CONTROLLED GEOMETRY VIA VOLUMES ON ALEXANDROV SPACES

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A dissertation submitted to the

Graduate School—New Brunswick

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

Xiaochun Rong

and approved by

New Brunswick, New Jersey

October, 2010

ABSTRACT OF THE DISSERTATION

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In this thesis we recall the basic definitions and properties for Alexandrov space and describe two geometry phenomenons controlled via volume (Hausdorff measure or rough volume) conditions. (1) For a path in $X \in \text{Alex}^n(\kappa)$ (the compact n-dimensional Alexandrov spaces with curvature $\geq \kappa$.), the sum of the length and the turning angle is bounded from below in terms of κ, n, diameter and volume of X. This generalizes a basic estimate by Cheeger on the length of a closed geodesic in closed Riemannian manifold ([Ch]). (2) Let Σ_p be the space of directions at $p \in X$ and the pointed radius $R = \inf\{r : X \subset B_r(p)\}$. If $X \in \text{Alex}^n(\kappa)$, then $\text{vol}(X) \leq \text{vol}(C_\kappa^R(\Sigma_p))$, where $C_\kappa^R(\Sigma_p)$ is the metric R-ball at the vertex in the κ-suspension $C_\kappa(\Sigma_p)$. We give an isometric classification of $X \in \text{Alex}^n(\kappa)$ whose volume achieves the maximal possible value $\text{vol}(C_\kappa^R(\Sigma_p))$. We also determine homeomorphic types of such X when X is a topological manifold. These results are natural extension of K. Grove and P. Petersen's work in 1992 ([GP 92]).

Acknowledgements

I would like to formally thank:

Professor *Xiaochun Rong*, my advisor, for his hard work and guidance through this wonderful journey and for believing my abilities. You have always been a great source of insight idea, motivation and inspiration. Thank you so much for the encouragement at every step.

Professor *Feng Luo*, *Zhangchao Han* and *Jianguo Cao*, for serving on my thesis committee. You have represented me an amazing world of mathematics. Thank you for your support and guidance.

Professor *Jeff Cheeger* and *Karsten Grove*, for the great discussion and help.

Professor *Xiaojun Huang* and *Yanyan Li*, for the wonderful courses and encouragement you gave me.

Shicheng Xu and *Brian Manning*, my geometry partner. I learned a lot from you.

Biao Yin, Yuan Yuan, Yuan Zhang, Ming Shi and all the Chinese math students in Rutgers, for warming the Chinese family in U.S. through this journey.

My parents, for your never-ending love and support throughout all my life.

My girl friend, *Xiaohua Zhuang*, for always being there for me through this journey.

Dedication

This thesis is dedicated to my great parents, *Yihua Li* and *Rong Wang*, who have raised and taught me to be the person I am today. You have been with me every step of the way, through good times and bad. Thank you for all the unconditional love and support. I love you!

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Chapter 1

Introduction

An n-dimensional Alexandrov space X with curvature $\geq \kappa$ (denoted as $X \in \text{Alex}^n(\kappa)$) is an n-dimensional complete length metric space such that any geodesic triangle looks 'fatter' than a comparison triangle in S^2_κ, the 2-dimensional space form of constant curvature κ. A basic motivation for studying Alexandrov spaces is that the Gromov-Hausdorff limit of a converging sequence of Riemannian manifolds with sectional curvature $\geq \kappa$ is an Alexandrov space with the same curvature lower bound, but an Alexandrov space in general may have geometrical or topological singularities. There followed throughout the 90's an explosion of work starting with a seminal paper [BGP] centered on Alexandrov geometry. Many important results have been obtained in understanding both local and global structures of an Alexandrov space and in applications ([BBI], [BGP], [Kap 07], [Pet 07] and references within). Because tools from Alexandrov geometry played a significant role in Perel'man's proof of the famous Poincare's conjecture, Alexandrov geometry has been getting a lot of attention lately.

The theory of Alexandrov geometry is not easy to apprehend, partially because it deals with the metric space with possibly both geometrical and topological singularities, and thus most conventional tools from differential geometry may not be applied. In *Chapter 2* we recall the definitions and basic prosperities on Alexandrov spaces. To make this thesis more self contained, we give proofs for most of the theorems, while the rest of them are referred to [BGP].

The main body of this thesis is in Chapter 3 and 4, in which we show some new results via the volume conditions. Theses are joint work with Xiaochun Rong and will be published in [LR 09, 10].

In *Chapter 3* we consider a loop c in $X \in \mathrm{Alex}^n(\kappa)$. There are two basic geometric invariants for a continuous curve, the length and the turning angle (which measures the closeness from being a geodesic, the definition can be found in Definition 3.0.5). For example, an m-broken geodesic γ_m has a finite turning angle $\Theta(\gamma_m) = \sum_{i=2}^m \theta_i$, where θ_i is the difference between π and the angle of the adjacent broken geodesics. If X is a Riemannian manifold, then any C^2-curve c on X has the turning angle $\Theta(c) = \int_0^1 |\nabla_{c'} c'| dt$. Let Haus_n denote the "normalized" n-dimensional Hausdorff measure such that $\mathrm{Haus}_n(I^n) = 1$, where I^n is the unit n-cube in $\mathbb{R}^n$. Let $\mathrm{sn}_\kappa(r) = \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} r$, r, $\frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa} r$ for $\kappa > 0$, $= 0$, < 0 respectively. For $r > 0$, let $r_* = \{t : \mathrm{sn}_\kappa(t)$ achieves its maxima in $[0, r]\}$, i.e. $r_* = \frac{\pi}{2\sqrt{\kappa}}$ for the case $\kappa > 0$ and $r > \frac{\pi}{2\sqrt{\kappa}}$; $r_* = r$ otherwise (an analog definition is applied for d_* given $d > 0$). One of our main results is the following estimate.

Theorem 3.A. *Let $X \in Alex^n(\kappa)$ $(n \geq 2)$, and let c be a loop at $p \in X$ and $c \subset B_r(p)$. Then the length $L(c)$ and the turning angle $\Theta(c)$ satisfy:*

$$L(c) + (n-1)r \cdot \Theta(c) \geq \frac{(n-1)Haus_n(B_r(p))}{vol(S_1^{n-2}) \cdot sn_\kappa^{n-1}(r_*)}.$$

This Theorem indicates that, for any loop, the sum of the length and the turning angle is bounded from below in terms of κ, the dimension, the radius and Hausdorff measure of a metric ball containing c. We also give an application on local injectivity radius estimate (see Theorem 3.B). When $B_r(p) = X$ is a closed Riemannian manifold, this generalizes a basic estimate by Cheeger on the length of a closed geodesic in [Ch]. (Note that when U is an open subset of an n-dimensional Riemannian manifold, $\mathrm{Haus}_n(U) = \mathrm{vol}(U)$) This is useful when one estimates the injectivity radius at a point where it is realized by a geodesic loop (see the following discussion).

Theorem 1.0.1 (J. Cheeger)**.** *Let M be a closed n-manifold $(n \geq 2)$ with sectional curvature $sec_M \geq \kappa$ $(\kappa \leq 0)$. For any closed geodesic γ,*

$$L(\gamma) \geq \frac{(n-1)vol(M)}{vol(S_1^{n-2}) \cdot sn_\kappa^{n-1}(diam(M))}.$$

The lower bound in Theorem 3.A is optimal in all dimensions; the inequality becomes an equality when c is a great circle in an n-dimensional spherical space form (note that $\mathrm{vol}(S_1^n) = \frac{2\pi}{n-1} \cdot \mathrm{vol}(S_1^{n-2})$, $n \geq 2$).

Corollary 1.0.2. *Let $X \in Alex^n(\kappa)$ $(n \geq 2)$, $diam(X) \leq d$ and $Haus_n(X) \geq v > 0$. If $c \subset X$ is a loop, then the sum of $L(c)$ and $\Theta(c)$ is bounded below by a constant,*

$$L(c) + \Theta(c) \geq c(n, k, d, v) > 0,$$

where $c(n, k, d, v) = \frac{v \cdot \min\{1, [(n-1)d]^{-1}\}}{vol(S_1^{n-2}) \cdot sn_\kappa^{n-1}(d_*)}$.

Corollary 1.0.2 reveals an interesting geometric property on the loop space over a compact Alexandrov space X: any short loop has turning angle not small, or equivalently, any loop with small turning angle is not short. For instance, given $0 \leq \epsilon < 1$, we call a loop, c, ϵ-close to a geodesic, if $\Theta(c) \leq \epsilon \cdot \frac{\mathrm{Haus}_n(X)}{d \cdot \mathrm{vol}(S_1^{n-2}) \cdot \mathrm{sn}_\kappa^{n-1}(d_*)}$. Theorem 3.A implies the following:

Corollary 1.0.3. *Let $X \in Alex^n(\kappa)$ $(n \geq 2)$. If c is a loop ϵ-close to a geodesic, then*

$$L(c) \geq (1 - \epsilon) \cdot \frac{(n-1) Haus_n(X)}{vol(S_1^{n-2}) \cdot sn_\kappa^{n-1}(d_*)}.$$

Theorem 3.A can be useful in analyzing local geometry concerning the *injectivity radius* of a point p in a complete Riemannian manifold M (e.g., $\sec_M$ has no upper bound). If $q \in M$ is a point such that $|pq| = \mathrm{injrad}_p < \infty$ (the injectivity radius at p), then either q is a conjugate point to p or there is a geodesic loop γ at p passing through q. In the later case, $2 \cdot \mathrm{injrad}_p = L(\gamma)$ and $\Theta(\gamma)$ satisfy Theorem 3.A. In the former case (e.g, no geodesic loop with $L(\gamma) = 2 \cdot \mathrm{injrad}_p$), using Theorem 3.A we can establish a similar relation.

To have the discussion also including an Alexandrov space X, we need the following notions: we call a point $p \in X$ a regular point, if there is a non-trivial minimal geodesic along any direction in the space of directions at p, Σ_p. As in the Riemannian case, we define the *cut locus*, C_p, at a regular point as the collection of points $q \in X$ such that q is the furthest point on a radial curve from p with arc length equal to $|pq|$. Let $q \in C_p$ such that $|pq| = |pC_p|$, call the injectivity radius of p, and denoted by injrad_p. Clearly, the gradient-exponential map is a homeomorphism on the ball of radius $< \mathrm{injrad}_p$. Let $\mathrm{geod}(p, q) = \{[pq]\}$ denote the set of minimal geodesic $[pq]$ from p to q. We call the following number in $[0, 2\pi]$,

$$\theta_p = 2\pi - \sup_{q \in C_p, |pq| = \mathrm{injrad}_p} \{\measuredangle(\dot\gamma_1(0), \dot\gamma_2(0)) + \measuredangle(-\dot\gamma_1(1), -\dot\gamma_2(1)),\ \gamma_1, \gamma_2 \in \mathrm{geod}(p, q)\},$$

the *geodesic angle* of p. Observe that $\theta_p = 0$ if and only if $2 \cdot \mathrm{injrad}_p$ is realized by the length of a closed geodesic at p, and $\theta_p = 2\pi$[1] if and only if there is a unique minimal geodesic $[pq]$ (e.g., a flat cone with angle $< \frac{\pi}{2}$, and p is close to the vertex). Hence, θ_p measures the existence of such a closed geodesic at p.

A consequence of Theorem 3.A is:

Theorem 3.B. *Let X be an n-dimensional Alexandrov space ($n \geq 2$) with curv $\geq \kappa$. If $p \in X$ is a regular point, then for any $r >$ injrad$_p$,*

$$injrad_p \geq \frac{n-1}{2} \cdot \left[\frac{Haus_n(B_r(p))}{vol(S_1^{n-2}) sn_\kappa^{n-1}(r_*)} - r \cdot \theta_p \right].$$

Theorem 3.B provides a local estimate for injrad_p in terms of local geometry when θ_p is relatively small (e.g., $\theta_p < \frac{\mathrm{Haus}_n(B_r(p))}{r \cdot \mathrm{vol}(S_1^{n-2}) \cdot \mathrm{sn}_\kappa^{n-1}(r)}$). On the other hand, θ_p not relatively small indicates that geodesics from p to q are confined in a narrow region.

Theorem 3.A substantially improves an analog of Theorem 1.0.1 in Alexandrov geometry by [BGP] (see Corollary 1.0.4 and Proposition 2.7.4), which gives an implicit lower bound on the length of an almost closed geodesic (when m fixed and $\delta \to 0$, δ_1 cannot be very small; see Remark 8.7 in [BGP]), implicitly in terms of k, n, d and the rough volume $V_{r_n}(X)$. However, because $\chi_m(\delta_1, \delta) \to \infty$ as $m \to \infty$, Proposition 2.7.4 fails to imply a lower bound on the length of an m-broken geodesic loop (of length, say one) with m large while $m\delta$ are very small (so both δ_1 and δ are small).

In view of the above, it is natural to ask if a similar estimate in Theorem 3.A holds in terms of the rough volume. First, the rough volume is not a measure since it's not countably additive (e.g., rationales in $[0, 1]$ has rough volume 1 while a point has rough volume 0). However, we find the equivalency of the two measures on open subsets (see Remark 3.3.7).

Theorem 3.C. *Let $X \in Alex^n(\kappa)$. Then*

$$V_{r_n}(X) = c(n) \cdot Haus_n(X),$$

where $c(n) = \frac{V_{r_n}(I^n)}{Haus_n(I^n)} = V_{r_n}(I^n)$ is a constant depending only on the dimension, and I^n denotes the Euclidean unit n-cube.

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[1] When X is a Riemannian manifold, $\theta_p = 2\pi$ implies that q is a conjugate point of p.

Theorem 3.C can be useful in practice; once proving a result involving $V_{r_n}(X)$ (which is easier to estimate than $\mathrm{Haus}_n(X)$), one gets automatically a result in terms of $\mathrm{Haus}_n(X)$. As for the value of $c(n)$, except $c(1) = 1$ and $c(2) \geq \frac{2}{\sqrt{3}}$, it seems hard to have an estimate in general.

A consequence of Corollary 1.0.3 and Theorem 3.C is:

Corollary 1.0.4. *Let X be a compact n-dimensional Alexandrov space ($n \geq 2$) with curvature $\geq \kappa$. If c is a loop ϵ-close to a geodesic, then*

$$L(c) \geq (1-\epsilon) \cdot \frac{V_{r_n}(X)}{C(n) \cdot sn_\kappa^{n-1}(d_*)},$$

where $C(n) = \frac{c(n) \cdot vol(S_1^{n-2})}{n-1}$ and $c(n)$ is the constant in Theorem 3.C.

Comparing Corollary 1.0.4 with Proposition 2.7.4; the former gives an explicit sharp estimate and applies to all m-broken geodesic loops with $m\delta$ relatively small.

In *Chapter 4* we describe a rigidity/almost rigidity phenomenon in Alexandrov geometry which is a natural extension of K. Grove and P. Petersen's work in 1992 ([GP 92]). Let M be a Riemannian manifold with sectional curvature $\geq \kappa$ and the radius of M be $\mathrm{rad}(M) = \inf\{r : \exists p \in X, X \subset B_r(p)\}$, then $\mathrm{vol}(M) \leq \mathrm{vol}(B_r(S_\kappa^n))$, where $B_r(S_\kappa^n)$ is the r-ball in the simply connected space S_κ^n with constant curvature κ. In the rest of the introduction we will assume $r \leq \frac{\pi}{2\sqrt{\kappa}}$ or $r = \frac{\pi}{\sqrt{\kappa}}$ for the case $\kappa > 0$ (because otherwise the above volume estimate is not optimal). For a sequence of M_i reaches the above maximal volume, the following theorem has been proved by Grove and Petersen.

Theorem 1.0.5. *[Grove-Petersen] Let M_i be a sequence of Riemannian manifold with sectional curvature $\geq \kappa$. Assume that $rad(M_i) \leq r$ and $vol(M_i) \to vol(B_r(S_\kappa^n))$. Then there is a subsequence of M_i which Gromove-Hausdorff converges to a metric space X, where $X = \bar{B}_r(S_\kappa^n)/x \sim \phi(x)$, and $\phi : \partial \bar{B}_r(S_\kappa^n) \to \partial \bar{B}_r(S_\kappa^n)$ is an antipodal map or a reflection by a totally geodesic hyperplane. Moreover, M_i is homeomorphic to S_1^n or $\mathbb{R}P^n$ for i large enough.*

On Alexandrov spaces, given $\Sigma \in \mathrm{Alex}^{n-1}(1)$, let $\mathcal{M}_\kappa^r(\Sigma) = \{X \in \mathrm{Alex}^n(\kappa) \mid \exists\, p \in X,\ \Sigma_p = \Sigma,\ \bar{B}_r(p) = X\}$, where Σ_p is the space of directions of p, namely, the equivalent class of geodesics from p (see Chapter 2.6). By Toponogov triangle comparison, it's not difficult to see that for $X \in \mathcal{M}^r_\kappa(\Sigma)$, $\mathrm{vol}(X) \le \mathrm{vol}(\bar{C}^r_\kappa(\Sigma_p))$, where $\bar{C}^r_\kappa(\Sigma)$ denotes the closed r-ball centered at the vertex of the κ-suspension $C_\kappa(\Sigma)$ (see Chapter 2.3.1) and 'vol' denotes the n-dimensional Hausdorff measure or the rough volume. The following is our result which gives an isometric classification for $X \in \mathrm{Alex}^n(\kappa)$ whose volume achieves the maxima above.

Theorem 4.A (relatively maximal volume)**.** *Let $X \in \mathcal{M}^r_\kappa(\Sigma)$. Then vol$(X) =$ vol$(\bar{C}^r_\kappa(\Sigma_p))$, if and only if both of the following are satisfied*

(1) $\kappa \le 0$ or $\kappa > 0$, $r \le \frac{\pi}{2\sqrt{\kappa}}$, $r = \frac{\pi}{\sqrt{\kappa}}$.

(2) X is isometric to $\bar{C}^r_\kappa(\Sigma))/x \sim f(x)$, where $f : \Sigma \times \{r\} \to \Sigma \times \{r\}$ is an isometric involution (which can be trivial).

A significant difference in Theorem 4.A than the classical volume rigidity discussion (using S^n_κ as the model space) is that, the isometric types rely on an arbitrary space of direction Σ (which has infinitely many types). Thereafter the isometric classification of Alexandrov spaces in $\mathcal{M}^r_\kappa(\Sigma)$ with relatively maximal volume reduces to a classification for equivariant isometric $\mathbb{Z}_2$-actions on Σ. When let $\Sigma = S^{n-1}_1$ and X be a limit of Riemannian manifolds, Theorem 4.A implies the rigidity part in Theorem 1.0.5 (the almost rigidity part can be implied by letting $\Sigma = S^{n-1}_1$ in Theorem 4.B and 4.C). When let $\Sigma = S^{n-1}_1$ and $r = \frac{\pi}{\sqrt{\kappa}}$ for $\kappa > 0$, Theorem 4.A implies the maximal volume rigidity theorem (see Theorem 2.7.5, which takes S^n_κ as the uniform model space) in Alexandrov geometry, which generalizes the maximal volume rigidity theorem in Riemannian geometry with an analogue conditions. In the proof of the "isometric involution", because of the lack of smooth structure, our proof relies on the elementary triangle comparisons. The significant difference in the proof of "open ball isometry" will be discussed in the comments of Theorem 4.D.

We also determine the homeomorphic types when $X \in \mathcal{M}^r_\kappa(\Sigma)$ achieves the maximal volume and is a topological manifold (such X, rather than a limit of Riemannian manifolds, may have large singularities).

Theorem 4.B. *Let $X \in \mathcal{M}_\kappa^r(\Sigma)$ with vol$(X) = v(\Sigma, \kappa, r)$. If X is a closed topological manifold, then X is homeomorphic to the unit sphere S_1^n or a real projective space $\mathbb{R}P^n$.*

An interesting point in Theorem 4.B is that Σ may not be a topological manifold (in particular, S_1^{n-1}), but its suspension $C_1(\Sigma)$ (which has the relatively maximal volume) is homeomorphic to a sphere (e.g. Σ is a spherical suspension of a homology 3-sphere). Indeed, we show that at any topological point $p \in X \in \text{Alex}^n(\kappa)$, Σ_p is homotopically equivalent to a sphere (see Lemma 4.4.1).

Using Theorem 4.A and the Perel'man's stability theorem, we obtain a homeomorphic classification for the Alexandrov spaces whose volumes are almost relatively maximal.

Theorem 4.C (Almost relatively maximal volume)**.** *There exists a constant $\epsilon = \epsilon(\Sigma, n, \kappa, r) > 0$ such that if $X \in \mathcal{M}_\kappa^r(\Sigma)$ satisfies that vol$(X) \geq v(\Sigma, \kappa, r) - \epsilon$, then X is homeomorphic to some element some element described in Theorem 4.A (2).*

A basic tool we developed in proving our rigidity results is a pointed version of Bishop-Gromov's relative volume comparison with open ball rigidity in Alexandrov geometry.

Theorem 4.D. *Let $X \in Alex^n(\kappa)$. For any $p \in X$, and $0 < t \leq r$,*

$$\frac{vol(B_t(p))}{vol(C_\kappa^t(\Sigma_p))} \geq \frac{vol(B_r(p))}{vol(C_\kappa^r(\Sigma_p))}, \qquad \lim_{t \to 0} \frac{vol(B_t(p))}{vol(C_\kappa^t(\Sigma_p))} = 1,$$

and " $=$ " holds if and only if the open metric ball $B_r(p)$ is isometric to $C_\kappa^r(\Sigma_p)$ with respect to the intrinsic distance.

When let $\Sigma = S_1^{n-1}$, this will imply Theorem 2.7.5. However, using an arbitrary Σ instead of S_1^{n-1} will cause a significant diffulty. We observe that the proof for rigidity Theorem 2.7.5 mentioned in [BGP], relies on an induction applied to the property that each cross section $S_r = \{x \in X : |px| = r\}$ achieves the maximal volume of $S_1^{n-1} \times \{t\}$, on which the maximal volume rigidity holds. (This method can be viewed as a singular case for the proof in Riemannian geometry.) In our case, the cross section can only be compared to the model space $\Sigma \times \{t\}$, on which rigidity may not holds. Comparing to the

Bishop-Gromov relative volume comparison in Alexandrov geometry (and Riemannian geometry) (see Theorem 2.7.6), the monotonicity for volume ratio is essentially same (a verification is not trivial, see Proposition 4.1.7). In our proof for the monotonicity for the volume ratio, we take an elementary (calculus) approach which relies on a right partition for applying triangle comparison; in particular it does not rely on a co-area formula for Hausdorff measure which is used in the proof of Theorem 2.7.6 in [BGP]. As a consequence, we show that the absolute rigidity is equivalent to the relative rigidity (respective to the radius, see Lemma 4.2.2).

Chapter 2

Definitions and Basic Properties

Our main goal in this Chapter is to recall the definitions of Gromov-Hausdorff distance, Alexandrov space and dimension, volume, burst point, space of directions, ect. We give proofs for most of the properties, while the rest of them are referred to [BGP]. We will use these properties in Chapter 3 and 4 frequently.

2.1 Gromov-Hausdorff distance

Let X, Y be bounded subsets in a metric space (Z, d). We let

$$d(X,Y) = \inf\{d(x,y) : x \in X, y \in Y\},$$

$$B_\epsilon(X) = \{x \in Z : d(x,X) < \epsilon\},$$

$$d_H(X,Y) = \inf\{\epsilon : X \subset B_\epsilon(X), Y \subset B_\epsilon(Y)\}.$$

It's clear that $d(X,Y)$ is small if a pair of points are close to each other; $d_H(X,Y)$ is small if X and Y almost cover each other, i.e. each point in X is close to some point in Y and vice versa.

Definition 2.1.1 (Gromov-Hausdorff distance)**.** Let X and Y be metric spaces of finite diameter. The Gromov-Hausdorff distance (GH-distance) of X and Y is

$$d_{GH}(X,Y) = \inf_{(Z,d)}\{d_H(X,Y) : \ X \text{ and } Y \text{ are isometrically embedded into } (Z,d)\}.$$

Let $\mathcal{M}et$ be the collection of isometric classes of compact metric spaces. By the following proposition, $(\mathcal{M}et, d_{GH})$ is a complete metric space, where $d_{GH}(\ ,\ )$ measures the distance of two metric spaces from being isometric to each other. We say that a sequence of compact metric spaces X_i converges in the sense of Gromov-Hausdorff to a compact metric space X if $d_{GH}(X_i, Y) \to 0$ as $i \to \infty$, and denote by $X_i \xrightarrow{d_{GH}} Y$.

Lemma 2.1.2.

(1) $d_{GH}(\ ,\ )$ satisfies the triangle comparison.

(2) $d_{GH}(X,Y)=0$ if and only if X is isometric to Y.

(3) The metric space $(\mathcal{M}et, d_{GH})$ is complete.

The d_{GH} defined above is not easy to calculate even for very simple spaces (for example, the GH-distance between a square and a disk). Now we recall an alternative formulation which is more convenience in the sense of convergence.

The map $f : X \to Y$ (is not necessarily continuous) is called a (Gromov-Hausdorff) ϵ-approximation if $||f(x_1)f(x_2)| - |x_1x_2|| < \epsilon$ for any $x_1, x_2 \in X$ and Y is contained in the ϵ-neighborhood $U_\epsilon(f(X))$.

Definition 2.1.3. Let X and Y be compact metric spaces, define

$$\hat{d}_{GH}(X,Y) = \inf\{\epsilon : \text{ there are GH } \epsilon\text{-approximations } f : X \to Y \text{ and } g : Y \to X\}.$$

Let X be a compact metric space, and $Y = \{p\}$, it's not hard to see that $d_{GH} = \operatorname{diam}(X)/2$ and $\hat{d}_{GH} = \operatorname{diam}(X)$. This shows that $\hat{d}_{GH} \neq d_{GH}$ in general. However, due to the following lemma, they are equivalent in the sense of convergence. An advantage to use $\hat{d}_{GH}$ is that one can measure the convergence by an ϵ-approximation, i.e. $X_i \xrightarrow{d_{GH}} Y$ if and only if $\hat{d}_{GH}(X_i, Y) \to 0$, or equivalently, for any small $\epsilon > 0$, there exists an ϵ-approximations $f_i : X_i \to X$ for large i.

Proposition 2.1.4. $\frac{2}{3} d_{GH} \leq \hat{d}_{GH} \leq 2 d_{GH}$.

We also can define the pointed GH-convergence, which is useful for the non-compact spaces. A pointed map, $f : (X,p) \to (Y,q)$, $f(p) = q$, is called an ϵ-pointed GH-approximation, if $||f(x_1)f(x_2)| - |x_1x_2|| < \epsilon$ for any $x_1, x_2 \in B_{\frac{1}{\epsilon}}(p)$ and $B_{\frac{1}{\epsilon}}(q) \subset B_\epsilon(f(B_{\frac{1}{\epsilon}}(p)))$. We say that a sequence (X_i, p_i) converges to (X,p), if there is a sequence of ϵ_i-pointed GH-approximation $f_i : (X_i, p_i) \to (X,p)$, with $\epsilon_i \to 0$.

Proposition 2.1.5. *(X_i, p_i) pointed converges to (X,p) if and only if $B_r(p_i)$ converges to $B_r(p)$ and $p_i \to p$ for all $r > 0$.*

2.2 Basic concepts

In this section we give the basic definitions of the Alexandrov spaces and show some equivalent definitions.

Definition 2.2.1. We call a metric space $(M, |\cdot,\cdot|)$ an *intrinsic metric* space if for any $x, y \in M$, $\epsilon > 0$ there is a sequence of points $x = z_0,\ z_1, \cdots, z_k = y$ such that $|z_i z_{i+1}| < \epsilon$ and $\sum_{i=0}^{k-1} |z_i z_{i+1}| < |xy| + \epsilon$. A (minimal) *geodesic* is a continuous curve whose length is equal to the distance between its ends. In a locally compact complete space with intrinsic metric any two points can be joined by a geodesic. A collection of three points $p, q, r \in M$ and three geodesics $\overline{pq}$, $\overline{pr}$, $\overline{qr}$ is called a *triangle* in M and is denoted by $\triangle pqr$.

For $\triangle pqr$ in M we may construct a triangle $\tilde{\triangle} pqr$ on S^2_κ with vertices $\tilde{p}$, $\tilde{q}$, $\tilde{r}$ and sides of lengths $|\tilde{p}\tilde{q}| = |pq|$, $|\tilde{p}\tilde{r}| = |pr|$, $|\tilde{q}\tilde{r}| = |qr|$ (if such triangle exists), where S^n_κ denotes the n-dimensional space form of constant sectional curvature κ. The triangle $\tilde{\triangle} pqr$ always uniquely exists up to a rigid shift for $\kappa \le 0$. For $\kappa > 0$ it exists only with the additional assumption that the perimeter of $\triangle pqr$ is less than $\frac{2\pi}{\sqrt{\kappa}}$. We let $\tilde{\measuredangle} pqr$ denote the angle at $\tilde{q}$ of the triangle $\tilde{\triangle} pqr$.

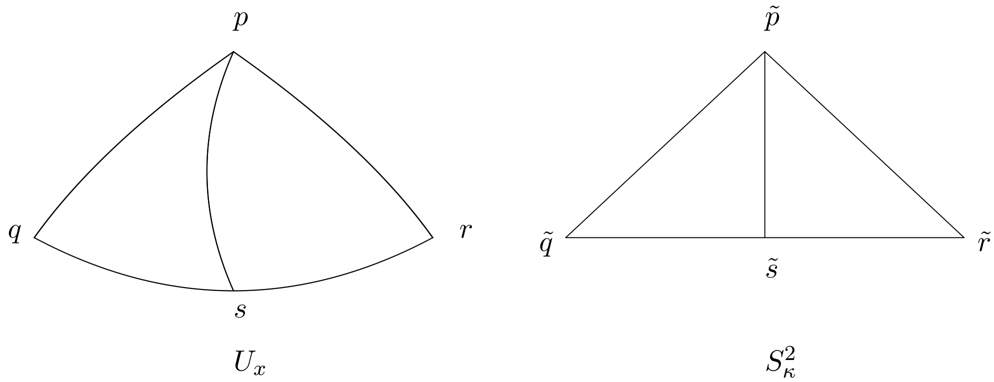
Definition 2.2.2. A locally complete space M with intrinsic metric is called an *Alexandrov space* with curvature $\ge \kappa$ (will be denoted by $\mathrm{Alex}\,(\kappa)$) if for any point $x \in M$ there exists a neighborhood U_x such that:

(D) For any four (distinct) points $(a; b, c, d)$ in U_x,

$$\tilde{\measuredangle} bac + \tilde{\measuredangle} bad + \tilde{\measuredangle} cad \le 2\pi.$$

Proposition 2.2.3. *Let space M be locally compact, then the condition (D) is equivalent to any of the following:*

(A) *for any triangle $\triangle pqr$ with vertices in U_x and any point s on the side $\overline{qr}$, we have $|ps| \ge |\tilde{p}\tilde{s}|$, where $\tilde{s}$ is the point on the side $\overline{\tilde{q}\tilde{r}}$ of the triangle $\tilde{\triangle} pqr$ corresponding to s, i.e. $|qs| = |\tilde{q}\tilde{s}|$, $|rs| = |\tilde{r}\tilde{s}|$.*



(B) *Let q, r be points on arbitrary geodesics γ, σ from the origin p, then the angle $\tilde{\measuredangle} qpr$ is non-increasing with respect to $|pq|$ and $|pr|$.*

(C)* and *(C_1)

> ***(C)*** *For any triangle $\triangle pqr$ contained in U_x, none of its angles is less than the corresponding angle of the triangle $\tilde{\triangle} pqr$ on S^2_κ.*
>
> ***(C_1)*** *If r is an interior point of the geodesic $\overline{pq}$, then for any point s, $\measuredangle srp + \measuredangle srq = \pi$.*

We will state some consequences of the above proposition and give the proof later.

Definition 2.2.4. If (B) is satisfied, the limit $\lim_{|pq|, |pr| \to 0} \tilde{\measuredangle} qpr$ (which does not depend on κ) exists. We call it the angle between γ, σ at p. It is easily verified that the angles between three geodesics with common origin satisfy the triangle inequality.

A consequence of the condition (C_1) is that geodesics do not bifurcate. Thus if a geodesic is extendable, the extension is unique. We list a few other properties of spaces of curvature bounded below which follow easily from (C) and (C_1).

Proposition 2.2.5.

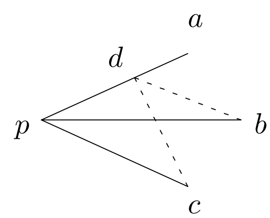
(1) If the geodesics $\overline{p_i q_i}$ converge to $\overline{pq}$ and the geodesics $\overline{p_i r_i}$ converge to $\overline{pr}$, then $\measuredangle pqr \le \lim_{i \to \infty} \inf \measuredangle q_i p_i r_i$. (follows by (C))

(2) If $\overline{pa}, \overline{pb}, \overline{pc}$ are geodesics, then $\measuredangle apb + \measuredangle bpc + \measuredangle cpa \le 2\pi$. (follows by (C) and ($C_1$))

Proof.

(1) By (C), this is obvious.

(2) If $d \in \overline{pa}$, then by (C_1),



$\measuredangle adb + \measuredangle adc + \measuredangle bdc \le (\measuredangle adb + \measuredangle bdp) + (\measuredangle adc + \measuredangle cdp) = 2\pi.$

To complete the proof using (1), it's sufficient to check if $\overline{pb}$, $\overline{pc}$ are unique. This can be guaranteed by taking b, c as the interior points of the geodesics. □

Proof of Proposition 2.2.3.

(1) To prove $(D) \Rightarrow (A)$ it's sufficient apply to the technique Lemma 2.2.6 on $(a; b, c, d)$.

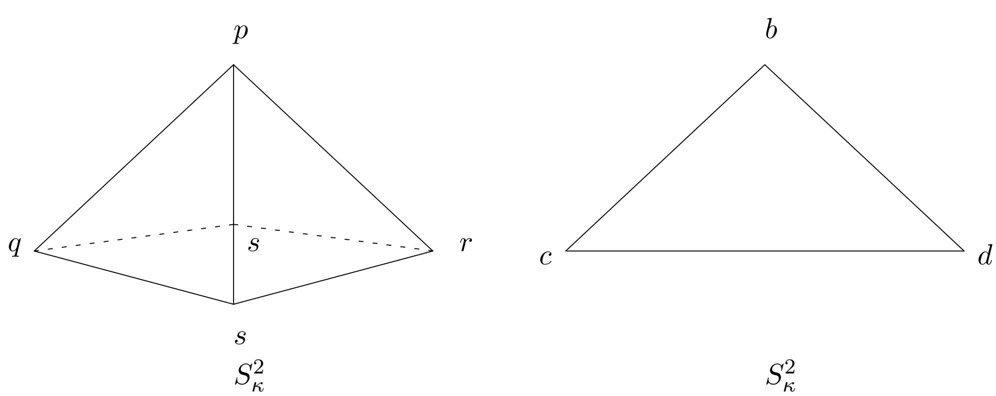
(2) (A) ⇔ (B). Just notice the property that $\measuredangle qpr \ge \measuredangle bac \Leftrightarrow |qr| \ge |bc|$ in the κ-plane provided $|pq| = |ab|$, $|pr| = |ac|$.

(3) (B) ⇒ (C) + (C_1). Obvious.

(4) (C) + (C_1) ⇒ (A). Let's use the graph in (A). By (C) + (C_1), $\tilde{\measuredangle} psq + \tilde{\measuredangle} psr \le \measuredangle psq + \measuredangle psr = \pi$. Then by Lemma 2.2.6, $\tilde{\measuredangle} pqs \ge \tilde{\measuredangle} pqr$, hence $|ps| \ge |\tilde{p}\tilde{s}|$.

(5) Proposition 2.2.5(2) + (C) ⇒ (D). Obvious. □

Lemma 2.2.6. *Let triangles $\triangle pqs$, $\triangle prs$ be given on a S^2_κ, which are exteriorly adjacent to each other with the common side $\overline{ps}$. Construct another triangle $\triangle bcd$ on S^2_κ, where $|bc| = |pq|$, $|bd| = |pr|$, $|cd| = |qs| + |sr|$, and $|bc| + |bd| + |cd| \le \frac{2\pi}{\sqrt{\kappa}}$ in the case $\kappa > 0$. Then $\measuredangle psq + \measuredangle psr \le \pi$ $(\ge \pi)$ if and only if $\measuredangle pqs \ge \measuredangle bcd$ and $\measuredangle prs \ge \measuredangle bdc$ (respectively, $\measuredangle pqs \le \measuredangle bcd$ and $\measuredangle prs \le \measuredangle bdc$).*



Proof. The proof can be easily produced by applying the cosine law on S^2_κ on the given triangles. □

Example 2.2.7 (Examples of Alexandrov space with curvature $\geq \kappa$).

(1) Riemannian manifolds without boundary or with locally convex boundary, whose section curvatures are not less than κ.

(2) The quotient space $M/G \in \mathrm{Alex}\,(\kappa)$ if M is an Riemannian manifolds with curv $\geq \kappa$ and G acts isometrically on M (see Chapter 2.3.2).

(3) The κ-suspension constructed in Chapter 2.3.1.

Some 2 dimensional simple examples:

(4) The 2-dimensional flat cone.

(5) The space produced by gluing two 2-dimensional unit disks via boundary isometric identification.

In the above we define the space with curvature bounded from below using local conditions. In general, the local conditions may not be satisfied globally. For example, a plane with a closed disk removed. If we add the completeness to the space, these conditions can be "globalized". This was first proved by A.D. Alexandrov for dimension 2. For Riemannian manifolds it is the well known Toponogov's Comparison Theorem. The argument in proofing Proposition 2.2.3 is still valid if the conditions are defined "globally". Hence it's enough to prove the globalization theorem for one of the local conditions.

Theorem 2.2.8. *Let M be a complete space satisfying condition (D). Then for any quadruple of points $(a; b, c, d)$ we have $\tilde{\measuredangle} bac + \tilde{\measuredangle} bad + \tilde{\measuredangle} cad \leq 2\pi$.*

The proof is fairly technique and we will omit it here. In the following, we will always assume that the geodesic exists, otherwise just make an easy modification.

2.3 Natural construction

2.3.1 κ-suspensions

We will construct metric cones from a given metric spaces, and list some propositions when the base space is an Alexandrov space without giving the proofs (c.f. [BGP]).

Definition 2.3.1 (Flat cone)**.** Let X be a metric space. The *flat cone* over X with vertex p is the quotient space $C_0(X) = X \times [0,\infty]/\sim$, where $(x_1, a_1) \sim (x_2, a_2) \sim p \Leftrightarrow a_1 = a_2 = 0$. Let $\Pi : C_0(X) - p \to X$ be the natural projection. The metric of the cone is defined from the cosine formula:

$$|\bar{x}_1\bar{x}_2|^2 = a_1^2 + a_2^2 - 2a_1a_2\cos(\min\{|x_1x_2|, \pi\}), \tag{2.1}$$

where $\bar{x}_1 = (x_1, a_1)$, $\bar{x}_2 = (x_2, a_2)$.

Proposition 2.3.2. *Let X be a complete metric space. The following two conditions are equivalent:*

(a) $X \in Alex(1)$.

(b) $C_0(X)$ is not a straight line and belongs to $Alex(0)$.

The construction of the cone can be more general by using the spherical or hyperbolic cosine formula S^2_κ instead of the Euclidean cosine formula. We call these cones κ-suspensions. In particular, the above is the case $\kappa = 0$ and the following are the cases $\kappa = 1$ and -1.

Definition 2.3.3 (Spherical suspension)**.** Let X be a metric space of diameter $\le \pi$. The *spherical suspension* is the quotient space $C_1(X) = X \times [0,\pi]/\sim$, where $(x_1, a_1) \sim (x_2, a_2) \Leftrightarrow a_1 = a_2 = 0$ or $a_1 = a_2 = \pi$. The metric is defined from the cosine formula:

$$\cos|\bar{x}_1\bar{x}_2| = \cos a_1 \cos a_2 + \sin a_1 \sin a_2 \cos|x_1x_2|, \tag{2.2}$$

where $\bar{x}_1 = (x_1, a_1)$, $\bar{x}_2 = (x_2, a_2)$.

Proposition 2.3.4. *Let X be a complete metric space of diameter $\le \pi$. Then the following two conditions are equivalent:*

(a) $X \in Alex(1)$.

(b) $C_1(X)$ is not a circle and belongs to $Alex(1)$.

Definition 2.3.5 (Hyperbolic Suspension)**.** Let X be a metric space of diameter $\le \pi$. The *elliptic cone* over X is the quotient space $C_{-1}(X) = X \times [0,\infty]/\sim$, where

$(x_1, a_1) \sim (x_2, a_2) \Leftrightarrow a_1 = a_2 = 0$. The metric is defined from the cosine formula:

$$\cosh |\bar{x}_1\bar{x}_2| = \cosh a_1 \cosh a_2 - \sinh a_1 \sinh a_2 \cos |x_1x_2|, \tag{2.3}$$

where $\bar{x}_1 = (x_1, a_1)$, $\bar{x}_2 = (x_2, a_2)$.

Proposition 2.3.6. *Let X be a complete metric space of diameter $\leq \pi$. Then the following two conditions are equivalent:*

(a) $X \in Alex(1)$ is a space with curvature ≥ 1.

(b) $C_{-1}(X)$ is not a straight line and belongs to $Alex(-1)$.

In Chapter 4, we will discuss more properties about the κ-suspensions and show that they (with a boundary gluing) shall be regarded as the model spaces who have the relatively maximal volume.

2.3.2 Quotient spaces

Proposition 2.3.7. *Let the group G act isometrically on a space $X \in Alex(\kappa)$ with curvature $\geq \kappa$. Then the quotient space $X/G \in Alex(\kappa)$, whose points correspond to the closure of the orbits of G.*

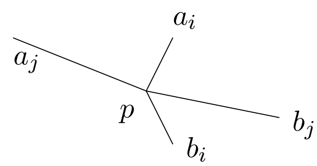
Proof. It's obvious that X/G is locally complete with respect to the intrinsic metric. We now check condition (D). For a quadruple $(\bar{a}; \bar{b}, \bar{c}, \bar{d})$ in X/G and the quadruple $(a; b, c, d)$ in X such that $\Pi(a) = \bar{a}, \ldots, \Pi(d) = \bar{d}$, where $\Pi : X \to X/G$ is a natural projection. Additionally, because the action is isometry, we can choose the points b, c, d such that $|ab|, |ac|, |ad|$ do not differ much from the corresponding distances $|\bar{a}\bar{b}|, |\bar{a}\bar{c}|, |\bar{a}\bar{d}|$. Since $|bc| \geq |\bar{b}\bar{c}|$, $|bd| \geq |\bar{b}\bar{d}|$, $|\bar{c}\bar{d}| \geq |cd|$, the angles with vertex a in X are not smaller than the angles with vertex $\bar{a}$ in X/G. Thus if X/G violates condition (D), so does X. $\square$

2.4 Burst points

At every point in a Riemannian manifold there exists a smooth coordinate system, however, Alexandrov spaces may have "singular" points. For example, the boundary and the vertex of a cone. We will give a constraint to describe the "non-singular" points and show that these points can be associated with a small neighborhood which is bi-Lipschitz homeomorphic to a ball in $\mathbb{R}^n$ (see Theorem 2.4.2). Moreover, the bi-Lipschitz constant is arbitrarily close to 1 (depending on the size of the neighborhood, see Theorem 2.8.4).

Definition 2.4.1. Let $M \in \text{Alex}(\kappa)$. A point $p \in M$ is called the (n, δ)-*burst point* if there are n-pairs (a_i, b_i), such that the following hold for all $1 \le i \ne\le j$:

$$\tilde{\measuredangle} a_i p b_i > \pi - \delta, \qquad \tilde{\measuredangle} a_i p a_j > \frac{\pi}{2} - \delta,$$
$$\tilde{\measuredangle} a_i p b_j > \frac{\pi}{2} - \delta, \qquad \tilde{\measuredangle} b_i p b_j > \frac{\pi}{2} - \delta, \tag{2.4}$$



The n-pair (a_i, b_i) is called an (n, δ)-*explosion* (or (n, δ)-*strainer* or simply an *explosion* or *strainer*) at the point p.

Together with condition (D), condition (2.4) also implies the upper bounds $\tilde{\measuredangle} a_i p a_j < \frac{\pi}{2} + 2\delta$, $\tilde{\measuredangle} a_i p b_i < \frac{\pi}{2} + 2\delta$, $\tilde{\measuredangle} b_i p b_j < \frac{\pi}{2} + 2\delta$. Clearly the set of (n, δ)-burst points is open. By condition (D), the explosion (a_i, b_i) can be chosen arbitrarily near to p if there exists one.

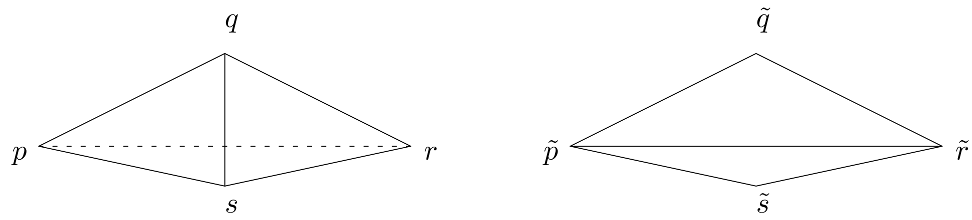
Theorem 2.4.2. *Let p be an (n, δ)-burst point with explosion (a_i, b_i), $i = 1, \dots, n$ and there is no $(n+1, 4\delta)$-burst points near p, where $\delta < \frac{1}{2n}$. Then the map $\varphi(q) = (|a_1 q|, |a_2 q|, \dots, |a_n q|)$ gives a bi-Lipschitz homeomorphism between a neighborhood of the point p and a domain in $\mathbb{R}^n$.*

To prove Theorem 2.4.2 we need the following lemma which will be useful later on. The proof of Theorem 2.4.2 (see [BGP] §5) is omitted here.

Lemma 2.4.3. *Let p, q, r, s be the points in $X \in Alex(\kappa)$. If $|qs| < \delta \min\{|pq|, |rq|\}$ and $\tilde{\measuredangle} pqr > \pi - \delta_1$, then*

$$|\tilde{\measuredangle} pqs + \tilde{\measuredangle} rqs - \pi| < 10\delta + \delta_1 \quad \text{and} \quad |\tilde{\measuredangle} psq + \tilde{\measuredangle} rsq - \pi| < 10\delta + \delta_1.$$

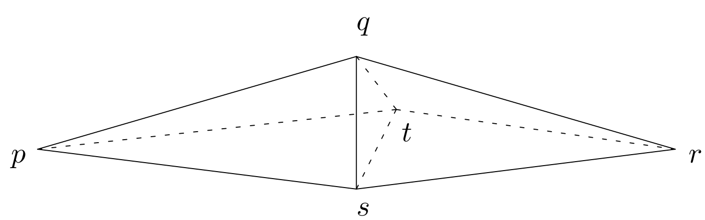
In particular, if geodesics exist, then the angles $\tilde{\measuredangle} pqs$, $\tilde{\measuredangle} rqs$ are little different from the corresponding angles $\measuredangle pqs$, $\measuredangle rqs$.



X $\qquad$ S^2_κ

Proof. The inequality $\tilde\measuredangle pqs + \tilde\measuredangle rqs - \pi < \delta_1$ follows directly from condition (D) for the quadruple $(q; p, s, r)$. Consider the triangles $\tilde\triangle prs$, $\tilde\triangle pqr$. Since $|qs| < \delta \min\{|pq|, |rq|\}$, we get $\tilde\measuredangle psr > \pi - 4\delta - \delta_1$. Then by condition (D), $\tilde\measuredangle psq + \tilde\measuredangle rsq - \pi < 4\delta + \delta_1$. Together with $\tilde\measuredangle pqs + \tilde\measuredangle psq \geq \pi - 2\delta$ and $\tilde\measuredangle rqs + \tilde\measuredangle rsq \geq \pi - 2\delta$. Then we have $\tilde\measuredangle pqs + \tilde\measuredangle rqs - \pi > -8\delta - \delta_1$ and $\tilde\measuredangle psq + \tilde\measuredangle rsq - \pi > -4\delta - \delta_1$. $\square$

Corollary 2.4.4. *Let p, q, r, s, t be points in $X \in Alex(\kappa)$ such that $|qs| < \delta \min\{|pq|, |rq|\}$, $\tilde\measuredangle pqr > \pi - \delta$, $||pq| - |ps|| < \delta |qs|$ and $\tilde\measuredangle qts > \pi - \delta$. Then each of the angles $\tilde\measuredangle ptq$, $\tilde\measuredangle pts$, $\tilde\measuredangle rtq$, $\tilde\measuredangle rts$ differs from $\frac{\pi}{2}$ less than 100δ.*



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Proof. Obviously, $|qs| < 2\delta \min\{|ps|, |rs|\}$ and $\tilde\measuredangle psr > \pi - 5\delta$. By Lemma 2.4.3,

$$|\tilde\measuredangle ptq + \tilde\measuredangle rtq - \pi| < 20\delta \quad \text{and} \quad |\tilde\measuredangle pts + \tilde\measuredangle rts - \pi| < 20\delta. \tag{2.5}$$

Since $\tilde\measuredangle qts > \pi - \delta$, by condition (D), it remains to show that $|\tilde\measuredangle ptq - \frac{\pi}{2}| < 20\delta$ or $|\tilde\measuredangle pts - \frac{\pi}{2}| < 20\delta$, which is equivalent to $||pt| - |pq|| < \delta|qt|$ or $||pt| - |st|| < \delta|st|$. Let $\alpha = \frac{||pt|-|pq||}{|qt|}$ and $\beta = \frac{||pt|-|st||}{|st|}$. Then

$$\alpha|qt| + \beta|st| \leq ||pq| - |st|| < \delta|qs| \tag{2.6}$$

Let $\lambda = \frac{|qt|}{|qs|}$, then $\frac{|st|}{|qs|} \geq \frac{|qs|-|st|}{|qs|} = 1 - \lambda$. Thus (2.6) becomes $\lambda\alpha + (1-\lambda)\beta < \delta$, which enforces that either $\alpha < \delta$ or $\beta < \delta$. $\square$

Using the construction idea in the proof of Theorem 2.4.2, one can see the following lemma, which is useful to show the dimension theorem in Chapter 2.5.

Lemma 2.4.5. *Any (n,δ)-burst point can be approached by a sequence of (n,δ')-burst with $\delta' > 0$ arbitrarily small, where $\delta < \frac{1}{8n}$.*

Corollary 2.4.6. *The set of (n,δ)-burst points is open dense in $X \in Alex^n(\kappa)$ for any $\delta > 0$.*

2.5 Dimension

For a space $X \in \mathrm{Alex}\,(\kappa)$, one can define the canonical Hausdorff dimension. Another idea is to take the maximal number n such that the (n,δ)-explosion exists for some point in X, or equivalently, the number n such that a neighborhood of burst point is homeomorphic to a region in $\mathbb{R}^n$. In the following we will first define the burst index and rough dimension (rough volume) and show that the they are the same as Hausdorff dimension for an $X \in \mathrm{Alex}\,(\kappa)$. In the rest of this thesis, we will use $\mathrm{Alex}^n(\kappa)$ to denote the n-dimensional space of curvature $\geq \kappa$.

Definition 2.5.1. Let $p \in X \in \mathrm{Alex}\,(\kappa)$. The number n is called the *burst index* near p if there are (n,δ)-burst points in any neighborhood of this point but the analogous condition with n replaced by $(n+1)$ is not satisfied (n is a natural number or 0). If there is no such n, then we suppose the burst index to be ∞.

Definition 2.5.2.

(1) The *α-dimensional rough volume* $V_{r_\alpha}(U)$ of a bounded set $U \subset X$ in a metric space is $\limsup_{\epsilon \to 0} \epsilon^\alpha \beta_U(\epsilon)$, where $\beta_U(\epsilon)$ is the largest number of points in U that are at least ϵ pairwise distance from each other (we call it ϵ-net). $\inf\{\alpha : V_{r_\alpha}(X) = 0\} = \sup\{\alpha : V_{r_\alpha}(X) = \infty\}$ is called the *rough dimension* of X (denoted as $\dim_r(X)$).

(2) The n-dimensional Hausdorff measure of a subset $A \subset X$ is defined as $H^n(A) = \lim_{\epsilon \to 0} H^n_\epsilon(A)$, where

$$H^n_\epsilon(A) = \inf\left\{\sum_{i=1}^{\infty} \mathrm{diam}(U_i)^n : \bigcup_{i=1}^{\infty} U_i \supset A, \mathrm{diam}(U_i) < \epsilon\right\}.$$

$\inf\{n : H^n(X) = 0\} = \sup\{n : H^n(A) = \infty\}$ is called the *Hausdorff dimension* $\dim_H(A)$.

Obviously the Hausdorff dimension $\dim_H(X) \le \dim_r(X)$. If $f : X \to Y$ is a Lipschitz map, then $\dim_H f(X) \le \dim_H(X)$ and $\dim_r f(X) \le \dim_r(X)$; if f is bi-Lipschitz, then $\dim_H f(X) = \dim_H(X)$ and $\dim_r f(X) = \dim_r(X)$.

Lemma 2.5.3. *Let $u, v \in X \in Alex(\kappa)$, and let U and V be their neighborhoods which are sufficiently small, then* $\dim_r(U) = \dim_r(V)$.

Proof. It's sufficient to prove for the case $\kappa = 0$. Assume $\limsup_{\epsilon\to 0} \epsilon^\alpha \beta_U(\epsilon) = \infty$, then for each i large there is an ϵ_i-net $x_1, \dots, x_{N_i} \in U$ such that $\epsilon_i^\alpha N_i \ge i$, where $\epsilon_i \to 0$ and $N_i = \beta_U(\epsilon_i)$.

We now construct an ϵ_i'-net in V. Let $R > 0$ small such that $B_v(R) \subset V$. Let y_j be the point on some geodesic $\overline{vx_j}$ so that $|vy_j| = \frac{R}{D}|vx_j|$, where $D = \sup\{|ux| : x \in V\}$. Clearly the points y_j are in B and for an $\epsilon_i' = \frac{R}{D} \cdot \epsilon$-net. Thus we have

$$(\epsilon_i')^\alpha \beta_V(\epsilon_i') \ge \left(\frac{R}{D}\right)^\alpha \epsilon_i^\alpha N_i \ge i \left(\frac{R}{D}\right)^\alpha.$$

We conclude that $V_{r_\alpha}(U) > 0$ and $\dim_r U \ge \dim_r V$. Similarly $\dim_r U \le \dim_r V$ by switching the position U and V. □

Remark 2.5.4. If the triangle comparison is only satisfied locally (such as a square from which remove a closed disk), the above proof still valids if the points can be passed through from each other by a sequence of intersected balls which satisfy the triangle comparison.

Proposition 2.5.5. *Let point $p \in X \in Alex(\kappa)$. Then for a sufficiently small neighborhood U of p, the burst index of X near p is equal to* $\dim_r U$ *and* $\dim_H U$. *In particular, the burst index is equal to* $\dim_r X$ *and equal to* $\dim_H X$.

Proof. Let the burst index of M near p be n and let n be a natural number (the case $n = 0$ is trivial - M is a point; the case $n = \infty$ is analogue). Then by definition of the burst index and Lemma 2.4.5 there are no $(n+1, \frac{1}{8(n+1)})$-burst points in some neighborhood $U \ni p$. Then by Theorem 2.4.2 there is a bi-Lipschitz homeomorphism from some neighborhood $U_1 \subset U$ of an $(n, \frac{1}{100n})$-burst point $p_1 \in U$ onto a domain in $\mathbb{R}^n$. Thus $\dim_r U_1 = \dim_H U_1 = n$. By Lemma 2.5.3 we get $\dim_r U = \dim_r U_1 = n$ and finally $\dim_H U = n$, since $\dim_H U_1 \leq \dim_H U \leq \dim_r U$. □

2.6 Tangent cones and space of directions

We will define the tangent cone for a point p in $X \in \text{Alex}^n(\kappa)$, which is a generalization of the tangent space in Riemannian geometry. One natural definition is the Hausdorff limit of the blow up metric in a small neighborhood of the point. Because of the singulary, the tangent cone metrically may not be an Euclidean space. However, we will show that it is a flat cone (0-suspension) over the space of directions, where the space of directions is the equivalent class of the geodesics from p (in fact, it is a space in $\text{Alex}^{n-1}(1)$). The space of directions is very useful to characterize the infinitesimal structure near the point. [BGP] §7 shows that the space of directions is continuous along the interior of a geodesic and semi-continuous up to the end points. In this section we will modify the proof and show that the space of directions is isometric along the interior of a geodesic. This was proved by A. Petrunin in [Pet 98].

2.6.1 Definitions and properties

As a formal definition, let's first define the space of directions, and construct the tangent cone as the flat cone over the space of directions, then show that such cone is a metric blow up near the point.

Definition 2.6.1 (Space of directions)**.** Let $p \in X \in \text{Alex}^n(\kappa)$. Geodesics with origin p are said equivalent if one is the extension of another. Let Σ'_p denote such equivalent class, associated with the distance (between two geodesics from p) as the angle at p between the two geodesics. The metric completion of Σ'_p is called the *space of directions* at the point p (denoted by Σ_p). We will use $\overrightarrow{pq}$ or briefly $[q]$ to represent the geodesic class $\overline{pq}$ (or one of the geodesics if they are multiple) in Σ_p. Note that by definition, $\Sigma'_p \subset \Sigma_p$ is also the collection of directions in which there is a geodesic goes out.

An important property for the space of directions is:

Theorem 2.6.2. *The space of directions at any point of $X \in Alex^n(\kappa)$ is compact.*

We will omit the proof of Theorem 2.6.2 but list a technical lemma which is required in the argument, since this lemma is useful in some other situations.

Lemma 2.6.3. *Let $\{\overline{pa_i}\}$ be a finite collection of geodesics in $X \in Alex^n(\kappa)$. Then for any $\delta > 0$ there is a neighborhood U of the point p (depending on δ and the collection of geodesics) such that the angles of all the triangles $\triangle pqr$ with vertices q, r on the parts of the geodesics $\overline{pa_i}$ in U differ from the corresponding angles of the triangles $\tilde{\triangle} pqr$ by no more than δ.*

Proof. It is sufficient to consider the case of two geodesics $\overline{pa}, \overline{pb}$. Let $R > 0$ small such that if $a_1 \in \overline{pa}$, $b_1 \in \overline{pb}$ with $|pa_1| \leq R$, $|pb_1| \leq R$, then $\measuredangle a_1pb_1 - \tilde{\measuredangle} a_1pb_1 < \delta/2$. Consider the $\triangle pa_1b_1$ with $a_1 \in \overline{pa}$, $b_1 \in \overline{pb}$ and $|pa_1| < (0.1)\delta R$, $|pb_1| < (0.1)\delta R$, we then have $\tilde{\measuredangle} a_1b_2b_1 \leq \delta/2$. Let the point $b_2 \in \overline{pb}$ be such that $|pb_2| = R$. Put the triangles $\tilde{\triangle} pa_1b_1$ and $\tilde{\triangle} a_1b_1b_2$ on the κ-plane externally along the side $\tilde{a}_1\tilde{b}_1$, then by comparing this with the triangle $\tilde{\triangle} a_1pb_2$ we get

$$\begin{aligned}
&\tilde{\measuredangle} a_1pb_1 + \tilde{\measuredangle} a_1b_2b_1 - \tilde{\measuredangle} a_1pb_2 - \tilde{\measuredangle} a_1b_2p \\
&= (\tilde{\measuredangle} pa_1b_2 - \tilde{\measuredangle} pa_1b_1 - \tilde{\measuredangle} b_1a_1b_2) + (\pi - \tilde{\measuredangle} pb_1a_1 - \tilde{\measuredangle} b_2b_1a_1) \\
&\geq \pi - \tilde{\measuredangle} pb_1a_1 - \tilde{\measuredangle} b_2b_1a_1.
\end{aligned}$$

Therefore

$$\begin{aligned}
0 \leq \measuredangle pb_1a_1 - \tilde{\measuredangle} pb_1a_1 &\leq (\measuredangle pb_1a_1 + \measuredangle b_2b_1a_1) - (\tilde{\measuredangle} pb_1a_1 + \tilde{\measuredangle} b_2b_1a_1) \\
&\leq (\tilde{\measuredangle} a_1pb_1 - \tilde{\measuredangle} a_1pb_2) + (\tilde{\measuredangle} a_1b_2b_1 - \tilde{\measuredangle} a_1b_2p) \\
&< \delta/2 + (\delta/2 - 0) < \delta.
\end{aligned}$$

Similarly we get $0 \leq \measuredangle pa_1b_1 - \tilde{\measuredangle} pa_1b_1 < \delta$. □

Definition 2.6.4. The tangent cone C_p at the point $p \in X \in \text{Alex}^n(\kappa)$ is the flat cone (see Chapter 2.3.1) over the space of directions Σ_p.

Up to this point we don't know if $\text{curv}(C_p) \geq 0$, or $\text{curv}(\Sigma_p) \geq 1$. However, by Proposition 2.3.2, they are equivalent to each other. The map $\exp_p : C'_p \to X$ is defined in the canonical way, but the domain C'_p is a star-shape subset of C_p (for example, p is the glued point on the glued two disks via boundary identification). The inverse map $\exp_p^{-1}$, considered as a multi-valued map, is defined on all X. For our purpose, in the rest of the thesis $\exp_p^{-1}$ will mean a single-valued function by choosing one direction of the geodesics. The map $\exp_p^{-1} : X \to C_p$ may not be onto or continuous, even in a small neighborhood of p. We also can construct $\exp_{\kappa,p}^{-1} : X \to C_\kappa(p)$ which will become a distance non-decreasing map using the natural map from the flat cone to the κ-suspension.

Theorem 2.6.5. *Let $(X, \rho) \in Alex^n(\kappa)$ and let $p \in X$. Then the spaces with base point $(X, p, \lambda\rho)$ Gromov-Hausdorff converge to the tangent cone C_p as $\lambda \to \infty$.*

Corollary 2.6.6.

(1) The tangent cone $C_p \in Alex(0)$. Thus if $\dim X > 1$, then $\Sigma_p \in Alex(1)$.

(2) $\dim \Sigma_p = \dim X - 1$, or equivalently, $\dim C_p = \dim X$.

Proof. (1) It's clear that $(X, \lambda\rho) \in \text{Alex}^n(\lambda^{-2}\kappa)$. Then as the limit space, the curvature of C_p is bounded from below by $0 = \lim_{\lambda\to\infty} \lambda^{-2}\kappa$.

(2) Because $\exp_{\kappa,p}^{-1} : X \to C_\kappa(p)$ is distance non-decreasing, and the map $C_\kappa(p) \to C_p$ is bi-Lipschitz, we get $\dim C_p \geq X$. We now prove $\dim \Sigma_p \leq \dim X - 1$ by lifting an n-explosion (a'_i, b'_i) for a point $q' \in \Sigma'_p$ to an $(n+1)$-explosion (a_i, b_i) in X. Select (a'_i, b'_i) arbitrarily close to q', i.e. $\measuredangle a_i pq < \epsilon$, $\measuredangle b_i pq < \epsilon$. Take q as the interior point of the geodesic q' and a_i, b_i on the geodesics a'_i, b'_i such that $|pa_i| = |pb_i| = |pq|$ for $1 \leq i \leq n$. It's easy to check that (a_i, b_i) form an n-explosion at q. We will get the $(n+1)$-explosion when take a_{n+1}, b_{n+1} as the points on geodesic $\overline{pq}$ with the opposite directions from q. $\square$

Remark 2.6.7. One may compare this argument to Lemma 4.2.9 (2).

2.6.2 The continuity of tangent cones

For compact metric spaces X and Y, we say $X \leq Y$ if there exists a non-contracting (not necessarily continuous) map $f : X \to Y$ i.e. $|f(x)f(y)|_Y \geq |xy|_X$. To show $X \leq Y$, it's sufficient to check the condition over a dense subset of X. It can be verified that if $X \le Y \le X$, then X and Y are isometric. We say $\liminf_{i\to\infty} X_i \ge X$ if the GH-limit space X' of any subsequence satisfies that $X' \ge X$. Similarly one can define the inequality $\limsup_{i\to\infty} X_i \le X$.

Proposition 2.6.8. *For compact metric spaces X and Y, if $X \le Y \le X$, then X and Y are isometric to each other.*

Proof. It suffices to show that if $f : X \to X$ is a non-contracting map, then f is an isometry. Let $\mathcal{A}$ be the collection of all ϵ-net (the most number is $\beta_X(\epsilon)$) of X and define a map $\phi : \mathcal{A} \to \mathbb{R}^+, \{x_i\} \mapsto \sum_{j<k} |x_j\, x_k|$. Note that any sequence of elements in $\mathcal{A}$ has point-wise convergent subsequence, and ϕ is bounded by $\beta_X(\epsilon)\mathrm{diam}X$, hence ϕ takes maximum at some element $\{a_i\}$. Together with

$$\phi(f(\{a_i\})) = \sum_{j<k} |f(a_j)\, f(a_k)| \ge \sum_{j<k} |a_j\, a_k| = \phi(\{a_i\}),$$

we get that f is isometric when restricted on $\{a_i\}$. Now it remains to show that $\{a_i\}$ is ϵ-dense in X. Because ϕ takes maximum at $\{a_i\}$, $\{a_i\}$ gets the maximal number of points as the ϵ-nets, and this implies the ϵ-density. □

Theorem 2.6.9 (The semicontinuity of tangent cones). *If q_i, p are points in $X \in Alex^n(\kappa)$ and $p_i \to p$, then $\liminf_{i\to\infty} \Sigma_{p_i} \ge \Sigma_p$.*

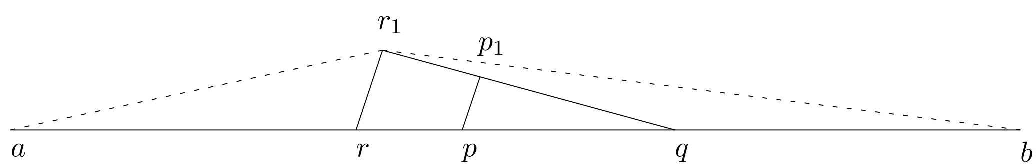
Proof. Not losing generality, we can assume $\Sigma_{p_i} \xrightarrow{H} \Sigma$. We will show that $\Sigma \ge \Sigma_p$. Take an ϵ-net $A_\epsilon = \{\overrightarrow{pa_1}, \dots, \overrightarrow{pa_m}\}$ in Σ_p'. By Proposition 2.2.5, we have $\liminf_{i\to\infty} \measuredangle a_j p_i a_{j'} \ge \measuredangle a_j p a_{j'}$, i.e., $\liminf_{i\to\infty} |\overrightarrow{p_ia_j}\, \overrightarrow{p_ia_{j'}}|_{\Sigma_{p_i}} \ge |\overrightarrow{pa_j}\, \overrightarrow{pa_{j'}}|_{\Sigma_p}$. Let $b_j' \in \Sigma$ be the limit points of the sequence $\overrightarrow{p_ia_j}$ as $i \to \infty$. Then $|\overrightarrow{p_ia_j}\, \overrightarrow{p_ia_{j'}}|_{\Sigma_{p_i}} - |b_j' b_{j'}'|_\Sigma \to 0$. Thus $|b_j' b_{j'}'|_\Sigma \ge |\overrightarrow{pa_j}\, \overrightarrow{pa_{j'}}|_{\Sigma_p}$ and we can define a non-contracting map $f_\epsilon : A_\epsilon \to \Sigma$ as $f_\epsilon(\overrightarrow{pa_j}) = b_j'$. □

Theorem 2.6.10. *Let p, r be interior points of the geodesic $\overline{ab}$ in $X \in Alex^n(\kappa)$. Then $\Sigma_p = \Sigma_r$.*

The above two theorems state that, the space of directions (as well as the tangent cones) doesn't change along the interior of a geodesic, but at the limit point it can be "smaller" (but not collapse). For example, consider the 2-dimensional flat cone with vertex p and $\Sigma_p = S(\frac{1}{2})$, where $S(r)$ denotes the circle with radius r. Let a_i be points on the geodesic $\overline{pq}$, hence $\Sigma_{a_i} = S(1)$. If $\lim\limits_{i\to\infty} a_i = a \neq p$, then $\Sigma_a = S(1) = \Sigma_{a_i}$. If $\lim\limits_{i\to\infty} a_i = p$, then $\Sigma_p = S(\frac{1}{2}) < S(1) = \Sigma_{a_i}$, because any geodesic can not pass through the vertex p.

In the approach in [BGP] (§7), the space of direction Σ_p is first reduced to a spherical suspension of Γ_p provided that q is an interior point of the geodesic. Then a technical lemma describing the "similar triangles" prosperities is established for the points p_1, r_1 near $\overline{ab}$ with that $\overline{pp_1}$, $\overline{rr_1}$ are almost perpendicular to $\overline{ab}$. In the following, we modify the original proof without using the condition "almost perpendicular". We show that the "similar triangles" properties almost hold (depending on the size and location of the triangles) for any shape of triangles along the interior of a geodesic.

Lemma 2.6.11 (Infinitesimal similar triangles). *Let the points r, p, q be points on the geodesic $\overline{ab}$ with the order: a, r, p, q, b and $|qr| < \delta \min\{|ar|, |bq|\}$. Let the point r_1 near r such that $|rr_1| < \delta^2 |rq|$, and p_1 be the point on $\overline{qr_1}$ so that $\dfrac{|r_1 q|}{|p_1 q|} = \dfrac{|rq|}{|pq|}$. Then*



(a) $\measuredangle r_1 q r < (\sin \measuredangle r_1 r q + 2\delta) \dfrac{|rr_1|}{|rq|}$,

(b) $\tilde{\measuredangle} r_1 r q > \measuredangle r_1 r q - 2\delta$,

(c) $\left| \dfrac{|pp_1|}{|rr_1|} \cdot \dfrac{|rq|}{|pq|} - 1 \right| < 15\delta$,

(d) $|\measuredangle p_1 p q - \measuredangle r_1 r q\,| < 3\delta$,

Proof. Due to the comment in Definition 2.2.4, it's sufficient to give a proof for $\kappa = 0$. For convenience, in the following we always assume that r_1 is not on the geodesic $\overline{ab}$.

(a) By applying condition (C) and the cosine formula in $\triangle arr_1$ we get

$$\begin{aligned}|ar_1| &\le (|ar|^2 + |rr_1|^2 + 2|ar||rr_1|\cos\measuredangle r_1rb)^{1/2}\\ &\le |ar| + |rr_1|\cos\measuredangle r_1rb + \frac{|rr_1|^2\sin^2\measuredangle r_1rb}{2(|ar| + |rr_1|\cos\measuredangle r_1rb)}\\ &\le |ar| + |rr_1|\cos\measuredangle r_1rb + \frac{1}{1-\delta^3}\cdot\frac{|rr_1|^2}{2|ar|}\sin^2\measuredangle r_1rb,\end{aligned} \tag{2.7}$$

since the function

$$\begin{aligned}f(x) &= (c^2 + x^2 \pm 2cx\cos\theta)^{1/2}\\ &= ((c \pm x\cos\theta)^2 + x^2\sin^2\theta)^{1/2}\\ &\le \left((c\pm x\cos\theta)^2 + x^2\sin^2\theta + \frac{x^4\sin^4\theta}{4(c\pm x\cos\theta)^2}\right)^{1/2}\\ &= c \pm x\cos\theta + \frac{x^2\sin^2\theta}{2(c\pm x\cos\theta)},\end{aligned} \tag{2.8}$$

provided $\frac{x}{c} \le \delta < 1$. Similarly, in $\triangle r_1rq$ we get

$$\begin{aligned}|r_1q| &\le (|rq|^2 + |rr_1|^2 - 2|rq||rr_1|\cos\measuredangle r_1rb)^{1/2}\\ &\le |rq| - |rr_1|\cos\measuredangle r_1rb + \frac{|rr_1|^2\sin^2\measuredangle r_1rb}{2(|rq| - |rr_1|\cos\measuredangle r_1rb)}\\ &\le |rq| - |rr_1|\cos\measuredangle r_1rb + \frac{1}{1-\delta^2}\cdot\frac{|rr_1|^2}{2|rq|}\sin^2\measuredangle r_1rb.\end{aligned} \tag{2.9}$$

In $\triangle r_1qb$ we get

$$\begin{aligned}|r_1b| &\le (|qb|^2 + |r_1q|^2 + 2|qb||r_1q|\cos\measuredangle r_1qr)^{1/2}\\ &\le |qb| + |r_1q| - \frac{|qb||r_1q|}{|qb| + |r_1q|}(1 - \cos\measuredangle r_1qr).\end{aligned} \tag{2.10}$$

Summing up inequalities (2.7)-(2.10) and taking in account that $|ar| + |rq| + |qb| = |ab| \le |ar_1| + |br_1|$, we get

$$\frac{|qb||r_1q|}{|qb| + |r_1q|}(1 - \cos\measuredangle r_1qr) \le \frac{1}{1-\delta^2}\left(\frac{1}{|ar|} + \frac{1}{|rq|}\right)\cdot\frac{1}{2}|rr_1|^2\sin^2\measuredangle r_1rb. \tag{2.11}$$

Thus

$$\begin{aligned}1 - \cos\measuredangle r_1qr &\le \frac{1}{1-\delta^2}\left(1 + \frac{|rq|}{|ar|}\right)\frac{|rq|}{|r_1q|}\left(1 + \frac{|r_1q|}{|qb|}\right)\cdot\frac{|rr_1|^2}{2|rq|^2}\sin^2\measuredangle r_1rb\\ &\le \frac{(1+\delta)(1+\delta^2)(1+\delta+\delta^3)}{1-\delta^2}\cdot\frac{|rr_1|^2}{2|rq|^2}\sin^2\measuredangle r_1rb\\ &\le (1+3\delta)\cdot\frac{|rr_1|^2}{2|rq|^2}\sin^2\measuredangle r_1rb,\end{aligned} \tag{2.12}$$

consequently,

$$\measuredangle r_1qr \le (1+2\delta)\sin\measuredangle r_1rb\cdot\frac{|rr_1|}{2|rq|} \le (\sin\measuredangle r_1rb + 2\delta)\cdot\frac{|rr_1|}{2|rq|}, \tag{2.13}$$

provided $\frac{|rr_1|}{|rq|} < \delta^2$ and $\measuredangle r_1qr$ is small in terms of δ by (2.12).

Remark 2.6.12. (1) The condition $|rq| < \delta|qb|$ (which controls the size of the triangle) can not be removed since we used $\triangle r_1qb$ and it will not work if use $\triangle r_1qa$ instead.

(2) In estimates (2.7) and (2.9), $|ar| + |rr_1|\cos\measuredangle r_1rb$ ($|rq| - |rr_1|\cos\measuredangle r_1rb$ respectively) is the "distance" from a (q respectively) to the projection point of r_1 on $\overline{ab}$.

(b) Let $\theta = \measuredangle r_1rq = \measuredangle r_1rb$, and not losing generality, assume $\theta > 2\delta$. If $\tilde\measuredangle r_1rq \le \theta - 2\delta$, then in $\tilde\triangle rr_1q$,

$$\begin{aligned} |r_1q| &= (|rq|^2 + |rr_1|^2 - 2|rq||rr_1|\cos\tilde\measuredangle r_1rq)^{1/2} \\ &\le |rq| - |rr_1|\cos\tilde\measuredangle r_1rq + \frac{1}{1-\delta^2}\cdot\frac{|rr_1|^2}{2|rq|}\sin^2\tilde\measuredangle r_1rq \\ &\le |rq| - |rr_1|\cos(\theta-2\delta) + \frac{\delta^2}{2(1-\delta^2)}\cdot|rr_1|. \end{aligned} \tag{2.14}$$

Similarly, in $\tilde\triangle arr_1$, $\tilde\measuredangle arr_1 \le \measuredangle arr_1 = \pi - \theta$.

$$\begin{aligned} |ar_1| &= (|ar|^2 + |rr_1|^2 - 2|ar||rr_1|\cos\tilde\measuredangle arr_1)^{1/2} \\ &\le |ar| - |rr_1|\cos(\pi-\theta) + \frac{\delta^2}{2(1-\delta^2)}\cdot|rr_1| \\ &= |ar| + |rr_1|\cos\theta + \frac{\delta^2}{2(1-\delta^2)}\cdot|rr_1|. \end{aligned} \tag{2.15}$$

Summing up (2.14) and (2.15), we get

$$\begin{aligned} |aq| &\le |r_1q| + |ar_1| \\ &\le |rq| + |ar| - |rr_1|\cdot(\cos\theta - \cos(\theta-2\delta)) + \frac{\delta^2}{1-\delta^2}\cdot|rr_1| \\ &= |aq| + |rr_1|\cdot(-2\sin(\theta-\delta)\sin\delta) + \frac{\delta^2}{1-\delta^2}\cdot|rr_1|, \end{aligned} \tag{2.16}$$

or equivalently,

$$2\sin(\theta-\delta)\sin\delta \le \frac{\delta^2}{1-\delta^2}. \tag{2.17}$$

This is a contradiction for small $\delta > 0$.

(c) To show the desired inequality, we need the following estimate:

$$0 \leq \widetilde{\measuredangle} p_1qp - \widetilde{\measuredangle} r_1qr < 5\delta \frac{|rr_1|}{|rq|}. \tag{2.18}$$

By (a) and condition (B) and (C),

$$\widetilde{\measuredangle} r_1qr \leq \widetilde{\measuredangle} p_1qp \leq \measuredangle r_1qr < (\sin \measuredangle r_1rq + 2\delta)\frac{|rr_1|}{|rq|}, \tag{2.19}$$

consequently,

$$\widetilde{\measuredangle} p_1qp + \widetilde{\measuredangle} r_1qr < (2\sin \measuredangle r_1rq + 4\delta)\frac{|rr_1|}{|rq|}. \tag{2.20}$$

On the other hand, in $\widetilde{\triangle} r_1qr$, $|r_1q| \sin \widetilde{\measuredangle} r_1qr = |rr_1| \sin \widetilde{\measuredangle} r_1rq$. Plugging (b) into this equation, we get

$$\begin{aligned}\widetilde{\measuredangle} r_1qr &\geq \sin \widetilde{\measuredangle} r_1qr = \sin \widetilde{\measuredangle} r_1rq \cdot \frac{|rr_1|}{|r_1q|} \\ &\geq \sin(\measuredangle r_1rq - 2\delta) \cdot \frac{|rr_1|}{|r_1q|} \geq (\sin \measuredangle r_1rq - 2\delta) \cdot \frac{|rr_1|}{|r_1q|}. \end{aligned} \tag{2.21}$$

Combining (2.19) and (2.21) we get (2.18). Now let $\dfrac{|pq|}{|rq|} = \dfrac{|p_1q|}{|r_1q|} = t$, then

$$\begin{aligned}|pp_1|^2 &= |pq|^2 + |p_1q|^2 - 2|pq||p_1q| \cos \widetilde{\measuredangle} p_1qp \\ &= t^2(|rq|^2 + |r_1q|^2 - 2|rq||r_1q| \cos \widetilde{\measuredangle} p_1qp) \\ &= t^2(|rr_1|^2 + 2|rq||r_1q| \cdot (\cos \widetilde{\measuredangle} r_1qr - \cos \widetilde{\measuredangle} p_1qp)), \end{aligned} \tag{2.22}$$

or equivalently,

$$\frac{|pp_1|^2}{t^2|rr_1|^2} - 1 = \frac{2|rq||r_1q|}{|rr_1|^2} \cdot (\cos \widetilde{\measuredangle} r_1qr - \cos \widetilde{\measuredangle} p_1qp). \tag{2.23}$$

By (2.18) and (2.20) and select δ such that $(\sin \measuredangle r_1rb + 2\delta)\frac{|rr_1|}{|rq|} < (1+2\delta)\delta^2 < \frac{\pi}{2}$, we get that

$$\begin{aligned}|\cos \widetilde{\measuredangle} r_1qr - \cos \widetilde{\measuredangle} p_1qp| &= 2\sin \frac{|\widetilde{\measuredangle} r_1qr + \widetilde{\measuredangle} p_1qp|}{2} \cdot \sin \frac{|\widetilde{\measuredangle} r_1qr + \widetilde{\measuredangle} p_1qp|}{2} \\ &< 2\sin\left((\sin \measuredangle r_1rb + 2\delta)\frac{|rr_1|}{|rq|}\right) \cdot \sin\left(3\delta\frac{|rr_1|}{|rq|}\right) \\ &\leq 2(1+2\delta)3\delta \cdot \frac{|rr_1|^2}{|rq|^2} < 7\delta \cdot \frac{|rr_1|^2}{|rq|^2}. \end{aligned} \tag{2.24}$$

Plugging (2.24) in to (2.23), we get

$$\left|\frac{|pp_1|}{t|rr_1|} - 1\right| \leq \left|\frac{|pp_1|^2}{t^2|rr_1|^2} - 1\right| < 14\delta \cdot \frac{|r_1q|}{|rq|} < 15\delta.$$

(d) The lower bound

$$\measuredangle p_1pq \geq \tilde{\measuredangle} p_1pq \geq \tilde{\measuredangle} r_1rq - \delta > \measuredangle r_1rq - 3\delta \tag{2.25}$$

follows easily from (b) and (c). It remains to show $\measuredangle p_1pq < \measuredangle r_1rq + 3\delta$. Apply condition (A) on $\triangle rpp_1$, we get

$$|rp_1|^2 \leq |rp|^2 + |pp_1|^2 + 2|rp||pp_1| \cos \measuredangle p_1pq. \tag{2.26}$$

Now consider the comparison triangle $\tilde{\triangle} rr_1q$ in $\mathbb{R}^2$. Take $\tilde{p}$ on $\overline{\tilde{r}\tilde{q}}$ and $\tilde{p}_1$ on $\overline{\tilde{r}_1\tilde{q}}$ such that $|\tilde{r}\tilde{p}| = |rp|$ and $|\tilde{r}_1\tilde{p}_1| = |r_1p_1|$. It's clear that $|\tilde{p}\tilde{p}_1| = |rr_1| \cdot \frac{|pq|}{|rq|}$, $\measuredangle \tilde{p}_1\tilde{p}\tilde{q} = \tilde{\measuredangle} r_1rq \leq \measuredangle r_1rq$ and $|rp_1| \geq |\tilde{r}\tilde{p}_1|$ by condition (A). Thus

$$\begin{aligned} |rp_1|^2 &\geq |\tilde{r}\tilde{p}_1|^2 = |\tilde{r}\tilde{p}|^2 + |\tilde{p}\tilde{p}_1|^2 + 2|\tilde{r}\tilde{p}||\tilde{p}\tilde{p}_1| \cos \measuredangle \tilde{p}_1\tilde{p}\tilde{q} \\ &\geq |rp|^2 + \left(|rr_1| \cdot \frac{|pq|}{|rq|}\right)^2 + 2|rp| \left(|rr_1| \cdot \frac{|pq|}{|rq|}\right) \cos \measuredangle r_1rq. \end{aligned} \tag{2.27}$$

Combine (2.26) and (2.27), and get

$$\frac{|pp_1|^2}{|rr_1|^2} + 2\frac{|rp||pp_1|}{|rr_1|^2} \cos \measuredangle p_1pq \geq \frac{|pq|^2}{|rq|^2} + 2\frac{|rp||pq|}{|rr_1||rq|} \cos \measuredangle r_1rq.$$

By (c),

$$(1+15\delta)^2 \frac{|pq|}{|rq|} + 2\frac{|rp|}{|rr_1|} \cos \measuredangle p_1pq \geq \frac{|pq|}{|rq|} + 2\frac{|rp|}{|rr_1|} \cos \measuredangle r_1rq,$$

or,

$$16\delta^3 \cdot \frac{|pq|}{|rp|} \geq 16\delta \cdot \frac{|pq||rr_1|}{|rq||rp|} \geq \cos \measuredangle r_1rq - \cos \measuredangle p_1pq.$$

If $\measuredangle p_1pq \geq \measuredangle r_1rq + 3\delta$, then

$$0 = \lim_{\delta \to 0} 16\delta^2 \cdot \frac{|pq|}{|rp|} \geq \lim_{\delta \to 0} \frac{\cos \measuredangle r_1rq - \cos(\measuredangle r_1rq + 3\delta)}{\delta} = \sin \measuredangle r_1rq,$$

a contradiction. □

Proof of Theorem 2.6.10. Let r, p, q be points on the geodesic $\overline{ab}$ such that $|qr| < \delta \min\{|ar|, |bq|\}$, and the points r_1, r_2 be near to r with $|rr_j| < \delta^2|rp|$, $j = 1, 2$. Let p_1, p_2 lie on the geodesics $\overline{qr_1}, \overline{qr_2}$ so that $\frac{|qp_1|}{|qr_1|} = \frac{|qp_2|}{|qr_2|} = \frac{|qp|}{|qr|}$. Then by Lemma 2.6.11(c), $\left|\frac{|pp_j|}{|rr_j|} \cdot \frac{|rq|}{|pq|} - 1\right| < 15\delta$, $j = 1, 2$. Therefore, since $\dfrac{|p_1p_2|}{|r_1r_2|} \geq \dfrac{|qp|}{|qr|}$ by condition (B), it's not difficult to check that

$$\frac{\cos \tilde{\measuredangle} p_1pp_2}{\cos \tilde{\measuredangle} r_1rr_2} \leq \left(\frac{1+15\delta}{1-15\delta}\right)^2.$$

Let $\delta \to 0$ (which also forces $r_j \to r$ along the geodesic $\overline{rr_j}$) we get $\measuredangle p_1pp_2 \geq \measuredangle r_1rr_2$, and this will imply $\Sigma_p \geq \Sigma_r$. When switch q to be between ar and apply an analogous setup, we get $\Sigma_p \leq \Sigma_r$. Thus $\Sigma_p = \Sigma_r$ by Proposition 2.6.8. □

2.6.3 Conventions and notations

We now summarize the notations we have used so far and introduce some new ones which will be frequently used in the rest of this thesis.

(1) Let $\Sigma_p' \subset \Sigma_p$ be the collection of directions in which there is a geodesic goes out. Let $\overrightarrow{pq} \in \Sigma_p$ denote one of the directions of the geodesics jointing p, q. If A is a subset, let $\Gamma_A^p \subset \Sigma_p$ denote the directions $\{\overrightarrow{pa} \in \Sigma_p : a \in A, a \neq p\}$.

(2) We let $c(a, b, \dots)$ denote positive constant depending on $a, b, \dots$. If just say c, it means a constant does not depending on anything, or determined arbitrary.

(3) We let $\chi(\delta, \sigma, \dots)$ denote the positive function of $\delta, \sigma, \dots$ (but may depend on other parameters), where $\chi(\delta, \sigma, \dots) \to 0$ as $\delta, \sigma, \dots \to 0$ for any fixed values of the other parameters.

(4) We let $\text{sn}_\kappa(r)$ denote the canonical trigonometric functions on S_κ^2, that is,

$$\text{sn}_\kappa(r) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} \cdot r), & \text{for } \kappa > 0, \\ r, & \text{for } \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa} \cdot r), & \text{for } \kappa < 0. \end{cases}$$

The cosine law for a triangle $\triangle pab$ in S_κ^2 is

$$\begin{cases} \kappa = 1, & \cos|ab| = \cos|pa| \cos|pb| + \sin|pa| \sin|pb| \cos \measuredangle apb; \\ \kappa = 0, & |ab|^2 = |pa|^2 + |pb|^2 - 2|pa||pb| \cos \measuredangle apb; \\ \kappa = -1 & \cosh|ab| = \cosh|pa| \cosh|pb| - \sinh|pa| \sinh|pb| \cosh \measuredangle apb, \end{cases}$$

or equivalently (by several steps of applying trigonometric identities),

$$\text{sn}_\kappa^2 \frac{|ab|}{2} = \text{sn}_\kappa^2 \frac{|pa| - |pb|}{2} + \sin^2 \frac{\measuredangle apb}{2} \text{sn}_\kappa |pa| \text{sn}_\kappa |pb|.$$

The second type is convenience in some comparison cases since it only consists of increasing functions.

2.7 Rough volume and Hausdorff measure

2.7.1 Rough volume

Rough volume is not a measure, since it may not have countable additivity. For example, let Q be the rational numbers in $[0,1]$. Then $V_{r_1}(Q) = 1$, but $V_{r_1}(x) = 0$ for any $x \in Q$. However, one of the reasons to define rough volume is that the Hausdorff measure is not easy to compute or estimate in the lack of smooth coordinates, but for a subset in $X \in \mathrm{Alex}^n(\kappa)$ one can give an upper bound of the n-dimensional rough volume depending on an arbitrary point p and the $(n-1)$-dimensional rough volume of the directions (a subset of Σ_p) from the point to the subset. To state the result, let's first introduce a function $\psi(\kappa, D)$, $D > 0$ defined as:

$$\psi(\kappa, D) = \max_{q,p,r \in S^2_\kappa} \left\{ \frac{|pr|}{\measuredangle pqr},\ |qp|, |qr|, |pr| \le D, |pr| \ge 2||qp| - |qr|| \right\}.$$

In fact, we have (see Lemma 3.3.3)

$$\frac{2}{3} \cdot \mathrm{sn}_\kappa(D) \le \psi(\kappa, D) \le 2 \cdot \mathrm{sn}_\kappa(D),$$

provided $D < \frac{\pi}{2\sqrt{\kappa}}$ when $\kappa > 0$. If $\kappa > 0$ and $D \ge \frac{\pi}{2\sqrt{\kappa}}$, it's easy to see that $\psi(\kappa, D) = \psi(\kappa, \frac{\pi}{2}) = \lim_{d \to \frac{\pi}{2}-} \psi(\kappa, d)$. We will often omit κ in the function ψ in this section.

Lemma 2.7.1. *Let $p \in M \in Alex^n(\kappa)$, $A \subset M$ and Γ^p_A be defined as in Chapter 2.6.3. Then*

$$V_{r_n}(A) \le V_{r_{n-1}}(\Gamma^p_A) \cdot 2D_1 \cdot \psi^{n-1}(D),$$

where $D = diam(A \cup \{p\})$, $D_1 = \max_{a \in A} |ap| - \min_{a \in A} |ap|$.

Remark 2.7.2. By a different approach, Theorem 3.C gives a better estimate but with some priori conditions on A.

$$V_{r_n}(A) \le c(n) \cdot V_{r_{n-1}}(\Gamma^p_A) \cdot \int_{d_1}^{d_2} \mathrm{sn}^{n-1}_\kappa(t)dt,$$

where $d_1 = \min_{a \in A} |ap|$, $d_2 = \max_{a \in A} |ap|$.

Proof of Lemma 2.7.1. Assume $\beta_A(\epsilon)$ is the maximal number of ϵ-net in A. Consider the distribution of these points between the balls $B_p(d_1 + j\epsilon)$, $j = 1, 2, \ldots, \frac{d_2}{\epsilon}$, where

$d_1 = \min_{a \in A} |ap|$ and $d_2 = \max_{a \in A} |ap|$. There are at least $\beta_A(\epsilon) \cdot \left(\frac{2D_1}{\epsilon} + 1\right)^{-1}$ of them such that their distance to p differ pairwise by not more than $\frac{\epsilon}{2}$. Thus by condition (C) we get $\beta_A(\epsilon) \cdot \left(\frac{2D_1}{\epsilon} + 1\right)^{-1}$ points in Γ_A^p at a pairwise distance (which is the angle between geodesics) of at least $\frac{\epsilon}{\psi(D)}$. Therefore we obtain the inequality

$$\beta_{\Gamma_A^p}\left(\frac{\epsilon}{\psi(D)}\right) \geq \beta_A(\epsilon)\left(\frac{2D_1}{\epsilon} + 1\right)^{-1}$$

or

$$\left(\frac{\epsilon}{\psi(D)}\right)^{n-1} \cdot \beta_{\Gamma_A^p}\left(\frac{\epsilon}{\psi(D)}\right) \geq \beta_A(\epsilon)\left(\frac{2D_1}{\epsilon} + 1\right)^{-1}\left(\frac{\epsilon}{\psi(D)}\right)^{n-1}.$$

Let $\epsilon \to 0$, we get the assertion of the Lemma. □

Corollary 2.7.3. *For $X \in Alex^n(\kappa)$, we have the bound $V_{r_n}(X) \leq c(n, \kappa, diam(X))$. In addition, $\beta_X(\epsilon) \leq c(n, \kappa, diam(X)) \cdot \epsilon^{-n}$ for all $\epsilon > 0$.*

The proof is carried out by the proof of Lemma 2.7.1 and an induction taking into account that diam$M \leq \frac{\pi}{\sqrt{\kappa}}$.

We will give a generalization in Chapter 3 of the following:

Proposition 2.7.4. *Let $X \in Alex^n(\kappa)$. If γ_m is an m-broken geodesic loop, then the n-dimensional rough volume,*

$$V_{r_n}(X) \leq \chi_m(\delta_1, \delta) \cdot d \cdot \psi^{n-1}(\kappa, d),$$

where $d = diam(X)$, $\delta_1 = \frac{1}{diam(X)} \max\{|p_i p_{i+1}|, 1 \leq i \leq m\}$, $\max_i\{\theta_i\} \leq \delta$ and $\chi_m(\delta_1, \delta)$ is a constant depending on m, δ_1 and δ such that $\chi_m(\delta_1, \delta) \to 0$ as $\delta_1, \delta \to 0$ (m fixed).

2.7.2 Some results on Hausdorff measure and Hausdorff dimension

Let Haus$_n$ denote the n-dimensional Hausdorff measure and $B_r(S_\kappa^n)$ be the open r-ball in S_κ^n. We now state some results without giving the detailed proofs. The proof can be found in [BGP].

Theorem 2.7.5. *Let $X \in Alex^n(\kappa)$. Then for any $p \in X$ and $r > 0$, $Haus_n(B_r(p)) \leq Haus_n(B_r(S_\kappa^n))$. The equality holds if the open ball $B_r(p))$ is isometric to $B_r(S_\kappa^n)$ in* *terms of their intrinsic metric. In particular, if* $\kappa > 0$ *and* $r = \frac{\pi}{\sqrt{\kappa}}$ *in which case* $Haus_n(X) = Haus_n(S^n_\kappa)$*, we have that* X *is isometric to* S^n_κ*.*

Theorem 2.7.6. *Let* p *be a point in* $X \in Alex^n(\kappa)$*. Then the ratio* $\dfrac{Haus_n(B_r(p))}{Haus_n(B_r(S^n_\kappa))}$ *is a non-increasing function of* $r > 0$*.*

We omit the proofs for these two theorem, however, we note that the proofs mentioned in [BGP] rely on a "singular" version of co-area formula. For the isometry part in Theorem 2.7.5, the co-area formula is used to reduce the n-dimensional Hausdorff measure to the $(n-1)$-dimensional Hausdorff measure on the cross section $S_r = \{x \in X : |px| = r\}$, so that the induction can be applied. However, this idea can not be carried out in our situation in Chapter 4.

If A itself is an Alexandrov space, then $\dim_H(A) = \dim_r(A)$ (see Proposition 2.5.5). For a subset A of $X \in \text{Alex}^n(\kappa)$, we only have $\dim_H(A) \le \dim_r(A)$.

Theorem 2.7.7. *Let* $X \in Alex^n(\kappa)$ *and* X^δ_m *denote the collection of all* (m, δ)*-burst points. Then* $\dim_H(X - X^\delta_m) \le m - 1$*.*

For technical reason, we define the boundary points in $X \in \text{Alex}^n(\kappa)$ inductively as the following way.

Definition 2.7.8. One dimensional Alexandrov space is a manifold (circle or interval), we define the boundary as the way in manifold. For $X \in \text{Alex}^n(\kappa)$, a point $p \in X$ is said to be a boundary point if Σ_p has boundary. If not so, the point p is called the interior point.

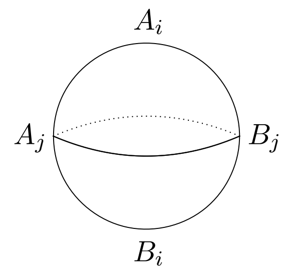
Theorem 2.7.9. *Let* $X \in Alex^n(\kappa)$ *and* $N^\delta_1 = \{q \in X - X^\delta_m$ *and* q *is an interior point*$\}$*. Then* $\dim_H(N^\delta_1) \le n - 2$*.*

2.8 A theorem on almost isometry at (n, δ)-burst points

Recall that the map $\varphi(q) = (|a_1q|, |a_2q|, \dots, |a_nq|)$ defined in a neighborhood U of the n-burst point p is a bi-Lipschitz homeomorphism between U and a domain in $\mathbb{R}^n$, provided that (a_i, b_i) is an (n, δ)-explosion for δ small (see Theorem 2.4.2). In this section we will show that such map is an almost isometry depending on δ, the size of the neighborhood and the diameter of the explosion.

Definition 2.8.1. We say that a complete space $X \in \mathrm{Alex}^{n-1}(1)$ has an (m,δ)-explosion (A_i, B_i), $1 \le i \le m \le n$, if $A_i, B_i \subset X$ are compact subsets such that

$$|A_i, B_i| > \pi - \delta, \qquad |A_i, B_j| > \frac{\pi}{2} - \delta,$$
$$|A_i, A_j| > \frac{\pi}{2} - \delta, \qquad |B_i, B_j| > \frac{\pi}{2} - \delta, \qquad (2.28)$$



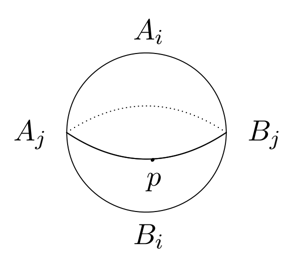
for any $i \neq j$.

Comparing to Definition 2.4.1 which is for a point, this defines explosion over the whole space and the maximal number of pairs can be up to $n = \dim(X) + 1$. Clearly a point $p \in X \in \mathrm{Alex}^n(\kappa)$ has an (m,δ)-explosion if and only if its space of directions $\Sigma_p \in \mathrm{Alex}^{n-1}(1)$ has an (m,δ)-explosion.

We list the technical lemmas needed in the following proofs. The proof of these assertions is based directly on the triangle comparison and some elementary spherical geometry (see the graphs).

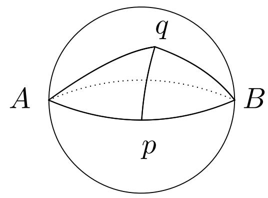
Lemma 2.8.2. *Let $X \in Alex^{n-1}(1)$.*

(1) Let the sets (A_i, B_i) form an (m,δ)-explosion in X; $p \in X$ such that $|pA_i| > \frac{\pi}{2} - \delta$, $|pB_i| > \frac{\pi}{2} - \delta$ for all i. Then the sets $(\Gamma^p_{A_i}, \Gamma^p_{B_i})$ form an $(m, \chi(\delta))$-explosion in Σ_p.

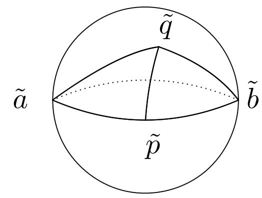


(2) Let (A, B) forms a $(1,\delta)$-explosion in X; $p, q \in X$ such that $|pA| > \frac{\pi}{2} - \delta$, $|pB| > \frac{\pi}{2} - \delta$. Let the points $\tilde{a}, \tilde{b}, \tilde{p}, \tilde{q}$ be given on the unit sphere S^2 such that $|\tilde{a}\tilde{b}| = \pi$, $|\tilde{p}\tilde{a}| = |\tilde{p}\tilde{b}| = \frac{\pi}{2}$, $||pq| - |\tilde{p}\tilde{q}|| < \delta$, and also

$$|\measuredangle \tilde{a}\tilde{p}\tilde{q} - |\Gamma^p_A \Gamma^p_q|| < \delta, \quad |\measuredangle \tilde{b}\tilde{p}\tilde{q} - |\Gamma^p_B \Gamma^p_q|| < \delta, \qquad (2.29)$$



$X \in Alex^{n-1}(1)$



S^2

where $\Gamma_A^p, \Gamma_B^p \subset \Sigma_p$ *are defined as in Chapter 2.6.3. Then*

$$||Aq| - |\tilde{a}\tilde{q}|| < \chi(\delta) \text{ and } ||Bq| - |\tilde{b}\tilde{q}|| < \chi(\delta). \tag{2.30}$$

(3) Let conditions (b) be all satisfied except for (2.29) and assume instead that

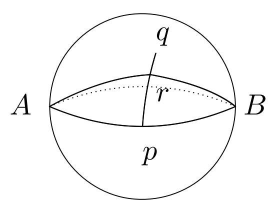
$$||Aq| - |\tilde{a}\tilde{q}|| < \delta, \quad ||Bq| - |\tilde{b}\tilde{q}|| < \delta. \tag{2.31}$$

Then either for the direction $\Gamma_q^p \in \Sigma_p$ *we have*

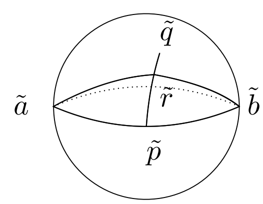
$$|\measuredangle \tilde{a}\tilde{p}\tilde{q} - |\Gamma_A^p \Gamma_q^p|| < \chi(\delta), \text{ and } |\measuredangle \tilde{b}\tilde{p}\tilde{q} - |\Gamma_B^p \Gamma_q^p|| < \chi(\delta), \tag{2.32}$$

or $|pq| > \pi - \chi(\delta)$.

(4) Let conditions (c) be satisfied and let there be given points r *on the geodesic* $\overline{pq}$ *and* $\tilde{r}$ *on* $\overline{\tilde{p}\tilde{q}}$ *so that* $||pr| - |\tilde{p}\tilde{r}|| < \delta$.



$X \in Alex^{n-1}(1)$



S^2

Then either

$$||Ar| - |\tilde{a}\tilde{r}|| < \chi(\delta) \text{ and } ||Br| - |\tilde{b}\tilde{r}|| < \chi(\delta), \tag{2.33}$$

or $|Ap| + |Aq| + |Ar| > 2\pi - \chi(\delta)$, *or* $|Bp| + |Bq| + |Br| > 2\pi - \chi(\delta)$.

Lemma 2.8.3. *Let* $X \in Alex^{n-1}(1)$ *have an* (n, δ)*-explosion* (A_i, B_i)*. Then for any point* $q \in X$*, we have*

$$\left| \sum_{i=1}^{n} \cos^2 |A_i q| - 1 \right| < \chi(\delta).$$

Proof. We use induction with respect to the dimension. If $n = 2$,

$$\begin{aligned}\sum_{i=1}^{2} \cos|A_i q| - 1 &= \cos^2|A_1 q| + \cos^2|A_2 q| - 1 \\ &= \frac{1+\cos 2|A_1 q|}{2} + \frac{1+\cos 2|A_2 q|}{2} - 1 \\ &= \frac{1}{2}(\cos 2|A_1 q| + \cos 2|A_2 q|) \\ &= \cos(|A_1 q| + |A_2 q|)\cos(|A_1 q| - |A_2 q|).\end{aligned}$$

Because $\dim X = 1$, X is either an interval or a circle, either $|A_1 q| + |A_2 q| = |A_1 A_2|$ or $|A_1 q| - |A_2 q| = |A_1 A_2|$, which is close to $\frac{\pi}{2}$.

Note that in the unit sphere S_1^{n-1}, if take (a_i, b_i) as an $(n-1, 0)$-explosion, i.e. $|a_i b_i| = \pi$, $|a_i b_j| = |a_i a_j| = |b_i b_j| = \frac{\pi}{2}$, $1 \le i \le n$, then for any $p \in S_1^{n-1}$, $\sum_{i=1}^{n} \cos^2|pa_i| = 1$. Now let point $p \in A_n$ and construct an $(n-1, 0)$-explosion $\tilde{A}_i, \tilde{B}_i$ $(1 \le i \le n-1)$ on the unit sphere S_1^{n-1}. Take $\tilde{p} \in S_1^{n-1}$ with $|\tilde{p}\tilde{A}_i| = |\tilde{p}\tilde{B}_i| = \frac{\pi}{2}$ for all i. Take $\tilde{q} \in S_1^{n-1}$ with $|\tilde{p}\tilde{q}| = |pq|$ and $|\measuredangle \tilde{A}_i \tilde{p}\tilde{q} - |\Gamma^p_{A_i}\Gamma^p_q|| < \chi(\delta)$ for all i by solving the following $(n-1)$-system: $|\tilde{p}\tilde{q}| = |pq|$, $\measuredangle \tilde{A}_i \tilde{p}\tilde{q} = |\Gamma^p_{A_i}\Gamma^p_q|$, $i = 1, 2, \ldots n-2$. By the inductive hypothesis, the $(n-1)$th inequality $|\measuredangle \tilde{A}_{n-1}\tilde{p}\tilde{q} - |\Gamma^p_{A_{n-1}}\Gamma^p_q|| < \chi(\delta)$ is automatically satisfied . If we also can show that $|\measuredangle \tilde{B}_i \tilde{p}\tilde{q} - |\Gamma^p_{B_i}\Gamma^p_q|| < \chi(\delta)$, then by Lemma 2.8.2 (2), we get that $||A_i q| - |\tilde{A}_i \tilde{q}|| < \chi(\delta)$, which implies that

$$\begin{aligned}\left|\sum_{i=1}^{n} \cos^2|A_i q| - 1\right| &= \left|\sum_{i=1}^{n} \cos^2|A_i q| - \left(\sum_{i=1}^{n-1} \cos^2|\tilde{A}_i \tilde{q}| + \cos^2|\tilde{p}\tilde{q}|\right)\right| \\ &= \left|\sum_{i=1}^{n-1} \cos^2|A_i q| - \sum_{i=1}^{n-1} \cos^2|\tilde{A}_i \tilde{q}|\right| < \chi(\delta).\end{aligned}$$

Now let's check $|\measuredangle \tilde{B}_i \tilde{p}\tilde{q} - |\Gamma^p_{B_i}\Gamma^p_q|| < \chi(\delta)$. By Lemma 2.8.2 (1), $(\Gamma^p_{A_i}, \Gamma^p_{B_i})$ form an $(n-2, \chi(\delta)$-explosion in Σ_p, in particular,

$$|\Gamma^p_{A_i}\Gamma^p_q| + |\Gamma^p_{B_i}\Gamma^p_q| \ge |\Gamma^p_{A_i}\Gamma^p_{B_i}| > \pi - \chi(\delta).$$

Plugging this into $|\Gamma^p_{A_i}\Gamma^p_q| + |\Gamma^p_{B_i}\Gamma^p_q| + |\Gamma^p_{A_i}\Gamma^p_{B_i}| \le 2\pi$, we get

$$\left||\Gamma^p_{A_i}\Gamma^p_q| + |\Gamma^p_{B_i}\Gamma^p_q| - \pi\right| < \chi(\delta).$$

Clearly, $\measuredangle \tilde{A}_i \tilde{p}\tilde{q} + \measuredangle \tilde{B}_i \tilde{p}\tilde{q} = \pi$. Thus $|\measuredangle \tilde{B}_i \tilde{p}\tilde{q} - |\Gamma^p_{B_i}\Gamma^p_q|| < \chi(\delta)$. □

Theorem 2.8.4. *Let $X \in Alex^n(\kappa)$ and $p \in X$ have an (n,δ)-explosion (a_i, b_i). Then the map $f : X \to \mathbb{R}^n$ given by $f(q) = (|a_1q|, \ldots, |a_nq|)$ maps a small neighborhood U of the point p almost isometrically onto a domain in $\mathbb{R}^n$, i.e. $\left|\frac{|f(q)f(r)|}{|qr|} - 1\right| < \chi(\delta, \delta_1)$ for any points $q, r \in U$, where*

$$\delta_1 = \max_{1\le i\le n} \left\{|pa_i|^{-1} diamU, \quad |pb_i|^{-1} diamU\right\}.$$

Proof. Let's first investigate the term $\frac{|f(q)f(r)|}{|qr|} = \sum_{i=1}^{n} \frac{||a_iq| - |a_ir||}{|qr|}$. Consider the triangle $\triangle a_irq$ and let $a = |rq|$, $b = |a_ir|$, and $c = |a_iq|$. Then $a < \delta_1 c$ and we have

$$\begin{aligned}\cos\tilde{\measuredangle}a_iqr &= \frac{a^2+c^2-b^2}{2ac}\\ &= \frac{c^2-b^2}{2ac} + \frac{a}{2c} = \frac{c-b}{a} - \frac{(c-b)^2}{2ac} + \frac{a}{2c}.\end{aligned}$$

Since $\frac{(c-b)^2}{2ac} + \frac{a}{2c} \le \frac{a^2}{2ac} + \frac{a}{2c} = \frac{a}{c} < \delta_1$, we get

$$\left|\cos^2\tilde{\measuredangle}a_iqr - \frac{(|a_iq| - |a_ir|)^2}{|qr|^2}\right| < \chi(\delta_1).$$

Thus it's sufficient to show that $\left|\sum_{i=1}^{n}\cos^2\tilde{\measuredangle}a_iqr - 1\right| < \chi(\delta, \delta_1)$. By Lemma 2.4.3 and selecting U small, we get $|\measuredangle a_iqr - \tilde{\measuredangle}a_iqr| < \chi(\delta, \delta_1)$. By Lemma 2.8.3 we have $\left|\sum_{i=1}^{n}\cos^2\measuredangle a_iqr - 1\right| < \chi(\delta, \delta_1)$. Thus the desired inequality holds. □

Chapter 3

Bounding Geometry of Loops in Alexandrov Spaces

The goal of this Chapter is to prove Theorem 3.A - 3.C.

Let's first define the turning angle.

Definition 3.0.5. Let $c : [0,1] \to X$ be a continuous curve. Given a partition, $P : 0 = t_1 < \cdots < t_{m+1} = 1$, let $p_i = c(t_i)$, and let $\gamma_m = \{[p_ip_{i+1}]\}_{i=1}^m$ denote an m-broken geodesic, $\gamma_m|_{[t_i,t_{i+1}]} = [p_ip_{i+1}]$, a minimal geodesic jointing p_i and p_{i+1}. We call the following number,

$$\Theta(c) = \lim_{m\to\infty} \sup_{|P|=m} \{\sum_{i=1}^m \theta_i\},$$

the turning angle of c, where $\theta_i = \pi - \measuredangle p_{i-1}p_ip_{i+1}$ and $\theta_1 = \measuredangle p_{m+1}p_1p_2$ for $p_{m+1} = p_1$ (the loop case) and $\theta_1 = 0$ otherwise. For convenience, we assign 2π as the turning angle of a trivial loop.

Clearly, a curve is a geodesic if and only if $\Theta(c) = 0$, and thus $\Theta(c)$ measures the closeness of a curve from a geodesic. An m-broken geodesic γ_m has a finite turning angle $\Theta(\gamma_m) = \sum_{i=2}^m \theta_i$. If M is a Riemannian manifold, then any C^2-curve c on M satisfies that $\Theta(c) = \int_0^1 |\nabla_{c'}c'|dt$. Because an Alexandrov space in general may not contain any closed geodesic (nor any m-broken geodesic loop with small turning angle; e.g., a flat cone), a loop with the minimal turning angle should be the counterpart of a closed geodesic on a (closed) Riemannian manifold.

We now give an indication for the proof of Theorem 3.A. First, it is worth to note that our arguments also imply a new (metric) proof for Theorem 1.0.1; which does not require a Riemannian structure. Our approach is different from the proof of Proposition 2.7.4 in [BGP] which follows the lines of the proof of Theorem 1.0.1 in [Ch]. Indeed, we found Theorem 3.A after an unsuccessful attempt to remove the dependence on m

from $\chi_m(\delta_1, \delta)$ in Proposition 2.7.4.

We take an elementary approach to estimate $\text{Haus}_n(X)$ (say the case $r = \text{diam}(X)$): expressing $\text{Haus}_n(X)$ as a 'Riemann sum', bounding each term and evaluating the "Riemann sum" of the bounds via identifying a proper integrant. Let $\gamma_m = \{[p_ip_{i+1}]\}_{i=1}^m$ be an m-broken geodesic loop approximating to a loop c in Theorem 3.A, and divide $X = \bigcup_{i=1}^m X_i$ such that $\text{Haus}_n(X) = \sum_{i=1}^m \text{Haus}_n(X_i)$, where $X_i = \{x \in X \ |xp_i| \le |xp_j|, \text{ for all } 1 \le j \ne i \le m\}$. Observe that if γ_m is a closed geodesic and $|p_ip_{i+1}|$ is sufficiently small, then X_i is like the 'union of normal slices' over $[p_ip_{i+1}]$ (when X is a Riemannian manifold). So in spirit, we are estimating $\text{Haus}_n(X)$ via a Riemann sum of a double integral: first over a normal slice at $\gamma_m(t)$, followed by integral over γ_m. To obtain a sharp estimate for $\text{Haus}_n(X_i)$, we establish a basic Hausdorff measure estimate (see Lemma 3.1.2), which bounds the Hausdorff measure of any subset $A \subseteq X$ in terms of the Hausdorff measure of the space of directions at any point $p \in X$, $|pA|$ and $\text{diam}(A \cup \{p\})$. Note that this result also substantially improves a basic rough volume estimate in [BGP] (Lemma 8.2 in [BGP]). The key point in our proof is an estimate of the maximal and minimal angles between some fixed direction and all directions in $\Gamma_{p_i} = \{[p_ix] \subseteq \Sigma_{p_i}(X),\ x \in X_i - \{p_i\}\}$, in which we find a (right) link between $\measuredangle xp_ip_{i+1}$ and $|xp_i|$ (see Lemma 3.1.3). The main ingredient in the proofs of Lemmas 3.1.2 and 3.1.3 is the cosine law in κ-space forms.

In Chapter 3.1, we will prove Theorem 3.A by assuming two technical lemmas.

In Chapter 3.2, we will complete the proof of Theorem 3.A by proving the two technical lemmas.

In Chapter 3.3, we will prove Theorem 3.C.

3.1 Proof of Theorem 3.A (I)

The goal in this section is to prove the following basic estimate modulo two technical results. The proofs of the technical results will be given in Chapter 3.3.

Theorem 3.1.1. *Let $X \in Alex^n(\kappa)$ $(n \ge 2)$. If γ_m is an m-broken geodesic loop at p*

such that $\gamma_m \subset B_r(p)$*, then*

$$Haus_n(B_r(p)) \le vol(S_1^{n-2}) \left[\frac{sn_\kappa^{n-1}(r_0)}{n-1} L(\gamma_m) + \Theta(\gamma_m)\int_0^r sn_\kappa^{n-1}(t)dt\right],$$

where $r_0 = r$ *for* $\kappa \le 0$ *and* $r_0 = \min\{r, \frac{\pi}{2\sqrt{\kappa}}\}$ *for* $\kappa > 0$*, and* $c(n)$ *is constant depending on* n*.*

Theorem 3.1.1 provides a sharp bound for $\mathrm{Haus}_n(B_r(p))$ explicitly in terms of $L(\gamma_m)$ and $\Theta(\gamma_m)$ (comparing to Proposition 2.7.4). Because the bound in Theorem 3.1.1 is independent of m, Theorem 3.1.1 easily implies Theorem 3.A.

Proof of Theorem 3.A by assuming Theorem 3.1.1. Since $p \in C \subset B_r(p))$, we may assume a sequence of m-broken geodesics, $p \in \gamma_m \subset B_r(p)$ (m large), such that $L(\gamma_m) \to L(c)$ and $\Theta(\gamma_m) \to \Theta(c)$, as $m \to \infty$. Applying Theorem 1.1 to γ_m, we get

$$\mathrm{Haus}_n(B_r) \le \mathrm{vol}(S_1^{n-2}) \left[\frac{sn_\kappa^{n-1}(r_0)}{n-1} L(\gamma_m) + \Theta(\gamma_m)\int_0^r \mathrm{sn}_\kappa^{n-1}(t)dt\right]. \tag{3.1}$$

Note that $\max\{\mathrm{sn}_\kappa(r)\} = \mathrm{sn}_\kappa(r_0)$. Then

$$\int_0^r \mathrm{sn}_\kappa^{n-1}(t)dt \le sn_\kappa^{n-1}(r_0)r. \tag{3.2}$$

Plugging (3.2) into (3.1), we derive

$$\mathrm{Haus}_n(B_r(p)) \le \mathrm{vol}(S_1^{n-2})\mathrm{sn}_\kappa^{n-1}(r_0)\left[\frac{L(\gamma_m)}{n-1} + \Theta(\gamma_m)r\right].$$

Taking limit as $m \to \infty$, we obtain the desired inequality. □

Given an m-broken geodesic loop, $p \in \gamma_m = \{[p_ip_{i+1}]\}_{i=1}^m \subset B_r(p)$, we will divide $B_r(p)$ into m subsets,

$$X_i = \{x \in B_r(p),\ \ |xp_i| \le |xp_j|,\ \text{for all } j \ne i\}, \qquad 1 \le i \le m.$$

Clearly, $X_i \subseteq B_r(p_i)$ for all i, $B_r(p) = \bigcup_i X_i$ and $V_{r_n}(B_r(p)) \le \sum_i V_{r_n}(X_i)$. In our estimate for $\mathrm{Haus}_n(X_i)$, we will use the following general estimate.

Lemma 3.1.2. *Let* $X \in Alex^n(\kappa)$*. Given any bounded subset* $A \subseteq X$*, and* $p \in X$*, then*

$$Haus_n(A) \le Haus_{n-1}(\Gamma)\int_{r_1}^{r_2} sn_\kappa^{n-1}(t)dt. \tag{3.3}$$

If A satisfies that $V_{r_n}(A) = V_{r_n}(\mathring{A})$ ($\mathring{A}$ denotes the interior of A), then

$$V_{r_n}(A) \leq b(n) \cdot V_{r_{n-1}}(\Gamma) \int_{r_1}^{r_2} sn_\kappa^{n-1}(t)dt, \tag{3.4}$$

where $r_2 = \max_{x \in A}\{|xp|\}$, $r_1 = \min_{x \in A}\{|px|\}$, $\Gamma_p = \{[px] \in \Sigma_p(X), x \in A - \{p\}\}$ and $b(n) = \frac{V_{r_n}(I^n)}{V_{r_{n-1}}(I^{n-1})}$.

Note that Theorem 3.C actually holds for any open subsets of X (see Remark 3.3.7), and thus (3.3) and (3.4) are equivalent on open subsets. One may compare (3.4) with Lemma 8.2 in [BGP] (see Lemma 3.3.2 in Chapter 3.4); the former gives an explicit sharp inequality.

We will further partition X_i into thin annulus A_{ij}, and use Lemma 3.1.2 to estimate $\mathrm{Haus}_n(A_{ij})$. To estimate $\mathrm{Haus}_{n-1}(\Gamma)$, we shall choose a direction in $\Gamma_{p_i}^j \subseteq \Sigma_{p_i}(X)$ and estimate the maximal and minimal angles of directions in Γ with and the fixed direction, where $\Gamma_{p_i}^j = \{[p_i x] \in \Sigma_{p_i}(X),\ x \in A_{ij} - \{p_i\}\}$. This will be done in the following lemma.

Lemma 3.1.3. *Let the assumptions be as in Theorem 3.1.1. For $\epsilon > 0$, there is $\eta > 0$ such that if $\max_i\{|p_i p_{i+1}|\} < \eta$, then for any $x \in X_i - \{p_i\}$, the following inequality holds:*

$$-\frac{e^\epsilon |p_i p_{i+1}|}{2\tan_\kappa |xp_i|} - \frac{36\eta^{\frac{3}{2}}}{|\tan_\kappa |xp_i||^{\frac{3}{2}}} \leq \measuredangle xp_ip_{i+1} - \frac{\pi}{2} \leq \frac{e^\epsilon |p_i p_{i-1}|}{2\tan_\kappa |xp_i|} + \frac{36\eta^{\frac{3}{2}}}{|\tan_\kappa |xp_i||^{\frac{3}{2}}} + \theta_i,$$

where $\tan_\kappa r = \frac{sn_\kappa r}{sn'_\kappa(r)}$, and when $\kappa > 0$ and $|xp_i| = \frac{\pi}{2\sqrt{\kappa}}$, the first term on the right of the inequality is zero.

It turns out that the inequality in Lemma 3.1.3 is in the right form; based on it we get the explicit sharp estimate in Theorem 3.A.

Using Lemmas 3.1.2 and 3.1.3, we will establish the following basic estimate. The proof of Lemmas 3.1.2 and 3.1.3 will be given in Chapter 3.3.

Proposition 3.1.4. *Let $B_r(p) \subset X \in Alex^n(\kappa)$, and let $[pq]$ denote a geodesic in X from p to q. Given $0 \leq \alpha \leq \pi$, $0 \leq \theta < \pi$ and $L_1 > L_2 > 0$, let*

$$A([pq], \alpha, L_1, L_2, \theta)$$
$$= \{x \in B_r(p) - \{p\},\ \frac{L_2}{\tan_\kappa |xp|} \leq \measuredangle xpq - \alpha + \frac{36\eta^{\frac{3}{2}}}{|\tan_\kappa |xp||^{\frac{3}{2}}} \leq \frac{L_1}{\tan_\kappa |xp|} + \theta\}. \tag{3.5}$$

Then the Hausdorff measure of $A = A([pq], \alpha, L_1, L_2, \theta)$ satisfies

$$Haus_n(A) \le vol(S_1^{n-2}) \left[\frac{(L_1+L_2)sn_\kappa^{n-1}(r_0)}{n-1} + \theta \cdot \int_0^r sn_\kappa^{n-1}(t)dt + O(\eta^{\frac{3}{2}}) \right]$$

where $r_0 = r$ for $\kappa \le 0$ and $r_0 = \min\{r, \frac{\pi}{2\sqrt{\kappa}}\}$ for $\kappa > 0$.

We will give a proof for Proposition 3.1.4 using Lemma 3.1.2.

Proof of Proposition. 3.1.4

Let $A = A([p,q], \alpha, L_1, L_2, \theta)$. Given a partition for $[0,1] : 0 = a_0 < a_1 < \cdots < a_N = 1$, let $r_j = a_j r$, $A_j = \{x \in A,\ r_j \le |xp| \le r_{j+1}\}$, $1 \le j \le N$. If $\kappa > 0$ and $d > \frac{\pi}{2\sqrt{\kappa}}$, we will chose $\{a_j\}$ such that some $r_j = \frac{\pi}{2\sqrt{\kappa}}$ (note that some A_j may be an empty set; for instance, if $\theta = 0$, then $A_j = \varnothing$ when $r_j > \frac{\pi}{2\sqrt{\kappa}}$ because otherwise, $\tan_\kappa |xp_i| < 0$).

For $x \in A_j$,

$$-\frac{L_2}{\tan_\kappa |xp|} - \frac{36\eta^{\frac{3}{2}}}{|\tan_\kappa |xp||^{\frac{3}{2}}} \le \measuredangle xpq - \alpha \le \frac{L_1}{\tan_\kappa |xp|} + \theta + \frac{36\eta^{\frac{3}{2}}}{|\tan_\kappa |xp||^{\frac{3}{2}}}$$

implies

$$-\frac{L_2}{\tan_\kappa(c_j)} - \frac{36\eta^{\frac{3}{2}}}{|\tan_\kappa |xp||^{\frac{3}{2}}} \le \measuredangle xpq - \alpha \le \frac{L_1}{\tan_\kappa(c_j)} + \theta + \frac{36\eta^{\frac{3}{2}}}{|\tan_\kappa |xp||^{\frac{3}{2}}}, \tag{3.6}$$

where $c_j = r_{j+1}$ when $\kappa \le 0$ or $\kappa > 0$ and $r_{j+1} \le \frac{\pi}{2\sqrt{\kappa}}$, otherwise $c_j = r_j$. Let $\Gamma_j = \{[xp] \in \Sigma_p(X),\ x \in A_j\}$. Because $\mathrm{curv}(\Sigma_{[pq]}(\Gamma_j)) \ge 1$, $\mathrm{vol}(\Sigma_{[pq]}(\Gamma_j)) \le \mathrm{vol}(S_1^{n-2})$, where $\Sigma_{[pq]}(\Gamma_j)$ denotes the space of directions of Γ_j at $[pq] \in \Gamma_j$. Applying Lemma 3.1.2 to Γ_j at $[pq]$, by $\mathrm{curv}(\Gamma_j) \ge 1$ and (1.4.1) we derive

$$\begin{aligned} \mathrm{Haus}_{n-2}(\Gamma_j) &\le \mathrm{vol}(\Sigma_{[pq]}(\Gamma_j)) \cdot \int_{\alpha - \frac{L_2}{\tan_\kappa(c_j)} - \frac{36\eta^{\frac{3}{2}}}{|\tan_\kappa |xp||^{\frac{3}{2}}}}^{\alpha + \frac{L_1}{\tan_\kappa(c_j)} + \theta + \frac{36\eta^{\frac{3}{2}}}{|\tan_\kappa |xp||^{\frac{3}{2}}}} \sin^{n-3}(t)dt \\ &\le \mathrm{vol}(S_1^{n-2}) \cdot \left(\frac{L_1+L_2}{\tan_\kappa(c_j)} + \theta + \frac{72\eta^{\frac{3}{2}}}{|\tan_\kappa(c_j)|^{\frac{3}{2}}} \right). \end{aligned} \tag{3.7}$$

For $\epsilon > 0$, when $\triangle_j = r_{j+1} - r_j$ is sufficiently small, we may assume that $\frac{sn_\kappa^{n-1}(r_{j+1})}{\mathrm{sn}_\kappa(r_j)} \le e^\epsilon sn_\kappa^{n-2}(r_j)$.

Case 1. Assume $\kappa \le 0$ or $\kappa > 0$ and $d \le \frac{\pi}{2\sqrt{\kappa}}$. By applying Lemma 3.1.2 to A_j: from (3.7) we get

$$\begin{aligned}
\mathrm{Haus}_n(A_j) &\le \mathrm{Haus}_{n-1}(\Gamma_j)\int_{r_j}^{r_{j+1}} \mathrm{sn}_\kappa^{n-1}(t)dt \\
&\le \mathrm{Haus}_{n-1}(\Gamma_j)(r_{j+1}-r_j)\mathrm{sn}_\kappa^{n-1}(c_j) \\
&\le \mathrm{vol}(S_1^{n-2})\left(\frac{L_1+L_2}{\tan_\kappa(c_j)}+\theta+\frac{72\eta^{\frac{3}{2}}}{|\tan_\kappa(c_j)|^{\frac{3}{2}}}\right)\mathrm{sn}_\kappa^{n-1}(c_j)\Delta_j \\
&\le e^\epsilon\cdot\mathrm{vol}(S_1^{n-2})\left[(L_1+L_2)\mathrm{sn}_\kappa^{n-2}(c_j)\mathrm{sn}'_\kappa(c_j)+\theta\cdot\mathrm{sn}_\kappa^{n-1}(c_j)\right. \\
&\quad\left.+72\eta^{\frac{3}{2}}\mathrm{sn}_\kappa^{n-\frac{5}{2}}(c_j)\cdot|\mathrm{sn}'_\kappa(c_j)|^{\frac{3}{2}}\right]\Delta_j. \qquad (3.8)
\end{aligned}$$

Then

$$\begin{aligned}
e^{-\epsilon}\cdot\mathrm{Haus}_n(A) &= e^{-\epsilon}\cdot\sum_{j=1}^{N}\mathrm{Haus}_n(A_j) \\
&\le \mathrm{vol}(S_1^{n-2})(L_1+L_2)\sum_{j=0}^{N}\mathrm{sn}_\kappa^{n-2}(c_j)\mathrm{sn}'_\kappa(c_j)\Delta_j \\
&\quad+\theta\sum_{j=0}^{N}\mathrm{sn}_\kappa^{n-1}(c_j)\Delta_j+72\eta^{\frac{3}{2}}\sum_{j=0}^{N}\mathrm{sn}_\kappa^{n-\frac{5}{2}}(c_j)\cdot|\mathrm{sn}'_\kappa(c_j)|^{\frac{3}{2}}\Delta_j. \qquad (3.9)
\end{aligned}$$

Finally, view (3.9) as Riemann sum of some integrals and let $N\to\infty$. Note that for $n=2$, $\int_0^r \mathrm{sn}_\kappa^{-\frac{1}{2}}(t)\cdot|\mathrm{sn}'_\kappa(t)|^{\frac{3}{2}}dt<\infty$ because $sn_\kappa^{-\frac{1}{2}}(t)=t^{-\frac{1}{2}}+o(t)$, we get

$$\begin{aligned}
\mathrm{Haus}_n(A) &\le e^\epsilon\cdot\mathrm{vol}(S_1^{n-2})\left[(L_1+L_2)\int_0^{r_0}\mathrm{sn}_\kappa^{n-2}(t)\mathrm{sn}'_\kappa(t)dt\right. \\
&\quad\left.+\theta\cdot\int_0^r\mathrm{sn}_\kappa^{n-1}(t)dt+72\eta^{\frac{3}{2}}\int_0^r\mathrm{sn}_\kappa^{n-\frac{5}{2}}(t)\cdot|\mathrm{sn}'_\kappa(t)|^{\frac{3}{2}}dt\right] \\
&= \mathrm{vol}(S_1^{n-2})\left[e^\epsilon\cdot\frac{(L_1+L_2)\mathrm{sn}_\kappa^{n-1}(r_0)}{n-1}+\theta\cdot\int_0^r\mathrm{sn}_\kappa^{n-1}(t)dt+O(\eta^{\frac{3}{2}})\right] \qquad (3.10)
\end{aligned}$$

Letting $\epsilon\to 0$, we see the desired result.

Case 2. Assume $\kappa>0$ and $d>\frac{\pi}{2\sqrt{\kappa}}$. For A_j with $c_j\le\frac{\pi}{2\sqrt{\kappa}}$, the estimate in (3.8) still valid. If $c_j>\frac{\pi}{2\sqrt{\kappa}}$, then we modify the estimate (3.7) by throwing out the negative term with "$\tan_\kappa(c_j)\le 0$", and obtain

$$\mathrm{Haus}_n(A_j)\le e^\epsilon\cdot\mathrm{vol}(S_1^{n-2})[\theta\cdot sn_\kappa^{n-1}(c_j)+72\eta^{\frac{3}{2}}\mathrm{sn}_\kappa^{n-\frac{5}{2}}(c_j)(\mathrm{sn}'_\kappa(c_j))^2]\triangle_i. \qquad (3.11)$$

Combining (3.8) and (3.11), we derive

$$\begin{aligned}\mathrm{Haus}_n(A) &= \sum_{j=1}^{N} V_{r_n}(A_j)\\ &\le e^{\epsilon}\cdot \mathrm{vol}(S_1^{n-2})(L_1+L_2)\sum_{j=0}^{r_{j+1}\le\frac{\pi}{2\sqrt{\kappa}}} \mathrm{sn}_\kappa^{n-2}(c_j)\mathrm{sn}'_\kappa(c_j)\Delta_j\\ &\quad + \theta\sum_{j=0}^{N}\mathrm{sn}_\kappa^{n-1}(r_j)\Delta_j + O(\eta^{\frac{3}{2}}). \qquad (3.12)\end{aligned}$$

In (3.12), letting $N\to\infty$ and $\epsilon\to 0$, we get

$$\begin{aligned}\mathrm{Haus}_n(A) &\le \mathrm{vol}(S_1^{n-2})\left[(L_1+L_2)\int_0^{r_0}\mathrm{sn}_\kappa^{n-2}(t)\mathrm{sn}'_\kappa(t)dt + \theta\int_0^{r}\mathrm{sn}_\kappa^{n-1}(t)dt\right]\\ &= \mathrm{vol}(S_1^{n-2})\left[\frac{(L_1+L_2)\mathrm{sn}_\kappa^{n-1}(r_0)}{n-1} + \theta\int_0^{r}\mathrm{sn}_\kappa^{n-1}(t)dt\right]. \qquad (3.13)\end{aligned}$$

As mentioned in the Introduction, we did not success in an early attempt to modify the proof of Proposition 2.7.4 in [BGP] in order to remove the dependence on m from $\chi_m(\delta_1,\delta)$ and factor out $L(\gamma_m)$ out from $\chi_m(\delta_1,\delta)$. We like to conclude this section by explaining the reason for this failure. The proof in [BGP] is, following the idea in [Ch], to divide X into two parts and estimate their rough volumes: one part, U_{δ_1}, is like a δ_1-tube around γ_m, and the other part, $X-U_{\delta_1}$. Since points in $X-U_{\delta_1}$ is a definite distance away from $\{p_i\}$, this allowed [BGP] to have an estimate for the diameter of the directions pointing to points in $X-U_{\delta_1}$, in terms of δ_1,δ and m. Unfortunately, the rough volumes of two parts in terms of δ_1 are in different order, that makes it impossible to remove the dependence on m, nor to factor $L(\gamma_m)$, from $\chi_m(\delta_1,\delta)$. □

3.2 Proof of Theorem 3.A (II)

In this section, we will give proofs for Lemmas 3.1.2 and 3.1.3, and thus complete the proof of Theorem 3.A. The main ingredient in the proof is the cosine law in the κ-space form.

For $\Sigma\in\mathrm{Alex}^{n-1}(1)$, one can construct an n-dimensional Alexandrov space $C_\kappa(\Sigma)$ with curvature $\ge\kappa$ (cf. [BGP]): for $\kappa\le 0$, let $C_\kappa(\Sigma)=(\Sigma\times\mathbb{R})/(\Sigma\times\{0\})$ denote a cone over Σ, and for $\kappa>0$, let $C_\kappa(\Sigma)=(\Sigma\times[0,\frac{\pi}{\sqrt{\kappa}}])/(\Sigma\times\{0\},\Sigma\times\{\frac{\pi}{\sqrt{\kappa}}\})$ denote the suspension over Σ. We define a metric d on $C_\kappa(\Sigma)$ via the cosine law in the space form of constant sectional curvature κ. For instance, if $\kappa = 0$, then for $(x,t),(x',t') \in (\Sigma \times \mathbb{R})/(\Sigma \times \{0\})$,

$$d((x,t),(x',t')) = t^2 + (t')^2 - 2tt' \cos |xx'|_\Sigma).$$

Note that for any $X \in \mathrm{Alex}^n(\kappa)$ and $p \in X$, the space of directions $\Sigma_p \in \mathrm{Alex}^{n-1}(1)$, and thus we get $C_\kappa(\Sigma_p) \in \mathrm{Alex}^n(\kappa)$ for a given κ. If $k > 0$, then $\mathrm{diam}(C_\kappa(\Sigma)) = \pi$.

Given $\Sigma \in \mathrm{Alex}^{n-1}(1)$ and $0 \le r_1 < r_2$, let

$$A_{r_1}^{r_2}(\Gamma) = \{x \in C_\kappa(\Sigma) : [px] \in \Gamma \text{ and } r_1 \le |px| \le r_2\},$$

where p is the vertex of the κ-cone $C_\kappa(\Gamma)$ which is a κ-suspension for $\kappa > 0$ (in particular, $r_2 \le \frac{\pi}{\sqrt{\kappa}}$ for $\kappa > 0$).

The following integral formula for the Hausdorff measure of an annulus in a κ-cone easily implies Lemma 3.1.2.

Lemma 3.2.1. *Let $A_{r_1}^{r_2}(\Gamma)$ be defined as in the above. Then*

$$Haus_n(A_{r_1}^{r_2}(\Gamma)) = Haus_{n-1}(\Gamma) \cdot \int_{r_1}^{r_2} sn_\kappa^{n-1}(t)dt. \tag{3.14}$$

Corollary 3.2.2.

$$Haus_n(B_r(C_\kappa(\Gamma)) = Haus_{n-1}(\Gamma) \cdot \int_0^r sn_\kappa^{n-1}(t)dt. \tag{3.15}$$

Let A and $\Gamma = \Gamma_p$ be as in Lemma 3.1.2. Consider the map, $\log_p : A \to A_{r_1}^{r_2}(\Gamma)$, defined by $x \in A$, $\log_p x = |xp| \cdot [px]$. Because $\log_p$ is a distance non-decreasing map, by Lemma 3.2.1 we can conclude Lemma 3.1.2:

$$\mathrm{Haus}_n(A) \le \mathrm{Haus}_n(A_{r_1}^{r_2}(\Gamma_p)) = \mathrm{Haus}_{n-1}(\Gamma_p) \cdot \int_{r_1}^{r_2} \mathrm{sn}_\kappa^{n-1}(t)dt.$$

Proof of Lemma 3.2.1. Note that for $\kappa > 0$, $C_\kappa(\Gamma)$ is a κ-suspension over Γ. If $r_1 \ge \frac{\pi}{2\sqrt{\kappa}}$, by the symmetry we see that $\mathrm{Haus}_n(A_{r_1}^{r_2}(\Gamma)) = \mathrm{Haus}_n(A_{\frac{\pi}{\sqrt{\kappa}}-r_2}^{\frac{\pi}{\sqrt{\kappa}}-r_1}(\Gamma))$. If $r_1 < \frac{\pi}{2\sqrt{\kappa}} < r_2$, then similarly we may identify

$$\mathrm{Haus}_n(A_{r_1}^{r_2}(\Gamma)) = \mathrm{Haus}_n(A_{r_1}^{\frac{\pi}{2\sqrt{\kappa}}}(\Gamma)) + \mathrm{Haus}_n(A_{\frac{\pi}{\sqrt{\kappa}}-r_2}^{\frac{\pi}{\sqrt{\kappa}}}(\Gamma)).$$

Hence, without loss of generality we may assume that $r_2 \le \frac{\pi}{2\sqrt{\kappa}}$.

We will divide $A_{r_1}^{r_2}(\Gamma)$ into small annulus and express $\mathrm{Haus}_n(A_{r_1}^{r_2}(\Gamma))$ as a Riemannian sum of the Hausdorff measure of these small annulus. The key in the proof is an estimate the Hausdorff measure of a small annulus in terms of the Hausdorff measure of a cross section and the width of the small annulus (one may view this as a local co-area formula estimate).

Let $\{t_i\}$ be an N-partition of $[r_1, r_2]$ and $\Delta t = \frac{r_2 - r_1}{N}$ be sufficiently small. By the above assumption, $\mathrm{sn}_\kappa(t)$ is increasing in each $[t_1, t_{i+1}]$. Let $S_t = \{x \in A : |px| = t\}$ and $A_{t_i}^{t_{i+1}} = \{x \in A : t_i \le |px| \le t_{i+1}\}$. Define the product metric $|(a,u),(b,v)| = \sqrt{|a,b|^2 + |u,v|^2}$ over $S_{t_i} \times [t_i, t_{i+1}]$. Because S_{t_i} is an Alexandrov space and the normalized Haus_n has countable additivity, we have

$$\frac{\mathrm{Haus}_n(S_{t_i} \times [t_i, t_{i+1}])}{\mathrm{Haus}_{n-1}(S_{t_i}) \cdot (t_{i+1} - t_i)} = \frac{\mathrm{Haus}_n(I^n)}{\mathrm{Haus}_{n-1}(I^{n-1}) \cdot \mathrm{Haus}_1(I^1)} = 1. \tag{3.16}$$

Consider the map $f : A_{t_i}^{t_{i+1}} \to S_{t_i} \times [r_1, r_2]$ defined as the following: for $x \in A_{t_i}^{t_{i+1}}$, let $x' \in S_{t_i}$ be the point on geodesic $[px]$ such that $|px'| = t_i$, then $f(x) = (x', |px|)$ and $|f(x_1)f(x_2)|^2 = |x_1'x_2'|^2 + (|px_1| - |px_2|)^2$.

For any $x_1, x_2 \in A_{t_i}^{t_{i+1}}$ Assume $|px_2| \ge |px_1|$. We will show that

$$\frac{|x_1x_2|}{|f(x_1)f(x_2)|} = 1 + O(\Delta t) \tag{3.17}$$

Applying the following version of cosine law (which can be easily derived) to the triangle $\triangle px_1x_2$ and $\triangle px_1'x_2'$, we get that

$$\begin{aligned} \mathrm{sn}_\kappa^2 \frac{|x_1x_2|}{2} &= \mathrm{sn}_\kappa^2 \frac{|px_1| - |px_2|}{2} + \sin \frac{\measuredangle x_1px_2}{2} \cdot \mathrm{sn}_\kappa|px_1|\mathrm{sn}_\kappa|px_2| \\ \mathrm{sn}_\kappa^2 \frac{|x_1'x_2'|}{2} &= \sin \frac{\measuredangle x_1'px_2'}{2} \cdot \mathrm{sn}_\kappa^2(t_i) \end{aligned}$$

Since $\measuredangle x_1px_2 = \measuredangle x_1'px_2'$,

$$\begin{aligned} \mathrm{sn}_\kappa^2 \frac{|x_1x_2|}{2} &= \mathrm{sn}_\kappa^2 \frac{|px_1| - |px_2|}{2} + \frac{\mathrm{sn}_\kappa|px_1|\mathrm{sn}_\kappa|px_2|}{\mathrm{sn}_\kappa^2(t_i)} \mathrm{sn}_\kappa^2 \frac{|x_1'x_2'|}{2} \\ &= \mathrm{sn}_\kappa^2 \frac{|px_1| - |px_2|}{2} + (1 + O(\Delta t))\mathrm{sn}_\kappa^2 \frac{|x_1'x_2'|}{2}. \end{aligned} \tag{3.18}$$

By the Taylor expansion of $(\mathrm{sn}_\kappa^{-1}(\sqrt{\mathrm{sn}_\kappa^2(x) + (1 + O(\Delta t))\mathrm{sn}_\kappa^2(y)}))^2$, we get that

$$\begin{aligned} |x_1x_2|^2 &= (|px_1| - |px_2|)^2 + |x_1'x_2'|^2 + O(\Delta t)|x_1'x_2'|^2 \\ &= |f(x_1)f(x_2)|^2 + O(\Delta t)|x_1'x_2'|^2. \end{aligned}$$

which leads to (3.17). By the cosine law, it's easy to see that

$$\mathrm{Haus}_{n-1}(S_{t_i}) = \mathrm{sn}_\kappa^{n-1}(t_i)\mathrm{Haus}_{n-1}(\Gamma_p). \tag{3.19}$$

Together with (3.16) and (3.17),

$$\begin{aligned}\mathrm{Haus}_n(A_{t_i}^{t_{i+1}}) &= (1+O(\Delta t))^n \mathrm{Haus}_n(S_{t_i}\times[r_1,r_2])\\ &= (1+O(\Delta t))^n \mathrm{Haus}_{n-1}(S_{t_i})\Delta t\\ &= (1+O(\Delta t))^n \mathrm{Haus}_{n-1}(\Gamma_p)\mathrm{sn}_\kappa^{n-1}(t_i)\Delta t.\end{aligned}$$

Summing up the above for $i = 0, 1, \cdots, N-1$ and letting $\max\{\Delta t\} \to 0$ we prove Lemma 3.2.1. □

Proof of Lemma 3.1.3. For $\epsilon > 0$, we may chose η small so that for all i, $\frac{|p_ip_{i+1}|}{2} < \eta$ implies that $\tan_\kappa \frac{|p_ip_{i+1}|}{2} \le e^\epsilon \cdot \frac{|p_ip_{i+1}|}{2}$. We first claim that

$$\cos\tilde{\angle}xp_ip_{i+1} \le \frac{e^\epsilon\cdot|p_ip_{i+1}|}{2\tan_\kappa(|xp_i|)}, \tag{3.20}$$

where $\tilde{\angle}xp_ip_{i+1}$ denotes the corresponding angle in the comparison triangle $\tilde{\triangle}xp_ip_{i+1} \subset S_\kappa^2$. The proof of the claim relies on the cosine law in the κ-space form, and is thus divided into three cases: $\kappa = 0, \kappa = -1$ and $\kappa = 1$.

Case 1. Assume $\kappa = 0$. By the cosine law and by the fact that $|xp_i| \le |xp_{i+1}|$, we derive

$$\begin{aligned}\cos\tilde{\angle}xp_ip_{i+1} &= \frac{|xp_i|^2+|p_ip_{i+1}|^2-|xp_{i+1}|^2}{2|xp_i|\cdot|p_ip_{i+1}|}\\ &\le \frac{|xp_i|^2+|p_ip_{i+1}|^2-|xp_i|^2}{2|xp_i|\cdot|p_ip_{i+1}|}\\ &= \frac{|p_ip_{i+1}|}{2|xp_i|} = \frac{|p_ip_{i+1}|}{2\tan_0(|xp_i|)}.\end{aligned} \tag{3.21}$$

Case 2. Assume $\kappa = -1$. By the cosine law and $|xp_i| \le |xp_{i+1}|$, we derive

$$\begin{aligned}\cos\tilde{\angle}xp_ip_{i+1} &= \frac{\cosh|xp_i|\cosh|p_ip_{i+1}|-\cosh|xp_{i+1}|}{\sinh|xp_i|\sinh|p_ip_{i+1}|}\\ &\le \frac{\cosh|xp_i|}{\sinh|xp_i|}\cdot\frac{\cosh|p_ip_{i+1}|-1}{\sinh|p_ip_{i+1}|}\\ &= \frac{\tanh\frac{|p_ip_{i+1}|}{2}}{\tanh|xp_i|} \le \frac{|p_ip_{i+1}|}{2\tan_\kappa|xp_i|}\end{aligned} \tag{3.22}$$

Case 3. Assume $\kappa = 1$. Again by the cosine law and $|xp_i| \le |xp_{i+1}|$, we derive:

$$\begin{aligned}
\cos \tilde{\measuredangle} xp_ip_{i+1} &= \frac{\cos|xp_{i+1}| - \cos|xp_i|\cos|p_ip_{i+1}|}{\sin|xp_i|\sin|p_ip_{i+1}|} \\
&\le \frac{\cos|xp_i| - \cos|xp_i|\cos|p_ip_{i+1}|}{\sin|xp_i|\sin|p_ip_{i+1}|} \\
&= \frac{\cos|xp_i| 2\sin^2\frac{|p_ip_{i+1}|}{2}}{\sin|xp_i| 2\sin\frac{|p_ip_{i+1}|}{2}\cos\frac{|p_ip_{i+1}|}{2}} \\
&= \frac{\tan\frac{|p_ip_{i+1}|}{2}}{\tan|xp_i|} \le \frac{e^{\epsilon}\cdot|p_ip_{i+1}|}{2\tan_\kappa|xp_i|}.
\end{aligned} \tag{3.23}$$

By now, (3.20) follows from (3.21)–(3.23). Next, we shall show that the inequality, $u \ge \cos\alpha$, implies

$$\alpha \ge \frac{\pi}{2} - u - 36|u|^{\frac{3}{2}}. \tag{3.24}$$

(this will give the left hand side inequality in Lemma 3.1.3.) Note that in our case, we may assume $0 \le \alpha \le \pi$. Thus, if $u \ge 1$ or $u \le -1$, then (3.24) holds. On the other hand, for $u \in (-1, 1)$, it's sufficient to show $\cos^{-1} u \ge \frac{\pi}{2} - u - 36|u|^{3/2}$, equivalently, the function

$$f(u) = u + 36|u|^{3/2} - \frac{\pi}{2} + \cos^{-1} u \ge 0.$$

By calculation,

$$f'(u) = 1 + 54\cdot\mathrm{sign}(u)|u|^{1/2} - \frac{1}{\sqrt{1-u^2}}, \qquad f''(u) = \frac{27}{|u|^{1/2}} - \frac{u}{(1-u^2)^{3/2}}.$$

It's easy to see that $f''(u) > 0$, for $-1 < u < \frac{5\sqrt{13}-1}{18}$ and $f''(u) < 0$ for $\frac{5\sqrt{13}-1}{18} < u < 1$. Hence $u = 0$ is the only critical point ($f'(u) = 0$) for $0 < u < \frac{5\sqrt{13}-1}{18}$. Together with $f(0) = 0$ and $f(1) > 0$, we get that $f(u) \ge 0$ for all $u \in (-1, 1)$. Plugging in (3.24) with $\alpha = \measuredangle xp_ip_{i+1}$ and $u = \frac{e^{\epsilon}\cdot|p_ip_{i+1}|}{2\tan_\kappa|xp_i|}$, we obtain

$$\begin{aligned}
\measuredangle xp_ip_{i+1} &\ge \frac{\pi}{2} - \frac{e^{\epsilon}|p_ip_{i+1}|}{2\tan_\kappa|xp_i|} - 36\left(\frac{e^{\epsilon}|p_ip_{i+1}|}{2|\tan_\kappa|xp_i||}\right)^{3/2} \\
&\ge \frac{\pi}{2} - \frac{e^{\epsilon}|p_ip_{i+1}|}{2\tan_\kappa|xp_i|} - \frac{36\eta^{3/2}}{|\tan_\kappa|xp_i||^{3/2}}.
\end{aligned} \tag{3.25}$$

Similarly applying $|xp_i| \le |xp_{i+1}|$,

$$\measuredangle xp_ip_{i-1} \ge \frac{\pi}{2} - \frac{e^{\epsilon}|p_ip_{i-1}|}{2\tan_\kappa|xp_i|} - \frac{36\eta^{3/2}}{|\tan_\kappa|xp_i||^{3/2}}. \tag{3.26}$$

Plugging (3.25), (3.26) and $\measuredangle p_{i-1}p_ip_{i+1} = \pi - \theta_i$ into

$$\measuredangle p_{i-1}p_ip_{i+1} + \measuredangle xp_ip_{i-1} + \measuredangle xp_ip_{i+1} \le 2\pi, \qquad \text{(the condition (B) in [BGP])}$$

we get the right hand side of the inequality in Lemma 3.1.3. □

3.3 Proof of Theorems 3.B and 3.C

Proof of Theorem 3.B. Let $q \in C_p$ such that $|pq| = \mathrm{injrad}_p$. We may assume $\gamma_1, \gamma_2 \in \mathrm{geod}(p, q)$ such that

$$\theta_p = 2\pi - \measuredangle(\gamma_1'(0), \gamma_2'(0)) + \measuredangle(-\gamma_1'(1), -\gamma_2'(1)).$$

(note that if $\mathrm{geod}(p, q) = \{\gamma\}$, then $\gamma_1 = \gamma_2 = \gamma$.) By Theorem 3.A, we have

$$\begin{aligned} 2 \cdot \mathrm{injrad}_p &= L(\gamma_1 * \gamma_2^{-1}) \\ &\ge (n-1) \cdot \left[\frac{\mathrm{Haus}_n(B_r(p))}{\mathrm{vol}(S_1^{n-2}) \cdot sn_\kappa^{n-1} r} - \Theta(\gamma_1 * \gamma_2^{-1}) r \right] \\ &= (n-1) \cdot \left[\frac{\mathrm{Haus}_n(B_r(p))}{\mathrm{vol}(S_1^{n-2}) \cdot sn_\kappa^{n-1} r} - \theta_p r \right]. \end{aligned}$$

□

Our proof of Theorem 3.C relies on the local structure of an Alexandrov space, which we briefly recall (see [BGP] for details). The notion of an (n, δ)-strainer maybe viewed as a counterpart of a normal coordinate on a Riemannian manifold, defined as follows: for $p \in X$, n-pairs of points $\{(p_i, q_i)\}_{i=1}^n$ is called an (n, δ)-strainer at p, if

$$\measuredangle p_ipp_j - \frac{\pi}{2} < \delta, \qquad \measuredangle p_ipq_i - \pi < \delta, \qquad \measuredangle q_ipq_j - \frac{\pi}{2} < \delta. \qquad (1 \le i \ne j \le n)$$

We call the number, $\rho = \min\{|pp_i|, |pq_i|\}$, the radius of the (n, δ)-strainer. By the continuity, the subset of points with an (n, δ)-strainer is open in X. Let S_δ denote the set of points admitting no (n, δ)-strainer. Then S_δ is a closed subset whose Hausdorff dimension $\dim_H(S_\delta) \le n - 1$. Recall that on a Riemannian manifold, the exponential map on a small r-ball is an e^ϵ-bi-Lipschitz map and $\epsilon \to 0$ as $r \to 0$. A similar property is true on a finite-dimensional Alexandrov space.

Lemma 3.3.1 ([BGP]). *Let $X \in Alex^n(\kappa)$. If $p \in X$ has an (n,δ)-strainer with radius $\rho > 0$, then there are $\epsilon = \epsilon(n,\delta,\rho) > 0$ and $\eta(n,\delta,\rho) > 0$ such that $B_\eta(p)$ is e^ϵ bi-Lipschitz to an open subset in $\mathbb{R}^n$. Moreover, $\epsilon \to 0$ as $\delta \to 0$.*

In the proof of Theorem 3.C, we will also need the following rough volume estimate in [BGP].

Lemma 3.3.2 (Lemma 8.2 in [BGP] or Lemma 2.7.1). *Let X be an n-dimensional Alexandrov space of curvature $\geq k$. Given any subset $A \subseteq X$, and $p \in M$,*

$$V_{r_n}(A) \leq V_{r_{n-1}}(\Gamma_p) 2 d_1 \psi^{n-1}(\kappa, d),$$

where $d_1 = diam(A \cup \{p\})$, $d = \max_{x\in A}\{|px|\} - \min_{x\in A}\{|px|\}$ and $\Gamma_p \subseteq \Sigma_p$ consists of geodesic $[pa]$ for every point $a \in A - \{p\}$.

Lemma 3.3.2 is used in our proof together with the following estimate for $\psi(\kappa, d)$.

Lemma 3.3.3. *The function $\psi(\kappa,d)$ satisfies the following inequalities:*

$$\frac{2}{3} \cdot sn_\kappa(d) \leq \psi(\kappa,d) \leq 2 \cdot sn_\kappa(d),$$

provided $d < \frac{\pi}{2\sqrt{\kappa}}$ when $\kappa > 0$, where the $sn_\kappa(r)$ is defined in Theorem 3.A.

Corollary 3.3.4. *Let $A \in Alex^n(\kappa)$, $p \in A$. Then for all $r \leq \min\{\frac{\pi}{2\sqrt{\kappa}}, 1\}$ when $\kappa > 0$, $V_{r_n}(B_r(p)) \leq c(n,\kappa) \cdot r^n$, where $c(n,\kappa) > 0$ is a constant depending only on n and κ.*

We will leave the proof of Lemma 3.3.3 at the end of this section.

Lemma 3.3.5. *Let $A \in Alex^n(\kappa)$. For $\delta > 0$, there is a sequence $\mu_i \to 0$, such that $V_{r_n}(B_{\mu_i}(S_\delta)) \to 0$ as $i \to \infty$.*

Proof. Recall that the Hausdorff dimension, $\dim_H(S_\delta) \leq n-1$ ([BGP]), and thus $\mathrm{Haus}_n(S_\delta) = 0$. We claim that $V_{r_n}(S_\delta) = 0$. Let B_j denote the j^{-1}-tubular neighborhood of S_δ. Then $B_1 \supset B_2 \supset \cdots$, and $S_\delta = \bigcap_j B_j$. Consequently, $\mathrm{Haus}_n(B_j) \to \mathrm{Haus}_n(S_\delta) = 0$. Assume $V_{r_n}(S_\delta) = \ell > 0$. By definition, there is a sequence, $\epsilon_i \to 0$, and ϵ_i-net $\{x_i^k\} \subset S_\delta$ such that $\epsilon_i^n \cdot |\{x_i^k\}| \to \ell$. Given any large j, choose $\epsilon_i \leq j^{-1}$, and we have

$$\bigcup_k B_{\frac{\epsilon_i}{2}}(x_i^k) \subseteq B_j, \qquad B_{\frac{\epsilon_i}{2}}(x_i^k) \cap B_{\frac{\epsilon_i}{2}}(x_i^l) = \emptyset, k \neq l$$

and thus

$$|\{x_i^k\}| \cdot \min_k \{\mathrm{Haus}_n(B_{\frac{\epsilon_i}{2}}(x_i^k))\} \leq \sum_k \mathrm{Haus}_n(B_{\frac{\epsilon_i}{2}}(x_i^k))$$
$$\leq \mathrm{Haus}_n(B_j) \to 0.$$

By the Bishop-Gromov relative volume comparison for Alexandrov space ([BGP]), we have, for any $p \in A$ and $r > 0$,

$$\mathrm{Haus}_n(B_r(p)) \geq \frac{\mathrm{Haus}_n(A)}{\mathrm{vol}(B^\kappa_{\mathrm{diam}(A)})} \cdot \mathrm{vol}(B_r^\kappa) = c(n,\kappa,A) \cdot r^n > 0.$$

In particular, $\mathrm{Haus}_n(B_{\frac{\epsilon_i}{2}}(x_i^k)) \geq c(n,\kappa,A) \cdot (\frac{\epsilon_i}{2})^n$, and thus

$$\frac{c(n,\kappa,A)}{2^n} \cdot \ell \approx \frac{c(n,\kappa)}{2^n} \epsilon_i^n \cdot |\{x_i^k\}| \leq \mathrm{Huas}_n(B_j) \to 0,$$

a contradiction.

Since $V_{r_n}(S_\delta) = 0$, we may assume a sequence of $\epsilon_i \to 0$ and a sequence of finite ϵ_i-net $\{x_i^k\}$ such that $\epsilon_i^n \cdot |\{x_i^k\}| \leq i^{-1}$. Since $\{B_{\epsilon_i}(x_i^k)\}$ is a finite open cover for S_δ, we may assume $0 < \mu_i < \epsilon_i$ such that

$$B_{\mu_i}(S_\delta) \subseteq \bigcup_k B_{\epsilon_i}(x_i^k),$$

and thus

$$V_{r_n}(B_{\mu_i}(S_\delta)) \leq \sum_k V_{r_n}(B_{\epsilon_i}(x_i^k)) \leq |\{x_i^k\}| \cdot \max_k \{V_{r_n}(B_{\epsilon_i}(x_i^k))\}.$$

By Corollary 3.3.4,

$$V_{r_n}(B_{\epsilon_i}(x_i^k)) \leq c(n,\kappa)\epsilon_i^n,$$

and thus

$$V_{r_n}(B_{\mu_i}(S_\delta)) \leq c(n,\kappa) \cdot (\epsilon_i^n \cdot |\{x_i^k\}|) \leq i^{-1}.$$

□

The following is special case of Theorem 3.C.

Lemma 3.3.6. *If $U \subset \mathbb{R}^n$ is a bounded region, then*

$$V_{r_n}(U) = c(n) \cdot Haus_n(U),$$

where $c(n) = \frac{V_{r_n}(I^n)}{Haus_n(I^n)}$ and I^n is an n-cube in $\mathbb{R}^n$.

Proof. Let $\partial U = \bar{U} - U$. Because ∂U is closed and bounded, ∂U is compact. Clearly, $\dim_H(\partial U) = 0$. Following the proof of Lemma 3.3.5, we may assume a sequence $\mu_i \to 0$ such that $V_{r_n}(B_{\mu_i}(\partial U)) \to 0$, as $i \to \infty$.

It is easy to check that $\mathrm{Haus}_n(I^n_r) = \mathrm{vol}(I^n_r) = r^n \cdot \mathrm{vol}(I^n)$ and $V_{r_n}(I^n_r) = r^n \cdot V_{r_n}(I^n)$, and thus $c(n) = \frac{V_{r_n}(I^n_r)}{\mathrm{vol}(I^n_r)}$. We may approximate U by U_j consisting of finitely many disjoint n-cube $I^n_{r_{j_k}} \subseteq U$: $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_i \subseteq \cdots$ and $\mathrm{Haus}_n(U - U_j) < j^{-1}$. Then

$$\begin{aligned}\mathrm{vol}(U) &= \lim_{j\to\infty} \mathrm{vol}(U_j) = \lim_{j\to\infty} \sum_k \mathrm{vol}(I^n_{r_{j_k}}) \\ &= \lim_{j\to\infty} \frac{1}{c(n)} \cdot \sum_k V_{r_n}(I^n_{r_{j_k}}) = \frac{1}{c(n)} \cdot \lim_{i\to\infty} V_{r_n}(U_i) \\ &= \frac{1}{c(n)} V_{r_n}(U) - \frac{1}{c(n)} \lim_{k\to\infty} V_{r_n}(U - U_j).\end{aligned}$$

Clearly, for each μ_i, we may assume that j large such that $U - U_j \subseteq B_{\mu_i}(\partial U)$, and thus $V_{r_n}(U - U_j) \le V_{r_n}(B_{\mu_i}(\partial U))$, and thus $\lim_{j\to\infty} V_{r_n}(U - U_j) = 0$. □

Proof of Theorem 3.C. Step 1. Fixing small $\delta > 0$, by Lemma 3.3.5 we may assume a sequence $\mu_i \to 0$ such that

$$V_{r_n}(X - B_{\mu_i}(S_\delta)) = V_{r_n}(X) - V_{r_n}(B_{\mu_i}(S_\delta)) \to V_{r_n}(X), \qquad i \to \infty. \tag{3.27}$$

For each $\mu = \mu_i$, by the compactness of $X - B_\mu(S_\delta)$ we can conclude that every point in $X - B_\mu(S_\delta)$ has an (n, δ)-strainer with radius $\rho = \rho(n, \delta, \mu) > 0$ (if not, then there is a sequence $x_i \in X - B_\mu(S_\delta)$ such that the (n, δ)-strainer at x_i has radius $\rho_i \to 0$. Passing to a subsequence, we may assume $x_i \to x \in X - B_\mu(S_\delta)$. Because the (n, δ)-strainer at x has radius $\rho > 0$, by definition we see that for large i, the (n, δ)-strainer at x_i has radius at least $\rho/2$, a contradiction).

By Lemma 3.1, we may assume that $\eta(\delta, \rho) > 0$ and $\epsilon > 0$ such that $B_\eta(p)$ is e^ϵ-bi-Lipschitz to an Euclidean region B^e_η, and $\epsilon \to 0$ as $\delta \to 0$ and $\eta \to 0$ (equivalently, $\delta \to 0$ and $\mu \to 0$).

Step 2. Decompose $X - B_\mu(S_\delta)$ into the disjoint small region, $X - B_\mu(S_\delta) = \bigcup_i U_i$, such that each U_i is contained in an $\frac{\eta}{10}$-ball. Let U^e_i be the corresponding subset in $\mathbb{R}^n$ (or equivalently, U^e_i denotes an Euclidean metric on U_i which is e^ϵ-bi-Lipschitz to U_i).

In particular,

$$e^{-\epsilon} \leq \frac{V_{r_n}(U_i)}{V_{r_n}(U_i^e)} \leq e^{\epsilon}, \qquad e^{-\epsilon} \leq \frac{\text{Haus}_n(U_i)}{\text{Haus}_n(U_i^e)} \leq e^{\epsilon},$$

together with Lemma 3.3.5 imply

$$e^{-2\epsilon}c(n) = e^{-2\epsilon} \cdot \frac{V_{r_n}(U_i^e)}{\text{Haus}_n(U_i^e)} \leq \frac{V_{r_n}(U_i)}{\text{Haus}_n(U_i)} \leq e^{2\epsilon}\frac{V_{r_n}(U_i^e)}{\text{Haus}_n(U_i^e)} = e^{2\epsilon}c(n).$$

Because V_{r_n} is finitely additive, we obtain

$$e^{-2\epsilon}c(n)\sum_i \text{Haus}_n(U_i) \leq \sum_i V_{r_n}(U_i) \leq e^{2\epsilon}c(n)\sum_i \text{Haus}_n(U_i),$$

and thus

$$e^{-2\epsilon}c(n) \cdot \text{Haus}_n(B_\mu(S_\delta)) \leq V_{r_n}(X - B_\mu(S_\delta)) \leq e^{2\epsilon}c(n) \cdot \text{Haus}_n(X - B_\mu(S_\delta)). \quad (3.28)$$

In (3.28), letting $\delta \to 0$ and $\mu \to 0$, we then have $\epsilon \to 0$, $V_{r_n}(X - B_\mu(S_\delta)) \to V_{r_n}(X)$ (see (3.27)) and $\text{Haus}_n(X - B_\mu(S_\delta)) \to \text{Haus}_n(X)$. By now we obtain the desired result. □

Proof of Lemma 3.3.3. We will first reduce the proof to the case when $|qp| = |qr|$ (see (3.29) below). We may assume that $|qp| \geq |qr|$, and let s be a point on the geodesic from q to p such that $|qs| = |qr| = x$. From the condition that $2(|qp| - |qr|) \leq |pr|$, we derive

$$|pr| - |rs| \leq |ps| = |qp| - |qr| \leq \frac{1}{2}|pr|,$$

and thus $|pr| \leq 2|rs|$. From

$$|rs| \leq |pr| + |ps| = |pr| + |qp| - |qr| \leq |pr| + \frac{1}{2}|pr|,$$

we get that $|pr| \geq \frac{2}{3}|rs|$, and therefore

$$\frac{2}{3}\frac{|rs|}{\theta} \leq \frac{|pr|}{\theta} \leq 2\frac{|rs|}{\theta},$$

where $\theta = \measuredangle pqr$. In the above inequality, taking maximum over $p, q, r \in S_\kappa^2$ under the conditions for $\psi(\kappa, d)$, we get

$$\frac{2}{3}\max_{q,r,s \in S_\kappa^2}\left\{\frac{|rs|}{\theta},\ |qs| = |qr| \leq d\right\} \leq \psi(\kappa, d) \leq 2\max_{q,r,s \in S_\kappa^2}\left\{\frac{|rs|}{\theta},\ |qr| = |qs| \leq d\right\}. \quad (3.29)$$

We claim that for each fixed x,

$$\max_{|rs|}\left\{\frac{|rs|}{\theta},\ |qr| = |qs| = x\right\} = \mathrm{sn}_\kappa x. \tag{3.30}$$

Clearly, Lemma 3.3.3 follows from (3.29) and (3.30). In the rest of the proof, we will verify (3.30).

Case 1. For $k < 0$, applying the cosine law to the triangle $\triangle qrs$ we derive

$$\begin{aligned}\cosh(\sqrt{-\kappa}|rs|) &= \cosh^2(\sqrt{-\kappa}x) - \sinh^2(\sqrt{-\kappa}x)\cos\theta \\ &= 1 + \sinh^2(\sqrt{-\kappa}x)(1 - \cos\theta) \\ &= 1 + 2\sinh^2(\sqrt{-\kappa}x)\sin^2\frac{\theta}{2},\end{aligned}$$

and thus

$$\sinh\frac{\sqrt{-\kappa}|rs|}{2} = \sin\frac{\theta}{2}\sinh(\sqrt{-\kappa}x). \tag{3.31}$$

Since $\sin z \le z$ and $z \le \sinh z$ for $z > 0$, from (3.31) we get

$$\frac{\sqrt{-\kappa}|rs|}{2} \le \sinh\frac{\sqrt{-\kappa}|rs|}{2} = \sin\frac{\theta}{2}\sinh(\sqrt{-\kappa}x) \le \frac{\theta}{2}\sinh(\sqrt{-\kappa}x),$$

and thus

$$\frac{|rs|}{\theta} \le \frac{\sinh(\sqrt{-\kappa}x)}{\sqrt{-\kappa}}.$$

On the other hand, $|rs| \to 0 \Leftrightarrow \theta \to 0$. Using (3.31), we derive

$$\lim_{\theta\to 0}\frac{|rs|}{\theta} = \lim_{\theta\to 0}\frac{|rs|}{\sinh\frac{\sqrt{-\kappa}|rs|}{2}}\cdot\frac{\sin\frac{\theta}{2}\sinh(\sqrt{-\kappa}x)}{\theta} = \frac{\sinh(\sqrt{-\kappa}x)}{\sqrt{-\kappa}}.$$

By now, we can conclude (3.30) for $k < 0$.

Case 2. For $k = 0$, applying the cosine law to $\triangle qrs$, we get that $|rs| = 2x\sin\frac{\theta}{2} \le \theta x$ and thus $\frac{|rs|}{\theta} \le x$. On the other hand,

$$\lim_{\theta\to 0}\frac{|rs|}{\theta} = \lim_{\theta\to 0}\frac{2x\sin\frac{\theta}{2}}{\theta} = x.$$

Similarly, we can conclude (3.30) for $k = 0$.

Case 3. For $\kappa > 0$, applying the cosine law to $\triangle qrs$, we get

$$\sin\frac{\sqrt{k}|rs|}{2} = \sin\frac{\theta}{2}\sin(\sqrt{k}x). \tag{3.32}$$

By (3.32), we get

$$\frac{|rs|}{\theta} = \frac{\frac{\sqrt{\kappa}|rs|}{2}}{\sin\frac{\sqrt{\kappa}|rs|}{2}} \cdot \frac{\sin\frac{\sqrt{\kappa}|rs|}{2}}{\sqrt{\kappa}\frac{\theta}{2}}$$
$$= \frac{\frac{\sqrt{\kappa}|rs|}{2}}{\sin\frac{\sqrt{\kappa}|rs|}{2}} \cdot \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}} \cdot \frac{\sin(\sqrt{\kappa}x)}{\sqrt{\kappa}}. \tag{3.33}$$

We claim that

$$\frac{\frac{\sqrt{\kappa}|rs|}{2}}{\sin\frac{\sqrt{\kappa}|rs|}{2}} \cdot \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}} \leq 1.$$

Because $\theta \to 0$ if and only if $|rs| \to 0$,

$$\lim_{\theta\to 0} \frac{\frac{\sqrt{\kappa}|rs|}{2}}{\sin\frac{\sqrt{\kappa}|rs|}{2}} \cdot \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}} = 1,$$

and consequently we conclude from (3.33) that (3.30) holds for $\kappa > 0$.

To see the claim, let $\lambda = \sin(\sqrt{\kappa}x)$, and rewrite (3.32) as

$$\sin\frac{\sqrt{\kappa}|rs|}{2} = \lambda \sin\frac{\theta}{2}, \qquad \frac{\sqrt{\kappa}|rs|}{2} = \sin^{-1}(\lambda\sin\frac{\theta}{2}).$$

Then

$$\frac{\frac{\sqrt{\kappa}|rs|}{2}}{\sin\frac{\sqrt{\kappa}|rs|}{2}} \cdot \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}} = \frac{\sin^{-1}(\lambda\sin\frac{\theta}{2})}{\lambda\sin\frac{\theta}{2}} \cdot \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}} = \frac{\sin^{-1}(\lambda\sin\frac{\theta}{2})}{\lambda\frac{\theta}{2}} \leq 1,$$

because for all $0 < \lambda \leq 1$ and $0 \leq \frac{\theta}{2} \leq \frac{\pi}{2}$, and thus $\lambda \sin\frac{\theta}{2} \leq \sin(\lambda\frac{\theta}{2})$. □

Remark 3.3.7. It is easy to see that the proof of Theorem 3.C goes through when replacing X with any open subset U of X (note that all we need is that $V_{r_n}(S_\delta \cap U) \leq V_{r_n}(S_\delta) = 0$).

Example 3.3.8. We will calculate an example showing that when $L(c) << 1$, the estimate for $\Theta(c)$ is not sharp.

Consider a sector of angle θ ($0 < \theta < \pi$) in a flat 2-disk of radius d. We obtain a flat cone, X^2, by identifying the two sides of the sector. Then $\mathrm{vol}(X^2) = \frac{1}{2}\theta d^2$. Let c denote a geodesic loop at a point near the vertex. Then $L(c) << 1$ and $\Theta(c) = \theta$. In this case, the inequality in Theorem 3.A reads:

$$L(c) + \Theta(c) \cdot d \geq \frac{(2-1)\cdot \mathrm{vol}(X^2)}{\mathrm{vol}(S_1^0)\cdot d} = \frac{\theta}{2}\cdot d.$$

Let B_d^m denote a closed ball of radius d in $\mathbb{R}^m$, and let $X^{m+1} = X^2 \times B_d^m$ be the metric product. Then X^{m+2} is compact Alexandrov space of cur ≥ 0, and

$$\operatorname{diam}(X^{m+2}) = \sqrt{2}d, \qquad \operatorname{vol}(X^{m+2}) = \operatorname{vol}(X^2)\cdot\operatorname{vol}(B_d^m) = \frac{\operatorname{vol}(S_1^{m-1})}{2(m+1)}\cdot\theta\cdot d^{m+2}.$$

Let $(p_i, x) \in X^{m+2} = X^2 \times B_d^m$ such that p_i converges to the vertex of X^2, and let $\gamma_i \subset X^2$ be a sequence of geodesic loops at p_i. Then $(\gamma_i, x) \subset X^{m+2}$ is a sequence of geodesic loops such that $L(\gamma_i, x) = L(\gamma_i) \to 0$ and $\Theta((\gamma_i, 0)) \equiv \theta$. Applying Theorem 3.A to $(\gamma_i, 0)$ and taking limit as $i \to \infty$, one gets (we also assume $m = 2s$ is even)

$$\begin{aligned}
\theta\cdot d &\geq \frac{(m+1)\cdot\operatorname{vol}(X^{m+2})}{(m-1)\cdot\operatorname{vol}(S_1^m)\cdot d^{m+1}}\\
&= \frac{\operatorname{vol}(S_1^{m-1})}{2(m-1)\cdot\operatorname{vol}(S_1^m)}\cdot\theta\cdot d\\
&= \frac{\frac{2^{\frac{m}{2}}\pi^{\frac{m-2}{2}}}{(m-1)!!}}{(m-1)\cdot\frac{\pi^{\frac{m}{2}}}{(\frac{m}{2})!}}\cdot\theta\cdot d\\
&= \frac{1}{\pi}\cdot\frac{1}{2s-1}\cdot\left[\frac{(2s)\cdot(2s-2)\cdots 4\cdot 2}{(2s-1)\cdot(2s-3)\cdots 3\cdot 1}\right]\cdot\theta\cdot d\\
&\geq \frac{1}{\pi(2s-1)}\cdot\theta\cdot d.
\end{aligned}$$

Chapter 4

Alexandrov Spaces of Relatively Maximal Volumes

The goal of this Chapter is to prove Theorem 4.A – 4.D.

Let's explain our approach to Theorem 4.A. Recall that given $\Sigma \in \mathrm{Alex}^{n-1}(1)$, we let $\mathcal{M}_\kappa^r(\Sigma) = \{X \in \mathrm{Alex}^n(\kappa) \mid \exists\, p \in X,\ \Sigma_p = \Sigma,\ \bar{B}_r(p) = X\}$, i.e. $X \in \mathcal{M}_\kappa^r(\Sigma)$, there is $p \in X$ such that $\Sigma_p = \Sigma$ and $X = \bar{B}_r(p)$. Our proof consists of two parts: using the maximal volume condition $\mathrm{vol}(X) = \mathrm{vol}(\bar{C}_\kappa^r(\Sigma_p))$, we first show that $\exp_p : \bar{C}_\kappa^r(\Sigma_p) \to X$ is well defined and the open ball $B_r(p)$ is isometric to $C_\kappa^r(\Sigma_p)$ via $\exp_p$ with respect to the intrinsic distance. (Note that the continuous non-expanding map $g\exp_P$ can be defined in the general case, c.f. [Pet 07] , and it's the same as $\exp_p$ provided the maximal volume condition in our case.) Thus $X = \bar{C}_\kappa^r(\Sigma_p)/\sim$, where the relation $\sim$ is over $\Sigma_p \times \{r\}$: $x \sim y$ if and only if $\exp_p(x) = \exp_p(y)$. Secondly, we will show that an equivalent class coincides with an orbit of an isometric $\mathbb{Z}_2$-action on $\Sigma_p \times \{r\}$.

By Bishop volume comparison, an r-ball in S_κ^n is characterized as the r-ball of (absolute) maximal volume, among all r-balls on any Riemannian n-manifold with sectional curvature $\geq \kappa$. This has been extended to Alexandrov spaces with curvature $\geq \kappa$ (10.13 in [BGP]), but still using S_κ^n as the model space. Obviously, the present Bishop (or Bishop-Gromov) volume comparison is inadequate for our purpose, and the original proof can not be carried on in our case since the induction (for volume rigidity) can not be applied on the cross section $S_r = \{x \in X : |px| = r\}$. Instead, the pointed version of the Bishop-Gromov relative volume comparison (Theorem 4.D) is required for the proof of Theorem 4.A.

A difficulty in proving the rigidity part in Theorem 4.D is: a distance non-increasing, volume preserving map (it's $\exp_p$ in our case) between Alexandrov spaces is not necessary to be isometry. For example, a gluing map from a flat sector to a flat cone.

However, we observe that the isometry holds over the sets of interior points in terms of their intrinsic metrics. Thus we prove that $\exp_p$ is an isometry by first showing that it is an almost isometry over the set of non-singular points ((n, δ)-burst points, near which exists a neighborhood e^ϵ-bi-Lipschitz ($\epsilon << 1$) to a ball in $\mathbb{R}^n$, see Theorem 2.8.4), and then continuously approximate $d_{B_R(p)}$ by piece-wised geodesics which bypass the singular points. The existence of such approximation is guaranteed by (1) $\exp_p$ maps singular points to the singular points (which excludes the previous counterexample); (2) the set of interior singular points has codimension at least 2.

To show that the equivalent class coincides with orbits of an isometric involution. We first show that if $x_1 \neq y_1 \in \Sigma_p \times \{r\}$ with $q_1 \sim q_2$, then the union of the two geodesics $\exp_p(\overline{pq_1})$ and $\exp_p(\overline{pq_2})$ forms a local geodesic near $q = \exp_p(q_1) = \exp_p(q_2)$. Because geodesics do not bifurcate, the equivalent classes defines an involution f : $\Sigma_p \times \{r\} \to \Sigma_p \times \{r\}$, and thus $X = \bar{C}^r_\kappa(\Sigma_p)/x \sim f(x), x \in \Sigma_p \times \{r\}$.

It remains to show that f is an isometry: assuming four distinct points, $x_1, x_2, y_1, y_2 \in \Sigma_p \times \{r\}$ such that $x_1 \sim x_2$ and $y_1 \sim y_2$. It's sufficient to show that when points a_i, b_i on $\overline{px}_i$ and $\overline{py}_i$ approaches to x_i and y_i respectively, the ratio of the corresponding distances (in $C^r_\kappa(\Sigma_p)$) $\dfrac{|a_1b_1|}{|a_2b_2|}$ approaches to 1. We observe that the desired property holds, if we are allowed to apply triangle comparison argument on the geodesic triangle formed by $\overline{px}_1 * \overline{px}_2$ and $\overline{py}_1 * \overline{py}_2$ (these are not minimal geodesics). We overcome the above trouble by proving that one can construct the above triangles (only for non-fixed points of f) in the doubling space $\widehat{X} = \bar{C}^r_\kappa(\Sigma_p^+) \cup_f \bar{C}^r_\kappa(\Sigma_p^-)$, in which $\overline{px}_1 * \overline{px}_2$ and $\overline{py}_1 * \overline{py}_2$ become minimal geodesics and the triangle comparison holds for the above structure. We need to show that the set of fixed points is closed, which guarantees that the above triangle comparison and that the isometry on non-fixed points can be extended to $\Sigma_p \times \{r\}$.

We also show that the space of directions of the glued fixed points (of f) in X has an analogue boundary "self-gluing" structure induced by f. Moreover, if $f \neq \mathrm{id}$, then $\dim(\mathrm{Fix}(f)) \leq n - 2$). It's worth to point out that our argument do not require the condition that f is an isometry.

In Chapter 4.1, we will prove the monotonicity part for the Bishop-Gromov relative volume comparison Theorem 4.D.

In Chapter 4.2, we will prove the open ball rigidity in Theorem 4.D

In Chapter 4.3, we will prove Theorem 4.A; the gluing map f is an isometric involution.

In Chapter 4.4, we will prove Theorem 4.B and 4.D.

4.1 Proof of Theorem 4.D (the monotonicity)

In this section, the 'vol' denote the Hausdorff measure or rough volume. For $p \in X$, let $B_r(p)$ denote the open r-ball in X centered at p and $A_R^r(p)$ denote the annulus $\{x \in X : r \leq |px| \leq R\}$, $0 \leq r < R$. Let $C_\kappa^r(\Sigma_p)$ denote the r-ball centered at the vertex in $C_\kappa(\Sigma_p)$ and $A_R^r(\Sigma_p)$ denote the corresponding annulus.

Theorem 4.1.1. *Let X be a complete n-dimensional Alexandrov space with curvature $cur(X) \geq k$. Then for $p \in X$ and $R_3 > R_2 > R_1 \geq 0$,*

$$\frac{vol(A_{R_2}^{R_1}(p))}{vol(A_{R_3}^{R_2}(p))} \geq \frac{vol(A_{R_2}^{R_1}(\Sigma_p))}{vol(A_{R_3}^{R_2}(\Sigma_p))}.$$

In particular,

$$\frac{vol(B_{R_2}(p))}{vol(B_{R_3}(p))} \geq \frac{vol(C_\kappa^{R_2}(\Sigma_p))}{vol(C_\kappa^{R_3}(\Sigma_p))}.$$

Lemma 4.1.2.

(1) For $\lambda \in [0,1]$ and $x \in [0,\pi]$, $\sin \lambda x \geq \lambda \sin x$.

(2) For $\lambda \in [0,1]$ and $x \geq 0$, $\sinh \lambda x \leq \lambda \sinh x$.

(3) For $\lambda \geq 0$ and $x \geq 0$, $\frac{\sin \lambda x}{\lambda \sin x} \geq 1 - (\lambda x)^2/6$.

(4) For $\lambda \geq 0$ and $x \geq 0$, $\frac{\sinh \lambda x}{\lambda \sinh x} \geq \frac{x}{\sinh x} \geq 1 - x$.

(5) Let $\triangle pab$ be a triangle in S_κ^2. The cosine law can be written as

$$sn_\kappa^2 \frac{|ab|}{2} = sn_\kappa^2 \frac{|pa| - |pb|}{2} + \sin^2 \frac{\measuredangle apb}{2} sn_\kappa |pa| sn_\kappa |pb|.$$

Proof. (1) Let $f(x) = \sin \lambda x - \lambda \sin x$, then

$$f'(x) = \lambda \cos \lambda x - \lambda \cos x = \lambda(\cos \lambda x - \cos x) \geq 0$$

since $0 \leq \lambda x \leq x \leq \pi$.

(2) Let $f(x) = \sinh \lambda x - \lambda \sinh x$, then

$$f'(x) = \lambda \cosh \lambda x - \lambda \cosh x = \lambda(\cosh \lambda x - \cosh x) \leq 0$$

since $0 \leq \lambda x \leq x$.

(3) For $x > 0$, one can show that $x \geq \sin x \geq x - x^3/6$. Then

$$\frac{\sin \lambda x}{\lambda \sin x} \geq \frac{\lambda x - (\lambda x)^3/6}{\lambda x} = 1 - (\lambda x)^2/6.$$

(4) The first equality is easy to see through $\sinh \lambda x \geq \lambda x$. Obviously. the second equality is true for $x \geq 1$. For $0 < x < 1$,

$$\sinh x = x + \frac{x^3}{6} + \cdots \leq x(1 + x + x^2 + \cdots) = \frac{x}{1-x}.$$

(5) By trigonometric metric identities. □

For $R - \delta > 0$ and $r - \lambda\delta > 0$, define a map $f : A_R^{R-\delta}(p) \to A_r^{r-\lambda\delta}(p)$, where $x \mapsto f(x) = x'$ is the point on a choice of minimal geodesic $\overline{px}$ such that

$$|px'| = r - \lambda(R - |px|).$$

Clearly, f is injective and well defined, since the geodesic does not branch. The following lemma shows that f behaves like a bi-Lipschitz function.

Lemma 4.1.3. *Let* $\lambda = \frac{sn_\kappa r}{sn_\kappa R}$, *for* $x, y \in A_R^{R-\delta}(\Sigma_p)$, *let* $x', y' \in A_r^{r-\lambda\delta}(\Sigma_p)$ *denote* $f(x)$, $f(y)$. *Then for small* $\delta > 0$ *independent of* r,

$$c_\kappa(\delta)\lambda \leq \frac{sn_\kappa \frac{|x'y'|}{2}}{sn_\kappa \frac{|xy|}{2}} \leq c_\kappa(\delta)^{-1}\lambda,$$

where $c_0 = 1$, $c_1(\delta) = \frac{\sin R-\delta}{\sin R+\delta} = 1 - \frac{2\delta}{\sin R+\delta}$ *and* $c_{-1}(\delta) = 1 - \delta \cdot \frac{\cosh R}{R}$, *for* $\delta > 0$ *sufficiently small.*

Remark 4.1.4. In the above inequalities, We only need the left half estimate in the purpose of the proof of monotonicity. However, the right half estimate is useful in calculating $\mathrm{vol}(C_\kappa^r(\Sigma_p))$ (Proposition 4.1.7) and concluding the monotonicity as the "vol" form (see the proof of Theorem 4.1.1).

Proof. It's sufficient to prove for $\kappa = 1, -1$. The case $\kappa = 0$ is straight forward.

(Case 1, $\kappa = 1$) Noting that

$$\frac{|px'| - |py'|}{|px| - |py|} = \frac{\lambda(|px| - |py|)}{|px| - |py|} = \lambda,$$

by Lemma 4.1.2(3) and $0 \le ||px| - |py|| \le \delta < \frac{1}{2}\sin R$, we have

$$\begin{aligned}
\sin\left(\frac{||px'| - |py'||}{2}\right) &= \sin\left(\lambda \cdot \frac{||px| - |py||}{2}\right) \\
&\ge \left(1 - \frac{(\lambda\delta)^2}{6}\right)\lambda \cdot \sin\left(\frac{||px| - |py||}{2}\right) \\
&\ge \left(1 - \frac{\delta^2}{6\sin^2 R}\right)\lambda \cdot \sin\left(\frac{||px| - |py||}{2}\right) \\
&\ge \left(1 - \frac{2\delta}{\sin R + \delta}\right)\lambda \cdot \sin\left(\frac{||px| - |py||}{2}\right) \\
&= c_1\lambda \cdot \sin\left(\frac{||px| - |py||}{2}\right).
\end{aligned}$$

Thus

$$c_1\lambda \le \frac{\sin\left(\frac{||px'|-|py'||}{2}\right)}{\sin\left(\frac{||px|-|py||}{2}\right)} \le \frac{\lambda\frac{||px|-|py||}{2}}{\sin\left(\frac{||px|-|py||}{2}\right)} \le \lambda \cdot \frac{\delta}{\sin\delta} \le c_1^{-1}\lambda. \tag{4.1}$$

For any $x \in A_R^{R-\delta}(\Sigma_p)$, by Lemma 4.1.2(1), we have

$$\sin|px'| \ge \frac{|px'|}{r}\sin r \ge \frac{r - \lambda\delta}{r}\sin r = \frac{r - \frac{\sin r}{\sin R}\delta}{r}\sin r \ge \left(1 - \frac{\delta}{\sin R}\right)\sin r,$$

Together with $\sin|px'| - \sin r = 2\sin\frac{|px'|-r}{2}\cos\frac{|px'|+r}{2} \le r - |px'| \le \lambda\delta$, we get

$$\left(1 - \frac{\delta}{\sin R}\right)\sin r \le \sin|px'| \le \sin r + \lambda\delta = \left(1 + \frac{\delta}{\sin R}\right)\sin r.$$

Similarly, $\sin|px| \ge \frac{|px|}{R}\sin R \ge \frac{R-\delta}{R}\sin R \ge \left(1 - \frac{\delta}{\sin R}\right)\sin R$ and $\sin|px| - \sin R = 2\sin\frac{|px|-R}{2}\cos\frac{|px|+R}{2} \le R - |px| \le \delta$, hence

$$\left(1 - \frac{\delta}{\sin R}\right)\sin R \le \sin|px| \le \sin R + \delta = \left(1 + \frac{\delta}{\sin R}\right)\sin R.$$

So

$$c_1\frac{\sin r}{\sin R} \le \frac{\sin|px'|}{\sin|px|} \le c_1^{-1}\frac{\sin r}{\sin R}. \tag{4.2}$$

Let $\theta = \measuredangle xpy$. Since $\frac{|xy|}{2} \le \frac{\pi}{2}$, by the cosine law and inequalities (4.1), (4.2),

$$c_1^2\lambda^2 \le \frac{\sin^2\frac{|x'y'|}{2}}{\sin^2\frac{|xy|}{2}} = \frac{\sin^2\frac{|px'|-|py'|}{2} + \sin^2\frac{\theta}{2}\sin|px'|\sin|py'|}{\sin^2\frac{|px|-|py|}{2} + \sin^2\frac{\theta}{2}\sin|px|\sin|py|} \le c_1^{-2}\lambda^2.$$

(Case 2, $\kappa = -1$) By Lemma 4.1.2(2), $\lambda\delta = \frac{\sinh r}{\sinh R} \cdot \frac{R}{\cosh R} < \frac{r}{R} \cdot R = r$. Together with Lemma 4.1.2(4), we get

$$\lambda \geq \frac{\sinh\left(\frac{||px'|-|py'||}{2}\right)}{\sinh\left(\frac{||px|-|py||}{2}\right)} = \frac{\sinh\left(\lambda \cdot \frac{||px|-|py||}{2}\right)}{\sinh\left(\frac{||px|-|py||}{2}\right)} \geq (1-\delta)\,\lambda \geq c_{-1}\lambda, \tag{4.3}$$

since $\frac{\cosh R}{R} \geq \frac{1+R^2/2}{R} > 1$. If $\delta < \frac{R}{\cosh R} < R$, then $\frac{\lambda\delta}{2r} < \frac{r}{R} \cdot \frac{\delta}{2r} = \frac{\delta}{2R} < 1$. Hence we can apply Lemma 4.1.2(2) with $\lambda = \frac{\sinh r}{\sinh R} \leq \frac{r}{R}$, to get

$$\frac{\sinh r - \sinh(r-\lambda\delta)}{\sinh r} \leq \frac{2\sinh(\lambda\delta/2)\cdot\cosh r}{\sinh r} \leq \frac{\lambda\delta}{r}\cdot\cosh r \leq \frac{\delta\cdot\cosh R}{R},$$

thus

$$\sinh(r-\lambda\delta) \geq \left(1 - \delta\cdot\frac{\cosh R}{R}\right)\sinh r.$$

For $x' \in A_r^{r-\lambda\delta}(\Sigma_p)$, $\left(1-\delta\cdot\frac{\cosh R}{R}\right)\sinh r \leq \sinh(r-\lambda\delta) \leq \sinh|px'| \leq \sinh r$. For $x \in A_R^{R-\lambda\delta}(\Sigma_p)$,

$$\frac{\sinh R - \sinh(R-\delta)}{\sinh R} \leq \frac{2\sinh(\delta/2)\cosh R}{\sinh R} \leq \frac{\delta\cdot\cosh R}{R},$$

and $\left(1-\delta\cdot\frac{\cosh R}{R}\right)\sinh R \leq \sinh(R-\lambda\delta) \leq \sinh|px| \leq \sinh R$. Then

$$c_{-1}\frac{\sinh r}{\sinh R} \leq \frac{\sinh|px'|}{\sinh|px|} \leq c_{-1}^{-1}\frac{\sinh r}{\sinh R} \tag{4.4}$$

By the cosine law and inequalities (4.3),(4.4),

$$c_{-1}^2\lambda^2 \leq \frac{\sinh^2\frac{|x'y'|}{2}}{\sinh^2\frac{|xy|}{2}} = \frac{\sinh^2\frac{|px'|-|py'|}{2} + \sin^2\frac{\theta}{2}\sinh|px'|\sinh|py'|}{\sinh^2\frac{|px|-|py|}{2} + \sin^2\frac{\theta}{2}\sinh|px|\sinh|py|} \leq c_{-1}^{-2}\lambda^2.$$

□

Lemma 4.1.5. Let U, V be subsets of $X \in Alex^n(\kappa)$ and $f : V \to U$ be an injection. If f satisfies $sn_\kappa\frac{|f(a)f(b)|}{2} \geq c \cdot sn_\kappa\frac{|ab|}{2}$, where c is a constant independent of a, b, then $vol(U) \geq c^n \cdot vol(V)$.

Proof. For the rough volume case, assume there is an ϵ-net $\{x_i\}$ in V, where $|\{x_i\}| = \beta_V(\epsilon)$. Note that when $\kappa > 0$, $\max\{\mathrm{diam}(V), \mathrm{diam}(U)\} \leq \frac{\pi}{\sqrt{\kappa}}$. Hence $\{f(x_i)\}$ becomes an $2\mathrm{sn}_\kappa^{-1}\left(c\cdot\mathrm{sn}_\kappa\frac{\epsilon}{2}\right)$-net in U. We get $\beta_U\left(2\mathrm{sn}_\kappa^{-1}\left(c\cdot\mathrm{sn}_\kappa\frac{\epsilon}{2}\right)\right) \geq \beta_V(\epsilon)$, or

$$\frac{\epsilon^n}{\left(2\mathrm{sn}_\kappa^{-1}\left(c\cdot\mathrm{sn}_\kappa\frac{\epsilon}{2}\right)\right)^n}\cdot\left(2\mathrm{sn}_\kappa^{-1}\left(c\cdot\mathrm{sn}_\kappa\frac{\epsilon}{2}\right)\right)^n\beta_U\left(2\mathrm{sn}_\kappa^{-1}\left(\mathrm{sn}_\kappa\frac{\epsilon}{2}\right)\right) \geq \epsilon^n\beta_V(\epsilon).$$

Let $\epsilon \to 0$, we get $\frac{1}{c^n}\mathrm{vol}(U) \geq \mathrm{vol}(V)$.

An analog proof is applied for the Hausdorff measure case. □

The following corollary easily follows by Lemma 4.1.5 and 4.1.3:

Corollary 4.1.6. *For the $f : A_R^{R-\delta}(p) \to A_r^{r-\lambda\delta}(p)$ defined above, where $\lambda = \frac{sn_\kappa r}{sn_\kappa R}$, we have*

$$\frac{vol(A_r^{r-\lambda\delta}(p))}{vol(A_R^{R-\delta}(p))} \geq c_\kappa^n \left(\frac{sn_\kappa r}{sn_\kappa R}\right)^n. \tag{4.5}$$

Proof of Theorem 4.1.1. By a rescaling, it's sufficient to prove for the case $\kappa = 1, 0$ and -1. The proof is based on the volume comparison estimate in Lemma 4.1.5. The key is to get the integral by taking Riemann sums of (4.5) in a right form. We only give a prove for the case $\kappa = 1$, other cases are analogous. We first claim that

$$\frac{\mathrm{vol}(A_{R_2}^{R_1}(p))}{\mathrm{vol}(A_{R_3}^{R_2}(p))} \geq \frac{\int_{R_1}^{R_2}(\sin t)^{n-1}dt}{\int_{R_2}^{R_3}(\sin t)^{n-1}dt}, \tag{4.6}$$

then by the following Proposition 4.1.7, we get the desired comparison theorem. Not losing generality, we may assume $0 < R_1 < R_2 < R_3 < \pi$. Let A_R^r be the shorthand of $A_R^r(p)$. We will show (4.6) by the following 2 steps.

STEP 1. For a fixed $r \in [R_2, R_3]$, take small $\delta < \frac{1}{2}\sin r$. Define a monotonic sequence in $[0,1]$: $a_0 = 1$, $a_{i+1} = a_i - \frac{\sin a_i r}{r \sin r}\delta$, $i = 0, 1, \cdots, \infty$. If assume $a_i \in [0,1]$, then

$$\frac{a_i\delta}{r} \leq \frac{\sin a_i r}{r\sin r}\delta \leq \frac{a_i r}{r \sin r}\cdot\frac{1}{2}\sin r \leq a_i,$$

which follows $0 \leq a_{i+1} \leq \left(1 - \frac{\delta}{r}\right)a_i$. Thus by induction $a_i \searrow 0$ as $i \to \infty$ for sufficiently small δ independent of r, provided $R_1 < R_2 \leq r \leq R_3$. (In the case $\kappa = -1$,

$$\frac{a_i\delta}{\sinh r} = \frac{a_i r}{r\sinh r}\delta \leq \frac{\sinh a_i r}{r \sinh r}\delta \leq \frac{a_i}{r}\delta \leq a_i,$$

and $0 \leq a_{i+1} \leq \left(1 - \frac{\delta}{\sinh r}\right)a_i$.)

Apply Corollary 4.1.6 for $A_{a_i r}^{a_{i+1} r}$ and $A_r^{r-\delta}$, where $\lambda_i = \frac{a_i r - a_{i+1} r}{r-(r-\delta)} = \frac{\sin a_i r}{\sin r}$:

$$\frac{\mathrm{vol}(A_{a_i r}^{a_{i+1} r})}{\mathrm{vol}(A_r^{r-\delta})} \geq \left(1 - \frac{2\delta}{\sin r}\right)^n \left(\frac{\sin a_i r}{\sin r}\right)^n. \tag{4.7}$$

Not losing generality, we can assume $a_N r = R_1$ for some j by taking a smaller δ. Summing up (4.7) for $i = 0, 1, \cdots, N-1$, we get

$$\frac{\mathrm{vol}(A_r^{R_1})}{\mathrm{vol}(A_r^{r-\delta})} \geq \left(1 - \frac{2\delta}{\sin r}\right)^n \frac{\sum_{i=0}^{N-1}(\sin a_i r)^n}{(\sin r)^n}. \tag{4.8}$$

Let $g(r) = \frac{\delta}{\sin r}\sum_{i=0}^{N-1}(\sin a_i r)^n$. Then

$$g(r) = \sum_{i=0}^{N-1}(\sin a_i r)^{n-1}(a_i r - a_{i+1}r) = \tau(\delta) + \int_{R_1}^{r}(\sin t)^{n-1}dt,$$

where $\tau(\delta)$ depends only on δ, R_1, R_2 and $\tau(\delta) \to 0$ as $\delta \to 0$, since $\Delta a_i r = a_i r - a_{i+1} r = \frac{\sin a_i}{\sin r}\delta \leq \frac{\delta}{\sin r} \to 0$ as $\delta \to 0$. (In the case $\kappa = -1$, $\Delta a_i r = \frac{\sinh a_i r}{\sinh r}\delta \leq \delta$.) Plugging this into (4.8), we get

$$\begin{aligned}\frac{\mathrm{vol}(A_r^{R_1})}{\mathrm{vol}(A_r^{r-\delta})} &\geq \left(1 - \frac{2\delta}{\sin r}\right)^n \frac{\tau(\delta) + \int_{R_1}^{r}(\sin t)^{n-1}dt}{\delta(\sin r)^{n-1}} \\ &= \left(1 - \frac{2\delta}{\sin r}\right)^n \frac{\int_{R_1}^{r}(\sin t)^{n-1}dt}{\delta(\sin r)^{n-1}}\left(1 + \frac{\tau(\delta)}{\int_{R_1}^{r}(\sin t)^{n-1}dt}\right) \\ &\geq (1+\tau(\delta))\frac{\int_{R_1}^{r}(\sin t)^{n-1}dt}{\delta(\sin r)^{n-1}}, \qquad (4.9)\end{aligned}$$

or equivalently,

$$\frac{\mathrm{vol}(A_r^{r-\delta})}{\mathrm{vol}(A_r^{R_1})} \leq (1+\tau(\delta))\frac{\delta(\sin r)^{n-1}}{\int_{R_1}^{r}(\sin t)^{n-1}dt}. \qquad (4.10)$$

STEP 2. Let $r_j = R_2 + j\delta$, $i = 0, 1, \cdots, m$ be a partition of $[R_2, R_3]$, where $m = \left[\frac{R_3 - R_2}{\delta}\right]$. Apply inequality (4.10) to such r_j:

$$\frac{\mathrm{vol}(A_{r_j}^{r_{j-1}})}{\mathrm{vol}(A_{r_j}^{R_1})} \leq (1+\tau(\delta))\frac{\delta(\sin r_j)^{n-1}}{\int_{R_1}^{r_j}(\sin t)^{n-1}dt} = o(\delta). \qquad (4.11)$$

Sum (4.11) for $j = 0, 1, \cdots, m$:

$$\sum_{j=0}^{m}\frac{\mathrm{vol}(A_{r_j}^{r_{j-1}})}{\mathrm{vol}(A_{r_j}^{R_1})} \leq (1+\tau(\delta))\sum_{j=0}^{m}\frac{\delta(\sin r_j)^{n-1}}{\int_{R_1}^{r_j}(\sin t)^{n-1}dt}. \qquad (4.12)$$

Using $-\log(1-x) = x + o(x^2)$ and (4.11), the left hand side of (4.12)

$$\begin{aligned}\sum_{j=0}^{m}\frac{\mathrm{vol}(A_{r_j}^{r_{j-1}})}{\mathrm{vol}(A_{r_j}^{R_1})} &= \sum_{j=0}^{m}\left(\log\frac{\mathrm{vol}(A_{r_j}^{R_1})}{\mathrm{vol}(A_{r_{j-1}}^{R_1})} + o(\delta^2)\right) \\ &= \log\frac{\mathrm{vol}(A_{R_3}^{R_1})}{\mathrm{vol}(A_{R_2}^{R_1})} + o(\delta). \qquad (4.13)\end{aligned}$$

To rewrite the right hand side of (4.12), let $\phi_\kappa(r) = \int_{R_1}^{r}(sn_\kappa t)^{n-1}dt$, (Here $\kappa = 1$ and $\phi(r) = \int_{R_1}^{r}(\sin t)^{n-1}dt$) then

$$\sum_{j=0}^{m}\frac{\delta(\sin r_j)^{n-1}}{\int_{R_1}^{r_j}(\sin t)^{n-1}dt} = \sum_{j=0}^{m}\frac{\delta\phi'(r_j)}{\phi(r_j)} = \int_{R_2}^{R_3}\frac{\phi'(t)}{\phi(t)}dt + \tau(\delta) = \log\frac{\phi(R_3)}{\phi(R_2)} + \tau(\delta). \qquad (4.14)$$

Combing (4.12), (4.13) and (4.14), we get

$$\log \frac{\mathrm{vol}(A_{R_3}^{R_1})}{\mathrm{vol}(A_{R_2}^{R_1})} + o(\delta) \leq (1+\tau(\delta)) \left(\log \frac{\int_{R_1}^{R_3} (sn_\kappa t)^{n-1} dt}{\int_{R_1}^{R_2} (sn_\kappa t)^{n-1} dt} + \tau(\delta) \right).$$

Letting $\delta \to 0$, we get the desired inequality (4.6). □

Proposition 4.1.7. *$vol(C_\kappa^R(\Sigma_p)) = \gamma \cdot vol(\Sigma_p) \int_0^R (sn_\kappa t)^{n-1} dt$, where γ is a constant depending only on Σ_p.*

Proof. Not losing generality, let's assume $r < R < \frac{\pi}{\sqrt{\kappa}}$ in the case $\kappa > 0$. By the above proof,

$$\frac{\mathrm{vol}(C_\kappa^r(\Sigma_p))}{\mathrm{vol}(C_\kappa^r(\Sigma_p))} \geq \frac{\int_0^r (\mathrm{sn}_\kappa t)^{n-1} dt}{\int_0^R (\mathrm{sn}_\kappa t)^{n-1} dt}.$$

Noting that in $C_\kappa(\Sigma_p)$, we can also consider the inverse function $f^{-1} : A_r^{r-\lambda\delta}(\Sigma_p) \to A_R^{R-\delta}(\Sigma_p)$. By an analog argument applied on the upper bound in Lemma 4.1.3, we can show that "$\leq$" also holds for the above inequality. Hence

$$\frac{\mathrm{vol}(C_\kappa^r(\Sigma_p))}{\mathrm{vol}(C_\kappa^R(\Sigma_p))} = \frac{\int_0^r (\mathrm{sn}_\kappa t)^{n-1} dt}{\int_0^R (\mathrm{sn}_\kappa t)^{n-1} dt} = \frac{\mathrm{vol}(\Sigma_p) \int_0^r (\mathrm{sn}_\kappa t)^{n-1} dt}{\mathrm{vol}(\Sigma_p) \int_0^R (\mathrm{sn}_\kappa t)^{n-1} dt}.$$

Let $r \to 0$ we get the desired equation, where

$$\gamma = \lim_{r \to 0} \frac{\mathrm{vol}(C_\kappa^r(\Sigma_p))}{\mathrm{vol}(\Sigma_p) \int_0^r (\mathrm{sn}_\kappa t)^{n-1} dt}.$$

□

4.2 Proof of Theorem 4.D (the open ball rigidity)

The goal of this section is to prove the following open ball rigidity Theorem 4.2.1. Comparing to the proof in [BGP], we avoid using co-area formula and induction on the cross sections, since the cross section is not known to be an Alexandrov space, and even if so, there is no maximal volume rigidity for the model space being $\Sigma_p \in \mathrm{Alex}^{n-1}(1)$. Let's briefly explain our approach. We first show that the $\exp_p : C_\kappa^R(\Sigma_p) \to B_R(p)$ is well defined and preserves the volume (see Lemmas 4.2.2 and 4.2.3). Given $a, b \in X$, the key point to establish the isometry using the volume preserving is to estimate the distance $|ab|$ via the volume of a small tubular neighborhood of geodesic $\overline{ab}$ (Lemma

4.2.6). Unfortunately, in Alexandrov spaces, this can only be done when $\overline{ab}$ is contained in the set of (n,δ)-burst points X^δ (on which exist neighborhoods almost isometric to a ball in $\mathbb{R}^n$). In fact (Lemma 4.2.7), we show that $\frac{|ab|}{|\exp_p(a), \exp_p(b)|} = 1 + \chi(\delta)$ if geodesics $\overline{ab}$ and $\overline{\exp_p(a)\exp_p(b)}$ are both contained in X^δ. Finally, we extend the above $\chi(\delta)$-isometry (in terms of the intrinsic metric over X^δ) to X using a basic property of the interior singular points in Alexandrov spaces (Lemma 4.2.8) and Lemma 4.2.9.

Theorem 4.2.1. *If* $\frac{vol(B_r(p))}{vol(C^r_\kappa(\Sigma_p))} = \frac{vol(B_R(p))}{vol(C^R_\kappa(\Sigma_p))}$*, for some* $R > r > 0$*, then* $B_R(p)$ *is isometric to* $C^R_\kappa(\Sigma_p)$ *respect to their intrinsic metric.*

Lemma 4.2.2. $\frac{vol(B_r(p))}{vol(C^r_\kappa(\Sigma_p))} = \frac{vol(B_R(p))}{vol(C^R_\kappa(\Sigma_p))}$ *for some* $0 < r < R$ *if and only if* $vol(B_R) = vol(C^R_\kappa(\Sigma_p))$.

Proof. ($\Leftarrow$) If $\frac{\mathrm{vol}(B_R(p))}{\mathrm{vol}(C^R_\kappa(\Sigma_p))} = 1$, then $\frac{\mathrm{vol}(B_r(p))}{\mathrm{vol}(C^r_\kappa(\Sigma_p))} = 1$ for any $0 < r < R$, since $\frac{\mathrm{vol}(B_r(p))}{\mathrm{vol}(C^r_\kappa(\Sigma_p))}$ is non-increasing and $\lim_{r\to 0} \frac{\mathrm{vol}(B_r(p))}{\mathrm{vol}(C^r_\kappa(\Sigma_p))} = 1$.

($\Rightarrow$) Assume $\frac{\mathrm{vol}(B_r(p))}{\mathrm{vol}(C^r_\kappa(\Sigma_p))} = \frac{\mathrm{vol}(B_R(p))}{\mathrm{vol}(C^R_\kappa(\Sigma_p))}$, for some $0 < r < R$, then

$$\frac{\mathrm{vol}(B_R(p))}{\mathrm{vol}(A^r_R(p))} = \frac{\mathrm{vol}(C^R_\kappa(\Sigma_p))}{\mathrm{vol}(A^r_R(\Sigma_p))}.$$

For any $0 < t < r$,

$$\frac{\mathrm{vol}(B_t(p))}{\mathrm{vol}(A^r_R(p))} + \frac{\mathrm{vol}(A^t_R(p))}{\mathrm{vol}(A^r_R(p))} = \frac{\mathrm{vol}(C^t_\kappa(\Sigma_p))}{\mathrm{vol}(A^r_R(\Sigma_p))} + \frac{\mathrm{vol}(A^t_R(\Sigma_p))}{\mathrm{vol}(A^t_R(\Sigma_p))}.$$

By the relative comparison Theorem 4.1.1, $\frac{\mathrm{vol}(B_t(p))}{\mathrm{vol}(A^r_R(p))} \geq \frac{\mathrm{vol}(C^t_\kappa(\Sigma_p))}{\mathrm{vol}(A^r_R(\Sigma_p))}$, and $\frac{\mathrm{vol}(A^t_R(p))}{\mathrm{vol}(A^r_R(p))} \geq \frac{\mathrm{vol}(C^t_\kappa(\Sigma_p))}{\mathrm{vol}(A^t_R(\Sigma_p))}$, hence $\frac{\mathrm{vol}(B_t(p))}{\mathrm{vol}(A^r_R(p))} = \frac{\mathrm{vol}(C^t_\kappa(\Sigma_p))}{\mathrm{vol}(A^r_R(\Sigma_p))}$ or equivalently, $\frac{\mathrm{vol}(B_t(p))}{\mathrm{vol}(C^t_\kappa(\Sigma_p))} = \frac{\mathrm{vol}(A^r_R(p))}{\mathrm{vol}(A^r_R(\Sigma_p))}$. Let $t \to 0$ we get $\mathrm{vol}(A^r_R(p)) = \mathrm{vol}(A^r_R(\Sigma_p))$. Thus $\mathrm{vol}(B_R(p)) = \mathrm{vol}(C^R_\kappa(\Sigma_p))$. $\square$

Lemma 4.2.3. *If* $vol(B_R(p)) = vol(C^R_\kappa(\Sigma_p))$*, then the exponential map* $\exp_p : C^R_\kappa(\Sigma_p) \to B_R(p)$ *is well defined. Moreover, it is a distance non-expanding bijection, and any geodesic in* $B_R(p)$ *from* p *can be extended. Consequently,* $\exp_p$ *is a homeomorphism and satisfies the following condition* $\exp_p^{-1}(B_y(r)) \supset B_{\exp_p^{-1}(y)}(r)$.

Proof. (1) Consider the distance non-distorting map $\exp_p^{-1} : B_R(p) \to C^R_\kappa(\Sigma_p)$ (If there is more than one image, we just select one) whose inverse map $\exp_p$ defined over $\exp_p^{-1}(B_R(p))$ is a distance non-expanding. We claim that $\exp_p^{-1}(B_R(p))$ is dense in $C^R_\kappa(\Sigma_p)$ then $\exp_p$ can be extended to a the map over $C^R_\kappa(\Sigma_p)$. If $\exp_p^{-1}(B_R(p))$ is not so, then $C^R_\kappa(\Sigma_p) - \exp_p^{-1}(B_R(p))$ contains an open ball, and $\mathrm{vol}(C^R_\kappa(\Sigma_p)) > \mathrm{vol}(\exp_p^{-1}(B_R(p))) \geq \mathrm{vol}(B_R(p))$, a contradiction.

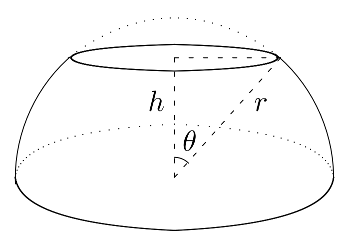
(2) We will show that any geodesic from p to $q \in B_R(p)$ can be extended longer, hence $\exp_p$ is a bijection. Let $q' = \exp_p(\tilde{q}')$ where $\tilde{q}' \in C^R_\kappa(\Sigma_p)$ is the extended point of the geodesic $\exp_p^{-1}(\overline{pq})$. Then $|pq| + |qq'| \leq |\tilde{p}\tilde{q}| + |\tilde{q}\tilde{q}'| = |\tilde{p}\tilde{q}'| = |pq'|$, which forces $\overline{pq} \cup \overline{qq'}$ being a geodesic. To show the bijection, assume $\exp_p(q_1') = \exp_p(q_2') = q$, then there are two geodesics $\overline{pq}_1$ and $\overline{pq}_2$ jointing p and q. Let's extend the geodesic $\overline{pq}_1 = \exp_p(\overline{p'q_1'})$ to q_1^* and take an interior point $x_2 \in \overline{pq}_2$. Note that $|\overline{pq}_1| = |\overline{pq}_2|$, then

$$|px_2| + |x_2q_1^*| < |px_2| + |x_2q| + |qq_1^*| = |\overline{pq}_2| + |qq_1^*| = |\overline{pq}_1| + |qq_1^*| = |\overline{pq_1^*}|,$$

which contradicts to that $\overline{pq^*}$ is a minimal geodesic. □

For a subset A in $X \in Alex^n(\kappa)$ and $\delta > 0$ small, let A^δ be the collection of points in A admitting (n, δ)-explosions. The following two lemmas are the preparations to calculate the volume of a tubular neighborhood (Lemma 4.2.6).

Lemma 4.2.4. Let $\mu = vol(T^n(1))$ be the rough volume (or Hausdorff measure) of the n dimensional cube with side length 1 in $\mathbb{R}^n$. Let H_0 be a half ball in $\mathbb{R}^n$ with a removed cap. Then $vol(H_0) = \mu(n) \cdot vol_0(H_0) = \mu(n) r \cdot vol_0(B_0^{n-1}(r)) \int_\theta^{\pi/2} \sin^n t dt$, where vol_0 is the Euclidean volume and $\mu(n)$ is a constant depending on n.



H_0

Proof. (1) For Hausdorff measure and Rough volume, we both have $\mathrm{vol}(H_0) = \mu(n)\mathrm{vol}_0(H_0)$. For any cube $T(l)$ in $\mathbb{R}^n$, by rescaling, $\mathrm{vol}(T(l)) = \mu(n) \cdot l^n = \mu(n) \cdot \mathrm{vol}_0(T(l))$. We can approximate H_0 by the union of finite many non-intersected cubes $T_i(l_i)$, $i = 1, 2, \cdots, N$, such that $\mathrm{vol}(H_0 - \cup_{i=1}^N (T_i(l_i))) \to 0$, as $N \to \infty$. Then

$$\mathrm{vol}(H_0) = \lim_{N \to 0} \mathrm{vol} \cup_{i=1}^N (T_i(l_i))) = \mu(n) \lim_{N \to 0} \mathrm{vol}_0 \cup_{i=1}^N (T_i(l_i))) = \mu(n) \cdot \mathrm{vol}_0(H_0).$$

(2) It remains to show that $\mathrm{vol}_0(H_0) = \mathrm{vol}_0(B_0^{n-1}(r)) \int_\theta^{\pi/2} \sin^n t dt$. Let $s \in [0, h]$ be the parameter for the height and $t \in [\theta, \frac{\pi}{2}]$ be the parameter for corresponding angle.

Then $s = r\cos t$ and

$$\begin{aligned}\mathrm{vol}_0(H_0) &= \int_0^h \mathrm{vol}_0(B_0^{n-1}(r\sin t))ds = \int_\theta^{\pi/2} \mathrm{vol}_0(B_0^{n-1}(r\sin t))r\sin t dt \\ &= r\cdot \mathrm{vol}_0(B_0^{n-1}(r))\int_\theta^{\pi/2} \sin^n t dt.\end{aligned}$$

□

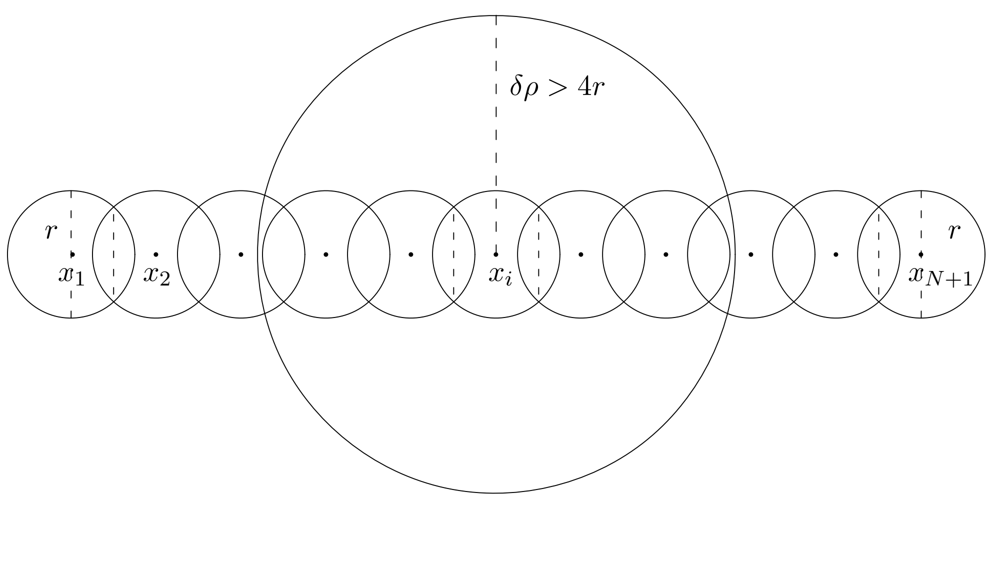
Lemma 4.2.5 (BGP Theorem 9.4). *For any $x \in X^\delta(\rho)$ associated with an (n,δ)-explosion (a_i, b_i), where $\rho = \min_i\{|xa_i|, |xb_i|\} > 0$. Then the map $f : M \to \mathbb{R}^n$ given by $f(q) = (|a_1q|, \ldots, |a_nq|)$ maps a small neighborhood U of the point x almost isometrically onto a domain in $\mathbb{R}^n$, i.e. $||f(q)f(r)| - |qr|| < \chi(\delta,\delta_1)|qr|$ for any $q, r \in U$. where $\delta_1 = \rho^{-1}\cdot diamU$. Particularly, $B_x(\delta\rho)$ is $\chi(\delta)$-isometric to $B_0^n(\delta\rho)$ in $\mathbb{R}^n$.*

In the following lemma we estimate the volume of the union of balls $\bigcup_{i=1}^{N+1} B_{x_i}(r)$ (a "tubular" neighborhood) in terms of r and $\sum_{i=1}^{N} |x_ix_{i+1}|$.

Lemma 4.2.6. *Let X be an n-dimensional metric space and $x_i \in X^\delta(\rho)$, $i = 1, 2, \cdots, N+1$. Let $0 < r < \delta\rho/4$ and $B_0^n(r)$ be the r-ball in $\mathbb{R}^n$. Assume $|x_ix_{i+1}| = l_i \le l \le 2r$ and $B_{x_i}(r)\cap B_{x_j}(r)\cap B_{x_k}(r) = \varnothing$, where $1 \le i \ne j \ne k \le N$. Then*

$$\begin{aligned}\mu^{-1}(n)(1+\chi(\delta))\cdot vol\left(\bigcup_{i=1}^{N+1} B_{x_i}(r)\right) &= vol(B_0^n(r)) + vol_0(B_0^{n-1}(r))\sum_{i=1}^{N} l_i \\ &\quad + O(r^{n+1})\sum_{i=1}^{N} l_i,\end{aligned}$$

where $\mu(n)$ is the constant in Lemma 4.2.4.



Proof. Since $l_i < 2r$ and $B_{x_i}(r) \cap B_{x_j}(r) \cap B_{x_k}(r) = \varnothing$,

$$\operatorname{vol}\left(\bigcup_{i=1}^{N+1} B_{x_i}(r)\right) = \operatorname{vol}(B^-_{x_1}(r)) + \operatorname{vol}(B^+_{x_{N+1}}(r)) + \sum_{i=2}^{N+1} \operatorname{vol}(H^-_i(r)) + \sum_{i=1}^{N} \operatorname{vol}(H^+_i(r))$$

where $B^{\pm}_{x_i}(r)$ denotes the left and right half balls and $H^{\pm}_i(r)$ denotes the left and right trapezoid ball with height $l_i/2$. By Lemma 4.2.5, every two adjacent balls $B_{x_i}(r) \cup B_{x_{i+1}}(r)$ in contained in a ball $B_{x_i}(\delta\rho)$ which is $1 + \chi(\delta)$-bi Lipschitz to a ball in $\mathbb{R}^n$, then

$$(1 + \chi(\delta)) \cdot \operatorname{vol}(H^{\pm}_i(r)) = \operatorname{vol}(H_0) = \mu(n) r \cdot \operatorname{vol}_0(B^{n-1}_0(r)) \sum_{i=1}^{N} \int_{\theta_i}^{\pi/2} \sin^n t dt,$$

where $r \cos\theta_i = l_i/2$ and $\mu(n$ is the constant in Lemma 4.2.4. Hence

$$\begin{aligned} &(1 + \chi(\delta)) \cdot \operatorname{vol}\left(\bigcup_{i=1}^{N+1} B_{x_i}(r)\right) \\ &= \mu(n)\operatorname{vol}_0(B^n_0(r)) + 2\mu(n) r \cdot \operatorname{vol}_0(B^{n-1}_0(r)) \sum_{i=1}^{N} \int_{\theta_i}^{\pi/2} \sin^n t dt. \end{aligned} \tag{4.15}$$

Let $f(l) = \displaystyle\int_{\theta}^{\pi/2} \sin^n t dt$, where $r\cos\theta = l/2$. Noting that $l = 0$ when $\theta = \pi/2$, we have $f(0) = 0$ and

$$df = -\sin^n\theta \cdot d\theta = -\sin^n\theta \cdot \frac{dl}{-2r\sin\theta} = \frac{\sin^{n-1}\theta}{2r} \cdot dl.$$

Using the Taylor expansion of $f(l)$ at $l = 0$:

$$f(l) = 0 + \frac{l}{2r} + O(l^2),$$

and $l_i < l$, we get

$$2r \cdot \sum_{i=1}^{N} \int_{\theta_i}^{\pi/2} \sin^n t dt = 2r \cdot \left(\sum_{i=1}^{N} \frac{l_i}{2r} + l_i \cdot O(l)\right) = \sum_{i=1}^{N} l_i + 2r \cdot \left(\sum_{i=1}^{N} l_i\right) O(l).$$

Together with (4.15), we get the desired estimate provided $l < 2r$. □

Now we can establish an almost isometry over the set of δ-burst points.

Lemma 4.2.7. Let $f : U^\delta \to V^\delta$ be a distance non-expanding surjection, where $U^\delta \subset X$, $V^\delta \subset Y$ are subsets containing only δ-burst points. Assume that for $\epsilon > 0$ small and any $A \subset V^\delta$,

$$\frac{vol(f^{-1}(A))}{vol(A)} \le 1 + \epsilon.$$

Then for sufficiently small $\delta > 0$,

(1) if $|ab| = r$ is sufficiently small, then $|f^{-1}(a)f^{-1}(b)| \le 2r$;

(2) if the geodesics $\overline{ab} \subset U^\delta$ and $\overline{f(a)f(b)} \subset V^\delta$, then

$$\frac{|ab|}{|f(a)f(b)|} \le 1 + \epsilon + \chi(\delta).$$

Proof. (1) For $a, b \in V^\delta = \cup_{\rho > 0} V^\delta(\rho)$, there exists $\rho > 0$ such that $a, b \in V^\delta(\rho)$ since $V^\delta(\rho_2) \subset V^\delta(\rho_1)$ if $\rho_1 < \rho_2$ and $V^\delta(\rho)$ are all open. We can also assume $f^{-1}(a), f^{-1}(b) \in U^\delta(\rho)$ by taking a smaller ρ (if $f^{-1}(x)$ contains more than one point, then only take one point).

If $|ab| = r$ but $|f^{-1}(a)f^{-1}(b)| > 2r$, consider the balls $B_a(r)$ and $B_b(r)$. By Lemma 4.2.6,

$$(1+\chi(\delta))\mathrm{vol}(B_a(r) \cup B_b(r)) = \mu(n)\mathrm{vol}_0 B_0^n(r) + 2\mu(n) r \cdot \mathrm{vol}_0(B_0^{n-1}(r)) \int_{\pi/3}^{\pi/2} \sin^n t dt + O(r^{n+1}).$$

Since $B_{f^{-1}(a)}(r) \cap B_{f^{-1}(b)}(r) = \varnothing$,

$$\begin{aligned}
&(1+\chi(\delta))\mathrm{vol}(B_{f^{-1}(a)}(r) \cup B_{f^{-1}(b)}(r)) \\
&= (1+\chi(\delta))(\mathrm{vol} B_{f^{-1}(a)}(r) + \mathrm{vol}(B_{f^{-1}(b)}(r))) \\
&= 2\mu(n)\mathrm{vol}_0 B_0^n(r).
\end{aligned}$$

We can take $r > 0$ small enough such that $B_a(r) \cup B_b(r) \subset V^\delta$, then $B_{f^{-1}(a)}(r) \cup B_{f^{-1}(b)}(r) \subset f^{-1}(B_a(r) \cup B_b(r))$ because f is a distance non-expanding surjection. Hence

$$\begin{aligned}
1 + \epsilon &\ge \frac{\mathrm{vol}(f^{-1}(A))}{\mathrm{vol}(A)} \\
&\ge (1-\chi(\delta)) \frac{2\mu(n)\mathrm{vol}_0 B_0^n(r)}{\mu(n)\mathrm{vol}_0 B_0^n(r) + 2\mu(n) r \cdot \mathrm{vol}_0(B_0^{n-1}(r)) \int_{\pi/3}^{\pi/2} \sin^n t dt + O(r^{n+1})} \\
&= (1-\chi(\delta)) \frac{2\int_0^{\pi/2} \sin^n t dt}{\int_0^{\pi/2} \sin^n t dt + \int_{\pi/3}^{\pi/2} \sin^n t dt + O(r)}.
\end{aligned}$$

This leads to a contradiction for sufficiently small r, ϵ, δ.

(2) Consider the geodesic $\overline{f(a)f(b)} \subset V^\delta = \cup_{\rho>0} V^\delta(\rho)$, there exists $\rho > 0$ such that $\overline{f(a)f(b)} \subset V^\delta(\rho)$. Select $\rho > 0$ such that $\overline{f(a)f(b)} \subset V^\delta(\rho)$ and $\overline{ab} \subset U^\delta(\rho)$. Let $\{y_i\}$ be an N-partition of $\overline{f(a)f(b)}$ with $|y_i y_{i+1}| = r = |f(a)f(b)|/N < \rho/4$ for a large $N \in \mathbb{N}$. We take a small r such that $B_{y_i}(r) \subset V^\delta$ for all y_i. To apply the estimate in Lemma 4.2.6 on $\bigcup_{i=1}^{N+1} B_{y_i}(r)$, we need to check if $B_{y_i}(r) \cap B_{y_j}(r) \cap B_{y_k}(r) = \varnothing$ for $i \neq j \neq k$. In this case it's sufficient to show that $B_{y_i}(r) \cap B_{y_{i+2}}(r) = \varnothing$. If $p \in B_{y_i}(r) \cap B_{y_{i+2}}(r)$, then $|py_i| < r$ and $|py_{i+2}| < r$, hence $2r = |y_i y_{i+1}| \leq |py_i| + |py_{i+1}| < 2r$, a contradiction. By Lemma 4.2.6,

$$\begin{aligned}
&\mu(n)^{-1}(1+\chi(\delta)) \cdot \operatorname{vol}\left(\bigcup_{i=1}^{N+1} B_{y_i}(r)\right) \\
&= \operatorname{vol}_0(B^n(r)) + \operatorname{vol}_0(B^{n-1}(r))Nr + O(r^{n+1})Nr \\
&= \operatorname{vol}_0(B^n(r)) + \operatorname{vol}_0(B^{n-1}(r))|f(a)f(b)| + O(r^{n+1})|f(a)f(b)| \\
&= \operatorname{vol}_0(B^{n-1}(r))|f(a)f(b)| + O(r^n).
\end{aligned}$$

Let $x_i = f^{-1}(y_i)$. By (1), $l_i = |x_i x_{i+1}| < 2r$. Because f is distance non-expanding, $B_{x_i}(r) \cap B_{x_{i+2}}(r) = \varnothing$. Thus by Lemma 4.2.6,

$$\mu(n)^{-1}(1+\chi(\delta)) \cdot \operatorname{vol}\left(\bigcup_{i=1}^{N+1} B_{x_i}(r)\right) = \operatorname{vol}_0(B^{n-1}(r)) \sum_{i=1}^{N} l_i + O(r^n).$$

Under the intrinsic metric of U,

$$\sum_{i=1}^{N} l_i = \sum_{i=1}^{N} |x_i x_{i+1}| \geq |f^{-1}(a) f^{-1}(b)|,$$

hence

$$\mu(n)^{-1}(1+\chi(\delta)) \cdot \operatorname{vol}\left(\bigcup_{i=1}^{N+1} B_{x_i}(r)\right) \geq \operatorname{vol}_0(B^{n-1}(r))|f^{-1}(a)f^{-1}(b)| + O(r^n).$$

Let $A = \bigcup_{i=1}^{N+1} B_{y_i}(r)$. Again because f is distance non-expanding, $\bigcup_{i=1}^{N+1} B_{x_i}(r) \subset f^{-1}(A)$. By the assumption,

$$\begin{aligned}
1+\epsilon &\geq \frac{\operatorname{vol}(f^{-1}(A))}{\operatorname{vol}(A)} \geq \frac{\operatorname{vol}\left(\bigcup_{i=1}^{N+1} B_{x_i}(r)\right)}{\operatorname{vol}\left(\bigcup_{i=1}^{N+1} B_{y_i}(r)\right)} \\
&\geq (1-\chi(\delta)) \cdot \frac{\operatorname{vol}_0(B^{n-1}(r))|f^{-1}(a)f^{-1}(b)| + O(r^n)}{\operatorname{vol}_0(B^{n-1}(r))|f(a)f(b)| + O(r^n)}, \\
&= (1-\chi(\delta)) \cdot \frac{|f^{-1}(a)f^{-1}(b)| + O(r)}{|f(a)f(b)| + O(r)}.
\end{aligned}$$

Let $r \to 0$, we get

$$1+\epsilon+\chi(\delta) \geq \frac{|f^{-1}(a)f^{-1}(b)|}{|f(a)f(b)|}.$$

□

For a subset A in $X \in \mathrm{Alex}^n(\kappa)$, let $\partial A = A \cap \partial X$ be the boundary of A as defined in [BGP]. In particular, $\partial C_\kappa^R(\Sigma_p) = \partial\Sigma_p \times [0,R)$. We let $A^o = A - \partial A$ and $N^\delta(A) = A - A^\delta$. Clearly, $A^o \supset A^\delta$ and $N^\delta(A) = N^\delta(A^o) \cup \partial A$, where $N^\delta(A^o)$ is the interior δ-singular points. The following two lemmas guarantee the extension of the intrinsic metric.

Lemma 4.2.8 ([BGP]). *Let $X \in Alex^n(\kappa)$, then* $\dim(N^\delta(X^o)) \leq n-2$. *Thus $d_{X^\delta}(x,y) = d_{X^o}(x,y)$ for $x,y \in X^\delta$.*

Lemma 4.2.9. *Let $q = \exp_p(\tilde{q})$.*

(1) If $q \in \partial B_R(p)$ then $\tilde{q}$ is not an (n,δ)-burst point.

(2) If q is an (n,δ)-burst point, then $\overrightarrow{pq}$ is an $(n-1,\chi(\delta))$-burst point in Σ_p. Thus $\tilde{q}$ is an $(n,\chi(\delta))$-burst point and $\exp_p^{-1}(B_R(p)^\delta) \subset C_\kappa^R(\Sigma_p)^{\chi(\delta)}$.

Proof. (1) Assume not so, then the ϵ-ball $B_\epsilon(\tilde{q})$ is $\chi(\delta)$-isometric to a ball B_ε^0 in $\mathbb{R}^n$ for $\epsilon > 0$ small. Since Σ_q has boundary, by induction, it's not hard to show that $\mathrm{vol}(\Sigma_q) \leq \frac{1}{2}\mathrm{vol}(S_1^{n-1})$, thus $\mathrm{vol}(B_\epsilon(q)) \leq \frac{1}{2}\mathrm{vol}(S_1^{n-1}) \cdot \int_0^\epsilon sn_k^{n-1}(t)dt$. Because $\exp_p^{-1}$ is distance non-decreasing and keeps the volume, $\mathrm{vol}(B_\epsilon(q)) = \mathrm{vol}(\exp_p^{-1}(B_\epsilon(q))) \geq \mathrm{vol}(B_\epsilon(\tilde{q})) = (1+\chi(\delta))\mathrm{vol}(B_\varepsilon^0) = (1+\chi(\delta))\mathrm{vol}(S_1^{n-1}) \cdot \int_0^\epsilon t^{n-1}dt$. We get that

$$(1+\chi(\delta))\mathrm{vol}(S_1^{n-1}) \cdot \int_0^\epsilon t^{n-1}dt \leq \frac{1}{2}\mathrm{vol}(S_1^{n-1}) \cdot \int_0^\epsilon sn_k^{n-1}(t)dt,$$

a contradiction as $\delta, \epsilon > 0$ small.

(2) Since geodesic $\overline{pq}$ can be extended and the interior points of a geodesic have the same space of direction, we can assume that q is in a neighborhood U_p of p in which any triangle with vertex p is δ-close to the comparison triangle. Because a neighborhood of q is almost to a small ball in $\mathbb{R}^n$, there exists an (n,δ)-explosion at q, where a_n, b_n are points on the extended geodesic $\overline{pq}$. In addition, we can assume $|qa_i|$, $|qb_i|$ to be short

such that $a_i, b_i \in U_p$ and $\measuredangle a_ipq, \measuredangle b_ipq < 2\delta$. We claim that $\{([a_i] = \overrightarrow{pa_i}, [b_i] = \overrightarrow{pa_i})\}_{i=1}^{n-1}$ forms an $(n-1,\delta)$-explosion at $[q] = \overrightarrow{pq}$.

It's easy to check that $\measuredangle a_ipq = \frac{|a_iq|}{|pq|} + \chi(\delta)$. Thus

$$\cos\tilde{\measuredangle}[a_i][q][x_j] = \frac{|a_iq|^2 + |x_jq|^2 - |a_ix_j|}{2|a_iq||x_jq|} + \chi(\delta) = \cos\tilde{\measuredangle} a_iqx_j + \chi(\delta),$$

where $i, j = 1, 2, \ldots, n-1$, $x_j = a_j$ or b_j. Then the claim is proved by the assumption that U_p is small. □

Proof of Theorem 4.2.1. Let $\tilde{a}, \tilde{b} \in C_\kappa^R(\Sigma_p)$ and $a = \exp_p(\tilde{a})$, $b = \exp_p(\tilde{b}) \in B_R(p)$. It's clear that the interior part of the geodesic $\overline{ab}$ either contains only boundary point or does not contain any boundary point. In any case, for $\delta > 0$ small, since $\dim(N^\delta(B_R(p)^o)) \le n-2$, there is a sequence of piece-wise geodesics $L_j = \bigcup \overline{x_ix_{i+1}} \subset B_R(p)^\delta$ such that

$$\left|\sum |x_ix_{i+1}| - |ab|_{B_R(p)}\right| < \frac{1}{j}.$$

By Lemma 4.2.9 (2), $\exp_p^{-1}(L_j)$ contains only $(n, \chi(\delta))$-burst points. Because $\exp_p$ is homeomorphic and, one can modify L_j such that in addition,

$$\left|\sum |\tilde{x}_i\tilde{x}_{i+1}| - |\tilde{a}\tilde{b}|_{C_\kappa^R(\Sigma_p)}\right| < \frac{1}{j},$$

where $\tilde{x}_i = \exp_p^{-1}(x_i)$. By Lemma 4.2.7 (2) and because $\exp_p$ is distance decreasing, $|\tilde{x}_i\tilde{x}_{i+1}| = (1+\chi(\delta))|x_ix_{i+1}|$. Let $\delta \to 0$, $j \to \infty$, then we get $|ab|_{B_R(p)} = |\tilde{a}\tilde{b}|_{C_\kappa^R(\Sigma_p)}$. □

4.3 Proof of Theorem 4.A

The aim of this section is to prove Theorem 4.A. Assume $X \in \mathcal{M}_\kappa(\Sigma, R)$ and $\mathrm{vol}(X) = \mathrm{vol}(C_\kappa^R(\Sigma_p))$. In this section, X always satisfies such maximal volume condition. By Theorem 4.D, the open ball $B_R(p)$ is isometric to $C_\kappa^R(\Sigma_p)$ in terms of their intrinsic metrics, hence $\exp_p : \bar{C}_\kappa^R(\Sigma_p) \to X$ can be viewed as a self gluing map along the "bottom" $\Sigma_p \times \{R\}$.

We now introduce some notations. Let p_o denote the vertex of $\bar{C}_\kappa^R(\Sigma_p)$. For $M \in \mathrm{Alex}^n(\kappa)$ and a point $p \in X$, let $L_p(M) = \{q \in M : |pq| \ge |px| \text{ for any } x \in M\}$. In particular, $L_{p_o}(\bar{C}_\kappa^R(\Sigma_p)) = \Sigma_p \times \{R\}$ and $L_p(X) = X - B_R(p)$ for the above X. In the following Lemma 4.3.1 we show that $\exp_p^{-1}(q)$ contains at most 2 points for any $q \in L_p(X)$, which implies that $R \le \frac{\pi}{2\sqrt{\kappa}}$ or $R = \frac{\pi}{\sqrt{\kappa}}$ for $\kappa > 0$. Let $L_p^i(X) = \{q \in L_p(X) : \exp_p^{-1}(q)$ has i points$\}$, $i = 1, 2$. Let $X^1 = X - L_p^2(X)$ denote the collection of points x in X such that $\exp_p^{-1}(x)$ is unique. Usually, we let x^c denote a point in $\bar{C}_\kappa^R(\Sigma_p)$. If $q \in L_p^2(X)$, we will say $\{q_+^c, q_-^c\} = \exp_p^{-1}(q)$. Let $\overline{qr}$ denote the geodesic jointing p and q in X and $\overline{qr}^c \subset \bar{C}_\kappa^R(\Sigma_p)$ is the lifting, if $\exp_p^{-1}(\overline{qr})$ is not broken. If $q, r \in X^1$, let $\overline{qr}_c = \exp_p(\overline{q^c r^c})$ be the projection of the geodesic jointing q^c and r^c in the cone $\bar{C}_\kappa^R(\Sigma_p)$. It's clear that $|\overline{q^c r^c}| = |q^c r^c|_{\bar{C}_\kappa^R(\Sigma_p)} \ge |qr|_X = |\overline{qr}|$. The equality holds if and only if $|qr|_X$ is realized by $\overline{qr}_c$. Let $\tilde{\triangle} pqr = \triangle_\kappa \tilde{p}\tilde{q}\tilde{r}$ denote the comparison triangle in S_κ^2, and $\tilde{\measuredangle} pqr = \measuredangle_\kappa \tilde{p}\tilde{q}\tilde{r}$ denote the comparison angle in S_κ^2. Let $\overrightarrow{qr} \in \Sigma_q$ denote the equivalent class of the geodesic $\overline{qr}$ in X.

Lemma 4.3.1. *Assume* $\exp_p(q_1^c) = \exp_p(q_2^c) = q \in L_p(X)$ *and* $q_1^c \ne q_2^c$. *Let* $\overline{pq}_i = \exp_p(\overline{p_o q^c})$ *denote the image of the geodesic* $\overline{p_o q_i^c}$, $i = 1, 2$. *Then the joint* $\overline{pq}_1 \cup \overline{pq}_2$ *forms a local geodesic in a small neighborhood of* q. *Therefore,* $\exp_p^{-1}(q)$ *contains at most 2 points.*

Proof. Let $x_i \in \overline{pq_i}$ and $x_i^c = \exp_p^{-1}(x_i)$, $i = 1, 2$. We first show that if x_1, x_2 are both close to q enough, the geodesic $\overline{x_1 x_2}$ intersects with $L_p(X)$. If not, then $\overline{x_1 x_2} \subset X - L_p(X) = B_R(p)$. By the open ball isometry (Theorem 4.2.1), $|x_1 x_2|_X = |x_1^c x_2^c|_{\bar{C}_\kappa^R(\Sigma_p)}$. This leads to a contradiction when let $x_1, x_2 \to q$ since $x_1^c \to q_1^c \ne q_2^c \leftarrow x_2^c$.

Let $a \in \overline{x_1 x_2} \cap L_p(X)$, it remains to show that $a = q$. If not, consider the triangles $\triangle_i pqa \subset X$ formed by $\overline{pq}_i$, $\overline{pa}$ and $\overline{qa}$, $i = 1, 2$. Let $\triangle \widetilde{p_1 q_1 a_1}$ and $\triangle \widetilde{p_2 q_2 a_2}$ be their comparison triangles in S_κ^2 . Take $\widetilde{x}_1 \in \overline{\widetilde{p}_1 \widetilde{q}_1}$, such that $|\widetilde{q}_1 \widetilde{x}_1| = |q x_1|$. By [BGP] condition (A), $|x_1 a| \ge |\widetilde{x}_1 \widetilde{a}_1| \ge |\widetilde{x}_1 \widetilde{q}_1| = |x_1 q|$. (The inequality $|\widetilde{x}_1 \widetilde{a}_1| \ge |\widetilde{x}_1 \widetilde{q}_1|$ holds for $|x_1 a|$ small even in the case $\kappa > 0$, $\frac{\pi}{2\sqrt{\kappa}} < R < \frac{\pi}{\sqrt{\kappa}}$.) By a same argument applied on $\triangle \widetilde{p_2 q_2 a_2}$, we will get that $|x_2 a| \ge |x_2 q|$. Thus $|x_1 x_2| = |x_1 a| + |x_2 a| \ge |x_1 q| + |x_2 q|$, and implies that $\overline{x_1 q x_2}$ is a geodesic. □

We now can define the self gluing map $f : \Sigma_p \to \Sigma_p$. Since $\Sigma_p = \{\overrightarrow{p_o q_\pm^c},\ q \in L_p(X)\}$, for $q \in L_p^2(X)$, let $f : \overrightarrow{p_o q_+^c} \rightleftharpoons \overrightarrow{p_o q_-^c}$ and $f\left(\overrightarrow{p_o q^c}\right) = \overrightarrow{p_o q^c}$ if $q \in L_p^1(X)$. Such f is naturally an involution and equivalent to a map f_R over $\Sigma_p \times \{R\} = L_p(\bar{C}^R_\kappa(\Sigma_p))$, and it's clear that $X = \bar{C}^R_\kappa(\Sigma_p)/x \sim f_R(x)$.

In the following we will carefully analyze the structure of Σ_q for $q \in L_p(X)$ (the clear result is in Lemma 4.3.7). Lemmas 4.3.1 to 4.3.6 are preparation to show Lemma 4.3.7, while Lemma 4.3.4 plays a key role in showing the self gluing structure of Σ_q for $q \in L^1_p(X)$. Lemma 4.3.8 plays a key role in the proof of isometry of f.

For $x \in \bar{C}^R_\kappa(\Sigma_p)$, let $\Gamma_{x^c} \in \text{Alex}^{n-2}(1)$ be the space of directions of $\overrightarrow{p_o x^c}$ in Σ_p. It's easy to check that if $|p_o x^c| < R$, then $\Sigma_{x^c} = C^\pi_1(\Gamma_{x^c})$. Extend the geodesic $\overline{p_o x^c}$ to q^c, where $|p_o q^c| = R$, then $\Gamma_{q^c} = \Gamma_{x^c}$ and $\Sigma_{q^c} = \bar{C}^{\frac{\pi}{2}}_1(\Gamma_{q^c})$. The following Corollary gives a necessary condition for the gluing points and immediately implies Lemma 4.3.7 (1).

Corollary 4.3.2. *If* $\{q^c_+, q^c_-\} = \exp_p^{-1}(q)$, *then* $\Sigma_{q^c_+} = \Sigma_{q^c_-}$.

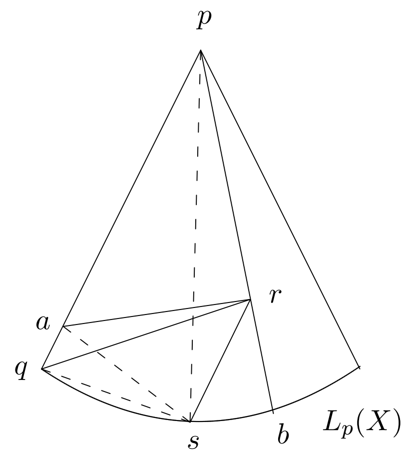
Proof. Because $\overline{pq}_+ \cup \overline{pq}_-$ is a local geodesic near q, there are $x_+ \in \overline{pq}_+$ and $x_- \in \overline{pq}_-$ such that $\Sigma_{x_+} = \Sigma_q = \Sigma_{x_-}$. Since $|px_+|, |px_-| < R$ and the open ball isometry, we get that $\Sigma_{x^c_+} = \Sigma_{x_+} = \Sigma_{x_-} = \Sigma_{x^c_-}$. Thus $\Gamma_{q^c_+} = \Gamma_{x^c_+} = \Gamma_{x^c_-} = \Gamma_{q^c_-}$ and $\Sigma_{q^c_+} = \Sigma_{q^c_-}$. □

The following corollary concludes that the estimate $\text{vol}(X) \le \text{vol}(C^R_\kappa(\Sigma_p))$ is not optimal in the case $\kappa > 0$ and $\frac{\pi}{2\sqrt{\kappa}} < R < \frac{\pi}{\sqrt{\kappa}}$.

Corollary 4.3.3. *Assume vol*$(X) =$ *vol*$(\bar{C}^R_\kappa(\Sigma_p))$ *and* $\kappa > 0$, *then* $R \le \frac{\pi}{2\sqrt{\kappa}}$ *or* $R = \frac{\pi}{\sqrt{\kappa}}$. *In the second case,* $X = C_\kappa(\Sigma_p)$ *which is the k-suspension of* Σ_p.

Proof. Assume $\frac{\pi}{2\sqrt{\kappa}} < R < \frac{\pi}{\sqrt{\kappa}}$. We claim that $L_p(X) = \{q\}$ has only one point. By Lemma 4.3.1, $\Sigma_p \times \{R\} = \exp_p^{-1}(q)$ contains at most 2 points, a contradiction. Let $a \ne b \in L_p(X)$, consider the triangle $\triangle pab$ and the compared triangle $\widetilde{\triangle} pab \in S^2_\kappa$. Take $c \in \overline{ab}$ and the corresponding $\tilde{c} \in \overline{\tilde{a}\tilde{b}}$ with $|ac| = |\widetilde{ac}|$. By the triangle comparison, $|pc| \ge |\tilde{p}\tilde{c}| > R$, a contradiction. □

Lemma 4.3.4. *Let* $q \in L_p^1(X)$ *and* $r \in X$. *If* $\overline{qr}_c = \exp_p(\overline{q^c r^c})$ *is the minimal geodesic jointing* q, r *in* X, *then for any* $a \in \overline{pq}$ *with* $|pa| \geq |pr|$, $\overline{ar}_c$ *is the minimal geodesic in* X. *Immediately, we get that* $\measuredangle pqr = \measuredangle p_o q^c r^c$ *for any* $q \in L_p^1(X)$.



Remark. This Lemma also holds for $q \in L_p^2(X)$ if we take a in the selected $\overline{pq}$, which is the image of $\overline{p_o q^c}$ and $\overline{p_o q^c}$ forms a hinge with $\overline{q^c r^c}$.

Proof. Argue by contradiction, assume $a \neq q$ and $\overline{ar}_c$ is not minimal, then the minimal geodesic $\overline{ar}$ has to intersect with $L_p(X)$. Not losing generality, we can assume that $s \in L_p(X)$ is the only intersection, i.e. $\overline{as}, \overline{sr} \subset B_R(p)$. Let $\overline{ps}$ be the geodesic such that its lifting $\overline{ps}^c$ forms a triangle with $\overline{p_o a^c}$ and $\overline{as}^c$ in $C_\kappa^R(\Sigma_p)$. Extend $\overline{pr}$ to $b \in L_p(X)$.

Since $\overline{as}_c$ is the minimal geodesic in X, we have $|as| = |\overline{as}_c|$ and by the cosine law,

$$sn_\kappa^2 \frac{|as|}{2} = sn_\kappa^2 \frac{|\overline{as}_c|}{2} = sn_\kappa^2 \frac{|aq|}{2} + \sin^2 \frac{\measuredangle spa}{2} \cdot sn_\kappa|pa| sn_\kappa|ps|.$$

In $\triangle pqs$, we have ($\overline{qs}$ may intersects with $L_p(X)$, but the following still holds)

$$sn_\kappa \frac{|qs|}{2} \leq \sin \frac{\measuredangle spq}{2} \cdot sn_\kappa|pq|.$$

Hence

$$sn_\kappa^2 \frac{|as|}{2} > \frac{sn_\kappa|pa|}{sn_\kappa|pq|} \cdot sn_\kappa^2 \frac{|qs|}{2}. \tag{4.16}$$

Since $|qr| = |\overline{qr}_c|$,

$$sn_\kappa^2 \frac{|qr|}{2} = sn_\kappa^2 \frac{|\overline{qr}_c|}{2} = sn_\kappa^2 \frac{|pq| - |pr|}{2} + \sin^2 \frac{\measuredangle rpq}{2} \cdot sn_\kappa|pq| sn_\kappa|pr|.$$

Together with

$$sn_\kappa^2 \frac{|\overline{ar}_c|}{2} = sn_\kappa^2 \frac{|pa| - |pr|}{2} + \sin^2 \frac{\measuredangle rpa}{2} \cdot sn_\kappa|pa| sn_\kappa|pr|,$$

we get

$$\begin{aligned} sn_\kappa^2 \frac{|qr|}{2} &= \frac{sn_\kappa|pq|}{sn_\kappa|pa|} \cdot sn_\kappa^2 \frac{|\overline{ar}_c|}{2} + \left(sn_\kappa^2 \frac{|pq| - |pr|}{2} - \frac{sn_\kappa|pq|}{sn_\kappa|pa|} \cdot sn_\kappa^2 \frac{|pa| - |pr|}{2} \right) \\ &> \frac{sn_\kappa|pq|}{sn_\kappa|pa|} \cdot sn_\kappa^2 \frac{|\overline{ar}_c|}{2}. \end{aligned} \tag{4.17}$$

The last inequality is verified by the following property: Let $x = |pr|$, $y = |pa|$ and $z = |pq|$. If $x \le y < z$, then

$$f(y) = \frac{sn_\kappa z}{sn_\kappa y} \cdot sn_\kappa^2 \frac{y-x}{2} < f(z) = sn_\kappa^2 \frac{z-x}{2}.$$

It's true because

$$\begin{aligned} f'(y) &= sn_k z \cdot \frac{sn_\kappa \frac{y-x}{2} cn_\kappa \frac{y-x}{2} sn_\kappa y - sn_\kappa^2 \frac{y-x}{2} cn_\kappa y}{sn_\kappa^2 y} \\ &= \frac{sn_\kappa z \cdot sn_\kappa \frac{y-x}{2}}{sn_\kappa^2 y} \cdot sn_\kappa \frac{y+x}{2} \ge 0. \end{aligned}$$

Let $t = \sqrt{\frac{sn_\kappa |pa|}{sn_\kappa |pq|}} \in (0,1)$ and rewrite (4.16) and (4.17) as

$$\begin{aligned} \frac{|as|}{2} &> sn_\kappa^{-1}\left(t \cdot sn_\kappa \frac{|qs|}{2}\right), \\ \frac{|qr|}{2} &> sn_\kappa^{-1}\left(\frac{1}{t} \cdot sn_\kappa \frac{|\overline{ar}_c|}{2}\right). \end{aligned} \tag{4.18}$$

Note that $|qs| + |sr| \ge |qr|$ and $|\overline{ar}_c| > |as| + |sr|$ by the assumption that $\overline{ar}_c$ is not minimal. we get $|qs| + |\overline{ar}_c| > |qr| + |as|$. Together with (4.18):

$$\frac{1}{2}\left(|qs| + |\overline{ar}_c|\right) > sn_\kappa^{-1}\left(t \cdot sn_\kappa \frac{|qs|}{2}\right) + sn_\kappa^{-1}\left(\frac{1}{t} sn_\kappa \cdot \frac{|\overline{ar}_c|}{2}\right). \tag{4.19}$$

Because $|aq| \le |rb|$, we have

$$\begin{aligned} |qs| &\le |as| + |aq| < |\overline{ar}_c| - |sr| + |aq| \\ &\le |\overline{ar}_c| - |br| + |aq| \le |\overline{ar}_c|. \end{aligned}$$

Let $u = sn_\kappa \frac{|qs|}{2} < v = sn_\kappa \frac{|\overline{ar}_c|}{2}$ and $g(t) = sn_\kappa^{-1}(tu) + sn_\kappa^{-1}(v/t)$. Then

$$g'(t) = \frac{u}{\sqrt{1 \pm (tu)^2}} - \frac{v}{t^2\sqrt{1 \pm (v/t)^2}} < 0.$$

Thus $g(t) > g(1) = \frac{1}{2}\left(|qs| + |\overline{ar}_c|\right)$, a contradiction to (4.19). □

Lemma 4.3.5. *Let $a, b \in C_\kappa(\Sigma_p)$ and $|pa| \ge |pb|$. For the case $\kappa > 0$, we assume $|pa| \le \frac{\pi}{2\sqrt{k}}$. Then $\measuredangle pab \le \frac{\pi}{2}$. In particular, if $|pa| < |pb|$, then $\measuredangle pab < \frac{\pi}{2}$.*

Proof. We argue by contradiction for the cases $\kappa = 0, 1, -1$. Assume $\measuredangle pab > \frac{\pi}{2}$. Extend the geodesic $\overline{pa}$ shortly to a' with $|aa'|$ sufficiently small, then $\tilde{\measuredangle} a'ab \le \measuredangle a'ab < \frac{\pi}{2}$. Then apply the cosine law to the triangles $\triangle aa'b$, $\triangle pa'b$ and $\triangle pab$.

Case 1, $\kappa = 0$.

$$\begin{aligned}
|a'b|^2 &= |aa'|^2 + |ab|^2 - 2|aa'||ab| \cos \tilde{\measuredangle} a'ab, \\
|a'b|^2 &= |pa'|^2 + |pb|^2 - 2|pa'||pb| \cos \measuredangle apb \\
&= (|pa| + |aa'|)^2 + |pb|^2 - 2(|pa| + |aa'|)|pb| \cos \measuredangle apb.
\end{aligned}$$

Together with $|ab|^2 = |pa|^2 + |pb|^2 - 2|pa||pb| \cos \measuredangle apb$, we get

$$0 = |ab| \cos \tilde{\measuredangle} a'ab + |pa| - |pb| \cos \measuredangle apb > 0,$$

a contradiction.

Case 2, $\kappa = 1$. In this case $|pb| \leq |pa| \leq \frac{\pi}{2}$.

$$\begin{aligned}
\cos |a'b| &= \cos |aa'| \cos |ab| + \cos \tilde{\measuredangle} a'ab \sin |aa'| \sin |ab| > \cos |aa'| \cos |ab|, \\
\cos |a'b| &= \cos |pa'| \cos |pb| + \cos \measuredangle apb \sin |pa'| \sin |pb|.
\end{aligned}$$

Together with $\cos |ab| = \cos |pa| \cos |pb| + \cos \measuredangle apb \sin |pa| \sin |pb|$, we get

$$\begin{aligned}
&\cos |pa'| \cos |pb| + \cos \measuredangle apb \sin |pa'| \sin |pb| \\
&> \cos |aa'|(\cos |pa| \cos |pb| + \cos \measuredangle apb \sin |pa| \sin |pb|),
\end{aligned}$$

i.e.

$$\begin{aligned}
&\cos |pb|(\cos |pa'| - \cos |aa'| \cos |pa|) \\
&> \cos \measuredangle apb \sin |pb|(\cos |aa'| \sin |pa| - \sin |pa'|).
\end{aligned}$$

Noting that $|pa'| = |pa| + |aa'|$, we get

$$\cos |pb|(-\sin |aa'| \sin |pa|) > \cos \measuredangle apb \sin |pb|(\sin |aa'| \cos |pa|).$$

Therefore,

$$\begin{aligned}
0 &> \cos |pb| \sin |pa| - \cos \measuredangle apb \sin |pb| \cos |pa| \\
&\geq \cos |pb| \sin |pa| - \sin |pb| \cos |pa| \\
&= \sin(|pa| - |pb|) \geq 0,
\end{aligned}$$

a contradiction.

Case 3, $\kappa = -1$. An analog proof.

$$\cosh|a'b| = \cosh|aa'|\cosh|ab| - \cos\tilde{\measuredangle}a'ab\sinh|aa'|\sinh|ab| < \cosh|aa'|\cosh|ab|,$$

$$\cosh|a'b| = \cosh|pa'|\cosh|pb| - \cos\measuredangle apb\sinh|pa'|\sinh|pb|.$$

Together with $\cosh|ab| = \cosh|pa|\cosh|pb| - \cos\measuredangle apb\sinh|pa|\sinh|pb|$, we get

$$\begin{aligned}&\cosh|pa'|\cosh|pb| - \cosh\measuredangle apb\sinh|pa'|\sinh|pb|\\ &< \cosh|aa'|(\cosh|pa|\cosh|pb| - \cos\measuredangle apb\sinh|pa|\sinh|pb|),\end{aligned}$$

i.e.

$$\begin{aligned}&\cosh|pb|(\cosh|pa'| - \cosh|aa'|\cosh|pa|)\\ &< \cosh\measuredangle apb\sinh|pb|(\sinh|pa'| - \cosh|aa'|\sinh|pa|).\end{aligned}$$

Noting that $|pa'| = |pa| + |aa'|$, we get

$$\cosh|pb|(\sinh|aa'|\sinh|pa|) < \cos\measuredangle apb\sinh|pb|(\sinh|aa'|\cosh|pa|).$$

Therefore,

$$\begin{aligned}0 &> \cosh|pb|\sinh|pa| - \cos\measuredangle apb\sinh|pb|\cosh|pa|\\ &\geq \cosh|pb|\sinh|pa| - \sinh|pb|\cosh|pa|\\ &= \sinh(|pa| - |pb|) \geq 0,\end{aligned}$$

a contradiction. □

Lemma 4.3.6. Let $X \in Alex^n(\kappa)$ and $\overline{qa}, \overline{qb}$ be geodesics in X. Take $a_i \in \overline{qa}$, $b_i \in \overline{qb}$ such that $a_i, b_i \to q$. Let c_i be points on the geodesics $\overline{a_ib_i}$. Then

$$\lim_{i\to\infty}\measuredangle a_iqc_i + \lim_{i\to\infty}\measuredangle b_iqc_i = \measuredangle aqb.$$

Proof. For $\epsilon > 0$ small, let U_p be the deleted neighborhood of p such that for any triangle $\triangle pqr$ with $q, r \in U_p$, each angle of $\triangle pqr$ differs from the corresponding angle of $\tilde{\triangle}pqr$ by less than ϵ (See [BGP] Lemma 11.2 for the existence of such U_p). For $a_i, b_i, c_i \in U_p$, consider the comparing triangles $\tilde{\triangle}a_iqc_i$ and $\tilde{\triangle}b_iqc_i$ which take $\tilde{q}\tilde{c}_i$ as the common side. Then $\tilde{\measuredangle} a_iqc_i + \tilde{\measuredangle} b_iqc_i \geq \measuredangle a_iqc_i + \measuredangle b_iqc_i - 2\epsilon = \pi - 2\epsilon$. Thus $|\tilde{a}_i\tilde{b}_i| \leq |\tilde{a}_i\tilde{c}_i| + |\tilde{b}_i\tilde{c}_i| \leq |\tilde{a}_i\tilde{b}_i| + 3\epsilon$. Together with $|\tilde{a}_i\tilde{c}_i| + |\tilde{b}_i\tilde{c}_i| = |a_ic_i| + |b_ic_i| = |a_ib_i|$, we get that $\tilde{\measuredangle} a_iqb_i$ differs from $\measuredangle \tilde{a}_i\tilde{q}\tilde{b}_i = \tilde{\measuredangle} a_iqc_i + \tilde{\measuredangle} b_iqc_i$ by less than 10ϵ. Again, by the property of U_p,

$$|\measuredangle a_iqb_i - (\measuredangle a_iqc_i + \measuredangle b_iqc_i)| < 20\epsilon.$$

Let $i \to \infty$ and $\epsilon \to 0$, we get the desired assertion. □

We now can give a structure of Σ_q for $q \in L_p(X)$.

Proposition 4.3.7. *Let $q \in L_p(X)$. By Corollary 4.3.2, let $\Gamma_q = \Gamma_{q^c}$ for $q \in L_p^1(X)$ and $\Gamma_q = \Gamma_{q_+^c} = \Gamma_{q_-^c}$ for $q \in L_p^2(X)$.*

(1) If $q \in L_p^2(X)$, then $\Sigma_q = C_1^{\pi}(\Gamma_q)$ is a spherical suspension of Γ_q.

(2) If $q \in L_p^1(X)$, then the open ball $B_{\frac{\pi}{2}}(\overrightarrow{qp}) = \Sigma_q - L_{\overrightarrow{qp}}(\Sigma_q)$ is isometric to $C_1^{\frac{\pi}{2}}(\Gamma_q)$, and $\Sigma_q = \bar{C}_1^{\frac{\pi}{2}}(\Gamma_q)/[x] \sim f_q([x])$ is produced by some self-gluing map f_q induced by $f : \Sigma_p \to \Sigma_p$ at q.

Proof. (1) Let $\{q_+^c, q_-^c\} = \exp_p^{-1}(q)$ and $x \in \overline{pq}_+$. By Lemma 4.3.1, x can be chosen as the interior point of the local geodesic $\overline{pq}_+ \cup \overline{pq}_-$ at q. Then $\Sigma_q = \Sigma_x = C_1^{\pi}(\Gamma_x) = C_1^{\pi}(\Gamma_q)$.

(2) First of all, by Lemmas 4.3.4 and 4.3.5, $\Sigma_q \subset \bar{B}_{\frac{\pi}{2}}(\overrightarrow{qp})$. Let $\overrightarrow{qa}, \overrightarrow{qb}$ be two points in Σ_q, where $\overline{qa}$, $\overline{qb}$ are the corresponding geodesics. Take $a_i \in \overline{qa}$, $b_i \in \overline{qb}$ such that $a_i, b_i \to q$ (see the graphs below). Assume that each of the geodesics $\overline{a_ib_i}$ intersects with $L_p(X)$ at c_i. By Lemma 4.3.4 and 4.3.5,

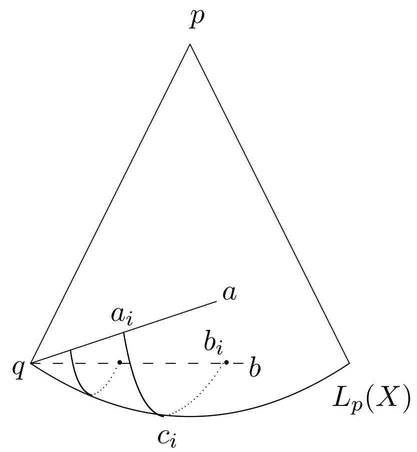
$$|\overrightarrow{qp}, \overrightarrow{qc_i}|_{\Sigma_q} = \measuredangle pqc_i = \measuredangle p_oq^cc_i^c \to \frac{\pi}{2}.$$

Thus $[c] = \lim\limits_{i\to\infty} \overrightarrow{qc_i}$ is a point in $L_{\overrightarrow{qa}}(\Sigma_q)$. By lemma 4.3.6,

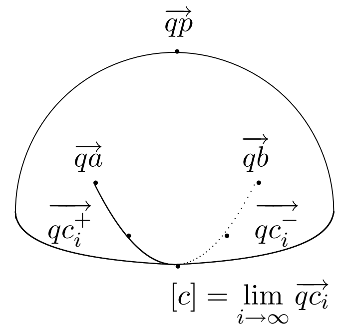
$$|\overrightarrow{qa}, \overrightarrow{qb}|_{\Sigma_q} = \measuredangle aqb = \lim_{i\to\infty} \measuredangle a_iqc_i + \lim_{i\to\infty} \measuredangle b_iqc_i = |\overrightarrow{qa}, [c]|_{\Sigma_q} + |[c], \overrightarrow{qb}|_{\Sigma_q}.$$

Therefore, $|\overrightarrow{qa}, \overrightarrow{qb}|_{\Sigma_q}$ is realized by a geodesic $\overline{\overrightarrow{qa}, [c]} \cup \overline{[c], \overrightarrow{qa}}$ which crosses $L_{\overrightarrow{qp}}(\Sigma_p)$.

The above argument implies that $|[\overline{qa}], [\overline{qb}]|_{B_{\frac{\pi}{2}}(\overrightarrow{qp})}$, the intrinsic distance in the open ball $B_{\frac{\pi}{2}}(\overrightarrow{qp})$, is realized by taking limit of $\tilde{\measuredangle} a_iqb_i$, where all $\overline{a_ib_i} \subset B_R(p)$, i.e., $B_{\frac{\pi}{2}}(\overrightarrow{qp})$ is isometric to $\Sigma_{q^c} = C_1^{\frac{\pi}{2}}(\Gamma_q)$.



$X = C_k^R(\Sigma_p)/x \sim f_R(x)$



$\Sigma_q = C_1^{\frac{\pi}{2}}(\Gamma_p)/[x] \sim f_q([x])$

□

Using Lemma 4.3.7, we can analyze the topology of the fix points of f.

Lemma 4.3.8.

(1) Fix(f) is closed in $\Sigma_p \times \{R\}$. Thus NFix(f) $= \Sigma_p -$ Fix(f) is open.

(2) If $L_p^1(X)$ contains a subset of dimension $> n-2$, then $X = \bar{C}_\kappa^R(\Sigma_p)$.

Proof. (1) If not, there is a sequence $q_i \in L_p^1(X)$ but $\lim_{i\to\infty} q_i = q \in L_p^2(X)$. Let $\overline{pq}_+$, $\overline{pq}_-$ be the two geodesics jointing p and q. Since there is a unique geodesic jointing pq_i, passing to a subsequence, we may assume $\overline{pq_i} \to \overline{pq}_+$. By Lemma 4.3.1, there exists $x_+ \in \overline{pq}_+$ and $x_- \in \overline{pq}_-$ such that $\overline{x_+q} \cup \overline{qx_-}$ forms a minimal geodesic. Since $q_i \to q$, we may assume $\overline{q_ix_-} \to \overline{qx_-}$. By Lemma 4.3.7(2) (or Lemmas 4.3.4 and 4.3.5), $\measuredangle pq_ix_- \leq \frac{\pi}{2}$. According to [BGP] 2.8.1, it follows that $\dfrac{\pi}{2} \geq \liminf_{i\to\infty} \measuredangle pq_ix^- \geq \measuredangle x_+qx_- = \pi$, a contradiction.

(2) It's sufficient to show that $\exp_p^{-1}(L_p^1(X)) = \Sigma_p \times \{R\}$. Not losing generality, we can assume $\partial\Sigma_p = \varnothing$. If not so, consider the open Alexandrov space $\Sigma_p' = \Sigma_p - \partial\Sigma_p$ and the corresponding $X' = X - \exp_p(\partial\Sigma_p \times [0, R])$. When we have $\exp_p -1(L_p^1(X')) = \Sigma_p' \times \{R\}$, we get that $\exp_p^{-1}(L_p^1(X)) = \Sigma_p \times \{R\}$ since $\exp_p^{-1}(L_p^1(X))$ is close. In the following we first show a claim in (a) and then prove the lemma in (b).

(a) If $\partial X \neq \varnothing$, then $L_p(X) = \Sigma_p \times \{R\}$. Because $\partial\Sigma_p = \varnothing$, for any $q \in L_p^2(X)$, Γ_q has no boundary and $\Sigma_q = C_1^\pi(\Gamma_q)$ has no boundary, thus $\partial X \subset L_p^1(X)$. Therefore

$\exp_p^{-1}(\partial X)$ is a closed subset in $\Sigma_p \times \{R\}$. We then show that $\exp_p^{-1}(\partial X)$ is open in $\Sigma_p \times \{R\}$ and get $X = \bar{C}_\kappa^R(\Sigma_p)$.

Argue by induction. For any $q \in \partial X$, $\Sigma_q = \bar{C}_1^{\frac{\pi}{2}}(\Gamma_q)/[x] \sim f_q([x])$ has boundary. Since Γ_q has no boundary, by induction hypothesis, $\Sigma_q = \bar{C}_1^{\frac{\pi}{2}}(\Gamma_q)$. Then one can find a neighborhood $U_q \subset X$ such that U_q is isometric to $\exp_p^{-1}(U_q)$. Thus $U_q \cap L_p^1(X)$ is an open neighborhood of q contains only fixed points.

The existence of such U_q is equivalent to the existence of a neighborhood in which any geodesic has no intersection with $L_p^1(X)$. Because Σ_q is compact, let $\{\overrightarrow{qx_i}\}$ be an ϵ-dense subset of Σ_q. Through the argument of Lemma 4.3.7(2), one can find a neighborhood U_q such that there is no pair of points on any of the above two directions jointing by a geodesic crossing $L_p(X)$. Since $\{\overrightarrow{qx_i}\}$ is dense for any ϵ small, the above property also holds for all points U_q.

(b) If $L_p^1(X)$ contains a subset of dimension $> n-2$, because the points in $L_p^1(X)$ admit no δ-explosion and the set of interior δ-explosions has dimension at most $n-2$ (see [BGP] Corollary 12.8), $L_p^1(X)$ has to contain boundary point of X. Then the assertion follows By (a). □

Now view $\widehat{f}$ as a map between Σ_p^+ and Σ_p^- and $f_R : \Sigma_p \times \{R\} \to \Sigma_p \times \{R\}$ as a map between the "bottom" of two copies of $\bar{C}_\kappa(\Sigma_p)$: $\bar{C}_\kappa(\Sigma_p^+)$, $\bar{C}_\kappa(\Sigma_p^-)$, namely, $\widehat{f_R} : \Sigma_p^+ \times \{R\} \to \Sigma_p^- \times \{R\}$. We construct a metric length space $\widehat{X} = \bar{C}_\kappa^R(\Sigma_p^+) \cup_{\widehat{f_R}} \bar{C}_\kappa^R(\Sigma_p^-)$ in terms of the intrinsic metric. For any $\hat{x} \in \widehat{X}$, it's not hard to see that $\overline{p^+\hat{x}p^-}$ is a minimal geodesic (Note that the extension of geodesic $\overline{px}$ is no longer minimal for $x \in L_p^2(X)$).

Lemma 4.3.9. *Assume $X \in Alex^n(\kappa)$. Let $[x], [y] \in \Sigma_p$ such that the geodesic $\overline{[x][y]} \subset$ NFix(f). Let $x^c, y^c \in X$ be the points on the geodesic $[x], [y]$ with $|px^c| = |py^c| = R$, and $\hat{x}, \hat{y}$ be the corresponding points in $\widehat{X}$ constructed as the above. Then the joint geodesics $\overline{p^+\hat{x}p^-}$ and $\overline{p^+\hat{y}p^-}$ satisfy the condition (B) for the same comparison curvature κ.*

Proof. This can be easily seen by the argument of Globalization Theorem ([BGP]) since NFix(f) is open and a small neighborhood of $\hat{q} \in \widehat{X}$ with $\overrightarrow{p^+\hat{q}} \in$ NFix$(\widehat{f})$ and $|p^+\hat{q}| = R$

is identical same as the one of the corresponding $q \in L_p^2(X)$ (which is a union of two small neighborhoods). $\square$

Proof of Theorem 4.A.

($\Rightarrow$) (1) has been proved as in Corollary 4.3.3. The involution is proved in Lemma 4.3.1. We now show that $f : \Sigma_p \to \Sigma_p$ is an isometry.

(i) For $[q], [r] \in \Sigma_p$, we first show that f performs an isometry, if the geodesic $\overline{[q][r]} \subset \mathrm{NFix}(f)$. Let $q, r \in X$ on the directions $[q], [r]$ such that $|pq| = |pr| = R$. Let $\widehat{X} = \bar{C}_\kappa^R(\Sigma_p^+) \cup_{\widehat{f_R}} \bar{C}_\kappa^R(\Sigma_p^-)$ be constructed as the above. Let $|ab|_{\widehat{X}}$ denote the distance in $\widehat{X}$ and $|ab|_\pm$ denote the distance in $C_\kappa^R(\Sigma_p^\pm)$ respectively. We shall show that $|qr|_+ = |\widehat{f_R}(q)\widehat{f_R}(r)|_-$. Let $\{x_i\}_{i=0}^N$, $x_0 = q$, $x_{N+1} = r$ be a partition of the geodesic $\overline{qr}$ in $\widehat{X}$ such that $\measuredangle x_i p^\pm x_{i+1} < \epsilon$ for all i.

Let a_i be the point on $\overline{p^+ x_i}$ such that $|p^+ a_i| = R - \sqrt{\epsilon}$. For ϵ small,

$$\begin{aligned} sn_\kappa \frac{|a_i a_{i+1}|_+}{2} &= \sin \frac{\measuredangle x_i p^+ x_{i+1}}{2} \cdot sn_\kappa(R - \sqrt{\epsilon}) \\ &< \sin \frac{\epsilon}{2} \cdot sn_\kappa(R - \sqrt{\epsilon}) < sn_\kappa(\sqrt{\epsilon}). \end{aligned}$$

Thus $|a_i a_{i+1}|_+ < 2\sqrt{\epsilon}$, so the minimal geodesic $\overline{a_i a_{i+1}} \subset C_\kappa^R(\Sigma_p^+)$ and $|a_i a_{i+1}|_{\widehat{X}} = |a_i a_{i+1}|_+$. Therefore,

$$\left| |qr|_+ - \sum_{i=0}^{N-1} |a_i a_{i+1}|_{\widehat{X}} \right| = \left| |qr|_+ - \sum_{i=0}^{N-1} |a_i a_{i+1}|_+ \right| \le 2\sqrt{\epsilon}. \tag{4.20}$$

Similarly, select $b_i \in \overline{p^- x_i}$ such that $|p^- b_i| = R - \sqrt{\epsilon}$. We get

$$\left| |\widehat{f_R}(q)\widehat{f_R}(r)|_- - \sum_{i=0}^{N-1} |b_i b_{i+1}|_{\widehat{X}} \right| = \left| |f_R(q) f_R(r)|_- - \sum_{i=0}^{N-1} |b_i b_{i+1}|_- \right| \le 2\sqrt{\epsilon}. \tag{4.21}$$

By Lemma 4.3.9, we can feel free to apply Toponogov's Triangle Comparison over the joint geodesics $\overline{p^+ \hat{x}_i p^-}$ and $\overline{p^+ \hat{x}_{i+1} p^-}$. For each i, because $\overline{p^+ x_i p^-}$ forms a minimal geodesic connecting p^+ and p^-, we have

$$\frac{sn_\kappa(R - \sqrt{\epsilon})}{sn_\kappa(R + 2\sqrt{\epsilon})} \le \frac{sn_\kappa \frac{|a_i a_{i+1}|_{\widehat{X}}}{2}}{sn_\kappa \frac{|b_i b_{i+1}|_{\widehat{X}}}{2}} \le \frac{sn_\kappa(R + 2\sqrt{\epsilon})}{sn_\kappa(R - \sqrt{\epsilon})},$$

i.e. $1 - o(\epsilon) < \frac{|a_i a_{i+1}|_{\hat{X}}}{|b_i b_{i+1}|_{\hat{X}}} < 1 + o(\epsilon)$. Summing up for $i = 0, 1, \cdots, N-1$, together with (4.20) and (4.21), we get

$$1 - o(\epsilon) < \frac{|qr|_+}{|\widehat{f_R}(q)\widehat{f_R}(r)|_-} < 1 + o(\epsilon).$$

Let $\epsilon \to 0$, we get $|qr|_+ = |\widehat{f_R}(q)\widehat{f_R}(r)|_-$.

(ii) f_R is continuous. Let $q_i^c \in \Sigma_p \times \{R\}$, and $q_i^c \to q^c$. By (i) and because $\mathrm{NFix}(f_R)$ is open, it's sufficient to prove for the case $q^c \in \mathrm{Fix}(f_R)$. We now show that $\lim\limits_{i\to\infty} f(q_i^c) = f(q^c) = q^c$. Consider the sequences $\exp_p(q_i^c)$, $\exp_p(f(q_i^c))$ in X. Because $\exp_p$ is distance decreasing, $\exp_p(q_i^c)$ and $\exp_p(f(q_i^c))$ converge to the same limit point x. Thus $\lim\limits_{i\to\infty} f(q_i^c) = f(q^c) = q^c = \lim\limits_{i\to\infty} q_i^c$, since $x = \exp_p(q^c) \in L_p^1(X)$,

(iii) Finally, we prove that f is an isometry. For any $x, y \in \Sigma_p$, because $\mathrm{NFix}(f)$ is open, the geodesic $\overline{xy}$ can be decomposed into the pieces and each piece contains only fixed point or no fixed point of f. Consequently, $|xy| = \mathrm{Length}(f(\overline{xy}))$. By (ii), $f(\overline{xy})$ is also a curve. Thus $|xy| \geq |f(x)f(y)|$. Since f is an involution, we also have $|f(x)f(y)| \geq |xy|$.

($\Leftarrow$) We only need to check that $X = C_k^R(\Sigma_p)/x \sim f_R(x)$ is an Alexandrov space, provided that $f : \Sigma_p \to \Sigma_p$ is an isometric involution. By the doubling theorem ([BGP]), $\widehat{X} \in \mathrm{Alex}^n(\kappa)$. Now we construct a $\mathbb{Z}_2$-isometric action $\mathbb{Z}_f^2$ (induced by f) on $\widehat{X}$ such that $X = \widehat{X}/\mathbb{Z}_f^2$. Then $X \in \mathrm{Alex}^n(\kappa)$. View $\widehat{f}$ as a map between Σ_p^+ and Σ_p^-. For any $x \in C_\kappa^R(\Sigma_p^+)$, let $\mathbb{Z}_f^2(x)$ be the point in $C_\kappa^R(\Sigma_p^-)$, such that $\overrightarrow{p^-\mathbb{Z}_f^2(x)} = \widehat{f}(\overrightarrow{p^+x})$ and $|p^+x| = |p^-\mathbb{Z}_f^2(x)|$. Parallel definition is applied for the case $x \in C_\kappa^R(\Sigma_p^-)$. □

4.4 Proof of Theorems 4.B and 4.C

Lemma 4.4.1. *Let $A \in Alex^n(\kappa)$. Assume that A is a topological manifold. Then for any $p \in A$, Σ_p is homotopically equivalent to a sphere S_1^{n-1}. In particular, Σ_p is a sphere if and only if Σ_p is a topological manifold.*

Proof. Let T_pX denote the tangent cone at p. Because p is a topological manifold point, T_pX is a flat cone homeomorphic to $\mathbb{R}^n$. In particular, an r-ball $B_r(o) \subset T_pX$ is homeomorphic to an Euclidean ball and thus $\partial B_r(o)$ is homeomorphic to S_1^{n-1}, where o is the vertex of T_pX. We may identify $C_\kappa(\Sigma_p)$ with an Alexandrov metric on T_pX, and we will construct a homotopy equivalence on T_pX, from a Euclidean sphere to Σ_p. Consider two Euclidean balls of radii $\epsilon < R$ such that $\Sigma_p \times \{r\}$ is contained in the annulus bounded by the two Euclidean balls. Starting with $\mathrm{id}_{S_R^{n-1}}$, and continuously deforms it into $\Sigma_p \times \{r\}$ (using the Alexandrov metric on T_pX). Then, using the Euclidean metric, continuously deforms Σ_p into $\partial B_\epsilon(o)$.

We now construct a deformation: $\phi : S_R^{n-1} \times [0,1] \to T_pX$ such that $\phi((s,x),0) = (s,x)$ and $\phi((s,x),1) = (r,x) \in \Sigma_p \times \{r\}$. Define

$$\phi((s,x),t) = (s-(s-r)t, x).$$

Similarly, using the Euclidean metric one can define a map, $\psi : \Sigma_p \times \{r\} \times [0,1] \to T_pX$ such that $\psi((s,x),0) = (s,x)$ and $\psi((s,x),1) = (\epsilon,x) \in S_\epsilon^{n-1}$. By the construction, we have $\psi \circ \phi \simeq \mathrm{id}_{S^{n-1}}$. □

Proof of Theorem 4.B. Let $X \in \mathcal{M}_\kappa^r(\Sigma)$ with $\mathrm{vol}(X) = v(\Sigma,\kappa,r)$. By Theorem A, X is isometric to $\bar{C}_\kappa^r(\Sigma))/x \sim f(x)$, $f : \Sigma \to \Sigma$ is an isometric involution. Recall that in the proof of Theorem A, we construct, unless $\kappa > 0$ and $r = \frac{\pi}{\sqrt{\kappa}}$ (in this case X is isometric to $C_\kappa(\Sigma)$), an Alexandrov space (double) $\widehat{X} = \bar{C}_\kappa^r(\Sigma)^+ \cup_f \bar{C}_\kappa^r(\Sigma))^-$. Since X is a topological manifold, $\widehat{X}$ is also a topological manifold. By Lemma 4.4.1, Σ is homotopically equivalent to S_1^{n-1} and thus Σ is simply connected. Because $\bar{C}_\kappa^r(\Sigma)$ is contractible, by Van-Kampen theorem we see that $\widehat{X}$ is simply connected, and from Mayer-Vietoris exact sequence of $(\bar{C}_\kappa^r(\Sigma)^+, \bar{C}_\kappa^r(\Sigma)^-)$ we see that $\widehat{X}$ is a homology sphere, and thus a homotopy sphere. By the Poincaré conjecture, $\widehat{X}$ is a homeomorphic sphere.

We now naturally extend the isometric $\mathbb{Z}_2$-action on Σ to an isometric $\mathbb{Z}_2$-action on $\widehat{X}$ such that $X = \widehat{X}/\mathbb{Z}_2$ and that the extended $\mathbb{Z}_2$ has the same fixed point set $F \subset \Sigma$. Then $\dim(F) \le n-2$. If the $\mathbb{Z}_2$-action on Σ is free, then X is homeomorphic to a real projective space $\mathbb{R}P^n$. Otherwise, $X = \widehat{X}/\mathbb{Z}_2$ is simply connected. Note that if $\dim(F) < n-2$, then X is not a homology manifold at a point $p \in F$, a contradiction. Thus $\dim(F) = n-2$. By the Smith theorem, the $\mathbb{Z}_2$-fixed point set F is connected and F is a $\mathbb{Z}_2$-homology sphere. In this case, it is easy to check that X is a homology sphere. □

Example 4.4.2. Let $N = S^3/\Gamma$ denote a homology sphere (Poincaré sphere) of constant curvature one, and let $\Sigma = C_1(N)$ denote the spherical suspension over N. Then Σ is not a topological manifold (only a homology manifold). It is known that the spherical suspension, $X = C_1(\Sigma)$ is homeomorphic to S^5_1. Note that $X \in \mathcal{M}_1^\pi(\Sigma)$ achieves the maximal volume.

Proof of Theorem 4.C. We argue by contradiction: assuming a sequence $X_i \in \mathcal{M}_\kappa^r(\Sigma)$ such that $\mathrm{vol}(C_\kappa^r(\Sigma))) < \mathrm{vol}(X_i) + \epsilon_i$ and $\epsilon_i \to 0$, but X_i is not homeomorphic to any element in $\mathcal{M}_\kappa^r(\Sigma)$ with the relatively maximal volume.

Let $p_i \in X_i$, $\Sigma_{p_i} = \Sigma$ and $X_i = \bar{B}_r(p_i)$ for all i. Since the sequence has a uniform lower bound on volumes, we may assume, passing to a subsequence if necessary, that $(X_i, p_i) \xrightarrow{d_{GH}} (X, p) \in \mathrm{Alex}^n(\kappa)$. By Perel'man's stability theorem, X_i is homeomorphic to X when i large. Taking limit as $i \to \infty$, $\mathrm{vol}(C_\kappa^r(\Sigma_{p_i}))) < \mathrm{vol}(X_i) + \epsilon_i$, we see that $\mathrm{vol}(C_\kappa^r(\Sigma_{p_i}))) \le \mathrm{vol}(X)$. By the volume comparison, $\mathrm{vol}(C_\kappa^r(\Sigma)) \ge \mathrm{vol}(X_i)$, and taking a limit,

$$\mathrm{vol}(C_\kappa^r(\Sigma)) \ge \lim_{i\to\infty} \mathrm{vol}(X_i) = \mathrm{vol}(X),$$

and therefore $\mathrm{vol}(X) = \mathrm{vol}(C_\kappa^r(\Sigma))$. We will show that $X \in \mathcal{M}_\kappa^r(\Sigma)$, and this, because X has the relatively maximal volume, leads to a contradiction.

We will first construct a distance non-increasing continuous onto map from $C_\kappa^r(\Sigma))$ to X. Since the two spaces have the same volume, following the proof of Theorem D we may conclude that $B_r(p)$ is isometric to $B_\ell(C_\kappa(\Sigma))$ with respect to the intrinsic metric. In particular, $\Sigma_p X$ is isometric to Σ (note that the boundary points, $\partial B_r(C_\kappa(\Sigma)) - \{x \in C_\kappa(\Sigma),\ d(p,x) = r\}$, have no self-gluing in X, and thus the interior isometry actually extends to this part).

Recall that $g\exp_{p_i} : B_\ell(C_\kappa(\Sigma_{p_i})) \to X_i$ is a continuous distance non-increasing map. Let $f_i : (X_i, p_i) \to (X, p)$ be an ϵ_i Gromov-Hausdorff approximation ($\epsilon_i \to 0$). Then $\phi_i = f_i \circ g\exp_{p_i} : B_\ell(C_\kappa(\Sigma_{p_i})) \to X$ is an ϵ_i distance non-increasing and ϵ_i-onto map.

Passing to a subsequence, we may assume $\phi_i \to \phi : B_\ell(C_\kappa(\Sigma)) \to X$. Clearly, ϕ is a distance non-increasing continuous onto map.

Finally, for $\kappa > 0$, it is clear that $\ell' = \operatorname{diam}(X) \leq \frac{\pi}{2\sqrt{\kappa}}$, or $\ell' = \operatorname{diam}(X) = \frac{\pi}{\sqrt{\kappa}}$, because for $\frac{\pi}{2\sqrt{\kappa}} < \ell' < \frac{\pi}{\sqrt{\kappa}}$, $B_{\ell'}(C_\kappa(\Sigma)) \notin \mathrm{Alex}^n(\kappa)$ since $B_{\ell'}(C_\kappa(\Sigma)) \subset C_\kappa(\Sigma)$ is not a convex subset. $\square$

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