STOCHASTIC ANALYSIS OF BIDDING IN SEQUENTIAL AUCTIONS

AND RELATED PROBLEMS

by

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ABSTRACT OF THE DISSERTATION

Stochastic Analysis of Bidding in Sequential Auctions and Related Problems

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In this thesis we study bidding in sequential auctions and taboo optimization criteria for Markov Decision Processes. In the second chapter we study the problem of sequentially bidding in N auctions of identical items. It is assumed that at each auction there is a sufficiently high price that if paid the item is won. The objective is to acquire a fixed number of these items at a minimum expected cost. In the third chapter we consider the problem of a firm ("the bidder") that in each period, of an infinite time horizon, buys items in auctions and sells the acquired items in a secondary market. We investigate optimal bidding strategies for the bidder that take into account the cost of acquiring the items, the random sale price and demand of the secondary market as well as pertinent salvage value or inventory holding costs. In the final chapter we consider Markovian systems where costs or rewards are unknown either in some states or in all states. For such cases we define taboo optimization criteria for a propitiously defined set of taboo states.

Dedication

I would like to dedicate this thesis to my grandfather Krishnamoorthy Alladi (1925-2010). He passed away just a couple of weeks before I could finish my journey as a PhD student. I know he would have been proud of me and I wish he were here to celebrate this achievement with me.

My grandfather was a mathematics professor who inspired me not only learn mathematics from a very young age but also take up teaching as my career. Without his constant encouragement I would never have gone to IIT-Bombay for my engineering degree and later to Rutgers Business School for my PhD. Thanks tata!

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there for me and I would not have been able to finish this without her support.

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Chapter 1

Introduction

In this thesis we study bidding in sequential auctions and taboo optimization criteria for Markov Decision Processes. In the second chapter we study the problem of sequentially bidding in N auctions of identical items. It is assumed that at each auction there is a sufficiently high price that if paid the item is won. The objective is to acquire a fixed number of these items at a minimum expected cost. We develop a Markov decision processes model for the most general case of this problem. We study structural properties of the optimal policies for several interesting cases. We prove that, under certain assumptions, the optimal value function and the optimal bid are decreasing functions of the number of remaining auctions , increasing functions of the number of opponents and decreasing functions of the inventory on hand.

In the third chapter we consider the problem of a firm (“the bidder”) that in each period, of an infinite time horizon, buys items in auctions and sells the acquired items in a secondary market. We investigate optimal bidding strategies for the bidder that take into account the cost of acquiring the items, the random sale price and demand of the secondary market as well as pertinent salvage value or inventory holding costs. The main results of this chapter are as follows. Here, we prove that, under a few assumptions, the optimal value function is an increasing function of the number of remaining auctions , a decreasing function of the number of opponents and a increasing function of the inventory on hand. while the optimal bid is a decreasing function of the number of remaining auctions, an increasing function of the number of opponents and a decreasing function of inventory on hand.

In the final chapter we consider Markovian systems where costs or rewards are unknown either in some states or in all states. For such cases we define taboo optimization criteria for a propitiously defined set of taboo states. Taboo states represent undesirable states for the system. We show that computing policies that maximize expected taboo first return rewards and mean taboo return times is in general a hard problem, for which well known methods from MDP theory can not be applied. However, it is shown herein that if certain monotonicity properties are satisfied then efficient computation of an optimal deterministic policy is possible.

1.1 Literature Survey

The literature on auction models can be divided into two categories, single auctions and multiple auctions. The literature on models where one item is sold in a single auction is well developed. Milgrom and Weber [18] and Engelbrecht-Wiggans [10] are the traditional references in this area. Klemperer [17] has complied a survey on auction models and mechanisms that has a detailed state of current research. One of the major results in auction theory is the Revenue Equivalence Theorem (Vickrey [29]) which was generalized twenty years later by Myerson [20] and Riley and Samuelson [23]. This theorem states that any allocation in an auction in which: (i) the bidder with the highest value always wins, (ii) the bidder with the lowest possible value expects zero surplus, (iii) all bidders are risk neutral and (iv) all bidders are drawn from a strictly increasing and atomless distribution, will lead to the same revenue for the seller. Bulow and Roberts [3] showed that from a seller's perspective it is optimal to sell the item to the bidder with the highest "marginal revenue'.' This result tied the idea of auction design with traditional market mechanism design which saw an explosion of auction design research.

The literature on multiple auctions is also substantial. Milgrom's book [19] is an excellent introduction to the current state of research in this area. Starting from Vickrey's [29] seminal work there has been a great deal of research in the field and many models for optimal auction design and bidding have been developed, c.f. [31, 30, 11]. Much of the literature is restricted to cases where the bidders have a unit demand and most of these models are analyzed using game theory.

The study of sequential auctions was initiated by Weber [30]. He studied an independent private values environment with each bidder having unit demand and with the seller having multiple items for sale. He considered the revenue generated from the sequence of auctions to the auctioneer as the optimization criterion and proved that under certain conditions the sequence of prices in each auction is a martinagale, *i.e.* the prices don't drift either up or down. Feng, and Chatterjee [12] also consider a similar problem but over a infinte horizon with time discounting. Many papers observed a price decline in sequential auctions, c.f., Ashenfelter [1] in wine auctions, Jones et al. [14] for wool auctions, van den Berg et al. [27] for flower auctions, Ashenfelter and Genesove [2] , for real estate right-to-choose auctions, and Kittsteiner, Nikutta and Winter [16]. This phenomenon has been termed "the decreasing price anomaly." Zeithammer [33] studied bidders forward-looking behavior in sequential auctions and finds that buyers underbid when they expect the seller to offer another auction in the near future, and the auctioneer decides to sell or not to sell based on bidders behavior. Ganuza [13] studied the impact of bidder ignorance on the revenue of an auctioneer. He finds that the auctioneer has incentive to release less information than is efficient in order to promote competition.

Rothkopf and Oren [22] characterize a sequential auction as a multi-stage control process where the state is represented by the competitor's strategy and state transitions represent the competitors' reaction to a strategy used by the the bidder. The control is the bidders strategy. They show the existence of an equilibrium policy in the case of identical bidders. They also consider various "response" functions and study how a bidders policy changes under each of the response functions. We also consider the sequential auction to be a multi-stage control process but our model significantly differs from their model in the respect that the state in our model represents the information available to the bidder, for example the number of auctions remaining in the current period, the number of items already won and the number of bidders participating in the auction. The state transitions at the end of each auction depend on whether the item is won or not and on the change in the number of bidders.

Literature on taboo optimization criteria is non-existent. The classical methods of optimization c.f., [7, 15, 24, 6] involve minimizing the average cost per unit time over an infinite horizon. An attempt to avoid use of a cost structure was done in Derman [8]. Derman considered the problem of finding the policy that maximizes the expected time between replacements subject to conditions that the probabilities of replacement through certain undesirable states are bounded by known numbers. Studying taboo measures as optimization criteria has not been attempted to the best of our knowledge.

Chapter 2

Fixed Demand Model

2.1 Introduction

We consider the problem of a firm ("the bidder") that in a given time period buys items in a sequence of auctions. The objective of the buyer is to minimize his expected total cost for the period, while acquiring a fixed number of items. It is assumed that there is a *buy-it-now-price* available at which the buyer can obtain the item outright at any auction or sometimes in the open market. In the current literature the problem where a fixed number of units have to be acquired has not been studied. Here the bidders' valuations derive from resale of good acquired in the auctions. We study several aspects of this problem including the number of opponents (fixed or random) and the demand(unit or multi-unit).

The main results of this chapter are as follows.

1. We obtain bidding strategies for the case where a bidder has to acquire a fixed number of items through a series of auctions at minimum expected cost. We model this problem as a Markov decision process.

2. We prove that, under certain assumptions, the optimal value function is a decreasing function of n, the number of remaining auctions , an increasing function of m, the number of opponents and a decreasing function of l, the inventory on hand.

3. We prove that the optimal bid is also a decreasing function of n, an increasing function of m, and a decreasing function of l.

2.1.1 Problem Definition

We study optimal bidding strategies for the buyer for the following auction procedure. There is a sequence of N auctions of identical items. Before each auction the number of opposing bidders (*opponents*) m is known. Every bidder submits a sealed bid. At the end of each auction the winning bid is announced and one of the highest bidders wins the auction. The objective of the buyer is to acquire L items at a minimum expected cost. To avoid trivialities we assume that $L < N$.

It is assumed that there is a *buy-it-now-price* available at which the buyer can obtain the item outright at any auction or sometimes in the open market.

After the buyer acquires all L items he does not bid in any of the remaining auctions if any. It is assumed that the the set of all bids available (to the buyer and all opponents) is a finite set $\{a_0, a_1, \dots a_p\}$ where $a_0 < a_1 < \cdots < a_p$. We assume that $a_0 = 0$ and a_p denotes the *buy-it-now-price*. For simplicity we will use the same symbol a to represent both the bid price a and the action of the buyer biding amount a.

We assume that there exist known probabilities $p_m(a)$ that correspond to the probability that the buyer wins an auction when his bid is a and there are m opponents present. We assume that $p_m(a_0) = 0$ and $p_m(a_p) = 1$.

The number of opponents in each auction is random. Let Z_n be the number of opponents participating in the n^{th} auction. It is assumed that Z_n for $n = 1, 2, \cdots, N$ is a discrete time Markov chain with transition probabilities:

$$q_{mm'}(n) = P(Z_{n+1} = m' | Z_n = m).$$

The initial distribution of the number of opponents is known and is denoted for simplicity by:

$$q_m(1) = P(Z_1 = m).$$

It is assumed that whenever there is a tie in an auction involving the buyer , then the buyer loses. This assumption is made to simplify the exposition. Other tie breaking procedures like deciding the winner randomly will not change the analysis but would complicate the exposition. This supposition leads to the following.

$$p_m(a) = P(\text{ all opponents' bids} < a). \tag{2.1}$$

The above problem is modeled as a Markov Decision process below.

1. The state space $\mathcal{S}$ is the set of triplets (n, m, l) where $n\,(1 \leq n \leq N)$ represents the number of auctions remaining, m represents the number of bidders participating in the auction and $l\,(0 \leq l \leq L)$ represents the number of items already acquired by the buyer . Since the buyer starts with no items and has to acquire L items there is the restriction: $L \leq n + l \leq N$, for any state (n, m, l) in $\mathcal{S}$.

2. In any state (n, m, l) the following action sets $A(n, m, l)$ are available.

   - $A(n, m, L) = \{a_0\}$,
   - $A(n, m, l) = \{a_p\}$, for all states (n, m, l) with $n + l = L$,
   - $A(n, m, l) = \{a_1, a_2, \dots a_p\})$, for all states (n, m, l) with $L < n + l \leq N + L$.

3. When an action $a \in A(n, m, l)$ is taken in state (n, m, l) the following transitions are possible.

   - If $l = L$ the only possible transition is back to the same state (n, m, L) with probability 1.

- If $l < L$ depending on whether or not the buyer wins the auction the next state is $(n-1, m', l+1)$ with probability $p_m(a)\, q_{mm'}(N-n)$ or $(n-1, m', l)$ with probability $(1 - p_m(a))\, q_{mm'}(N-n)$.

4. The following costs are incurred.

   - In states (n, m, L) there is no cost.
   - In states (n, m, l) with $L + 1 < n + l \leq N + L$, a cost is incurred only if the item is won in the auction. The expected cost when action a is taken is $a\, p_m(a)$.

Let $a^*_{n,m,l}$ denote the optimal action in the state (n, m, l). Let $v(n, m, l)$ denote the value function in state (n, m, l) and $w(n, m, l; a)$ denote the expected remaining cost when action a is taken in state (n, m, l) and an optimal policy is followed thereafter. Note that $v(n, m, l) = w(n, m, l; a^*_{n,m,l})$.

The dynamic programming equations are

$$v(n, m, l) = \min_{a \in A}\{w(n, m, l; a)\} \tag{2.2}$$

where,

$$\begin{aligned}
w(n, m, l; a) &= a\, p_m(a) + \sum_{m'=1}^{\infty} \{p_m(a)\, q_{mm'}(N-n) v(n-1, m', l+1) \\
&\quad + \sum_{m'=1}^{\infty} \{(1 - p_m(a)) q_{mm'}(N-n) v(n-1, m', l)\}\}, \text{ if } n + l > L, \\
&= n a_p, \quad \text{if } n + l = L,\ l < L \\
&= 0, \qquad \text{otherwise.}
\end{aligned}$$

The above dynamic programming equations can be solved to develop a bidding strategy for the general model. In the subsequent sections we consider structural properties of the optimal policies for several interesting cases.

In the sequel we make the following assumptions.

Assumption A. For any fixed m, $p_m(a)$ is an increasing function of a.

Assumption B. For any fixed a, $p_m(a)$ is a decreasing function of m.

Assumption C. There exists a function G with $\sum_{i=-\infty}^{\infty} G(i) = 1$ such that:

$$q_{mm'}(n) = \begin{cases} G(m-m') & \text{if } m' > 1, \\ \\ \sum_{k=-\infty}^{-m+1} G(k) & \text{if } m' = 1. \end{cases} \tag{2.3}$$

2.2 The Single Item Case With a Constant Number of Opponents

In this section we consider the case where $L = 1$ and there is a constant number of opponents, $m^0 \geq 1$, in all auctions. The state space is the set $\{(n, m^0, 0)\,,\ (n, m^0, 1)\}_{n=1,\ldots,N}$. The action sets are $A(n, m^0, 0) = \{a_1, \ldots, a_p\}$ and $A(n, m^0, 1) = \{a_0\}$. Note that $q_{mm'}(n) = 1$ if $m = m' = m^0$ and 0 otherwise.

Note that if the buyer has acquired the item there is no decision problem. Thus, all information relevant for the decision problem of the buyer at any time is the number of auctions remaining before the single item is acquired.

We obtain a simplified MDP where the state n corresponds to the number of remaining auctions before the item is acquired by the buyer , and all action sets are equal to $A = \{a_1, \ldots, a_p\}$. When action a is taken in state n by the buyer he either wins the auction with probability $p(a)$, in which case he does not bid in the remaining auctions, or he loses and transitions to state $n-1$ with probability $p(a)$. The expected costs for taken action a in any state is $a\, p(a)$. The dynamic programming equations of Eq. (2.2) reduce to the following.

$$v(n) = \min_{a \in A}\{w(n; a)\} \tag{2.4}$$

where,

$$\begin{aligned} w(n; a) &= a\, p(a) + (1 - p(a))\, v(n-1), \text{ if } n > 1 \ , \\ &= a_p, \quad \text{if } n = 1 \ . \end{aligned}$$

We next state and prove theorem 2.2.1.

Theorem 2.2.1. *Under assumption A the following relationships hold for all n.*

$$v(n+1) \leq v(n), \tag{2.5}$$

$$a^*_{n+1} \leq a^*_n. \tag{2.6}$$

Proof:

We prove both Eqs. (2.5) and (2.6) simultaneously by induction on n. For $n = 1$, since $v(1) = a_p$ both inequalities are true. Assuming they hold for $n - 1$, *i.e.*,

$$\begin{aligned} v(n) &\leq v(n-1), \\ a_n^* &\leq a_{n-1}^*, \end{aligned}$$

we prove that

$$\begin{aligned} v(n+1) &\leq v(n), \\ a_{n+1}^* &\leq a_n^*. \end{aligned}$$

The inequality $v(n+1) \leq v(n)$ is true because from the definition of $w(n; a)$ and the induction assumption we have

$$v(n{+}1) = \min_{a\in A}\{a\,p(a){+}(1{-}p(a))v(n)\} \leq \min_{a\in A}\{a\,p(a){+}(1{-}p(a))v(n{-}1)\} = v(n).$$

To complete the induction for Ineq. (2.6) by contradiction we assume that $a_{n+1}^* > a_n^*$. Since a_{n+1}^* is the optimal action in state $n{+}1$ any other action $\hat{a}_{n+1}$ will be such that $w(n+1, \hat{a}_{n+1}) \geq v(n+1)$. Consider specifically the action $\hat{a}_{n+1} = a_n^*$. From the previous argument we have that $w(n{+}1; a_n^*) > v(n{+}1)$. This inequality, and the definition of $w(\cdot)$ imply the following.

$$a_n^* p(a_n^*) + (1 - p(a_n^*))v(n) > a_{n+1}^* p(a_{n+1}^*) + (1 - p(a_{n+1}^*))v(n).$$

Rearranging the terms of the above inequality we obtain:

$$\frac{a^*_{n+1}p(a^*_{n+1}) - a^*_n p(a^*_n)}{p(a^*_{n+1}) - p(a^*_n)} < v(n). \tag{2.7}$$

Similarly, in state n action a^*_n is optimal, so $v(n) < w(n, a^*_{n+1})$. Expanding this and rearranging the terms we obtain,

$$\frac{a^*_{n+1}p(a^*_{n+1}) - a^*_n p(a^*_n)}{p(a^*_{n+1}) - p(a^*_n)} > v(n-1). \tag{2.8}$$

Combining inequalities (2.7) and (2.8) we have

$$v(n) > \frac{a^*_{n+1}p(a^*_{n+1}) - a^*_n p(a^*_n)}{p(a^*_{n+1}) - p(a^*_n)} > v(n-1). \tag{2.9}$$

The above inequalities contradict the induction assumption. This completes the proof of theorem 2.2.1.

2.2.1 Computational example

The properties of $v(n)$ and a^*_n as stated in Theorem 2.2.1 are illustrated by the graphs of Figs. 2.1 and 2.2 respectively for the case with $N = 200$, $m^0 = L = 1$ and $A = \{1 \ldots, 50\}$. The winning probability $p(a)$ is calculated assuming the single opponent chooses bids from A with equal probabilities.

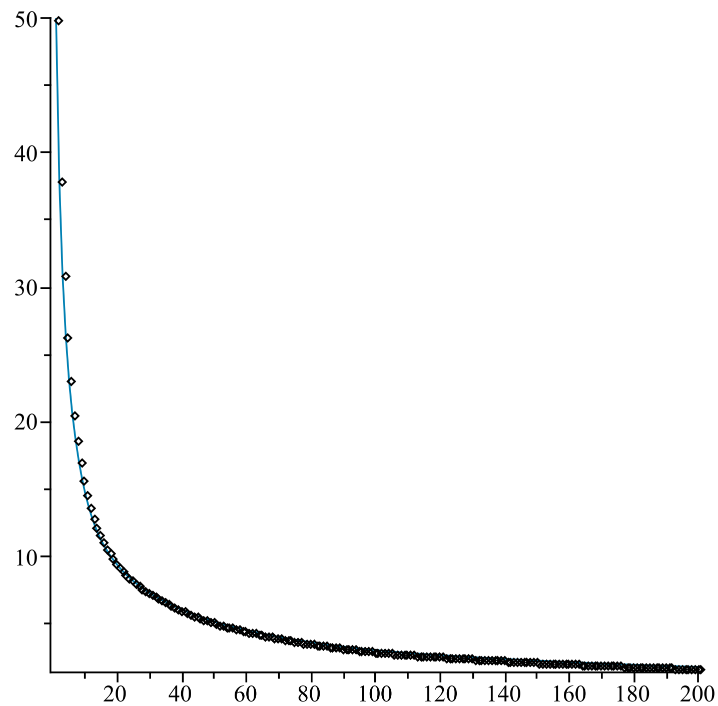

Figure 2.1: $v(n)$ *vs* n.

2.3 The Multi-Item Case With a Constant Number of Opponents

In this section we consider the problem where $L > 1$ and there is a constant number of opponents, $m^0 \geq 1$, in all auctions. The state space is the set $\{(n, m^0, l)\}$, $n = 1, \ldots, N$ and $l = 1, \ldots, L$. The action sets are $A(n, m^0, l) = \{a_1, \ldots, a_p\}$ for $l < L$ and $A(n, m^0, L) = \{a_0\}$. Note that $q_{mm'}(n) = 1$ if $m = m' = m^0$ and 0 otherwise.

Since the number of opponents is fixed we can simplify the above MDP.

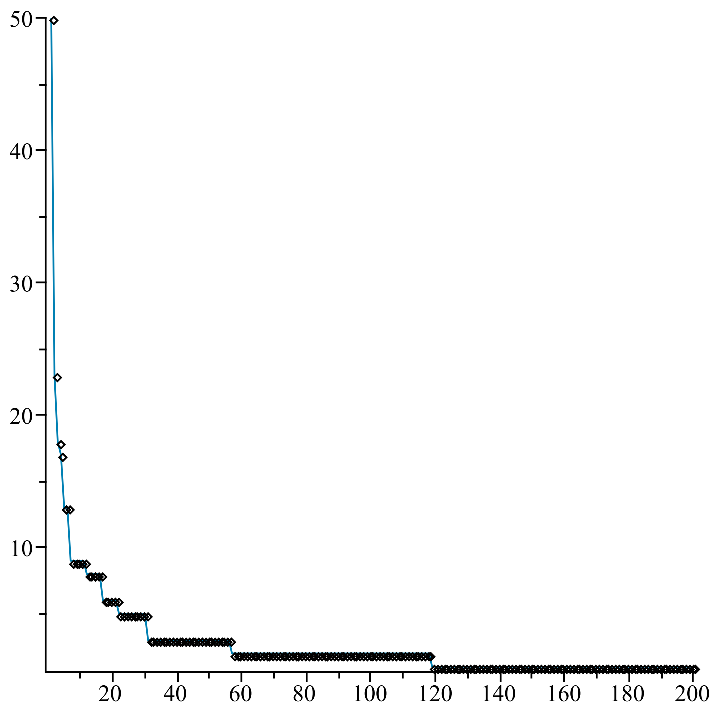

Figure 2.2: a_n^* *vs* n.

The state is now (n, l) which represents the number of remaining auctions and the number of items already acquired respectively. The action sets are $A(n, l) = \{a_1, \ldots, a_p\}$ if $l < L$, and $A(n, L) = \{a_0\}$. The transition probabilities and expected costs also simplify. The dynamic programming equations of Eq. (2.2) reduce to the following.

$$v(n, l) = \min_{a \in A}\{w(n, l; a)\} \tag{2.10}$$

where

$$
\begin{aligned}
w(n,l;a) &= a\,p(a) + p(a)v(n-1,l+1) \\
&\quad +(1-p(a))v(n-1,l) \quad \text{if } L+1 \leq n+l \leq N, \\
&= na_p, \quad \text{if } n+l = L, \\
&= 0, \qquad \text{if } l = L.
\end{aligned}
$$

We next state and prove Theorem 2.3.1

Theorem 2.3.1. *Under assumption A the following relationships hold true for all n and l.*

$$v(n+1,l) \leq v(n,l) \tag{2.11}$$

$$v(n,l+1) \leq v(n,l) \tag{2.12}$$

Proof. The proofs of both inequalities (2.11) and (2.12) are by induction on n. For $n = 1$ the inequalities $v(1,l+1) \leq v(1,l)$ and $v(2,l) \leq v(1,l)$ follow from Eq. (2.10) since $v(n,L) = 0$ and $v(n,L-n) = na_p$.

To complete the induction step of Ineq. (2.11) we assume that $v(n,l) \leq v(n-1,l)$ and prove that $v(n+1,l) \leq v(n,l)$. From the induction assumption and the definition of $w(n,l;a)$ we can conclude that $w(n+1,l;a) \leq w(n,l;a)\ \forall a$. This concludes the induction step because the last inequality implies that

$$v(n+1,l) = \min_{a \in A} w(n+1,l;a) \leq \min_{a \in A} w(n,l;a) = v(n,l).$$

To complete the induction step of Ineq. (2.12) we assume that $v(n,l) \leq v(n,l-1)$ and prove that $v(n,l+1) \leq v(n,l)$. From the definition of $w(n,l;a)$ and the induction assumption we can conclude that $w(n,l+1;a) \leq w(n,l;a)$. This concludes the induction step because

$$v(n,l+1) = \min_{a \in A} w(n,l+1;a) \leq \min_{a \in A} w(n,l;a) = v(n,l).$$

□

Theorem 2.3.2. *Under assumptions A and B the following relationships hold for all n and l.*

$$a^*_{n,l+1} \leq a^*_{n,l} \tag{2.13}$$

$$a^*_{n+1,l} \geq a^*_{n,l}. \tag{2.14}$$

Proof. The proofs of both the inequalities (2.13) and (2.14) are through induction. For $n = 1$ the inequalities $a^*_{1,l+1} \leq a^*_{1,l}$ and $a^*_{1,l} \geq a^*_{2,l}$ follow from Eq. (2.10) since $a_{n,L} = a_0$ and $a_{n,L-n} = a_p$.

To complete the induction of Ineq. (2.13) we assume that $a^*_{n-1,l} \leq a^*_{n-1,l-1}$ and prove that $a^*_{n,l} \leq a^*_{n,l-1}$. To prove this we assume that $a^*_{n,l} > a^*_{n,l-1}$, and prove that it produces a contradiction.

Since $a^*_{n,l}$ is the optimal action in state (n,l) and $a^*_{n,l-1}$ is the optimal action in state $(n,l-1)$ we have $v(n,l) < w(n,l;a^*_{n,l-1})$ and $v(n,l+1) \leq w(n,l-1;a^*_{n,l})$. Simplifying the inequalities and combining the results we obtain $2\,v(n-1,l) > v(n-1,l-1)+v(n-1,l+1)$. This implies the following.

$$w(n,l;a^*_{n-1,l-1})+w(n,l;a^*_{n-1,l+1}) > v(n-1,l-1)+v(n-1,l+1). \quad (2.15)$$

We notice that the induction assumption implies the following.

$$2\,v(n-2,l) \leq v(n-2,l-1)+v(n-2,l+1). \quad (2.16)$$

Inequalities (2.15) and (2.16) together imply

$$2\,v(n-2,l) > v(n-2,l-1)+v(n-2,l+1) \quad (2.17)$$

which contradicts Ineq. (2.16).

We next complete the induction step for Ineq. (2.14). We assume that $a^*_{n-1,l} \leq a^*_{n-2,l}$ and prove that $a^*_{n,l} \leq a^*_{n-1,l}$. We prove this by contradiction. We assume that $a^*_{n,l} > a^*_{n-1,l}$ and show that this produces a contradiction.

From the definition of $v(n,l)$ and $w(n,l;a)$ we have $v(n,l) \leq w(n,l;a^*_{n-1,l})$ and $v(n-1,l) \leq w(n-1,l;a^*_{n,l})$. Simplifying these inequalities and combining the results results in the following : $v(n-1,l)+v(n-2,l+1) > v(n-2,l)+v(n-1,l+1)$. From the last inequality we can conclude that

$$w(n-1,l;a^*_{n-1,l+1})+v(n-2,l+1) > v(n-2,l)+v(n-1,l+1).$$

Using the definitions of $w(n-1,l;a^*_{n-1,l+1}$ and $v(n-1,l+1)$ the above inequality simplifies to the following inequality

$$v(n-2,l+2) > v(n-2,l)$$

which contradicts Ineq. (2.11). □

2.3.1 Computational Example

The properties of $a^*_{n,l}$ as stated in Theorem 2.3.2 are illustrated by the graphs of Fig. 2.3 for the case with $N = 20$, $m^0 = 4$ and $A = \{1\ldots, 10\}$. The winning probability $p(a)$ is calculated assuming the four opponents choose bids from A with equal probabilities.

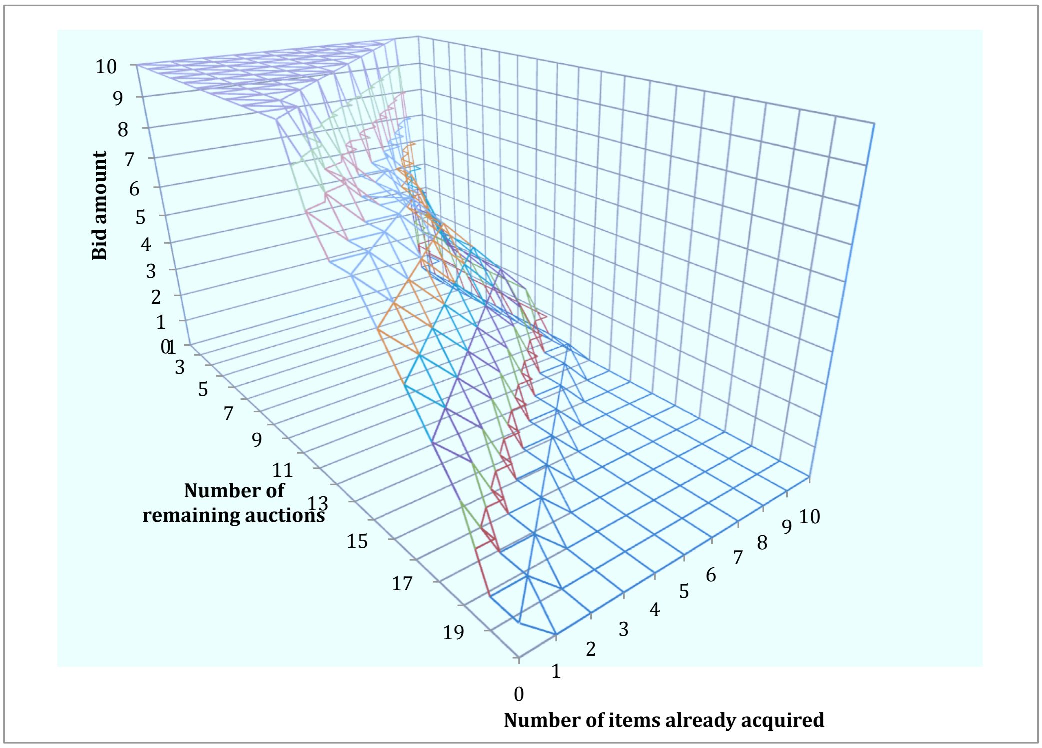

Figure 2.3: $a^*_{n,l}$ *vs* n *vs* l.

2.4 Single Item Case With Randomly Varying Number Of Opponents

In this section we consider the problem where $L = 1$ and the number of opponents may change with each auction as described in the section 2.1.1. We make the following assumption about the probabilities $q_{mm'}(n)$.

Assumption C. There exists a function G with $\sum_{i=-\infty}^{\infty} G(i) = 1$ such that:

$$q_{mm'}(n) = \begin{cases} G(m' - m) & \text{if } m' > 1, \\ \sum_{k=m-1}^{\infty} G(k) & \text{if } m' = 1. \end{cases} \tag{2.18}$$

In this case the state space is the set $\{(n, m, 0)\,,\ (n, m, 1)\}$, $n = 1, \ldots, N$ and $m = 1, 2, \cdots$. The action sets are $A(n, m, 0) = \{a_1, \ldots, a_p\}$ and $A(n, m, 1) = \{a_0\}$.

Once the buyer wins an auction there is no decision problem. The relevant information at any time for the decision problem of the buyer in this case is the number of auctions remaining before the single item is acquired and the number of opponents. This simplifies the above MDP. The state (n, m) now represents the number of remaining auctions before the item is acquired by the buyer and the number of opponents present. All action sets are equal to $A = \{a_1, \ldots, a_p\}$. The transition probabilities and expected costs also simplify and the dynamic programming equations of Eq. (2.2) reduce to the following.

$$v(n,m) = \min_{a \in A}\{w(n,m;a)\} \tag{2.19}$$

with,

$$\begin{aligned} w(n,m;a) &= a\,p_m(a) + (1-p_m(a))E(n-1,m) \text{ for } n>1 \text{ and } \forall m, \\ &= a_p \text{ for } n=1 \text{ and } \forall m. \end{aligned}$$

where for notational simplicity we have defined:

$$E(n-1,m) = \sum_{m'} q_{mm'}(N-n)v(n-1,m').$$

We now state and prove the theorem 2.4.1.

Theorem 2.4.1. *Under assumptions A and B the following relationships hold for all n.*

$$v(n,m) \le v(n-1,m) \tag{2.20}$$

$$v(n,m-1) \le v(n,m). \tag{2.21}$$

Proof. The proof of inequality (2.20) is by induction on n. For $n=1$, the inequalities $v(1,m) \ge v(2,m)$ for all m, follow from the Eq. (2.19) since $v(1,m) = a_p$. Assuming that $v(n-1,m) \ge v(n,m)$ for all m, we now prove that $v(n,m) \ge v(n+1,m)$ for all m. We know that $w(n,m;a) = a\,p_m(a) + (1-p_m(a))E(n-1,m)$ is a convex combination of a and $E(n-1,m)$. From the induction assumption it follows that $E(n-1,m) \ge E(n,m)$. Thus, we can conclude that the inequality below holds.

$$w(n, m; a) \geq w(n+1, m; a) \quad \forall a \in A$$

The above inequality implies that

$$v(n, m) = \min_{a \in A} w(n, m; a) \geq \min_{a \in A} w(n+1, m; a) = v(n+1, m)$$

completing the induction.

The proof of inequality (2.21) is also by induction on n. For $n = 1$, the inequalities $v(1, m) \geq v(1, m-1)$ hold for all m since $v(1, m) = a_p$ for all m. Assuming that $v(n-1, m) \geq v(n-1, m-1)$ for all m, we now show that $v(n, m) \geq v(n, m-1)$ for all m. We prove last inequality by contradiction. We assume that $v(n, m) < v(n, m-1)$ and prove that this produces a contradiction. The last inequality implies that $v(n, m) < w(n, m-1; a^*_{n,m})$ which is the same as the following inequality.

$$\begin{aligned} & a^*_{n,m}\, p_m(a^*_{n,m}) + (1 - p_m(a^*_{n,m}))E(n-1, m) \\ < \quad & a^*_{n,m}\, p_{m-1}(a^*_{n,m}) + (1 - p_{m-1}(a^*_{n,m}))E(n-1, m-1) \end{aligned}$$

From assumption B we know that $p_m(a^*_{n,m}) \leq p_{m-1}(a^*_{n,m})$. Let $p_m(a^*_{n,m}) = p_{m-1}(a^*_{n,m}) - \delta$. Substituting $p_m(a^*_{n,m})$ in the above inequality yields the following.

$$\begin{aligned} & a^*_{n,m}\, p_{m-1}(a^*_{n,m}) + (1 - p_m(a^*_{n,m}))E(n-1, m) + \delta(E(n-1, m) - a^*_{n,m}) \\ < \quad & a^*_{n,m}\, p_{m-1}(a^*_{n,m}) + (1 - p_{m-1}(a^*_{n,m}))E(n-1, m-1) \end{aligned}$$

$v(n,m)$ is a convex combination of $a^*_{n,m}$ and $E(n,m)$. Note that $E(n-1,m) > a^*_{n,m}$ because if we assume the contrary we will have $w(n,m,a_0) < v(n,m)$ which is a contradiction. This implies that

$$\begin{aligned} & a^*_{n,m}\, p_{m-1}(a^*_{n,m}) + (1 - p_m(a^*_{n,m}))E(n-1,m) \\ < \;& a^*_{n,m}\, p_{m-1}(a^*_{n,m}) + (1 - p_{m-1}(a^*_{n,m}))E(n-1,m-1) \end{aligned}$$

which contradicts the induction assumption. This concludes the induction of Ineq. (2.21) and the proof of the theorem. □

Theorem 2.4.2. *Under assumptions A, B and C the following relationships hold for all n.*

$$a^*_{n,m} \leq a^*_{n,m+1} \tag{2.22}$$

$$a^*_{n,m} \geq a^*_{n-1,m}. \tag{2.23}$$

Proof. To prove Ineq. (2.22) we assume that $a^*_{n,m-1} > a^*_{n,m}$ and prove that this produces a contradiction. Since $a^*_{n,m}$ is the optimal action in state (n,m) we have that $w(n,m;a^*_{n,m-1}) > v(n,m)$. Simplifying the inequality using the fact that $a^*_{n,m-1} > a^*_{n,m}$ we obtain

$$T_m > E(n-1,m). \tag{2.24}$$

where,

$$T_m = \frac{a^*_{n,m-1} p_m(a^*_{n,m-1}) - a^*_{n,m} p_m(a^*_{n,m})}{p_m(a^*_{n,m-1}) - p_m(a^*_{n,m})}.$$

Similarly, in state $(n, m-1)$ action $a^*_{n,m-1}$ is optimal, so $v(n, m-1) < w(n, m-1, a^*_{n,m})$. Simplifying the last inequality we obtain,

$$T_{m-1} < E(n-1, m-1). \tag{2.25}$$

We next show that

$$T_m < T_{m-1}. \tag{2.26}$$

Indeed, using the definitions of T_m and T_{m-1}, the above simplifies to the following equivalent inequality

$$p_m(a^*_{n,m})p_{m-1}(a^*_{n,m-1}) < p_m(a^*_{n,m-1})p_{m-1}(a^*_{n,m}),$$

which is true under assumption B since we have assumed that $a^*_{n,m-1} > a^*_{n,m}$.

Inequalities (2.24), (2.25) and (2.26) together imply that

$$E(n-1, m) < E(n-1, m-1).$$

Under assumption C this inequality can be expressed as follows.

$$\sum_{i=-m+1}^{\infty} [G(i)(v(n-1, m+i+1) - v(n-1, m+i))] + G(-m+1)v(n-1, 1) < 0.$$

The above inequality is a contradiction since from Ineq. (2.21) we know that $v(n-1, m+i+1) - v(n-1, m+i) \geq 0$ and by assumption C, $G(i) \geq 0$.

Next, to prove $a^*_{n,m} \leq a^*_{n-1,m}$ we show that $a^*_{n,m} > a^*_{n-1,m}$ leads to a contradiction. Since $a^*_{n,m}$ is the optimal action in state (n, m) we have that $w(n, m; a^*_{n-1,m}) > v(n, m)$. Simplifying this inequality we obtain

$$T < E(n-1,m). \tag{2.27}$$

where,

$$T = \frac{a^*_{n,m} p_m(a^*_{n,m}) - a^*_{n-1,m} p_m(a^*_{n-1,m})}{p_m(a^*_{n,m}) - p_m(a^*_{n-1,m})}$$

Similarly, in state $(n-1, m)$ action $a^*_{n-1,m}$ is optimal, so $w(n-1, m, a^*_{n,m}) > v(n-1, m)$. Simplifying the last inequality we obtain,

$$T > E(n-2,m). \tag{2.28}$$

Inequalities (2.27) and (2.28) together imply that

$$E(n-2,m) < E(n-1,m).$$

Under assumption C the above inequality can be expressed as follows.

$$\sum_{i=-m+1}^{\infty} \{G(i)(v(n-2,m) - v(n-1,m))\} < 0.$$

The above inequality is a contradiction since from Ineq. (2.21) we know that $v(n-2, m) - v(n-1, m) \geq 0$ and by assumption C, $G(i) \geq 0$.

$\square$

2.4.1 Computational Example

The properties of $a^*_{n,m}$ as stated in Theorem 2.4.2 are illustrated by the graph of Fig. 2.4 for the case with $N = 20$, $1 \leq m \leq 20$ and $A = \{1 \ldots, 10\}$. We assume that $G(i) = 1/39$ for $i = -19 \ldots, 0, \ldots 19$. The winning probability $p_m(a)$ is calculated assuming that each of the opponents choose bids from A with equal probabilities.

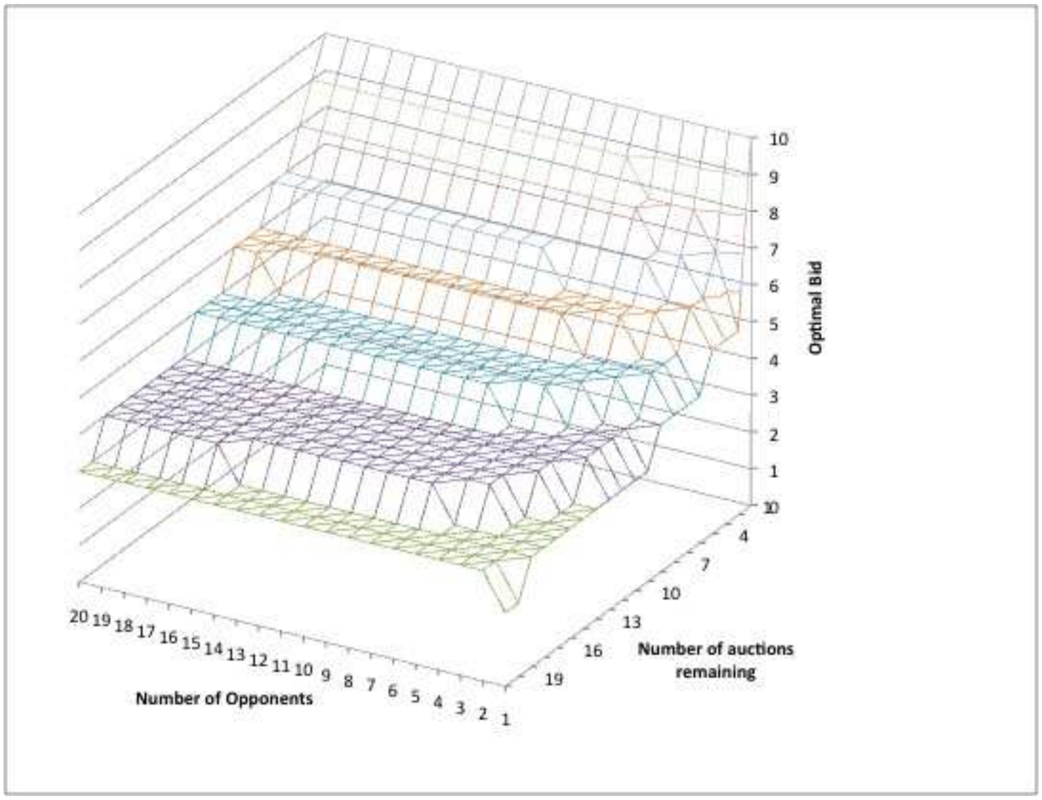

Figure 2.4: $a^*_{n,m}$ vs n vs m.

2.5 The Multi-Item Case With Randomly Varying Number of Opponents

In this section we consider the most general version of the problem as described in section 2.1.1. Recall that the dynamic programming equations are as follows.

$$v(n,m,l) = \min_{a \in A}\{w(n,m,l;a)\}$$

with,

$$\begin{aligned} w(n,m,l;a) &= a\,p_m(a) + p_m(a)E(n-1,m,l+1) \\ &\quad +(1-p_m(a))E(n-1,m,l) \quad \text{if } L+1 \le n+l \le N, \\ &= na_p, \quad \text{if } n+l = L, \\ &= 0, \qquad \text{if } l = L \end{aligned}$$

where for notational simplicity we have defined:

$$E(n-1,m,l) = \sum_{m'=1}^{\infty} q_{mm'}(N-n)v(n-1,m',l).$$

We state and prove the following theorems.

Theorem 2.5.1. *Under assumption A the following relationships hold true for all n, m and l.*

$$v(n+1,m,l) \le v(n,m,l) \tag{2.29}$$

$$v(n,m,l) \le v(n,m,l-1) \tag{2.30}$$

$$v(n,m,l) \ge v(n,m-1,l). \tag{2.31}$$

Proof. The proof of all three inequalities are by induction on n. For $n = 1$, the inequalities, $v(2, m, l) \leq v(1, m, l)$, $v(1, m, l) \leq v(1, m, l-1)$ and $v(1, m-1, l) \leq v(1, m, l)$ follow from Eq. (2.2) since $v(n, m, L) = 0$ and $v(n, m, L-n) = na_p$.

To complete the induction of Ineq. (2.29) we assume that $v(n-1, m, l) \leq v(n-2, m, l)$ and prove that $v(n, m, l) \leq v(n-1, m, l)$. From the induction assumption and it follows that $E(n-1, m, l) \leq E(n-2, m, l)$. Thus we can conclude that $w(n, m, l; a) \leq w(n-1, m, l; a)\ \forall\ a$. This concludes the induction step because the last inequality implies that

$$v(n, m, l) = \min_{a \in A} w(n, m, l; a) \leq \min_{a \in A} w(n-1, m, l; a) = v(n-1, m, l).$$

To complete the induction of Ineq. (2.30) we assume that $v(n, m, l-1) \leq v(n, m, l-2)$ and prove that $v(n, m, l) \leq v(n, m, l-1)$. The induction assumption implies that $E(n, m, l-1) \leq E(n, m, l-2)$. From this and the definition of $w(n, m, l; a)$ we can conclude that $w(n, m, l; a) \leq w(n, m, l-1; a)\ \forall\ a$. This concludes the induction because

$$v(n, m, l) = \min_{a \in A} w(n, m, l; a) \leq \min_{a \in A} w(n, m, l-1; a) = v(n, m, l-1).$$

To complete the induction of Ineq. (2.31) we assume that $v(n-1, m, l) \geq v(n-1, m-1, l)$ and prove that $v(n, m, l) \geq v(n, m-1, l)$. To prove the last inequality we assume that $v(n, m-1, l) > v(n, m, l)$ and proceed to show that this produces a contradiction. This assumption implies that $w(n,m-1,l;a^*_{n,m,l}) > v(n,m,l)$. Using the definition of $v(n,m,l)$ and $w(n,m,l;a)$ in the last inequality we obtain,

$$\begin{aligned} & p_{m-1}(a^*)(a^* + E(n-1,m-1,l+1)) + (1-p_{m-1}(a^*))E(n-1,m-1,l) \\ > \; & p_m(a^*)(a^* + E(n-1,m,l+1)) + (1-p_m(a^*))E(n-1,m,l). \end{aligned}$$

where $a^* = a^*_{n,m,l}$. Condition B states that $p_m(a^*) < p_{m-1}(a^*)$. Let $p_m(a^*) = p_{m-1}(a^*) - \delta$, where $\delta \geq 0$. Using this in the above inequality we obtain

$$\begin{aligned} & p_{m-1}(a^*)(a^* + E(n-1,m-1,l+1)) + (1-p_{m-1}(a^*))E(n-1,m-1,l) \\ > \; & p_{m-1}(a^*)(a^* + E(n-1,m,l+1)) + (1-p_{m-1}(a^*))E(n-1,m,l) \\ & +\delta(E(n-1,m,l) - (a^* + E(n-1,m,l+1))). \end{aligned}$$

From its definition we know that $v(n,m,l)$ is a convex combination of $E(n-1,m,l)$ and $a^* + E(n-1,m,l+1)$. Note that $a^* + E(n-1,m,l+1) \leq E(n-1,m,l)$ because if we assume that contrary then $w(n,m,l;a_0) < v(n,m,l)$, which is a contradiction. This implies that the term multiplying δ in the above inequality is positive. Hence

$$\begin{aligned} & p_{m-1}(a^*)(a^* + E(n-1,m-1,l+1)) + (1-p_{m-1}(a^*))E(n-1,m-1,l) \\ > \; & p_{m-1}(a^*)(a^* + E(n-1,m,l+1)) + (1-p_{m-1}(a^*))E(n-1,m,l) \end{aligned}$$

which contradicts the induction assumption.

$\square$

Theorem 2.5.2. *Under assumption A, B and C the following relationships hold true for all n, m and l.*

$$a^*_{n,m,l+1} \leq a^*_{n,m,l} \tag{2.32}$$

$$a^*_{n,m,l} \leq a^*_{n,m+1,l} \tag{2.33}$$

$$a^*_{n,m,l} \leq a^*_{n-1,m,l}. \tag{2.34}$$

Proof. The proofs of all three inequalities is by induction on n. For $n = 1$, the inequalities, $a^*_{2,m,l} \leq a^*_{1,m,l}$, $a^*_{1,m,l} \leq a^*_{1,m+1,l}$ and $a^*_{1,m,l} \leq a^*_{1,m,l-1}$ ($L \leq 2 + l \leq N$) hold from Eq. (2.2) since $a^*_{n,m,L} = 0$ and $a^*_{n,m,L-n} = a_p$.

To complete the induction of Ineq. (2.32) we assume that $a^*_{n-1,m,l} \leq a^*_{n-1,m,l-1}$ and prove that $a^*_{n,m,l+1} \leq a^*_{n-1,m,l}$. To prove this part we assume that $a^*_{n,m,l+1} > a^*_{n,m,l}$ and show that this produces a contradiction. From the definitions of $v(n,m,l)$ and $w(n,m,l;a)$ we have $v(n,m,l) < w(n,m,l;a^*_{n,m,l+1})$ and $v(n,m,l+1) < w(n,m,l+1;a^*_{n,m,l})$. Simplifying and combining the results of the last two inequalities we obtain

$$E(n-1,m,l) + E(n-1,m,l+2) > 2E(n-1,m,l+1).$$

This can be rewritten as

$$\sum_{m'} q_{mm'}[v(n-1,m',l) + v(n-1,m',l+2)] > 2\sum_{m'} q_{mm'}v(n-1,m',l+1).$$

This implies that

$$\sum_{m'} q_{mm'}[w(n-1,m',l;a^*)+w(n-1,m',l+2;a^*)] > 2\sum_{m'} q_{mm'}v(n-1,m',l+1). \tag{2.35}$$

where $a^* = a^*_{n-1,m',l+1}$.

Notice that the induction assumption implies that

$$E(n-2,m,l)+E(n-2,m,l+2) > 2E(n-2,m,l+1). \tag{2.36}$$

Simplifying Ineq. (2.35) using assumption C and Ineq. (2.36) leads to the inequality

$$\begin{aligned} &\sum_{m'} q_{mm'}[E(n-2,m',l+1)-E(n-2,m',l+2)] \\ > \; &\sum_{m'} q_{mm'}[E(n-2,m',l+1)-E(n-2,m',l+2)] \end{aligned}$$

which is a contradiction because both sides of the strict inequality are identical.

To complete the induction of Ineq. (2.34) we assume that $a^*_{n,m,l} \geq a^*_{n,m-1,l}$ and prove that $a^*_{n,m+1,l} \geq a^*_{n,m,l}$. To prove this part we assume that $a^*_{n,m,l} > a^*_{n,m+1,l}$ and show that this produces a contradiction. Since $a^*_{n,m,l}$ is the optimal action in state (n,m,l) we have that $v(n,m,l) < w(n,m,l,a^*_{n,m-1,l})$. This simplifies to:

$$T_m + E(n-1,m,l+1) < E(n-1,m,l). \tag{2.37}$$

where

$$T_m = \frac{a^*_{n,m,l} p_m(a^*_{n,m,l}) - a^*_{n,m+1,l} p_m(a^*_{n,m+1,l})}{p_m(a^*_{n,m,l}) - p_m(a^*_{n,m+1,l})}.$$

Similarly in state $(n, m+1, l)$ we have $v(n, m+1, l) < w(n, m+1, l, a^*_{n,m,l})$ which simplifies to

$$T_{m+1} + E(n-1, m+1, l+1) > E(n-1, m+1, l). \tag{2.38}$$

Note that

$$T_{m+1} < T_m. \tag{2.39}$$

Indeed, from the definitions of T_m and T_{m+1} the last inequality simplifies to the following inequality

$$p_m(a^*_{n,m,l}) p_m(a^*_{n,m-1,l}) < p_{m-1}(a^*_{n,m,l}) p_{m-1}(a^*_{n,m-1,l}),$$

which is true under assumption B since we have assumed that $a^*_{n,m,l} > a^*_{n,m+1,l}$.

Inequalities (2.37), (2.38), (2.39) together imply that

$$E(n-1, m, l) - E(n-1, m+1, l) < E(n-1, m, l+1) - E(n-1, m+1, l+1).$$

This implies that

$$\begin{aligned} & \sum_{i=-m+1}^{\infty} \{G(i)[v(n-1, m+i, l) - v(n-1, m+i+1, l)]\} \\ & + G(-m)(v(n-1, 1, l) - v(n-1, 1, l+1)) \\ < & \sum_{i=-m+1}^{\infty} \{G(i)[v(n-1, m+i, l+1) - v(n-1, m+i+1, l+1)].\} \end{aligned}$$

Since $v(n-1,1,l) - v(n-1,1,l+1) \geq 0$, the above inequality implies that

$$
\begin{aligned}
& \sum_{i=-m+1}^{\infty} \{G(i)[v(n-1,m+i,l) - v(n-1,m+i+1,l)]\} \\
< & \sum_{i=-m+1}^{\infty} \{G(i)[v(n-1,m+i,l+1) - v(n-1,m+i+1,l+1)].\}
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
& \sum_{m'=1}^{\infty} \{G(m'-m)[v(n-1,m'+1,l+1) - v(n-1,m',l+1)]\} \\
< & \sum_{m'=1}^{\infty} \{G(m'-m)[v(n-1,m'+1,l) - v(n-1,m',l)].\}
\end{aligned}
$$

From the definition of $w(n,m,l;a)$ the above inequality implies that

$$
\begin{aligned}
& \sum_{m'=1}^{\infty} \{G(m'-m)[v(n-1,m'+1,l+1) - w(n-1,m',l+1;a_1^*)]\} \\
< & \sum_{m'=1}^{\infty} \{G(m'-m)[w(n-1,m'+1,l;a_2^*) - v(n-1,m',l)].\} \qquad (2.40)
\end{aligned}
$$

where $a_1^* = a^*_{n-1,m'+1,l+1}$ and $a_2^* = a^*_{n-1,m',l}$ From assumption B we have that $p_m(a) \geq p_{m+1}(a)$. Let $p_m(a_1^*) = p_{m+1}(a_1^*) + \delta_1$ and $p_m(a_2^*) = p_{m+1}(a_2^*) + \delta_2$ Note that the induction assumption implies that

$$
E(n-2,m+1,l+1) - E(n-2,m,l+1) > E(n-2,m+1,l) - E(n-2,m,l). \qquad (2.41)
$$

Simplifying Ineq. (2.40) using assumption C and Ineq. (2.41) leads to the following inequality

$$
\begin{aligned}
&\sum_{m'=1}^{\infty}\{G(m'-m)[E(n-2,m'+1,l+1)-E(n-2,m',l+1)]\} \\
&+\delta_1[E(n-2,m'+1,l+1)-(a_1^*+E(n-2,m'+1,l+2))] \\
&+\delta_2[E(n-2,m',l)-(a_2^*+E(n-2,m',l+1))] \\
<\ &\sum_{m'=1}^{\infty}\{G(m'-m)[E(n-2,m'+1,l+1)-E(n-2,m',l+1)]\}
\end{aligned}
$$

From the proof of Theorem 2.5.1 we know that the terms multiplying δ_1 and δ_2 are both positive. This leads to the inequality

$$
\begin{aligned}
&\sum_{m'=1}^{\infty}\{G(m'-m)[E(n-2,m'+1,l+1)-E(n-2,m',l+1)]\} \\
<\ &\sum_{m'=1}^{\infty}\{G(m'-m)[E(n-2,m'+1,l+1)-E(n-2,m',l+1)]\}
\end{aligned}
$$

which is a contradiction because both sides of the strict inequality are identical.

To complete the induction of Eq. (2.32) we assume that $a^*_{n,m,l} \leq a^*_{n-1,m,l}$ and prove that $a^*_{n+1,m,l+1} \leq a^*_{n,m,l}$. To prove this part we assume that $a^*_{n+1,m,l+1} > a^*_{n,m,l}$ and prove that it produces a contradiction. From the definitions of $v(n,m,l)$ and $w(n,m,l;a)$ we have $v(n,m,l) < w(n,m,l;a^*_{n+1,m,l})$ and $v(n+1,m,l) < w(n+1,m,l;a^*_{n,m,l})$. Simplifying and combining the results of the last two inequalities we obtain

$$E(n-1,m,l+1)-E(n,m,l+1) > E(n-1,m,l)-E(n,m,l).$$

The above inequality is equivalent to the following inequality.

$$\sum_{m'} q_{mm'}[v(n-1,m',l+1)-v(n,m',l+1)] > \sum_{m'} q_{mm'}[v(n-1,m',l)-v(n,m',l)].$$

Simplifying the above inequality using assumption C, the induction assumption and the fact $v(n-1,m',l+1) < w(n-1,m',l+1;a)$ leads to the inequality

$$\begin{aligned} & \sum_{m'} q_{mm'}[E(n-2,m',l+1) - E(n-1,m',l+1)] \\ > & \sum_{m'} q_{mm'}[E(n-2,m',l+1) - E(n-1,m',l+1)] \end{aligned}$$

which is a contradiction because both sides of the strict inequality are identical.

$\square$

2.6 Conclusions

In this chapter we considered the problem the problem of a firm ("the bidder") that in a given time period buys items in a sequence of auctions. The objective of the buyer is to minimize his expected total cost for the period, while acquiring a fixed number of items. It is assumed that there is a *buy-it-now-price* available at which the buyer can obtain the item outright at any auction or sometimes in the open market. We formulated the problem as a Markov Decision process and proved that, under a few assumptions, the optimal value function and the optimal bid are decreasing functions of the number of remaining auctions, increasing functions of the number of opponents and decreasing functions of the inventory on hand.

Extensions of this model include relaxing the condition that the bid distribution is constant through all the auctions. A model where the distribution is updated using a Bayesian framework can also be constructed and studied. Other extensions include batch sales, where in each auctions more than one item is sold. In such a model the batch size can also become a decision variable if bids can be placed for batches of items.

Chapter 3

Integrated Auction Inventory Model

3.1 Introduction

We consider the problem of a firm ("the bidder") that in each period, of an infinite time horizon, buys items in auctions and sells the acquired items in a secondary market. We investigate optimal bidding strategies for the bidder that take into account the cost of acquiring the items, the random sale price and demand of the secondary market as well as pertinent salvage value or inventory holding costs. The objective of the buyer is to maximize his expected infinite horizon discounted reward. In the current literature the topics of biding in auctions and inventory control of a firm are treated as though they are essentially independent. Yet there is often considerable interaction between the bidding policy in auctions and the inventory control problem of a firm. This chapter presents and studies models in which purchasing, via bidding in auctions, and inventory decisions are made dynamically and interact directly with each other. In this way the bidders' valuations derive from resale of good acquired in the auctions. We study several aspects of this problem including the number of opponents (fixed or random) and inventory accumulation (i.e., salvaging units at the end at a known price or carrying unsold items at the end of a period to the next while incurring a known cost.)

The main results of this chapter are as follows.

1. We provide new models that combine purchasing decisions made via bidding in auctions with inventory management decisions. We formulate these problems as Markov decision processes.

2. We prove that, under a few assumptions, the optimal value function is an increasing function of n, the number of remaining auctions , a decreasing function of m, the number of opponents and a increasing function of x, the inventory on hand.

3. We prove that the optimal bid is a decreasing function of n, an increasing function of m, and a decreasing function of x.

3.1.1 General Problem Definition

We consider the problem of buying items via a sequence of N auctions during each time period and then selling them in a secondary market. The demand D in the secondary market is random with a known discrete distribution. Let $p_D(d) = P(D = d)$, $P_D(d) = P(D \leq d)$, and $\bar{P}_D(d) = 1 - P_D(d)$. The sales price R is also a random variable with a known distribution. Let $r = E(R) < \infty$. Excess demand is assumed to be lost and the penalty of losing sales of x units is $\delta(x)$.

We consider two cases of this problem. In the first case any items that are unsold at the end of a time period are salvaged at a known price $s < r$. In the second case any items that are unsold at the end of a period are carried over as inventory to the next period with an inventory carrying cost of h per item per period.

We study optimal bidding strategies for the buyer for the following auction procedure. There is a sequence of $N \geq 1$ auctions of identical items in each time period. Before each auction the number of opposing bidders (opponents) m is known. Every bidder submits a sealed bid. The highest bidder wins the auction. At the end of each auction the winning bid is announced. The objective of the buyer is to maximize his expected infinite horizon discounted reward.

It is assumed that the set of all bids available (to the buyer and all opponents) is a finite set $\{a_0, a_1, \dots a_p\}$ where $a_0 < a_1 < \dots < a_p$. For simplicity we will use the same symbol a to represent both the bid price and the action of the buyer bidding amount a. We assume that $a_0 = 0$ represents the action of not bidding.

It is assumed that there exist known probabilities $p_m(a)$ that correspond to the probability that the buyer wins an auction when his bid is a and there are m opponents present. For consistency of notation we assume that $p_m(a_0) = 0$. For convenience let $\bar{p}_m(a) = 1 - p_m(a)$.

The number of opponents in each auction is random. Let Z_n be the number of opponents participating in the n^{th} auction. It is assumed that Z_n for $n = 1, 2, \cdots, N$ is a discrete time Markov chain with transition probabilities:

$$q_{mm'}(n) = P(Z_{n+1} = m' | Z_n = m).$$

The initial distribution of the number of opponents is known and is denoted for simplicity by:

$$q_m(1) = P(Z_1 = m).$$

We assume that whenever there is a tie in an auction involving the buyer , then the buyer loses. This assumption is made to simplify the exposition. Other tie breaking procedures like deciding the winner randomly will not change the analysis but would complicate the exposition. This supposition leads to the following.

$$p_m(a) = P(\text{ all opponents' bids} < a). \tag{3.1}$$

In the sequel we make the following assumptions.

Assumption A. For any fixed m, $p_m(a)$ is an increasing function of a.

Assumption B. For any fixed a, $p_m(a)$ is a decreasing function of m.

Assumption C. There exists a function G with $\sum_{i=-\infty}^{\infty} G(i) = 1$ such that:

$$q_{mm'}(n) = \begin{cases} G(m'-m) & \text{if } m' > 1, \\ \\ \sum_{k=m-1}^{\infty} G(k) & \text{if } m' = 1. \end{cases} \tag{3.2}$$

Assumption D. $\delta(x)$ is an increasing convex function of x and $\delta(x) = 0$ if $x \leq 0$.

3.2 The Salvage Case

3.2.1 Model Definition

In this section we consider the version of the problem where any items that are unsold at the end of a period are salvaged at a known price $s < r$. We model this problem as a Markov Decision process.

1. The state space $\mathcal{S}$ in this case is the set $\{(n, m, x),\ n = 0, \ldots, N,\ m = 1, \ldots,\ x = 0, 1, \ldots\}$, where n represents the number of auctions remaining during the current epoch, m represents the number of bidders participating in the current auction, $x \geq 0$ represents the inventory level at the beginning of the current $(N - n)$ auction. Note that:

- If $n = 0$ then $m = 0$.
- State $(0, 0, x)$ represents the state of the system at the end of an epoch when all auctions are over.
- Possible states at the beginning of an epoch, prior to the start of the N auctions, are of the form $(N, m, 0)$, for all $m = 1, \ldots$.

2. In any state (n, m, x) the following action sets $A(n, m, x)$ are available.
   - $A(0, 0, x) = \{a_0\}$.
   - $A(n, m, x) = \{a_0, \ldots, a_p\}$ for $n > 0$.

3. When an action $a \in A(n, m, x)$ is taken in state (n, m, x) the following transitions are possible.
   - If $n = 0$, then starting from state $(0, 0, x)$ the next state is $(N, m, 0)$ with probability $q_m(1)$.
   - If $n > 0$ then depending on whether or not the buyer wins the current auction the next state is $(n-1, m', x+1)$ with probability $p_m(a)\, q_{mm'}(N-n)$ or state $(n-1, m', x)$ with $\bar{p}_m(a)\, q_{mm'}(N-n)$.

4. When an action $a \in A(n, m, x)$ is taken in state (n, m, x) the expected reward $r_a(n, m, x)$ is as follows.
   - $r_a(0, 0, x) = \sum_{d=0}^{x}(rd + s(x-d))\, p_D(d) + \sum_{d=x+1}^{\infty}(rx - \delta(d - x))p_D(d)$

- $r_a(n,m,x) = -a\,p_m(a)$ if $n > 0$.

Lemma 3.2.1. *The expected reward function in state* $(0,0,x)$, $r_a(0,0,x)$ *is an increasing function of* x *i.e.*

$$r_a(0,0,x) \leq r_a(0,0,x+1). \tag{3.3}$$

Proof. The proof is evident from the fact that the difference $r_a(0,0,x+1) - r_a(0,0,x)$ can be simplified to

$$\sum_{x+1}^{\infty} rp_D(d) + \sum_{d=0}^{x} sp_D(d) + \sum_{d=x+1}^{\infty} (\delta(d-x) - \delta(d-x-1))p_D(d),$$

which is non-negative because $\delta(\cdot)$ is an increasing function. □

Let $a^*_{n,m,x}$ denote the optimal action in the state (n,m,x). Let $v(n,m,x)$ denote the value function in state (n,m,x) and $w(n,m,x;a)$ denote the expected future reward when action a is taken in state (n,m,x) and an optimal policy is followed thereafter. Note that $v(n,m,x) = w(n,m,x;a^*_{n,m,x})$.

The dynamic programming equations are

$$v(n,m,x) = \max_{a \in A}\{w(n,m,x;a)\} \tag{3.4}$$

where,

$$\begin{aligned} w(n,m,x;a) &= r_a(0,0,x) + \beta \sum_{m=1}^{\infty} q_m(1) v(N,m,0) \text{ if } n = 0 \\ &= r_a(n,m,x) + p_m(a)E(n-1,m,x+1) \\ &\quad +\bar{p}_m(a)E(n-1,m,x) \text{ if } n > 0, \end{aligned}$$

$E(n-1, m, x) = \sum_{m'=1}^{\infty} q_{mm'}(N-n)v(n-1, m', x)$ and β is the discount factor.

The above dynamic programming equations can be solved to develop a bidding strategies for general settings. We now consider structural properties of the optimal policies for some interesting cases.

3.2.2 Constant Number of Opponents Case

In this section we consider the problem where there is a constant number of opponents, $m^0 \geq 1$, in all auctions. The state space is the set $\{(n, m^0, x)\}_{n=1...N}$. The action sets are $A(n, m^0, x) = \{a_0, \ldots, a_p\}$. Note that $q_{mm'}(n) = 1$ if $m = m' = m^0$ and 0 otherwise.

During any auction the buyer makes a decision based on the number of auctions remaining and the number of items already acquired. We obtain a simplified MDP where the state (n, x) represents the number of remaining auctions and the number of items already acquired. The action sets are $A(n, x) = \{a_0, \ldots, a_p\}$. The transition probabilities and expected rewards also simplify analogously and the dynamic programming equations in Eq. (3.4) reduce to the following.

$$v(n, x) = \max_{a \in A}\{w(n, x; a)\} \tag{3.5}$$

where

$$\begin{aligned} w(n,x;a) &= r_a(0,x) + \beta v(N,m,0) \text{ if } n = 0 \\ &= r_a(n,x) + p(a)E(n-1,x+1) + \bar{p}(a)E(n-1,x) \text{ if } n > 0. \end{aligned}$$

We now state and prove the theorem 3.2.1.

Theorem 3.2.1. *Under assumption A the following is true for all $n \geq 0$ and all $x \geq 0$.*

$$v(n,x) \leq v(n,x+1) \tag{3.6}$$

$$v(n,x) \leq v(n+1,x). \tag{3.7}$$

Proof. We first prove Ineq. (3.6) by induction on n. For $n = 0$ we have $v(0,x+1) - v(0,x) = r_a(0,x+1) - r_a(0,x) \geq 0$ from Lemma 3.2.1. For $n = 1$ the difference $w(1,x+1;a) - w(1,x;a)$ simplifies to

$$p(a)(v(0,x+2) - v(0,x+1)) + \bar{p}(a)(v(0,x+1) - v(0,x)). \tag{3.8}$$

which is non-negative. This implies the following inequality.

$$v(1,x+1) = \max_{a \in A} w(1,x+1;a) \geq \max_{a \in A} w(1,x;a) = v(1,x).$$

The induction assumption is $v(n,x+1) \geq v(n,x)\ \forall\ x$. We now show that $v(n+1,x+1) \geq v(n+1,x)$. From the induction assumption and the definition of $w(n,x;a)$ we can conclude that $w(n+1,x+1;a) \geq w(n+1,x;a)$. This concludes the induction because,

$$v(n+1,x+1) = \max_{a\in A} w(n+1,x+1;a) \geq \max_{a\in A} w(n+1,x;a) = v(n+1,x).$$

We next prove Ineq. (3.7) by contradiction. We assume that $v(n,x) > v(n+1,x)$ and prove that it produces a contradiction. The previous inequality implies that $v(n,x) > w(n+1,x;a^*_{n,x})$. Since $v(n+1,x)$ is a convex combination of $v(n,x)$ and $v(n,x+1) - a^*_{n+1,x}$ the last inequality will imply $v(n,x) > v(n,x+1) - a^*_{n+1,x}$, which in turn implies that $w(n+1,x;a_0) \geq v(n+1,x)$. Since $v(n,x)$ is the optimality value function, the last inequality must imply $a^*_{n+1,x} = a_0$ which in turn implies that $v(n,x) > v(n,x+1)$ which contradicts Ineq. (3.6). □

Theorem 3.2.2. *Under assumptions A and B the following relationships hold.*

$$a^*_{n,x} \geq a^*_{n,x+1} \qquad \forall\, n \geq 0, \tag{3.9}$$

$$a^*_{n,x} \geq a^*_{n+1,x} \qquad \forall\, n > 0. \tag{3.10}$$

Proof. The proof of Ineq. (3.9) is by induction on n. For $n = 0$ the inequality is obviously true because $a^*_{0,x} = a_0,\ \forall\, x$. For $n = 1$ to prove that $a^*_{1,x} \geq a^*_{1,x+1}$ we assume $a^*_{1,x} < a^*_{1,x+1}$ and prove that it produces a contradiction. Since $a^*_{1,x}$ is the optimal action in state $(1,x)$ and $a^*_{1,x+1}$ is the optimal action in state $(1,x+1)$ we have that $v(1,x) > w(1,x;a^*_{1,x+1})$ and $v(1,x+1) >$

$w(1, x+1; a^*_{1,x})$. Simplifying the inequalities and combining the results we obtain

$$v(0, x+1) - v(0, x) < v(0, x+2) - v(0, x+1).$$

The above inequality simplifies to $r(0, x+1) - r(0, x) < r(0, x+2) - r(0, x+1)$, which from the definition of $r(0, x)$ further simplifies to

$$(r-s)p_D(x+1) + \sum_{d=x+1}^{\infty} p_D(d)[(\delta(d-x) - \delta(d-x-1)) - (\delta(d-x-1) - \delta(d-x-2))] < 0.$$

The above inequality is a contradicts assumption D since we also assumed that $r > s$. To complete the induction we assume that $a^*_{n-1,x} \geq a^*_{n-1,x+1}$ and prove that $a^*_{n,x} \geq a^*_{n,x+1}$. To prove this we assume that $a^*_{n,x} < a^*_{n,x+1}$ and prove that this produces a contradiction.

Since $a^*_{n,x}$ is the optimal action in state (n, x) and $a^*_{n,x+1}$ is the optimal action in state $(n, x+1)$ we have $v(n, x) \geq w(n, x; a^*_{n,x+1})$ and $v(n, x+1) \geq w(n, x+1; a^*_{n,x})$. Simplifying these inequalities and combining the results we obtain the following.

$$v(n-1, x+1) - v(n-1, x) < v(n-1, x+2) - v(n-1, x+1). \tag{3.11}$$

From the definition of $v(n, x)$ the above inequality implies the following.

$$w(n-1, x+1; a^*_{n-1,x}) - v(n-1, x) < v(n-1, x+2) - w(n-1, x+1; a^*_{n-1,x+2}). \tag{3.12}$$

Note that the induction assumption implies the following inequality.

$$v(n-2,x+2) - v(n-2,x+1) < v(n-2,x+1) - v(n-2,x). \quad (3.13)$$

Using Ineq. (3.13) and lemma 3.2.1, Ineq. (3.12) simplifies to

$$v(n-2,x+2) - v(n-2,x+1) > v(n-2,x+1) - v(n-2,x) \quad (3.14)$$

which contradicts Ineq. (3.13).

The proof of Ineq. (3.10) is also by induction on n. For $n = 1$ to prove that $a^*_{1,x} \geq a^*_{2,x}$, we assume that $a^*_{1,x} < a^*_{2,x}$ and show that it produces a contradiction. From the definitions of $v(1,x)$ and $v(2,x)$ we know that $v(1,x) > w(1,x;a^*_{2,x})$ and $w(2,x;a^*_{2,x}) > w(2,x;a^*_{1,x})$. Simplifying these inequalities and combining the results we obtain $v(1,x+1) - v(1,x) > v(0,x+1) - v(0,x)$. The last inequality implies that $v(1,x+1) - w(1,x;a^*_{1,x+1}) < v(0,x+1) - v(0,x)$, which from the definition of $w(n,x;a)$ and Ineq. (3.7) simplifies to $v(0,x+1) - v(0,x) > v(0,x+1) - v(0,x)$, a contradiction as both sides of a strict inequality are identical. Now we assume that $a^*_{n-1,x} \geq a^*_{n,x}$ and prove that $a^*_{n,x} \geq a^*_{n+1,x}$. To prove this we assume that $a^*_{n+1,x} > a^*_{n,x}$ and show that this produces a contradiction. From the definitions of $v(n,x)$ and $v(n+1,x)$ we have $v(n,x) > w(n,x;a^*_{n+1,x})$ and $v(n+1,x) > w(n+1,x;a^*_{n,x})$.

Simplifying the above inequalities and combining the results gives the following inequality.

$$v(n-1,x) + v(n,x+1) > v(n,x) + v(n-1,x+1).$$

From the definition of $w(n, x; a)$ the above inequality implies that

$$v(n-1, x) + v(n, x+1) > w(n, x; a^*_{n,x+1}) + v(n-1, x+1).$$

From the definition of $v(n, x+1)$ and $w(n, x; a)$ the above inequality simplifies to the following inequality

$$v(n-1, x+1) - v(n-1, x) > v(n-1, x+1) - v(n-1, x),$$

which is a contradiction because both sides of the strict inequality are identical. □

3.2.3 Varying Number of Opponents Case

In this section we consider the problem where the number of opponents may change with each auction as described in the section 3.2.1.

Recall that the dynamic programming equations are as follows.

$$v(n, m, x) = \max_{a \in A} \{w(n, m, x; a)\} \tag{3.15}$$

where,

$$\begin{aligned} w(n, m, x; a) &= r_a(0, 0, x) + \beta \sum_{m=1}^{\infty} q_m(1) v(N, m, 0) \text{ if } n = 0, \\ &= r_a(n, m, x) + p_m(a) E(n-1, m, x+1) \\ &\quad + \bar{p}_m(a) E(n-1, m, x) \text{ if } n > 0. \end{aligned}$$

We next state and prove the following theorems 3.2.3 and 3.2.4.

Theorem 3.2.3. *Under assumptions A, B and C the following relationships hold.*

$$v(n,m,x) \leq v(n,m,x+1) \; \forall \; n \geq 0, \tag{3.16}$$

$$v(n,m,x) \leq v(n+1,m,x) \; \forall \; n \geq 0, \tag{3.17}$$

$$v(n,m,x) \geq v(n,m+1,x) \; \forall \; n > 0. \tag{3.18}$$

Proof. We first show that Ineq. (3.16) holds. For $n = 0$ the inequality $v(0,0,x) \leq v(0,0,x+1)$ is true from the definition of $v(0,0,x)$ and Lemma 3.2.1. For $n = 1$ we show that $v(1,m,x) \leq v(1,m,x+1)$ by contradiction. If we assume the contrary we have that $v(1,m,x) > v(1,m,x+1)$, which implies that $v(1,m,x) > w(1,m,x+1;a^*_{1,m,x})$. The last inequality simplifies to

$$p_m(a^*_{1,m,x})(v(0,0,x+2)-v(0,0,x+1))+\bar{p}_m(a^*_{1,m,x})(v(0,0,x+1)-v(0,0,x)) < 0,$$

which contradicts the previous step of this induction. Now, we assume that $v(n-1,m,x) \leq v(n-1,m,x+1) \; \forall x$ and prove that $v(n,m,x) \leq v(n,m,x+1)$. The induction assumption implies that $E(n-1,m,x) \leq E(n-1,m,x)$. From this fact and the definition of $w(n,m,x;a)$ we can conclude that $w(n,m,x;a) \leq w(n,m,x+1;a) \; \forall \; a$ which in turn implies the following:

$$v(n,m,x) = \max_{a \in A} w(n,m,x;a) \leq \max_{a \in A} w(n,m,x+1;a) = v(n,m,x+1).$$

This completes the induction.

Next, we prove that Ineq. (3.17) holds by induction on n. For $n = 0$ to show that the inequality $v(0,0,x) \leq v(1,m,x)$, holds we assume that $v(0,0,x) > v(1,m,x)$, and show that it produces a contradiction. Since $v(1,m,x)$ is a convex combination of $v(0,0,x)$ and $v(0,0,x+1)-a^*_{1,m,x}$. From the assumption that $v(0,0,x) > v(1,m,x)$ we can conclude that $v(0,0,x) > v(0,0,x+1) - a^*_{1,m,x}$, which in turn implies that $w(1,m,x;a_0) \geq v(1,m,x)$. The last inequality implies that $a^*_{1,m,x} = a_0$ which in turn implies that $v(0,0,x) > v(0,0,x+1)$ which contradicts Ineq. (3.16)

For $n = 1$ from the fact that $E(1,m,x) \geq v(0,0,x)$ for all x we can conclude $w(1,m,x;a) \leq w(2,m,x;a)$ which implies that

$$v(1,m,x) = \max_{a \in A} w(1,m,x;a) \leq \max_{a \in A} w(2,m,x;a) = v(2,m,x).$$

The induction assumption in this case will be $v(n-1,m,x) \leq v(n,m,x)$, using which we prove that $v(n,m,x) \leq v(n+1,m,x)$. From the induction assumption and it follows that $E(n-1,m,x) \leq E(n,m,x)$. From this fact and the definition of $w(n,m,x;a)$ we can conclude that $w(n-1,m,x;a) \leq w(n,m,x;a)\ \forall\ a$. The last inequality implies that

$$v(n,m,x) = \max_{a \in A} w(n,m,x;a) \leq \max_{a \in A} w(n-1,m,x;a) = v(n-1,m,x),$$

which completes the induction.

We now show that Ineq. (3.18) holds. For $n = 1$ to prove that $v(1, m, x) \geq v(1, m+1, x)$ we assume $v(1, m, x) < v(1, m+1, x)$ and show that it produces a contradiction. From the definition of $w(n, m, x; a)$ the last inequality implies that $w(1, m, x; a^*_{1,m+1,x}) < v(1, m+1, x)$. Simplifying this inequality we obtain $v(0, 0, x+1) - a^*_{1,m,x+1} < v(0, 0, x)$. Since $v(1, m, x+1)$ is a convex combination of $v(0, 0, x+1) - a^*_{1,m,x+1}$ and $v(0, 0, x)$, the last inequality implies that $w(1, m, x+1; a_0) \geq v(1, m, x+1)$. The last inequality implies that $a^*_{1,m,x+1} = a_0$, which leads to $v(0, 0, x) > v(0, 0, x+1)$ which contradicts Ineq. (3.16).

We complete the induction of Ineq. (3.18) along similar lines. We assume that $v(n-1, m, x) \geq v(n-1, m+1, x)$ and prove that $v(n, m, x) \geq v(n, m+1, x)$. To prove the last inequality we assume that $v(n, m, x) < v(n, m+1, x)$ and proceed to show that this produces a contradiction. This assumption implies that $w(n, m, x; a^*_{n,m+1,x}) < v(n, m+1, x)$. Simplifying the previous inequality using assumption B leads to $E(n, m, x+1) - a^*_{n,m,x+1} < E(n, m, x)$. From the definitions of $v(n+1, m, x)$ the last inequality implies that $w(n+1, m, x; a_0) \geq v(n+1, m, x)$, from which we can conclude that $a^*_{n,m,x+1} = a_0$. This fact implies that $E(n, m, x+1) < E(n, m, x)$, which is a contradiction.

□

Theorem 3.2.4. *Under assumption A, B and C the following relationships* *hold true for all* n, m *and* l.

$$a^*_{n,m,x} \geq a^*_{n,m,x+1} \quad \textit{for } n \geq 0, \tag{3.19}$$

$$a^*_{n,m,x} \geq a^*_{n+1,m,x} \quad \textit{for } n > 0, \tag{3.20}$$

$$a^*_{n,m,x} \leq a^*_{n,m+1,x} \quad \textit{for } n \geq 0. \tag{3.21}$$

Proof. We prove Ineq. (3.19) by induction on n. For $n = 0$ the inequality is obviously true because $a^*(0, 0, x) = a_0$ for all x. For $n = 1$, we show that the inequality $a^*_{1,m,x} \geq a^*_{1,m,x+1}$ holds by contradiction. We assume that $a^*_{1,m,x} < a^*_{1,m,x+1}$ and show that it leads to a contradiction. We know that $v(1, m, x) > w(1, m, x; a^*_{1,m,x+1})$ and $v(1, m, x - 1) > w(1, m, x - 1; a^*_{1,m,x})$. Simplifying these inequalities and combining the results we obtain

$$v(0, 0, x + 1) - v(0, 0, x) < v(0, 0, x + 2) - v(0, 0, x + 1).$$

The above inequality simplifies to $r(0, 0, x + 1) - r(0, 0, x) < r(0, 0, x + 2) - r(0, 0, x + 1)$, which further simplifies to

$$(r-s)p_D(d)+\sum_{d=x+1}^{\infty} p_D(d)[(\delta(d-x)-\delta(d-x-1))-(\delta(d-x-1)-\delta(d-x-2))] < 0.$$

This contradicts assumption D because we have also assumed $r > s$.

To complete the induction of Ineq. (3.19) we assume that $a^*_{n-1,m,x} \geq a^*_{n-1,m,x+1}$ and prove that $a^*_{n,m,x} \geq a^*_{n,m,x+1}$. To prove last inequality we assume that $a^*_{n,m,x+1} > a^*_{n,m,x}$ and show that this produces a contradiction.

From the definitions of $v(n,m,x)$ and $w(n,m,x;a)$ we have $v(n,m,x) < w(n,m,x;a^*_{n,m,x+1})$ and $v(n,m,x+1) < w(n,m,x+1;a^*_{n,m,x})$. Simplifying and combining the results of the last two inequalities we obtain

$$E(n-1,m,x+1) - E(n-1,m,x) > E(n-1,m,x+2) - E(n-1,m,x+1).$$

This can be rewritten as

$$\sum_{m'} q_{mm'}[2v(n-1,m',x+1) - v(n-1,m',x) - v(n-1,m',x+2)] > 0.$$

The above inequality implies the following:

$$\begin{aligned} & \sum_{m'} q_{mm'}[v(n-1,m',x+1) - w(n-1,m',x;a^*_{n-1,m',x+1})] \\ > & \sum_{m'} q_{mm'}[w(n-1,m',x+2;a^*_{n-1,m',x+1}) - v(n-1,m',x+1)] \end{aligned} \tag{3.22}$$

Notice that the induction assumption implies the following for all $m \geq 1$.

$$E(n-2,m,x) + E(n-2,m,x+2) > 2E(n-2,m,x+1). \tag{3.23}$$

Simplifying Ineq. (3.22) using lemma 3.2.1, assumption C and Ineq. (3.23) leads to the inequality

$$\begin{aligned} & \sum_{m'} q_{mm'}[E(n-2,m',x+1) - E(n-2,m',x+2)] \\ > & \sum_{m'} q_{mm'}[E(n-2,m',x+1) - E(n-2,m',x+2)] \end{aligned}$$

which is a contradiction because both sides of the strict inequality are identical.

The proof of Ineq. (3.20) is also by induction on n. For $n = 1$ we need to prove that $a^*_{1,m,x} \geq a^*_{2,m,x}$. To prove this we assume $a^*_{1,m,x} < a^*_{2,m,x}$ and show that it produces a contradiction. We know that $v(1, m, x) > w(1, m, x; a^*_{2,m,x})$ and $v(2, m, x) > w(2, m, x; a^*_{1,m,x})$. Simplifying the inequalities and combining the results leads to $v(0, 0, x+1) - v(0, 0, x) < E(1, m, x+1) - E(1, m, x)$ or equivalently

$$v(0, 0, x + 1) - v(0, 0, x) < \sum_{m'} q_{mm'}[v(1, m', x + 1) - v(1, m', x)].$$

The above inequality implies that

$$v(0, 0, x + 1) - v(0, 0, x) < \sum_{m'} q_{mm'}[v(1, m', x + 1) - w(1, m', x; a^*_{1,m',x+1})].$$

The above inequality simplifies to

$$v(0, 0, x + 1) - v(0, 0, x) < v(0, 0, x + 1) - v(0, 0, x)$$

which is a contradiction. For the next step in the induction we assume that $a^*_{n-1,m,x} \leq a^*_{n,m,x}$ and prove that $a^*_{n,m,x+1} \leq a^*_{n+1,m,x}$. To prove the last inequality we assume that $a^*_{n,m,x+1} > a^*_{n+1,m,x}$ and show that it produces a contradiction. From the definitions of $v(n, m, x)$ and $w(n, m, x; a)$ we have $v(n, m, x) > w(n, m, x; a^*_{n+1,m,x})$ and $v(n + 1, m, x) > w(n + 1, m, x; a^*_{n,m,x})$. Simplifying and combining the results of the last two inequalities we obtain

$$E(n, m, x + 1) - E(n, m, x) < E(n - 1, m, x + 1) - E(n - 1, m, x).$$

The above inequality is equivalent to the following inequality.

$$\sum_{m'} q_{mm'}[v(n,m',x+1)-v(n,m',x)] < \sum_{m'} q_{mm'}[v(n-1,m',x+1)-v(n-1,m',x)].$$

The above inequality implies the following:

$$\begin{aligned} & \sum_{m'} q_{mm'}[w(n,m',x+1;a_1^*)-v(n,m',x)] \\ < & \sum_{m'} q_{mm'}[v(n-1,m',x+1)-w(n-1,m',x;a_2^*)]. \end{aligned}$$

where $a_1^* = a^*_{n,m',x}$ and $a_2^* = a^*_{n-1,m',x+1}$.

Simplifying the above inequality using assumption C and the induction assumption leads to the inequality

$$\begin{aligned} & \sum_{m'} q_{mm'}[E(n-2,m',x+1)-E(n-2,m',x)] \\ < & \sum_{m'} q_{mm'}[E(n-2,m',x+1)-E(n-2,m',x)], \end{aligned}$$

which is a contradiction because both sides of the strict inequality are identical.

The proof of Ineq. (3.21) is also by induction on n. For $n = 0$ the inequality is obviously true because $a^*_{0,0,x} = a_0$ for all x. For $n = 1$ we have to prove $a^*_{1,m,x} \le a^*_{1,m+1,x}$. We assume that $a^*_{1,m,x} > a^*_{1,m+1,x}$ and show that it leads to a contradiction. We know that $v(1,m,x) > w(1,m,x;a^*_{1,m+1,x})$ and $v(1,m+1,x) > w(1,m+1,x;a^*_{1,m,x})$. Simplifying the inequalities we obtain $v(0,0,x+1)-v(0,0,x) > T_{1,m}$ and $v(0,0,x+1)-v(0,0,x) < T_{1,m+1}$ where,

$$T_{n,m} = \frac{a^*_{n,m,x}p_m(a^*_{n,m,x}) - a^*_{n,m+1,x}p_m(a^*_{n,m+1,x})}{p_m(a^*_{n,m,x}) - p_m(a^*_{n,m+1,x})}.$$

This is a contradiction because from assumption B we have $T_{1,m} \geq T_{1,m+1}$ because we assumed $a^*_{1,m,x} > a^*_{1,m+1,x}$.

To complete the induction of we assume that $a^*_{n-1,m+1,x} \geq a^*_{n-1,m,x}$ and prove that $a^*_{n,m+1,x} \geq a^*_{n,m,x}$. To prove this part we assume that $a^*_{n,m+1,x} < a^*_{n,m,x}$ and show that this produces a contradiction. Since $a^*_{n,m,x}$ is the optimal action in state (n, m, x) we have that $v(n, m, x) < w(n, m, x, a^*_{n,m+1,x})$. This simplifies to:

$$E(n-1, m, x+1) - E(n-1, m, x) > T_{n,m}. \tag{3.24}$$

Similarly in state $(n, m+1, x)$ we have $v(n, m+1, x) < w(n, m+1, x, a^*_{n,m,x})$ which simplifies to

$$E(n-1, m+1, x+1) - E(n-1, m+1, x) < T_{n,m+1}. \tag{3.25}$$

Note that

$$T_{n,m+1} < T_{n,m}. \tag{3.26}$$

Indeed, from the definitions of T_m and T_{m+1} the last inequality simplifies to the following inequality

$$p_m(a^*_{n,m,x})p_m(a^*_{n,m-1,x}) < p_{m-1}(a^*_{n,m,x})p_{m-1}(a^*_{n,m-1,x}),$$

which is true under assumption B since we have assumed that $a^*_{n,m,x} > a^*_{n,m+1,x}$.

Inequalities (3.24), (3.25), (3.26) together imply that

$$E(n-1,m+1,x+1)-E(n-1,m+1,x) < E(n-1,m,x+1)-E(n-1,m,x).$$

This implies that

$$\begin{aligned} & \sum_{i=-m+1}^{\infty}\{G(i)[v(n-1,m+i+1,x+1)-v(n-1,m+i+1,x)].\} \\ & +G(-m)(v(n-1,1,x+1)-v(n-1,1,x)) \\ < & \sum_{i=-m+1}^{\infty}\{G(i)[v(n-1,m+i,x+1)-v(n-1,m+i,x)]\}. \end{aligned}$$

From Theorem 3.2.3 we have $v(n-1,1,x+1)-v(n-1,1,x) \geq 0$. Hence, the above inequality implies that

$$\begin{aligned} & \sum_{m'=1}^{\infty} q_{mm'}[v(n-1,m'+1,x+1)-v(n-1,m'+1,x)] \\ < & \sum_{m'=1}^{\infty} q_{mm'}[v(n-1,m',x+1)-v(n-1,m',x)]. \end{aligned}$$

From the above inequality we obtain the following.

$$\begin{aligned} & \sum_{m'=1}^{\infty} q_{mm'}[w(n-1,m'+1,x+1;a_1^*)-v(n-1,m',x+1)] \\ < & \sum_{m'=1}^{\infty} q_{mm'}[v(n-1,m'+1,x)-w(n-1,m',x;a_2^*)], \end{aligned} \tag{3.27}$$

where $a_1^* = a^*_{n-1,m',x+1}$ and $a_2^* = a^*_{n-1,m'+1,x}$. From assumption B we have $p_m(a) \geq p_{m+1}(a)$, so, let $p_m(a_1^*) = p_{m+1}(a_1^*)+\delta_1$ and $p_m(a_2^*) = p_{m+1}(a_2^*)+\delta_2$. Note that the induction assumption implies that

$$E(n-2,m+1,x+1)-E(n-2,m,x+1) > E(n-2,m+1,x)-E(n-2,m,x). \tag{3.28}$$

Simplifying Ineq. (3.27) using assumption C and Ineq. (3.28) leads to the inequality

$$\begin{aligned}
&\sum_{m'=1}^{\infty} q_{mm'}[E(n-2,m'+1,x+1)-E(n-2,m',x+1)] \\
&+\delta_1[-a_1^*+E(n-2,m'+1,x+1)-E(n-2,m'+1,x)] \\
&+\delta_2[-a_2^*+E(n-2,m',x+2)-E(n-2,m',x+1)] \\
< \; &\sum_{m'=1}^{\infty} q_{mm'}[E(n-2,m'+1,x+1)-E(n-2,m',x+1)].
\end{aligned}$$

From the proof of Ineq. (3.18) we know that the terms multiplying δ_1 and δ_2 are positive. Hence the above inequality implies

$$\begin{aligned}
&\sum_{m'=1}^{\infty} q_{mm'}[E(n-2,m'+1,x+1)-E(n-2,m',x+1)] \\
< \; &\sum_{m'=1}^{\infty} \{q_{mm'}[E(n-2,m'+1,x+1)-E(n-2,m',x+1)].
\end{aligned}$$

which is a contradiction because both sides of the strict inequality are identical.

□

3.3 Inventory Case

3.3.1 Model Definition

In this section we consider the version of the problem where any items that are unsold at the end of a period are carried over as inventory to the next period with an inventory carrying cost of h units per item per period. We model this problem as a Markov Decision process.

1. The state space $\mathcal{S}$ in this case is the set $\{(n, m, x),\ n = 0, \ldots, N,\ m = 1, \ldots,\ x = 0, 1, \ldots\}$, where n represents the number of auctions remaining during the current epoch, m represents the number of bidders participating in the current auction, $x \geq 0$ represents the inventory level at the beginning of the current $(N - n)$ auction. Note that:

    - If $n = 0$ then $m = 0$.
    - State $(0, 0, x)$ represents the state of the system at the end of an epoch when all auctions are over.
    - Possible states at the beginning of an epoch, prior to the start of the N auctions, are of the form (N, m, x), for all $m = 1, \ldots$ and $x = 0, 1, \ldots$.

2. In any state (n, m, x) the following action sets $A(n, m, x)$ are available.

    - $A(0, 0, x) = \{a_0\}$.
    - $A(n, m, x) = \{a_0, \ldots, a_p\}$ for $n > 0$.

3. When an action $a \in A(n, m, x)$ is taken in state (n, m, x) the following transitions are possible.

    - If $n = 0$, then starting from state $(0, 0, x)$ the next state is $(N, m, (x-$

$d)^+)$ with probability $q_m(1)p_D(d)$, if $x > d$ and $q_m(1)\bar{P}_D(d)$, otherwise, where $d = 0, 1, \ldots$.

- If $n > 0$ then depending on whether or not the buyer wins the current auction the next state is $(n-1, m', x+1)$ with probability $p_m(a)\, q_{mm'}(N-n)$ or state $(n-1, m', x)$ with $\bar{p}_m(a)\, q_{mm'}(N-n)$.

4. When an action $a \in A(n, m, x)$ is taken in state (n, m, x) the expected reward $r_a(n, m, x)$ is as follows.

$$r_a(n,m,x) = \begin{cases} \sum_{d=0}^{\infty}(r(d \wedge x) - h(x-d)^+ - \delta(d-x)^+)\, p_D(d) & \text{if } n = 0, \\ \\ -a\, p_m(a) & \text{if } n > 0. \end{cases}$$

where $d \wedge x = \min\{d, x\}$ and $d \vee x = \max\{d, x\}$.

Let $a^*_{n,m,x}$ denote the optimal action in the state (n, m, x). Let $v(n, m, x)$ denote the value function in state (n, m, x) and $w(n, m, x; a)$ denote the expected future reward when action a is taken in state (n, m, x) and an optimal policy is followed thereafter. Note that $v(n, m, x) = w(n, m, x; a^*_{n,m,x})$.

The dynamic programming equations are

$$v(n, m, x) = \max_{a \in A}\{w(n, m, x; a)\} \tag{3.29}$$

where,

$$
\begin{aligned}
w(n,m,x;a) &= r(0,0,x) + \beta \sum_{d=0}^{x} E_1(N,m,(x-d)\vee 0)p_D(d) \text{ if } n=0 \\
&= r_a(n,m,x) + p_m(a)E(n-1,m,x+1) \\
&\quad + \bar{p}_m(a)E(n-1,m,x) \text{ if } n>0,
\end{aligned}
$$

$E(n-1,m,x) = \sum_{m'=1}^{\infty} q_{mm'}(N-n)v(n-1,m',x)$, $E_1(N,m,x) = \sum_{m=1}^{\infty} q_m(1)v(N,m,x)$ and β is the discount factor.

The above dynamic programming equations can be solved to develop a bidding strategy for a general setting. We now consider structural properties of the optimal policies for some interesting cases.

In the following sections we consider the above problem with a fixed inventory capacity. Let x_* represent the fixed inventory capacity. We make the following assumption about x_*.

Assumption E. $P(D \le x_*) \le \frac{r}{r+h}$

Remark 1: Assumption E is not very restrictive because in most situations $h << r$.

Remark 2: Assumption E can be replaced with the less restrictive assumption

$$\sum_{d=x+1}^{\infty} (r + \delta(d-x) - \delta(d-x-1))p_D(d) \ge \sum_{d=0}^{x} hp_D(d),$$

but this assumption may be more difficult to verify in practice.

Lemma 3.3.1. *Under assumption E the expected reward function in state* $(0,0,x)$, $r_a(0,0,x)$ *is an increasing function of* x *i.e.*

$$r_a(0,0,x) \leq r_a(0,0,x+1). \tag{3.30}$$

Proof. The proof is evident from the fact that the difference $r(0,0,x+1) - r(0,0,x)$ can be simplified to

$$rP_D(x+1) - h\bar{P}_D(x+1) + \sum_{d=x+1}^{\infty} (\delta(d-x) - \delta(d-x-1))p_D(d),$$

which under assumption E is non-negative. $\square$

3.3.2 Constant Number of Opponents with Fixed Inventory Capacity.

In this section we consider the problem where there is a constant number of opponents, $m^0 \geq 1$, in all auctions. The state space is the set $\{(n, m^0, x)\}$ where $n = 1 \ldots N$ and $x = 0 \ldots x_*$. The action sets are $A(0,0,x) = \{a_0\}$, $A(n, m^0, x_*) = \{a_0\}$ and $A(n, m^0, x) = \{a_0, \ldots, a_p\}$ otherwise. Note that $q_{mm'}(n) = 1$ if $m = m' = m^0$ and 0 otherwise.

During any auction the buyer makes his decision on how much to bid based on the number of auctions remaining and the number of items already acquired. We obtain a simplified MDP where the state (n, x) represents the maximum possible number of remaining auctions and the number of items already acquired. The action sets are $A(0, x) = \{a_0\}$, $A(n, x_*) = \{a_0\}$ and $A(n, x) = \{a_0, \ldots, a_p\}$. The transition probabilities and expected rewards also simplify analogously and the dynamic programming equations in Eq. (3.29) reduce to the following.

$$v(n,x) = \max_{a\in A}\{w(n,x;a)\} \tag{3.31}$$

where

$$\begin{aligned} w(n,x;a) &= r(0,x) + \beta \sum_{d=0}^{x} v(N,(x-d)\vee 0))p_D(d) \text{ if } n=0, \\ &= r_a(n,m,x) + p(a)v(n-1,m,x+1) \\ &\quad +\bar{p}(a)v(n-1,x) \text{ if } n>0 \text{ and } x<x_*, \\ &= v(n-1,x_*) \text{ if } n>0 \text{ and } x=x_*. \end{aligned}$$

Let $v^k(n,x)$ denote the kth step of the successive approximation process used to find the value function $v(n,x)$. The successive approximation equations can be written as follows.

$$v^k(n,x) = \max_{a\in A}\{w^k(n,x;a)\}$$

where

$$\begin{aligned} w^k(n,m,x;a) &= r(0,x) + \beta \sum_{d=0}^{x} v^{k-1}(N,(x-d)\vee 0))p_D(d) \text{ if } n=0, \\ &= r_a(n,x) + p(a)v^{k-1}(n-1,m,x+1) \\ &\quad +\bar{p}(a)v^{k-1}(n-1,x) \text{ if } n>0, \\ &= v^{k-1}(n-1,x_*) \text{ if } n>0 \text{ and } x=x_*. \end{aligned}$$

Let $a^*_{n,x}(k)$ denote the optimal action in state (n,x) during the kth step of the successive approximation process. Note that $\lim_{k\to\infty} v^k(n,x) = v(n,x)$ and $\lim_{k\to\infty} a^*_{n,x}(k) = a^*_{n,x}$.

We next state and prove the theorems 3.3.1 and 3.3.2.

Theorem 3.3.1. *Under assumptions A and D the following inequalities hold for all n and x.*

$$\begin{aligned} v(n,x) &\leq v(n,x+1) \text{ for } n \geq 0 \text{ and } x < x_*, \quad &(3.32)\\ v(n,x) &\leq v(n+1,x) \text{ for } n \geq 0 \text{ and } x \leq x_*. \quad &(3.33) \end{aligned}$$

Proof. We first prove Ineq. (3.32) by induction. We start with the functions $v^k(n,x)$. By definition $v^0(n,x)$ is an increasing function of x. For $k=1$, when $n=0$ we have $v^1(0,x) = r(0,x)$ which by Lemma 3.3.1 is an increasing function of x. For $n=1$ the difference $w^1(1,x+1;a) - w^1(1,x;a)$ simplifies to

$$p(a)(v^1(0,x+2) - v^1(0,x+1)) + \bar{p}(a)(v^1(0,x+1) - v^1(0,x)) \geq 0, \quad (3.34)$$

which implies that $v^1(1,x+1) \geq v^1(1,x)$. The argument proceeds by induction on n establishing $v^1(n,x+1) \geq v^1(n,x)$. Assuming that $v^{k-1}(n,x)$ is an increasing function of x we now show that $v^k(n,x)$ is an increasing function of x. For $n=0$ we have $v^k(0,x) = r(0,x) + \beta\sum_{d=0}^{\infty} v^{k-1}(N,m,(x-d)\vee 0)$ which by Lemma 3.3.1 and the induction assumption is an increasing function of

x. For $n = 1$ the difference $w^k(1, x+1; a) - w^k(1, x; a)$ simplifies to

$$p(a)(v^k(0, x+2) - v^k(0, x+1)) + \bar{p}(a)(v^k(0, x+1) - v^k(0, x)) \geq 0, \quad (3.35)$$

which implies that $v^k(1, x+1) \geq v^k(1, x)$. Proceeding using induction on n we establish $v^k(n, x+1) \geq v^k(n, x)$. Taking the limit as k goes to infinity we complete the proof of Ineq. (3.32).

We next prove Ineq. (3.33) by contradiction. We assume that $v(n, x) > v(n+1, x)$ and prove that it produces a contradiction. Note that $v(n+1, x)$ is a convex combination of $v(n, x)$ and $v(n, x+1) - a^*_{n+1,x}$. From the assumption that $v(n, x) > v(n+1, x)$ we can conclude that $v(n, x) > v(n, x+1) - a^*_{n+1,x}$. The last inequality implies that $w(n+1, x; a_0) \geq v(n+1, x)$. The last inequality implies that $a^*_{n+1,x} = a_0$ which in turn implies that $v(n, x) > v(n, x+1)$ which contradicts Ineq. (3.32). $\square$

Theorem 3.3.2. *Under assumptions A and D the following relationships hold.*

$$a^*_{n,x} \geq a^*_{n,x+1} \text{ for } n \geq 0 \text{ and } x < x_*, \quad (3.36)$$

$$a^*_{n,x} \geq a^*_{n+1,x} \text{ for } n > 0 \text{ and } x \leq x_*. \quad (3.37)$$

Proof. We first prove Ineq. (3.36). For $n = 0$ the inequality is obviously true because $a^*_{0,x} = a_0, \ \forall\, x$. For $n > 0$ the proof of Ineq. (3.36) is by induction on k using the functions $a^*_{n,x}(k)$. For $k = 0$ the inequality is obviously true as all actions are equivalent. For $k = 1$ to prove that $a^*_{1,x}(1) \geq a^*_{1,x+1}(1)$ assume $a^*_{1,x}(1) < a^*_{1,x+1}(1)$ and prove that it produces a contradiction. We know that $v^1(1, x) > w^1(1, x; a^*_{1,x+1}(1))$ and $v^k(1, x + 1) > w^k(1, x + 1; a^*_{1,x}(1))$. Simplifying the inequalities and combining the results we obtain

$$v^1(0, x + 1) - v^1(0, x) < v^1(0, x + 2) - v^1(0, x + 1).$$

The above inequality simplifies to $r(0, x + 1) - r(0, x) < r(0, x + 2) - r(0, x + 1)$, which from the definition of $r(0, x)$ further simplifies to

$$(r+h)p_D(x+1)+\sum_{d=x+1}^{\infty} p_D(d)[(\delta(d-x)-\delta(d-x-1))-(\delta(d-x-1)-\delta(d-x-2))] < 0,$$

which contradicts assumption D. The argument proceeds by induction on n establishing $a^*_{n,x}(1) \geq a^*_{n,x+1}(1)$. Now we assume that $a^*_{n,x}(k-1) \geq a^*_{n,x+1}(k-1)$ and prove that $a^*_{n,x}(k) \geq a^*_{n,x+1}(k)$. To prove this we assume that $a^*_{n,x}(k) < a^*_{n,x+1}(k)$ and prove that this produces a contradiction.

We know that $v^k(n, x) > w^k(n, x; a^*_{n,x+1}(k))$ and $v^k(n, x + 1) > w(n, x + 1; a^*_{n,x}(k))$. Simplifying these inequalities and combining the results we obtain the following.

$$v^{k-1}(n-1, x+1)-v^{k-1}(n-1, x) < v^{k-1}(n-1, x+2)-v^{k-1}(n-1, x+1). \quad (3.38)$$

From the definition of $v^k(n, x)$ the above inequality implies the following.

$$\begin{aligned} & v^{k-1}(n-1, x+1) - w^{k-1}(n-1, x; a^*_{n-1,x+1}(k-1)) \\ < \ & w^{k-1}(n-1, x+2; a^*_{n-1,x+1}(k-1)) - v^{k-1}(n-1, x+1). \quad (3.39) \end{aligned}$$

Note that the induction assumption implies the following inequality.

$$v^{k-2}(n-2,x+2)-v^{k-2}(n-2,x+1) > v^{k-2}(n-2,x+1)-v^{k-2}(n-2,x). \quad (3.40)$$

Using Ineq. (3.40) and lemma 3.3.1 Ineq. (3.39) simplifies to

$$v^{k-2}(n-2,x)-v^{k-2}(n-2,x+1) < v^{k-2}(n-2,x)-v^{k-2}(n-2,x+1) \quad (3.41)$$

which contradicts Ineq. (3.40). This proves that $a^*_{n,x}(k) \geq a^*_{n,x+1}(k)$ for all k. The proof is complete by taking limit as k goes to infinity of the last inequality.

The proof of Ineq. (3.37) is by induction on n. For $n = 1$ to prove that $a^*_{1,x} \geq a^*_{2,x}$, we assume that $a^*_{2,x} > a^*_{1,x}$ and show that it produces a contradiction. From the definitions of $v(1,x)$ and $v(2,x)$ we know that $v(1,x) > w(1,x;a^*_{2,x})$ and $v(2,x) > w(2,x;a^*_{1,x})$. Simplifying the above inequalities and combining the results we obtain $v(1,x+1) - v(1,x) > v(0,x+1) - v(0,x)$. The last inequality implies that $v(1,x+1) - w(1,x;a^*_{1,x+1}) > v(0,x+1) - v(0,x)$, which from the definition of $w(n,x;a)$ and Ineq. (3.33) simplifies to $v(0,x+1) - v(0,x) > v(0,x+1) - v(0,x)$ which is a contradiction. Now we assume that $a^*_{n-1,x} \geq a^*_{n,x}$ and prove that $a^*_{n,x} \geq a^*_{n+1,x}$. To prove this we assume that $a^*_{n,x} < a^*_{n+1,x}$ and show that this produces a contradiction. From the definitions of $v(n,x)$ and $v(n+1,x)$ we have $v(n,x) > w(n,x;a^*_{n+1,x})$ and $v(n+1,x) > w(n+1,x;a^*_{n,x})$.

Simplifying the above inequalities and combining the results gives the following inequality.

$$v(n-1, x+1) - v(n-1, x) < v(n, x+1) - v(n, x).$$

From the definition of $w(n, x; a)$ the above inequality implies that

$$v(n-1, x+) - v(n, x+1) > w(n, x; a^*_{n,x+1}) + v(n-1, x+1).$$

From the definition of $v(n, x+1)$ and $w(n, x; a)$ the above inequality simplifies to the following inequality

$$v(n-1, x+1) - v(n-1, x) < v(n-1, x+1) - v(n-1, x),$$

which is a contradiction because both sides of the strict inequality are identical. □

3.3.3 Varying Number of Opponents with Fixed Inventory Capacity.

In this section we consider the problem where the number of opponents may change with each auction as described in the section 3.3.1. We also assume that x_* represents the inventory capacity.

The dynamic programming equations are as follows.

$$v(n, m, x) = \max_{a \in A}\{w(n, m, x; a)\} \tag{3.42}$$

where,

$$\begin{aligned} w(n,m,x;a) &= r(0,0,x) + \beta \sum_{d=0}^{x} E_1(N,m,(x-d)\vee 0)p_D(d) \text{ if } n=0, \\ &= r_a(n,m,x) + p_m(a)E(n-1,m,x+1) \\ &\quad +\bar{p}_m(a)E(n-1,m,x) \text{ if } n>0 \text{ and } x<x_*, \\ &= E(n-1,m,x_*) \text{ if } x=x_*. \end{aligned}$$

Let $v^k(n,m,x)$ denote the kth step of the successive approximation process used to find the value function $v(n,m,x)$. Let $E^k(n,m,x) = \sum_{m'=1}^{\infty} q_{mm'}v^k(n,m',x)$ and $E_1^k(N,0,x) = \sum_{m'=1}^{\infty} q_{m'}(1)v^k(N,m',x)$. The successive approximation equations can be written as follows.

$$v^k(n,m,x) = \max_{a\in A}\{w^k(n,m,x;a)\}$$

where

$$\begin{aligned} w^k(n,m,x;a) &= r(0,0,x) + \beta \sum_{d=0}^{x} E_1^{k-1}(N,0,(x-d)\vee 0)p_D(d) \text{ if } n=0, \\ &= r_a(n,m,x) + p_m(a)E^{k-1}(n-1,m,x+1) \\ &\quad +\bar{p}_m(a)E^{k-1}(n-1,m,x) \text{ if } n>0 \text{ and } x<x_*, \\ &= E^{k-1}(n-1,m,x_*) \text{ if } x=x_*. \end{aligned}$$

Let $a^*_{n,m,x}(k)$ denote the optimal action in state (n,m,x) during the kth step of the successive approximation process. Note that $\lim_{k\to\infty} v^k(n,m,x) = v(n,m,x)$ and $\lim_{k\to\infty} a^*_{n,m,x}(k) = a^*_{n,m,x}$.

We now state and prove the theorems 3.3.3 and 3.3.4.

Theorem 3.3.3. *Under assumptions A, B, C and D the following relationships hold.*

$$v(n,m,x) \le v(n,m,x+1) \ \forall \ n \ge 0 \ \textit{and} \ x < x_*, \tag{3.43}$$

$$v(n,m,x) \le v(n+1,m,x) \ \forall \ n \ge 0 \ \textit{and} \ x \le x_*, \tag{3.44}$$

$$v(n,m,x) \ge v(n,m+1,x) \ \forall \ n \ge 0 \ \textit{and} \ x \le x_*. \tag{3.45}$$

Proof. We first show that Ineq. (3.43) holds. To prove this we use the functions $v^k(n,m,x)$. For $k = 0$ the inequality is obviously true because $v^0(n,m,x) = 0$. For $k = 1$ the inequality $v^1(0,0,x) \le v^1(0,0,x+1)$ is true from the definition of $v^1(0,0,x)$ and Lemma 3.3.1. For $n = 1$ we show that $v^1(1,m,x) \le v^1(1,m,x+1)$ by contradiction. If we assume the contrary we have $v^1(1,m,x) > v^1(1,m,x+1)$, which implies that $v^1(1,m,x) > w^1(1,m,x+1;a^*_{1,m,x})$. Simplifying the last inequality leads to

$$p_m(a^*_{1,m,x})(v(0,0,x+2)-v(0,0,x+1))+\bar{p}_m(a^*_{1,m,x})(v(0,0,x+1)-v(0,0,x)) < 0$$

which is a contradiction. The argument proceeds by induction on n and we obtain $v^1(n,m,x) \le v^1(n,m,x+1)$. Now, we assume that $v^{k-1}(n,m,x) \le v^{k-1}(n,m,x+1)$ and prove that $v^k(n,m,x) \le v^k(n,m,x+1)$. The induction assumption implies that $E^{k-1}(n,m,x) \le E^{k-1}(n,m,x)$. From this and the definition of $w(n,m,x;a)$ we can conclude that $w(n,m,x;a) \le w(n,m,x+$

$1; a)$ $\forall$ a which in turn implies that

$$v^k(n,m,x) = \max_{a \in A} w^k(n,m,x;a) \leq \max_{a \in A} w^k(n,m,x+1;a) = v^k(n,m,x+1).$$

This implies that $v^k(n,m,x) \leq v^k(n,m,x+1)$ for all k. Taking the limit as k goes to infinity completes the proof of Ineq. (3.43).

Next, we prove that Ineq. (3.44) holds by induction on n. For $n = 0$ to show that the inequality $v(0,0,x) \leq v(1,m,x)$, holds we assume that $v(0,0,x) > v(1,m,x)$, and show that it produces a contradiction. Since $v(1,m,x)$ is a convex combination of $v(0,0,x)$ and $v(0,0,x+1) - a^*_{1,m,x}$. From the assumption that $v(0,0,x) > v(1,m,x)$ we can conclude that $v(0,0,x) > v(0,0,x+1) - a^*_{1,m,x}$, which in turn implies that $w(1,m,x;a_0) \geq v(1,m,x)$. The last inequality implies that $a^*_{1,m,x} = a_0$ which further implies that $v(0,0,x) > v(0,0,x+1)$ which contradicts Ineq. (3.43).

For $n = 1$ from the fact that $E(1,m,x) \geq v(0,0,x)$ we can conclude that $w(1,m,x;a) \leq w(2,m,x;a)$ which implies that

$$v(1,m,x) = \max_{a \in A} w(1,m,x;a) \leq \max_{a \in A} w(2,m,x;a) = v(2,m,x).$$

The induction assumption in this case will be $v(n-1,m,x) \leq v(n,m,x)$, using which we prove that $v(n,m,x) \leq v(n+1,m,x)$. From the induction assumption and it follows that $E(n-1,m,x) \leq E(n,m,x)$. From this fact and the definition of $w(n,m,x;a)$ we can conclude that $w(n-1,m,x;a) \leq$

$w(n, m, x; a)\ \forall\ a$. The last inequality implies that

$$v(n, m, x) = \max_{a \in A} w(n, m, x; a) \leq \max_{a \in A} w(n-1, m, x; a) = v(n-1, m, x),$$

which completes the induction.

We now show that Ineq. (3.45) holds. For $n = 1$ to prove that $v(1, m, x) \geq v(1, m+1, x)$ we assume $v(1, m, x) < v(1, m+1, x)$ and show that it produces a contradiction. From the definition of $w(n, m, x; a)$ the last inequality implies that $v(1, m, x; a^*_{1,m+1,x}) < w(1, m+1, x)$. Simplifying this inequality we obtain $v(0, 0, x+1) - a^*_{1,m,x+1} < v(0, 0, x)$. Since $v(1, m, x+1)$ is a convex combination of $v(0, 0, x+1) - a^*_{1,m,x+1}$ and $v(0, 0, x)$, the last inequality implies that $w(1, m, x+1; a_0) \geq v(1, m, x+1)$ which further implies that $v(0, 0, x) > v(0, 0, x+1)$ which is a contradiction.

We complete the induction of Ineq. (3.45) along similar lines. We assume that $v(n, m, x) \geq v(n, m+1, x)$ and prove that $v(n, m, x) \geq v(n, m+1, x)$. To prove the last inequality we assume that $v(n, m, x) < v(n, m+1, x)$ and proceed to show that this produces a contradiction. This assumption implies that $w(n, m, x; a^*_{n,m+1,x}) < v(n, m+1, x)$. Simplifying the previous inequality leads to $E(n, m, x+1) - a^*_{n,m,x+1} < E(n, m, x)$. From the definitions of $v(n+1, m, x)$ the last inequality implies that $w(n+1, m, x; a_0) \geq v(n+1, m, x)$. From the last inequality we can conclude that $a^*_{n+1,m,x} = a_0$ which implies $E(n, m, x+1) < E(n, m, x)$, which is a contradiction. $\square$

Theorem 3.3.4. *Under assumption A, B, C and D the following relationships hold true for all n, m and x.*

$$a^*_{n,m,x} \geq a^*_{n,m,x+1} \quad \textit{for } n \geq 0, \tag{3.46}$$

$$a^*_{n,m,x} \geq a^*_{n+1,m,x} \quad \textit{for } n > 0, \tag{3.47}$$

$$a^*_{n,m,x} \leq a^*_{n,m+1,x} \quad \textit{for } n \geq 0. \tag{3.48}$$

Proof. We first prove Ineq. (3.46). For $n = 0$ the inequality is obviously true because $a^*_{0,0,x} = a_0,\ \forall x$. For $n > 0$ the proof of Ineq. (3.46) is by induction on k using the functions $a^*_{n,m,x}(k)$. For $k = 0$ the inequality is obviously true as all actions are equivalent. For $k = 1$ to prove that $a^*_{1,m,x}(1) \geq a^*_{1,m,x+1}(1)$ assume $a^*_{1,m,x}(1) < a^*_{1,m,x+1}(1)$ and prove that it produces a contradiction. We know that $v^1(1,m,x) \geq w^1(1,x;a^*_{1,m,x+1}(1))$ and $v^k(1,m,x+1) \geq w^k(1,x+1;a^*_{1,m,x}(1))$. Simplifying the inequalities and combining the results we obtain

$$v^1(0,0,x+1) - v^1(0,0,x) < v^1(0,0,x+2) - v^1(0,0,x+1).$$

The above inequality simplifies to $r(0,0,x+1) - r(0,0,x) < r(0,0,x+2) - r(0,0,x+1)$, which from the definition of $r(0,0,x)$ implies that

$$\sum_{d=x+1}^{\infty} p_D(d)[(\delta(d-x) - \delta(d-x-1)) - (\delta(d-x-1) - \delta(d-x-2))] < 0,$$

which contradicts assumption D. The argument proceeds by induction on n establishing $a^*_{n,m,x}(1) \geq a^*_{n,m,x+1}(1)$ Now we assume that $a^*_{n,m,x}(k-1) \geq$

$a^*_{n,m,x+1}(k-1)$ and prove that $a^*_{n,m,x}(k) \geq a^*_{n,m,x+1}(k)$. To prove this we assume that $a^*_{n,m,x}(k) < a^*_{n,m,x+1}$ and prove that this produces a contradiction.

We know that $v^k(n,m,x) \geq w^k(n,m,x;a^*_{n,m,x+1}(k))$ and $v^k(n,m,x+1) \geq w(n,m,x+1;a^*_{n,m,x}(k))$. Simplifying these inequalities and combining the results we obtain the following.

$$\begin{aligned} & E^{k-1}(n-1,m,x+1) - E^{k-1}(n-1,m,x) \\ < \; & E^{k-1}(n-1,m,x+2) - E^{k-1}(n-1,m,x+1). \end{aligned} \tag{3.49}$$

From the definition of $v^k(n,m,x)$ the above inequality implies the following.

$$\begin{aligned} & \sum_{m'} q_{mm'}[v^{k-1}(n-1,m',x+2) - w^{k-1}(n-1,m',x+1;a^*_{2m'})] \\ > \; & \sum_{m'} q_{mm'}[w^{k-1}(n-1,m',x+1;a^*_{1m'}) - v^{k-1}(n-1,m',x)]. \end{aligned} \tag{3.50}$$

where $a^*_{1m'} = a^*_{n-1,m',x+2}(k-1)$ and $a^*_{2m'} = a^*_{n-1,m',x}(k-1)$. Note that the induction assumption implies the following inequality.

$$\begin{aligned} & \sum_{m'} q_{mm'}[E^{k-2}(n-2,m',x+2) - E^{k-2}(n-2,m',x+1)] \\ > \; & \sum_{m'} q_{mm'}[E^{k-2}(n-2,m',x+1) - E^{k-2}(n-2,m',x)]. \end{aligned} \tag{3.51}$$

Using Ineq. (3.51) and lemma 3.3.1, Ineq. (3.50) simplifies to

$$\begin{aligned} & \sum_{m'} q_{mm'}[E^{k-2}(n-2,m',x+1) - E^{k-2}(n-2,m',x)] \\ > \; & \sum_{m'} q_{mm'}[E^{k-2}(n-2,m',x+2) - E^{k-2}(n-2,m',x+1)]. \end{aligned} \tag{3.52}$$

which contradicts Ineq. (3.51). This proves that $a^*_{n,m,x}(k) \geq a^*_{n,m,x+1}(k)$ for all k. The proof is complete by taking limit as k goes to infinity of the last inequality.

The proof of Ineq. (3.47) is also by induction on n. For $n = 1$ we need to prove that $a^*_{1,m,x} \geq a^*_{2,m,x}$. To prove this we assume $a^*_{1,m,x} < a^*_{2,m,x}$ and show that it produces a contradiction. We know that $v(1, m, x) > w(1, m, x; a^*_{2,m,x})$ and $v(2, m, x) > w(2, m, x; a^*_{1,m,x})$. Simplifying the inequalities and combining the results leads to $v(0, 0, x+1) - v(0, 0, x) < E(1, m, x+1) - E(1, m, x)$ or equivalently

$$v(0, 0, x + 1) - v(0, 0, x) < \sum_{m'} q_{mm'}[v(1, m', x + 1) - v(1, m', x)].$$

The above inequality implies that

$$v(0, 0, x + 1) - v(0, 0, x) < \sum_{m'} q_{mm'}[v(1, m', x + 1) - w(1, m', x; a^*_{1,m',x+1})].$$

The above inequality simplifies to

$$v(0, 0, x + 1) - v(0, 0, x) > v(0, 0, x + 1) - v(0, 0, x)$$

which is a contradiction. For the next step in the induction we assume that $a^*_{n-1,m,x} \geq a^*_{n,m,x}$ and prove that $a^*_{n,m,x+1} \geq a^*_{n+1,m,x}$. To prove the last inequality we assume that $a^*_{n+1,m,x+1} > a^*_{n,m,x}$ and show that it produces a contradiction. From the definitions of $v(n, m, x)$ and $w(n, m, x; a)$ we have

$v(n,m,x) < w(n,m,x;a^*_{n+1,m,x})$ and $v(n+1,m,x) < w(n+1,m,x;a^*_{n,m,x})$.

Simplifying and combining the results of the last two inequalities we obtain

$$E(n,m,x+1) - E(n,m,x) > E(n-1,m,x+1) - E(n-1,m,x).$$

The above inequality is equivalent to the following inequality.

$$\sum_{m'} q_{mm'}[v(n,m',x+1) - v(n,m',x)] > \sum_{m'} q_{mm'}[v(n-1,m',x+1) - v(n-1,m',x)].$$

The above inequality implies that

$$\begin{aligned} & \sum_{m'} q_{mm'}[v(n,m',x+1) - w(n,m',x;a^*_1)] \\ > & \sum_{m'} q_{mm'}[w(n-1,m',x+1;a^*_2) - v(n-1,m',x)]. \end{aligned}$$

where $a^*_1 = a^*_{n,m',x+1}$ and $a^*_2 = a^*_{n-1,m',x}$.

Simplifying the above inequality using assumption C and the induction assumption leads to the inequality

$$\begin{aligned} & \sum_{m'} q_{mm'}[E(n-2,m',x+1) - E(n-2,m',x)] \\ > & \sum_{m'} q_{mm'}[E(n-2,m',x+1) - E(n-2,m',x)] \end{aligned}$$

which is a contradiction because both sides of the strict inequality are identical.

The proof of Ineq. (3.48) is also by induction on n. For $n = 0$ the inequality is obviously true because $a^*_{0,0,x} = a_0$ for all x. For $n = 1$ we have to prove $a^*_{1,m,x} \leq a^*_{1,m+1,x}$. We assume that $a^*_{1,m,x} > a^*_{1,m+1,x}$ and show that it leads to a contradiction. We know that $v(1,m,x) > w(1,m,x;a^*_{1,m+1,x})$ and $v(1,m+1,x) > w(1,m+1,x;a^*_{1,m,x})$. Simplifying the inequalities we obtain $v(0,0,x+1) - v(0,0,x) > T_{1,m}$ and $v(0,0,x+1) - v(0,0,x) < T_{1,m+1}$ where,

$$T_{n,m} = \frac{a^*_{n,m,x} p_m(a^*_{n,m,x}) - a^*_{n,m+1,x} p_m(a^*_{n,m+1,x})}{p_m(a^*_{n,m,x}) - p_m(a^*_{n,m+1,x})}.$$

This is a contradiction because from assumption B we have $T_{1,m} \geq T_{1,m+1}$ because we assumed $a^*_{1,m,x} > a^*_{1,m+1,x}$.

To complete the induction of Ineq. (3.48) we assume that $a^*_{n-1,m+1,x} \geq a^*_{n-1,m,x}$ and prove that $a^*_{n,m+1,x} \geq a^*_{n,m,x}$. To prove this part we assume that $a^*_{n,m,x} > a^*_{n,m+1,x}$ and show that this produces a contradiction. Since $a^*_{n,m,x}$ is the optimal action in state (n,m,x) we have that $v(n,m,x) < w(n,m,x,a^*_{n,m+1,x})$. This simplifies to:

$$E(n-1,m,x+1) - E(n-1,m,x) > T_{n,m}. \tag{3.53}$$

Similarly in state $(n,m+1,x)$ we have $v(n,m+1,x) < w(n,m+1,x,a^*_{n,m,x})$ which simplifies to

$$E(n-1,m+1,x+1) - E(n-1,m+1,x) < T_{n,m+1}. \tag{3.54}$$

Note that

$$T_{n,m+1} < T_{n,m}. \tag{3.55}$$

Indeed, from the definitions of T_m and T_{m+1} the last inequality simplifies to the following inequality

$$p_m(a^*_{n,m,x})p_m(a^*_{n,m-1,x}) < p_{m-1}(a^*_{n,m,x})p_{m-1}(a^*_{n,m-1,x}),$$

which is true under assumption B since we have assumed that $a^*_{n,m,x} > a^*_{n,m+1,x}$.

Inequalities (3.53), (3.54), (3.55) together imply that

$$E(n-1,m+1,x+1)-E(n-1,m+1,x) < E(n-1,m,x+1)-E(n-1,m,x).$$

This implies that

$$\begin{aligned} & \sum_{i=-m+1}^{\infty}\{G(i)[v(n-1,m+i+1,x+1)-v(n-1,m+i+1,x)].\} \\ & +G(-m)(v(n-1,1,x+1)-v(n-1,1,x)) \\ < & \sum_{i=-m+1}^{\infty}\{G(i)[v(n-1,m+i,x+1)-v(n-1,m+i,x)]\} \end{aligned}$$

From Theorem 3.3.3 we have $v(n-1,1,x+1)-v(n-1,1,x) \geq 0$. Hence, the above inequality implies that

$$\begin{aligned} & \sum_{m'=1}^{\infty} q_{mm'}[v(n-1,m'+1,x+1)-v(n-1,m'+1,x)] \\ < & \sum_{m'=1}^{\infty} q_{mm'}[v(n-1,m',x+1)-v(n-1,m',x)]. \end{aligned}$$

This inequality implies the following.

$$\begin{aligned} & \sum_{m'=1}^{\infty} q_{mm'}[w(n-1,m'+1,x+1;a_1^*)-v(n-1,m',x+1)] \\ < & \sum_{m'=1}^{\infty} q_{mm'}[v(n-1,m'+1,x)-w(n-1,m',x;a_2^*)] \end{aligned} \tag{3.56}$$

where $a_1^* = a^*_{n-1,m',x+1}$ and $a_2^* = a^*_{n-1,m'+1,x}$. From assumption B we have $p_m(a) \geq p_{m+1}(a)$, so, let $p_m(a_1^*) \geq p_{m+1}(a_1^*) + \delta_1$ and $p_m(a_2^*) \geq p_{m+1}(a_2^*) + \delta_2$. Note that the induction assumption implies that

$$E(n-2, m+1, x+1) - E(n-2, m, x+1) > E(n-2, m+1, x) - E(n-2, m, x). \tag{3.57}$$

Simplifying Ineq. (3.56) using assumption C and Ineq. (3.57) leads to the inequality

$$\begin{aligned}
& \sum_{m'=1}^{\infty} q_{mm'}[E(n-2, m'+1, x+1) - E(n-2, m', x+1)] \\
& +\delta_1[-a_1^* + E(n-2, m'+1, x+1) - E(n-2, m'+1, x)] \\
& +\delta_2[-a_2^* + E(n-2, m', x+2) - E(n-2, m', x+1)] \\
< & \sum_{m'=1}^{\infty} q_{mm'}[E(n-2, m'+1, x+1) - E(n-2, m', x+1)].
\end{aligned}$$

From the proof of Ineq. (3.45) we know that the terms multiplying δ_1 and δ_2 are positive. Hence the above inequality implies

$$\begin{aligned}
& \sum_{m'=1}^{\infty} q_{mm'}[E(n-2, m'+1, x+1) - E(n-2, m', x+1)] \\
< & \sum_{m'=1}^{\infty} \{q_{mm'}[E(n-2, m'+1, x+1) - E(n-2, m', x+1)].
\end{aligned}$$

which is a contradiction because both sides of the strict inequality are identical.

□

3.4 Conclusions

In this chapter we considered the problem the problem of "the bidder" who in each period, of an infinite time horizon, buys items in auctions and sells acquired items in a secondary market. We formulated the problem as a Markov Decision process and proved that, under a few assumptions, the optimal value function and the optimal are decreasing functions of the number of remaining auctions, increasing functions of the number of opponents and decreasing functions of the inventory on hand.

This model can be extended to the case where the items are sold not at the end of the time period but also after each auction. We are currently working on this model and expect to finish it soon. The condition that the bid distribution is constant through all the auctions can be relaxed and a model where the distribution is updated using a Bayesian framework can also be constructed and studied.

Chapter 4

Optimizing Taboo Criteria in Markov Decision Processes

4.1 Introduction

The optimization of Markovian systems is often based on costs, or rewards, associated with the states of the system. However, in many applied problems it may be difficult or even impossible to determine costs, or rewards, associated with some of the states. We consider two cases. In the first case rewards and costs are known for some states only. For this case we show that one can define a taboo first passage reward which can be used as an optimization criterion. The first example shows how this can be achieved in the context of an inventory control problem, similar to the models discussed in [28], [24] and [34]. In the second case none of the costs or rewards associated with the states are known. In this situation a meaningful optimization criterion is to maximize the taboo mean return time to a fixed state. The second example models such a case in the context of the classical replacement problem, [8]. Taboo states represent undesirable states for the system see examples below and also [26].

We show that computing policies that maximize expected taboo first return rewards and mean taboo return times is in general a hard problem, for which well known methods from MDP theory can not be applied. However, it is shown herein that if certain monotonicity properties are satisfied then efficient computation of an optimal deterministic policy is possible.

4.1.1 Notation and definitions

We will use the following notation. Let X_n denote the state of the system at transition n and let $\mathcal{S} = \{i = 1, 2, \ldots, S\}$ denote the finite state space of the system. For each $i \in \mathcal{S}$, there is a finite set A_i of actions available. When the system is in state i and action $a \in A_i$ is taken the system moves to state $j \in \mathcal{S}$ with probability $p_{ij}(a)$. For a detailed classification of policies we refer to [9, 32]. In what follows we use the notation $\pi(i)$ to denote the unique action assigned by a deterministic policy π in state i.

Let G and B be two disjoint subsets of $\mathcal{S}$, where the former represents a set of target (good) states while the later represents a set of taboo (bad) states and let C denote the set of remaining states of $\mathcal{S}$. We will assume that all of these sets are nonempty. Let Bj denote the set $B \cup \{j\}$ and let BG denote the set $B \cup G$.

Remark. When B is empty this analysis reduces to the well known expected first passage problem, cf, [5, 24].

For any given policy π, and fixed states i, j where $i \in C \cup G$ and $j \in G$ let

$$\tau_{ij} = \inf\{n : X_n = j\,,\, X_\nu \notin Bj,\ \text{for } \nu \leq n-1\,,\ X_0 = i\}.$$

We will use the following notation.

$$\begin{aligned}
{}_Bf_{ij}^n(\pi) &= \mathbf{P}_\pi(\tau_{ij} = n \,|\, X_0 = i),\\
{}_Bf_{ij}(\pi) &= \mathbf{P}_\pi(\tau_{ij} < \infty \,|\, X_0 = i),\\
{}_Bm_{ij}(\pi) &= \mathbf{E}_\pi(\tau_{ij}).
\end{aligned}$$

Further, for a given subset G of $\mathcal{S}$, we will use the following notation.

$$\tau_{iG} = \inf\{n : X_n \in G\,,\, X_\nu \notin B \cup G,\ \text{for } \nu \leq n-1\,,\ X_0 = i\}$$

and

$$\begin{aligned}
{}_Bf_{iG}^n(\pi) &= \mathbf{P}_\pi(\tau_{iG} = n \,|\, X_0 = i),\\
{}_Bf_{iG}(\pi) &= \mathbf{P}_\pi(\tau_{iG} < \infty \,|\, X_0 = i),\\
{}_Bm_{iG}(\pi) &= \mathbf{E}_\pi(\tau_{iG}).
\end{aligned}$$

4.2 Taboo mean first passage reward

In this section we consider the problem where only some of the costs or rewards associated with the system are know. For such a system we study the problem of maximization of the taboo mean first passage reward.

Let r_k denote the reward when the state of the system is k. Note that rewards need only be known for all states in $\mathcal{S} \setminus B := \{k \in \mathcal{S}, k \notin B\}$. Extending the definitions given in the previous section, we define the following quantities.

$$
\begin{aligned}
{}_BV_{ij}^n &= \sum_0^{n-1} r_{X_t}\,\mathbf{1}_{\{X_t \notin Bj\,,\ 0 \leq \nu < n \text{ and } X_n = j\}},\\
{}_Bv_{ij}^N &= \mathbb{E}_\pi(\sum_{n=1}^{N} {}_BV_{ij}^n),\\
{}_Bv_{ij} &= \lim_{n\to\infty} {}_Bv_{ij}^N
\end{aligned}
$$

and

$$
\begin{aligned}
{}_BV_{iG}^n &= \sum_0^{n-1} r_{X_t}\,\mathbf{1}_{\{X_t \notin\ B\cup G\,,\ 0 \leq t < n \text{ and } X_n \in G\}},\\
{}_Bv_{iG}^N &= \mathbb{E}_\pi(\sum_{n=1}^{N} {}_BV_{iG}^n),\\
{}_Bv_{iG} &= \lim_{n\to\infty} {}_Bv_{iG}^N,
\end{aligned}
$$

where $\mathbf{1}_B$ is the indicator function of the set B.

We next establish theorem 4.2.1 and discuss the computation of the optimal solution.

Theorem 4.2.1. *For a given policy π the following relations hold true.*

$$ {}_Bv_{iG}(\pi) \;=\; r_i\, {}_Bf_{iG}(\pi) + \sum_{l \notin BG} p_{il}(\pi)\, {}_Bv_{lG}(\pi) \tag{4.1} $$

Proof. The proof of the theorem is by recursion. Dropping the symbol π for simplicity, we have

$$ {}_Bv^1_{iG} = r_i \sum_{j \notin B} {}_Bp_{ij} \; = r_i \sum_{j \notin B} {}_Bf^1_{ij} = r_i\, {}_Bf^1_{iG} $$

and

$$ {}_Bv^N_{iG} = r_i \sum_{n=1}^{N} {}_Bf^n_{iG} + \sum_{k \notin BG} p_{ik}\, {}_Bv^{N-1}_{kG} \quad \text{for } N \geq 2. $$

Taking the limit as N goes to infinity on both the sides we have

$$ {}_Bv_{iG}(\pi) = r_i\, {}_Bf_{iG}(\pi) + \sum_{l \notin BG} p_{il}(\pi)\, {}_Bv_{lG}(\pi). \tag{4.2} $$

If the limit of ${}_Bv^N_{iG}$ as N goes to infinity exists then the theorem is proved. To prove that the limit exists we first show that the value of ${}_Bv^N_{iG}$ increases with N and that it is bounded for every N.

Let $r^* = \max_k r_k$. From the definition of ${}_Bv^N_{iG}$ we have that

$$ {}_Bv^1_{iG} = r_i\, {}_Bf^1_{iG} \leq r^*\, {}_Bf^*_{iG}. $$

Continuing the recursion we can write,

$$ \begin{aligned} {}_Bv^N_{iG} &= E\left(\textstyle\sum_{t=1}^{N} r_{X_t} \mathbf{1}_{\{X_t \notin BG,\ 0 \leq t < N \text{ and } X_N \in G\}}\right) \\ &\leq r^* \left({}_Bf^1_{iG} + {}_Bf^2_{iG} + \ldots + {}_Bf^N_{iG}\right) \\ &\leq r^*\, {}_Bf^*_{ij}. \end{aligned} $$

This proves that ${}_B v_{iG}^N$ is increasing and is bounded for every N, hence equation (4.2) is valid. This completes the proof of theorem 4.2.1.

Let ${}_B v_{iG}^* = \sup_\pi \{ {}_B v_{iG} \}$ denote the value function. The problem is to establish the existence of and find a policy π^* such that ${}_B v_{iG}^* = {}_B v_{iG}(\pi^*)$.

The above problem can not be solved using standard MDP methods since the rewards, $r_i {}_B f_{iG}(\pi)$, depend on the policy π, and not just on the state action pairs. Moreover, an uniformly optimal policy does not exist for all $i \in C \cup G$, *i.e.* the policy that maximizes ${}_B v_{iG}$ need not be the same as the policy that maximizes ${}_B v_{kG}$ for $i \neq k$. As an illustrative example consider the replacement problem discussed in a later section, with $L = 5, l = 4, G = \{ 1 \}, B = \{ 5 \}$ and transition probability matrix

$$P = \begin{pmatrix} 0.41 & 0.01 & 0.05 & 0.17 & 0.36 \\ 0.27 & 0.20 & 0.01 & 0.05 & 0.47 \\ 0.23 & 0.03 & 0.29 & 0.01 & 0.53 \\ 0.06 & 0.09 & 0.03 & 0.20 & 0.62 \\ 0.20 & 0.20 & 0.20 & 0.20 & 0.20 \end{pmatrix}$$

For this problem the deterministic policy that maximizes ${}_5 m_{11}$ is not to replace in states 1 and 2 and to replace in states $3, 4$ and 5. While the policy that maximizes ${}_5 m_{31}$ is not to replace in states $1, 2$ and 3 and to replace in states 4 and 5.

In the next section we consider an inventory control example and show that when certain monotonicity properties hold then the optimal deterministic policy can be easily computed.

4.2.1 An inventory control application.

We study the following inventory control problem, where a sequence of ordering decisions is to be made at the beginning of a number of periods of equal duration. The demand in each period is assumed to be an observation of a random variable with a known discrete distribution function $p_D(d) = P(\mathrm{D} = d)$, with a maximum possible value $d = L$. We assume that the demand distribution function is an increasing function of d, *i.e.* $P(\mathrm{D} = d) \leq P(\mathrm{D} = d + 1)$.

The state is the discrete amount of inventory available at the beginning of every time period: $\{-L + 1, \ldots, 0, 1, \ldots, S\}$, where S is the inventory capacity of the system. At each period a decision on whether or not to order is taken. Let $a = 0$ denote the action "do not order" and $a = 1$ denote the action "order up to S".

Whenever the system enters a state $i > 0$ and the decision to order is taken, then an ordering cost is incurred and the state changes to state S. If the system enters a state $i \leq 0$ then an order must be placed. If the decision not to order is taken the next state will be state j with probability p_{ij}, where $p_{ij} = P\{\mathrm{D} = i - j\}$ for $i \geq j$ and $p_{ij} = 0$ for $i < j$.

The standard cost - reward structure for this model is as follows. There is a fixed ordering cost C_o, that does not depend on the number of units ordered; there is a holding cost $h(j) = h \cdot j$ when j units are at hand at the end of a period; and a shortage or penalty cost $p(k)$, where k is the amount of shortage at the end of a period. Finally, there is a revenue function $r(j) = r \cdot j$ where j is the number of units sold. We assume that $r > h$.

This model has been studied with the assumption of a known penalty cost for lost sales [21] and with a constraint on the probability of stockout in [25]. We shall refer to the problem of finding the policy that maximizes the mean first return reward to the "full capacity state" S, with known penalty costs as the classical "lost sales problem".

We consider the case where the penalty cost is unknown. Unlike the holding or ordering costs, the penalty cost represents many intangibles such as an opportunity cost for lost sales, a loss in customer goodwill etc., making it impossible to ascertain in many cases. In this case, we consider the criterion of maximizing the taboo first return reward ${}_Hv_{SS}$ to the "full capacity" state S, when we define as the taboo set H the set of all states where a penalty cost will be incurred, i.e., $H = \{-L+1, \ldots, 0\}$.

We prove that the optimal deterministic policy for this problem is a "control limit" policy, *i.e.*, there exists an integer s^* such that it is optimal to order only when the inventory level becomes smaller than s^*. In inventory theory terminology this type of policy is commonly known as a (s, S) policy.

To prove the above we consider the finite horizon version of the above problem. The finite horizon mean taboo first passage reward function can be written as

$$
{}_B v_{iS}^N = \begin{cases} \max\left\{ {}_B v_{iS}^N(0), {}_B v_{iS}^N(1) \right\} & \text{if } i > 0\,, \\ {}_B v_{iS}^N(1) & \text{if } i \leq 0\,, \end{cases} \tag{4.3}
$$

where ${}_B v_{iS}^N(0) = r_S \sum_{n=1}^{N} {}_B f_{iS}^n + \sum_{k \notin Bj} p_{ik} {}_B v_{kS}^{N-1}$ and ${}_B v_{iS}^N(1) = r_S - C_o$.

In the above equations ${}_B v_{iS}^N(1)$ is the $N-$ period mean taboo reward when optimal policy was employed for the first $N-1$ periods and action $a = 1$, is taken at time N. Note that ${}_B v_{iS}^N(0)$ is the corresponding quantity when action $a = 0$ is taken at time N. The expected reward is as follows:

$$
r_i = \sum_{d=0}^{i-1} r(d) P\{\mathrm{D} = d\} - \sum_{d=0}^{i-1} h(i-d) P\{\mathrm{D} = d\}.
$$

In the sequel the optimal policy for the $N-$ period taboo reward problem will be denoted by π_N^*.

Lemma 4.2.1. *r_i is an increasing function of i.*

Proof. Consider the difference $r_{i+1} - r_i$. It simplifies to

$$
r\, i\, P\{D = i\} - h\, P\{D \leq i-1\}
$$

which is greater than 0 because we assumed $r > h$ and $P\{D = i\} \geq P\{D = i-1\}$. □

For any finite discrete demand distribution the following two theorems hold.

Theorem 4.2.2. *For any fixed k the function $\rho_k(i) = \sum_{j=k}^{S} p_{ij}$ is increasing in $i = 0, 1, \ldots S$.*

Proof: It suffices to show that, for any $0 < i_1 \leq i_2$, $\sum_{j=k}^{S} P\{\mathrm{D} = i_1 - j\} \leq \sum_{j=k}^{S} P\{\mathrm{D} = i_2 - j\}$. Simplifying the terms the inequality can be written as $\sum_{j=0}^{i_1} P\{\mathrm{D} = j\} \leq \sum_{j=0}^{i_2} P\{\mathrm{D} = j\}$ which is true for $i_1 \leq i_2$ and the proof is complete.

Theorem 4.2.3. *For any stochastic matrix $P = (p_{ij})$, the following two conditions are equivalent:*

Condition A: *For any increasing function h on $\mathcal{S} \setminus H = \{1, \ldots, S\}$, the function*

$$\xi(i) = \sum_{j=1}^{S} p_{ij} h(j)$$

is a increasing function of i.

Condition B: *For each $k = 1, \ldots, S$, the function*

$$\rho_k(i) = \textstyle\sum_{j=k}^{S} p_{ij} \;, \quad i = 1, \ldots S \tag{4.4}$$

is increasing in i.

Proof: For the proof see [7].

The next theorem contains the main result for this inventory control model.

Theorem 4.2.4. *The optimal deterministic policy for the problem of maximization of the taboo first passage reward is a "control limit" policy.*

Proof:

We first prove that ${}_Bv_{iS}^N(0)$ is an increasing function of i for all N.To show this we first prove, using induction, that $\sum_{n=0}^{N} {}_Bf_{iS}^n$ is a increasing function of i for all N. For $n = 0$, ${}_Bf_{iS}^0 = 0$ is an increasing function of i by definition. Hence, from condition A we have that ${}_Bf_{iS}^1$ is an increasing function of i. The induction can be completed using the following,

$$ {}_Bf_{iS}^n(\pi) = \sum_{l \notin Bj} p_{il}\, {}_Bf_{lS}^{n-1}(\pi) \tag{4.5} $$

and condition A. Since ${}_Bf_{iS}^n$ is an increasing function of i, $\sum_{n=0}^{N} {}_Bf_{iS}^n$ is also an increasing function of i for all N.

Now consider

$$ {}_Bv_{iS}^N(0) = r_i \sum_{n=1}^{N} {}_Bf_{iS}^n + \sum_{k \notin Bj} p_{ik}\, {}_Bv_{kS}^{N-1}. \tag{4.6} $$

For $N = 0$ we have that ${}_Bv_{iS}^0(0) = 0$ is a increasing function of i by definition. For $N = 1$, since $P_{iS} = 0$ ${}_Bv_{iS}^1(0) = 0$ is again an increasing function of i. The induction can be completed using condition A, the fact that $\sum_{n=0}^{N} {}_Bf_{iS}^n$ is an increasing function of i for all N and Eq. (4.6). Hence, we can conclude that ${}_Bv_{iS}^N(0)$ is increasing in i.

Since ${}_Bv_{iS}^N(0)$ is increasing in i, and ${}_Bv_{iS}^N(1)$ is independent of i, there exists an s_N^* such that an optimal policy π_N^* for ${}_Bv_{iS}^N$ is specified by the actions: $\pi_N^*(i) = 1$, for $1 \le i \le s_N^*$ and $\pi_N^*(i) = 0$ for $i > s_N^*$.

For the infinite horizon case, let ${}_Bv_{iS}(0)$ and ${}_Bv_{iS}(1)$ denote the taboo first passage reward when in state i the action is $a = 0$ and $a = 1$ respectively.

Note that the following hold,

$$ {}_B v_{iS} = \lim_{N\to\infty} {}_B v_{iS}^N, \tag{4.7} $$

and

$$ {}_B v_{iS} = \begin{cases} \max\left\{ {}_B v_{iS}(0), {}_B v_{iS}(1) \right\} & \text{if } i > 0, \\ {}_B v_{iS}(1) & \text{if } i \le 0\,, \end{cases} \tag{4.8} $$

where

$$ {}_B v_{iS}(0) = r_i \, {}_B f_{iS} + \sum_{l \notin Bj} p_{il} \, {}_B v_{lS}, $$

and

$$ {}_B v_{iS}(1) = r_S - C_o. $$

From Eq. (4.7) and the fact that $\sum_{n=0}^N {}_B f_{iS}^n$ and ${}_B v_{iS}^N$ are increasing functions of i and for all N, it follows that ${}_B v_{iS}$ is increasing in i. Using this, Eq.(4.8), and condition A it follows that there exists a number $1 \le s^* \le S$ such that ${}_B v_{iS} = {}_B v_{iS}(0)$ if $i > s*$ and ${}_B v_{iS} = {}_B v_{iS}(1)$ if $i \le s^*$.

The proof of theorem 4.2.4 is complete.

Remark 1: We illustrate these ideas with the following example. The inventory capacity is $S = 10$, the holding cost is 10 per unit held, the ordering cost is 10 and the penalty cost is linear: $p(k) = p \cdot k$. The demand has a discrete distribution between 1 and 20. Given the optimal value of s^* for the taboo mean first return reward problem, one can solve the corresponding lost sales problem with a penalty cost p and by adjusting the value of p obtain the same optimal control limit s^*, for a range of values of the penalty cost.

We call this range the "range of imputed penalties" and we will denote it with $ROIP$. In table (4.1) we list values of the imputed penalty cost and the value of the common control limit as we change the value of the unit reward.

Remark 2: The above results can be extended to the case where the optimality criterion is the taboo discounted mean first passage reward.

4.3 Taboo mean first passage times

In this section we consider the problem where none of the costs or rewards associated with the states are known. For such a system we study the problem of maximizing the taboo mean first passage times

The problem of maximizing the taboo mean first passage times is a special case of the problem considered in the previous section, with the reward $r_i = 1\ \forall i \in \mathcal{S}$ and $i \notin H$ and $r_i = 0$ otherwise. Using the results of the previous section we have that the optimal Markov policy is deterministic. Further we can also prove that under similar monotonicity conditions on the p_{ij}'s the optimal deterministic policy is a "control limit" policy .

We next establish Theorem 4.3.1 below. Its proof follows as a special case of theorem, 4.2.1, however we provide a direct proof that is a generalization of some of the results in section 12 of Chung [4] pg. 66.

Theorem 4.3.1. *For any state $i \notin B$ and a given policy π the following* *relations hold.*

$$ {}_Bf_{iG}(\pi) = p_{ij} + \sum_{l \notin BG} p_{il}\, {}_Bf_{lG}(\pi). \tag{4.9} $$

$$ {}_Bm_{iG}(\pi) = {}_Bf_{iG} + \sum_{l \notin BG} p_{il}\, {}_Bm_{iG}. \tag{4.10} $$

Proof. For notational simplicity we omit π in the proof. We first prove Eq. (4.10). The proof of Eq. (4.9) follows along similar lines.

$$
\begin{aligned}
{}_Bm_{iG} &= \textstyle\sum_{n=1}^{\infty} n\, \mathbf{P}(X_n \in G, X_t \notin BG | X_0 = i) \\
&= \textstyle\sum_{n=1}^{\infty} n \sum_{j \in G} \mathbf{P}(X_n = j, X_t \notin BG | X_0 = i) \\
&= \textstyle\sum_{j \in G} \sum_{n=1}^{\infty} n\, \mathbf{P}(X_n = j, X_t \notin BG | X_0 = i) \\
&= \textstyle\sum_{j \in G}\, {}_{BG}m_{ij} \\
&= \textstyle\sum_{j \in G} \{\, {}_{BG}f_{ij} + \sum_{l \notin Bj} p_{il}\, {}_{BG}m_{lj}\} \\
&= {}_Bf_{iG} + \textstyle\sum_{l \notin BG} p_{il} \sum_{j \in G}\, {}_{BG}m_{lj} \\
{}_Bm_{iG} &= {}_Bf_{iG} + \textstyle\sum_{l \notin BG} p_{il}\, {}_Bm_{iG}.
\end{aligned}
$$

□

Let ${}_Bm^*_{ij} = \sup_\pi\{{}_Bm_{ij}(\pi)\}$, we have:

$$ {}_Bm^*_{ij} = \sup_\pi \{{}_Bf_{ij}(\pi) + \sum_{l \notin Bj} p_{il} l\, {}_Bm_{lj}(\pi)\}. $$

The above formulation also appears similar to the optimality equations used when maximizing the first passage reward but the term corresponding to the reward ${}_Bf_{ij}(\pi)$ depends on the policy and not solely on the current state - action pair. To find an optimal policy, the quantities ${}_Bf_{ij}(\pi)$ and

${}_Bm_{ij}(\pi)$ must be computed for every policy since policy improvement type optimality conditions do not hold.

4.3.1 The replacement problem

We study the following model for a system the state of which is continuously observed. The set of possible states is a finite set $\{1, \ldots, D\}$, where larger values represent increased states of deterioration from the "new condition" represented by state 1 to the "totally inoperative condition" of state D. Let $\{d, \ldots, D\}$ denote the set of inoperative states. Whenever the system changes state a decision has to be made as to whether it is replaced or not. Whenever the system enters a state $i < d$ and the decision to replace is taken, then a certain cost is incurred and its state, immediately, changes to state 1. If the system enters a state i, for $d \leq i < D$, then it must be replaced at an increased cost. There is no cost whenever the decision not to replace is taken in a state $i < d$; in this case the next state will be state j with probability p_{ij}, where $p_{ij} \geq 0$ for $j \geq i$ and $p_{ij} = 0$ otherwise.

When the above problem is formulated as a cost minimization problem Derman [7] proved that "control limit policies" are optimal, under certain sufficient conditions. Furthermore, to deal with the issue of unknown costs, Derman [8] considers the above problem and models it as maximization of the mean return time to state 1 with a constraint on the probability of being in the undesirable states. For such a system we can consider the problem of maximizing the taboo mean first return time to state 1, $({}_Bm_{11})$ where the taboo set is $B = \{d, \ldots, D\}$.

Using similar arguments as those from the inventory control example of the previous section we can prove that under the same sufficient conditions as those in Derman [7] the optimal deterministic policy for this problem is a "control limit" policy *i.e.* there exists an i^* such that the optimal policy π^* is specified by the actions $\pi^*(i) = 0$, (do not replace) for $i \leq i^*$ and $\pi^*(i) = 1$ (replace) for $i > i^*$.

4.4 Conclusions

In this chapter we investigated the problem of maximization of taboo first passage reward and taboo mean first passage times in Markov decision processes. We showed that in general it is impossible to compute optimal policies using existing dynamic programming methods. However, for certain problems with additional structure, such as satisfying monotonicity conditions, efficient computation is possible.

The optimization criteria studied in this chapter can be applied in many models where some or all of the costs are not available.

Reward	Control limit	$ROIP$
30	2	73 to 80
40	4	121 to 134
60	5	203 to 228
80	7	347 to 396
100	7	435 to 497
150	9	749 to 866
1000	9	5021 to 5810

Table 4.1: Imputed costs for increasing rewards.

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Curriculum Vitae

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