Make-to-Stock Production-Inventory Systems with Compound Poisson Demands, Constant Continuous Replenishment and Lost Sales

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Abstract of the Thesis

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Supply contracts are designed to minimize inventory costs or to hedge against undesirable events (e.g., shortages) in the face of demand or supply uncertainty. In particular, replenishment terms stipulated by supply contracts need to be optimized with respect to overall costs, profits, service levels, etc. This thesis considers a continuous-review, single-product Make-to-Stock production-inventory system with infinite base-stock level, compound Poisson demands and constant continuous replenishment under the lost-sales policy, in which the inventory is subject to a cost function consisting of holding costs and lost-sale penalties. The main objective is to minimize pertinent inventory cost functions (the expected discounted cost and the time average cost) with respect to the replenishment rate.

For the expected discounted cost case, we first derive an integro-differential equation system for the expected discounted cost incurred up until the first loss occurrence, conditioned on an initial inventory level, from which we obtain the Laplace transform for the conditional expectation of the discounted cost over an infinite time horizon. For a system starting from an arbitrary initial inventory level, we obtain a closed form formula for the expected discounted cost via the inversion of its Laplace transform. For the special cases of constant or proportional penalty function and exponentially distributed demand sizes, we exhibit an explicit expression for the conditional expectation of the discounted cost. Finally, we minimize the cost function with respect to the replenishment rate and provide an algorithm to compute the attendant optimal replenishment rate. We further obtain a closed form formula for the time-average cost under a suitable stability condition. For exponentially distributed demand sizes, we exhibit explicit solutions for the optimal replenishment rate for both the expected discounted cost function conditioned on initial empty inventory, as well as the time-average cost function.

For each case, numerical studies are conducted to illustrate our results and investigate further properties of the system.

Keywords and Phrases: Make-to-Stock Production-Inventory Systems, Compound Poisson, Conditional Expected Discounted Cost, Time-Average Cost, Lost-Sales, Constant Continuous Replenishment.

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Chapter 1   Introduction

Supply contracts are designed to minimize inventory costs or to hedge against undesirable events (e.g., shortages) in the face of demand or supply uncertainty [Simchi-Levy et al. (2008)]. In particular, replenishment terms stipulated by supply contracts need to be optimized with respect to overall costs, profits, service levels, financing costs, etc. In this thesis, we consider a Make-to-Stock (MTS) continuous-review single-product inventory system with infinite base-stock level, compound Poisson demands and constant replenishment, subject to the lost-sales rule. In this system, unsatisfied demand can be partially fulfilled from on-hand inventory (if any) and excess demand (shortage) is lost. The excess demand is referred to as the lost-sale size. Replenishment is continuous at a constant (deterministic) rate, which in our model can also be interpreted as a production rate. The system incurs a cost function consisting of two types of costs: holding cost and lost-sales cost. A holding cost is incurred as a function of the inventory on hand and assessed at a constant rate per unit of on-hand inventory per unit time. A lost-sales cost is a penalty imposed at each loss occurrence, and is assumed to be a function of the quantity of the unsatisfied demand lost-sale size. We consider two kinds of cost functions: discounted costs and time-average costs. For the first kind, the cost function is the expected costs discounted to time 0, conditioned on the initial inventory level, and as such the time value of cash flows is accounted for, while for the second kind, the cost function is not undiscounted but rather time averaged.

The objective of this thesis is twofold: first, to derive expressions for the computation of the aforementioned cost functions, and second, to derive the optimal replenishment rate that minimizes the respective cost functions.

Throughout the thesis, we use the following notational conventions and terminology. Let $\mathbb{R}$ denote the set of real numbers. For any $x \in \mathbb{R}$, $x^+ = \max\{x, 0\}$. The indicator function of set A is denoted by 1_A. For a random variable X, its probability density function (pdf) is denoted by $f_X(x)$, cumulative distribution function (cdf) by $F_X(x)$ and the complementary cdf by $\bar{F}_X(x) = 1 - F_X(x)$. If real functions $f(x)$ and $g(x)$ are defined on $[0, \infty)$, then the convolution function of $f(x)$ and $g(x)$ is given by

$$\left\langle f * g \right\rangle (u) = \int_0^u f(u-x)\, g(x)\, dx .$$

The n-th fold convolution of function $f(\cdot)$ is denoted by $f^{*(n)}(\cdot)$. The Laplace transform of a function $f(x)$ is defined by

$$\mathcal{L}\left[f\right](z) = \tilde{f}(z) = \int_0^\infty e^{-zx} f(x)\, dx ,$$

and the corresponding inverse Laplace transform is denoted by the following contour integral

$$f(x) = \mathcal{L}^{-1}\left[\tilde{f}\right](x) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\gamma - iR}^{\gamma + iR} e^{zx}\, \tilde{f}(z)\, dz , \tag{1.1}$$

where γ is any real number that exceeds the real part of all the singularities of $\tilde{f}(z)$ [Widder, (1959)]. Throughout this thesis, we restrict the Laplace domain to the real line unless otherwise specified.

An arrival process $\{A_i : i \geq 0\}$ is a random sequence, such that for all $i \geq 0$, $A_i \geq 0$ and $A_{i+1} > A_i$ with probability 1 (w.p.1). A real-valued process $\{X(t) : t \geq 0\}$ has conditionally stationary increments with respect to process $\{A(t) : t \geq 0\}$, if for any $n > m \geq 0$ and any real $t \geq 0$ and $x \geq 0$,

$$\mathbb{P}\{X(A_n+t)-X(A_n)\leq x \mid X(A_n)=u\}= \\ \mathbb{P}\{X(A_m+t)-X(A_m)\leq x \mid X(A_m)=u\} \tag{1.2}$$

A real-valued process $\{Z(t): t\geq 0\}$ is said to have conditionally stationary discounted increments with respect to $\{X(t): t\geq 0\}$ and $\{A_i : i\geq 0\}$, if for any $n>m\geq 0$ and any real x, u and $t\geq 0$,

$$\mathbb{P}\{Z(A_n+t)-Z(A_n)\leq x e^{-rA_n} \mid X(A_n)=u\}= \\ \mathbb{P}\{Z(A_m+t)-Z(A_m)\leq x e^{-rA_m} \mid X(A_m)=u\} \tag{1.3}$$

In the sequel, $\{X(t): t\geq 0\}$ will stand for the inventory process, $\{A_i : i\geq 0\}$ for the demand arrival time process, $\{Z(t): t\geq 0\}$ for the discounted cost process and $N_A(t)=\sup\{n: A_n\leq t\}$ is the number of arrivals up to time t.

Finally, we shall make repeated use of the following relation for any non-negative valued random variable X,

$$\tilde{\bar{F}}_X(z)=\int_0^\infty e^{-zx}\bar{F}_X(x)\,dx=\frac{1}{z}\left[1+\int_0^\infty e^{-zx}d\bar{F}_X(x)\right]=\frac{1}{z}\left[1-\tilde{f}_X(z)\right] \tag{1.4}$$

where the second equality follows by integration by parts.

This thesis consists of two parts, each treating the production-inventory system described in Chapter 3. The first part studies expected discounted costs, conditioned on the initial inventory level, while the second part studies time-average costs. The rest of this thesis is organized as follows. Chapter 2 provides literature review. Chapter 3 formulates the production-inventory model under study. Chapter 4 studies the expected discounted cost function and its optimization, while Chapter 5 treats the optimization for the time average cost function. Some further discussions of the system are provided in Chapter 6. Consequently, Chapter 7 concludes this thesis and provides some discussion for future research.

Chapter 2 Literature Review

There is a large body of literature addressing the management of inventory systems with compound Poisson demands, that is, demand arrivals follow a Poisson process, and the corresponding demand sizes follows an iid arbitrary distribution, independent of arrivals. Early papers on inventory process include Richards (1975), Thompstone and Silver (1975), Archibald and Silver (1978), Feldman (1978), and Federgruen and Schechner (1983). Tijms (1972), Sahin (1979, 1983), and Federgruen and Schechner (1983) generalize the compound Poisson assumption to a general compound renewal processes, in which both the demand inter-arrival times and demand sizes have arbitrary distributions. Ohno and Ishigaki, (2001) considers a multi-item continuous-review inventory system with compound Poisson demands under a general cost structure. Presman and Sethi (2006) provides a detailed literature review with a comprehensive reference list. The aforementioned papers assume various replenishment policies, but exclude continuous replenishment.

Production-inventory systems with constant replenishment and various demand processes have been previously studied in literature. Gavish and Graves (1980) studies a one-product production-inventory problem where demand is governed by a Poisson process and unsatisfied demand is backordered. The system is subject to a fixed setup cost, a liner inventory holding cost and a linear backorder cost. To minimize the expected cost per time unit, the paper treats the problem as an M/D/1 queueing system and proves that the optimal policy is a two-critical-number policy. Graves and Keilson (1981) considers a one-product, one-machine production-inventory problem, where the demand process is governed by a compound Poisson process with exponential demand sizes and the system is subject to a (r,R) policy with a constant replenishment rate. The paper analyses the cost optimization problem as a constrained Markov process using the compensation method and the optimal policy is obtained via a search of the policy space. Graves (1982) presents two models for inventory systems with constant production rate of perishable items. For each of these models, the paper derives analytical expressions for the steady-state distribution of system inventory, using a queuing-theoretic approach. The steady-state results are then used to evaluate various system performance metrics. De Kok (1985) deals with a one-product production/inventory model with compound Poisson demands and lost-sales, where the production rate can be dynamically adjusted in order to cope with random fluctuations in demand. The paper considers the average number of lost-sales occurrences per unit time and the fraction of lost demand as service level measures. For a two-critical-number control rule, it derives practically useful approximations for the switch-over level in order to achieve a prescribed service level. Gullu and Jackson (1993) considers a one-product inventory problem with a constant production rate and a demand process with stationary and independent increments, where the replenishment policy is produce-up-to-S. The paper derives the stationary distribution of the inventory level by extending existing results for dam systems, and then optimizes the time-average cost of the system, by exhibiting a closed form formula for the optimal policy.

A number of papers consider production-inventory problems for an integrated supply chain system, transportation and distribution. Lei, et al (2006) studies the integrated production, inventory, and distribution routing problem (PIDRP). Optimally solving such an integrated problem is generally difficult due to its combinatorial nature, especially when transporter routing is involved. The authors propose a two-phase solution approach to this problem, which can simultaneously coordinate the production, inventory, and transportation operations over the entire planning horizon, without the need to aggregate demand or relax constraints on transportation capacities. Armstrong et al (2008) studies the zero-inventory production and distribution problem with a single transporter and a fixed sequence of customers, where a subset of the customers is chosen from the given sequence to receive deliveries so as to maximize the total demand satisfied, without violating the product lifespan, the production/distribution capacity, and the delivery time window constraints.

Discounted costs are also frequently addressed in the inventory management literature. The pioneering work in Hadley (1964) offers a simple comparison of optimal order quantities computed using average costs with those computed using discounted costs. Constantinedes and Richard (1977) offers an infinite-horizon, continuous time, discounted cash management model with fixed and proportional transaction costs and linear holding and penalty costs. Federgruen and Schechner (1983) considers a single-item continuous review inventory model with a fixed delivery lag and compound renewal demand under the backlog policy. The paper presents cost formulas for the expected discounted inventory cost as a function of the inventory position just after a replenishment decision points. Wee and Law (2001), Bose, Goswami and Chaudhuri (1995), and Ray and Chaudhuri (1997) present variations of a deteriorating inventory system with price-dependent demand model taking into account the time-value of money. Presman and Sethi (2006) considers inventory models with compound Poisson demands under discounted and long-run average cost structures. This paper connects two classical inventory results: the EOQ formula and the optimality of an (s,S) policy in stochastic inventory models with a fixed ordering cost. Under the two different valuation frameworks, i.e., the discounted cost and time average cost, the paper proves that the optimal ordering level s is unique, but the order-up-to-level S may not be. They also provide a detailed literature review on the optimal ordering policy for Poisson demand processes and fixed ordering cost.

A number of papers study the derivation of optimal or near-optimal inventory replenishment, which minimize the time-average or expected discounted costs. Springael and Nieuwenhuyse (2005) studies a lost-sales inventory model with a compound Poisson demand process, in which replenishment lead times are negligible. On-hand inventory is managed according to a $(0,B^*)$

policy, namely, when on-hand inventory drops to 0, the retailer instantaneously gets a fixed amount of B^* units from the central stockroom as replenishment. The paper analyzes the time-average cost of the system and provides a steepest-descent-based algorithm to calculate the optimal B^* parameter. In a similar vein, Minner and Silver (2007) studies an inventory system with compound Poisson demands and negligible replenishment lead times. The paper formulates the optimization problem as a Markov-decision-problem, which can be applied to inventory systems with a small number of products. For a large number of products, the paper proposes several heuristics for the optimal reorder points and reorder quantities. Zhao and Katehakis (2006) studies a single-item stochastic inventory system with a minimum order quantity (MOQ) over finite and infinite time horizons under the discounted cost criterion. The paper characterizes the optimal ordering policy everywhere in the state space outside of a state interval for each time period, and develops an upper bound and a lower bound for these intervals. Zhou, et al. (2007) considers a model of single-item periodic-review inventory system with stochastic demand, and linear ordering cost, where in each time period, the system must order either no items or at least MOQ items. The paper studies the performance of a simple heuristic policy, easily implementable in practice, and develops an algorithm to compute optimal parameters. For additional literature, refer to Yang, (2004), Yang and Yu (2002), and Yang and Qi (2010).

MTS production-inventory systems have also been studies via other techniques. Zhao and Melamed (2006, 2007) apply the stochastic fluid model (SFM) paradigm to a class of single-stage, single-product MTS production-inventory systems with stochastic demand and random production capacity, where unsatisfied demand is either lost or backordered. The authors derive formulas for infinitesimal perturbation analysis (IPA) derivatives of sample-path time averages of inventory level and lost sales, as well as backorder levels, with respect to the base-stock level and a parameter of the production rate process. It is further shown that all IPA derivatives under study are unbiased and fast to compute, thereby providing the theoretical basis for online adaptive control of MTS production-inventory systems.

Production-inventory systems with constant replenishment rates are commonly seen in the manufacturing industry and in service organizations. For example, a pharmaceutical manufacturer (or a chemical industry) often set up production lines to satisfy incoming demands from customers. As a consequence of high setup times and costs, no modification for the production line is done after the process has been started. The importance of an optimal production rate as a decision in the production planning stage can be seen as follows. If the production rate is high there will be extra inventory held in stock and high carrying costs will be incurred. On the other hand, if the production rate is low there will be high penalty costs. Such models can also be applied in service organizations, such as blood banks and food companies etc.

From the managerial point of view, inventory models are treated differently from those of queueing. However, similarities between the mathematical formalisms of inventory models and queueing have been observed from a fairly early stage of their development. The linkage between those two areas has been studied by Prabhu (1965). For other related recent work in the broader area of service systems we refer the reader to Adan, et al. (2005), Perry and Stadje (2003), and Li and Glazebrook (2010).

We are not aware of any previous work on stochastic models with continuous replenishment and discounted or time-averaged cash flows. For a good recent survey of related Markovian demand inventory models and theory, we refer the reader to Beyer, et al (2010) and references therein.

Chapter 3 The Inventory Process

All random processes in this section are defined over a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider a single-product continuous-review production-inventory system, subject to the lost sales rule. The demand arrival stream constitutes a compound Poisson process with rate λ and arrival times $\{A_i : i \geq 0\}$, where time $A_0 = 0$ by convention. Thus, the corresponding sequence of interarrival times, $\{T_i : i \geq 1\}$, where $T_i = A_i - A_{i-1}$, is exponentially distributed and the sequence is identically independently distributed (iid). The corresponding demand sizes form an iid sequence $\{D_i : i \geq 1\}$ with a common density function $f_D(x)$ and common mean demand, $\mu_D = \mathbb{E}[D] < \infty$, where demand size D_i arrives at time A_i. Replenishment occurs at a constant (deterministic) rate $\rho \geq 0$. Let $\{I(t) : t \geq 0\}$ denote the right-continuous inventory process, given by

$$I(t) = I(0) + \rho t - \sum_{i=1}^{N_A(t)} [D_i - L(A_i)], \tag{3.1}$$

where $N_A(t)$ is the number of demands arriving over $(0, t]$ and

$$L(t) = \begin{cases} [D_i - I(A_i -)]^+, \ t = A_i, \ i = 1, 2, ... \\ 0, \ \text{otherwise} \end{cases} \tag{3.2}$$

is the lost-sales size (excess demand that cannot be satisfied from on-hand inventory under the lost-sale rule). Let $\{\tau_i : i \geq 0\}$ be the sequence of loss occurrence times, given by

$$\tau_i = \begin{cases} \inf\{t > \tau_{i\text{-}1} : L(t) > 0\}, \text{ if there exists } t > \tau_{i\text{-}1} \text{ s.t.} L(t) > 0 \\ \infty, \ \text{otherwise} \end{cases} \tag{3.3}$$

where $\tau_0 = 0$. Throughout this thesis, we focus our interest on the case where $\tau_i < \infty$. Let $\{J_k : k = 1, 2, ...\}$ be the sequence of random arrival indexes at which loss occurs, namely, $\tau_k = A_{J_k}$.

Figure 3.1 illustrates the evolution of the inventory process with lost-sales over an infinite time horizon.

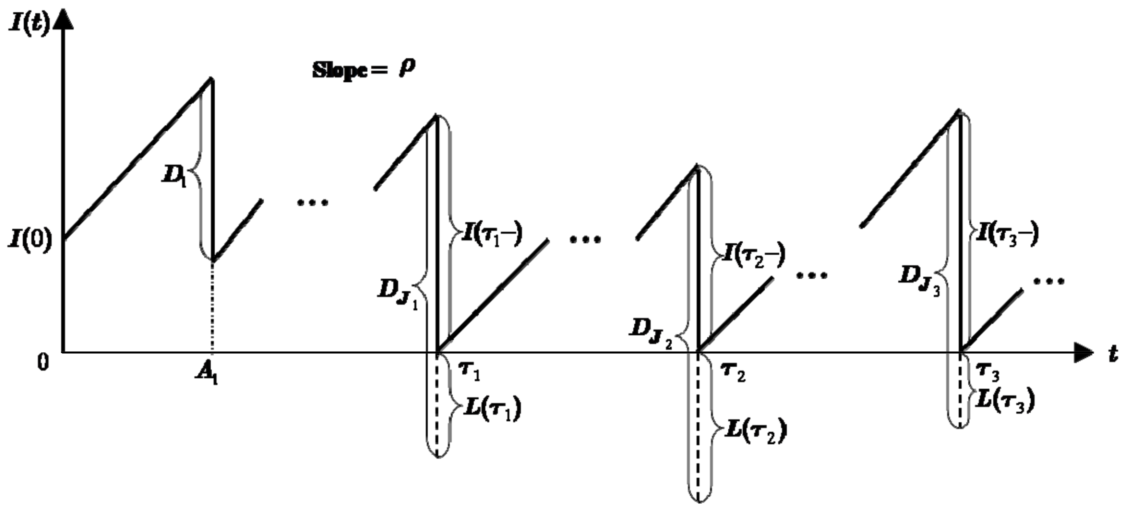

Figure 3.1. A Sample Path of the Inventory Level Process, $\{I(t)\}$

Figure 3.2 depicts the detailed evolution of a sample path of the inventory process over the interval $[0, \tau_1]$.

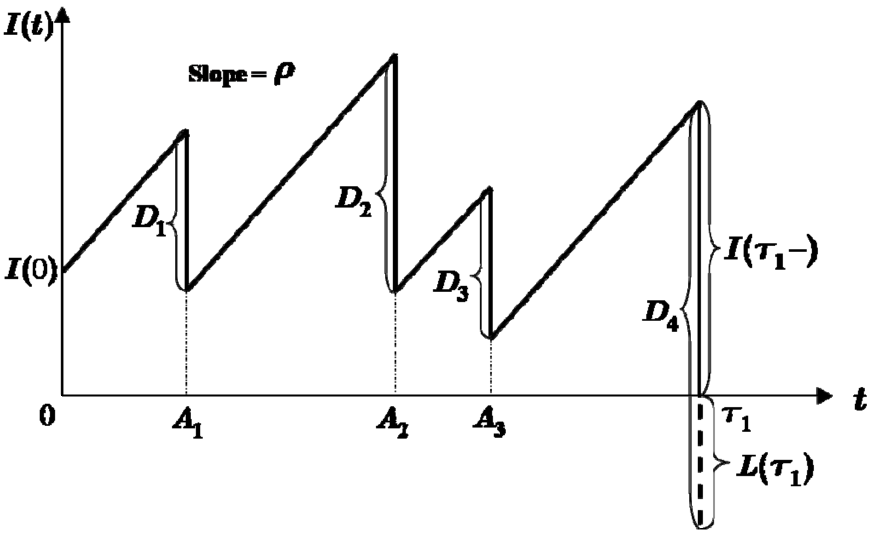

Figure 3.2. A Sample Path of the Inventory Level Process over the Interval $[0, \tau_1]$

We next proceed to study some properties of the system.

Proposition 3.1

The inventory process, $\{I(t)\}$ given by Eq. (3.1) has conditionally stationary increments with respect to $\{A_i\}$.

Proof.

Follows directly from Eqs. (3.1) and (3.2), since the process $\{I(t) : t \geq 0\}$ is a function of a given initial state, a Poisson arrival process, an iid demand size process and a deterministic replenishment rate. □

Note that Proposition 3.1 implies that the inventory process, $\{I(t)\}$ is Markov renewal process, as imbedded with respect to the arrival times or lost-sale occurrence.

Next, consider the following auxiliary function

$$\psi(z) = \lambda \tilde{f}_D(z) + \rho z - \lambda - r\,, \tag{3.4}$$

where constant $r \geq 0$ is the interest rate.

The following Lemma provides some properties for the roots of the equation $\psi(z) = 0$.

Lemma 3.1

For any $r > 0$, the equation $\psi(z) = 0$ has two distinct roots, ξ and θ, where $\xi > 0$ and $\theta < 0$.

Proof.

We first prove that the function $\psi(z)$ is convex by computing its first and second derivatives,

$$\frac{d}{dz}\psi(z) = \rho - \lambda \int_0^\infty x\, e^{-z\,x} f_D(x)\, dx\,, \tag{3.5}$$

$$\frac{d^2}{dz^2}\psi(z) = \lambda \int_0^\infty x^2\, e^{-z\,x} f_D(x)\, dx\,. \tag{3.6}$$

Since the case of zero demand with probability 1 is precluded, it follows from Eq. (3.6) that

$$\frac{d^2}{dz^2}\psi(z) > 0\,. \tag{3.7}$$

Note that, for $r > 0$, $\psi(0) < 0$, $\psi(\infty) = \infty$ and $\psi(-\infty) = \infty$. Consequently, by the continuity of $\psi(z)$, there must be exactly one positive root and exactly one negative root for $\psi(z) = 0$, and the proof is complete. □

In view of Lemma 3.1, we have

$$\lambda \tilde{f}_D(\xi) + \rho \xi - \lambda - r = 0, \tag{3.8}$$

$$\lambda \tilde{f}_D(\theta) + \rho \theta - \lambda - r = 0. \tag{3.9}$$

Figure 3.3 outlines the key features of the function $\psi(z)$ and the roots of the equation $\psi(z) = 0$.

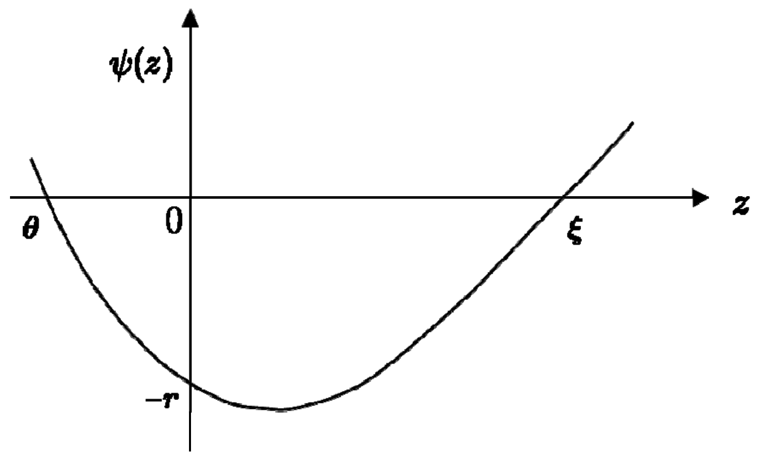

Figure 3.3. Illustration of the Function $\psi(z)$ and its Root Structure

Lemma 3.2

The following relations hold:

$$\rho = \frac{r}{\xi} + \lambda \tilde{\bar{F}}_D(\xi) \tag{3.10}$$

where the function $\xi = \xi(\rho)$, implicitly defined by Eq. (3.10), is strictly decreasing in ρ.

$$\lim_{\rho \to 0} \xi(\rho) = \infty \tag{3.11}$$

$$\lim_{\rho \to 0} \rho \xi(\rho) = \lambda + r. \tag{3.12}$$

Proof.

To prove Eq. (3.10) note first that by Eq. (3.8),

$$\rho = \frac{r}{\xi} + \frac{\lambda}{\xi}\left[1 - \tilde{f}_D(\xi)\right], \tag{3.13}$$

Eq. (3.10) now follows from the above equation with the aid of Eq.(1.4).

To prove that $\xi = \xi(\rho)$ is decreasing in ρ,we differentiate Eq. (3.10) with respect to ρ, yielding

$$1 = -\frac{d}{d\rho}\xi(\rho)\left[\frac{r}{\xi^2(\rho)} + \lambda\int_0^\infty x\ e^{-x\xi(\rho)}\overline{F}_D(x)\,dx\right].$$

The equation above implies $\frac{d}{d\rho}\xi(\rho) < 0$ since the second term in the square bracket on the right hand side is strictly positive for all $\rho \geq 0$, which in turn implies the result.

To prove Eq. (3.11) sending $\rho \downarrow 0$ on both sides of Eq. (3.10) implies

$$\lim_{\rho\to 0}\left[\frac{r}{\xi(\rho)} + \lambda\tilde{\overline{F}}_D(\xi(\rho))\right] = 0\,.$$

Since both terms are non-negative, the only way each term vanishes in the limit is for Eq. (3.11) to hold.

To prove Eq. (3.12), note that Eq. (3.8) can be rewritten as

$$\rho\,\xi = \lambda + r - \lambda\,\tilde{f}_D(\xi)\,.$$

Eq. (3.12) now follows by taking limit of $\rho \downarrow 0$ in the above equation and noting that $\lim_{\rho\to 0}\tilde{f}_D(\xi) = 0$ by Eq. (3.11) . □

Next, define the function

$$V_\rho(z) = \frac{(z-\xi)(z-\theta)}{\psi(z)} \tag{3.14}$$

where the values of $V_\rho(\xi)$ and $V_\rho(\theta)$ are defined as the corresponding limits above as $z \to \xi$ and $z \to \theta$, respectively. Then, setting $z = 0$ in Eq. (3.14) yields

$$\xi\theta = -r\,V_\rho(0)\,. \tag{3.15}$$

Furthermore, denote

$$\eta_\xi(x) = \mathcal{L}^{-1}\left[\frac{1}{\psi(z)}\right](x)\,. \tag{3.16}$$

Then, we derive a closed form for $\eta_\xi(x)$ defined in Eq. (3.16).

Lemma 3.3

$$\eta_\xi(u) = \frac{V_\rho(\xi)}{\xi-\theta}e^{\xi u} + \frac{V_\rho(\theta)}{\theta-\xi}e^{\theta u}\,. \tag{3.17}$$

Proof.

We prove the result by the Contour Integration and Residue Theorem [Churchill (1971)]. By Eq.(3.14), one has

$$\frac{1}{\psi(z)} = \frac{V_\rho(z)}{(z-\xi)(z-\theta)}, \tag{3.18}$$

where ξ and θ are the only singularities of $\dfrac{1}{\psi(z)}$. Substituting Eq. (3.18) into Eq. (1.1), we have,

$$\mathcal{L}^{-1}\left[\frac{1}{\psi(z)}\right](x) = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\gamma-iR}^{\gamma+iR} e^{zx}\,\frac{V_\rho(z)}{(z-\xi)(z-\theta)}\,dz\,,$$

for any real $\gamma > \xi$. To this end, define for $R > \gamma - \theta$ a counter-clock contour path $C_R = H_R \cup L_R$ (see Figure 3.4 below), where

$$H_R = \left\{(x, iy) : (x-\gamma)^2 + y^2 = R^2,\ \gamma - R \le x \le \gamma\right\}$$
$$L_R = \left\{(x, iy) : x = \gamma,\ -R \le y \le R\right\}$$

Hence, the contour integral can be written as

$$\frac{1}{2\pi i}\int_{C_R} e^{zx}\frac{1}{\psi(z)}dz = \frac{1}{2\pi i}\int_{H_R} e^{zx}\frac{1}{\psi(z)}dz + \frac{1}{2\pi i}\int_{\gamma-iR}^{\gamma+iR} e^{zx}\frac{1}{\psi(z)}dz \quad (3.19)$$

Now, by the Residue Theorem and Eq. (3.18), the left hand side of Eq. (3.19) becomes

$$\frac{1}{2\pi i}\int_{C_R} e^{zx}\frac{1}{\psi(z)}dz = \frac{V_\rho(\xi)}{\xi-\theta}e^{\xi u} + \frac{V_\rho(\theta)}{\theta-\xi}e^{\theta u}. \quad (3.20)$$

The result now follows by substituting Eq. (3.20) into Eq. (3.19) and sending $R \uparrow \infty$, since the first term on the right-hand side of vanishes. To see this, note that for any $x > 0$, we have

$\lim_{R\to\infty}\int_{H_R} e^{zx}\frac{1}{\psi(z)}dz \le \lim_{R\to\infty}\frac{\pi R}{\psi(-R)e^{Rx}} = 0$, [cf. Saff and Snider (1993)]. □

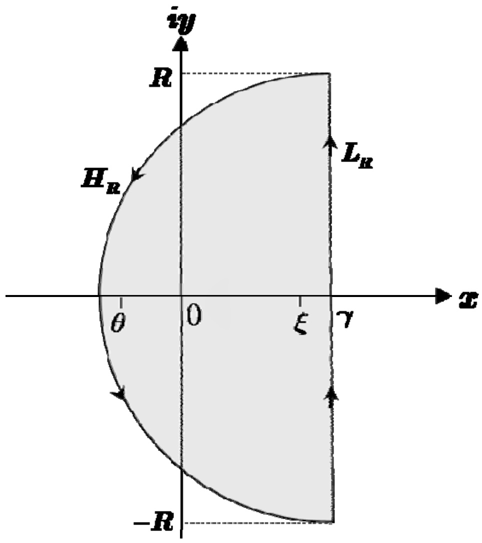

Figure 3.4. Contour Integral for the Inverse Laplace Transform.

Chapter 4 Expected Discounted Cost

This chapter provides a formula for the expected discounted cost function (consisting of holding costs and lost-sales penalties), conditioned on the initial inventory level, so as to minimize the aforementioned cost function with respect to the replenishment rate. To this end, we derive a system of integro-differential equations based on a renewal argument that decomposes the total cost into the partial cost up until the first demand arrival and the residual cost thereafter. From this system of equations, we obtain a closed form formula for the Laplace transform of the expectation of the discounted cost function, conditional on the initial inventory level, where the Laplace transform is taken with respect to the initial inventory level. Moreover, a closed form formula is exhibited for the expected discounted cost function, conditioned on zero initial inventory. We then provide an algorithm for a numerical computation of the optimal replenishment rate which minimizes the aforementioned cost function. In particular, we consider two special cases of lost-sales penalty functions: constant penalty and loss-proportional penalty. Furthermore, for the special case of exponential demand sizes, we exhibit closed form formulas for the expected discounted cost function, conditioned on any initial inventory level, and its optimal replenishment rate. Finally, some numerical studies are carried out to illustrate the results and investigate further properties of the system.

Throughout this chapter, we assume continuously compounded discounting at rate, $r > 0$. Accordingly, the present value of a future cash flow Y at time t is Ye^{-rt}.

The following result will be used to derive expected discounted cost functions, conditioned on the initial inventory level.

Proposition 4.1

Let $\{X(t): t \geq 0\}$ has conditionally stationary increments with respect to an arrival process $\{A_i : i \geq 0\}$, and let the process $\{Z(t): t \geq 0\}$ be given by

$$Z(t) = \int_0^t e^{-rz} g_1(X(z))dz + \sum_{i=1}^{N_A(t)} e^{-rA_i} g_2(A_i)$$

where $g_1(x)$ is a real-valued integrable function, $g_2(x)$ is a real-valued measurable function. Then $\{Z(t): t \geq 0\}$ has conditionally stationary discounted increments with respect to $\{X(t): t \geq 0\}$ and $\{A_i : i \geq 0\}$, and satisfies for any $n > m \geq 0$ and any real $t \geq 0$

$$\mathbb{E}\left[Z(A_n + t) - Z(A_n) | X(A_n) = u\right] = e^{-r(A_m - A_n)} \mathbb{E}\left[Z(A_m + t) - Z(A_m) | X(A_m) = u\right]. \tag{4.1}$$

Proof.

It suffices to prove for $Z_1(t) = \int_0^t e^{-rz} g_1(X(z))dz$ since the proof for $Z_2(t) = \sum_{i=1}^{N_A(t)} e^{-rA_i} g_2(A_i)$ is analogous. Accordingly, for any $n > m \geq 0$ and any real $t \geq 0$ and $x,\ u \geq 0$,

$$\begin{aligned}
&\mathbb{P}\{Z_1(A_n + t) - Z_1(A_n) \leq x e^{-rA_n} \mid X(A_n) = u\} \\
&\quad = \mathbb{P}\{\int_{A_n}^{A_n+t} e^{-rz} g_1(X(z))dz \leq x e^{-rA_n} \mid X(A_n) = u\} \\
&\quad = \mathbb{P}\{\int_{A_m}^{A_m+t} e^{-r(v+A_n-A_m)} g_1(X(v + A_n - A_m))dv \leq x e^{-rA_n} \mid X(A_n) = u\} \\
&\quad = \mathbb{P}\{\int_{A_m}^{A_m+t} e^{-rv} g_1(X(v + A_n - A_m))dv \leq x e^{-rA_m} \mid X(A_n) = u\} \\
&\quad = \mathbb{P}\{\int_{A_m}^{A_m+t} e^{-rz} g_1(X(v))dv \leq x e^{-rA_m} \mid X(A_m) = u\} \\
&\quad = \mathbb{P}\{Z_1(A_m + t) - Z_1(A_m) \leq x e^{-rA_m} \mid X(A_m) = u\}
\end{aligned}$$

where the first and last equations follow from the definition of $Z_1(t)$, the second equation results from the change of variable $z \to v + A_n - A_m$, the third equation holds by multiplying both sides of the inequality by $e^{r(A_n - A_m)}$, and the fourth equation follows from Eq. (1.3) by assumption. This completes the proof. □

4.1 Discounted Cost Functions

The production-inventory system under study incurs costs in the form of holding costs and lost-sales penalties. These cost components are described below.

- **Discounted holding costs**. While there is inventory on hand, a holding cost is incurred at rate h per unit time and per inventory unit. Accordingly, the discounted holding cost process $H_\rho = \{H_\rho(t) : t \geq 0\}$ is given by

$$H_\rho(t) = h \int_0^t e^{-rz} I(z)\, dz \,. \tag{4.2}$$

- **Lost-sales penalties**. Whenever a customer's demand cannot be fully satisfied from on-hand inventory, a penalty $w(x)$ is incurred as a non-decreasing function of the lost-sale size, x, with the proviso that $w(0) = 0$. In particular, we shall consider a linear penalty function (to be studied in Section 4.6) of the form

$$w(x) = 1_{(0,\infty)} K_0 + K_1 x \,, \tag{4.3}$$

where $K_0 \geq 0$ is a constant penalty per lost-sale occurrence, $K_1 \geq 0$ is a constant penalty per unit of lost sales, and the two constants do not vanish simultaneously. Accordingly, the discounted penalty process $W_\rho = \{W_\rho(t) : t \geq 0\}$ is given by

$$W_\rho(t) = \sum_{i=1}^{N_A(t)} e^{-rA_i} w(L(A_i)) \,. \tag{4.4}$$

The discounted inventory cost process $C_\rho = \{C_\rho(t) : t \geq 0\}$ is given by

$$C_\rho(t) = H_\rho(t) + W_\rho(t). \tag{4.5}$$

Of particular interest is the inventory discounted cost until and including the first lost-sale occurrence, given by

$$C_\rho(\tau_1) = h\int_0^{\tau_1} I(z)e^{-rz}\,dz + w(L(\tau_1))e^{-r\tau_1} \tag{4.6}$$

and its associated conditional expected discounted cost function, given by

$$c_\rho(u) = \mathbb{E}[C_\rho(\tau_1) \mid I(0) = u]. \tag{4.7}$$

Furthermore, define the auxiliary function

$$d_\rho(u) = \mathbb{E}[e^{-r\tau_1} \mid I(0) = u], \tag{4.8}$$

Next, the conditional expected discounted cost function over the interval $(0, t]$ is given by

$$\Phi_\rho(t \mid u) = \mathbb{E}[C_\rho(t) \mid I(0) = u]. \tag{4.9}$$

Proposition 4.2

For any $u \geq 0$, $\Phi_\rho(t \mid u)$ is non-decreasing in t and uniformly bounded by

$$\Phi_\rho(t \mid u) \leq \frac{h}{r}\left(u + \frac{\rho}{r}\right) + \frac{\lambda}{r}\mathbb{E}\left[w(D_i)\right], \tag{4.10}$$

independent of t.

Proof.

$\Phi_\rho(t \mid u)$ is non-decreasing in t by (4.9). To prove Eq. (4.10), we first write

$$\begin{aligned}\Phi_\rho(t \mid u) &\leq \int_0^t h\left[u + \rho\, z\right]e^{-rz}\,dz + \mathbb{E}\left[\sum_{i=1}^{N_A(t)} w(D_i)\, e^{-rA_i}\right] \\ &= \frac{h}{r}\left(u + \frac{\rho}{r}\right)\left(1 - e^{-rt}\right) - \frac{\rho h t}{r}e^{-rt} + \mathbb{E}\left[w(D_i)\right]\mathbb{E}\left[\sum_{i=1}^{N_A(t)} e^{-rA_i}\right]\end{aligned} \tag{4.11}$$

where the inequality holds by Eqs. (4.2) and (4.4) and the facts that $I(t) \le u + \rho t$ by Eq. (3.1) and $D_i \ge L(A_i)$ $i = 1,2,\ldots$, by Eq. (3.2). Furthermore, by the property that $\Phi_\rho(t \mid u)$ is increasing over t, sending t to infinity, Eq. (4.11) reduces to

$$\Phi_\rho(t \mid u) \le \frac{h}{r}\left(u + \frac{\rho}{r}\right) + \mathbb{E}\left[w(D_i)\right]\mathbb{E}\left[\sum_{i=1}^{\infty} e^{-rA_i}\right] \tag{4.12}$$

Denoting $\varphi = \mathbb{E}\left[\sum_{i=1}^{\infty} e^{-r A_i}\right]$, we obtain the equation

$$\begin{aligned}
\varphi &= \mathbb{E}\left[e^{-rT_1} + \sum_{i=2}^{\infty} e^{-rA_i}\right] \\
&= \mathbb{E}\left[e^{-rT_1}\right] + \mathbb{E}\left[e^{-rT_1}\,\mathbb{E}\left[\sum_{i=2}^{\infty} e^{-r(A_i - T_1)} \mid T_1\right]\right] \\
&= \mathbb{E}\left[e^{-rT_1}\right] + \mathbb{E}\left[e^{-rT_1}\,\varphi\right] \\
&= \mathbb{E}\left[e^{-rT_1}\right]\left(1 + \varphi\right)
\end{aligned} \tag{4.13}$$

But by the independence of the processes $\{A_i : i \ge 1\}$ and $\{D_i : i \ge 1\}$ and the Jensen inequality on the exponential as a convex function, and noting that $\mathbb{E}[A_i] = i\,\mathbb{E}[T_1]$, we deduce

$$\varphi \le \sum_{i=1}^{\infty} e^{-r\mathbb{E}[A_i]} = \frac{1}{1 - e^{-r\mathbb{E}[T_1]}} < \infty. \tag{4.14}$$

Since all quantities in Eq. (4.13) are finite, we obtain

$$\varphi = \frac{\tilde{f}_T(r)}{1 - \tilde{f}_T(r)} = \frac{\lambda}{r} \tag{4.15}$$

where the second equality holds by Eq. (4.34). The result now follows by substituting Eq. (4.15) into Eq.(4.12). □

Proposition 4.2 guarantees the existence of the asymptotic conditional expected discounted cost function, given by

$$\Phi_\rho(u) = \lim_{t \to \infty} \Phi_\rho(t \mid u). \tag{4.16}$$

Lemma 4.1

The cost process $\{C_\rho(t)\}$ has conditionally stationary discounted increments with respect to $\{I(t)\}$ and $\{A_i : i \geq 0\}$, and for any s, t and $u \geq 0$,

$$\mathbb{E}[C_\rho(s+t) - C_\rho(s) \mid I(s) = u] = e^{-rs}\, \Phi_\rho(t \mid u). \tag{4.17}$$

Proof.

The property that $\{C_\rho(t)\}$ has conditionally stationary discounted increments follows immediately from Proposition 4.1 by setting $g_1(x) = 1_{[0,\infty)}(x)\, h\, x$ and $g_2(x) = w(L(x))$. Eq. (4.17) immediately follows from this property and the independent increment property of the compound Poisson process. □

In this chapter we derive closed form formulas for the conditional expected discounted cost function $\Phi_\rho(u)$ of Eq. (4.16). To this end, we shall need the following structural result in the sequel.

Theorem 4.1

For any given initial inventory $u \geq 0$, $\Phi_\rho(u)$ and $c_\rho(u)$ satisfy the following equation,

$$\Phi_\rho(u) = c_\rho(u) + d_\rho(u)\, \Phi_\rho(0). \tag{4.18}$$

Proof.

For any $t \geq 0$,

$$\begin{aligned}
&\mathbb{E}\left[C_\rho(\tau_1+t)-C_\rho(\tau_1)\,|\,I(0)=u\right]\\
&\quad=\mathbb{E}_{\tau_1}\left[\mathbb{E}\left[C_\rho(\tau_1+t)-C_\rho(\tau_1)\,|\,\tau_1, I(0)=u\right]\right]\\
&\quad=\mathbb{E}\left[e^{-r\tau_1}\,\Phi_\rho(t\,|\,0)\,|\,I(0)=u\right]\\
&\quad=\Phi_\rho(t\,|\,0)\;\mathbb{E}\left[e^{-r\tau_1}\,|\,I(0)=u\right]\\
&\quad=\Phi_\rho(t\,|\,0)\;d_\rho(u)
\end{aligned} \tag{4.19}$$

where $\mathbb{E}_{\tau_1}$ is the expectation operator with respect to the measure induced by τ_1. Here, the second equality holds by Eq. (4.17), and the last equality holds by Eq. (4.8).

Next, decomposing the infinite time horizon as $(0,\infty)=(0,\tau_1]\cup(\tau_1,\infty)$ yields

$$\begin{aligned}
\Phi_\rho(u)&=\mathbb{E}[C_\rho(\tau_1)\,|\,I(0)=u]+\lim_{t\to\infty}\mathbb{E}[\left(C_\rho(\tau_1+t)-C_\rho(\tau_1)\right)|\,I(0)=u]\\
&=c_\rho(u)+d_\rho(u)\,\lim_{t\to\infty}\Phi_\rho(t\,|\,0)
\end{aligned}$$

from which Eq. (4.18) readily follows. □

4.2 Equations for $c_\rho(u)$

In this section we derive an integro-differential equation for $c_\rho(u)$ from which we obtain a closed form formulas for its Laplace transform and $c_\rho(0)$.

For any given initial inventory level $u \geq 0$, and a time interval $(0,s]$, where $s>0$ is small, consider the following disjoint events and the corresponding discounted cost function, $c_\rho(u)$.

(1) On the event $\{A_1 > s\}$, the corresponding cost is

$$\begin{aligned}
&\mathbb{E}[C_\rho \mathbf{1}_{\{A_1>s\}} \mid I(0)=u] \\
&\qquad = \int_s^\infty \lambda e^{-\lambda t}\left[h\int_0^s (u+\rho z)e^{-rz}dz + c_\rho(u+\rho s)e^{-rs}\right]dt \\
&\qquad = e^{-\lambda s}\left[h\int_0^s (u+\rho z)e^{-rz}dz + c_\rho(u+\rho s)e^{-rs}\right]
\end{aligned}, \quad (4.20)$$

where the first term in the sums above is the discounted carrying cost over $(0,s]$, and the second is the discounted residual cost over $(s,\tau_1]$, since $\{A_1 > s\} \subset \{s \le \tau_1\}$.

(2) On the event $\{A_1 \le s\}$, the corresponding cost is

$$\mathbb{E}[C_\rho \mathbf{1}_{\{A_1 \le s\}} \mid I(0)=u] = \int_0^s \lambda e^{-\lambda t} M(u,t)\,dt \quad (4.21)$$

where $M(u,t) = \mathbb{E}[C_{\rho,r} \mid A_1 = t, I(0)=u]$ is given by

$$\begin{aligned}
M(u,t) = {} & h\int_0^t (u+\rho z)e^{-rz}\,dz \\
& + e^{-rt}\int_0^{u+\rho t} f_D(x)\,c_\rho(u+\rho t - x)\,dx \\
& + e^{-rt}\int_{u+\rho t}^\infty f_D(x)\,w(x-(u+\rho t))\,dx
\end{aligned} \quad (4.22)$$

So that

$$M(u,0) = \left\langle f_D * c_\rho \right\rangle(u) + \int_u^\infty f_D(x)\,w(x-u)dx\,. \quad (4.23)$$

Thus, adding Eqs. (4.20) and (4.21) yields

$$c_\rho(u) = e^{-\lambda s}\left[h\int_0^s (u+\rho z)e^{-rz}\,dz + c_\rho(u+\rho s)e^{-rs}\right] + \int_0^s \lambda e^{-\lambda t} M(u,t)\,dt \quad (4.24)$$

Next, differentiating Eq. (4.24) respect to s, and setting $s=0$, we have

$$0 = h\,u - (\lambda + r)\,c_\rho(u) + \rho \frac{\partial}{\partial u} c_\rho(u) + \lambda\, M(u, 0). \tag{4.25}$$

Finally, substituting Eq. (4.23) into Eq. (4.25) yields after rearranging terms

$$\rho \frac{\partial}{\partial u} c_\rho(u) - (\lambda + r)\,c_\rho(u) + \lambda \left\langle f_D * c_\rho \right\rangle(u) = -g(u), \tag{4.26}$$

where

$$g(u) = h\,u + \lambda \int_u^\infty f_D(x)\,w(x-u)dx. \tag{4.27}$$

It is convenient to decompose the function above into $g(u) = g_1(u) + g_2(u)$, where

$$g_1(u) = h\,u, \tag{4.28}$$

$$g_2(u) = \lambda \int_u^\infty f_D(x)\,w(x-u)dx. \tag{4.29}$$

Thus, $g_1(u)$ corresponds to the carrying cost component, while $g_2(u)$ corresponds to the lost-sales penalty component.

Next, we proceed to solve Eq. (4.26) for $c_\rho(u)$. To this end, we take Laplace transform on both sides of that equation to get

$$\rho[z\ \tilde{c}_\rho(z) - c_\rho(0)] - (\lambda + r)\,\tilde{c}_\rho(z) + \lambda\,\tilde{f}_D(z)\ \tilde{c}_\rho(z) = -\tilde{g}(z), \quad z > 0 \tag{4.30}$$

Rearranging and simplifying the above equation yields

$$[\lambda\,\tilde{f}_D(z) + \rho z - \lambda - r\,]\ \tilde{c}_\rho(z) - \rho\ c_\rho(0) = -\tilde{g}(z), \quad z > 0, \tag{4.31}$$

In view of Eq. (3.4), Eq. (4.31) can now be written as

$$\psi(z)\ \tilde{c}_\rho(z) - \rho\ c_\rho(0) = -\tilde{g}(z), \quad z > 0, \tag{4.32}$$

We are now in a position to derive a closed form formula for $c_\rho(0)$.

Proposition 4.3

For $c_\rho(0)$, the following is true,

$$c_\rho(0) = \begin{cases} \dfrac{1}{\rho}\tilde{g}(\xi), & \text{if } \rho > 0 \\ \dfrac{\lambda}{\lambda + r}\mathbb{E}[w(D)], & \text{if } \rho = 0 \end{cases} \tag{4.33}$$

Proof.

For $\rho > 0$, the result follows by setting $z = \xi$ in Eq. (4.32)and noting that its first term now vanishes by Lemma 3.1.

For $\rho = 0$, note that given $I(0) = 0$, $I(t) = 0$ for $t \le \tau_1$, $\tau_1 = A_1 = T_1$ and $L(\tau_1) = D_1$, w.p.1,. By Eq. (4.7), we have

$$\begin{aligned} c_\rho(0) &= \mathbb{E}[w(D_1)e^{-r\tau_1} \mid I(0) = 0] \\ &= \mathbb{E}[w(D_1)]\ \mathbb{E}[e^{-r\tau_1} \mid I(0) = 0] \\ &= \mathbb{E}[w(D_1)]\ \mathbb{E}[e^{-rT_1}] \end{aligned}$$

Furthermore, by the assumption that the inter-arrival time is exponentially distributed, one has

$$\tilde{f}_T(z) = \frac{\lambda}{\lambda + z}. \tag{4.34}$$

Finally, the result readily follows by substituting Eq.(4.34) with $z = r$ into the equation for $c_\rho(0)$ above. □

Next, we proceed to derive a closed form formula for the Laplace transform of $c_\rho(u)$.

Corollary 4.1

$$\tilde{c}_\rho(z) = \frac{\tilde{g}(\xi) - \tilde{g}(z)}{\psi(z)}, \; z \neq \xi \,. \tag{4.35}$$

Proof.

Follows immediately by substituting Eq. (4.33) into Eq. (4.32) and dividing the resultant equation by $\psi(z) \neq 0$. □

4.3 The Function $d_\rho(u)$

In this subsection, we derive a closed form formula for $d_\rho(0)$ in Proposition 4.4, and provide a closed form formula for $\tilde{d}_\rho(z)$ in Proposition 4.5.

Proposition 4.4

The following holds

$$d_\rho(0) = \frac{\lambda}{\rho}\,\tilde{\bar{F}}_D(\xi) = 1 - \frac{r}{\rho\,\xi}. \tag{4.36}$$

Proof.

To prove Eq. (4.36), consider the special case $h = 0$ and $w(x) = 1_{(0,\infty)}(x)$. Then, Eqs. (4.6) and (4.7) imply

$$c_\rho(u) = d_\rho(u). \tag{4.37}$$

Furthermore, Eq. (4.27) becomes

$$g(u) = \lambda \int_u^\infty f_D(x)\,dx = \lambda\,\bar{F}_D(u) \tag{4.38}$$

and in view of Eqs. (4.37) and (4.38), Eq. (4.33) becomes

$$c_\rho(0) = d_\rho(0) = \frac{\lambda}{\rho}\tilde{\bar{F}}_D(\xi)\,. \tag{4.39}$$

By Eqs. (1.4) and (3.4), we have

$$\tilde{\bar{F}}_D(z) = \frac{\rho\, z - r - \psi(z)}{\lambda\, z}, \tag{4.40}$$

so setting $z = \xi$ above, noting that $\psi(\xi) = 0$ and substituting the resultant Eq. (4.40) into Eq. (4.39) yield Eq. (4.36). □

Proposition 4.5

For $\rho > 0$,

$$\tilde{d}_\rho(z) = \frac{r}{\psi(z)}\left[\frac{1}{z} - \frac{1}{\xi}\right] + \frac{1}{z}, \quad z \neq \xi \tag{4.41}$$

Proof.

To prove Eq. (4.41), take the Laplace transform of Eq. (4.38) and substitute it into Eq.(4.35), yielding

$$\tilde{d}_\rho(z) = \frac{\lambda}{\psi(z)}\left[\tilde{\bar{F}}_D(\xi) - \tilde{\bar{F}}_D(z)\right],\ z \neq \xi\,. \tag{4.42}$$

Finally, substituting Eq. (4.40) into Eq. (4.42) yields Eq. (4.41) □

4.4 The Function $\Phi_\rho(u)$

Recall that our goal is to compute an optimal replenishment rate, ρ^* of the expected discounted cost $\Phi_\rho(u)$. However, for $u > 0$, it is not possible to derive a closed form representation of $\Phi_\rho(u)$ as function of ρ. Nevertheless, it is possible to derive a closed form representation for

$\Phi_\rho(u) = \Phi_{\rho(\xi)}(u)$ as function of ξ. Using the fact that the mapping $\rho \mapsto \xi$ (implicitly defined by Eq. (3.10)) is 1-1, one can optimize $\Phi_\rho(u) = \Phi_{\rho(\xi)}(u)$ with respect to ξ. We can then map back the optimal ξ^* to determine the corresponding optimal ρ^*. The main results in this subsection are given in Theorem 4.2 and Theorem 4.4.

In the sequel, for notational simplicity, we will use $\Phi_\xi(u)$, $\Phi_{\rho(\xi)}(u)$ interchangeably. The same convention will be adopted for other quantities such as $c_{\rho(\xi)}$, $d_{\rho(\xi)}$, $\alpha_{\rho(\xi)}$, $\beta_{\rho(\xi)}$, $\delta_{\rho(\xi)}$, etc.

Theorem 4.2

$$\Phi_\xi(0) = \xi\, c_\rho(0) = \frac{\xi}{r}\,\tilde{g}(\xi). \tag{4.43}$$

Proof.

Setting $u = 0$ in Eq. (4.18) and rearranging yield

$$\Phi_{\rho(\xi)}(0) = \frac{c_{\rho(\xi)}(0)}{1 - d_{\rho(\xi)}(0)}. \tag{4.44}$$

Eq. (4.43) now follows by substituting Eqs. (4.33) and (4.36) into Eq. (4.44). □

Proposition 4.6

$$\Phi_\xi(u) = c_\xi(u) + \frac{\xi}{r}\tilde{g}(\xi)\, d_\xi(u), \quad u \ge 0 \tag{4.45}$$

$$\tilde{\Phi}_\xi(z) = \xi\,\tilde{g}(\xi)\left[\frac{1}{r\,z} + \frac{1}{z\,\psi(z)}\right] - \frac{\tilde{g}(z)}{\psi(z)}, \quad z \ne \xi \tag{4.46}$$

Proof.

Eq. (4.45) follows readily by substituting Eq. (4.43) into Eq. (4.18). Eq. (4.46) obtains by taking Laplace transforms of both sides of Eq. (4.45) and substituting $\tilde{c}_\rho(z)$ in Eq. (4.35) and $\tilde{d}_\rho(z)$ in Eq. (4.41) . □

We next obtain an alternate representation of $\Phi_\xi(u)$ by inverting Eq. (4.46). To this end, define

$$G_\xi(x) = \xi\,\tilde{g}(\xi) - g(x), \tag{4.47}$$

Theorem 4.3

$$\Phi_\xi(u) = \Phi_\xi(0) + \left\langle G_\xi * \eta_\xi \right\rangle(u), \quad u \geq 0 \tag{4.48}$$

where $\Phi_\xi(0)$ is given by Eq. (4.43) and $G_\xi(x)$ given by Eq. (4.47).

Proof.

Eq. (4.46) can be rewritten as

$$\tilde{\Phi}_\xi(z) = \frac{\xi\,\tilde{g}(\xi)}{r} \times \frac{1}{z} + \frac{1}{\psi(z)}\left[\frac{\xi\,\tilde{g}(\xi)}{z} - \tilde{g}(z)\right].$$

Eq. (4.48) now follows by inverting the equation above, noting that $\frac{\xi\,\tilde{g}(\xi)}{r} = \Phi_\xi(0)$ by Eq. (4.43) and $\frac{\xi\,\tilde{g}(\xi)}{z} - \tilde{g}(z) = \tilde{G}_\xi(z)$ by Eq. (4.47). □

Theorem 4.4

$$\begin{aligned}\Phi_{\xi}(u) &= \frac{\xi\,\tilde{g}(\xi)}{r} + \frac{V_{\rho(\xi)}(\xi)}{\xi-\theta}\left[e^{\xi u}\int_u^{\infty} g(x)\,e^{-\xi x}\,dx - \tilde{g}(\xi)\right] \\ &\quad + \frac{V_{\rho(\xi)}(\theta)}{\xi-\theta}\left[e^{\theta u}\int_0^{u} g(x)\,e^{-\theta x}\,dx - \frac{\xi\,\tilde{g}(\xi)}{\theta}\left(e^{\theta u}-1\right)\right]\end{aligned} \tag{4.49}$$

where $V_{\rho(\xi)}(x)$ is given by Eq. (3.14) and $g(x)$ is given by Eq. (4.27).

Proof.

By Eq. (3.17), the convolution term in Eq. (4.48) becomes

$$\left\langle G_{\xi} * \eta_{\xi}\right\rangle(u) = \frac{V_{\rho(\xi)}(\xi)}{\xi-\theta}e^{\xi u}\int_0^{u} G_{\xi}(x)\,e^{-\xi x}\,dx + \frac{V_{\rho(\xi)}(\theta)}{\theta-\xi}e^{\theta u}\int_0^{u} G_{\xi}(x)\,e^{-\theta x}\,dx\,, \tag{4.50}$$

where $G_{\xi}(x)$ is given by Eq. (4.47).

Next, substituting Eqs. (4.43) and (4.50) into Eq. (4.48) yields

$$\Phi_{\xi}(u) = \frac{\xi\,\tilde{g}(\xi)}{r} + \frac{V_{\rho(\xi)}(\xi)}{\xi-\theta}e^{\xi u}\int_0^{u} G_{\xi}(x)\,e^{-\xi x}\,dx + \frac{V_{\rho(\xi)}(\theta)}{\theta-\xi}e^{\theta u}\int_0^{u} G_{\xi}(x)\,e^{-\theta x}\,dx\,, \tag{4.51}$$

Eq. (4.49) now follows by substituting Eq. (4.47) into Eq.(4.51). □

In the next two subsections, we shall specialize Eq. (4.49) to two penalty function structures: constant lost-sale penalty and loss-proportional penalty.

4.4.1 Constant Lost-Sale Penalty

In this subsection, we consider the constant lost-sales penalty function, $w(x) = K_0\,\mathbf{1}_{(0,\infty)}(x)$, provided $\rho > 0$. Then Eq. (4.27) becomes

$$g(u) = h\,u + \lambda\,K_0\bar{F}_D(u), \quad u \geq 0, \tag{4.52}$$

and taking Laplace transforms above yields

$$\tilde{g}(z) = \frac{h}{z^2} + \lambda\,K_0\tilde{\bar{F}}_D(z). \tag{4.53}$$

Next, setting $z = \xi$ and substituting $\tilde{\bar{F}}_D(\xi)$ from Eq. (3.10) into Eq. (4.53), we have

$$\tilde{g}(\xi) = \frac{h}{\xi^2} + K_0\left[\rho - \frac{r}{\xi}\right], \tag{4.54}$$

and substituting Eq. (4.54) into Eq. (4.43) yields

$$\Phi_\xi(0) = \frac{h}{r\,\xi} + K_0\left(\frac{\rho\,\xi}{r} - 1\right). \tag{4.55}$$

Furthermore, substituting Eqs. (4.52) and (4.54) into Eq. (4.49) yields

$$\Phi_\xi(u) = \Phi_\xi(0) + \frac{V_{\rho(\xi)}(\xi)}{\xi - \theta}\phi_1^c(u,\xi) + \frac{V_{\rho(\xi)}(\theta)}{\xi - \theta}\phi_2^c(u,\theta), \tag{4.56}$$

where $\Phi_\xi(0)$ is given by Eq. (4.55), and

$$\phi_1^c(u,\xi) = \frac{h}{\xi}u + \lambda\,K_0\,e^{\xi u}\int_u^\infty \bar{F}_D(x)e^{-\xi x}\,dx - \frac{1}{\xi}K_0\left(\rho\,\xi - r\right),$$

$$\phi_2^c(u,\theta) = -\frac{h}{\theta}u + \lambda\,K_0 e^{\theta u}\int_0^u \bar{F}_D(x)e^{-\theta x}\,dx - \frac{1}{\theta}\left(r\Phi_\rho(0) - \frac{h}{\theta}\right)\left(e^{\theta u} - 1\right).$$

4.4.2 Loss-Proportional Penalty

In this subsection, we consider the loss-proportional penalty, $w(x) = K_1 x$, provided $\rho > 0$. Then Eq. (4.27) becomes

$$g(u) = h\,u + \lambda\,K_1\int_u^\infty x\,f_D(x)\,dx, \quad u \geq 0, \tag{4.57}$$

and the corresponding Laplace transform is given by

$$\tilde{g}(z) = \frac{h}{z^2} + \lambda K_1 \left[\frac{\mu_D}{z} - \frac{1 - \tilde{f}_D(z)}{z^2} \right]. \tag{4.58}$$

Next, setting $z = \xi$ in Eq. (4.58) and substituting further $\tilde{f}_D(\xi)$ given in Eq. (3.8) , we have

$$\tilde{g}(\xi) = \frac{h}{\xi^2} + \lambda K_1 \left[\frac{\mu_D}{\xi} - \frac{\rho}{\lambda \xi} + \frac{r}{\lambda \xi^2} \right], \tag{4.59}$$

and substituting Eq. (4.59) into Eq. (4.43) yields

$$\Phi_\xi(0) = \frac{h}{r\xi} + K_1 \left[\frac{\lambda \mu_D - \rho}{r} + \frac{1}{\xi} \right], \tag{4.60}$$

Furthermore, substituting Eqs. (4.57) and (4.59) into Eq. (4.49) yields

$$\Phi_\xi(u) = \Phi_\xi(0) + \frac{V_{\rho(\xi)}(\xi)}{\xi - \theta} \phi_1^p(u, \xi) + \frac{V_{\rho(\xi)}(\theta)}{\xi - \theta} \phi_2^p(u, \theta), \tag{4.61}$$

where $\Phi_\xi(0)$ is given by Eq. (4.60), and

$$\phi_1^p(u, \xi) = \frac{h}{\xi} u + \lambda K_1 e^{\xi u} \int_u^\infty \int_x^\infty z f_D(z) e^{-\xi x} \, dz \, dx + \frac{1}{\xi} \left(\frac{h}{\xi} - r\Phi_\rho(0) \right),$$

$$\phi_2^p(u, \theta) = -\frac{h}{\theta} u + \lambda K_1 e^{\theta u} \int_0^u \int_x^\infty z f_D(z) e^{-\theta x} \, dz \, dx + \frac{1}{\theta} \left(\frac{h}{\theta} - r\Phi_\rho(0) \right) \left(e^{\theta u} - 1 \right).$$

4.5 Optimal Replenishment Rate

In this subsection, we optimize the expected discounted cost of $\Phi_\rho(u)$ with respect to the replenishment rate, ρ, via an optimization of $\Phi_{\rho(\xi)}(u)$ with respect to ξ. We first provide a general structural result for an optimal replenishment rate, ρ^* (admitting the possibility of multiple optimal replenishment rates), and then describe a computational method for finding the optimal solutions.

While the cost function, $\Phi_\xi(u)$, given by Eq. (4.49), is expressed in terms of the two roots, θ and ξ, we shall express $\Phi_\xi(u)$ as a function of ξ alone. To this end, we write with the aid Eq. (3.15)

$$\theta = -r V_\rho(0) / \xi . \tag{4.62}$$

Accordingly, substituting Eq. (4.62) into Eq. (4.49) yields

$$\begin{aligned}\Phi_\xi(u) = &\\ &\frac{\xi \tilde{g}(\xi)}{r} + \frac{\xi V_\xi(\xi)}{\xi^2 + r V_\xi(0)}\left[e^{\xi u}\int_u^\infty g(x) e^{-\xi x}\,dx - \tilde{g}(\xi)\right] \\ &+ \frac{\xi V_\xi\left(-r V_\xi(0)/\xi\right)}{\xi^2 + r V_\xi(0)}\left[e^{-r V_\xi(0)/\xi u}\int_0^u g(x) e^{r V_\xi(0)/\xi x}\,dx + \frac{\xi^2 \tilde{g}(\xi)}{r V_\xi(0)}\left(e^{-r V_\xi(0)/\xi u} - 1\right)\right]\end{aligned} \tag{4.63}$$

The boundedness of $\Phi_\xi(u)$, implied by Proposition 4.2, guarantees the existence of the global minimizing point, $\xi^* = \underset{\xi>0}{\operatorname{argmin}}\,\{\Phi_\xi(u)\}$. However, the function $\Phi_\xi(u)$ is not convex in general. The following example illustrates the non-convexity of the functions, $\Phi_\xi(u)$ and $\Phi_\rho(u)$.

Consider the production-inventory system with constant demand size $d = 30$ and the following parameters: $\lambda = 1$, $h = 1$, $u = 0$ $K_0 = 100$, $K_1 = 0$, and $r = 0.1$.

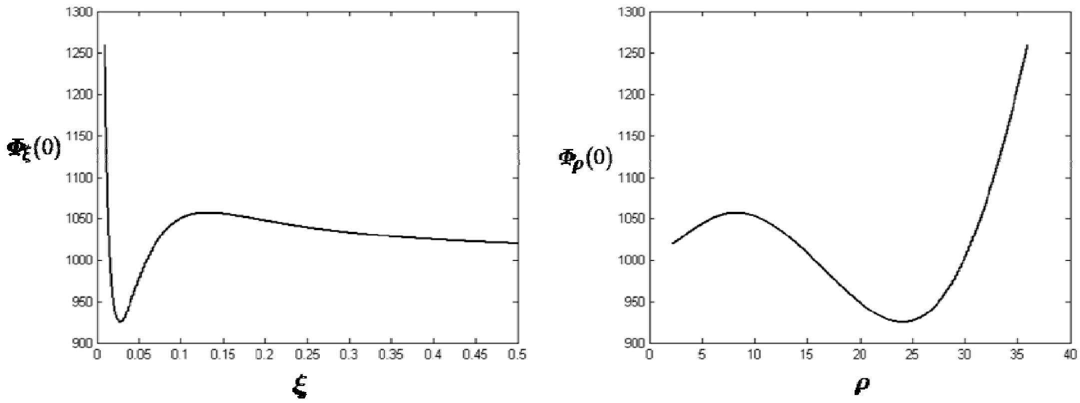

Figure 4.1. The Functions $\Phi_\xi(0)$ (Left) and $\Phi_\rho(0)$ (Right) for Constant Demand Size

Figure 4.1 depicts the functions $\Phi_\xi(0)$ (left) and $\Phi_\rho(0)$ (right). It shows that in this case, the expected discounted cost is not convex in ξ or ρ. In fact, it is theoretically challenging to prove uniqueness of the global minimizing point, which remains as an open problem.

In light of Theorem 4.4, a minimizing point, ξ^*, can be calculated in several ways. A straightforward but relatively time consuming method is global search. However, when $\Phi_\xi(u)$ is convex, the availability of derivatives of the objective function $\Phi_\xi(u)$ with respect to ξ allows us to apply the relatively fast Newton's Method, where successive approximations of the minimum are started with any $\xi_0 > 0$, and given by the iterative scheme

$$\xi_{n+1} = \xi_n - \frac{\dfrac{\partial}{\partial \xi}\Phi_{\xi_n}(u)}{\dfrac{\partial^2}{\partial \xi^2}\Phi_{\xi_n}(u)}, \quad n=0,1,\ldots. \tag{4.64}$$

We next state, for completeness, the following theorem without proof.

Theorem 4.5

Given $I(0)=u$, the optimal replenishment rates for $\Phi_\rho(u)$ are given by

$$\rho^* = \frac{r+\lambda\left[1-\tilde{f}_D(\xi^*)\right]}{\xi^*}. \tag{4.65}$$

□

4.6 Special Cases

In this subsection we study special cases of production-inventory system with selected demand-size distributions, subject to two specialized lost-sales penalty structures: constant lost-sales penalty and loss-proportional penalty. We shall assume here that $\rho > 0$ and $r \geq 0$, and $V_\rho(x) = V_{\rho(\xi)}(x)$ is given by Eq. (3.14) and $G_\xi(x)$ is given by Eq. (4.47).

4.6.1 Example: Computation of $\Phi_\rho(u)$ for Exponential Demand Size Distribution

In this subsection, we illustrate the derivation of the function $\Phi_\xi(u)$, subject to each penalty function, for the case of exponentially distributed demand size with rate $\beta > 0$, that is,

$$f_D(x) = \beta e^{-\beta x}, \quad x \geq 0 \tag{4.66}$$

$$\tilde{f}_D(z) = \frac{\beta}{\beta + z}, \quad z \geq 0 \tag{4.67}$$

Then, substituting Eq. (4.67) into Eq. (3.4) yields

$$\psi(z) = \frac{\lambda \beta}{\beta + z} + \rho z - \lambda - r = \frac{(z-\theta)(z-\xi)}{V_\rho(z)}, \tag{4.68}$$

where,

$$V_\rho(z) = \frac{z+\beta}{\rho}. \tag{4.69}$$

Hence, in agreement with Lemma 3.1, the roots of the equation $\psi(z) = 0$ are given by

$$\xi = \frac{\lambda + r - \rho\beta + \sqrt{\left(\lambda + r - \rho\beta\right)^2 + 4r\rho\beta}}{2\rho} \geq 0 \tag{4.70}$$

$$\theta = \frac{\lambda + r - \rho\beta - \sqrt{\left(\lambda + r - \rho\beta\right)^2 + 4r\rho\beta}}{2\rho} \leq 0. \tag{4.71}$$

4.6.2 Constant Lost-Sale Penalty

In this case, Eq. (4.63) can be written as

$$\Phi_\rho(u) = a_0 + a_1 u + a_2 e^{\theta u} \tag{4.72}$$

where

$$a_0 = \frac{h}{r}\left(\frac{1}{\xi} + \frac{1}{\beta} + \frac{1}{\theta}\right) \tag{4.73}$$

$$a_1 = \frac{h}{r} \tag{4.74}$$

$$a_2 = \frac{\lambda K_0 \xi}{r\left(\beta + \xi\right)} - \frac{h}{r}\left(\frac{1}{\beta} + \frac{1}{\theta}\right) \tag{4.75}$$

4.6.3 Loss-Proportional Penalty

In this case, Eq. (4.63) can be written as

$$\Phi_\rho(u) = a_0 + a_1 u + a_4 e^{-\beta u} + a_5 e^{\theta u}, \tag{4.76}$$

where a_0 and a_1 are given by Eqs. (4.73) - (4.74), and

$$a_4 = \frac{\lambda K_1}{\rho}\left[\frac{1}{(\xi-\theta)(\beta+\theta)} - \frac{1}{\theta(\beta+\xi)}\right] - \frac{h}{r}\left(\frac{1}{\beta}+\frac{1}{\theta}\right)$$

$$a_5 = -\frac{\lambda K_1}{\rho(\beta+\xi)(\beta+\theta)}.$$

A numerical study of $\Phi_\rho(u)$ with exponential demand distribution is described next.

4.7 Optimal Replenishment Rate under Delayed Replenishment

In practice, the system starts with an arbitrary inventory level, $I(0) = u > 0$. Suppose the system operates under delayed replenishment such that replenishment starts only after the first lost-sale occurrence. For example, suppose the system has an initial setup time during which replenishment is unavailable (e.g., the production facility requires a setup time to gear up for production). Accordingly, the corresponding expected discounted cost, $\hat{\Phi}_\rho(u)$, over an infinite time horizon can be expressed as

$$\hat{\Phi}_\rho(u) = c_0(u) + d_0(u)\,\Phi_\rho(0). \tag{4.77}$$

From Eq.(4.77), it is readily seen that minimizing $\hat{\Phi}_\rho(u)$ with respect to ρ is equivalent to minimizing $\Phi_\rho(0)$ with respect to ρ, since only the latter term is a function of ρ.

4.7.1 Constant Lost-Sales Penalty

In this case, $w(x) = 1_{\{x>0\}} K_0$, where $K_0 > 0$ is a constant, and $\Phi_\xi(0)$ is given by Eq. (4.55). In view of Eq. (3.13), Eq. (4.55) can be written as

$$\Phi_{\xi}(0) = \frac{h + \lambda \xi K_0 \left[1 - \tilde{f}_D(\xi)\right]}{r \xi}, \tag{4.78}$$

By Eq.(4.78), the optimal ξ^* is given by

$$\xi^* = \underset{\xi>0}{\textbf{argmin}} \left\{ \frac{h}{\xi} - \lambda K_0 \tilde{f}_D(\xi) \right\}. \tag{4.79}$$

Table 4.1 exhibits ξ^*, ρ^* and $\Phi_{\rho^*}(0)$ in closed form formulas, when available, for selected demand distributions; detailed derivations are given in Section A.1 of Appendix A.

Table 4.1. Optimal Quantities for Production-Inventory Systems Subject to Constant Penalty and Various Demand Distributions

Distribution	ξ^*	ρ^*	$\Phi_{\rho^*}(0)$
$D = d$ $d > 0$	$\underset{\xi>0}{\text{argmin}} \left\{ \frac{h}{\xi} - \lambda K_0 e^{-\xi d} \right\}$	$\frac{r + \lambda\left[1 - e^{-\xi^* d}\right]}{\xi^*}$	$\frac{h + K_0 \xi^* \left(\rho^* \xi^* - r\right)}{r \xi^*}$
$D \sim \text{Exp}(\beta)$ $\beta > 0$	$\begin{cases} \frac{\beta\sqrt{h}}{\sqrt{\lambda\beta K_0} - \sqrt{h}}, & \text{if } \beta\lambda K_0 > h \\ \infty, & \text{otherwise} \end{cases}$	$\begin{cases} \frac{\lambda}{\beta} - \frac{\sqrt{\lambda h}}{\beta\sqrt{\beta K_0}}, & \text{if } \beta\lambda K_0 > h \\ 0, & \text{otherwise} \end{cases}$	$\begin{cases} \frac{2\sqrt{\lambda\beta K_0 h} - h}{r\beta}, & \text{if } \beta\lambda K_0 > h \\ \frac{\lambda K_0}{r}, & \text{otherwise} \end{cases}$
$D \sim U(a,b)$ $0 \le a < b$	$\underset{\xi>0}{\text{argmin}} \left\{ \frac{h}{\xi} - \lambda K_0 \frac{e^{-a\xi} - e^{-b\xi}}{(b-a)\,\xi} \right\}$	$\frac{r}{\xi^*} + \frac{\lambda}{\xi^*}\left[1 - \frac{e^{-a\xi^*} - e^{-b\xi^*}}{(b-a)\,\xi^*}\right]$	$\frac{h + K_0 \xi^* \left(\rho^* \xi^* - r\right)}{r \xi^*}$
$D \sim \Gamma(\alpha,\beta)$ $\alpha, \beta > 0$	$\underset{\xi>0}{\text{argmin}} \left\{ \frac{h}{\xi} - \lambda K_0 (1 + \xi/\beta)^{-\alpha} \right\}$	$\frac{r}{\xi^*} + \frac{\lambda}{\xi^*}\left[1 - \left(1 + \xi^*/\beta\right)^{-\alpha}\right]$	$\frac{h + K_0 \xi^* \left(\rho^* \xi^* - r\right)}{r \xi^*}$

In the table above and elsewhere, the argmin operation corresponds to a search for the optimal argument, where a closed form formula is unavailable or not readily available; for exponential demand distributions, the optimal solution is given in a closed form formula. Furthermore, the condition $\beta\lambda K_0 > h$ ensures a positive optimal replenishment rate; otherwise, it is optimal to have zero replenishment and bear the repeated penalty costs (degenerate case).

4.7.2 Loss-Proportional Penalty

In this case, $w(x) = 1_{\{x>0\}} K_1 x$, where $K_1 > 0$ is constant, and $\Phi_\xi(0)$ is given by Eq. (4.60). In view of Eq. (3.13), Eq. (4.60) can be written as

$$\Phi_\xi(0) = \frac{1}{r}\left[\frac{h}{\xi} - \lambda K_1 \frac{1-\tilde{f}_D(\xi)}{\xi}\right] + \frac{\lambda K_1 \mu_D}{r}, \tag{4.80}$$

where $\mu_D = \mathbb{E}[D]$. Consequently, by Eq. (4.80), the optimal ξ^* is given by

$$\xi^* = \operatorname*{argmin}_{\xi>0}\left\{\frac{h}{\xi} - \lambda K_1 \frac{1-\tilde{f}_D(\xi)}{\xi}\right\}. \tag{4.81}$$

Table 4.2 exhibits ξ^*, ρ^* and $\Phi_{\rho^*}(0)$ in closed form formulas, when available, for selected demand distributions; detailed derivations are given in Section A.2 of Appendix A.

Table 4.2. Optimal Quantities for Production-Inventory Systems Subject to Loss-Proportional Penalty and Various Demand Distributions

Distribution	ξ^*	ρ^*	$\Phi_{\rho^*}(0)$
$D = d$ $d > 0$	$\operatorname*{argmin}_{\xi>0}\left\{\frac{h}{\xi} - \lambda K_1 \frac{1-e^{-\xi d}}{\xi}\right\}$	$\frac{r+\lambda\left[1-e^{-\xi^* d}\right]}{\xi^*}$	$\frac{1}{r}\left[\frac{h}{\xi^*} - \lambda K_1 \frac{1-e^{-\xi^* d}}{\xi^*} + \lambda K_1 d\right]$
$D \sim \mathrm{Exp}(\beta)$ $\beta > 0$	$\begin{cases} \frac{\beta\sqrt{h}}{\sqrt{\lambda K_1}-\sqrt{h}}, & \text{if } \lambda K_1 > h \\ \infty, & \text{otherwise} \end{cases}$	$\begin{cases} \frac{\sqrt{\lambda K_1}-\sqrt{h}}{\beta}\left[\frac{r}{\sqrt{h}}+\sqrt{\frac{\lambda}{K_1}}\right], & \text{if } \lambda K_1 > h \\ 0, & \text{otherwise} \end{cases}$	$\begin{cases} \frac{2\sqrt{\lambda K_1 h}-h}{r\beta}, & \text{if } \lambda K_1 > h \\ \frac{\lambda K_1}{r\beta}, & \text{otherwise} \end{cases}$
$D \sim U(a,b)$ $0 \le a < b$	$\operatorname*{argmin}_{\xi>0}\left\{\frac{h}{\xi} - \frac{\lambda K_1}{\xi}\left[1-\frac{e^{-a\xi}-e^{-b\xi}}{(b-a)\,\xi}\right]\right\}$	$\frac{r}{\xi^*}+\frac{\lambda}{\xi^*}\left[1-\frac{e^{-a\xi^*}-e^{-b\xi^*}}{(b-a)\,\xi^*}\right]$	$\frac{1}{r}\left[\frac{h-\lambda K_1}{\xi^*}+\lambda K_1\frac{e^{-a\xi^*}-e^{-b\xi^*}}{(b-a)\,\xi^{*2}}+\frac{\lambda K_1(b-a)}{2}\right]$
$D \sim \Gamma(\alpha,\beta)$ $\alpha, \beta > 0$	$\operatorname*{argmin}_{\xi>0}\left\{\frac{h}{\xi} - \lambda K_1 \frac{1-(1+\xi/\beta)^{-\alpha}}{\xi}\right\}$	$\frac{r}{\xi^*}+\frac{\lambda}{\xi^*}\left[1-(1+\xi^*/\beta)^{-\alpha}\right]$	$\frac{1}{r}\left[\frac{h}{\xi^*} - \lambda K_1 \frac{1-(1+\xi^*/\beta)^{-\alpha}}{\xi^*} + \lambda K_1 \alpha\beta\right]$

Again, for an exponential demand distribution, the condition $\lambda K_1 > h$ ensures a positive optimal replenishment rate; otherwise, it is optimal to have zero replenishment and bear the repeated penalty costs (degenerate case).

4.8 Numerical Study

This section contains two numerical studies of production-inventory systems with selected demand-size distributions, subject to a constant lost-sales penalty. Both studies were conducted with the following common parameters: $\lambda = 1$, $h = 1$, $K_0 = 100$, and $r = 0.1$. Recall that only the exponential demand-size distribution gives rise to closed-form optimal solutions when conditioned on $I(0)=0$; in all other cases, optimal solutions were obtained by a search.

4.8.1 Optimal Numerical Solutions for Empty Initial Inventory

In this subsection we compute and compare the numerical values of $\Phi_{\rho^*}(0)$ for increasing mean demand sizes, and under the following demand-size distributions: constant, exponential, uniform and Gamma.

Table 4.3 displays ρ^* and $\Phi_{\rho^*}(u)$ as functions of the mean demand, $\mathbf{E}[D]=1/\beta$, with the four aforementioned demand-size distributions.

Table 4.3. Optimal Values for Selected Demand-Size Distributions

	$D=1/\beta$		$D \sim \mathbf{Exp}(\beta)$		$D \sim U(0,2/\beta)$		$D \sim \Gamma(4,1/(4\beta))$	
$\lambda \mathbf{E}[D]$	ρ^*	$\Phi_{\rho^*}(0)$	ρ^*	$\Phi_{\rho^*}(0)$	ρ^*	$\Phi_{\rho^*}(0)$	ρ^*	$\Phi_{\rho^*}(0)$
0.05	0.27	44.47	0.27	44.22	0.27	44.39	0.27	44.41
0.10	0.41	62.74	0.40	62.25	0.41	62.58	0.41	62.62
0.15	0.53	76.71	0.52	75.96	0.52	76.45	0.52	76.52
0.20	0.63	88.44	0.62	87.44	0.63	88.10	0.63	88.19
0.30	0.82	108.03	0.80	106.54	0.82	107.53	0.82	107.66
0.80	1.62	174.82	1.54	170.89	1.59	173.47	1.59	173.83
1.30	2.30	221.40	2.16	215.04	2.25	219.19	2.26	219.79
1.80	2.93	259.11	2.72	250.33	2.85	256.04	2.87	256.88
2.30	3.51	291.50	3.23	280.32	3.41	287.56	3.45	288.64
2.80	4.07	320.24	3.72	306.66	3.95	315.43	3.99	316.76
3.30	4.63	346.27	4.20	330.32	4.47	340.58	4.53	342.18
3.80	5.17	370.19	4.62	351.87	4.99	363.62	5.02	365.48
4.30	5.69	392.40	5.05	371.73	5.43	384.95	5.51	387.08
4.80	6.17	413.20	5.48	390.18	5.92	404.85	6.01	407.26
5.30	6.66	432.79	5.87	407.43	6.37	423.54	6.47	426.23
5.80	7.15	451.34	6.22	423.66	6.82	441.20	6.94	444.17
6.30	7.64	468.98	6.61	439.00	7.22	457.96	7.35	461.20
6.80	8.08	485.84	6.95	453.54	7.68	473.90	7.83	477.44
7.30	8.59	501.97	7.28	467.37	8.08	489.12	8.24	492.95
7.80	9.03	517.44	7.61	480.57	8.48	503.69	8.65	507.82
8.30	9.47	532.35	7.94	493.19	8.87	517.68	9.06	522.11
8.80	9.92	546.72	8.27	505.30	9.27	531.14	9.47	535.88
9.30	10.36	560.62	8.60	516.92	9.67	544.11	9.88	549.15
9.80	10.71	574.05	8.84	528.10	10.07	556.64	10.30	562.00
10.00	10.95	579.30	9.02	532.46	10.19	561.50	10.43	566.99
15.00	14.86	693.46	11.58	624.60	13.41	666.27	13.98	675.10
20.00	18.37	784.52	13.61	694.43	16.28	747.54	16.88	760.16
25.00	21.23	860.50	15.00	750.00	18.21	813.34	19.60	830.15
30.00	23.87	925.43	16.14	795.45	19.81	867.66	21.50	889.22

From Table 4.3 it can be seen that the respective ρ^* and the corresponding $\Phi_{\rho^*}(0)$ increase in this order of distributions: exponential, uniform, Gamma and constant. Note that as the average demand increases, ρ^* and $\Phi_{\rho^*}(0)$ increase as expected. Furthermore, for each selected demand-size distribution, we observe that $\rho^* > \lambda \mathbf{E}[D]$ for $\lambda \mathbf{E}[D] < 7$, whereas $\rho^* < \lambda \mathbf{E}[D]$ for $\lambda \mathbf{E}[D] > 15$. This can be explained heuristically as follows: in the former case the optimal cost is dominated by its penalty component due to a relatively high inventory level, while in the latter case, it is dominated by its carrying cost component due to a relatively low inventory level.

4.8.2 Optimal Numerical Solutions for Arbitrary initial Inventory Levels

In this subsection we compute and compare the numerical values of ξ^*, ρ^* and $\Phi_{\rho^*}(u)$ for selected demand-size distributions (constant, exponential and uniform) with increasing initial inventory levels and low, medium and large average demands.

Table 4.4-6 display ρ^*, ξ^* and $\Phi_{\rho^*}(u)$ for sample low, medium and high demand as functions of the initial inventory level $I(0) = u$.

Table 4.4. Optimal Quantities for Selected Demand-Size Distributions under a Low Demand with

$\lambda\,\mathbb{E}[D] = 1/\beta = 2$

	$D = 1/\beta$			$D \sim \mathbf{Exp}(\beta)$			$D \sim U(0, 2/\beta)$		
$I(0) = u$	ξ^*	ρ^*	$\Phi_{\rho^*}(u)$	ξ^*	ρ^*	$\Phi_{\rho^*}(u)$	ξ^*	ρ^*	$\Phi_{\rho^*}(u)$
0	0.076	3.169	272.590	0.082	2.936	262.840	0.078	3.087	269.170
5	0.130	2.530	157.450	0.113	2.515	193.450	0.113	2.614	172.160
10	0.194	2.173	150.260	0.155	2.171	181.490	0.177	2.165	163.060
15	0.301	1.835	175.720	0.208	1.893	191.810	0.215	1.996	177.840
20	0.372	1.679	197.900	0.273	1.660	212.640	0.367	1.569	204.800
25	0.513	1.445	229.660	0.357	1.447	239.160	0.387	1.528	232.330
30	0.547	1.399	262.250	0.466	1.250	269.040	0.469	1.383	264.410
35	0.717	1.202	295.100	0.610	1.065	301.060	0.674	1.119	297.570
40	1.160	0.864	330.040	0.812	0.885	334.520	1.065	0.816	329.030
45	1.714	0.623	364.120	1.109	0.712	368.970	1.439	0.644	362.650
50	7.598	0.145	392.380	1.591	0.541	404.150	3.055	0.333	396.290

Table 4.5. Optimal Quantities for Selected Demand-Size Distributions under a Medium Demand

with $\lambda\,\mathbb{E}[D] = 1/\beta = 10$

	$D = 1/\beta$			$D \sim \mathbf{Exp}(\beta)$			$D \sim U(0, 2/\beta)$		
$I(0) = u$	ξ^*	ρ^*	$\Phi_{\rho^*}(u)$	ξ^*	ρ^*	$\Phi_{\rho^*}(u)$	ξ^*	ρ^*	$\Phi_{\rho^*}(u)$
0	0.038	10.940	579.290	0.046	9.014	532.460	0.041	10.175	561.500
5	0.042	10.537	538.850	0.048	8.831	510.550	0.040	10.281	529.820
10	0.042	10.537	473.330	0.051	8.575	495.870	0.045	9.783	503.520
15	0.046	10.181	473.980	0.054	8.338	487.280	0.047	9.603	486.570
20	0.051	9.787	473.100	0.058	8.046	483.780	0.050	9.350	483.970
25	0.051	9.787	472.220	0.062	7.779	484.590	0.055	8.966	483.810
30	0.051	9.787	480.220	0.067	7.475	489.050	0.061	8.556	486.500
35	0.059	9.242	491.380	0.072	7.198	496.600	0.061	8.556	494.790
40	0.076	8.315	503.470	0.077	6.944	506.800	0.073	7.860	506.230
45	0.076	8.315	517.540	0.084	6.621	519.270	0.078	7.606	519.590
50	0.076	8.315	536.250	0.091	6.331	533.700	0.090	7.066	532.690

Table 4.6. Optimal Quantities for Selected Demand-Size Distributions under a High Demand with

$$\lambda\, \mathbb{E}[D] = 1 / \beta = 20$$

	$D = 1/\beta$			$D \sim \mathrm{Exp}(\beta)$			$D \sim U(0, 2/\beta)$		
$I(0) = u$	ξ^*	ρ^*	$\Phi_{\rho^*}(u)$	ξ^*	ρ^*	$\Phi_{\rho^*}(u)$	ξ^*	ρ^*	$\Phi_{\rho^*}(u)$
0	0.030	18.299	784.500	0.041	13.519	694.430	0.034	16.187	747.520
5	0.031	18.251	774.650	0.041	13.356	684.430	0.034	16.187	736.510
10	0.032	17.943	759.150	0.043	13.147	676.910	0.034	16.187	726.210
15	0.032	17.943	736.390	0.044	12.928	671.680	0.037	15.709	716.850
20	0.032	17.943	705.290	0.045	12.685	668.580	0.037	15.709	709.080
25	0.035	17.324	705.480	0.047	12.437	667.440	0.039	15.203	703.600
30	0.036	16.994	707.410	0.049	12.171	668.130	0.039	15.203	700.800
35	0.036	16.994	705.130	0.051	11.889	670.500	0.042	14.642	698.260
40	0.040	16.303	706.100	0.053	11.609	674.430	0.042	14.642	698.590
45	0.040	16.303	709.320	0.055	11.317	679.790	0.045	14.073	701.890
50	0.042	15.929	715.630	0.058	11.007	686.490	0.045	14.073	707.640

Table 4.4 to Table 4.6 above reveal a similar behavior of ρ^* and $\Phi_{\rho^*}(u)$ as functions of $I(0) = u$. For each demand-size distribution in each of these tables, ρ^* decrease as $I(0) = u$ increases, while the corresponding $\Phi_{\rho^*}(u)$ first decreases and then increases as function of u. Furthermore, $\Phi_{\rho^*}(u)$ attains its minimum for each demand-size distribution in a narrow range of $I(0) = u$: around $u = 10$ in Table 4.4, in the range [20,25] in Table 4.5, and in the range [25,35] in Table 4.6. We also observe that in each of these tables, ρ^* decreases in the demand-size distribution in this order: constant, uniform and exponential; this, however, does not generally hold for $\Phi_{\rho^*}(u)$.

4.9 Additional Properties: Renewal-Type Solution for Cost Functions

In this section, we derive an alternative representation for the cost functions $c_\rho(u)$, $d_\rho(u)$ and $\Phi_\rho(u)$. However, instead of attempting the inversion of $\tilde{c}_\rho(z)$ in Eq. (4.35), we shall proceed to derive a closed form formula for $c_\rho(u)$ by developing a renewal-type equation for it with the aid of the positive root, ξ. To this end, define

$$c_{\rho,\xi}(z) = e^{-\xi z} c_\rho(z), \;\; z \geq 0. \tag{4.82}$$

Substituting the representation $c_\rho(u) = e^{\xi u} c_{\rho,\xi}(u)$ into Eq. (4.26) and rearranging terms yields

$$\rho \frac{\partial}{\partial z} c_{\rho,\,\xi}(z) - (\lambda + r - \rho\,\xi)\, c_{\rho,\xi}(z) + \lambda \int_0^z e^{-\xi\, x} c_{\rho,\xi}(z-x) f_D(x)\, dx = -e^{-\xi\, z} g(z). \tag{4.83}$$

Next, substituting $\lambda + r - \rho\,\xi = \lambda\,\tilde{f}_D(\xi)$ by appeal to Eq. (3.8) into Eq. (4.83) yields after rearrangement

$$\rho \frac{\partial}{\partial z} c_{\rho,\,\xi}(z) = \lambda\,\tilde{f}_D(\xi)\, c_{\rho,\xi}(z) - \lambda \int_0^z e^{-\xi\, x} c_{\rho,\xi}(z-x) f_D(x)\, dx - e^{-\xi\, z} g(z). \tag{4.84}$$

For any $u \geq 0$, integrating both sides of Eq. (4.84) with respect to z over the interval $[0, u]$ yields

$$\begin{aligned}
&\rho\Big[c_{\rho,\,\xi}(u) - c_{\rho,\,\xi}(0)\Big] \\
&\quad = \lambda\,\tilde{f}_D(\xi) \int_0^u c_{\rho,\xi}(z)\, dz - \lambda \int_0^u \int_0^z e^{-\xi\, x} c_{\rho,\xi}(z-x) f_D(x)\, dx\, dz - \int_0^u e^{-\xi\, z} g(z)\ dz \\
&\quad = \lambda \int_0^u \left[\int_{u-x}^\infty e^{-\xi\, y} f_D(y)\, dy\right]\ c_{\rho,\xi}(x)\ dx - \int_0^u e^{-\xi\, z} g(z)\ dz
\end{aligned} \tag{4.85}$$

The second equality above holds because

$$\int_0^u \int_0^z e^{-\xi x} c_{\rho,\xi}(z-x) f_D(x)\, dx\, dz$$
$$= \int_0^u \int_0^z e^{-\xi (z-y)} c_{\rho,\xi}(y) f_D(z-y)\, dy\, dz$$
$$= \int_0^u \int_y^u e^{-\xi (z-y)} f_D(z-y)\, dz\ c_{\rho,\xi}(y)\, dy$$
$$= \int_0^u \left[\int_0^{u-y} e^{-\xi x} f_D(x)\, dx \right] c_{\rho,\xi}(y)\, dy$$

where the first equality holds by the variable change $z - x \to y$, the second equality holds by changing the order of integration, and the third equality holds by the variable change $z - y \to x$.

Next, for $\rho > 0$, substituting $c_\rho(0)$ from Eq. (4.33) into Eq. (4.82) yields

$$c_{\rho,\xi}(0) = \frac{1}{\rho} \tilde{g}(\xi), \tag{4.86}$$

while substituting Eq. (4.86) into Eq. (4.85) and rearranging yields

$$\rho\, c_{\rho,\xi}(u) = \lambda \int_0^u \left[\int_{u-x}^{\infty} e^{-\xi y} f_D(y)\, dy \right] c_{\rho,\xi}(x)\ dx + \int_u^{\infty} e^{-\xi y} g(y)\ dy\,. \tag{4.87}$$

Dividing both sides of Eq. (4.87) by $\rho e^{-\xi u}$, we have the following functional equation in $c_\rho(u)$ for $u > 0$,

$$c_\rho(u) = \frac{\lambda}{\rho} \int_0^u \left[\int_{u-x}^{\infty} e^{\xi [(u-x)-y]} f_D(y)\, dy \right]\ c_\rho(x)\ dx + \frac{1}{\rho} \int_u^{\infty} e^{\xi (u-y)} g(y)\, dy\,. \tag{4.88}$$

Note that Eq. (4.88) is consistent with Eq. (4.33) by setting $u = 0$ in the former.

We now proceed to solve for $\tilde{c}_\rho(z)$ and $c_\rho(u)$. To this end, define the following two auxiliary functions,

$$\alpha_\rho(x) = \frac{\lambda}{\rho} \int_x^{\infty} e^{\xi (x-y)} f_D(y)\, dy = \frac{\lambda}{\rho} \int_0^{\infty} e^{-\xi y} f_D(x+y)\, dy, \quad x \geq 0 \tag{4.89}$$

$$\beta_\rho(x) = \frac{1}{\rho} \int_x^{\infty} e^{\xi (x-y)} g(y)\ dy, \quad x \geq 0\,. \tag{4.90}$$

Proposition 4.7

The following renewal-type equation holds

$$c_\rho(u) = \left\langle \alpha_\rho * c_\rho \right\rangle(u) + \beta_\rho(u), \quad u \geq 0. \tag{4.91}$$

Proof.

Follows readily by rewriting Eq. (4.88) in terms of Eqs. (4.89) and (4.90). □

Corollary 4.2

The following results hold:

$$\tilde{c}_\rho(z) = \frac{\tilde{\beta}_\rho(z)}{1 - \tilde{\alpha}_\rho(z)}. \tag{4.92}$$

$$c_\rho(u) = \sum_{n=0}^{\infty} \left\langle \beta_\rho * \alpha_\rho^{*(n)} \right\rangle(u) \tag{4.93}$$

where $\alpha_\rho^{*(n)}$ is the n -fold convolution of $\alpha_\rho(x)$ with itself.

Proof.

Taking Laplace transforms on both sides of Eq. (4.91) readily yields

$$\tilde{c}_\rho(z) = \tilde{\alpha}_\rho(z)\ \tilde{c}_\rho(z) + \tilde{\beta}_\rho(z). \tag{4.94}$$

Eq. (4.92) readily follows by solving Eq. (4.94) for $\tilde{c}_\rho(z)$. Finally, Inverting Eq. (4.92) term by term yields Eq. (4.93). □

Note that the terms in Eq. (4.93) can be readily computed recursively, since $\beta_\rho * \alpha_\rho{}^{*(n)} = \beta_\rho * \alpha_\rho^{*(n-1)} * \alpha_\rho$.

For notational convenience, define

$$\delta_\rho(x) = \frac{\lambda}{\rho} e^{\xi(\rho)x} \int_x^\infty e^{-\xi(\rho)y} \bar{F}(y)\,dy, \quad x \geq 0\,. \tag{4.95}$$

Note that $\delta_\rho(x)$ can be obtained from Eq. (4.90) by setting $h = 0$ and $w(x) = 1_{(0,\infty)}(x)$.

Proposition 4.8

The following holds

$$d_\rho(u) = \left\langle \alpha_\rho * d_\rho \right\rangle(u) + \delta_\rho(u), \quad u \geq 0\,. \tag{4.96}$$

where $\alpha_\rho(x)$ is given by Eq. (4.89).

Proof.

Readily follows from Proposition 4.7 by setting $h = 0$ and $w(x) = 1_{(0,\infty)}(x)$. □

Corollary 4.3

$$\tilde{d}_\rho(z) = \frac{\tilde{\delta}_\rho(z)}{1 - \tilde{\alpha}_\rho(z)}\,. \tag{4.97}$$

$$d_\rho(u) = \sum_{n=0}^\infty \left\langle \delta_\rho * \alpha_\rho^{*(n)} \right\rangle(u) \tag{4.98}$$

Proof.

Follows readily from Corollary 4.2 by setting there $h = 0$ and $w(x) = 1_{(0,\infty)}(x)$. □

Next, we shall derive a renewal formula for $\Phi_\xi(u)$. To this end, define

$$\sigma_\xi(x) = \beta_\xi(x) + \frac{\xi}{r}\tilde{g}(\xi)\;\delta_\xi(x)\,, \qquad x \geq 0\,. \tag{4.99}$$

Lemma 4.2

$$\boldsymbol{\Phi}_{\xi}(\boldsymbol{u}) = \sum\nolimits_{n=0}^{\infty} \left\langle \boldsymbol{\sigma}_{\xi} * \boldsymbol{\alpha}_{\xi}^{*(n)} \right\rangle (\boldsymbol{u}) \tag{4.100}$$

$$\tilde{\boldsymbol{\Phi}}_{\xi}(z) = \frac{\tilde{\boldsymbol{\sigma}}_{\xi}(z)}{1 - \tilde{\boldsymbol{\alpha}}_{\xi}(z)}. \tag{4.101}$$

Proof.

Eq. (4.100) follows by substituting Eqs. (4.93), (4.98) and (4.43) into Eq. (4.18) and simplifying with the aid of Eq. (4.99). Eq. (4.101) immediately follows by taking the Laplace transform of Eq. (4.100). □

The renewal-type formulas for $\boldsymbol{c}_{\rho}(\boldsymbol{u})$, $\boldsymbol{d}_{\rho}(\boldsymbol{u})$ and $\boldsymbol{\Phi}_{\rho}(\boldsymbol{u})$, given in Eqs. (4.91), (4.96) and (4.100), respectively, can be solved numerically [Tortorella (2005)]. This thesis is concerned with their analytical solutions by Laplace transform techniques rather than their numerical solution. For further discussion on renewal-type equations, refer to Burton (2005), Miller, (1971), and Linz (1987).

Chapter 5 Time-Average Cost

In this chapter, we address the time-average cost function of MTS production-inventory systems. Some results follow immediately from Chapter 4 by setting $r = 0$ in their discounted counterparts.

5.1 Time-Average Cost Functions

The inventory cost function under study is incurred by carrying costs and lost-sales penalties. These cost components are described as below.

- **Carrying costs.** While there is inventory on hand, a carrying cost is incurred at rate h per unit time and per inventory unit. Accordingly, the carrying cost process $H_\rho = \{H_\rho(t) : t \geq 0\}$ is given by

$$H_\rho(t) = h \int_0^t I(z)\, dz \tag{5.1}$$

- **Lost-sales penalties.** Whenever a customer's demand cannot be satisfied from on-hand inventory, a penalty of the form $w(x)$ is incurred as a non-decreasing function of the lost-sale size, x, with the proviso that $w(0) = 0$. In particular, we shall consider a linear penalty function (to be studied in Section 5.5 as a special case) of the form

$$w(x) = 1_{\{x>0\}} K_0 + K_1\, x, \tag{5.2}$$

where $K_0 \geq 0$ is a constant penalty per lost-sale occurrence, $K_1 \geq 0$ is a constant penalty per unit of lost-sales, and the two constants do not vanish simultaneously. Accordingly, the penalty process $W_\rho = \{W_\rho(t) : t \geq 0\}$ is given by

$$W_\rho(t) = \sum_{i=1}^{N_A(t)} w(L(A_i)) \tag{5.3}$$

The *inventory cost* process $C_\rho = \{C_\rho(t) : t \geq 0\}$ is given by

$$C_\rho(t) = H_\rho(t) + W_\rho(t) = h\int_0^t I(z)\, dz + \sum_{i=1}^{N_A(t)} w(L(A_i)). \tag{5.4}$$

The infinite-horizon *time-average inventory cost* is defined by

$$\bar{c}_\rho = \lim_{t\to\infty} \frac{\mathbb{E}[C_\rho(t) \,|\, I(0) = u]}{t}. \tag{5.5}$$

In a similar vein, the infinite-horizon time-average carrying cost is defined by

$$\bar{h}_\rho = \lim_{t\to\infty} \frac{\mathbb{E}[H_\rho(t) \,|\, I(0) = u]}{t}, \tag{5.6}$$

and the infinite-horizon time-average penalty by

$$\bar{w}_\rho = \lim_{t\to\infty} \frac{\mathbb{E}[W_\rho(t) \,|\, I(0) = u]}{t}. \tag{5.7}$$

Thus, we have

$$\bar{c}_\rho = \bar{h}_\rho + \bar{w}_\rho. \tag{5.8}$$

Throughout this chapter, replenishment occurs at a constant (deterministic) rate $\rho > 0$ and the system is subject to the stability condition

$$\rho < \lambda \mu_D. \tag{5.9}$$

By Prabhu (1965), this stability condition implies that a lost sale occurs with probability 1 over an infinite time horizon.

5.2 Properties of ξ

Consider again Eq. (3.4) and denote

$$\psi_r(z) = \lambda \tilde{f}_D(z) + \rho z - \lambda - r \tag{5.10}$$

as a function of $r \geq 0$. In view of Lemma 3.1, Eq. (5.10) has two roots, denoted by $\xi_r \geq 0$ and $\theta_r \leq 0$. We then have the following result.

Lemma 5.1

For $r = 0$, the equation $\psi_0(z) = 0$ has two distinct roots, θ_0 and ξ_0, satisfying the following

(a) if $0 < \rho < \lambda \mathbb{E}[D]$, then $\theta_0 = 0$ and $\xi_0 > 0$;

(b) if $\rho = \lambda \mathbb{E}[D]$, then $\theta_0 = \xi_0 = 0$;

(c) if $\rho > \lambda \mathbb{E}[D]$, then $\theta_0 < 0$ and $\xi_0 = 0$.

Proof.

Letting $r = 0$ in Eq. (5.10) and setting the resulting equation to be zero, we have

$$\psi_0(z) = \lambda \tilde{f}_D(z) + \rho z - \lambda = 0. \tag{5.11}$$

By Lemma 3.1, Eq. (5.11) has two roots. Clearly, $z_0 = 0$ is a root of the above equation, since $\tilde{f}_D(0) = 1$. To study the other root, substitute the representation of $\tilde{f}_D(z)$ from Eq. (5.11) into Eq.(1.4), yielding

$$\tilde{\tilde{F}}_D(z) = \frac{\rho}{\lambda}. \tag{5.12}$$

Since any Laplace transform of a non-negative function is strictly decreasing in z from infinity to zero, Eq. (5.12) has a unique root z_1. In the remainder of the proof, we shall use the fact that $\tilde{\tilde{F}}_D(0) = \mathbb{E}[D]$.

To prove part (a), note that its premise implies $0 < \frac{\rho}{\lambda} < \mathbb{E}[D] = \tilde{\bar{F}}_D(0)$, so by the monotone decreasing property of $\tilde{\bar{F}}_D(z)$, we must have $\xi_0 = z_1 > 0$, and furthermore, $\theta_0 = z_0 = 0$.

To prove part (b), note that its premise implies $\frac{\rho}{\lambda} = \tilde{\bar{F}}_D(0)$, so by the monotone decreasing property of $\tilde{\bar{F}}_D(z)$, we must have $z_0 = z_1 = \theta_0 = \xi_0 = 0$.

Finally, to prove part (c), note that its premise implies $\frac{\rho}{\lambda} > \tilde{\bar{F}}_D(0)$, so by the monotone decreasing property of $\tilde{\bar{F}}_D(z)$, we must have $\theta_0 = z_1 < 0$, and furthermore, $\xi_0 = z_0 = 0$. □

For the case $r = 0$, Figure 5.1 outlines the key features of the function $\psi_0(z)$ and its roots.

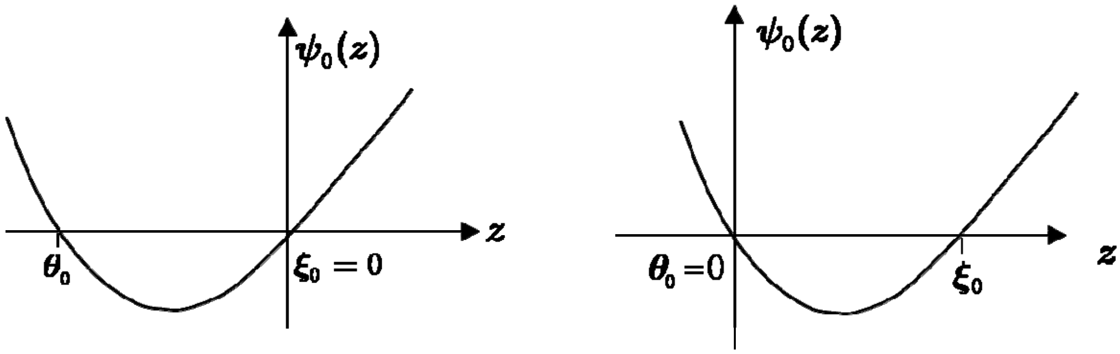

Figure 5.1. Illustration of the Function $\psi_0(z)$ and its Root Structure for the Case $r = 0$ when $\rho > \lambda\mathbb{E}[D]$ (Left) and $\rho < \lambda\mathbb{E}[D]$ (Right)

We then have the following result.

Lemma 5.2

For any $r \geq 0$, under the stability condition in Eq. (5.9), the equation $\psi_r(z) = 0$ has two distinct roots, ξ_r and θ_r as follows:

(a) If $r = 0$, then $\theta_0 = 0$ and $\xi_0 > 0$.

(b) If $r > 0$, then $\theta_r < 0$ and $\xi_r > 0$.

Proof.

We first prove that the function $\psi_r(z)$ is convex by computing its first and second derivatives,

$$\frac{\partial}{\partial z}\psi_r(z) = \rho - \lambda\int_0^\infty x\,e^{-zx} f_D(x)\,dx\,, \tag{5.13}$$

$$\frac{\partial^2}{\partial z^2}\psi_r(z) = \lambda\int_0^\infty x^2\,e^{-zx} f_D(x)\,dx\,. \tag{5.14}$$

Since the case of zero demand with probability 1 is precluded, it follows from Eq. (5.14) that

$$\frac{\partial^2}{\partial z^2}\psi_r(z) > 0\,. \tag{5.15}$$

To prove part (a) for $r = 0$, we have $\psi_0(0) = 0$, namely, zero is a root of $\psi_0(z) = 0$. It remains to show the existence of exactly one more positive root. First, note that $\frac{\partial}{\partial z}\psi_0(z)\,|_{z=0} < 0$ by Eqs. (5.13) and (5.9). Therefore, there exists $z' > 0$ such that $\psi_0(z') < 0$ But since $\psi_0(\infty) = \infty$, it must have a positive root. Second, we prove by contradiction that $\psi_0(z) = 0$ cannot have more than two roots. Otherwise, by Rolle's Theorem, there must be more than one z^*, such that $\frac{\partial}{\partial z}\psi_0(z^*) = 0$. This contradicts the fact that there is at most one z^* such that $\frac{\partial}{\partial z}\psi_0(z^*) = 0$ by Eq. (5.13), thereby establishing part (a).

To prove part (b) for $r > 0$, note that we have $\psi_r(0) < 0$, $\psi_r(\infty) = \infty$ and $\psi_r(-\infty) = \infty$. Consequently, $\psi_r(z) = 0$ must have at least one positive root and one negative root. An argument similar to that in part (a) establishes that there cannot be more than two roots as required. □

Figure 5.2 illustrates the key features of the function $\psi_r(z)$ and the root structure for the equation and $\psi_r(z) = 0$.

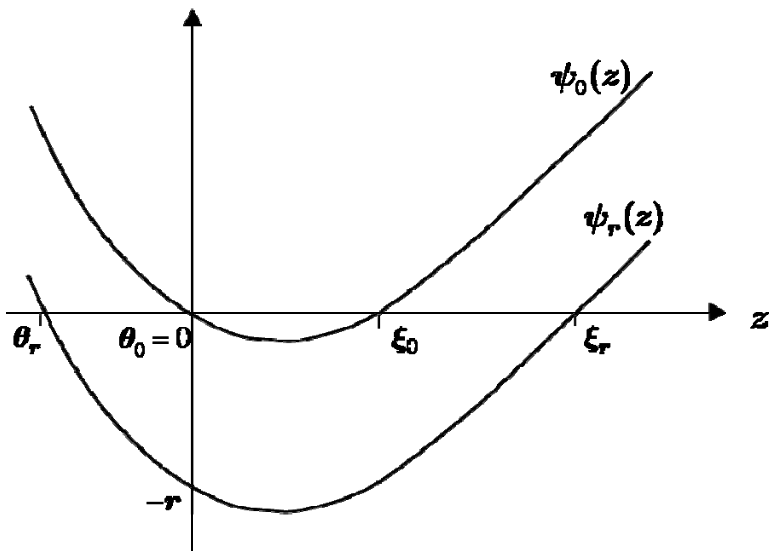

Figure 5.2. Illustration of the Function $\psi_r(z)$ and its Root Structure for $r = 0$ and $r > 0$

In particular, for $r = 0$, we denote

$$\xi = \xi_0 .$$

Accordingly, in view of Lemma 5.2, we can write,

$$\lambda \tilde{f}_D(\xi) + \rho \xi - \lambda = 0, \tag{5.16}$$

and in general, for $r \geq 0$,

$$\lambda \tilde{f}_D(\xi_r) + \rho\, \xi_r - \lambda - r = 0. \tag{5.17}$$

Lemma 5.3

(a) For $\rho \geq 0$ and $\xi = \xi(\rho)$,

$$\rho = \lambda \tilde{\bar{F}}_D(\xi). \tag{5.18}$$

(b) For $\rho \geq 0$, the mapping $\rho \mapsto \xi(\rho)$ is strictly monotone decreasing in ρ.

Proof.

To prove part (a), note first that by Eq. (5.16),

$$\rho = \frac{\lambda\left[1 - \tilde{f}_D(\xi)\right]}{\xi}. \tag{5.19}$$

Eq. (5.18) now follows by Eqs. (1.4) and (5.19).

To prove part (b), we differentiate Eq. (5.16) with respect to ρ, yielding

$$1 = -\lambda\, \xi'(\rho) \int_0^\infty x\; e^{-x\,\xi(\rho)} \bar{F}_D(x)\, dx\,.$$

The equation above implies $\xi'(\rho) < 0$ since the integral on the right-hand side is strictly positive for all $\rho \geq 0$, which in turn implies the result. □

Corollary 5.1

For $r = 0$,

$$\lim_{\rho \to 0} \rho\, \xi = \lambda, \tag{5.20}$$

where $\xi = \xi(\rho)$.

Proof.

Sending $\rho \downarrow 0$ on both sides of Eq. (5.18) implies $\lim_{\rho\to 0} \tilde{\tilde{F}}_D(\xi(\rho)) = 0$, which in turn implies

$$\lim_{\rho\to 0} \xi(\rho) = \infty \,. \tag{5.21}$$

Furthermore, Eq. (5.16) can be written as

$$\rho\,\xi = \lambda - \lambda\,\tilde{f}_D(\xi)\,. \tag{5.22}$$

The proof immediately follows via sending $\rho \downarrow 0$ in Eq. (5.22) and the fact $\lim_{\rho\to 0} \tilde{f}_D(\xi) = 0$. □

5.3 Computing the Time-Average Cost Function

In this section we derive closed form formulas for the time-average cost functions.

5.3.1 The Function $\bar{c}_\rho$

To derive the time average cost function $\bar{c}_\rho$ we first consider the inventory cost until and including the first lost-sale occurrence, which is given by

$$C_\rho(\tau_1) = h\int_0^{\tau_1} I(z)\,dz + w(L(\tau_1)) \quad . \tag{5.23}$$

Its expected value, conditioned on $I(0) = u$, is denoted by

$$c_\rho(u) = \mathbb{E}[C_\rho(\tau_1) \,|\, I(0) = u]\,. \tag{5.24}$$

Note that the inventory process over intervals of the form $(\tau_i, \tau_{i+1}]$ is a renewal process and the corresponding incurred cost process can be regarded as a renewal reward process. Consequently, by Theorem 3.6.1 in Ross (1996), with probability 1, the time-average cost in Eq. (5.5) is independent of the initial inventory level, and given by

$$\bar{c}_\rho = \frac{c_\rho(0)}{\mathbb{E}[\tau_1 \,|\, I(0)=0]}. \tag{5.25}$$

In the following, we shall obtain a formula for $\bar{c}_\rho$, by deriving $c_\rho(0)$ and $\mathbb{E}[\tau_1 \,|\, I(0)=0]$.

To derive $c_\rho(0)$, we shall apply the formula for $c_\rho(u)$, given in Eq. (4.33), by setting there $r=0$. We then have the following result.

Lemma 5.4

$$c_\rho(0) = \frac{\tilde{g}(\xi)}{\rho}. \tag{5.26}$$

Proof.

Follows from Eq. (4.33) and the fact that $\lim_{r\to 0} \xi_r = \xi$. □

Next, to derive $\mathbb{E}[\tau_1 \,|\, I(0)=u]$, define

$$d_{\rho,r}(u) = \mathbb{E}[e^{-r\tau_1} \,|\, I(0)=u]. \tag{5.27}$$

Finally, we shall need the following lemma.

Lemma 5.5

Under the stability condition (5.9), the conditional expected time to the first lost-sale occurrence is

$$\mathbb{E}[\tau_1 \,|\, I(0)=u] = -\frac{\partial}{\partial r} d_{\rho,r}(u)\Big|_{r=0}. \tag{5.28}$$

Proof.

To prove Eq. (4.18), write

$$\begin{aligned}\lim_{r\to 0}\frac{\partial}{\partial r}d_{\rho,r}(u) &= \lim_{r\to 0}\frac{\partial}{\partial r}\int_0^\infty e^{-rt} f_{\tau_1|I(0)}(t\,|\,u)\,dt = \lim_{r\to 0}\int_0^\infty \frac{\partial}{\partial r}e^{-rt} f_{\tau_1|I(0)}(t\,|\,u)\,dt \\ &= -\lim_{r\to 0}\int_0^\infty t\; e^{-rt} f_{\tau_1|I(0)}(t\,|\,u)\,dt = -\int_0^\infty \lim_{r\to 0} t\; e^{-rt} f_{\tau_1|I(0)}(t\,|\,u)\,dt \\ &= -\int_0^\infty t\; f_{\tau_1|I(0)}(t\,|\,u)\,dt = -\mathbb{E}[\tau_1 \mid I(0)=u]\end{aligned}$$

where the second equality holds by the Leibniz integral rule, while the fourth one holds by the Dominated Convergence Theorem, because $\left|t e^{-rt} f_{\tau_1|I(0)}(t\,|\,u)\right| \le t f_{\tau_1|I(0)}(t\,|\,u)$ such that

$\int_0^\infty t\; f_{\tau_1|I(0)}(t\,|\,u)\,dt = \mathbb{E}[\tau_1 \mid I(0)=u] < \infty$ [Prabhu (1965)]. □

By Eq. (4.36), we have

$$d_{\rho,r}(0) = 1 - \frac{r}{\rho\,\xi_r}\,. \tag{5.29}$$

Theorem 5.1

$$\mathbb{E}[\tau_1 \mid I(0)=0] = \frac{1}{\rho\,\xi}\,. \tag{5.30}$$

Proof.

In view of Eqs. (4.18) and (4.36), we have

$$\begin{aligned}\mathbb{E}[\tau_1 \mid I(0)=0] &= -\frac{\partial}{\partial r}d_{\rho,r}(0)\,|_{r=0} = -\lim_{r\to 0}\frac{\partial}{\partial r}\left[1-\frac{r}{\rho\,\xi_r}\right] \\ &= \frac{1}{\rho}\lim_{r\to 0}\left\{\frac{1}{\xi_r}-\frac{r}{[\xi_r]^2}\xi'_r\right\} = \frac{1}{\rho\,\xi}\end{aligned}$$

Here, the first equality holds by Eq. (4.18) at $u = 0$, the second equality holds by Eq. (5.29). The fourth equality is due to the fact that $\lim_{r\to 0} \frac{r}{[\xi_r]^2} \xi_r' = 0$, and to show that it suffices to prove that $\lim_{r\to 0} \frac{\xi_r'}{[\xi_r]^2}$ exists and is finite. To see that, we first note that $\lim_{r\to 0} [\xi_r]^2 = \xi^2$. Secondly, since $r \mapsto \xi_r$ is a one-to-one mapping by Eq. (5.17), one has $\lim_{r\to 0} \xi_r' = \frac{1}{\lim_{\xi_r \to \xi_0} r'(\xi_r)}$. Furthermore, by Eq. (5.17), one has

$$r'(\xi_r) = \frac{\partial}{\partial z} \psi_r(z)\Big|_{z=\xi_r} .$$

Therefore, by continuity of $\frac{\partial}{\partial z} \psi_r(z)$ at $z = \xi_r$, shown in the proof for Lemma 5.2, we have

$$\lim_{\xi_r \to \xi} r'(\xi_r) = \frac{\partial}{\partial z} \psi_0(z)\Big|_{z=\xi} > 0 , \tag{5.31}$$

again by the proof for Lemma 5.2. We conclude that $\lim_{r\to 0} \frac{\xi_r'}{[\xi_r]^2}$ is finite as claimed. □

5.3.2 Closed form formula for $\bar{c}_\rho$

The following theorem provides computable representations for the infinite-horizon time-average total cost and its components (carrying cost and lost-sales penalty).

Theorem 5.2

$$\bar{c}_\rho = \xi\, \tilde{g}(\xi) , \tag{5.32}$$

$$\bar{h}_\rho = \xi\, \tilde{g}_1(\xi) , \tag{5.33}$$

$$\bar{w}_\rho = \xi\, \tilde{g}_2(\xi) \tag{5.34}$$

where $\xi = \xi(\rho)$, functions $g(\cdot)$, $g_1(\cdot)$ and $g_2(\cdot)$ are given by Eqs. (4.27), (4.28) and (4.29), respectively.

Proof.

To prove Eq. (4.43), substitute Eqs. (5.26) and (5.30) into Eq. (5.25). Eqs. (5.33) and (5.34) readily follow by noting that $g = g_1 + g_2$ implies $\tilde{g} = \tilde{g}_1 + \tilde{g}_2$. □

Denote by $\bar{I}$ the infinite-horizon time-average inventory, namely,

$$\bar{I} = \lim_{t \to \infty} \frac{1}{t} \int_0^t I(z)\, dz . \tag{5.35}$$

Corollary 5.2

$$\bar{h}_\rho = \frac{h}{\xi} , \tag{5.36}$$

$$\bar{I} = \frac{1}{\xi} . \tag{5.37}$$

Proof.

Note that $\tilde{g}_1(z) = \dfrac{h}{z^2}$ by Eq. (4.28). Eq. (5.36) now readily follows by substituting $\tilde{g}_1(\xi)$ into Eq. (5.33). Finally, Eq. (5.37) follows immediately from Eq. (5.36) by noting that the inventory time average is equivalent to the time-average carrying cost with $h = 1$. □

5.4 Cost Function Properties

In this section, we study properties of the cost function $\bar{c}_\rho$ given by Eq. (5.32) and its components, $\bar{h}_\rho$ and $\bar{w}_\rho$. To this end, we first provide some asymptotic results of the cost functions, and then demonstrate the existence and uniqueness of its minimum.

We first rewrite Eq. (5.32) as

$$\bar{c}_\rho = \mathcal{L}\left[g'\right](\xi) + g(0+) = \mathcal{L}\left[g'\right](\xi) + \lambda\,\mathbb{E}[w(D)], \tag{5.38}$$

where the first equality holds by a property of the Laplace transform [Widder (1959)], the second equality follows from Eq. (4.27), and $g'(u)$ is given by

$$g'(u) = h + g_2'(u) = h - \lambda\left[\int_u^\infty f_D(x)\,w'(x-u)\,dx + f_D(u)\,w(0+)\right]. \tag{5.39}$$

Lemma 5.6

(a) $\bar{h}_\rho$ is monotone increasing and convex in $\rho \geq 0$, and has the following asymptotes

$$\lim_{\rho \to 0} \bar{h}_\rho = 0 \tag{5.40}$$

$$\lim_{\rho \to \infty} \bar{h}_\rho = \infty \tag{5.41}$$

(b) $\bar{w}_\rho$ is monotone decreasing and concave in $\rho \geq 0$, and has the following asymptotes

$$\lim_{\rho \to 0} \bar{w}_\rho = \lambda\,\mathbb{E}[w(D)] \tag{5.42}$$

$$\lim_{\rho \to \infty} \bar{w}_\rho = 0 \tag{5.43}$$

(c) $\bar{c}_\rho$ has the following asymptotes

$$\lim_{\rho \to 0} \bar{c}_\rho = \lambda\,\mathbb{E}[w(D)] \tag{5.44}$$

$$\lim_{\rho \to \infty} \bar{c}_\rho = \infty \tag{5.45}$$

Proof.

Part (a) readily follows from Eq. (5.36).

To prove part (b), we first prove Eq. (5.42) by writing

$$\lim_{\rho\to 0}\bar{w}_\rho = \lim_{\xi\to\infty}\xi\,\tilde{g}_2(\xi) = \lim_{u\to 0} g_2(u) = \lambda\,\mathbb{E}[w(D)]\,.$$

Here, the first equality follows from Eq. (5.34) and the monotone decreasing relation between ρ and ξ exhibited in Eq. (3.10); the second equality holds by the Initial Value Theorem of the Laplace transform [Widder (1959)]; and the third equality holds by Eq.(4.29).

Next, to prove Eq. (5.43) we write

$$\lim_{\rho\to\infty}\bar{w}_\rho = \lim_{\xi\to 0}\xi\,\tilde{g}_2(\xi) = \lim_{u\to\infty} g_2(u) = \infty\,.$$

Here, the first equality holds by Eq. (5.34) and the decreasing monotone relation between ρ and ξ exhibited in Eq. (3.10); the second equality holds by the Final Value Theorem of the Laplace transform [Widder (1959)]; and the last equality holds by Eq.(4.29).

We next show that the monotonicity and concavity of $\bar{w}_\rho$ follow from its first and second derivatives, respectively. To this end, we write

$$\bar{w}_\rho = \xi\,\tilde{g}_2(\xi) = \mathcal{L}\left[g_2'\right](\xi) + g_2(0+) = \int_0^\infty e^{-\xi x} g_2'(x)\,dx + \lambda\,\mathbb{E}[w(D)], \qquad (5.46)$$

where the first equality holds by Eq. (5.34); the second equality holds by a property of the Laplace transform [Widder (1959)]; the first term in the third equality holds by definition; and the second term in the third equality holds by Eq. (4.29). Differentiating Eq. (5.46) now yields

$$\frac{\partial}{\partial\xi}\bar{w}_\rho = -\int_0^\infty x\,e^{-\xi x} g_2'(x)\,dx \geq 0 \qquad (5.47)$$

$$\frac{\partial^2}{\partial\xi^2}\bar{w}_\rho = \int_0^\infty x^2\,e^{-\xi x} g_2'(x)\,dx \leq 0 \qquad (5.48)$$

Here, we use the fact that Eq. (4.29) implies

$$g_2'(u) = -\lambda\left[\int_u^\infty f_D(x)\,w'(x-u)\,dx + f_D(u)\,w(0+)\right] \le 0\,, \tag{5.49}$$

since the equality holds by the generalized Leibniz's integral rule, and the inequality holds in view of $f_D(u) \ge 0$ and the fact that the inequalities $w(0+), w'(x) \ge 0$ hold by assumption. This completes the proof for part (b).

Finally, Eqs. (5.44) and (5.45) follow by adding Eq. (5.40) to Eq. (5.42), and adding Eq. (5.41) to Eq. (5.43), respectively. . □

We are now in a position to study the existence and uniqueness of the minima of $\bar{c}_\rho$. We mention that it is straightforward to prove the existence of minima; however the proof of uniqueness is much challenging. Still, we can prove uniqueness for some important cost functions. To this end, we need the following general result.

Proposition 5.1

Let $f(x)$ be a continuous function, not identically zero, satisfying

$$\int_0^\infty f(x)\,dx = 0\,, \tag{5.50}$$

and there exists a constant $x_0 > 0$ such that $f(x) \le 0$ for $0 \le x \le x_0$, and $f(x) \ge 0$ for $x > x_0$. Then, $\tilde{f}(z) = 0$ if and only if $z = 0$.

Proof.

The proof of the necessary condition is trivial. To prove the sufficient condition, we first write Eq. (5.50) as

$$\int_{x_0}^{\infty} f(x)\,dx = -\int_0^{x_0} f(x)\,dx\,. \tag{5.51}$$

Next, for all $z > 0$,

$$\begin{aligned}\tilde{f}(z) &= \int_0^{x_0} e^{-zx} f(x)\,dx + \int_{x_0}^{\infty} e^{-zx} f(x)\,dx \\ &\leq \int_0^{x_0} e^{-zx} f(x)\,dx + e^{-zx_0}\int_{x_0}^{\infty} f(x)\,dx \\ &= \int_0^{x_0} e^{-zx} f(x)\,dx - e^{-zx_0}\int_0^{x_0} f(x)\,dx \\ &= \int_0^{x_0}\left[e^{-zx} - e^{-zx_0}\right] f(x)\,dx < 0\end{aligned}$$

because the first inequality holds by $e^{-zx} \leq e^{-zx_0}$ for $x \geq x_0$, the second equality holds by Eq. (5.51), and the last inequality holds by the relations of $e^{-zx} > e^{-zx_0}$ and $f(x) \leq 0$ but not identically zero for $0 \leq x \leq x_0$. This completes the proof. □

The following Lemma provides results for the case $w(0+) = 0$.

Lemma 5.7

For $w(0+) = 0$,

(a) if $h = 0$, then $\bar{c}_\rho$ attains a unique minimum at $\rho^* = \lambda\,\mathbb{E}[D]$, where $\xi^* = 0$;

(b) if

$$0 < h < \lambda\,\mathbb{E}[w'(D)] < \infty, \tag{5.52}$$

then $\bar{c}_\rho$ has a unique and finite minimum at $\rho^* = \lambda \tilde{\bar{F}}_D(\xi^*)$, where $\xi^* > 0$;

(c) if $h \geq \lambda\,\mathbb{E}[w'(D)] > 0$, then $\bar{c}_\rho$ attains a unique minimum at $\rho^* = 0$, where $\xi^* = \infty$.

Proof.

If $w(0+) = 0$, then Eq. (5.39) implies

$$g'(u) = h - \lambda \int_u^\infty f_D(x)\, w'(x-u)\, dx = R(u)\,, \tag{5.53}$$

where $R(u)$ is an increasing function of u. Furthermore, Eqs. (5.38) and (5.53) jointly imply

$$\bar{c}_\rho = \tilde{R}(\xi) \; + \lambda\, \mathbb{E}[w(D)]\,. \tag{5.54}$$

Eq. (5.54) shows that minimizing $\bar{c}_\rho$ in ρ is equivalent to minimizing $\tilde{R}(\xi)$ in ξ.

To prove Part (a), observe that $h = 0$ implies that $R(u) < 0$ because $w(x)$ is a non-decreasing function (of the loss) by assumption, and consequently $\tilde{R}(\xi)$ is strictly increasing. Part (a) now follows since $\tilde{R}(\xi)$ attains a unique minimum at $\xi^* = 0$.

To prove Part (b), the existence of the minimum follows from the continuity of $\bar{c}_\rho$ and Part (c) in Lemma 5.6. It remains to prove the uniqueness of the minimum. To this end, differentiate Eq. (5.54) with respect to ξ, and set the derivative to zero, yielding

$$\frac{\partial}{\partial \xi}\bar{c}_\rho(\xi) = -\int_0^\infty x\, e^{-\xi x} R(x)\, dx = \int_0^\infty f_\xi(x)\, dx = 0\,. \tag{5.55}$$

where $f_\xi(x) = -x\, e^{-\xi x} R(x)$. Next, Eq. (5.53) implies

$$\lim_{u\to\infty} R(u) = h > 0\,. \tag{5.56}$$

Furthermore, the assumption $h < \lambda\, \mathbb{E}[w'(D)] < \infty$ and Eq. (5.53) imply $R(0+) < 0$. Using the two limits above and the continuity and monotonicity of $R(u)$, it follows that there exists a constant $u_0 > 0$, such that $R(u) \le 0$ for $0 \le u \le u_0$, while $R(u) \ge 0$ for

$u \geq u_0$. Consequently, we conclude that for any $\xi \geq 0$, one has $f_\xi(x) \geq 0$ for $0 \leq u \leq u_0$, while, $f_\xi(x) \leq 0$ for $u \geq u_0$. Letting ξ^* denote a solution of Eq.(5.55), we next prove its uniqueness by contradiction. Suppose there exists another solution ξ' of Eq.(5.55), such that without loss of generality, $\xi^* < \xi'$. Then, by Eq. (5.55), one has

$$\frac{\partial}{\partial \xi}\bar{c}_\rho(\xi') = \int_0^\infty f_{\xi'}(x)\,dx = \int_0^\infty e^{-(\xi'-\xi^*)x} f_{\xi^*}(x)\,dx = 0.$$

In view of Proposition 5.1, we must have $\xi' - \xi^* = 0$ in contradiction to the assumption $\xi^* < \xi'$, which completes the proof for Part (b).

Finally, to prove Part (c), if $h \geq \lambda\,\mathbb{E}[w'(D)]$, then $R(u) \geq 0$ by Eq. (5.53). It follows that $\tilde{R}(\xi)$ is non-increasing, which completes the proof for Part (c). □

Figure 5.3 illustrates a typical $\bar{c}_\rho$ as function of the original domain variable (the replenishment rate, ρ), and a Laplace domain variable (the positive root, ξ); recall that ρ and ξ are related by Eq.(5.18).

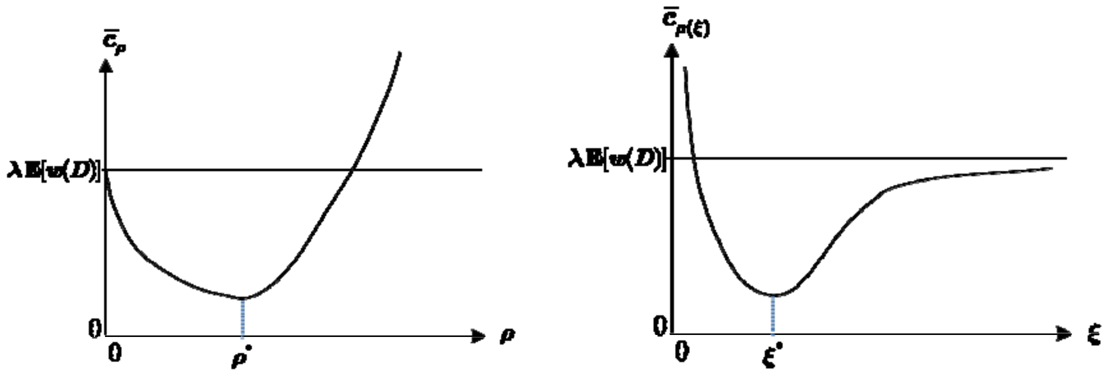

Figure 5.3. A Typical $\bar{c}_\rho$ as Function of ρ (Left) and ξ (Right)

5.5 Optimal Replenishment Rate

In this section, we optimize the time-average cost of Eq. (5.5) with respect to the replenishment rate, ρ. We first provide a general structural result for the optimal replenishment rates, ρ^*, and then we study some special cases. Note that we admit the possibility of multiple optimal replenishment rates.

Theorem 5.3

The optimal replenishment rates for Eq. (5.5) are given by

$$\rho^* = \lambda\ \tilde{\bar{F}}_D(\xi^*), \tag{5.57}$$

where

$$\xi^* = \operatorname*{argmin}_{\xi>0} \{\xi\ \tilde{g}(\xi)\}. \tag{5.58}$$

Proof.

In view of Eq. (4.43), minimizing $\bar{c}_\rho = \xi(\rho)\,\tilde{g}(\xi(\rho))$ with respect to ρ is equivalent to minimizing $\bar{c}_\rho = \xi\,\tilde{g}(\xi)$ with respect to the nonnegative variable ξ. To this end, we first compute Eq. (5.58), namely, perform optimization on $\bar{c}_\rho = \xi\,\tilde{g}(\xi)$ in the Laplace domain to find the optimal values ξ^*. Next, by Lemma 5.3(b), $\rho \mapsto \xi(\rho)$ is 1-1, and consequently, we can invert each $\xi^* = \xi(\rho^*)$ via Eq. (4.65) to obtain the corresponding optimal replenishment rate, ρ^*. □

The minimum values, ξ^*, given in Eq. (5.58), can be calculated in several ways. A straightforward but relatively time consuming method is global search. However, when ξ^* is unique, the availability of derivatives of $\bar{c}_\rho(\xi)$ with respect to ξ allows us to apply the relatively fast Newton's Method, where successive approximations of the minimum are given by the iterative scheme,

$$\xi_{n+1} = \xi_n - \frac{\frac{\partial}{\partial \xi}\bar{c}_\rho(\xi_n)}{\frac{\partial^2}{\partial \xi^2}\bar{c}_\rho(\xi_n)}, \quad n = 0, 1, \ldots. \tag{5.59}$$

We next proceed to study production-inventory systems with specialized lost-sales penalty structures, specifically the constant lost-sales penalty and the loss-proportional penalty. Under each penalty structure, we study the optimal average costs, subject to particular demand distributions, such as constant, uniform, Exponential and Gamma distributions.

5.5.1 Constant Lost-Sales Penalty

In this case, $w(x) = 1_{\{x>0\}} K_0$, where $K_0 > 0$ is a constant. Then, Eq. (4.27) becomes

$$g(u) = h\,u + \lambda\,K_0 \int_u^\infty f_D(x)\,dx = h\,u + \lambda\,K_0 \bar{F}_D(u), \tag{5.60}$$

and the corresponding Laplace transform is given by

$$\tilde{g}(z) = \frac{h}{z^2} + \lambda\,K_0 \tilde{\bar{F}}_D(z) = \frac{h}{z^2} + \lambda\,K_0 \frac{1 - \tilde{f}_D(z)}{z},$$

where the second equality holds by Eq. (1.4). In view of Eq. (4.43), we now have

$$\bar{c}_\rho = \xi\,\tilde{g}(\xi) = \frac{h}{\xi} + \lambda K_0 \left[1 - \tilde{f}_D(\xi)\right] = \frac{h}{\xi} + \rho\,K_0\,\xi, \tag{5.61}$$

where the last equality holds by Eq. (5.19). By Eq. (5.58), the optimal ξ^* is given by

$$\xi^* = \operatorname*{argmin}_{\xi>0} \left\{ \frac{h}{\xi} - \lambda\,K_0\,\tilde{f}_D(\xi) \right\}. \tag{5.62}$$

We mention that ξ^* is a monotonically decreasing function of $\frac{\lambda K_0}{h}$. To see this, Eq. (5.62) can be rewritten as $\xi^* = \underset{\xi>0}{\textbf{argmin}}\left\{\frac{1}{\xi} - \frac{\lambda K_0}{h}\ \tilde{f}_D(\xi)\right\}$, so that the derivative of the rewritten objective function with respect to $\frac{\lambda K_0}{h}$ is $-\tilde{f}_D(\xi) < 0$, which implies the result. It follows that ρ^* is a monotonically increasing function of $\frac{\lambda K_0}{h}$, because ξ^* is a monotonically decreasing function of $\frac{\lambda K_0}{h}$, while $\xi = \xi(\rho)$ is monotonically decreasing in ρ.

Table 5.1 exhibits the formulas for ξ^*, ρ^* and $\bar{c}_{\rho^*}$ for selected demand distribution with detailed derivations given in Section A.3 of Appendix A.

Table 5.1. Optimal Quantities for Production-Inventory Systems Subject to Constant Penalty and Various Demand Distributions

Distribution	ξ^*	ρ^*	$\bar{c}_{\rho^*}$
$D = d$ $d > 0$	$\underset{\xi>0}{\operatorname{argmin}}\left\{\frac{h}{\xi} - \lambda K_0 e^{-\xi d}\right\}$	$\frac{\lambda}{\xi^*}\left[1 - e^{-\xi^* d}\right]$	$\frac{h}{\xi^*} + K_0 \rho^* \xi^*$
$D \sim \mathrm{Exp}(\beta)$ $\beta > 0$ $\beta\lambda K_0 > h$	$\frac{\beta\sqrt{h}}{\sqrt{\lambda\beta K_0} - \sqrt{h}}$	$\frac{\lambda}{\beta} - \frac{\sqrt{\lambda h}}{\beta\sqrt{\beta K_0}}$	$2\sqrt{\frac{h\lambda K_0}{\beta}} - \frac{h}{\beta}$
$D \sim U(a,b)$ $0 \le a < b$	$\underset{\xi>0}{\operatorname{argmin}}\left\{\frac{h}{\xi} - \lambda K_0 \frac{e^{-a\xi} - e^{-b\xi}}{(b-a)\xi}\right\}$	$\frac{\lambda}{\xi^*}\left[1 - \frac{e^{-a\xi^*} - e^{-b\xi^*}}{(b-a)\xi^*}\right]$	$\frac{h}{\xi^*} + K_0 \rho^* \xi^*$
$D \sim \Gamma(\alpha,\beta)$ $\alpha, \beta > 0$	$\underset{\xi>0}{\operatorname{argmin}}\left\{\frac{h}{\xi} - \lambda K_0 \left(1 + \frac{\xi}{\beta}\right)^{-\alpha}\right\}$	$\frac{\lambda}{\xi^*}\left[1 - \left(1 + \frac{\xi^*}{\beta}\right)^{-\alpha}\right]$	$\frac{h}{\xi^*} + K_0 \rho^* \xi^*$

5.5.2 Loss-Proportional Penalty

In this case, $w(x) = 1_{\{x>0\}} K_1 x$ where $K_1 > 0$ is constant. Then, Eq. (4.27) becomes

$$g(u) = h\,u + \lambda K_1 \int_u^{\infty} (x-u) f_D(x)\,dx\,, \tag{5.63}$$

and the corresponding Laplace transform is given by

$$\tilde{g}(z) = \frac{h}{z^2} + \lambda K_1 \left[\frac{\mu_D}{z} - \frac{1 - \tilde{f}_D(z)}{z^2}\right]. \tag{5.64}$$

In view of Eq. (4.43), we now have

$$\bar{c}_\rho = \frac{h}{\xi} + \lambda K_1 \left[\mu_D - \frac{1 - \tilde{f}_D(\xi)}{\xi}\right], \tag{5.65}$$

where $\mu_D = \mathbb{E}[D]$. Note that $\lambda\,\xi^{-1}\left[1-\tilde{f}_D(\xi)\right] = \rho$ by Eq. (3.8), we also have

$$\bar{c}_\rho = \frac{h}{\xi} - K_1\rho + \lambda\, K_1\, \mu_D \tag{5.66}$$

Consequently, by Eq. (4.80), the optimal ξ^* is given by

$$\xi^* = \underset{\xi>0}{\operatorname{argmin}}\left\{\frac{h}{\xi} - \lambda\, K_1 \frac{1-\tilde{f}_D(\xi)}{\xi}\right\}. \tag{5.67}$$

We mention that ξ^* is a monotonically decreasing function of $\dfrac{\lambda\, K_1}{h}$. To see this, by Eq.(5.67), we have $\xi^* = \underset{\xi>0}{\operatorname{argmin}}\left\{\dfrac{1}{\xi} + \dfrac{\lambda\, K_1}{h}\left[\mu_D - \dfrac{1-\tilde{f}_D(\xi)}{\xi}\right]\right\}$, so that the derivative of the rewritten objective function with respect to $\dfrac{\lambda\, K_1}{h}$ is

$$\mu_D - \frac{1-\tilde{f}_D(\xi)}{\xi} = \mu_D - \tilde{\bar{F}}_D(\xi) = \mu_D - \int_0^\infty e^{-\xi\, x}\bar{F}_D(x)dx < \mu_D - \int_0^\infty \bar{F}_D(x)dx = 0\,,$$

which implies the result. It follows that ρ^* is a monotonically increasing function of $\dfrac{\lambda\, K_1}{h}$, because ξ^* is a monotonically decreasing function of $\dfrac{\lambda\, K_1}{h}$, while $\xi = \xi(\rho)$ is monotonically decreasing in ρ.

Table 5.2 exhibits the expressions for ξ^*, ρ^* and $\bar{c}_\rho^{\ *}$ for selected demand distribution with detailed derivations given in Section A.4 of Appendix A.

Table 5.2. Optimal Quantities for Production-Inventory Systems Subject to Loss-Proportional Penalty and Various Demand Distributions

Distribution	ξ^*	ρ^*	$\bar{c}_{\rho^*}$
$D = d$ $d > 0$	$\underset{\xi>0}{\operatorname{argmin}}\left\{\frac{h}{\xi} - \lambda K_1 \frac{1-e^{-\xi d}}{\xi}\right\}$	$\frac{\lambda}{\xi^*}[1 - e^{-\xi^* d}]$	$\frac{h}{\xi^*} - k\rho^* + \lambda K_1 d$
$D \sim \mathrm{Exp}(\beta)$ $\beta > 0$ $\lambda K_1 > h$	$\frac{\beta\sqrt{h}}{\sqrt{\lambda K_1} - \sqrt{h}}$	$\frac{\lambda}{\beta} - \frac{1}{\beta}\sqrt{\frac{h\lambda}{K_1}}$	$\frac{2}{\beta}\sqrt{h\lambda K_1} - \frac{h}{\beta}$
$D \sim U(a,b)$ $0 \le a < b$	$\underset{\xi>0}{\operatorname{argmin}}\left\{\frac{h}{\xi} - \frac{\lambda K_1}{\xi}\left[1 - \frac{e^{-a\xi} - e^{-b\xi}}{(b-a)\xi}\right]\right\}$	$\frac{\lambda}{\xi^*}\left[1 - \frac{e^{-a\xi^*} - e^{-b\xi^*}}{(b-a)\xi^*}\right]$	$\frac{h}{\xi^*} - K_1\rho^* + \frac{\lambda K_1[b\text{-}a]}{2}$
$D \sim \Gamma(\alpha,\beta)$ $\alpha, \beta > 0$	$\underset{\xi>0}{\operatorname{argmin}}\left\{\frac{h}{\xi} - \lambda K_1 \frac{1 - \left(1 + \frac{\xi}{\beta}\right)^{-\alpha}}{\xi}\right\}$	$\frac{\lambda}{\xi^*}\left[1 - \left(1 + \frac{\xi^*}{\beta}\right)^{-\alpha}\right]$	$\frac{h}{\xi^*} - K_1\rho^* + \lambda K_1 \mu_D$

5.5.3 Exponential Demand: Relationship between the Optimal and Cost-Balanced Rates

In this section, we assume that demand is exponential, and under this assumption we relate the optimal replenishment rate, ρ^*, and the corresponding *cost-balanced replenishment* rate, $\hat{\rho}$, which is the replenishment such that

$$\bar{h}_{\hat{\rho}} = \bar{w}_{\hat{\rho}}\,. \tag{5.68}$$

Let $\beta > 0$ be the rate parameter of the exponential demand distribution, so

$$f_D(x) = \beta\, e^{-\beta x}, \quad x \ge 0\,. \tag{5.69}$$

and

$$\tilde{f}_D(z) = \frac{\beta}{\beta + z}. \tag{5.70}$$

Accordingly, $\psi_0(z)$ becomes

$$\psi_0(z) = \rho z - \lambda + \frac{\lambda \beta}{\beta + z}.$$

And the equation $\psi_0(z) = 0$ can be written as

$$\left[z - \left(\frac{\lambda}{\rho} - \beta \right) \right] z = 0. \tag{5.71}$$

Hence, the positive root of Eq. (5.71) is given by

$$\xi = \frac{\lambda}{\rho} - \beta. \tag{5.72}$$

We then have the following result.

Proposition 5.2

Let the demand distribution be exponential, and assume that the penalty function is of the form $w(x) = 1_{\{x>0\}} K_0$ or $w(x) = \mathbf{1}_{\{x>0\}} K_1 x$. Then, for any $\rho > 0$,

$$\hat{\rho} > \rho^* \tag{5.73}$$

$$\bar{h}_{\rho^*} \le \bar{w}_{\rho^*} \tag{5.74}$$

Proof.

Assume first that the lost-sale penalty is of the form $w(x) = 1_{\{x>0\}} K_0$. Substituting Eq. (5.72) into Eq. (5.61), we have

$$\bar{c}_\rho = \frac{\rho h}{\lambda - \rho \beta} + K_0 [\lambda - \rho \beta] = \bar{h}_\rho + \bar{w}_\rho, \tag{5.75}$$

where the time average carrying cost is

$$\bar{h}_\rho = \frac{\rho h}{\lambda - \rho\beta}, \tag{5.76}$$

and the time average lost-sales penalty is

$$\bar{w}_\rho = K_0[\lambda - \rho\beta]. \tag{5.77}$$

Next, equate Eqs. (5.76) and (5.77) and solve for $\hat{\rho}$, yielding

$$\hat{\rho} = \frac{\lambda}{\beta} - \sqrt{\left(\frac{h}{2K_0\beta^2}\right)^2 + \frac{\lambda h}{K_0\beta^3}} + \frac{h}{2K_0\beta^2}. \tag{5.78}$$

Letting $a = \frac{h}{2K_0\beta^2} > 0$ and $b = \sqrt{\frac{\lambda h}{K_0\beta^3}} > 0$ above, and noting that $\sqrt{a^2 + b^2} < a + b$, we get

$$\sqrt{\left(\frac{h}{2K_0\beta^2}\right)^2 + \frac{\lambda h}{K_0\beta^3}} - \frac{h}{2K_0\beta^2} < \sqrt{\frac{\lambda h}{K_0\beta^3}}. \tag{5.79}$$

Eqs. (5.78) and (5.79) readily imply

$$\hat{\rho} > \frac{\lambda}{\beta} - \sqrt{\frac{\lambda h}{K_0\beta^3}} = \rho^*, \tag{5.80}$$

where the equality in Eq. (5.80) follows from the exponential case in Table 5.1. This completes the proof of Eq. (5.73).

To prove Eq. (5.74), first note that $\bar{h}_\rho$ is an increasing function of ρ by Lemma 5.6(a) while $\bar{w}_\rho$ is a decreasing function of ρ by Lemma 5.6(b). Second, the aforementioned monotonicity of $\bar{h}_\rho$ and $\bar{w}_\rho$ in conjunction with Eq. (5.80) imply $\bar{h}_{\hat{\rho}} \geq \bar{h}_{\rho^*}$ and $\bar{w}_{\hat{\rho}} \leq \bar{w}_{\rho^*}$. Eq. (5.74) now follows from the last two inequalities together with Eq. (5.68).

Finally, the corresponding proofs for the case $w(x) = 1_{\{x>0\}} K_1 x$ are readily seen to be analogous to the proofs above for $w(x) = 1_{\{x>0\}} K_0$, but with K_0 replaced by $\dfrac{K_1}{\beta}$. □

A numerical study illustrating the relationships in Eqs. (5.73) and (5.74) appears in next section.

5.6 Numerical Study

In this section, we study three special cases with constant lost-sales, with $\lambda = 0.5$, $h = 1$ and $K_0 = 100$ in all cases. As a check on accuracy, we performed paired evaluations of the requisite cost functions: by analytical formulas developed earlier and by simulation. Accordingly, in the figures below, curves are paired as follows: those with circles correspond to analytical results, while those with asterisks correspond to their simulation counterparts.

In the first case, we study the average total cost, $\overline{c}_\rho$, as function of the replenishment rate, ρ, under three demand distributions: constant, exponential and uniform. To ensure that these systems are comparable, we let $\mu_D = 2$ be the common mean of all the aforementioned demand distributions.

Figure 5.4 depicts $\overline{c}_\rho$ as a function of ρ for each demand distribution. Here, curve styles correspond to demand distributions: solid curves to the constant distribution, dashed curves to the exponential distribution and dotted curves to the uniform distribution. Figure 5.4 shows a good agreement between all pairs of analytical and simulation results. Furthermore, the system with constant demand has the largest optimal replenishment rate, while its exponential counterpart has the smallest one.

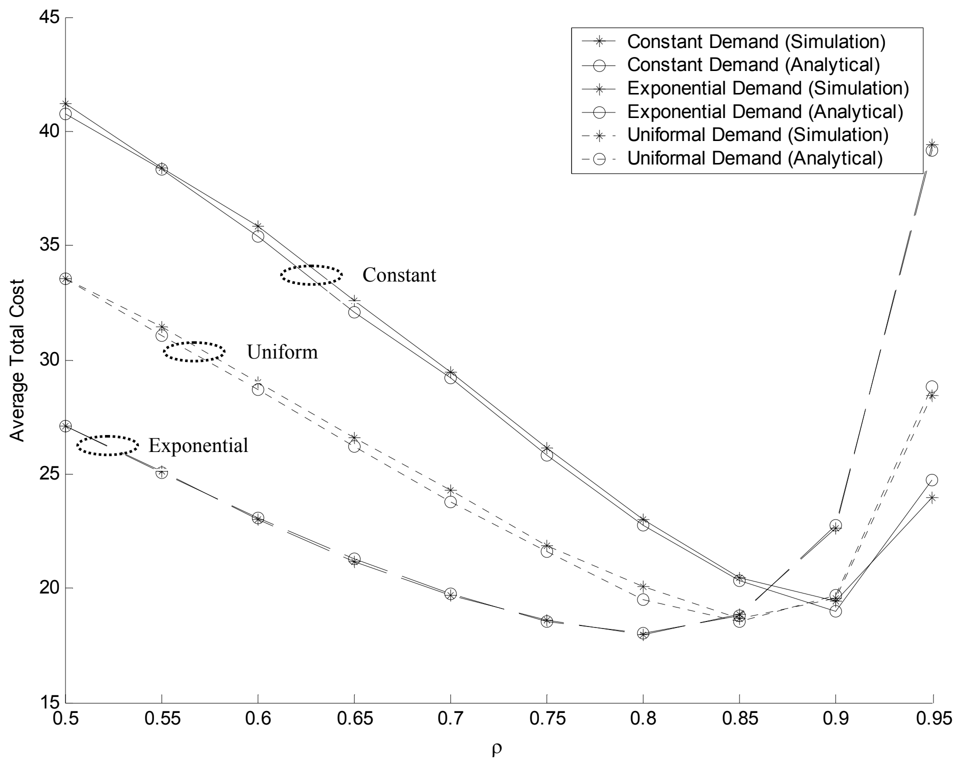

Figure 5.4. Average Costs for Inventory Systems with Various Demand Distributions as Functions of the Replenishment Rate

In the second case, we study the average total cost, $\overline{c}_\rho$, and its components (average carrying cost, $\overline{h}_\rho$, and average penalty, $\overline{w}_\rho$) as functions of the replenishment rate, ρ, under an exponential demand distribution with rate parameter, $\beta = 0.5$.

Figure 5.5 depicts the $\overline{c}_\rho$, $\overline{h}_\rho$ and $\overline{w}_\rho$ as functions of ρ. Here, curve styles correspond to cost types: solid curves to the average total costs, dashed curves average carrying costs and dotted curves to average penalties. Figure 5.5 shows a good agreement between all pairs of analytical and simulation results. Furthermore, the optimal solution P1 (with replenishment rate ρ^*) for the average total costs differs from its cost-balanced counterpart, P2 (with replenishment rate $\hat{\rho}$), such that $\hat{\rho} > \rho^*$, in agreement with Eq. (5.73).

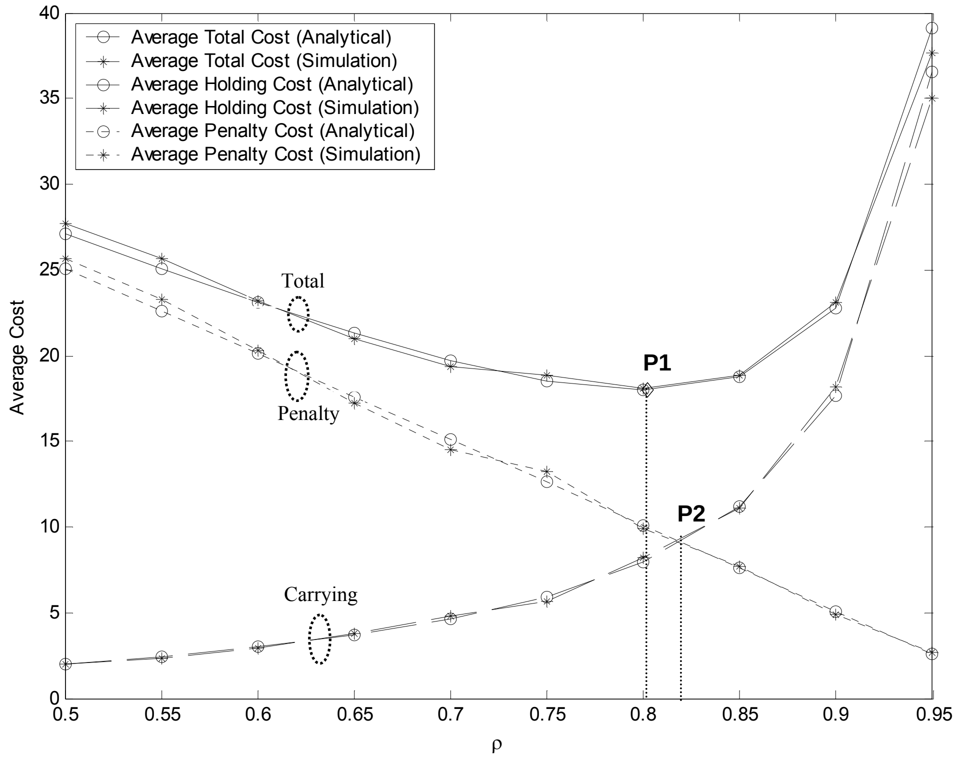

Figure 5.5. Average Total Costs, Carrying Costs and Penalties as Functions of the Replenishment Rate under Exponential Demand

In the third case, we study analytically-computed quantities associated with the optimal solution, $(\rho^*, \bar{c}_{\rho^*})$, under various demand distributions: constant, exponential, uniform and Gamma.

Table 5.3 displays ρ^* and ξ^* as functions of the mean demand, β^{-1}, with the four aforementioned demand distributions. From Table 5.3 it can be seen that the respective optimal replenishment rates increase in this order of distributions: constant, uniform, Gamma and exponential. Note that as the average demand decreases (i.e., β gets larger), the optimal replenishment rate approaches zero for all demand distributions, as it should be, since the optimal replenishment must be zero in the absence of demand.

Table 5.3. Analytically-Computed Optimal Quantities under Various Demand Distributions

	$D=\frac{1}{\beta}$		$\mathrm{Exp}(\beta)$		$U\left(0,\frac{2}{\beta}\right)$		$\Gamma\left(4,\frac{1}{4\beta}\right)$	
β	ξ^*	ρ^*	ξ^*	ρ^*	ξ^*	ρ^*	ξ^*	ρ^*
0.1	0.060	3.798	0.080	2.806	0.068	3.367	0.065	3.515
0.6	0.121	0.762	0.133	0.689	0.125	0.736	0.124	0.743
1.1	0.159	0.427	0.170	0.398	0.163	0.417	0.162	0.420
1.6	0.189	0.298	0.200	0.281	0.193	0.292	0.192	0.293
2.1	0.215	0.229	0.226	0.217	0.218	0.225	0.217	0.226
2.6	0.238	0.186	0.249	0.177	0.241	0.183	0.240	0.184
3.1	0.258	0.156	0.269	0.150	0.262	0.154	0.261	0.155
3.6	0.277	0.135	0.288	0.130	0.281	0.133	0.280	0.134
4.1	0.295	0.119	0.306	0.115	0.299	0.117	0.298	0.118
4.6	0.312	0.106	0.323	0.103	0.316	0.105	0.315	0.105
5.1	0.328	0.096	0.339	0.093	0.332	0.095	0.331	0.095
5.6	0.343	0.087	0.354	0.085	0.347	0.087	0.346	0.087
6.1	0.358	0.080	0.369	0.078	0.361	0.080	0.361	0.080
6.6	0.372	0.074	0.382	0.072	0.375	0.074	0.374	0.074
7.1	0.385	0.069	0.396	0.067	0.389	0.069	0.388	0.069
7.6	0.398	0.065	0.409	0.063	0.402	0.064	0.401	0.064
8.1	0.411	0.061	0.421	0.059	0.414	0.060	0.413	0.060
8.6	0.423	0.057	0.433	0.056	0.426	0.057	0.426	0.057
9.1	0.435	0.054	0.445	0.053	0.438	0.054	0.437	0.054
9.6	0.446	0.051	0.457	0.050	0.450	0.051	0.449	0.051
10.1	0.457	0.049	0.468	0.048	0.461	0.049	0.460	0.049
10.6	0.468	0.047	0.479	0.046	0.472	0.046	0.471	0.046
11.1	0.479	0.045	0.490	0.044	0.483	0.044	0.482	0.044
11.6	0.489	0.043	0.500	0.042	0.493	0.042	0.492	0.042
12.1	0.500	0.041	0.510	0.040	0.503	0.041	0.502	0.041
12.6	0.510	0.039	0.520	0.038	0.513	0.039	0.512	0.039
13.1	0.520	0.038	0.530	0.037	0.523	0.038	0.522	0.038
13.6	0.529	0.036	0.540	0.036	0.533	0.036	0.532	0.036
14.1	0.539	0.035	0.549	0.034	0.542	0.035	0.541	0.035
14.6	0.548	0.034	0.558	0.033	0.551	0.034	0.550	0.034

Chapter 6 Further Properties

The chapter investigates additional properties of the production-inventory system under study.

6.1 Expected Stockout Time

In the following, we consider $\psi_r(z) = \psi(z)$, $\xi = \xi_r$, $\theta = \theta_r$ and $d_\rho(u) = d_{\rho,r}(u)$ as functions of $r \geq 0$.

The next proposition studies the expectation

$$\Gamma(u) = \mathbb{E}[\tau_1 \mid I(0) = u] = -\frac{\partial}{\partial r} d_{\rho,r}(u)\,|_{r=0}, \quad u \geq 0, \tag{6.1}$$

where the second equality follows from Eq. (4.8).

Proposition 6.1

(a) For $0 < \rho < \lambda \mathbb{E}[D]$,

$$\tilde{\Gamma}(z) = \frac{1}{\lambda \tilde{f}_D(z) + \rho z - \lambda}\left[\frac{1}{\xi_0} - \frac{1}{z}\right], \quad z \geq 0 \tag{6.2}$$

(b) For $\rho \geq \lambda \mathbb{E}[D]$,

$$\Gamma(u) = \infty, \quad u \geq 0 \tag{6.3}$$

Proof.

To prove part (a), note that by Eq. (6.1), one has

$$\tilde{\Gamma}(z) = -\frac{\partial}{\partial r} \tilde{d}_{\rho,r}(z)\,|_{r=0} \tag{6.4}$$

where the interchange above of the Laplace integral transform and differentiation holds by Leibnitz's rule.

Next, substituting Eq. (4.41) into Eq. (6.4) yields

$$\begin{aligned}\tilde{\Gamma}(z) &= \lim_{r\to 0}\frac{\partial}{\partial r}\left\{\frac{r}{\psi_r(z)}\left[\frac{1}{\xi_r}-\frac{1}{z}\right]\right\} \\ &= \lim_{r\to 0}\frac{1}{\psi_r(z)}\left[\frac{1}{\xi_r}-\frac{1}{z}\right]+\frac{1}{\psi_0^2(z)}\lim_{r\to 0}\frac{r}{\xi_r}-\frac{1}{\psi_0(z)}\lim_{r\to 0}\frac{r}{\xi_r^2}\frac{\partial}{\partial r}\xi_r\end{aligned} \tag{6.5}$$

where $\psi_0(z)$ is given by Eq. (5.11).

It remains to show that the two last terms on the right-hand side of Eq. (6.5) both vanish. Since $\xi_0 > 0$ by part (a) of Lemma 5.1, it suffices to show that $\frac{\partial}{\partial r}\xi_r$ is bounded. To this end, differentiate Eq. (3.8) with respect to r yielding

$$1 = \left[\rho - \lambda\int_0^\infty x\, e^{-\xi_r\, x} f_D(x)\, dx\right]\frac{\partial}{\partial r}\xi_r\,. \tag{6.6}$$

Observe that

$$\rho - \lambda\int_0^\infty x\, e^{-\xi_0\, x} f_D(x)\, dx < \rho - \lambda\,\mathbb{E}[D] < 0\,, \tag{6.7}$$

where the first inequality holds by the fact that $\xi_0 > 0$ and the second inequality holds by assumption. Eqs. (6.6) and (6.7) readily imply that $\frac{\partial}{\partial r}\xi_r$ is bounded, which completes the proof of Eq. (6.2).

Finally, part (b) follows from the fact that $\tau_1 = \infty$ with positive probability [Prabhu (1965)]. □

Corollary 6.1

For $\rho \geq 0$,

$$\Gamma(0) = \frac{1}{\rho \xi_0} = \frac{1}{\lambda - \lambda \tilde{f}_D(\xi_0)}, \tag{6.8}$$

Proof.

We consider two cases.

(a) Assume that $0 < \rho < \lambda \mathbb{E}[D]$. The first equality in Eq. (6.8) follows by substituting Eq. (6.2) into the representation $\Gamma(0) = \lim_{z \to \infty} z \tilde{\Gamma}(z)$ [Widder, (1959)], and then sending $z \uparrow \infty$. The second equality readily follows by further substituting $\rho \xi_0 = \lambda - \lambda \tilde{f}_D(\xi_0)$ which is obtained by setting $z = \xi_0$ in Eq. (5.11).

(b) Assume that $\rho \geq \lambda \mathbb{E}[D]$. Eq. (6.8) now follows trivially, since $\Gamma(0) = \infty$ by part (b) of Proposition 6.1, and $\xi_0 = 0$ by part (b) and (c) of Lemma 5.1. □

6.2 Bound on the Probability of the Inventory Exceeding a Given Value

Since our model's base-stock level is infinite, an excursion of the inventory process can generally reach arbitrarily large levels. The following lemma provides a bound on the probability of the inventory level exceeding a given value.

Lemma 6.1

For a given replenishment rate, ρ, consider the corresponding inventory process in the steady state regime. Then, for $s > 0$,

$$P\{I(t) \geq s\} \leq \frac{1}{s \xi}, \quad t \geq 0. \tag{6.9}$$

Proof.

By Markov inequality [Karr (1993)],

$$P\{I(t) \geq s\} \leq \frac{\mathbb{E}[I(t)]}{s} = \frac{\bar{I}}{s}$$

The result now follows from Eq. (5.37). □

6.3 Demand Fill Rate

Let π denote the fill rate (limiting fraction of demand arrivals that can be immediately satisfied from inventory on hand), and let the lost-sales rate be denoted by $\bar{\pi} = 1 - \pi$. Then,

$$\bar{\pi} = \lim_{t \to \infty} \frac{N_B(t)}{N_A(t)},$$

where $N_A(t)$ and $N_B(t)$ are the numbers of demand arrivals and lost-sale, respectively, in the interval $(0, t]$. Alternatively, $\bar{\pi}$ can be represented as [cf. Ross (1996), Theorem 3.4.4]

$$\bar{\pi} = \frac{\mathbb{E}[T_1]}{\mathbb{E}[\tau_1 \mid I(0) = 0]}. \tag{6.10}$$

Proposition 6.2

For $\rho \geq 0$,

$$\pi = \tilde{f}_D(\xi_0) = 1 - \frac{\rho \xi_0}{\lambda} \tag{6.11}$$

Proof.

For $\rho \geq 0$,

$$\bar{\pi} = \frac{1}{\lambda \Gamma(0)} = \frac{\rho \xi_0}{\lambda} = 1 - \tilde{f}_D(\xi_0), \tag{6.12}$$

where the first equality follows from Eq. (6.10) and the facts that $\Gamma(0) = \mathbb{E}[\tau_1 \,|\, I(0) = 0]$ and $\mathbb{E}[T_1] = 1/\lambda$, and the second equation holds by further substituting the representation of $\Gamma(0)$ in Eq. (6.8). Eq. (6.11) now follows immediately from Eq. (6.12). □

We mention in passing that sending $\rho \downarrow 0$ on both sides of Eq. (6.11) yields the intuitive result $\pi = 0$ by substituting Eq. (3.12) with $r = 0$ into Eq. (6.11). This is consistent with the fact that for $\rho = 0$, one has $\mathbb{E}[\tau_1 \,|\, I(0) = 0] = 1/\lambda$, since in a system without replenishment, started with zero initial inventory, each demand arrival results in a lost sale, and consequently, the mean time between losses is coincides with the mean interarrival time, $\mathbb{E}[T_1] = 1/\lambda$.

Chapter 7 Conclusion

In this thesis, we investigated a single-product continuous-review production-inventory system with infinite base-stock level, compound Poisson demands and lost-sales, and constant replenishment, subject to holding costs and lost-sale penalties. We derived closed-form formulas for both the expected discounted cost as a function of arbitrary initial inventory level, as well as the time average cost. The resultant cost functions were optimized with respect to the replenishment rate via a simple search in the Laplace transform domain.

The results can be readily generalized to an optimization of cost functions under study with respect to the replenishment rate, subject to a given minimal service level, e.g., a fill rate π [de Kok (1985)]. For this problem, one can apply Eq. (6.11) to compute the critical value ξ' such that $\pi = \tilde{f}_D(\xi')$. It can be readily shown that the cost optimization problem with the constraint that the fill rate is at least π can be solved by a search in the Laplace domain restricted to the interval $0 < z \leq \xi'$, as opposed to the original search space, $z > 0$.

Although optimal replenishment rate may not be unique in general, it is highly likely that the optimal replenishment rate is unique under fairly general conditions. More general conditions that ensure such uniqueness are the subject of future research.

Further, the research presented in this thesis may be extended in several directions. First, one may consider more general cost functions, e.g., with nonlinear terms. Second, the Poisson assumption can be relaxed to a general renewal arrival process. Finally, one may consider a continuous-replenishment inventory with a finite base-stock level (where replenishment is suspended when the inventory level reaches or is at the base-stock level) rather than one with unlimited capacity.

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Appendix A

A.1 Proofs for Table 4.1 Formulas

Constant Demand Size. Consider the first distribution row of Table 4.1, where $D = d > 0$ is a constant, so that

$$\tilde{f}_D(z) = \exp\{-zd\}\,. \tag{8.1}$$

The corresponding ξ^* follows by substituting Eq. (8.1) into Eq. (4.79), the corresponding ρ^* follows by substituting this ξ^* and Eq. (8.1) into Eq. (4.65), and the corresponding $\Phi_{\rho*}(0)$ follows by substituting these ξ^* and ρ^* into Eq. (4.78).

Exponentially-Distributed Demand Size. Consider the second distribution row of Table 4.1, where $D \sim \text{Exp}(\beta)$. Substituting Eq. (4.67) into Eq. (4.79) yields,

$$\xi^* = \underset{\xi>0}{\mathbf{argmin}}\left\{\frac{h}{\xi} - \frac{\beta\lambda\, K_0}{\xi+\beta}\right\}. \tag{8.2}$$

Finally, the corresponding ξ^* is obtained from Eq. (8.2) by first derivative test with respect ξ, the corresponding ρ^* follows by substituting such ξ^* into Eq. (4.65), and the corresponding $\Phi_{\rho*}(0)$ follows by substituting ξ^* into Eq.(4.78).

Uniformly-Distributed Demand Size. Consider the third distribution row of Table 4.1, where $D \sim U(a,b)$, so that

$$\tilde{f}_D(z) = \frac{e^{-a\,z} - e^{-b\,z}}{(b-a)\,z}\,. \tag{8.3}$$

The corresponding ξ^* follows by substituting Eq. (8.3) into Eq. (4.79), the corresponding ρ^* follows by substituting this ξ^* and Eq. (8.3) into Eq. (4.65), and the corresponding $\Phi_{\rho^*}(0)$ follows by substituting these ξ^* and ρ^* into Eq. (4.78).

Gamma-Distributed Demand Size. Consider the fourth distribution row of Table 4.1, where $D \sim \Gamma(\alpha, \beta)$, so that

$$\tilde{f}_D(z) = \left(1 + \frac{z}{\beta}\right)^{-\alpha}. \tag{8.4}$$

The corresponding ξ^* follows by substituting Eq. (8.4) into Eq. (4.79), the corresponding ρ^* follows by this ξ^* and Eq. (8.1) into Eq. (4.65), and the corresponding $\Phi_{\rho^*}(0)$ follows by substituting these ξ^* and ρ^* into Eq. (4.78). □

A.2 Proofs for Table 4.2 Formulas

Constant Demand Size. Consider the first distribution row of Table 4.2, where $D = d$. Then, the corresponding ξ^* follows by substituting Eq. (8.1) into Eq. (4.81), the corresponding ρ^* follows by substituting this ξ^* and Eq. (8.1) into Eq. (4.65), and the corresponding $\Phi_{\rho^*}(0)$ follows by substituting these ξ^* and ρ^* into Eq. (4.60).

Exponentially-Distributed Demand Size. Consider the second distribution row of Table 4.2, where $D \sim \text{Exp}(\beta)$. Substituting Eq. (4.67) into Eq. (4.81) yields,

$$\xi^* = \operatorname*{argmin}_{\xi>0} \left\{ \frac{h}{\xi} - \frac{\lambda K_1}{\xi + \beta} \right\}. \tag{8.5}$$

Finally, the corresponding ξ^* is obtained from Eq. (8.5) by first derivative test with respect ξ, the corresponding ρ^* follows by substituting such ξ^* into Eq. (4.65), and the corresponding $\Phi_{\rho*}(0)$ follows by substituting ξ^* into Eq. (4.60).

Uniformly-Distributed Demand Size. Consider the third distribution row of Table 4.2, where $D \sim U(a,b)$. Then, the corresponding ξ^* follows by substituting Eq. (8.3) into Eq. (4.81), the corresponding ρ^* follows by substituting this ξ^* and Eq. (8.3) into Eq. (4.65), and the corresponding $\Phi_{\rho*}(0)$ follows by substituting these ξ^* and ρ^* into Eq.(4.60).

Gamma-Distributed Demand Size. Consider the fourth distribution row of Table 4.2, where $D \sim \Gamma(\alpha,\beta)$. Then, the corresponding ξ^* follows by substituting Eq. (8.4) into Eq. (4.81), the corresponding ρ^* follows by substituting this ξ^* and Eq. (8.4) into Eq. (4.65), and the corresponding $\Phi_{\rho*}(0)$ follows by substituting these ξ^* and ρ^* into Eq. (4.60). □

A.3 Proofs for Table 5.1 Formulas

Constant Demand Size. Consider the first distribution row of Table 5.1, where $D = d > 0$ is a constant, so that

$$\tilde{f}_D(z) = \exp\{-zd\}. \tag{8.6}$$

The corresponding ξ^* follows by substituting Eq. (8.1) into Eq. (4.79), the corresponding ρ^* follows by substituting this ξ^* and Eq. (8.1) into Eq. (5.19), and the corresponding $\bar{c}_{\rho^*}$ follows by substituting these ξ^* and ρ^* into Eq. (5.61).

Exponentially-Distributed Demand Size. Consider the second distribution row of Table 5.1, where $D \sim \mathrm{Exp}(\beta)$. Substituting Eq. (5.70) into Eq. (4.78) yields,

$$\bar{c}_\rho = \lambda K_0 + \frac{h}{\xi} - \frac{\lambda K_0 \beta}{\beta + \xi}. \tag{8.7}$$

Finally, the corresponding ξ^* is obtained by straightforward minimization of Eq. (8.7) in ξ, the corresponding ρ^* follows by substituting this ξ^* into Eq. (5.72), and the corresponding $\bar{c}_{\rho^*}$ follows by substituting ξ^* into Eq. (8.7).

Uniformly-Distributed Demand Size. Consider the third distribution row of Table 5.1, where $D \sim U(a,b)$, so that

$$\tilde{f}_D(z) = \frac{e^{-a z} - e^{-b z}}{(b-a)\, z}. \tag{1.8}$$

The corresponding ξ^* follows by substituting Eq. (8.3) into Eq. (4.79), the corresponding ρ^* follows by substituting this ξ^* and Eq. (8.3) into Eq. (5.19), and the corresponding $\bar{c}_{\rho^*}$ follows by substituting these ξ^* and ρ^* into Eq. (5.61).

Gamma-Distributed Demand Size. Consider the fourth distribution row of Table 5.1, where $D \sim \Gamma(\alpha, \beta)$, so that

$$\tilde{f}_D(z) = \left(1 + \frac{z}{\beta}\right)^{-\alpha}. \tag{1.9}$$

The corresponding ξ^* follows by substituting Eq. (8.4) into Eq. (4.79), the corresponding ρ^* follows by substituting this ξ^* and Eq. (8.4) into Eq. (5.19), and the corresponding $\bar{c}_{\rho^*}$ follows by substituting these ξ^* and ρ^* into Eq. (5.61). □

A.4 Proofs for Table 5.2 Formulas

Constant Demand Size. Consider the first distribution row of Table 5.2, where $D = d$. Then, the corresponding ξ^* follows by substituting Eq. (8.1) into Eq. (4.81), the corresponding ρ^* follows by substituting this ξ^* and Eq. (8.1) into Eq. (5.19), and the corresponding $\bar{c}_{\rho^*}$ follows by substituting these ξ^* and ρ^* into Eq.(5.66).

Exponentially-Distributed Demand Size. Consider the second distribution row of Table 5.2, where $D \sim \text{Exp}(\beta)$. Substituting Eq. (5.70) into Eq. (4.80) yields,

$$\bar{c}_\rho = \frac{h}{\xi} - \frac{\lambda K_1}{\beta + \xi} + \lambda K_1 \mu_D \quad . \tag{8.10}$$

Finally, the corresponding ξ^* is obtained by straightforward minimization of Eq.(8.10), the corresponding ρ^* follows by substituting this ξ^* into Eq. (5.72), and the corresponding $\bar{c}_{\rho^*}$ follows by substituting this ξ^* into Eq. (8.10).

Uniformly-Distributed Demand Size. Consider the third distribution row of Table 5.2, where $D \sim U(a,b)$. Then, the corresponding ξ^* follows by substituting Eq. (8.3) into Eq. (4.81), the corresponding ρ^* follows by substituting this ξ^* and Eq. (8.3) into Eq. (5.19), and the corresponding $\bar{c}_{\rho^*}$ follows by substituting these ξ^* and ρ^* into Eq. (5.66).

Gamma-Distributed Demand Size. Consider the fourth distribution row of Table 5.2, where $D \sim \Gamma(\alpha, \beta)$. Then, the corresponding ξ^* follows by substituting Eq. (8.4) into Eq. (4.81), the corresponding ρ^* follows by substituting this ξ^* and Eq. (8.4) into Eq. (5.19), and the corresponding $\bar{c}_{\rho^*}$ follows by substituting these ξ^* and ρ^* into Eq. (5.66). □

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