EMBEDDING SPANNING SUBGRAPHS INTO LARGE DENSE GRAPHS

by

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ABSTRACT OF THE DISSERTATION

Embedding Spanning Subgraphs into Large Dense Graphs

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In this thesis we are going to present some results on embedding spanning subgraphs into large dense graphs.

Spanning Trees

Bollobás conjectured that if G is a graph on n vertices, $\delta(G) \geq (1/2 + \epsilon)n$ for some $\epsilon > 0$, and T is a bounded degree tree on n vertices, then T is a subgraph of G. The problem was solved in the affirmative by Komlós, Sárközy and Szemerédi for large graphs. They then strengthened their result, and showed that the maximum degree of T need not be bounded: there exists a constant c such that T is a subgraph of G if $\Delta(T) \leq cn/\log n$, $\delta(G) \geq (1/2 + \epsilon)n$ and n is large. Both proofs are based on the Regularity Lemma-Blow-up Lemma Method. Recently, using other methods, it was shown that bounded degree trees embed into graphs with minimum degree $n/2 + C \log n$, where C is a constant depending on the maximum degree of T. Here we show that in general $n/2 + O(\Delta(T) \cdot \log n)$ is sufficient for every $\Delta(T) \leq cn/\log n$. We also show that this bound is tight for the two extreme values of m i.e. when $m = C$ and when $m = cn/\log n$.

Powers of Hamiltonian Cycles

In 1962 Pósa conjectured that if $\delta(G) \geq \frac{2}{3}n$ then G contains the square of a Hamiltonian cycle. Later, in 1974, Seymour generalized this conjecture: if $\delta(G) \geq (\frac{k-1}{k})n$ then G contains the $(k-1)$th power of a Hamiltonian cycle. In 1998 the conjecture was proved by Komlós, Sárközy and Szemerédi for large graphs using the Regularity Lemma. We present a "deregularised" proof of the Pósa-Seymour conjecture which results in a much lower threshold value for n, the size of the graph for which the conjecture is true. We hope that the tools used in this proof will push down the threshold value for n to around 100 at which point we will be able to verify the conjecture for every n.

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During the course of my schooling, a lot of teachers affected and influenced me, but none more so than my friend and mentor Sarmad Abbasi. He gave me my first glimpse of The Book and, in a sense, is to blame for all that transpired as its consequence.

Dedication

This thesis is dedicated to my parents who devoted the prime of their lives to the education of their children.

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Chapter 1

Introduction

Broadly speaking, this thesis focuses on structural problems in extremal graph theory. The unifying theme of the questions that have been tackled therein involves the application of the probabilistic method to investigate properties required of graphs with a certain given structure. This area of combinatorics has a distinctly Hungarian flavor and as Bollobás (see [6], Preface) has said: *"Extremal graph theory, in its strictest sense, is a branch of graph theory developed and loved by Hungarians."* The impact of these kind of results has been felt in many areas of computer science and is not restricted just to some abstract domains (see for example [2, 4, 10, 46, 55]).

This thesis considers two generalizations, in different directions, of a celebrated theorem of Dirac on graphs. First let us fix the notation. We will always talk about simple, undirected graphs $G(V, E)$ and by $deg(v)$, we denote the degree of the vertex $v \in V$. The minimum degree of a vertex in G is denoted by $\delta(G)$ whereas $\Delta(G)$ denotes the maximum degree of a vertex. A cycle of length j is denoted by C_j and a cycle that contains all the vertices of the graph is called a Hamiltonian cycle. A graph is called Hamiltonian if it contains a Hamiltonian cycle.

1.1 Generalizations of Dirac's Theorem

One of the basic theorems of graph theory is Dirac's Theorem [15] giving a sufficient condition for the existence of a Hamiltonian cycle in a graph.

Theorem 1 (Dirac 1952)**.** *Let G be a simple graph on $n \geq 3$ vertices. If each vertex has degree at least $n/2$ then G is Hamiltonian.* □

This result, along with Turán's Theorem, is a quintessential example of the type of

questions considered in extremal graph theory.

In 1975 one of the most important tools in extremal graph theory, the Regularity Lemma, was introduced by Szemerédi [54]. The remarkable lemma asserts that, roughly speaking, any large enough dense graph can be approximated by a random-looking graph so that one may use the properties of random graphs to prove certain facts easily. However, the main drawback of the result is that it is true only for astronomically large graphs. Hence, one recent trend has been to avoid the use of the Regularity Lemma (see for example [43, 12]). The use of elementary random arguments instead, drastically brings down the threshold size of graphs for which the results become true.

The work in this doctoral dissertation has focused on proving two generalizations of Dirac's Theorem without resorting to the use of the Regularity Lemma. The questions investigated have had a very rich history and a lot of prominent mathematicians have worked on them. It is pertinent to note that the proofs given using probabilistic techniques are inherently algorithmic and are relatively straightforward to derandomize to give deterministic procedures that output the required structure in the graph.

1.1.1 Spanning Trees

One possible generalization of the Dirac's Theorem might be to look for a degree condition on a graph if instead of looking for a specific spanning tree (a Hamiltonian path is a spanning tree with maximum degree two) one looks for a general spanning tree with some degree bounds. In this vein Bollobás conjectured that if G is a graph on n vertices, $\delta(G) \geq (1/2 + \epsilon)n$ for some $\epsilon > 0$, and T is a bounded degree tree on n vertices, then T is a subgraph of G. The conjecture was resolved in the affirmative by Komlós, Sárközy and Szemerédi [39] for large graphs. They then strengthened their result [41], and showed that the maximum degree of T need not be bounded by a constant: there exists a constant c such that T is a subgraph of G if $\Delta(T) \leq cn/\log n$, $\delta(G) \geq (1/2 + \epsilon)n$ and n is large. Both proofs are based on the Regularity Lemma-Blow-up Lemma Method [36]. Recently, using other methods, it was shown by Csaba, Levitt, Nagy-György and Szemerédi [12] that bounded degree trees embed into graphs with minimum degree $n/2 + C \log n$, where C is a constant depending on the maximum

degree of T.

In this thesis we provide a substantial generalization of these results.

Theorem 2. *There exist two positive constants c and K such that the following holds. Assume that $T = (V, E(T))$ is a tree on n vertices with $\Delta(T) = m \leq cn/\log n$ and $G = (W, E(G))$ is a graph on n vertices with minimum degree $\delta(G) \geq n/2 + Km\log n$. Then there exists an n_0 such that $T \subset G$ for $n \geq n_0$.* $\square$

We show that, apart from the Dirac's Theorem condition of $n/2$ just another $Km\log n$ extra minimum degree is required to ensure that any tree on n vertices with maximum degree $m \leq cn/\log n$ is a subgraph of G.

As in many questions of this type, the proof is divided into the so-called extremal and the non-extremal case. Basically, the non-extremal case is where the edges of the graph are distributed relatively evenly all over the graph, whereas in the extremal case we may be able to find induced subgraphs that are either almost complete or almost empty. Different methods are used to solve the problem in these different cases.

A variety of techniques are used to solve the problem depending on the type of tree that needs to be embedded. We make a distinction between height two trees and larger height trees. In the non-extremal case, a greedy embedding method is combined with an "augmenting path" type argument at the end to embed the height two tree. If the tree is of larger height we check whether the tree has many leaves, or only a few. In both cases we leave out some vertices of the tree and first embed the resulting smaller subtree in a particular way. This partial embedding enables us to finish the embedding of the tree using a König-Hall type argument at the end.

The case when G is extremal is handled by an involved averaging argument in conjunction with a generalized matching theorem. We note that the lowerbound on the degree of G is tight in this case.

1.1.2 Powers of Hamiltonian Cycles

A second quite natural generalization of the Dirac's Theorem might be to look for structures stronger than *simple* Hamiltonian cycles in a given graph. Let C be a cycle

in a graph G on $n \geq 3$ vertices. Then the graph C^k is called the kth *power* of C if for every pair of vertices $u, v \in V(C)$, uv is an edge in C^k if and only if the distance between u and v is at most k in C. Obviously powers of cycles are much stronger structures than simple cycles. A question extending Dirac's Theorem was asked by Pósa (see [20]) in 1962:

Conjecture 3 (Pósa 1962)**.** *Let G be a graph on n vertices. If $\delta(G) \geq \frac{2}{3}n$, then G contains the square of a Hamiltonian cycle.*

This conjecture was further generalized by Seymour [51] in 1974:

Conjecture 4 (Seymour 1974)**.** *Let G be a graph on n vertices. If $\delta(G) \geq \frac{k-1}{k}n$, then G contains the $(k-1)^{th}$ power of a Hamiltonian cycle.*

Substantial amount of work has been done on these problems. A long series of papers made several steps towards the resolution of the conjectures (see for example [21, 23, 24, 25, 27]).

Then using the Regularity Lemma–Blow-up Lemma method first in [37] Komlós, Sárközy and Szemerédi proved Conjecture 4 in asymptotic form, then in [38] and [40] they proved both conjectures for $n \geq n_0$. The proofs used the Regularity Lemma [54], the Blow-up Lemma [36] and the Hajnal-Szemerédi Theorem [32]. Since the proofs used the Regularity Lemma the resulting n_0 is very large (it involves a tower function). The use of the Regularity Lemma was removed by Levitt, Sárközy and Szemerédi in a new proof of Pósa's Conjecture in [43]. In this dissertation we give another proof of the Pósa-Seymour conjecture that avoids the use of the Regularity Lemma, thus resulting in a much smaller threshold size of G as compared to the bound given in [40].

Theorem 5. *There exists a natural number n_0 such that if G is a graph on $n \geq n_0$ vertices and $\delta(G) \geq (\frac{k-1}{k})n$ then G contains the $(k-1)^{th}$ power of a Hamiltonian cycle.*

The proof uses a number of probabilistic techniques in extremal graph theory. The proof again treats the extremal and the non-extremal cases separately. The basic step in finding the $(k-1)^{th}$ power path is to go around a k-partite complete graph picking one vertex in each color class until there are no vertices left. What we end up with

is a $(k-1)^{th}$ power of a path. Our main thrust, therefore, is to try to cover the vertices of the graph with balanced k-partite complete graphs. These complete graphs are then connected together in a cycle using the **Connecting Lemma**. This lemma is a powerful result of independent interest and what it asserts is that we can find a very short $(k-1)^{th}$ power path (of size about $8k^2 + o(k^2)$) between any two disjoint $(k-1)$-cliques in the non-extremal graph with $\delta(G) \geq \frac{k-1}{k}n$. This is a major improvement over the similar statements in other related papers (see the Connecting Lemma's in [43, 49, 48]) where the length of the connecting path had been a huge constant. In particular, for $k = 3$ (Pósa's Conjecture) the Connecting Lemma gives us a procedure to construct a square path of length 16 between any two edges of the graph.

So in the end, by "unfolding" the connected balanced k-partite complete graphs into $(k-1)^{th}$ power of a paths, we get our $(k-1)^{th}$ power of a Hamiltonian cycle.

Future Work: Completely Resolving Pósa's Conjecture

After bringing down the threshold value n_0 for which the Pósa-Seymour conjecture has been shown to be true, our next goal is, to solve Pósa's Conjecture completely. The major improvement has been the much more efficient Connecting Lemma. We think we can further improve this slightly, so that we can push the n_0 down to around one hundred, after which, we may check for all the cases by computer search. This is possible since the very strict structure that is required of the graphs with such a high degree and the ideas presented in the thesis preclude many graphs, so that the brute force search for the remaining cases becomes feasible.

1.1.3   Future Work: Generalizations of Ore's Theorem

A result closely related to Dirac's Theorem is the following:

Theorem 6 (Ore 1960)**.** *Let G be a simple graph on $n \geq 3$ vertices. If $deg(u) + deg(v) \geq n$ for every pair of non-adjacent vertices $u, v \in V(G)$ then G is Hamiltonian.* $\square$

We believe that the methods used to generalize Dirac's Theorem can be very useful to make advances on similar generalizations for Ore's Theorem e.g. we can look for

lowerbounds on the sum of the degrees of non-adjacent vertices of a graph that guarantee existence of certain spanning trees or powers of Hamiltonian cycles.

Chapter 2

Embedding Spanning Trees

2.1   Introduction

We will only deal with simple graphs, without loops or multiple edges. Given a graph $F = (V, E)$, and $U \subset V$, the subgraph of F induced by the vertices of U will be denoted by $F|_U$. The degree of a vertex $v \in V(F)$ is denoted by $deg_F(v)$, or by $deg(v)$ if F is clear from the context. The minimum degree and maximum degree of F will be denoted by $\delta(F)$ and $\Delta(F)$, respectively. We will let $N(v)$ represent the neighborhood of a vertex v, hence $deg(v) = |N(v)|$. Let $A \subset V(F)$. Then we denote $|N(v) \cap A|$ by $deg(v, A)$. The number of vertices in a graph F is written $v(F)$, and the number of its edges is $e(F)$. If $A, B \subset V(F)$, then $e(A, B)$ is the number of edges of F with one endpoint in A and the other endpoint in B, and $e(A) = e(A, A) = e(F|_A)$. For a tree T we denote the set of its leaves by $\ell(T)$.

Given a bipartite graph H, a proper 2-coloring of H is a partition of the vertices of H into two color classes such that adjacent vertices of H are assigned to different color classes.

For graphs J and F, we will write $J \subset F$ if and only if there exists an adjacency preserving injection $I : V(J) \rightarrow V(F)$. If $I(x) = v$ then we say that x *maps onto* v and that v *is covered by* x. If a vertex of F is not covered by any vertex of J then we call it *uncovered.*

Throughout the chapter we use $a \ll b$ to denote that a is sufficiently smaller than b so that the ensuing calculation holds true. Our goal is to show the following:

Theorem 7. *There exist two positive constants c and K such that the following holds. Assume that $T = (V, E(T))$ is a tree on n vertices with $\Delta(T) = m \leq cn/\log n$ and*

$G = (W, E(G))$ *is a graph on n vertices with minimum degree* $\delta(G) \geq n/2 + Km \log n$. *Then there exists an* n_0 *such that* $T \subset G$ *for* $n \geq n_0$.

Bollobás [6] conjectured that if G is a graph on n vertices, $\delta(G) \geq (1/2 + \epsilon)n$ for some $\epsilon > 0$, and T is a bounded degree tree on n vertices, then $T \subset G$. The problem was solved in the affirmative by Komlós, Sárközy and Szemerédi [39] for large graphs. They then strengthened their result (see [41]), and showed that $\Delta(T)$ need not be bounded: there exists a constant c such that $T \subset G$ if $m \leq cn/\log n$, $\delta(G) \geq (1/2 + \epsilon)n$ and n is large. Notice that their result does not tell if a smaller minimum degree is sufficient when $m \ll cn/\log n$. Both proofs are based on the Regularity Lemma – Blow-up Lemma Method. Recently, using other methods, it was shown in Csaba et al. [12] that bounded degree trees embed into graphs with minimum degree $n/2 + C \log n$, where C is a constant depending on the maximum degree of T. In this chapter we show that in general $n/2 + O(m \log n)$ is sufficient for every $m \leq cn/\log n$.

Throughout the chapter we reserve the letters u, v, w (sometimes with subscripts) for vertices of G, and use the letters x, y, z for vertices of T. We may assume that G is *minimal with respect to edge deletion,* retaining the property that $\delta(G) \geq n/2 + Km \log n$. In particular, there are no edges $uv \in E(G)$ such that *both* vertices u, v have degree larger than $n/2 + Km \log n$.

Let us remark that throughout the chapter we make no attempt to optimize the absolute constants.

2.1.1 Outline of the embedding method

We divide the problem into two subproblems, depending on whether G is *extremal* or *non-extremal.* We call G γ-extremal for $\gamma > 0$ if either G or its complement $\overline{G}$ contains a subgraph on $n/2$ vertices with at most γn^2 induced edges. Otherwise G is γ-non-extremal. Observe that since G is minimal with respect to deleting edges, if $A, B \subset W$ such that $|A| = |B| = n/2$ then $e(A, B) \geq \gamma n^2$. Set $\gamma = 2^{-20}$ and $K = 2^{\gamma^{-50}}$. We first classify all the graphs as either γ-non-extremal or γ-extremal for some suitable value of γ and then prove the statement of our theorem for each of these two classes.

In each case we will construct the adjacency preserving mapping $I : V \to W$ step by step. It is worth noting that while at any time in the course of the embedding we have an adjacency preserving partial mapping, the final I is not necessarily an extension of it. It is possible that we modify the original mapping of certain subtrees.

If G is non-extremal, we will have three cases. The first case is when T has essentially height two, that is, most vertices are close to the root. In the other two cases first we find a subtree T_0 that has about $\gamma^5 n$ vertices. If T_0 has many leaves, then it is *broad*, otherwise it is *long*. Depending on T_0 we will choose different methods when finishing the embedding of T. In both cases we leave out some vertices of T, and first embed the resulting smaller subtree in a particular way. This partial embedding enables us to finish the embedding of T. More details as follows.

Case 1: T has essential height two

In this case T has a root r and almost all vertices are at a distance at most two from the root. We mostly apply a greedy embedding method and heavily use the minimum degree condition for G, in particular the $m \log n$ additive term.

Case 2: T_0 is a broad subtree

We leave out several leaves from T, and embed the resulting smaller tree in the first phase. During the embedding we use a linear size random set M for connecting the already embedded parts with newly embedded forests. Most of this random set will remain intact during the first phase, and therefore can be used in the second phase for mapping the unmapped leaves by a matching procedure. We apply two basic methods for mapping large chunks of T in the first phase.

One of these, the CM procedure is used for mapping the so called *narrow* forests. A rooted forest is called *narrow* if the set of vertices at a distance at most two from the roots is small compared to the size of the forest, otherwise it is *wide*. It is possible that a narrow forest has to be remapped during the embedding. The CM procedure is very similar to the Main Mapping Procedure in [12].

For embedding wide forests we use another method that we refer to as the LL procedure, which maps level after level, using matchings. Unlike in the case of narrow

forests, we do not remap wide subtrees but map them once and for all. We remark that there are mixed cases: a wide forest may contain a narrow sub-forest, and a narrow forest may contain a wide sub-forest. In such cases we will divide the mixed forests in a *preprocessing procedure*. The fact that T has many leaves is only used at the very end, when finishing the embedding by the matching procedure.

If T has a large portion in wide forests, then we will divide T_0 into subforests as well. These new forests will be handled as the other forests in the decomposition. If there are at least $n/2$ vertices in narrow forests, then we need T_0 at the end in order to finish the embedding. The main idea here is that after the almost embedding only very few vertices (about $\gamma^{13}n$) can be left out. If we put this 'noise' to the 'large' $I(T_0)$, this results in a subgraph of G into which we can embed T_0.

We remark that a simplified version of Case 1 is used when embedding wide forests, it is one of the basic building blocks of the LL procedure.

Case 3: T_0 is a long subtree

This time T_0 is long, and therefore contains many long paths $p_0 p_1 \ldots p_k$ such that $deg_T(p_i) = 2$ for $1 \le p_i \le k$. We call such paths *induced paths*. We leave out several vertices from these paths, and embed the resulting smaller tree with the method of phase one of Case 2. Then we re-insert the left out vertices by a matching procedure. This method is very similar to the one discussed in [12].

When G is γ-extremal we have two cases. In the first case G is very close to the union of two complete graphs, both on $n/2$ vertices. In the second case $G$is very close to a complete bipartite graph, that has color classes of size $n/2$. The embedding algorithms in these cases are similar to each other, and based on the method for the non-extremal case. As in the case where T has height two, here we need the full strength of the minimum degree condition on G.

2.1.2   On the minimum degree bound of G

In [12] it was shown that $\delta(G) - n/2 = \Omega(\log n)$ for T to be embedded into G where the maximum degree of T is bounded by a constant. Let T be a complete ternary tree

on n vertices. An explicit n-vertex graph G was constructed with minimum degree $\delta(G) = n/2 + (\log_3 n)/16 - 1$ such that $T \not\subset G$.

In [41] it is proved that for every constant $\epsilon > 0$ there exist constants $c > 0$ and n_0 such that if $m \leq cn/\log n$ and $\delta(G) \geq n/2 + \epsilon n$ then $T \subset G$. They also showed that their result is optimal up to a constant factor. Their construction is as follows. Let T be rooted tree with root r. Let $deg(r) = (\log n)/c_1$, where c_1 is a sufficiently large constant, and assume that the degrees of the children of r are as equal as possible. If G is an Erdős-Rényi random graph on n vertices with edge probability 0.9, say, then with high probability $T \not\subset G$, while $\delta(G) > 0.8n$ with high probability.

The reader will see from the proof of Theorem 7 that in case G is non-extremal and $m = o(\sqrt{n})$ then there exists an $\epsilon > 0$ constant such that even $\delta(G) = (1/2 - \epsilon)n$ is sufficiently large in order to embed a spanning tree T with maximum degree m. This resembles the constant degree case considered in [12]. However, when $m = \Omega(\sqrt{n})$ our proof requires that $\delta(G) - n/2 = \Omega(m \log n)$, and we conjecture that this is indeed the true bound.

2.2 Some tools for the embedding

Definition 1. *Assume that we are given a rooted tree F with root ρ. Let x and y be any two vertices of F. We say that y is below x if the simple path connecting y with ρ goes through x. Let $F(x)$ denote the subtree rooted at x containing every vertex which is below x. Sometimes we will call x the tip of the subtree $F(x)$.*

We can decompose a rooted tree into levels: the ith level contains those vertices of F which are at a distance $i - 1$ from r. We denote the ith level set of F by $L_i(F)$ or L_i, if F is clear from the context. We say that F has height two if it has three levels. The first one includes ρ, the second level contains the neighbors of r, and the third level contains all the rest. We say that F has essentially height two, if there are at most $\gamma^4 v(F)$ vertices below the third level.

Observe that if G is not γ-extremal, then it has many triangles: there are at least γn^2 edges in the neighborhood of every vertex, since these neighborhoods are of size

larger than $n/2$. As any such edge is in a triangle, summing over vertices will count each triangle three times. We get that there are at least $\gamma n^3/3$ triangles in G.

We record here a simple statement, which is also very useful throughout the proof:

Lemma 8. *Let H be a tree of size m. If J is a graph with $\delta(J) \geq m$, then there is an embedding of H into J. If $J' = J'(A, B)$ is a bipartite graph such that every $b \in B$ has at least $m/2$ neighbors in A and every $a \in A$ has at least m neighbors in B then $H \subset J'$.*

Proof: We leave the proof of the first part to the reader. For the second part we remark that H is 2-colorable, and one of its color classes has size at most $m/2$. □

We will frequently apply the following folklore statement (proof is omitted):

Lemma 9. *Every graph H has a subgraph H' such that $\delta(H') \geq e(H)/v(H)$.* □

Remark 1. *We will frequently use the above two lemmas in the following way: whenever we find a dense subgraph in the uncovered part of W, we can find a subgraph of it with large minimum degree into which we can map a large subtree of T.*

Let $F = F(A, B)$ be a bipartite graph satisfying the following requirements:

- $|A| = t$ and $|B| = \gamma^{-10}t$,

- $\gamma^{-4} \ll t$,

- every $b \in B$ has at least $(1/2 + \gamma^4)t$ neighbors in A.

Then we have the following *Cleaning Lemma.*

Lemma 10 (Cleaning lemma - first version). *F has a subgraph $F' = F'(A', B')$ such that $A' \subset A$, $B' \subset B$, every $b \in B'$ has at least $(1/2 + \gamma^4/2)t$ neighbors in A' and every $a \in A'$ has at least t neighbors in B'.*

Proof: First we assume, that every vertex in B has exactly $(1/2 + \gamma^4)t$ neighbors in A – if necessary, we discard edges incident to those vertices of B which have larger degrees. Then we have $e(F) = (1/2 + \gamma^4)\gamma^{-10}t^2$. We will find the desired subgraph F' step-by-step in the following way.

Let $A_1 = \{a \in A : \ |N(a) \cap B| < t\}$, and let $B_1 = \{b \in B : \ |N(b) \cap (A - A_1)| \leq (1/2 + \gamma^4/2)t\}$. In the first *cleaning step* we delete the vertices of A_1 from A and the vertices of B_1 from B. Removing these vertices deletes at most $|A_1|t + 2|A_1|t(1/2+\gamma^4)/\gamma^4$ edges from F.

In the ith cleaning step we first identify a subset $A_i \subset A - \cup_{j=1}^{i-1}A_j$ and a subset $B_i \subset B - \cup_{j=1}^{i-1}B_j$:

$$A_i = \{a \in A - \cup_{j=1}^{i-1}A_j : \ |N(a) \cap (B - \cup_{j=1}^{i-1}B_j)| < t\}$$

and

$$B_i = \{b \in B - \cup_{j=1}^{i-1}B_j : \ |N(b) \cap (A - \cup_{j=1}^{i}A_j)| < (1+\gamma^4)t/2\}.$$

Clearly $e(A_i, B - \cup_{j=1}^{i-1}B_j) < |A_i|t$, and moreover,

$$|B_i| \leq 2\frac{e(A_i, B - \cup_{j=1}^{i-1}B_j)}{\gamma^4 t} < 2\frac{|A_i|}{\gamma^4}.$$

Deleting the vertices of A_i and B_i therefore removes at most $|A_i|t + 2|A_i|(1/2+\gamma^4)t/\gamma^4$ edges. Let us assume that after k cleaning steps the cleaning process stops: either every vertex left satisfies the degree requirements of the lemma, or there are no vertices left. The total number of edges we lose from F is at most

$$\sum_{1 \leq i \leq k} |A_i|(t + 2t(1/2 + \gamma^4)/\gamma^4) < t^2(1 + \frac{1 + 2\gamma^4}{\gamma^4}) = t^2(3 + 1/\gamma^4) \ll \gamma^{-10}t^2.$$

We must have the former case, as most of the edges are still present when the process stops. The induced subgraph on $A - \cup_1^k A_i$ and $B - \cup_1^k B_i$ will be denoted by F', and is easily seen to satisfy the requirements of the lemma. □

For the second version of the Cleaning Lemma we assume that F satisfies the following:

- $|A| = t$ and $|B| = \gamma^{-10}t$,

- $\gamma^{-3} \ll t$,

- every $b \in B$ has at least $(1/2 - \gamma^3/2)t$ neighbors in A.

While the setup is somewhat different, the proof of the lemma below is very similar to that of the first version, we omit the details:

Lemma 11 (Cleaning lemma - second version)**.** *F has a subgraph $F' = F'(A', B')$ such that $A' \subset A$, $B' \subset B$, every $b \in B'$ has at least $(1/2 - \gamma^3)t$ neighbors in A' and every $a \in A'$ has at least $\gamma^{-5}t$ neighbors in B'.* $\qquad\square$

Random methods play an important role in the thesis. We will frequently use Azuma's inequality, in a bit more general form than how it is usually considered. Let $(\Omega, \mathcal{A}, P)$ be a finite probability space with the filtration (in this case a sequence of partitions of Ω)

$$(\emptyset, \Omega) = \mathcal{A}_0 \subset \mathcal{A}_1 \subset \ldots \subset \mathcal{A}_t = \mathcal{A}.$$

Let X be a measurable random variable. For each $1 \leq i \leq t$ we define the *martingale difference $d_i = \mathbb{E}(X|\mathcal{A}_i) - \mathbb{E}(X|\mathcal{A}_{i-1})$*, and assume that $|d_i| \leq \sigma_i$. We have the following important inequality, see e.g. in [3, 45].

Theorem 12 (Azuma's inequality)**.** *For all $a > 0$*

$$P(|X - \mathbb{E}X| \geq a) \leq 2e^{-a^2/2\sigma^2}$$

where $\sigma^2 = \sum_{i=1}^{t} \sigma_i^2$.

Throughout the chapter we will use Azuma's inequality in the following context: X will be the sum of t not necessarily independent variables such that σ_i, the absolute value of the martingale difference will never be larger than $t/(k \log n)$ for some real number k. We let $a = t/\sqrt{k}$. Then the probability that $|X - \mathbb{E}X| \geq t/\sqrt{k}$ will be at most n^{-k}. Usually k will be some positive power of K.

Throughout the embedding we will frequently use random subsets of W for various purposes. We will mostly use the well known fact that the degree of a vertex of G to a random set R is about $|R|/2$ with high probability, if R is not very small. We will also need that non-extremality and bipartite non-extremality are inherited by random subsets with high probability if the random sets are not very small, with parameter $\gamma/2$ instead of γ.

Let us define when we call a bipartite graph non-extremal. Given $R_1, R_2 \subset W, R_1 \cap R_2 = \emptyset$, the induced bipartite subgraph on R_1 and R_2 is $\gamma/2$-non-extremal if for every $A \subset R_1$, $B \subset R_2$, where $|A| = |R_1|/2$ and $|B| = |R_2|/2$ we have that $e(A, B) \geq \gamma|R_1||R_2|/2$. The proof of the statement below is fairly standard, and can be found in [12].

Lemma 13. *Let $R_1, R_2 \subset W$ with $R_1 \cap R_2 = \emptyset$ chosen randomly such that $|R_1|, |R_2| \geq K \log n$. Then for every $v \in W$ we have that $deg(v, R_i) \geq (1 - \gamma^{20})|R_i|/2$ with high probability. Furthermore, the induced subgraphs on R_1, R_2 and the bipartite subgraph with color classes R_1 and R_2 will be $\gamma/2$-non-extremal with high probability.*

Let $J(P, Q)$ be a bipartite graph, with $|P| = a$ and $|Q| = b$. Let γ be a small positive real number. We assume that $a > \gamma^{-4}$, $deg(x) \geq (1/2 - \gamma^4)b$ for every $x \in A$ and $deg(y) \geq (1/2 - \gamma^4)a$ for every $y \in B$. We further assume that J is $\gamma/2$-non-extremal, that is, if $P' \subset P$, $Q' \subset Q$ such that $|P'| = a/2$ and $|Q'| = b/2$, then $e(P', Q') \geq \gamma ab/2$.

Let $f : P \cup Q \to \mathbf{N}$ be a function. For a set $S \subset V(J)$ let $f(S) = \sum_{z \in S} f(z)$. The following conditions guarantee the existence of an f-factor, see the Lovász problem book [44].

(i) $f(P) = f(Q)$

(ii) $\forall X \subset P, \ Y \subset Q$ we have that $f(X) \leq e(X, Y) + f(Q - Y)$

When the degrees of the vertices are all in a small range, and Condition (i) above is satisfied, there exists an f-factor. More precisely, we have the following:

Lemma 14. *Set $q = b/a$. Let us assume that f satisfies condition (i), and $q(1 - \gamma^5) \leq f(x) \leq q(1 + \gamma^5)$ for every $x \in P$ and $f(y) = 1$ for every $y \in Q$. Then condition (ii) is satisfied as well. Hence, F has an f-factor.*

Proof: We assume that condition (i) is satisfied, and verify that condition (ii) holds by dividing the problem into several cases, depending on the size of X and Y.

Case 1. $1 \le |X| \le (1 - 2\gamma^4)a/2$. This case is divided into two sub-cases.

Case 1.1. $|Y| \le (1 + \gamma^4)b/2$. This case is easy to check, since $f(Y) = b - |Y|$ will be at least $(1 - \gamma^4)b/2$. This is an upper bound for $f(X)$, since $(1 - 2\gamma^4)(1 + \gamma^5) < (1 - \gamma^4)$.

Case 1.2. Here we assume that $|Y| > (1 + \gamma^4)b/2$. We have the following estimation for the number of edges between X and Y:

$$e(X, Y) \ge |X|(|Y| - (1 + \gamma^4)b/2).$$

Using the above bound for $f(X)$ it is sufficient to check the following inequality:

$$(1 - \gamma^4)b/2 \le b - |Y| + |X|(|Y| - (1 + \gamma^4)b/2),$$

which is equivalent to

$$0 \le (1 + \gamma^4)b/2 - |Y| + |X|(|Y| - (1 + \gamma^4)b/2).$$

Setting $s = |Y| - (1 + \gamma^4)b/2$ we get

$$0 \le -s + |X|s = s(|X| - 1),$$

which clearly holds.

Case 2. $(1 - 2\gamma^4)a/2 < |X| \le (1 + \gamma^4)a/2$. Then $f(X) \le (1 + \gamma^4)(1 + \gamma^5)b/2 < (1 + 1.5\gamma^4)b/2$. Again, we consider sub-cases.

Case 2.1. $|Y| \le (1 - 2\gamma^4)b/2$. This case can be verified very similarly to Case 1.1.

Case 2.2. $(1-2\gamma^4)b/2 < |Y| \le (1+\gamma/4)b/2$. This is the case when the non-extremality of J plays an important role: $e(X,Y) \ge \gamma ab - 3\gamma^4 ab > \gamma ab/2$. Since $f(Y) = b - |Y| \ge (1-\gamma/4)b/2$ we are done if

$$(1 + 1.5\gamma^4)b/2 \le (1-\gamma/4)b/2 + \gamma ab/2,$$

which easily seen to hold because $a > \gamma^{-4}$.

Case 2.3 $|Y| > (1+\gamma/4)b/2$. In this case it is sufficient if

$$(1 + 1.5\gamma^4)b/2 \le b - |Y| + |X|(|Y| - (1+\gamma^4)b/2).$$

Set $s = |Y| - (1+\gamma/4)b/2$, then the above is equivalent to

$$(1.5\gamma^4 + \gamma/4)b/2 + s \le |X|(s + (\gamma/4 - \gamma^4)b/2).$$

This last inequality holds, since $|X| \approx a/2$.

Case 3. $|X| > (1 + \gamma^4)a/2$. Set $s = f(X) - (b - |Y|)$. Note that $s \le |Y|$ always. If $s \le 0$, then we are done, so assume that $s > 0$. Using that

$$e(X,Y) \ge |Y|(|X| - (1+\gamma^4)a/2),$$

we check the following inequality:

$$f(X) \le f(X) - s + |Y|(|X| - (1+\gamma^4)a/2).$$

This is easily seen to hold since $s \le |Y|$ and $|X| > (1 + \gamma^4)a/2$.

$\square$

At certain points we will use another result that is based on finding f-factors in bipartite graphs satisfying conditions (i) and (ii). The following lemma is a special case of the main result of Csaba [11].

Lemma 15. *Let H be a bipartite graph having two color classes of size N and assume that $\delta(H) > N/2$. Then H has a spanning $N/4$-regular subgraph H_r.* $\square$

Another useful result is the following:

Lemma 16. *Assume that G is non-extremal and $u, v \in W$. Then there are at least $\gamma n/5$ vertex disjoint paths of length 3 connecting u and v.*

Proof: Set $a = |N(u) \cap N(v)|$. If $a \geq n/2 - \gamma n/2$ then, by the non-extremality of G, there are at least $\gamma n^2/2$ edges in $N(u) \cap N(v)$. One can therefore find $\gamma n/2$ vertex disjoint edges in $N(u) \cap N(v)$, determining $\gamma n/2$ vertex disjoint paths of length 3 between u and v. At the other extreme, if $a \leq \gamma n/2$ then there are at least γn^2 edges in $N(u)$ induced by $\overline{G}$, again since G is non-extremal. If v' is the endpoint of an edge in $\overline{G}|_{N(u)}$, then it has a neighbor in $N(v)$. It follows that there are at least $\gamma n^2/2$ edges between $N(u)$ and $N(v)$, and we can easily find the vertex disjoint paths of length 3 between u and v.

Assume now that $\gamma n/2 \leq a \leq n/2 - \gamma n/2$. Then the number of edges connecting $N(u) \cap N(v)$ with $N(u) \cup N(v)$ is at least $a(n/2 - a) \geq \gamma n^2/5$. From these one can easily choose $\gamma n/5$ vertex disjoint edges to yield the desired vertex disjoint paths of length 3. $\square$

2.3 T has essential height two

In this section we consider the case when T has essentially height 2, that is, when at most $\gamma^4 n$ vertices of T are farther than 2 from the root. First we consider the case when there are no vertices at level four or higher, then we discuss how to modify that method to embed trees with a few vertices at higher levels.

Notice, that in these cases m, the maximum degree of T has order $\Omega(\sqrt{n})$. The vertices adjacent to r are denoted by x_i and we let $X = \cup x_i$. The leaves of T are denoted by y_j, and we let $Y = \cup y_j$. We also assume without loss of generality that $deg_T(x_i) \geq deg_T(x_j)$ if $i < j$.

We first define two auxiliary graphs. Let u be an arbitrary vertex of G. We assume that $deg(u) = n/2 + Km \log n$ – if it is larger, then discard edges arbitrarily. The first auxiliary graph is the bipartite graph AG. We will construct an embedding of $T - r$ into AG. This embedding after some modifications will provide the embedding of T into G. The first color class of AG is $N(u)$, the other color class is W, the vertex set of

G. There is an edge $vw \in E(AG)$ if $v \in N(u), w \in W$ and $vw \in E(G)$.

In order to discover the structure of AG we define another auxiliary graph F. We define F as follows. The vertex set of F is $N(u)$, and $v_1 v_2 \in E(F)$ for $v_1, v_2 \in N(u)$ if $|N_G(v_1) \cap N_G(v_2)| \geq \gamma^3 n$. It is easy to see that if $v_1, v_2, v_3 \in N(u)$ then at least one of the $N(v_i) \cap N(v_j)$, $i \neq j$ sets has at least $n/6$ vertices, hence, every three element subset of $N(u)$ spans at least one edge in F. This implies that even after deleting a few vertices from $N(u)$ (which will be done later), F will have at most two components. Observe, that F cannot have an induced path of length four or longer.

We call F highly connected if deleting at most $\gamma^3 n/10$ vertices of $N(u)$ cannot make it disconnected such that both components have size at least $\gamma^3 n$. In this case the vertex u is called non-extremal, otherwise it is extremal. We will show that if u is an extremal vertex, then $N(u)$ contains many non-extremal vertices. Our goal is to map r, the root of T onto a non-extremal vertex.

Assume, that u is an extremal vertex, that is, after deleting at most $\gamma^3 n/10$ vertices from $N(u)$ the leftover of F falls apart into two components, A and B. We assume that $|A| \geq |B| \geq \gamma^3 n$. The following simple fact is given without a proof.

Fact 1. *Let $v_1, v_2 \in A$ and $w \in B$. Then*

$$|N(v_i) \cap N(w)| \leq \gamma^3 n$$

for $i = 1, 2$, and

$$|N(v_1) \cap N(v_2)| \geq n/2 - 2\gamma^3 n.$$

$\square$

Let $v \in A$ be arbitrary and let $C \subset N(v)$ be defined as follows: $C = \{w \in N(v) : |N(w) \cap A| \geq |A|(1 + 5\gamma^3)/2\}$. That is, every two vertices of C have many common neighbors in A. One can prove the following fact by counting the edges between A and $N(v)$.

Fact 2. $|N(v) - C| \leq 5\gamma^3 n.$

Notice that every two vertices of C have a common neighborhood of size at least $\gamma^3 n$. Hence, $N(v)$ can be divided into two sets, C and $N(v) - C$ such that $|C| \geq n/2 - 5\gamma^3 n$.

If u is non-extremal then we let $u = I(r)$. If u is extremal, then we choose an arbitrary vertex $v \in A$. For simpler notation we rename v to u, and let $u = I(r)$. Moreover, we set $A = C$ and $B = N(v) - C$.

Our first goal is to embed the second and third level of T into AG such that the second level is mapped into a subset of $N(u)$ and the third level is mapped into W. Later we need to modify this mapping in order to have the embedding of T into G. For that we need to introduce some randomness when embedding T into AG. Let us consider the first t for which

$$\sum_{i \leq t} deg_T(x_i) \geq \gamma^2 n.$$

Recall, that the x_is are in degree-decreasing order, hence t is a well-defined number. Next, we choose a random set R having exactly t elements $\{v_1, v_2, \ldots, v_t\}$. Observe that $t \gg \log n$ even in case $m = cn/\log n$ if c is sufficiently small. If deleting a few vertices would not make F disconnected then R is chosen from $N(u)$, otherwise $R \subset A$. We have the following.

Claim 17. *With high probability the subgraph of F spanned by R cannot be made disconnected by deleting less than $\gamma^3 t/9$ vertices.*

Proof: Assume to the contrary that by deleting at most $\gamma^3 t/9$ vertices the leftover will be disconnected. First, the number of components is two, since every three element subsets of F spans at least one edge. Denote these components by A and B. Let $v' \in A$ and $v'' \in B$ be arbitrary vertices. Let $v \in N(u) - R$. Then we have three possibilities: 1. $vv' \in E(F)$, but $vv'' \notin E(F)$, 2. $vv' \notin E(F)$, but $vv'' \in E(F)$, 3. $vv' \in E(F)$ and $vv'' \in E(F)$. Denote the set of those vs for which the third case holds by D. Removing the set D cuts F into two parts, hence $|D| \geq \gamma^3 n/10$. Using Azuma's inequality and the fact that $t \gg \log n$ we get that with high probability R will contain at least $\gamma^3 t/9$ vertices from D. $\square$

We will also make use of the following property of F.

Observation 1. *Every vertex of F has degree at least $\gamma^3 n/10$.*

Proof: If there were a vertex with less neighbors, then by deleting those neighbors we would disconnect F. $\square$

Given the set R we let $v_i = I(x_i)$ for $1 \leq i \leq t$ and $W = W - R$. Then we repeat the following procedure for mapping the neighbors of the x_is. Randomly choose a set of size $deg_T(x_i) - 1 - d_i$ from $N_{AG}(v)$. Here the *deficiency* d_i is a non-negative integer, we say more about the d_is later. Map the y_j neighbors of x_i onto them arbitrarily. Then delete the recently covered vertices from W and $N(u)$ and repeat the above for x_{i+1} if $i < t$.

As it was indicated above the role of R is to help to find an embedding of T into G, given an embedding into AG. It is also used to finish the embedding of T into AG at the very end. That is why we need the deficiencies. We let $\sum d_i = 2m/\gamma$, and assume that the deficiencies are distributed evenly, that is, $|d_i - d_j| \leq 1$ for every i, j.

A greedy embedding procedure

Throughout the embedding a greedy embedding procedure will play an important role. Let $U \subset W$ be an arbitrary subset. Every $w \in U$ has at least $Km \log n - m$ neighbors in $N(u)$ even if m vertices of $N(u)$ are covered and therefore deleted. Hence the average number of neighbors a vertex of $N(u)$ has is at least $|U|Km \log n/(2n)$. That is, unless $|U| < 2n/(K \log n)$, we can find a vertex $v \in N(u)$ that has m uncovered neighbors in U. Then we pick the leftmost unmapped x from the second level of T, let $v = I(x)$, and map the at most m neighbors of x onto the above found neighbors of v. After this we delete v from $N(u)$ and its covered neighbors from U.

Denote S the set of those vertices of W which has less than γn neighbors in $N(u)$. By the non-extremality of G we have that $|S| \leq n/2 - \gamma n$. We let $L = W - S$. We will first take care of the vertices of S, this means that we will cover them right away, and at the end we will have only such vertices which have degree at least γn into $N(u)$. Observe that if $v \in L$ then $|N(v) \cap A| \geq (\gamma - 5\gamma^3)n$, thus, v has many neighbors in R with high probability.

Covering the vertices of S

We will use the greedy embedding procedure for covering most of S, at most $2n/(K \log n)$ vertices of S will be left out. Let $h_1 = n/(K \log n)$. Observe that if $|S| > h_1$ then we can find a vertex $v \in N(u)$ having at least $m/2$ neighbors in S.

Since $|S| \leq n/2 - \gamma n$ we can cover at least h_1 vertices of S, in each step using at least $m/2$ vertices from S and at most $m/2$ more from the rest of W. This follows from the minimum degree condition of G and that for every $m/2$ vertices of S we use up at most $m/2$ vertices from W. Hence, at the end the degree of any $v \in N(u)$ will be at least $n/2 + Km \log n - |S| - h_1 \geq \gamma n - h_1$. Next we will reduce the number of vertices of S to $h_2 = h_1/2$. By the greedy procedure we will always find a vertex $v \in N(u)$ having at least $m/4$ neighbors in S. Those neighborhoods will be complemented by at most $3m/4$ uncovered neighbors of v in W. This is again possible since the degree of v into W is at least $\gamma n - h_1 - 3h_2$ even at the end. In general, we can reduce the size of S to $h_q = 2^{-q}h_1$ in a similar fashion. At every embedding step we use up at least $m2^{-q}$ vertices from S, and at most $(1 - 2^{-q})m$ vertices from W. That is, at the end of the qth iteration, we have that every $v \in N(u)$ has at least $\gamma n - h_1 - 3h_2 - \ldots - (2^q - 1)h_q$ neighbors left in W. Noticing that $K \gg 1/\gamma$ we get that the number of neighbors is always at least $\gamma n/2$, hence, this procedure will not get stuck. Moreover, we use up at most $n/2 - \gamma n/2$ vertices from W in order to cover every vertex of S.

Covering most of L

It is easy to cover most of L by the help of the greedy embedding procedure. This time notice that in the beginning every vertex of L had at least γn neighbors in $N(u)$. Since $|B| < 5\gamma^3 n$, vertices of L have more than $\gamma n/2$ neighbors in A. Therefore, we can build the tree further in such a way that the majority of the vertices of L are matched to vertices of A.

Since $\gamma n/2$ is much larger than $Km \log n$, the greedy procedure will stop later. Simple averaging argument shows that if $|L| > 2m/\gamma$ then one can find a vertex $v \in A$ with m neighbors from L. This fact can be used to cover most of L greedily. We will need to be a bit more precise, however. We always proceed in a left to right order when embedding the second level of T, in particular, when covering R we used the x_is with the t largest degree. Set $m_1 = deg_T(x_t) - 1$. It is easy to see that when the greedy algorithm gets stuck, L will have at most $2m_1/\gamma$ vertices.

Finishing the embedding

Our strategy will be as follows. First, there can be at most m *repeated* vertices in AG which are used in the second and in the third level as well. In the first phase we will take care of the vertices having a repeated leaf. Then, in the second phase, we insert those vertices of L that could not be inserted with the greedy embedding procedure. In both cases we will use R and structural properties of F.

We will erase the leaves which are repeated in the second level. Let us first assume that $v \in B$ (recall, that B is the small component in F) has a repeated leaf w and $w = I(y)$. Since $|L| \geq n/2 + \gamma n/2$, every vertex has several neighbors in it. Moreover, the vast majority of L have been covered by some leaf of T such that its parent is mapped onto a vertex of A. Let $w' \in L$ be adjacent to v, and assume that x is the parent of $I^{-1}(w')$ and $v' = I(x) \in A$. We change I and let $w' = I(y)$. Iterating this procedure we can achieve that no vertex of B will have a repeated leaf at the end.

From now on we assume that only vertices of A lost leaves when deleting repeated leaves. We will use the fact that F and $F|_R$ are highly connected. Say that a vertex $v \in A$ lost some leaves. By Observation 1 and Azuma's inequality we conclude, that v has at least $\gamma t/11$ neighbors in $F|_R$. We will find many paths in $F|_R$ of length at most 3 which go from v to vertices of $I^{-1}(y)$ where $I(x) \in R$ and $xy \in E(T)$. Then v will *steal* the leave from $I(x)$.

Stealing goes as follows. Say, $vv_i \in E(F)$. Then the probability that a neighbor of x_i is mapped onto a neighbor of v, is at least $\gamma^3/2$ (at the end of mapping R we may lose a few vertices from the intersection). By Azuma's inequality at least $\gamma^3 m_1/3$ neighbors of x_i are mapped onto a neighbor of v. Such a neighbor can be stolen by deleting it from the subtree originating at v_i and inserting it to the subtree of v. Stealing can also be done on a *directed $v - v_3$ stealing path*, say. Assume, that $vv_1, v_1v_2, v_2v_3 \in E(F)$. Then v can steal from v_1, the missing leaf of v_1 will be stolen from v_2, and finally v_2 will steal from v_3. At the end only v_3 will miss a leaf. Notice that overall at most m leaves will be stolen at this point, and the number of stealing paths is at most $3/\gamma^3$.

Second, we have to take care of the leftover vertices of L. Since R is a random set of size t, every vertex of L will be adjacent to at least $\gamma t/2$ vertices from R with very high probability. If $v \in L$ is adjacent to $v_i \in R$ then we *insert* v into the subtree originating

from v_i. Since every $v \in L$ has $\gamma t/2$ choices, we can do this insertion evenly. Simple computation shows that no more than $2m_1/(\gamma^2 t)$ will be inserted into the subtree of any v_i. If $t > 1/\gamma^4$ then at most $\gamma^2 m_1$ new vertices will be inserted into a subtree at this step, that is, a very small proportion of the size of the subtrees. Since the subtrees below $x_1, \ldots, x_t$ has a total of about $\gamma^2 n$ vertices, $t \gg 1/\gamma^4$ if n is sufficiently large.

When we are done with the above, there will be vertices in R with more neighbors than needed, some vertices with less neighbors and there will be vertices with the required number of leaves. Also, if $v \notin R$ and is covered by some x from the second level, then the subtree of v has the right number of leaves after stealing. Then we will find many $v_i - v_j$-paths of length at most 3 where v_i has less leaves and v_j has more leaves than needed, and $v_i, v_j \in R$. After stealing we can reduce the discrepancies. Observe that we need a total of at most $2m_1/\gamma$ stealing paths, and no vertex will be in many stealing paths, except perhaps those which are the first vertices of such a path. Hence, this procedure will not get stuck before inserting every vertex of L, since the sizes of the intersections are large enough to find these paths. This finishes the embedding of T in case it has height two.

The following remark plays a crucial role later, when we embed trees with larger height.

Remark 2. *Assume that we have a subgraph $G' \subset G$ on at most $t(1 + \gamma^2)$ vertices, where a t element set H_1 comes randomly from W and the rest, at most $\gamma^2 t$ vertices, denoted by H_2, are chosen arbitrarily from $W - H_1$. Then we can embed a tree T' on $v(G')$ vertices having height two into G' as follows: We choose u from H_1, and let $N'(u) = N(u) \cap H_1$. The small set H_2 will be taken care of first. Since any of these will have at least $Km \log t$ neighbors in $N'(u)$, we can handle these greedily, the same way as we did with the vertices of S. Then we will continue as before.*

2.3.1 T has a few vertices at a distance at least three from the root

Let us now discuss the case when T has essentially height two, but there are at most $\gamma^4 n$ vertices at a distance at least three from the root.

Let us denote the number of vertices farther than two from r by N. By our assumption $N \leq \gamma^4 n$. Since the embedding method is very similar to the one for embedding trees of height two, we will focus on the differences.

- As before, we start from a non-extremal vertex $u = I(r)$.

- We choose R as before, but embed less leaves, more precisely we leave out an additional ℓ leaves, where $\ell = \max\{\sqrt{n}, 2N/\gamma\}$.

- We identify L and S, and then choose a random set $L_R \subset L$ with size $|L_R| = \ell$. Since $|L| \geq n/2 + \gamma n$ and $L_R \subset L$ is random and not very small, every vertex of W has at least $\gamma\ell/2$ neighbors in L_R with high probability.

- Embed T, except those vertices from the higher levels, and the leaves we left out.

- Let $x \in V(T)$ be any vertex at distance two from r which has a non-empty subtree $T(x)$ originating from it. We will embed $T(x)$ in a greedy way into L_R, using the fact that every vertex has more neighbors in L_R than the total number of vertices below the third level.

- The leftover of L_R will be inserted as leaves to the vertices of R. This is easy, since $L_R \subset L$, hence, all these vertices has many neighbors in R.

2.4 T has larger height and T_0 is a broad forest

In this section we assume that at least $\gamma^{20} n$ vertices of T belong to higher levels of T than the first three levels, and that T_0 has at least $\gamma^7 n$ leaves. Our first goal is to present an algorithm for embedding most of T into G, a forest of size at most $\gamma^{20} n$ can be left out. Later we will extend this "almost embedding" into a proper one. The almost embedding algorithm comprises two main parts.

First, we will discuss a method for decomposing T into subtrees/forests of linear size. The set of these subtrees/forests is divided into three parts, that is, *classified* depending on certain properties. If the first three level sets of a forest is small, then it is a *narrow* forest, we also say it is *remappable*. Forests of the second type have wide

levels, we will call them *wide* or *lossless* forests. Finally, there can be very small forests which will be embedded at the very end greedily, since the total number of vertices in small forests is very small.

The embedding will go as follows. Wide forests will be embedded into randomly chosen subsets, level by level, using matchings. This method is called the LL procedure. The word "lossless" refers to the fact that these forests can be embedded into a random set of the same size, using perhaps a little help from a universal random subset chosen at the beginning. Narrow forests will be embedded by the CM procedure, using a dense subset in the uncovered part of G, very similarly to the Main Mapping Procedure of [12]. We do not always have such a dense subset, but already embedded subtrees of this type can be remapped in such a way that we gain a dense subset in G. For applying this latter mapping procedure we may need many more vertices than their actual size. Finally, very small forests will be easy to embed into a random subset.

2.4.1 Finding T_0

In the first step we find a forest T_0 with size about $\gamma^5 n$. Observe first that if T has very few leaves, at most $\gamma^2 n$, then we can always find several long induced paths, their union will be T_0. If T has more leaves then starting from the root we either pick some neighbors of the root, or go down in the tree, until we find a forest having size in between $\gamma^5 n$ and $2\gamma^5 n$. This forest will be T_0. If the number of leaves in T_0 is very small, less than $\gamma^7 n$, then T_0 is a *long* forest, otherwise it is *broad*. Clearly, if T_0 is the union of long induceds path then it is a long forest. The role of T_0 is to help turn the almost embedding into a proper one at the end.

In this section we consider the case when T_0 is broad, in the next section we will consider the case when T_0 is long. As we will see, we may not need T_0 if at least $n/2$ vertices of T are in wide forests. More details follows below.

2.4.2 Decomposing T into small forests

An important component of the decomposition is the following folklore result.

Lemma 18. *Let J be any tree on t vertices. Then J has a split vertex $x \in V(J)$ such that it is possible to group the vertices of $J - x$ into two forests, J_1 and J_2 such that $t/3 \leq v(J_1), v(J_2)(\leq 2t/3)$ and there is no edge connecting J_1 and J_2 in $J - x$.*

We first repeatedly apply Lemma 18 until we get a decomposition of $T - T_0$ into a set of forests $\{F_1, F_2, \ldots, F_\ell\}$ such that $\gamma^{40} n \leq v(F_i) \leq 2\gamma^{40} n$ for every i. It is easy to see that for this we need to cut out at most γ^{-40} split vertices. A forest of the decomposition may contain many components. However, all forests are connected to the rest of T through at most γ^{-40} vertices.

We associate a tree ST with this decomposition, the *skeleton* of $T - T_0$. ST is given by the split vertices and the paths connecting them in $T - T_0$. It is immediate, that the leaves in ST are split vertices, but not every split vertex is necessarily a leaf of ST. It is also clear, that every forest of the decomposition is connected to ST via at most γ^{-40} vertices.

Classification of forests

Let us assume that F is a forest with $v(F) = \Omega(n)$ containing the subtrees $\widetilde{T}_j$, $j = 1, 2 \ldots, s$ and denote the roots of these subtrees by r_j. We let $L_1 = \cup r_j$, and $L_i =$ the children of L_{i-1}, the L_is are the *level sets* of F. We call the forest F *narrow* if

$$|L_1 + L_2 + L_3| \leq \gamma^{20} v(F),$$

otherwise, if the second and third levels contain several vertices we call F *wide*. Observe, that F is always narrow if $m = o(\sqrt{n})$. For embedding narrow forests we will use the CM procedure and wide forests will be embedded with the LL procedure. Both methods will be discussed later. In order to apply these methods we need a preprocessing step for both kind of forests.

Preprocessing of narrow forests

Let F be a forest with a single root r'. Let us color red every vertex x of F for which $v(F(x)) \geq \gamma^3 v(F)$. Consider a 'last red vertex' y, that is, y has the property that on

the $r' - y$ path every vertex is red, and no child of y is red. Denote the parent of y by x, it is the red vertex just above y on the $r' - y$ path. Denote the children of y by $z_1, z_2, \ldots, z_l$. Since the trees below every z_i are small, we may assume that the sum of the subtrees below the first s vertices, $z_1, \ldots, z_s$ is at least $\gamma^3 v(F)$, but not larger than $2\gamma^3 v(F)$. The subtree rooted at y containing $z_1, \ldots, z_s$ will be called a *separable* subtree. During the embedding we may have to embed this part separately. Observe that this subtree is remappable: the first level of it contains x, the second level of it contains y, and the third level has the children of y, this is $o(n)$ vertices.

It is easy to see that the union of narrow forests is also a narrow forest: let F_1, F_2 be two narrow forests with roots r_1 and r_2, respectively. Then $F_1 \cup F_2$ with roots r_1 and r_2 is a narrow forest, since the second and third level is small compared to the total number of vertices of F. This remark will prove to be useful when embedding narrow forests.

Preprocessing of wide forests

This case is considerably more involved than the previous one, since we may have to decompose F_i into several forests, some wide and some narrow or very small.

Recall that a forest F is wide if $|L_1 + L_2 + L_3| \geq \gamma^{20} v(F)$. If the number of subtrees (and hence the number of roots) is a constant, then this implies that $m = \Omega(\sqrt{n})$. Let us call *broom* the forest of the first three levels of F. In general, a broom is a subtree that has height two and one root. The second level of a broom will not necessarily contain every children of its root, but the third level contains every children of the second level.

In order to prepare for finishing the embedding of T, we may delete some leaves of wide forests temporarily. Assume that the number of leaves in wide forests is at least half the number of leaves in T, hence, it is more than $\gamma^3 n/2$. Let $z \in F$ be a vertex not in the broom and assume that it has l leaf neighbors. Then we delete $\gamma^4 l$ leaves out of these. At the end of the embedding we will insert these leaves back easily using a matching argument, since the z vertices will be mapped using a random procedure.

If most of the vertices of F are concentrated in L_3, then we can use the method for embedding a tree having height essentially 2. Assuming that F has several levels we call it partly narrow if there exist an $i \geq 4$ such that

$$|L_i| < \gamma^{40} v(F)$$

and there are at least $\gamma^{20} v(F)$ vertices in levels below L_i. In such a case we recognize a narrow subforest $F' \subset F$ as follows. Denote the parents of L_i by $P(L_i)$, and the grandparents by $P^2(L_i)$. Clearly, both these sets will be small. Then F' will be the forest which is rooted at the grandparents $P^2(L_i)$. These roots are part of the wide forest, while we cut off the subforest F'. We need to preprocess F' as is described above for narrow trees.

If below a certain level the number of vertices is at most $\gamma^{20} v(F)$ then the lower levels will be taken care of using the random set M (more details follow later). Since we also chop off levels from F if it has a small level set, we can assume that the depth of a wide forest is at most γ^{-40}, and that every level of it has at least $\gamma^{40} v(F)$ vertices.

Partitioning the level sets of F

We give a partitioning Cl of the set $\{0, 1, 2, \ldots, n\}$ as follows. The integer $l \in \{1, 2, \ldots, n\}$ will belong to Cl_i if $(1+\beta)^{i-1} \leq l < (1+\beta)^i$ where $\beta = \gamma^{60}$. We let $Cl_0 = \{0\}$. Clearly, there are $O(\log n)$ classes.

We repeatedly use a *partitioning method* for bipartite graphs. Let H be a bipartite graph with vertex classes A and B. Assume that B has a partitioning $\Lambda = \{\Lambda_1, \Lambda_2, \ldots, \Lambda_k\}$, and denote the elements of A by $x_1, x_2, \ldots, x_t$.

We introduce the relation '$\sim$' on A: $x \sim x'$ if for every Λ_j we have that $deg(x, \Lambda_j)$ and $deg(x', \Lambda_j)$ belong to the same class in Cl, that is, if x and x' have about the same number of neighbors in every $\Lambda_j \in \Lambda$. It is easy to see, that '$\sim$' induces a partitioning $\Pi = \{\Pi_1, \ldots, \Pi_s\}$ on A.

In the important special case when Λ is trivial, that is, when $|\Lambda| = 1$, A will have at most $O(\log|B|)$ partition sets, since the degree from A to B is bounded above by $|B|$. We also have the following.

Remark 3. *One can give an upper bound on $|\Pi|$ based upon $|\Lambda|$ and the maximum degree of H. In particular, if $|\Lambda|$ and $\Delta(H)$ are constants, then $|\Pi|$ is also a constant.*

Given a wide forest F, we will repeatedly apply the above partitioning method. Let L_ℓ be the last level set having size at least $\gamma^{40}v(F)$. We assume that its partition is trivial (has one partition set), and in a bottom-up manner we construct the partitions of every level starting from L_ℓ.

Decomposition of wide forests

Let F be a wide forest rooted at r' (the discussion is very similar in case there are more roots). We define the *weight* of a vertex $x \in V(F)$ to be $v(F(x))$, that is, the number of vertices of F below x.

We begin with an observation. Assume that some $x \in L_2$ has weight at least $\sqrt{K}m$. Since

$$\sum_{x \in L_2} v(F(x)) = v(F) - 1$$

there can be at most $v(F)/(\sqrt{K}m)$ such vertices. Since the total number of neighbors of such vertices is at most $mv(F)/(\sqrt{K}m)$ we get that the forest originating at the root of F containing those xs at its second level that have such a large subtree is a narrow forest, since its third level has at most $v(F)/\sqrt{K}$ vertices. Hence, we can deal with this narrow forest separately using the CM procedure, and assume that below every $x \in L_2$ we have a relatively small subtree.

Denote the partition of L_3 by $\Pi = \{\Pi_1, \Pi_2, \ldots, \Pi_t\}$, and let $\pi_i = |\Pi_i|$ for all i. Notice the following fact: using Remark 3 there are at most a finite number of partition sets that contain vertices of weight at most K^2. Denote this number by $\mathcal{C}$. Observe that partition sets that contain at most $\gamma^{20}|L_3|/(K^3\mathcal{C})$ vertices with low weight (at most K^2) have at most $v(F)/K$ vertices combined.

We call a partition set Λ *heavy* if its vertices have weight more than K^2, where Λ is partition set in some level of F. We also call a vertex heavy if its weight is larger than K^2. Otherwise we call the vertex and the partition set containing it *light.* Heavy

partition sets will be very small. In the LL procedure we have to deal with heavy partition sets separately.

Assume that the union of heavy partition sets at some level has at most $K \log n$ vertices. Then the total number of their children is at most $Kcn \log n / \log n \ll \gamma^{40} v(F)$. Hence, the forest originating at their parents having these heavy partitions at the second level with everything below them will be a narrow forest, and will be separated from the wide part.

Consider the brooms rooted at the heavy vertices at some level L_i of F with second level of the broom being the heavy children. We call these *heavy brooms*. If the total number of the vertices in the third level of the heavy brooms is at most $\gamma^{40} v(F)$ then we again can find a narrow forest. This implies that on the average a heavy broom has at least K times more vertices in its third level than in its second level. Assume that the total number of heavy vertices is t in some L_i. Let $x \in L_i$ be a heavy vertex. If the third level of the heavy broom of x contains at most $|L_{i+2}|/(Kt)$ vertices, then we will separate the subtree originating at x from F. Either it contains at least n/K vertices and hence this subtree will be a narrow subtree, or it is very small, and will be dealt with as such. If a heavy broom rooted at x contains at least $v(F)/(\sqrt{K} \log n)$ vertices, then we will call x *very heavy.* Thus, we may assume that the heavy brooms all have about average size or larger in a wide forest at every level, if there are any.

Summarizing, beginning with L_3 there will be light and heavy partition sets. The light sets are large, each has linear size, while the heavy partitions are small. Every vertex in a large partition set will have weight at most K^2, while heavy vertices in the heavy partitions have weight more than K^2. Clearly, no children of a light vertex can be heavy, while a heavy vertex can have heavy and light children at the same time. Moreover, every heavy broom will have many vertices in its third level, at least K times more than in the second level. The procedure is depicted in Figure 2.1.

When embedding, we will map the heavy brooms using a randomized version of the greedy mapping procedure that was applied when embedding a tree with height 2. The light partition sets will be embedded using matchings.
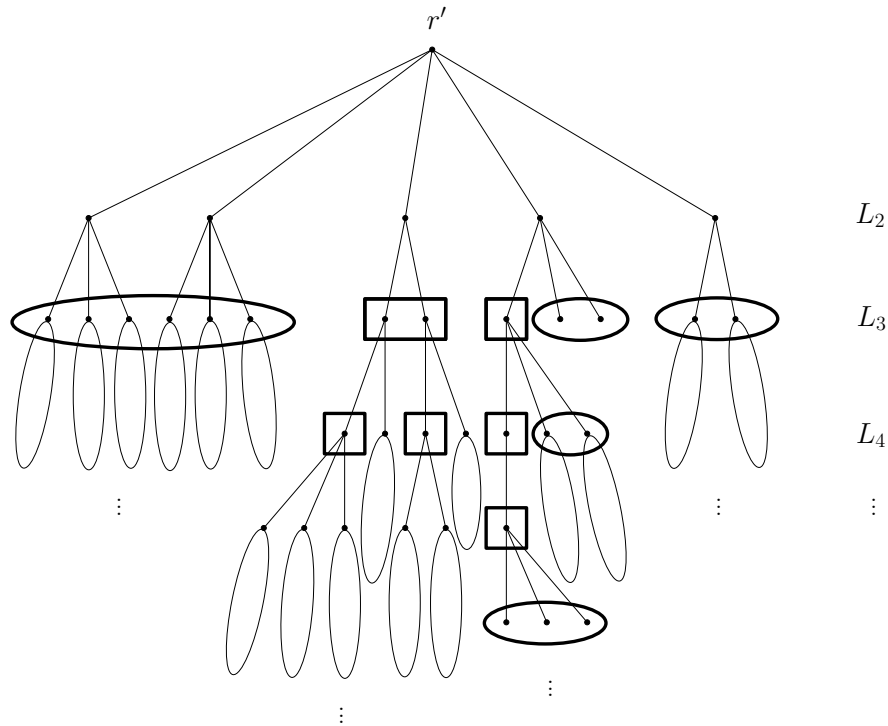
Figure 2.1: Decomposition of a wide forest

Forests in the decomposition that have more than one root

So far we have considered forests which are hanging from a split vertex. But ST may contain several vertices other than split vertices, and there can be very small or large subtrees hanging from them. Below we will consider the case of forests which has vertices from the skeleton ST. By cutting out at most $\gamma^{60}n$ vertices from the forest we will achieve that every component of the leftover of the forest will be connected to at most one split vertex.

Denote the split vertices by $x_1, x_2, \ldots, x_\tau$, where $\tau \leq \gamma^{-40}$. Assume the leaves of the forest F_i are the split vertices $x_1, x_2, \ldots, x_s$. Start a path from each of these, at time t reaching those vertices of ST which are at most distance t from the leaves, always staying in F_i. These paths at any time will give us a sub-forest of F_i. If we find an induced path of length 3 such that the subforests hanging from the endpoints of the edge have at most $\gamma^{100}n$ then we cut out the middle edge with its endpoints and the subforests. The cut out forest is very small, we will embed it at the very end using the random reservoir M. By averaging if we can find an induced path of length at least
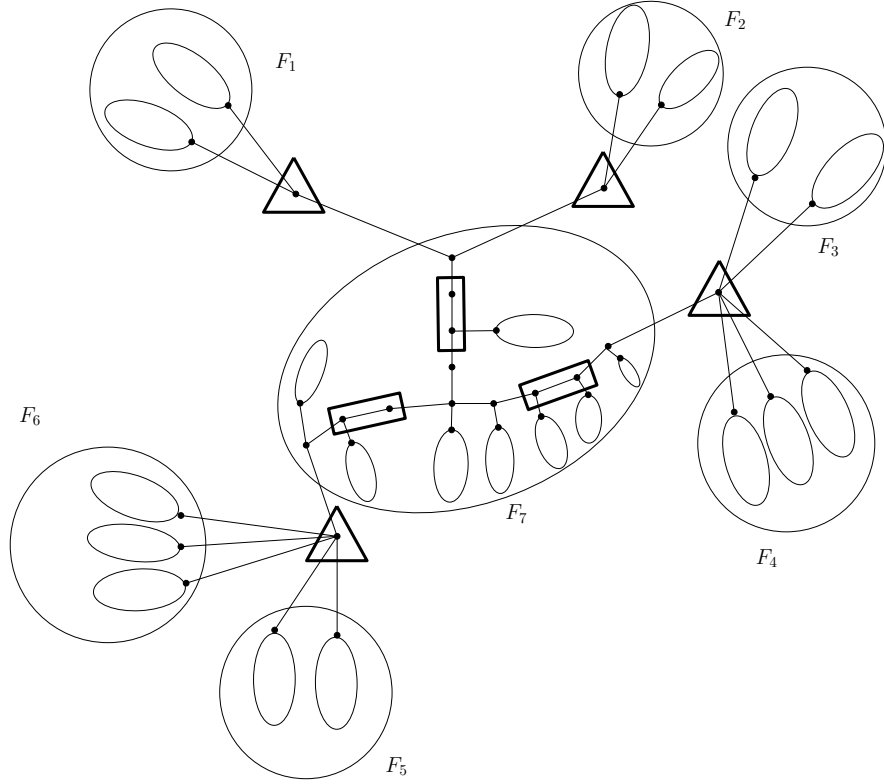
Figure 2.2: Decomposition of a tree

$2\gamma^{-60}$ then we can find an edge that can be cut out.

This way we can separate some split vertices from the rest of F_i. Clearly, if a component of ST has $4\gamma^{-100}$ vertices in ST, it will contain an edge that can be cut out. Continue the process only for those components which are not separated from the leaves $x_1, x_2, \ldots, x_s$. At the end we will get some components of size at most $4\gamma^{-100}$ having the split vertices, and perhaps large *independent* components. We choose an arbitrary vertex in every independent component, that will be the root. Finally, if a component contains a split vertex, then that split vertex will be the root. The subtrees hanging from these components of ST will give the F_i forest together with the vertices of the ST component. Notice that the number of components is not more than 3τ. With this we are prepared to apply the classification and preprocessing procedures. On Figure 2.2 the triangles indicate split vertices, and the rectangles are the cut-out edges. One can see that a split vertex can be the root of several forests, and that by cutting out edges we can achieve that no forest in the decomposition will have many roots. Moreover,

the connected parts of ST that have at least one split vertex will have a constant total number of vertices.

Observe that the number of small forests that arise this way is at most 3τ small, each having at most $\gamma^{100}n$ vertices, the total size is at most $3\gamma^{20}n$. This many vertices will be easily taken care of at the end with the help of the random reservoir M.

2.4.3 Preparations for the embedding

Before starting the actual embedding, we choose our random reservoir set $M \subset W(G)$ with $|M| = \gamma^{13}n$. By Lemma 13 every $v \in M$ will have about $|M|/2$ neighbors in M with high probability, and moreover, $G|_M$ will be a $\gamma/2$-non-extremal subgraph with high probability. The set M will be used as a *reservoir.* We will use up a few vertices of M during the embedding, but not more than $\gamma^2|M|$, hence, even at the very end it can be considered as a random set for our purposes.

Then for every wide forest F we will choose a random set $W_F \subset W$ with $|W_F| = v(F)$. Denote $W' = \cup W_F$. As in the case of M, with high probability we will have a minimum degree condition and non-extremality for every W_F. The set of uncovered vertices of $W - W'$ during the embedding will be denoted by Q. If Q is dense, that is, contains many edges, then it is not hard to embed a narrow subtree.

In the next step we decide if we need T_0 for finishing the embedding. If the wide forests contain at least γ^2n then we don't need T_0. We decompose T_0 into small forests the same way as we did for $T - T_0$. If not, we keep T_0, and its root will be the root of T. Next we embed the skeleton ST. The forests of the decomposition will be embedded using the LL and CM procedures, as follows below.

There is an important step here: in case the wide forests contains at least γ^2n leaves then we randomly, with probability γ^4, discard a few leaves from every wide forest. If these are from a light partition set, at the end it will be easy to map these leaves finding factors with Lemma 14 since their parents will be embedded randomly. If it is a leaf in a heavy broom then we will map the parent randomly: we randomly find a neighbor of the image of the grandparent, and then randomly choose some of its neighbors for the leaves. Then at the end we will insert the leftover leaves using the stealing paths, the

same way we did in case T had height 2. Otherwise, we will extend a partial embedding of T_0 in order to finish the embedding of T. More details will follow after we discussed the main embedding methods of forests.

2.4.4 Embedding wide forests with the LL procedure

We will make use of the partitions of the level sets of the forest F to be mapped. Assume that F is rooted at r', and we are given a $u \in W$ such that $u = I(r')$. The case when F has more (but a constant number) of roots can be dealt with easily by repeating the method for every root. Recall that we have a random set W_F reserved exclusively for embedding F. The LL procedure consists of three parts. The first part is on embedding the broom of F that is rooted at r'. In the second part we will map the heavy brooms with a different method, but that also has similarities with the one we considered for embedding a tree with height 2. In the third part, for mapping light vertices we will use matchings.

Initializing – Blowing-up F

We begin with 'blowing-up' the forest F with a factor of $(1 + \gamma^{20})$. This is a technical detail, which is helpful when embedding, and it means only a very small number of extra vertices. This means the following: Consider the next-to-last level of F. Assume that it is partitioned into the sets $\Pi_1, \Pi_2, \ldots, \Pi_s$. For every Π_i we take the maximum degree of the vertices of Π_i, and if a vertex has a smaller degree, then we will add 'imaginary' children to it. This way we will get that every vertex that belongs to the same partitions set of that level will have the same number of neighbors. We continue this procedure in a bottom-up manner. It is possible that we add not only imaginary neighbors, but a whole imaginary subtree, in order to achieve that if two vertices, x, y belong to the same partition set then $F(x)$ and $F(y)$ are isomorphic subtrees. Notice that we will increase the total size but at most

$$\sum_{i=0}^{\ell-2}(1 + \gamma^{60}) \sum_{j=0}^{i} |L_{\ell-j}| \le \gamma^{20} v(F)$$

where $\ell \leq \gamma^{-40}$ is the number of levels of F. We take $\gamma^{20}v(F)$ vertices at random from M and make W_F larger such that we will have enough vertices to embed the blown-up forest. During the embedding of F we will embed the blown-up forest. When we will be done, we delete those vertices that are not needed for F and set this leftover aside. We will take care of these vertices at the end, when finishing the embedding. Observe that overall (for every wide forest) we need at most $\gamma^{20}n$ vertices from M in order to perform the blow-up procedure.

Embedding the broom rooted at r' 'almost randomly' – First part of the LL procedure

Embedding the first three levels of F is very similar to embedding a tree with height 2, but there are differences. The main obstacle is that we want to embed the third level of F in such a way that "sufficient randomness" is involved.

The method is the following. First, take a random set $U \subset W_F$ such that $|U| = |L_2| + |L_3|$, and then take at random $2|U|/K$ vertices from M, and put these to U. Hence, U contains a little bit more vertices than what the second and third level of F has. Recall that we embed the blown-up F. Hence, L_2 is partitioned into at most $O(\log n)$ sets where the ith set, X_i contains those vertices that have the same degree which is about $(1 + \gamma^{60})^i$. Notice that if $|N(X_i)| < |L_3|/(\sqrt{K}\log n)$ then the number of vertices right below X_i is very small, and the forest with root r' and second level X_i is a narrow forest, and can be dealt with the CM procedure. So we will assume that $|N(X_i)| \geq |L_3|/(\sqrt{K}\log n)$. We divide U randomly into the subsets U_i where $|U_i| = |X_i + N(X_i)|(1 + 2/K)$.

Then we construct a bipartite graph with color classes $A_i = N(u) \cap U_i$ and $B_i = U_i$ for every i and connect $v \in A_i$ and $v' \in B_i$ by an edge if $vv' \in E(G)$. Since U_i is large, G restricted to U_i is 2γ-non-extremal with high probability, and every $v \in B_i$ will have at least $\sqrt{K}m/2$ neighbors in A_i (recall, that $m = \Omega(\sqrt{n})$). As before when T had height 2, we will divide B_i into two parts: $B_i = S \cup L$, where S contains those vertices which have less than $2\gamma|U_i|$ neighbors in A_i.

Recall that we greedily found vertices from $N(u)$ in order to insert most of S into the tree T in case T had height 2. We basically repeat that method: embed random parts first, then take care most of S but this time we allow a small error, and don't want to embed all of S. Then we insert most of the vertices of L, and finally eliminate the repeated leaves using the random parts. We are able to cover most of U_i, less than $2|U_i|/K$ vertices will be left out. That is, we embed a bit more than what we have in $N(X_i)$. Then we randomly decide how the children and grandchildren of r' will be mapped. First, we choose randomly a v for a children x, then we randomly distribute the children of x among those neighbors of v that we have found.

We repeat this procedure for every i thereby finding a random-like mapping of $\cup(X_i + N(X_i))$. A quasi-random property will be satisfied. Roughly speaking, if $S \subset U$ is a fixed subset with $|S| \gg |L_3|/K$ and $y \in L_3$ then the probability that $I(y) \in S$ is very close to $|S|/|L_3|$. More precisely, we will have that

$$\frac{|S| - |L_3|/K}{|L_3|} \leq \mathbf{P}(I(y) \in S) \leq \frac{|S|}{|L_3|}.$$

Let Π be a light partition set that have size $\pi \geq \gamma^{60} v(F)/(K^3 \mathcal{C})$. Let S be the set of the heavy vertices in L_3. We have the following lemma.

Lemma 19. *Let $v \in G$ be an arbitrary vertex. Then*

$$(i) \quad \mathbf{P}(|deg_G(v, \Pi) - \pi/2| > 2\pi/K) < \frac{1}{n^2}$$

and

$$(ii) \quad \mathbf{P}(|deg_G(v, S) - |S|/2| > |S|/K^{1/4}) < \frac{1}{n^2}.$$

Proof: We apply Azuma's inequality with $X = deg_G(v, \Pi)$ for (i). Since the probability that $I(y) \in N(v)$ is almost $1/2$, Π is large and $\sigma_i \leq cn/\log n$ for every i we get the inequality of the lemma. Very similarly we can show that every $v \in G$ will be neighboring to about half of the heavy vertices with high probability, since the number of heavy vertices is at least $\sqrt{K} \log n$. $\qquad\qquad\square$

Embedding heavy brooms – the second part

In this section we focus on the heavy vertices, we will discuss how to embed the children of light vertices in the next section. If a vertex is very heavy, it will be easy to handle it, too, using greedy embedding directly.

After mapping the broom rooted at r' we have a random set for embedding the leftover of F, denote it W_F'. Assume that a total of N vertices belong to heavy but not very heavy brooms in F. Here a heavy broom originates at a heavy vertex, it has every heavy children of the heavy root but no light children, and every children (light and heavy) of the second level of the broom. We choose at random $\gamma^{30} N$ vertices from M, and randomly choose N vertices from W_F'. Denote the union of these random sets by A, and set $N = |A|$.

We construct an auxiliary bipartite graph $H = H(A, B, E)$ as follows. The first color class is A, and the second class, B is a copy of A, i.e., $B = A$. We will have an edge between $v \in A$ and $w \in B$ if $deg_G(w, N(v)) \geq \gamma n/2$ and $v \neq w$. It is easy to see that $\delta(H) > (1 + \gamma)N/2$ since G is γ-non-extremal and we choose A randomly. Hence, by Lemma 15 we get that H has a spanning $N/4$-regular subgraph H_r.
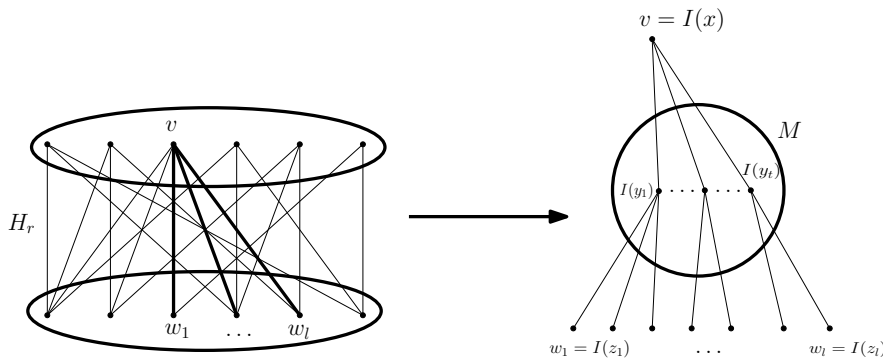
Figure 2.3: Embedding the heavy brooms using the random greedy method

The outline of embedding the heavy brooms is as follows. Given $I(y) = v \in A$ and its neighbors in B in H_r, we build several copies of the heavy broom rooted at y with the greedy embedding method using vertices of M for the second level of the broom. Notice that at most $3m/\gamma$ vertices will be left out from the neighborhood in H_r, because the degrees into $N_G(v)$ are large. The vertices that are left out at this point will be called

discarded vertices. Then we randomly select among the copies of the heavy broom. We will embed the heavy broom onto the selected copy. Since the second level contains only heavy vertices, we use up a very small number of vertices from M. We call this method *random greedy.* We will apply several rounds of the random greedy method.

After embedding a heavy broom we delete the newly covered vertices from A and B. We will show that an almost regular property of H_r can be maintained, since we select the images of the heavy brooms randomly. In fact we will have a similar quasi-random property to the one we had when embedding the broom rooted at r'.

We need one more lemma. For that let $J(D_1, D_2, E)$ be a bipartite graph with $t = |D_1| = |D_2|$. We also assume that $e(J)(1 - 1/q)/t \leq deg_J(v) \leq e(J)(1 + 1/q)/t$ for some large q. That is, J is 'almost regular'. Then if we choose a random neighbor of an almost randomly chosen vertex of D_1, the distribution of the neighbor is almost uniform in D_2, as the lemma shows below.

Lemma 20. *Assume that there is a probability $p(v)$ associated with every vertex $v \in D_1$ and $(1 - \epsilon)/t \leq p(v) \leq (1 + \epsilon)/t$, where $\epsilon > 0$. Choose a vertex $v \in D_1$ randomly according to the above probabilities. Then choose a random neighbor of v from D_2. The probability that we choose a given $w \in D_2$ is at least $(1 - \epsilon)(1 - 1/q)/t(1 + 1/q)$ and at most $(1 + \epsilon)(1 + 1/q)/t(1 - 1/q)$.*

Proof: Let us first assume that $p(v) \equiv 1/t$. Notice that in this case we choose from edges of J uniformly. Easy computation shows that the probability that one endpoint of a random edge is a fixed $w \in D_2$ is in between $(1 - 1/q)/t(1 + 1/q)$ and $(1 + 1/q)/t(1 - 1/q)$. Perturbing the probabilities of choosing a vertex with $(1 \pm \epsilon)$ will affect on the chance of choosing one endpoint of an edge. Edges will not be uniformly chosen anymore, however, the probability of choosing a particular edge will still be between $(1 - \epsilon)/e(J)$ and $(1 + \epsilon)/e(J)$, that is, almost uniform. From this the statement of the lemma follows easily. $\square$

We will use Lemma 20 repeatedly when embedding heavy brooms. Observe that if we choose the s leaves of a heavy broom with the random greedy method, and then randomly choose which leaf will be mapped onto which vertex of the randomly selected

broom, then the above lemma applies.

In the beginning H_r is $N/4$-regular. Recall that there are at most m but at least $\sqrt{K} \log n$ heavy vertices. Delete these vertices from A and B, and keep the name H_r. It is still almost $N/4$-regular, every degree will be between $(1 - 1/K^{1/4})N$ and $N/4$ since $N = O(n)$. In the first round we use the random greedy method for mapping the vertices of L_5 that are grandchild of a heavy vertex in L_3.

Before mapping $n/K^{1/4}$ vertices (there are many more to be mapped in the first round) H_r is almost regular since the degrees are much larger. Then we can apply Azuma's inequality as in Lemma 20 since the leftover degrees will depend on at least $n\sqrt{K} \log n / K^{1/4} n = K^{1/4} \log n$ randomly chosen heavy vertices and conclude that with high probability the leftover of H_r is almost regular – except the discarded vertices. However, one can see that there are only a very few vertices that are discarded several times. Overall their number is $o(n)$, hence if we delete those vertices that have degree larger $(1 + 2/K^{1/4})$ than the average, at the end we lose a very a few vertices, and the graph will be almost regular. Observe that since H_r is almost regular at every point of time and the heavy vertices are almost randomly distributed, we almost uniformly map the leaves of the heavy brooms.

We repeat the above method for the second round, when we map the heavy brooms that are rooted at vertices of L_5. The H_r graph is almost regular even at the end of the first round, hence, the heavy vertices of L_5 are almost uniformly distributed in A. Hence, H_r will be almost regular throughout the second round, every vertex will have degree $(1 \pm 4/K^{1/4})$ times the average – except those that were discarded several times, which will be deleted from H_r.

In the jth round we begin with a graph H_r in which every degree is $(1 \pm 2^j/K^{1/4})$ times the average. We will embed the heavy brooms that are rooted at the heavy vertices of L_{2j+1} and have leaves in L_{2j+3}. Since H_r has $\gamma^{30}N$ more vertices and the number of discarded vertices is very small, we can maintain the almost regularity of H_r till the end of every round, when every degree will be $(1 \pm 2^j/K^{1/4})$ times the average. We have at most $\gamma^{40}/2$ rounds, since there are at most γ^{-40} levels of the forest.

Embedding very heavy brooms

Assume that $x \in L_3$ is a very heavy vertex. Recall that a very heavy vertex has at least $n/(\sqrt{K} \log n)$ leaves in its broom. This can be embedded directly into a random set, using the same randomized procedure we used for embedding the broom of the root r' of F. The degrees will be large enough, since the very heavy broom and hence the random set is large. Then we continue the same procedure, starting from the third level of the broom of x. This finishes the part of the LL procedure which handles the heavy brooms.

Matching level-by-level for the light vertices – the third part of the LL procedure

By now we considered how to embed a heavy or a very heavy broom. A light vertex may belong to L_3, it can be below a light vertex, or it is the children of a heavy vertex. We will use different methods for mapping in the latter two cases. Notice that if a light vertex is in L_3 then we have already mapped it when embedding the broom of r'.

Assume that Π_j is a light partition set, and either it is in the 4th level or it is below a heavy partition. The outline of this case is as follows. The T_i subtrees rooted at the vertices of Π_j are isomorphic (recall the blow-up of F), and the parents of Π_j are randomly mapped. We will build the T_i subtrees into random sets level-by-level in a top-down fashion, and then match the roots of the subtrees with the already mapped parents. This latter matching is possible to find since both sets are random. Notice that we don't have to make distinction between the subtrees of the blown-up F rooted at the vertices of Π_j. Thus a matching argument works. Finally, we delete the unnecessary vertices from the mapped blown-up forest.

A more detailed description is as follows. For embedding the T_is first recall that every light partition is large. Therefore, we have a large number of trees to embed, and every level of the union of these trees is also large. We choose random subsets for the levels. Since these are large, by Lemma 13 every two consecutive random level sets will be a non-extremal bipartite graph with large minimum degree. Then we apply

Lemma 14 in order to build the levels of these trees. Finding a new level amounts to finding an f-factor between to consecutive random level sets, for some f. Assume that x belongs to level s of T_i, and x has f children. Then by Lemma 14 we find stars with f leaves centered at the images of the sth level of the T_is. Given $I(x)$ and its f neighbors in the factor, we randomly assign the f neighbors of x to the f neighbors of $I(x)$ in the factor. This way we can keep the randomness, and we can continue building the trees further with the next level.

When we are done with every level of the T_is, we have to connect them to their parents. These parents are either in L_3 or are heavy vertices. In both cases, we used the random greedy method in order to map these parents. We can apply Lemma 19 in order to conclude that the condition of Lemma 14 are satisfied, hence, we can connect the roots of the subtrees (the vertices of Π_j) to their parents. It is easy to see that repeating this method will lead to embedding every subtree that contain light vertices. This finishes the embedding of blown-up wide forests.

In order to find the embedding of the original F, we have to delete at most $\gamma^{20}v(F)$ vertices that belong to the blown-up F but not to F. That is a total of at most $\gamma^{20}n$ vertices.

2.4.5   Embedding narrow forests

A rough description of this case is as follows. We will embed most of the narrow forests into $W_N = W - \cup W_F - M - I(T_0)$ in case the wide forests has only a few leaves. Otherwise there is no T_0 and $W_N = W - \cup W_F - M$. However, when the narrow forests occupy only $\gamma^{14}n$ or less vertices of T, we will take care of the narrow forests greedily. We say more about this when we discuss how to finish the embedding.

The CM procedure goes smoothly until Q, the set of uncovered vertices of W_N has many edges. Then we find an already mapped narrow forest H such that many vertices of Q will have at least $(1/2 + \gamma^4)v(H)$ neighbors in $I(H)$. This is very similar to the method in [12]. We may not always find any forest that can be remapped this way such that many vertices of Q has more than half degree in their images. However, because

of the minimum degree condition it follows that in this case almost every vertex of Q is adjacent to almost half of the vertices of the images of almost all narrow forests. This, and the fact that in this case there will be many edges between almost all pairs of narrow forests, helps in finding an uncovered dense subgraph in G. When the size of Q drops below $\gamma^{15}n$, we stop with the CM procedure. The details are as follows.

Embedding T_0

Recall that in case wide forests has a very few leaves and T_0 is a broad forests then we need T_0 in order to finish the embedding. We will have a few cases depending on the structure of T_0.

- If T_0 has essential height 2 then we find a good vertex for the root of T_0 in W_N. Then randomly reserve $v(T_0) - \gamma^{13}n$ vertices for T_0. We will embed T_0 into this set plus the leftover we will have after the almost embedding of $T - T_0$. Using Remark 2 we will be able to embed T_0.

- T_0 has essential height 3. Randomly reserve a set S for L_4 that has a bit less vertices then the size of L_4, $|S|/|L_4| \approx 1 - \gamma^5$. Then we map the first three levels of T_0 using the random method for embedding the broom of a wide forest. In this case $|L_3(T_0)| > 10 \log n$, otherwise T_0 would have more levels. Then we can continue with the LL procedure for light vertices - however, we don't map the last level of T_0. We will do it using Lemma 14 at the end, when we put the leftover vertices to S.

- T_0 has larger height. This will be done very similarly to the case when we embed the heavy brooms of a wide forest. The goal, as before, is to map the parents of many leaves to a random set, and then either using Lemma 14 or stealing path finish the embedding even with a little noise, that is, a few leftover vertices added in the end.

  We may assume that the width T_0 is increasing, if we go 4 levels deeper we see at least twice as many vertices. If not, then T_0 will contain an induced path of

length at least 3. It is easy to connect any two vertices with a path of length at least 3 using W_N, and then we will choose the endpoint of the path randomly. As $v(T_0) \ll \gamma n$ and G restricted to W_N is non-extremal, we never get stuck.

We take the tip of T_0 which is defined to be the upper part of T_0 ending at a level L_i such that $|L_{i+1}| > 10 \log n$. Notice that even L_{i+2} is much smaller then the number of leaves in this case. We use the random embedding method of brooms to map the height-2 subtrees originating at the vertices of L_i.

Then we construct a bipartite graph $H(A, B)$ as follows. The set of so far un-covered vertices of W_N will be A and B. We will connect $u \in A$ and $v \in B$ if $|N(u) \cap N(v)| \geq \gamma n/2$. Since G is non-extremal, $\delta(H) \geq |A|/2$. Hence we can find a dense regular spanning subgraph $H_r \subset H$. Then we continue the same way as we did for wide forests.

The first CM procedure

This is very similar to the Main Mapping Procedure of [12]. We assume that the root r' of F is already mapped, and that Q spans at least $2\gamma^{40}n^2$ edges. If Q has this many edges, then it contains a subgraph spanned by some $Q' \subset Q$ with minimum degree at least $2\gamma^{40}n$. Let $N(r, M) = N(r) \cap M$, then $|N(r, M)| \geq (1 - \gamma^2)|M|/2$. Every $v \in Q'$ has at least $(1 - \gamma^2)|M|/2$ neighbors in M, hence, M has at least $(1 - 2\gamma^2)|M|/2$ vertices such that each has at least $2\gamma^{40}n$ neighbors in Q'. Denote the set of these vertices by M_Q. Notice, that there are at least $\gamma|M|^2/2$ edges between $N(r, M)$ and M_Q. Then the embedding of F will go as follows. We map the vertices of L_2 on those vertices of $N(r, M)$ which have many neighbors in M_Q. Then L_3 will be mapped onto arbitrary vertices of $N(I(L_3)) \cap M_Q$. The rest of F can be embedded greedily, since the minimum degree in Q' is larger than F. If F has more roots (not connected), we repeat the above for every subtree.
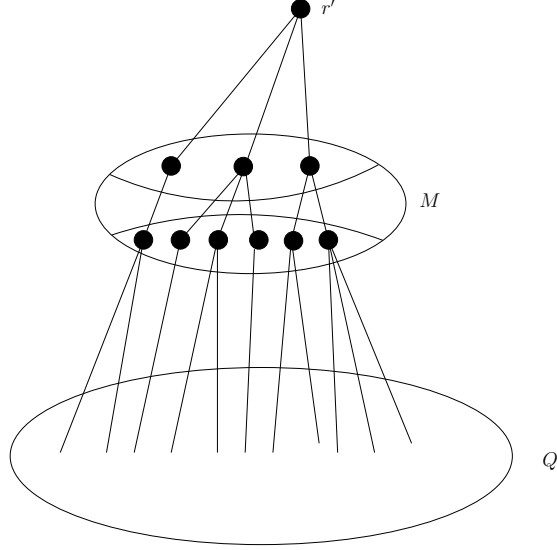
Figure 2.4: Embedding levels L_2, L_3 and L_4 of F with the CM Procedure

The second CM procedure

Assume that we want to map the narrow forest F that has t vertices. Let Q denote the set of those vertices of W_N that have not yet been covered by a vertex of T. We allow Q to be very sparse, but assume that $|Q| > \gamma^{15}n$. The goal is to replace some of the covered vertices of G by those in Q in such a way that we gain a dense subgraph in the new Q.

Let $T' \subset T$ denote the portion of T that has already been embedded. Assume that T' contains k narrow forests $\hat{T}_i$ in the decomposition such that the number of vertices of Q with degree at least $\sum_i (1/2 + \gamma^4)t_i$ into $\hat{T} = \cup_i \hat{T}_i$ is at least $\gamma^{-15}t$, and $\gamma^4 v(\hat{T})/2 \leq t \leq \gamma^4 v(\hat{T})/2$. Observe that the union of narrow forests is also a narrow forest having several roots. Thus, $\hat{T}$ is a narrow forest.

If the above is satisfied, we will first remap $\hat{T}$ in the following way. Let $Z = \{z \in Q : |N(z) \cap I(\hat{T})| \geq (1/2 + \gamma^4)v(\hat{T})\}$, and construct the bipartite graph $H = H(A, B)$, where $A = I(\hat{T}_i)$ and $B \subset Z$ with $|B_i| = \gamma^{-10}v(\hat{T})$. We connect $a \in A$ and $b \in B$ by an edge if they are adjacent in G. We apply the first version of the Cleaning Lemma to find a subgraph $H'(A', B') \subset H(A, B)$ such that $A' \subset A$, $B' \subset B$, every $b \in B'$ has at least $(1 + \gamma^4)v(\hat{T})/2$ neighbors in A' and every $a \in A'$ has at least $v(\hat{T})$ neighbors in B'.

We apply Lemma 8 to find a mapping of $\hat{T}$ onto H' such that at most $v(\hat{T})/2$ vertices of the new mapping cover vertices from $A = I(\hat{T})$ and the rest cover vertices from B. We re-embed $\hat{T}$ using vertices from M for the second and third levels, and then greedily using H' as in the first CM procedure, but we have not mapped F so far - seemingly it is a loss. On the other hand, there is a leftover bipartite graph $H''(A'', B'') \subset H(A, B)$ where $A'' \subset A$ and $B'' \subset B$, $|A''| \geq v(\hat{T})/2$ and $|B''| \geq \gamma^{-10}v(\hat{T})$, and every vertex of B'' has at least $\gamma^4 v(\hat{T})$ neighbors in A''. This dense bipartite subgraph can then be used to map F using the first CM procedure. Notice that even the smaller vertex class of H" is larger than F itself, hence, we can proceed greedily.

The third CM procedure

As before, we denote the new narrow forest to be embedded by F, and we let $t = v(F)$. As in the second mapping method we assume that Q is very sparse and $|Q| > \gamma^{15}n$. Denote the narrow forests of the decomposition by $\hat{T}_1, \hat{T}_2, \ldots, \hat{T}_\ell$, all having size $t_i = v(\hat{T}_i) \approx \gamma^{-4}t$ (here we again take union of narrow forests if needed, as in the second CM procedure). Assume that only a few vertices in Q have many neighbors in these forests, and so we cannot apply the second CM procedure. More precisely, for every i there are less than $\gamma^{-10}t_i$ vertices having at least $(1/2 + \gamma^4)t_i$ neighbors in $\hat{T}_i$.

We look for narrow forests that can be *weakly remapped*. That is, we look for forests $\hat{T}_j$ for which there are at least $\gamma^{20}n$ vertices in Q each having at least $(1/2 - \gamma^3/2)t_j$ neighbors in $\hat{T}_j$.

Assume that $\hat{T}_j$ can be weakly remapped, and apply the second version of the Cleaning Lemma. We get a bipartite graph $H'_j(A'_j, B'_j)$ where $A'_j \subset V(\hat{T}_j)$, $B'_j \subset B_j \subset Q, |B_j| = \gamma^{-10}t_j$, and every $a \in A'_j$ has at least $\gamma^{-5}t$ neighbors in B'_j and every $b \in B'_j$ has at least $(1/2 - \gamma^3)t_j$ neighbors in A'_j.

If the sizes of the color classes of $\hat{T}_j$ differ by at least $2\gamma^3 t_j$, then $\hat{T}_j \subset H'_j$. Moreover, after embedding $\hat{T}_j$ into H'_j we would get a leftover dense subgraph of H'_j in which every b has at least $\gamma^3 t_j$ neighbors in the leftover of A'_j. Such dense subgraphs can then be used to embed F. Hence, if $\hat{T}_j$ is weakly remappable but we cannot proceed the way we described above, then the sizes of its color classes differ by at most $2\gamma^3 t_j$, the forest is

approximately balanced.

Another important observation is that if $\hat{T}_j$ is weakly remappable, then we can embed most of it into H'_j, a subtree $\widetilde{T}_j$ of size at most $\gamma^3 t_j$ will be left out, this will be chosen to be the separable subtree we found in the preprocessing.

We will see, that we can find pairs $(\hat{T}_j, \hat{T}_i)$s such that about half of the smaller color classes of H_i and H_j will become vacant, moreover, there are a lot of edges in between these parts. These are large sets since these forests are approximately balanced. Then we will use this dense bipartite graph to re-embed the separable subtrees of $\hat{T}_j$ and $\hat{T}_i$, and then embed F. The details of this procedure are as follows.

From now on we assume that the color classes of the weakly remappable subtree $\hat{T}_j$ are roughly of equal size, and apart from a few vertices, the neighbors of the vertices of B' are concentrated in a subset of A' of size $(1/2 - 2(\gamma^4 + \gamma^3))t_j$.

Set $q = |Q|$ and let $n' = |W_N|$. We omit the proof of the following fact.

Lemma 21. *If Q does not have a subset Q' such that the induced subgraph on Q' has minimum degree at least $\gamma^{20} n'$, then Q has at most $\gamma^{10} q$ vertices which have more than $\gamma^{10} n'$ neighbors in Q.*

We claim that most of the subtrees are weakly remappable if none of them can be remapped using the second CM procedure. More precisely:

Lemma 22. *In the above setup if none of the good $\hat{T}_j$ forests can be remapped then at least $(1 - \gamma/2)n'$ vertices are in weakly remappable forests.*

Proof: First we consider the case when $n' \geq n/2$. Observe that most vertices in Q have at most $5\gamma^5 n'$ neighbors out of $n'/2$ that are are not in the union of the $\hat{T}_i$. This follows from Lemma 21 and the fact that T_0 has size at most $2\gamma^5 n$. Set $q'' = |Q''|$ where Q'' is the subset of Q containing those vertices having at most $\gamma^{10} n'$ neighbors in Q.

Assume that $\hat{T}_j$ is not a weakly remappable subtree. Then there are at most $\gamma^{20} n'$ vertices in Q'' which have more than $(1/2 - \gamma^3/2)t_j$ neighbors in $\hat{T}_j$, with the remaining $q'' - \gamma^{20} n'$ vertices having fewer. On the other hand, the number of edges going between Q'' and the union of the good subtrees is at least $(1/2 - 5\gamma^5)q"n'$.

Putting these together we get the following inequality:

$$(\gamma^{20}n'+(q''-\gamma^{20}n')(1/2-\gamma^3/2))c+(\gamma^{20}n'+(q''-\gamma^{20}n')(1/2+\gamma^4))(n'-c) \geq q''n'(1/2-5\gamma^5),$$

where c denotes the number of vertices in subtrees which are not weakly remappable. The dominating term on the left is $-q''\gamma^3 c$, while the dominating term on the right is $-q''n'\gamma^4$. Therefore $c < \gamma n/2$, which implies the statement of the lemma in case n' is large. If $\gamma^8 n \leq n' < n/2$ then we have a very similar computation except that there is no T_0 and the defect on the degrees on the left-hand side of the above inequality is at most $\gamma^{10}n'$ as is given by Lemma 21. Hence, the lemma follows in this case as well. $\square$

Without loss of generality let $\{\hat{T}_1, \hat{T}_2, \ldots, \hat{T}_k\}$ be the set of already embedded weakly remappable forests. By the second version of the Cleaning Lemma there is a partial remapping of $\hat{T}_i$ which leaves a subtree of size at most $\gamma^3 t_i$ unmapped. Let $R_i \subset I(\hat{T}_i)$ denote the vertices in the image of $\hat{T}_i$ that are not used in the remapping. Let $r_i = |R_i|$. Observe that r_i is roughly half of t_i.

Let us assume that we cannot remap any of $\hat{T}_1, \hat{T}_2, \ldots, \hat{T}_k$ such that a dense subgraph is left in Q. Then (R_i, Q) is sparse for every i, and $n'/2 > \sum r_i \geq (1/2 - 2\gamma/3)n'$.

Consider the set $\cup R_i$. It has many edges, $e(\cup R_i) \geq \gamma n^2/3$, since G is non-extremal. We will show that there is a pair (R_i, R_j) which has many edges.

Lemma 23. *There exists i, j such that $e(R_i, R_j) \geq \frac{\gamma}{2} r_i r_j$.*

Proof: Clearly,

$$\gamma n'^2/3 \leq e(\cup R_i) \leq \gamma/4 \sum_{i \neq j} r_i r_j + \sum_i e(R_i).$$

It is easy to see that $\sum_i e(R_i) \leq \gamma^{40} n^2$, hence, there exist a pair (R_i, R_j) which is very dense. $\square$

Putting these together, we see that if we fail to apply the second CM procedure, after remapping we can always find a large dense subgraph (R_i, R_j) on vacant vertices (although these were previously covered), unless Q is very small. Then we can greedily embed the new narrow forest F. Observe, that we use up at most $4\gamma^{-4}\gamma^{20}t$ vertices from M in order to embed t vertices, hence, we don't use up M fast.

Embedding a very small forest F

This time we don't have to be cautious. Every very small forest will be embedded in the end into M greedily. Recall, that the total number of vertices in small forests is at most $3\gamma^{-40}\gamma^{60}n \ll \gamma^{15}n$.

Making connections between subtrees

In the beginning we will embed the skeleton ST except the independent components and the cut-out edges. Notice, that this part is of constant size, hence it is easy to embed. Then the forests which are connected to a split vertex in ST will be embedded using the above procedure. We do the same with the independent components, if these are large enough. Given an independent component, it is easy to connect it to the rest, since it was separated by cutting an edge. Assume that everything is embedded except the cut-out edges and the forests hanging from them. Let xy be a cut-out edge, and assume that $z_1x, z_2y \in E(T)$. (there are no other neighbors, since a cut-out edge is the middle edge in an induced path of length 3) By now we have mapped z_1 to $I(z_1)$ and z_2 to $I(z_2)$. Then we connect z_1 and z_2 using M with the CM procedure. If the forest below xy is large, has at least $\gamma^{60}n$ vertices, then we have already determined, if it is wide or remappable, and we will embed it according to this classification. If it is very small, we embed it greedily into M.

Finishing the embedding

Earlier we discussed most of the details on how to finish the embedding, so here we just sketch the method. We assume that T_0 is not a long forest, perhaps it is not even used. We are at the end, the leftover is M and perhaps some other vertices.

- If the wide forests contain several leaves then we don't have T_0. We will insert the leftover as leaves into the wide forests. We prepared for this mapping such that the parents are mapped onto randomly chosen vertices. Then we can finish the embedding either using stealing paths or Lemma 14.

- T_0 has essential height 2. Then the root of T_0 has been mapped already. We put the leftover to the random set that was reserved for T_0 and using Remark 2 we can embed T_0 and hence finish the embedding of T.

- T_0 has larger height. In this case we basically used the method of embedding a wide forest for T_0, except that we left out several leaves. The parents of these leaves are mapped onto randomly chosen vertices. The leftover vertices are put into the random set reserved for the left out leaves. Using stealing paths or Lemma 14.

2.5 T has only a few leaves

In this section we discuss the case of embedding T when there is a subtree $T_0 \subset T$ with about $\gamma^5 n$ vertices and at most $\gamma^7 n$ leaves. We first give a sketch of the procedure.

First we note that the existence of T_0 follows from a simple averaging argument: split T into forests of size about $\gamma^5 n$, there will be one with only a few leaves, if T has a few leaves. A *line* in a tree T is a path P such that every vertex on P apart from possibly the endpoints has degree 2 in T. In this case, T_0 must contain several long lines. We find a set of $2\gamma^{10} n$ lines, each of length $1/(5\gamma^2)$. We will cut out the midpoint from each of these lines, set aside the cut-out vertex, and glue the two new endpoints together. The result is a tree $\widehat{T}$ on $(1 - 2\gamma^{10})n$ vertices.

We prepare G for the embedding of $\widehat{T}$. First we reserve a random set $W_F \subset W$ for every wide forest F. The union $M_2 = \cup W_F$ contains only a small number of vertices since T has only a few leaves. Then we find the vertices to be covered by the long lines by the help of a randomized procedure. Then we apply the embedding method of the previous section for embedding $\widehat{T}$ into G. Since $\widehat{T}$ is smaller than G, the embedding goes easily, even with the restriction that the long lines of $\widehat{T}$ are mapped onto the pre-determined long paths of G.

Finally we address the cut-out vertices. We will extend the embedding of the long lines by finding a perfect matching in an appropriately defined bipartite graph in G. With high probability, this procedure finds an embedding of T into G.

2.5.1 Preparations for the embedding

Finding random paths in G

In the first step we pick two random subsets $M, M_2 \subset W(G)$. M will have size $\gamma^{15}n$ as before, while M_2 is reserved for the wide forests of T. Then we find the paths U_i by the following randomized procedure:

Set $k = 2\gamma^9 n$. Pick randomly and independently $\frac{k}{20\gamma^2}$ edges from $G - M - M_2$ with replacement. From these, form k sets of size $1/(20\gamma^2)$. We can connect the edges of each subset into a path of length $\frac{1}{5\gamma^2}$ by way of the paths of length three guaranteed by Lemma 16. Denote the paths by $U_1, U_2, \ldots, U_k$. We discard those paths containing repeated vertices, where a vertex is repeated if either it appears twice in some U_i or it is contained by U_i and U_j for $i \neq j$.

Let v be any vertex in G. Let X_v denote the number of occurrences of v in the randomly chosen edges, then $\mathbb{E}X_v < \gamma^7$. This number follows a Poisson distribution,

$$\Pr(X_v > 2) \leq \sum_{i \geq 2} \frac{\gamma^{7i} e^{-\gamma^7}}{i!} \ll \gamma^{13}.$$

That is, on average we expect to have less than $\gamma^{13}n$ repeated vertices. The probability that there are more than $10\gamma^{13}n$ repeated vertices is by Markov's inequality at most $1/10$. We call a path *bad* if either it contains a vertex at least twice, or it contains a vertex which appears in another path. In the latter case we call only one of those a bad path. After discarding the bad paths, the vast majority of the paths remain. With probability at least 90% we have at most $\frac{10\gamma^{13}n}{8\gamma^2} < 2\gamma^{11}n$ bad paths.

A vertex v can be inserted into a path U_i if v has two consecutive neighbors on the path. We also say that v is adjacent to U_i. Let us estimate the probability that v is not adjacent to a given path U_i. There are at least γn^2 edges in the neighborhood of v, since G is non-extremal. The probability that none of the $\frac{1}{20\gamma^2}$ randomly chosen edges is in the neighborhood of v is at most

$$(1 - \gamma)^{\frac{1}{20\gamma^2}} \leq e^{-\frac{1}{20\gamma}} < \gamma^{20}.$$

It is easy to upper bound the number of those paths for which there are less than

$(1 - \gamma^{11})n$ adjacent vertices. By Markov's inequality, the probability that one cannot insert more than $(1 - \gamma^{11})n$ vertices into a path is γ^9. Using Markov's inequality again the number of such paths is at most $\gamma^{10}n$ with probability $\geq 1 - \gamma/3$.

Hence, one can find $k' = \gamma^9 n$ paths such that these do not contain repeated vertices, and one can insert at least $(1 - \gamma^{11})n$ vertices into any of them.

Finding long lines in T

First we show that if F is a tree having only a few leaves then it contains many long lines.

Lemma 24. *Let F be a tree on t vertices with ct leaves for some $0 < c \ll 1$. Then it is possible to find $s = ct$ vertex disjoint lines $p_1, \ldots, p_s$ in F such that $|p_i| = 1/(4c)$.*

Proof: Choose a root ρ. Substitute every maximal line in F by one edge (but we keep ρ as is). Then every vertex in the new tree F' will have degree 1 or at least 3 except possibly the root. Since in a tree the number of leaves is bounded by the number of vertices with degree at least 3 plus 2, we have at most $2ct + 1$ vertices and $2ct$ edges in F'. There were $t - 1$ edges in F, hence an average edge of F' corresponds to a line of length a least $1/(2c)$. Cut out a part of this line of length $1/(4c)$ and glue together the resulting two endpoints. Repeat this procedure: construct a new F' as before, and then find a long path again by an averaging argument. We can continue this way, and find many paths of length $1/(4c)$, as long as the leftover number of edges in F is larger than $t/2$. □

Apply the tree decomposition for $T - T_0$. We return edges of the lines to T as necessary in order to ensure that any two are at a distance of at least three from each other and that they are exactly $1/(5\gamma^2)$ long. We denote the resulting lines by $P_1, P_2, \ldots, P_{k'}$.

Let P_i be a long line. Let y_i be the midpoint of P_i and x_i and z_i its neighbors. Remove y_i from the path, and include the $x_i z_i$ edge. This yields a new line $\widehat{P}_i$ and the cut-out vertex y_i. We repeat this for every $1 \leq i \leq k'$, after which we arrive at a tree $\widehat{T}$ which has $(1 - k')n$ vertices. Let $\widehat{T}_0$ denote the subtree of $\hat{T}$ which arises by cutting out the midpoints.

2.5.2 Embedding $\widehat{T}$ into G

Our next goal is to embed $\widehat{T_0}$ into G, this is done greedily. Since the minimum degree of G is large, it is easy to embed the P_i lines into the randomly constructed paths of G, and then finish the embedding with the rest of $\widehat{T_0}$. Recall, that the P_i lines are at a distance at least three from each other.

After we are done with $\widehat{T_0}$, the embedding procedure of the previous section is able to proceed as long as there are enough vertices in G that are uncovered. As we have retained $k' \gg |M|$ vertices to embed at the very end, this condition is always maintained.

2.5.3 Finishing the embedding

Recall that the probability that a vertex v cannot be inserted into any of the $\frac{1}{20\gamma^2}$ randomly chosen edges is at most

$$(1 - \gamma)^{\frac{1}{20\gamma^2}} \approx e^{-\frac{1}{20\gamma}}.$$

Since we choose the edges and then form the edge sets randomly, by Chernoff's inequality every vertex v can be inserted to at least $(1 - e^{-\frac{1}{20\gamma}})\gamma^9 n - o(n)$ paths with probability $\geq 1 - 1/n^2$.

After embedding $\widehat{T}$, we have the uncovered $A \subset W(G)$ such that $|A| = k'$. If we can insert exactly one vertex of A to each of the U_i paths, then we are done with the embedding of T into G. For showing that these insertions are possible we will apply the König-Hall marriage theorem.

Construct a bipartite graph $F(A, B)$ where A is as above and the vertices of B correspond to the randomly chosen U_i paths. We connect $a \in A$ and $b \in B$ if the corresponding vertex can be inserted to the the path of b. We have $\gamma^9 n$ vertices in A and B. Recall that we discarded those paths into which less than $(1 - \gamma^{11})n$ vertices could be inserted. Furthermore, with high probability every vertex in A is adjacent to most of the paths, thus, the minimum degree in F is large with high probability. Hence, the König-Hall conditions are satisfied with high probability. Therefore, we can find a perfect matching in F. This finishes the proof of the second non-extremal case. $\qquad \square$

2.6   G is extremal and close to $K_{n/2} \cup K_{n/2}$

In this case $W(G)$ is partitioned into A and B such that $|A| = |B| = n/2$, and $e(G|_A)$ and $e(G|_B)$ is at most $\binom{n/2}{2} - \gamma n^2$. We state a simple lemma on the degrees of the vertices of A and B, we omit the proof.

Lemma 25. *Let $0 \leq \eta \leq 1/2$. Then the number of vertices of A having less than $n/2 - \eta n$ neighbors in A is at most $\frac{2\gamma n}{\eta}$. Analogous is true for B.* $\qquad\square$

The case of extremal G is divided into subcases depending on the structure of T. First, we will separately deal with the case when T has height 2, then we consider the embedding of trees with height at least 3.

2.6.1 T has height two

In this case T is a height-two tree rooted at r. The embedding method of this case is very similar to the one we used for embedding height-two trees in the non-extremal case, although there are differences as well.

Preprocessing

The vertices in the second level are denoted by x_i and we call them *parents*. The vertices in the third level of T are the leaves, and we denote them by y_j. We also assume that $deg_T(x_i) \geq deg_T(x_j)$ if $i < j$. We say that a vertex $v \in W$ is *good* if at least half the vertices of the graph have more than $\frac{n}{4} - 4\gamma n$ neighbors in $N(v)$.

Lemma 26. *If G is γ-extremal and close to $K_{n/2} \cup K_{n/2}$ then G contains a good vertex v.*

Proof: Let A and B be the two given partitions of the extremal graph G. Let $a \in A$ be the vertex with the highest degree in A and similarly let $b \in B$ be the vertex with the highest degree in B. By Lemma 25 we know that $deg(a, A), deg(b, B) \geq n/2 - 4\gamma n$. For every vertex v of the graph the degree of v in $N(a) \cup N(b)$ is greater than $\frac{n}{2} - 8\gamma n$. By the pigeonhole principle at least $\frac{n}{2}$ vertices of the graph have at least $\frac{n}{4} - 4\gamma n$ neighbors in $N(a)$ or have $\frac{n}{4} - 4\gamma n$ neighbors in $N(b)$. $\qquad\square$

Without any loss of generality we can assume that the *good* vertex came from A and we set $I(r) = a$. We then exchange the at most $4\gamma n$ non-neighbors of a in A with the neighbors of a in B and we call the adjusted partitions A and B as well. The exchange of vertices between the partitions could have resulted in an additional $2\gamma n^2$ missing edges within each of the partitions so that our new partitions may be treated as being part of a 3γ-extremal graph. Moreover, at least half the vertices of G have a large intersection with A which is now a subset of $N(a)$. Now we define the set $C = \{v \in A : deg(v, A) \leq \frac{1}{100}n\}$. By Lemma 25 we know that $|C| \leq 13\gamma n$.

It is possible, however, that C is very small, when $|C| < Km$. We will follow similar procedures, but there will be differences.

C is large

Assume first that C is large. Define $L = \{v \in V(G) : deg(v, A) \geq \frac{n}{4} - 4\gamma n\}$. We choose only as many vertices from B for L so as to make the size of $(A \setminus C) \cup L$ exactly equal to $n/2$ (recall that a is a good vertex). Next consider those vertices in $B \setminus L$ that have low degree into C. Call that set Z and define it as $Z = \{z \in B \setminus L : deg(z, C) \leq \frac{9}{10}|C|\}$. Using the fact that the degree in B of any vertex in C is more than $\frac{49}{100}n$, simple edge-counting argument shows that $|Z| \leq \frac{n}{10} = \frac{1}{5}|B|$. Call all the remaining set Z' i.e. $Z' = (B \setminus L) \setminus Z$.

C is small

In this case we will find $|C|$ vertices from $B \cap L$ and interchange these with C. Since we move only at most Km vertices, the number of edges between the two sets is still at most $Kmn/2 + 3\gamma n^2 < 4\gamma n^2/2$, that is, the partition is $4\gamma n$-extremal. Furthermore, vast majority of A is adjacent to a and all vertices of A has at least $n/100$ neighbors in A.

C is large

Having done the preprocessing and defining all the appropriate sets, we now start the actual embedding process for T. All the *parents* should come from $A \cup L$. The embedding of parents is done in decreasing order of the indices of the x_is.

- Choose a random set $R_C \subset A - C$ and $R_Z \subset A - C - R_C$, both having size $n/50$.

- We begin with covering the vertices of Z'. All the vertices of Z' have at least $\frac{9}{10}|C|$ degree into C. We use the familiar greedy strategy for embedding a height-2 tree. Since the degrees to C are large, we can proceed as long as there are at least $5m/4$ vertices left in Z'. For the leftover of Z' we will find parents in $(A - C) \cup L$. This latter set has exactly $n/2$ elements, therefore, every vertex has at least $Km \log n$ neighbors in it. If needed, we will choose "outside leaves" from $A - C - R_C - R_Z$, this means at most $m + 2m \log n \ll Km \log n$ vertices that we cover from $(A - C) \cup L$.

- For the set C we look for parents in $(A - C) \cup L$, as we did for Z'. If we have some vertices left, we find the outside leaves in R_C.

- We follow the same procedure for the set Z, we find parents in $(A - C) \cup L$, and if necessary, we find the outside leaves in R_Z.

- The leftover of $R_C \cup R_Z$ is put back to A. Observe, that we have no parent-leaf conflict. Every uncovered vertex is from the set $(A - C) \cup L$. The minimum degree in this set is larger than $n/200$, since only a very few vertices were covered so far, and C had the vertices with smaller degrees in A. Also, a very few vertices can have degree smaller than $0.55n/2$ in $A - C$. We will first cover them right away. After this only such vertices are left that are each adjacent to at least 51% of the uncovered vertices. Therefore we can embed the leftover using the method of Section 2.3.

C is small

This case is easier than the previous one. In this case we have no set Z', and we may assume that the set $B = Z$. We first take care of B using the greedy strategy. There is no problem until we have at most $O(n/\log n)$ vertices left in B. Then we can follow the familiar strategy: if there is not enough neighbors in B, we take neighbors from A.

At this point we could use up all the neighbors of the leftover in B, hence we have to be careful. We will choose the outside leaves from A randomly. Since every vertex of A has at least $n/100$ neighbors in A and we choose at most m neighbors randomly at a step, we can apply Azuma's inequality and get that the probability that we do not use up more than $\frac{K}{2} m \log n$ vertices from the neighborhood in A of a vertex of B is larger than $1 - 1/n^3$. Thus, we don't get stuck and can cover every vertex of B this way.

When we are done with B, it is easy to continue as before. First we take care of those vertices in A that have less than $0.55n/2$ neighbors in A and those that are not neighbors of A (recall, that we can have a few, but each of these have several neighbors in A). We can use the same greedy procedure what we used in case C was large. Then covering the large degree vertices of A is easy. This finishes the case when T has height two.

2.6.2 T has essentially height two

Define $L_{\geq 4} = L_4 \cup L_5 \cup \ldots$, and let $L'_3 \subset L_3$ denote the leaves in L_3. We say that T has essentially height two if $|L_{\geq 4}| \leq n/500$. Clearly, this implies that $|L'_3| \geq 499n/500 - m$. It turns out that a very similar procedure to the above will embed T with some minor modifications. We will not go into the details but will just outline the procedure briefly.

- We choose a *good* image for the root r as before.

- Then we define the sets L, C, Z and Z' in the sets A and B as was done in the previous case. We distinct the cases whether C is large or small.

- If C is large:

- we find the random sets $R_C, R_Z \subset A - C$ as above. Also, we find another random set $R_{Z'} \subset A - C - R_C - R_Z$, each having $n/300$ vertices.

- We take care of the vertices of Z' as above, that is, parents come from $(A - C) \cup L$, but we find outside leaves and parts from $L_{\geq 4}$ in $R_{Z'}$.

- Similarly, we find parents for the vertices of C from $(A - C) \cup L$, and find outside leaves and parts from $L_{\geq 4}$ in R_C.

- We follow a similar procedure for Z. That is, find parents from $(A - C) \cup L$, and find outside leaves and parts from $L_{\geq 4}$ in R_C.

- Put back the leftover of the random sets to $A - C$. Only vertices from $(A - C) \cup L$ are left. First we take care of those vertices that are adjacent to at most 55% of $(A - C) \cup L$. Then every uncovered vertex is adjacent to at least 52% of the other uncovered vertices. We can use the tree embedding procedures of Sections 2.3 and 2.4 in order to embed leftover.

- If C is small:

  - In this case we will find $|C|$ vertices from $B \cap L$ and interchange these with C.

  - We find a random set $R_B \subset A$ such that $|R_B| = n/300$.

  - We apply the greedy strategy for the vertices of B, but use R_B for the outside leaves and the parts in $L_{\geq 4}$. Then put back the leftover of R_B to A.

  - As in the previous case, take care of the small degree uncovered vertices of A first, then apply the tree embedding procedures of Sections 2.3 and 2.4 in order to finish the embedding.

2.6.3 T has larger height

In this case we assume that there are more than $n/500$ vertices at a distance more than two from the root, r, of the tree. In the previous extremal cases we began with preprocessing G and then embedded T using the decomposition of G and the structural properties of the essentially height-2 tree T. Since the structure of T could be much

more subtle here, we have to decompose T. Moreover, we will apply another kind of decomposition for G.

Decomposition of G

First we discuss the decomposition of G, assuming that we already have the sets A and B. We define the *inner degree* of $v \in A$ as $|N_G(v) \cap A|$ and the *outer degree* of $v \in A$ as $|N_G(v) \cap B|$. The inner and outer degree of $u \in B$ is defined similarly.

Assume that $u \in A$ and $v \in B$ are such that $e(A - u + v) + e(B - v + u) > e(A) + e(B)$. Switch u to B and v to A, and look for another pair of vertices which can be switched. At every switching step the total number of edges inside A and B is increased, therefore this process will stop in a finite number of steps. Observe that if there exists $u \in A$ and $v \in B$ such that both have inner degrees of at most $n/4$, then it is still possible to perform a switching step. Hence, at the end of the switching procedure we have that every vertex in one of the parts, say A, has an inner degree of at least $n/4$. Observe, that after the switching even the sparser set will contain at least $n^2/4 - 2\gamma n^2$ edges, so both sets will be 2γ-extremal. There may be up to $5\gamma n$ vertices in B which have at most $n/100$ neighbors in B (see Lemma 25). Denote the set of these vertices with low degree in B by B'. We adjust the $A - B$ partitioning of G, we let $B := B - B'$ and $A := A + B'$.

Decomposition of T

Let I and II be two sets that are originally empty. During the decomposition we will put the vertices of T either into I or into II. In the end we will have that $|I| = |A|$ and $|II| = |B|$. The vertices of I will be embedded onto vertices of A, the vertices of II will be embedded onto vertices of B. We start from the root r, and as we proceed downwards we assign subtrees either to the set I, II, or will further decompose the subtree. We say that an edge is *separated* if one of its endpoints is put into I, the other is put into II.

There is always an *active* vertex x, in the beginning r is the active vertex. Denote the children of the active vertex by $x_1, x_2, \ldots, x_k$. We consider several cases.

1 **Subtrees that fit into** II. If for some i we have that $|B| - |II| > v(T(x_i))$ then we put the vertices of $T(x_i)$ into II, and look for another child of the active vertex that satisfies this property.

2 **Subtrees that do not fit into** II. Assume that there are more than one children that do not fit into II. Let $v(T(x_j))$ be minimal among these. Then we let x_j to be the new active vertex, and the previous active one is put into II. If there was at least one subtree that fitted into II then we call x_j a *bridge.*

3 **Height-2 subtrees.** If one of the subtrees that do not fit into II is essentially a height-2 tree, then we can finish the decomposition: It is easy to cut the subtree into two pieces such that one piece has size $|B| - |II|$, and there will be at most $O(m)$ newly separated edges and bridges.

4 **Finishing the decomposition.** If $k = 1$ and $v(T(x_1)) = |B| - |II|$, then we put the vertices of $T(x_1)$ into II. The decomposition finished.

Notice that since the new active vertex we always comes from the smallest subtree that does not fit into II, $|A| - |I|$ decreases fast, and we finish in at most $2 \log n$ steps, and there will be at most $2 \log n$ bridges and $2m \log n$ separated edges since every separated edge is incident to a bridge. Also, every bridge vertex belongs to II.

This is not necessarily the final decomposition of T. If the vast majority of I are coming from a subtree with root in II that has essentially height 2, then we put the root into I, and one leaf of that subtree is put into II. Observe that in such a case most vertices that are a distance at least 3 from the root are in II.

Covering vertices with small inner degree

Recall, that every vertex of B has inner degree at least $n/100$. Assume that there is a subtree in II such that below level 2 it contains at least $100\gamma n$ vertices. If it has essentially height 2, then we do the following. Use the greedy embedding method with the small inner degree vertices being used up as leaves. We will perhaps cover a few other vertices as well.

If the subtree is higher, we use a version of the above greedy method. We decompose its levels into 3 sets, S_1, S_2 and S_3. Here S_i contains the levels that has index i mod 3. Clearly, one of S_i will be large. Say, that S_3 is the largest.

Every vertex has at least $n/100$ neighbors in B, hence we can apply the following method. We begin with a vertex $v \in S_1$ on the actual level (we start from the root). It has a very large neighborhood since it will be mapped onto a vertex with large inner degree. Every small inner degree vertex has several neighbors in $N(v)$. This determines a height-2 subtree, that will be embedded such that S_3 is mapped onto leaves of this subtree we have found. Then we continue with other large degree vertices of the level of v. If there are no more such vertices left, we continue with the level that is adjacent to the recently mapped S_3 vertices. We make sure that these vertices have large inner degree. This is possible since even the small inner degree is much larger than the number of vertices with small inner degree. Then we can continue the above procedure with mapping two new levels, the second one being used to map the vertices of S_3.

If the subtree is much larger than what we need in order to cover small inner degree vertices, we find a smaller subtree of it, that will be used for the covering.

The actual embedding method

Our goal will be to map vertices of I into A and vertices of II into B. First, we map the *skeleton* that contains the bridge vertices, and those that are adjacent to two bridge vertices. Clearly, the skeleton has size at most $4m \log n$. We also fix the neighbors of the skeleton vertices in A. Observe that this is possible since A is larger, therefore, every vertex has at least $Km \log n$ neighbors in A.

If there is a subtree having essential height 2 such that except its root all its vertices are in A then we apply the random greedy method to embed that subtree. The second level of the subtree may come from A or from B. For every vertex of the second level for which we use a vertex from B we will map a leaf of this subtree into B. There can be at most one such subtree in I.

After this only such subtrees are assigned to I that are higher. Start the embedding of such a subtree (or subtree of a subtree or the union of subtrees) that contains about

$100\gamma n$ vertices. We will cover every vertex of A that has inner degree at most $0.6n/2$ when embedding this subtree. Then the rest that is mapped into A using the methods of Sections 2.3 and 2.4. This is possible since the leftover of A is large and therefore non-extremal with high minimum degree.

Covering B is very similar. First, we cover the vertices with small inner degrees. Since we have subtrees in II with their roots that have height at least 2, this is doable. Then the subgraph spanned by B will be non-extremal with high minimum degree, and we finish as in the previous case.

2.7 G is extremal and close to $K_{n/2,n/2}$

In this case $V(G)$ is partitioned into A and B such that $|A| = |B| = n/2$ and $e(A) + e(B) \leq \gamma n^2$. We state a fact very similar to Lemma 25

Lemma 27. *Let $0 < \eta < 1/2$. Then the number of vertices of A having less than $n/2 - \eta n$ neighbors in B is at most $\frac{2\gamma n}{\eta}$. Analogous is true for B.*

Embedding procedure in this case is very similar to the one used in the previous extremal case. We will outline the procedure briefly.

- Using Lemma 26 we choose a vertex a such that $I(r) = a$ that has large set, L, with high degree into $N(a)$. Only this time a large proportion of L would lie in B.

- Define a set $C = \{c \in B : deg_A(c) \leq \frac{1}{100}n\}$.

- This implies that $|A \cap L| \geq |C|$.

- Define $Z = \{z \in A \setminus L : deg_{A \cap L}(z) \leq \frac{9}{10}A \cap L\}$ and $Z' = (A \setminus L) \setminus Z$.

- First cover C with *parents* from A using at most $\frac{n}{K}$ leaves from outside C to completely cover it. The "outside" leaves come from A or $B \setminus C$ depending on where the parent is embedded.

- Next cover the vertices in $|Z|$ completely in a similar manner.

- Then cover vertices in Z' completely using parents from $A \cap L$.

- Resolve the *parent-leaf* conflicts.

- Cover vertices of $A \cap L$ greedily using parents from $A \setminus Z$.

- In the end we cover $B \setminus C$ by reserving some randomly chosen parents from A, partitioning them into classes according to the number of their children, and finding children for each of the sets of the parents separately as we did in the previous extremal case.

2.7.1 T has larger height

In this extremal case T will have an essentially larger height. The embedding method despite some differences is very similar: first we get rid of the noise, and then apply embedding into a very dense graph.

Here again we will partition the tree into two parts, but now the parts are representing color classes. We have to find an almost proper 2-coloring of the vertices of a tree on n vertices, such that there are at most $O(m \log n)$ edges which connect two vertices with the same color.

Lemma 28. *Let J be a tree on n vertices, n being an even integer, and let $1/3 \leq \rho \leq 2/3$. Then J can be divided into vertex disjoint subtrees $J_1, J_2, \ldots, J_s$ plus at most $2\log_2 n$ split vertices, such that there is a proper 2-coloring of the vertices of the subtrees with such that one color class has size ρn and the other has size $(1-\rho)n$.*

We sketch the embedding method in this case. First, we perform a switching in order to have that one color class, say A, has vertices with outer degree at least $0.5n/2$. Then the vertices of the other class, B having large inner degree will be put to A. Then we partition the tree T such that the almost proper coloring of it reflects the sizes of A and B. Finally, we first get rid of vertices having small outer degree, then the leftover can be embedded easily.

Chapter 3

A New Proof of the Pósa-Seymour Conjecture

3.1 Introduction

3.1.1 Notations and Definitions

$V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G. (A, B, E) denotes a bipartite graph $G = (V, E)$, where $V = A \cup B$, and $E \subset A \times B$ such that A and B are disjoint. For a graph G and a subset U of its vertices, $G|_U$ is the restriction of G to U. $N(v)$ is the set of neighbors of v in V, and $N_S(v)$ is the set of neighbors of v in S. Hence the size of $N(v)$ is $|N(v)| = deg(v) = deg_G(v)$, the degree of v. $\delta(G)$ stands for the minimum and $\Delta(G)$ for the maximum degree of a vertex in G. $K_r(t)$ is the balanced complete r-partite graph with color classes of size t. We write $N(p_1, p_2, \ldots, p_l) = \bigcap_{i=1}^{l} N(p_i)$ for the set of common neighbors of $p_1, p_2, \ldots, p_l$. When A and B are subsets of $V(G)$, we denote by $e(A, B)$ the number of edges of G with one endpoint in A and the other in B. In particular, we write $deg_U(v) = e(\{v\}, U)$ for the number of edges from v to U. For non-empty A and B,

$$d(A, B) = \frac{e(A, B)}{|A||B|}$$

is the *density* of the graph between A and B. In particular, we write $d(A) = d(A, A)$. A graph G on n vertices is γ-*dense* if it has at least $\gamma\binom{n}{2}$ edges. A bipartite graph $G(A, B)$ is γ-*dense* if it contains at least $\gamma|A||B|$ edges. Throughout the chapter log denotes the base 2 logarithm.

3.1.2 Powers of Cycles

Let C be a cycle on a vertex set $V(C)$. Then the kth power of C, denoted by C^k, is defined as follows: $V(C^k) = V(C)$ and uv is an edge in C^k if and only if the distance

between u and v in C is at most k. The kth power of a path P is defined in an analogous manner. For notational convenience we call the kth power of a path a k-path. A classical result of Dirac [15] asserts that if $\delta(G) \geq n/2$, then G contains a Hamiltonian cycle. A natural question generalizing Dirac's theorem was asked by Pósa (see Erdős [20]) in 1962:

Conjecture 29 (Pósa). *Let G be a graph on n vertices. If $\delta(G) \geq \frac{2}{3}n$, then G contains the square of a Hamiltonian cycle.*

This conjecture was further generalized by Seymour [51] in 1974:

Conjecture 30 (Seymour). *Let G be a graph on n vertices. If $\delta(G) \geq (\frac{k-1}{k})n$, then G contains the $(k-1)^{th}$ power of a Hamiltonian cycle.*

Substantial amount of work has been done on these problems. Jacobson (unpublished) first established that the square of a Hamiltonian cycle can be found in any graph G given that $\delta(G) \geq 5n/6$. Later Faudree, Gould, Jacobson and Schelp [27] improved the result, showing that the square of a Hamiltonian cycle can be found if $\delta(G) \geq (3/4 + \varepsilon)n$. The same authors further relaxed the degree condition to $\delta(G) \geq 3n/4$. Fan and Häggkvist lowered the bound first in [21] to $\delta(G) \geq 5n/7$ and then in [22] to $\delta(G) \geq (17n + 9)/24$. Faudree, Gould and Jacobson [26] further lowered the minimum degree condition to $\delta(G) \geq 7n/10$. Then Fan and Kierstead [23] achieved the almost optimal $\delta(G) \geq \left(\frac{2}{3} + \varepsilon\right)n$. They also proved in [24] that already $\delta(G) \geq (2n - 1)/3$ is sufficient for the existence of the square of a Hamiltonian *path*. Finally, they proved in [25] that if $\delta(G) \geq 2n/3$ and G contains the square of a cycle with length greater than $2n/3$, then G contains square of a Hamiltonian cycle.

For Conjecture 30, in the above mentioned paper of Faudree *et al* in [27], it is proved that for any $\varepsilon > 0$ and positive integer k there is a C such that if graph G, on n vertices, satisfies

$$\delta(G) \geq \left(\frac{2k - 1}{2k} + \varepsilon\right)n,$$

then G contains the kth power of a Hamiltonian cycle.

Using the Regularity Lemma – Blow-up Lemma method first in [37] Komlós, Sárközy and Szemerédi proved Conjecture 30 in asymptotic form, then in [38] and [40] they

proved both conjectures for $n \geq n_0$. The proofs used the Regularity Lemma [54], the Blow-up Lemma [36, 35] and the Hajnal-Szemerédi Theorem [32]. Since the proofs used the Regularity Lemma the resulting n_0 is very large (it involves a tower function). The use of the Regularity Lemma was removed by Levitt, Sárközy and Szemerédi in a new proof of Pósa's conjecture in [43]. The purpose of this thesis is to present another proof of the Pósa-Seymour conjecture that avoids the use of the Regularity Lemma, thus resulting in a simpler proof and a much smaller n_0.

Theorem 31. *There exists a natural number n_0 such that if a graph G has order n with $n \geq n_0$ and*

$$\delta(G) \geq (\frac{k-1}{k})n \tag{3.1}$$

then G contains the $(k-1)^{th}$ power of a Hamiltonian cycle.

3.2 Outline of the Proof

Our proof is divided into two main cases, the extremal case when G satisfies the following so-called extremal condition and the non-extremal case when this condition is not satisfied.

Extremal Condition (EC) with parameter α: *There exists an $A \subset V(G)$ such that*

- *$(\frac{1}{k} - \alpha)n \leq |A| \leq (\frac{1}{k} + \alpha)n$ and*

- *$d(A) < \alpha$*

In this case we say that the set A and the graph G are α-extremal, otherwise we say that they are α-non-extremal.

Non-Extremal Case: The basic fact that we make use of in the non-extremal case is that if we go around a complete k-partite graph picking vertices from each of the color classes sequentially we end up with a $(k-1)$-path. We use this fact repeatedly throughout the chapter. We first try to cover a constant fraction of the vertices in G by $K_{k+1}(O(\log n))$'s and then the maximum number of the remaining vertices with

$K_k(O(\log n))$'s. We refer to the set of $K_{k+1}(O(\log n))$'s and $K_k(O(\log n))$'s by $\mathcal{C}$ and $\mathcal{K}$ respectively. We would inevitably be left with a set $\mathcal{I}$ consisting of few vertices that cannot be covered in such a manner. However, we show that the number of such vertices is small. Then, using a Connecting Lemma, we "connect" the cliques by $(k-1)$-paths of length at most $9k^2$ to get a cycle of cliques. This process would force us to move a small number of vertices from $\mathcal{C} \cup \mathcal{K}$ to $\mathcal{I}$. As observed before, we can now get our required Hamiltonian cycle if we go around each of the cliques in $\mathcal{C} \cup \mathcal{K}$ in a sequential manner. However, we need to accommodate the vertices in $\mathcal{I}$. Hence we perform the "going around" process with a little more care and incorporate the vertices in $\mathcal{I}$ into the paths that we are constructing.
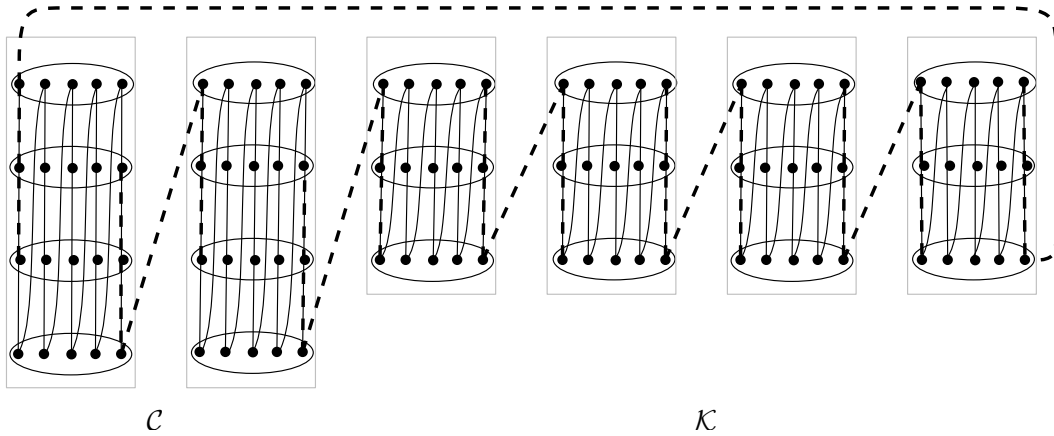

Figure 3.1: Non-Extremal Case: Dashed lines represent the $(k-1)$-paths constructed using the Connecting Lemma. Vertices from $\mathcal{I}$ are incorporated into the $(k-1)$-paths made inside the cliques as described in Section 3.4.2

Extremal Case: The extremal case uses a relatively simple König-Hall type argument in order to find the $(k-1)^{th}$ power of a Hamiltonian cycle in G. The details are left to Section 3.5.

3.3 Main Tools

We shall assume that n is sufficiently large and use the following main parameters:

$$0 < \eta \ll \alpha \ll 1, \tag{3.2}$$

where $a \ll b$ means that a is sufficiently small compared to b. In order to present the results transparently we do not compute the actual dependencies, although it could be done.

3.3.1 Complete k-Partite Subgraphs

In [40] the Regularity Lemma [54] was used to prove the Pósa-Seymour conjecture, however, here we use more elementary methods using only the Bollobás-Erdős-Simonovits bound [42].

Lemma 32 (Theorem 3.1 on page 328 in [6]). *There is an absolute constant $\beta_1 > 0$ such that if $0 < \epsilon < 1/s$ and we have a graph G with*

$$|E(G)| \geq \left(1 - \frac{1}{s} + \epsilon \right) \frac{n^2}{2}$$

then G contains a $K_{s+1}(t_1)$, where

$$t_1 = \left\lfloor \frac{\beta_1 \log n}{s \log 1/\epsilon} \right\rfloor.$$

The following two observations will be useful later on.

Fact 3. *If $G(A, B)$ is an η-dense bipartite graph, then there must be at least $\eta|B|/2$ vertices in B for which the degree in A is at least $\eta|A|/2$.*

Indeed, otherwise the total number of edges would be less than

$$\frac{\eta}{2}|A||B| + \frac{\eta}{2}|A||B| = \eta|A||B|,$$

a contradiction with the fact that $G(A, B)$ is η-dense.

Lemma 33. *Let the sets $A_1, A_2, \ldots, A_k$ form a complete k-partite graph, and for $1 \leq i \leq k$, $|A_i| = c_1 \log n$, and let B be a set of vertices such that $|B| = c_2 n$. If for every $b \in B$ $deg_{A_i}(b) \geq \eta|A_i|/2$, then we can find a complete $(k+1)$-partite graph $G(A'_1, A'_2, \ldots, A'_k, B')$ such that $A'_i \subset A_i, B' \subset B, |A'_i| \geq \eta|A_i|/2^k$ and $|B'| \geq c_2 n^{(1-kc_1)}$.*

The proof of the above statement uses standard counting arguments and we omit the details.

Lemma 34. *There exist constants n_0 and $\beta_2 > 0$ such that if G is a non-extremal graph on $n \geq n_0$ vertices with $\delta(G) \geq (\frac{k-1}{k} - \sqrt{\eta})n$, then G contains a $K_{k+1}(t)$, where $t = \lfloor \beta_2 \log n \rfloor$.*

Proof. We apply Lemma 32 on G to get k disjoint sets $A_1, A_2, \ldots, A_k$ each of size $t_1 = \left\lfloor \frac{\beta_1 \log n}{s \log 1/\epsilon} \right\rfloor = O(\log n)$ such that they form a complete balanced k-partite graph. Define $A := \bigcup_{i=1}^{k} A_i$ and let $B \subset V(G) \setminus A$ be the set of vertices that have more than $\eta|A_i|$ degree into each A_i. Our first observation is that we can assume that $|B| \leq \eta^2 n$. Indeed otherwise by Lemma 33 we get our desired $K_{k+1}(t)$.

Let $C = V(G) \setminus (A \cup B)$ and for $1 \leq i \leq k$ let $C_i = \{c \in C : deg_{A_i}(c) < \eta|A_i|\}$. By definition of B it follows that $C = \bigcup_{i=1}^{k} C_i$. From the minimum degree condition and the definition of C_i we have that for every i:

$$\left(\left(\frac{k-1}{k} - \sqrt{\eta} \right) n - |B| - |A| \right) |A_i| \leq e(A_i, C) \leq \eta|A_i||C_i| + |A_i|(|C| - |C_i|)$$

which gives us that $|C_i| \leq (1 + 3\sqrt{\eta})n/k$.

This together with the definition of C implies that $|C_i \cap C_j| < 4\sqrt{\eta}n$ for all $i \neq j$ hence $|C_i| \geq (1 - \sqrt{\eta})n/k$. Since G is α-non-extremal $e(G|_{C_i}) > \alpha(n/k)^2$ for every i. Consider any C_j and group the vertices in it by their neighborhood in $A \setminus A_j$. There can be at most $2^{t_1} < n^{k\beta_1}$ groups. We can safely ignore those groups which have at most $\eta n^{k\beta_1}$ vertices in them since they can contain at most $\eta n^{2k\beta_1}$ vertices in them. Similarly those groups may be disregarded which are connected to less than $2/3$ fraction of any of the A_i's. Indeed, from the minimum degree condition and the size of C_j we have that $e(A_i, C_j) \geq (1 - k\eta)n|A_i|$ for any $i \neq j$, and thus the total number of vertices in such groups is at most $20k\eta n$. Then either two such groups form a dense bipartite graph between them or one of the groups is internally dense. An application of Lemma 32 ensures that in either case we will find a $K_2(O(\log n))$ in C_j that together with $A \setminus A_j$ forms the required $K_{k+1}(t)$. $\square$

We will also use the following simple fact on the size of a maximum set of vertex disjoint paths in G (see [6]).

Lemma 35. *In a graph G on n vertices, we have*

$$\nu_1(G) \geq \max\{\delta(G), \delta(G)\frac{n}{4\Delta(G)}\} \text{ and } \nu_2(G) \geq (\delta(G) - 1)\frac{n}{6\Delta(G)}$$

where $\nu_i(G)$ denotes the size of maximum set of vertex disjoint paths of length i in G.

3.3.2   The Connecting Lemma

Given a non-extremal graph $G(V, E)$ on n vertices with $\delta(G) \geq (\frac{k-1}{k})n$ we can make the following observation:

Fact 4. *For every $v \in V$ there are more than $\frac{(k-1)!}{4}(\frac{n}{k})^{2k-2}$ $(k-2)$-paths of length $(2k-3)$ in $N(v)$.*

We call a $(k-2)$-path of length $2k-3$ *bad* if it is contained in the neighborhoods of at most $\alpha^{10k}n$ vertices of G. It is called *good* otherwise. A vertex $f \in V$ is said to be *feasible* if there are at most $\alpha^{4k}n^{2k-2}$ *bad* paths in $N(f)$.

Claim 36. *At most $\alpha^{6k}n$ vertices of G are not* feasible.

Proof. The proof uses a straightforward counting argument and the details are left to the reader.   □

We now state the Connecting Lemma:

Lemma 37 (Connecting Lemma)**.** *For every two disjoint ordered $(k-1)$-cliques of G, $A = (a_1, a_2, \ldots, a_{k-1})$ and $B = (b_{k-1}, \ldots, b_2, b_1)$ there is a $(k-1)$-path of length at most $9k^2$ which connects A and B even if $o(n)$ of the vertices of G are forbidden to be used in the path.*

Proof. Our general strategy will be to first specify a *feasible* vertex f and then to start building a $(k-1)$-path from A towards $N(f)$. Once we extend the path by $k-1$ vertices into $N(f)$ we use the same procedure to get inside $N(f)$ from B. Then we connect the two ends by a $(k-2)$-path using vertices only from $N(f)$. Finally, to convert the entire path into a $(k-1)$-path we make use of the fact that since v is feasible, $N(f)$ contains very few *bad* path, and hence we should be able to find a lot

of vertices that may be placed at $k-1$ intervals of the path to make it into a $(k-1)$-path.

Fix a *feasible* vertex v in G and let $F = N(f)$. If needed, we ignore a few vertices of F so that $|F| = (\frac{k-1}{k})n$. We will construct a short $(k-1)$-path:

$$(a_1, a_2, \ldots, a_{k-1}, w_1, w_2, \ldots, w_{k-1}, x_1, x_2, \ldots,$$

$$x_{k-1}, \ldots, y_{k-1}, \ldots, y_2, y_1, z_{k-1}, \ldots, z_2, z_1, b_{k-1}, \ldots, b_2, b_1)$$

between the two given ordered cliques that the x_i's and the y_i's come from the set F.

Extending the path by $k-1$ vertices into F

Let $W = N(a_1, a_2, \ldots, a_{k-1})$ which is of size at least n/k. Then we can make the following claim:

Claim 38. *We may assume that $W \cap F \le \alpha n/10$ otherwise we can extend our $(k-1)$-path into F by $k-1$ vertices with at least $\alpha n/10$ choice for each vertex.*
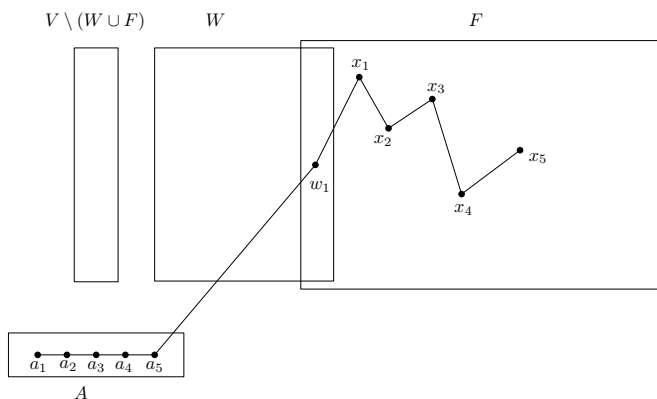

Figure 3.2: The $(k-1)$-path may be extended if $W \cap F$ is large

Proof. Pick any vertex from W and call it w_1. We now consider $N_F(a_2, a_3, \ldots, a_{k-1}, w_1)$. If this intersection is of size less than $\alpha n/10$ then we can simply assume that our starting clique was $(a_2, a_3, \ldots, a_{k-1}, w_1)$ instead of A (we call this new starting clique A as well). If, on the other hand, the intersection size is more than $\alpha n/10$ we can pick any vertex from the intersection and call it w_2. We now consider $N_F(a_3, a_4, \ldots, a_{k-1}, w_1, w_2)$. If this intersection is of size less than $\alpha n/10$ then we can again assume that our starting

clique was $(a_3, a_4, \ldots, a_{k-1}, w_1, w_2)$ instead of A (and we again call this new starting clique A). Otherwise we can choose a w_3 from amongst the intersection. We then consider $N_F(a_4, a_5, \ldots, a_{k-1}, w_1, w_2, w_3)$ and repeat the process. If at each step we end up with an intersection of size more than $\alpha n/10$ then after $k-1$ steps we will end up with a $(k-1)$-clique entirely inside F (with at least $\alpha n/10$ choices at each step). $\quad\square$

Therefore we can assume that $W \cap F < \alpha n/10$. Let W_{low} be the set of vertices in W that have low degree inside W. More precisely $W_{low} = \{w \in W : deg(w, W) \leq (\frac{1}{k} - \frac{\alpha}{10})n\}$.

Claim 39. *If $|W_{low}| > \alpha^{10}n$ then we can extend our $(k-1)$-path into F by $k-1$ vertices with at least $\alpha n/10$ choice for each vertex.*
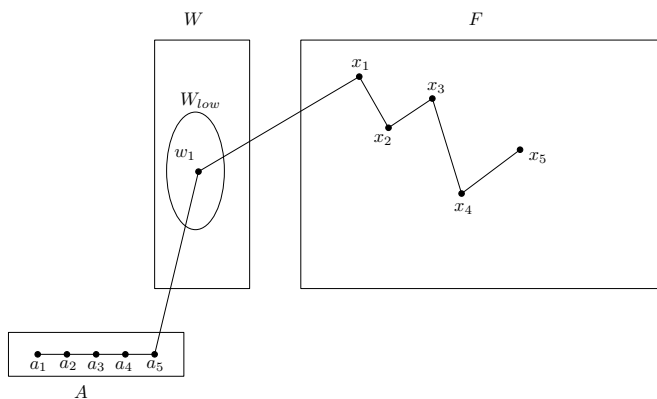

Figure 3.3: The $(k-1)$-path may be extended if W_{low} is large

Proof. We note that any $w \in W_{low}$ has at least $\alpha n/10$ neighbors in $N_F(a_2, a_3, \ldots, a_{k-1})$. In fact, w has at least $\alpha n/10$ neighbors in any n/k sized subset of F. So one can choose any vertex from W_{low} to be w_1 and any of its neighbors in $N_F(a_2, a_3, \ldots, a_{k-1})$ to be x_1. Then x_2 may be chosen from $N_F(a_3, a_4, \ldots, a_{k-1}, x_1, w_1)$ which is also of size about $\alpha n/10$ by the previous observation. We can continue extending the $(k-1)$-path since $x_3, x_4, \ldots, x_{k-1}$ may be chosen from F and we are always guaranteed to have at least $\alpha n/10$ choices for each of the x_i's until we end up with a $(k-1)$-clique that lies entirely inside F. $\quad\square$

Hence we may assume that $|W_{low}| \leq \alpha^{10}n$, which in turn implies that $d(G|_W) > (1 - \alpha/10)$, i.e. $G|_W$ is an almost complete graph. We find a $(k-1)$-clique $W_1 = (w_1, w_2, \ldots, w_{k-1})$ inside the set W which is easy to find since we have already established that $G|_W$ is an almost complete graph. Let $U_1 = \bigcup_{w \in W_1} N_F(w)$ and $I_1 = \bigcap_{w \in W_1} N_F(w)$. The minimum degree condition implies that $|U_1| \geq (\frac{k-2}{k})n$. In fact we can make an even stronger assertion:

Claim 40. *If $|U_1| < (\frac{k-2}{k} + \frac{\alpha}{k})n$ then we can extend our $(k-1)$-path into F by $k-1$ vertices with at least $\alpha n/10$ choice for each vertex.*

Proof. If $|U_1| < (\frac{k-2}{k} + \frac{\alpha}{k})n$ it means that U_1 and I_1 almost completely coincide and $|I_1| > (\frac{k-2}{k} - \alpha)n$ and so one can proceed as follows. Pick any vertex from I_1 and call it x_1. We can choose x_2 from $I_2 = I_1 \cap N(x_1)$ which is of size at least $(\frac{k-3}{k} - \alpha)n$. In general, let $I_i = I_{i-1} \cap N(x_{i-1})$ for $2 \leq i \leq k-3$ so that $|I_i| \geq (\frac{(k-1)-i}{k} - \alpha)n$. Hence we can choose x_i's from respective I_i's for $1 \leq i \leq k-3$ and for each x_i we have a lot of choices. We still need to choose x_{k-2} and x_{k-1}. For that we observe that $|I_{k-3} \cap N(x_{k-3})| > (\frac{1}{k} - \alpha)n$ and since we are not in the extremal case we can find $\alpha(n/k)^2$ edges in this set. We can choose any of these edges to be our required x_{k-2} and x_{k-1} and thus we have extended our $(k-1)$-path by $k-1$ vertices that lie completely inside F. $\square$
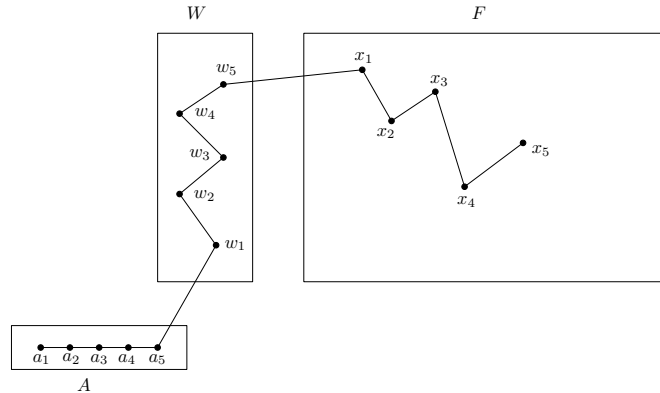

Figure 3.4: Extending the $(k-1)$-path when $G|_W$ is almost complete and W is disjoint from F

In case $|U_1| \geq (\frac{k-2}{k} + \frac{\alpha}{k})n$ for every $(k-1)$-clique W_1 inside W we can show that

there exists a $(k-1)$-clique W_2 such that $|\bigcap_{w\in W_2} N_F(w)| \geq \alpha n/10$. Pick a k-clique $W' = (w_1, w_2, \ldots, w_k)$ inside W. If for some pair of vertices, say $w_1, w_2 \in W'$, it is true that $|N_F(w_1, w_2)| \geq (\frac{k-3}{k} + \alpha/10)n$ then the clique $W_2 = (w_1, w_2, \ldots, w_{k-1})$ serves our purpose since by the minimum degree condition we know that $|N_F(w_3, w_4, \ldots, w_{k-1})| \geq (\frac{2}{k})n$ which intersects $N_F(w_1, w_2)$ by at least $\alpha n/10$. On the other hand, if there is no such pair then clearly $N_F(w_1) \cup N_F(w_2) \geq |F| - \alpha n/10$. Since $|N_F(w_3, w_4, \ldots, w_k)| \geq n/k$ hence either $N_F(w_1, w_3, w_4, \ldots, w_k)$ or $N_F(w_2, w_3, w_4, \ldots, w_k)$ is at least $\alpha n/10$.

For clarity of calculations let us set $U_2 = \bigcup_{w\in W_2} N_F(w)$ and $m = |F| = \frac{k-1}{k}n$ so that $|U_2| = (\frac{k-2}{k-1} + c_1)m = \frac{(k-2)+c_1(k-1)}{k-1}m$ for $\alpha/(k-1) < c_1 \leq 1/(k-1)$. The following series of facts are then easily deducible from the minimum degree condition and Claim (40):

Fact 5. *For any j-subset W' of W_2,*

$$|N(W') \cap U_2| \geq \frac{(k-2) - (j-1)(k-1)c_1}{(k-2) + (k-1)c_1}|U_2|. \quad \square$$

Fact 6. *For any i-subset F' of F,*

$$|N(F') \cap U_2| \geq \frac{((k-2) - i) + (k-1)c_1}{(k-2) + (k-1)c_1}|U_2|.$$

Actually, this fact holds for any *i-subset of $V(G)$.* $\quad \square$

Fact 7. *For every j, such that $3 \leq j \leq k-1$, there exists j-subset W'_j of W_2 such that*

$$|N(W'_j) \cap U_2| \geq \frac{(k-2) - (j-1)(k-1)c_1}{(k-2) + c_1(k-1)}|U_2| + \epsilon_1 m. \quad \square$$

We have the $(k-1)$-clique W_2 whose vertices we now rename as $(w_1, w_2, \ldots, w_{k-1})$. We choose our x_1 from amongst I_1. We note that x_2 has to be chosen from the common neighborhood of x_1 and a $(k-2)$-subset of $(w_1, w_2, \ldots, w_{k-1})$. We consider W'_{k-2}, say $(w_2, w_3, \ldots, w_{k-1})$ that has $\epsilon_1 m$ extra neighborhood as guaranteed by Fact 7. As we saw earlier, any vertex in F, and in particular x_1, has about ϵm common neighborhood with the neighborhood of W'_{k-2} in U_2. Therefore x_2 may be chosen from amongst this overlap. Similarly x_3 will have to come from the combined neighborhood of x_1, x_2, and $W'_{k-3} \subset W'_{k-2} = (w_2, w_3, \ldots, w_{k-1})$. For notational convenience we assume that this $W'_{k-3} = (w_3, w_4, \ldots, w_{k-1})$ and in general $W'_{k-j} = (w_j, w_{j+1}, \ldots, w_{k-1})$ for j, such that

$2 \leq j \leq k-2$. We can go on choosing the x_i's upto x_{k-3} making use of Fact 7. It is pertinent to note that the actual order of the w_i's in W_2 is determined by the set W'_{k-j} that we find at each step.

Now x_{k-2} and x_{k-1} need to be chosen from $U_2 \cap N(w_{k-2}, w_{k-1}, x_1, x_2, \ldots, x_{k-3})$. As previously x_{k-2} may be chosen with $\epsilon_1 m$ choices by Fact 7 so that by Fact 6 $|N(x_1, x_2, \ldots, x_{k-2}) \cap U_2| \geq \frac{k-2}{(k-2)+(k-1)c_1} + \alpha |U_2|$. There are now two cases to consider:

Case 1: $|U_2 \cap N(w_{k-2}, w_{k-1})| \geq \frac{k-2}{(k-2)+(k-1)c_1} + \alpha |U_2|$

In this case

$$N(x_1, x_2, x_{k-2}) \cap (N(w_{k-2}) \cup N(w_{k-1})) \geq \alpha |U_2|$$

hence without loss of generality we can claim that $U_2 \cap N(x_1, x_2, \ldots, x_{k-2}) \cap N(w_2) \geq \alpha |U_2|/2$ and x_{k-1} may be chosen from amongst this set. The final order of the vertices in the path would be

$$a_1, a_2, \ldots, a_{k-1}, w_1, w_2, \ldots, w_{k-3}, w_{k-1}, w_{k-2}, x_1, x_2, \ldots, x_{k-2}, x_{k-1}, \ldots$$

Case 2: $|U_2 \cap N(w_1, w_2)| < \frac{k-2}{(k-2)+(k-1)c_1} + \alpha |U_2|$

In this case

$$|(N(w_{k-2}, w_{k-1})| \geq (\frac{k-2}{(k-2)+(k-1)c_1} - \alpha)|U_2|$$

A simple calculation shows that

$$|N(x_1, x_2, \ldots, x_{k-3}, w_{k-2}) \cap N(w_{k-1})| \geq \frac{n}{k} - \alpha n$$

hence by the non-extremality condition this set has $\alpha(n/k)^2$ edges. We can pick any of these edges, say x'_{k-2}, x'_{k-1} and so our path will look as follows:

$$a_1, a_2, \ldots, a_{k-1}, w_1, w_2, \ldots, w_{k-3}, w_{k-2}, w_{k-1}, x_1, x_2, \ldots, x_{k-3}, x'_{k-2}, x'_{k-1}, \ldots$$

It is clear that we can extend the path from B into F using the same reasoning so that we have $k-1$ vertices inside F such that the path from the other end looks as follows:

$$b_1, b_2, \ldots, b_{k-1}, z_1, z_2, \ldots, z_{k-3}, z_{k-2}, z_{k-1}, y_1, y_2, \ldots, y_{k-2}, y_{k-1}, \ldots$$

where $y_i \in F$ for every i. Let us define two ordered $(k-2)$-cliques as follows: $X = (x_2, x_3, \ldots, x_{k-1})$ and $Y = (y_2, y_3, \ldots, y_{k-1})$. Out next task will be to connect X and Y using a $(k-1)$-path that lies entirely inside F.

Connecting X and Y inside F

We note that $\delta(G|_F) \geq (\frac{k-2}{k-1})|F|$ and that $G|_F$ is non-extremal by virtue of G being non-extremal. Hence by induction we can find many $(k-2)$-paths connecting X and Y. Once we have fixed a $(k-2)$-path, P', connecting X and Y we convert this path into a $(k-1)$-path, P, which can be accomplished as follows: after every $(k-1)$ vertices of P' we insert a vertex which is adjacent to the $(k-1)$ vertices on its either side on P'. Since f was chosen to be a feasible vertex, it contains at most $\alpha^{4k} n^{2k-2}$ *bad* paths, but each subpath of P', on $2k-2$ vertices can be chosen in at least $\alpha^{2k-2} n^{2k-2}$ different ways, thus ensuring that each of these subpaths may be chosen so that it is *good*. Since each *good* path is contained in the neighborhoods of at least $\alpha^{10k} n$ vertices of G we can find a distinct vertex to insert after every $k-1$ vertices of P'. Hence we end up with a $(k-1)$-path connecting the original $(k-1)$-cliques A and B.

Let L_k be the length of the $(k-1)$-path connecting two given $(k-1)$-cliques as detailed above. Then the following recurrence relation holds with the initial value $L_2 \leq 2$:

$$L_k \leq 8(k-1) + L_{k-1} + ((L_{k-1})/(k-1) - 1) = 8k^2 + o(k^2)$$

□

3.4   The Non-Extremal Case

Before we start the actual construction of the Hamiltonian cycle we need some preparation in the graph. Throughout this section we assume that we have a graph G and that the Extremal Case does not hold for G, i.e. there exists no $A \subset V(G)$ such that $(\frac{1}{k} - \alpha)n \leq |A| \leq (\frac{1}{k} + \alpha)n$ and $d(A) < \alpha$.

3.4.1 The Optimal Cover

We are going to work with a cover $(\mathcal{C}, \mathcal{K}, \mathcal{I})$ where $\mathcal{C}$ is a collection of balanced complete $(k+1)$-partite graphs, $\mathcal{K}$ is a collection of balanced complete k-partite graphs disjoint from $\mathcal{C}$ and $\mathcal{I}$ is the set of remaining vertices. By $V(\mathcal{C})$ and $V(\mathcal{K})$ we represent the set of vertices covered by the $(k+1)$-partite and k-partite graphs in $\mathcal{C}$ and $\mathcal{K}$ respectively. Our goal is to find the optimal cover, where we cover ηn vertices of G by $\mathcal{C}$ and the maximum number of remaining vertices by $\mathcal{K}$ such that we cannot significantly increase the number of vertices covered by $\mathcal{K}$. For $1 \leq i \leq k$ we say that a vertex v is i-sided to a $K_j \in \mathcal{K}$ if we have $d(v, K_j) \geq ((i-1)/k + \eta)$ i.e. v has a large degree to at least i color classes of K_j. Similarly a vertex v is i-sided to a $C_j \in \mathcal{C}$ if we have $d(v, C_j) \geq ((i-1)/(k+1) + 2\eta)$.

By Lemma 34 we know that we can find a $K_{k+1}(O(\log n))$ in G. However, we note that the conditions for the Lemma hold until the number of vertices covered by these $(k+1)$-cliques becomes $\sqrt{\eta} n$, at which point we do not have the minimum degree guarantee, hence we conclude that at least $\sqrt{\eta} n$ vertices can be covered by $(k+1)$-cliques. In fact, we just cover exactly $\sqrt{\eta} n$ vertices in G and no more.

Still the minimum degree is at least $(\frac{k-1}{k} - \sqrt{\eta})n$ in G', the remaining induced graph. We can apply Lemma 32 on G' since $\delta(G') \geq (\frac{k-1}{k} - \sqrt{\eta})n > (\frac{k-1}{k} - \sqrt{\eta})|V(G')|$ and can also show that almost all of the remaining vertices can be covered by k-cliques. Using Lemma 32 we find a collection of $K_k(t)$'s , where $t = O(\log n)$, such that it covers the maximum number of vertices in G'. Let $\mathcal{I}$ be the vertices of G' that are not part of any $K_k(t)$. Since we can no longer apply Lemma 32 with $s = k-1$ to increase $|V(\mathcal{K})|$ we either have that:

$$|E(G'|_{\mathcal{I}})| < \left(\frac{k-2}{k-1} + \eta \right) \frac{|\mathcal{I}|^2}{2}. \tag{3.3}$$

It is easy to see that in the optimal cover $d(\mathcal{I}, \mathcal{K}) < (\frac{k-1}{k} + \eta)$. Indeed otherwise we can significantly increase the size of $\mathcal{K}$. Assume that $d(\mathcal{I}, \mathcal{K}) \geq (\frac{k-1}{k} + \eta)$ and let each k-partite graph $K_i \in \mathcal{K}$ be composed of the color classes $V_i^1, V_i^2, \ldots, V_i^k$, and let $V^j = \bigcup_i V_i^j$ for $1 \leq j \leq k$. Since $\mathcal{I}$ is at least η-dense to every V^j, it follows that it is also η-dense to most of the individual color classes V_i^j of the cliques in $\mathcal{K}$. However,

if $\mathcal{I}$ is k-sided to more than η-fraction of the cliques in $\mathcal{K}$, a modified form of Lemma 33 gives us a large complete $(k+1)$-partite graph. This $(k+1)$-partite graph may be split into $k+1$ separate graphs that can be added to $\mathcal{K}$ hence significantly increasing its size, a contradiction to the optimality of the cover.

On the other hand, by the minimum degree of G' we have

$$e(\mathcal{I},\mathcal{K}) \geq \left(\frac{k-1}{k} - \eta\right)|V(G')||\mathcal{I}| - \left(\frac{k-2}{k-1} + \eta\right)\frac{|\mathcal{I}|^2}{2}. \tag{3.4}$$

From (3.4) and the above observation on $d(\mathcal{I},\mathcal{K})$ we get that $|\mathcal{I}| < 4\eta n$.

3.4.2 Dealing with the vertices in $\mathcal{I}$

Consider the cover $(\mathcal{C},\mathcal{K},\mathcal{I})$. We will insert the vertices in $\mathcal{I} = \{a_1, a_2, \ldots, a_{4\eta n}\}$ one by one into a $(k-1)$-path that we are constructing within each clique in $\mathcal{C} \cup \mathcal{K}$. We will first prove an easy consequence of the degree condition in G:

Lemma 41. *Every vertex $a \in \mathcal{I}$ is k-sided to at least η fraction of the cliques in $\mathcal{C} \cup \mathcal{K}$.*

Proof. For contradiction, assume that we are given a vertex $a \in \mathcal{I}$ that is not at least k-sided to an η fraction of the cliques in $\mathcal{C} \cup \mathcal{K}$. Then

$$deg_G(a) < |\mathcal{I}| + \eta n + (|V(\mathcal{C})| - \eta n)\left(\frac{k-1}{k+1} + 2\eta\right) + |\mathcal{K}|\left(\frac{k-1}{k} + \eta\right)$$
$$< \frac{k-1}{k}n$$

A contradiction to the minimum degree condition. $\square$

Case 1: When a is k-sided to a $(k+1)$-clique, say $C_j = (V_j^1, \ldots, V_j^{k+1}) \in \mathcal{C}$, we assume that a has high degree to all the color classes of C_j except V_j^b. Assume that that $(k-1)$-path that we are making within C_j has its end point $v_{0.k+1}$ in V_j^{k+1} which we want to extend now, incorporating a into it. We will proceed as follows: we continue constructing the path in the natural order, that is: $\ldots v_{0.k+1}, v_{1.1}, v_{1.2}, v_{1.3} \ldots$ where $v_{x.y}$ is any arbitrary vertex in V_j^y that has not yet been used in our path. However, we use a instead of the vertex $v_{1.b}$ in our path and continue on till $v_{1.k+1}$. However treating a as a vertex in V_j^b, creates an imbalance in sizes of the color classes of C_j, that is, there is one extra vertex in V_j^b. We remedy this by successively skipping one vertex each from
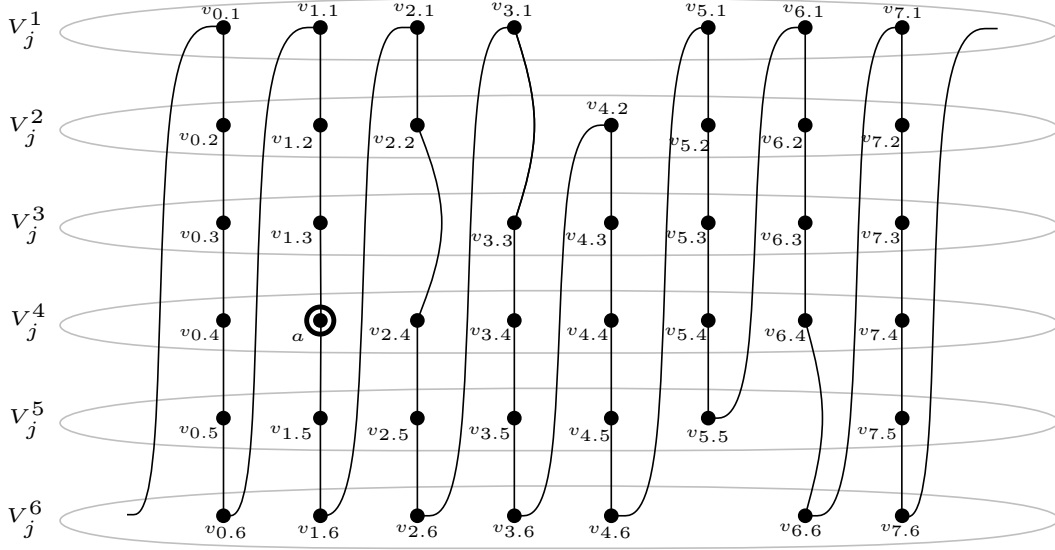
Figure 3.5: Inserting a into the $(k-1)$-path being constructed in the complete balanced $(k+1)$-partite graph C_j, where $k=5$ and $b=4$

V_j^i, $(i \neq b)$ in the next k iterations in a cyclic manner. The process of inserting a into our $(k-1)$-path is depicted in Figure 3.5, for $k=5$ and $b=4$. The final $(k-1)$-path in the figure is as follows:

$$(\ldots v_{0.5}, v_{0.6}, v_{1.1}, v_{1.2}, v_{1.3}, a, v_{1.5}, v_{1.6}, v_{2.1}, v_{2.2}, v_{2.4}, v_{2.5}, v_{2.6}, v_{3.1}, v_{3.3}, v_{3.4}, v_{3.5},$$

$$v_{3.6}, v_{4.2}, v_{4.3}, v_{4.4}, v_{4.5}, v_{4.6}, v_{5.1}, v_{5.2}, v_{5.3}, v_{5.4}, v_{5.5}, v_{6.1}, v_{6.2}, v_{6.3}, v_{6.4}, v_{6.6}, v_{7.1}, v_{7.2}, \ldots)$$

Insertion of any vertex $a \in \mathcal{I}$ and the subsequent rebalancing takes a total of $k+1$ iterations over the color classes hence we use up only $O(k^2)$ vertices to insert a into our path while regaining a balanced $(k+1)$-clique. Since a is k-sided to many cliques in $\mathcal{C}$ and $\mathcal{K}$ we can evenly distribute the vertices from $\mathcal{I}$ among different cliques so that the insertion is always possible.

Case 2: When a is k-sided to a k-clique, that is, to a $K_j = (V_j^1, V_j^2, \ldots, V_j^k) \in \mathcal{K}$ we can simply insert a into the path anywhere and continue in the natural order. It is easy to verify that all the required edges are present. Thus we have placed a into our path.

Using the procedure outlined above we can insert all the outside vertices into the $(k-1)$-paths inside the cliques.

3.4.3 Finding the Cycle

Using the Connecting Lemma we first connect together the cliques in $\mathcal{C} \cup \mathcal{K}$ through $(k-1)$-paths. For each vertex in the connecting paths that comes from $\mathcal{C} \cup \mathcal{K}$ we need to discard at most k other vertices to maintain the balance in the color classes of the cliques in the cover. Hence, in total at most $9k^3 \times O(n/\log n) = o(n)$ vertices would need to be moved into $\mathcal{I}$. Then we start constructing a $(k-1)$-path inside each of the cliques in the cover by going around each of the color classes in a sequential manner. During this construction we can incorporate all the vertices in $\mathcal{I}$ into the paths using the procedure outlined in the previous section. When we finish covering up the vertices in $\mathcal{C} \cup \mathcal{K}$ we end up with the required $(k-1)^{th}$ power of a Hamiltonian cycle.

3.5 The Extremal Case

In this case, the graph G satisfies the extremal condition. We take the maximum number of disjoint α-extremal sets $A_1, A_2, \ldots, A_l$. We assume for $1 \le i \le l$, $|A_i| = \lfloor \frac{n}{k} \rfloor$ because if some $|A_i| > \lfloor \frac{n}{k} \rfloor$, then since $\delta(G|_{A_i}) \ge |A_i| - \lfloor \frac{n}{k} \rfloor$, there exists a matching of size $|A_i| - \lfloor \frac{n}{k} \rfloor$ in $G|_{A_i}$ by Lemma (35). We may contract every $e = \{u, v\}$ of these matched edges into a vertex x_e, such that x_e is connected to the common neighbors of u and v. We denoting the resulting set by A_i as well. We let $B = V(G) \setminus (A_1 \cup \cdots \cup A_l)$ for $l \le k$. Furthermore, we say that $v \in A_i$ is *bad* if we have

$$deg_{A_i}(v) \ge \alpha^{1/3}|A_i|. \tag{3.5}$$

Note that by the fact that $d(A_i) < \alpha$, there are at most $\alpha^{2/3}|A_i|$ *bad* vertices in any A_i. A vertex $v \in A_i$ (or B) is *exceptional for A_j* (for $j \ne i$) if $deg_{A_j}(v) < \alpha^{1/3}|A_j|$. By (3.1), for each $v \in A_i$ (or B), there can be at most one $j \ne i$, such that v is *exceptional* for A_j. We denote the set of vertices in A_i (or B) that are *exceptional for A_j* by $E_i(j)$ (or $E_B(j)$). By the minimum degree condition a vertex can be in $E_i(j)$ for at most one $j \ne i$. The following two remarks are easy to deduce.

Remark 4. *If a vertex v is in $E_i(j)$ for some j then it is* bad, *indeed $deg_{A_i}(v) > (1 - \frac{\alpha^{1/3}}{2})|A_i|$.*

Remark 5. *Switching a* bad *vertex in A_i with a vertex in $E_j(i)$ reduces the number of* exceptional *vertices. Hence we may assume that either there are no* bad *vertices in A_i or $E_j(i)$ is empty for every $j \neq i$.*

3.5.1 Finding the Cycle

To convey the basic idea of the proof we deal separately with cases when $l = k$ and when $l < k$.

G has k extremal sets

In this case the vertex set V can be partitioned into $A_1, A_2, \ldots, A_k$ such that $|A_i| = \lfloor \frac{n}{k} \rfloor$ and $d(A_i) < \alpha$ for $1 \leq i \leq k-1$, that is, $l = k$ (and hence $B = \phi$). We will further subdivide this case into two subcases.

The Clean Case: There are no *bad* or *exceptional* vertices in any A_i, (hence $E_i(j)$ is empty for all i, j by Remark 4). We will cover $A_1 \cup \cdots \cup A_k$ with k-cliques such that every clique uses a vertex from each A_i. For each $v \in A_i$, we have $deg_{A_j}(v) \geq (1 - \alpha^{1/3})|A_j|$ for all $j \neq i$. Furthermore, since in this case there are no *bad* vertices it is relatively straightforward to find k-cliques by a simple greedy procedure that uses the König-Hall theorem as follows. We first find a perfect matching M_1 between A_1 and A_2. Then we find a perfect matching between M_1 and A_3, such that $e = \{x, y\} \in M_1$ is matched with a vertex $z \in N(x, y) \cap A_3$. We can continue this process to find the desired k-cliques. Indeed, let M_{k-2} be the $(k-1)$-cliques made so far, from $A_1, A_2, \ldots, A_{k-1}$. For any clique $(x_1, x_2, \ldots, x_{k-1})$, $x_i \in A_i$ we have that $|N(x_1, x_2, \ldots x_{k-1}) \cap A_k| \geq (1 - \alpha^{1/4})|A_k|$, therefore, by König-Hall theorem there exists a perfect matching between the $(k-1)$-cliques and vertices in A_k, therefore we can extend these $(k-1)$-cliques to k-cliques. Call this clique cover $C_k = \{c_1, c_2, \ldots, c_{\lfloor \frac{n}{k} \rfloor}\}$.

Let $c_1 = (x_1, x_2, \ldots, x_k)$ and $c_2 = (y_1, y_2, \ldots, y_k)$ be any two such k-cliques in C_k (note that $x_i, y_i \in A_i$). We say that c_1 *precedes* c_2 if x_i is connected to $y_1, \ldots, y_{i-1}$ for $1 \leq i \leq k$. c_1 *precedes* c_2 basically means that $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$ is a $(k-1)$-path. We say that $\{c_1, c_2\}$ is a *good* pair, if c_1 *precedes* c_2 and c_2 *precedes* c_1. By
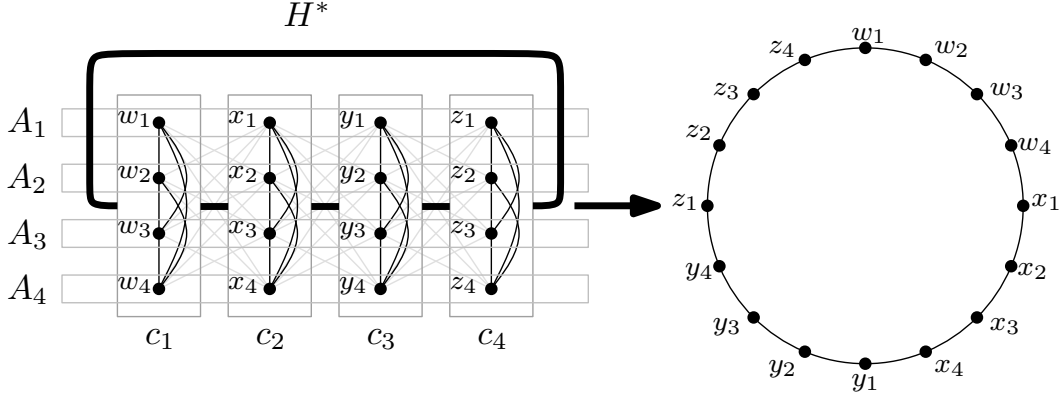
Figure 3.6: Unfolding the cliques in the order defined by H^* gives us the required power of a Hamiltonian cycle

the degree conditions above, any $c_i \in C_k$ makes a *good* pair with at least $(1 - \alpha^{1/5})|C_k|$ other cliques in C_k.

We define an auxiliary graph G^* in the following way: the vertex set of the graph G^* is $C_k = \{c_1, c_2, \ldots, c_{\lfloor \frac{n}{k} \rfloor}\}$ and $\{c_i, c_j\}$ is an edge in G^* if and only if $\{c_i, c_j\}$ is a *good* pair. By the above observation $\delta(G^*) > |C_k|/2$, hence there exists a Hamiltonian cycle H^* in G^*. If we take the cliques in the order of H^* and unfold individual cliques in the natural order defined by $A_1, A_2, \ldots, A_k$, it is easy to see that this gives us the $(k-1)^{th}$ power of a Hamiltonian cycle in G.

Handling the *Exceptional* Vertices: In this case we have some $E_j(i)$'s that are non-empty. The main idea is to reduce this case to the clean case where there are no *exceptional* vertices. Handling *bad* vertices is similar to (and simpler than) handling the *exceptional* vertices and we omit the details for that case.

Define X_i to be the set of all the vertices that are *exceptional* for A_i, that is, $X_i = \bigcup_{j=1}^{k} E_j(i)$.

Case 1: If $|X_i| > 1$, we would want to find paths of length 2 with endpoints in A_i and centers either inside A_i or at *exceptional* vertices in $E_j(i)$ for some j. For this purpose we note that $\delta(G|_{A_i \cup X_i}) \geq |X_i|$ by the minimum degree condition. Furthermore, since $d(A_i) < \alpha$ it follows that $\Delta(G|_{A_i \cup X_i}) \leq \alpha^{1/3}|A_i| + |X_i|$. Thus by Lemma 35 we can find more than $|X_i|$ vertex disjoint paths of length two. However, not all such paths may have their endpoints in A_i or their centers in X_i. This can easily be handled by noting
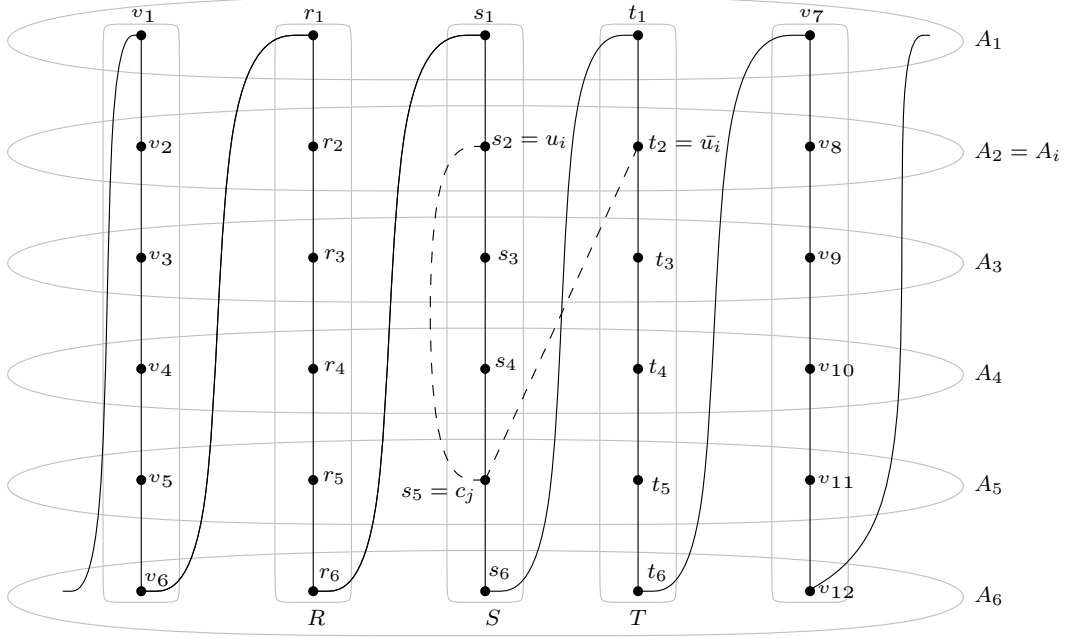
Figure 3.7: Finding the *exceptional clique* when $|X_i| > 1$

that any vertex in X_i may be switched with any of the vertices in A_i and the exchanged vertices become *exceptional* or not *bad* in their respective new sets. Therefore we may assume that there is a set, P_i, of $|X_i|$ disjoint paths of length 2, such that the two endpoints of each path are vertices in A_i and the center is an *exceptional* vertex in some $E_j(i)$.

We embed each of these paths in a distinct unit of three k-cliques as follows: let $(u_i, c_j, \bar{u}_i) \in P_i$ be one of the paths such that $u_i, \bar{u}_i \in A_i$, and $c_j \in A_j$. Select a clique in the natural order $S = (s_1, s_2, \ldots, s_k)$ such that $s_i = u_i$ and $s_j = c_j$, (so we use the $\{u_i, c_j\}$ edge). Now we select another clique $T = (t_1, t_2, \ldots, t_k)$ such that $t_i = \bar{u}_i$ and S *precedes* T. Then we select a clique $R = (r_1, r_2, \ldots, r_k)$ which *precedes* S.

It is easy to see that there are many cliques with the given restrictions such that only s_i is the *bad* vertex among all the three cliques. The cliques, unfolded in the order R, S, T, make a $(k-1)$-path. We replace this set of three k-cliques by a single k-clique (which we call an *exceptional clique*) with one vertex each from $A_1, \ldots, A_k$. The new vertex of the *exceptional clique* in A_m is connected to all the common neighbors of r_m and t_m. Since r_m and t_m are not *bad* vertices for $1 \leq m \leq k$, therefore these new

vertices have high degree in all the sets A_i where $i \neq m$. We deal with all the *exceptional* vertices in this manner and get *exceptional cliques* for each of them. In the remaining graph, we use the procedure described in the previous section to find a cover consisting of k-cliques and add the *exceptional cliques* to the cover. Then, as previously, we find a Hamiltonian cycle of the cliques in the cover and unfold the vertices in the cliques in the order defined by the cycle to get $(k-1)^{th}$ power of a Hamiltonian cycle. In Figure 3.7 the relevant portion in the final $(k-1)^{th}$ power of a Hamiltonian cycle looks as follows: $(\ldots, v_5, v_6, r_1, r_2, r_3, r_4, r_5, r_6, s_1, s_2, s_3, s_4, s_5, s_6, t_1, t_2, t_3, t_4, t_5, t_6, v_7, v_8, \ldots)$

Case 2: When $|X_i| = 1$, we may not be able to find the length 2 path as above, for
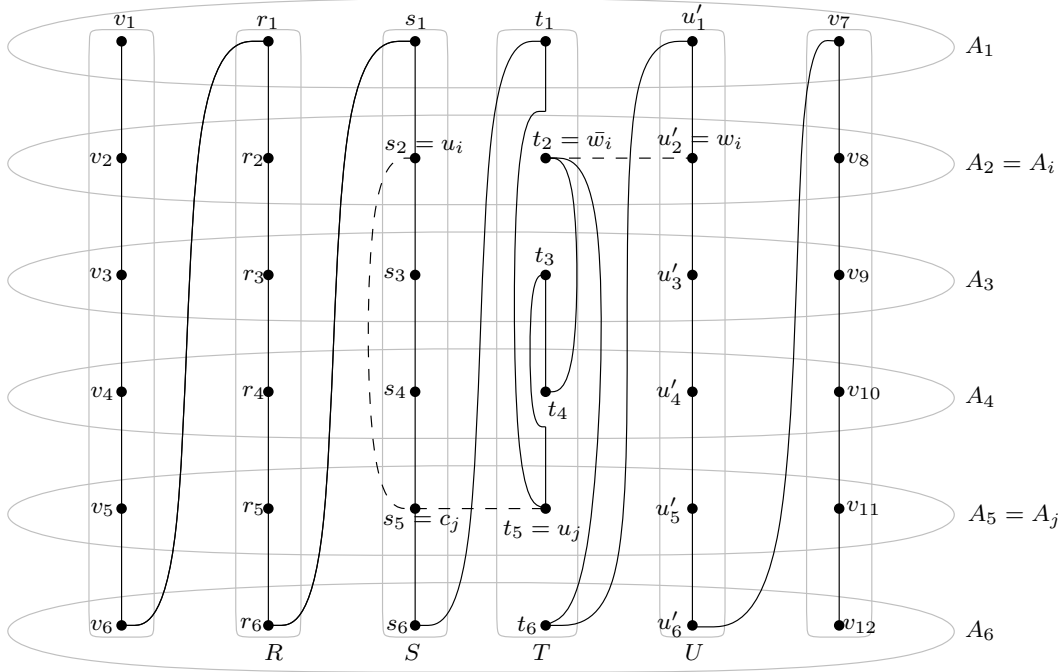

Figure 3.8: Finding the *exceptional clique* when $|X_i| = 1$

example, when the *exceptional* vertex $c_j \in E_j(i)$ for some j has exactly one neighbor $y \in A_i$ (it has to have at least one neighbor). Then all the vertices in A_i (except y) may have exactly one neighbor inside A_i. Note that by the minimum degree condition this case can only happen if $|X_i| = 1$. Therefore we find a path $p_i = (u_i, c_j, u_j)$ of length 2, where $u_i \in A_i$ and $c_j, u_j \in A_j$ such that c_j is an *exceptional* vertex for A_i. Additionally, we select an edge $\{w_i, \bar{w}_i\}$ inside A_i disjoint from all the length 2 paths p_j for $1 \leq j \leq k$.

Select a clique in the natural order $S = (s_1, s_2, \ldots, s_k)$ such that $s_i = u_i$ and $s_j = c_j$ so that we use the $\{u_i, c_j\}$ edge. Now select another clique $T = (t_1, t_2, \ldots, t_k)$ such that $t_i = w_i$ and $t_j = u_j$. However, we are going to consider T in the following order:

$$T' = (t_1, t_2, \ldots, t_{i-1}, t_j = u_j, t_{i+1}, \ldots, t_{j-1}, t_i = w_i, t_{j+1}, \ldots t_k)$$

(i.e. this order switches the positions of t_i and t_j). Note that T' utilizes the $\{c_j, u_j\}$ edge of p_i and that S *precedes* T'.

Next we find a clique U such that T' *precedes* U. Such a clique exists, because we can utilize the edge $\{w_i, \bar{w}_i\}$ and w_i and $\bar{w}_i$ are not *bad* vertices. There are many cliques $U = (u_1', \ldots, u_k')$, and $u_i' = \bar{w}_i$, such that T' *precedes* U. Then we find another clique R which *precedes* S. We replace this set of four k-cliques by a single k-clique (the *exceptional clique*) with one vertex each from $A_1, \ldots, A_k$. As previously, the new vertex of the *exceptional clique* in A_m is connected to all the common neighbors of r_m and u_m'. Since r_m and u_m' are not *bad* vertices for $1 \leq m \leq k$, therefore these new vertices have high degree in all the sets A_i where $i \neq m$. We deal with all the *exceptional* vertices in this manner and get *exceptional cliques* for each of them. We get $(k-1)^{th}$ power of a Hamiltonian cycle using the same method as was done in the previous cases. In Figure 3.8 the relevant portion in the final $(k-1)^{th}$ power of a Hamiltonian cycle looks as follows:

$$(\ldots, v_5, v_6, r_1, r_2, r_3, r_4, r_5, r_6, s_1, s_2, s_3, s_4, s_5, s_6,$$

$$t_1, t_5, t_3, t_4, t_2, t_6, u_1', u_2', u_3', u_4', u_5', u_6', v_7, v_8, \ldots)$$

G has less than k extremal sets

We first assume that $A_1, A_2, \ldots, A_l$ are the extremal sets where $l < k$ and we let $A = \bigcup_{i=1}^{l} A_i$ and $B = V(G) \setminus A$. We say that $v \in B$ is *bad* if $deg_A(v) \leq (1 - \alpha^{1/3})|A|$. While the *bad* vertices in A_i's are defined exactly as before. Then

$$\delta(G|_B) \geq \left(\frac{k-1}{k}\right)n - \left(\frac{l}{k}\right)n \geq \left(\frac{k-l-1}{k}\right)n \geq \left(\frac{k-l-1}{k-l}\right)|B|$$

and since there is no extremal set $A' \subset B$ with $|A'| = \lfloor \frac{n}{k} \rfloor = \lfloor (\frac{1}{k-l})|B| \rfloor$ therefore, $G|_B$ does not satisfy the extremal condition.

The Clean Case: For now we assume that there are no *bad* or *exceptional* vertices. By the non-extremality of $G|_B$ and its minimum degree $\delta(G|_B)$, using the procedure given in previous sections on the non-extremal case, we can find $(k-l-1)^{th}$ power of a Hamiltonian cycle $H = (p_1, p_2, \ldots, p_{|B|})$ in B. We will insert l vertices after every $k-l$ vertices in H such that we get $(k-1)^{th}$ power of a Hamiltonian cycle.

For this purpose we divide H into consecutive intervals of $k-l$ vertices each. We define $B' = \{b_1, b_2, \ldots, b_{\lfloor \frac{n}{k} \rfloor}\}$ as follows: b_1 corresponds to $\{p_1, p_2, \ldots, p_{(k-l)}\}$; b_2 corresponds to $\{p_{(k-l)+1}, p_{(k-l)+2}, \ldots, p_{2(k-l)}\}$ and so on, and $b_{\lfloor \frac{n}{k} \rfloor}$ corresponds to the set $\{p_{(\lfloor \frac{n}{k} \rfloor - 1)(k-l)+1}, \ldots, p_{|B|}\}$. We also have that $|A_1| = |A_2| = \cdots = |A_l| = |B'| = \lfloor \frac{n}{k} \rfloor$.
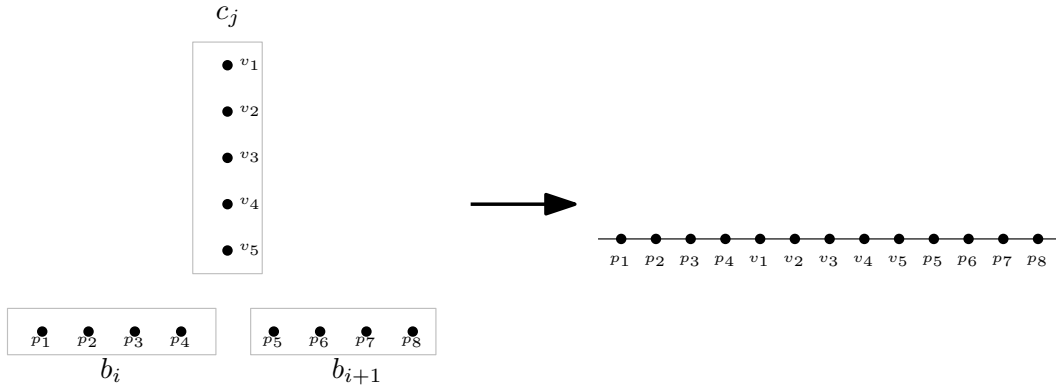

Figure 3.9: Unfolding the vertices in a *compatible* triplet gives us a $(k-1)$-path

Next we define an auxiliary graph G^* where $V(G^*) = A \cup \{b_1, b_2, \ldots, b_{\lfloor \frac{n}{k} \rfloor}\}$. If $u, v \in A$ form an edge in G then $\{u, v\}$ is also in $E(G^*)$. Additionally, every b_i has edges to all the vertices in

$$N_G(p_{(i-1)(k-l)}, p_{(i-1)(k-l)+1}, \ldots, p_{i(k-l)}) \cap A.$$

That is, b_i is connected to all the common neighbors in A of the vertices it represents. We cover $A_1 \cup \cdots \cup A_l$ with l-cliques, each of which uses a vertex from each A_i. Since for each $v \in A_i$, we have that $deg_{A_j}(v) \geq (1 - \alpha^{1/3})|A_j|$ for all $j \neq i$ and there are no *bad* vertices, it is straightforward to find l-cliques by a simple greedy procedure that uses the König-Hall theorem. We call this clique cover C_l.

Let $c_1 = (y_1, y_2, \ldots, y_l)$ and $c_2 = (z_1, z_2, \ldots, z_l)$ be any two cliques in C_l. Note that by the degree conditions above any given $c_i \in C_l$ is *good* for at least $(1 - \alpha^{1/4})|C_k|$
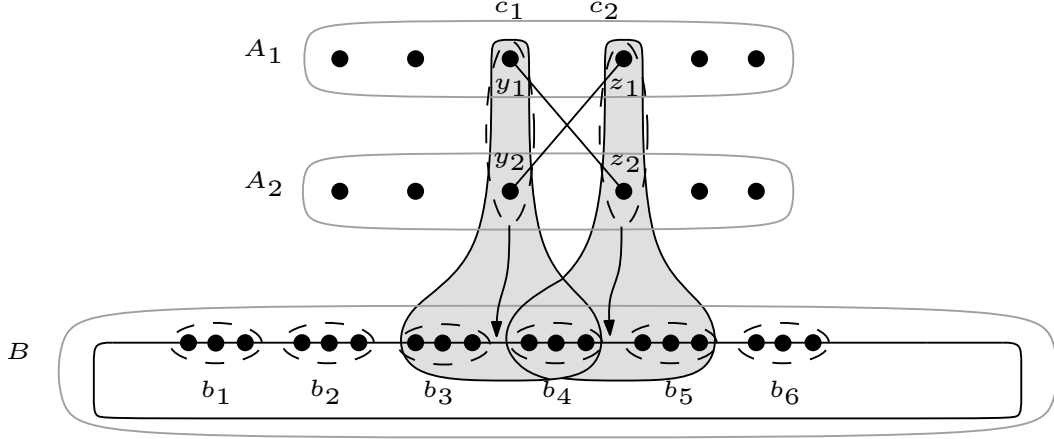
Figure 3.10: The shaded region indicates the *compatible* triplets. The l-cliques c_1 and c_2 are a *good* pair. Here $k = 5$ and $l = 2$

l-cliques in C_l. Furthermore, we say that an adjacent pair $\{b_i, b_{i+1}\}$ and a $c_j \in C_l$ is a *compatible* triplet if the vertices in b_i, c_j, b_{i+1} make a $(k-1)$-path when unfolded in the natural order. Since for every pair $\{b_i, b_{i+1}\}$ in B', $N_{G^*}(b_i, b_{i+1}) \cap A_i \geq (1 - \alpha^{1/4})|A_i|$ it follows that every adjacent pair of vertices in B' forms a *compatible* triplet with at least $(1 - \alpha^{1/5})$-fraction of the l-cliques in C_l. So we match every adjacent pair $\{b_i, b_{i+1}\}$, such that i is odd, with an l-clique, $c_j \in C_l$, to make *compatible* triplets greedily.

For the leftover adjacent pairs $\{b_i, b_{i+1}\}$ where i is even, we also match them to l-cliques with the added restriction that $\{b_i, b_{i+1}\}$ form a *compatible* triplet with an unused l-clique c_j and c_j is *good* for c_{j-1} and c_{j+1}. From the observations above the König-Hall criteria is easily satisfied hence we may find the required matching. We can now unfold the *compatible* triplets giving us the required $(k-1)^{th}$ power of the Hamiltonian cycle.

Handling the *Exceptional* Vertices: Most of the techniques and methods involved in this case have already been covered in the previous sections. We will refer to these methods extensively in solving this case. There is a small set of *exceptional* vertices $\bigcup_{i=1}^{l} E_B(i) = X_B \subset B$ and some $E_j(i)$'s are non-empty (handling *bad* vertices is simpler and the details are omitted). This time we define X_i to be the set of all the vertices in A that are *exceptional* for A_i, that is, $X_i = \bigcup_{j=1}^{l} E_j(i)$.

We find length 2 paths (and one more required edge if needed) and units of three

or four l-cliques in A (as was done for *exceptional* vertices in Section 3.5.1) for each of the *exceptional* vertices in $X_i \cup X_B$. However, this time we do not replace them by *exceptional l-cliques*. We cover the rest of A with l-cliques by a greedy approach that uses the König-Hall Theorem. Since B is non-extremal we find can an optimal cover in $B^* = B \setminus X_B$ that consists of balanced $K_{k-l+1}(O(\log n))$'s, $K_{k-l}(O(\log n))$'s and a small set of left over vertices, represented by $\mathcal{C}, \mathcal{K}$ and $\mathcal{I}$. This is possible since even after the removal of the *exceptional* vertices the minimum degree of the vertices in $G|_{B^*}$ is still greater than $(1 - \frac{1}{k-l-1} + \epsilon)|B^*|$ for some small $\epsilon > 0$.
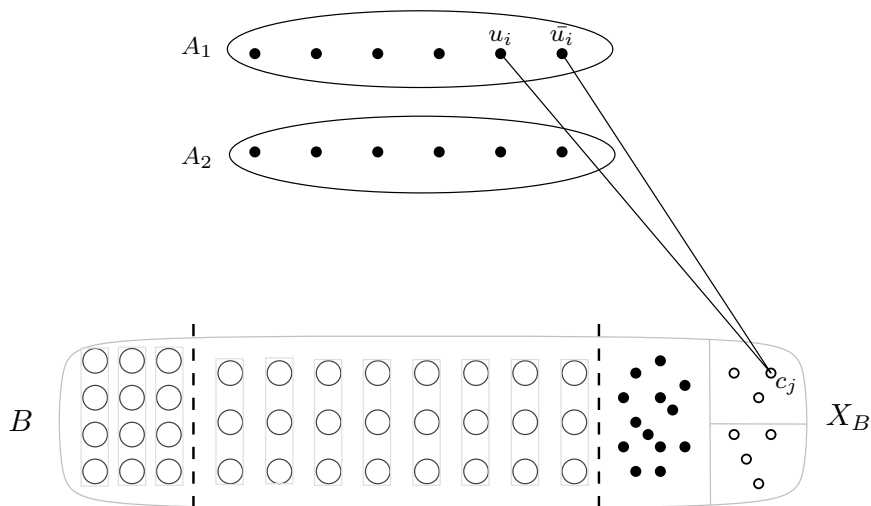

Figure 3.11: Handling the exceptional vertices in the non-clean case

We start making the $(k - l - 1)^{th}$ power of a Hamiltonian cycle in B^* using the procedure in Section 3.4 on the non-extremal case. However, we note that since all the vertices in X_B are almost completely connected to B^*, any such vertex is completely connected to almost all the cliques in $\mathcal{C}$. Therefore we insert the vertices in X_B into the $(k - l)$-paths that are constructed within every $K_{k-l+1}(O(\log n))$ in $\mathcal{C}$ so that each such vertex is far apart (at least k distance on the $(k - l)$-path) from each other. Due to the minimum degree condition, we can thus assign the *exceptional* vertices in X_B to cliques in $\mathcal{C}$ in a balanced manner by a greedy approach.

After we have made the $(k-l-1)^{th}$ power of a Hamiltonian cycle, $H' = (p_1, \ldots, p_{|B|})$ in B, we divide the cycle into consecutive intervals of $k - l$ vertices as before. As was previously done, we define $B' = \{b_1, b_2, \ldots, b_{\lfloor \frac{n}{k} \rfloor}\}$ as follows: b_1 corresponds to
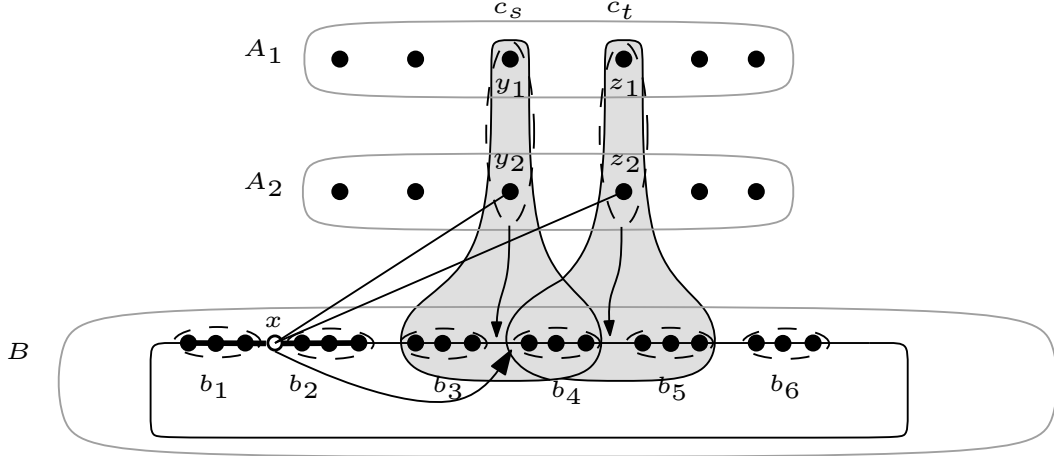
Figure 3.12: The shaded region indicates where the l-cliques from A will be inserted into the $(k - l - 1)^{th}$ power of a Hamiltonian cycle in B. The heavy edges represent the portion of the path that is actually a $(k - l)$-path. Here $k = 5$ and $l = 2$.

$\{p_1, p_2, \ldots, p_{(k-l)}\}$; b_2 corresponds to $\{p_{(k-l)+1}, p_{(k-l)+2}, \ldots, p_{2(k-l)}\}$ and so on, and $b_{\lfloor \frac{n}{k} \rfloor}$ corresponds to the set $\{p_{(\lfloor \frac{n}{k} \rfloor - 1)(k-l)+1}, \ldots, p_{|B|}\}$. We note that all the vertices from X_B are located in those sections of H' that are actually $(k - l)$-paths and due to the way we distributed these vertices, at most one vertex can appear in any single interval b_i of H'.

Suppose an exceptional vertex $x \in E_B(i) \subset X_B$, and $|E_B(i)| > 1$ (the case $|E_B(i)| = 1$ is dealt with analogously) is the center of the length 2 path that has its endpoints in the l-cliques c_s and c_t, then we find two compatible triplets $\{b_{i-1}, c_s, b_i\}$ and $\{b_i, c_t, b_{i+1}\}$. We can move x from its original position in H' to before the b_i interval. This relocation is possible since deleting x from its original position still leaves behind a $(k-l-1)$-path and since x is connected to all of B, we can find a pair of consecutive intervals b_{i-1}, b_i and b_i, b_{i+1} that form compatible triplets with c_s and c_t and at the same time form a $(k - l - 1)$-path when taken in the order of $b_{i-1}, c_s, x, b_i, c_t, b_{i+1}$. All the exceptional vertices in X_B are thus associated with two (or three) l-cliques in A and relocated in H'.

Next we find consecutive pairs of intervals b_{j-1}, b_j and b_j, b_{j+1} for each of the exceptional vertices in $x' \in X_i$ (if there is only one vertex in X_i we have a similar procedure in which x_i is associated with three l-cliques in A), for every i, such that they form

compatible triplets with c_s and c_t (the associated l-cliques of x'). This is again possible because of the high minimum degree of the graph. Finally we find *compatible triplets* for all of the remaining consecutive pairs of intervals in H' in two stages as we did in the clean case. To get the final $(k-1)^{th}$ power of a Hamiltonian cycle we unfold all the *compatible triplets*.

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