, so-
01:01:42	Romina:	So then four with four colors, right?
01:01:44	Ankur:	And to each one of these you could put a-
01:01:46	Romina:	Well, no, because don't you have-
01:01:47	Ankur:	Wait.
01:01:48	Romina:	-four different colors when it's 4?
01:01:49	Jeff:	From each one here, it would be, you can go four, four, it would be six.
01:01:54	Ankur:	No. You can't put four, because you got to have-
01:01:56	Romina:	Isn't it a whole different-
01:01:59	Jeff:	You mean you-
01:02:00	Romina:	Just answer a question. Is it a different color? When there are four, is there four different colors?
01:02:02	Ankur:	All right, look. Write four spaces.
01:02:03	R1:	Yes, Romina.
01:02:04	Jeff:	Yes.
01:02:04	Romina:	OK.
01:02:05	Ankur:	It's, it's-
01:02:06	Michael:	Factorial.
01:02:08	Jeff:	Wait. so-

Time	Speaker	Transcript
01:02:10	Ankur:	This one's factorial. Because look. Can you see this?
01:02:16	Romina:	Yeah. It is.
01:02:17	Ankur:	'Cause you have four spaces. [Ankur draws four dashes.] You have-
01:02:19	Romina:	So it would be four.
01:02:20	Ankur:	-to use all four colors. For the first one, you have four choices. For the second one, you only have three because you used one here. Two one. [Refer to Figure H17.]

<u>4</u> <u>3</u> <u>2</u> <u>1</u>

Figure H17. Ankur illustrates 4!

01:02:24	Jeff:	Yeah. Three. And then two and one. And so what-
01:02:29	Romina:	So then four, four, ten, twelve.
01:02:31	Jeff:	So we show for the three.
01:02:33	Romina:	Twenty-five, is that what you're saying?
01:02:33	Jeff:	It would be three.
01:02:34	Ankur:	Twenty-four. And then times five would be the next one.
01:02:35	Jeff:	Wait. Wait. Wait a minute. Wait a second. Are we doing-
01:02:36	Romina:	Oh, no. I'm adding wrong. Sorry.
01:02:37	Jeff:	Aren't you supposed to- wouldn't you be adding them?
01:02:39	Romina:	No.
01:02:40	Jeff:	No. You'd multiply.
01:02:40	Romina:	Factorial is multiplying.
01:02:41	Jeff:	Three times two times one.
01:02:42	Ankur:	Uh-huh.
01:02:42	Jeff:	All right.
01:02:43	Romina:	Twenty-four, right?
01:02:43	Jeff:	Twenty-four.
01:02:44	Romina:	And then five would be-
01:02:46	Jeff:	And then the next one would be, uh, five times four-
01:02:48	Romina:	Twenty-
01:02:49	Ankur:	Twenty-four times five.
01:02:50	Jeff:	Well, twenty-five times five would be one twenty-five, minus-
01:02:52	Ankur:	Minus five.
01:02:53	Jeff:	Five. One twenty?
01:02:54	Michael:	Why do you minus five?
01:02:55	Ankur:	Because we did twenty-five times four, instead of twenty-four times four. [Note: He meant times five.]

Time	Speaker	Transcript
01:03:00	Jeff:	'Cause it's easier. [Jeff completes the table; refer to Figure H18. Ankur laughs.]

1	1
2	2
3	6
4	24
5	120

Figure H18. Jeff's table for number of towers *n* tall with *n* different colors

01:03:04	R1:	So, *n* tall with choices of *n* different colors?
01:03:07	Ankur:	It would be *n* factorial.
01:03:08	Romina:	*n* factorial.
01:03:08	R1:	You all believe that?
01:03:10	Jeff:	Yes.
01:03:10	Romina:	Yeah.
01:03:11	R1:	You're absolutely convinced?
01:03:11	Jeff:	I don't want to write that.
01:03:12	Ankur:	And so are you-
01:03:12	Michael:	What's the sign for that? Just pick it up.
01:03:13	Romina:	It's an exclamation point.
01:03:14	Jeff:	Exclamation point.
01:03:15	Michael:	Is it upside down or something?
01:03:16	Ankur:	No.
01:03:16	Romina:	No.
01:03:17	Ankur:	It's just an exclamation mark.
01:03:18	Jeff:	It would just be me! If it was upside down. [Jeff writes *n*!]
01:03:20	Romina:	That [the *n*] looks like a *u*.
01:03:22	R1:	Michael, are you convinced? Michael?
01:03:23	Michael:	Yeah.
01:03:24	R1:	You're all convinced, right?
01:03:26	Michael:	It makes sense.
01:03:27	Romina:	Yup.
01:03:27	R1:	OK. Well, you've all done a, worked very very hard. Does it start to come together a little bit? [There are discussions about their current mathematics class and about scheduling the next meeting.]
01:07:48	R1:	Then I'm gonna, then I'm gonna, I'm gonna challenge you with a very classical famous problem. OK? All of you together. And

Time	Speaker	Transcript
		maybe, um, [your classroom teacher] will even let you think about it in class. I suggest you think about it by not, by thinking about either your binomials or using the towers. In fact, if you want us to leave you a bag, we will, if you'd like to use them in class. But you see these coefficients here?
01:08:13	Jeff:	Yes.
01:08:14	R1:	You showed me why the addition rule worked. You might really want to think about that and make sure that that really is a strong understanding you have of that. OK? And why that works. Because in general, if I were asking you to do this, remember the other notation I showed you, that, um- Let's start with this row, I'm gonna use the pen and another piece of paper. You see that one, two, one?
01:08:38	Ankur:	Each number represents-
01:08:39	R1:	Right. That means you're building two high.
01:08:41	Ankur:	Uh-huh.
01:08:42	R1:	So that one means maybe you're selecting no blues. Right?
01:08:46	Ankur:	Uh-huh.
01:08:46	R1:	That was the one. And this one means that you're selecting-
01:08:50	Ankur:	One.
01:08:51	Jeff:	One blue out of two choices.
01:08:53	Ankur:	One blue, two-
01:08:53	R1:	And?
01:08:54	Jeff:	And one red.
01:08:56	R1:	Why?
01:08:57	Jeff:	'Cause you picked the one blue?
01:08:58	R1:	And then this is 2 2. Right?
01:09:00	Jeff:	That means you picked two blues.
01:09:01	R1:	OK. And these turn out to be ones. And then we could have gone to three high. Right? OK?
01:09:07	Ankur:	Three one. Three two. 3 3.
01:09:09	R1:	Exactly.
01:09:10	Jeff:	It would be three one -
01:09:10	R1:	Right.
01:09:10	Jeff:	Three two. Yeah. And so on.
01:09:12	R1:	Remember that notation? [Researcher 1 writes rows 2 and 3 of Pascal's Triangle. Refer to Figure H19.] And, and, and you know what that number is. You're gonna be playing with those ideas later at some point. Um. And so if we kept doing this, right? OK. To the, let's say, n^{th} row. That would be n.

Time	Speaker	Transcript

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

Figure H19. Rows 2 and 3 of Pascal's Triangle in combinatorics notation

01:09:27	Ankur:	Over zero.
01:09:28	R1:	Uh-huh.
01:09:28	Jeff:	*m* one.
01:09:30	Romina:	*n*.
01:09:30	Jeff:	*m* [Pause.]. *m m*? *m*- [Jeff is reading the *n*'s as *m*'s.]
01:09:37	Ankur:	It would be 1 still.
01:09:38	Jeff:	1. Yeah. 1 *m*.
01:09:39	Ankur:	2.
01:09:40	Jeff:	*m* 2.
01:09:40	Ankur:	Until the bottom number equals-
01:09:41	Michael:	The bottom number equals *n*.
01:09:42	Ankur:	*n*.
01:09:42	Romina:	Yeah.
01:09:43	Jeff:	'Till whatever, yeah, whatever the, yeah.
01:09:45	R1:	OK. So let's take a general term, an *r* term in there, right?
01:09:49	Jeff:	Yeah.
01:09:49	Ankur:	Uh-huh.
01:09:50	R1:	OK. One of the things I want you to tell me, if this is *r*, right?
01:09:55	Jeff:	Uh-huh.
01:09:55	R1:	The r^{th} term, the general term, right? How can you tell me, in general, where that r^{th} term comes from? From this before. Because if this is *n*, right, you could make this *n* plus 1 if you want and you could make the row before it *n*. Is that right? This is the n^{th} row. [Refer to Figure H20.]

Time	Speaker	Transcript

$$\binom{n}{0}\binom{n}{1}\binom{n}{2}\binom{n}{r}\binom{n}{n}$$

Figure H20. Row n of Pascal's Triangle as written by Researcher 1

01:10:12	Ankur:	It goes-
01:10:13	R1:	This is the n minus 1 row. [Researcher 1 indicates the position above row n.]
01:10:14	Ankur:	Yeah.
01:10:14	Romina:	Uh-huh.
01:10:15	R1:	Do you understand that? So I want you to tell me that, but I also want you to tell me how you can predict the numbers, how they grow this way? [Horizontally; refer to Figure H21.]

$$\binom{3}{0}\rightarrow\binom{3}{1}\rightarrow\binom{3}{2}\rightarrow\binom{3}{3}$$

Figure H21. Predicting how the numbers in Pascal's Triangle grow horizontally

01:10:24	Jeff:	Well, it's just, every time we move up, you more, every time this- Whatever number this, it starts from 0 and goes to that number. [Jeff points to Figure H21.]
01:10:30	R1:	Well.
01:10:31	Ankur:	It grows by-
01:10:32	R1:	I want you-
01:10:33	Jeff:	Like down.
01:10:33	Ankur:	Like you have one red and then two red.
01:10:36	R1:	I want you to, I want you to be able to explain that to me with the towers for- that you can show me for a particular row how that works. In other words, if you started with this 1 [the 1 at the top of Pascal's Triangle]. We're talking about towers three tall, with let's say three whites, right? Isn't that what that might be? Possibly? [Researcher 1 points to the first 1 in row 3.]
01:10:53	Ankur:	Yes.
01:10:53	Jeff:	Mm.
01:10:53	R1:	OK. Or three blues, I don't care how you start. Now you're going to get three towers. Right? And they're all going to be three tall, right?
01:11:00	Ankur:	They're gonna be-

Time	Speaker	Transcript
01:11:00	R1:	What are they going to look like? [Pause.]
01:11:05	Ankur:	One blue, two white.
01:11:07	R1:	OK. So what will all three of them look like? One blue, two white?
01:11:13	Ankur:	One blue, two white.
01:11:13	Romina:	Yeah, going-
01:11:14	R1:	Well, how, how would you get that from here [the two-tall tower with both white cubes]? The one blue?
01:11:18	Romina:	Be-
01:11:19	Ankur:	Because you're changing, because that's no blues.
01:11:24	R1:	Uh-huh.
01:11:25	Ankur:	Then the three would be one blue.
01:11:27	R1:	OK. How do, how is it?
01:11:29	Ankur:	There's three spaces the blue can go.
01:11:30	R1:	OK. So there's three spaces the blue can go. Right? It could go-
01:11:33	Ankur:	Uh-huh.
01:11:35	R1:	-here, right? It can go here.
01:11:35	Ankur:	There or there.
01:11:36	R1:	Or it can go here. Agreed? OK.
01:11:37	Ankur:	And then next there's two blues.
01:11:39	R1:	Well, I want you to think about that. And I want you to tell, try to come up with a general way of-
01:11:45	Jeff:	Coming up with that.
01:11:46	R1:	-of of coming up with some kind of a generalization, like you did here. Think about how that works with the towers. Do you understand my question? We may have to say it again.
01:11:54	Jeff:	Yeah. I, I don't.
01:11:55	Romina:	Yeah.
01:11:56	R1:	I think you have a pretty good idea. It might be helpful to have the plastic cubes.
01:12:02	Jeff:	Yeah. My little cousin just got 'em for, um, he was, for his math homework now.
01:12:07	R1:	Is that right?
01:12:07	Jeff:	Yeah. He's in first grade and I was like, wow. That when we started to do that kind of stuff. And I was all excited.
01:12:11	Romina:	Did we play with these in school? I remember.
01:12:13	R1:	Would you like to keep a set here and someone be responsible for keeping-
01:12:16	Ankur:	Romina.
01:12:17	Romina:	I knew that was coming.

Time	Speaker	Transcript
01:12:18	Jeff:	Yup.
01:12:19	R1:	Someone trustworthy and responsible.
01:12:21	Jeff:	She is.
01:12:21	Ankur:	She wouldn't do anything wrong.
01:12:22	R1:	And would remember to bring them. Will you have a chance to think about that over the next couple of weeks?
01:12:27	Jeff:	Yeah.
01:12:28	R1:	OK. So why don't we put these in there?
01:12:31	Romina:	I'll bring these to math class on Monday.
01:12:33	Ankur:	Great!
01:12:35	R1:	You all worked very hard. You've done some very nice mathematics. You know that. When did you start playing with these?
01:12:44	Jeff:	First grade.
01:12:45	R1:	First grade. When did you start building towers? Do you remember?
01:12:47	Jeff:	That's when we-
01:12:47	Romina:	Fourth grade.
01:12:48	R1:	Fourth grade.
01:12:49	Jeff:	-were doing that.
01:12:49	Michael:	Yeah, but the whole grade was doing it, right?
01:12:51	Romina:	Yeah, like-
01:12:52	Michael:	Everyone, not, not just like you know.
01:12:54	Jeff:	Mm. Yeah.
01:12:55	Romina:	Yeah. Everyone was.
01:12:55	R1:	Some of you were building towers in grade three. We have some videotapes-
01:12:59	Jeff:	I've been, I've been building-
01:13:00	R1:	-of some of you building in-
01:13:02	Jeff:	-towers mentally for many years now.
01:13:03	R1:	For many years. OK. How far can you go with block towers?
01:13:08	Romina:	Forever.
01:13:11	Ankur:	As far as you can take us.
01:13:12	Romina:	I think any math would be
01:13:14	Jeff:	I feel that, just, 'cause-
01:13:16	Romina:	If you could, if you broke it down enough, it could be with blocks.
01:13:20	R1:	You think it could all be blocks. OK. So we will see you in two weeks then. And, uh, we'll double check the dates.
01:13:27	Romina:	Yeah.

Time	Speaker	Transcript
01:13:29	R1:	If you think there's something that where you can't be here, you know, let us know.
01:13:31	Jeff:	We'll let you know.
01:13:32	R6:	Do you have phone numbers for us?
01:13:33	Jeff:	Um.
01:13:35	Romina:	We'll tell, we'll tell [your classroom teacher] and I'm sure he knows how to get in touch with you. Or is that not right?
01:13:39	R6:	No. Fine.
01:13:39	R1:	That's a very good way. A very good way. I would love you to write up what you think you know about block towers.
01:13:46	Michael:	We're gonna need notes.
01:13:49	R1:	It would really be nice, you know. If you had to write up an essay, let's say, for a college admission application about what you know about block towers.
01:13:56	Romina:	Oh, wow.
01:13:58	Jeff:	I'd laugh.
01:13:59	R3:	They do sometimes.

June 12, 1998

Time	Speaker	Transcript
00:00:00	–	Jeff, Ankur, Romina, and Brian are sitting at a table with Researcher 1. For about 7 minutes, Researcher 1 talks about the longitudinal study – how students invented notations and how the notations were windows into the students' thinking.
00:06:55	R1:	Well, you know what I thought I would, um, maybe spend a little time here, just chatting with you about, seeing what you want to do from here. Because you've been spending a lot of time with what's called counting or combinatorics problems. You've been exploring combinations. You explored really what is called the binomial theorem. Have you studied that in school yet? [Romina and Jeff shake their heads.] But, but that was, um, when you were looking the coefficients of Pascal's Triangle. Remember that? [Romina nods.] As a review. I don't, I don't- Let's just do a quick kind of review. *a* plus *b*. You remember?
00:07:33	Brian:	Yeah. Yeah.
00:07:33	R1:	You remember that? That's called the binomial, right? You raise it to the 0 power, that's 1. Right. Uh, and then if you raise

Time	Speaker	Transcript
		it to the first power, right?
00:07:42	Jeff:	*a* plus *b*.
00:07:45	R1:	That's $1a$, right? And $1b$. Right? And if you raise it to the second power, that's $1a^2$? Plus $2ab$ plus b^2. You remember that? [Ankur nods.]
00:07:55	Jeff:	Yeah.
00:07:56	R1:	And *a* plus *b* quantity cubed. Right? Do you know what that is? How many *a* cubes?
00:08:04	Jeff:	Two, three. No, one.
00:08:06	Ankur:	Three.
00:08:12	R1:	Three what?
00:08:13	Ankur:	Three *a*-
00:08:15	Romina:	a^2b.
00:08:17	R1:	That's a reasonable conjecture. You can check it later.
00:08:20	Ankur:	Three b^2a.
00:08:22	R1:	Three ab^2.
00:08:24	Jeff:	And, uh-
00:08:25	R1:	One.
00:08:25	Jeff:	*b* squ- *b* to the cubed.
00:08:27	R1:	Does that look at all familiar? [Brian nods.] Well, you know. You can actually multiply *a* plus *b* quantity squared, $a^2+2ab+b^2$ squared times the binomial $a+b$ and actually do all that. But you notice, we talked about some of these coefficients, right?
00:08:48	Jeff:	Yeah. It was 1, 1 1.
00:08:49	R1:	So you can guess what this is going to be, without doing it maybe. Do you see this pattern?
00:08:56	Jeff:	1.
00:08:56	Romina:	1.
00:08:59	Jeff:	1 a^3.
00:09:01	Romina:	4.
00:09:04	R1:	*a* to the third.
00:09:07	Jeff:	*a* to the third *b*. Um, 4.
00:09:12	Ankur:	6.
00:09:12	Jeff:	Wait, 6.
00:09:14	R1:	What's that?
00:09:15	Jeff:	3 and 3.
00:09:17	Brian:	That's a^2b^2.
00:09:19	R1:	a^2b^2.
00:09:20	Jeff:	And, uh, $4ab^3$ and $1b^4$.

Time	Speaker	Transcript
00:09:28	R1:	Of course, those are all pluses in between here.
00:09:31	Brian:	Yeah.
00:09:33	R1:	OK, so, what, what's really important is to spend some reflective time to think about, is, um, what these coefficients. You can see how you generate it this way. You saw that little piece of clip with, um, Researcher 5 and Stephanie. She was actually using block towers to show why 3 and 3 give you six. Why-
00:09:55	Jeff:	//That's what we did.
00:09:55	R1:	//Why 3 a^2. You were working on it. Remember that. Why do $3a^2b$ and $3ab^2$ give you $6a^2b^2$? [Brian nods.]
00:10:04	Romina:	Yeah.
00:10:05	R1:	And you were thinking about that. So if this [$3a^2b$] were a tower, right? What would the tower look like? You can talk about this as a class of towers. a^2b. And this whole class of towers [$3ab^2$]. Right? Can you imagine them in front of you, right? Three of a certain class of towers, and three of a certain class of towers here. And then you're making them taller. And you're going to end up with six of a certain class of towers. Can you imagine this in your head? Do you understand my question? Why don't you talk about that for a minute? This is your view. Before you move on to something else, it might helpful to come back to this idea. This 3 a^2b plus $3ab^2$, what does that mean in terms of towers? And why?
00:11:03	Jeff:	Well, the *ab* cubed is 1.
00:11:05	Ankur:	It's like two of one.
00:11:07	Jeff:	Well, yeah, like so it's a^2b having two of one and one of another.
00:11:18	Ankur:	You're gonna have one light one, so you have two and two. [They are working with drawing of light and dark cubes.]
00:11:19	Jeff:	And then you could have another one. Like this would be dark and those two are light. And then you have the next one on top would be light or dark, and that's it. [Brian nods.]
00:11:36	R1:	Do you agree with that, Brian?
00:11:38	Brian:	Yeah.
00:11:39	R1:	My question is, where did the six come from?
00:11:42	Ankur:	Three. Because you have three with 2 *a*'s and a *b*.
00:11:47	Jeff:	You have, like, the 3 has like that, like 1.
00:11:51	Romina:	It's like that.
00:11:54	Jeff:	One's like this, like this, like that. [Jeff is drawing circles and boxes.] And that one's open. Like there's three different

Time	Speaker	Transcript
		combinations, right? [Inaudible.] You know what I'm saying?
00:12:01	R1:	I'm, I'm not sure. Um, what do the circle and the box represent?
00:12:05	Jeff:	That would be like-
00:12:06	Ankur:	The different colors.
00:12:07	Jeff:	-the different colors.
00:12:08	R1:	OK. So, so what are you using for dark and what are you using for light?
00:12:14	Ankur:	Colors.
00:12:14	Jeff:	[Inaudible.]
00:12:17	Ankur:	When you add one up, it's gonna go with, it's either gonna match, it's gonna match one of the two colors. If it matches with the one that only has one, then it will be a^2, and if it doesn't, then it will- [Inaudible.] 'Cause you have three of that and it's like you have- [Inaudible.]
00:12:29	R1:	OK. But-
00:12:31	Ankur:	You come in here. And you have-
00:12:32	R1:	Let's be very particular and assume a very slow learner.
00:12:34	Ankur:	And you have three with an a and two b.
00:12:35	R1:	Uh-huh.
00:12:36	Ankur:	If you add an a-
00:12:37	Jeff:	You have an a^2b^2.
00:12:39	Ankur:	You can put it on those three, and you get $3a^2b^2$. If you add a b-
00:12:44	Jeff:	A b. You get $3a^2b^2$.
00:12:45	Ankur:	$3a^2b^2$.
00:12:46	Jeff:	And then together, that makes $6a^2b^2$.
00:12:49	R1:	OK. And, and these three are different from these three. [Romina nods.] You can imagine in your head how these three, what these look like, how they are unique.
00:12:57	Jeff:	Yes.
00:12:58	R1:	You see these? [Romina nods.] What do you see when you imagine these in your heads?
00:13:02	Jeff:	That kind of like those.
00:13:04	Ankur:	Two dark and one light.
00:13:04	Jeff:	Uh.
00:13:05	R1:	//Two dark and one light.
00:13:05	Jeff:	//Two dark and one light.
00:13:07	R1:	And how of those there are exactly?
00:13:08	Ankur:	Three.
00:13:08	R1:	Three. You could imagine them. OK? And this would be?

Time	Speaker	Transcript
00:13:12	Romina:	One light.
00:13:13	R1:	One light and two dark.
00:13:15	Jeff:	And that would lead up to this.
00:13:18	R1:	So can you see how the tower images can come to help you make sense of these ideas? So, so this is called the binomial expansion. And, um, these are called the coefficients, these numbers that tell you how many you're going to have. And you could imagine these where there would only be three with exactly one color and two other colors. You can see that in your head, you can visualize that. [Brian nods.] Um, and, and in some ways you can think of these as, uh, a whole distribution of these towers. And you know how many of a particular kind.
00:13:54	R1:	[They turn to a discussion of probability.]

December 14, 1998

Time	Speaker	Transcript
00:00:00	–	[For the first 30 minutes, Michael and Researcher 1 discuss some probability problems, including probability problems related to towers problems. Michael says that if upside-down towers could be considered the same as right-side-up ones, the towers problems would be different from pizzas. But if you impose an orientation, they are the same.]
00:30:56	R1:	And then if that's the case [impose a top and bottom], is it [the towers problem] like pizza or is it not like pizza?
00:31:00	Michael:	It's like pizza.
00:31:01	R1:	Exactly like pizza?
00:31:02	Michael:	Pretty much.
00:31:03	R1:	What's the pretty much mean?
00:31:05	Michael:	It would be it exactly, I mean. Yes, it would be exactly.
00:31:08	R1:	It would be exactly.
00:31:09	Michael:	That phrase what I just said before.
00:31:10	R1:	Now tell me why again. Just for the record.
00:31:12	Michael:	Why? Because [Pause.] Um. Because, like, with this binary, whatever, these numbers.
00:31:23	R1:	Uh-huh.
00:31:24	Michael:	They, they- This 111 is an eight [Sic; it's seven.] in binary.
00:31:27	R1:	Uh-huh.
00:31:28	Michael:	That, that- between the 000 and the 111, there's every possibility

Time	Speaker	Transcript
		you could, you, you give me one; it's in there.
00:31:37	R1:	Uh-huh.
00:31:38	Michael:	For three. So that's, that's what I like about the binary system, that everything's in there.
00:31:43	R1:	Uh-huh.
00:31:43	Michael:	So you take, and each one is a number, so this [000] is zero, this [111] is eight. [Sic; it's seven.] Therefore, there's eight. One. [Pause.] Yeah, eight.
00:31:52	R1:	Uh-huh.
00:31:53	Michael:	'Cause there's every single possibility in there.
00:31:56	R1:	Right.
00:31:57	Michael:	That way, it's, it has to be in there.
00:32:00	R1:	Uh-huh. OK. OK. Um, so. My next question for you. In these eight [the eight binary numbers from 000 to 111; refer to Figure H22.], right, with the pizza problem. You said you start with the 0's, which is no toppings?

000	100
001	101
010	110
011	111

Figure H22. Michael's list of binary numbers from 0 to 7

Time	Speaker	Transcript
00:32:16	Michael:	Uh-huh.
00:32:17	R1:	And all the 1's, which is.
00:32:19	Michael:	Everything on it.
00:32:20	R1:	Everything on it. So if I were to ask you to, to talk about, from none, no toppings, to three toppings. We could talk about no toppings, we could talk about one topping, we could talk about two toppings, or we could talk about three. OK? Can you, can you give me the distribution of those kinds of toppings? How many with none, how many with one, how many with two, and how many with three? Can you tell me what that looks like?
00:32:45	Michael:	Yeah, I could. Um. [Pause.] I could tell you what- you ever hear of, uh- you probably have- that, that triangle?
00:32:56	R1:	Pascal's?
00:32:56	Michael:	Yeah. That has a lot to do with it. You- I mean-
00:32:59	R1:	Show me.
00:33:00	Michael:	I forgot how it goes. It's like one on the side. Tell me if I'm

Time	Speaker	Transcript
		wrong. [Michael draws Pascal's Triangle; row 0 is missing and row 1 is incomplete. Refer to Figure H23.] It's a 1 over here, it's 4, 6, 4, 1. And so on and so on.

				1					
			1		2		1		← Row 2
		1		3		3		1	← Row 3
	1		4		6		4	1	← Row 4

Figure H23. Michael's version of Pascal's Triangle ; row numbers added

Time	Speaker	Transcript
00:33:16	R1:	OK.
00:33:17	Michael:	And that would tell you exactly which pizzas have four toppings, which pizzas have- not four, which pizzas have- Let's say you have a pizza with- you can only have- let's say you have a pizza with one topping only. [He actually discusses the two topping case here.] Therefore you have two possibilities. So only one pizza would have no toppings, one pizza would have both, and two of 'em would have one. One. You know what I'm saying?
00:33:48	R1:	Uh-huh.
00:33:49	Michael:	And this one [row 3], how many toppings is it, three?
00:33:51	R1:	Uh-huh.
00:33:53	Michael:	It's- One has- with all the 0's. One has all the 1's. And there's, uh, six of them-
00:34:00	R1:	Uh-huh.
00:34:01	Michael:	With, with two.
00:34:02	R1:	Uh-huh. How do, how do you distinguish these 3 and these 3?
00:34:06	Michael:	Um. How do I- um- [Pause.] One, two, three. Oh, the, the- if you have 3 toppings, like this [first 3 in 1331] has mushrooms and peppers. [Michael writes *mp* under the 3.] Um. And- No. [Michael crosses out the *mp*. Pause.] Now this is with two toppings or three? [Michael indicates row 3.] This one has three. Right?
00:34:33	R1:	OK, this is-
00:34:34	Michael:	This one is with three, with-
00:34:35	R1:	This is-
00:34:36	Michael:	-with three toppings. Right?
00:34:37	R1:	OK, you have eight.
00:34:38	Michael:	Or this is actually with two toppings, because you could- plain would be a topping, you know.

Time	Speaker	Transcript
00:34:42	R1:	OK, are we talking about this row here?
00:34:43	Michael:	Yeah, this row right here. [Michael circles 1331.]
00:34:44	R1:	OK, this row you have 1, 3, 3, and 1.
00:34:46	Michael:	Yeah.
00:34:47	R1:	For a total of eight. Right? Eight?
00:34:49	Michael:	Yeah.
00:34:50	R1:	One, two, three, four, five, six, seven, eight. Right?
00:34:53	Michael:	All right. Let's say you had to have toppings. So it's three toppings.
00:34:55	R1:	So this 1 [first 1 in row 3] tells you- which one would this be?
00:34:58	Michael:	I guess it, it would- The one? It would be this one. No, no, no, it would be this one [000].
00:35:03	R1:	OK, that would be no- that would be, that doesn't have any toppings.
00:35:05	Michael:	Yeah.
00:35:06	R1:	And this one [last 1 in 1331] could be- has-
00:35:08	Michael:	Has-
00:35:08	R1:	All toppings, right?
00:35:09	Michael:	-has, has two toppings.
00:35:11	R1:	The last one.
00:35:12	Michael:	Yeah, the last one has two toppings.
00:35:13	R1:	OK. The last one has two toppings?
00:35:14	Michael:	Yeah.
00:35:15	R1:	What's the 111?
00:35:17	Michael:	Um, the 111 is, uh, or actually- cause it- oh, no, all right, that one has three.
00:35:24	R1:	OK. So this [first 1 in 1331] has one top- this has //no toppings-
00:35:26	Michael:	//Nothing.
00:35:27	R1:	This [last 1 in 1331] has three//
00:35:27	Michael:	//Three.
00:35:28	R1:	So if this has none, this [first 3 in 1331] would have to be how many toppings?
00:35:30	Michael:	Has to have one.
00:35:33	R1:	One.
00:35:33	Michael:	The next one has two.
00:35:34	R1:	Two. OK.
00:35:35	Michael:	So zero, one, two, three.
00:35:36	R1:	OK. So if this is, this [first 3 in 1331] has one, has one topping-
00:35:38	Michael:	One topping.
00:35:40	R1:	OK. So what would that be?

Time	Speaker	Transcript
00:35:41	Michael:	Where are they?
00:35:41	R1:	Uh-huh.
00:35:42	Michael:	One, two, three. [Michael checks off 001, 010, and 100 in his list of binary numbers.]
00:35:45	R1:	OK. And then this one has two toppings.
00:35:47	Michael:	One, two, three. [Michael checks off 011, 101, and 110.]
00:35:49	R1:	OK.
00:35:50	Michael:	They're all in there.
00:35:51	R1:	OK. I follow what you said so far.
00:35:53	Michael:	Uh-huh.
00:35:54	R1:	Now. Could we pursue this a little bit more or are you getting tired?
00:35:56	Michael:	No, you could go on.
00:35:56	R1:	Great. OK. Now let's talk about this row is [row 4 of Pascal's Triangle].
00:36:02	Michael:	All right. This row is, uh, four toppings. This one has zero, this one has one, this one has two toppings, this one has three toppings, this one has four. [Michael writes 0, 1, 2, 3, and 4 underneath 1 4 6 4 1.]
00:36:09	R1:	OK. Now. So you, you think you could also find those for me if I asked you to.
00:36:17	Michael:	Um. Yeah.
00:36:17	R1:	OK. Just, just for, for fun, all right.
00:36:25	Michael:	You want me to find them for you?
00:36:26	R1:	Write them out. It shouldn't take you long.
00:36:28	Michael:	All right.
00:36:29	R1:	So I want you to have both lists, I want you to have the three topping lists-
00:36:31	Michael:	All right, so how much is this?
00:36:32	R1:	-and-
00:36:32	Michael:	This is ten. Six, sixteen, so. That's not, that's not that much. [Michael writes 000 and 001.] All right, so, 0 1.
00:36:43	R1:	No, wait. You want to have four.
00:36:44	Michael:	Oh, yeah. [Michael writes 0's in front of 000 and 001. Michael continues to write 4-digit binary numbers.] That's one, two, three [0011], uh, I can't remember binary. Uh, this was four. [Michael writes 0100.] Five [0101]. Six [0110]. Seven [0111]. Eight [1000]. [Michael writes 1001 twice.] No, I got- [Michael crosses out the second 1001 and writes 1010.] That's ten, right? [Michael writes 1011.]

Time	**Speaker**	**Transcript**
00:37:31	R1:	You did eight before. One, two, three, four, five, six, seven, eight. Wait.
00:37:35	Michael:	There's zero.
00:37:36	R1:	Zero, one, two, three, four, five, six, seven, eight, nine. [Researcher 1 points at the numbers in the list, stopping at 1010.]
00:37:36	Michael:	That's ten. [Michael points to 1010.] Eleven. [Michael writes 1110.] No, that's not it. [Michael crosses out 1110.] This one's eight. [Michael writes 1.] What am I on now, eleven? Or twelve? Twelve. [Michael writes 1010. Michael points to 1010.] Ten. [Michael changes 1010 to 1100 and then writes 1101, 1110, and 1111.] I hope that's sixteen. [Michael points to each number in the list in turn, enumerating seventeen numbers.]
00:38:24	R1:	You counted this one. [Researcher 1 points to a number that was crossed out.]
00:38:25	Michael:	Oh. [Michael counts again.] Yeah, sixteen. [Refer to Figure H24.]

0000	0100	1000	1100
0001	0101	1001	1101
0010	0110	1010	1110
0011	0111	1011	1111

Figure H24. Binary numbers from 0 to 15

00:38:31	R1:	OK. There's a way to check it just by checking these back from binary to the regular numbers, right? That's what you were doing?
00:38:42	Michael:	Yeah, those, those are, those are how you express the numbers.
00:38:44	R1:	That's how, that's how you were checking it. So let's just do a quick random check. So what one would this one be and why? [Researcher 1 points to 1101.]
00:38:52	Michael:	That's [Pause.] twelve. No, not twelve. Is it?
00:38:57	R1:	What-
00:38:58	Michael:	Eight plus four?
00:38:59	R1:	Is twelve-
00:38:59	Michael:	Yeah, that's twelve.
00:38:59	R1:	-and one is-
00:39:00	Michael:	Oh, there's a one there? I didn't see the one. That's thirteen.
00:39:01	R1:	It's thirteen. OK. So this [1101] would be thirteen?
00:39:04	Michael:	Uh-huh.

Time	Speaker	Transcript
00:39:05	R1:	Is that, is that right?
00:39:06	Michael:	Yeah.
00:39:07	R1:	And this is a- [Researcher 1 points to 1110.]
00:39:07	Michael:	Fourteen. [Researcher 1 points to 1111.] And fifteen.
00:39:10	R1:	[Pause.] OK. Where's sixteen?
00:39:16	Michael:	There is no sixteen. You can't do it with four.
00:39:18	R1:	Good.
00:39:19	Michael:	There's sixteen possible.
00:39:20	R1:	Now this is my question to you. OK? You know one of the characteristics of this triangle, um, that if you, let's say, were to, um, let's go here. This 1 and this 3. [Researcher 1 indicates the 13 from 1331.]
00:39:39	Michael:	Equal that four. [Michael indicates the first 4 from 14641.]
00:39:40	R1:	Equals this four.
00:39:41	Michael:	That's how I made it.
00:39:42	R1:	Right. That's how you made it. Now could you explain to me-
00:39:47	Michael:	Why?
00:39:48	R1:	-with your pizzas, how these, which tell you how many of them have- Here you're making it with three toppings. [Researcher 1 points to row 3 of Pascal's Triangle.] Right?
00:39:56	Michael:	Uh-huh.
00:39:57	R1:	And, and these have none and these have one. [Researcher 1 points to the first 1 and 3 of row 3.] Right? Now you're adding an extra topping. Right? OK. And this 4 [the first 4 in row 4], right? Tells you how many you're going to have, right?
00:40:13	Michael:	With, with, uh, one.
00:40:14	R1:	With exactly one topping when you can select from four toppings. Right? So how, how, how does that work? Can you explain to me how this- the one that you get here, right? [Researcher 1 points to first 1 in row 3.] Which means-
00:40:27	Michael:	None.
00:40:28	R1:	So, so, so you, you know which, you know which pizza this is, don't you? [Researcher 1 points to the first 1 in row 3.]
00:40:32	Michael:	Yeah. That is, with that [000].
00:40:36	R1:	OK, you know that, right? And you know which pizzas these are, you told me that. [Researcher 1 points to the first 3 in row 3.]
00:40:39	Michael:	Yeah, I know what those are.
00:40:41	R1:	Right?
00:40:41	Michael:	One, two, three.
00:40:42	R1:	How, how does that, that addition of that other topping-

Time	Speaker	Transcript
00:40:45	Michael:	All right. Uh.
00:40:46	R1:	-end up with that. Would you explain that to me?
00:40:48	Michael:	Let me write, let me write down those-
00:40:49	R1:	Well, start, can we start another piece of paper?
00:40:51	Michael:	-those three pizzas.
00:40:52	R1:	Sure.
00:40:53	Michael:	[Michael starts to write.] Uh, it's one topping. [Michael writes 100, 010, and 001.] All right. Um, the reason why it's 4. The next time around.
00:41:06	R1:	Uh-huh.
00:41:06	Michael:	With one topping, would be 'cause, in those four, you get, let's just say these are pizzas again. [Michael writes 100, 010, and 001 again.] In those four, you're going to have another, I guess, either a 1 or a 0 added on to it.
00:41:23	R1:	Uh-huh.
00:41:23	Michael:	On, I guess on this side [the left side]. And this one I guess has to be zero. [Michael adds 0's to the left of the three binary numbers, giving 0100, 0010, and 0001.] Has to be zero, has to be zero, but that, that pizza with nothing on it, could have a 1 now. [Michael writes 1000.]
00:41:38	R1:	Uh-huh.
00:41:38	Michael:	Now it- And that means, that, that's my, my explanation.
00:41:40	R1:	OK, so you're not adding any toppings-
00:41:42	Michael:	On the ones with it //because they already have it.
00:41:43	R1:	-because //they already have a topping.
00:41:44	Michael:	But the one with that, with that, without anything, has one.
00:41:47	R1:	OK, so let's do another one, just to be sure I understand. So suppose, can you also show me then how the 3 plus the 3 gives me the 6?
00:41:54	Michael:	Um, all right, uh.
00:41:58	R1:	Take another piece of paper. I find this fascinating.
00:42:00	Michael:	I, I, I'll explain, let me explain with the 1 and the 3, first cause it's-
00:42:03	R1:	Sure.
00:42:03	Michael:	You had-
00:42:04	R1:	Absolutely.
00:42:05	Michael:	Um, I guess, these three pizzas are the root pizzas,
00:42:12	R1:	Uh-huh.
00:42:13	Michael:	And this one I guess also. So when you add some- when you do the next time around, the next one is zero. It, it's- You take the- [Pause.] Uh, it- I have, I know it in my head, I see it in my head.

Time	Speaker	Transcript
00:42:29	R1:	I know.
00:42:30	Michael:	I can't like say it.
00:42:31	R1:	Yes you can.
00:42:31	Michael:	-Cause I-
00:42:31	R1:	I mean, you always, you always say do say it, Michael. Go ahead.
00:42:32	Michael:	Um.
00:42:33	R1:	Go for it.
00:42:34	Michael:	It's those four pizzas-
00:42:37	R1:	Uh-huh.
00:42:37	Michael:	-that will give you the one, the next one with only one. Um. [Pause.]
00:42:44	R1:	Right. OK.
00:42:45	Michael:	Like, all right, um.
00:42:46	R1:	I understand that. And you just explained really well, and you told me where they came from.
00:42:50	Michael:	They came from-
00:42:51	R1:	You, you said this one. [Researcher 1 indicates row 3 of Pascal's Triangle.]
00:42:52	Michael:	This one.
00:42:52	R1:	This one had the, all the 0's, no toppings.
00:42:54	Michael:	Yeah.
00:42:54	R1:	You had to add 1. Right? And these already had a topping, you didn't have to add any.
00:42:59	Michael:	Yeah.
00:43:00	R1:	And so all of these are the ones with 1. [Researcher 1 indicates the first 4 in row 4.] And they, you showed me where they came from.
00:43:03	Michael:	All right.
00:43:04	R1:	That's what you told me. Now I'm saying to you, here we have those pizzas, right, that have one topping? [Researcher 1 indicates the first 3 in row 3.]
00:43:12	Michael:	And these are the-
00:43:13	R1:	And two toppings. [Researcher 1 indicates the second 3 in row 3.]
00:43:13	Michael:	There's three. All right. I'll explain that.
00:43:15	R1:	Good.
00:43:16	Michael:	These three. The original three. The ones that got added, that didn't get anything added, went into this category. [Michael points to the first 4 in row 4.] The ones that had another topping, got another topping added, went into this one. [Michael indicates

Time	Speaker	Transcript
		the 6 in row 4.] And these 3 [the second 3 in row 3] originally didn't have anything added to them. So they still have two.
00:43:31	R1:	Uh-huh.
00:43:31	Michael:	So these guys, these, these three, these three over here [first 3 in row 3].
00:43:34	R1:	Uh-huh.
00:43:35	Michael:	Got a topping.
00:43:36	R1:	Uh-huh.
00:43:36	Michael:	So they, they moved up, I guess.
00:43:38	R1:	Uh-huh.
00:43:39	Michael:	And they stayed, these, these guys over here [second 3 in row 3] they just stayed where they were.
00:43:43	R1:	Uh-huh. That's interesting.
00:43:44	Michael:	And this, so on for the next one.
00:43:46	R1:	Very interesting. Michael, I have a real, real special request from you. Can you write this up?
00:43:56	Michael:	Write this, this //one right here?
00:43:57	R1:	//The explanation of why the-
00:43:58	Michael:	Why-
00:43:58	R1:	-Pascal's Triangle //works.
00:43:59	Michael:	//One plus three equals four.
00:44:01	R1:	Yeah. Take another row, and would you write it up for me? 'Cause I want to share it with some people.
00:44:05	Michael:	When, when's the next time I'll see you?
00:44:07	R1:	Oh, I'm going to see you on the fifteenth, but I want your email. [A discussion of schedules and emails follows.]

APPENDIX I: TRANSCRIPTS OF 1999 SESSIONS

January 22, 1999 World Series Session

Time	Speaker	Transcript
00:00:00	–	[The students are presented with the World series problem. They work on it for about 55 minutes. Michael remarks to Ankur that this "obviously" has something to do with Pascal's Triangle. He notes that the A's and B's that the other students are using to indicate winning games for the two World Series teams are like 0's and 1's.]
00:55:59	R1:	OK, um. Don't go away, Michael.
00:56:02	Michael:	I won't.
00:56:02	R1:	Because one of the other questions I wanted to ask is that Michael showed me something last time, um, when we talked ourselves, that I guess you all didn't hear and I wasn't sure how much of it he was going to share with this. And I guess a little bit of it I connect to from what you talked about the last time. But, but what you explained to me, Michael, I remember the last time, is- you see that addition of 10, you know, the 6 and 4?
00:56:28	Michael:	Yeah. No, no. Why do you add them together? Then why is-
00:56:28	R1:	Or the 20. Why you add them together and you had an explanation.
00:56:32	Michael:	Yeah.
00:56:32	R1:	And you were using pizzas to explain it to me. Any of you ever heard this before?
00:56:37	Ankur:	Yes.
00:56:37	R1:	You were talking about toppings on pizzas.
00:56:39	Romina:	Oh, yeah, I remember that.
00:56:40	R1:	Is that right?
00:56:42	Michael:	Why are they adding on-
00:56:42	R1:	And so, um, you took a row there-
00:56:45	Michael:	Yeah, uh-
00:56:47	R6:	[To Jeff, who is leaving.] I'll see you next week.
00:56:49	Jeff:	Yes, I'm very sorry.
00:56:50	R1:	At twelve.
00:56:51	Jeff:	Twelve o'clock. We're on. Twelve or twelve-thirty?
00:56:54	R1:	Uh, twelve. We'll have lunch.
00:56:55	Jeff:	All right.
00:56:56	R6:	Good luck.
00:56:57	R3:	Good luck in your game.

Time	Speaker	Transcript
00:56:58	R1:	Yeah.
00:56:58	Jeff:	Thank you very much.
00:56:59	R1:	Thank you.
00:57:00	Michael:	All right. Um.
00:57:02	R1:	Do you remember?
00:57:03	Michael:	Yeah, I remember. All right. You have, what? Three toppings and this one has four?
00:57:05	R1:	OK, so which one is this?
00:57:06	Michael:	Three toppings.
00:57:07	R1:	If you're thinking of pizzas.
00:57:08	Michael:	[Michael points to row 3 of Pascal's Triangle – 1 3 3 1.] This is like a three-topping pizza. There will be one with, uh- [Pause. Michael knocks on the board.]
00:57:15	Ankur:	Plain?
00:57:15	Romina:	Plain.
00:57:16	Michael:	Plain, right? Three with just two toppings, three with-
00:57:19	Ankur:	Just one.
00:57:20	Michael:	-uh, just one topping, //three with just two.
00:57:21	Ankur:	//Three with-
00:57:22	Michael:	And one with all toppings. And when you have that one pizza, what- if you don't add on a, a topping, it'll still stay in that zero place. But they you add, you add a mush- if you do add a topping, that, those ones will become into four different pizza pies, where- it, you-
00:57:43	R1:	Show me the 1 and the 3 giving you the 4, in terms of pizzas. Can you tell me that?
00:57:47	Michael:	All right, all right. Wait. Hold on.
00:57:50	R1:	Sure. Just tell me that in pizza toppings.
00:57:51	Michael:	In pizza toppings.
00:57:52	R1:	How 1 plus 3 equals 4.
00:57:53	Michael:	I'm trying to think. I, I had it. Last time I talked to you, I had, had it so good.
00:58:00	Ankur:	[Ankur laughs.] Why don't you just roll the tape?
00:58:02	Michael:	Yeah, you got it on tape?
00:58:03	R1:	Well, why don't you help him figure it out? Yeah. Let's, let's go back and, and think of what that means. Can you show me, can you show the 1 and the 3 being a 4, so everyone knows what we're focusing on, Michael?
00:58:12	Michael:	What, what are you talking about?
00:58:13	R1:	//Draw the lines.

Time	Speaker	Transcript
00:58:13	Ankur:	//How the one-
00:58:14	R1:	[Michael draws the lines.] OK. Now, why- I'm asking you why that works, with pizzas. [Refer to Figure I1.]
00:58:20	Michael:	All right. You're going to add a topping to every single pizza on there, right? There's going to be twice as many pizzas. But these three pizzas- three of them got a topping, went there [to the right], and three of them didn't, went there [to the left]. One of them had a topping, right there, and one of them didn't, went there. [Romina and Ankur both nod.] 'Cause these three pizzas are going to turn into six pizzas. Now I got it, right? And three of them, which had three toppings and gained another one, are in the next category [indicated by the arrow from 3 to 6]. They moved a step up. These guys stayed in the same place 'cause they didn't get one [indicated by the arrow from 3 to 4].

Figure I1. Michael indicates addition from row 3 to row 4 on Pascal's Triangle

Time	Speaker	Transcript
00:58:54	Brian:	Yeah.
00:58:55	Michael:	That's why they had-
00:58:56	Ankur:	Uh-huh.
00:58:58	Michael:	Did I get that, like the last time?
00:58:59	R1:	I don't know. Brian, does that make any sense to you?
00:59:01	Brian:	I grasped it. Yeah.
00:59:01	Michael:	This pizza-
00:59:02	R1:	Can you tell it to me, then, with some toppings? Make up some toppings and see what it says.
00:59:05	Michael:	All right, you got, um- cheese pizza, no, plain pizza, and you got a pizza with mushrooms. You're going to add a topping. [Slight pause.] Is that right? [Another pause, as Michael taps his pencil.] You have three toppings in this one [row 3 of Pascal's Triangle –1 3 3 1]. You have, uh, you can have one plain, one with peppers and mushrooms and sausage. And then you're going to add another topping. I don't know, more cheese? Uh, you're gonna, you're either gonna add more cheese to, to one of them, either, you're going to, you're going to add another topping. [Ankur laughs.] You're gonna, like- the possibilities are going to

Time	Speaker	Transcript
		double because some of them you're going to add, you're going to add cheese to them, and some of them you're not. Like, you're going to have this one pizza, it's plain. The one that's not going to get another topping is gonna go here. The one that's going to get the cheese is going to go here. And three of these which only had the mushrooms, there are going to get cheese. They're going to go here.
01:00:08	R1:	Wait a minute. Is cheese the plain?
01:00:10	Michael:	Cheese, well, extra cheese.
01:00:12	Ankur:	Extra cheese.
01:00:12	R1:	OK.
01:00:13	Michael:	Anchovies. Whatever you want.
01:00:13	R1:	That helps a lot.
01:00:14	Michael:	These, these with the mushrooms. If they get anchovies, they're gonna be three different, three different pizzas with two toppings. They're gonna go here [toward the 6]. If they don't, they're still going to be three different pizzas-
01:00:24	R1:	Uh-huh.
01:00:24	Michael:	-that have one topping [following the arrow to the 4]. So, you understand? Like, um, when you add a, when you add a topping or whatever, uh, this many is gonna, is gonna like, if you think of it as steps, like one, two- this one's going to, like, move up a step 'cause it's-
01:00:48	Brian:	Get a topping.
01:00:48	Michael:	-either gonna get one or not, all right? So there's twice as many possibilities now. And three of them will have, and three then will not have, that extra one.
01:00:59	R1:	Does that make sense?
01:01:00	Romina:	Yeah, we understood this.
01:01:01	Brian:	You got something, you move up one.
01:01:02	Michael:	I, I don't know if that makes sense.
01:01:03	Brian:	If you don't, you join up with the one of those over there.
01:01:07	R1:	Could you say that again, Brian? That was good. One more time. If you've got something-
01:01:11	Romina:	If you move up.
01:01:11	Ankur:	If you got something, you go on.
01:01:13	Brian:	If you get something, you move up [to the right], and if you don't, then you just join the other one [add with the number to the left].
01:01:16	Romina:	You join in.

Time	Speaker	Transcript
01:01:16	Michael:	You stay, you stay.
01:01:17	Ankur:	That's what you were trying to say?
01:01:18	Michael:	[Laughter.] Yeah, I, but then-
01:01:20	Romina:	You could have helped him out there, Brian.
01:01:22	Brian:	That wouldn't have done anything.
01:01:24	Michael:	Now with the, with the 1 3 3 1, that circled one is, I guess you win those three games in a row. [They return to a discussion of the World Series problem.]

January 29, 1999 Interview

Time	Speaker	Transcript
00:00:00	–	As the tape begins, we see a test pattern and hear the voice of Researcher 10.
00:00:01	R10:	'Cause I remember the binary numbers.
00:00:02	Michael:	If you had a, if you had a choice of zero toppings.
00:00:04	R10:	Uh-huh.
00:00:05	Michael:	There's only one pizza you could make.
00:00:07	R10:	Uh-huh.
00:00:08	Michael:	If you have a choice of, uh, one topping, you can have a pizza with the topping or without it.
00:00:13	R10:	Uh, so this is 1, 2, and 1? [The test pattern goes away. We see that Researcher 10 and Michael are looking at a copy of his email.]
00:00:16	Michael:	This is- you have a choice of two toppings. No. Yeah, two toppings. There's how many pizzas you made with as many toppings-
00:00:23	R10:	This is, this is with two toppings? [She writes "2 toppings."]
00:00:26	Michael:	Yes. Two toppings. I might have to correct myself at the end.
00:00:29	R10:	Yeah.
00:00:30	Michael:	Um.
00:00:31	R10:	We had 1 1 up here.
00:00:33	Michael:	Right.
00:00:35	R10:	OK? I don't understand this 1 up here [at the top of Pascal's Triangle]. Where does it come from?
00:00:37	Michael:	That's, you don't have- All right, you go to a pizza place. "I want a pizza." You don't have any toppings to choose from. You can only make one pizza. A pizza without anything.
00:00:45	R10:	Um. So this is like a 0.

Time	Speaker	Transcript
00:00:48	Michael:	Yeah, it's a 0. Like everything in this column [the leftmost number in each row of Pascal's Triangle] has no toppings. Everything in this column has-
00:00:53	R10:	It's just a plain cheese pizza.
00:00:55	Michael:	Yeah, a plain cheese. Now, when the pizza guys says, "all right, you can put a mushroom on that pizza."
00:01:01	R10:	Oh, so this is like the first one.
00:01:03	Michael:	You can either have a plain one or a mushroom one. That's it. You can't have anything else.
00:01:06	R10:	OK. So this [first 1 in row 1] is without mushrooms and this [second 1] is with mushrooms.
00:01:09	Michael:	With. Now he goes, "All right, I got mushrooms and peppers." You could, you could still order that cheese pizza.
00:01:13	R10:	Yeah.
00:01:14	Michael:	You could order one with mushrooms or one without- or one with peppers.
00:01:18	R10:	Oh.
00:01:19	Michael:	Or you can have it with both.
00:01:20	R10:	I understand.
00:01:20	Michael:	And there's two with one topping.
00:01:23	R10:	OK. So these guys here [the 2 in row 2]. One of them is the mushroom and-
00:01:25	Michael:	They're two different one, but they have the same-
00:01:29	R10:	OK, I got it. OK. And this one down here?
00:01:30	Michael:	All right. The first, the 1 right there?
00:01:32	R10:	1 3-
00:01:33	Michael:	This is with three toppings.
00:01:33	R10:	-3 1 is three toppings.
00:01:36	Michael:	The 1 will have no, that one always on the left-
00:01:39	R10:	All right, so this is mushrooms and peppers and sausage.
00:01:41	Michael:	Yeah.
00:01:41	R10:	OK.
00:01:42	Michael:	The 1 on the left would have no toppings on it.
00:01:47	R10:	OK.
00:01:47	Michael:	The one right over it [the next number in the row, 3] is three different pizzas, one with a different topping a different topping on each one.
00:01:52	R10:	Which is one topping, right?
00:01:53	Michael:	Yeah.
00:01:53	R10:	OK.

Time	Speaker	Transcript
00:01:53	Michael:	And the third, the next one is, the pizzas with two, two toppings, mushroom and peppers.
00:02:00	R10:	Oh, so it could be these, these, or these.
00:02:02	Michael:	Yeah. Or- yeah.
00:02:03	R10:	I got it.
00:02:04	Michael:	Last one is with all three.
00:02:05	R10:	OK. I understand that. And I'm sure I could understand the way it kept going from there. [Michael nods.] You know what I found the most interesting? When I was reading yours [your email]?
00:02:14	Michael:	Uh-huh.
00:02:14	R10:	Is that underlined stuff. [Michael nods.] How you went from- how you, how you got-
00:02:14	Michael:	Like this stuff right there. [Refer to Figure I2.]

1
1 1
1 **2** 1
1 **3** 3 1
1 4 6 4 1
1 5 10 10 5 1

Figure I2. Michael's email diagram of the addition rule

Time	Speaker	Transcript
00:02:23	R10:	Yeah, it was this thing here.
00:02:24	Michael:	Yeah.
00:02:25	R10:	And, and it was when you underlined the 1 here.
00:02:29	Michael:	Uh-huh.
00:02:30	R10:	And you underlined the 2 here. And somehow you got to the 3 here.
00:02:35	Michael:	Yeah. All right.
00:02:36	R10:	I don't completely understand that.
00:02:37	Michael:	This 1, um, has no toppings on it, all right?
00:02:43	R10:	This 1 at the end?
00:02:44	Michael:	Has no toppings on it.
00:02:44	R10:	OK. OK.
00:02:45	Michael:	When that pizza man says, "All right, yeah. I give you, I'll give you, uh, another topping to choose from."
00:02:52	R10:	Uh-huh.
00:02:52	Michael:	All right. That plain pizza you just made, you could either put that mushroom on it, or you can't.

Time	Speaker	Transcript
00:02:56	R10:	Oh, OK.
00:02:57	Michael:	Let's say you already made that one pizza.
00:02:58	R10:	Yeah.
00:02:58	Michael:	He goes, "Oh. I just got some mushrooms." You could either put it on there, or don't. So then-
00:03:02	R10:	Oh.
00:03:03	Michael:	You could make two different pizzas with that.
00:03:04	R10:	OK.
00:03:05	Michael:	And the one without it is that 1, and the one with the mushroom that you just put on, is part of that 3.
00:03:10	R10:	Oh, wait, wait just a minute. I almost lost you. Uh, so say that again.
00:03:15	Michael:	You made that one pizza already, with no, no toppings.
00:03:16	R10:	This, this one up here.
00:03:16	Michael:	Yeah.
00:03:17	R10:	OK.
00:03:18	Michael:	You made it already.
00:03:18	R10:	Uh-huh.
00:03:19	Michael:	He just says, "I got, I just got some mushrooms. You can either put it or not. Put it on there or not."
00:03:24	R10:	Uh-huh.
00:03:24	Michael:	If you do put it on there, it's gonna have two top- it's gonna have one topping.
00:03:28	R10:	Uh-huh.
00:03:28	Michael:	So it will be in this category. [Michael points to row 2 of Pascal's Triangle.]
00:03:30	R10:	Oh. I see.
00:03:31	Michael:	If you don't, it's still gonna be in the same category or whatever you call it.
00:03:33	R10:	Oh. OK. OK. And so this pizza, you don't put anything else on top of it.
00:03:38	Michael:	It will be, it'll, it'll stay where it is, or, or-
00:03:42	R10:	Moves- It moves on down this way.
00:03:43	Michael:	Yeah. If you, if you do put something on it, it moves to the right. And-
00:03:48	R10:	Oh.
00:03:48	Michael:	-the same thing with the two. If you don't put anything on it, it's still gonna have two toppings, so it's gonna stay in, right there. That's why you add these two together.
00:03:56	R10:	Oh. Well, well, then, where does this come from? [Researcher

Time	Speaker	Transcript
		10 circles the second 3 in row 3.]
00:04:00	Michael:	That's from the 2, the 2 that had one, if you decided to put the mushrooms on it.
00:04:06	R10:	Uh-huh.
00:04:06	Michael:	It's gonna be two of them that go there and have, and have two toppings now.
00:04:10	R10:	So they'll have two toppings now.
00:04:11	Michael:	And this one which has both toppings on it, if you don't put that mushroom on there, it's gonna stay where it is.
00:04:17	R10:	Oh. OK. So this one [2 from row 2] got another topping.
00:04:21	Michael:	So it moved up, right, in the world of pizzas.
00:04:23	R10:	And it went that way [to the right]. If it didn't get another topping?
00:04:26	Michael:	It would have gone there [to the left]. If both of them didn't get another topping.
00:04:30	R10:	OK. So with no top- no more.
00:04:33	Michael:	Yeah. If it doesn't get a topping, it goes right. If it does, it goes left.
00:04:37	R10:	Left.
00:04:37	Michael:	But since there's-
00:04:38	R10:	Now if it got, if it got-
00:04:39	Michael:	If it gets one, it goes right.
00:04:41	R10:	Uh-huh.
00:04:41	Michael:	If it doesn't, it goes, it goes left. But it's not really-
00:04:43	R10:	That happens here too?
00:04:45	Michael:	Yeah. Like it, it- that one pizza. Um, you, if you put a topping on it, it would be part of this category.
00:04:54	R10:	OK.
00:04:54	Michael:	If you didn't, it would be part of this.
00:04:55	R10:	And so, and so that's the reason that this one got bigger?
00:04:58	Michael:	Yeah.
00:04:59	R10:	Help me to understand that.
00:05:00	Michael:	Well, um. The reason why you add, would add these two together.
00:05:04	R10:	Uh-huh.
00:05:05	Michael:	Because obviously with these three pizzas, you're gonna have six of them then.
00:05:12	R10:	Uh-huh.
00:05:13	Michael:	Six, you're gonna- By adding one, you're gonna- the one you have now, plus another three with it.

Time	Speaker	Transcript
00:05:18	R10:	OK, you're saying this 2 and this 1.
00:05:20	Michael:	Yeah. Out of, out of those three pizzas, you can make six more with another topping.
00:05:24	R10:	OK.
00:05:25	Michael:	You understand that? 'Cause-
00:05:26	R10:	Yeah.
00:05:26	Michael:	If you have a pizza with something, you can make two more pizzas with it. You can make one with or one, or one that stayed the same.
00:05:33	R10:	OK. OK. So these three that you started with.
00:05:35	Michael:	Yeah.
00:05:36	R10:	When you move to the next row.
00:05:38	Michael:	There's gonna be twice as many of those pizzas.
00:05:40	R10:	Because you've added this other topping.
00:05:41	Michael:	Yeah.
00:05:41	R10:	OK.
00:05:42	Michael:	Um.
00:05:43	R10:	OK. And so-
00:05:44	Michael:	When you, when you double it-
00:05:45	R10:	OK. Where the six? I'm confused. Now you've lost me.
00:05:47	Michael:	All right, the six. You have three pizzas, right?
00:05:49	R10:	Yeah. Right here. These two and this one, OK.
00:05:51	Michael:	Uh-huh.
00:05:52	R10:	Show me the six down underneath.
00:05:54	Michael:	Uh, all right.
00:05:55	R10:	'Cause I see a lot.
00:05:56	Michael:	The six underneath would be, um, two from this one, two from this one- three from this one- [He repeats.] Two from this one, three from this one, and this one. [Refer to Figure I3.]

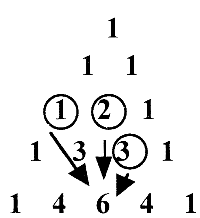


Figure I3. Michael discusses the question: Where did the 6 come from?

00:06:09	R10:	That's one?

Time	Speaker	Transcript
00:06:09	Michael:	One plus three plus two. Six.
00:06:11	R10:	OK. Show me that again.
00:06:12	Michael:	You'll have one from this one.
00:06:14	R10:	Uh-huh.
00:06:15	Michael:	All of these.
00:06:16	R10:	Oh.
00:06:18	Michael:	And two from these.
00:06:19	R10:	Oh. Oh. OK. So from, you're saying from these three-
00:06:25	Michael:	They disperse into these three categories.
00:06:27	R10:	OK. But they're, but they're not all of them.
00:06:29	Michael:	No, 'cause, um, this one-
00:06:32	R10:	OK. OK. From these three, you say I've got to make six.
00:06:37	Michael:	Uh-huh.
00:06:39	R10:	OK. Now show them to me again. I think I almost got it.
00:06:41	Michael:	All right. There's one of this.
00:06:42	R10:	Uh-huh.
00:06:44	Michael:	All of these.
00:06:45	R10:	And what are all these?
00:06:47	Michael:	It'll be the- it'll be this one with a topping-
00:06:50	R10:	Uh-huh
00:06:50	Michael:	-and these two without one.
00:06:52	R10:	OK.
00:06:53	Michael:	And two that came from this guy.
00:06:55	R10:	And what were they?
00:06:56	Michael:	It's these two that both had another topping added.
00:06:58	R10:	That have another topping. OK. And so that's where the six came from?
00:07:02	Michael:	Uh-huh.
00:07:03	R10:	OK. Then, what about the rest of them?
00:07:05	Michael:	All right. Well, like before-
00:07:07	R10:	Because in, in total, don't we have eight?
00:07:09	Michael:	Yeah.
00:07:09	R10:	OK.
00:07:10	Michael:	So then there's a 1-
00:07:12	R10:	Uh-huh.
00:07:12	Michael:	-that, that got an extra topping.
00:07:14	R10:	Uh-huh.
00:07:14	Michael:	So now it's part of the "two" category.
00:07:16	R10:	Uh-huh.
00:07:17	Michael:	It has two toppings now.

Time	Speaker	Transcript
00:07:19	R10:	Uh-huh.
00:07:19	Michael:	This one [the last 1 in row 2] has two, and it didn't get one.
00:07:21	R10:	Uh-huh.
00:07:21	Michael:	So it goes here [to the right].
00:07:23	R10:	Uh-huh
00:07:23	Michael:	So now there's 3 with two.
00:07:25	R10:	Uh-huh.
00:07:26	Michael:	And that, the one that, that didn't get a- that got, that got a topping on this one-
00:07:30	R10:	Uh-huh.
00:07:31	Michael:	-has now three toppings.
00:07:32	R10:	Oh. OK. So you're saying that this [3] doubled to six?
00:07:37	Michael:	Uh-huh.
00:07:38	R10:	And this one?
00:07:40	Michael:	Has two more, one in there and one in there.
00:07:42	R10:	Oh, he's what makes the eight?
00:07:42	Michael:	Uh-huh.
00:07:43	R10:	Could you talk me- What's the next row gonna be?
00:07:46	Michael:	It would be 1 4-
00:07:49	R10:	Uh-huh.
00:07:49	Michael:	-6 4-
00:07:50	R10:	Uh-huh.
00:07:50	Michael:	-uh, 1.
00:07:51	R10:	OK. Talk me through it again so I'll understand.
00:07:54	Michael:	This one pizza. This pizza came from this one, which didn't get one. [1+0=1.]
00:07:57	R10:	Uh-huh.
00:07:58	Michael:	These four came from one of these that got one.
00:08:01	R10:	Uh-huh.
00:08:02	Michael:	And three of these that didn't. [1+3=4.]
00:08:03	R10:	Uh-huh.
00:08:03	Michael:	The six came from these three that got a topping.
00:08:07	R10:	Uh-huh.
00:08:08	Michael:	And these three that didn't. [3+3=6.]
00:08:09	R10:	OK.
00:08:09	Michael:	These four came from the three that got a topping and one that didn't. [3+1=4.]
00:08:12	R10:	Uh-huh.
00:08:13	Michael:	And this one came from that one that got, that got a topping.
00:08:16	R10:	Oh. That makes sense to me. But you must admit that

Time	Speaker	Transcript
		sometimes it helps to have somebody talk you through it.
00:08:23	Michael:	Yeah. [They discuss how it helps to talk it over.]
00:08:31	R10:	That's true for the pizzas. Are there any other problems you've done that you could think about in that sort of doubling kind of Pascal way? To know how you got from one row to the next?
00:08:43	Michael:	Um, like in this one I didn't really use the pizza. I used binary.
00:08:50	R10:	Uh-huh.
00:08:51	Michael:	'Cause when you add, when you add another, another place right there.
00:08:56	R10:	Uh-huh.
00:08:57	Michael:	That number would either, either be that or that.
00:08:58	R10:	Uh-huh.
00:08:59	Michael:	You understand?
00:09:00	R10:	Um. Yeah.
00:09:00	Michael:	Like that number- If you, if you say, all right, you can only have three places.
00:09:06	R10:	Uh-huh. Oh!
00:09:07	Michael:	Then you said all right, you're going to have four now. You can make it to two different numbers.
00:09:09	R10:	Oh. I see. It's sort of that tree thing.
00:09:11	Michael:	Yeah. A tree. It branches out.
00:09:13	R10:	Yeah.
00:09:14	Michael:	Each one comes in. Each, each one of those-
00:09:15	R10:	Uh-huh.
00:09:16	Michael:	-combinations will become, will become two.
00:09:19	R10:	Yeah. But then you went back, and you said you could have done it this way.
00:09:23	Michael:	What, what do you mean?
00:09:24	R10:	You went back after you did it the binary way and said you could have done it differently.
00:09:30	Michael:	In this explanation [his email]?
00:09:30	R10:	Yeah.
00:09:31	Michael:	I was just like- This was just explaining how 1 becomes, can come into 2.
00:09:35	R10:	Uh-huh.
00:09:37	Michael:	And this explains how those, those doubled combinations, where they fall into, which part of the triangle they go.
00:09:43	R10:	Yeah.
00:09:44	Michael:	And why. Like-
00:09:45	R10:	Yeah. Yeah. I get it. I was just wondering if any of the other

Time	Speaker	Transcript
		kinds of problems that you've done that seemed to have this doubling-
00:09:53	Michael:	Uh-huh.
00:09:53	R10:	-rule, um, could have been described with the Pascal Triangle in the way you're doing it now or not?
00:10:04	Michael:	Um. Well, obviously I just did the World Series one. It's probably-
00:10:08	R10:	You want- but, but, could it be, could you use Pascal to think about the World Series problem?
00:10:14	Michael:	I did. I, personally, I used it.
00:10:15	R10:	Oh, really? Oh.
00:10:16	--	[They discuss the probabilities related to the World Series problem for about three minutes.]

January 29, 1999 World Series Session

Time	Speaker	Transcript
00:00:00		Ankur, Michael, Romina, and Jeff discuss the World Series problem with Researcher 5 for about 12 minutes.
00:12:43	Michael:	I want to explain something.
00:12:44	Romina:	There's the away team.
00:12:45	Michael:	You're familiar with, uh, Pascal's Triangle, right? Right? You know those.
00:12:49	R5:	Yeah. Yeah.
00:12:50	Michael:	I had- there's a way to, um, connect that and binary numbers together.
00:12:56	R5:	Yes.
00:12:57	Michael:	You are familiar with that?
00:12:58	R5:	Uh-huh.
00:12:59	Michael:	All right. So I used that to find a solution to that problem. Right? Let me just-
00:13:05	R5:	How does it go?
00:13:08	Michael:	[Michael writes five rows of Pascal's Triangle on his paper..] All right. Well, the connection would be, like, this is a two number. [Michael points to row 2.] Like the binary number with two numbers in two places.
00:13:27	R5:	Yes.
00:13:28	Michael:	The first one is the amount, the total number, that only have, that have no 1's in it. The second one would be with one. And the

Time	Speaker	Transcript
		third place would be the amount that had two, right?
00:13:39	R5:	So you're looking at the row 1 2 1.
00:13:41	Michael:	Well, well, I'm just going to give you an example.
00:13:42	R5:	Yeah. OK.
00:13:43	Michael:	You understand?
00:13:43	R5:	Yes.
00:13:44	Michael:	This column has, has 1 with just whole 0's in it. [Michael writes 00.]
00:13:45	R5:	Uh-huh.
00:13:46	Michael:	This one has- there's two in that, those four combinations that have the 1 and the-
00:13:50	R5:	What are the two?
00:13:51	Michael:	This one and that one. [Michael writes 10 and 01.]
00:13:52	R5:	Uh-huh.
00:13:52	Michael:	All right. And the last one, all 1's. [Michael writes 11.]
00:13:56	R5:	Uh-huh.
00:13:57	Michael:	And this holds true for the next one. [Michael points to row 3.] This one has, there's only one with all 0's.
00:14:00	R5:	Yes.
00:14:01	Michael:	There's three with, uh, with one.
00:14:03	Ankur:	One.
00:14:04	Michael:	And there's-
00:14:04	R5:	Why are there three with one?
00:14:07	Michael:	'Cause there's three places. There's only, there's only three ways to do it. And it's like that. [Michael writes 001 010 100.]
00:14:16	R5:	OK.
00:14:17	Michael:	The next one would be three, all right, with two 1's.
00:14:20	R5:	Uh-huh.
00:14:21	Michael:	And the last one would be one with all three.
00:14:23	R5:	Why are there three with two 1's? Just three?
00:14:26	Michael:	Why with-
00:14:26	Jeff:	It's the same situation as the one, except you're switching, you're switching-
00:14:28	Ankur:	The 0.
00:14:29	Jeff:	The 0 would be in the 1's spot.
00:14:30	Michael:	I kind of wrote an explanation about this.
00:14:32	R5:	Oh, you mean if it had two 1's, it-
00:14:33	Ankur:	It's the same thing.
00:14:34	Jeff:	It's the same as having two 0's in that situation.
00:14:36	Ankur:	'Cause you're having two of one number and one of the other.

Time	Speaker	Transcript
00:14:39	Michael:	Uh.
00:14:40	R5:	Oh, because the ones with two 1's in them, if, if they have three digits, have one 0? I see.
00:14:49	Michael:	Use the board.
00:14:49	R5:	[Researcher 5 reads a note from the film crew.] It says, OK. The, the message from on high is, "Use the board."
00:14:53	Michael:	I guess I should go up.
00:14:55	R5:	Yeah, OK. OK, and this is, and, and here we're just following through one of the ways. Well, there's an easer down there and chalk by the, the tray. [To Michael.] Um-
00:15:09	Michael:	[Michael erases the board.] I'll, I'll explain to you real quick why you add 'em together, which is like the, kind of saying, why is there three. And it's the same probability.
00:15:17	R5:	OK.
00:15:18	Michael:	I wrote that up, and I gave it to Dr. Maher, if she's here, and I don't know what she did with it.
00:15:23	Romina:	She's probably, probably reading it.
00:15:24	Michael:	All right. [Michael writes rows 0 through 5 of Pascal's Triangle on the board. Row 1 is missing. Refer to Figure I4.] All right. Um, in the, in the previous, in this category or whatever, you have two numbers. One of them has noth- nothing in it. [Michael moves around to get out of the way of the camera.] Am I the wrong way? One of them has nothing in it, like no 1's in it.

```
          1           ← Row 0
        1   2   1        ← Row 2
      1   3   3   1      ← Row 3
    1   4   6   4   1    ← Row 4
  1   5  10  10   5   1  ← Row 5
```

Figure I4. Michael's drawing of Pascal's Triangle (without row 1); row numbers added

Time	Speaker	Transcript
00:15:59	R5:	Uh-huh.
00:16:00	Michael:	The second one has, there's two with just one. The last one, there's, all spaces are filled up, or whatever. [Michael is pointing to row 2 of Pascal's Triangle.]
00:16:07	R5:	OK.
00:16:08	Michael:	Then when you add another place to it-
00:16:10	R5:	Yes.
00:16:11	Michael:	Each combination in there could turn into two different ones, you understand? One will have another, one will have a 0 or 1 to the

Time	Speaker	Transcript
		end of it.
00:16:18	R5:	Could you show me that? Yeah.
00:16:21	Michael:	[Michael turns back to the board.] You have-
00:16:22	R5:	I want to be just certain that I understand exactly what you're seeing here.
00:16:25	Michael:	Take this first number, which is zero. [Michael writes 00.] Uh, second. This is one. [Michael writes 01.] This would be two. [Michael writes 10.] And this is three. [Michael writes 11.]
00:16:34	R5:	Uh-huh.
00:16:35	Michael:	It's binary numbers. When you add another place to it, now you have three spots to fill up. The, these two, this 0, 0 still stays, could become the same value or whatever. [Michael draws three dashes under the 00 and writes 0's over them.] And this 1 could become 1. So now, two are formed from every single com- possibility. The same thing will go with this [01]. It's either this-
00:16:54	R5:	Uh-huh.
00:16:55	Michael:	-or this. [Michael writes 010 011. Refer to Figure I5.] Right? You understand?

00	01	10	11
<u>000</u>	<u>010</u>		
<u>001</u>	<u>011</u>		

Figure I5. Michael describes an instance of the addition rule using binary notation

Time	Speaker	Transcript
00:16:58	R5:	I do.
00:16:59	Michael:	Now-
00:16:59	Jeff:	That was like adding the extra slot.
00:17:01	Ankur:	That's the extra slot with two possibilities.
00:17:02	Michael:	Now when you, when you add an extra slot, obviously one of them is still going to have one. [Michael points to 010.] And one of them is going to have two. [Michael points to 011.] So the 2 right here [in Row 2], there's going to be two, one from each- that, that still have one.
00:17:19	R5:	Uh-huh.
00:17:19	Michael:	That's why you add it to that. And this one will have a, this one will add, get one added. [Michael points from the first 1 in row 2 to the first 3 in row 3.] So that's 1. So there's your 3. They come together like that. This one [Michael points to the 2 in row 2.], there will be the other two of the- and now in terms of four

Time	**Speaker**	**Transcript**
		combinations, the other two will have, uh, two. And the ones that don't get anything added on will have one. You know what I'm talking about? [Refer to Figure I6 for a diagram of where Michael was pointing.]

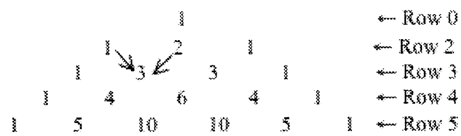


Figure I6. Michael shows one case of the addition rule; row numbers added

00:17:41	R5:	Yes, I, I think so.
00:17:42	Michael:	Um, so they all, that's why they add together.
00:17:44	R5:	Uh-huh.
00:17:44	Michael:	Um, and then that would explain why there's a 3 there [the second 3 in row 3], why there's a 4 [the first 4 in row 4]. Because these 3 are ones that have an- a 1 added on.
00:17:53	R5:	Uh-huh.
00:17:54	Michael:	They go into this category [the 4]; the ones that don't get anything added go into that one [the 6]. [Refer to Figure I7.]

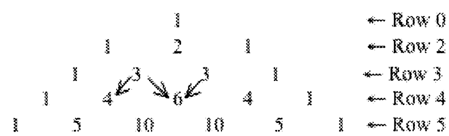


Figure I7. Michael shows other cases of the addition rule; row numbers added

00:17:56	R5:	Are you saying, for example, that that 6 comes up by adding?
00:18:01	Michael:	Yeah. Like you add, there's, there's- now there's, with these 3 there, with these two [both 3's], these six possibilities, there's twelve of them.
00:18:10	R5:	Uh-huh.
00:18:11	Michael:	Six of them get a 1 added on to them.
00:18:12	R5:	Right.
00:18:13	Michael:	There. [Michael forms a line with his hand from the first 3 in row 3 to the 6 in row 4.] That way that would give you, bring it to three. Three ones in the, in the series, whatever, and these three, uh, half of them don't get, don't get a 1 added onto them. That

Time	**Speaker**	**Transcript**
		would, that would leave them in the same category like with just three.
00:18:29	R5:	Uh-huh.
00:18:30	Michael:	That's why you, that would add them.
00:18:32	R5:	Uh-huh.
00:18:33	Michael:	You understand that?
00:18:34	R5:	OK.
00:18:34	Michael:	Um. [He is interrupted by the buzzer.]
00:18:40	R5:	And the whole triangle is built that way?
00:18:42	Michael:	Yeah. By adding.

APPENDIX J: TRANSCRIPT OF SESSION OF MAY 12, 1999

Time	Speaker	Transcript
00:00:00	–	[Romina, Jeff, and Michael are sitting at a table talking with Researcher 1. At the start, the display says 02:01:01.27. No sounds can be heard. Researcher 2 walks across the room, opens the door, and goes out. The camera shows the rest of the room. Researcher 2 returns.]
00:01:05	R1:	-mentioned some of what went on. I have, I don't have a clue. Can you sort of tell me about it and how some of you suggested it's connected to other things you had done? I'm really curious. Feel free to use the board and show me and tell me.
00:01:16	Jeff:	Well.
00:01:17	R1:	But I don't know what went on.
00:01:17	Romina:	When, when we came up with that thing that almost was like the Pascal's Triangle. What was it with *e*? What were we doing?
00:01:21	Jeff:	It was, um-
00:01:22	Romina:	Ten, ten percent of a hundred.
00:01:25	Jeff:	Is that what it was?
00:01:26	Romina:	Yeah.
00:01:28	Jeff:	Um, I'm not sure. I don't know.
00:01:31	Romina:	Can we have a calculator? Are we allowed to have one?
00:01:33	R1:	Sure. You know where they are?
00:01:36	R4:	Yeah. [Researcher 4 gets the calculators.]
00:01:37	R1:	I mean, I want to have them around anyway. [Researcher 4 gives the calculators to Researcher 3.]
00:01:43	Jeff:	This is going to be our first time ever using calculators.
00:01:45	Romina:	Yeah.
00:01:46	R1:	Wow, first time.
00:01:46	Jeff:	We usually- [Researcher 3 distributes calculators.]
00:01:46	R1:	You never use them?
00:01:47	Romina:	Thank you.
00:01:48	R3:	You're welcome.
00:01:52	Michael:	Are there any games on this? [Students start using calculators.]
00:01:58	Jeff:	We said one times or one minus. [Jeff talks as he punches buttons on the calculator.]
00:01:59	Romina:	What, what am I doing?
00:02:00	Jeff:	Oh that's good right there.
00:02:01	Romina:	Yeah. [Romina laughs.]
00:02:02	Jeff:	One. Is it one minus one over a hundred?
00:02:02	Romina:	Hm.

Time	Speaker	Transcript
00:02:06	Jeff:	Hundred raised to- [The camera is focused on Jeff's calculator, but the display is not visible.]
00:02:11	Romina:	OK, why am I- Oh, okay. I know why it's wrong.
00:02:14	Jeff:	Um.
00:02:17	Romina:	Was it like, weren't we doing this? [Romina shows her calculator to Jeff.]
00:02:18	Jeff:	Yeah, that's for the last one.
00:02:20	Romina:	Yeah.
00:02:21	Michael:	That's this- [unintelligible; chair is moving].
00:02:23	Romina:	OK, this is scary. Look when we were- Because we were discussing like percentages. [Romina holds out her calculator.] And, uh, like an increase and we did a hundred and we took ten percent of it and that's one two one, that's one three three one, and, you know, that is-
00:02:38	Jeff:	Yeah, we kept going in-
00:02:39	Romina:	It doesn't come out yet. After a while it goes.
00:02:41	Jeff:	It kind of makes you think. After a while it stops, but we were, uh-
00:02:44	Romina:	We really thought that was it, look. [Romina shows the calculator to Researcher 1.]
00:02:45	Jeff:	We were into it.
00:02:48	R1:	Oh. So what does it mean?
00:02:50	Jeff:	Uh, we didn't, we didn't know.
00:02:51	Romina:	We didn't know because then it stops, though.
00:02:53	Jeff:	Yeah, but it was interesting for, for a while.
00:02:55	Romina:	While it was going on it was very-
00:02:57	Jeff:	We were kind of, uh-
00:02:59	Michael:	[Michael returns and sits down.] Are we going to [Inaudible.]?
00:03:00	Jeff:	But um, what was the question? What were you-
00:03:02	Romina:	We wanted to know what we did in class today.
00:03:04	Jeff:	Um, we were looking a lot at, at working at *e* and, and the equation for it.
00:03:09	Michael:	And how it, how it, how it connects with *ln* and-
00:03:11	Jeff:	Yeah, um-
00:03:11	Romina:	And we were also trying to find like, you know how we had when we had *a* plus *b* to the *n*? We want to know what- And we had like numbers before it when we got to big numbers, we want to know, you figure out what the numbers were, like in front of the *a*, you know, cubed.
00:03:24	Jeff:	You know, that's, that's like, um-

Time	**Speaker**	**Transcript**
00:03:25	R1:	You could use the board too.
00:03:26	Jeff:	Uh, we just- [Jeff writes on his paper; refer to Figure J1.] Like if you were looking, if we were looking for like *a* plus *b*-

$$(a+b)^{10}$$
$$1a^{10} \quad 10a^{9}b^{1}\ \_45$$

Figure J1. Jeff's discussion of the binomial expansion

00:03:33	Romina:	To the tenth.
00:03:33	Jeff:	To the tenth say, um, obviously it- Was the first one ten? Was it //one *a* to the tenth and //then ten-
00:03:39	Michael:	//No it's one, yeah.
00:03:40	Romina:	//*b*. Oh no, you're right. Sorry.
00:03:42	Jeff:	Ten. Ten *a* to the ninth *b* to the first, right?
00:03:45	Romina:	Mm hm.
00:03:45	Jeff:	And then how to find out //this number.
00:03:47	Romina:	//What the next one was.
00:03:48	Michael:	It's forty-five.
00:03:49	Jeff:	And it was forty-five but we were working on how to figure it out when we were doing it. We knew it was the choose thing, whatever that means. The- You do a forty- What was it? Ten choose two?
00:03:58	Michael:	Yeah.
00:03:58	Romina:	Uh-huh.
00:03:58	Jeff:	You know what I'm talking about? Like, uh, was it N-C-R- actually that's supposed to be lower case. Two- is that how you do it? [Jeff writes 10 NCr 2.] Right?
00:04:05	Michael:	Yeah, it's one of these things like that.
00:04:06	Jeff:	And that equals forty-five and that's the answer. You know. I'm not, we're not really sure how all this works but it's like, what is that, if-
00:04:13	Romina:	We, we learned that, we learned that with her. [Romina points to Jeff's paper, where it says "10 NCr 2."]
00:04:15	Jeff:	Yeah. Yeah the- Yeah, we, we went, we went over that, remember that? With the total-
00:04:19	Romina:	We tried to go over that. [Romina laughs.]
00:04:20	Jeff:	If you have ten different, what was it? Ten different things.
00:04:24	Michael:	You have-

Time	**Speaker**	**Transcript**
00:04:25	Romina:	//Ten high. //Ten high.
00:04:26	Jeff:	//Ten high. How many-
00:04:26	Romina:	//How many would have two reds, only two reds.
00:04:27	Jeff:	//How many would have two, two reds.
00:04:29	R1:	One more time.
00:04:31	Jeff:	If you had towers// of ten high.
00:04:32	Michael:	//If you had like towers.
00:04:32	R1:	Towers.
00:04:35	Jeff:	If you have towers with ten high //and two colors.
00:04:35	Michael:	//How many different places can you put two reds in there?
00:04:36	Jeff:	Yeah.
00:04:37	Romina:	Yeah.
00:04:37	Jeff:	And like *a* would be one color and *b* would be blue, um, *b* would be the other color. Then how many would you have, *a* being two in the whole thing? And that would be forty-five and that's, that's what this number would be. [Jeff points to his paper; refer to Figure J1.]
00:04:50	R1:	And these towers are how tall?
00:04:52	Jeff:	Ten tall.
00:04:53	Romina:	Ten.
00:04:54	Jeff:	That'd be the ten there.
00:04:54	Romina:	Mm hm.
00:04:54	Jeff:	The two would be the two colors and then, right?
00:04:58	Michael:	No.
00:04:58	Romina:	You have two of one color.
00:04:59	Jeff:	No, ten would be the two of the one color and the two is implied that there's two, only two colors? Or-
00:05:04	Michael:	The two is the-
00:05:04	Romina:	It's only *a* plus *b*.
00:05:06	Jeff:	Yeah but in the, when you write this, I mean is it implied that there's only two colors?
00:05:10	Romina:	I believe it is but-
00:05:12	Jeff:	Is that, is it implied?
00:05:14	Romina:	I, I'll go with the yeah. I don't know. [Romina laughs.]
00:05:16	Michael:	Uh, it's only this?
00:05:17	Jeff:	Yeah.
00:05:18	Michael:	//It, it, it's-
00:05:18	Romina:	//Is that like-
00:05:19	Jeff:	Is that one, the only one works for-
00:05:20	Michael:	No, it, it's just like you have ten things where, how many different

Time	Speaker	Transcript
		places can you put these two? That's all.
00:05:25	Jeff:	Yeah, I know but-
00:05:25	Michael:	You know what I'm saying?
00:05:25	Jeff:	But if there's, oh, yeah, two. All right, I see what you're saying.
00:05:25	Michael:	That's all.
00:05:28	Jeff:	There could be a hundred colors but it would still-
00:05:31	Michael:	You pick two things out of those ten.
00:05:32	Jeff:	Yeah.
00:05:33	Michael:	How many different places can you put them?
00:05:34	Jeff:	Put them. All right. All right.
00:05:35	Michael:	Forty-five.
00:05:37	R1:	So, so you're saying that's forty-five and what if I wanted eight red? Eight red ones or eight *a*'s?
00:05:41	Jeff:	Then it would be ten-
00:05:41	Michael:	Um. [Michael starts working on calculator.]
00:05:42	Romina:	Choose eight.
00:05:43	Jeff:	Choose eight, yeah.
00:05:44	Michael:	A smaller number.
00:05:45	Jeff:	Because //that would be how many different spots can you move those eight of one color in the tower of ten.
00:05:47	Romina:	//Now how do you-
00:05:50	Michael:	It's forty-five also.
00:05:51	R1:	Why?
00:05:52	Romina:	Like how do //you, how do you, how do you do that on a calculator?
00:05:53	R1:	//How'd you do that so fast Michael?
00:05:53	Jeff:	Um.
00:05:54	Michael:	No, I did it in my head, that's all.
00:05:54	Jeff:	You go to, uh, math. [Jeff and Romina start working on calculators.]
00:05:56	R1:	Tell us how you did it. [Researcher 1 is speaking to Michael.]
00:05:57	Michael:	Um.
00:05:57	Jeff:	Probability.
00:05:58	Michael:	There's a button that-
00:06:00	Jeff:	N-C-R.
00:06:00	Michael:	Take ten, that button then eight.
00:06:02	Romina:	And math.
00:06:03	Michael:	And it comes out forty-five.
00:06:05	Jeff:	Why is that the case?
00:06:07	Romina:	Hm.

Time	Speaker	Transcript
00:06:09	Michael:	Well if you take like on the-
00:06:10	Romina:	Well because-
00:06:12	Michael:	You know how on Pascal's Triangle.
00:06:13	Romina:	That's like the two.//You have eight left over.
00:06:14	Jeff:	//Oh, cause you could switch them all around. [Jeff moves his arms up and down.] is that, is that, I guess you're counting.
00:06:17	R1:	I don't know. Tell me.
00:06:17	Michael:	Cause then you would have-
00:06:18	Romina:	//Is that the same thing as that because, like, the eight left over to get to the ten, right?
00:06:18	Michael:	//It'll be- It would be the same thing.
00:06:22	Jeff:	Exactly.
00:06:23	Romina:	It's like almost switching colors.
00:06:23	Jeff:	Yeah.
00:06:24	Romina:	It'd be like two of the other color.
00:06:25	Jeff:	And then, and then, yeah, exactly.
00:06:26	R1:	Say that one more time, Romina.
00:06:28	Romina:	It'd be two of the other color instead of, like say you started with red for this two. That was for the reds and then when you//make red eight.
00:06:33	Michael:	//That would be the other eight.
00:06:34	Romina:	The, like, say the blues have two.
00:06:36	Jeff:	And it's seven. And then obviously three should be the same as that.
00:06:39	Romina:	Yeah. [Jeff pushes calculator buttons and then nods.] Yeah.
00:06:47	R1:	So, you're pressing the calculator, you have a new command that gets you those numbers.
00:06:53	Romina:	We'd know how to do it, I mean it's not-
00:06:54	R1:	But if you didn't have the calculator?
00:06:57	Romina:	We'd write them out.
00:06:57	Jeff:	You'd have to write them all out.
00:06:58	Michael:	Well, Bob-
00:07:00	R1:	Because Alex wants to know how you do that without a calculator. [Alex is the cameraman.]
00:07:03	Jeff:	Well, I obviously if the calculator-
00:07:04	R1:	Can you, can you help him understand that?
00:07:06	Jeff:	Well we would make a, say, tower of ten. [Jeff draws a 10-high tower; refer to Figure J2.]
00:07:10	Michael:	Can I say something? //All right, um-
00:07:10	Romina:	//I don't know.

Time	Speaker	Transcript
00:07:11	Jeff:	Bob. Yeah go for it.
00:07:12	Michael:	No, I'm talking, I just wanted to say that Bob had like an actual formula to write the equals and-
00:07:17	Jeff:	Do we know, do we know what it is? Or-
00:07:19	Michael:	I think so. I don't know if I can remember it.
00:07:19	R1:	Why don't I leave you a few minutes and think about explaining this to us.
00:07:24	Michael:	It depends on-
00:07:24	Jeff:	Well it's not, it's not hard to explain.
00:07:25	Michael:	You remember if? I forgot it.
00:07:26	R1:	OK, I'll stay then.
00:07:28	Alex:	Well, actually no.
00:07:29	Romina:	It's not right.
00:07:30	Michael:	All right.
00:07:31	R1:	He has a bad memory.
00:07:32	Michael:	I got to like do trial and error to see if I can figure it out what it was.
00:07:35	R1:	OK.
00:07:35	Jeff:	All right, say that's ten? [Jeff indicates box; refer to Figure J2.] Then, um, you would just have to find- Say you had, uh, one was one color and two was the other color.

Figure J2. Jeff's diagram of 10-tall towers

00:07:43	Romina:	Why, why don't you show her how to do it for like three. Show them how we can get all of them. You're going to have to draw the tower. [Jeff draws a three-tall tower. Romina laughs.]
00:07:48	Jeff:	You have two colors. And out of this tower of three you'd have to find out all the different places you could put those two colors in. So you could put it there and there. [Jeff draws; refer to Figure J3.] Or you could put it, uh, there and there. Or, am I missing any? Yes, I am?

Time	Speaker	Transcript
00:08:04	R1:	I understand.
00:08:05	Romina:	You could just do like-
00:08:06	Jeff:	Yeah.

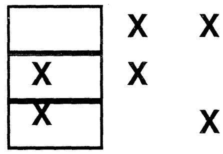


Figure J3. Jeff's diagram of places to put two colors in 3-tall towers

00:08:06	Romina:	Do you want to look for another one?
00:08:07	Jeff:	No, go for it.
00:08:08	Romina:	No, you could just do, you could do like our blue, blue, blue. [Romina writes BBB.]
00:08:12	Jeff:	You gonna write every one? [Romina writes BBR and BRB.]
00:08:14	Romina:	Well, there wasn't that many. No I'm just like giving you an example.
00:08:15	Jeff:	Yeah.
00:08:16	Romina:	And then you just kind of move it through. [Romina writes RBB.] And that's how we figure them out when we have to write them out.
00:08:23	R1:	So you're saying there's a way of getting these without the calculator.
00:08:30	Jeff:	Yeah. And there's a, there's a formula that somebody-
00:08:34	Romina:	Not too-
00:08:34	Jeff:	-had come up with but I don't know, I don't know how it, how it goes. I'm really not sure.
00:08:38	Romina:	I've seen it.
00:08:39	Jeff:	I don't remember it.
00:08:40	Romina:	Yeah, there's some-
00:08:41	Michael:	Yeah.
00:08:41	Romina:	Something to that effect. [Romina points to Michael's paper; refer to Figure J4.]
00:08:42	Michael:	It was this guy.
00:08:43	Romina:	That's it?
00:08:43	Michael:	Yeah.
00:08:44	Jeff:	It's this right here? [Jeff points to Michael's paper.]
00:08:45	Michael:	Yeah.
00:08:46	R1:	Why don't you show us up here, Michael.

Time	Speaker	Transcript
00:08:48	Michael:	Oh, man. I, I didn't come up with this, so don't ask me how to do it. [Michael goes up to the board.]
00:08:49	R1:	It doesn't matter that you came up with it.
00:08:52	Michael:	If you would have like *n* choose *x*. [Michael copies from his paper to the board; refer to Figure J4.]

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

Figure J4. Michael shows the formula for *n* choose *x*

00:09:02	Romina:	That's on, that's on the division, *n* to the *x*, or is that just like your-
00:09:05	Jeff:	Um, yeah, the [Inaudible.].
00:09:07	Michael:	No, that's, that's choose to the, that's how you write it I think. [Michael points to the left side of Figure J4.] I think that's how you write it.
00:09:08	Romina:	That's just, that's what it is?
00:09:11	R1:	Do you want an equals sign there?
00:09:13	Michael:	No. That's, that's not in- Yeah. [Michael writes the equals sign] Yeah, I could do that. Times *x*. [Michael writes the *x*!.] That, that would be the number.
00:09:24	R1:	OK. Hi, Ankur. Come on in.
00:09:25	Ankur:	Hi. Sorry I'm late.
00:09:27	R1:	We're glad you're here.
00:09:29	Jeff:	Didn't you go with them?
00:09:30	Ankur:	No, I didn't go with them. I went with Steve.
00:09:34	Jeff:	That's dirty.
00:09:34	R1:	Hi, did you eat?
00:09:36	Ankur:	No.
00:09:37	R1:	Are you hungry? Yes.
00:09:39	Ankur:	Yeah I guess so. But it's all right. It's all right.
00:09:39	Romina:	You can, uh-
00:09:39	R1:	I'll tell you what. I, I-.
00:09:41	Michael:	I hate stopping- [Ankur gets something to eat. Michael, Jeff, and Romina stay at the table working. There is a gap in the tape, and there is talking in the background.]
00:09:58	Jeff:	All right, what are we going to do?
00:09:59	Michael:	Oh. Oh yeah, um-
00:10:01	Romina:	What, what does that get? That gets you, like-

Time	Speaker	Transcript
00:10:03	Michael:	That gives you that choose thing.
00:10:04	Jeff:	That gives you-
00:10:05	Michael:	I don't, I don't know what it means.
00:10:09	Romina:	I was working with him one day when he brought that up in class but he lost me.
00:10:12	Jeff:	That was the day that me and, and my table were doing the, uh, finding the square roots without a calculator.
00:10:18	Michael:	Yeah, but he did that before like-
00:10:20	Romina:	Not with me.
00:10:21	Michael:	-in, in class when he was talking about choosing.
00:10:22	Jeff:	Who?
00:10:22	Romina:	I was in your group.
00:10:23	Jeff:	Oh, Romina, you have an eyelash.
00:10:26	Michael:	No, in class when he was talking about choosing. He figured it out.
00:10:30	Jeff:	All right, um, well you figure, say you do, uh, say you're doing three, right? So that would be three times two times one. [Jeff writes 3*2*1.] That would be each space, I imagine.
00:10:42	Romina:	And *x* is how many, how many you want of the color?
00:10:43	Michael:	Yeah, I guess that would show combinations.
00:10:45	Jeff:	Yeah, that would, because that would give you the total number of-
00:10:47	Romina:	Yeah. Yeah that would be it, yeah.
00:10:47	Jeff:	Total number of combinations?
00:10:49	Michael:	I guess, yeah.
00:10:50	Jeff:	All right. So then that would be, say, six factorial. [Jeff writes 6!] Divided by. [Jeff writes a division bar and left parenthesis.] That would be- [Jeff writes 3.] Why would you- //why, where does that work?
00:10:59	Michael:	//No wait. //This is, that would be-
00:11:00	Romina:	//Come on, X is the number we want and X is like the number we want to get, like the choose number.
00:11:05	Michael:	That's the, yeah, choosing one.
00:11:07	Jeff:	And-
00:11:08	Romina:	So I guess this would-
00:11:10	Michael:	He was telling me like this was-
00:11:12	Romina:	That would like take away all the, all the ones we would choose.
00:11:15	Michael:	He said something about repeats. One would take away //the repeats.
00:11:15	Jeff:	//Yeah, this would take away the repeats, right?

Time	Speaker	Transcript
00:11:17	Michael:	I guess.
00:11:19	Romina:	And will this, and this will give you-
00:11:20	Michael:	And this will, this will take away the-
00:11:21	Jeff:	This will take away all the other ones?
00:11:22	Michael:	All the ones that, that you don't care where they are.
00:11:22	Romina:	Like the ones that are higher than-
00:11:24	Jeff:	Yeah. Yeah.
00:11:25	Michael:	You only care about, you only care about the two that are moving. Not the other-
00:11:26	Jeff:	Yeah, exactly.
00:11:28	Michael:	Not the other, uh, four. It's just-
00:11:29	Jeff:	And then, and X-
00:11:30	Romina:	That makes sense.
00:11:32	Jeff:	Yeah. That make, let's see if it works. So say-
00:11:35	Romina:	Where's factorial? [Jeff and Romina work on a calculator.]
00:11:36	Jeff:	Where's the exclamation point? [Jeff punches buttons on the calculator.]
00:11:37	Michael:	Math.
00:11:39	Jeff:	Ah.
00:11:39	Michael:	Probability 4.
00:11:41	Romina:	What are we doing, three or six?
00:11:41	Michael:	Just hit four.
00:11:42	Jeff:	All right.
00:11:43	Michael:	I mean would you-
00:11:45	Jeff:	I don't even know why I did that. That was stupid. Uh, quit that. Six divide- Mm. Six divide-
00:11:53	Romina:	Where is the little, where is that? I don't know.
00:11:56	Jeff:	Uh.
00:11:57	Michael:	Are you doing. why are you doing the six only? Oh, because three factorial is six, right?
00:12:01	Jeff:	You got it. That's not six factorial. Um, divided by three minus, what's X? We don't know?
00:12:09	Michael:	Do two.
00:12:10	Jeff:	Minus two.
00:12:14	Michael:	Times-
00:12:16	Romina:	You're a lot farther on that than I am.
00:12:20	Michael:	You have to put a parenthesis around that whole thing, too. [Michael points to Jeff's calculator display.] Later on that.
00:12:24	Jeff:	Then times.
00:12:24	Michael:	No, you got to, at the beginning of that and the end of this then.

Time	Speaker	Transcript
		[Jeff pushes calculator buttons.]
00:12:31	Michael:	Get rid of that one there. [Michael points to Jeff's calculator display.]
00:12:35	Jeff:	Don't we have to close that in, though?
00:12:36	Michael:	No, you don't have to worry about it.
00:12:37	Jeff:	Oh. All right. So do I have to delete that other one? No.
00:12:39	Michael:	No, leave that like that. Two.
00:12:37	Jeff:	Oh. All right. So do I have to delete that other one? No.
00:12:39	Michael:	No, leave that like that. Two.
00:12:42	Jeff:	So divided by two factorial. [Jeff pushes calculator buttons] Um.
00:12:50	Michael:	Let's see. And do it the other way and it should come out. [Calculator display is: 6/((3-2)!*2!), followed by 3.]
00:12:53	Jeff:	What, what, what was the other way?
00:12:55	Romina:	Did, did it work? [Michael pushes buttons on Jeff's calculator.] Four.
00:13:00	Michael:	No, it's three.
00:13:01	Romina:	Oh.
00:13:01	Jeff:	Oh, that's all right. [The calculator display is **3 nCr 2**.]
00:13:03	Michael:	Yeah, it works.
00:13:04	Jeff:	All right. That's the case then. All right, so when Ankur's done eating, you can explain all this. And we can explain it to Ankur? [Inaudible.] Ankur missed the beginning, so explain it to him.
00:13:15	Michael:	There was really no beginning, though.
00:13:16	Jeff:	It doesn't matter. We're still going to have to. I didn't know. I forgot that we were, uh, we have to explain things.
00:13:23	Michael:	I thought it was just recess or something. [Romina laughs.]
00:13:25	Jeff:	Yeah, I know. I'm like that. And, uh-
00:13:34	Ankur:	Where do you want me to sit?
00:13:37	Alex:	You can go ahead and sit anywhere you want. //That's probably a good spot.
00:13:37	Romina:	//I saw you out of the corner of my eye. You confused me. I'm like what is he doing? [Break in video; then Jeff and Romina return.] I drove him.
00:13:41	Jeff:	Ah.
00:13:45	Romina:	I drove him here.
00:13:45	Ankur:	By yourself?
00:13:52	Jeff:	Yeah, right, um. Do you want to hear our explanation of, of this and why this works?
00:14:00	R1:	That would be really great, but it would help us enormously if you would use the board.

Time	Speaker	Transcript
00:14:02	Michael:	I wrote that, so you don't have to. [Jeff goes to the board.]
00:14:04	R1:	Would you mind, Jeff?
00:14:05	Jeff:	All right, I need help because-
00:14:06	R1:	They'll help you.
00:14:07	Jeff:	I don't want to get stuck. I got stuck up there last time by myself and I was looking like an idiot. All right, the reason why this works- We have no chalk. Right there in the side. All right, the reason why this works, first you get all the total number of the, this was going to cover all the total possibilities of your tower.
00:14:26	Romina:	//The factorial.
00:14:27	Jeff:	//Say in terms of a tower. The factorial right here. So say you're doing towers of three. You have the three factorial and that'll cover all of the different combinations that you could put three in with two colors. [Jeff waves his left hand.] All right? [There is a pause. Jeff waves his hands.] Yes?
00:14:45	Ankur:	Sounds good.
00:14:48	R1:	Why don't you go through and when you're all done I'll ask my question. Just go.
00:14:51	Michael:	Like, you should use the explanation like she used. [Michael points in the direction of Romina and Researcher 1.] Like the people on the line. That's better because you have like the first one. Then you have-
00:14:57	Romina:	Two spaces.
00:14:58	Jeff:	All right, I'll do people on the line.
00:14:59	Michael:	Two spaces. You have two people left so that's times two.
00:15:00	Jeff:	All right, say we're doing us three right here.
00:15:01	Michael:	Yeah, on the line.
00:15:02	Jeff:	This, us three, um-
00:15:04	Romina:	There's three different people to fill in the first spot.
00:15:07	Jeff:	Yeah. Then there's, after, then once one goes there, there's only two people left to fill in this spot. And then-
00:15:10	Romina:	So you multiply three and two.
00:15:12	Jeff:	Three times two and then once, once someone goes in the other, there's only one person left. And they get the last spot, so that's times the one.
00:15:18	Romina:	And that's everyone.
00:15:19	Jeff:	That make more sense?
00:15:20	R1:	Well I'm, I didn't mind your other example here.
00:15:23	Jeff:	Yeah, I, I just like the okay through the way so I could move- You know, steady progress.

Time	Speaker	Transcript
00:15:27	R1:	Um. But I guess, so why are you multiplying?
00:15:30	Romina:	We don't like that question.
00:15:30	Jeff:	Ah.
00:15:31	R1:	You don't like that question.
00:15:32	Romina:	No. That one gets us all the time.
00:15:34	R1:	Why aren't you adding?
00:15:35	Jeff:	Uh, because you don't add. It's just, you don't do it. There's no adding going on it any where anymore. That's like out of style. [Romina laughs.]
00:15:41	R1:	That's not the answer.
00:15:42	Jeff:	I know that doesn't, that doesn't work. Um, you do it because, uh-
00:15:52	Michael:	I can't help you on this one.
00:15:53	Jeff:	Yeah, I know.
00:15:54	Romina:	Yeah, we're-
00:15:56	Michael:	That's a good question.
00:15:57	R1:	OK, I'll leave you to tell me.
00:15:58	Michael:	Why do you multiply?
00:15:59	R1:	You'll figure that out. [Researcher 1 leaves the table.]
00:16:00	Romina:	We never know this one.
00:16:02	Jeff:	Yeah it's like the //eternal question.
00:16:03	Ankur:	//Yeah it's cause, if you have three things, there's three things you put here, right?
00:16:03	Romina:	Mm hm.
00:16:09	Ankur:	And red, white and blue. And then there's only-
00:16:09	Romina:	Uh, are we [Inaudible.].
00:16:10	Ankur:	-two things.
00:16:12	Michael:	//And if there's two more-
00:16:12	Ankur:	//Out of that two-
00:16:13	Romina:	//We're doing just two colors. We're doing two colors.
00:16:14	Jeff	Yeah, just do- No, we're- Yeah.
00:16:16	Michael:	//If you have like-
00:16:17	Romina:	//To explain it, maybe you want to do three different colors?
00:16:18	Jeff:	No. Yeah, all right, maybe we can do that. All right, how you saying this?
00:16:22	Ankur:	The red, white and blue, right?
00:16:25	Romina:	OK.
00:16:26	Ankur:	You take, if red goes over here, that means you only have, with red there could go either go white and blue.
00:16:32	Romina:	Mm hm.
00:16:33	Ankur:	Like it's each one of those three goes with two more. You know

Time	Speaker	Transcript
		what I mean? There's three things-
00:16:40	Michael:	You could see how you got this.
00:16:41	Ankur:	-here and then there's two things here.
00:16:39	Michael:	Because you see you have-
00:16:40	Jeff:	All right, yeah.
00:16:41	Ankur:	Each one of those, those three goes with //two other.
00:16:42	Jeff:	//Those three things go with-
00:16:43	Romina:	//OK, like with our line thing.
00:16:44	Ankur:	//So it's three times two.
00:16:45	Jeff:	All right.
00:16:45	Romina:	Like our line thing.
00:16:47	Michael:	Or you could say like you have two more colors to add on. So you could do, you could make these into two different combinations.
00:16:52	Ankur:	Yeah.
00:16:53	Michael:	So that's two.
00:16:53	Jeff:	Yeah. That's- Yeah, that's why. All right.
00:16:54	Michael:	That's like times. That's why you multiply.
00:16:55	Ankur:	That's how you-
00:16:56	Michael:	That's just why. All right? Don't ask us anymore.
00:16:59	Jeff:	All right, so then, all right. Uh, //Researcher 1. [Jeff gestures.]
00:17:00	Romina:	//Researcher 1. [laughs.]
00:17:03	Jeff:	All right, I think we're good with this.
00:17:06	R1:	I'll stay here. Explain it to me on the board.
00:17:07	Jeff:	All right, the reason- here, Ankur.
00:17:10	Ankur:	Just do it; you're right there.
00:17:11	Romina:	You could just say it.
00:17:13	Jeff:	Um, just do it with three colors?
00:17:15	Ankur:	Yeah.
00:17:16	Jeff:	All right, say you have three colors, red, white and blue. [Writes R W and B in a column on the board.] Uh, here you do it.
00:17:21	Ankur:	Yeah, one of those colors goes in the first.
00:17:22	Jeff:	All right. [Jeff turns to the board.]
00:17:23	Ankur:	One of those colors goes in the first spot.
00:17:24	Jeff:	So, say you have your three spots. [Jeff draws three dashes in a column on the board.] Say red goes in the first one, all right? [Jeff writes R on the first dash.] Then you could do-
00:17:28	Ankur:	Either one of them-
00:17:29	Romina:	Draw the line through the white and blue.
00:17:31	Ankur:	One, one color goes in the first spot, so there's two colors left. So

Time	Speaker	Transcript
		there's three different colors that can go in the first spot and each of those colors can go with two other colors.
00:17:39	Jeff:	Two other ones. So this is either going to be a white and blue or a blue and a white. Right? And then- [Jeff writes on the board.] Or the white could to up to the first thing and this could be one of the two other colors or the blue's going to go here and it's going to be, the other two are going to be the combination either way of the other one. So that's why you multiply.
00:17:57	Romina:	Make that a [Inaudible.].
00:17:57	Jeff:	I used to have a bandaid on and now I got chalk stuck to my finger.
00:18:00	Michael:	It's impressive, huh? [Romina laughs.]
00:18:03	R1:	What does this have to do with the towers and what you were showing me about *a* plus *b* to the *n*?
00:18:11	Michael:	Well, you asked us why you multiply.
00:18:12	R1:	And why, why they would be-
00:18:13	Ankur:	We just answered why we multiply.
00:18:14	Michael:	Yeah.
00:18:15	Jeff:	Yeah. We're not there yet.
00:18:16	R1:	OK.
00:18:17	Romina:	We're still working on that.
00:18:19	Jeff:	Yeah, all right.
00:18:20	Michael:	All right. So that's why you multiply.
00:18:22	Jeff:	All right. Moving on. So that's why that's three factorial. So that's, all right, that's good.
00:18:26	Romina:	That's all your combinations right there.
00:18:27	Jeff:	Yeah. All right. [Jeff erases board.] All right, now we're going to put that number over, um, *n* minus *x*, um- [Jeff writes $(n\text{-}x)!$ in denominator; numerator is 3!]
00:18:41	Romina:	Factorial.
00:18:42	Michael:	Explain that part.
00:18:44	Jeff:	All right. This, the *n* would be the number you were-
00:18:49	Michael:	You want to, you know, choosing from. Like, let's say two. This is three choose two. You want to know how many different places you could put those three.
00:18:55	Jeff:	Yeah. So that's where the *n* comes in.
00:18:57	Michael:	So you-
00:18:58	Jeff:	So you're going to take a number so that would be, that would be three and the same reason the three's up there, it's coming down here.

Time	Speaker	Transcript
00:19:03	Michael:	Minus, minus the x.
00:19:04	Jeff:	Minus-
00:19:06	Michael:	Then it'll give you one.
00:19:07	Jeff:	The reason you subtract, that's why you're raising the, how come the x is there? Because you're raising it to two, um. That's it. Right?
00:19:17	Michael:	Right.
00:19:17	Jeff:	That's why it's there.
00:19:18	Romina:	And then-
00:19:18	Jeff:	The x.
00:19:18	Romina:	Multiply-
00:19:19	Jeff:	And then that subtracted will give you, will give you-
00:19:24	Michael:	If this was, if this was a higher number like five choose two, you, that n minus x would be like a three and-
00:19:30	Jeff:	And the factorial-
00:19:32	Michael:	Those-
00:19:32	Jeff:	-will eliminate all the other ones-
00:19:34	Michael:	Yes, and those-
00:19:34	Jeff:	-that you don't want.
00:19:35	Michael:	Those three, it doesn't, you don't want to know where, It doesn't matter where they are. That's why you want, you want, you know, eliminate them. Because you only, you're only worrying about the two. How many different combinations that you could put those two in. It's five choose two. You're only worried about like the, the two. Like I said, the people on the line. Five people on the line, you want to know how many different places you could put those, those two people.
00:19:59	Jeff:	All the other ones where-
00:20:00	Michael:	Now there's going to be, there's going to be a lot of, a lot of repeats because you're also going to count by those other three people where they're going to be and you're not worried about those other three people.
00:20:08	Jeff:	So that only-
00:20:10	Michael:	So that's, that's why you would divide, to get rid of the, to get rid of them.
00:20:12	Jeff:	To subtract them.
00:20:15	Michael:	No, it's divide. Why divide that, n minus x?
00:20:18	Jeff:	Oh, that's the way. All right. All right. OK, all right.
00:20:22	Michael:	And so, you say the next part. I don't know. [Inaudible.]
00:20:24	Jeff:	All right. And, why, why do we want-

Time	Speaker	Transcript
00:20:26	Michael:	I don't know. I don't, I, no it's times, huh.
00:20:28	Romina:	Times.
00:20:29	Jeff:	[Jeff continues to write on the board; refer to Figure J5.] There's supposed to be another one in there. Why is, why-

$$\frac{3!}{(n-x)!*x!}$$

Figure J5. Jeff's preliminary equation for ways to arrange three people on a line

00:20:37	Romina:	Didn't, didn't you guys say something about repeats?
00:20:40	Michael:	Yeah, that's what Bob said. I don't know. I don't trust that kid.
00:20:41	Romina:	That gets like the repeats out.
00:20:44	Michael:	But it worked. It works. That's all.
00:20:45	Jeff:	All right, we don't know where the, the final *x* comes from.
00:20:47	R1:	Why don't you, um-
00:20:47	Michael:	Why don't we think about it?
00:20:48	Jeff:	Work on it?
00:20:49	R1:	You need to work out a piece of the problem and see if you can tell me. Ankur's not convinced. He's looking at me, not being convinced.
00:20:55	Jeff:	What, what um-
00:20:56	Michael:	You're not convinced, Ankur?
00:20:57	Jeff:	Like I, I mean how-
00:20:59	R1:	Are you convinced, Ankur, about this?
00:21:00	Jeff:	Yeah but say, all right, say we're doing five choose two, right, with this. Then we go five factorial. [Jeff writes on the board.] Which is what?
00:21:07	Michael:	That'll give you all the combinations they put everybody in.
00:21:09	Jeff:	Uh, twenty times three.
00:21:11	Ankur:	OK. Sixty.
00:21:12	Jeff:	Would be sixty times two.
00:21:14	Ankur:	One-twenty.
00:21:14	Jeff:	One-twenty? That would be; it's one-twenty, right, Romina?
00:21:18	Romina:	Yeah.
00:21:20	Jeff:	We're faster than the calculator, right? [Romina laughs.] We're good like that. So that'd be one-twenty.
00:21:24	Michael:	And, and if you're doing choose two, obviously there's going to be a lot of times where those two are going to be in the same spot as the other three are going to be-

Time	Speaker	Transcript
00:21:30	Romina:	What are you doing, five choose two?
00:21:31	Michael:	-you know, I guess moving around different spots.
00:21:31	Jeff:	Yeah.
00:21:31	Michael:	That's why you want to get rid of the, the n minus x thing.
00:21:35	Jeff:	Yeah, we got, that makes sense.
00:21:36	Michael:	Yeah, that, that makes sense to you?
00:21:37	Jeff:	That, that part right here, is this all good? Up to this point? Do you understand why this is all happening?
00:21:44	R1:	I'm waiting for the whole thing.
00:21:47	Michael:	Whole thing? Oh we're not done with that yet.
00:21:49	Jeff:	Then, um, then you multiply. Well, at this point here you have three.
00:21:54	Romina:	That's six.
00:21:54	Jeff:	Yeah, it's six. So you have one-twenty over six times five factorial. [Jeff writes this on the board.]
00:22:03	Romina:	No isn't it-
00:22:03	Michael:	Oh I guess the repeats-
00:22:04	Jeff:	Or-
00:22:04	Michael:	Would, would be like-
00:22:05	Romina:	Isn't it three factorial, two factorial?
00:22:07	Jeff:	Three factorial. Oh two, oh, it's act-, all right, yeah. Two.
00:22:10	Michael:	I guess the, the x-
00:22:12	Jeff:	That's the number you were raising-
00:22:14	Michael:	That's x.
00:22:15	Jeff:	-and, and five choose x, say and there was-
00:22:15	Michael:	Since you- Mm hm.
00:22:16	Jeff:	And this was-
00:22:18	Ankur:	I get it. I get it. I get it. I get it. [Romina laughs.]
00:22:21	Michael:	I, I got it now.
00:22:23	Jeff:	Like that. [Jeff finishes writing on the board.]
00:22:25	Michael:	All right, then the last number would be-
00:22:26	Jeff:	Because this just gives you the number.
00:22:28	Michael:	You have- Yeah. Yeah.
00:22:29	Jeff:	You're going to multiply by the number.
00:22:29	Michael:	Those, those, you want to get rid of those. The, all the combinations that the three are moved around and those, those two aren't.
00:22:33	Jeff:	Yeah, they-
00:22:35	Michael:	But then those two themselves will be repeat-
00:22:36	Jeff:	Yeah-

Time	Speaker	Transcript
00:22:37	Michael:	-will be mixed up.
00:22:38	Jeff:	You're repeating that's what you-
00:22:39	Michael:	That's why you want to get rid of that, too.
00:22:40	Jeff:	Exactly. And then, so that would be just two.
00:22:42	Michael:	Yeah.
00:22:43	Jeff:	So it would be one-twenty divided by twelve and you get ten. Is that what it is?
00:22:55	Michael:	Yeah it is. Do you get like why we divide by the n minus x and the x? You know, you, you get that?
00:23:07	R3:	I don't get that. Could you [Inaudible.]?
00:23:07	Michael:	You don't get that?
00:23:08	R1:	Ankur, did you have that?
00:23:09	Jeff:	What, what part [Inaudible.].
00:23:10	R1:	I wonder if Ankur has that? I wonder if Ankur could explain it.
00:23:11	Romina:	I don't think the x [Inaudible.].
00:23:15	Michael:	All right. The top thing, the n to the, the n to the, uh, factorial was going to give you how many?
00:23:21	Romina:	That's all the combinations.
00:23:22	Michael:	That's every single combination.
00:23:23	Romina:	I got that. That I got.
00:23:24	Michael:	Right? Now you're, you're only worried about them, those two people in that line. So there's going to be some instances where those two people are going to be in the same place and those three-
00:23:32	Jeff:	Are the ones changing.
00:23:33	Michael:	Will be, you know, will be switch, you know, changing.
00:23:34	Jeff:	And that's-
00:23:35	Michael:	So that's, that would be the, the three factorial. You want to, you want to get rid of that. You want to get rid of them.
00:23:40	Ankur:	It's a negative?
00:23:41	Romina:	Hold on. Well, we-
00:23:41	Michael:	Don't worry about that three, we're doing like five.
00:23:43	Romina:	No, we're doing this one so the two-
00:23:43	Ankur:	All right, so you have the five minus two, is that what you're explaining on there?
00:23:46	Romina:	Five minus two, that's-
00:23:46	Michael:	So you have the hundred and twenty different combinations.
00:23:46	Ankur:	Yeah.
00:23:47	Jeff:	Total.
00:23:49	Michael:	All right. But you don't think like when those two people are

Time	Speaker	Transcript
		going to be in these two spots-
00:23:52	Jeff:	And everyone else is changing.
00:23:54	Michael:	-not those other three.
00:23:54	Jeff:	And those are, those are, those make no difference because all we're worried about are where those two people are.
00:23:56	Romina:	Oh like when, oh, oh, okay, okay, okay.
00:23:58	Michael:	All right, those two people are going to be moving around and it- you know, they're like-
00:23:59	Jeff:	These people are going to stay the same and all the every, all the three people, they're just going-
00:24:00	Michael:	-the two people staying in the same place. So that's why you get rid of that.
00:24:02	Jeff:	You know, going nuts.
00:24:02	Michael:	But then those two people themselves could switch places too.
00:24:06	Ankur:	Yeah. [Ankur nods.]
00:24:07	Michael:	You know what I'm saying?
00:24:07	Ankur:	Um-huh.
00:24:08	Michael:	Or if-
00:24:08	Ankur:	So you got to get rid of those, too.
00:24:08	Michael:	-there were three that could go on. [Michael crosses his arms, with index fingers up.]
00:24:10	Jeff:	So that's why you get rid of the three.
00:24:11	Ankur:	That's why you do the x.
00:24:12	Michael:	Then you get rid of the, you know-
00:24:14	Jeff:	The other one.
00:24:15	Ankur:	Yeah, so you get rid of those.
00:24:16	Romina:	OK.
00:24:17	Jeff:	And then-
00:24:17	Romina:	Oh, there you go. That makes sense.
00:24:19	Michael:	Because you're not worried about every, each person.
00:24:20	Romina:	Just the two.
00:24:21	Michael:	Just worry about two, right.
00:24:22	Jeff:	Just those two. Exactly.
00:24:23	Romina:	Yeah, we have, I got it. I'm good.
00:24:24	Michael:	Extension?
00:24:26	R1:	Ankur? Can you explain this because poor Researcher 3 is trying to understand this, and she's not following Michael.
00:24:36	Ankur:	Something like, I understood it but-
00:24:39	Jeff:	Just go through it.
00:24:40	Ankur:	All right. The top number is five factorial, that's the total number

Time	Speaker	Transcript
		of possibilities for, for five, for five people. And then the five minus two comes, comes in where you're not worried about everyone, you're just worried about two people at a time. So we need to subtract the five minus two. Those get, that gives you and you do factorial, that gives you all the possibilities of just two people, right?
00:25:05	Romina:	Three people.
00:25:06	Jeff:	No, three extras.
00:25:07	Michael:	No. The three that you don't, you're not worried about.
00:25:08	Jeff:	That's going to eliminate everyone except the two people you're worried about.
00:25:12	Ankur:	OK. Everyone except the two people you're worried about. And then the x factorial eliminates, except the-
00:25:18	Michael:	When the two people-
00:25:19	Romina:	Two people, yeah.
00:25:20	Ankur:	Yeah. When the two people are switched back and forth when you have the same ones over again.
00:25:25	Jeff:	OK, [Inaudible.].
00:25:26	R3:	It's, it's getting better. It's getting better. So they switch back and forth you're saying and with your fingers. [Researcher 3 wiggles her fingers.] I think I'm getting- So could you give me an example?
00:25:38	Ankur:	Like when you have, when you have like person A over here.
00:25:41	Michael:	You want to stand up and show them?
00:25:41	Ankur:	And person B over here. And then you have person B and person A.
00:25:42	Michael:	You want to be in a line and we'll show them?
00:25:43	R1:	Michael, start from the beginning very slow.
00:25:45	Michael:	All right. You have five people.
00:25:46	R1:	Stand up and show us.
00:25:47	R3:	Stand up and show us.
00:25:48	Michael:	All right. [Michael goes to the board.]
00:25:50	Jeff:	All right, I'm going to sit in your seat.
00:25:53	Michael:	I'm going to write it nice and clear so it [Inaudible.]. [Michael erases and then writes 5! divided by 3! times 2! on the board.] All right. You got five people in a line. You agree with me that's how many different combinations you can put those five people.
00:26:10	R3:	That part I understand.
00:26:11	Michael:	All right.
00:26:11	R3:	When you did multiplication that you showed.

Time	Speaker	Transcript
00:26:14	Michael:	Now and you're only worried, you want to know how many different places you can put those two people. All right? So in all the combinations you're going to have, they're going to be repeated a lot. A lot. When you have like, the two people in a certain place and you know, those three. If the three are, are like this. [Michael points to the board and moves his hands.] And then one of them switches, that's another combination. And you get a lot of repeats like that.
00:26:37	R3:	Oh, I see. OK.
00:26:39	Michael:	So by eliminating that, you eliminate the combinations that repeat by the three people moving around.
00:26:45	R3:	Uh-hum.
00:26:45	Michael:	Then let's say you just have those two people in, in any given combination. If, if one, if this guy switches the place with this guy it's the, they're different combinations, but in this we're not worried about where they are. We just, you understand?
00:27:00	R3:	Mm hm.
00:27:02	Michael:	That's why we get rid of the, the two factorial to, to, um, eliminate the amount like as many times as you could, as many combinations as you could put those two people. Right? Like the three would, would be to eliminate the combinations you could put those three people that you're not worried about. Then the two, they would repeat because those people, they move around. They, they could, they move around in the, in the line also. And then when, when, when you're done with all that, you just get, um, you get how many places you can just put that two. Like you're not worried if, like you don't care who they are. You don't care like if this guy has a switch with this guy. You understand like why you would eliminate, how that eliminates.
00:27:41	R1:	OK. I don't want to say those are people. I want to think of the tower now. Isn't that what Jeff said? And now I'm thinking of towers that are five tall?
00:27:50	Jeff:	Yeah. We just- [Mike sits.]
00:27:51	R1:	And we're talking of those that have two reds?
00:27:54	Jeff:	Yeah. Well. [Inaudible.]
00:27:54	R1:	Explain it to me with that.
00:27:55	Jeff:	All right. Say, say we're doing, we're doing towers that were, were five tall. Towers of five tall with two different colors in it. Then that's the total amount of possibilities is the five factorial that you could have. All right, in, with, with five high

Time	Speaker	Transcript
		combinations. So that's where, that's the five factorial on top. Then the three factorial on the bottom would be five different, five different spots minus the two spots that you're concerned about, leaving you with the three other spots-
00:28:26	Romina:	You could say-
00:28:26	Jeff:	-that you don't care about. That's going to eliminate all of them.
00:28:29	Romina:	That's like, if you say like the reds. Let's say reds are our two colors that they stay in the same place, and like-
00:28:34	Jeff:	Reds.
00:28:34	Romina:	There was. Like yeah, the two stay in the same place and then the other three are just switching while they're staying in the same place.
00:28:39	Jeff:	Yeah, they're staying in the same spot.
00:28:40	Romina:	But we're not concerned with those.
00:28:41	Jeff:	That's why you're not concerned with those.
00:28:43	Michael:	It's going to repeat like six times.
00:28:44	Jeff:	Yeah. So that's where the three factorial comes from, and you're multiplying that by the two factorial. Those are what you're-
00:28:50	Romina:	That's to say like the first place and the third place and then they just switch.
00:28:51	Michael:	Yeah, like-
00:28:52	Jeff:	Exactly.
00:28:54	Michael:	They just don't have a name on them so the, they're the same thing.
00:28:56	Romina:	Yeah.
00:28:57	Jeff:	And then that's where the bottom number comes from and then you divide them by each other and that gives you what we're looking for.
00:29:04	R1:	OK, so I think I follow what you said. But why were we doing this?
00:29:09	Jeff:	Uh, you, we don't-
00:29:11	Michael:	We were talking about-
00:29:12	Romina:	We want, you wanted us to explain the choose.
00:29:13	Michael:	The choose that we, all right, whoa-
00:29:15	Romina:	Which goes back to Pascal's Triangle and see where *a* plus *b*-
00:29:19	Michael:	Yeah.
00:29:19	Romina:	-to the *n*.
00:29:20	Michael:	All right. Over here, you wanted, the *a* plus *b* to the *n* thing, you wanted to know how we got the choose thing. What does that mean?

Time	Speaker	Transcript
00:29:24	Jeff:	Yeah, how we got the third number.
00:29:25	Romina:	Yeah.
00:29:26	Jeff:	And that's how we got off to, to here.
00:29:30	R1:	OK, so what did that have to do with what you did in class today?
00:29:32	Romina:	That's how we would get the number.
00:29:33	Jeff:	We were looking at, we were doing this in class today. That's what we were doing. We were looking at *a* plus *b*-
00:29:37	Romina:	We're going to be-
00:29:38	Michael:	It was like in Pascal's Triangle things go like, by that. Like this choose this. Like, um, if you go to the one, three, three, one part of it, it would be, um-
00:29:47	R1:	Show me on the board, Michael. [Michael goes to the board.]
00:29:49	Jeff:	Go get 'em, Mike.
00:29:52	Michael:	[Michael writes rows 0 to 3 of Pascal's triangle; refer to Figure J6.] This would be like, all right, this would be like three choose one. How many different places you put that one, that one guy. There's only one place. There's only, oh, I'm wrong. What am I doing?

1
1 1
1 2 1
1 3 3 1

Figure J6. Michael draws rows 0 to 3 of Pascal's Triangle

Time	Speaker	Transcript
00:30:18	Romina:	That's when you only have like, it's all one color.
00:30:20	Michael:	No, there, there's a way it has something to do with- I think that would be three choose zero, I guess. No. All right, and then the next one would be three choose three. Obviously three different places.
00:30:32	R1:	Three choose what? What was the next one?
00:30:34	Michael:	Three choose one. The next would be three choose two, which we just figured that out. There's three. And last one is three choose three. You can only put those three people in those three places. You can't, you know, no more places to put them.
00:30:48	R1:	OK so that's really interesting. That's really very interesting. So you've put something else together. I have another question. You could write more rows of that triangle.

Time	Speaker	Transcript
00:30:58	Michael:	Yeah.
00:30:59	R1:	And now you're telling me you can write them as the choose way, you've called that. So can you take, let's say another row or two and show me the addition rule and what it looks like with your new notation.
00:31:17	Michael:	You mean the addition rule.
00:31:18	R1:	For a particular, for a particular row.
00:31:20	Michael:	Add this and this [Michael points to Pascal's Triangle on the board.] and go like that?
00:31:21	R1:	Sure, or three and three, six. Show me what that looks like with that new notation. Do you understand my question?
00:31:29	Michael:	Uh, I don't really.
00:31:29	Romina:	I don't understand.
00:31:29	Ankur:	Instead of writing three you write-
00:31:33	R1:	Write your next row, Michael. [See Figure J7.] Now some time ago you, you had a reason. You explained to me-

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Figure J7. Michael adds row 4 to Pascal's Triangle

Time	Speaker	Transcript
00:31:45	Michael:	Why you add.
00:31:46	R1:	Why you add.
00:31:47	Michael:	Yeah.
00:31:48	R1:	You remember that? You, might, might be useful for folks who haven't heard it to hear it whatever way you want to explain it.
00:31:53	Michael:	I don't think I can explain it too good. Um.
00:31:55	R1:	Um, you know, however you want to explain it. You've had it a few ways.
00:32:00	Michael:	Um, I can't, I can't remember too well. I know why you add, if I explain it, I don't think anyone will understand.
00:32:13	R1:	Try.
00:32:15	Michael:	I didn't. Didn't I tell you guys like last time I came here?
00:32:18	Jeff:	Well, go for it, dude, just-

Time	Speaker	Transcript
00:32:20	Romina:	You could try.
00:32:20	Michael:	You don't have that paper, do you? You can just hand them, hand that out.
00:32:21	Romina:	You started talking about toppings. I think something-
00:32:24	Michael:	Hand that out instead.
00:32:25	Jeff:	Just-
00:32:27	Michael:	Um, all right. If, all right, let's go to, let's go to this one. This would be like three different places I guess. [Michael points to the 1331 row of Pascal's triangle.] And um-
00:32:37	Jeff:	Which one we looking at?
00:32:38	Michael:	That one right there. [Michael points to 1331.] You have three-
00:32:41	Jeff:	That would be *a* plus *b* to the third.
00:32:42	Michael:	All right, let's say you have like, here's a number, all right? [Michael writes 000.] Zero means no toppings. One would, this, one would be-
00:32:51	Romina:	This one would be a topping.
00:32:51	Michael:	One would be a topping. So first category is everything with no toppings. [Michael points to the first 1 in the 1331 row.] And that's, you can't make, that's, that's your number for that one. [Michael points to the first 0 in 000.] That's like, like binary numbers or something. Next would be- [Michael writes 001, then 010, then 101.] There's all the, the ones that have one topping.
00:33:12	Jeff:	Right, you got to write that 0 at the end. You messed up.
00:33:14	Michael:	What?
00:33:14	Jeff:	Last one should be a hundred, not a hundred and one.
00:33:15	Michael:	I knew that. [Michael changes 101 to 100.] There's your, um, your 3 choose 1. [Michael points to 001.] And there's three different combinations you could put that. Um, I can go on forever doing this. But, um, when you have a new, when you add another place, another topping- [Michael draws three dashes next to the numbers already there.]
00:33:34	Jeff:	That could be one or the other, one or the other-
00:33:36	Michael:	So it could be one or the other.
00:33:36	Jeff:	-one or the other.
00:33:37	Michael:	It could be a 0 or 1, a 0 or 1. [Michael writes 0 and 1 above each dash; refer to Figure J8.]

Time	Speaker	Transcript

000^{0}_{1}

001^{0}_{1}

010^{0}_{1}

100^{0}_{1}

Figure J8. Binary representations of pizzas with no toppings and one topping

00:33:38	Jeff:	Yeah. All right.
00:33:39	Michael:	A zero or one. So all these threes would either move up a step onto the next category and, uh, have two toppings. [Michael points to the 6 in row 4.] Or they might stay behind and still only have one if they have the zero. [Michael points to the 4 in row 4.] So three, three will get a topping, go to this one. [Michael draws lines from the first 3 in row 3 to the 4 and 6 in row 4, and then Michael points to the 6.] And three won't, will stay. [Michael points to the 4.] And obviously this guy's going to get a topping. [Michael draws a line from the first 1 in row to 3 to the first 4 in row 4.] That's why you add this one. [Michel points to the first 1 in row 3.]
00:34:03	Jeff:	Uh-huh.
00:34:05	Michael:	So now this guy's going to have, without toppings. You're going to add a topping on to him. That's going to be one topping. [Michael points to the first 4 in row 4.] These three [Michael points to the first 3 in row 3.] with one topping won't get one so, you know-
00:34:14	Jeff:	That's their four.
00:34:15	Michael:	You can put them in the same category as this one. [Michael points to the first 3 in row 3.]
00:34:17	Jeff:	Yeah.
00:34:17	Michael:	That's four.
00:34:17	Jeff:	Those are your four.
00:34:18	Michael:	And you know-
00:34:19	Ankur:	Three.
00:34:19	Jeff:	Those three.
00:34:20	Michael:	[Michael draws a line from the second 3 in row 3 to the 6 in row 4; refer to Figure J9.] The three that had two toppings won't get any.

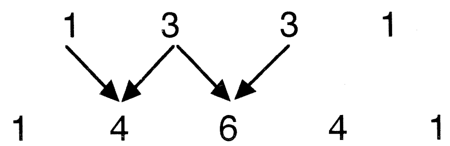


Figure J9. Indicating addition on Pascal's Triangle

Time	Speaker	Transcript
00:34:23	Jeff:	Yeah. So they'll go to [Inaudible.].
00:34:23	Michael:	And you could put them in together with the ones that did get something. That's why you would add. Keep on adding.
00:34:28	R3:	What do you mean by toppings?
00:34:29	Michael:	Pizza toppings.
00:34:30	R3:	Um, for example-
00:34:31	Michael:	Like here you have a choice of three different ones. Here you have a choice of five and like the ones would be like the mushrooms, the peppers, the whatever, just by going like- The one would indicate you have it or not.
00:34:46	R1:	OK. OK. I remember.
00:34:48	Michael:	You remember.
00:34:49	R1:	I remember this. But now I don't want to think of the numbers in that triangle, I want to think of those as chooses. [Researcher 1 walks to the board.] So for example, let's just take this row. One, three, three, one.
00:35:11	Michael:	Mm hm.
00:35:13	R1:	All right. [Researcher 1 writes 1 3 3 1.] If I wrote these as chooses the way you're writing them-
00:35:19	Michael:	3 choose 0, 3 choose 1.
00:35:20	R1:	This is 3 choose 0. [Researcher 1 writes this.]
00:35:21	Michael:	Yeah.
00:35:22	R1:	This is 3 choose 1. [Researcher 1 writes this.]
00:35:23	Jeff:	Choose 1. Same thing.
00:35:24	R1:	Three choose-
00:35:25	Michael:	Two and three choose, then three choose, three choose three. [Researcher 1 writes this; refer to Figure J10.]

Time	Speaker	Transcript

$$1 \quad 3 \quad 3 \quad 1 \qquad \binom{3}{0} \binom{3}{1} \binom{3}{2} \binom{3}{3}$$

Figure J10. Researcher 1 writes row 3 of Pascal's Triangle in two different ways

00:35:28	R1:	Right.
00:35:29	Jeff:	So that's how you get it. It's like the same thing, cause like 3 and 0 is like 3 and 3, right? And then 3 2.
00:35:32	R1:	OK, so-
00:35:34	Michael:	You want us to write the triangle looking like that?
00:35:36	R1:	I would, I would, I would like you to do that and then tell me what the general rule is.
00:35:41	Jeff:	All right.
00:35:42	R1:	With this notation. Do you understand my question? I'll leave you to work on that. So, so I'd like you to write out some of the rows with the triangle, and then I'd like-
00:35:51	Michael:	So use it like, like that. Like the next one would be, uh, four choose zero.
00:35:55	Jeff:	Yeah.
00:35:56	Romina:	Four choose -
00:35:56	Michael:	The four choose zero then //four choose one, four choose two- [Michael writes this on the board.]
00:35:57	Jeff:	//Four choose one, four choose two.
00:35:58	Ankur:	Four choose three.
00:36:00	Michael:	We're in a bad place. [Michael has run out of room on the board.]
00:36:02	R1:	Right. You probably want to use this. [Researcher 1 indicates parentheses.]
00:36:03	Michael:	Yeah. [Michael erases and rewrites; refer to Figure J11.]

$$\binom{4}{0} \binom{4}{1} \binom{4}{2} \binom{4}{3} \binom{4}{4}$$

Figure J11. Row 4 of Pascal's Triangle

00:36:03	R1:	So that people can read it.
00:36:04	Michael:	Um.
00:36:05	Alex:	Ask your question one more time.
00:36:06	R1:	OK, so I'd like you to rewrite your triangle if you like.

Time	Speaker	Transcript
		[Researcher 1 approaches the board.]
00:36:09	Michael:	From top to bottom?
00:36:10	R1:	Top to bottom.
00:36:11	Romina:	Do you want the ones and like-
00:36:13	Jeff:	All right. So what-
00:36:14	R1:	I want everything-
00:36:14	Jeff:	What would-
00:36:14	R1:	I want everything written in this form. [Researcher 1 points to what Michael has written.] Do you understand?
00:36:16	Ankur:	Uh-huh. [Ankur nods.]
00:36:17	Michael:	That's, that's easy. [Michael erases the board.]
00:36:18	R1:	And then I would like the general row.
00:36:19	Jeff:	Is that one?
00:36:19	R1:	What would the general row look like? [Michael writes rows 0 through 2 of Pascal's Triangle in choose notation.] Where you have towers? [Researcher 1 forms a tower with her hands.]
00:36:24	Romina:	That's a 0, no, 0, 3.
00:36:27	Michael:	*X* high.
00:36:28	R1:	Something like that.
00:36:29	Jeff:	All right, well that's-
00:36:30	R1:	Ankur understands. So he can tell you. [Researcher 1 returns to her seat.]
00:36:37	Romina:	See, like that? [Romina points to her paper.]
00:36:38	Michael:	So it would be, um, like *N* over, not two.
00:36:42	Ankur:	It would be-
00:36:43	Michael:	*N* choose-
00:36:44	Ankur:	It would be-
00:36:46	Romina:	Well, and *N*, make *N* like your height or something.
00:36:49	Jeff:	All right, so say-
00:36:50	Romina:	N equals height.
00:36:52	Jeff:	Well that would-
00:36:52	Ankur:	Well, write the *X*. Write *a* plus *b* to the whatever it is next to it.
00:36:57	Jeff:	Yeah.
00:36:58	Ankur:	You know what I mean?
00:36:59	Jeff:	Yeah. So right. That would be *a* plus *b* to the-
00:37:00	Michael:	This would be nothing, you know, it would be adding.
00:37:02	Jeff:	Yeah, zero, one, two. So *a* plus *b* to the second.
00:37:05	Romina:	Well, it'd be like *N* over *N* minus, but what?
00:37:07	Jeff:	Yeah, well, *a* plus *b* to the second, so it would be if, or *a* plus *b* to the n^{th}.

Time	Speaker	Transcript
00:37:13	Romina:	To the-
00:37:14	Ankur:	No, all you need is like-
00:37:14	Romina:	*n* is factorial.
00:37:14	Jeff:	It'd be *n*, *n* over-
00:37:16	Michael:	*n*, fa-
00:37:18	Jeff:	*n* mi-
00:37:18	Romina:	No, that's just like- No, it's not right. I'm just saying like-
00:37:21	Jeff:	It would be-
00:37:23	Romina:	You would have to multiply it.
00:37:24	Jeff:	*n* over-
00:37:28	Michael:	Well, if you had an *n*, it would be, uh-
00:37:30	Ankur:	To the height of the tower which is *n*, right?
00:37:32	Michael:	You'd have a bunch of *n*'s.
00:37:33	Jeff:	Yeah, and it'd be over, just-
00:37:34	Michael:	There'd be *n* plus one *n*'s going this way. [Michael points toward the right.]
00:37:37	Jeff:	Yeah. If-
00:37:38	Michael:	All right?
00:37:38	Jeff:	it would be *n* over 0.
00:37:39	Michael:	So if *n* was three, you'd have four *n*'s going this way.
00:37:42	Jeff:	Yeah.
00:37:42	Michael:	And the bottom numbers would be just going from 0 to-
00:37:44	Jeff:	Just-
00:37:45	Michael:	To-
00:37:45	Jeff:	Yeah. Well, yeah.
00:37:46	Michael:	0 to *n*. [Michael writes on the board; refer to Figure J12.]

$$\binom{n}{0}\binom{n}{1}\binom{n}{2}\binom{n}{3}$$

Figure J12. First four entries of Row *n* of Pascal's Triangle

00:37:50	Jeff:	Exactly.
00:37:51	Michael:	To *n*.
00:37:51	Jeff:	To *n*. Whatever *n* equals.
00:37:53	Romina:	Is there a way to write that, you know how [Inaudible.]?
00:37:58	Ankur:	Guess.
00:37:59	Jeff:	Yeah, so how do you, yeah, now that makes sense but, so it would be *n* over 0 to the n^{th}. And whatever-

Time	Speaker	Transcript
00:38:08	Michael:	Zero, what are you talking about?
00:38:09	Jeff:	Wherever you're looking for.
00:38:09	Ankur:	What are you talking about, 0 to the *n*?
00:38:11	Michael:	0 minus *n*? That would be negative.
00:38:13	Jeff:	No, not minus, like that's to whatever *n* is. *n* over 0, *n* over 1.
00:38:18	Romina:	1.
00:38:19	Jeff:	Not divided by like *n*, 1, *n*, uh, 2, *n*, 3.
00:38:25	Michael:	That was-
00:38:26	Jeff:	All the way until *n* could be over *n*. You know what I'm saying?
00:38:28	Michael:	Yeah.
00:38:29	Jeff:	Not, not divided by. I was using bad, uh, bad looking things there. But-
00:38:34	Michael:	Each of those would be a number-
00:38:35	Jeff:	Yeah, it's what, 0 to *n*.
00:38:37	Ankur:	And *n* represents the height of the tower?
00:38:39	Romina:	The height of the tower, yup.
00:38:42	Michael:	Yeah, *n*, *n* represents-
00:38:43	R1:	Do you want that divided sign here?
00:38:39	Romina:	The height of the tower, yup.
00:38:42	Michael:	Yeah, *n*, *n* represents-
00:38:43	R1:	Do you want that divided sign here?
00:38:45	Michael:	No.
00:38:45	R1:	On this one?
00:38:46	Jeff:	No.
00:38:46	Ankur:	No. Cross that off.
00:38:46	Romina:	No.
00:38:46	Jeff:	I was using it to separate, and that was, that's a habit of mine, it looks bad.
00:38:49	Michael:	Sorry about that. [Michael erases the board.] It would be, uh, as many, it's like height of the tower with two colors. You have two numbers.
00:38:59	Jeff:	Yeah. How do you, how are you, can you write that to get this?
00:39:04	Romina:	Like that's what I meant. Like I didn't mean factorial. I meant like when we used four first and like three first. I don't know how to write that, though.
00:39:10	R1:	So you go 0, 1, 2, 3, dot, dot, dot, up to *n*.
00:39:16	Jeff:	Yeah.
00:39:16	Michael:	Mm hm.
00:39:17	R1:	Can we get one in the middle there, like *n* choose *r*?
00:39:22	Jeff:	Like how would you just go right to *n* choose 3? Or *n* choose *r*?

Time	Speaker	Transcript
		that number. It's not, it's never going to change.
00:40:35	R1:	[Researcher 1 walks to the board.] OK. I'll buy that. But something in here could be an *n* choose *r*. Right? Something in here could be an *n* choose *r*.
00:40:41	Romina:	Mm hm.
00:40:42	R1:	That's what I heard you say, Jeff?
00:40:43	Jeff:	Yes.
00:40:43	R1:	Sort of a general one in here, *n* choose *x*.
00:40:46	Jeff:	That's what-
00:40:47	R1:	Whatever you choose to use.
00:40:47	Jeff:	Yeah, that's what that is. So, yeah.
00:40:49	R1:	OK. OK, so this is my question to you. You've written out two rows and you have the third one there.
00:40:55	Jeff:	Mm hm.
00:40:56	R1:	Maybe somebody will come up here and write these up nicely.
00:40:59	Jeff:	Is that what you want?
00:41:01	R1:	Yes. Because then I want to ask, I want; after you do that I have a question to ask you. Thanks. [Researcher 1 hands chalk to Jeff.]
00:41:06	Michael:	You want to erase those? [Jeff walks to the board.]
00:41:17	Jeff:	[Jeff writes rows 0 through 3 of Pascal's Triangle on the board.] You want to make that the line so bad. [Jeff draws and then erases a fraction bar.] I know.
00:41:19	Michael:	No, don't do that.
00:41:30	Ankur:	How far do you want him to go?
00:41:34	Michael:	One more.
00:41:34	Jeff:	I want to, uh. You want one more for good measure?
00:42:02	Michael:	No. Don't worry about it.
00:42:03	R1:	Go to the n^{th} one, then.
00:42:06	Jeff:	Would that just be-
00:42:07	R1:	Dot, dot, dot.
00:42:08	Jeff:	*N* 0.
00:42:10	Michael:	Dot, dot, dot, *N* to *N*. [Jeff writes the first and last entries of row *N*.]
00:42:20	R1:	And the last one, Jeff. Is the last one *N N*?
00:42:24	Michael:	Yeah.
00:42:25	Romina:	Mm hm.
00:42:25	Jeff:	Yeah.
00:42:26	R1:	Don't you want to put it at the end?
00:42:28	Michael:	Yeah, put it at the end, make it nice. [Jeff puts it at the end.]
00:42:30	R1:	What's the middle one there? What would you, how would you

Time	Speaker	Transcript
		show the middle one?
00:42:31	Jeff:	Uh, actually, you could put N, X. [Jeff writes this; refer to Figure J14.]

$$\binom{N}{0}\cdots\binom{N}{X}\cdots\binom{N}{N}$$

Figure J14. Jeff writes row *N* of Pascal's Triangle with a general entry

Time	Speaker	Transcript
00:42:33	R1:	OK. *N* choose *X*, *N* choose *N*.
00:42:40	Jeff:	Those are dots because you can't really make a dot. Now you can. [Jeff sits down.]
00:42:44	R1:	OK, now, now, show me, show me, while you're up there, Jeff, just show me, uh, an addition rule of Pascal's Triangle. Let's say from, give me an example from the third, fourth row to the fifth row.
00:42:55	Jeff:	[Jeff goes back to the board.] Fourth row to this? [Jeff points to row *N*.]
00:42:57	R1:	Fourth row to the fifth.
00:42:59	Michael:	The three to the four.
00:43:00	Jeff:	Oh, fourth row. All right. Um.
00:43:02	R1:	Show me that three plus three is six. Which ones would it be?
00:43:07	Jeff:	That would, like you're saying from here [3 choose 1] to here [3 choose 2] going to there [4 choose 2]?
00:43:10	Michael:	Uh-huh.
00:43:10	R1:	OK, show me. How would you draw your little arrow to show that?
00:43:15	Michael:	[Michael stands up and indicates where arrows should be drawn.] This one and that one.
00:43:16	Jeff:	OK, is that it? [Jeff draws arrow on board as indicated by Michael; refer to Figure J15.] Is that all, that's all you want?

Time	Speaker	Transcript

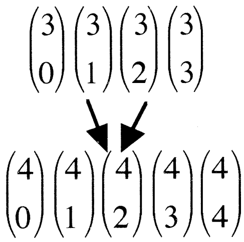


Figure J15. Jeff and Michael indicate 3+3=6 in choose notation

00:43:18	Michael:	Yeah.
00:43:18	R1:	Is that true? Do you believe that?
00:43:20	Jeff:	Yeah.
00:43:20	Michael:	Yeah, I believe so.
00:43:21	R1:	You all believe that?
00:43:22	Romina:	Yeah.
00:43:22	Michael:	Uh-huh.
00:43:22	R1:	No one could persuade you otherwise?
00:43:23	Michael:	No.
00:43:25	R1:	OK, so you're saying three choose one, plus //three choose two equals four choose two. Right?
00:43:27	Jeff:	//Three choose two should equal four choose two.
00:43:30	Romina:	Look at all the numbers added up.
00:43:32	R1:	OK. So what's four choose two plus four choose three?
00:43:35	Jeff:	Four choose two plus four choose three? That would be, [Michael laughs.] that would be five-
00:43:40	Michael:	Oh, five-
00:43:41	Ankur:	Five choose-
00:43:43	Michael:	Five choose three.
00:43:44	Ankur:	Yeah.
00:43:46	Michael:	Right?
00:43:47	Ankur:	Yeah.
00:43:48	Jeff:	Yeah.
00:43:48	R1:	I don't know if Romina's convinced.
00:43:50	Jeff:	Why is he five choose three?
00:43:52	R1:	Yeah, I don't think Jeff is either.
00:43:52	Jeff:	Is this here-
00:43:53	Romina:	Yeah, I don't really-
00:43:53	Ankur:	Because it's, it's always the one on the right.
00:43:55	Michael:	Because, see, this guy gets another topping, I guess, so he turns,

Time	Speaker	Transcript
		he would be a two.
00:44:01	Jeff:	Uh huh.
00:44:02	Michael:	Whatever it is in here. And this guy doesn't, so it stays two.
00:44:03	Jeff:	Ah, it doesn't, it stays two.
00:44:04	Michael:	So-
00:44:05	Jeff:	It was that.
00:44:06	Michael:	Because he's moving up, this bottom number's going to change.
00:44:09	Jeff:	Oh, all right.
00:44:09	R1:	Explain that one more time, Michael, please.
00:44:11	Michael:	Um, wherever this guy goes, wherever this guy goes he's going to get another topping because he's moving this way. [Michael points to the right.]
00:44:18	Romina:	Um-hm.
00:44:18	Jeff:	So that turns into a two.
00:44:19	Michael:	So this bottom number's going to change to two. [Michael points to 3 choose 1.] This guy's not going anywhere. [Michael points to 3 choose 2.] The bottom number stays the same, so it's going to be five. Because you know the next one's going to be five and it, it has to be a two because- You understand why you add? All right. Good.
00:44:33	Romina:	I'm with you.
00:44:34	R1:	OK, so that's really very interesting. Let me ask you to explain that to Brian for a minute, but we'll let him eat first. Did you eat, Brian?
00:44:40	Brian:	No.
00:44:40	R1:	Just help yourself. You can wash up.
00:44:43	Jeff:	We don't get another break?
00:44:45	R1:	All right, Brian, just eat. You can.
00:44:46	Brian:	I don't think you want to know what I went through.
00:44:48	Ankur:	Well at least you got a tux.
00:44:49	R1:	We're glad you're here.
00:44:50	Brian:	Neither did I. I didn't.
00:44:52	Ankur:	I didn't either.
00:44:52	Romina:	What happened?
00:44:52	Ankur:	[Inaudible.] what happened to my coat.
00:44:53	Brian:	The coat is like fit for a midget. [Break in tape.]
00:44:57	Alex:	Keep going. [Jeff is erasing the board. Various off-topic conversations are in progress.]
00:44:57	Michael:	All right.
00:44:59	R1:	[Side conversation.] OK, sure, why not.

Time	Speaker	Transcript
00:45:00	Alex:	OK. Good.
00:45:01	Jeff:	All right. Well, all right.
00:45:02	Ankur:	[Inaudible.] you remember.
00:45:04	Jeff:	All right, we're looking at, we're looking at this right here. You guys got to pay attention to it.
00:45:08	R1:	Erase it better, Jeff, before you start, because-
00:45:09	Ankur:	Yeah, remember and you didn't pay me back for like three months.
00:45:12	Brian:	I had it there the whole time.
00:45:14	Ankur:	Yeah, but it cancelled out.
00:45:15	Jeff:	All right. Say we have this row right here. We got um, *N* choose 0. [Jeff writes this on the left side of the board.] And over here we have *N* choose *X*. [Jeff writes this in the middle of the board.] And then over here we have *N* choose *N*. [Jeff writes on the right side of the board]. All right? Then this right here [Jeff points to *N* choose *X*.] would be- Oh, we're explaining the general addition, the addition rule using this type of, to fill out the triangle. Using chooses to fill out the triangle and this here would be *N* choose *X* plus one and then *N*, *N* choose *X* plus two and so on to whatever *N* equals. Right there we got the- I didn't, I didn't leave enough room. And this here would be *X* minus one and then-
00:46:02	Ankur:	You did that one man.
00:46:03	Jeff:	What?
00:46:04	Ankur:	Nothing.
00:46:05	Jeff:	That'd be *X* minus two and so on each way. Right? So it'd be that.

$$\binom{N}{0}\cdots\binom{N}{X-2}\binom{N}{X-1}\binom{N}{X}\binom{N}{X+1}\binom{N}{X+2}\cdots\binom{N}{N}$$

Figure J16. Expanded version of row *N* of Pascal's Triangle

Time	Speaker	Transcript
00:46:10	Ankur:	Can I see the row below that?
00:46:12	Jeff:	And the row above this would be *N* minus one, right? Yeah.
00:46:17	Michael:	Mm hm.
00:46:19	Jeff:	Um, choose zero. This again would be *N*, *N* minus one choose *X* and then-
00:46:29	Michael:	*N* minus one.

Time	Speaker	Transcript
00:46:30	Jeff:	*N* minus one, *N* minus one. [Refer to Figure J17.] That's a one. Um, how do you want me to, to- Where do you want me to go from here?

$$\binom{N-1}{0}\cdots\binom{N-1}{X}\cdots\binom{N-1}{N-1}$$

$$\binom{N}{0}\cdots\binom{N}{X-2}\binom{N}{X-1}\binom{N}{X}\binom{N}{X+1}\binom{N}{X+2}\cdots\binom{N}{N}$$

Figure J17. Rows *N*-1 and *N* of Pascal's Triangle

00:46:40	R1:	Well, you know, um, Brian wasn't here, so you might want to give him some background to what you've been doing.
00:46:46	Jeff:	Start at the beginning? We did, we worked for an hour and a half getting to this point. Explaining this, doing this. All right, um.
00:46:54	R1:	But Brian's a quick study.
00:46:54	Brian:	That's what I am.
00:46:56	Jeff:	All right. We did, uh, this is Pascal's Triangle using-
00:47:02	Brian:	The whole choose thing.
00:47:03	Jeff:	-the choose situation. That's what this is.
00:47:04	Michael:	You know how choose works, like one, three, three, one.
00:47:06	Brian:	Yeah.
00:47:07	Jeff:	Yeah.
00:47:07	Michael:	Three choose zero, three choose one-
00:47:08	Brian:	One, four, six-
00:47:09	Michael:	Yeah. It's all like chooses of something.
00:47:11	Jeff:	All right. So, um, I don't- Um, how would you like to, uh, how do you want to do this? How do you want to-
00:47:19	Michael:	We're just-
00:47:20	Jeff:	Well, tell him what we did.
00:47:21	Michael:	-replacing the three in the chooses by *N*'s and *X*'s.
00:47:24	Jeff:	Yeah, exactly. And rather doing, like, uh, rather- Say this is the, uh-
00:47:29	Michael:	If N was three.
00:47:30	Jeff:	Yeah, say if N was the third row, it would be three choose zero. [Jeff writes 3 choose 0.] That would give you one. [Jeff writes 1.]
00:47:36	Ankur:	Like, you'd have one, three, three, one.

Time	Speaker	Transcript
00:47:38	Jeff:	Three choose one. [Jeff writes 3 choose 1.]
00:47:39	Michael:	That'd be three.
00:47:39	Jeff:	That would give you the three. [Jeff writes 3.] The three choose two. [Jeff write 3 choose 2.]. That would give you the other three. [Jeff writes 3.] That's equal to three and then three choose three. [Jeff writes 3 choose 3.] That equals the other one. [Jeff writes 1.] And like that's filling out this part of the triangle and so on. And that's what, that's what we're doing now. We went, other stuff we did we did the whole, we found that equation to find out choose.
00:48:01	Michael:	What choose means.
00:48:02	Jeff:	Yeah, we did all that.
00:48:03	Romina:	And choose.
00:48:04	Jeff:	But you missed out on all that. That's the choose equation.
00:48:05	Romina:	That's the choose equals.
00:48:08	Jeff:	And we spent time explaining. That's what we spent the bulk, bulk of the thing, trying to figure out how to explain that. And-
00:48:14	Brian:	What's that little exclamation point?
00:48:15	Michael:	//Factorial.
00:48:16	Romina:	//Factorial.
00:48:16	Ankur:	//Factorial.
00:48:16	Jeff:	Factorial.
00:48:17	Brian:	That's what it is?
00:48:17	Romina:	Yeah.
00:48:17	Jeff:	Yeah.
00:48:18	Brian:	All right.
00:48:18	Jeff:	It was really excited, like *N*!
00:48:20	Romina:	You want to know what this is? That's all the combinations. [Romina points to her paper; refer to Figure J18.] That's minusing. You know how like they're saying-

$$\frac{n!}{(n-x)!\cdot x!}$$

Figure J18. Romina shows factorial notation to Brian

00:48:26	Brian:	Yeah.
00:48:26	Romina:	-three choose two.
00:48:27	Brian:	Yeah.

Time	Speaker	Transcript
00:48:27	Romina:	We don't care about the three, so that's like when the threes are switching, not the twos. And that's when the twos are like in the first place and the third place, and they just switch and nothing else moves.
00:48:35	Brian:	So this-
00:48:35	Romina:	It's basically the same thing.
00:48:35	Brian:	Is this, is that this over this? [Brian points to the expression from Figure J17.]
00:48:37	Michael:	Yeah.
00:48:38	Romina:	It's *N*, *N* factorial over *N* minus *X* factorial times *X* factorial. [Romina writes the expression from Figure J17 again.]
00:48:45	Michael:	And that equals *N* choose *X*.
00:48:46	Romina:	Like this is when the- the things we don't know- [Romina circles $(n\text{-}x)!$] No, I'm just saying using the things that we don't care about when they- they switch and this is one of the things we do care about, just switch in the same place and everything stays the same.
00:48:57	Brian:	All right.
00:48:58	Romina:	And that's all of them. [Romina laughs.]
00:49:00	Ankur:	The Reader's Digest version.
00:49:01	Romina:	Yeah.
00:49:01	R1:	What was that, Ankur?
00:49:02	Ankur:	No, I just said like the Reader's Digest version or something.
00:49:05	R1:	The Reader's Digest version?
00:49:07	Jeff:	Yeah. So where, where do you want to go with, with this?
00:49:10	R1:	Well, I want you to show me how the addition rule works in general.
00:49:14	Jeff:	All right. Well that's how- //[Inaudible.].
00:49:16	R1:	So you showed me what *N* minus one choose *X*-
00:49:17	Michael:	Go from, go from *N X* and *X* plus one.
00:49:19	Jeff:	Wait, this is, this is one.
00:49:21	Ankur:	Yeah, add that in terms of *X*. Like below it, you know what I mean?
00:49:23	Michael:	Add these two. What are these two going to equal?
00:49:26	Jeff:	All right, well that's gonna be-
00:49:27	Michael:	We want the-
00:49:28	Jeff:	-*N* plus one over-
00:49:30	Michael:	-plus *N* plus one.
00:49:30	Ankur:	*X* plus one.
00:49:33	Jeff:	*X* plus one? [Jeff draws lines; refer to Figure J19.]

Time	**Speaker**	**Transcript**

$$\begin{tikzcd}[column sep=small] \binom{N}{X} \arrow[dr] & & \binom{N}{X+1} \arrow[dl] \\ & \binom{N+1}{X+1} & \end{tikzcd}$$

Figure J19. Diagram of the general addition rule

00:49:33	Michael:	*N*.
00:49:34	Ankur:	Yeah. I think. Uh-huh.
00:49:37	Jeff:	That's what these two are going to come into?
00:49:39	Ankur:	Mm hm.
00:49:40	Jeff:	Right?
00:49:41	Michael:	Yeah.
00:49:41	Ankur:	Yeah.
00:49:40	Jeff:	And that's cause-
00:49:41	R1:	Can you write it, can you write it as an equation? Just like you wrote three plus three equals six.
00:49:46	Jeff:	Um, that would-
00:49:48	Ankur:	*N* plus, just that plus that. [Ankur points to the board.]
00:49:50	R1:	Why don't you do it on the side?
00:49:51	Jeff:	Just *N*. Oh, would it be-
00:49:51	Michael:	Oh, *N* choose *X*.
00:49:52	Jeff:	*N* choose *X*, um, plus-
00:49:53	Ankur:	Plus.
00:49:54	Jeff:	-*N* choose *X* plus one. [Jeff writes on the board.]
00:49:57	Michael:	Equals that.
00:50:00	Jeff:	Plus one equals that right there. [Jeff finishes the equation; refer to Figure J20.]

Time	Speaker	Transcript

$$\binom{N}{X}+\binom{N}{X+1}=\binom{N+1}{X+1}$$

Figure J20. Students' version of Pascal's Identity

00:50:02	Jeff:	Then, well, that's, that's because this would be gaining an *X* [Jeff points to *N* choose *X*.] and going into the *X* plus one. [Jeff points to *N*+1 choose *X*+1.]
00:50:14	Michael:	Yeah.
00:50:15	Jeff:	And this would be losing an *X*. [Jeff points to *N* choose *X*+1.]
00:50:16	Michael:	No, no, not losing, not getting anything.
00:50:16	Ankur:	Staying the same.
00:50:17	Romina:	No.
00:50:18	Ankur:	It's not getting anything.
00:50:18	Jeff:	That would be staying the same and that's-
00:50:19	Ankur:	That's, yeah, the plus that.
00:50:20	Jeff:	-is the *X* plus one.
00:50:22	Michael:	And the top numbers have changed because you have more.
00:50:24	Jeff:	Because you're adding, you have more things.
00:50:25	Ankur:	One more.
00:50:25	Jeff:	One more-
00:50:27	Michael:	Topping or-
00:50:28	R1:	Say it so Brian can follow it because he wasn't here for the earlier pizza discussion.
00:50:31	Michael:	You can follow it.
00:50:32	Brian:	I'll just sit in the back and watch.
00:50:33	R1:	Go ahead, Brian. Don't be easy on them, Brian, make them work.
00:50:35	Jeff:	What, what we're doing is the next line of the triangle- Remember how today in class the other triangle was one, two-
00:50:40	Brian:	Yeah.
00:50:41	Jeff:	-three, that whole row there? Well, that's the increase in *N*, and then the *X* plus one. If you added another topping onto your whole. Say we're doing pizzas.
00:50:50	Brian:	All right.
00:50:51	Jeff:	If you add another topping onto it?
00:50:53	Romina:	You know how we get the triangle and how we go one two one and add those two together.
00:50:56	Brian:	Yeah.
00:50:56	Jeff:	Yeah.

Time	Speaker	Transcript
00:50:57	Romina:	That's what we're doing right there.
00:50:57	Jeff:	Yeah. Well, that's what we're doing.
00:50:58	Ankur:	We're just adding it.
00:50:58	Michael:	You know why, do you know why we add, though?
00:50:58	Brian:	That's all you're all doing?
00:50:59	Romina:	That's all we're doing.
00:51:02	Jeff:	We, we were explaining why you add.
00:51:03	Brian:	All right, keep going.
00:51:03	Jeff:	And why do, because when you add another topping like onto it, this one- Say the toppings were one and zero.
00:51:10	Brian:	Uh huh.
00:51:11	Jeff:	If it gets a topping, that's why it goes up to the X plus one. [Jeff points to N+1 choose X+1.] And since it doesn't get anything, it'll stay the same. And in this one, it's staying the same, right? [Jeff points to N choose X+1 and looks at Michael.]
00:51:20	Michael:	Yeah.
00:51:21	Jeff:	And that's why it's going there. Like saying that's the zero.
00:51:25	Brian:	OK.
00:51:26	Jeff:	And going to there. Make sense?
00:51:28	Brian:	Yes. It actually does.
00:51:30	Jeff:	So, so that would be the general addition rule in this case? That's it?
00:51:34	R1:	Are you impressed?
00:51:35	Jeff:	Impressed?
00:51:37	R1:	Mm hm.
00:51:37	Michael:	Not really.
00:51:37	Jeff:	Not really. I don't think we did anything that spectacular.
00:51:42	Michael:	Yeah, that's all.
00:51:43	R1:	Well, you might be.
00:51:44	Ankur:	Nothing more than we ever did before.
00:51:45	R1:	You might pick up a probability book in-
00:51:46	Jeff:	Is this all in-
00:51:47	R1:	-freshman college and see if you recognize this.
00:51:51	Jeff:	I mean, I don't know. It just, just seems like-
00:51:52	Romina:	We just-
00:51:53	R1:	If someone said to you, why does this work and this is a rule and you've shown me things with factorials, you can probably write those in factorial notations. I bet you could. In fact, I wish someone would do it on the board on the right there. Write that addition statement using factorial notations.

Time	Speaker	Transcript
00:52:11	Jeff:	All right. Um, you want to do it. Want to do it?
00:52:14	Michael:	Just that thing real quick?
00:52:15	Jeff:	We're writing this right here?
00:52:16	R1:	Sure.
00:52:16	Jeff:	The addition rule in factorial notation? [Michael replaces Jeff at the board.]
00:52:19	R1:	That's another form, isn't it?
00:52:20	Jeff:	Yeah. [Michael writes on the board.]
00:52:22	R1:	Brian would like to know that, I know he would.
00:52:25	Romina:	[Someone sneezes.] Bless you.
00:52:27	Brian:	Right.
00:52:27	Ankur:	Oh, yeah.
00:52:28	Michael:	That whole thing plus-
00:52:31	Ankur:	Plus.
00:52:35	Michael:	-over. [Michael writes $(n-1)!$ in the numerator.]
00:52:39	Michael:	No.
00:52:40	Ankur:	No, it's just N.
00:52:41	Jeff:	Yeah, N factorial.
00:52:42	Michael:	I just, I just saw that. [Michael changes $(n-1)!$ to $n!$] Um.
00:52:48	Ankur:	Over, just do everything it is.
00:52:50	Michael:	*N* minus *X*.
00:52:53	Ankur:	*X*, parenthesis.
00:52:54	Michael:	Plus one.
00:52:58	Ankur:	Yeah. And then *N* and do the *X* factorial. Put that all in parentheses.
00:53:04	Jeff:	It's not an *X*, it's not *X*. Yeah, there you go. There you go.
00:53:10	Ankur:	No, it's not the top.
00:53:12	Michael:	Yeah, the whole thing.
00:53:13	Ankur:	Plus one? Do you have that plus one on the bottom?
00:53:17	Michael:	Yeah.
00:53:18	Michael:	Equals. Um. [Michael writes the numerator and then laughs.] Um, this whole thing on the bottom, um.
00:53:30	Ankur:	It's the same, it's the same thing. Just copy it.
00:53:33	Jeff:	Yeah.
00:53:34	Ankur:	*N*.
00:53:35	Jeff:	*N*.
00:53:35	Ankur:	Minus *X*.
00:53:36	Jeff:	Minus *X* plus, exactly. You know how like intimidating this equation must be, like if you just pick up a book and look at that? There you go. [Michael continues to write on the board.] Yeah.

Time	Speaker	Transcript
00:53:57	Michael:	[Michael finishes writing; refer to Figure J21.] That's what you want, I think. [Michael sits.]

$$\left(\frac{n!}{(n-x)!x!}\right)+\left(\frac{n!}{(n-x+1)!(x+1)!}\right)=\frac{(n+1)!}{(n-x+1)!(x+1)!}$$

Figure J21. Factorial form of the addition rule (first pass)

00:54:03	R1:	Do you all agree?
00:54:04	Jeff:	Yeah. I got chalk all over my pants like Dr. Zabrower.
00:54:11	Michael:	That means like-
00:54:12	Jeff:	That's-
00:54:13	Michael:	It's too confusing?
00:54:14	R3:	Is that the same thing?
00:54:15	Michael:	Yeah.
00:54:15	Ankur:	It is the same thing.
00:54:17	R3:	It is?
00:54:17	Michael:	Yeah. *N*.
00:54:17	Ankur:	As that. Yeah.
00:54:18	Michael:	This thing, all right, you see how that is that? [Michael points from choose notation to factorial notation.]
00:54:20	R1:	Mm hm.
00:54:22	Michael:	You know how- I'll go up there again. [Michael goes to the board.]
00:54:27	Jeff:	We just wrote out the, yeah, exactly, we wrote out the equation, how to find *N* choose, exactly.
00:54:33	Michael:	That's, that's, I guess that's what you want. [Michael writes the addition rule again; refer to Figure J20.]
00:54:37	Jeff:	Yeah. It's exact- We just wrote, we instead of writing-
00:54:39	Michael:	You agree with this? [Michael points to the board.] Right? So we just wrote, we wrote that-
00:54:45	Jeff:	We wrote it in the, in the form.
00:54:45	Ankur:	In that form.
00:54:45	Michael:	It still doesn't look, it doesn't look too good.
00:54:47	Jeff:	Yeah. It looks kind of mean.
00:54:49	Michael:	We wrote that like that. [Michael points to the equations shown in Figure J20 and Figure J21.]
00:54:53	R1:	Did you all very carefully check that arithmetic?
00:54:55	Michael:	You think we're wrong? [Michael turns towards the board.]

Time	Speaker	Transcript
00:54:57	Ankur:	What, you found an error?
00:54:58	Jeff:	All right. Well what's, what, go to the, uh, write the regular equation down.
00:55:02	Romina:	Here's a-
00:55:02	Ankur:	There it is, right there. [Ankur points.]
00:55:02	R1:	Why don't you get a piece of paper and-
00:55:04	Jeff:	Where is it?
00:55:05	Ankur:	It's right above *N* over *X*.
00:55:05	Michael:	Oh, yeah. Never mind.
00:55:06	Jeff:	All right.
00:55:06	Romina:	You found it?
00:55:06	Jeff:	Yeah.
00:55:06	Ankur:	The first one.
00:55:14	Michael:	[Michael writes the new version of the equation on the board; refer to Figure J22.] There you go.

$$\left(\frac{n!}{(n-x)!x!}\right)+\left(\frac{n!}{(n-x+1)!(x+1)!}\right)=\frac{(n+1)!}{((n+1)-(x+1))!(x+1)!}$$

Figure J22. Factorial form of the addition rule (corrected)

Time	Speaker	Transcript
00:55:17	Jeff:	Yeah, all right.
00:55:19	R1:	Sure?
00:55:21	Michael:	Yeah, I'm sure. You got anything else? Yeah, I guess.
00:55:24	R1:	Did you check it?
00:55:26	Michael:	What do you mean? Is it wrong?
00:55:29	R1:	Now that, that's really, really very frightening.
00:55:32	Michael:	Yeah.
00:55:32	R1:	What do you think? Is that foreboding?
00:55:35	Jeff:	I guess.
00:55:36	R1:	I wonder if there's a way of simplifying it.
00:55:39	Jeff:	Of what?
00:55:39	Michael:	Simplifying it. Hey!
00:55:40	Ankur:	Yeah, you could [Inaudible.]; that's simplifying.
00:55:42	Jeff:	Yeah that's, that's pretty-
00:55:44	R1:	That's a way to simplify it. But you know I see N plus one parenthesis minus parenthesis X plus one. That looks like that could be a little simpler. See that N plus one parenthesis that Michael just put there.

Time	Speaker	Transcript
00:55:59	Michael:	Yeah.
00:56:00	R1:	Minus the expression X plus one. Suppose you distributed that minus one.
00:56:07	Jeff:	So you want, all right, so- All right.
00:56:10	Michael:	Why would you want to do that?
00:56:10	Jeff:	So distributing, say over there, right? You'd have, you'd have N plus one minus X minus one factorial? [Jeff writes on his paper.]
00:56:19	Romina:	Mm hm.
00:56:20	Jeff:	Um, that would be in, in parenthesis.
00:56:24	Michael:	Oh yeah, yeah, there you go.
00:56:24	Jeff:	And then, well, that-
00:56:27	Romina:	Why don't you get another piece? [Jeff starts a new piece of paper.]
00:56:31	Jeff:	So, all right, so it'd be *N* plus one factorial divided by, um, *N* plus one in parentheses minus *X* minus one factorial. All right? And then, well, that's, that's pretty much all you can do there. Then *X* plus one factorial, so you could actually can, you can cancel out? Can you cancel that out? The *X*, minus *X* minus one and the *X* plus one? Or-
00:57:04	R1:	That's what I'm asking you to think about. Not right, not now necessarily, but, um-
00:57:06	Jeff:	Yeah, can you, I mean, can you cross out factorials or is that the first factorial on the bottom of the one all the way to the right? Does that affect, that's affecting the N plus one too, so can you, are you allowed to cross out like that? Cross these both out?
00:57:20	R1:	What a good question. What do you all think?
00:57:22	Jeff:	Well, can we throw in numbers and see?
00:57:25	Romina:	Would we be able to cross out the *N* plus ones?
00:57:27	Jeff:	Well then what are you left with?
00:57:29	Romina:	Yeah. Yeah. It doesn't-
00:57:30	Jeff:	Factorial divided by factorial?
00:57:33	Michael:	Now wouldn't that just be, uh-
00:57:35	Jeff:	Now I'm saying you could.
00:57:36	Michael:	But you're talking about simplifying, wouldn't that just be, uh-
00:57:38	Jeff:	Yeah.
00:57:39	Romina:	I don't, would that, this whole thing be-
00:57:41	Jeff:	Yeah then it would be nothing, right?
00:57:43	Romina:	Yeah.
00:57:44	Jeff:	Then that would cross out and that would cross out.
00:57:45	Romina:	You get two factorials.

Time	Speaker	Transcript
00:57:47	Ankur:	You can't do that.
00:57:47	Michael:	You know.
00:57:48	Jeff:	Yeah.
00:57:47	Michael:	She's talking about simplifying, and you just like, you know, put that negative in there and it would be just *N* minus *X*?
00:57:56	Jeff:	Where? Where's this at?
00:57:57	Michael:	Right at *N* minus. minus, that one right there. [Michael points to the blackboard.]
00:57:59	Romina:	The one all the way to the side.
00:58:01	Jeff:	Oh yeah, and then the, all right, so you, so you do that, *N* minus *X* factorial.
00:58:01	Michael:	[Michael goes to the board.] That could be-
00:58:01	Jeff:	*N* minus. Yeah exactly.
00:58:04	Michael:	Uh, I'm not too good with my uh-
00:58:07	Jeff:	Simplification.
00:58:08	Michael:	Yeah.
00:58:08	Jeff:	Yeah, because that, it would be- You got the plus one.
00:58:11	Michael:	I'm just wondering. Wouldn't you, wouldn't that equal *N* plus one minus *X* minus one? [Michael writes this on the board.]
00:58:19	Jeff:	Yes, then the plus one and the minus one-
00:58:19	Michael:	Are gone.
00:58:19	Jeff:	So it would be *N* minus *X* factorial.
00:58:20	Michael:	*N* minus *X* so-
00:58:21	Jeff:	It'd be *N* minus *X* factorial, um, times *X* plus one factorial? Right? Yeah.
00:58:34	Michael:	[Michael writes on the board; refer to Figure J23.] A little simpler. I still don't like it.

$$\left(\frac{n!}{(n-x)!x!}\right)+\left(\frac{n!}{(n-x+1)!(x+1)!}\right)=\left(\frac{(n+1)!}{(n-x)!(x+1)!}\right)$$

Figure J23. Simplified version of the factorial form of the addition rule

Time	Speaker	Transcript
00:58:37	Jeff:	Then, but then you could cross out, OK, could you cross out?
00:58:39	Michael:	Which are you talking about?
00:58:40	Jeff:	Up, no, the bottom and the top.
00:58:42	Romina:	The top.
00:58:42	Jeff:	Oh, that's plus one. All right, my bad, [Inaudible.] I didn't.
00:58:45	Michael:	Anything else?

Time	Speaker	Transcript
00:58:49	Jeff:	Well, if *X* equals negative one, just-
00:58:51	Ankur:	And can't you do that on the other side too?
00:58:51	Michael:	Um.
00:58:54	Jeff:	That would be, um-
00:58:56	Ankur:	It would be *N* minus one.
00:58:56	Jeff:	*N* minus *X* minus one factorial. No.
00:59:01	Michael:	No, it'll still be the same number.
00:59:02	Jeff:	Yeah. And it'll be *X* plus one.
00:59:03	Michael:	Don't even bother.
00:59:05	Jeff:	Factorial.
00:59:08	R1:	I'm, I'm impressed that twenty of ten you're doing this arithmetic. Um, you know, of course the next thing to do is to learn how to do the algebra of factorials so that you indeed could do the addition.
00:59:23	Michael:	[Inaudible.].
00:59:23	Jeff:	[Inaudible.] the factorial.
00:59:24	R1:	Would you like to know how to do that? Would you like to know how to do the algebra factorials? I bet you know how to do a little bit already. I'll just show you one thing that I know you know and I'll leave you to think about this because everyone is getting tired, but let's just take something like this, right? [Researcher 1 walks to the board and writes 6 choose 2.] Six choose two, right? And you know, you, you told me you could write that how? [Researcher 1 writes =.] As-
00:59:55	Michael:	Um, six factorial over-
00:59:57	R1:	Six factorial. [Researcher 1 writes 6! in the numerator.]
00:59:59	Michael:	Three fact, four factorial times two factorial.
01:00:03	R1:	Times two factorial, right? [Researcher 1 writes 4! times 2! in the denominator.]
01:00:05	Romina:	Mm hm.
01:00:06	R1:	And you know what six factorial is, right? Six times five. [Researcher 1 writes 6 times 5.]
01:00:11	Michael:	Times one-twenty.
01:00:12	Jeff:	Thirty. Yeah.
01:00:13	R1:	I'm not going to do that though. I don't like to. I don't like to do multiplication. [Researcher 1 writes 4! next to 6 times 5.] I'm very lazy. I'm just going to write six times five times four factorial. Is that okay?
01:00:21	Jeff:	That's, that's simplifying is great, then you can- [Students all talk at once.]

Time	Speaker	Transcript
01:00:24	R1:	But can I do that?
01:00:26	Romina:	Yeah.
01:00:26	Michael:	And then you could cross out the four factorials and-
01:00:27	Romina:	Oh.
01:00:28	R1:	[Researcher 1 writes 4! and 2! in denominator. Refer to Figure J24.] Oh, then I can cross out the four factorials.

$$\binom{6}{2} = \frac{6!}{4!\,2!} = \frac{6 \cdot 5 \cdot 4!}{4!\,2!}$$

Figure J24. Simplifying 6 Choose 2

01:00:28	Jeff:	Oh, all right, that makes sense.
01:00:29	R1:	Right?
01:00:31	Jeff:	So you just get thirty divided by, you get thirty divided by two.
01:00:33	R1:	Yeah. Look at all the time that will save you in an SAT question.
01:00:35	Jeff:	That'd be big.
01:00:37	R1:	But, but if you think about this-
01:00:39	Jeff:	She broke, she broke it down farther.
01:00:40	Romina:	Oh yeah she just-
01:00:42	Jeff:	Like rather than say you have six factorial-
01:00:43	Ankur:	Mm hm.
01:00:43	Jeff:	She broke it down until she got a number that she got that she wanted.
01:00:45	Romina:	She had two numbers.
01:00:47	Jeff:	That matched the number on the bottom.
01:00:48	Ankur:	All right. Yeah.
01:00:50	Jeff:	Then you end up like with the two factorial and then cross out and that's thirty over the two factorial and that's two. So it's just 15.
01:00:51	Michael:	But then it would probably be even longer than that. If N is a big number- [Michael and Researcher 1 are carrying on a separate conversation in the background.]
01:00:55	R1:	Does it matter?
01:00:59	Michael:	-you'd have to write, you would have to write N times N minus one times N minus- [Tape ends.]

APPENDIX K: TRANSCRIPT OF SESSION OF MAY 5, 2000

Time	Speaker	Transcript
00:00:41	–	[The students are introduced to the taxicab problem. They work for about six minutes.]
00:06:02	Jeff:	This is hard.
00:06:05	Romina:	Two-
00:06:10	Jeff:	How many was there? For, um, for the blue dot. How many different ways.
00:06:15	Brian:	Five.
00:06:30	Romina:	Ha. I already lost count.
00:06:36	Jeff:	How many you got for red so far? [Jeff is talking to Brian.]
00:06:38	Romina:	Well, I'm saying like if you go all the way over.
00:06:39	Brian:	Two, three.
00:06:40	Romina:	And then //you go all the way// over and leave only one space.
00:06:40	Michael:	//Yeah. One, two, three. Yeah, one, two, three, four, five, six. Six going like that.
00:06:42	Brian:	One, two, three, four.
00:06:43	Jeff:	You only got five?
00:06:43	Brian:	No. I'm just-
00:06:46	Jeff:	Oh, I can't. //I can't keep //track of what I'm doing.
00:06:48	Michael:	//Six this way. //Then you got-
00:06:49	Jeff:	You know what I'm //saying?
00:06:51	Michael:	//possibility of doing this. //One, two.
00:06:52	Romina:	//Yeah. How do we get that?
00:06:53	Michael:	Three, four. Oh, got one. But then you got- Ah, this is a lot.
00:07:01	Romina:	//Yeah, you could do this.
00:07:03	Michael:	You guys want to do the green? We'll do the blue.
00:07:04	Jeff:	No that's all right. //We already did the blue.
00:07:05	Brian:	//We already did the blue.
00:07:06	Romina:	Yeah, the blue is fine.
00:07:06	Brian:	We're doing red.
00:07:07	Romina:	OK, we can't count. Like we need a, can't we, can't we do towers on this?
00:07:13	Jeff:	That's what I'm saying. Look, all right, you go to here-
00:07:15	Romina:	Towers. 'Cause they're like blocks.
00:07:16	Jeff:	All right, you go to here and you got a choice of going there or

Time	Speaker	Transcript
		there. Right? [Indicating a choice of across or down at an intersection point of the grid on his problem sheet.] So then you pick one of those and then you got a choice of there or there. When you get to- you know what I'm saying? Maybe we can add all those up or something and get like a whole.
00:07:30	Romina:	All right.
00:07:35	Michael:	There's a lot.
00:07:38	Romina:	OK, for ours there's ten.
00:07:42	Michael:	There's more than ten.
00:07:46	Romina:	No. I mean there's ten blocks. Like ten lines to get to that thing, right?
00:07:49	Michael:	Yeah, six by five.
00:07:52	Romina:	So if there's ten, ten could be like the number of blocks we have in the tower.
00:07:58	Michael:	This is one.
00:08:00	Romina:	How do we do that? Two to the *n*?
00:08:05	Michael:	How, how many? This was five they said?
00:08:09	Romina:	Yeah.
00:08:10	Michael:	How much you guys get for the red? Still doing that one?
00:08:15	Romina:	How could-
00:08:16	Michael:	It's got to be some kind of pattern.
00:08:19	Romina:	OK, there's ten lines, ten lines.
00:08:22	Michael:	Ten ways of getting there. So you can do. Like you got to-
00:08:27	Romina:	There's ten different lines to get there.
00:08:28	Michael:	Think of the possibilities of doing this and then doing that.
00:08:30	Romina:	Well how many- OK, there's ten. How many lines end up in the thing?
00:08:32	Brian:	What are you doing, man?
00:08:33	Jeff:	I'm just, I'm not, uh, trying to-
00:08:35	Romina:	Two, three, four, five, six, seven, eight.
00:08:36	Michael:	Three, four, five.
00:08:36	Jeff:	-get easier.
00:08:38	Michael:	There's thirty plus, I have like thirty. About sixty I think.
00:08:59	Michael:	You might want to-
00:09:01	Romina:	So. It couldn't be like a block ten high in six different colors, type deal? That would be-
00:09:09	Michael:	There's like, there's ten line, there's ten like lines in here, and the answer was five. So I'm waiting for them. Maybe, it would be nice if it was like a half or something.
00:09:18	Romina:	So maybe it's thirty?

Time	Speaker	Transcript
00:09:20	Michael:	It'd be nice if it was.
00:09:22	Romina:	How many are there in here? One, two, three, four, twelve, twenty. You guys got at least twenty-four yet?
00:09:35	Jeff:	Uh, which, wait a sec.
00:09:37	Brian:	I'm at eight. What do you think? [To Jeff.] What are you guys thinking? [To Romina and Michael.]
00:09:48	Romina:	To get to this one, there could also be five times two but there's ten lines.
00:09:53	Brian:	I've counted it.
00:09:55	Romina:	And there's five different ways to go.
00:09:56	Brian:	Wait, for five?
00:09:58	Romina:	For the blue one.
00:09:59	Brian:	There's ten lines?
00:10:00	Romina:	//[Inaudible.]
00:10:01	Michael:	//You got eight for the red. I only have nine ways.
00:10:03	Jeff:	//No but I'm like-
00:10:05	Michael:	//You have eight?
00:10:07	Brian:	I'm drawing them. I'm not stumped; I'm just like not speeding through it. You know. Did you count the middle lines?
00:10:08	–	They work for 34 minutes.
00:44:01	Romina:	Couldn't we just do something like in towers where like lines over are like the color and the lines down are the, um, number of blocks?
00:44:11	Jeff:	All right. And that would?
00:44:12	Romina:	Because, OK, lines over- because what is it- the number of blocks to the number of colors?
00:44:22	Jeff:	I don't know what you're- What, what's that?
00:44:25	Romina:	Two to the *n*. Two is the amount of blocks or the colors?
00:44:28	Michael:	For what? Like towers on them?
00:44:30	Romina:	Yeah.
00:44:31	Jeff:	Colors. *n* is the number of blocks. I think. I don't know. I'm not sure.
00:44:37	Michael:	Well you figure a block has this, you got two, two tower like this. Or two colors actually. I think it's, uh, the colors and *n* is the blocks.
00:44:48	Romina:	Color two- //right.
00:44:49	Jeff:	//Same thing.
00:44:50	Romina:	All right, here we have one color. Nah; it doesn't work for the first one.
00:45:07	Romina:	Scratch that idea.

Time	Speaker	Transcript
00:45:08	Jeff:	Well, why, you know, what happened to, to what we were doing?
00:45:11	Romina:	No, I know. Just keep on going. [Jeff, Brian, and Michael continue working silently.]
00:45:13	Jeff:	All right.
00:45:19	Romina:	You're right. [Inaudible.] Three by two.
00:45:21	Jeff:	Can you help me out?
00:45:24	Romina:	What, what [Inaudible.] //by two of this sheet?
00:45:27	Brian:	//That's what I got so far.
00:45:29	Romina:	//You need one? [Inaudible.]
00:45:31	Brian:	//That's how far right there. It's on the board. //The board.
00:45:35	Romina:	//I know, I'm looking for-
00:45:39	Brian:	Mike do you see anything that I'm not getting?
00:45:46	Romina:	//Three by three.
00:45:51	Michael:	//Which one you doing?
00:46:05	Brian:	Two by three.
00:46:10	Romina:	Three by two. All right, here. This is what we got.
00:46:19	Jeff:	It's really hot in here.
00:46:20	Romina:	All right, we got down two over three. Over three, down two.
00:46:28	Brian:	//OK.
00:46:32	Romina:	//That's one of those? The first one.
00:46:41	Brian:	[Inaudible.]
00:46:46	Romina:	All right we got those. Got a down one over three.
00:46:52	Romina:	Except they don't have one, one, one, one, one, that one.
00:46:57	Jeff:	That's one we don't have?
00:47:00	Romina:	We don't have his last one over there. Check. I think that was the only one. So that nine does equal ten.
00:47:16	Jeff:	I don't see uh- Um-
00:47:18	Romina:	Because we don't have that one?
00:47:23	Jeff:	No, we don't have that one. [Inaudible.]
00:47:25	Romina:	All right. It's, um, it's Pascal's Triangle.
00:47:31	Michael:	What is that? Two by three?
00:47:32	Jeff:	It is?
00:47:32	Romina:	Yeah.
00:47:34	Jeff:	Let me see.
00:47:35	Romina:	All right. Yeah, it is.
00:47:36	Michael:	What?
00:47:38	Romina:	It's Pascal's Triangle.
00:47:40	Michael:	Two, three.
00:47:41	Romina:	No, it's not. It doesn't work out.

Time	Speaker	Transcript
00:47:43	Jeff:	See look at- Here, Mike.
00:47:44	Romina:	Because twelve, that doesn't-
00:47:45	Jeff:	Mike look, just look at it in this thing. You got the 6 and the 4 and the 6 are the 10. That should be a 15. //That should be a 20.
00:47:53	Romina:	//But that's not a 15. That is a 12 because he even got the 12.
00:47:55	Jeff:	Well, that should, that should be a 20 right there.
00:47:59	Romina:	[Inaudible.]
00:48:00	Michael:	Up to here is been a one, one, one, one and-
00:48:03	Jeff:	Huh.
00:48:04	Brian:	So what's wrong?
00:48:07	Michael:	It should be six, fifteen.
00:48:09	Romina:	Do, do a four by two.
00:48:11	Michael:	Yeah.
00:48:13	Jeff:	You do the four by two, and it should put us, uh, in business.
00:48:16	Brian:	All right.
00:48:17	Romina:	And then, because we'll compare it to all-
00:48:20	Jeff:	If this comes through it just-
00:48:21	Romina:	If it's Pascal's Triangle, it'll just give us problems.
00:48:25	Jeff:	No but it, it's just nice how, you start, like, when you start from nothing. You know what I'm saying? Like we have no clue what we're doing.
00:48:32	Romina:	But he even got twelve when he did it.
00:48:34	Michael:	I might be missing two.
00:48:35	Jeff:	It could be, it's not hard to miss three, right?
00:48:37	Michael:	Two.
00:48:38	Jeff:	Three.
00:48:41	Romina:	So for the next one, Jeff, we missed five?
00:48:41	Jeff:	It's very easy. I mean, there's a lot of things going on.
00:48:43	Michael:	That's kind of a lot.
00:48:45	Jeff:	We like blew like a lot of these. You know what I'm saying?
00:48:48	Romina:	Uh-huh. Yeah. I think we, uh, got a few wrong somewhere.
00:48:51	Jeff:	That's what I'm saying. So why- like it wouldn't be totally out of control.
00:48:54	Brian:	Oh.
00:48:55	Romina:	Do, do it the other way. Just turn it around. That'll make our life- that- because that's we did. It's the same thing but-
00:49:07	Brian:	Is that the air that just turned on?
00:49:09	Jeff:	Yeah. But, it don't work though.
00:49:11	Romina:	I'll be right back. [Romina leaves the room.]
00:49:43	Jeff:	So how do you do your deal? I don't know how to do your deal.

Time	Speaker	Transcript
00:49:50	Brian:	It's nothing, the ones with two moves, the ones with three moves so I just go like three moves over- starting over first. Over three down two boom, boom boom, boom boom. Then, then I go to over down, over down. This row gets eliminated pretty much. [Jeff nods his head at Brian.]
00:50:05	Jeff:	All right. But you're not going to get there. I hear you. [Jeff and Brian work silently.]
00:50:43	Jeff:	It'd be so much easier if some of us were lefties.
00:50:48	Brian:	Why?
00:50:49	Jeff:	You'd just block like, uh, you try to see what someone does, and it's just like- I mean like what is Mike looking at when I'm writing right now. //You know what I'm saying?
00:50:54	Michael:	//Oh yeah.
00:50:56	Jeff:	It's like, what- Which one are you doing, man?
00:50:58	Michael:	We're looking for fifteen for that one, right?
00:51:00	Jeff:	Yeah. Did you get that?
00:51:02	Michael:	Hm. Don't know, yet.
00:51:04	Brian:	What number are we looking for, for this one?
00:51:08	Jeff:	Fifteen. How many you got?
00:51:10	Brian:	Eight.
00:51:52	Jeff:	[Jeff is talking about an ice cream truck.] When it rained he went home and as soon as it stopped he came back out. Annoying bastard. I can't take it no more.
00:52:09	Michael:	What is that?
00:52:11	Brian:	Zero.
00:52:12	Michael:	I got twelve so far.
00:52:21	Jeff:	Yeah, twelve's like the number that we got stuck at last time.
00:52:54	Michael:	I think I got it.
00:52:58	Brian:	[Inaudible.] It's fifteen.
00:53:09	Jeff:	All right. All right, what if we even went- Let me know when you're done. All right. Because there's an easier way to- [Inaudible.] Listen to me for one sec.
00:53:22	Michael:	Go ahead.
00:53:23	Jeff:	All right. If- all right. Say in a situation where it's like, uh a two by four.
00:53:29	Michael:	Uh-huh.
00:53:29	Jeff:	All right. If we know that in a four-by-four, it's six if you figure out all the ways to get to the beginning parts of this, this would all just be six different ways to get from here to here. So you figure out all the ways to get there and you could just add six.

Time	Speaker	Transcript
		You know.
00:53:46	Michael:	If you have the two, you could find out how many ways it's to get to here and add that where every two is.
00:53:53	Jeff:	You know what I'm saying? So like from- from-
00:53:58	Brian:	I got fifteen.
00:53:59	Jeff:	You did?
00:54:00	Brian:	Yeah.
00:54:01	Jeff:	All right. 'Cause from there to there you have six different ways. And then from there, there's one way. To there there's one way and from there- //
00:54:18	Brian:	//Haaa. Tell me when you're done.
00:54:20	Jeff:	Sure. One, two; there's three ways. Um.
00:54:34	Michael:	I got fifteen also.
00:54:36	Brian:	Yeah, Mike. [Inaudible.] [Brian leaves the room.]
00:54:42	Michael:	So what does that mean?
00:54:44	Jeff:	It means that it is the triangle. Right here?
00:54:47	Michael:	Uh-huh.
00:54:48	Jeff:	You have fifteen there?
00:54:49	Michael:	I got fifteen.
00:54:49	Jeff:	That's good. Yeah, because then- Yeah. This, then in a three by three it should be twenty. That'll be, uh-
00:55:04	Michael:	Is nine blocks for that one?
00:55:04	Jeff:	In the nine block, it should be twenty. [Inaudible.]
00:55:50	Michael:	Where'd they go?
00:55:51	Jeff:	What?
00:55:51	Michael:	Where'd they go?
00:55:52	Jeff:	Well, Brian's just standing there. I don't know. I need to go to the bathroom though. I'm like going to pass out. [Jeff leaves.]
00:56:09	Brian:	So what are we doing now?
00:56:12	Michael:	No idea.
00:56:15	Brian:	Did you figure out the, uh, five by five?
00:56:17	Michael:	Five by five? I'm doing three by three right now.
00:56:19	Brian:	It's the green. If we already know what it is then why don't we just figure out-
00:56:23	Michael:	I just want to make sure that's twenty first, so-
00:57:07	Michael:	I'm missing two. That's probably right though.
00:57:13	Brian:	Did you get the, uh, staircase one?
00:57:15	Michael:	Which one? For the three by three?
00:57:20	Brian:	Yeah. [Romina returns.] All right, I got twelve, fifteen, right.
00:57:27	Romina:	Oh, you guys went and wrote on this didn't you?

Time	Speaker	Transcript
00:57:30	Michael:	I didn't do it.
00:57:32	Romina:	All right.
00:57:35	Brian:	Did Jeff tell you?
00:57:36	Romina:	What?
00:57:37	Brian:	I got fifteen for this one.
00:57:39	Romina:	For which one?
00:57:40	Michael:	//For-
00:57:41	Brian:	//Four by two.
00:57:41	Romina:	So you did get fifteen? So now it's working? And then the two by four has to be fifteen too. Now if we do three by three and that's twenty, then we're done.
00:57:50	Brian:	That's what he's doing.
00:57:52	Romina:	What?
00:57:53	Brian:	He said he was off by two. You can just get another one.
00:58:01	Romina:	I'll just turn this around.
00:58:02	Brian:	It's only a couple of numbers.
00:58:14	Romina:	Did it again. You got twelve for this one? Fifteen, I mean?
00:58:28	Brian:	Yep. Now, which one are you expecting to be twenty? Three by three?
00:58:38	Brian:	I guess I'll do it. Check it out.
00:58:42	Romina:	I don't think- it's here- he has- He was just doing three by three wasn't he?
00:58:49	Brian:	Yeah. It's no big deal.
01:00:36	Romina:	I'm already stuck.
01:00:37	Jeff:	You shouldn't be. Where you going?
01:00:40	Romina:	Three by three.
01:00:42	Jeff:	You said F making the, the boxes.
01:00:47	Michael:	Yeah, I got twenty for that one.
01:00:50	Jeff:	For three by three?
01:00:52	Michael:	Yeah.
01:00:53	Jeff:	All right well then, I mean, can't we explain why we think- Well, all right.
01:00:58	Michael:	//They're going to ask us.
01:00:58	Jeff:	//All right then the next question is why- //why-
01:01:00	Romina:	//Now-
01:01:00	Michael:	//How do you know?
01:01:01	Romina:	//OK. Just relate this back to the blocks.
01:01:02	Jeff:	Wait. Why is this, why does the Pascal's Triangle work for this is the question.
01:01:06	Romina:	//Exactly. Relate it to the blocks.

Time	Speaker	Transcript
01:01:07	Michael:	//Just think first, how do you know it's twenty? You know, how do you know it's not nothing else?
01:01:10	Jeff:	Well F that. If we could explain-
01:01:12	Romina:	Stop saying that.
01:01:13	Jeff:	Why, why this is the Pascal's Triangle up to here, we don't need to explain how we're positive this is twenty. //You know what I'm saying?
01:01:19	Romina:	How does it go? This is-
01:01:20	Jeff:	One.
01:01:23	Michael:	It should be ones on all the sides.
01:01:27	Romina:	Yeah, right. So.
01:01:30	Jeff:	That's six.
01:01:32	Romina:	This is just one, two; this is three. //So-
01:01:38	Jeff:	//What's that?
01:01:39	Romina:	With one, //there's only one possibility.
01:01:42	Michael:	//All right, how, how is he getting them?
01:01:43	Romina:	Two.
01:01:44	Michael:	How are you getting yours? Maybe the way you're doing will give us-
01:01:46	Jeff:	Has some kind of- Yeah, we can work something out.
01:01:48	Brian:	[Inaudible.]
01:01:50	Michael:	Do you just like- are you guessing or do you have some kind of pattern?
01:01:52	Brian:	I'm just- doing it man. I'm just- you know-
01:01:57	Michael:	Ah.
01:01:58	Brian:	I know there's a way to make two and get there in two moves. I know there's a way to make it in three moves. Four moves.
01:02:03	Michael:	So you're going by the moves, right?
01:02:04	Brian:	Yeah.
01:02:04	Jeff:	Don't use that one.
01:02:06	Romina:	Hold on. For the Pascal's Triangle-
01:02:08	Michael:	Yeah.
01:02:09	Jeff:	You're making thumbprints again.
01:02:10	Romina:	The one, two, one.
01:02:10	Jeff:	Bringing it back to eighty-six.
01:02:11	Romina:	That's with what? With?
01:02:12	Michael:	Um-
01:02:13	Romina:	Two colors. It's, it's two to the x, like that?
01:02:19	Michael:	Yeah it's two.
01:02:19	Romina:	So it's two colors.

Time	Speaker	Transcript
01:02:21	Michael:	Think of it as zero, one, two. You only have two colors of choices. Zero, one, two. Three.
01:02:25	Romina:	Huh
01:02:26	Michael:	Three toppings on a pizza.
01:02:28	Romina:	Yeah, like, so then how could this, this is two what? Two? Two different ways, like.
01:02:37	Michael:	Two. Uh, it's the total. One, two, three. That's, that's the total length that you can get, have to get there, to get there.
01:02:45	Romina:	Yeah, OK.
01:02:45	Michael:	You know?
01:02:46	Romina:	So for this one, the total length is three.
01:02:49	Michael:	But then this one is one, two, three, four, five and you get ten. You know?
01:02:53	Romina:	But you're in the second row.
01:02:56	Michael:	Yeah. Right. This is one, two, three, four, five, six and you get twenty.
01:03:09	Romina:	All right.
01:03:13	Brian:	All right.
01:03:18	Romina:	I'm going to write it this way because I'm having a- I don't know about you people but- How does this go? It's not like in the blocks, is it?
01:03:31	Jeff:	What? For the thing?
01:03:32	Romina:	Yeah.
01:03:34	Jeff:	Yeah, it'll fit. Why- why don't you start like in the middle like here.
01:03:39	Romina:	Yeah.
01:03:40	Jeff:	Or why don't you use a different transparency?
01:03:43	Romina:	Well I just want it like- I'm just doing it so I can see it.
01:03:45	Michael:	Why don't you do it like //that that way we can see it.
01:03:46	Jeff:	//Why do you keep- //you're starting all the way over on the side every time.
01:03:48	Brian:	//All right. There's twenty.
01:03:48	Michael:	Start like this. It's easier to figure out like a two by two box. Over here. [Inaudible.]
01:03:53	Romina:	No, I know. It's just- it's just so I can see it so that's one block, two block, three block, OK.
01:03:56	Jeff:	All right.
01:03:58	Michael:	That would be seven, twenty-one, thirty-five.
01:04:05	Jeff:	All right. You want to, um- You want to try and explain this and then wherever we get like confused along the way, you know

Time	Speaker	Transcript
		maybe that's how we'll be able to- as we talk through it we could even- Oh sorry. I tried to stop it from hitting your leg. I don't even see it.
01:04:20	Michael:	It's a wet erase marker. It will come off with water. [They discuss the mark on Romina's sweater.]
01:06:38	Michael:	This takes a long time to figure this out.
01:06:39	Brian:	You know probably- if we just think about it what do we work on every single //time.
01:06:44	Jeff:	//Yeah I know, but we go to-
01:06:46	Michael:	We got to explain it.
01:06:47	Jeff:	You got to figure it out. You know what I'm saying? You got to go through it.
01:06:49	Brian:	She's going to look at these, and she's like I have no idea what you're doing.
01:06:52	Jeff:	I'm out- I've got to leave //in five minutes.
01:06:54	Romina:	//No she's going to go like this. You're still on this?
01:06:57	Jeff:	She won't say that.
01:07:02	Michael:	No they got, they got a different problem than us.
01:07:05	Jeff:	They have the same, do they have the same problem down, down there?
01:07:07	Romina:	Are they done?
01:07:09	R1:	Um, they're working on a different problem.
01:07:10	Jeff:	They have a different problem?
01:07:11	Brian:	All right.
01:07:12	Romina:	Like they didn't get this one to work on?
01:07:13	Michael:	We, we can't justify our answer but we're, we're, uh-
01:07:15	Jeff:	We're going to talk through it.
01:07:17	Romina:	//[Inaudible.]
01:07:18	Jeff:	//And we want to see where that takes us. I'm going to have to leave in five minutes though.
01:07:23	R1:	Oh, so, you've got to talk fast.
01:07:25	Jeff:	So we're going to talk fast. All right.
01:07:29	R1:	OK, the problem I really wanted to give you was for all the points on the grid.
01:07:34	Romina:	Oh good. //We did that.
01:07:35	Jeff:	//Oh yeah. That's what we did.
01:07:36	Romina:	Why don't we do points-
01:07:37	Brian:	We got it. We got it.
01:07:39	Jeff:	You know-
01:07:39	Romina:	Points up to-

Time	Speaker	Transcript
01:07:40	R1:	All right. So tell me, tell me.
01:07:41	Brian:	Pens are flying now.
01:07:42	R1:	Yeah. Did you like the problem?
01:07:44	Romina:	No. Nah, it was OK.
01:07:46	Jeff:	It's just, doing all this kind of stuff really hurts your brain, but other than that, it was all right.
01:07:50	Romina:	It- your eyes. All right. What we did is we, we took it-
01:07:54	Jeff:	We broke it down.
01:07:55	Romina:	Yeah, we just went from point to point on the thing.
01:07:58	Jeff:	Yeah. Like we even- we'll just say we started making the box like that. How many different ways can you get from this point to this point? You know, make an easier problem. Like the basic math deal.
01:08:07	Romina:	So we did like up to this point there's two. Up to this point there's three, four, six, three. So that, those are our numbers. Those are up to the points like down and diagonal. And what we got is Pascal's Triangle.
01:08:21	Jeff:	Yeah. We started, you know, and then as we started, you know, like it takes two to get to there. Three to, you know, to get there as Romina just went through and did. And then as we started filling it out we noticed that if you tilt it like that, and throw ones on the outside and a one on top, I mean you're looking at Pascal's Triangle. And so we stopped at this point because I mean making, you know, like thirty plus different things like this it gets, it just gets confusing you know.
01:08:46	R1:	Hm.
01:08:47	Jeff:	And so Brian had a, Brian was get, like doing, you know, we were, some of us were drawing out all the ways. Brian had another method of finding out the ways to do it. You know. And we just-
01:08:56	Romina:	And then we just compared them. And like //whatever he didn't have-
01:08:59	Jeff:	//-brought it all together and then that's kind of what we're looking at right now.
01:09:03	R1:	So you found those numbers, all of them, by counting?
01:09:07	Romina:	Yeah. //The ones we have written. Yeah.
01:09:07	Jeff:	//Well up, up to here. Right. What is written we counted through them.
01:09:11	R1:	OK. So is there any way you can justify if I were to say to pick, you said these are like rows, like so this one, two, one would be

Time	Speaker	Transcript
		what row? These points here //of the triangle?
01:09:26	Jeff:	//What? Um, I'm not-
01:09:28	R1:	You put ones on the side I noticed.
01:09:31	Jeff:	Yeah.
01:09:31	R1:	So if you were to look at //one, two, one.
01:09:33	Michael:	//Do you mean like this row?
01:09:34	R1:	Well, pick any row.
01:09:34	Jeff:	All right. All right. We'll say one, two, one, because that's an easy place to start from.
01:09:36	R1:	Right.
01:09:37	Jeff:	What's the question though?
01:09:38	R1:	Right. So-
01:09:39	Brian:	//One, two, three, one.
01:09:39	R1:	//-that's the second row.
01:09:40	Jeff:	//Yeah.
01:09:40	Michael:	//I mean I guess we're saying-
01:09:41	Jeff:	That's the second- yeah.
01:09:42	Michael:	Things with, uh, one, one block.
01:09:44	R1:	OK.
01:09:45	Michael:	Two blocks, three blocks, four blocks.
01:09:47	Jeff:	And then this would be five blocks then.
01:09:51	Michael:	Not four. That doesn't make sense.
01:09:53	R1:	How would that be?
01:09:57	Michael:	Six, you could, um, things that- I don't know.
01:10:10	Jeff:	See I'm still not exactly sure what you're asking.
01:10:11	Michael:	//Yeah, I don't know.
01:10:12	Romina:	//Yeah.
01:10:13	R1:	I didn't ask anything yet.
01:10:13	Jeff:	I'm all-
01:10:14	R1:	I was- I was saying that you-
01:10:14	Jeff:	What are you trying to-
01:10:14	R1:	-you're showing me that's Pascal's Triangle but I don't see it. Help me see it.
01:10:18	Jeff:	You don't see it?
01:10:18	R1:	Right. Can you show it to me?
01:10:19	Jeff:	All right, well, here. The one, one two one, one three three one, one four six four one.
01:10:27	R1:	OK let- show me again where's the four?
01:10:31	Jeff:	All right. We're going- all right. The, that one right there-
01:10:35	R1:	Uh-huh.

Time	Speaker	Transcript
01:10:35	Jeff:	-that we added in and this two is the three. The two in that one right there is a three and there's two ones on the outside.
01:10:41	R1:	Uh-huh.
01:10:41	Jeff:	So you get one three three one. And then the one, the one and the three for the four. Three and the three for the six. The three and the one for the other four and then the other one on the end and then continuing through the four and the one together is the five. The four and the six is the ten. Six and the four is the other ten. Four and the one is the five. Do you see it?
01:10:58	R1:	OK. So can you explain to me, for instance, why that works? Why, where this ten comes from when you know, you're just saying well there's a pattern here because you found them, but is there a way where you haven't found them that makes sense to predict the number of paths of one you haven't found?
01:11:17	Jeff:	Well like to here, I mean we //would say-
01:11:19	R1:	//You understand my question?
01:11:20	Jeff:	Well like to here we would say it was thirty-five.
01:11:22	R1:	Right. How would you, how, where //would the thirty-five come from?
01:11:24	Michael:	//You can't justify it because- You can justify these because you can say you counted. You can't justify that because you can't say you counted.
01:11:29	Jeff:	Yeah because we didn't count it. We're saying we're following the pattern.
01:11:32	R1:	Right.
01:11:32	Jeff:	That's, that is our justification as of now.
01:11:34	Romina:	[Inaudible.]
01:11:35	R1:	Right.
01:11:35	Jeff:	That we're just following the pattern.
01:11:36	R1:	Do you understand my next question Jeff? What I'm asking?
01:11:37	Jeff:	Yeah.
01:11:38	Romina:	What if three, what if Pascal-
01:11:39	R1:	Because so you notice this pattern and the pattern fits Pascal's Triangle.
01:11:46	Brian:	So does that mean there's //thirty-five for the red one?
01:11:48	Michael:	//Only these are zeros. This is like one topping, you know, on the pizza?
01:11:50	Romina:	Yeah, one topping, two toppings.
01:11:52	BRAIN	Remember how- Mike you had thirty-four for the red one, right?
01:11:55	Michael:	Um- Yeah I think that was the problem.

Time	Speaker	Transcript
01:11:58	Brian:	It's thirty-five.
01:11:58	Jeff:	Yeah, it's thirty-five.
01:11:59	Michael:	Oh, I probably missed one.
01:12:02	Jeff:	Good, uh, deduction.
01:12:04	R1:	So, so you counted thirty-four by brute force //and you're saying that by this pattern, um, you would feel more comfortable with the pattern in saying thirty-five.
01:12:09	Jeff:	//Yeah.
01:12:12	Brian:	But-
01:12:13	R1:	Right?
01:12:13	Romina:	Did you actually get thirty-five?
01:12:14	Michael:	I got //thirty-four.
01:12:15	Brian:	//He got thirty-four but you know he's been off by like one cause like you know. Yeah, it could- it could of //been one.
01:12:20	Romina:	//Natural tendencies? Um.
01:12:22	Michael:	Stop that.
01:12:23	R1:	OK. So why is- why is that-
01:12:24	Romina:	All right.
01:12:25	R1:	Why do you think that, why do those number seem to work? How could you explain those numbers? That's, that's really, isn't that interesting?
01:12:34	Jeff:	Yeah. It, it hurts though. It really does.
01:12:37	Romina:	All right. Hold on. Yeah, I'm having trouble seeing Pascal's Triangle.
01:12:43	R1:	It's hard to see the other way, isn't it?
01:12:52	Romina:	All right. So for this one the two comes from when there's-
01:12:58	Jeff:	One block.
01:12:59	Romina:	One-
01:13:00	Jeff:	Block.
01:13:01	Romina:	Is that-
01:13:01	Jeff:	One block.
01:13:02	Romina:	Isn't that two blocks?
01:13:03	Jeff:	One, two.
01:13:06	Romina:	No. Um, let's go back to towers. The two comes from- this is one block. This is two blocks with two colors.
01:13:16	Jeff:	I have to leave. I'm kind of out.
01:13:25	Romina:	Hold on. How's this go? Just tell me where this comes from.
01:13:26	Michael:	What happened?
01:13:27	Romina:	OK. This is with- with just one block?
01:13:29	Michael:	This is nothing.

Time	Speaker	Transcript
01:13:30	Romina:	This is nothing? This is one block?
01:13:32	Michael:	This is like- yeah, //one. All right.
01:13:33	Romina:	//One block, two- this one tells how many blocks.
01:13:35	Michael:	One block. Two blocks. Not two blocks but like- [Inaudible.]
01:13:40	Romina:	One block, two blocks, three blocks. Oh no, this is zero block, one block, two block?
01:13:45	Michael:	For one block you get two. Right? Or two blocks-
01:13:49	Romina:	All right.
01:13:51	Michael:	Three- three- three blocks. One. So you can't really say it because there's three for three and then you get four here. You can't really, I don't think you can use that. That, that row thing.
01:14:04	Romina:	All right. Yeah. I know. I'm just trying to- because for like-
01:14:18	Michael:	There's got to be some type of, you know, way. It would be nice if I could see it.
01:14:29	Romina:	Can't you just go one, two, three, four?
01:14:32	Michael:	Uh-huh.
01:14:34	Romina:	That's what it goes, one, two, three, four? Because then, OK for this one, for the three. If we name all the ones going horizontal *A*'s and ones going down same with *B*. And this would be with two *A*'s and one *B* there's three and then there's two *B*'s with one *A*, three. And for this one remember, like two *A*'s two *B*'s, six.
01:14:54	Michael:	You could say, um-
01:14:56	Romina:	Like, do you understand what I'm saying?
01:14:57	Michael:	Like, yeah, you could say like this row is everything with perimeter two. I mean I half the perimeter, like.
01:15:02	Romina:	Well, no I'm saying so to get that-
01:15:03	Michael:	In order to get to that point, you have to go over one and down, uh, one or down one and over one. Just like that row. Everything in this row, over two and down two and over one.
01:15:14	Romina:	Yeah but like I'm just saying like if she were to pick anything like right there we could say it's like eight, eight *A*'s and like six *B*'s. You know like, and then we could tell you where you it is in this one.
01:15:29	Michael:	Well, you could say all, everything in this row, the shortest route is two. Everything in this row shortest route is three. This one shortest route is four.
01:15:42	Romina:	Yeah.
01:15:43	Michael:	The shortest route is five, six, and so on. So that's how you could, you know, name them. This is row six because it has everything in the row has shortest route of six.

Time	Speaker	Transcript
01:15:55	Romina:	No, I understand. I'm just saying like-
01:15:58	Michael:	There's a, you know-
01:15:59	Romina:	To get it-
01:15:59	Michael:	//To- to say it like, oh I'll pick this block-
01:16:00	Romina:	//Because isn't that how, isn't that how we get like these? Like doesn't the two, there's, that I mean, that's one. That means it's one of *A* color, one of *B* color. Here's one, it's either one, either way you go. It's one of across and one down. And for three that means there's two *A* color and one *B* color, so here it's two across, one down or the other way you can get three is two down.
01:16:26	Michael:	You mean like one *A* color and two-
01:16:26	Romina:	Yeah.
01:16:27	Michael:	This is one-
01:16:28	Romina:	Like two blues, one red. Two across, one down, or this is two reds, one blue, two down, one across. And that's how we would get the Pascal's Triangle.
01:16:40	Michael:	But there's like- you know, there's got to be a way that we could just say, all right this one's three. //So five down this has to be this because of some kind of-
01:16:45	Romina:	//I know, I'm just saying-
01:16:48	Romina:	So if it were-
01:16:48	Michael:	Pattern- I mean like, you know, reasoning. You can't just say I counted them.
01:16:53	Romina:	I know. I'm just saying so like, and then that could relate back to this, but that is this, so it's believable, and for-
01:17:02	Michael:	So what, what are you looking for right now?
01:17:05	Romina:	Yeah like-
01:17:06	R1:	I think Romina knows what I'm looking for. I think she's said it very articulately. That if I were to pick any point right on-
01:17:12	Michael:	Uh-huh.
01:17:13	R1:	If I were to make a larger grid, right, Brian? I think he knows what I'm looking for.
01:17:16	Brian:	Yeah.
01:17:17	R1:	She's looking for a way to come up with a particular pattern that she's identifying that. I think I'm hearing you say, you're trying to look at blocks.
01:17:32	Romina:	Uh-huh.
01:17:33	R1:	Colors?
01:17:33	Romina:	Yeah.

Time	Speaker	Transcript
01:17:34	R1:	And then you're doing *A*'s and *B*'s.
01:17:35	Romina:	Uh-huh.
01:17:36	R1:	That's what I'm hearing you say? And you were trying to say maybe that could get you to some general point. Why don't you try saying that again? I, I thought I followed you but I'm not so sure that Brian and Michael followed what you said.
01:17:48	Romina:	Like why-
01:17:49	R1:	So say it again. What you were-
01:17:50	Romina:	Like why this and this are related?
01:17:51	R1:	Yeah.
01:17:51	Romina:	Well,
01:17:52	R1:	Throw out your idea //again for them so we can hear it.
01:17:54	Romina:	//When we look, whenever we do this we always, we always talk about towers and how this is like a tower of two high with two different colors and there's one, one tower you can make it makes all one color and one and one and then all the other color. And, and then for this one it's three high, and this is all one color. There's two of one color and one of the other, whatever. And for this it's basically the same thing because this is, let's see. This is, this is two, but usually you go one across and one down, so there's two different ways to get to that one. And for this one there's going to be two across and one down. Or to go down here it's two down and one across which is basically the same thing and it just goes on. Do you understand? Understand? Was that good? Or do you want more?
01:18:44	Brian:	Yeah.
01:18:45	Romina:	Or do you want more?
01:18:46	R1:	I don't know. I don't know if Michael-
01:18:47	Brian:	Mike, do you understand?
01:18:47	Michael:	Yeah, I understand what you're talking about.
01:18:48	Romina:	Yeah. Yeah.
01:18:48	Michael:	This would be, um, one, like we'll think of it as pizzas because that's like the thing I like but-
01:18:56	Romina:	Do the towers.
01:18:56	Michael:	Or towers. I mean this will be a tower of three-
01:18:59	R1:	Think of it as pizzas.
01:19:00	Michael:	A pizza. A pizza with, um, three possible choices for toppings and- I like the tower.
01:19:08	Romina:	Yeah, the tower is easier.
01:19:09	Michael:	You have, you have a tower of three, and you have, you know,

Time	Speaker	Transcript
		two colors. So one, it's either you know, color x and two of color y. Well this is direction x and two, two directions of y, you know-
01:19:19	Romina:	Yeah.
01:19:20	Michael:	-of y. So that makes, that makes sense.
01:19:23	Romina:	So for like the three, it would be two x, one y or two y, one x.
01:19:27	Michael:	Yeah, I got that.
01:19:28	Romina:	And this would be-
01:19:28	R1:	OK. Well, where I'm still having a little trouble is, um, OK, so you're talking about these blocks, right?
01:19:37	Romina:	Uh-huh.
01:19:37	R1:	So what are you labeling them? These blocks? Which is the A and which is the B and why is it OK to call them A's and B's?
01:19:43	Romina:	We'll do it. How about x and y?
01:19:45	R1:	Sure.
01:19:46	Romina:	x will be the ones that go horizontal.
01:19:48	R1:	OK.
01:19:49	Romina:	And y will be the ones that go up and down. Basic graphing skills.
01:19:52	R1:	Does that make any sense, Brian?
01:19:54	Brian:	Yeah.
01:19:55	R1:	Brian, do you think so?
01:19:57	Brian:	I think so, yeah. I'm hanging out. I'm doing good now. I was like, when is that research paper.
01:20:04	Michael:	English.
01:20:07	R1:	Researcher 12, Researcher 3, do you have any questions?
01:20:09	R12:	Well I mean I have very similar questions. That is, it's still not clear to me how, how they know that the, to get to any particular corner corresponds to one of the numbers in Pascal's Triangle.
01:20:29	Romina:	You see, I haven't done that either yet.
01:20:32	R1:	OK, why don't you think about that for a couple of minutes?
01:20:33	Romina:	All right let's say-
01:20:35	R1:	Let me just leave you be while you think.
01:20:36	Romina:	What would that be anyway?
01:20:37	Brian:	We'll say thirty-five there.
01:20:46	Romina:	You know, why don't we do this one?
01:20:47	Brian:	Thirty there.
01:20:48	Romina:	This is thirty?
01:20:49	Michael:	No, no that's uh-//
01:20:50	Romina:	No.

Time	Speaker	Transcript
01:20:51	Brian:	No //twenty-one.
01:20:53	Romina:	//This is twenty?
01:20:53	Romina:	No, you know, why don't we do it this way.
01:20:54	Brian:	Twenty-one.
01:20:54	Michael:	That should be-
01:20:53	Brian:	Twenty-one. That one right there should be twenty-one.
01:20:57	Romina:	One, six.
01:20:58	Brian:	And that should be a six. Fifteen plus six, twenty-one. And twenty.
01:21:03	Romina:	Like that? Is that one of them? One //six-
01:21:04	Michael:	//No. The next one. The next one.
01:21:06	Romina:	All right so that's one, seven.
01:21:10	Michael:	Twenty-one.
01:21:11	Romina:	OK, I'm not just- I, I'm doing-
01:21:13	Brian:	Thirty-five and twenty-one.
01:21:15	Romina:	And one. Seven and one.
01:21:20	Brian:	Seven.
01:21:24	Michael:	Like we know it is that. But-
01:21:26	Romina:	OK. So this-
01:21:27	Michael:	-without, without just saying, oh, it follows the pattern.
01:21:29	Brian:	Why.
01:21:30	Michael:	He wants to know why. Yeah.
01:21:31	Romina:	So this one is- is that thirty-five again? Or no, this one's thirty-five.
01:21:37	Brian:	This one's thirty-five.
01:21:38	Romina:	This one's thirty-five so then this one is?
01:21:40	Brian:	Twenty-one.
01:21:42	Romina:	Twenty-one. So let's see. One, two, three, four, five; one, two. I don't know. I see how it would go. Because-
01:22:00	Michael:	I, I know, we know it follows a pattern but he wants to know.
01:22:04	Romina:	OK. Five.
01:22:04	Michael:	Without saying oh it just follows a pattern. //Why is it-
01:22:04	Romina:	//OK, five and two- five and two, just add that. That's how many blocks there are. That's seven. So you got to go one, two- no. One, two, three, four, five, six, seven. Gets you down to the seven. And five of one thing and two of another thing, so you just- you don't count- we won't count the one because that doesn't involve that.
01:22:24	Michael:	What do you mean five and seven?
01:22:25	Romina:	What?

Time	Speaker	Transcript
01:22:26	Michael:	What are you talking about five and seven?
01:22:27	Brian:	Five across and two down.
01:22:28	Romina:	Five across and two down. Like you just count in. It goes- that's with one of one color and that's with two of two- of another color. That's with three, that's with four, that's with five. So it's either the two or the five. Both of them are the same thing. Yeah, we can explain this. Right? If anyone you pick like this one, you know it's one, two, three, four, five, six, seven. You know it's seven and it's going to be one, two, three- six of one color so it's going to be seven.
01:23:02	Michael:	Are you saying five across, one, two, three, four, five; one, two.
01:23:07	Romina:	So, either way, no, but it's seven blocks. It's five plus two. That's how many blocks you had. For seven blocks you go down. Go one, two, three, four, five, six to the seventh row. And now you know it's five by two so there means there's five of one color, two of another color so if I go to the second one this has to- this is all one color. This is one with one color this is two. So it's either twenty-one or there's three of one color, there's four of one color, and this is five of one color or twenty-one again.
01:23:31	Michael:	But suppose you were to say not colors but like //ups and downs, you know-
01:23:36	Romina:	//Or like that- this is //with two- two-
01:23:39	Michael:	//But why- you know, why is it thirty-five? If you go- Or why is it- let's go- go a little easier. Why is it, you know, four if- of, um-
01:23:45	Romina:	All right. Four, right? Four is, all right, why don't we do six? Six is a little harder. Six is one two- the one with six. There's one, two, three, four. You know there's four.
01:23:58	Michael:	It's two and two. All right. Two, four.
01:24:00	Romina:	This one.
01:24:01	Michael:	One, two, three, four.
01:24:04	Romina:	It's 'cause it's four blocks. No matter how you go there you had to take four spaces. And any direction you take has to be four spaces, right? So that means it's four- it's four blocks high. So you go to the fourth one. So you know it's in here. And it's, to get here, it's two across and two down. So whatever, like you know- Do you understand? Whatever route you take you'll end up two across two down. So it means there's-
01:24:24	Michael:	Two across and two down. That would be this one, because this

Time	Speaker	Transcript
		would be one across and two down, and this is two across and two down, and this is, wait, two down, two down and one across. One across and two down, and this is two across and one down.
01:24:36	Romina:	No, this is three across one down.
01:24:38	Michael:	Oh whatever. Three.
01:24:39	Romina:	And this is three down-
01:24:40	Michael:	No it's impossi. It doesn't make sense.
01:24:41	Romina:	Three across.
01:24:41	Michael:	Three across would be at, you'd be in, you'd be somewhere else.
01:24:43	Romina:	No you won't. Three across, one down is still in that row.
01:24:46	Michael:	Yeah but you, you're doing this, this square right here, right? Two and two.
01:24:52	Romina:	I'm doing the six, right? You want me to do the six?
01:24:54	Michael:	Yeah. That square right there.
01:24:55	Romina:	That's still a four.
01:24:55	Michael:	Mm hum.
01:24:56	Romina:	That's two across, two down. That's four, so you're in the four blocks. And then it's this- to get to here, the only way you can get to here is somewhere you got to go two across and two down. So there's two of one color and two of another. This is all one color. This is one and three. Two and two. Three and one.
01:25:11	Michael:	All right. Yeah. That makes sense.
01:25:12	Romina:	All one color. And the, the four is still three and one but then it's three across and one down so it means it's three of one color or the one of the other color.
01:25:19	Michael:	That, that's a pretty good explanation.
01:25:22	Brian:	It's cool.
01:25:26	Romina:	So. Who's calling them in?
01:25:28	Brian:	Don't call them in yet. Let's hang out. I'm going to go home. I'm going to weigh a hundred and ten pounds.
01:25:33	Romina:	Does it look better?
01:25:34	Michael:	Yeah.
01:25:41	Brian:	You didn't have to get them.
01:25:43	R1:	Oh. So.
01:25:45	Romina:	We're ready for his question.
01:25:47	Brian:	Romina's got something good.
01:25:48	R1:	OK, ready for your question.
01:25:49	Romina:	Come on down.

Time	Speaker	Transcript
01:25:50	R1:	Researcher 12.
01:25:51	Brian:	Researcher 12.
01:25:53	R12:	All right.
01:25:53	Romina:	He's our summer buddy.
01:25:56	Michael:	All right. Ask, ask your question again so we know what we're-
01:25:59	Romina:	Exactly what you're saying.
01:26:00	R12:	Uh, my question was you said that you found Pascal's Triangle-
01:26:06	Romina:	Uh-huh.
01:26:07	R12:	-here and um, it wasn't clear to me that if you go, let's take-
01:26:12	Michael:	You want a like reason why- how it relates?
01:26:15	R12:	Yeah.
01:26:16	Romina:	OK.
01:26:17	Michael:	Not because it looks like it? You want to know why.
01:26:18	R12:	Uh-huh.
01:26:18	Romina:	Now we just picked any point. Let's say we picked this point. No matter how you get to this point, you're-
01:26:22	Michael:	Do the six one. The six one-
01:26:23	Romina:	Well we'll do the six and the four.
01:26:24	Michael:	All right.
01:26:24	Romina:	OK, to this point you know you need to take at least- you have to take four moves. That's the shortest amount of moves because just like a simple one, two, three, four. So that means it's- let's say you we're relating back to this four moves equals four blocks. So I'd have to go down to the four block area. So that's one, two, three, four. And now here you're going three across and one down. Or so.
01:26:44	Michael:	There's no possible way you could-
01:26:45	Romina:	Do anything else.
01:26:46	Michael:	You have to, no matter how or which way you go you have to go three and then one.
01:26:49	R12:?	Uh-huh.
01:26:50	Romina:	No matter, in any move you're going one down and three across no matter- in any direction you take. So the three across and one down, that relates to three colors and then-
01:26:58	Michael:	Of one-
01:26:58	Romina:	Three of one color and one of another. So you go and you look in here. Say- OK, here's with all one color. This is with one of one color-
01:27:05	Michael:	That's- that's nothing.
01:27:07	Romina:	No that's all one color but we're not using that because you can't

Time	Speaker	Transcript
		all go all in the same direction. That's all one color. That's with one of one color and three of the other. So that's four and that's what we have and if you go down to here this is two and two and this is three and one which is the same thing. So there's your other four. And if you go to the sixth, the only way you can get there again is by four moves. It goes one, one, two, three, four. So you're in the four block again but this time you have to take, no matter what you do, you go two across and two down anyway you do it. So that would be two and two which is your six but you're still in like the four block area.
01:27:43	Michael:	Like you know what the, um, let me write this down. Like when you write the Pascal's Triangle, this is really like, like. All right, let's say-
01:28:00	Romina:	[Inaudible.]
01:28:01	Michael:	Let's say it's like, uh, I don't know how to say it, like, um, like a pizza or something. All right, you have choice of four toppings.
01:28:09	R12:	OK.
01:28:10	Michael:	All right. This one is the pizza with nothing. So you'll only, there's only one possibility without any toppings on the pizza.
01:28:18	R12:	Uh-huh.
01:28:18	Michael:	Now if you have one choice of topping you get four. I, I see it but I don't know how to like say it.
01:28:26	R12:	Maybe you can help me see how you're relating the number of toppings and the number of blocks.
01:28:33	Michael:	To this?
01:28:33	Romina:	You just can't do blocks, can you?
01:28:34	R12:	Yeah. To the- get- getting to any- to a particular corner.
01:28:39	Michael:	I like see something and I, if I say it'll-
01:28:41	Romina:	Just use blocks.
01:28:42	Michael:	- it'll make it a lot clearer but I just don't, don't know how to say it.
01:28:46	R12:	Why don't you just try saying it?
01:28:50	Michael:	All right. Well- I'm trying to think of like a, a way.
01:28:58	Romina:	Mike, if we were to use pizzas could you explain this 'cause I don't know how to do this, OK, that means you have four toppings.
01:29:02	Michael:	This is, um, yeah, four toppings.
01:29:03	Romina:	Plain.
01:29:04	Michael:	//You have one topping, you're going to make //four different kinds of pizzas.

Time	Speaker	Transcript
01:29:05	Romina:	//One topping. //Two toppings.
01:29:06	Michael:	//Two toppings.
01:29:07	Romina:	//Three toppings.
01:29:08	Michael:	//You can make six.
01:29:08	Romina:	All toppings.
01:29:09	Michael:	Yeah.
01:29:09	Romina:	All right. So, you could do that. Just do-
01:29:11	Michael:	Don't know where to go from there though. Like how to relate toppings to that.
01:29:14	Romina:	Just the same way I just did with the blocks. Same thing.
01:29:23	Michael:	All right, think of a topping as like, um, being able to go across so if you're only able to go across one time then you could do it four different ways and-
01:29:35	Romina:	That's one topping.
01:29:36	Michael:	Here. You could do this- This- this one right here. Go across this time and go down this time and go down this time and that time. The rest is all going down. The rest of your moves are all going down.
01:29:52	R12:	So you're saying one topping.
01:29:54	Michael:	Yeah. Yeah, one topping would be like you're only able to walk across or go across or drive across actually it's a taxi, one time, one block.
01:30:01	R12:	OK.
01:30:02	Michael:	Now the six would mean you're able to drive two blocks across and two down. Um, four would be you're able to drive three across and the last, and this one right here is you're able to drive, wait, four, um, you're able to drive four across which, I mean, drive four down, no, nothing across. I'm trying, I'm trying to say, I can't really say.
01:30:35	Brian:	Good job.
01:30:36	Michael:	Yeah, this would mean you would drive nothing across. It wouldn't even get you to that, wouldn't even get you there. So, that's why, you know, the ones don't really count in our, in our like model. Like-
01:30:47	Romina:	The ones, the ones //would be if you just could-
01:30:48	Michael:	//The only thing-
01:30:48	Romina:	-if you're going just to this point because it's only, you're only going in one direction. Like you can't get to any of the inside points because you have to use two directions.
01:30:57	Michael:	Yeah. So on the odd do you see like four-

Time	Speaker	Transcript
01:31:02	R12:	What I understood you say, you're saying is that the number of toppings related to-
01:31:07	Michael:	To the number of times you go across.
01:31:09	R12:	OK. So that one that you have at the corner there-
01:31:14	Michael:	This one right there?
01:31:15	R12:	Uh-huh. How many toppings is that one?
01:31:17	Michael:	That's all the toppings. But you really, you can't get there by going all, you know acro- um.
01:31:22	Romina:	Those would be like the across toppings.
01:31:25	Michael:	Yeah. This one actually, this would be, uh, all toppings, which would really mean all down.
01:31:29	R1:	So are you telling me that some of those are across and some of those are down?
01:31:33	Michael:	Yeah, like how I was saying it.
01:31:36	Romina:	This one would be two across.
01:31:39	Michael:	No, no. This would be one across and-
01:31:42	Romina:	One across, yeah.
01:31:43	Michael:	-and three down. All right? That's-
01:31:45	Romina:	No-
01:31:45	Michael:	No, one across and three down.
01:31:48	Romina:	Yeah, that one-
01:31:50	Michael:	All right, this one you go two across and two down and three across and one, and one down.
01:31:58	R1:	So how does that work with the *A*'s and the *B*'s and the toppings? So I see what you mean by across and down but now if I'm thinking of *A*'s and *B*'s or *x*'s and *y*'s, right. Would you say that just one more time? I know that you've said it.
01:32:12	Michael:	I, I said it?
01:32:14	R1:	No. Somehow it came out of the conversation. Somebody said it.
01:32:17	Brian:	Romina was bringing it up.
01:32:18	Romina:	Um, I'm sorry. What, what am I trying?
01:32:20	Brian:	*x*'s and *y*'s, like-
01:32:21	R1:	I think it was Romina who did it, yes. She used *x*'s and *y*'s for acrosses and downs earlier.
01:32:26	Romina:	OK.
01:32:27	R1:	Didn't you, Romina?
01:32:27	Romina:	So if we're doing the same one with, um, with no- no *x*'s then you'd have to go four down- four *y*'s down and that would be this one. But you're not going to get there. Whatever. But if you're

Time	Speaker	Transcript
		trying to get there, it's one *x* and then you go three *y*'s. So that's your four. If you're trying to get to this one over here, it's two *x*'s, two *y*'s then three *x*'s, one *y* and they all, they all equal four but they all have different amounts of *x*'s and *y*'s, and that's how we get this. Yes? No?
01:33:02	R12:	And the *x*'s and *y*'s. What does the *x* correspond to again?
01:33:06	Romina:	*x* is across.
01:33:07	Brian:	Going across. And *y* is down.
01:33:09	Romina:	Or a topping.
01:33:10	R12:	Uh-huh.
01:33:10	Romina:	Or a color.
01:33:11	R12:	Uh-huh.
01:33:12	Romina:	All the same thing. And all our *y*'s are down, topping, color.
01:33:15	R12:	Uh-huh.
01:33:17	R1:	Could you have used zeros and ones?
01:33:19	Romina:	Sure.
01:33:20	R1:	How does that work?
01:33:22	Romina:	That's his area. [Romina points toward Michael.]
01:33:23	Michael:	I don't believe it.
01:33:25	Brian:	Come on Mike.
01:33:25	R1:	Is that Michael's area?
01:33:26	Romina:	Come on, Mike. Zero, one.
01:33:27	Brian:	//Break out the binary.
01:33:29	R1:	//Does that work with zeros and ones?
01:33:31	Michael:	Uh man, I haven't seen that in a while. Uh, I really- don't remember.
01:33:39	Romina:	Well just- the same thing-
01:33:41	Michael:	Oh like-
01:33:42	Romina:	One would be every time across-
01:33:42	Michael:	Yeah, one-
01:33:43	Romina:	Zero would be every time down.
01:33:47	Michael:	Just- All right, this- right there. This group is, you know, everything that has one one and two zeros. [Writing binary codes: 100, 010, and 001.]
01:33:57	R1:	Uh-huh.
01:33:57	Michael:	That's that. The next one would be- [Writing binary codes: 110,011 and 101] two ones and one zero. That's this. And I guess the one you could call going across and two down. Across and two down. Twice and down. You know you go two ones-
01:34:20	R1:	//Uh-huh.

Time	Speaker	Transcript
01:34:21	Michael:	//or two acrosses and one down, there's a zero. That's a, is that good?
01:34:26	R1:	I don't know. Is that another way?
01:34:27	Romina:	Yeah.
01:34:28	Michael:	Do you- like do you see how you can relate the zeros across and down.
01:34:30	Brian:	The same thing.
01:34:31	R1:	Brian. Brian thinks-
01:34:32	Michael:	The one moving across and the zero would mean down.
01:34:33	R1:	Romina?
01:34:33	Romina:	Yeah, see I can't work like that. I work in, um, towers.
01:34:36	R1:	You're working in towers.
01:34:37	Romina:	He works in pizzas and binary.
01:34:40	R1:	Brian are you- work both ways Brian?
01:34:43	Brian:	No. No I'm totally not a binary kid. I don't-
01:34:47	Romina:	We- see me and Brian were absent when we did binaries in like sixth grade.
01:34:50	Brian:	I missed a week.
01:34:51	Romina:	We obviously weren't there.
01:34:52	R1:	Is that right?
01:34:53	Brian:	What class was that?
01:34:54	Michael:	Seventh grade. Twice.
01:34:55	Romina:	Seventh grade. We weren't there.
01:34:56	Brian:	I wasn't in that class all year man.
01:34:57	Romina:	I was in surgery.
01:34:58	Brian:	I was playing basketball all year in that class.
01:34:59	R1:	Wow. That's really neat. Do you have anything else to add?
01:35:02	Brian:	Um, no. I mean I'm-
01:35:05	Michael:	I mean- I mean did that convince you? Or sort of?
01:35:08	R1:	Well I see- I see how you get the numbers. I see how you get those numbers.
01:35:10	Michael:	All right, you figure-
01:35:11	R1:	I guess my- my question still is suppose I wanted to get just a general number there, um-
01:35:17	Romina:	OK, that-
01:35:19	R1:	How would you talk about some general numbers?
01:35:20	Romina:	All right. We'll just pick this one.
01:35:23	R1:	Uh-huh.
01:35:24	Michael:	We've proved to you that you understand why it relates to the Pascal's Triangle.

Time	Speaker	Transcript
01:35:27	Romina:	Yeah.
01:35:27	R1:	Oh yeah.
01:35:28	Michael:	So you give us a general number, we look at the triangle.
01:35:29	Romina:	You pick a general number.
01:35:30	Michael:	That's basically-
01:35:31	R1:	But-
01:35:31	Romina:	To get the simplest way you're going to go all your overs and all your downs at one time so that'll tell you this is going to be one, two, three, four, five, five across, so one, and then one, two, three, four, five and five down.
01:35:41	Michael:	And five down.
01:35:44	Romina:	So you know there's going to be a total of ten blocks.
01:35:45	R12:	Uh-huh.
01:35:47	Romina:	And then- so you've got your ten block row and then you're going to know it's five of one color and five of the other color.
01:35:54	Michael:	There's going to be one right in the middle.
01:35:56	Romina:	There's going to be a number. Yeah, it's going to be the one right in the middle. It's going to, well, I don't know. I don't know what it's going to be but-
01:36:00	Michael:	The one that like-
01:36:01	Romina:	The one right in the middle of everything.
01:36:02	Michael:	I don't- That's, that's-
01:36:03	R12:	Which, which row of Pascal's?
01:36:04	Michael:	-that's way up there. That's-
01:36:05	Romina:	It's going to be the tenth row because you took ten moves to get there. So you're going to go down to the tenth row.
01:36:08	Michael:	Yeah, it's going to be the tenth row because you have-
01:36:10	Romina:	And the tenth row that has five of one color and five of the other color, that's your number and that's how many ways you can get to that point.
01:36:16	Michael:	Which that one will be in the middle.
01:36:17	Romina:	Uh-huh.
01:36:17	Michael:	Because just the way it's set up. That one will end up in the middle.
01:36:21	Romina:	Plus it's like an even, the square.
01:36:23	Michael:	One. Yeah, it's, no, it's an odd number. That's why it's in the middle.
01:36:24	Romina:	Yeah. It's a square.
01:36:25	Michael:	Even numbers. There is no-
01:36:27	R12:	How do you know it's the tenth row?

Time	Speaker	Transcript
01:36:28	R1:	Yeah.
01:36:28	Romina:	Because it took us five moves to get, uh, ten moves to get there.
01:36:32	Michael:	Because you have ten spots. Ten toppings and-
01:36:34	Romina:	Because you know you can always-
01:36:36	Michael:	Ten different places to put these numbers.
01:36:37	Romina:	Yeah.
01:36:37	Michael:	Which is ten.
01:36:38	Romina:	And you know, and this ten there's, there's only ten moves you can take because this is like the simplest way. You go all the way across and all the way down.
01:36:44	R12:	Uh-huh.
01:36:44	Romina:	And that's going to be like your, the simplest way and that's going to mean that's the shortest way to get there. Like-
01:36:49	R12:	Maybe help me understand that by running us through-
01:36:52	Romina:	OK //like-
01:36:53	R12:	//-each story //from the first row-
01:36:54	Romina:	//-this one?
01:36:55	R1:	Yeah, what's the first row?
01:36:55	R12:	-of Pascal's Triangle.
01:36:57	Romina:	This one, there's only two moves you can get to this one. You go over one down one. Two moves.
01:37:01	Michael:	You mean like the first row that would be-
01:37:02	Romina:	To the second row because there's two high in block terms. And for this one it's two across and one down-
01:37:10	Michael:	I mean, like I said before, the rows correspond to the shortest distance.
01:37:12	Romina:	Yeah.
01:37:13	Michael:	I mean the //shortest route.
01:37:14	Romina:	//Yeah. So this is //three moves.
01:37:15	Michael:	//Everything in this row, two.
01:37:16	Romina:	Third row.
01:37:17	Michael:	And this one three. So that's how-
01:37:19	R12:	Say it again please.
01:37:20	Romina:	OK, this one. There's three moves.
01:37:21	Brian:	One, two, three.
01:37:22	Romina:	And then this is the third row.
01:37:24	R12:	So the-
01:37:25	Romina:	This one's four moves, fourth row.
01:37:26	Michael:	If the shortest route is ten, then it- then it's //in the tenth row.
01:37:28	Romina:	//Tenth row.

Time	Speaker	Transcript
01:37:29	R12:	I'm still a little confused.
01:37:32	Michael:	All right. If you pick any point on-
01:37:34	R12:	Start, start from the very first row please.
01:37:35	Michael:	The first, the first one.
01:37:35	Romina:	The first.
01:37:36	Michael:	All right.
01:37:37	Romina:	No moves. There's only, you're stationary there. That's one. Just one.
01:37:42	R12:	So it's the top row of-
01:37:43	Romina:	Yeah, that's just your Pascal's.
01:37:44	R12:	-Pascal's Triangle?
01:37:46	Romina:	Yeah. You go down to here. There. You're going to go over one, down one. There's only two moves.
01:37:52	R12:	Two.
01:37:53	Romina:	That's the simplest way you can go.
01:37:53	R12:	Uh-huh.
01:37:54	Romina:	So that's Pascal's like second row, two blocks, two toppings, whatever you want to say.
01:38:00	R12:	Uh-huh.
01:38:00	Romina:	And this one, you're going over two and down one so that's a total of three moves. The simplest moves so that's the third row and you can go-
01:38:08	R12:	So it's the second going over two blocks-
01:38:10	Romina:	Yeah.
01:38:11	R12:	-and it's which row of Pascal's Triangle?
01:38:12	Michael:	//That's in the third row.
01:38:13	Romina:	//The third row.
01:38:14	Michael:	//Because it takes three to get there.
01:38:14	Romina:	//Because you have two and one. And you're going over two over one. You're doing three complete moves. And that move just happens to be two and one. [Inaudible.]
01:38:24	R12:	Uh-huh.
01:38:25	Romina:	And, and this one here you're making, you're going over three and down one so that's a total of four moves. That's the fourth row.
01:38:32	R1:	So, what about the r^{th} row?
01:38:34	Michael:	Would be-
01:38:36	Romina:	The r^{th} row would be r moves.
01:38:42	Michael:	Yeah, r moves r shortest distance. Whatever-
01:38:46	Romina:	Yeah.

Time	Speaker	Transcript
01:38:46	Michael:	-you want to call it.
01:38:47	R12:	Uh-huh.
01:38:48	Michael:	*r* half the perimeter whichever, you know.
01:38:55	R12:	OK.
01:38:57	R1:	Are you convinced?
01:38:59	R12:	Yeah.
01:39:00	R1:	It's really very interesting. Interesting problem. Did you ever do anything like this before?
01:39:09	Michael:	No, no I've never seen it before in my life.
01:39:11	Romina:	We just discovered Pascal's Triangle.
01:39:12	Brian:	Didn't we have to, didn't we have to do something in Pantozzi's class with the subway?
01:39:15	R1:	What's that?
01:39:17	Romina:	Yeah but we didn't do it, though.
01:39:18	Brian:	Uh, no. Something, I don't know, somewhere like-
01:39:22	Romina:	We can-
01:39:24	Brian:	If a person is let off at like this subway station and they want to go to this building what's the shortest way to go or something?
01:39:29	Michael:	No, it was like, no, it was a bunch of subway stops.
01:39:31	Romina:	Yeah.
01:39:32	Michael:	And there's some subway stop is three blocks away from this building. Something, which stop should he get off at?
01:39:37	Brian:	Something like this.
01:39:38	Michael:	In order to get there. And then, cause like he had to go to a couple of buildings.
01:39:40	Brian:	It wasn't exact. It wasn't exact so we're not going to get into it.
01:39:46	R1:	So some of the same kind of-
01:39:47	Michael:	Yeah.
01:39:47	R1:	-reasoning you used.
01:39:48	Michael:	Yeah. That was last year though.
01:39:51	R1:	You, you are wonderful for staying and working this hard. …

APPENDIX L: TRANSCRIPTS OF 2002 INTERVIEWS

Interview with Michael on April 3, 2002

Time	Speaker	Transcript
00:00:19	R1:	-and the time you were young. All the way through.
00:00:22	Michael:	I remember, I-
00:00:23	R1:	So if you could remember differences.
00:00:23	Michael:	Yeah, I don't remember much from the young, when I was younger.
00:00:28	R1:	Uh-huh. But what you remember might be helpful as it comes about.
00:00:30	Michael:	OK. Like from the high school, when we did those things in high school, when it was just like the small group of, it was Romina, Jeff, Ankur, and Brian, and me. I mean, when we were given problems, I wasn't like the one who was going to, you know, start shouting out answers and stuff, and I guess everyone could, you know, I would just sit there and just do my own thing and then when something comes up, I'll, I'll join in. I mean, that's the way I've really always been with, like, schoolwork and stuff. Um, um, that's how like, Bobby's kind of like that too. And that's how, like, you know, we kind of like have this, you know, sort of connection, cause we both understand, like, things on the same level. Um, like Romina and Jeff and, like, Brian, they're all like good friends so you know they went about it differently. They would, um. Those just are, you know, like, um, you know, calling out answers. Not just calling out, but like you, you could just tell that there was a different way of learn, like there's a different thing, else, like for them and me, happens differently. Um, I mean, it's all good. In the end, we all start sharing ideas, and, and it works out fine. Um, I mean, overall, I'm comfortable with the whole, you know. Even though we're not all the same. You know, we do, over the course of those, you know, however many years, we develop a, like a connection between everybody. You know, I felt comfortable learning like that. It wasn't, it wasn't, you know, awkward or anything like that. Uh.
00:02:05	R1:	OK. So, um, let's think, um, a little bit more specifically.
00:02:16	Michael:	OK.
00:02:17	R1:	And I'm thinking now at the night session that was the night before your prom. And, um, you brought a lot to that session,

Time	Speaker	Transcript
		um, that took it in a direction which, which went back from, uh, some interviews we had done with you earlier, where you had begun to think about, uh, Pascal's Triangle. And you had to think about how one could look at that triangle, um, from, um, from the lens of some of the problems you had worked on, like towers and pizzas.
00:02:51	Michael:	Uh-huh.
00:02:53	R1:	And, and had some ideas that you brought to that session. Do you remember that at all?
00:02:56	Michael:	Was that the, um, session that we were in that small room?
00:02:59	R1:	We were in, right-
00:02:59	Michael:	By, by the offices?
00:03:00	R1:	By Mr. Pantozzi's room.
00:03:01	Michael:	OK.
00:03:02	R1:	And it was very late. We didn't start till about 7. And people were getting their tuxedos that night, uh, for the prom. In fact, Brian came in late because his tuxedo was the wrong size.
00:03:12	Michael:	OK, I think I remember that. I don't remember, um, I do remember what you were talking about the Pascal, like how I looked at it, um, sort of like, uh, each thing is like relating to probability. Is that, is that what you're talking about? Like um, like what she had on her sheet right there? With the choose? [Michael is referring to the copy of Pascal's Triangle shown in Figure L1.]

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

Figure L1. Pascal's Triangle in combinatorics notation

Time	Speaker	Transcript
00:03:30	R1:	This was looking at how-

Time	Speaker	Transcript
00:03:31	Michael:	Cause I don't remember the specific night.
00:03:33	R1:	Right. It was looking at how the triangle grew. [Michael nods.] And, um, that was the question. How, how can you talk about how that triangle grows.
00:03:43	Michael:	I mean.
00:03:43	R1:	And you had used the, uh, example of pizzas to think about-
00:03:47	Michael:	Oh, OK.
00:03:48	R1:	-the movement from one row of the triangle to the other. And at that session, um, the more general question was, was posed to the group. Um, any row of the triangle. How would you move to the next one. And, um, so some of your ideas came into that session, and I wondered if you remember that.
00:04:08	Michael:	Yeah, I, I do remember. I'd probably be better writing stuff down, too.
00:04:10	R1:	Yeah, sure. [Researcher 7 gives him pen and paper.] It's a long time ago.
00:04:14	Michael:	Yeah.
00:04:14	R1:	A few years ago.
00:04:15	Michael:	I still remember. I think I still remember it.
00:04:17	R7:	Michael, as you're writing, might I ask you if you remember the prom and the tux.
00:04:22	Michael:	Well, if it, if it was meant, the junior prom, I didn't, I didn't go to the junior prom. I went to the senior prom, so-
00:04:27	R1:	OK. [Researcher 1 nods.]
00:04:27	Michael:	So, I didn't-
00:04:28	R7:	Was it the junior prom?
00:04:28	Michael:	I don't know.
00:04:29	R1:	Junior. So not everybody-
00:04:31	Michael:	So, no, I didn't, I didn't go to the junior prom.
00:04:33	R7:	Ah. [Michael is writing.]
00:04:34	Michael:	I'm not into the whole prom thing.
00:04:36	R7:	I see.
00:04:38	Michael:	You have to go to the senior prom, but-
00:04:38	R1:	It's optional, whether you go to the prom.
00:04:40	Michael:	Yeah. It's not as big as a deal, you know. [Michael continues to write.] All right. I'm going to draw a small little triangle. I do remember, um, I kind of, uh, related each row. [Michael looks away.] Let's see if I can- [to himself] -with a, a certain pizza, uh, topping situation. [Michael looks off into space again.] OK, I think it was, um, I'll just pick a row three, for

Time	Speaker	Transcript
		example. One three three and one. So, I don't know if that was, um, two toppings, or-?
00:05:20	R1:	Well, think about it.
00:05:21	Michael:	Um, yeah, I'm trying, I'm trying to, you know, get it back. [Michael looks at his paper.] Um. [Michael writes some more.] OK, I think I, I think this row, um, was like, um, you have, uh, two toppings to choose, and, um, like different, uh, no, that's not gonna- hold on a second. No, it would have to be, it would have to be more than that. [There is a pause while Michael looks at his paper.] Uh, can you refresh the pizza thing? Was it you have a certain amount of pizzas?
00:06:20	R1:	Sure.
00:06:21	Michael:	Or a certain amount of toppings?
00:06:23	R1:	A certain amount of toppings.
00:06:24	Michael:	OK.
00:06:24	R1:	OK. So the question is, if, um, you can select from two toppings.
00:06:32	Michael:	How many different pizzas.
00:06:33	R1:	How many different pizzas can you make?
00:06:34	R7:	Does the cheese count as a topping, or just the-
00:06:37	R1:	No, that, that's plain.
00:06:38	Michael:	All right.
00:06:39	R7:	That's plain?
00:06:39	R1:	That's plain.
00:06:40	R7:	So it doesn't count, that, that counts as empty.
00:06:42	R1:	No. Unless you, unless you offered as a topping extra cheese
00:06:46	R7:	OK. So you, so you get two toppings to put on with that one pizza. [Michael continues to write.]
00:06:51	R1:	Uh-huh.
00:06:52	Michael:	OK. I think I found my place, found where I was supposed to be. Um, like you said, the two, two toppings, um. [There is a pause.] I just- [Michael shakes his head.]
00:07:04	R1:	[Inaudible.]. Feel free to pick them. [Researcher 11 laughs.] Feel free to pick the two toppings.
00:07:10	R7:	Um, mushroom.
00:07:10	R1:	Make them up.
00:07:10	R7:	Mushrooms. I like mushrooms.
00:07:11	Michael:	OK. If you had no toppings, that would be one pizza.
00:07:15	R1:	OK. So which, where is that on the triangle?
00:07:17	Michael:	Well, I'm going to just draw it.

Time	Speaker	Transcript
00:07:17	R1:	Sure.
00:07:18	Michael:	And then we'll find it. You know. If you're having only, using just one topping, you can make two possible pizzas with that. [Michael writes.] And then if you have all, all the toppings, that's one. Right. And then automatically you, I see that, that relates to this row. [Michael points to his paper; Researcher 1 nods.] And I'm pretty sure it would go down, this is like a third topping, and a fourth topping. Now I think the way I, um, thought about it is, like, the row on the outside would be your plain pizza. And there's only one way to make a plain pizza. And the next, you know, from then on, the next one over would be how many pizzas you could make, um, using only one topping, and then so on until you get to the last row which is, um, all your toppings. And, once again, you can only make one pizza out of that. And um, it turns out that you, um, like we just mapped that one out. If you, if we go to the next row, it would just follow. And, um, all right. So, you see, like a, we see like the physical connection, we can see the numbers that- Now, like, why do they? I remember, um. [Michael points to the paper shown in Figure L1.] I don't know if I should do that cause I kind of saw it, if that's cheating.
00:08:33	R1:	No, no it isn't cheating. This is notation that you used in the past and you can use it again. But why don't you use the notation explicitly from one row to the next.
00:08:41	Michael:	Yeah.
00:08:42	R1:	Rather than the general, to show us.
00:08:43	Michael:	So, yeah, I'm not going to-
00:08:43	R1:	Does that make sense?
00:08:44	Michael:	Yeah. I'm not going to use the general one.
00:08:45	R1:	We'll look at the general rule later.
00:08:47	Michael:	Well, we'll take the 1 2 1 row. It's basically written, um, in that choose. I don't know what it's called. I just-
00:08:53	R1:	Choose is fine.
00:08:54	Michael:	Choose format. [Researcher 11 laughs.]
00:08:56	R1:	That's OK.
00:08:56	R7:	Choose is good.
00:08:57	Michael:	It would be, um, 2 choose 0?
00:09:00	R1:	OK.
00:09:01	Michael:	If I'm writing it correctly. Do you have to, do I have to put the brackets around it?

Time	Speaker	Transcript
00:09:04	R1:	That's the way we do it.
00:09:05	Michael:	OK.
00:09:05	R1:	That's-
00:09:06	Michael:	And it would be 2 choose 1, 2 choose 1, and then 2 choose 2. That's a tough row to say. [Michael writes row 2 of Pascal's Triangle in choose notation.] And the next row would be the same, which is with three and you have another, you know, you can go all, as far as, up to three. So it's going to be four on the bottom. [Michael writes row 3 of Pascal's Triangle.] OK. Now to connect the two rows together. See if I can remember that. [There is a pause as Michael looks at his papers.] I mean, I kind of understand why the rows, why, like each row is made of the other two combined.
00:10:11	R1:	Why is that?
00:10:12	Michael:	Well, you're starting off with this, you know, this group of pizzas that has no toppings. And this group of pizzas that has one. So when you, when you go up, you have the choice of adding one more. So to those two, well, actually it's one. To that, to that one that had nothing, you could either not give it that extra topping.
00:10:33	R1:	Uh-huh.
00:10:34	Michael:	You know, the one that brings you from one step to the next, or you can. [Researcher 1 nods.] So for those pieces that you do give that extra topping, it moves to the right. And for the others it moves to the left. And that's kind of why it doubles. The amount doubles each time.
00:10:47	R1:	Um-huh.
00:10:47	Michael:	And then, so, so that's where you get the, you know, not the half but the portion comes from this side. And the same goes for the one right next to it. The one that doesn't get the topping, you know, still stays in the one topping category. So then they combine. [Michael draws arrows from row 2 to row 3; refer to Figure L2.]

Time	Speaker	Transcript

$$\begin{tikzcd}[column sep=small]
 & \binom{2}{0} \arrow[dl] \arrow[dr] & & \binom{2}{1} \arrow[dl] \arrow[dr] & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{tikzcd}$$

Figure L2. Michael's illustration of the addition rule

00:11:03	R1:	Uh-huh.
00:11:04	Michael:	That, that's how I see how the, the two-
00:11:06	R7:	[Inaudible.] This one, there's only one of those, isn't there?
00:11:08	Michael:	Yes. Just think of-
00:11:10	R7:	There's nothing. So you're saying, put one on there. Is that what you're saying?
00:11:11	Michael:	Yes. And put one on there and keep one plain. So they kind of multiply.
00:11:15	R7:	This one [points to Michael's paper] already has one.
00:11:16	Michael:	Yeah.
00:11:17	R7:	So don't put one there.
00:11:18	Michael:	Yeah, well, yeah. There's two here in this, in this little, you know.
00:11:21	R7:	That's right, sorry, two.
00:11:21	Michael:	Two. And it, um, for those two, you can make four more pizzas by adding another topping on it. You can keep, you know, have those two the same way they were. Or just add, you know, the topping on those two. So half of, half of them go to the left.
00:11:37	R7:	Yeah.
00:11:38	Michael:	Which would keep it in the one topping category. And half would go to the right, which would-
00:11:41	R7:	Go to the next one.
00:11:42	Michael:	Yeah.
00:11:43	R7:	[Inaudible.]
00:11:44	Michael:	Give it, you know.
00:11:44	R1:	OK, so that, that's, that was the idea that we brought in at that session, and, um, the idea came, was attributed to you at that session. And, um, at that session, what the students did, I asked them to write an equation to show, for instance, how that might happen from one row to the next. Um, so can you just do that, write.

Time	Speaker	Transcript
00:12:06	Michael:	See if I can write? OK.
00:12:08	R1:	Right. To show, um.
00:12:10	Michael:	Like a general equation?
00:12:11	R1:	Well, um, that was what I was going for ultimately. Now you just said like a general equation. Um, and eventually, uh, that's what you all gave me.
00:12:20	Michael:	To, uh, give an amount for any spot.
00:12:22	R1:	Right.
00:12:22	Michael:	In the-
00:12:23	R1:	Right.
00:12:24	Michael:	All right, so I guess we'll give, uh, these, you know, the row a name.
00:12:29	R1:	Uh-huh.
00:12:30	Michael:	Um, call that *r*. And, um, I guess the spot in the row, like, you know, zero topping, one topping.
00:12:39	R1:	Uh-huh.
00:12:40	Michael:	Call that, *n* sounds fine.
00:12:44	R1:	So, um.
00:12:46	Michael:	Just, just to like pick, you know, one spot, and then see what-
00:12:49	R1:	OK.
00:12:50	Michael:	OK? Um. [There is a pause.] I'm just going to like work this out in my head and see if it actually works. [Inaudible. Michael writes the left side of the equation.]
00:13:31	R7:	[Inaudible.]
00:13:40	Michael:	All right. Yeah. Go ahead.
00:13:42	R7:	So these are the chooses.
00:13:44	Michael:	Yeah, the chooses.
00:13:46	R7:	*r* is the row, like how many-
00:13:47	Michael:	Yeah.
00:13:48	R7:	This would be- [Inaudible.]
00:13:48	Michael:	Um.
00:13:49	R7:	No. That's how many possible toppings.
00:13:50	Michael:	Um. Yeah.
00:13:54	R7:	And *n* is how many actual. So *r* choose *n*. Or is it *r* choose *n* +1?
00:14:00	Michael:	Equals. [Michael writes the rest of the equation. Refer to Figure L3.]
00:14:00	R7:	$r + 1$.

Time	Speaker	Transcript

$$\binom{r}{n} + \binom{r}{n+1} = \binom{r+1}{n+1}$$

Figure L3. Michael's equation for Pascal's Identity

00:14:04	R1:	OK, so, um, it, it's interesting to me that you reversed the notation a little bit than-
00:14:11	R7:	Yeah.
00:14:12	R1:	-than what you used then, which I think is very nice.
00:14:14	R7:	Absolutely.
00:14:15	R1:	Because it's evidence that you've rebuilt it and not memorized the formula.
00:14:18	Michael:	Was, was it-
00:14:19	R7:	It shows how much flexibility.
00:14:21	R1:	So we're happy about that.
00:14:22	R11:	The standard one is to write *n* choose *r*.
00:14:24	R1:	To do it the other way. But the way you labeled it and the way you explained it was very, very nice. I mean I, I didn't expect it.
00:14:31	R11:	I think it's actually much more sensible from that notation.
00:14:33	R1:	Yes, it is.
00:14:35	R11:	Exactly.
00:14:36	Michael:	*n* is number of toppings.
00:14:36	R1:	Yeah. Right.
00:14:37	R7:	Can I just ask a quick question that popped into my head?
00:14:38	Michael:	Sure.
00:14:39	R7:	It was a question about um, Norbert Weiner said of Richard Feynman. He said something about, oh, you thought something through, and then you wrote it down. Feynman said, no, I didn't. The writing is the thinking. To what extent is the writing-
00:14:55	Michael:	Is that? I mean.
00:14:57	R7:	To what extent did you, is the writing part of the thinking?
00:15:00	Michael:	Um. Well, it definitely is part because when you start writing things down, you kind of, um, like, it, it, I didn't have it all in my head and then put it down. You, you saw I paused, you know. And I, you have, you have to stare at it for a while. And, and, you know, cause you only have so much in your head, and you can, when it's on paper, you can make sense of it. So.
00:15:19	R1:	Well, that, that comes to the, to the next question. And if this doesn't satisfy Researcher 8, she can, she'll probably want to

Time	Speaker	Transcript
		talk to you some more.
00:15:27	Michael:	Uh-huh.
00:15:28	R1:	But one of the other questions that seemed to come up with some of the folks who have been looking at the later tapes was, when you were asked to do a, a specific, um, solve a specific question, let's say for rows four and five. Um, when you presented your solution, just like in the taxicab problem, you were asked to solve the problem for particular points. You didn't do that. You ended up solving the general problem. But you weren't asked to solve the general problem. That was not what the group was asked to do.
00:15:59	Michael:	Uh-huh.
00:16:00	R1:	But you did that. Do you, do you, can you talk about that a little bit?
00:16:04	Michael:	Like, solving a general problem.
00:16:06	R1:	You solved a general, although that was not the problem posed to you.
00:16:09	Michael:	Yeah. I. I-
00:16:10	R1:	How, how would you explain that? Is my question clear?
00:16:13	Michael:	Uh, I mean.
00:16:15	R7:	[Inaudible.] Why did you go to the general question?
00:16:16	Michael:	I mean, I don't remember that specifically, but I know, when I, um, I haven't, I haven't taken math courses in a while, but usually when I, the way I think of, uh, make everything into a problem first, OK? For me, it's easier to understand if you just take one, you know, specific case, it doesn't look, you know. You see the r's and the n's and you can't combine them, so you know, I don't know if you understand. You have a big equation, you could see each part. It's like specifically. It's not like, um, if I start writing, let's say something was a 2 and a 3 and I combine them into a 5. It doesn't, it doesn't mean anything, I mean when it's a number. But, like, um, I feel comfortable putting every, using a general form, putting in equations. That's the way I, like, I feel comfortable with.
00:17:05	R1:	OK, so.
00:17:05	Michael:	Dealing with problems like that.
00:17:06	R1:	So am I hearing you say, I don't want to put words into your mouth. But am I hearing you say that when you see the general, you also see the particular ones in there?
00:17:14	Michael:	Yes.

Time	Speaker	Transcript
00:17:14	R1:	That's what I thought I heard you say.
00:17:15	Michael:	You see the relationship between.
00:17:17	R1:	Of, of many particulars.
00:17:18	Michael:	Of many, yeah.
00:17:19	R7:	Yeah, I feel along [Inaudible.]. I don't want to put words into his mouth, so.
00:17:24	R1:	Be careful how you ask it then.
00:17:25	R7:	Yeah. Well, I'm trying to-
00:17:26	R11:	He knows that. He's a bright boy.
00:17:27	R1:	We're only teasing him.
00:17:30	R7:	Well, it's the issue of-
00:17:31	R1:	He won't agree with you if he doesn't, I can assure you.
00:17:33	R7:	No, but you will agree with me. I'm not sure if we're constructing the meanings as we go along. But what I'm hearing you say is, what I think I'm hearing you say is, the specific numbers can sometimes obscure patterns.
00:17:42	Michael:	[Michael nods.] Exactly. Yeah.
00:17:43	R7:	That the general form-
00:17:46	Michael:	Yeah.
00:17:46	R7:	There isn't anything left but the pattern.
00:17:50	Michael:	[Michael nods.] That, that's exactly like, that's the, you know, when you see something, there's a pattern, you're not going to get it if it's just all numbers. If you see, you know, the symbols and - [Researcher 7 nods.]
00:17:58	R1:	But this you did as a group, generally. It wasn't just you, it was other people, too, when you called me to give me, or explain something, often it was the more general solution. Why do you think that happened as sort of-
00:18:14	Michael:	I mean, maybe it's more convincing.
00:18:17	R1:	Say more about that.
00:18:18	Michael:	It's more convincing to propose a general solution, because someone, a solution to a specific problem, and then give you, well, is it gonna work for this, you know, this other. And if we, you know, if we just gave you an answer for one problem, then we have to step back and redo, you know, the whole process with that other number. If we have it, a general problem posed, there's only so much you can ask. You can, you know, just verify that it's right. And once it is, it'll work for, you know.
00:18:42	R1:	OK, so convincing was a piece of this.
00:18:43	Michael:	Yeah.

Time	Speaker	Transcript
00:18:45	R1:	Can you say a little bit about convincing, um, the role of convincing?
00:18:48	Michael:	Well, it's important to convince, you know. When you're proposed with a problem, if you're not, if someone's not fully convinced, then, it's like you failed. That's the point, to convey your, um, your thinking in a way that people understand it.
00:19:05	R1:	Researcher 7, were you going to say something?
00:19:06	R7:	Yeah, it's a question that comes, Michael, from remembering you, I think it was either grade 5 or grade 6 when you were doing the tower of Hanoi.
00:19:13	R1:	Grade six.
00:19:13	R7:	Grade six, doing the Tower of Hanoi problem. Um, Dr. Davis was running the board, and people were doing different things. I think Stephanie's group was working on- Matt was sitting alone, so you were working with Jeff. [Inaudible.]
00:19:27	Michael:	I saw that tape, too.
00:19:28	R7:	Yeah, and then you jumped up and said, "Oh, I've got it, I've got it." Uh, and when Dr. Davis said what do you got, you said it's twice the number before, plus the one. So you, you basically-
00:19:38	Michael:	I think I was wrong, though. But I was-
00:19:40	R7:	No.
00:19:41	Michael:	Was that right? Because I remember the tape, and I remember looking at it. I was like, oh, there's something wrong and it must have been a different problem.
00:19:45	R7:	No, you actually showed what it was. You actually picked up a bunch and you moved them. [Researcher 7 moves his hands as if moving towers.]
00:19:48	Michael:	[Michael moves imaginary towers, too.] Yeah, twice the number, plus the one. Twice. Yeah.
00:19:50	R7:	My question is, how does that relate to this issue of general case?
00:19:54	Michael:	Well, cause then you can bring, cause actually there's three things going on there.
00:19:58	R7:	All right.
00:19:59	Michael:	You're grabbing the, in the previous task, moving that.
00:20:02	R7:	Yeah.
00:20:03	Michael:	Adding one more, and then moving that tower again. And if you just give it as one lump, you know, number, then it doesn't, and you don't see the process why it's doubling plus one. You could see if I give that lump of towers at the other end. To

Time	Speaker	Transcript
		move that, it takes *n* plus one and then another.
00:20:20	R7:	So the process why is important?
00:20:22	Michael:	You can see, um, more of what's going behind the scenes than just, you know, an answer. [Michael waves to someone off camera.]
00:20:30	R7:	I'm- what they're doing. You have a sort of tendency to want to see what's going on.
00:20:35	Michael:	Yeah. [Interview continues on another topic.]

Interview with Romina on July 22, 2002

Time	Speaker	Transcript
00:00:00	R8:	-before, probably. If not, I'll tell you about it.
00:00:03	Romina:	OK.
00:00:03	R8:	They did- this just became very interesting to me because this was where actually there was some formalism that happened. Um, this was the one, um, we reminded Brian about it by telling him this was the night before the Junior Prom or something, and he came in very late, talking about how his tux didn't fit, and all the time that it took to get it.
00:00:22	Romina:	And baseball.
00:00:24	R8:	Yeah. Um, he came in late, and actually you guys had been working for like an hour and a half.
00:00:31	Romina:	Uh-huh.
00:00:31	R8:	And basically Jeff kind of gave him- we talked about the Reader's Digest. Ankur called it the Reader's Digest version-
00:00:37	Romina:	Yeah.
00:00:38	R8:	-of what you had been doing.
00:00:39	Romina:	I kind of remember that.
00:00:39	R8:	Yeah. And, and Jeff and Michael did the writing on the board, but all of you kind of participated in creating this formal notation for Pascal's Triangle addition rule.
00:00:51	Romina:	Uh-huh.
00:00:51	R8:	And, and part of it was interesting because Researcher 1 actually pushed a little more than she usually does.
00:00:57	Romina:	Yeah. [Romina nods.]
00:00:58	R8:	So that was interesting to me, too, how you guys did these things where she was pushing, can you write it this way, can

Time	Speaker	Transcript
		you write it that way. So, I wanted to talk to you about that.
00:01:07	Romina:	OK.
00:01:09	R8:	That's basically where I wanted to go. So are we on camera now?
00:01:13	Dan	Yeah, we're good.
00:01:15	R8:	OK. So that was the introduction that wasn't gonna be on camera. So. What I, I kind of gave you, where we are going, um, what Michael did in that session, and what we did later was, we looked at Pascal's Triangle and then, um, you guys wrote it in this form, the choose notation, you called it. You said you know, about combinatorics? Or combinations?
00:01:41	Romina:	All right. Three choose two, yeah.
00:01:43	R8:	I guess it's combinations or permutations, so this would be combinations. Um, and he always talked about using pizzas. Do you remember?
00:01:51	Romina:	Yes.
00:01:52	R8:	Brian was always, Brian. Michael was explaining, would explain things by talking about, you're going from a number of pizza toppings to a different number of toppings and how the addition rule works there. And you pretty much talked about how- you seemed to think of towers as the primary image -
00:02:06	Romina:	Uh-huh.
00:02:07	R8:	-in Pascal's Triangle. [Researcher 8 shows Romina two diagrams of Pascal's Triangle; refer to Figures L1 and L4.] So can you tell me anything? I sort of refreshed your memory a little bit. Do you remember anything from how you guys worked on this or how, you know, the addition rule would apply here [Figure L1.]? Or you can just tell me in this one [Figure L4.] how you-

Time	Speaker	Transcript

1

1 1

1 2 1

1 3 3 1

Figure L4. Pascal's Triangle referred to in interview with Romina

00:02:21	Romina:	Right.
00:02:22	R8:	-how you did the addition rule.
00:02:24	Romina:	For, for this one? I look at, I mean I did something similar to this, and I think this is what, how many, how many toppings, like the top number, like the one choose one or one choose zero would be how many toppings. Or I mean if we were talking about towers-
00:02:39	R8:	OK.
00:02:41	Romina:	-how many high and this would be like with zero reds, with one red. This would be two high [Romina points to row 2.] with zero reds, one red, two reds. And it just keeps going like three high, zero reds, one red, two, then three red. So it would be like three high and like of those, you choose how many blocks of each color.
00:02:59	R8:	OK. So a three-high tower that has three red cubes, you can make one like that.
00:03:07	Romina:	Uh-huh.
00:03:09	R8:	OK. And that's what I see here.
00:03:10	Romina:	Yeah.
00:03:10	R8:	OK. You can make one tower. So you're saying this one [Researcher 8 points to 3 choose 2.] is three high with two red cubes.
00:03:17	Romina:	You can make three. Yes.
00:03:18	R8:	You can make three. OK. And the two red would be one blue-
00:03:21	Romina:	Uh-huh.
00:03:21	R8:	So I see how you could make one blue, in each of the three places. OK. Now how would the addition rule work in terms of towers? I know the addition rule, one plus two

Time	Speaker	Transcript
		equals three, and so on. Does, does it make sense to talk about in terms of either towers or pizzas? You know what I'm saying? Either with this one [Figure L1] or with this one [Figure L4]?
00:03:45	Romina:	It goes- this was Michael's area of expertise.
00:03:52	R8:	Yeah.
00:03:52	Romina:	So, I know if you have two here [Romina points to the 2 in the center of row 2 of Figure L4.], this would be, we're in this one. [Romina indicates 2 choose 1 in Figure L1.]
00:03:57	R8:	Yeah.
00:03:58	Romina:	So. OK, the way it goes, it's like this two [2 choose 1], you can take, you can take either direction with it. You're either gonna add another, add another red, say, so it goes to two red [Romina points from 2 choose 1 to 3 choose 2.], or you're gonna add another blue, which stays with one red. [Romina points from 2 choose 1 to 3 choose 1.]
00:04:16	R8:	OK.
00:04:16	Romina:	And then, That's it, so it branches out.
00:04:20	R8:	OK. So.
00:04:21	Romina:	I, I don't think I'm sure what you're asking.
00:04:24	R8:	Actually, it sounds like you did answer exactly-
00:04:26	Romina:	All right.
00:04:27	R8:	-what I was asking. Why don't you just write on it? Um, show me the branches.
00:04:30	Romina:	Like this, this one goes off of that to either of those. [Romina draws lines. Refer to Figure L5.]

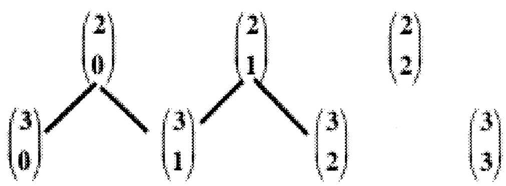


Figure L5. Romina's illustration of the addition rule

Time	Speaker	Transcript
00:04:34	R8:	And when it goes out to the left-
00:04:38	Romina:	This is, we're adding a, a blue, say, and this one [Romina points to 3 choose 2.] we're adding another red, so this changes to two.
00:04:44	R8:	OK. And then, so this one also goes to that [Researcher 8

Time	Speaker	Transcript
		points to 2 choose 0.]
00:04:47	Romina:	Uh-huh. Cause this one, yeah, you have the same [Romina draws a line from 2 choose 0 to 3 choose 1.], This one you could either add a red-
00:04:55	R8:	OK. Or not add a red.
00:04:56	Romina:	Or you add another blue. [Romina draws a line from 2 choose 0 to 3 choose 0.] Yeah.
00:04:57	R8:	OK. Which is adding a blue instead. OK. [Pause.] And, OK, so explain to me. This [Researcher 8 points to 2 choose 0.] is two tall-
00:05:14	Romina:	Uh-huh.
00:05:14	R8:	-towers.
00:05:15	Romina:	Yes.
00:05:15	R8:	And so you get to the three-tall level by adding something.
00:05:19	Romina:	You're adding. Yeah.
00:05:20	R8:	OK. And you go to the right when you add.
00:05:24	Romina:	The red one.
00:05:25	R8:	Or that we're counting it essentially.
00:05:26	Romina:	Uh-huh.
00:05:27	R8:	You're saying. Like the 0 meant red, you told me. [Researcher 8 points to 2 choose 0.]
00:05:29	Romina:	Yes, my way, that's just the way-
00:05:31	R8:	OK. The two reds. So when you add the number that's represented here- the color that's represented by this number, you're telling me you go to the right.
00:05:39	Romina:	Uh-huh.
00:05:40	R8:	OK. And when you add the other color-
00:05:41	Romina:	Yeah, you go kind of to the left. Yeah.
00:05:44	R8:	OK. OK. And that explains- all right, so I see why this number goes to the higher - because you're adding something. And then, what do these numbers at the bottom come from again? Can you give me- I think you sort of gave that to me.
00:06:04	Romina:	I'm, the number on the bottom would, say, would- The number, OK, this, for this, the number on the top indicates how many high it is.
00:06:10	R8:	OK.
00:06:11	Romina:	So it would be three, and the number on the bottom would be, like, if we chose a color, say, red, how many reds are in each of these towers three high.

Time	Speaker	Transcript
00:06:18	R8:	OK. OK. So when this one goes to the left [Researcher 8 points to 2 choose 1.] here [Researcher 8 points to 3 choose 1], you're saying, you told me you didn't add any reds.
00:06:25	Romina:	Uh-huh.
00:06:26	R8:	That's why that stayed to the one. You added one tow- one, uh,
00:06:29	Romina:	One block.
00:06:30	R8:	-one color to the tower.
00:06:31	Romina:	It was a blue.
00:06:32	R8:	But it was blue.
00:06:32	Romina:	Uh-huh.
00:06:32	R8:	OK, so that one always goes up. OK. And then, this 0 [Researcher 8 points to 2 choose 0.] came over to this 1 [Researcher 8 points to 3 choose 1.] because this didn't have any reds and you added a red to it.
00:06:41	Romina:	Uh-huh.
00:06:42	R8:	OK. And that works, all right, that explains why these two numbers come up to here.
00:06:46	Romina:	Uh-huh.
00:06:46	R8:	OK. Now do you remember, um, Brian's generating the general rule for that? Can you tell me a general rule, or-?
00:07:00	Romina:	Brian's general rule?
00:07:01	R8:	Not Brian's.
00:07:02	Romina:	Uh, Michael's?
00:07:02	R8:	Sorry.
00:07:07	Romina:	Can, I don't know, I don't remember his-
00:07:10	R8:	I'm not sure I remember his specific notation either. Like, oh, I do remember a little bit, like, um, suppose we're down here to some row, and I can't remember. He might have called it row *R*.
00:07:26	Romina:	OK.
00:07:27	R8:	Cause it's the *R* row. He was saying the *R* row-. Let's see, this is the 0 row, the first row, the second row, so the *R* row would have an *R* on top.
00:07:37	Romina:	OK.
00:07:37	R8:	Right. And it would go *R* 0, *R* 1
00:07:43	Romina:	Uh-huh.
00:07:44	R8:	-over to *R*. [Researcher 8 draws this.] Well, I guess what Researcher 1 was pushing you to do that time was to write the next row and write how the addition rule works, you

Time	Speaker	Transcript
		know, for *R*, and maybe they wrote something in here, like *R N*. [Researcher 8 writes this.] Does that sound familiar to you?

$$\binom{R}{0}\binom{R}{1}\cdots\binom{R}{N}\cdots\binom{R}{R}$$
$$\binom{R+1}{0} \qquad\qquad \binom{R+1}{R+1}$$

Figure L6. Rows *R* and *R*+1 of Pascal's Triangle

00:08:03	Romina:	Kind of. I mean, it sort, it kind of does, but I have to use *R* and *N*?
00:08:12	R8:	Or whatever-
00:08:13	Romina:	Well, I'm saying-
00:08:15	R8:	-letters that you guys picked. You may have put an x in there, actually.
00:08:18	Romina:	So-
00:08:19	R8:	It, it would-
00:08:21	Romina:	-it would be, would it be like a, that this one would be *R* plus over, and then we would have-[Romina writes *R*+1 choose 0 on the side of the paper.].
00:08:25	R8:	Yeah. Actually, I was starting to write that one down, because I sort of remember, you know, if you go from the row would be *R* 0, *R* 1, and so on, and the next one would obviously be *R*+1 0 over to *R*+1. [Researcher 8 writes row *R*+1. Refer to Figure L6.]
00:08:37	Romina:	So wouldn't they all be *R* +1 over, *R* +1 over 0, then *R* +1. I don't understand. Like, wouldn't it be like that?
00:08:48	R8:	Yeah.
00:08:52	Romina:	-over two, [Romina writes I+1 choose 2.] where, where would you want the *N* to come in?
00:08:57	R8:	Yeah, that, well, that's a good question. Um, well you told me that this one plus this one and up to this one here.
00:09:06	Romina:	OK.
00:09:07	R8:	So that, that was the first thing that Researcher 1 had you write. Can you write that, and that's the same as writing one plus two equals three. So could you write the one plus two equals three in choose notation?

Time	Speaker	Transcript
00:09:22	Romina:	You mean, all right, hold on. I'm not sure I'm understanding here. Uh. [Pause.] How would I, how would I get, combine these two to equal that one?
00:09:33	R8:	Yeah. If you were just writing it in choose notation. That's what Michael called it, choose notation.
00:09:42	Romina:	[Pause.] I'm not really sure. Like, it, you know, I'm gonna- [Romina draws two 2-tall towers.] So if I say this one was red [Romina shades in the first tower.] and someone was, no, these were both red, and then the-oh, now, I'm really, now-
00:10:00	R8:	Yeah.
00:10:00	Romina:	I'm saying, how would I make this the ones with- [Romina draws a 3-tall tower, then pauses.] 'Cause there's three of these, right? [Romina draws two more 3-tall towers.] How would I combine these two to make these three? I'm not-
00:10:21	R8:	Well, that's, that isn't actually what I was asking.
00:10:22	Romina:	OK.
00:10:24	R8:	But that's an interesting way to put it. How would you, well, no, maybe I should, maybe I should clarify my question instead. [Romina laughs.] All I'm saying is, she said, um, she said, you could say 1 plus 2 equals 3.
00:10:38	Romina:	Uh-huh.
00:10:38	R8:	But we don't have a 1 here, we have a 2choose 0. And we don't have a 2 there, we have a 2 choose 1.
00:10:43	Romina:	2 choose 1.
00:10:44	R8:	So she's saying, instead of writing 1+2=3 [Researcher 8 writes this.], you're going to write this [Researcher 8 points to 1.] in choose notation and this [Researcher 8 points to 2.] in choose notation and this [Researcher 8 points to 3.] in choose notation. Cause this three isn't really a 3 , it's a 3 choose 1 And this one isn't really a 1, it's a 2choose 0. So you know what I'm saying?
00:11:03	Romina:	Yeah, I know it. I don't know, I don't know how to, I don't know. [pause.] Hm.
00:11:15	R8:	I know it's a little unfair to ask you to remember-
00:11:17	Romina:	Yeah, I don't-
00:11:17	R8:	-from all those years ago.
00:11:20	Romina:	So you mean, *N* choose *R*, like that? [Romina writes *R* choose *N*.] Plus-
00:11:22	R8:	Yeah, well that was what she was going for eventually.

Time	**Speaker**	**Transcript**
00:11:24	Romina:	R choose $N+1$equals $R+1$ over- I'm not even sure if it would be $N+1$ [Romina completes the equation. Refer to Figure L7.]

$$\binom{R}{N}+\binom{R}{N+1}=\binom{R+1}{N+1}$$

Figure L7. Romina writes Pascal's Identity

00:11:38	R8:	Yeah, oh! Yeah, that's very nice.
00:11:40	Romina:	I don't, I'm not sure it that's right, though.
00:11:44	R8:	Yeah, well, let me show you actually,
00:11:45	Romina:	-do-
00:11:46	R8:	-what I was asking first, which-
00:11:48	Romina:	Uh-huh.
00:11:48	R8:	-was the very simple version. I have a tendency to ask simple questions that people don't, they're so simple that people don't realize that I'm asking anything like that. [Researcher 8 writes 2 choose 0 plus 2 choose 1 = 3 choose 1.] That was what I was going for first.
00:12:00	Romina:	Oh, OK. [Romina laughs.]
00:12:02	R8:	That this plus this equals that. Right?
00:12:04	Romina:	OK.
00:12:04	R8:	It was too simple. I made it sound too complicated. And then the next thing that Researcher 1 did in that session was, well, how do you go from this to this? And you pretty much already did that.
00:12:15	Romina:	Uh-huh.
00:12:16	R8:	So you anticipated my next step, was to go to this, from this row to the following row, using this as kind of a pattern. So, yeah, that's what I was asking, if you could kind of remember how you did those things there.
00:12:26	Romina:	Is that right? Is that what you were asking for?
00:12:29	R8:	Yeah. I was asking to see if you remember-
00:12:30	Romina:	OK.
00:12:31	R8:	-that kind of notation.
00:12:32	Romina:	Uh-huh.
00:12:32	R8:	And yeah. It seems like you did. Or, did you remember or did you reconstruct it? Or a little bit of both?
00:12:39	Romina:	Uh. I reconstructed, I guess. I mean, I remember Michael

Time	Speaker	Transcript
		doing something. I remember when Researcher 1 was asking us to do something like that. But I don't remember what I did.
00:12:49	R8:	OK. But you can sort of tell me how this relates to this-
00:12:54	Romina:	Uh-huh.
00:12:54	.R8:	-it looks like, because you used this-
00:12:55	Romina:	It's the same.
00:12:56	R8:	You used this idea to generate that. So tell me what, what these letters would be here.
00:13:01	Romina:	That- the *R* would be the- how high.
00:13:04	R8:	All right.
00:13:05	Romina:	And then if you have- well, you're saying if we're getting from the 2to the 3-
00:13:12	R8:	OK.
00:13:13	Romina:	This is just showing you that, if you had this, this one. I guess this one would be easier to look at. [Romina indicates Pascal's Triangle as shown in Figure L1.]
00:13:20	R8:	OK.
00:13:20	Romina:	If you had this *R* with 0 and this, you can get, you get one more, um, one more *N*.
00:13:28	R8:	OK.
00:13:28	Romina:	So that's why you have like the, the *N* plus 1. You, you go up one because-I don't know, it's kind of hard to explain. The *R* plus 1, obviously, you know because you added a block to it, so-
00:13:39	R8:	You added one to it.
00:13:39	Romina:	-this became one higher.
00:13:40	R8:	Yeah. OK.
00:13:40	Romina:	And the, the, the choose, because it can't be, you can't choose a, like, the same, the zero, because you had to have gone up, because you added a color.
00:13:51	R8:	Yeah. You added a color. Right.
00:13:53	Romina:	That's where, like, your *N* plus 1 comes from, I think.
00:13:56	R8:	OK. And then this *N*+1 is the one- [Researcher 8 points to *R* choose *N*+1]-
00:13:59	Romina:	-because you already had that, the one color. And that's when you add, when you add no color, you have one color.
00:14:04	R8:	Right. So you didn't- you added the *R* to this one, but you didn't add the color to this one.
00:14:07	Romina:	Uh-huh.

Time	Speaker	Transcript
00:14:09	R8:	OK. That was actually my main question-
00:14:14	Romina:	OK. [Romina laughs.]
00:14:16	R8:	-about Pascal's Triangle. Um. But since it went so fast, maybe [Romina laughs.] I could ask you another one-.
00:14:19	Romina:	OK.
00:14:20	R8:	-if that's OK. Um. Now, you did this with towers. Do you remember how Michael did it with pizzas? You said- There was two other things you talked about, that he did it with pizzas and he did it with binary.
00:14:31	Romina:	Yeah.
00:14:31	R8:	Um. Can you look at this in terms of pizzas too?
00:14:39	Romina:	Well.
00:14:39	R8:	The way Michael does?
00:14:40	Romina:	I, I'm not, I don't, that's not like my preference. But I mean, he does, if you have two toppings, if you have a possibility of two toppings, in this one, you don't have any toppings. And this one, you have one topping.
00:14:51	R8:	OK, so you can pick from two toppings, but you don't put any on.
00:14:53	Romina:	Yeah, you don't have to necessarily put it. You have two toppings to pick from. And then, what he did on this one [Romina points to 4 choose 0.] is either you could, now, when you add a third toppings to your pizza.
00:15:00	R8:	OK.
00:15:00	Romina:	Like you have three options, you could either not add anything to the pizza.
00:15:04	R8:	OK.
00:15:04	Romina:	Or you could just add one more topping.
00:15:06	R8:	OK.
00:15:07	Romina:	So I think that's how he does it.
00:15:09	R8:	All right. So when you said "add one more topping," or "not add one more topping," how do you- can you relate that to red and blue?
00:15:18	Romina:	You either, it's either, like, you add one more red block, or you just keep it consistent and add another blue.
00:15:28	R8:	OK
00:15:28	Romina:	So blue would be like nothing, like not an ingredient, and red would be an ingredient.
00:15:33	R8:	OK. OK. So adding a blue topping is like, or adding a blue block is like-

Time	Speaker	Transcript
00:15:38	Romina:	Is like
00:15:39	R8:	-not adding a topping.
00:15:39	Romina:	Uh-huh. And adding a red one.
00:15:40	R8:	Is, OK.
00:15:41	Romina:	Like, his binary, it does the same thing. A 0 would be blue or no topping.
00:15:46	R8:	Uh-huh
00:15:46	Romina:	And a red one, which would be a 1, would be a topping.
00:15:50	R8:	OK. So if I pointed to any one of these, you would be able to say, like say this one. [Researcher 8 points to 3 choose 3.]
00:15:59	Romina:	Yeah. He has-
00:16:00	R8:	In terms of pizzas.
00:16:01	Romina:	-three toppings, and he put all three toppings on the pizza.
00:16:05	R8:	OK. And then so this [Researcher 8 points to 3 choose 2.] would be three toppings and he-
00:16:07	Romina:	Uh-huh.
00:16:07	R8:	-put two of them on the pizza.
00:16:08	Romina:	Yeah. [Romina nods.]
00:16:09	R8:	OK.
00:16:09	Romina:	And like, the reason he does that is, with the towers, for two-high, you could either have a three-high in two colors, you could have it, both, red-red-blue, or you could have it red-blue-red, or you could have blue-red-red. The red's toppings, like, but those are all consistent with there's two colors, and three high. The toppings, it's not so much the order you put them in, but whether they're on or not.
00:16:31	R8:	OK.
00:16:31	Romina:	So it's not like you put pepperoni, then sausage. It's, you have pepperoni and sausage.
00:16:35	R8:	Right.
00:16:36	Romina:	That's why the pizza is sometimes easier when you're dealing with Pascal's Triangle.
00:16:39	R8:	OK. So blue-red-red like you said would be-
00:16:43	Romina:	Like, if
00:16:44	R8:	-pepperoni yes, to-
00:16:46	Romina:	Yeah. And like in this-
00:16:47	R8:	-whatever.
00:16:48	Romina:	In this one, this one, there's three of them, three different types when it comes to towers.

Time	Speaker	Transcript
00:16:53	R8:	Uh-huh.
00:16:54	Romina:	So I mean, it is with two reds, but there's three different ways to put the two reds in the three high.
00:17:00	R8:	OK. Just like there's three different ways you can put two toppings on the pizza.
00:17:04	Romina:	Yeah, but we don't necessarily, that doesn't matter much 'cause it's all on the pizza.
00:17:08	R8:	Yeah. OK. Now you also were telling Researcher 12 that this has something to do with the taxicab problem too.
00:17:16	Romina:	Yeah. [Romina smiles.]
00:17:18	R8:	Now, I'm not sure I understood how the addition rule works with the taxicab problem.
00:17:26	Romina:	Hm, I'm not sure either. The, the taxicab problem, the top number would be the, the, the actual blocks that he has to go to get from point A to point B if it's a grid. [Romina draws a 4 by 2 grid.]
00:17:39	R8:	OK.
00:17:40	Romina:	No matter what, to get to this point, you're gonna have to go one, two, three, four, five six. At all, six. So that would be our top number in it.
00:17:48	R8:	OK.
00:17:49	Romina:	And, I mean, the bottom number, I don't know, I didn't really get into it that much. But we were trying, when you do, like, this, five, four over and down two. How many ways, like that, there-I'm not sure, exactly, how they did it. Like we, I don't even know, I don't even know if you can do it like that. But we do know that the top number would be-
00:18:09	R8:	So we don't, so there's no six on there. [i.e. There is no row 6 on the Pascal's Triangle shown in Figure L1].
00:18:10	Romina:	No. We didn't get that.
00:18:11	R8:	So.
00:18:12	Romina:	I mean, if, we did, uh-
00:18:13	R8:	OK.
00:18:14	Romina:	-did like [Romina draws another grid.] one, two, three four. So we'd be on this line. [Points to row 3 of Pascal's Triangle.]
00:18:21	R8:	OK. [Pause.] It wouldn't be the next one? [Researcher 8 points to row 4 of Pascal's Triangle.]
00:18:26	Romina:	Is it the next one?
00:18:27	R8:	Like four on top?

Time	Speaker	Transcript
00:18:28	Romina:	Oh, yeah, that's what I meant. I was just doing lines. Yeah.
00:18:30	R8:	OK.
00:18:31	Romina:	Cause there's a four one two. Yeah.
00:18:32	R8:	OK. So for this one [Researcher 8 points to the grid in Note 8.], you have to go two, no matter how you do it, you have to go over two and down two.
00:18:39	Romina:	Yeah. Uh-huh.
00:18:39	R8:	You, OK, so that would be-is that-
00:18:43	Romina:	Like, you could-
00:18:44	R8:	-a specific place on the, on Pascal's Triangle? I mean, I see,- let me think about it.
00:18:49	Romina:	I think-
00:18:52	R8:	Because I also heard you say that right means red.
00:18:54	Romina:	Yeah. I'm trying to. [Pause.] See, it works, I just don't know how I would do it for the, the three. Or the four. Cause I mean [Romina retraces the lines on the grid in Note 8.], and you'd always have to turn, so [pause] this would be saying, I think this would more work, if it was a line of like [Romina draws a 4-by-1 grid.] No, that doesn't work. And [Romina draws a 3-by-1 grid.] does that work? One, two, I think that would work more.
00:19:31	R8:	OK.
00:19:31	Romina:	I'm-
00:19:32	R8:	[Inaudible.]
00:19:33	Romina:	No, that still wouldn't work because, no matter what, I don't, the way I'm thinking about it, it's 0's if you could just go in one direction.
00:19:39	R8:	OK.
00:19:40	Romina:	But you always, you have to, you have to go down one. So that would bring you to there. So I'm not exactly sure how it, how it, it would relate where this line is, the-
00:19:49	R8:	Actually, I thought this one you drew might relate to that. If you started here.
00:19:54	Romina:	And then-
00:19:55	R8:	And then you go over four.
00:19:56	Romina:	And down one.
00:19:56	R8:	And don't go-Well, for the zero, just go over four.
00:19:59	Romina:	Yeah.
00:20:00	R8:	Would that work?
00:20:00	Romina:	Yeah. And that's why it goes on the, I think it goes like that

Time	Speaker	Transcript
		on- [Romina draws a small right triangle.]
00:20:03	R8:	Right.
00:20:04	Romina:	-the taxicab problem.
00:20:05	R8:	I saw that, yeah, on the tape. Yeah, your Pascal's Triangle looks kind of sideways.
00:20:08	Romina:	So.
00:20:09	R8:	OK.
00:20:10	Romina:	And how- I don't- Is there a problem, is there a copy of the taxicab problem? Brian?
00:20:19	R8:	[Pause. Researcher 8 stands up.] I didn't have one in my notes- [Someone is speaking off to the side.]
00:20:29	Romina:	Oh, that's OK. But I mean, I think it's on the, like it goes on the, on the diagonal like that. How we did. [Romina shuffles papers.] Where's the, how we did, uh, [Researcher 8 shuffles papers.] You can probably recreate it.
00:20:47	R8:	I'll let you recreate it.
00:20:50	Romina:	What did we do, was two, three, three. [Romina draws another grid. Refer to Figure L8.]

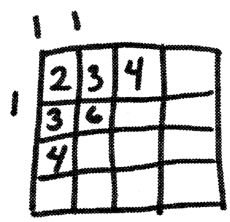


Figure L8. Romina's first taxicab grid

Time	Speaker	Transcript
00:20:55	R8:	Yes, I, I was starting to draw it. I guess that's what you have here-
00:20:59	Romina:	I, I guess-
00:21:00	R8:	-the numbers going diagonally here.
00:21:03	Romina:	[Romina sighs]. I'm get-
00:21:06	R8:	I have slightly different numbers, OK. Oh, maybe I don't. OK. I see what you're saying.
00:21:08	Romina:	No. Like, OK, this would be the one one one. [Romina darkens lines on the grid.]
00:21:11	R8:	Right.
00:21:12	Romina:	To this-
00:21:12	R8:	Yeah.

Time	Speaker	Transcript
00:21:13	Romina:	-there's either, you take one- For this, you could take one red. [Romina darkens a line down one box from the origin.] Or one blue. [Romina darkens a line to the right one box from the origin.]
00:21:17	R8:	OK.
00:21:18	Romina:	To get to the next level. But to this one [Romina indicates a point one down and one to the right of the origin.] you either take one red one blue or one blue one red.
00:21:22	R8:	OK.
00:21:23	Romina:	To get to the two.
00:21:24	R8:	So.
00:21:24	Romina:	So there's one of each color-
00:21:26	R8:	Blue. OK.
00:21:27	Romina:	Like, it would be my one two one. There's one is red red, my two is red blue or blue blue. [Romina writes 1 2 1; under the 1 she writes RR and under the 2 she writes RB|BB.]

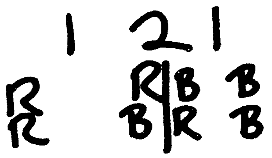


Figure L9. Romina's diagram relating towers and row 2 of Pascal's Triangle

00:21:34	R8:	OK.
00:21:35	Romina:	And then this would be blue blue [Romina adds BB. Refer to Figure L9.], or up down, up, I mean, to right and down. And then to get-
00:21:43	R8:	OK
00:21:44	Romina:	-to here-
00:21:45	R8:	OK
00:21:46	Romina:	-the next line, it's, you either have all red, all blue, or you go two reds one blue, and then-
00:21:57	R8:	[Inaudible.]
00:21:57	Romina:	-two blues one red.
00:21:57	R8:	OK. OK. Now I think a question is beginning to formulate itself in my mind.
00:22:03	Romina:	Uh-huh.

Time	Speaker	Transcript
00:22:04	R8:	But it might take a while. Um. This point here. [Researcher 8 points to the position one down and one to the right of the origin on the grid in Figure L8.] You said that was a red and a blue. Or a blue and a red.
00:22:11	Romina:	Uh-huh.
00:22:13	R8:	And that belongs to. Show me that on that triangle here. [Researcher 8 points to Figure L1.] That would be, sorry. OK, on this one. [Researcher 8 points to the origin of the Figure L8 grid.] Let me say that again. A red and a blue or a blue and a red. That's a two-tall tower.
00:22:30	Romina:	Uh-huh.
00:22:31	R8:	Red and a blue and a blue and a red. So I would say that would be that point there. [Researcher 8 points to 2 choose 1.]
00:22:33	Romina:	Yeah.
00:22:34	R8:	Do you agree with that?
00:22:35	Romina:	Yes.
00:22:36	R8:	OK.
00:22:36	Romina:	And why that that's only a one, I don't know. [Romina points to the position one block to the right of the origin on the Figure L8 grid.]
00:22:39	R8:	Well, that's that, that was my, that was my thinking. Maybe that, I mean, I see a 1 [Researcher 8 points to the origin.] and a 1 1. [Researcher 8 points to the 1 under the origin and the 1 to the right of the origin.] So it almost looks like these 1's belong up there. [Researcher 8 points to row 2 of Pascal's Triangle.]
00:22:49	Romina:	Uh-huh.
00:22:50	R8:	You know what I'm saying? And then maybe the thing that goes with that 2- 'cause you did say it was diagonal. And when you said it was diagonal. I was thinking of lines this way. [Researcher 8 waves her pen diagonally across the Figure L8 grid.] So I'm thinking this 1 [Researcher 8 indicates the 1 down from the origin.] actually belongs with that 2. [Researcher 8 indicates the point one over and one down from the origin.]
00:23:00	Romina:	See, I, I would-
00:23:00	R8:	You know what I'm saying?
00:23:01	Romina:	Yeah. I was looking at it as-
00:23:04	R8:	-a different kind of diagonal.

Time	Speaker	Transcript
00:23:05	Romina:	Uh-huh.
00:23:05	R8:	I was looking at this one.
00:23:06	Romina:	Yeah, yeah, I guess. Oh, and then we do, that would make more sense. And then we put the 2 there. [Romina adds a 2 and two 1s to the grid.] Yeah, that would make more sense.
00:23:13	R8:	So then, this one would be a two-tall tower.
00:23:15	Romina:	Uh-huh.
00:23:16	R8:	Cause it's two blues, and this would be a two-tall tower cause it's two-
00:23:18	Romina:	Yeah.
00:23:19	R8:	-two reds.
00:23:19	Romina:	I was looking at, I was looking at the diagonal shifted up a little bit more.
00:23:22	R8:	OK. All right. Well, I don't know about you, but I feel that clarified it. OK, so, so I can see-
00:23:33	Romina:	Uh-huh.
00:23:34	R8:	-Pascal's Triangle here now. Now, does the addition rule make any sense, looking at it that way?
00:23:40	Romina:	Yeah, because it shows you the different, you could either, when you move from this point [Romina indicates the point one down and one to the right of the origin] to this point [Romina indicates a point one to the right of the first point.] you, see you're adding. You could either go across one [Romina traces a path from the origin to that new point by going two to the right and one down], or when you go get to this point. [Romina traces a path from the origin: one to the right, one down, one to the right.] Do you see what I'm saying?
00:24:00	R8:	I think so.
00:24:00	Romina:	At, like-
00:24:00	R8:	I think, all right. To get here, you're saying- [Researcher 8 points to Romina's selected point – two to the right and one down from the origin.]
00:24:03	Romina:	Uh-huh.
00:24:04	R8:	You either get from there. [Researcher 8 indicates the point one down and one to the right of the origin.] Or you get from there. [Researcher 8 indicates the point two to the right of the origin.]
00:24:06	Romina:	Yeah. Those are your only.
00:24:08	R8:	OK. OK, and this one is a, is a two actually. [Researcher 8

Time	Speaker	Transcript
		indicates the point one down and one to the right of the origin.]
00:24:12	Romina:	Uh-huh.
00:24:13	R8:	OK. And this one is a 1. [Researcher 8 indicates the point two places to the right of the origin.]
00:24:15	Romina:	Yeah, we should really draw it. [Romina starts to draw a new grid..]
00:24:17	R8:	So you're telling me, yeah. And, actually, this number down here is a 3. [Romina continues drawing the grid.] OK.
00:24:23	Romina:	So say, this is, put our 2 there. [Romina writes a 2 on the grid.]
00:24:26	R8:	Right.
00:24:27	Romina:	A 1 And our 1. [Romina writes 1's on the grid.] Then put our, is that right? Then, no, that, we put a three here. [Romina adds a 3 to the grid.] A 3 here. [Romina continues adding to the grid. Refer to Figure L10.] And our 1, a 1, and a 6 there.

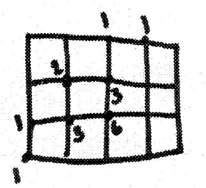


Figure L10. Romina's second taxicab grid

00:24:39	R8:	OK.
00:24:40	Romina:	So to get to this 3, we were either here. [Romina indicates the point one down and one to the right of the origin.] Or we were here. [Romina indicates the point two down from the origin.] We were either, we either had two blues. [Romina indicates the point two down.]
00:24:47	R8:	Right.
00:24:48	Romina:	Or we had one, kind, one- OK, hold on. We had one and one of each. [Romina indicates the point one down and one over.]
00:24:55	R8:	Right. Red and a blue. OK.
00:24:58	Romina:	Actually, this way it would be the same thing. [Romina indicates the two possible paths to the point one down and

Time	Speaker	Transcript
		one over.] Red and a blue. Red and a blue. Or we had two blues. So this, we either added another blue or added a red.
00:25:06	R8:	OK. And that would be where on that triangle?
00:25:10	Romina:	On the-
00:25:11	R8:	We're going to a 3-
00:25:12	Romina:	Yeah.
00:25:12	R8:	-from the 1 and a 2. OK.
00:25:13	Romina:	Here.
00:25:14	R8:	So that's just the one we were doing.
00:25:15	Romina:	Yes.
00:25:16	R8:	In fact.
00:25:16	Romina:	Uh-huh.
00:25:17	R8:	OK. So I think you've showed me three ways to do that now.
00:25:21	Romina:	[Romina laughs.]
00:25:22	R8:	You did, OK, you did towers, which is adding a tower-
00:25:24	Romina:	Uh-huh.
00:25:24	R8:	-of the same color or different color. And then you did pizzas. which is a topping on or off. And now, OK.
00:25:31	Romina:	Taxicab.
00:25:32	R8:	One more time on this. On the taxicab, it's- all right, you went, you can get to here [the point two down and one over] either from there [the point two down]-
00:25:38	Romina:	Uh-huh.
00:25:39	R8:	-or from there [the point one over and one down]. OK.
00:25:40	Romina:	We either, we either got to this three from an all blue tower [Romina indicates the point two down.] or a tower that had one blue and one red. [Romina indicates the point one down and one over.]
00:25:45	R8:	OK.
00:25:46	Romina:	And then we just added one, so we either added another blue or we added a red.
00:25:50	R8:	OK. Or else we went from two downs and added a right, or we went from a right and a down and we added a down. OK.
00:25:56	Romina:	Yeah.
00:25:58	R8:	OK. I think we're done.
00:26:02	Romina:	[Romina laughs.]
00:26:05	R8:	Do, well, actually, one other question that I just thought about. Um. Researcher 3 did ask you a little bit about this.

Time	Speaker	Transcript
		Um. About your tutoring. Um. You did sort of allude to using things something like this in your tutoring? [They discuss Romina's tutoring.]

Interview with Ankur on July 31, 2002

Time	Speaker	Transcript
00:00:00	–	The tape begins with Ankur talking about his schooling and future plans. He is a student at Kean studying physical therapy. Then Researcher 8 and Ankur view a videotape of the night session. They watch as the students on the video work on Pascal's Triangle. Brian is present during this interview, sitting off to the side.
00:11:26	R8:	Looking at the chooses in a different way, looking at them in terms of the towers or something.
00:11:31	Ankur:	I remember that, with the towers. Now it's, now it's recalled.
00:11:34	R8:	Yeah. Oh, OK. Do you want to-
00:11:36	Ankur:	Um, I don't remember exactly, but I remember like something- [Pause.]
00:11:42	R8:	You can have the pen if you want.
00:11:43	Ankur:	Uhhh. No, I'm trying to think. I remember something. Um. We, we used like these numbers to represent like the various towers, I think. Some- something towards- of that nature.
00:11:55	R8:	OK.
00:11:55	Ankur:	Something like with what- Say there was like a tower of 3 tall with red and blue, like-
00:12:01	R8:	OK.
00:12:02	Ankur:	Like, if the first one represented only one red, or no reds.
00:12:03	R8:	OK.
00:12:04	Ankur:	One red, two reds.
00:12:04	R8:	OK. So. No reds, one. All right, so 3 tall would be-
00:12:11	Ankur:	This was no reds, one red, two reds, three reds.
00:12:13	R8:	OK. OK. //So that was-
00:12:15	Ankur:	//And those numbers would match here. [Ankur indicates row 3 of Pascal's Triangle.]
00:12:17	R8:	OK. So-
00:12:18	Ankur:	There would be one with no red, three with one red.

Time	Speaker	Transcript
00:12:22	R8:	OK.
00:12:22	Ankur:	Three with two red and-
00:12:24	R8:	OK.
00:12:25	Ankur:	-one with //3 reds.
00:12:26	R8:	OK. //That makes sense. OK.
00:12:28	Ankur:	That's what we were using those for.
00:12:29	R8:	OK. And then the 3 choose 0, all right, that agrees with that.
00:12:33	Ankur:	Uh-huh.
00:12:34	R8:	Um. I think later on, you get- [She starts to write.]
00:12:41	Ankur:	I remember what we were thinking, but I don't remember the videotape really. [They watch more of the videotape for about 2 minutes.]
00:14:22	R8:	So if you had dot, dot, dot. Um, what would you put down here? This is row 4 and this is row 3. What are you going to put down here for row *N*?
00:14:29	Ankur:	With- The top would be *N*. And that's the-
00:14:31	R8:	Yeah. Why don't you write it?
00:14:34	Ankur:	[Ankur takes the pen.] I guess-
00:14:34	R8:	Oh, you're left-handed too.
00:14:35	Ankur:	Yeah. Unfortunately. That would be the height of the tower.
00:14:38	R8:	OK.
00:14:39	Ankur:	I don't know, the bottom would be the- Are you talking about what the whole row would look like, or what?
00:14:45	R8:	Yeah, well, I guess you were working on both. The whole row and like a random position in the row. So you could- could you start writing the row? Like what would the first number be in the row?
00:14:55	Ankur:	The first number would be 0.
00:14:56	R8:	OK. So. You, can you write that down?
00:14:58	Ankur:	Yeah.
00:14:59	R8:	And then the second one over-
00:15:00	Ankur:	It would be 1.
00:15:01	R8:	It would be *N* 1.
00:15:02	Ankur:	All the way to the *N* over *N*.
00:15:04	R8:	OK. So that would be-
00:15:06	Ankur:	That would be at the end.
00:15:06	R8:	OK. [Ankur writes the first and last entries in the general row.] Now I noticed you put the lines there, too, which

Time	Speaker	Transcript
		you guys put there.
00:15:13	Ankur:	Oh, that's just a habit.
00:15:15	R8:	Yeah.
00:15:16	Ankur:	Just a habit.
00:15:16	R8:	Um. OK. So that was the point. And the second one you told me was N 1. [Ankur adds this entry; refer to Figure L11.]

$$\binom{N}{0}\binom{N}{1} \quad \binom{N}{N}$$

Figure L11. Ankur writes row N of Pascal's Triangle

Time	Speaker	Transcript
00:15:21	Ankur:	Uh-huh. [They watch the videotape for 9 minutes, as the students on the videotape write row N, including a general entry in the row. Then the students on the videotape start to talk about the addition rule.]
00:24:38	R8:	OK. Can, can you write that next row for me?
00:24:40	Ankur:	OK.
00:24:41	R8:	So we can work on that.
00:24:43	Ankur:	[Ankur adds row 5 to Pascal's Triangle.] Um.
00:24:55	R8:	[Inaudible.]
00:24:56	Ankur:	[Ankur is speaking softly to himself.] 5, 3.
00:25:05	R8:	OK, so what did she say? What's 4 choose 2 plus 4 choose 3?
00:25:08	Ankur:	Did I write that right? I don't know if I did.
00:25:10	R8:	Yeah, it looks good.
00:25:12	Ankur:	Yeah.
00:25:12	R8:	It goes from 0 to 5 and it's, it's lined up a little better than mine was. All right, so what's 4 choose 2 plus 4 choose 3?
00:25:20	Ankur:	5 choose 3.
00:25:21	R8:	OK. And there it is right there.
00:25:23	Ankur:	Uh-huh.
00:25:23	R8:	So can you draw those arrows, too? [Ankur does so. Refer to Figure L12.]

Time	**Speaker**	**Transcript**

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

Figure L12. Ankur shows an example of the addition rule

00:25:26	R8:	4 choose 3, 4 choose 3. That would be, that would be 5. OK. They look at the video some more.
00:25:33	Ankur:	I think I see where this is leading. [We hear Michael on the tape say 5 choose 3.]
00:25:38	R8:	Yeah. OK. Do you see where it's leading? Do you remember, or do you just sort of see now where it's leading?
00:25:44	Ankur:	Um, I could probably remember what I thought.
00:25:47	R8:	OK.
00:25:47	Ankur:	I probably thought if you gave me these two. [Ankur points to 5 choose 2 and 5 choose 3.]
00:25:51	R8:	OK.
00:25:51	Ankur:	That would lead to 6 choose 3.
00:25:53	R8:	OK. OK.
00:25:54	Ankur:	Now that's, that's probably what I thought.
00:25:56	R8:	OK. Um, well, maybe you could just write the 6, squeeze the 6 choose 3 under there so we could put the arrows in. [Ankur does so.] Yeah, actually, I'm not sure she got to that specific one.
00:26:07	Ankur:	I just used a random one there, that's why I-
00:26:08	R8:	Yeah. OK. But- [There is a pause as they listen to Jeff on the tape saying, "Why is he 5 choose 3?" and Ankur responding, "It's always the one on the right."]
00:26:17	Ankur:	That's what I was just thinking right now.
00:26:21	R8:	So, it's always, tell me what that means. It's always the one on the right.
00:26:23	Ankur:	It's the, like, the-
00:26:27	R8:	Oh, I see. I think I see. But tell me anyway.
00:26:29	Ankur:	[Ankur laughs.] If, like, for instance, right here. 2 choose, 2 choose 0 and 2 choose 1 leads to 3 choose 1.

Time	Speaker	Transcript
		The one on the right refers to this 1 [in 2 choose 1].
00:26:38	R8:	OK. So the- you, you go to the next level-
00:26:41	Ankur:	And it carries-
00:26:42	R8:	-which means the 2 always becomes a 3.
00:26:43	Ankur:	And then it carries the-
00:26:45	R8:	And it takes the number from the number on the right.
00:26:46	Ankur:	Uh-huh.
00:26:47	R8:	OK. All right. So that's how you knew it was gonna be 6 choose 3.
00:26:50	Ankur:	Yeah. Whatever that's-
00:26:52	R8:	OK.
00:26:52	Ankur:	Whatever the reasoning is behind it, which I probably couldn't tell you right now.
00:26:55	R8:	OK. But you feel like, it sounds like you could do any two of them. I mean-
00:27:01	Ankur:	Yeah.
00:27:01	R8:	Even if they weren't on the triangle.
00:27:02	Ankur:	Yeah.
00:27:03	R8:	You know, like if I said, you know.
00:27:05	Ankur:	Uh-huh. That's probably-
00:27:06	R8:	7 choose 4 plus 7 choose 5.
00:27:07	Ankur:	That's probably what led us to write the formula.
00:27:10	R8:	Yeah. OK. All right.
00:27:14	Ankur:	Using that. We probably used the formula.
00:27:15	R8:	OK.
00:27:16	Ankur:	I don't know [Inaudible.] or not.
00:27:16	R8:	Yeah. [They continue looking at the video.] And here's Michael's talking about why this works because of toppings. Pizza toppings. [They listen as Dr. Maher says, "explain that one more time, Michael," and Michael responds. They smile. As Michael finishes his explanation, Ankur laughs.]
00:27:47	Ankur:	I was reading ahead. [They listen to Dr. Maher talk about explaining to Brian.]
00:27:54	R8:	... and Brian shows up.
00:27:59	Ankur:	You just showed up and you had to eat. [Ankur turns to Brian and laughs. He laughs again as he listens to Brian on the tape talk about his rental tux, which doesn't fit.]
00:28:11	R8:	OK. And then there was a little break here. See, we transcribe everything. [They continue listening as Jeff

Time	Speaker	Transcript
		begins to talk. Ankur laughs. On the tape, Jeff says, "say we have this row right here."]
00:28:33	Ankur:	This just sums up everything that we're probably doing, right?
00:28:35	R8:	Yeah. He's [Pause.], here's where you get to the Reader's Digest version. [They listen to Jeff say, "and over here we have *N* choose *N*."] So he's explaining to Brian-
00:28:44	Ankur:	To Brian.
00:28:45	R8:	-everything that happened so far. It doesn't take that long. [Jeff says, "and this here would be *N* choose *X*+1 and then *N* choose *X*+2."] So he was still in the [Inaudible.] things, *X*+1, *X*+2 there.
00:29:09	Ankur:	Uh-huh.
00:29:10	R8:	OK. OK. [Researcher 8 starts to add entries to row *N*.] I can't write it sideways. [She hands the pen to Ankur.]
00:29:17	Ankur:	Just showing-
00:29:18	R8:	You can do it for me. Thanks. [Jeff is saying, "that would be *X*-2."] OK. In fact, maybe- [Ankur writes.] And he did it on the other side, too, so-
00:29:31	Ankur:	So *N*. [Ankur adds entries to row *N*. Refer to Figure L13.]

$$\binom{N}{0}\binom{N}{1}\binom{N}{X-1}\binom{N}{X}\binom{N}{X+1}\binom{N}{X+2}\binom{N}{N}$$

Figure L13. Expanded version of row *N* of Pascal's Triangle

00:29:32	R8:	So *N* choose- yeah, OK, just so we're sort of keeping up with what he wrote. [They watch some more of the video. Jeff is talking about the row "above this."] Now you asked him for the row below that. But he didn't actually do that. He's doing the row above it. Um. Do we have enough room? Could you, could you write some of the row above the *N* row? Like he was doing?
00:29:56	Ankur:	Yeah, I guess. Um.
00:29:59	R8:	That's, that's like the *N*-1 row.
00:30:02	Ankur:	Yeah.
00:30:03	R8:	OK.
00:30:04	Ankur:	So I guess it's just- how would you write- Would I just write, would I just write 0 down here, or-

Time	Speaker	Transcript
00:30:08	R8:	Yeah. Right? I mean-
00:30:09	Ankur:	Yeah.
00:30:10	R8:	They all start with 0, right.
00:30:10	Ankur:	Uh-huh. [Ankur writes.] And I guess you want me to skip the *X*.
00:30:15	R8:	Yeah, probably. [She turns toward the computer screen.] Um. Yeah, right. You see, he's got an *N*-1 choose *X*. So that's gonna be somewhere up there.
00:30:24	Ankur:	The middle?
00:30:25	R8:	Yeah. So where would it be? Um-
00:30:30	Ankur:	*N*-1 choose *X*?
00:30:30	R8:	Yeah. I'm looking at the 4 and the 5 to sort of confirm where it would be. Would it be to this side [left] of the *X* or that side [right] of the *X*?
00:30:37	Ankur:	What are you- I don't understand what you're saying, cause-
00:30:39	R8:	Yeah, I'm not sure.
00:30:41	Ankur:	Because this row is just the *N* row, right?
00:30:42	R8:	Yeah, this is the *N* row, *N* choose *X*, but you know each row is sort of offset diagonally from the row below it.
00:30:46	Ankur:	Uh-huh.
00:30:47	R8:	So the 4 choose 0 is like above the 4 choose 5- [sic; she means 5 choose 0.]
00:30:49	Ankur:	Yeah.
00:30:50	R8:	-and to the right. So the 4 choose 2 is above the 5 choose 2 and to the right. So the *N*-1 choose *X* is gonna be over there [to the left] or over there [to the right]?
00:30:59	Ankur:	Ah, *N*-1 choose *X* would be [pause] over there [to the right].
00:31:05	R8:	Yeah. Yeah. I, I agree with that.
00:31:10	Ankur:	It just took a second to register. [Ankur writes some entries in row *N*-1.]
00:31:11	R8:	OK. And then-
00:31:13	Ankur:	And this would be, I guess if the 6 was not there.
00:31:17	R8:	Pretend that's not there.
00:31:18	Ankur:	That would be *X*-1.
00:31:21	R8:	OK. Yeah. Yeah, I think that's what they're doing here. [They look at the video.] He [Jeff] just wrote the first one. *N* choose *X*. Yeah, he put the last one over here, *N*-1, *N*-1.

Time	Speaker	Transcript
00:31:43	Ankur:	Uh-huh.
00:31:44	R8:	That makes sense, right? That's the last one.
00:31:45	Ankur:	Uh-huh. [They continue viewing the video. Jeff is asking, "where do you want me to go from here?" Ankur laughs and nods at Brian.] Brian's a quick study. That's what Dr. Maher said. That's what I am. [Jeff and Brian are talking about "the choose situation" on the video.] He has no idea what they're talking about.
00:32:27	R8:	He said some numbers there, 1 4 6.
00:32:31	Ankur:	He did know. [They watch Jeff talk about the third row: 3 choose 0, etc.]
00:32:45	R8:	So he's going over again.
00:32:46	Ankur:	Uh-huh.
00:32:47	R8:	This is that. Which you-
00:32:50	Ankur:	Comparing to-
00:32:51	R8:	Yeah.
00:32:52	Ankur:	Uh-huh. [They continue looking at the video as Jeff talks about the choose equation.] This is weird. We all said it at once.
00:33:22	R8:	The exclamation point. I guess you had factorial notation written-
00:33:26	Ankur:	Uh-huh.
00:33:27	R8:	-at some point. [Researcher 8 and Ankur laugh when Jeff says *N*! They continue listening as Romina begins to talk.] She's explaining why you have the factorials here.
00:33:42	Ankur:	Uh-huh. [They continue watching the video.]
00:33:55	R8:	So she's explaining there. Can you see that? That's *n* factorial over *n-x* factorial *x* factorial. That's *n* choose *x*.
00:34:02	Ankur:	Uh-huh. [Ankur nods.]
00:34:02	R8:	So. She's sort of giving a real quickie version of why that works. [They watch the video as the students write on the board.]
00:34:19	Ankur:	[Inaudible.]
00:34:20	R8:	The Reader's Digest Version. [We hear Jeff say, "so where do you want to go with this?" Researcher 1 responds, "Tell me how the addition rule works in general."] OK. So I think you were the first one to figure out exactly what she was asking you to do there. Do you remember?
00:34:50	Ankur:	Isn't that what we were just talking about?

Time	Speaker	Transcript
00:34:53	R8:	Yeah! I mean, well, there were-
00:34:55	Ankur:	We were just talking about adding the, adding any two of these [numbers in Pascal's Triangle] and seeing what comes up after that.
00:35:01	R8:	Now she's asking you to do it with the N-1 row and the N row. But actually, I'm not sure that that's how they did it. Um.
00:35:09	Ankur:	They had that in terms of X, like, yeah, that's like with this row. And what would come?
00:35:15	R8:	Yeah. What would come after that? So, is that something- You want to see some more, or do you think you could show me what-
00:35:19	Ankur:	Add, what, add this and this? [Ankur points to N choose X and N choose X+1.]
00:35:21	R8:	Yeah.
00:35:22	Ankur:	Well, it would- [Ankur picks up a pen and starts to write.] I guess this would be N+1, and that would be- [Pause.]
00:35:33	R8:	Yeah. That- what's going to be underneath?
00:35:36	Ankur:	Oh, X+1. [Ankur writes N+1 choose X+1 underneath N choose X and N choose X+1.]
00:35:37	R8:	OK. N+1 X+1, 'cause you were telling me the number on-
00:35:41	Ankur:	OK.
00:35:42	R8:	The bottom number, you were saying-
00:35:43	Ankur:	Uh-huh.
00:35:43	R8:	-always comes from the right.
00:35:43	Ankur:	Uh-huh.
00:35:44	R8:	OK. So, now, the next thing she asked for, um- I won't ask you to do it now- is, if you were going to write this one out, I mean, without the arrows, you could write 3+3=6.
00:35:59	Ankur:	Uh-huh.
00:36:00	R8:	You know, just a regular addition equation.
00:36:01	Ankur:	Yeah.
00:36:02	R8:	And you could also write that as 3 choose 1 plus// 3 choose 2 equals-
00:36:06	Ankur:	//3 choose 2 equals
00:36:07	R8:	-equals //4 choose 2
00:36:07	Ankur:	//4 choose 2. Uh-huh.
00:36:08	R8:	So now she's gonna want to see this [the general terms] as

Time	Speaker	Transcript
		just a regular equation with an equals sign in it instead of arrows. You know what I mean?
00:36:17	Ankur:	So this plus this?
00:36:21	R8:	Yeah. That's all.
00:36:23	Ankur:	Um, I understand what you're saying, but I don't understand what you're looking for.
00:36:27	R8:	OK. An equation with a plus sign and an equals in it? Like, let me show you what I'm-
00:36:33	Ankur:	Yeah. [Ankur laughs.]
00:36:34	R8:	OK. So this-
00:36:35	Ankur:	I already got to that point.
00:36:36	R8:	OK. So instead of writing it as Pascal's Triangle, I'm just writing a real simple equation, 3+3=6. Only now I'm changing this 3 into- what does this 3 equal? Um-
00:36:47	Ankur:	3 choose-
00:36:48	R8:	Right.
00:36:49	Ankur:	3 choose 1.
00:36:50	R8:	3 choose 1 plus- what's the next thing that we want?
00:36:54	Ankur:	3 choose 2.
00:36:55	R8:	Right. OK. 2. [She write this.] I hope I'm getting this right.
00:36:58	Ankur:	Equals 4 choose 2.
00:36:59	R8:	Right. There you go. You write the rest of it if you would. [She hands the pen to Ankur.] So now she say all right, do these arrows but write an equation.
00:37:02	Ankur:	That's just- [Ankur writes the equation shown in Figure L14.] Like that?

$$\binom{N}{X} + \binom{N}{X+1} = \binom{N+1}{X+1}$$

Figure L14. Ankur writes Pascal's Identity

Time	Speaker	Transcript
00:37:22	R8:	OK. Yeah, I think.
00:37:25	Ankur:	That's what we said on there [the video].
00:37:25	R8:	That's just what we said.
00:37:27	Ankur:	In words, I guess.
00:37:28	R8:	In words, but she was kind of pushing to see an actual equation there. [They watch some more of the video. Jeff is writing the equation that Ankur has just written.]

Time	Speaker	Transcript
		X+1, he just got it, OK. [We hear Ankur's voice.] Right. And you-
00:38:00	Ankur:	That's when I told him.
00:38:01	R8:	You're telling him what to do there.
00:38:05	Ankur:	Easier said than done. Understood.
00:38:07	R8:	Yeah. [They continue to watch as Jeff finishes the equation.] OK. [Jeff says to Brian, "we're explaining why you add."] OK. Well, that's pretty much the end of-
00:40:12	Ankur:	Uh-huh.
00:40:12	R8:	-this explanation of all this. Um. I'm not sure at this point if I followed all of the explanation about pizza toppings. Do you remember?
00:40:20	Ankur:	Neither do I. [Ankur laughs.] 'Cause-
00:40:23	R8:	Um.
00:40:24	Ankur:	Umm.
00:40:25	R8:	Yeah. Let me
00:40:26	Ankur:	I don't remember. Mike related it somehow, and-
00:40:31	R8:	Well, you did relate, you told me this one was, was like the three pizza topping problem. Actually we started with this, I guess.
00:40:37	Ankur:	Um.
00:40:38	R8:	This is two pizza toppings, right.
00:40:39	Ankur:	Um, I guess.
00:40:41	R8:	Yeah. Let me think. Um, or no, actually, when you explained it-
00:40:44	Ankur:	I did it with the towers.
00:40:45	R8:	-it was towers.
00:40:46	Ankur:	I don't remember the pizza problem really.
00:40:47	R8:	Yeah.
00:40:48	Ankur:	I remember the towers though.
00:40:49	R8:	So, OK, so this was towers again.
00:40:52	Ankur:	Uh-huh.
00:40:52	R8:	And this was three-tall towers.
00:40:55	Ankur:	With no red, like
00:40:56	R8:	Let's do this one.
00:40:57	Ankur:	With no red, one red, two red-
00:40:58	R8:	OK.
00:40:58	Ankur:	Three red.
00:40:59	R8:	So how would this be in terms of towers? This would be-
00:41:02	Ankur:	A tower N tall

Time	Speaker	Transcript
00:41:04	R8:	*N* tall.
00:41:04	Ankur:	With one-
00:41:06	R8:	OK
00:41:07	Ankur:	-of one color, and then- With none of one color.
00:41:09	R8:	Right.
00:41:10	Ankur:	One of one color.
00:41:11	R8:	So it's gonna be zero reds, one red, all the way up to-
00:41:13	Ankur:	Up to N.
00:41:14	R8:	All reds.
00:41:14	Ankur:	All reds. [Ankur nods.]
00:41:15	R8:	Yeah. OK. N reds. OK. Does it make sense to explain this addition rule in terms of adding, in terms of going to the next row of towers?
00:41:28	Ankur:	I mean, you could do it the same way. It's-
00:41:31	R8:	All right, so this one is a tower N tall.
00:41:34	Ankur:	With X amount of reds.
00:41:36	R8:	With X reds
00:41:38	Ankur:	And then, you add it with the tower the same height with one more red.
00:41:43	R8:	One more red. And how do you use that to produce-
00:41:46	Ankur:	And you get that a tower one taller.
00:41:48	R8:	OK. So you have two towers and you get one one taller. And how do you get to the one taller one from here?
00:41:53	Ankur:	It's *N* plus 1.
00:41:55	R8:	OK. Um. OK. And what is the- I'm trying to visualize how that would work with towers. How that, you know.
00:42:04	Ankur:	Uh-huh.
00:42:04	R8:	OK, so, they're both the same height. And then-
00:42:06	Ankur:	They're both the same height. One has one more red than the other.
00:42:08	R8:	OK, and one has one more red than the other one. And then you go to the next row. It's one taller.
00:42:14	Ankur:	It's one taller.
00:42:15	R8:	OK. So, say, these become one taller, so you add a cube to them.
00:42:22	Ankur:	Uh-huh.
00:42:23	R8:	And these become one taller, so you add a cube to them, too.
00:42:25	Ankur:	Uh-huh.
00:42:27	R8:	Now how does the- but this one is an *X*. But this one is

Time	Speaker	Transcript
		an *X* plus one. But this one is an *X* plus one and that one stays an *X* plus one.
00:42:34	Ankur:	Uh-huh.
00:42:35	R8:	How does that relate to-
00:42:37	Ankur:	Why is that like that?
00:42:39	R8:	Yeah.
00:42:40	Ankur:	Ask Brian. [Laughs.]
00:42:44	Brian:	No.
00:42:47	Ankur:	Um.
00:42:48	R8:	OK.
00:42:48	Ankur:	I probably couldn't tell you right now.
00:42:51	R8:	OK, well let's see, um, well, maybe we can look at it in terms of this. Maybe it's easier to look at it specifically. This [3 choose 1] is a 3-tall tower with one red.
00:43:03	Ankur:	Uh-huh.
00:43:05	R8:	So it's got- it's 3 tall, so it's got one red and two blues.
00:43:10	Ankur:	Uh-huh.
00:43:11	R8:	And then you changed that into a 4-tall tower with two reds.
00:43:16	Ankur:	Uh-huh.
00:43:17	R8:	So how are you gonna do that?
00:43:18	Ankur:	With 2 reds? You just add a red.
00:43:20	R8:	You have to put a red on it.
00:43:21	Ankur:	Uh-huh.
00:43:22	R8:	OK. But this one's [3 choose 2] a 3-tall tower that already has two reds. And it goes to a 4-tall tower that has 2 reds.
00:43:27	Ankur:	So you got to add a blue.
00:43:28	R8:	So you're gonna add a blue to that one.
00:43:29	Ankur:	Uh-huh.
00:43:30	R8:	OK. So this one combines with this one in the sense that-
00:43:36	Ankur:	You add all blues to this one and all reds to that one.
00:43:36	R8:	OK. In fact, you only have to add one-
00:43:39	Ankur:	Uh-huh.
00:43:40	R8:	-because you're going from 3 to 4.
00:43:41	Ankur:	Yeah.
00:43:41	R8:	So you add a blue to this one to get here and a red to that one to get here.
00:43:43	Ankur:	Uh-huh.
00:43:45	R8:	OK.

Time	Speaker	Transcript
00:43:46	Ankur:	You add a red to all those.
00:43:47	R8:	All 3.
00:43:48	Ankur:	All 3 of those.
00:43:48	R8:	Right.
00:43:49	Ankur:	A red to all 3 of those and a blue to all 3 of those and that's how you get-
00:43:51	R8:	Yeah. OK. That makes sense.
00:43:52	Ankur:	That's why you have the extra- that's why the bottom number's *X* plus 1.
00:43:57	R8:	OK. So-
00:43:58	Ankur:	Because these you added the same color.
00:44:00	R8:	OK. So I think you already told me the general one then. You already went to the general. So tell me again with the general one. All right, here's an *X*-tall or an *N*-tall-
00:44:09	Ankur:	Uh-huh.
00:44:10	R8:	-with *X* reds [*N* choose *X*].
00:44:11	Ankur:	Uh-huh.
00:44:11	R8:	And how are you gonna get down there [to *N*+1 choose *X*+1]?
00:44:12	Ankur:	You're gonna add a red.
00:44:13	R8:	You're gonna add a red to all of these.
00:44:14	Ankur:	Uh-huh.
00:44:15	R8:	And you go from there [*N* choose *X*+1] to there [*N*+1 choose *X*+1]?
00:44:17	Ankur:	By adding the other color.
00:44:19	R8:	The blue or something.
00:44:20	Ankur:	Uh-huh.
00:44:20	R8:	OK. All right. That makes a lot of sense.
00:44:23	Ankur:	I guess so.
00:44:24	R8:	OK. Thank you. Um. So in fact, I think, that was the main thing I wanted to explore with you, how you guys went from this Pascal's Triangle to this Pascal's Triangle and this addition rule to this addition rule to the general row to the general addition rule.
00:44:46	Ankur:	Uh-huh.
00:44:47	R8:	And what I had intended to ask was, how you remembered it in terms of what Michael did in terms of pizzas and binary and so on. You didn't remember that, but you did manage to tell me another way to look at it. So, um, thank you for that and unless you want to look at

Time	Speaker	Transcript
		any more of this, I'm good with the questions I wanted to ask.
00:45:10	Ankur:	I'm OK.

REFERENCES

Anderson, I. (1989). *A First Course in Combinatorial Mathematics*. Oxford: Clarendon Press.

Anning, N. (1929). The Pascal triangle. *The American Mathematical Monthly*, 37, 22-25.

Askey, R. (1996). A brief introduction to the world of *q*. In *CRM Proceedings and Lecture Notes, Volume 9* (pp. 13-20). Centre de Recherches Mathématiques: American Mathematical Society.

Bellisio, C. (1999). *A study of elementary students' ability to work with algebraic notation and variables*. Unpublished doctoral dissertation, Rutgers University, NJ.

Bridger, M. & Bridger, M. (2001). Mapping diagrams: Another view of functions. In A. Cuoco & F. Curcio (Eds.), *The Roles of Representation in School Mathematics, 2001 Yearbook* (pp. 103-116). Reston, VA: National Council of Teachers of Mathematics.

Cajori, F. (1928a). *A History of Mathematical Notations, Volume I: Notations in Elementary Mathematics*. Chicago: The Open Court Publishing Company.

Cajori, F. (1928b). *A History of Mathematical Notations, Volume II: Notations Mainly in Higher Mathematics*. Chicago: The Open Court Publishing Company.

Charles, R. & Thompson, A. (1996). *Secondary Math, An Integrated Approach: Focus on Algebra*. New York: Addison-Wesley Publishing Company.

Coulombe, W. & Berenson, S. (2001). Representations of patterns and functions: Tools for learning. In A. Cuoco & F. Curcio (Eds.), *The Roles of Representation in School Mathematics, 2001 Yearbook* (pp. 166-172). Reston, VA: National Council of Teachers of Mathematics.

Davis, G. & Tall, D. (2002). What is a scheme? In D. Tall & M. Thomas (Eds.), *Intelligence, Learning and Understanding in Mathematics: A Tribute to Richard Skemp* (pp. 131-150). Flaxton, Australia: PostPressed.

Davis, R. (1984). *Learning Mathematics: The Cognitive Science Approach to Mathematics Education.* Norwood, NJ: Ablex Publishing.

Davis, R. (1992). Understanding 'Understanding.' *Journal of Mathematical Behavior* 11, 225-241.

Davis, R. & Maher, C. (1990). What do we do when we "do mathematics"?. In *Journal for Research in Mathematics Education Monograph No. 4: Constructivist Views on the Teaching and Learning of Mathematics* (pp. 65-78). Reston, VA: National Council of Teachers of Mathematics.

Davis, R. & Maher, C. (1996). A new view of the goals and means for school mathematics. In M. Pugach & C. Warger (Eds.), *Curriculum Trends, Special Education, and Reform.* New York: Columbia University.

Davis, R. & Maher, C. (1997). How students think: The role of representations. In L. English (Ed.), *Mathematical Reasoning: Analogies, Metaphors, and Images* (pp. 93-116). Mahwah, NJ: Lawrence Erlbaum Associates.

Davis, R., Maher, C., & Martino, A. (1992). Using videotapes to study the construction of mathematical knowledge by individual children working in groups. *Journal of Science Education and Technology*, pp. 177-189.

Dörfler, W. (2000). Means for meaning. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and Communicating in Mathematics Classrooms* (pp. 99-132). Mahwah, NJ: Lawrence Erlbaum Associates.

Edwards, A. (1987). *Pascal's Arithmetic Triangle.* London: Charles Griffin & Company Limited.

English, L. (1997). Analogies, metaphors, and images: Vehicles for mathematical reasoning. In L. English (Ed.), *Mathematical Reasoning: Analogies, Metaphors, and Images* (pp. 3-20). Mahwah, NJ: Lawrence Erlbaum Associates.

Friedlander, A. & Tabach, M. (2001). Promoting multiple representations in algebra. In A. Cuoco & F. Curcio (Eds.), *The Roles of Representation in School Mathematics, 2001 Yearbook* (pp. 173-185). Reston, VA: National Council of Teachers of Mathematics.

Furth, H. (1981). *Piaget and Knowledge.* Chicago: The University of Chicago Press.

Glass, B. (2001). *Mathematical problem solving and justification with community college students.* Unpublished doctoral dissertation, Rutgers University, NJ.

Goldin, G. (1990). Epistemology, constructivism, and discovery learning in mathematics. In R. Davis, C. Maher, & N. Noddings (Eds.), *Journal for Research in Mathematics Education Monograph No. 4: Constructivist Views on the Teaching and Learning of Mathematics* (pp. 31-47). Reston, VA: National Council of Teachers of Mathematics.

Goldin, G. (1998). Representational systems, learning, and problem solving in mathematics. *Journal of Mathematical Behavior* 17(2), 137-165.

Goldin, G. & Shteingold, N. (2001). Systems of representations and the development of mathematical concepts. In A. Cuoco & F. Curcio (Eds.), *The Roles of Representation in School Mathematics, 2001 Yearbook* (pp. 1-23). Reston, VA: National Council of Teachers of Mathematics.

Greer, B. & Harel, G. (1998). The role of isomorphisms in mathematical cognition. *Journal of Mathematical Behavior* 17(1), 5-24.

Janvier, C. (1987). Representation and Understanding: The notion of function as an example. In C. Janver (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 67-71). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kamii, C., Kirkland, L., & Lewis, B. (2001). Representations and abstraction in young children's numerical reasoning. In A. Cuoco & F. Curcio (Eds.), *The Roles of Representation in School Mathematics, 2001 Yearbook* (pp. 24-34). Reston, VA: National Council of Teachers of Mathematics.

Kaput, J. (1987). Representation systems and mathematics. In C. Janvier (Ed.), *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 19-26). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kiczek, R. (2000). *Tracing the development of probabilistic thinking: Profiles from a longitudinal study.* Unpublished doctoral dissertation, Rutgers University, NJ.

Kiczek, R., Maher, C., & Speiser, R. (2001). Tracing the origins and extensions of Michael's representation. In A. Cuoco, & F. Curcio (Eds.), *The Roles of Representation in School Mathematics, 2001 Yearbook* (pp. 201-214). Reston, VA: National Council of Teachers of Mathematics.

Lakoff, G. & Núñez, R. (2000). *Where Mathematics Comes From.* New York, NY: Basic Books.

Lamon, S. (2001). Presenting and representing: From fractions to rational numbers. In A. Cuoco & F. Curcio (Eds.), *The Roles of Representation in School Mathematics, 2001 Yearbook* (pp. 146-165). Reston, VA: National Council of Teachers of Mathematics.

Lehrer, R., Schauble, L. Carpenter, S., & Penner. D. (2000). The interrelated development of inscriptions and conceptual understanding. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and Communicating in Mathematics Classrooms* (pp. 325-360). Mahwah, NJ: Lawrence Erlbaum Associates.

Lesh, R., Behr, M., & Post, T. (1987). Rational number relations and proportions. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 33-40). Hillsdale, NJ: Lawrence Erlbaum Associates.

Maher, C. (1998). *Can Teachers Help Children Make Convincing Arguments? A Glimpse into the Process.* Rio de Janeiro, Brazil: Universidade Santa Úrsula.

Maher, C. (2002). How students structure their own investigations and educate us: What we have learned from a fourteen year study. In A. Cockburn & E. Nardi (Eds.), *Proceedings of the Twenty-sixth Annual Meeting of the International Group for the Psychology of Mathematics Education* (PME26) (Vol. 1, pp. 31-46). Norwich, England: School of Education and Professional Development, University of East Anglia.

Maher, C., & Davis, R. (1995). Children's explorations leading to proof. In C. Hoyles & L. Healy (Eds.), *Justifying and Proving in School Mathematics* (pp. 87-105). London: Mathematical Sciences Group, Institute of Education, University of London.

Maher, C., Martino, A., & Alston, A. (1993). Children's construction of mathematical ideas. Sixteenth Annual Conference of the Mathematics Education Research Group of Australia (MERGA-16) (pp. 13-37). Brisbane, Australia.

Maher, C. & Martino, A. (1999). Teacher questioning to promote justification and generalization: What research practice has taught us. *Journal of Mathematical Behavior*, 18, 53-78.

Maher, C. & Speiser, R. (1997). How far can you go with block towers? Stephanie's intellectual development. *Journal for Research in Mathematics Education*, 16(2), 125-132.

Martino, A. (1992). *Elementary Students' Construction of Mathematical Knowledge: Analysis by Profile.* Unpublished doctoral dissertation, Rutgers University, NJ.

Muter, E. (1999). *The Development of Student Ideas in Combinatorics and Proof: A Six-Year Study.* Unpublished doctoral dissertation, Rutgers University, NJ.

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Pascal, B. (1952). Treatise on the arithmetical triangle. In M. Adler (Ed.), R. Scofield (Trans.), *Great Books of the Western World: Vol. 33, Pascal* (pp. 447-473). New York: Encyclopaedia Britannica, Inc. (Original work published 1654)

Perlman, G. (1982). Making mathematical notation more meaningful. *Mathematics Teacher*, 75(7), 462-466.

Pirie, S. & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics, 26*(2-3), 165-190.

Powell, A. (2003). *"So let's prove it!": Emergent and elaborated mathematical ideas and reasoning in the discourse and inscriptions of learners engaged in a combinatorial task.* Unpublished doctoral dissertation, Rutgers University, NJ.

Powell, A., Francisco, J., & Maher, C. (2003). Analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *Journal of Mathematical Behavior.* 22, 405-435.

Pimm, D. (1987). *Speaking Mathematically.* New York: Routledge & Kegan Paul, Inc.

Putz, J. (1986). The Pascal polytope: An extension of Pascal's triangle to *n* dimensions. *The College Mathematics Journal.* 17(2), 144-155.

Seife, C. (2000). *Zero: The Biography of a Dangerous Idea.* New York: Penguin Books.

Sfard, A. (2000). Symbolizing mathematical reality into being – or how mathematical discourse and mathematical objects create each other. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and Communicating in Mathematics Classrooms* (pp. 37-98).

Smith, S. (2003). Representations in school mathematics: Children's representations of problems. In J. Kilpatrick, W. Martin, & D. Schifer (Eds.), *A Research Companion to Principles and Standards for School Mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Skemp, R. (1987). *The Psychology of Learning Mathematics.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Skemp, R. (2002). Instrumental understanding and relational understanding. In D. Tall & M. Thomas (Eds.), *Intelligence, Learning and Understanding in Mathematics: A Tribute to Richard Skemp* (pp. 131-150). Flaxton, Australia: PostPressed. (Original work published 1976)

Speiser, R., Walter, C., & Maher, C. (2003). Representing motion: An experiment in learning. *Journal of Mathematical Behavior, 22*, 1-35.

Tarlow, L. (2004). Tracing students' development of ideas in combinatorics and proof. (Doctoral dissertation, Rutgers, The State University of New Jersey, 2004). *Dissertation Abstracts International*, 64/12, 4398.

Uptegrove, E. (2003). Understanding Pascal's triangle. In N. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of PME and PMENA* (Vol. 1, p. 261). Honolulu, HI.

Uptegrove, E. & Maher, C. (2004). Students building isomorphisms. In M. Høines & A. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 352-360). Bergen, Norway.

Uptegrove, E. & Maher, C. (2004). From concrete representations to abstract symbols. In D. McDougall & J. Ross (Eds.), *Proceedings of the 26th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp 577-584). Toronto, Canada.

Warner, L. (2005). *Behaviors that indicate mathematical flexible thought*. Unpublished doctoral dissertation, Rutgers University, NJ.