COST ANALYSES ON WARRANTY POLICIES FOR SYSTEMS SUBJECT TO TWO TYPES OF WARRANTY PERIODS

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ABSTRACT OF THE DISSERTATION

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In general, a warranty is an obligation attached to products that requires the manufacturers to provide compensation for customer (buyer) according to the warranty terms when the warranted products fail to perform their intended functions [179]. A warranty is important to the manufacturer as well as the customer of any commercial product since it provides protection to both parties. As for the customer, a warranty provides a resource for dealing with items that fail due to the uncertainty of the product's performance and unreliable products. For the manufacturer, it provides protection since the warranty terms explicitly limit the responsibility of a manufacturer in terms of both time and type of product failure. Because of the role of the warranty, manufacturers have developed various types of warranty policy to grab the interest of the customers. However, manufacturers cannot extend the warranty period without limit and maximize warranty benefits because of the cost related to it.

Many researchers have investigated on the topic of warranty modeling and policy and expanded their studies of warranty in various different conditions, i.e., maintenance policies.

In this dissertation, we focus on the developments of warranty cost models with various maintenance policies as well as the warranty policy with post warranty periods for single-component and multi-component systems including parallel-series, series-parallel and *k-out-of-n* systems from the perspectives of consumer and manufacturer, maintenance policies and repair policies. First, the role, concept and other factors of the warranty policies, are introduced. We conduct the literature review and present the selected mathematical background that will be used throughout the dissertation.

We develop several warranty cost models and derive reliability measures for various systems including series-parallel, parallel-series, and *k-out-of-n* configurations based on the proposed alter- and mixed- quasi-renewal processes. We focus on the warranty cost analysis including repairable products with a given warranty period using the induction method. Additionally, we use the non-homogenous Poisson process and minimal repair to develop warranty cost models for the *k-out-of-n* systems in the warranty period subject to failure times and repair times (two dimensional model). We combine maintenance policies and several warranty policies such as failure repair/replacement warranty, pro-rata warranty and combination warranty into the cost analysis. Additionally, we investigate the maintenance policies with warranty period and post warranty period based on two dimensions such as failure times and repair times. Finally, we present concluding remarks and future research topics.

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DEDICATION

To my wife Jennifer Jaeyoung Kim

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LIST OF ABBREVIATIONS

ARP	age replacement policy
BRP	block replacement policy
BED	bivariate exponential distribution
cdf	cumulative distribution function
CM	corrective maintenance
CMW	combination warranty
CPM	corrective maintenance combined with preventive maintenance
CR	corrective replacement
CV	coefficient of variation
Exp (λ)	Exponential distribution with parameter λ
FR	failure replacement
FRW	free repair warranty
Gamma (n, λ)	Gamma distribution with parameters n and λ
GBRP	Generalized Block Replacement Policy
IFR	Increasing Failure Rate
k-out-of-n	a system is working iff at least k out of n components are working
i.i.d.	identical and independent distributed
MBRP	Modified Block Replacement Policy
MCMC	Markov Chain Monte Carlo
MLE	Maximum likelihood estimates
MTBF	Mean time between failures
MTTR	Mean time to replacement
NHPP	non-homogeneous Poison process
Normal (μ,σ^2)	Normal distribution with μ and σ^2
pdf	probability density function
PM	preventive maintenance
pmf	probability mass function

PR	preventive replacement
PRW	pro-rata warranty
QRP	Quasi-renewal processes
r.v.	random variable
SD	Standard deviation
Uniform (a, b)	Uniform distribution with parameters a and b
Weibull (a, b)	Weibull distribution with parameters a and b
w.r.t	with respect to

Chapter 1

Introduction

As the market becomes competitive and diversified, it is hard for the manufacturers to differentiate its product to consumers with only quality and an eye catching design. Also with the massive information available to consumers regarding the product manufacturers need to find a better way to communicate with its customers to differentiate and to inform its product. In order to achieve this goal, many companies promote the warranty policy as an effective tool to attract consumers.

Hyundai Motor Company (HMC) first introduced its new model “Excel” car to the US in 1986 [179]. The company used low price strategy to penetrate into the competitive market. However, company was not as successful as they had hoped since the perception of Hyundai car was "affordable but low quality" compared to the same size vehicles of its US and Japanese competitors. Trying to overcome this perception, HMC invested heavily into the brand, design and quality. Also it launched the 10-year or 100,000 miles warranty program for the cars sold in US in 1998 [1, 179]. This warranty policy was non-precedent and was more than enough to successfully promote the improvement of its quality to many potential buyers. With this program, the company was able to communicate with the consumers more effectively on their commitment to the product and show confidence of the quality of its cars. Consequently now the perception from the consumer towards HMC is different from when it first entered the market.

Recently service industry is trying to adopt the warranty system from the manufacturing industry, such as computer, automotive manufacture and television set manufacture. For example, to deliver better service, a hospital group, Geisinger Health System, in Pennsylvania has conducted an experiment in February 2006 for elective heart bypass surgery, the hospital charges a flat fee service that includes a warranty of 90 days of follow-up treatment [91, 97]. Under this program, after a surgery patients need not pay for any further service such as treatment from complications, follow up visits and etc. The hospital normally charges a flat fee for the surgery and the 50% of the estimated cost for any potential treatments for the 90 days during the warranty period using historical data. This implies that any additional cost incurred the hospital needs to bear. This system provides hospital with the incentives to improve its service to its patients while they are in treatment and take close care for any follow up treatments to minimize the future cost. As for the patients, they are receiving better quality service from the doctors and hospitals compared with previously where the service was focused more on the frequency. As a result, patients have been less likely to return to intensive care and have spent fewer days in the hospital before they were discharged. Now the hospital is known for its superior service and the follow up treatment resulting from the warranty policy.

In summary, the warranty policy can be utilized to benefit a company in many ways. The above two examples well illustrate its role as a communicational, promotional tool and also incentive to improve the quality of the product or service.

1.1. Concept of the Warranty Policy

Warranty policy is a guarantee or an obligation to repair or replace a defective product or parts when the product does not perform its expected function during a given time period. This is a contract between the customer and the manufacturer upon the point when the policy is sold.

Warranty benefits both the consumer and the manufacturer as it is set to protect both parties. The consumer is protected as it guarantees a resource to deal with any defects or errors while using the product. Similarly, the manufacturer is protected because the warranty terms explicitly limit the responsibility in terms of both time and type of product failure. The warranty policy is an obligation attached to products that require the manufacturer to provide compensation for consumers according to the warranty terms when the warranted products fail to perform their intended functions [179].

As for a manufacturer, with the increase in demand for better quality warranty, it tries to develop an appealing policy and strategically use it as a promotional/marketing tool. Companies often emphasize on the benefits received under the policy such as details of the compensation for the defects, the charge or the period of the warranty. However, given that any service under the warranty policy is a potential cost item for a company, drafting a policy which is economically optimal so that it minimizes the cost but maximizes the satisfaction of the consumer is critical.

In summary, the warranty policy concept is to protect both the consumer and the manufacturer. The consumer is provided a resource for dealing with items that fail to function properly, i.e., unreliable products. Whereas the manufacturer is provided protection because warranty terms explicitly limits the responsibility in terms of both time and type of product failure. When products are getting more complicated, it would be difficult for customers to make a purchasing decision. So, the warranty policy would provide one of the criteria for products' quality and reliability. And the longer warranty period cost more expenses for the sellers. When a manufacturer wants to provide better warranty condition than their other competitive sellers, they are supposed to provide better quality of products. Otherwise, they couldn't save their warranty cost. Such trade-offs would make the warranty policy be a strong marketing tool to increase the sales rate and to advertise the quality of products.

1.2. Warranty Policies

There are various characteristics which categorize the warranty policy separately. These characteristics include the number of warranty dimensions, the renewability of a warranty and the warranty compensation methods. You can refer more details to [20, 21].

One and Two Dimensional Policies

First, consider the number of warranty dimensions. Most warranties in practice are one dimensional for which the warranty terms are based on product age or product usage, but not both. Compared to one dimensional warranty, two dimensional warranties are more complex since the warranty obligation depends on both product age and product usage as well as the potential interaction between them. Two dimensional warranties are often seen in automobile industry. As mentioned in section 1.1, HMC is currently offering 10 years with 100,000 miles warranty on the power train for most of their new models. Several researchers [20, 95] have studied the warranty policy based on the automobile industry's data.

Renewing Warranty and Non-renewing Warranty

One of the basic characteristics of warranties is whether they are renewable or not. For a regular renewable policy with warranty period, whenever a product fails in the warranty period, a customer is compensated according to the terms of the warranty contract and the warranty policy is renewed for another period. As a result, a warranty cycle starting from the point of sale, ending at the warranty expiration date, is a random variable whose value depends on the warranty period, the total number of failures under the warranty and the actual failure inter-arrival times. The majority of warranties in the market are non-renewable for which the warranty cycle, which is the same as the warranty period, is not random, but pre-determined since the warranty obligation will be terminated as soon as warranty period unit of time passes after the sale. These types of policies are also known as fixed period warranties.

Free Replacement Warranty, Pro-rata Warranty and Combination warranty

According to the methods of compensation specified in a warranty contract upon premature failures, there are three basic types of warranties: free replacement/repair warranty (FRW), pro-rata warranty (PRW) and combination warranty (CMW) . Under FRW, a failed item is replaced/repaired at no cost to the buyer if the failure occurs in the warranty period. On the other hand, under PRW, warranty services are not provided free of charge, but are provided at a pro-rated cost with the proration depending on the amount of usage or service time provided by the item prior to its failure [20]. In Chapter 7, alternative PRW is suggested. In the alternative PRW, customers will have to pay partial repairing service cost depending on the failure time. If the replacement costs are more expensive than the repair costs, manufacturers would provide to repair the failed parts/products instead of replacement and vice versa. Accordingly, whenever the product is failed, the manufacturers would commonly provide repair services than replacement services. So, alternative PRW is more easily applicable than original PRW because alternative PRW handles a repair service, not a replacement service. Combination warranty contains both features of FRW and PRW, which often contains two warranty periods, a free replacement period followed by a pro-rata period. Full-service warranty also known as preventive maintenance warranty, is a policy that may be offered for expensive deteriorating complex products such as automobiles. Under these type of policies, consumers not only receive free repairs upon premature failures, but also free preventive maintenance.

1.3. Organization of the Study

In this dissertation, we study warranty cost analysis and maintenance policies under various conditions with factors such as different types of warranty policies from the perspectives of consumer and manufacturer, maintenance policies and repair policies. For the cost analysis, we obtain the expected warranty cost and develop related cost models. To conduct warranty analysis, we also explore the characteristics of the warranty policy. In Chapter 1, we briefly discuss the concept of warranty and review the overall information about the warranty policy such as warranty's role, concept, different types and purpose. In Chapter 2, we conduct the literature review about the research on warranty and warranty related maintenance. We also briefly discuss basic concepts on counting processes such as renewal process, quasi-renewal process, non-homogenous Poisson process, compound and marked Poisson process and bivariate exponential distribution that will later be used in this research. For the literature review, we will focus on research articles which have been published relatively recently since 2000 and also briefly review papers that published before the year 2000. The research objectives are described and summarized in Chapter 3.

In Chapter 4, we introduce two altered quasi renewal processes based on the ordinary quasi-renewal process. The first is called altered quasi-renewal process with random parameter and the second is a mixed quasi-renewal process with considerations of replacements and repairs strategies. Based on the proposed alter- and mixed- quasi-renewal processes, we develop several warranty cost models and also derive reliability measures for various systems including series-parallel, parallel-series, and *k-out-of-n* configurations. The results of this study using mixed and altered quasi-renewal processes can be found helpful for practitioners to analyze the system warranty cost in practice.

In Chapter 5, warranty cost models for various systems subject to imperfect repair based on the quasi-renewal processes and exponential distribution are developed. This chapter focuses on the warranty cost analysis including repairable products with a given warranty period considering conditional probabilities and renewal theory.

In Chapter 6, we develop a modified block replacement model for *k-out-of-n* systems and determine optimum policies of both a threshold level for the number of failed components to prevent the system's failures and the maintenance cycle that minimizes the expected total system cost. To overcome the existing block replacement policies' drawbacks which are rather wasteful if a preventive replacement happens just after a failure replacement, in our developed policy, replacement service for a failure is provided when *m* number of failed components occur. We also take into considerations downtime period of each failed component using the order statistics for life time and age distributions for *k-out-of-n* systems.

In Chapter 7, warranty period and post warranty period are considered. We use non-homogenous Poisson process (NHPP) and minimal repair to develop warranty cost models for *k-out-of-n* systems in the warranty period subject to one dimension and two dimensions. The relationships between current inter-failure interval and the next inter-failure interval are investigated. Using the optimized warranty period, we obtain the expected values of n^{th} inter-failures intervals. For the post warranty period, we obtain total expected cost and total expected duration with respect to maintenance policies such as corrective maintenance and preventive maintenance. Then, we obtain a long-run expected cost per unit time and determine optimum warranty periods and periodical maintenance periods.

In Chapter 8, we develop a two-dimensional warranty policy with repair times and failure times which are statistically correlated in bivariate distributions. Based on our developed approaches, we investigate the property of the bivariate renewal function and obtain the number of warranty services in a warranty period using the field data.

Numerical examples are discussed in each Chapter to demonstrate the results and proposed models derived for Chapters 4-8.

The last Chapter, Chapter 9, presents concluding remarks and future research topics. For the future research, there are two interesting problems as follows: The warranty cost models can be developed considering two maintenance policies such as age replacement policy and block replacement policy under different warranty policies in the warranty period and post warranty peirod. The other future research topic is to develop warranty cost models considering the non-renewable warranty policies with different lengths of warranty periods.

Chapter 2

Background and Literature Review

2.1. Warranty Cost Analysis

This chapter discusses about the research on warranty policies and related topics that many researchers [7, 20, 38-40, 50, 53, 85, 117, 120, 121, 123, 146, 182, 190] have been done in the literature by several different categorized groups. General descriptions of various types of warranty policies and mathematical models can be found in Blischke and Murthy [20, 21].

2.1.1. One Dimensional (= Attribute) Warranty and Two Dimensional Warranty

One dimensional warranty is characterized by the warranty period, which is defined in terms of a single variable. Single variable could be time, age or usage. In the case of two-dimensional warranties, there are two dimensions to express warranty polices. One is representing time and the other representing item usage. As a result, many different types of warranties may be defined based on the characteristics of warranty policies [20]. And many researchers have studied the cost analysis based on two dimensional warranty [10, 32, 42, 43, 70, 71, 83, 107, 197]. Yun and Kang [197] examine new warranty servicing strategy, considering imperfect repair with a two-dimensional warranty. Baik *et al.*[10] study two-dimensional failure modeling for a system where degradation is due to age and usage with minimal repair. Most of the products have one of two attributes with some exceptions, for example, a vehicle. Several researchers [20, 95] have studies the warranty policy based on the automobile industry’s data. Compared to one-attribute warranties, two-attribute warranties are more complex [4-8]. Chun and Tang [44] propose several decision models that estimate the expected total cost incurred under various types of two-attribute warranty policies. Kim and Rao [85] consider two-attribute warranty policies for non-repairable items and the item failures are described in terms of a bivariate exponential distribution. Jiang and Ji [76] study a multiple attribute value model based on four attributes such as cost, availability, reliability and lifetime. Samatli-Pac and Taner [146] develop and investigate different repair strategies for one- and two-dimensional warranties with the objective of minimizing manufacturer’s expected warranty cost using QRP. Other researchers [10, 32, 42, 43, 70, 71, 83, 197] have also developed warranty models by considering two-dimensional warranty strategies.

2.1.2. Renewing Warranty and Non-renewing Warranty

Under a renewing warranty, the product which fails during its warranty period is replaced by a new one at a cost to the manufacturer or at a pro-rated cost to the user and the warranty is renewed. Under a non-renewing warranty, the manufacturer guarantees a satisfactory service only during the original warranty period. Renewable warranties are usually given to the non-repairable and inexpensive products such as home appliances and so on. Compared to the renewable warranties, the period of non-renewable warranties is relatively longer. So this might be one of possible reasons why such policies are not as popular as non-renewable ones for warranty issuers [6]. Jung *et al.* [81] investigate the optimal replacement policies following the expiration of warranty such as renewing warranty and non-renewing warranty. Chukova and Hayakawa [38, 39] evaluate the warranty costs over the warranty period under non-renewing and renewing warranty policies over the life cycle of the product. Sahin and

Polatoglu [144] prove that the cost rate function is psedo-convex under a fixed-maintenance period policy under non-renewing and renewing warranty policies. Chen and Chien [30] investigate a model to study the effect of PM carried out by the buyer on items sold under a renewing FRW.

2.1.3. Warranty Period and Post Warranty Period

During warranty period, as mentioned above, there are several kinds of warranty polices such as FRW, PRW or CMW. However, during post warranty period, customers have to repair or replace the failure product at their own expenses. Jung and Park [80] consider two types of warranty policies such as renewing warranty and non-renewing warranty with warranty period and post warranty period. They derive the expressions for the expected maintenance costs for the periodic preventive maintenance during post warranty period. Jung *et al.*[81] study the optimal replacement policies during post warranty period considering the expected downtime per unit time and the expected cost rate per unit time. Jung [79] consider the optimal period for the periodic PM during the post warranty period which minimize the expected long-run maintenance cost per unit time.

2.1.4. Warranty Reserve

Warranty reserve is one of important factors which would be considered for the warranty policies. Therefore, several researchers [26, 72, 73, 126, 161, 184] have considered the warranty reserve for the cost anlaysis. Patankar and Mitra [126] investigate the effect of warranty execution on the expected warranty reserves of a linear pro rata rebate plan. Ja *et al.* [72, 73] consider a policy where warranty is not renewed on product failure within the warranty period but the product is minimally repaired by the manufacturer with the warranty reserves.

2.2. Reliability and Warranty

The relationship between warranty policies and products' reliability is very closely related. If the product's reliability is good, then the product's warranty could be extended. Otherwise, the product's warranty should be considered again. However, there are some exceptions. To increase a product's sales, some providers extend the product's warranty period. They use the warranty policy as a marketing tool. The reliability of product is determined by several important factors such as product's design, development, manufacturing stages and so on. It depends on the selection of suppliers and their cooperation in quality efforts as well. This implies that several important factors must take into account the interaction between warranty and reliability. A company either gives a warranty that is far shorter than the expected life of their item or increases the cost to a very high level to cover expected warranty costs. Therefore, a product's reliability is one of important measures to investigate the warranty cost analysis [106]. In the other hand, Percy [127] presents some new ideas for improving a product's reliability by adopting Bayesian methodology.

2.3. Maintenance Policies and Warranty

Many researchers [30-32, 37, 51, 59, 77, 79, 80, 82, 90, 92, 102, 109, 115, 118, 125, 130, 144, 145, 160, 168, 170, 173, 174, 176-178, 181, 193] have published studies on maintenance polices. Jhang and Sheu [75] derive the expected long-run cost per unit time for each policy. Sheu [153] considers a two-typed failures system which is subject to shocks what arrive by a

NHPP with the ARP and the BRP. Wang [174] summarizes, classifies and compares various existing maintenance policies for both single-unit and multi-unit systems. Pham and Wang [129] also summarize various treatment methods and optimal policies on the imperfect maintenance. Jung and Park [80] develop the optimal periodic PM policies following the expiration of warranty. Garbatov and Soares [59] plan the maintenance from an economic point of view so as to minimize maintenance costs but satisfying a minimum reliability level.

The maintenance objectives are to minimize the maintenance related operating costs, to maximize equipment availability and reliability or prolong equipment lifetime [76]. For deteriorating complex products, it is essential to perform preventive maintenance to achieve satisfactory reliability performance. Maintenance involves planned and unplanned actions carried out to retain a system at or restore it to an acceptable operating condition. Planned maintenance is usually referred as preventive maintenance while unplanned maintenance is labeled as corrective maintenance or repair [179]. Two well-known preventive maintenance policies are block replacement policy and age replacement policy. Barlow and Hunter [13] suggest these two types of preventive maintenance. Since then, a lot of research have been done regarding maintenance polices. Jhang and Sheu [75] derive the expected long-run cost per unit time for each policy. Sheu [153] considers a two-typed failures system which is subject to shocks what arrive by a NHPP with age and block replacement policy. Wang [174] summarized, classified and compared various existing maintenance policies for both single-unit and multi-unit systems. Also, Pham and Wang [129] summarize various treatment methods and optimal policies on the imperfect maintenance. Jung and Park [80] develop the optimal periodic preventive maintenance policies following the expiration of warranty.

Garbatov and Soares [59] plan the maintenance from an economic point of view so as to minimize maintenance costs but satisfying a minimum reliability level.

2.3.1. Age Replacement Polices

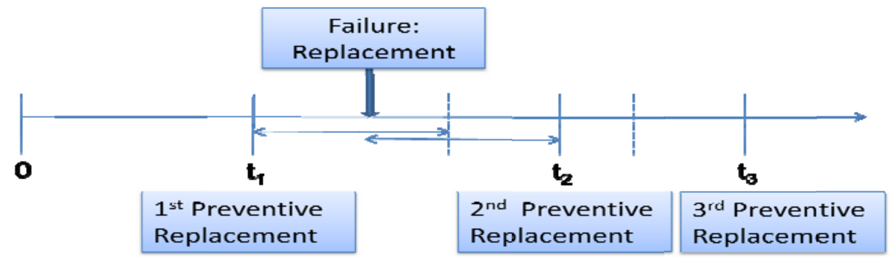


Figure 2.3.1 Age Replacement Policies

In the age replacement policy, a preventive replacement is performed after a given continuous operation time T without failure, and a failure replacement is performed if the system fails before T [76]. This model has been generalized by many researchers [5, 33-36, 40, 44, 50, 51, 75, 76, 90, 99, 104, 117, 126, 133, 140, 153, 154, 156, 185, 188, 192]. In Figure 2.3.1, a product is replaced at a certain age t, or upon failure, whichever occurs first. And if the failure replacement happened then the next preventive replacement is rescheduled from the time of failure replacement. Sheu and Chien [154] consider a generalized age-replacement policy of a system subject to shocks, which arrived by NHPP, with random lead-time. Bai and Yun [5] propose a generalized replacement policy based on the system age and the minimal repair which is similar to the age replacement policy. Kumar and Westberg [90] develop maintenance model under age replacement policy using proportional hazards model and TTT-plotting. Yeh et al. [192] investigate the effects of a renewing FRW on the age replacement policy for a nonrepairable product and compare maintenance polices with warranty and without warranty. Chien [35] investigates the effects of an imperfect renewing FRW on the age replacement policy for a product with an increasing failure rate. Sheu *et al.* [158] propose a generalized replacement policy where a system has two types of failures and is replaced at the minor failure or catastrophic failure or at age T, whichever occurs first.

Cost model [13, 76, 130]

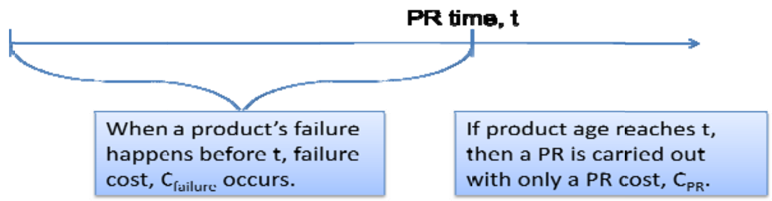


Figure 2.3.2 Cost model based on the age replacement policy

In Figure 2.3.2, it presents the basic cost model based on the age replacement policy. Let PR be preventive replacement and C_{PR} and $C_{failure}$ stand for preventive replacement cost and failure cost, respectively. If a random variable x is a failure time, a cost coefficient is defined as

$$C(t) = \begin{cases} C_{failure} & if \;\; x < t \\ C_{PR} & if \;\; x \geq t \end{cases}$$

$E\left(T(t)\right)$ is the expected duration and the expected cost rate is given by

$$Expected \;\; cost \;\; rate = \frac{E\left(C(t)\right)}{E\left(T(t)\right)} = \frac{C_{PR}R(t) + C_{failure}F(t)}{\int_0^t R(x)\,dx}$$

Availability model [54, 76]

In a similar way as deriving the cost model, the availability model based on the age replacement policy is given by

$$A(t) = \frac{1}{1 + \dfrac{T_f F(t) + T_p R(t)}{\int_0^T R(t)\,dt}}$$

MTBF stands for a mean time between failures and MTTR stands for a mean time to replacement. T_f is a time of performing a failure replacement and T_p is a time of performing a preventive replacement.

Reliability model [76]

There are several reliability models. One of them is explained here. The PR occurrence rate is just the number of PR over total replacement by time t. And higher occurrence rate is more reliable. The reliability model based on the age replacement policy is given by

$$\begin{aligned} Occurrence\ rate\ \ for\ \ PR &= \frac{\#\ \ of\ \ PR\ by\ T}{\#\ \ of\ \ FR\ by\ T + \#\ \ of\ \ PR\ by\ T} \\ &= \frac{\#\ \ of\ \ PR\ by\ T}{\#\ \ of\ \ total\ \ replacement} \end{aligned}$$

2.3.2. Block Replacement Policies

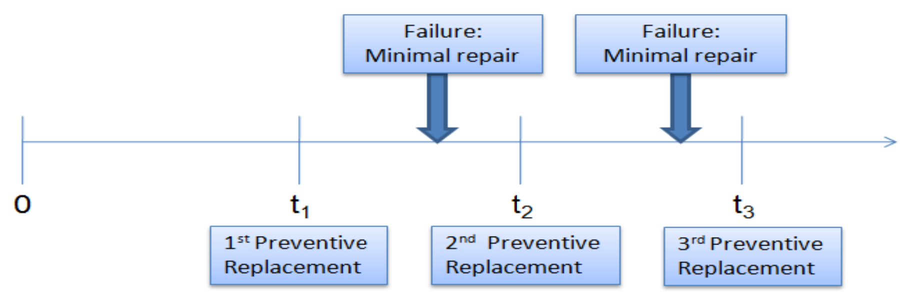


Figure 2.3.3 Block replacement policies

In the block replacement policy, an operating system is replaced by a new one at times kT, k=1,2,… and at failures. In Figure 2.3.3, preventive maintenance is performed after it has been operating time t regardless of the number of intervening failures. One of drawbacks of block replacement policy is that it is rather wasteful because sometimes almost-new systems are replaced at planned replacement times [152]. Many research [16, 17, 75, 84, 108, 114, 119, 150-153, 157, 166] have been done regarding this block replacement policy too. Sheu and

Griffith [151] consider an extended block replacement policy with used items and shock models with two types of failures.

Age replacement policy is useful in maintaining simple equipment. In the other hands, block replacement policy is useful in maintaining large and complex equipment. For the age replacement policy, between maintenance periods, a failed component/system is replaced at the moment. However, in the block replacement policy, between maintenance periods, a failed component/system is repaired minimally.

Cost model [3, 14]

In a similar way of cost model in age replacement policy, let C_{PR} and C_{CR} stand for preventive replacement cost and corrective replacement cost, respectively. Consider a single component system. The system is replaced on failure and preventively at times T, 2T, …, etc. Let H(t) denotes the mean number of replacements in the interval (0, t) of a unit(system).

$E\left(T\left(t\right)\right)$ is the expected duration and the expected cost rate is given by

$$Expected\ cost\ rate = \frac{C_{PR} + C_{CR}H\left(t\right)}{T}$$

Modified cost model 1 [119]

Park and Yoo [119] propose the modified block replacement policy where a block replacement is performed at failure k, counting after the pre-determined individual failure-replacement interval $(0, \tau]$. They called this policy as the block replacement policy based on idle count. C_d is downtime cost per unit. Additionally, M(t) represents the mean number of failures replacements during $(0, \tau]$ and $R^{(i)}\left(\tau\right)$ is the time-to-failure i from τ for the fleet. $C_d D$ is the mean downtime cost per unit. Let $G_\tau\left(t\right)$ be the cdf of the residual life at τ.

$$Expected\ \ cost\ \ rate = \frac{C_{PR} + C_{CR}M\left(\tau\right) + C_d D}{\tau + E\left\{R^{(k)}\left(\tau\right)\right\}}$$

$$where\ D = \sum_{j=1}^{k-1} \frac{j}{N} \binom{N}{j} \int_0^{\infty} \left[G_{\tau}\left(t\right)\right]^i \left[1 - G_{\tau}\left(t\right)\right]^{N-i}\ dt$$

Modified cost model 2 [114]

Nakagawa [114] propose another modified block replacement policy with an idle period, units are replaced at failure until a fixed time *T* and then follows an idle period *d*, during which failed units are left idle. *I(d)* is the mean downtime per unit during *d*.

$$Expected\ \ cost\ \ rate = \frac{C_{PR} + C_{CR}M\left(T\right) + C_d I\left(d\right)}{\tau + d}$$

$$where\ \ I\left(d\right) = \int_0^d G_T\left(t\right) dt$$

2.3.3. Maintenance Cost Analysis

Boland and Proschan [24] investigate a model for the minimal repair-periodic replacement policy and consider the problem of determining the period which minimizes the total expected cost of repair and replacement. Park *et al.* [118] consider the situation where each PM relieves stress temporarily and hence slows the rate of system degradation, while the hazard rate of the system remains monotonically increasing. Canfield [27] obtains the cost optimization of the PM intervention interval by determining the average cost-rate of system operation. Wang and Pham [178] investigate availability, maintenance cost and optimal maintenance polices of the series system with n constituting components under the general assumption that each component is subject to correlated failure and repair, imperfect repair, shut-off rule and arbitrary distributions of times to failure and repair.

2.3.4. Maintenance Policies and Warranty

The maintenance objectives are to minimize the maintenance related operating costs, to maximize equipment availability and reliability or prolong equipment lifetime [76]. For deteriorating complex products, it is essential to perform preventive maintenance to achieve satisfactory reliability performance. Maintenance involves planned and unplanned actions carried out to retain a system at or restore it to an acceptable operating condition. Planned maintenance is usually referred as preventive maintenance while unplanned maintenance is labeled as corrective maintenance or repair [179]. Two well-known preventive maintenance policies are block replacement policy and age replacement policy. Barlow and Hunter [13] suggest these two types of preventive maintenance. Since then, a lot of research have been done regarding maintenance polices. Jhang and Sheu [75] derive the expected long-run cost per unit time for each policy. Sheu [153] considers a two-typed failures system which is subject to shocks what arrive by a NHPP with age and block replacement policy. Wang [174] summarize, classify and compare various existing maintenance policies for both single-unit and multi-unit systems. Also, Pham and Wang [129] summarize various treatment methods and optimal policies on the imperfect maintenance. Jung and Park [80] develop the optimal periodic preventive maintenance policies following the expiration of warranty. Garbatov and Soares [59] plan the maintenance from an economic point of view so as to minimize maintenance costs but satisfying a minimum reliability level. Also, several researchers [31, 77, 160] investigate the maintenance policies based on the Bayesian approach. Chen and Popova [31] propose two kinds of Bayesian maintenance polices. Additionally, a set of maintenance policies which consist of minimal repair and preventive maintenance is analyzed for the case of known and unknown failure parameters of the item’s lifetime distribution. Sheu *et al.*[160]

and Juang and Anderson [77] consider a Bayesian theoretic approach to determine an optimal adaptive preventive maintenance policy with minimal repair. A Bayesian approach is established to formally express and update the uncertain parameters for determining an optimal adaptive preventive maintenance policy. Stephens and Crowder [164] analyze the discrete time warranty data based on the Markov Chain Monte Carlo (MCMC) model.

2.4 Other Topics

2.4.1 Burn-in Process and Warranty

The burn-in process is a part of the production process whereby manufactured products are operated for a short period of time before release [128]. Burn-in is used to improve product quality pre-sale. Particularly for products with an initially high failure rate sold under warranty, burn-in can be used to reduce the warranty cost [155]. Several researchers [29, 34, 128, 155, 159, 163, 169, 186, 187, 189, 198, 199] have investigated the warranty policy using the burn-in process. Wu *et al.* [186] develop a cost model to determine the optimal burn-in time and warranty length for non-repairable products under the fully renewing FRW and PRW policy. In Chang's paper [29], the optimal burn-in decision has to take both the critical time and its post-burn-in mean residual life into considerations for improving reliability due to the features of unimodal failure rate function and its upside down unimodal mean residual life. Rangan and Khajoui [136] construct a new stochastic model which treats burn-in, warranty and maintenance strategies together in order to define coordinated strategies for system design and management. Wu and Clements-Croome [189] consider a product with a long time dormant period and investigate two burn-in policies, which incur different burn-in costs and different burn-in effects on the products. Sheu and Chien [155] consider a general repairable product sold under warranty and determine the burn-in time required before the product is put on sale. Burn-in time is optimized to minimize the expected total cost under various warranty policies. In Yun *et al*'s papers [198, 199], optimal burn-in time to minimize the total mean cost, which is the sum of manufacturing cost with burn-in and cumulative warranty cost, is studied under cumulative FRW and PRW .

2.4.2 Software Reliability and Warranty

On the other hand, based on various software systems, many researchers [51, 55, 132, 133, 140, 143, 145, 165, 182] have investigated and studied the warranty policy considering several factors such as maintenance and upgrade of software models. Using software reliability, Pham and Zhang [132] develop cost models with warranty cost, time to remove each error detected in the software system and risk cost due to software failure. Sahin and Zahedi [143, 145] present a framework and develop a Markov decision model to analyze warranty, maintenance and upgrade decisions for software packages under different market conditions. Voas [171] presents several methodologies according to the specific needs of the organization requesting assurances about the software's integrity and the peculiarities of that type of software. Williams [182] suggest an approach to calculating the delivery cost of a software product when warranty is to be provided with an imperfect debugging phenomenon.

2.4.3 Bayesian Approach and Warranty

The Bayesian decision method is another approach for the warranty analysis. In this section, we investigate many papers [25, 31, 56, 63, 68, 77, 78, 93, 96, 104, 127, 128, 160, 164, 172] which cover the warranty policy and the maintenance policy based the Bayesian decision method.

In order to set up the warranty policy, a policy maker should have some information about a product's failure. For example, there are past failure data, experimental data regarding the product's failure, intuition of the product's failure. The Bayesian decision approach is a way to incorporate this information into the decision making process [31]. Jung and Han [78] determine an optimal replacement policy for a repairable system with warranty period based on the Bayesian approach in case of renewing FRW and renewing PRW. Huang and Zhuo [67] propose a Bayesian decision model for determining the optimal warranty policy for repairable products. Fang and Huang [56] present an approach along with Bayesian process to tackle a complex decision problem and based on that approach, the optimal prior and posterior decisions of pricing scheme, production plan and warranty policy can be determined simultaneously. Gutierrez-Pulido *et al.* [63] provide an approach for the determination of warranty length that takes into account the following aspects: choice of a good estimate of the failure-time model of the product and the use of a utility function that incorporates different considerations of costs, marketing and quality. Chukova *et al.* [41] design a procedure for estimating the degree of repair as well as other modeling parameters by Markov Chain Monte Carlo (MCMC) methods.

Also, several researchers [31, 77, 160] investigate the maintenance policies based on the Bayesian approach. Chen and Popova [31] propose two kinds of Bayesian maintenance polices. Additionally, a set of maintenance policies which consist of minimal repair and preventive maintenance is analyzed for the case of known and unknown failure parameters of the item's lifetime distribution. Sheu *et al.*[160] and Juang and Anderson [77] consider a

Bayesian theoretic approach to determine an optimal adaptive preventive maintenance policy with minimal repair. A Bayesian approach is established to formally express and update the uncertain parameters for determining an optimal adaptive preventive maintenance policy. Stephens and Crowder [164] analyze the discrete time warranty data based on the MCMC model.

2.5 Mathematical Background

In this subsection, we investigate several backgrounds to study warranty analysis mathematically. Several processes have been considered to stand for failure intervals. Amongst them, two types of stochastic processes, renewal processes and non-homogeneous Poisson processes [74, 88, 133, 180] are very useful for warranty cost modeling. We study renewal process [89, 142], quasi-renewal process [173, 175] and its extensions and bivariate distributions. When Poisson process' parameter λ is constant, it is Poisson process. However, when the parameter is not constant, it is non-homogenous Poisson process. And there are two more applications such as combined Poisson process and marked Poisson process [167].

2.5.1 Renewal Processes [45, 66, 89, 142]

Consider a counting process for which the times between successive events are independent and identically distributed with an arbitrary distribution. Such a counting process is called a renewal process. Let $\{N(t), t \geq 0\}$ be a counting process and let X_n denote the time between the (n-1)st and the nth event of this process, $n \geq 1$. If the sequence of nonnegative r.v. $\{X_1, X_2, \cdots\}$ is independent and identically distributed, then the counting process

$\{N(t), t \geq 0\}$ is said to be a renewal process. The probability theories were used to model the failure times for the warranty policy. Assuming that successive failure times form a renewal process, Balcer and Sahin [12] derive moments of the total replacement cost for PRW policy and FRW policy. Phelps [134] and Balachandran *et al.* [11] use a Markovian approach for the cost analysis under warranty.

2.5.2 Quasi-renewal Processes [40, 120, 122-124, 131, 138, 139, 146, 175]

Wang and Pham [175] introduce the quasi-renewal processes (QRP). Additionally, Wang and Pham [175] propose a quasi renewal process (QRP) which is motivated by imperfect repair processes of hardware which are used in many studies [40, 120, 122-124, 131, 138, 139, 146, 175]. Let X_n be the inter-occurrence time between the (n-1)$^{\text{th}}$ and n^{th} events of the process. Let $f_i(x), F_i(x)$ *and* $h_i(x)$ be the pdf, cdf, and failure rate of random variable X_i, respectively. We say $\{N(t),\ t > 0\}$ is a *quasi-renewal process* (QRP) associated with the distribution F and the parameter α , $\alpha > 0$ a constant, if $X_n = \alpha^{n-1} \cdot Z_n$, n =1,2,… where Z_ns are iid and $Z_n \sim F$, where $\{N(t),\ t > 0\}$ is a counting process. The pdf, cdf and failure rate, respectively, for n = 2,3,4,… are given by

$$
\begin{aligned}
f_n(x) &= \frac{1}{\alpha^{n-1}} f_1\left(\frac{1}{\alpha^{n-1}} x\right) \\
F_n(x) &= F_1\left(\frac{1}{\alpha^{n-1}} x\right) \\
h_n(x) &= \frac{1}{\alpha^{n-1}} h_1\left(\frac{1}{\alpha^{n-1}} x\right)
\end{aligned}
\tag{2.1}
$$

QRP is described in Figure 2.5.1. W stands for warranty period and X_i is the ith inter-failure interval. Then $X_2 = \alpha \cdot X_1$ and $X_3 = \alpha^2 \cdot X_1$. Eventually, X_m is equal to $\alpha^{m-1} \cdot X_1$.

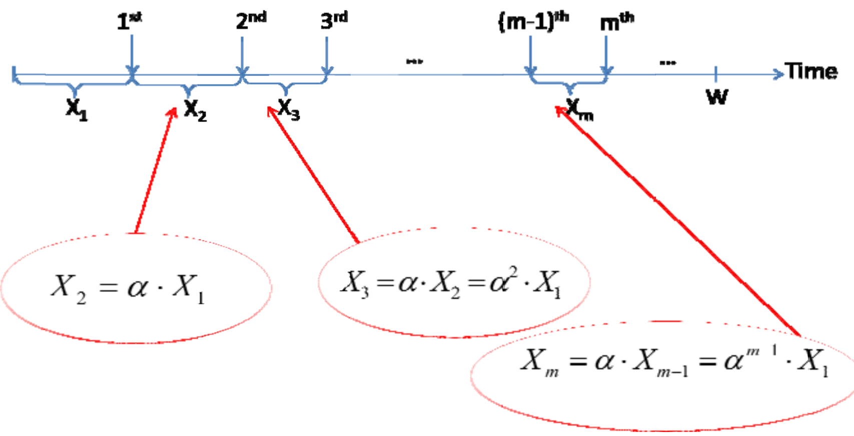


Figure 2.5.1 Quasi-renewal process

2.5.3 Extensions of Quasi-renewal Processes

Bai and Pham [6, 8] suggest two extensions of QRP such as truncated quasi-renewal process and censored quasi-renewal process. They omit a certain range of possible values for the truncated QRP. After rescaling of the pmf makes it possible to satisfy the necessary condition of distribution which summation of probability is equal to one. The truncated QRP above m means that for a given t, N(t) can only take values of 0,1,…,m. For such N(t), pmf is given by

$$P\{N(t)=i\} = \frac{G^{(i)}(t) - G^{(i+1)}(t)}{1 - G^{(m+1)}(t)}, \qquad i = 0, 1, \cdots, m$$

Where $G^{(i)}(t)$ is the convolution of the inter-occurrence times $X_1, X_1, \cdots, X_i$ and $G^{(0)}(t) = 1$. So truncated QRP's first moment and second moment are obtained by

$$
\begin{aligned}
E\left(N(t)\right) &= \sum_{i=0}^{m} i \cdot P^{T}\left\{N(t)=i\right\} \\
&= \sum_{i=0}^{m} i\left(\frac{G^{(i)}(t)-G^{(i+1)}(t)}{1-G^{(m+1)}(t)}\right) \\
&= \frac{\sum_{i=1}^{m} G^{(i)}(t)-mG^{(m+1)}(t)}{1-G^{(m+1)}(t)}
\end{aligned}
$$

and

$$
\begin{aligned}
E\left(N^{2}(t)\right) &= \sum_{i=0}^{m} i^{2} \cdot P^{T}\left\{N(t)=i\right\} \\
&= \sum_{i=0}^{m} i^{2}\left(\frac{G^{(i)}(t)-G^{(i+1)}(t)}{1-G^{(m+1)}(t)}\right) \\
&= \frac{\sum_{i=1}^{m+1} (2i-1) G^{(i)}(t)-m^{2}G^{(m+1)}(t)}{1-G^{(m+1)}(t)}
\end{aligned}
$$

If we use these two moments, we obtain the variance of N(t). There is another extension of QRP, censored QRP. It is similar to truncated QRP but it is slightly different. If there are above a certain number of failures, they would be transformed into the last number of failures. It means that any observed failure above a certain number, m, are transformed into a single value m. Censored QRP's first moment and second moment are given by

$$
\begin{aligned}
E\left(N(t)\right) &= \sum_{i=0}^{m-1} i\left(G^{(i)}(t)-G^{(i+1)}(t)\right)+mG^{(m)}(t) \\
&= \sum_{i=1}^{m-1} iG^{(i)}(t)-\sum_{j=2}^{m}(j-1)G^{(j)}(t)+mG^{(m)}(t) \\
&= \sum_{i=1}^{m} G^{(i)}(t)
\end{aligned}
$$

and

$$
\begin{aligned}
E\left(N^2(t)\right) &= \sum_{i=0}^{m-1} i^2\left(G^{(i)}(t)-G^{(i+1)}(t)\right)+m^2G^{(m)}(t) \\
&= \sum_{i=1}^{m-1} i^2G^{(i)}(t)-\sum_{j=2}^{m}(j-1)^2 G^{(j)}(t)+m^2G^{(m)}(t) \\
&= \sum_{i=1}^{m}(2i-1)G^{(i)}(t)
\end{aligned}
$$

2.5.4 Non-homogeneous Poisson Processes [74, 89, 141, 142]

$\{N(t), t \geq 0\}$ is said to be a non-homogeneous Poisson process with intensity function $\lambda(t)$ if satisfies

- $N(t)=0$
- $\{N(t), t \geq 0\}$ has independent increments
- Pr(exactly 1 event in (t, t+h)) = $\lambda(t)h+o(h)$
- Pr(more than 1 event in (t, t+h)) = $o(h)$

Then $\Pr\left(N(t)=n\right)=e^{-m(t)}\dfrac{m(t)^n}{n!},\ n \geq 0$ $\quad$ *where* $m(t)=\int_0^t \lambda(s)\,ds$

N(t) has a Poisson distribution with mean m(t) which is the mean value function of the process.

2.5.5 Compound Poisson Processes and Marked Poisson Processes [167]

Both compound Poisson and marked Poisson processes appear often as models of physical phenomena. Given a Poisson process $N(t)$ of rate $\lambda > 0$, suppose that each event has associated with it a random variable, possibly representing a value, an interval. The successive values $X_1, X_2, X_3, \cdots$ are assumed to be independent random variables. Then, a compound Poisson process is the cumulative value process defined by

$$Z(t) = \sum_{k=1}^{N(t)} X_k \qquad for\ t \geq 0$$

If $\lambda > 0$ is the rate for the process $N(t)$ and $\mu = E(X_1)$ and $\sigma^2 = Var(X_1)$ are the common mean and variance for $X_1, X_2, X_3, \cdots$, then the moments of $Z(t)$ can be determined as follows:

$$E(N(t)) = \lambda\mu t;\ Var(N(t)) = \lambda(\sigma^2 + \mu^2)t$$

A marked Poisson process is the sequence of pairs $(W_1, X_1), (W_2, X_2), \cdots,$ *where* $W_1, W_2, \cdots$ are the waiting times or event times in the Poisson process $N(t)$.

Theorem 2.1 [167] Let $(W_1, X_1), (W_2, X_2), \cdots$ be a marked Poisson process where $X_1, X_2, X_3, \cdots$ are the waiting times in a Poisson process of rate λ and $X_1, X_2, X_3, \cdots$ are independent identically distributed continuous random variables having probability density function $f(x)$. Then $(W_1, X_1), (W_2, X_2), \cdots$ form a two-dimensional non-homogeneous Poisson point process where the mean number of points in a region A is given by

$$\mu(A) = \iint_A \lambda f(x)\, dxdt$$

2.5.6 Bivariate Exponential Distribution

The bivariate distributions have been investigated by many researchers [22, 46, 52, 58, 60-62, 65, 85, 103, 110-113, 149, 162, 195, 196] for the reliability applications. Specifically, some researchers have studied for the cases of bivariate gamma distributions [46, 110, 112, 195], bivariate exponential distributions [22, 23, 52, 61, 65, 85, 97, 113, 117, 149], bivariate logistic distributions [28, 62] and others [60, 200].

Among various bivariate distributions, bivariate exponential distributions (BED) are one of the most common distributions applied in reliability engineering. The BEDs have also attracted many practical applications in reliability problems. However, unfortunately, there is no clear and explicit form for the BED unlike bivariate normal distributions. Therefore, a lot of researchers [22, 23, 52, 61, 65, 85, 97, 113, 117, 149] have tried to develop various types of BEDs.

Chapter 3

Research Objectives

In this dissertation, we study various warranty models with explicit consideration of the characteristics of warranty, the system structure, the impact of repairs, and the value of time. There are many papers published over the last several decades on warranty and its related topics. As a result of Chapter 2, Literature review, we summarize several special areas which not many papers have covered and investigated and we want to investigate those areas and list the objectives which this dissertation will cover.

Many researchers investigate the expected warranty cost for warranty cost analysis. While expected warranty cost is a good measure on the overall cost of warranty, it provides little information of the risk contained in a warranty program. Therefore, it is not sufficient enough to express the data using only the expected values. If we obtain the variance of warranty cost with the expected value, the result would provide a better accurate cost analysis. The objective is to develop warranty cost models for single and multi-component systems including series-parallel, parallel-series and *k-out-of-n* systems by calculating the expected value and the variance of the warranty cost.

More specific, in Chapter 4, we introduce the concepts of altered quasi renewal based on the ordinary quasi-renewal process. Based on the proposed altered quasi-renewal processes, we develop the warranty cost models and also derive reliability measures for various systems including series-parallel, parallel-series, and *k-out-of-n* configurations.

In Chapter 5, using altered quasi-renewal process, we develop cost models by obtaining the expected warranty cost and variance of warranty cost and determine the distribution of number of failures by induction method.

In Chapter 6, we develop a modified block replacement model for *k-out-of-n* systems and determine optimum policies of both a threshold level for the number of failed components to prevent the system's failures and the maintenance cycle that minimizes the expected total system cost.

In Chapter 7, in the warranty period, we present warranty cost analyses using NHPP, and marked Poisson process. In the post warranty period, we consider the maintenance policies with preventive maintenance and corrective maintenance. After developing cost models with the warranty period and the post warranty period, we obtain the expected values of total cost and total duration.

In Chapter 8, we conduct warranty cost anslysis using the two-dimensional warranty model when failure times and repair times are dependent. Field data is considered for real applications.

Concluding remarks are presented in Chapter 9, in which, we summarize the contributions of the study, and discuss some future research directions.

Chapter 4

Altered Quasi-renewal Concepts for Modeling Renewable Warranty Costs with Imperfect Repairs[1]

4.1 Introduction

In this chapter, we focus on the analysis of warranty cost and the distribution function of number of product failures within a warranty period w. Using the warranty cost model, manufacturers can make the appropriate decisions related to the warranty policy.

In order to set up the warranty policy, a policy maker should have some information about a product's failure. For example, there are past failure data, experimental data regarding the product's failure, intuition of the product's failure. We investigate several backgrounds to study warranty analysis mathematically. Several processes have been considered to stand for failure intervals. Amongst them, renewal process is one of frequently used models. You can refer renewal process to Section 2.5.1.

In this chapter, a replacement service is assumed to be possible during the warranty period by introducing two alternate quasi renewal concepts based on the QRP. You can refer QRP to Section 2.5.2. The first is called an altered QRP. In the QRP model, upon each imperfect repair, the time to failure will be reduced to a fraction α of the immediately previous interval

[1] M. Park and H. Pham, "Altered quasi-renewal concepts for modeling renewable warranty costs with imperfect repairs" *Mathematical and Computer Modelling* (2010), doi:10.1016/j.mcm.2010.05.028

and be dependent of all previous intervals. The altered QRP, however, does not necessarily have the same patterns. The second concept is called a mixed QRP considering both repairs and replacements subject to imperfect strategies. These strategies would allow an immediate replacement in case of a premature or severe failure despite number of failures is less than a threshold level within a warranty period. The concept of the above alternative quasi renewal models and other important properties will be discussed in Section 4.4.

Using these two concepts, we develop cost models for multi-component systems. For our warranty cost analysis, we obtain a distribution function of the number of failures assuming that the policy is renewable. In other words, whenever a product fails within warranty period *w*, the warranty policy is renewed for another period of *w* if the product is restored after the failure. Note that repair and replacement do not happen simultaneously.

Chapter 4 is organized as follows. In Section 4.2, a literature review is discussed. Section 4.3 consists of a description of the distribution of number of failures and the explanations of the assumptions used in this study. Section 4.4 focuses on two altered QRPs. Section 4.5 derives two different approaches for the warranty cost analysis. In Section 4.6, we discuss a real application. A numerical example is given in Section 4.7 to illustrate the two concepts with different conditions and finally, concluding remarks are discussed in Section 4.8.

4.1.1 Nomenclature

w : length of a warranty period

T : r.v. time

λ : constant failure rate

h : an upper limit on the number of repairs under warranty

α, β: parameters for QRP and altered QRP, respectively

n_a, n_b: the number of repairs and replacements within a warranty period, respectively

n : the number of system failures within the warranty period, i.e. $n = n_a + n_b$

$N_a(t), N_b(t), N_s(t)$: the number of repairs, the number of replacements and the number of system failures at r.v. time t, respectively

X_i: the renewal inter-failure time of repairs between the (i-1)$^{\text{th}}$ and i^{th} events of the process

Y_i: the renewal inter-failure time of replacements between the (i-1)$^{\text{th}}$ and i^{th} events of the process

$f_s(\cdot), F_s(\cdot), R_s(\cdot)$: pdf, cdf and reliability function of system failure times within a warranty period w, respectively

$f_{ij}(\cdot), F_{ij}(\cdot), R_{ij}(\cdot)$: pdf, cdf and reliability function of component j's failure times after (i-1)$^{\text{th}}$ repair/replacement within a warranty period w, respectively

$f_{is}(\cdot)$, $F_{is}(\cdot)$, $R_{is}(\cdot)$: pdf, cdf and reliability function of system failure times after (i-1) repair/replacement within a warranty period w, respectively

c_a, c_b: warranty cost for repairs and replacements within a warranty period w, respectively

4.2 Literature Review

Recently, Bai and Pham [8] develop the truncated and censored QRPs from the concept of QRP and introduce *repair-limit risk-free* warranty policies where it considers a number of system failures within a warranty period, and thereafter the failed product would be replaced instead of being repaired. They assumed that only repair service would occur in the upper limit of the number of repairs under warranty. Park and Pham [122] develop the warranty cost models, reliability and other measures for *k-out-of-n* systems using applications of the QRP. Rehmert [138, 139] develops expressions for the point availability using the QRP. Recently, Wu and Li [190] elaborate warranty cost models for repairable products with a dormant mode. Chukova and Hayakawa [38] conduct a cost analysis for non-renewing warranty with repair time. Samatli-Pac and Taner [146] develop and investigate different repair strategies for one- and two-dimensional warranties with the objective of minimizing manufacturer's expected warranty cost using the QRP. Zuo *et al.* [201] study a warranty servicing policy for a class of multi-state deteriorating and repairable products using minimal repairs.

Under a renewing warranty, for the product which fails during its warranty period, it is replaced by a new one at a cost borne by the manufacturer or at a pro-rated cost charged to the user then the warranty is renewed. Under a non-renewing warranty, the manufacturer guarantees a satisfactory service only during the original warranty period. Renewable warranties are usually given to non-repairable and inexpensive products. Compared to the renewable warranties, the period of non-renewable warranties is relatively longer, which provides one possible reason as to why such policies are not as popular as non-renewable ones for warranty issuers [6]. Jung *et al.* [81] investigate the optimal replacement policies following the expiration of warranty such as renewing warranty and non-renewing warranty. Chukova and Hayakawa [38, 39] evaluate the warranty costs over the warranty period under non-renewing and renewing warranty policies over the life cycle of the product. Sahin and Polatoglu [144] prove that the cost rate function is pseudo-convex under a fixed-maintenance period policy under non-renewing and renewing warranty policies. Chen and Chien [30] investigate a model to study the effect of PM carried out by the buyer on items sold under a renewing FRW.

The impact of a repair service on product reliability is one of the most significant factors in warranty cost. One can categorize repairs into three subcategories based on the effort of repairs or the condition of repaired items; as-good-as-new repair, minimal repair and imperfect repair. As-good-as-new repair assumes that after repair, the restored system functions like new such that the failure time distribution is the same as that of a new product. Minimal repair, which is also called as-bad-as-old repair, assumes that the failure rate of a repaired system equals that of the system just before the most recent failure. Imperfect repair refers to the case where a repair action responds to a system neither as-good-as-new nor as-bad-as-old but to a level in between. Most maintenance and warranty models using renewal theory were actually based on as-good-as-new assumption. A few researchers [13-19] consider imperfect repair for the warranty cost. Chukova *et al.* [39] classify the type of repair according to the depth of repair. Dimitrov *et al.* [50] evaluate the expected warranty costs for repairable product associated with linear pro-rata, non-renewing free replacement and renewing free replacement warranties considering several type of warranty services including the imperfect repair. Chukova *et al.* [41] present an approach to modeling imperfect repairs under the warranty policy. Yanez *et al.* [191] investigate a robust solution to a probabilistic model that is applicable to several types of repairs including imperfect repairs. Mettas and Zhao [101] explore the general renewal processes to model and analyze complex repairable systems with various degrees of repair. Wu and Clements [188] deal with the reliability modeling of failure processes for repairable systems where the failure intensity shows a bathtub-type non-monotonic behavior.

Clearly, as-good-as-new repair and as-bad-as-old repair represent two extreme types of repair services. So far the studies in warranty literature mostly focus on as-good-as-new repair scenario. It is well-known that after repair the system may not be as-good-as-new in maintenance practice. In reality, most repair actions are somewhere between these extremes, and we aim to conduct the study based on both imperfect repair and perfect repair. In this chapter, we suggest two alternative QRPs for the imperfect repair and through the proposed approach, replacement service which is same as perfect repair is considered with the imperfect repair, together.

Throughout the chapter, we consider the expected value and variance of the warranty cost together. The expected warranty cost has been mainly investigated for warranty cost analysis. While the expected warranty cost is a good measure on the overall cost of warranty, it provides little information of the risk contained in a warranty program. Therefore, it is not sufficient enough to express the data using only the expected values. The variance and the standard deviation provide us a numerical measure of the scatter of a data. These measures are useful for making comparisons between data sets that go beyond simple visual appearances. While measures of central tendency (i.e. expectation) are used to estimate 'normal' values of a dataset, measures of dispersion (i.e. variance) are important for describing the spread of the data, or its variation around a central value. Two distinct samples may have the same mean or median, but completely different levels of variability, or vice versa. A proper description of a set of data should include both of these characteristics [71].

For example, meteorologists often use variance to help classify abnormal climatic conditions. They use variance to describe the abnormality of a data value. Because we obtain the variance of warranty cost with the expected value, the result provides a more accurate cost analysis. We believe the models will help warranty policy makers to make optimal decisions with the objective of downsizing manufacturers' warranty cost.

4.3 Model Consideration

4.3.1 Problem Descriptions

This chapter presents the warranty model which helps policy makers to have appropriate and reasonable decisions related to the warranty policies. Using the proposed approaches, we obtain the expected warranty cost and the variance of the warranty cost. Two modified QRPs are suggested based on the original QRP. The first one is the altered QRP and the second one is the mixed QRP.

Imperfect repair is a repair service that noticeably improves the performance of the product. It can also be defined in terms of the degree to which the operating condition of an item is restored through maintenance [101, 177]. In collaboration, It is a maintenance action that restores the system operating state to be somewhere between as-good-as-new and as-bad-as-old. In this chapter, imperfect repairs are considered such that after each repair, the system is between the states of new and old, and each repair will be modeled by a QRP. The QRP is characterized by a scaling parameter α that alters the time until next failure after each renewal. If $0 < \alpha < 1$, then a repair service is imperfect. This parameter indicates the degree of the product's deterioration.

When we propose the warranty cost models using two alternative QRPs, the following assumptions are needed:

- Repair and replacement do not occur simultaneously.
- The time for both repair and replacement service are negligible.
- All warranty claims are valid and executed.
- Repairs are imperfect and the repair process can be modeled by a QRP.

- When the product fails to function, the manufacturer makes a decision between repair and replacement based on the failed product's condition of the product.
- The time to perform an inspection in which to determine whether the failed component needs a repair or a replacement is negligible.
- The warranty period is renewable.

4.3.2 Renewable Warranty

The warranty policy can be categorized into renewable warranty or nonrenewable warranty based on its renewability. Under the renewable warranty, we investigate the cost analysis with a pre-specified warranty period denoted by w. After a repair or replacement, the restored system will renew the warranty period again as for the original one due to the renewable nature of the warranty. Let warranty cycle T be a time interval starting from the point of sale and ending at the point of expiration of the warranty. In case of a non-renewable warranty, a warranty cycle T coincides with an original warranty period w. However, for a renewable policy, T is a r.v. whose value depends on the extended warranty periods as well as the original warranty period. Let t_i be the i^{th} failure time. If the i^{th} failure occurs during the warranty period, the product would have same length of warranty period from t_i. The warranty cycle T is composed of the failure intervals and the original warranty period from the last failure time under the renewable warranty policy. However, under the non-renewable warranty policy, the warranty cycle is same as the original warranty period. In next section, using the property of the renewable warranty policy, we obtain the distribution of the number of failures with imperfect repairs.

4.3.3 Distribution of N

Given N is the number of system failures, we obtain the distribution of N and derive several statistical properties of warranty cost function per cycle or per product sold. Let N be the number of system failures within a warranty period w. The warranty policy is assumed to be renewable after each repair/replacement. This subsection derives the distribution of the number of system failures within the warranty period. Under the perfect repair assumption, we can obtain the pmf of N as follows [9]:

$$P\left[N=n\right]=\left[F_s\left(w\right)\right]^n R_s\left(w\right),\ \ for\ n=1,2,3,\cdots$$

Lemma 1: Under the imperfect repair, every $F_{is}\left(w\right)$ is different, so the pmf of N is given by

$$P\left[N=n\right]=\left(\prod_{i=1}^{n}\left(F_{is}\left(w\right)\right)\right)\left(R_{(n+1)s}\left(w\right)\right) \tag{4.3.1}$$

Proof of Lemma 1:

$$P\left[N\geq n\right]=P\left(\min\left\{i;X_i>w\right\}\geq n+1\right)=P\left(\bigcap_{i\leq n}\left(X_i\leq w\right)\right)=\prod_{i=1}^{n}P\left(X_i\leq w\right)=\prod_{i=1}^{n}\left(F_{is}\left(w\right)\right)$$

$$\begin{aligned}P\left[N=n\right]&=P\left[N\geq n\right]-P\left[N\geq n+1\right]\\&=\prod_{i=1}^{n}\left(F_{is}\left(w\right)\right)-\prod_{i=1}^{n+1}\left(F_{is}\left(w\right)\right)\\&=\prod_{i=1}^{n}\left(F_{is}\left(w\right)\right)-\left(\prod_{i=1}^{n}\left(F_{is}\left(w\right)\right)\right)\left(F_{(n+1)s}\left(w\right)\right)\\&=\left(\prod_{i=1}^{n}\left(F_{is}\left(w\right)\right)\right)\left(R_{(n+1)s}\left(w\right)\right)\end{aligned}$$

4.4 Two Suggestd Quasi-renewal Processes

In this section, we first present two alternate QRPs: the altered QRP and mixed QRP. The altered QRP differs from the original QRP as it is changing parameter in the equation. The mixed QRP mixes the repair intervals and the replacement intervals.

4.4.1 Altered Quasi-renewal Process

From the original QRP, each inter-failure interval has a common pattern like $X_2 = \alpha \cdot X_1, X_3 = \alpha^2 \cdot X_1, X_4 = \alpha^3 \cdot X_1, \cdots, X_n = \alpha^{n-1} \cdot X_1$ where X_1 represents the first failure interval. It means that the failure time will be reduced with a fraction α if α is less than 1. In reality, such a specific pattern would be difficult to observe. We now discuss an altered QRP that does not need to assume such pattern.

Definition: A counting process $\{N(t),\ t>0\}$ is said to be the altered QRP associated with the distribution F and the random parameter β_n, $\beta_n > 0$, a constant, if $X_n = \beta_n \cdot X_1$, n = 2, 3,… where $X_1 \sim F$, and β_n, where $n = 2,3,\cdots$ are not necessarily equal. Define f_i and F_i be the pdf and cdf of X_i. As for altered QRP, the cdf and pdf for n = 2, 3, 4,… are, respectively, given by

$$F_n(x) = F_1\left(\frac{1}{\beta_n}x\right),\ f_n(x) = \frac{1}{\beta_n} f_1\left(\frac{1}{\beta_n}x\right) \tag{4.4.1}$$

Additionally, the component j's cdfs and pdfs of inter-failure interval i, respectively, for i = 2, 3, 4,… are given by

$$F_{ij}(x) = F_{1j}\left(\frac{1}{\beta_{ij}}x\right),\ f_{ij}(x) = \frac{1}{\beta_{ij}} f_{1j}\left(\frac{1}{\beta_{ij}}x\right) \tag{4.4.2}$$

4.4.2 Mixed Quasi-renewal Processes

Bai and Pham [8] proposed a warranty model addressing the relationships between the number of repairs and replacements. Let h be an upper limit of the number of repairs under the warranty. Assuming that N_a and N_b are correlated and $\operatorname{cov}(N_a, N_b) \neq 0$ then

$$N_a(w) < h \to N_b(w) = 0,\ N_b(w) > 0 \to N_a(w) = h \tag{4.4.3}$$

This implies that if there are fewer than h failures, the replacements would not occur, instead, only repair services would be provided. If there are h failures, a replacement would occur. In other words, if there are more than h system failures within w, the failed product will be replaced instead of being repaired again. As previously mentioned, *repair-limit risk-free* warranty policies of a fixed period w, where a replacement only happens after h failures under a warranty period, are suggested. However, in this chapter, we consider that the replacement would happen if the system failure is premature and severe because the likelihood of it occurring in real life is higher. Therefore, the relation (4.4.3) was altered as (4.4.4) for the mixed QRP as follows:

$$N_a(w) < h \rightarrow N_b(w) \geq 0,\ \ N_b(w) > 0 \rightarrow N_a(w) \leq h \tag{4.4.4}$$

In this model, even if there are fewer than h failures, the replacement strategies can be happen using mixed QRP. In the case of a failure of function of a product, either repair or replacement may be provided based on the failure condition, and the manufacturer should select which services would be provided. Additionally, in general, the superposition of any two renewal processes is not a renewal process. Since we assume that the repairs and replacements would not happen simultaneously, they could be the renewal processes.

4.5 System Warranty Cost Analysis

In this section, warranty cost analyses are conducted by computing the expected warranty cost as well as the variance of the warranty. We now discuss the cost analyses considering two approaches. The first approach is to use the mixed QRP and conditional probabilities for the warranty analysis. Failure time is assumed to follow exponential distribution. The second is to use the altered QRP with inter-failure intervals instead of exponential distribution and conditional probabilities.

4.5.1 Warranty Cost Modeling using the Mixed QRP

Let T be an exponential random variable time having failure rate λ, i.e. $T \sim \exp(\lambda)$ independent of renewal process. Then, its memoryless property for the calculation of expectation can be applied. Let $N_a(t)$ and $N_b(t)$ be the number of repairs and the number of replacement at r.v. time *t*. Denote c_a as the repair cost per failure and c_b the replacement cost per unit. Then for the warranty cost per product sold *C*, the expected value *E*(*C*) and variance *Var*(*C*) of the system warranty cost within a warranty period *w* can be obtained as follows, respectively,

$$E(C) = c_a E(N_a(t)) + c_b E(N_b(t))$$
$$Var(C) = (c_a)^2 Var(N_a(t)) + (c_b)^2 Var(N_b(t)) + 2c_a c_b \operatorname{cov}(N_a(t), N_b(t)) \qquad (4.5.1)$$

The expected number of repair services is $E[N_a(t)]$. We use the conditional probabilities to derive the expected function of the number of repairs given the first failure time, $X_1 = x$.

$$E[N_a(t)] = E[E[N_a(t)|X_1 = x]] = \int E[N_a(t)|X_1 = x] f_1(x)\, dx \qquad (4.5.2)$$

We separate two cases such as $T \geq x \text{ and } T < x$. Let T be *r.v.* time and could be any time interval, for example, a warranty period. If the warranty period *w* (*T* = *w*) is less than the first repair, then the number of repair services within the warranty period is zero. On the other hand, if the warranty period is larger than and equal to the first repair time, then the expected number of repair services given that $T \geq x$, is the expected number of repair services plus one. Generally, it would be adding one to the expected number of repair services during (*T*-*x*) period. Since *T* is assumed to be exponential distributed and using the memoryless property, the distribution of number of repairs during *T*-*x* period is the same as the distribution of the number of repairs during *T* period. For interested readers, it is referred to [142]. Therefore,

$$E[N_a(t)|X_1 = x, T < x] = 0 \qquad (4.5.3)$$

$$E\left[N_a(t)\middle|X_1 = x, T \geq x\right] = 1 + E\left[N_a(t)\right] \tag{4.5.4}$$

From eqs. (4.5.3) and (4.5.4) and conditional probabilities, the expected warranty cost is given by

$$\begin{aligned} E\left[N_a(t)\middle|X_1 = x\right] &= E\left[N_a(t)\middle|X_1 = x, T < x\right] P\left(T < x\middle|X_1 = x\right) \\ &\quad + E\left[N_a(t)\middle|X_1 = x, T \geq x\right] P\left(T \geq x\middle|X_1 = x\right) \\ &= E\left[N_a(t)\middle|X_1 = x, T \geq x\right] P\left(T \geq x\middle|X_1 = x\right) \\ &= \left(1 + E\left[N_a(t)\right]\right) P\left(T \geq x\right) \end{aligned} \tag{4.5.5}$$

From eqs. (4.5.2) and (4.5.5), we obtain

$$\begin{aligned} E\left[N_a(t)\right] &= \int E\left[N_a(t)\middle|X_1 = x\right] f_1(x)\,dx = \int \left(1 + E\left[N_a(t)\right]\right) e^{-\frac{x}{\lambda}} f_1(x)\,dx \\ &= \left(1 + E\left[N_a(t)\right]\right) \int e^{-\frac{x}{\lambda}} f_1(x)\,dx \end{aligned} \tag{4.5.6}$$

After simplifications, we obtain the expected number of repair services as follows:

$$E\left[N_a(t)\right] = \frac{\int e^{-\frac{x}{\lambda}} f_1(x)\,dx}{1 - \int e^{-\frac{x}{\lambda}} f_1(x)\,dx} \tag{4.5.7}$$

Similarly, let Y_i be the inter-failure interval of a replacement service between the $(i-1)^{th}$ and i^{th} events of the processes and that under a renewal process, replacement service is the same as perfect repair. We then obtain the expected number of replacements.

$$E\left[N_b(t)\right] = \frac{\int e^{-\frac{y}{\lambda}} f_1(y)\,dy}{1 - \int e^{-\frac{y}{\lambda}} f_1(y)\,dy} \tag{4.5.8}$$

Therefore, the expected warranty system cost is given by

$$E(C) = c_a E(N_a(t)) + c_b E(N_b(t)) = c_a \frac{\int e^{-\frac{x}{\lambda}} f_1(x)\,dx}{1 - \int e^{-\frac{x}{\lambda}} f_1(x)\,dx} + c_b \frac{\int e^{-\frac{y}{\lambda}} f_1(y)\,dy}{1 - \int e^{-\frac{y}{\lambda}} f_1(y)\,dy} \quad (4.5.9)$$

To obtain the variance of the warranty system cost, we first need to calculate the second moment. Similarly to the first moment, we consider the first failure during the warranty period. We separate the two cases such as $T \geq x$ *and* $T < x$. Then

$$E\left[(N_a(t))^2 | X_1 = x, T < x\right] = 0,\ E\left[(N_a(t))^2 | X_1 = x, T \geq x\right] = E\left[(1 + N_a(t))^2\right] \quad (4.5.10)$$

Therefore,

$$\begin{aligned} E\left[(N_a(t))^2 | X_1 = x\right] &= E\left[(N_a(t))^2 | X_1 = x, T < x\right] P(T < x | X_1 = x) \\ &\quad + E\left[(N_a(t))^2 | X_1 = x, T \geq x\right] P(T \geq x | X_1 = x) \\ &= E\left[(N_a(t))^2 | X_1 = x, T \geq x\right] P(T \geq x | X_1 = x) \quad (4.5.11) \\ &= E\left[(1 + N_a(t))^2\right] P(T \geq x) \\ &= \left(1 + 2E(N_a(t)) + E\left(N_a(t)^2\right)\right) e^{-\frac{x}{\lambda}} \end{aligned}$$

Using eqs. (4.5.10) and (4.5.11), we obtain the second moment as follows:

$$\begin{aligned} E\left[(N_a(t))^2\right] &= E\left[E\left[(N_a(t))^2 | X_1 = x\right]\right] \\ &= \int E\left[(N_a(t))^2 | X_1 = x\right] f_1(x)\,dx \\ &= \int \left(1 + 2E(N_a(t)) + E\left(N_a(t)^2\right)\right) e^{-\frac{x}{\lambda}} f_1(x)\,dx \\ &= \left(1 + 2E(N_a(t)) + E\left(N_a(t)^2\right)\right) \int e^{-\frac{x}{\lambda}} f_1(x)\,dx \end{aligned} \quad (4.5.12)$$

The second moment is given by

$$E\left(N_a(t)^2\right)=\frac{\left(1+2E\left(N_a(t)\right)\right)\int e^{-\frac{x}{\lambda}}f_1(x)\,dx}{\left(1-\int e^{-\frac{x}{\lambda}}f_1(x)\,dx\right)}=\frac{\left(1+2\frac{\int e^{-\frac{x}{\lambda}}f_1(x)\,dx}{1-\int e^{-\frac{x}{\lambda}}f_1(x)\,dx}\right)\int e^{-\frac{x}{\lambda}}f_1(x)\,dx}{\left(1-\int e^{-\frac{x}{\lambda}}f_1(x)\,dx\right)} \tag{4.5.13}$$

Therefore, the variance of the number of repair services is given by

$$\begin{aligned}Var\left(N_a(t)\right)&=E\left[\left(N_a(t)\right)^2\right]-\left[E\left(N_a(t)\right)\right]^2\\&=\frac{\left(1+2\frac{\int e^{-\frac{x}{\lambda}}f_1(x)\,dx}{1-\int e^{-\frac{x}{\lambda}}f_1(x)\,dx}\right)\int e^{-\frac{x}{\lambda}}f_1(x)\,dx}{\left(1-\int e^{-\frac{x}{\lambda}}f_1(x)\,dx\right)}-\left(\frac{\int e^{-\frac{x}{\lambda}}f_1(x)\,dx}{1-\int e^{-\frac{x}{\lambda}}f_1(x)\,dx}\right)^2\end{aligned} \tag{4.5.14}$$

Similarly, the variance of the number of replacement services is given by

$$Var\left(N_b(t)\right)=\frac{\left(1+2\frac{\int e^{-\frac{y}{\lambda}}f_1(y)\,dy}{1-\int e^{-\frac{y}{\lambda}}f_1(y)\,dy}\right)\int e^{-\frac{y}{\lambda}}f_1(y)\,dy}{\left(1-\int e^{-\frac{y}{\lambda}}f_1(y)\,dy\right)}-\left(\frac{\int e^{-\frac{y}{\lambda}}f_1(y)\,dy}{1-\int e^{-\frac{y}{\lambda}}f_1(y)\,dy}\right)^2 \tag{4.5.15}$$

Next, we will derive the covariance of the repair services and the replacement services. The covariance of those are given by

$$Cov\left(N_a,N_b\right)=E\left[N_aN_b\right]-E\left[N_a\right]E\left[N_b\right] \tag{4.5.16}$$

Using eq. (4.3.1), we obtain

$$E[N_aN_b] = \sum_{n_b=0}^{\infty}\sum_{n_a=0}^{h} n_an_bP[N_a = n_a, N_b = n_b]$$
$$= \sum_{n_b=0}^{\infty}\sum_{n_a=0}^{h} n_an_bP[N = n]$$
$$= \sum_{n_b=0}^{\infty}\sum_{n_a=0}^{h} n_an_b\left(\prod_{i=1}^{n_a+n_b}(F_{is}(w))\right)\left(R_{(n_a+n_b+1)s}(w)\right) \tag{4.5.17}$$

Therefore, the covariance of the repair services and the replacement services is given by

$$Cov(N_a, N_b)$$
$$= \sum_{n_b=0}^{\infty}\sum_{n_a=0}^{h} n_an_b\left(\prod_{i=1}^{n_a+n_b}(F_{is}(w))\right)\left(R_{(n_a+n_b+1)s}(w)\right) - \left(\frac{\int e^{-\frac{x}{\lambda}}f_1(x)dx}{1-\int e^{-\frac{x}{\lambda}}f_1(x)dx}\right)\left(\frac{\int e^{-\frac{y}{\lambda}}f_1(y)dy}{1-\int e^{-\frac{y}{\lambda}}f_1(y)dy}\right) \tag{4.5.18}$$

From eqs. (4.5.14), (4.5.15) and (4.5.18), the variance of the warranty system cost is given by

$$Var(C) = c_a^{\,2}Var(N_a) + c_b^{\,2}Var(N_b) + 2c_ac_bCov(N_a, N_b)$$
$$= (c_a)^2\left(\frac{\left(1+2\frac{\int e^{-\frac{x}{\lambda}}f_1(x)dx}{1-\int e^{-\frac{x}{\lambda}}f_1(x)dx}\right)\int e^{-\frac{x}{\lambda}}f_1(x)dx}{\left(1-\int e^{-\frac{x}{\lambda}}f_1(x)dx\right)} - \left(\frac{\int e^{-\frac{x}{\lambda}}f_1(x)dx}{1-\int e^{-\frac{x}{\lambda}}f_1(x)dx}\right)^2\right)$$
$$+(c_b)^2\left(\frac{\left(1+2\frac{\int e^{-\frac{y}{\lambda}}f_1(y)dy}{1-\int e^{-\frac{y}{\lambda}}f_1(y)dy}\right)\int e^{-\frac{y}{\lambda}}f_1(y)dy}{\left(1-\int e^{-\frac{y}{\lambda}}f_1(y)dy\right)} - \left(\frac{\int e^{-\frac{y}{\lambda}}f_1(y)dy}{1-\int e^{-\frac{y}{\lambda}}f_1(y)dy}\right)^2\right)$$
$$+2c_ac_b\left(\sum_{n_b=0}^{\infty}\sum_{n_a=0}^{h} n_an_b\left(\prod_{i=1}^{n_a+n_b}(F_{is}(w))\right)\left(R_{(n_a+n_b+1)s}(w)\right) - \left(\frac{\int e^{-\frac{x}{\lambda}}f_1(x)dx}{1-\int e^{-\frac{x}{\lambda}}f_1(x)dx}\right)\left(\frac{\int e^{-\frac{y}{\lambda}}f_1(y)dy}{1-\int e^{-\frac{y}{\lambda}}f_1(y)dy}\right)\right) \tag{4.5.19}$$

4.5.2 Warranty Cost Modeling using Inter-failure Intervals and Alternative QRPs

The reliability function of the i^{th} inter-failure interval for the system using altered QRP is given by

$$R_{is}(w) = 1 - F_{is}(w) = 1 - \int f_{is}(x)\,dx$$
$$= 1 - \int \frac{1}{\beta_{is}} f_{1s}\left(\frac{1}{\beta_{is}} x\right) dx \tag{4.5.20}$$

Therefore, the reliability function of a system with n failures is given by

$$R_s(w) = \prod_{i=1}^{n} \left(R_{is}(w)\right)$$
$$= \left(1 - \int f_{1s}(x)\,dx\right) \prod_{i=2}^{n} \left(1 - \int \frac{1}{\beta_{is}} f_{is}\left(\frac{1}{\beta_{is}} x\right) dx\right) \tag{4.5.21}$$

For the cost analysis, we now derive the expected warranty cost and the variance of the warranty cost. First, using eq. (4.4.1), the pmf of the number of system failures within a warranty period is given by

$$P(N = n) = \left(\prod_{i=1}^{n} \left(F_{is}(w)\right)\right)\left(R_{(n_s+1)s}(w)\right)$$
$$= \left(\int f_{1s}(x)\,dx\right)\left(\prod_{i=2}^{n}\left(\int \frac{1}{\beta_{is}} f_{is}\left(\frac{1}{\beta_{is}} x\right) dx\right)\right)\left(1 - \int \frac{1}{\beta_{(n+1)s}} \cdot f_{1s}\left(\frac{1}{\beta_{(n+1)s}} \cdot x\right) dx\right) \tag{4.5.22}$$

The expected number of repair warranty services can be obtained as follows:

$$E(N_a) = \sum_{n_a=1}^{\infty} n_a \cdot \left(\int f_{1s}(x)\,dx\right) \prod_{i=2}^{n_a} \left(\int \frac{1}{\beta_{is}} f_{is}\left(\frac{1}{\beta_{is}} x\right) dx\right)\left(1 - \int \frac{1}{\beta_{(n+1)s}} \cdot f_{1s}\left(\frac{1}{\beta_{(n+1)s}} \cdot x\right) dx\right) \tag{4.5.23}$$

We next derive the variance of the number of repair services. First we calculate the second moment. And the variance of the number of repair services is given by

$$
\begin{aligned}
Var\left(N_a\right) &= E\left(N_a{}^2\right)-\left[E\left(N_a\right)\right]^2 \\
&= \sum_{n_a=1}^{\infty} n_a{}^2 \cdot \left(\int f_{1s}(x)\,dx\right)\prod_{i=2}^{n_a}\left(\int \frac{1}{\beta_{is}} f_{is}\left(\frac{1}{\beta_{is}}x\right)dx\right)\left(1-\int \frac{1}{\beta_{(n+1)s}}\cdot f_{1s}\left(\frac{1}{\beta_{(n+1)s}}\cdot x\right)dx\right) \\
&\quad -\left(\sum_{n_a=1}^{\infty} n_a \cdot \left(\int f_{1s}(x)\,dx\right)\prod_{i=2}^{n_a}\left(\int \frac{1}{\beta_{is}} f_{is}\left(\frac{1}{\beta_{is}}x\right)dx\right)\left(1-\int \frac{1}{\beta_{(n+1)s}}\cdot f_{1s}\left(\frac{1}{\beta_{(n+1)s}}\cdot x\right)dx\right)\right)^2
\end{aligned}
\tag{4.5.24}
$$

Similarly to eq. (4.5.23), we obtain the expected number of replacement warranty services:

$$
E\left(N_b\right)=\sum_{n_b=1}^{\infty} n_b \cdot \left(\prod_{i=1}^{n_b}\left(\int f_{is}(y)\,dy\right)\right)\left(1-\int f_{is}(y)\,dy\right) \tag{4.5.25}
$$

Similarly, the variance of the number of replacement services is given by

$$
\begin{aligned}
Var\left(N_b\right) &= E\left(N_b{}^2\right)-\left[E\left(N_b\right)\right]^2 \\
&= \sum_{n_b=1}^{\infty} n_b{}^2 \cdot \left(\prod_{i=1}^{n_b}\left(\int f_{is}(y)\,dy\right)\right)\left(1-\int f_{is}(y)\,dy\right)-\left(\sum_{n_b=1}^{\infty} n_b \cdot \left(\prod_{i=1}^{n_b}\left(\int f_{is}(y)\,dy\right)\right)\left(1-\int f_{is}(y)\,dy\right)\right)^2
\end{aligned}
\tag{4.5.26}
$$

Let c_a and c_b are the repair cost per failure and replacement cost per failure, respectively. As for the warranty cost per product sold C, the expected warranty cost is given by

$$
\begin{aligned}
E(C) &= c_a E\left(N_a\right)+c_b E\left(N_b\right) \\
&= c_a \sum_{n_a=1}^{\infty} n_a \cdot \left(\int f_{1s}(x)\,dx\right)\prod_{i=2}^{n_a}\left(\int \frac{1}{\beta_{is}} f_{is}\left(\frac{1}{\beta_{is}}x\right)dx\right)\left(1-\int \frac{1}{\beta_{(n+1)s}}\cdot f_{1s}\left(\frac{1}{\beta_{(n+1)s}}\cdot x\right)dx\right) \\
&\quad + c_b \sum_{n_b=1}^{\infty} n_b \cdot \left(\prod_{i=1}^{n_b}\left(\int f_{is}(y)\,dy\right)\right)\left(1-\int f_{is}(y)\,dy\right)
\end{aligned}
\tag{4.5.27}
$$

From eqs. (4.5.24) and (4.5.26), the variance of warranty system cost can be obtained

$$
\begin{aligned}
Var(C) &= c_a^{\,2} \cdot Var(N_a) + c_b^{\,2} \cdot Var(N_b) + 2c_a c_b \cdot Cov(N_a, N_b) \\
&= c_a^{\,2} \cdot \left(\begin{array}{l} \displaystyle\sum_{n_a=1}^{\infty} n_a^{\,2} \cdot \left(\int f_{1s}(x)dx \right) \prod_{i=2}^{n_a} \left(\int \frac{1}{\beta_{is}} f_{is}\left(\frac{1}{\beta_{is}} x \right) dx \right) \left(1 - \int \frac{1}{\beta_{(n+1)s}} \cdot f_{1s}\left(\frac{1}{\beta_{(n+1)s}} \cdot x \right) dx \right) \\ -\left(\displaystyle\sum_{n_a=1}^{\infty} n_a \cdot \left(\int f_{1s}(x)dx \right) \prod_{i=2}^{n_a} \left(\int \frac{1}{\beta_{is}} f_{is}\left(\frac{1}{\beta_{is}} x \right) dx \right) \left(1 - \int \frac{1}{\beta_{(n+1)s}} \cdot f_{1s}\left(\frac{1}{\beta_{(n+1)s}} \cdot x \right) dx \right) \right)^2 \end{array} \right) \\
&+ c_b^{\,2} \cdot \left(\sum_{n_b=1}^{\infty} n_b^{\,2} \cdot \left(\prod_{i=1}^{n_b} \left(\int f_{is}(y)dy \right) \right) \left(1 - \int f_{is}(y)dy \right) - \left(\sum_{n_b=1}^{\infty} n_b \cdot \left(\prod_{i=1}^{n_b} \left(\int f_{is}(y)dy \right) \right) \left(1 - \int f_{is}(y)dy \right) \right)^2 \right) \\
&+ 2c_a c_b \cdot \left(\begin{array}{l} \displaystyle\sum_{n_b=0}^{\infty} \sum_{n_a=0}^{h} n_a n_b \left(\int f_{1s}(x)dx \prod_{i=2}^{n_a+n_b} \left(\int \frac{1}{\beta_{is}} f_{is}\left(\frac{1}{\beta_{is}} x \right) dx \right) \right) \left(1 - \int \frac{1}{\beta_{(n_a+n_b+1)s}} f_{1s}\left(\frac{1}{\beta_{(n_a+n_b+1)s}} x \right) dx \right) \\ -\displaystyle\sum_{n_a=1}^{\infty} n_a \cdot \left(\int f_{1s}(x)dx \right) \prod_{i=2}^{n_a} \left(\int \frac{1}{\beta_{is}} f_{is}\left(\frac{1}{\beta_{is}} x \right) dx \right) \left(1 - \int \frac{1}{\beta_{(n+1)s}} \cdot f_{1s}\left(\frac{1}{\beta_{(n+1)s}} \cdot x \right) dx \right) \\ \displaystyle\sum_{n_b=1}^{\infty} n_b \cdot \left(\prod_{i=1}^{n_b} \left(\int f_{is}(y)dy \right) \right) \left(1 - \int f_{is}(y)dy \right) \end{array} \right)
\end{aligned}
$$

(4.5.28)

Similarly, we may develop cost models in case of series, parallel, series-parallel and parallel-series systems based on the inter-occurrence intervals. The cost analyses for multi-component systems are similar to the cost analysis of the system and could be easily obtained using the proposed approaches.

4.6 An Industrial Application

This section shows real-life bench marking examples using the proposed approaches and existing data from Samatli-Paç and Taner [146]. They suggest a real life example for their warranty cost model using QRP. A leading beverage company runs its own repair facilities for the industrial refrigerators used in its retail outlets in Turkey. The company performs three different kinds of repairs on failed refrigerators. Type 1 repair is the simplest one which only the part causing the failure is repaired depending on whichever is applicable to that specific part. Type 2 repair is adding a preventive maintenance service on the most critical part affecting the lifetime after the type 1 repair. Type 3 is an ultimate refurbishment operation in which all critical parts are checked or tested and deteriorated parts are cleaned, repaired or replaced. The replacement service is not considered in this example.

Data indicated that repair types have average cost of 10, 28 and 59 YTL (new Turkish Liras). Statistical analyses were performed on a particular brand and model of refrigerator with data on 2150 refrigerators. Of these, 285 failed at least once during the observation period. The maximum likelihood estimation for the repair type 2 of the 285 repaired refrigerators yields that the time between the first and second failures is Weibully distributed with the shape parameter of 1.02 and the scale parameter of 127.57 and α is 0.91. Similarly, it yields that the time until first failure closely follows a Weibull distribution with parameters of (scale, shape) = (158.24, 1.68). Time limits of the region within which a repair will be attempted are set in view of the distribution that characterizes the time until first failure. In particular, the limits are set systematically first at around a conservative value of 5, 7, 9, 10, 15 & 20 years. So we obtain expected warranty cost and the variance of the warranty cost based on these different years. After the product's first failure, they provide three different repair services. Table 4.6.1 summarizes their repair costs, α values and inter-failure intervals' shape parameters and scale parameters from Samatli-Paç and Taner [146].

Table 4.6.1 Weibull distribution parameters and repair costs

Inter-failure interval	Repair type	shape	scale	α	cost*
1st	N/A	1.68	158.24	N/A	N/A
	1	1.55	38.38	0.25	10
2nd	2	1.02	127.57	0.91	28
	3	1.11	140.51	0.98	59

*new Turkish Liras

Using eqs (4.5.27) & (4.5.28) Table 4.6.2 is obtained. It shows the expected warranty cost for one product (unit: new Turkish Liras) and the variance of the expected warranty cost under the three different repair types for several choices of different warranty periods.

In Fig. 4.6.1, there is no big difference between three repair types based on expected warranty cost and the standard deviation in the earlier warranty period. In Table 4.6.2, if they select the repair type 2, it makes smaller expected warranty cost than when we choose other repair types although there is no big difference. However, as warranty time goes on, the warranty cost of the repair type 2 and 3 are increased higher than the repair type 1. In the later warranty period, the repair type 1 is much better than others in terms of the expected warranty cost.

These real-world benchmarking examples showed how the proposed model can be used for the warranty cost analysis. The next section covers for a various choices of parameters in terms of several systems for sensitivity analysis.

Table 4.6.2 Warranty cost analysis by different repair types

Year	Repair Type	E(C)	SD	CV
5	1	0.08	1.63	20.44
	2	0.04	1.94	48.23
	3	0.05	3.26	60.39
7	1	0.32	3.88	12.04
	2	0.19	4.98	26.22
	3	0.27	8.55	32.21
9	1	0.79	6.87	8.73
	2	0.60	10.07	16.69
	3	0.87	17.53	20.20
10	1	1.09	8.51	7.84
	2	0.98	13.50	13.82
	3	1.42	23.66	16.62
15	1	2.49	15.79	6.33
	2	6.14	41.46	6.75
	3	9.43	74.60	7.91
20	1	2.57	18.51	7.20

2	22.16	90.94	4.10
3	35.46	167.01	4.71

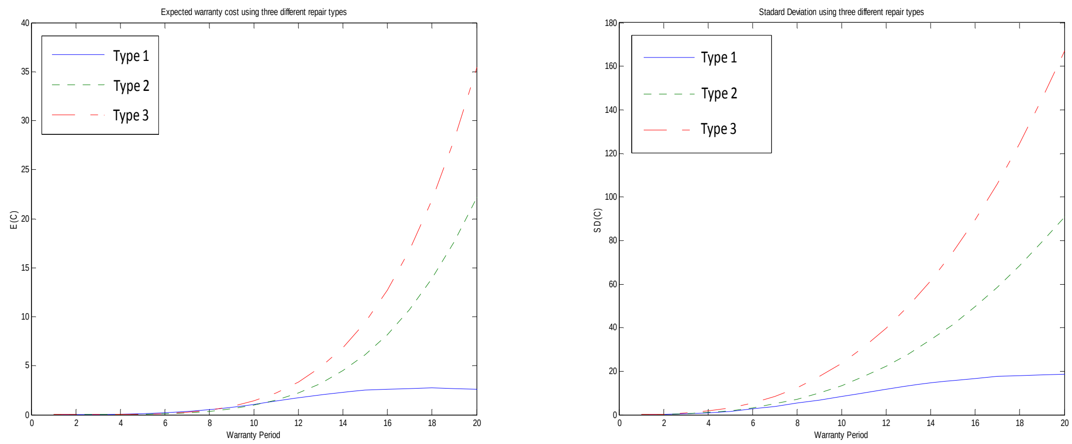


Fig. 4.6.1 Warranty cost analysis by different repair

4.7 Numerical Example and Sensitivity Analysis

In this section, a numerical example is presented using the warranty cost model. We consider a system assuming that a system's failure rate follows Weibull distribution. We investigate the warranty cost for the system and select the appropriate warranty service type among several different warranty service types based on the result of the warranty cost analysis. In other words, the cost analysis results help the policy maker select the appropriate warranty service among several choices. The warranty cost analysis is investigated using eqs. (5.27) and (5.28). We assume that the failure time of the system follows the Weibull distribution with different parameters. The reliability of the product is affected by the parameters of the Weibull distribution which is widely used in reliability engineering because other distributions such as exponential, Rayleigh, and normal are special cases of the Weibull distribution. It is also because its flexibility that allows accurate representation of a variety of lifetime distributions.

First, we consider Weibull distribution with shape parameter 1 and scale parameter 0.5. Let β_i be i^{th} inter-failure interval's parameter for the altered QRP. Given h=2, the warranty costs and the parameters of the altered QRP are assumed as follows:

$$c_a = \$20, c_b = \$30, \beta_i = 0.9, \beta_{n+1} = 0.8$$

The results of the expected warranty cost, $E(C)$, standard deviation, $SD(C)$, of warranty cost and coefficient of variation (CV) are listed in Table 4.7.1. For the sensitivity analysis purpose, we include in Table 4.7.1 a 10-warranty period unit which starts at 0.1 and finishes at 1.0 and *r.v.* T would be assumed as the total warranty cycle.

Table 4.7.1 $E(C)$, $SD(C)$ and CV

w	$E(C)$	$SD(C)$*	CV*	$SD(C)$**	CV**
0.1	34.07	10.87	0.32	20.45	0.13
0.2	38.23	21.89	0.57	40.80	0.17
0.3	42.39	31.47	0.74	59.55	0.18
0.4	46.45	39.75	0.86	75.42	0.19
0.5	50.33	46.85	0.93	87.54	0.19
0.6	53.95	52.88	0.98	95.51	0.18
0.7	57.26	57.97	1.01	99.46	0.17
0.8	60.21	62.24	1.03	99.88	0.17
0.9	62.78	65.82	1.05	97.53	0.16
1	64.96	68.81	1.06	93.25	0.15

* : Repair and replacement are independent.
** : Repair and replacement are dependent.

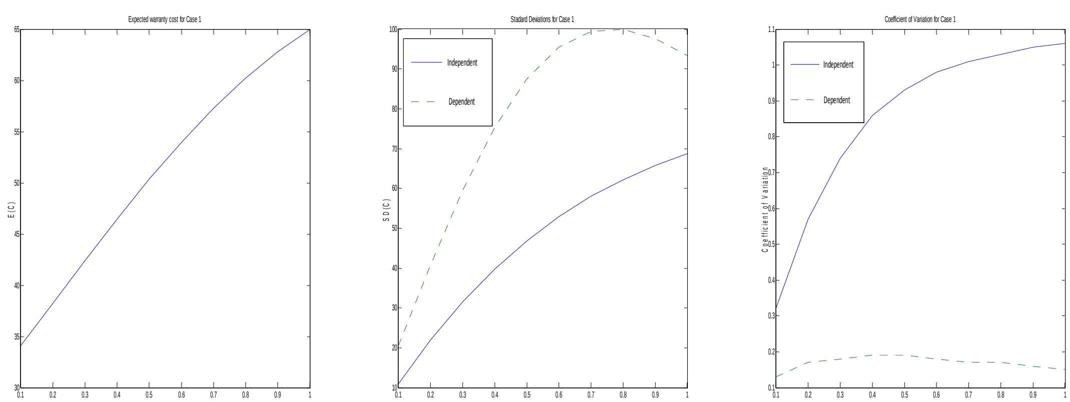


Fig. 4.7.1 E(C), SD(C) and CV

In Table 4.7.1, we separate the cases repair and replacement are independent and dependent. In both cases, the expected warranty cost increases as warranty time increases. When a repair and a replacement service are independent, it means that there is no covariance between repair and replacement. It is worth noting that the standard deviation of warranty cost is larger when repair and replacement are dependent than when repair and replacement are independent. In Fig. 4.7.1, when repair and replacement are independent, it can be seen that both the expectation and the standard deviation of warranty cost functions increase monotonically over *w*. We can observe that standard deviation is larger than the expected warranty cost as the warranty period goes on. The coefficient of variation (*CV*) is the ratio of the standard deviation to the mean and describes the magnitude sample values and the variation within them. When the warranty time progresses, the *CV* is decreasing. It indicates smaller variations. When the manufacturers have to make very important decisions for the company's revenue, they can use the proposed models to make important decisions. If we assume that there are several different warranty services, then one warranty service type should be taken among them. The first and the second inter-failure interval's parameters for the altered QRP are β_1 and β_2, respectively. In Table 4.7.2, the parameters of the first inter-failure rate for the altered QRP are changed as 0.6, 0.7 and 0.8 with other conditions which shown in the first row. Similarly, in Table 4.7.3, the parameters of the second inter-failure rate for the altered QRP are changed as 0.5, 0.6, 0.7, 0.8 and 0.9 with other parameter conditions for the sensitivity analysis. The scale parameters of the Weibull distribution for first inter-failure rate are changed as 5, 10, 15, 20 and 25 in Table 4.7.4. In Table 4.7.5, the warranty repair costs of the failure are changed as \$10, \$20, \$30, \$40 and \$50 with other parameters. Finally, in Table 4.7.6, the warranty replacement costs of the failure are changed as \$20, \$30, \$40, \$50 and \$60.

For simplicity, we consider the first failure and the second failure only. For Tables 4.7.2-4.7.6, we obtain the expected warranty cost, the variance of the warranty cost and the coefficient of variation based on different conditions.

Table 4.7.2 warranty cost analysis with different parameters for the first inter-failure interval

β_2=0.6, c_1=\$20, c_2=\$30, $\lambda_i = 2, k_i = 1$									
	Type1: β_1=0.6			Type2: β_1=0.7			Type3: β_1=0.8		
w	E(C)	SD	CV	E(C)	SD	CV	E(C)	SD	CV
0.1	34.13	13.79	0.40	34.10	12.58	0.37	34.08	11.57	0.34
0.2	38.43	26.56	0.69	38.34	24.50	0.64	38.27	22.80	0.60
0.3	42.80	37.21	0.87	42.59	34.52	0.81	42.44	32.28	0.76
0.4	47.08	45.96	0.98	46.73	42.84	0.92	46.49	40.21	0.87
0.5	51.15	53.04	1.04	50.66	49.67	0.98	50.32	46.77	0.93
0.6	54.89	58.68	1.07	54.29	55.18	1.02	53.85	52.14	0.97
0.7	58.25	63.10	1.08	57.54	59.59	1.04	57.02	56.47	0.99
0.8	61.15	66.54	1.09	60.37	63.07	1.04	59.79	59.93	1.00
0.9	63.60	69.20	1.09	62.77	65.79	1.05	62.15	62.67	1.01
1	65.59	71.23	1.09	64.75	67.89	1.05	64.11	64.81	1.01

Table 4.7.3 warranty cost analysis with different parameters for the second inter-failure interval

β_1=0.9, c_1=\$20, c_2=\$30, $\lambda_i = 2, k_i = 1$															
	Type1: β_2=0.5			Type2: β_2=0.6			Type3: β_2=0.7			Type4: β_2=0.8			Type5: β_2=0.9		
w	E(C)	SD	CV	E(C)	SD	CV	E(C)	SD	CV	E(C)	SD	CV	E(C)	SD	CV
0.1	34.07	10.60	0.31	34.07	10.72	0.31	34.07	10.81	0.32	34.07	10.87	0.32	34.07	10.92	0.32
0.2	38.20	20.93	0.55	38.21	21.35	0.56	38.22	21.66	0.57	38.23	21.89	0.57	38.24	22.07	0.58
0.3	42.28	29.51	0.70	42.32	30.37	0.72	42.36	30.99	0.73	42.39	31.47	0.74	42.41	31.84	0.75
0.4	46.20	36.56	0.79	46.30	37.95	0.82	46.38	38.97	0.84	46.45	39.75	0.86	46.50	40.36	0.87
0.5	49.87	42.27	0.85	50.06	44.27	0.88	50.20	45.73	0.91	50.33	46.85	0.93	50.43	47.73	0.95
0.6	53.22	46.83	0.88	53.51	49.47	0.92	53.75	51.40	0.96	53.95	52.88	0.98	54.11	54.05	1.00
0.7	56.21	50.42	0.90	56.62	53.70	0.95	56.96	56.12	0.99	57.26	57.97	1.01	57.51	59.42	1.03
0.8	58.81	53.17	0.90	59.34	57.12	0.96	59.81	60.03	1.00	60.21	62.24	1.03	60.56	63.97	1.06
0.9	61.01	55.22	0.91	61.67	59.85	0.97	62.26	63.24	1.02	62.78	65.82	1.05	63.24	67.82	1.07
1	62.82	56.69	0.90	63.61	61.99	0.97	64.32	65.88	1.02	64.96	68.81	1.06	65.54	71.07	1.08

Table 4.7.4 warranty cost analysis with different scale parameters

β_1=0.9, β_2=0.6, c_1=\$20, c_2=\$30, $k_1=1, \lambda_2=2, k_2=1$															
	Type1: λ_1=5			Type2: λ_1=10			Type3: λ_1=15			Type4: λ_1=20			Type5: λ_1=25		
w	E(C)	SD	CV	E(C)	SD	CV	E(C)	SD	CV	E(C)	SD	CV	E(C)	SD	CV
0.1	31.64	7.14	0.23	30.82	5.13	0.17	30.55	4.21	0.14	30.41	3.66	0.12	30.33	3.28	0.11
0.2	33.36	14.72	0.44	31.69	10.71	0.34	31.13	8.82	0.28	30.85	7.68	0.25	30.68	6.89	0.22
0.3	35.15	21.72	0.62	32.61	15.98	0.49	31.75	13.22	0.42	31.31	11.53	0.37	31.05	10.35	0.33
0.4	37.01	28.16	0.76	33.58	20.98	0.62	32.40	17.43	0.54	31.81	15.23	0.48	31.45	13.69	0.44
0.5	38.93	34.08	0.88	34.61	25.72	0.74	33.10	21.46	0.65	32.34	18.79	0.58	31.88	16.91	0.53
0.6	40.90	39.50	0.97	35.69	30.19	0.85	33.85	25.30	0.75	32.91	22.20	0.67	32.33	20.01	0.62
0.7	42.89	44.43	1.04	36.83	34.42	0.93	34.64	28.98	0.84	33.52	25.48	0.76	32.83	23.00	0.70
0.8	44.90	48.90	1.09	38.03	38.39	1.01	35.49	32.48	0.92	34.17	28.63	0.84	33.36	25.88	0.78
0.9	46.90	52.92	1.13	39.28	42.12	1.07	36.39	35.81	0.98	34.87	31.65	0.91	33.94	28.66	0.84
1	48.88	56.51	1.16	40.58	45.60	1.12	37.35	38.97	1.04	35.63	34.54	0.97	34.56	31.33	0.91

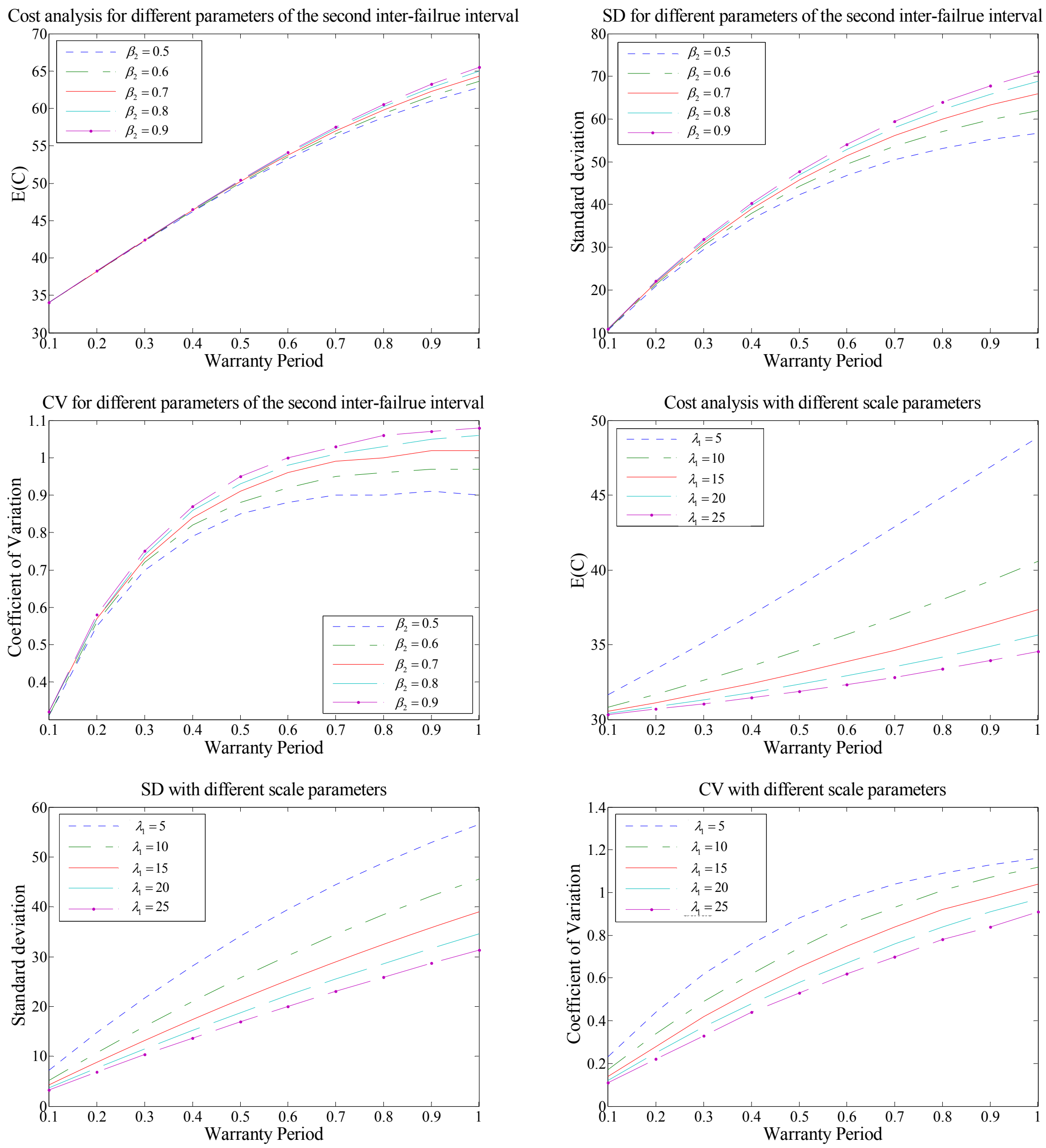
Table 4.7.5 warranty cost analysis with different repair costs

β_1=0.9, β_2=0.6, c_2=\$60, $\lambda_1=2, k_1=1, \lambda_2=2, k_2=1$															
	Type1: c_1=10			Type2: c_1=20			Type3: c_1=30			Type4: c_1=40			Type5: c_1=50		
w	E(C)	SD	CV	E(C)	SD	CV	E(C)	SD	CV	E(C)	SD	CV	E(C)	SD	CV
0.1	66.52	9.53	0.14	67.06	14.23	0.21	67.60	18.04	0.27	68.14	21.44	0.31	68.68	24.60	0.36
0.2	72.99	19.75	0.27	74.13	29.04	0.39	75.28	36.35	0.48	76.43	42.71	0.56	77.57	48.49	0.63
0.3	79.26	28.26	0.36	81.05	41.51	0.51	82.85	51.83	0.63	84.65	60.74	0.72	86.44	68.78	0.80
0.4	85.20	35.31	0.41	87.67	51.94	0.59	90.14	64.83	0.72	92.60	75.91	0.82	95.07	85.87	0.90
0.5	90.71	41.09	0.45	93.84	60.59	0.65	96.98	75.64	0.78	100.11	88.53	0.88	103.24	100.11	0.97
0.6	95.68	45.78	0.48	99.46	67.69	0.68	103.24	84.53	0.82	107.02	98.93	0.92	110.80	111.84	1.01
0.7	100.07	49.55	0.50	104.46	73.45	0.70	108.85	91.77	0.84	113.24	107.41	0.95	117.63	121.41	1.03
0.8	103.82	52.56	0.51	108.78	78.09	0.72	113.73	97.61	0.86	118.69	114.25	0.96	123.64	129.12	1.04
0.9	106.94	54.96	0.51	112.41	81.79	0.73	117.88	102.27	0.87	123.34	119.70	0.97	128.81	135.26	1.05
1	109.44	56.85	0.52	115.36	84.71	0.73	121.29	105.93	0.87	127.22	123.98	0.97	133.15	140.08	1.05

Table 4.7.6 warranty cost analysis with different replacement costs

β_1=0.9, β_2=0.6, c_1=\$20, $\lambda_1=2, k_1=1, \lambda_2=2, k_2=1$															
	Type1: c_2=20			Type2: c_2=30			Type3: c_2=40			Type4: c_2=50			Type5: c_2=60		
w	E(C)	SD	CV	E(C)	SD	CV	E(C)	SD	CV	E(C)	SD	CV	E(C)	SD	CV
0.1	23.07	9.20	0.40	34.07	10.72	0.31	45.07	12.02	0.27	56.06	13.18	0.24	67.06	14.23	0.21
0.2	26.24	17.96	0.68	38.21	21.35	0.56	50.19	24.23	0.48	62.16	26.77	0.43	74.13	29.04	0.39
0.3	29.41	25.41	0.86	42.32	30.37	0.72	55.23	34.55	0.63	68.14	38.22	0.56	81.05	41.51	0.51
0.4	32.51	31.69	0.97	46.30	37.95	0.82	60.09	43.22	0.72	73.88	47.83	0.65	87.67	51.94	0.59
0.5	35.46	36.93	1.04	50.06	44.27	0.88	64.65	50.43	0.78	79.25	55.80	0.70	93.84	60.59	0.65
0.6	38.19	41.24	1.08	53.51	49.47	0.92	68.83	56.35	0.82	84.15	62.35	0.74	99.46	67.69	0.68

0.7	40.67	44.76	1.10	56.62	53.70	0.95	72.56	61.18	0.84	88.51	67.67	0.76	104.46	73.45	0.70
0.8	42.86	47.59	1.11	59.34	57.12	0.96	75.82	65.07	0.86	92.30	71.97	0.78	108.78	78.09	0.72
0.9	44.76	49.85	1.11	61.67	59.85	0.97	78.58	68.18	0.87	95.50	75.39	0.79	112.41	81.79	0.73
1	46.36	51.61	1.11	63.61	61.99	0.97	80.86	70.62	0.87	98.11	78.09	0.80	115.36	84.71	0.73



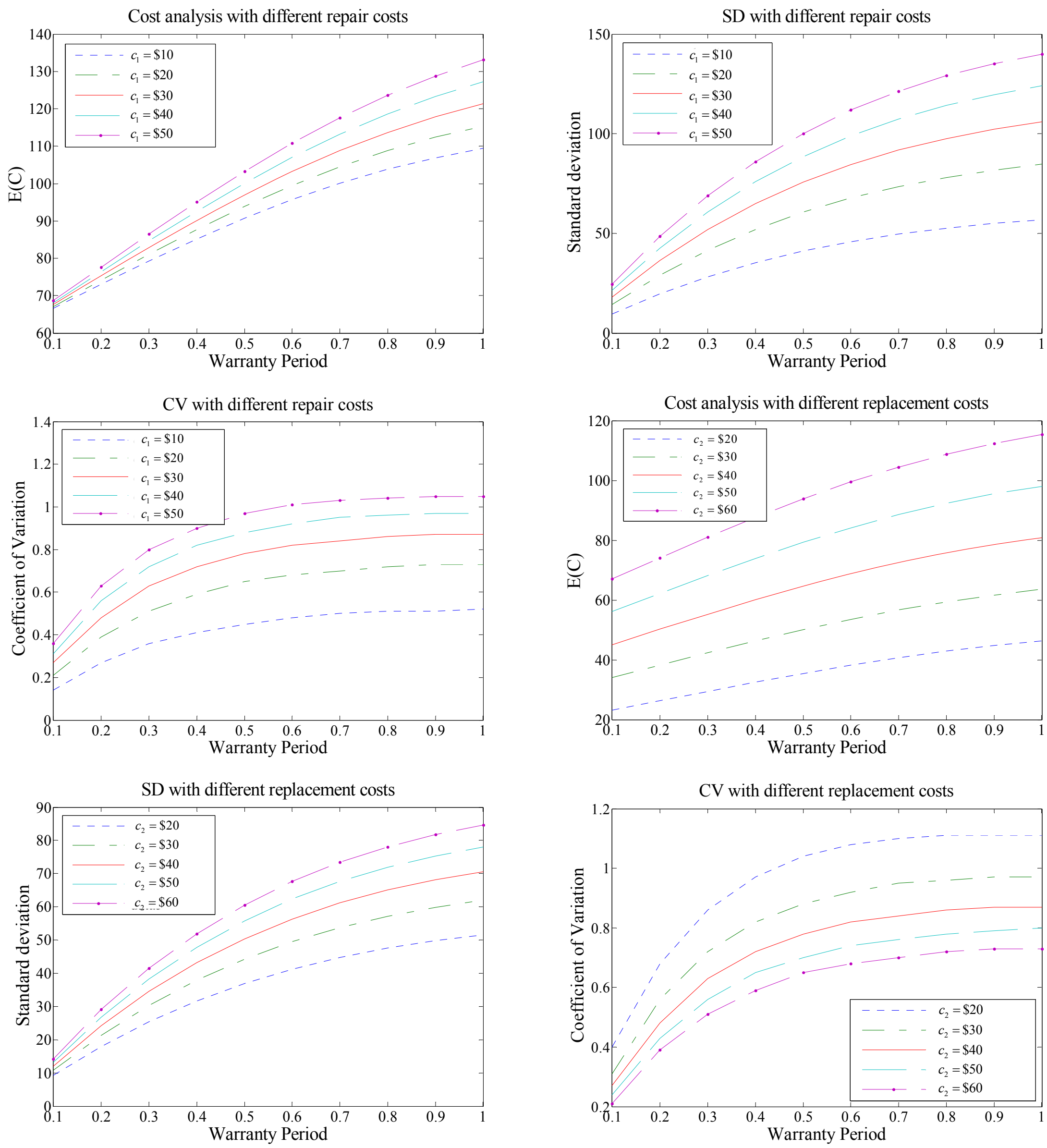


Fig. 4.7.2 warranty cost analysis with different parameters

For each table, we consider different warranty periods from 0.1 to 1.0. Then the expected warranty cost, variance of the warranty cost and coefficient of variation are obtained. When we change the altered QRP's parameter of the first inter-failure intervals, β_1, we obtain cost analysis results as in Table 4.7.2. In Table 4.7.3, as the second inter-failure interval parameters, β_2 changes, the expected warranty cost increases because smaller percentages indicate that the next failure will happen faster. In Table 4.7.4, as the scale parameter increases, the expected warranty cost and the standard deviation decrease. For Table 4.7.5, as the repair cost increases, the expected warranty cost and standard deviation increase. Based on the tables, we find that the warranty cost is easily affected by replacement cost and repair cost among many other factors. Also, as the replacement cost per service increases, the expected warranty cost and standard deviation increase too.

4.8 Concluding Remarks

In this chapter, we introduced two alternative quasi-renewal processes. They are: altered quasi-renewal and mixed quasi-renewal processes. These processes can be found in many practical applications such as the one described in Section 4.6. We obtained the expected value of warranty cost, covariance of warranty cost and variance of warranty cost for the warranted product. We presented sensitivity analysis and numerical examples. Based on this study, we believe that these mixed and altered quasi-renewal processes are useful for warranty cost analysis. The results of reliability and warranty cost functions can be easily applied and would be helpful for marketing purpose.

Chapter 5

Warranty Cost Analyses using Induction Method with Imperfect Repair[2]

5.1 Introduction

Given that the warranty is a post sale service, potential expenses could occur for manufacturer. This potential cost is difficult to estimate and is known to be a significant portion of the total cost. Therefore, the warranty cost could be a serious negative factor which affects the company's profit. Many companies use the warranty policy to increase their sales but also try to minimize the related cost. For example, if a warranty period is scheduled too long without considering the qualities of products, then it increases the risk of potential costs for the manufacturers. The range of warranty cost analysis would not only need to consider the characteristics of warranty policies and replacement/repair cost but also the distribution of the number of product failures.

Using a renewal process with an imperfect repair, Wang and Pham [175] introduce quasi renewal process (QRP) and apply QRP for software engineering successfully [131]. After the QRP was introduced by Wang and Pham [175], several researchers [40, 120, 122-124, 131,

[2] M. Park and H. Pham, "Warranty cost analyses using quasi-renewal processes for multi-component systems" *IEEE Transactions on Systems, Man and Cybernetics, Part A: Systems and Humans* (Accepted for Publication)

138, 139, 146, 175] have applied the QRP for their research topics. Rehmert [138, 139] develops expressions for the point availability using the QRP.

This chapter focuses on the cost analysis for the repairable models with a fixed warranty period, which the warranty policy starts from the original date of purchase and is effective until the warranty ends. Using the QRP, we consider the imperfect repair for the multi-component system as well as the single component system to develop the warranty cost model. Using the property of exponential distribution which is frequently used for the modeling, we propose new approach for the warranty cost analysis. The proposed approach is expected to be beneficial to the manufacturer to estimate the warranty cost accurately.

For many cases, repairable warranty policies have fixed warranty periods. It is common to find the products which have fixed warranty periods, for example, vehicles and electronic appliances such as laptops, desktop, DVD, HDTV and etc. This seems more realistic and applicable to our daily life. Cost analysis is conducted mainly using the QRP and exponential distribution to obtain the mean and variance. Exponential distribution has several properties that are useful, such as memoryless property. Further, property that the summation of exponential distributions follows gamma distribution was also used, however, there is a limitation for this property, it is valid when every parameters of exponential distributions are equal.

The outline of the chapter is as follows. Section 5.1 explains nomenclature. In Section 5.2, we describe the problem and several assumptions. Section 5.3 focuses on the distribution of the number of components' failures based on the QRP. Section 5.4 presents cost analysis using single component system and multi component systems such as series, parallel and parallel-series. In Section 5.5, we discuss the implications of both research and practice. Several numerical examples and sensitivity analyses are given in Section 5.6 to illustrate the methodologies derived in this chapter for single component system and four-component parallel-series system, concluding remarks are discussed in Section 5.7.

5.1.1 Nomenclature

$f_i(x), F_i(x)$ *and* $h_i(x)$: probability density function (pdf), cumulative distribution function (cdf), and failure rate of random variable X_i, respectively

$F(\cdot), R(\cdot)$: r.v. inter-failure time's cdf, reliability function, respectively

$F_j(\cdot), R_j(\cdot)$: cdf and reliability function of j^{th} component's r.v. inter-failure time, respectively

$F_s(\cdot), R_s(\cdot)$: cdf and reliability function of system's r.v. inter-failure time, respectively

w: prefixed warranty period

N: number of component's failures in the warranty period w

N_s: number of system's failures in the warranty period w

$N(t)$: number of component's failures by time t

C: system's warranty cost in warranty period w

c: warranty cost per one system failure in the warranty period w

S_n : arrival time of n^{th} renewal

5.2 Problem Description

We consider the following assumptions in this study:

- The inter-occurrence failure intervals follow exponential distribution
- The Repair time is negligible
- The failed products are repairable
- Inter-occurrence failure intervals are independent to each other, and
- The review time which examines whether the failed components need the repair services, is negligible.

With respect to warranty cost analysis, we consider imperfect repairs based on a QRP. When a repair is imperfect, the inter-failure intervals are specified by a parameter α that alters the random variables corresponding to time until next failures after each renewal. In other words, this parameter indicates the degree of repair. For example, if the parameter is less than 1, it indicates imperfect repair. If it is greater than 1, it indicates an improvement after repair.

5.3 Distribution of *N*

In this section, we first present the definition of QRP which introduced by Wang & Pham [175]. The impact of a repair service on product reliability is one of the most significant factors in warranty cost. One can divide repairs into three categories based on the effort of repairs or the condition of repaired items; as good as new repair, minimal repair and imperfect repair.

- As good as new repair - assumes that after repair the restored system functions like new such that the failure time distribution is the same as that of a new product.
- Minimal repair (also called as bad as old repair) - assumes that the failure rate of a repaired system equals that of the system just before the most recent failure.
- Imperfect repair - refers to the situation where a repair action responds to a system neither as good as new nor as bad as old but to a level in between.

Most maintenance models using renewal theory were actually based on ‘as good as new’. However, it is well-known that the imperfect repair is more common in real life compared to ‘as good as new’ repair. Wang and Pham [175] develop the QRP to model failure times when the repair is imperfect. You can refer QRP to section 2.5.2.

5.3.1 Repairable Warranty Policy with Fixed Warranty Period

According to the type of compensation upon products’ failures, there are three basic types of warranties such as FRW, PRW and CMW. We would normally purchase products with fixed warranty period, for examples, like electronic appliances, vehicles and etc. And if a system fails during the warranty period, products would be repaired. If we pay more money for longer warranty period, then warranty period can be extended. The pdf of inter-failure interval is following exponential distribution, i.e. $f_i(x) \sim \exp(\lambda)$. Then we conduct the cost analysis using QRP and exponential distribution for repairable warranty policy with fixed warranty period w. To derive the statistical properties of warranty cost per cycle or per product sold, it is necessary to obtain the distribution of N, the number of failures within w. We present repairable warranty polices with a warranty period and distribution of the number of system failures. Using this distribution of the number of component failures, the expected warranty system cost and the variance of the warranty system cost is obtained in the next section.

5.3.2 Distribution of N

It is clearly known that the $N(t)$ of “renewals” that has occurred up to time t and the arrival time of the n^{th} renewal, S_n, have the following relationship:

$$N(t) \geq n \leftrightarrow S_n \leq t \tag{5.3.2}$$

$N(t)$ is at least n if and only if the n^{th} renewal occurs prior to or at time t [142]. We assume that there are n system failures in the warranty period. X_i is the inter-arrival interval and i.i.d. r.v. from exponential distribution. If there are n failures and fixed warranty period w, then the sum of inter-arrival interval by n is less or equal to w and the sum of inter-arrival interval by $n+1$ is larger than w. That is,

$$\begin{aligned} &X_1 + X_2 + \cdots + X_n \le w \\ &X_1 + X_2 + \cdots + X_{n+1} > w \end{aligned} \tag{5.3.3}$$

Figure 5.3.1 Warranty model with fixed warranty period w and n failures

If Y_n denotes $Y_n = \sum_{i=1}^{n} X_i$, then

$$\begin{aligned} &Y_n \le w \\ &Y_n + X_{n+1} > w \end{aligned} \tag{5.3.4}$$

Using these two equations, we obtain the distribution of number of failures based on QRP and exponential distribution. A well-known characteristic of the exponential distribution is that if each r.v. X_i follows $Exp(\lambda)$ then the sum of n functions, Y_n follows the gamma distribution, $Gamma(n,\lambda)$. That is,

$$\begin{aligned} &X_i \sim Exp(\lambda), \\ &Y_n \sim Gamma(n,\lambda) \end{aligned} \tag{5.3.5}$$

Using the renewal processes, the corresponding pdfs are given by

$$f_X = \frac{1}{\lambda} e^{-\frac{1}{\lambda}x},$$

$$f_Y = \frac{1}{\Gamma(n)\lambda^n} y^{n-1} e^{-\frac{y}{\lambda}} \tag{5.3.6}$$

It is noted that the above results do not apply the QRP. Then, the pdf of the number of component's failures is given by

$$\begin{aligned} P(N=n) &= P(Y_n \le w,\ Y_n + X_{n+1} > w) \\ &= \int_0^w \int_{w-y}^{\infty} f_{Y_n}(y) f_{X_{n+1}}(x)\, dx \cdot dy \\ &= \int_0^w \int_{w-y}^{\infty} \frac{1}{\Gamma(n)\lambda^n} y^{n-1} e^{-\frac{y}{\lambda}} \frac{1}{\lambda} e^{-\frac{x}{\lambda}} dx \cdot dy \\ &= \frac{1}{n!}\left(\frac{w}{\lambda}\right)^n e^{\left(-\frac{w}{\lambda}\right)} \end{aligned} \tag{5.3.7}$$

The expected number of failures can be obtained as follow:

$$\begin{aligned} E(N) &= \sum_{n=1}^{\infty} nP(N=n) \\ &= \sum_{n=1}^{\infty} n \frac{1}{n!}\left(\frac{w}{\lambda}\right)^n e^{\left(-\frac{w}{\lambda}\right)} \\ &= \frac{w}{\lambda} \end{aligned} \tag{5.3.8}$$

Next we will derive the variance of the number of failures. First we calculate the second moment.

$$\begin{aligned} E(N^2) &= \sum_{n=1}^{\infty} n^2 \frac{1}{n!}\left(\frac{w}{\lambda}\right)^n e^{\left(-\frac{w}{\lambda}\right)} \\ &= e^{\left(-\frac{w}{\lambda}\right)} \sum_{n=1}^{\infty} \frac{n}{(n-1)!}\left(\frac{w}{\lambda}\right)^n \end{aligned} \tag{5.3.9}$$

Therefore, the variance of the number of failures is

$$
\begin{aligned}
Var(N) &= E\left(N^2\right) - \left(E(N)\right)^2 \\
&= e^{\left(-\frac{w}{\lambda}\right)} \sum_{n=1}^{\infty} \frac{n}{(n-1)!} \left(\frac{w}{\lambda}\right)^n - \left(\frac{w}{\lambda}\right)^2
\end{aligned} \tag{5.3.10}
$$

If a repair service is imperfect and the inter-occurrence interval follows QRP, the pdf of the n^{th} inter-occurrence interval can be written as;

$$
f_n(x) = \frac{1}{\alpha^{n-1}} f_1\left(\frac{1}{\alpha^{n-1}} x\right) \tag{5.3.11}
$$

Then for the $(n+1)^{th}$ function, $f_{X_{n+1}}(x)$, we assume that each function follows exponential distribution with parameter λ. We obtain

$$
f_{X_{n+1}}(x) = \frac{1}{\alpha^n} f_1\left(\frac{1}{\alpha^n} x\right) = \frac{1}{\alpha^n} \frac{1}{\lambda} e^{\left(-\frac{1}{\lambda}\frac{1}{\alpha^n} x\right)} \tag{5.3.12}
$$

Every exponential parameter for each interval is not same when the QRP are used. When parameter λ is different, $f_{Y_n}(y)$ could be obtained by using the method of induction. We begin with the case $n = 2$ assuming X_1 *and* X_2 are independent.

$$
\begin{aligned}
f_{X_1+X_2}(y) &= \int_0^y f_{X_1}(s) f_{X_2}(y-s)\, ds \\
&= \int_0^y \frac{1}{\lambda_1} e^{-\frac{s}{\lambda_1}} \frac{1}{\lambda_2} e^{-\frac{(y-s)}{\lambda_2}} ds \\
&= \frac{1}{\lambda_1 - \lambda_2}\left(e^{-\frac{y}{\lambda_1}} - e^{-\frac{y}{\lambda_2}}\right)
\end{aligned} \tag{5.3.13}
$$

We can rewrite eq. (5.3.13) as follows:

$$
f_{X_1+X_2}(y) = \left(\frac{\frac{1}{\lambda_1}}{\frac{1}{\lambda_1} - \frac{1}{\lambda_2}}\right) \frac{1}{\lambda_2} e^{-\frac{1}{\lambda_2} y} + \left(\frac{\frac{1}{\lambda_2}}{\frac{1}{\lambda_2} - \frac{1}{\lambda_1}}\right) \frac{1}{\lambda_1} e^{-\frac{1}{\lambda_1} y} \tag{5.3.14}
$$

Similarly, in the case of $n=3$,

$$f_{X_1+X_2+X_3}(y)=\sum_{i=1}^{3}\frac{1}{\lambda_i}e^{-\frac{1}{\lambda_i}y}\prod_{j\neq i}\left(\frac{\frac{1}{\lambda_j}}{\frac{1}{\lambda_j}-\frac{1}{\lambda_i}}\right) \tag{5.3.15}$$

In general, the pdf of the n^{th} terms can be obtained using the method of induction [142]:

$$f_{X_1+\cdots+X_n}(y)=\sum_{i=1}^{n}\frac{1}{\lambda_i}e^{-\frac{1}{\lambda_i}y}\prod_{j\neq i}\left(\frac{\frac{1}{\lambda_j}}{\frac{1}{\lambda_j}-\frac{1}{\lambda_i}}\right) \tag{5.3.16}$$

Also we can obtain the following

$$f_{X_{n+1}}=\frac{1}{\lambda_{n+1}}e^{-\frac{1}{\lambda_{n+1}}x}$$

$$f_{Y_n}=f_{X_1+\cdots+X_n}(y)=\sum_{i=1}^{n}\frac{1}{\lambda_i}e^{-\frac{1}{\lambda_i}y}\prod_{j\neq i}\left(\frac{\frac{1}{\lambda_j}}{\frac{1}{\lambda_j}-\frac{1}{\lambda_i}}\right) \tag{5.3.17}$$

Let C_{in} be $\prod_{j\neq i}\frac{\frac{1}{\lambda_j}}{\frac{1}{\lambda_j}-\frac{1}{\lambda_i}}$ with n failures. The probability mass function (pmf) of the number of failures is as follows:

$$
\begin{aligned}
P(N=n) &= P(Y_n \le w,\ Y_n + X_{n+1} > w) \\
&= \int_0^w \int_{w-y}^{\infty} \sum_{i=1}^{n} \frac{1}{\lambda_i} e^{-\frac{1}{\lambda_i}y} \left(\prod_{j \ne i} \frac{\frac{1}{\lambda_j}}{\frac{1}{\lambda_j} - \frac{1}{\lambda_i}} \right) \left(\frac{1}{\lambda_{n+1}} e^{-\frac{1}{\lambda_{n+1}}x} \right) dx \cdot dy \\
&= \int_0^w \left(\sum_{i=1}^{n} \frac{1}{\lambda_i} e^{-\frac{1}{\lambda_i}y} \cdot C_{in} \right) \left(\frac{1}{\lambda_{n+1}} (-\lambda_{n+1}) e^{-\frac{1}{\lambda_{n+1}}x} \Big|_{w-y}^{\infty} \right) dy \\
&= \int_0^w \left(\sum_{i=1}^{n} \frac{1}{\lambda_i} e^{-\frac{1}{\lambda_i}y} \cdot C_{in} e^{-\frac{(w-y)}{\lambda_{n+1}}} \right) dy \\
&= \sum_{i=1}^{n} \frac{C_{in}}{\lambda_i} e^{-\frac{w}{\lambda_{n+1}}} \int_0^w e^{-\left(\frac{1}{\lambda_i} - \frac{1}{\lambda_{n+1}}\right)y} dy \\
&= \sum_{i=1}^{n} \frac{C_{in} \lambda_{n+1}}{\lambda_i - \lambda_{n+1}} \left(e^{-\frac{w}{\lambda_i}} - e^{-\frac{w}{\lambda_{n+1}}} \right)
\end{aligned}
$$

Because λ_{n+1} is same as $\lambda\alpha^n$, the last term may be rewritten by

$$
\sum_{i=1}^{n} \frac{C_{in} \cdot \alpha^n}{\alpha^{i-1} - \alpha^n} \left(e^{-\frac{w}{\lambda \cdot \alpha^{i-1}}} - e^{-\frac{w}{\lambda \cdot \alpha^n}} \right) \tag{5.3.18}
$$

5.4 Warranty Cost Analysis

In this section, warranty cost analyses are conducted by computing the expected warranty cost as well as the variance of the warranty. We also obtain reliability functions and derive several statistical properties of warranty cost function per cycle or per product sold.

5.4.1 Single Component System

The goal of cost analysis is to obtain the expected value of warranty cost and the variance of warranty cost. If the inter-occurrence intervals follow the regular renewal processes, then the expected warranty cost and the variance of warranty cost are, respectively, given by

$$
E(C) = cE(N) = c\frac{w}{\lambda} \tag{5.4.1}
$$

$$Var(C) = c^2 Var(N) = c^2 \left(e^{\left(-\frac{w}{\lambda}\right)} \sum_{n=1}^{\infty} \frac{n}{(n-1)!} \left(\frac{w}{\lambda}\right)^n - \left(\frac{w}{\lambda}\right)^2 \right) \tag{5.4.2}$$

If they follow the QRP, then the expected number of failures and variance of the number of failures are:

$$E(N) = \sum_{n=1}^{\infty} nP(N=n) = \sum_{n=1}^{\infty} n \sum_{i=1}^{n} \frac{C_{in} \cdot \alpha^n}{\alpha^{i-1} - \alpha^n} \left(e^{-\frac{w}{\lambda \cdot \alpha^{i-1}}} - e^{-\frac{w}{\lambda \cdot \alpha^n}} \right) \tag{5.4.3}$$

$$E(N^2) = \sum_{n=1}^{\infty} n^2 P(N=n) = \sum_{n=1}^{\infty} n^2 \sum_{i=1}^{n} \frac{C_{in} \cdot \alpha^n}{\alpha^{i-1} - \alpha^n} \left(e^{-\frac{w}{\lambda \cdot \alpha^{i-1}}} - e^{-\frac{w}{\lambda \cdot \alpha^n}} \right) \tag{5.4.4}$$

and

$$Var(N) = \sum_{n=1}^{\infty} n^2 \sum_{i=1}^{n} \frac{C_{in} \cdot \alpha^n}{\alpha^{i-1} - \alpha^n} \left(e^{-\frac{w}{\lambda \cdot \alpha^{i-1}}} - e^{-\frac{w}{\lambda \cdot \alpha^n}} \right) - \left(\sum_{n=1}^{\infty} n \sum_{i=1}^{n} \frac{C_{in} \cdot \alpha^n}{\alpha^{i-1} - \alpha^n} \left(e^{-\frac{w}{\lambda \cdot \alpha^{i-1}}} - e^{-\frac{w}{\lambda \cdot \alpha^n}} \right) \right)^2 \tag{5.4.5}$$

Therefore, the expected warranty system cost is

$$\begin{aligned} E(C) &= cE(N) \\ &= c \sum_{n=1}^{\infty} n \sum_{i=1}^{n} \frac{C_{in} \cdot \alpha^n}{\alpha^{i-1} - \alpha^n} \left(e^{-\frac{w}{\lambda \cdot \alpha^{i-1}}} - e^{-\frac{w}{\lambda \cdot \alpha^n}} \right) \end{aligned} \tag{5.4.6}$$

The variance of the warranty system cost is given by

$$Var(C) = c^2 \left(\sum_{n=1}^{\infty} n^2 \sum_{i=1}^{n} \frac{C_{in} \cdot \alpha^n}{\alpha^{i-1} - \alpha^n} \left(e^{-\frac{w}{\lambda \cdot \alpha^{i-1}}} - e^{-\frac{w}{\lambda \cdot \alpha^n}} \right) - \left(\sum_{n=1}^{\infty} n \sum_{i=1}^{n} \frac{C_{in} \cdot \alpha^n}{\alpha^{i-1} - \alpha^n} \left(e^{-\frac{w}{\lambda \cdot \alpha^{i-1}}} - e^{-\frac{w}{\lambda \cdot \alpha^n}} \right) \right)^2 \right) \tag{5.4.7}$$

5.4.2 Multi-component System

In this subsection, we derive the distribution, the first and second moments of the warranty cost and analyze the warranty cost by computing the expected warranty cost as well as the variance of the warranty for multi-component systems such as series, parallel and parallel-series based on QRP and exponential distribution.

Series system

For the engine of a car to run we need several different parts to operate in sequence. If any of the parts in the sequence fails to operate we will not be able to start the engine. As such in general, a series system functions successfully only when all the components in the system properly functions. In other words, every component should work successfully in order for the system to work. Consider a series system consisting of q components as shown in Figure 5.4..1. We derive the series system's pmf, cdf and reliability functions. Then we obtain the expected warranty cost and variance of warranty cost.

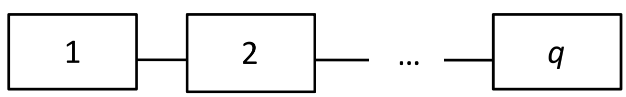


Figure 5.4.1. Series system with q components

Define $\Omega \equiv \{1,2,\cdots,q\}$. Let $F(n)$ and $R(n)$ be the cdf and the reliability function of the r.v. inter-failure interval of series system with n failures, respectively. Let C_{ijk} be $\prod_{j \neq h} \frac{\frac{1}{\lambda_j}}{\frac{1}{\lambda_j} - \frac{1}{\lambda_h}}$ of i^{th} failure for component j which has k failures totally. For the series system shown in Figure 5.4.1, the cdf of component j is given by

$$
\begin{aligned}
F_j(n) &= P(N_j \le n) \\
&= \sum_{k=1}^{n} P(N_j = k) \\
&= \sum_{k=1}^{n} \left(\sum_{i=1}^{k} \frac{C_{ijk} \cdot \alpha^k}{\alpha^{i-1} - \alpha^k} \left(e^{-\frac{w}{\lambda_j \cdot \alpha^{i-1}}} - e^{-\frac{w}{\lambda_j \cdot \alpha^k}} \right) \right)
\end{aligned}
\tag{5.4.8}
$$

By [54], the system reliability function $R(n)$ is same with $\prod_{j=1}^{q} R_j(n)$. Then,

$$
\begin{aligned}
R(n) &= \prod_{j=1}^{q} R_j(n) = \prod_{j=1}^{q} \left(1 - F_j(n)\right) \\
&= \prod_{j=1}^{q} \left(1 - \sum_{k=1}^{n} \left(\sum_{i=1}^{k} \frac{C_{ijk} \cdot \alpha^k}{\alpha^{i-1} - \alpha^k} \left(e^{-\frac{w}{\lambda_j \cdot \alpha^{i-1}}} - e^{-\frac{w}{\lambda_j \cdot \alpha^k}} \right) \right) \right)
\end{aligned}
\tag{5.4.9}
$$

And we obtain the pmf of series system.

$$
\begin{aligned}
P(N_s = n) &= P(N > n-1) - P(N > n) \\
&= R(n-1) - R(n)
\end{aligned}
$$

$$
= \prod_{j=1}^{q} \left(1 - \sum_{k=1}^{n-1} \left(\sum_{i=1}^{k} \frac{C_{ijk} \cdot \alpha^k}{\alpha^{i-1} - \alpha^k} \left(e^{-\frac{w}{\lambda_j \cdot \alpha^{i-1}}} - e^{-\frac{w}{\lambda_j \cdot \alpha^k}} \right) \right) \right) - \prod_{j=1}^{q} \left(1 - \sum_{k=1}^{n} \left(\sum_{i=1}^{k} \frac{C_{ijk} \cdot \alpha^k}{\alpha^{i-1} - \alpha^k} \left(e^{-\frac{w}{\lambda_j \cdot \alpha^{i-1}}} - e^{-\frac{w}{\lambda_j \cdot \alpha^k}} \right) \right) \right) \tag{5.4.10}
$$

For the cost analysis, we now wish to obtain the function of expected cost and variance cost. The expected number of failures is given by

$$
E(N) = \sum_{n=1}^{\infty} n P(N_s = n) \tag{5.4.11}
$$

where $P(N_s = n)$ is as in eq. (5.4.10). We now derive the variance of the number of failures. We can easily derive the second moment as follows:

$$
E(N^2) = \sum_{n=1}^{\infty} n^2 P(N_s = n) \tag{5.4.12}
$$

The variance of the number of failures is given by

$$Var(N)=E\left(N^2\right)-E\left(N\right)^2 \tag{5.4.13}$$

where $E\left(N\right)$ and $E\left(N^2\right)$ are given as eqs. (5.4.11) and (5.4.12).

The expected and variance of the system warranty cost for the series system are obtained as follows:

$$\begin{aligned} E(C)&=cE(N)\\ &=c\sum_{n=1}^{\infty}n\left(\prod_{j=1}^{q}\left(1-\sum_{k=1}^{n-1}\left(\sum_{i=1}^{k}\frac{C_{ijk}\cdot\alpha^k}{\alpha^{i-1}-\alpha^k}\left(e^{-\frac{w}{\lambda_j\cdot\alpha^{i-1}}}-e^{-\frac{w}{\lambda_j\cdot\alpha^k}}\right)\right)\right)-\prod_{j=1}^{q}\left(1-\sum_{k=1}^{n}\left(\sum_{i=1}^{k}\frac{C_{ijk}\cdot\alpha^k}{\alpha^{i-1}-\alpha^k}\left(e^{-\frac{w}{\lambda_j\cdot\alpha^{i-1}}}-e^{-\frac{w}{\lambda_j\cdot\alpha^k}}\right)\right)\right)\right)\end{aligned} \tag{5.4.14}$$

$$Var(C)=c^2\left(\begin{array}{l}\displaystyle\sum_{n=1}^{\infty}n^2\left(\begin{array}{l}\displaystyle\prod_{j=1}^{q}\left(1-\sum_{k=1}^{n-1}\left(\sum_{i=1}^{k}\frac{C_{ijk}\cdot\lambda_j\cdot\alpha^k}{\lambda_j\cdot\alpha^{i-1}-\lambda_j\cdot\alpha^k}\left(e^{-\frac{w}{\lambda_j\cdot\alpha^{i-1}}}-e^{-\frac{w}{\lambda_j\cdot\alpha^k}}\right)\right)\right)\\ \displaystyle-\prod_{j=1}^{q}\left(1-\sum_{k=1}^{n}\left(\sum_{i=1}^{k}\frac{C_{ijk}\cdot\lambda_j\cdot\alpha^k}{\lambda_j\cdot\alpha^{i-1}-\lambda_j\cdot\alpha^k}\left(e^{-\frac{w}{\lambda_j\cdot\alpha^{i-1}}}-e^{-\frac{w}{\lambda_j\cdot\alpha^k}}\right)\right)\right)\end{array}\right)\\ \displaystyle-\left(\sum_{n=1}^{\infty}n\left(\begin{array}{l}\displaystyle\prod_{j=1}^{q}\left(1-\sum_{k=1}^{n-1}\left(\sum_{i=1}^{k}\frac{C_{ijk}\cdot\lambda_j\cdot\alpha^k}{\lambda_j\cdot\alpha^{i-1}-\lambda_j\cdot\alpha^k}\left(e^{-\frac{w}{\lambda_j\cdot\alpha^{i-1}}}-e^{-\frac{w}{\lambda_j\cdot\alpha^k}}\right)\right)\right)\\ \displaystyle-\prod_{j=1}^{q}\left(1-\sum_{k=1}^{n}\left(\sum_{i=1}^{k}\frac{C_{ijk}\cdot\lambda_j\cdot\alpha^k}{\lambda_j\cdot\alpha^{i-1}-\lambda_j\cdot\alpha^k}\left(e^{-\frac{w}{\lambda_j\cdot\alpha^{i-1}}}-e^{-\frac{w}{\lambda_j\cdot\alpha^k}}\right)\right)\right)\end{array}\right)\right)^2\end{array}\right) \tag{5.4.15}$$

Parallel system

In contrast with the series system where a single failure of a component will result in failure of the whole system, in the parallel system, for the system to fail operating all parts of the system must fail. Similar to the analysis of series systems, we can obtain the warranty cost function for parallel systems where there are *q* components connected in parallel as shown in Figure 5.4.2. We derive the parallel system's pmf, cdf and reliability functions, respectively. Then we obtain the expected warranty cost and variance of warranty cost.

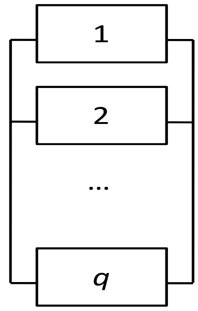


Figure 5.4.2 Parallel system with q components

Define $\Omega \equiv \{1, 2, \cdots, q\}$. Let $F(n)$ and $R(n)$ be the cdf and the reliability function of the r.v. inter-failure interval of parallel system with n failures, respectively. Then, for the parallel system shown in Figure 5.4.2, the cdf of a component j obtained from eq. (5.4.8). By [54], the parallel system reliability function $R(n)$ is same with $1-\prod_{j=1}^{q}\left(1-R_j(n)\right)$. Then, the parallel system reliability function is given by

$$
\begin{aligned}
R(n) &= 1-\prod_{j=1}^{q}\left(1-R_j(n)\right) \\
&= 1-\prod_{j=1}^{q}\left(\sum_{k=1}^{n}\left(\sum_{i=1}^{k}\frac{C_{ijk}\cdot\alpha^k}{\alpha^{i-1}-\alpha^k}\left(e^{-\frac{w}{\lambda_j\cdot\alpha^{i-1}}}-e^{-\frac{w}{\lambda_j\cdot\alpha^k}}\right)\right)\right)
\end{aligned}
\tag{5.4.16}
$$

Also we can obtain the pmf of the system and expected warranty cost and standard deviation of warranty cost.

$$
\begin{aligned}
P(N_s=n) &= P(N>n-1)-P(N>n) \\
&= R(n-1)-R(n) \\
&= 1-\prod_{j=1}^{q}\left(\sum_{k=1}^{n-1}\left(\sum_{i=1}^{k}\frac{C_{ijk}\cdot\alpha^k}{\alpha^{i-1}-\alpha^k}\left(e^{-\frac{w}{\lambda_j\cdot\alpha^{i-1}}}-e^{-\frac{w}{\lambda_j\cdot\alpha^k}}\right)\right)\right)-\left(1-\prod_{j=1}^{q}\left(\sum_{k=1}^{n}\left(\sum_{i=1}^{k}\frac{C_{ijk}\cdot\alpha^k}{\alpha^{i-1}-\alpha^k}\left(e^{-\frac{w}{\lambda_j\cdot\alpha^{i-1}}}-e^{-\frac{w}{\lambda_j\cdot\alpha^k}}\right)\right)\right)\right)
\end{aligned}
\tag{5.4.17}
$$

The first moment and the second moment of the number of system failures can be similarly obtained from eqs. (5.4.11) and (5.4.12) where $P(N_s=n)$ is given as eq. (5.4.17). Therefore, the variance of the number of system failures can be obtained using the first moment and the second moment. The expected warranty cost for parallel systems is given by

$$E(C) = cE(N) = c\sum_{n=1}^{\infty} n\left(\prod_{j=1}^{q}\left(\sum_{k=1}^{n}\left(\sum_{i=1}^{k}\frac{C_{ijk}\cdot\alpha^{k}}{\alpha^{i-1}-\alpha^{k}}\left(e^{-\frac{w}{\lambda_j\cdot\alpha^{i-1}}}-e^{-\frac{w}{\lambda_j\cdot\alpha^{k}}}\right)\right)\right)-\prod_{j=1}^{q}\left(\sum_{k=1}^{n-1}\left(\sum_{i=1}^{k}\frac{C_{ijk}\cdot\alpha^{k}}{\alpha^{i-1}-\alpha^{k}}\left(e^{-\frac{w}{\lambda_j\cdot\alpha^{i-1}}}-e^{-\frac{w}{\lambda_j\cdot\alpha^{k}}}\right)\right)\right)\right) \tag{5.4.18}$$

The variance of warranty cost is given by

$$\begin{aligned} Var(C) &= c^2\left(\begin{array}{l} \sum_{n=1}^{\infty} n^2\left(\begin{array}{l}\prod_{j=1}^{q}\left(\sum_{k=1}^{n}\left(\sum_{i=1}^{k}\frac{C_{ijk}\cdot\alpha^{k}}{\alpha^{i-1}-\alpha^{k}}\left(e^{-\frac{w}{\lambda_j\cdot\alpha^{i-1}}}-e^{-\frac{w}{\lambda_j\cdot\alpha^{k}}}\right)\right)\right) \\ -\prod_{j=1}^{q}\left(\sum_{k=1}^{n-1}\left(\sum_{i=1}^{k}\frac{C_{ijk}\cdot\alpha^{k}}{\alpha^{i-1}-\alpha^{k}}\left(e^{-\frac{w}{\lambda_j\cdot\alpha^{i-1}}}-e^{-\frac{w}{\lambda_j\cdot\alpha^{k}}}\right)\right)\right)\end{array}\right) \\ -\left(\sum_{n=1}^{\infty} n\left(\begin{array}{l}\prod_{j=1}^{q}\left(\sum_{k=1}^{n}\left(\sum_{i=1}^{k}\frac{C_{ijk}\cdot\alpha^{k}}{\alpha^{i-1}-\alpha^{k}}\left(e^{-\frac{w}{\lambda_j\cdot\alpha^{i-1}}}-e^{-\frac{w}{\lambda_j\cdot\alpha^{k}}}\right)\right)\right) \\ -\prod_{j=1}^{q}\left(\sum_{k=1}^{n-1}\left(\sum_{i=1}^{k}\frac{C_{ijk}\cdot\alpha^{k}}{\alpha^{i-1}-\alpha^{k}}\left(e^{-\frac{w}{\lambda_j\cdot\alpha^{i-1}}}-e^{-\frac{w}{\lambda_j\cdot\alpha^{k}}}\right)\right)\right)\end{array}\right)\right)^2 \end{array}\right) \end{aligned} \tag{5.4.19}$$

Parallel-series system

Consider a parallel-series system consisting of r parallel routes where each route has q units connected in series as shown in Figure 5.4.3. We derive the parallel-series system's pmf, cdf and reliability functions, then we obtain the expected warranty cost and variance of warranty cost.

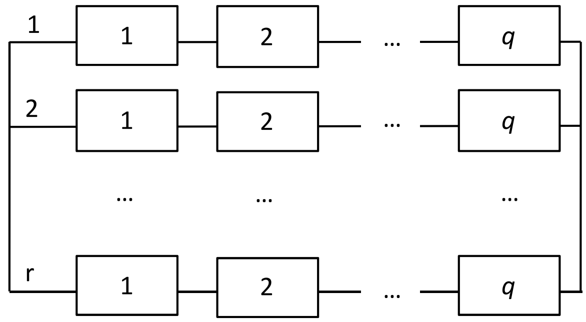


Figure 5.4.3 Parallel-series system with $r \cdot q$ components

Let $F_{jl}(n)$ and $R_{jl}(n)$ be the cdf and the reliability function of the inter-failure interval of component j in the row of l, respectively. Additionally, let C_{ijkl} be $\prod_{j \neq h} \frac{\frac{1}{\lambda_j}}{\frac{1}{\lambda_j} - \frac{1}{\lambda_h}}$ of i^{th} failure for component j in the l^{th} row with k failures. Then for the parallel-series system shown in Figure 5.4.3, the cdf of component j in the row of l is given by

$$F_{jl}(n) = \sum_{k=1}^{n} \left(\sum_{i=1}^{k} \frac{C_{ijkl} \cdot \alpha_{jl}^{k}}{\alpha_{jl}^{i-1} - \alpha_{jl}^{k}} \left(e^{-\frac{w}{\lambda_{jl} \cdot \alpha_{jl}^{i-1}}} - e^{-\frac{w}{\lambda_{jl} \cdot \alpha_{jl}^{k}}} \right) \right) \tag{5.4.20}$$

$$where\ j = 1, 2, \cdots q\ and\ l = 1, 2, \cdots r$$

The system reliability function is

$$\begin{aligned} R_s(n) &= 1 - \prod_{l=1}^{r} \left[1 - \prod_{j=1}^{q} \left(1 - F_{jl}(n) \right) \right] \\ &= 1 - \prod_{l=1}^{r} \left[1 - \prod_{j=1}^{q} \left(1 - \sum_{k=1}^{n} \left(\sum_{i=1}^{k} \frac{C_{ijkl} \cdot \alpha_{jl}^{k}}{\alpha_{jl}^{i-1} - \alpha_{jl}^{k}} \left(e^{-\frac{w}{\lambda_{jl} \cdot \alpha_{jl}^{i-1}}} - e^{-\frac{w}{\lambda_{jl} \cdot \alpha_{jl}^{k}}} \right) \right) \right) \right] \end{aligned} \tag{5.4.21}$$

Also, we can obtain the pmf of the system and expected warranty cost and standard deviation of warranty cost, respectively.

$$\begin{aligned}P(N=n)&=P(N>n-1)-P(N>n)\\&=R(n-1)-R(n)\\&=1-\prod_{l=1}^{r}\left[1-\prod_{j=1}^{q}\left(1-\sum_{k=1}^{n-1}\left(\sum_{i=1}^{k}\frac{C_{ijkl}\cdot\alpha_{jl}^{k}}{\alpha_{jl}^{i-1}-\alpha_{jl}^{k}}\left(e^{-\frac{w}{\lambda_{jl}\cdot\alpha_{jl}^{i-1}}}-e^{-\frac{w}{\lambda_{jl}\cdot\alpha_{jl}^{k}}}\right)\right)\right)\right]\\&\quad-\left(1-\prod_{l=1}^{r}\left[1-\prod_{j=1}^{q}\left(1-\sum_{k=1}^{n}\left(\sum_{i=1}^{k}\frac{C_{ijkl}\cdot\alpha_{jl}^{k}}{\alpha_{jl}^{i-1}-\alpha_{jl}^{k}}\left(e^{-\frac{w}{\lambda_{jl}\cdot\alpha_{jl}^{i-1}}}-e^{-\frac{w}{\lambda_{jl}\cdot\alpha_{jl}^{k}}}\right)\right)\right)\right]\right)\end{aligned}\tag{5.4.22}$$

Using the pmf of the system, the expected number of parallel-series system failures, $E(N)$ and the second moments of parallel-series system failures, $E(N^2)$ can be obtained. The expected warranty cost and variance of the system warranty cost for the parallel-series system are given as follows, respectively;

$$\begin{aligned}E(C)&=cE(N)\\&=c\sum_{n=1}^{\infty}n\left(\begin{array}{l}\prod_{l=1}^{r}\left[1-\prod_{j=1}^{q}\left(1-\sum_{k=1}^{n}\left(\sum_{i=1}^{k}\frac{C_{ijkl}\cdot\alpha_{jl}^{k}}{\alpha_{jl}^{i-1}-\alpha_{jl}^{k}}\left(e^{-\frac{w}{\lambda_{jl}\cdot\alpha_{jl}^{i-1}}}-e^{-\frac{w}{\lambda_{jl}\cdot\alpha_{jl}^{k}}}\right)\right)\right)\right]\\-\prod_{l=1}^{r}\left[1-\prod_{j=1}^{q}\left(1-\sum_{k=1}^{n-1}\left(\sum_{i=1}^{k}\frac{C_{ijkl}\cdot\alpha_{jl}^{k}}{\alpha_{jl}^{i-1}-\alpha_{jl}^{k}}\left(e^{-\frac{w}{\lambda_{jl}\cdot\alpha_{jl}^{i-1}}}-e^{-\frac{w}{\lambda_{jl}\cdot\alpha_{jl}^{k}}}\right)\right)\right)\right]\end{array}\right)\end{aligned}\tag{5.4.23}$$

$$Var(C)=c^2\left(\begin{array}{l}\sum_{n=1}^{\infty}n^2\left(\begin{array}{l}\prod_{l=1}^{r}\left[1-\prod_{j=1}^{q}\left(1-\sum_{k=1}^{n}\left(\sum_{i=1}^{k}\frac{C_{ijkl}\cdot\alpha_{jl}^{k}}{\alpha_{jl}^{i-1}-\alpha_{jl}^{k}}\left(e^{-\frac{w}{\lambda_{jl}\cdot\alpha_{jl}^{i-1}}}-e^{-\frac{w}{\lambda_{jl}\cdot\alpha_{jl}^{k}}}\right)\right)\right)\right]\\-\prod_{l=1}^{r}\left[1-\prod_{j=1}^{q}\left(1-\sum_{k=1}^{n-1}\left(\sum_{i=1}^{k}\frac{C_{ijkl}\cdot\alpha_{jl}^{k}}{\alpha_{jl}^{i-1}-\alpha_{jl}^{k}}\left(e^{-\frac{w}{\lambda_{jl}\cdot\alpha_{jl}^{i-1}}}-e^{-\frac{w}{\lambda_{jl}\cdot\alpha_{jl}^{k}}}\right)\right)\right)\right]\end{array}\right)\\-\left(\sum_{n=1}^{\infty}n\left(\begin{array}{l}\prod_{l=1}^{r}\left[1-\prod_{j=1}^{q}\left(1-\sum_{k=1}^{n}\left(\sum_{i=1}^{k}\frac{C_{ijkl}\cdot\alpha_{jl}^{k}}{\alpha_{jl}^{i-1}-\alpha_{jl}^{k}}\left(e^{-\frac{w}{\lambda_{jl}\cdot\alpha_{jl}^{i-1}}}-e^{-\frac{w}{\lambda_{jl}\cdot\alpha_{jl}^{k}}}\right)\right)\right)\right]\\-\prod_{l=1}^{r}\left[1-\prod_{j=1}^{q}\left(1-\sum_{k=1}^{n-1}\left(\sum_{i=1}^{k}\frac{C_{ijkl}\cdot\alpha_{jl}^{k}}{\alpha_{jl}^{i-1}-\alpha_{jl}^{k}}\left(e^{-\frac{w}{\lambda_{jl}\cdot\alpha_{jl}^{i-1}}}-e^{-\frac{w}{\lambda_{jl}\cdot\alpha_{jl}^{k}}}\right)\right)\right)\right]\end{array}\right)\right)^2\end{array}\right)\tag{5.4.24}$$

Similarly to the parallel-series systems, we can obtain the warranty cost functions for the series-parallel system.

5.5 Discussion

Implications for Research

We consider the expected value and variance of the warranty cost together. The expected warranty cost has been mainly investigated for warranty cost analysis. While expected warranty cost is a good measure on the overall cost of warranty, it provides little information of the risk contained in a warranty program. Therefore, it is not sufficient enough to express the data using only the expected values. Because we obtain the variance of warranty cost with the expected value, the result provides a more accurate cost analysis. Recently, many research [9, 64, 147, 148, 181] have been published on multi-component systems. Sarhan [147, 148] derived the reliability equivalence factors of multi-component systems including a series-parallel system and a parallel-series system. However, few researchers [9] have investigated the multi-component system under warranty. While several researchers have investigated simple systems under warranty, our proposed approach is to consider complicated systems as well as simple system to conduct the warranty cost analysis in detail.

Additionally, so far the studies in warranty literature mostly focus on “as good as new” repair scenario. Our study in this chapter aims to conduct the warranty cost analysis based on both the imperfect repair as well as perfect repair.

In summary, we develop warranty cost models for repairable systems from the stand point of both the manufacturer and the customer. They are very useful for warranty policy makers to make decisions regarding the warranty policy such as making warranty cost reserve, estimating the warranty period and calculating the warranty expenses.

Implications for Practice

The developed models in this chapter can be used in various ways. If the time for inter-failure interval can be an exponentially distributed random variable with parameter λ, our model can be applied for the manufacturing products such as electronic appliances and agricultural tractors, transportations such as airplanes and vehicles or power plant generators. If the warranty cost can be estimated accurately, then it makes critical role for both the manufacturing companies and the customers. Under the manufacturers' point of view, engineers can estimate the warranty cost and set up the warranty reserve approximately. Under the customers' point of view, the proposed model results can be used as a tool to help potential customers to select appropriate warranty options which are suitable for purchasing the products.

Samatli-Paç and Taner [146] suggest a real life example for their warranty cost model using a QRP. A leading beverage company runs its own repair facilities for the industrial refrigerators used in its retail outlets in Turkey. The company performs three different kinds of repairs on failed refrigerators. Data indicated that repair types 2 have average cost of 28 YTL (new Turkish Liras). Statistical analyses were performed on a particular brand and model of refrigerator with data on 2150 refrigerators. Of these, 285 failed at least once during the observation period. Maximum likelihood estimation for the repair type 2 of the 285 repaired refrigerators yields that the time between the first and second failures is Weibully distributed with the shape parameter of 1.02 and the scale parameter of 127.57 and α is 0.91.

If the shape parameter is closely near to 1, i.e. the Weibull distribution with shape parameter 1 and the scale parameter 127.57 is same as the exponential distribution with the exponential parameter 127.57. Similarly, it yields that the time until first failure closely follows an exponential distribution with a parameter of 158.24. Time limits of the region within which a repair will be attempted are set in view of the distribution that characterizes the time until first failure. In particular, the limits are set systematically first at around a conservative value of 1, 3 and 5 years. So we obtain expected warranty cost and the variance of the warranty cost based on these different years. Table 5.5.1 shows the expected warranty cost for one product (unit: new Turkish Liras) and the variance of the expected warranty cost under the repair type 2 with 1, 3 and 5 year warranty periods.

Table 5.5.1 Warranty cost analysis using Repair type 2 with $\lambda_1 = 158.24$, $\lambda_2 = 127.57$ and $\alpha = 0.91$

Warranty Period	Expected warranty cost	Standard Deviation	Coefficient of Variation
1	3.772	9.560	2.534
3	11.168	13.710	1.228
5	18.367	13.301	0.724

These real-world benchmarking examples showed how the proposed model can be used for the warranty cost analysis. The next section covers for a various choices of parameters in terms of several systems including multi-component systems.

5.6 Numerical Example and Sensitivity Analysis

In this section, two numerical examples are presented to illustrate the analyses of system cost functions. We first consider a single component system, and then a parallel-series system is investigated. For the numerical examples, because the computations have the infinite sum to obtain the expected number of failures and their variances, a numerical method is used in the calculation. The approximate numerical values are obtained using the *Mathematica*. We sample a certain number of terms in the series, and then run an extrapolation to estimate the contributions of other terms. There are two approaches to estimating this contribution. The first uses the Euler-Maclaurin method, and is based on approximating the sum by an integral. The second method, known as the Wynn epsilon method, samples a number of additional terms in the sum, and then tries to fit them to a polynomial multiplied by a decaying exponential [183].

Case I.

Consider a single component system which is the simplest case. We derive the expected warranty cost (E(C)), standard deviation of warranty cost (SD(C)) and coefficient of variation (CV) for a 20 warranty period units which start at 0.1 and finish at 2.0. We change the λ values as 1 and 0.5, and α values as 0.99, 0.95 and 0.90. We assume that α is less than 1 because the repair is imperfect. Other assumptions used are as followings;

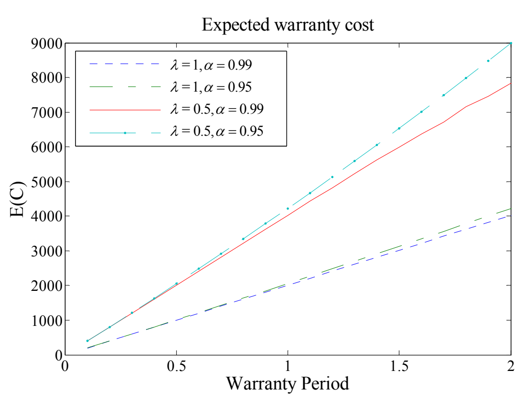
- The warranty cost per a failure is c = $2000
- There are *n* failures for each component.
- The given warranty period unit is fixed period j for each component;

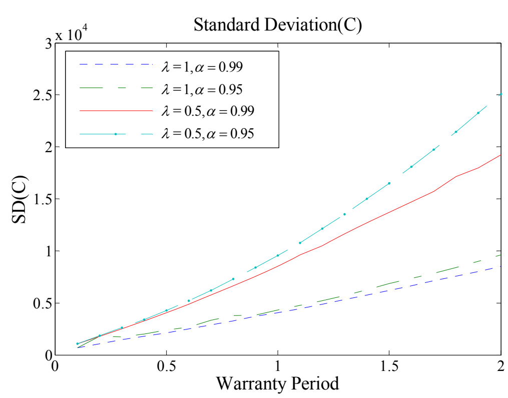
  $j \in \{0.1, 0.2, 0.3, 0.4, \cdots, 1.8, 1.9, 2.0\}$

Using eqs. (5.4.6) and (5.4.7) we obtain the expected warranty system cost and variance of warranty system cost. The expected warranty cost of single component system plots with λ =1 and λ = 0.5 are in Figure 5.6.1. For each case, α s have three different values 0.99, 0.95 and 0.90. As the warranty periods are longer, the expected warranty cost increases. If λ s are equal, the expected warranty cost would increase when they have smaller α. Smaller α means that the next failure time would come faster and there are more failed components. So, the cost would increase as more repairing service is needed. Also, as λ increases, the expected cost decreases. The variance and standard deviation of warranty cost in Figure 5.6.1 shows they increase as warranty time goes on. We see that the pattern of the standard deviation is similar to the expected cost patterns by changing α and λ. Using standard deviation and expected values, we obtain the CV in Figure 5.6.1. The CV shows that they have high variance and the plots are unstable.

Table 5.6.1 Cost analysis for single component system

	α	W	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\lambda = 1$	$\alpha = 0.99$	E(C)	198	400	601	802	1003	1204	1405	1606	1811	2010
		SD(C)	693	1080	1432	1781	2136	2498	2871	3249	3692	4046
		CV	3.497	2.697	2.382	2.221	2.131	2.076	2.043	2.023	2.038	2.013
	$\alpha = 0.95$	E(C)	201	403	605	809	1013	1219	1427	1636	1844	2055
		SD(C)	702	1786	1719	2007	2406	2687	3307	3751	3837	4340
		CV	3.500	4.432	2.842	2.482	2.374	2.204	2.318	2.293	2.081	2.112
	$\alpha = 0.90$	E(C)	201	405	610	818	1029	1242	1458	1676	1897	2121
		SD(C)	709	1112	1500	1894	2302	2729	3176	3644	4135	4649
		CV	3.524	2.750	2.458	2.314	2.238	2.197	2.179	2.174	2.180	2.192
$\lambda = 0.5$	$\alpha = 0.99$	E(C)	400	802	1204	1606	2010	2413	2818	3220	3624	4021
		SD(C)	1080	1781	2498	3249	4046	4869	5749	6643	7591	8522
		CV	2.697	2.221	2.076	2.023	2.013	2.018	2.040	2.063	2.094	2.119
	$\alpha = 0.95$	E(C)	402	809	1219	1635	2055	2479	2909	3343	3782	4225
		SD(C)	1092	1827	2591	3407	4282	5217	6213	7271	8387	9560
		CV	2.715	2.260	2.125	2.084	2.084	2.104	2.136	2.175	2.218	2.263
	$\alpha = 0.90$	E(C)	405	818	1242	1676	2121	2577	3045	3526	4021	4530
		SD(C)	1112	1894	2729	3644	4649	5752	6959	8275	9709	11269
		CV	2.750	2.314	2.197	2.174	2.192	2.232	2.285	2.347	2.415	2.487





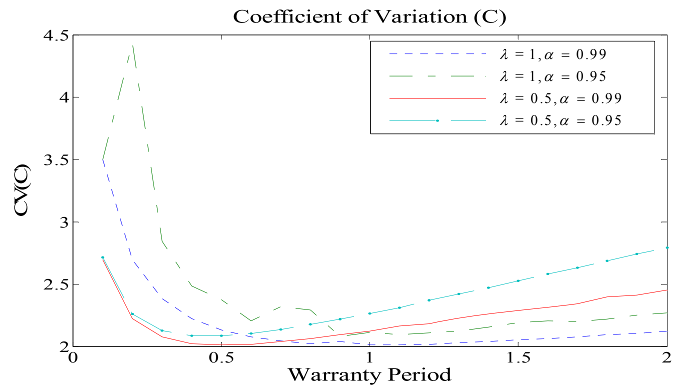


Figure 5.6.1 E(C), Standard deviation and Coefficient of Variation for the single system

Case II.

We consider the parallel-series system with four components shown in Figure 5.6.2 under the fixed warranty period policy assuming that for component *i, i*=1,2,3,4, the warranty period and λs are the same only in each interval, but λs between intervals are different.

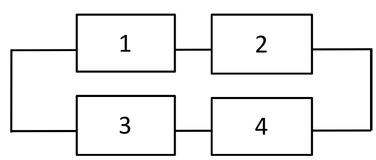


Figure 5.6.2 Parallel-series system with four components

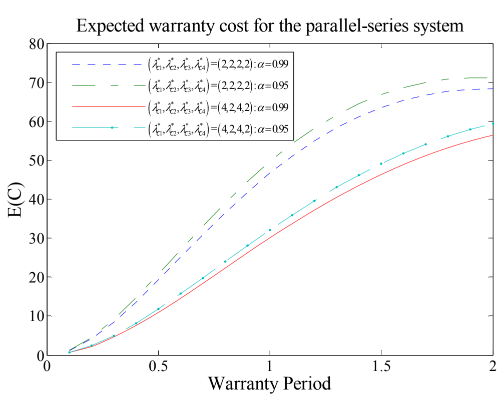
Let $\lambda^*_{Ci}, i=1,2,3,4$ be an exponential parameter for component *i*. In Case II, we use three cases, when $\left(\lambda^*_{C1},\lambda^*_{C2},\lambda^*_{C3},\lambda^*_{C4}\right)=\left(2,2,2,2\right),\left(1,2,3,4\right)$ and $\left(4,2,4,2\right)$. We vary α values as 0.99, 0.95 and 0.90. We assume that α is less than 1 because the repair is imperfect. Other assumptions used are as followings;

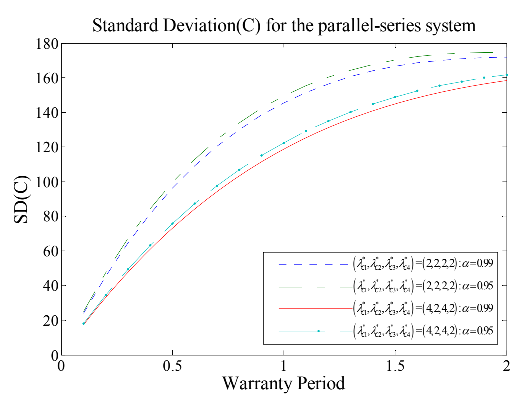
- The warranty cost per a failure is c=$500
- There are *n* failures for each component.
- The given warranty period unit is fixed period *j* for each component;
  $j \in \{0.1, 0.2, 0.3, 0.4, \cdots, 1.8, 1.9, 2.0\}$

Using eqs. (5.4.23) and (5.4.24), the cost analyses are shown in Table 5.6.2. The expected warranty costs have similar patterns as single component system's E(C). The coefficient of variation in Figure 5.6.3 shows that the data are stable.

Table 5.6.2 Cost analysis for the parallel-series system

$(\lambda^*_{C1},\lambda^*_{C2},\lambda^*_{C3},\lambda^*_{C4})$	α	W	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
(2,2, 2,2)	0.99	E(C)	6.91	24.93	50.51	80.79	113.51	146.94	179.81	211.16	240.38	267.02
		SD(C)	101.59	191.77	270.58	338.61	396.71	445.90	487.22	521.69	550.26	573.79
		CV	14.70	7.69	5.36	4.19	3.49	3.03	2.71	2.47	2.29	2.15
	0.95	E(C)	7.49	26.95	54.47	86.90	121.79	157.27	191.96	224.88	255.38	283.01
		SD(C)	105.73	199.25	280.60	350.43	409.69	459.53	501.09	535.49	563.78	586.88
		CV	14.12	7.39	5.15	4.03	3.36	2.92	2.61	2.38	2.21	2.07
	0.9	E(C)	8.32	29.84	60.10	95.56	133.46	171.74	208.91	243.92	276.09	304.99
		SD(C)	111.41	209.45	294.18	366.34	427.05	477.61	519.34	553.52	581.30	603.71
		CV	13.39	7.02	4.89	3.83	3.20	2.78	2.49	2.27	2.11	1.98
(4,2, 4,2)	0.99	E(C)	3.55	13.13	27.34	44.94	64.90	86.35	108.57	130.99	153.14	174.65
		SD(C)	72.85	139.74	200.66	255.72	305.18	349.38	388.68	423.47	454.15	481.11
		CV	20.54	10.64	7.34	5.69	4.70	4.05	3.58	3.23	2.97	2.75
	0.95	E(C)	3.84	14.22	29.54	48.47	69.86	92.78	116.44	140.22	163.63	186.27
		SD(C)	75.84	145.33	208.41	265.24	316.09	361.33	401.37	436.66	467.62	494.68
		CV	19.73	10.22	7.06	5.47	4.52	3.89	3.45	3.11	2.86	2.66
	0.9	E(C)	4.27	15.77	32.68	53.48	76.90	101.86	127.52	153.17	178.29	202.46
		SD(C)	79.96	152.97	218.98	278.15	330.81	377.38	418.35	454.20	485.44	512.54
		CV	18.71	9.70	6.70	5.20	4.30	3.70	3.28	2.97	2.72	2.53
(4,4, 4,4)	0.99	E(C)	1.82	6.91	14.77	24.93	36.97	50.51	65.22	80.79	96.96	113.51
		SD(C)	52.20	101.59	148.12	191.77	232.57	270.58	305.89	338.61	368.84	396.71
		CV	28.70	14.70	10.03	7.69	6.29	5.36	4.69	4.19	3.80	3.49
	0.95	E(C)	1.97	7.49	15.99	26.95	39.92	54.47	70.24	86.90	104.17	121.79
		SD(C)	54.37	105.73	154.03	199.25	241.41	280.60	316.90	350.43	381.31	409.69
		CV	27.55	14.12	9.63	7.39	6.05	5.15	4.51	4.03	3.66	3.36
	0.9	E(C)	2.20	8.32	17.73	29.84	44.12	60.10	77.37	95.56	114.34	133.46
		SD(C)	57.34	111.41	162.11	209.45	253.44	294.18	331.76	366.34	398.05	427.05
		CV	26.12	13.39	9.14	7.02	5.74	4.89	4.29	3.83	3.48	3.20





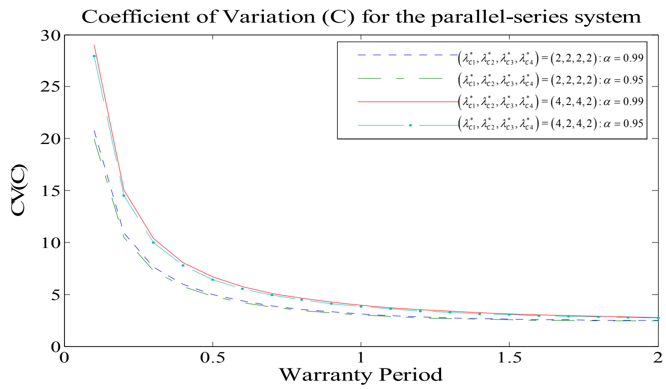


Figure 5.6.3 E(C), Standard deviation and coefficient of variation for the parallel-series system

5.7 Concluding Remarks

This research focuses on the cost analysis including the repairable models with a fixed warranty period. Not much research has been done using QRP, especially on warranty cost analysis although Wang and Pham [175] introduced the QRP in 1996. Based on the property of exponential distribution which is frequently used for the modeling, we develop the warranty cost model which is applicable in real life. We show a real-world benchmarking example for Section 5.5 and other computational experimentation for Section 5.6. The proposed approach is helpful for the manufacturer to estimate the warranty cost accurately.

Based on QRP and exponential distribution, we obtained the expected warranty cost, standard deviation of the warranty cost and coefficient of variation changing several parameters such as λ and α. λ is used in the exponential distribution and α s are used in the QRP. Cost analysis was used to draw figures and compare several warranty costs and their standard deviation with different parameters. Expected warranty costs are generally increased as the warranty periods go on. As α increases, the expected warranty cost decreases. Also with the increase of λ s, the expected warranty costs are decreased too.

Chapter 6

A Generalized Block Replacement Policy for a *k-out-of-n* System with Respect to Threshold Number of Failed Components and Risk Cost

6.1 Introduction

The objective of the maintenance policy is to find ways to minimize the expected total system costs or maximize the product's reliability. For a complicated product that deteriorating in function, it is essential to perform the preventive maintenance service to achieve the systems' satisfactory reliability performance. Maintenance involves planned or unplanned actions carried out to retain or to restore the system to an acceptable operating condition. Planned maintenance is usually referred as preventive maintenance (PM) while unplanned maintenance is corrective maintenance (CM) or repair. Furthermore, two well-known PM policies are block replacement policy (BRP) and age replacement policy (ARP) from Barlow and Hunter [13]. For the BRP, an operating system is replaced with a new system periodically and at failures. The strength of the BRP is its simplicity because we don't need to keep detailed records on the failure times. Nonetheless the weakness is that despite the product being practically new, we can replace at planned replacement times. In the classical BRP [14, 15] for a group of operating units, each unit is individually replaced on failure or they are replaced preventively at a fixed maintenance cycle τ. But such a policy is somewhat wasteful if a unit which is almost new are replaced as a block replacement. Many researchers [16-18, 84, 108, 114, 166] have developed modified block replacement policies (MBRP) based on classical BRP [13-15] and have tried to overcome several drawbacks [115] including:

- When a failure occurs just before the maintenance cycle τ, it left idle until τ without the failure replacement (FR) [19, 47, 49, 114, 115];
- When a failure occurs just before the maintenance cycle τ, it is replaced with an used one [18, 84, 108, 166];
- A young operating unit is not replaced at the maintenance cycle τ and remains in service [4, 16, 17].

Berg and Epstein [17] study a MBRP which overcomes the drawback at planned replacement times where one may be replacing a practically new item. Archibald and Dekker [4] extend the MBRP of Berg and Epstein [17] and consider a discrete time frame work for multi-component systems. Nakagawa [114, 115] proposes the MBRP with an idle period, where units are replaced at failures until a fixed time and then follows an idle period, during which failed units are left idle. There are two extreme examples under this MBRP: All units may fail and remain idle during the idle period, which is undesirable; all units may be operating but will be replacing them all during the preventive maintenance period. Park and Yoo [119] try to overcome these extreme cases and propose another MBRP where a block replacement is performed at a certain number of failures, counting after the pre-determined individual failure-replacement interval $(0, \tau]$. Despite both Nakagawa and Park & Yoo's effort to overcome the classical BRP's drawbacks, they were not successful to overcome completely as they assume that units are replaced at failure until a fixed time. In Park and Yoo's policy [119], their model has certain period that uses the classical BRP, meaning in some parts their policy has the classical policy's properties.

Multi-component systems are more realistic when its compare to a single-component system in real life. However, not many researchers [168] have studied the MBRP for multi-component system. Archibald and Dekker [4] develop the MBRP for multi-component systems. The surveys of maintenance models for multi-component systems have been accomplished [37, 168]. Pham and Wang [130] study the opportunistic maintenance of a *k-out-of-n* system with imperfect PM and minimal repair where partial failure is allowed. They develop opportunistic maintenance models with two decision variables subject to reliability requirements. Recently Park and Pham [29] develop cost models based on the quasi-renewal processes for multi-component systems. Li and Pham [92] develop optimal inspection-maintenance policies, consisting of the time sequence for inspection & preventive maintenance threshold levels for both degradation processes. They assumed that the system failure is only detected by individual inspection. However, in several other papers [130, 176], researchers assumed that the system failure is monitored and detected immediately.

In this chapter, we develop a generalized block replacement policy (GBRP) for a *k-out-of-n* system with respect to a threshold level for the number of failed components (*m*) and risk costs where m < (n-k+1). If there are less than *m* numbers of failures in the maintenance cycle τ, then the failure replacement (FR) services for failed units and the preventive replacement (PR) services for other deteriorating units will be provided at the end of the periodic time $k \cdot \tau,\ (k = 1, 2 \cdots)$. If there are *m* numbers of failures, then the failure replacement services will be provided immediately at that point. Note that the *k-out-of-n* system functions if and only if at least k of the components must function. In other words, the system will not fail as long as the number of failed components would be less than (n-k+1). Although there is no need to do anything when there are exactly *m* numbers of failures since the system still hasn't failed, we wouldn't want to take a risk to wait without any FR action if the number of failed components would be close to a critical upper threshold level (n-k+1).

The remainder of this chapter is organized as follows. Section 6.2 explains the structure of the GBRP. Section 6.3, we obtain the expected cost rates of the proposed policies. The downtime period is calculated by the life time distribution and the age distribution. We discuss several numerical examples in Section 6.4. Concluding remarks are discussed in Section 6.5.

6.1.1 Nomenclature

ECR(t) : Expected Cost Rate based on time t

c_1 : failure replacement cost

c_2 : risk cost per unit time

c_3 : PR cost

c_4 : system failure cost

N_τ : total number of failures in a *k-out-of-n* system during $[0,\ \tau]$

T_i : time to failure of the i^{th} component in a *k-out-of-n* system, $i = 1, 2, \cdots, n$

L : number of failed components after the last FR service until the next maintenance cycle

$N(t)$: total number of failures for a component during $[0,\ t]$

m : threshold level for failed components

n : number of components in the system

$f(t), F(t)$: pdf, cdf of time to failure of a component, respectively

$R(t)$: reliability function of a component

$M(t)$: $E\left[N(t)\right]$, renewal function associated with $F(t)$

$m(t)$: $\dfrac{dM(t)}{dt}$, intensity function

$U^{(i)}(n)$: i^{th} order statistics of $(T_1, T_2, \cdots, T_n)$ for a *k-out-of-n* system

$Z(t)$: time from *t* since the last renewal, i.e. the age of the component in use at time *t*

$\lfloor\ \rfloor$: integer part of the number

6.1.2 Assumptions

- Failure replacement times and preventive replacement times are negligible.
- Component are i.i.d.
- The FR cost for a failed component and the PR cost are costlier than the risk cost per unit time.
- The FR cost, the PR cost, the risk cost and the system failure cost are constant.
- If a failed component is not replaced at the moment and left idle condition by the next maintenance cycle, it incurs the risk cost based on its idle period.

6.2 Generalized Block Replacement Policy

The threshold level of failures which is set up to prevent the inefficiency and the failure of the system is denoted as m. When the total numbers of failures is less than the threshold number of failed components, *i.e.* $N_\tau < m$, the policy is described in Figure 6.2.1. When the total numbers of failures is greater than or equal to the threshold number of failed units, *i.e.* $N_\tau \geq m$, the total numbers of failures, N_τ, could be composed of several sets of m numbers of failures and remaining left idle failed units, L, by the periodic maintenance time τ as described in Figure 6.2.2. We provide the PR service for the deteriorating units and the FR service for the failed units at times $k \cdot \tau$ $(k = 1, 2, \cdots)$. The failed unit is not replaced at the moment, and it remains in failed state during the time interval from an individual failure to m number of failures jointly. This can be applied to the maintenance model where a component is monitored continuously, and its failures can be detected immediately when a failure happens. The company can save the maintenance expense by using this policy. We separate two cases: $N_\tau < m$ and $N_\tau \geq m$. In case of $N_\tau < m$, there is no failure replacement by time τ and failed units would be left idle to decrease the maintenance cost by time τ in Figure 6.2.1. After that, when it is the periodic maintenance cycle τ, we provide the FR service and the PR service. If the number of failed units does not reach m, we don't need to provide the FR service during the maintenance period. But at the time τ, we provide the FR services for failed units and the PR services for deteriorating units. The circles represent units' failures and dotted red lines represent the idle period in Figure 6.2.1.

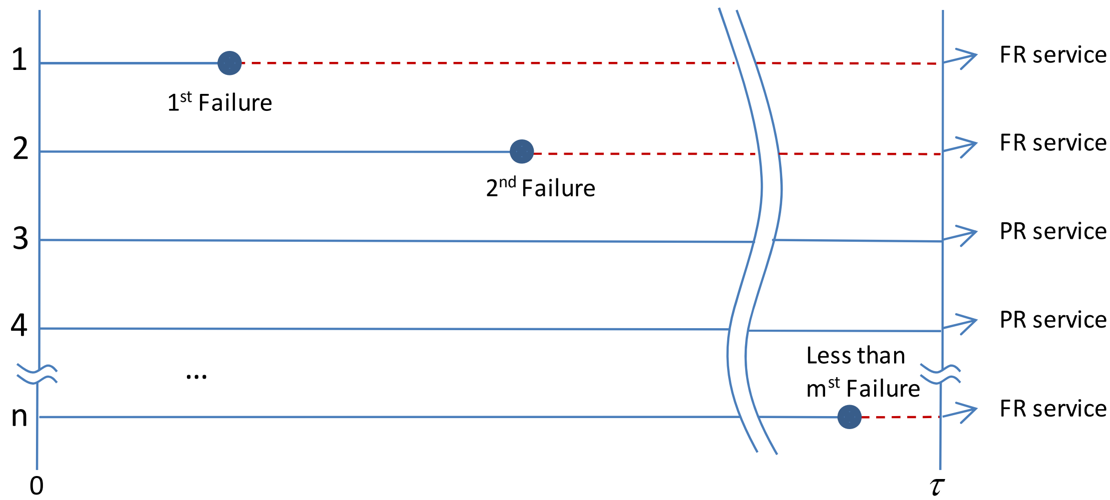


Figure 6.2.1: A *k-out-of-n* system with the number of failed components is less than *m,* $N_\tau < m$

When N_τ is greater than or equal to m, we provide the failure replacements at the moment. During one maintenance cycle τ, there may be more than one failure replacement. It means that there can be several sets of *m* numbers of failures. This model for a *k-out-of-n* system with a threshold level *m* is described in Figure 6.2.2. In Figure 6.2.2, there are *n* numbers of components and periodical maintenance cycles $k \cdot \tau$, $(k = 1, 2, \cdots)$. When m^{th} failure occurs, the FR services for failed components are provided. If we set up the threshold level *m* regarding the number of failures, then FR service for every failed component at the time of m^{th} failed components are provided. At each maintenance cycles $k \cdot \tau$, $(k = 1, 2, \cdots)$, the FR services for failed components and the PR services for other deteriorating components periodically are provided.

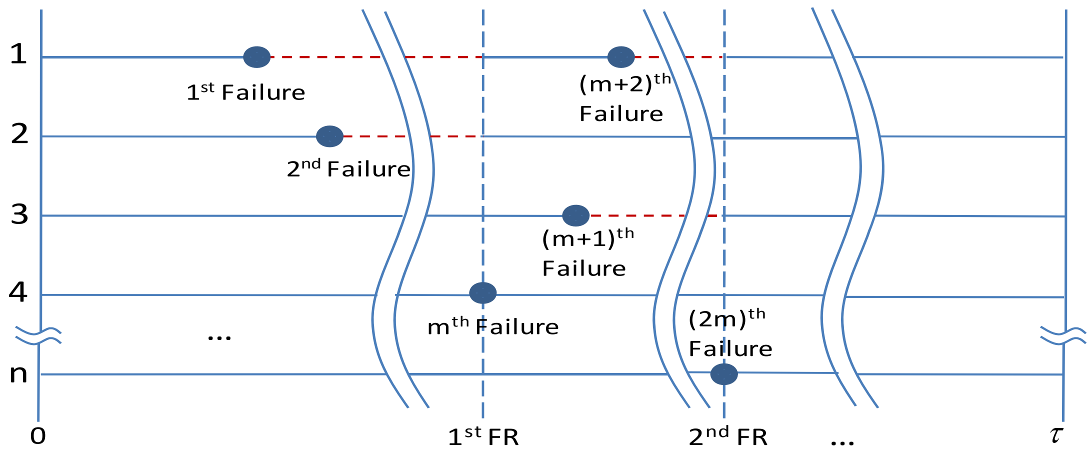


Figure 6.2.2: A *k-out-of-n* system when the number of failed components is larger than and equal to *m*, $N_\tau \geq m$.

6.3 Expected Cost Rates

Nakagawa [114, 115] propose the idle period to save the maintenance cost and Park and Yoo [119] use the idle period and suggest the idle count for each unit. They obtain the expected cost rate and compare their MBRPs to other existing MBRP and classical BRP. In their policies, units are replaced at failure until a fixed time T and then follow an idle period during which failed units are left idle. Compared to those existing MBRPs, we do not consider the first fixed time before the idle period. Instead, we consider the idle count for each component for *k-out-of-n* systems. Let $N(\tau)$ be the total number of failures for one unit during $[0,\ \tau]$. Let *m, n* and L be the threshold number of failed components, total component number in the system and the number of failed components that remain idle until the next maintenance, respectively. The number of failed components, N_τ, is given by

$$N_\tau = n \cdot N(\tau) = m \cdot \left\lfloor \frac{n \cdot N(\tau)}{m} \right\rfloor + L \tag{6.3.1}$$

When the total number of failed components is less than *m*, we provide the FR services for the failed components and the PR services for other deteriorating components at the maintenance cycle τ. As for the expected cost, let c_1 *and* c_2 be a FR cost and a risk cost per unit time, respectively. If the number of failures, N_τ is less than a threshold level of component failures, *m*, then the total failure replacement cost is $c_1 \cdot N_\tau$. These FR services take place only at the time of τ because N_τ is not larger than *m* (see Figure 6.2.1). The expected FR cost is given by $c_1 \cdot E(N_\tau) = c_1 \cdot n \cdot M(\tau)$. Also, the number of non-failed components are $n - N_\tau$. The PR cost, c_3 is multiplied by the number of non-failed components at the maintenance cycle τ. So their PR cost is $c_3(n - N_\tau)$, and the corresponding expected cost is given by $c_3 \cdot E(n - N_\tau) = c_3(n(1 - M(\tau)))$.

The risk cost can be derived by using the order statistics based on either the life time distributions or age distributions. For the *k-out-of-n* systems, even though there are *m* failures which are less than $n - k + 1$, the system still would not fail. Despite this we consider the risk cost because we believe that if some parts of the system are not functioning, it would make it less efficient and likely incur cost. If a machine in a factory, for example, is not properly running and would need to be repaired, it will also incur cost during the repair period with a less productivity. In this chapter, we consider the risk cost for failed components of *k-out-of-n* systems. To obtain the downtime period, we consider $F(t)$ and $Z(t)$. $F(t)$ is the cdf of the life time distribution and $Z(t)$ is the time to *t* since the last renewal.

Downtime period using the life time distribution, $F(t)$

The r.v. $U^{(i)}(n)$ is the i^{th} order statistics of n independent random samples for $i = 1, 2, \cdots, n$ that represents the time to i^{th} failure of n components while $(i-1)^{th}$ failed components are left idle. The survival function of $U^{(i)}(n)$ can be obtained [119]:

$$P\left\{U^{(i)}(n) > t\right\} = \sum_{j=0}^{i-1} \binom{n}{j} \left[F(t)\right]^j \left[1 - F(t)\right]^{n-j} \tag{6.3.2}$$

Let i be the number of failed components. Then from eq. (6.3.2), the mean time to the i^{th} failure $E\left\{U^{(i)}(n)\right\}$ can be obtained as follows:

$$\begin{aligned} E\left\{U^{(i)}(n)\right\} &= \int_0^\infty P\left\{U^{(i)}(n) > t\right\} dt \\ &= \int_0^\infty \left(\sum_{j=0}^{i-1} \binom{n}{j} \left[F(t)\right]^j \left[1 - F(t)\right]^{n-j} \right) dt \end{aligned} \tag{6.3.3}$$

The total risk cost can also be obtained

$$c_2 \cdot \sum_{j=1}^{N_\tau} \left(\tau - U^{(j)}(n) \right) \tag{6.3.4}$$

On the other hand, when the number of failures is larger than or equal to the threshold level of component failures, we provide the FR service for the failed components at the time when m^{th} failure happens. In Figure 6.2.2, there may be more than just one failure replacement service during one maintenance cycle, τ. Therefore, total cost is composed of a FR cost which multiplied by total number of failures for a *k-out-of-n* system, a risk cost which multiplied by idle times and a PR cost which multiplied by total number of non-failed components. Total FR cost is obtained by the FR cost, c_1, multiplied by the number of failed components during the maintenance cycle $(0,\ \tau]$. The risk cost per unit time, c_2, is multiplied by the downtime period. The downtime period composed of a certain number of failure replacement intervals and the last interval. There are $\left\lfloor \frac{N_\tau}{m} \right\rfloor$ number of failure replacement intervals. In a failure replacement interval, there are $m-1$ number of failed components because they provide the failure replacement service when the m^{th} component fails. Therefore, the downtime period excluding the last interval is given by

$$\sum_{i=1}^{\left\lfloor \frac{N_\tau}{m} \right\rfloor} \sum_{j=1}^{m-1} \left(U^{(i \cdot m)}(n) - U^{(j+m(i-1))}(n) \right) \tag{6.3.5}$$

The downtime period in the last interval is given by

$$\sum_{j=\left\lfloor \frac{N_\tau}{m} \right\rfloor \cdot m+1}^{N_\tau} \left(\tau - U^{(j)}(n) \right) \tag{6.3.6}$$

When we use the life time distribution and $N_\tau < m$, the expected downtime period is given by

$$\begin{aligned} E\left(D_{N_\tau < m}\right) &= E\left(\sum_{j=1}^{N_\tau} \left(\tau - U^{(j)}(n) \right) \right) \\ &= \tau \cdot n \cdot M(\tau) - \int_0^\infty \left(\sum_{n_\tau=1}^{\infty} \frac{1}{n_\tau} \sum_{j=1}^{n_\tau} \sum_{k=0}^{j-1} \binom{n}{k} \left[F(t)\right]^k \left[1-F(t)\right]^{n-k} \right) dt \end{aligned} \tag{6.3.7}$$

When the system is monitored continuously and $N_\tau < m$, using $U^{(j)}(n)$, the expected downtime period is obtained by

$$
\begin{aligned}
E\left[\sum_{j=1}^{N_\tau}\left(\tau-U^{(j)}(n)\right)\right] &= \tau\cdot E\left[N_\tau\right]-E\left[\sum_{j=1}^{N_\tau}U^{(j)}(n)\right] \\
&= \tau\cdot n\cdot M(\tau)-E\left[E\sum_{j=1}^{N_\tau}U^{(j)}(n)\middle|N_\tau=n_\tau\right] \\
&= \tau\cdot n\cdot M(\tau)-\sum_{n_\tau=1}^{\infty}E\left[\sum_{j=1}^{N_\tau}U^{(j)}(n)\middle|N_\tau=n_\tau\right]\cdot P\left[N_\tau=n_\tau\right] \\
&= \tau\cdot n\cdot M(\tau)-\sum_{n_\tau=1}^{\infty}\frac{1}{n_\tau}\sum_{j=1}^{n_\tau}E\left\{U^{(j)}(n)\right\} \\
&= \tau\cdot n\cdot M(\tau)-\int_0^\infty\left(\sum_{n_\tau=1}^{\infty}\frac{1}{n_\tau}\sum_{j=1}^{n_\tau}\sum_{k=0}^{j-1}\binom{n}{k}\left[F(t)\right]^k\left[1-F(t)\right]^{n-k}\right)dt
\end{aligned}
$$

Also when $N_\tau \geq m,$ the expected downtime period is given by

$$
\begin{aligned}
E\left(D_{N_\tau\geq m}\right) &= E\left(\sum_{i=1}^{\left\lfloor \frac{N_\tau}{m}\right\rfloor}\sum_{j=1}^{m-1}\left(U^{(i\cdot m)}(n)-U^{(j+m(i-1))}(n)\right)\right)+E\left(\sum_{j=\left\lfloor \frac{N_\tau}{m}\right\rfloor\cdot m+1}^{N_\tau}\left(\tau-U^{(j)}(n)\right)\right) \\
&= \tau(m-1)\left(\int_0^\infty\left(\sum_{n_\tau=1}^{\infty}\frac{1}{n_\tau}\sum_{i=1}^{\left\lfloor \frac{n_\tau}{m}\right\rfloor}\sum_{k=0}^{i\cdot m-1}\binom{n}{k}\left[F(t)\right]^k\left[1-F(t)\right]^{n-k}\right)dt\right) \\
&\quad -\sum_{j=1}^{m-1}\left(\int_0^\infty\left(\sum_{n_\tau=1}^{\infty}\frac{1}{n_\tau}\sum_{i=1}^{\left\lfloor \frac{n_\tau}{m}\right\rfloor}\sum_{k=0}^{m(i-1)+j-1}\binom{n}{k}\left[F(t)\right]^k\left[1-F(t)\right]^{n-k}\right)dt\right) \\
&\quad +\tau\cdot\left(n\cdot M(\tau)-\left\lfloor\frac{n\cdot M(\tau)}{m}\right\rfloor\cdot m\right)-\int_0^\infty\left(\sum_{n_\tau=1}^{\infty}\frac{1}{n_\tau}\sum_{j=\left\lfloor \frac{n_\tau}{m}\right\rfloor\cdot m+1}^{n_\tau}\sum_{k=0}^{j-1}\binom{n}{k}\left[F(t)\right]^k\left[1-F(t)\right]^{n-k}\right)dt
\end{aligned}
\tag{6.3.8}
$$

Downtime period using the age distribution, $Z(t)$

Let $Z(t)$ be the time from t since the last renewal. This is called the age distribution at t [141]. Then the function $H(t)$ is given by

$$H(t) = P\{Z(t) \le x\} = \begin{cases} F(t) - \int_0^{t-x} \left[1 - F(t-y)\right] dm(y) & x \le t \\ 1 & x > t \end{cases} \tag{6.3.9}$$

The r.v. $Q^{(i)}(n)$, is the i^{th} order statistics of n independent random samples for $i = 1, 2, \cdots, n$ that represents the time from i^{th} failure to τ while $(i-1)^{th}$ failed components are left idle [119]. Similarly, the survival function of $Q^{(i)}(n)$, can be obtained

$$P\left\{Q^{(i)}(n) > t\right\} = \sum_{j=0}^{i-1} \binom{n}{j} \left[H(t)\right]^j \left[1 - H(t)\right]^{n-j} \tag{6.3.10}$$

Let i be the number of failed components. From eq. (6.3.10), the mean time from failure i to time t, $E\left\{Q^{(i)}(n)\right\}$ is given by

$$\begin{aligned} E\left\{Q^{(i)}(n)\right\} &= \int_0^\infty P\left\{Q^{(i)}(n) > t\right\} dt \\ &= \int_0^\infty \left(\sum_{j=0}^{i-1} \binom{n}{j} \left[H(t)\right]^j \left[1 - H(t)\right]^{n-j} \right) dt \end{aligned} \tag{6.3.11}$$

If the number of failures, N_τ, is less than a threshold level of component failures, m, the failure replacement services then take place at the time of τ. To obtain the downtime period, we add each failed component idle period. So the total downtime period is given by

$$\sum_{j=1}^{N_\tau} \left(Q^{(j)}(n) \right) \tag{6.3.12}$$

On the other hand, if the number of failures, N_τ, is greater than a threshold level of failures, m, the failure replacement services may take place during the maintenance cycle. The following first term in eq. (6.3.13) covers the downtime period excluding the last period using the age distribution. The second term covers the last interval's downtime period. When $N_\tau \geq m$, the total downtime period is as follows:

$$\sum_{i=1}^{\left\lfloor \frac{N_\tau}{m} \right\rfloor} \sum_{j=1}^{m-1} \left(Q^{(j+m(i-1))}(n) \right) + \sum_{j=\left\lfloor \frac{N_\tau}{m} \right\rfloor \cdot m+1}^{N_\tau} \left(Q^{(j)}(n) \right) \tag{6.3.13}$$

The downtime period could be obtained using the age distribution for $N_\tau < m$ and $N_\tau \geq m$. When $N_\tau < m$, the expected downtime period is given by

$$\begin{aligned}
E\left(D_{N_\tau < m}\right) &= E\left(\sum_{j=1}^{N_\tau} \left(Q^{(j)}(n) \right) \right) \\
&= E\left[E \sum_{j=1}^{N_\tau} Q^{(j)}(n) \middle| N_\tau = n_\tau \right] \\
&= \sum_{n_\tau=1}^{\infty} E\left[\sum_{j=1}^{N_\tau} Q^{(j)}(n) \middle| N_\tau = n_\tau \right] \cdot P\left[N_\tau = n_\tau\right] \\
&= \sum_{n_\tau=1}^{\infty} \frac{1}{n_\tau} \sum_{j=1}^{n_\tau} E\left\{ Q^{(j)}(n) \right\} \\
&= \int_0^\infty \left(\sum_{n_\tau=1}^{\infty} \frac{1}{n_\tau} \sum_{j=1}^{n_\tau} \sum_{k=0}^{j-1} \binom{n}{k} \left[H(t)\right]^k \left[1-H(t)\right]^{n-k} \right) dt
\end{aligned} \tag{6.3.14}$$

Similarly, when $N_\tau \geq m$, the expected downtime period is obtained by

$$\begin{aligned}
E\left[\sum_{i=1}^{\left\lfloor \frac{N_\tau}{m} \right\rfloor} \sum_{j=1}^{m-1} \left(Q^{(j+m(i-1))}(n) \right) \right] &= \sum_{j=1}^{m-1} E\left(\sum_{i=1}^{\left\lfloor \frac{N_\tau}{m} \right\rfloor} Q^{(j+m(i-1))}(n) \right) \\
&= \sum_{j=1}^{m-1} \left(\int_0^\infty \left(\sum_{n_\tau=1}^{\infty} \frac{1}{n_\tau} \sum_{i=1}^{\left\lfloor \frac{n_\tau}{m} \right\rfloor} \sum_{k=0}^{m(i-1)+j-1} \binom{n}{k} \left[H(t)\right]^k \left[1-H(t)\right]^{n-k} \right) dt \right)
\end{aligned}$$

$$E\left[\sum_{j=\left\lfloor\frac{N_\tau}{m}\right\rfloor\cdot m+1}^{N_\tau} Q^{(j)}(n)\right] = E\left[E\sum_{j=\left\lfloor\frac{N_\tau}{m}\right\rfloor\cdot m+1}^{N_\tau} Q^{(j)}(n)\middle| N_\tau = n_\tau\right]$$

$$= \sum_{n_\tau=1}^{\infty} E\left[\sum_{j=\left\lfloor\frac{N_\tau}{m}\right\rfloor\cdot m+1}^{N_\tau} Q^{(j)}(n)\middle| N_\tau = n_\tau\right]\cdot P\left[N_\tau = n_\tau\right]$$

$$= \sum_{n_\tau=1}^{\infty}\frac{1}{n_\tau}\sum_{j=\left\lfloor\frac{n_\tau}{m}\right\rfloor\cdot m+1}^{n_\tau} E\left\{Q^{(j)}(n)\right\}$$

$$= \int_0^\infty\left(\sum_{n_\tau=1}^{\infty}\frac{1}{n_\tau}\sum_{j=\left\lfloor\frac{n_\tau}{m}\right\rfloor\cdot m+1}^{n_\tau}\sum_{k=0}^{j-1}\binom{n}{k}\left[H(t)\right]^k\left[1-H(t)\right]^{n-k}\right)dt$$

Therefore, when $N_\tau \geq m,$ the expected downtime period is given by

$$E\left(D_{N_\tau\geq m}\right) = E\left(\sum_{i=1}^{\left\lfloor\frac{N_\tau}{m}\right\rfloor}\sum_{j=1}^{m-1}\left(Q^{(j+m(i-1))}(n)\right)+\sum_{j=\left\lfloor\frac{N_\tau}{m}\right\rfloor\cdot m+1}^{N_\tau}\left(Q^{(j)}(n)\right)\right)$$

$$= \sum_{j=1}^{m-1}\left(\int_0^\infty\left(\sum_{n_\tau=1}^{\infty}\frac{1}{n_\tau}\sum_{i=1}^{\left\lfloor\frac{n_\tau}{m}\right\rfloor}\sum_{k=0}^{m(i-1)+j-1}\binom{n}{k}\left[H(t)\right]^k\left[1-H(t)\right]^{n-k}\right)dt\right)$$

$$+\int_0^\infty\left(\sum_{n_\tau=1}^{\infty}\frac{1}{n_\tau}\sum_{j=\left\lfloor\frac{n_\tau}{m}\right\rfloor\cdot m+1}^{n_\tau}\sum_{k=0}^{j-1}\binom{n}{k}\left[H(t)\right]^k\left[1-H(t)\right]^{n-k}\right)dt$$

(6.3.15)

If we use $U^{(i)}(n)$ and life time distribution $F(t)$, the total expected cost of $N_\tau < m$ is as follows:

$$
\begin{aligned}
&c_1 \cdot E\left(N_\tau\right) + c_2 \cdot E\left(D_{N_\tau < m}\right) + c_3\left(n - E\left(N_\tau\right)\right) \\
&= c_1 \cdot n \cdot M(\tau) + c_3\left(n\left(1 - M(\tau)\right)\right) + c_2 \cdot \left(\tau \cdot n \cdot M(\tau) - \int_0^\infty \left(\sum_{n_\tau=1}^{\infty} \frac{1}{n_\tau} \sum_{j=1}^{n_\tau} \sum_{k=0}^{j-1} \binom{n}{k} \left[F(t)\right]^k \left[1 - F(t)\right]^{n-k}\right) dt\right)
\end{aligned}
$$

(6.3.16)

When $N_\tau \geq m,$ the expected total cost is as follows:

$$
\begin{aligned}
&c_1 \cdot E\left(N_\tau\right) + c_2 \cdot E\left(D_{N_\tau \geq m}\right) + c_3 \cdot E\left(n - \left(N_\tau - \left(\left\lfloor \frac{N_\tau}{m} \right\rfloor \cdot m\right)\right)\right) \\
&= c_1 \cdot n \cdot M(\tau) + c_3 \cdot \left(n - \left(n \cdot M(\tau) - \left(\left\lfloor \frac{n \cdot M(\tau)}{m} \right\rfloor \cdot m\right)\right)\right) \\
&+ c_2 \cdot \left(
\begin{array}{l}
\tau(m-1)\left(\int_0^\infty \left(\sum_{n_\tau=1}^{\infty} \frac{1}{n_\tau} \sum_{i=1}^{\left\lfloor \frac{n_\tau}{m} \right\rfloor} \sum_{k=0}^{i \cdot m - 1} \binom{n}{k} \left[F(t)\right]^k \left[1 - F(t)\right]^{n-k}\right) dt\right) \\
-\sum_{j=1}^{m-1} \left(\int_0^\infty \left(\sum_{n_\tau=1}^{\infty} \frac{1}{n_\tau} \sum_{i=1}^{\left\lfloor \frac{n_\tau}{m} \right\rfloor} \sum_{k=0}^{m(i-1)+j-1} \binom{n}{k} \left[F(t)\right]^k \left[1 - F(t)\right]^{n-k}\right) dt\right) \\
+\tau \cdot \left(n \cdot M(\tau) - \left\lfloor \frac{n \cdot M(\tau)}{m} \right\rfloor \cdot m\right) - \int_0^\infty \left(\sum_{n_\tau=1}^{\infty} \frac{1}{n_\tau} \sum_{j=\left\lfloor \frac{n_\tau}{m} \right\rfloor \cdot m + 1}^{n_\tau} \sum_{k=0}^{j-1} \binom{n}{k} \left[F(t)\right]^k \left[1 - F(t)\right]^{n-k}\right) dt
\end{array}
\right)
\end{aligned}
$$

(6.3.17)

If r.v. L follows F_L distribution, we can obtain the expected cost rate, $ECR(\tau),$ as follows:

$$ECR(\tau) = \frac{\left\{ \begin{array}{l} \left[c_1 \cdot n \cdot M(\tau) + c_2 \cdot E\left(D_{N_\tau < m}\right) + c_3\left(n\left(1 - M(\tau)\right)\right)\right] \cdot F_L\left(m\left(1 - \left\lfloor \frac{n \cdot N(\tau)}{m} \right\rfloor\right)\right) \\ + \left[c_1 \cdot n \cdot M(\tau) + c_2 \cdot E\left(D_{N_\tau \geq m}\right) + c_3 \cdot \left(n - \left(n \cdot M(\tau) - \left(\left\lfloor \frac{n \cdot M(\tau)}{m} \right\rfloor \cdot m\right)\right)\right)\right] \cdot \bar{F}_L\left(m\left(1 - \left\lfloor \frac{n \cdot N(\tau)}{m} \right\rfloor\right)\right) \end{array} \right\}}{\tau}$$

(6.3.18)

We consider several costs other than the system failure cost. The threshold number of failures is less than $n-k+1,$ so before the system failure, they provide the FR for failed components and the PR service for the other deteriorating components. When the number of failed components is $n-k+1,$ the system would fail.

6.4 Numerical Example

If the threshold number of failures are fewer than $(n-k+1)$ and we use the life time distribution $F(t)$, then we can obtain the expected cost rate using eq. (6.3.18). Due to a complex nonlinear function, we use the Nelder-Mead downhill simplex method [92, 137] to obtain an optimum solution for the above optimization problem, with a hope to obtain the global solution. In this study, we also have used other non-linear optimization approaches such as random search method, differential evolution method and simulated annealing method. Among them, Nelder-Mead downhill simplex method seems to be the most popular direct search method for obtaining the optimum solution of a nonlinear function, which does not require the calculation of derivatives. Using the Nelder-Mead approach, we optimize the expected cost rate of the model. For the numerical example below, we apply our GBRP using

$F(t)$ in case of $m < n-k+1$. We wish to find the optimal period τ^* and the optimal threshold number of failures, m^*, which minimizes the expected cost rate, given in eq. (6.3.18). In summary, from eq. (6.3.18), we can formulate the following optimization problem:

$$\text{Find } (m^*, \tau^*): \qquad \text{Minimize } ECR(m,\tau)$$

In Fig. 6.4.1, a 2-out-of-10 system with a threshold level $m=3$ for GBRP is described. At the maintenance cycle τ, they will provide the FR service for all failed components and the PR service for other deteriorating components. Before the m^{th} failure, there is no need to take any action for failed components. Obviously it would save the cost. The failed components left idle before the m^{th} failure.

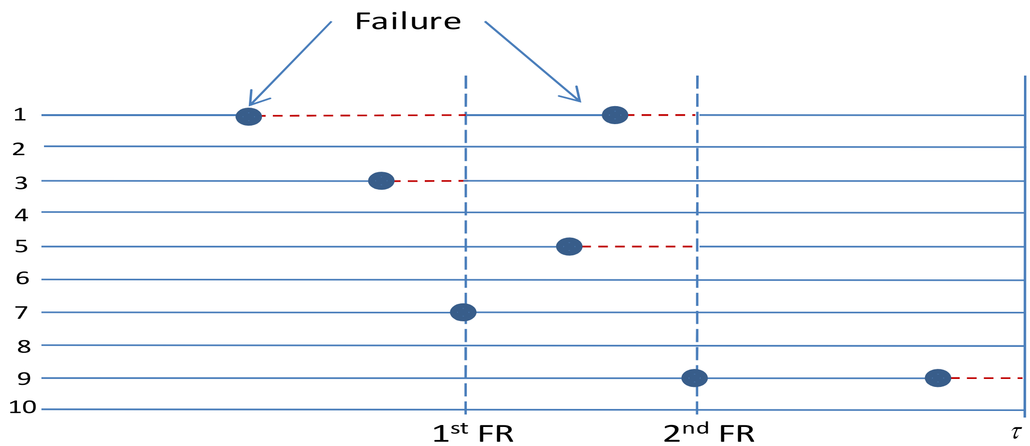


Fig. 6.4.1 *2-out-of-10* system with a threshold level *m*=3 for GBRP

Suppose that the failure time of a system follows an exponential distribution with parameter λ_1. That is, $F(t) = 1 - e^{-\frac{1}{\lambda_1}t}$ and $M(t) = \frac{t}{\lambda_1}$. Given various cost parameters such as FR cost, PR cost, risk cost, and the system failure cost, using eq. (6.3.18) we can obtain the results to illustrate an example as given in Figure 6.4.4. The number of failed components that remain idle until the next maintenance, L, is assumed to follow the truncated exponential distribution with parameter λ_2. The pdf of L, $0 < L \le m$ is given by

$$f_L(l) = \frac{\frac{1}{\lambda_2}\exp\left(-\frac{1}{\lambda_2}l\right)}{1-\exp\left(-\frac{m}{\lambda_2}\right)} \tag{6.4.1}$$

Table 6.4.1 shows that as $\frac{1}{\lambda_1}$ increases, the values of τ decrease. However, the expected cost rate increases monotonically and the number of threshold level, m increases as $\frac{1}{\lambda_1}$ increases.

Table 6.4.1: Expected cost rate for various values of λ_1 and number of components for a 2-out-of-n system when $c_1 = \$300,\ c_2 = \$10\ and\ c_3 = \$50$.

$\frac{1}{\lambda_1}$	$n=5$			$n=7$			$n=10$			$n=15$			$n=20$		
	m	τ	ECR	m	τ	ECR	m	τ	ECR	m	τ	ECR	m	τ	ECR
1/100	1	20.0	27.50	1	20.0	32.50	1	10.0	80.00	1	10.0	105.00	1	9.9	131.31
1/50	1	10.0	55.00	1	10.0	65.00	1	9.9	80.81	2	10.0	135.00	2	9.9	161.62
1/20	1	7.9	69.62	2	10.0	95.00	2	7.9	139.24	4	10.0	220.00	4	7.9	310.13
1/10	2	7.9	107.60	2	5.7	166.67	4	7.9	246.84	7	9.3	387.10	10	9.9	505.05
1/5	4	7.9	215.19	5	7.1	330.99	8	7.9	430.38	13	8.6	627.91		N/A	

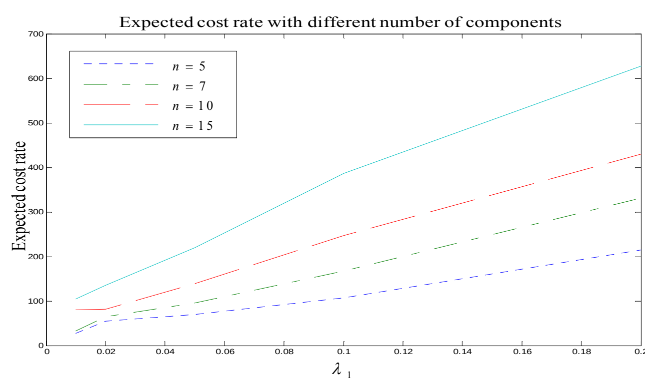


Figure 6.4.2: Expected cost rate for various λ_1 and the number of components for 2-out-of-n system when $c_1 = \$300,\ c_2 = \$10\ and\ c_3 = \$50.$

As the value of $\frac{1}{\lambda_1}$ increases, it implies that the number of failed components increases. From Table 6.4.2, we observe that m and the ECR increase. In Table 6.4.2, we use different cost values from Table 6.4.1. It is also interesting to note that the expected cost rate increases monotonically as $\frac{1}{\lambda_1}$ increases with other parameters fixed. When the number of components becomes greater, it appears that the expected cost rate increases.

Table 6.4.2 Expected cost rate for various λ_1 and number of components for 2-out-of-n system when $c_1 = \$300,\ c_2 = \$50\ and\ c_3 = \$10.$

$\frac{1}{\lambda_1}$	$n=5$			$n=7$			$n=10$			$n=15$			$n=20$		
	m	τ	ECR	m	τ	ECR	m	τ	ECR	m	τ	ECR	m	τ	ECR
1/100	1	20.0	17.50	1	20.0	18.50	1	10.0	40.00	1	10.0	45.00	1	9.9	50.51
1/50	1	10.0	35.00	1	10.0	37.00	1	9.9	40.40	1	6.6	68.18	2	9.9	80.81
1/20	1	7.9	44.30	1	5.7	64.91	2	7.9	88.61	2	5.3	141.51	2	3.9	205.13
1/10	2	7.9	82.28	2	5.7	117.54	2	3.9	179.49	2	2.6	288.46	2	1.9	421.05
1/5	2	1.9	157.90	2	2.8	239.29	2	1.9	368.42	13	8.6	572.09		N/A	

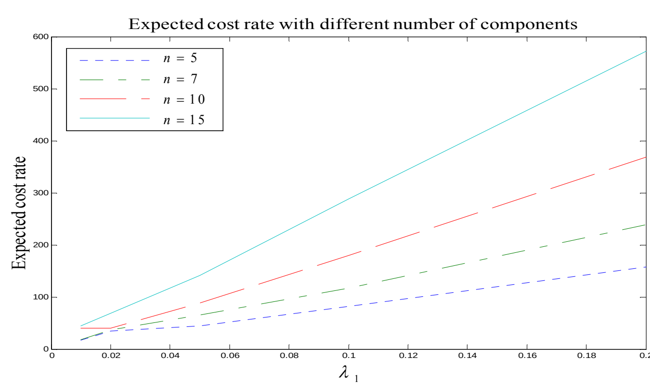


Fig. 6.4.3. Expected cost rate for various λ_1 and number of components for 2-out-of-n system when $c_1 = \$300,\ c_2 = \$50\ and\ c_3 = \$10.$

Using eq. (6.3.18), Table 6.4.3 gives the values of m, τ and the expected cost rate for various choices of c_1, c_2 *and* c_3 based on different n values. Let c_1, c_2 *and* c_3 be the FR cost, the risk cost and the PR cost. In Table 6.4.3, we present the results of sensitivity analysis for various cost coefficients and other values to illustrate the effects on the ECR and m. The first case in Table 6.4.3 is $c_1 = \$300, c_2 = \$50$ and $c_3 = \$10$. The number of components are n=5, 7, 10, 15 and 20. $\lambda_1 = 10$ is the parameter of the exponential distribution for the time to failure of one component. $\lambda_2 = 10$ is another parameter of the truncated exponential distribution for the number of failed components in the last interval after the last FR service. For each case, we use several different costs in Table 6.4.3. As we would expect, the expected cost rate increases and the maintenance cycle gets longer as the PR cost increases.

Table 6.4.3: Expected cost rate for various cost coefficients and number of components for a 2-out-of-n system with $c_1 = \$300$ *and* $c_2 = \$50$

PR Cost	$n=5$			$n=7$			$n=10$			$n=15$			$n=20$		
	m	τ	ECR	m	τ	ECR	m	τ	ECR	m	τ	ECR	m	τ	ECR
10	2	7.9	82.3	2	5.7	117.5	2	3.9	179.5	2	2.6	288.5	9	9.0	418.9
20	2	7.9	88.6	2	5.7	129.8	2	3.9	205.1	6	7.9	336.7	9	9.0	437.8
30	2	7.9	94.9	2	5.7	142.1	4	7.9	224.1	6	7.9	353.2	9	9.0	456.7
40	2	7.9	101.3	2	5.7	154.4	4	7.9	235.4	6	7.9	369.6	9	9.0	475.6
50	2	7.9	107.6	2	5.7	166.7	4	7.9	246.8	6	7.9	386.1	9	9.0	494.4
60	2	7.9	113.9	2	5.7	178.9	4	7.9	258.2	7	9.3	400.0	9	9.0	513.3
70	2	7.9	120.3	2	5.7	191.2	5	9.9	268.7	7	9.3	412.9	9	9.0	532.2
80	2	7.9	126.6	4	10.0	198.0	5	9.9	276.8	7	9.3	425.8	9	9.0	551.1
90	2	7.9	132.9	4	10.0	204.0	5	9.9	284.8	8	10.0	438.0	10	9.9	569.7
100	2	7.9	139.2	4	10.0	210.0	5	9.9	292.9	8	10.0	450.0	10	9.9	585.9
110	2	7.9	145.6	4	10.0	216.0	5	9.9	301.0	8	10.0	462.0	10	9.9	602.0
120	2	7.9	151.9	4	10.0	222.0	5	9.9	309.1	8	10.0	474.0	10	9.9	618.2
130	2	7.9	158.2	4	10.0	228.0	5	9.9	317.2	8	10.0	486.0	10	9.9	634.3
140	2	7.9	164.6	4	10.0	234.0	5	9.9	325.3	8	10.0	498.0	10	9.9	650.5
150	2	7.9	170.9	4	10.0	240.0	5	9.9	333.3	8	10.0	510.0	10	9.9	666.7
160	2	7.9	177.2	4	10.0	246.0	5	9.9	341.4	8	10.0	522.0	10	9.9	682.8
170	2	7.9	183.5	4	10.0	252.0	5	9.9	349.5	8	10.0	534.0	10	9.9	699.0

180	2	7.9	189.9	4	10.0	258.0	5	9.9	357.6	8	10.0	546.0	10	9.9	715.2
190	2	7.9	196.2	4	10.0	264.0	5	9.9	365.7	8	10.0	558.0	10	9.9	731.3
200	2	7.9	202.5	4	10.0	270.0	5	9.9	373.7	8	10.0	570.0	10	9.9	747.5

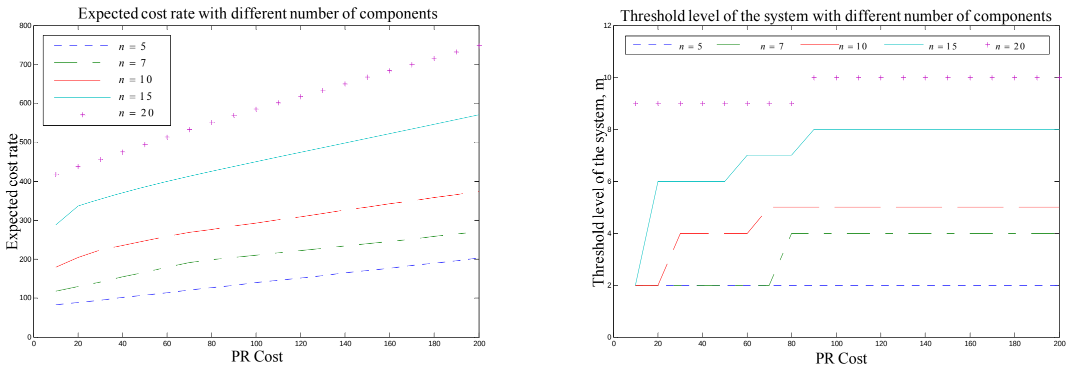


Fig. 6.4.4: Expected cost rate and the threshold level, m for various cost parameters and the number of components for a 2-out-of-n system

6.5 Concluding Remarks

We develop a GBRP for a *k-out-of-n* system based on the idle period aspects and obtain the optimal policy by determining the threshold number of failures and the periodic maintenance time interval. To minimize the cost, we use idle periods instead of immediate FR services. Until there are m numbers of failures, we did not provide the FR service, instead, at the time of m^{th} failures, we provide the FR service. At the time of periodic maintenance time τ, we provide the FR service for failed components and the PR service for other deteriorating components. Our model has also taken into account the risk cost where the cost coefficients can be varied and are not subject to any restrictions, although the risk cost is, intuitively, assumed to be less than the FR cost and the PR cost in practice. We set up the threshold number of failures as fewer than or greater than N_τ. For each case, we develop the expected cost rate and obtain the optimized maintenance cycle and the optimized number of threshold level of the system in order to minimize the cost rate.

Chapter 7

Warranty Cost Analysis and Optimization with Imperfect PM and Two Types of Warranty Periods

7.1 Introduction

Warranty policies are important factors in the process of decision making for both the consumer and manufacturer when buying a product. Companies, on one hand, use warranty policies as a marketing tool with hopes to increase the sales whilst to minimize the related warranty cost. Therefore, an appropriate warranty period, if not the best, is an important measure that the manufacturers seek to minimize the warranty costs. For example, if a warranty period is too long, then manufacturers are vulnerable to more claims, responsibilities resulting in higher cost. If the warranty period is too short, it could be a weakest link to attract customers to purchase the product. As such, warranty becomes an important factor for both consumer and manufacturer. Minimal repair is defined as the failure rate of a repaired product is same as just before the most recent repair. This concept of minimal repair was introduced by Barlow and Proschan [14]. If repairs are minimal, a failure rate then follows non-homogeneous Poisson processes (NHPP). Many researchers [88, 133, 180] have developed several warranty models based on the NHPP. Krivtsov [88] proposes to capitalize on the fact that NHPP's rate of occurrence of failures formally coincides with the hazard function of the underlying lifetime distribution. Yue and Cao [194], Finkelstein [57] and Murthy [105] consider a system with minimal repairs which is almost equivalent to a NHPP and develop their approaches. Ja *et al*. [72] derive the mean and the variance for the manufacturers' total discounted warranty cost of a single sale for single component items under several warranty policies. Bai and Pham [7] extend this analysis for the minimally repaired series systems and their application can be seen in warranty design, warranty reserve determination and risk analysis. Duchesne and Marri [53] show how risk adjustment principles considered in the economics and actuarial science literature can be applied to the determination of a warranty reserve while they much more extended those obtained by Bai & Pham [7].

The range of warranty cost analysis needs to consider not only the characteristic of the warranty policy and replacement/repair cost, but also the distribution of the number of product failures. Different models [20, 21] have been studied in order to provide guidance in selecting the optimal warranty plans. One of the main interests that arise from the warranty policy is to obtain the optimal warranty period and its corresponding warranty cost analysis.

Several researchers [7, 20, 38-40, 50, 53, 85, 117, 120, 121, 123, 146, 182, 190] have studied the warranty cost analysis. However, the published literature offers few approaches to determine the warranty period. Chien [33] determines the optimal warranty period and the optimal out-of-warranty replacement age while minimizing the corresponding cost functions. Menezes and Currim [100] focus on aiding decisions on how long the warranty period is for three genetic warranty types considering a product' price, costs and failure rates. Gutierrez-Pulido *et al.* [63] propose a methodology with the determination of warranty length that takes into account the reliability of the product, the consumer appreciation of the competitiveness of the warranty scheme, the effect on the image of the company when the product fails under the warranty period, and the costs that the manufacturer incurs to fulfill the warranty. Wu [185] provides a theoretical development and empirical study on determining price, warranty period and production rate.

In this chapter, a warranty period and a post warranty period are considered simultaneously. For warranty period, we assume that every failed component would be repaired minimally on the point of failure. In other words, a customer would have minimal repair services because of their warranty policy whenever a component fails. If it is repaired minimally, then the failure rate is well-known to follow NHPP. So, NHPP is used for cost analyses in the warranty period. The relationships between current inter-failure interval and next inter-failure interval are also studied with considerations of two-dimensional NHPP. The discussion of the methodology to conduct warranty cost analyses using NHPP will also be presented. The time limitation of repair services is considered for the customer's satisfaction. In other words, if the failed product comes to the warranty service centers, after repair the product should be returned back to the customer as soon as possible. If the product cannot be fixed within a threshold time of repair services, they should provide replacement services instead of repair services for the customer's satisfaction. Two-dimensional NHPP is used to model two warranty services for the methodology. Unlike many researchers [32, 71, 83] who used a product's usage and age/ time for two dimensions, we use the repair time and failure time as two dimensions for the warranty analysis.

The chapter is organized as follows. In Section 7.2, the research problems are described. In Section 7.3, we develop several cost analyses and their corresponding decision variables within a warranty period including two-dimensional NHPP (failure times and repair times). It presents the long run expected cost per unit time. Section 7.4 presents optimization problems to minimize the long run expected cost and obtain three decision variables such as the warranty period, repair time limit and the maintenance cycle. We discuss numerical examples in Section 7.5. The concluding remarks are presented in Section 7.6.

7.1.1 Nomenclature

T_w : Time of warranty period being finished

X_{Ψ} : Number of failures in the area Ψ

Y_n: Waiting time between (n-1)$^{\text{th}}$ and n^{th} failure

S_n: Waiting time until the n^{th} event, $S_n = \sum_{k=1}^{n} Y_k$

$N(\Psi)$: Number of failures occurred in the area Ψ

$f_R(t), F_R(t)$: pdf and cdf of the repair times, respectively

$m(t)$: Rate of NHPP which each failure time follows

c: warranty cost per a system failure

c_{cpm}, c_{pm} : Cost of CPM and cost of PM, respectively

A : Limitation of the warranty reserve

TC_i^W : Total cost incurred during the warranty period when $i=1$: aspect of a customer, $i=2$: aspect of a manufacturer

$TC_i(w_1, w_2, T)$: Total cost with three decision variables (w_1, w_2, T) when $i=1$: aspect of a customer, $i=2$: aspect of a manufacturer

$TD(w_1, T)$: Total duration with two decision variables (w_1, T)

p, q: probability that a PM alone is imperfect is p, and $q = 1 - p$

$TC_1(w_1, w_2, T | p, q)$: Total cost with three decision variables (w_1, w_2, T) under the customer's point of view when the PM is imperfect with the probability p and $q = 1 - p$

$TC_2(w_1, w_2, T | p, q)$: Total cost with three decision variables (w_1, w_2, T) under the manufacturer's point of view when the PM is imperfect with the probability p and $q = 1 - p$

$TD(w_1, T | p, q)$: Total duration with two decision variables (w_1, T) when the PM is imperfect with the probability p and $q = 1 - p$

TC^{PW} : Total cost incurred during post warranty period

$TC_{p,q}^{PW}$, $TD_{p,q}^{PW}$: Total cost and total duration in the post warranty period, respectively, when the PM is imperfect with probability p and $q = 1 - p$

TD^W, TD^{PW} : Total duration in the warranty period and in the post warranty period, respectively

$L_i(w_1, w_2, T)$: Long run expected cost per unit time with three decision variables (w_1, w_2, T) when $i = 1$, it is under the customers' point of view and when $i = 2$, it is under the manufacturers' point of view.

$L_1(w_1, w_2, T | p, q)$: Long run expected cost per unit time with three decision variables (w_1, w_2, T) under the customer's point of view when the PM is imperfect with the probability p and $q = 1 - p$

$L_2(w_1, w_2, T | p, q)$: Long run expected cost per unit time with three decision variables (w_1, w_2, T) under the manufacturer's point of view when the PM is imperfect with the probability p and $q = 1 - p$

y_1, y_2: Times to perform CPM and PM, respectively

7.2 Problem Description

We develop the warranty cost model considering warranty period and post warranty period for the *k-out-of-n* system. The structure of warranty period and post warranty period is described in Fig. 7.2.1. During the warranty period, whenever a product fails, only minimal repair services are provided. Corrective maintenance (CM) has the same meaning with minimal repair. However, during the post warranty period, it will consider two different kinds of services: corrective maintenance combined with preventive maintenance (CPM) and preventive maintenance (PM) itself. Perform CM on the failed components together with PM on all unfailed but deteriorating ones (which is CPM) once exactly m components are idle, or perform PM on the whole system once the total operating time reaches $T_i, i = 1, 2, \cdots$, whichever occurs first. That is, if m components fail in the post warranty period, CPM is performed; if less than m components fail in the post warranty period, then PM is carried out at time $T_i, i = 1, 2, \cdots$. m is assumed to be a predetermined positive integer, where $1 \le m \le n - k + 1$. As long as m is less than $n - k + 1$, the system will not fail and continue to operate. PM service is provided periodically in the post warranty period and upon at least a certain fixed number of failures which is m with CM.

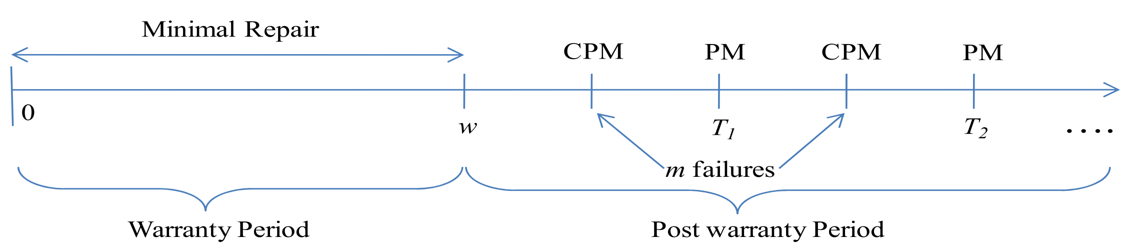


Fig. 7.2.1 Warranty period and post warranty period

Fig. 7.2.1 indicates that when there are less than a certain number of failures m, in the post warranty period, PM would be provided at the time of $T_i, i = 1, 2, \cdots$. When there are m number of failures, CM would be provided for failed components and PM for the deteriorating components which is CPM. After each CPM, the interval T_i starts again periodically as a new cycle.

Customers would not be interested in PM during the warranty period, because the warranty will enable them to receive a repair service. However, during the post warranty period, the customers have to bear the cost for any failure of either the components or the system, causing them to be cautious of the condition of the product and possible cost related to any failures. Thus, customers may be interested to purchase a PM service when the maintenance cost is much less than a failure cost. This cost analysis of post warranty period extends the work by Pham and Wang [130].

During the warranty period, in relation to the compensation to a failure of a product, there are three basic types of warranty policies: free repair/replacement warranty (FRW), pro-rata warranty (PRW) and combination warranty (CMW). FRW is a warranty which provides a free repair or replacement service for the customers. PRW would enable partial refund of purchase cost depending on the failure time. However, in the alternative PRW, customers need to pay service cost depending on the failure time. CMW contains both features of FRW and PRW. In the chapter, CMW is referred to as a combination of FRW and alternative PRW.

We develop warranty cost models considering both the warranty period and the post warranty period with minimal repairs. We then obtain the optimal warranty period, the repair time limitation and maintenance cycle to minimize the long run expected cost per unit time when warranty reserve are enough. Additionally, when the warranty reserve is limited, we try to obtain the optimal warranty period.

7.2.1 Assumptions

1) Each failure of a component of the system during the warranty period is immediately detected.
2) Inter-occurrence failure intervals are independent to each other.
3) The probability of having more than one failed component simultaneously is zero.
4) A repair time is not included in the warranty period. It is not only because the repair times relatively short compared to the warranty period but also because it increases the customer's satisfaction.
5) *k-out-of-n* system consists of *n* statistically independent and identically distributed components.

7.3 Warranty Cost Modeling

In this section, we develop two-dimensional warranty model with failure times and repair times during the warranty period and investigate the optimal warranty period and repair time limitation. Three decision policies (warranty periods, repair time limitation, and the maintenance cycles) are obtained to minimize the long run expected cost during the warranty period and the post warranty period.

7.3.1 Two-dimensional Warranty Model in the Warranty Period

During the warranty period, whenever a system fails, it would be repaired minimally on the point of failure. Additionally, two kinds of warranty services are considered. One is a repair service and the other is a replacement service. In general, when there is a failed component, the repair service is considered first, and replaced only when it cannot be repairable. Also, if failed products were delivered to the warranty service centers, they should return back to the customer within a certain threshold of time for customers' satisfaction. Therefore, repair warranty service time limitations are considered. Because the repair service is minimal, the repair service rate follows NHPP. Since there are two kinds of warranty services, repair and replace, we call it a two-dimensional NHPP warranty, referred to [167].

Fig. 7.3.1 describes the warranty services time model for a *k-out-of-n* system. The warranty policy is non-renewable. While trying to repair a failed system, if a repair time exceeds the time limit, w_2, then it is replaced, rather than continuing for repair. w_1 represents the warranty period and w_2 represents a time limitation for the repair service. The horizontal axis is the failure time $T_{1i}, i = 1, 2, \cdots,$ in a NHPP of rate $m(t)$ and the vertical axis is the repair time $T_{2i}, i = 1, 2, \cdots,$ which is assumed to be identically distributed and independent of the NHPP. We consider repair times which are less than the repair time limit and repair times are not included in the warranty period for the customer's satisfaction.

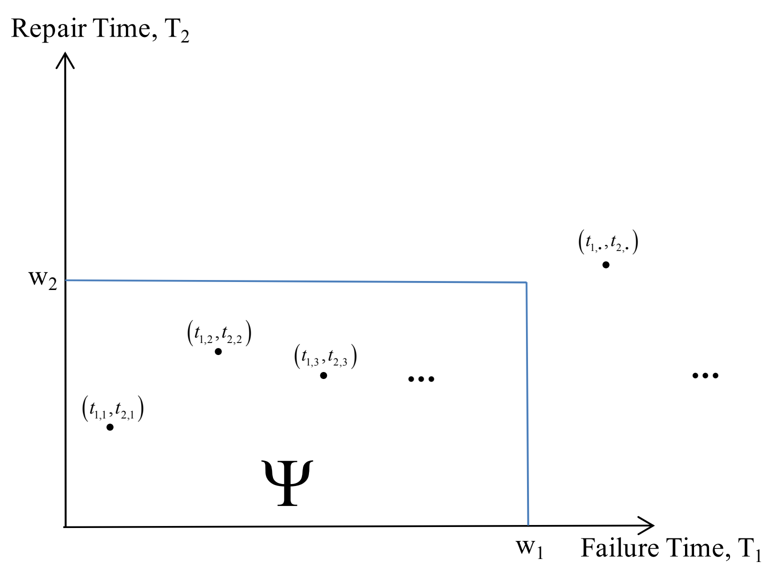


Fig. 7.3.1 Warranty services model using two-dimensional NHPP

In Fig. 7.3.1, $t_{1,1}, t_{1,2}, \cdots$, are the failure times in a NHPP of rate $m(t_1)$. Then a point is placed in the $(t_{1,\cdot}, t_{2,\cdot})$ place for $i = 1, 2, \cdots$. A *r.v.* repair time is assumed to be *i.i.d.* and independent of the Poison process. If $T_{1,1}, T_{1,2}, \cdots$ are failure times in a non-homogenous Poisson process of rate $m(t_1)$ and repair times $T_{2,1}, T_{2,2}, \cdots$ are independent identically distributed continuous r.v. having pdf $f_R(t_2)$. Then $(T_{1,1}, T_{2,1}), (T_{1,2}, T_{2,2}), \cdots$ form a two-dimensional NHPP in the (T_1, T_2) plane, where the mean number of points in a region Ψ is given by [167]

$$\mu(\Psi) = \iint_{\Psi} m(t_1) \cdot f_R(t_2)\, dt_2 dt_1 \tag{7.3.1}$$

The number of points $N(\Psi)$ falling in the area Ψ, has a Poisson distribution with mean $\mu(\Psi)$ in eq. (7.3.1). If X_Ψ denotes the number of failures in the area Ψ, then we obtain the reliability function of the component, in terms of number of failures, as follows:

$$\begin{aligned} R(x) &= P(X_\Psi \geq x) \\ &= 1 - P\{X_\Psi < x\} \\ &= 1 - \sum_{i=0}^{x-1} \frac{e^{-\mu(\Psi)}\left[\mu(\Psi)\right]^i}{i!} \\ &= 1 - \sum_{i=0}^{x-1} \frac{e^{-\iint_{\Psi} m(t_1)\cdot f_R(t_2)dt_2 dt_1}\left[\iint_{\Psi} m(t_1)\cdot f_R(t_2)\, dt_2 dt_1\right]^i}{i!} \end{aligned} \tag{7.3.2}$$

The expected number of failures can be obtained as follows:

$$\begin{aligned} E(X_\Psi) &= \sum_{x=1}^{\infty} P\{X_\Psi \geq x\} \\ &= \sum_{x=1}^{\infty}\left(1 - \sum_{i=0}^{x-1} \frac{e^{-\iint_{\Psi} m(t_1)\cdot f_R(t_2)dt_2 dt_1}\left[\iint_{\Psi} m(t_1)\cdot f_R(t_2)\, dt_2 dt_1\right]^i}{i!}\right) \end{aligned} \tag{7.3.3}$$

This is the expected warranty cost for a component. Then, we consider a *k-out-of-n* system where there are *n* components in the system. For the *k-out-of-n* system, we need to have at least *k-out-of-n* components in the system to work. Assuming that all units have identical and independent life distributions and the reliability function of a component is obtained by eq. (7.3.2). Then, the probability of having exactly *k* functioning units out of *n* is given by

$$\Pr\left(n,k,R(x)\right)=\binom{n}{k}R(x)^{k}\left(1-R(x)\right)^{n-k} \qquad k=0,1,2,\cdots,n \tag{7.3.4}$$

where $R(x)$ is given by eq. (7.3.2).

Since this is a *k-out-of-n* system, the probability that a system is not working is given by

$$\Pr\left(system\ is\ not\ working\right)=\sum_{i=0}^{k-1}\binom{n}{i}R(x)^{i}\left(1-R(x)\right)^{n-i} \tag{7.3.5}$$

where $R(x)$ is given by eq. (7.3.2).

If the expected number of system failures multiplied by the warranty cost, then the expected warranty cost $E_w(C)$ is obtained by

$$E_w(C)=c\cdot\sum_{x_s=1}^{\infty}x_s\left\{\sum_{i=0}^{k-1}\binom{n}{i}R(x)^{i}\left(1-R(x)\right)^{n-i}\right\}^{x_s} \tag{7.3.6}$$

where $R(x)$ is given by eq. (7.3.2) and let c be the warranty cost per a system failure and x_s be the number of system failures.

There are three basic types of warranty policies FRW, PRW and CMW. Under a FRW, customers would receive a repair/replacement warranty service for free. Under a PRW, customers would receive a refund depending on the failure time. The basic notion of a pro-rata warranty is that replacements are not provided free of charge, but at a prorated cost, depending on the amount of usage or service time provided prior to its failure [20]. In this chapter, we consider that customers have to pay partial repairing service cost depending on the failure time, this policy called alternative PRW. If the replacement cost exceeds the repair cost, manufacturers would provide repair service rather than replacement service. Accordingly, whenever the product fails, the manufacturer would end up providing the repair service more often than the replacement service. So, alternative PRW is more easily applicable than original PRW. CMW contains both features of FRW and alternative PRW. We look at cost analyses with both sides: perspectives of the customers and of the manufacturers. We use the expected warranty cost of the two-dimensional warranty, eq. (7.3.6) to obtain the long run expected cost per unit time. The repair service and the replacement service are considered together.

Let $E_w(C)$ be the expected warranty cost in the warranty period. And *i.i.d. r.v.* (t_{1i}, t_{2i}) is a pair of a failure time and a repair time of i^{th} failure, i=1,2,…. Under the alternative PRW, if the failure occurs under warranty, then customers have to pay partial repair cost, $g(t_{1i}, t_{2i})$. Therefore, from the customer's perspective, the total warranty cost in the warranty period TC_1^W is given by

$$TC_1^W = \begin{cases} E\left(\sum_{i=1}^{N(\Psi)} g(t_{1i}, t_{2i})\right) & \textit{Alternative PRW} \\ 0 & \textit{FRW} \\ p_1 \cdot \left(E\left(\sum_{i=1}^{N(\Psi)} g(t_{1i}, t_{2i})\right)\right) + q_1 \cdot 0 & \textit{CMW} \end{cases} \tag{7.3.7}$$

$$\textit{where } p_1 + q_1 = 1$$

From the manufacturer's perspective, the total warranty cost TC_2^W is given by

$$TC_2^W = \begin{cases} E_w(C) - E\left(\sum_{i=1}^{N(\Psi)} g(t_{1i}, t_{2i})\right) & \textit{Alternative PRW} \\ E_w(C) & \textit{FRW} \\ p_2 \cdot \left(E_w(C) - E\left(\sum_{i=1}^{N(\Psi)} g(t_{1i}, t_{2i})\right)\right) + q_2 \cdot E_w(C) & \textit{CMW} \end{cases} \tag{7.3.8}$$

where $p_2 + q_2 = 1$ and $E_w(C)$ is given in eq. (7.3.6).

Using these warranty policies (FRW, alternative PRW and CMW), we consider warranty cost analyses under the aspects of both the customers and manufacturers.

For the cost function, there may be several options [20]. Among them, the customer cost function for proportional linear co-pay [20] is assumed to be selected as follows:

$$g(t_{1i}, t_{2i}) = \begin{cases} \left(1 - \frac{t_{1i}}{w_1}\right)\left(1 - \frac{t_{2i}}{w_2}\right) & \textit{if } (t_{1i}, t_{2i}) \in \Omega \\ 0 & \textit{if } (t_{1i}, t_{2i}) \notin \Omega \end{cases} \tag{7.3.9}$$

Usually, $g(t_{1i}, t_{2i})$ would be any non-increasing positive function in $(t_{1i}, t_{2i}) \in \Omega,\ i = 1, 2, \ldots.$ $g(t_{1i}, t_{2i})$ decreases to zero as t_{1i} and t_{2i} increase.

7.3.2 Optimal Warranty Period When Warranty Reserve is Limited

In this section, we try to obtain the optimal warranty period when there is a limitation for warranty reserve. If the first failure time is larger than w_1, then the number of failure in w_1 is zero. $Y_i, i = 1, 2, \cdots$, denotes a waiting time between (n-1)$^{\text{th}}$ and n^{th} failure. Using the mean function, the probability that the first failure time is larger than w_1 is given by

$$P\{Y_1 > w_1\} = P(N(\Psi) = 0) = e^{-\iint_{\Psi} m(t_1) \cdot f_R(t_2) dt_2 dt_1}$$

Similarly, if $n \geq 2$, then

$$P\left\{Y_1+Y_2+Y_3+\cdots+Y_n>w_1\right\}=P\left(N\left(\Psi\right)\le n-1\right)$$
$$=\sum_{i=0}^{n-1}\frac{e^{-\iint_{\Psi}m(t_1)\cdot f_R(t_2)dt_2dt_1}\left[\iint_{\Psi}m\left(t_1\right)\cdot f_R\left(t_2\right)dt_2dt_1\right]^i}{i!} \tag{7.3.10}$$

So, the probability that the n^{th} waiting time is larger than w_1 is given by

$$P\left\{S_n>w_1\right\}=\sum_{i=0}^{n-1}\frac{e^{-\iint_{\Psi}m(t_1)\cdot f_R(t_2)dt_2dt_1}\left[\iint_{\Psi}m\left(t_1\right)\cdot f_R\left(t_2\right)dt_2dt_1\right]^i}{i!} \tag{7.3.11}$$

From eqs. (7.3.10) & (7.3.11), we obtain the expected waiting time for n^{th} failure.

$$E\left(S_n\right)=E\left(Y_1+Y_2+Y_3+\cdots+Y_n\right)$$
$$=\int_0^{\infty}P\left(S_n>w_1\right)dw_1 \tag{7.3.12}$$
$$=\sum_{i=0}^{n-1}\int_0^{\infty}\frac{e^{-\iint_{\Psi}m(t_1)\cdot f_R(t_2)dt_2dt_1}\left(\iint_{\Psi}m\left(t_1\right)\cdot f_R\left(t_2\right)dt_2dt_1\right)^i}{i!}\,dw_1$$

Also, the expected length time of n^{th} failure interval is given by:

$$E\left(Y_n\right)=\sum_{i=0}^{n-1}\int_0^{\infty}\frac{e^{-\iint_{\Psi}m(t_1)\cdot f_R(t_2)dt_2dt_1}\left(\iint_{\Psi}m\left(t_1\right)\cdot f_R\left(t_2\right)dt_2dt_1\right)^i}{i!}\,dt$$
$$-\sum_{i=0}^{n-2}\int_0^{\infty}\frac{e^{-\iint_{\Psi}m(t_1)\cdot f_R(t_2)dt_2dt_1}\left(\iint_{\Psi}m\left(t_1\right)\cdot f_R\left(t_2\right)dt_2dt_1\right)^i}{i!}\,dt \tag{7.3.13}$$

For example, if c = \$ 500 and warranty cost reserve is \$ 2,500, and $w_2=10$, then we can have up to 5 failures, and the expected waiting time until 5 failures is given by

$$E\left(Y_1+Y_2+Y_3+Y_4+Y_5\right)=\sum_{i=1}^{4}\int_0^{\infty}\frac{e^{-\iint_{\Psi}m(t_1)\cdot f_R(t_2)dt_2dt_1}\left(\iint_{\Psi}m\left(t_1\right)\cdot f_R\left(t_2\right)dt_2dt_1\right)^i}{i!}\,dw_1 \tag{7.3.14}$$
$$=34.0719$$

where the rate of NHPP which each failure time follows, is $m(t_1)=\frac{t_1^2}{2}$ and the pdf of the repair times follows the Weibull distribution with a shape parameter $k=2$ and a scale parameter $\lambda=200$ whose pdf is given by $f_R(t_2)=\frac{k}{\lambda}\left(\frac{t_2}{\lambda}\right)e^{-\left(\frac{t_2}{\lambda}\right)^k}=\frac{1}{100}\left(\frac{t_2}{200}\right)e^{-\left(\frac{t_2}{200}\right)^2}, t_2\geq 0.$

And using same rate of NHPP and pdf of the repair times, the expected waiting time until 6 failures is given by

$$\begin{aligned} E\left(Y_1+Y_2+\cdots+Y_6\right) &= \sum_{i=1}^{5}\int_0^{\infty}\frac{e^{-\iint_{\Psi}\frac{t_1^2}{2}\cdot\frac{1}{100}\left(\frac{t_2}{200}\right)e^{-\left(\frac{t_2}{200}\right)^2}dt_2dt_1}\left(\iint_{\Psi}\frac{t_1^2}{2}\cdot\frac{1}{100}\left(\frac{t_2}{200}\right)e^{-\left(\frac{t_2}{200}\right)^2}dt_2dt_1\right)^i}{i!}dw_1 \\ &= 37.5373 \end{aligned} \tag{7.3.15}$$

If we can have up to 5 failures and don't have sufficient warranty cost reserve for the 6th failures, then the warranty period would be located in the time unit interval between 34.0719 and 37.5373. The time unit can be chosen based on previous data from manufacturers. If each unit represents a month then the warranty period would be between 34.0719 months and 37.5373 months. So roughly a 35 or 36 or 37 months warranty period would be recommended for the manufacturers' warranty policy.

7.3.3 Imperfect Maintenance Service in the Post Warranty Period

We will consider the post warranty cost in the cost analyses in this section. For the cost analysis during warranty period, we use the cdf of failure rate which follows NHPP. On the contrary, for the cost analysis for post warranty period, we use the cdf of failure time. There are two services in the post warranty period, one is the PM and the other is CPM. PM is one type of maintenance services to prevent the system failure. CPM is a mixture between PM and CM. In the warranty period, a failed component would be repaired minimally whenever a failure occurs using its warranty. Additionally, the customer didn't consider PM in the warranty period because of their warranty. However, in post warranty period, there are repeated periodic intervals $T_i,\ i=1,2,\cdots$. If there are less than a certain number of failures, PM services would be provided at the time of $T_i,\ i=1,2,\cdots$. A threshold number of failure is named *m*. If there are *m* failures, they provide CPM services. After either PM service or CPM service is provided, another interval *T* is set up for next period's PM. Additionally, we consider that there are two kinds of PM; one is perfect PM and the other is imperfect PM. Usually, it is assumed that imperfect maintenance restores the unit's operating state to somewhere between "as good as new" and "as bad as old". In other words, this implies that after PM, a system is good as new with probability *p* and is bad as old with probability 1-*p*. The post warranty cost consists of two parts, CPM cost and PM cost. A renewal cycle is completed either by any CPM or by a imperfect PM where a system is good as new with probability *p*. Let T_p be the first perfect PM alone time point. $F_m(\cdot)$ and $R_m(\cdot)$ denotes the cdf and reliability function of exactly *m* component failures. The probability of CPM is given by

$$\begin{aligned} P(CPM) &= P\left[CPM \middle| T_p = T\right]\cdot E\left(I_{[T]}\left(T_p\right)\right) + P\left[CPM \middle| T_p = 2T\right]\cdot E\left(I_{[2T]}\left(T_p\right)\right) + \cdots \\ &= p\cdot F_m\left(T-w_1\right) + qp\cdot F_m\left(2T-w_1\right) + q^2 p\cdot F_m\left(3T-w_1\right) + \cdots \\ &= p\sum_{j=1}^{\infty} q^{j-1} F_m\left(jT-w_1\right) \end{aligned} \tag{7.3.16}$$

Similarly, the probability of PM is given by

$$\begin{aligned} P(PM) &= P\left[PM \middle| T_p = T\right]\cdot E\left(I_{[T]}\left(T_p\right)\right) + P\left[PM \middle| T_p = 2T\right]\cdot E\left(I_{[2T]}\left(T_p\right)\right) + \cdots \\ &= R_m\left(T-w_1\right) + q\cdot R_m\left(2T-w_1\right) + q^2\cdot R_m\left(3T-w_1\right) + \cdots \\ &= \sum_{j=1}^{\infty} q^{j-1} R_m\left(jT-w_1\right) \end{aligned} \tag{7.3.17}$$

Let c_{cpm} be the cost of CPM and let c_{pm} be the cost of PM. The total cost during a post warranty period with imperfect PM $TC_{p,q}^{PW}\left(w_1,T\right)$ is given by

$$\begin{aligned} TC_{p,q}^{PW}\left(w_1,T\right) &= c_{cpm}\cdot P\left(CPM\right)+c_{pm}\cdot P\left(PM\right) \\ &= c_{cpm}\cdot\left(p\sum_{j=1}^{\infty}q^{j-1}F_m\left(jT-w_1\right)\right)+c_{pm}\cdot\sum_{j=1}^{\infty}q^{j-1}R_m\left(jT-w_1\right) \end{aligned} \tag{7.3.18}$$

We obtain total cost during warranty period and post warranty period with perfect PM and imperfect PM. Under customers' point of view, the total cost with imperfect PM $TC_1\left(w_1,w_2,T\middle|p,q\right)$ is given by

$$TC_1\left(w_1,w_2,T\middle|p,q\right)=TC_1^W+TC_{p,q}^{PW}\left(w_1,T\right) \tag{7.3.19}$$

where TC_1^W and $TC_{p,q}^{PW}\left(w_1,T\right)$ are given in eq. (7.3.7) and (7.3.18), respectively.

As a special case when the PM is assumed to be perfect. It is easy to ascertain the total maintenance cost of CPM and PM individually. Then, total cost for post warranty period is given by

$$\begin{aligned} TC^{PW} &= c_{cpm}\cdot E\left(I_{[0,T-w)}\left(X\right)\right)+c_{pm}\cdot E\left(I_{[T-w,\infty)}\left(X\right)\right) \\ &= c_{cpm}\cdot F_m\left(T-w_1\right)+c_{pm}\cdot\left(1-F_m\left(T-w_1\right)\right) \\ &= F_m\left(T-w_1\right)\left(c_{cpm}-c_{pm}\right)+c_{pm} \end{aligned} \tag{7.3.20}$$

As the customers' point of view, the total cost is given by

$$\begin{aligned} TC_1\left(w_1,w_2,T\right) &= TC_1^W+TC^{PW} \\ &= TC_1^W+F_m\left(T-w_1\right)\left(c_{cpm}-c_{pm}\right)+c_{pm} \end{aligned} \tag{7.3.21}$$

where TC_1^W is given in eq. (7.3.7).

Under manufacturers' point of view, the total cost is given by

$$TC_2\left(w_1,w_2,T\right)=TC_2^W \tag{7.3.22}$$

where TC_2^W is given in eq. (7.3.8).

7.3.4 Long Run Expected Cost

In this section, we obtain the long run expected cost per unit time using the expected duration and the expected cost. We obtain the expected duration of a renewal cycle in a similar way to obtain the expected cost. The duration is illustrated in Fig. 7.3.2. Let y_1 be the time to perform CPM and y_2 be the time to perform PM.

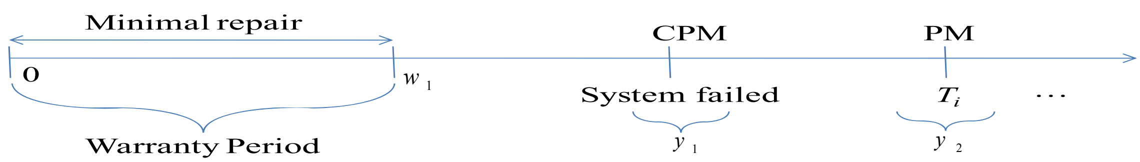


Fig. 7.3.2 expected duration structure

PM is assumed to be imperfect. So the total duration of a renewal cycle with decision variables (w_1, T) is given by

$$TD(w_1, T | p, q) = TD^W + TD_{p,q}^{PW} \tag{7.3.23}$$

We already obtain the $P(CPM)$ and $P(PM)$ from eqs. (7.3.16) & (7.3.17). When PM is imperfect, the post warranty duration $TD_{p,q}^{PW}$ is given by

$$\begin{aligned}
TD_{p,q}^{PW} &= \left\{ p\left[\int_0^{T-w_1} R_m(t)\,dt\right] + qp\left[\int_0^{2T-w_1} R_m(t)\,dt\right] + q^2 p\left[\int_0^{3T-w_1} R_m(t)\,dt\right] + \cdots \right\} + y_1 \cdot P(CPM) + y_2 \cdot P(PM) \\
&= \left\{ \begin{array}{l} p\int_0^{T-w_1} R_m(t)\,dt + qp\int_0^{T-w_1} R_m(t)\,dt + qp\int_{T-w_1}^{2T-w_1} R_m(t)\,dt + q^2 p\int_0^{T-w_1} R_m(t)\,dt + q^2 p\int_{T-w_1}^{2T-w_1} R_m(t)\,dt \\ +q^2 p\int_{2T-w_1}^{3T-w_1} R_m(t)\,dt + \cdots \end{array} \right\} \\
&\quad + y_1 \cdot P(CPM) + y_2 \cdot P(PM) \\
&= \left\{ p\left(1+q+q^2+\cdots\right)\int_0^{T-w_1} R_m(t)\,dt + pq\left(1+q+q^2+\cdots\right)\int_{T-w_1}^{2T-w_1} R_m(t)\,dt + \cdots \right\} \\
&\quad + y_1 \cdot P(CPM) + y_2 \cdot P(PM) \\
&= \left\{ \int_0^{T-w_1} R_m(t)\,dt + q\int_{T-w_1}^{2T-w_1} R_m(t)\,dt + q^2\int_{T-w_1}^{2T-w_1} R_m(t)\,dt + \cdots \right\} + y_1 \cdot P(CPM) + y_2 \cdot P(PM) \\
&= \int_0^{T-w_1} R_m(t)\,dt + \sum_{j=2}^{\infty} q^{j-1}\int_{(j-1)T-w_1}^{jT-w_1} R_m(t)\,dt + y_1 \cdot P(CPM) + y_2 \cdot P(PM)
\end{aligned}$$

(7.3.24)

The expected total duration of a renewal cycle when PM is imperfect $TD\left(w_1, T \middle| p, q\right)$ is given by

$$TD\left(w_1, T \middle| p, q\right)$$
$$= w_1 + \int_0^{T-w_1} R_m(t)\,dt + \sum_{j=2}^{\infty} q^{j-1} \int_{(j-1)T-w_1}^{jT-w_1} R_m(t)\,dt + y_1 \cdot \left(p \sum_{j=1}^{\infty} q^{j-1} F_m(jT - w_1) \right) + y_2 \cdot \sum_{j=1}^{\infty} q^{j-1} R_m(jT - w_1) \qquad (7.3.25)$$

We assume that PM at time T is perfect in this section. A renewal cycle consists of maintenance time and w_1 duration. Let TD^W and TD^{PW} be total duration of warranty period and post warranty period, respectively. The duration of post warranty period can be obtained as follows:

$$\begin{aligned} TD^{PW}(w_1, T) &= \int_0^{T-w_1} R_m(t)\,dt + y_1 \cdot E\left(I_{[0, T-w_1)}(X)\right) + y_2 \cdot E\left(I_{[T-w_1, \infty)}(X)\right) \\ &= \int_0^{T-w_1} R_m(t)\,dt + y_1 \cdot F_m(T - w_1) + y_2 \cdot \left(1 - F_m(T - w_1)\right) \\ &= \int_0^{T-w_1} R_m(t)\,dt + F_m(T - w_1)(y_1 - y_2) + y_2 \end{aligned} \qquad (7.3.26)$$

Total duration is given by

$$\begin{aligned} TD(w_1, T) &= TD^W + TD^{PW}(w_1, T) \\ &= w_1 + \int_0^{T-w_1} R_m(t)\,dt + F_m(T - w_1)(y_1 - y_2) + y_2 \end{aligned} \qquad (7.3.27)$$

On the other hand, the long run expected cost per unit time is obtained separately when PM is perfect or imperfect, when they are under the customer' point of view or under the manufacturer' point of view, when warranty polices are used among FRW, alternative PRW or CMW.

When PM is imperfect $(0 \le p \le 1)$ and under the customer' point of view, the long run expected cost per unit time is given by

$$L_1\left(w_1,w_2,T\middle|p,q\right)=\frac{TC_1\left(w_1,w_2,T\middle|p,q\right)}{TD\left(w_1,T\middle|p,q\right)}$$

$$=\frac{TC_1^W+c_{cpm}\cdot\left(p\sum_{j=1}^{\infty}q^{j-1}F_m\left(jT-w_1\right)\right)+c_{pm}\cdot\sum_{j=1}^{\infty}q^{j-1}R_m\left(jT-w_1\right)}{w_1+\int_0^{T-w_1}R_m(t)\,dt+\sum_{j=2}^{\infty}q^{j-1}\int_{(j-1)T-w_1}^{jT-w_1}R_m(t)\,dt+y_1\cdot\left(p\sum_{j=1}^{\infty}q^{j-1}F_m\left(jT-w_1\right)\right)+y_2\cdot\sum_{j=1}^{\infty}q^{j-1}R_m\left(jT-w_1\right)}$$

(7.3.28)

where TC_1^W is given in eq. (7.3.7).

When PM is perfect, that means p = 1, then eq. (7.3.28) becomes

$$L_1\left(w_1,w_2,T\right)=\frac{TC_1\left(w_1,w_2,T\right)}{TD\left(w_1,T\right)}=\frac{TC_1^W+F_m\left(T-w_1\right)\left(c_{cpm}-c_{pm}\right)+c_{pm}}{w_1+\int_0^{T-w_1}R_m(t)\,dt+F_m\left(T-w_1\right)\left(y_1-y_2\right)+y_2} \qquad (7.3.29)$$

When PM is imperfect and under the manufacturer' point of view, the long run expected cost is given by

$$L_2\left(w_1,w_2,T\middle|p,q\right)=\frac{TC_2\left(w_1,w_2,T\middle|p,q\right)}{TD\left(w_1,T\middle|p,q\right)}$$

$$=\frac{TC_2^W}{w_1+\int_0^{T-w_1}R_m(t)\,dt+\sum_{j=2}^{\infty}q^{j-1}\int_{(j-1)T-w_1}^{jT-w_1}R_m(t)\,dt+y_1\cdot\left(p\sum_{j=1}^{\infty}q^{j-1}F_m\left(jT-w_1\right)\right)+y_2\cdot\sum_{j=1}^{\infty}q^{j-1}R_m\left(jT-w_1\right)}$$

(7.3.30)

where TC_2^W is given in eq. (7.3.8).

When PM is perfect and under the manufacturer' point of view, the long run expected cost per unit time becomes:

$$L_2\left(w_1,w_2,T\right)=\frac{TC_2\left(w_1,w_2,T\right)}{TD\left(w_1,T\right)}=\frac{TC_2^W}{w_1+\int_0^{T-w_1}R_m(t)\,dt+F_m\left(T-w_1\right)\left(y_1-y_2\right)+y_2} \qquad (7.3.31)$$

7.4 Optimization Problem

In Sections 7.3, we obtain the long run expected cost per unit time using the expected warranty cost and the expected duration during the warranty period and the post warranty period. For the long run expected cost per unit time, we use two-dimensional warranty analysis and imperfect PM. From time to time, people need an optimal policies in which while warranty cost reserve is larger than the total warranty cost and while the maintenance interval in the post warranty period happens after the warranty period, the long run expected cost per unit time is minimized both under the customer's point of view and under the manufacturer's point of view.

The optimization problem can be formulated in terms of decision variables w_1, w_2 and T. Under customers' point of view, we minimize the function $L_1\left(w_1, w_2, T \middle| p, q\right)$ with some constraints based on the three types of warranty policy. Under customers' point of view,

$$\textit{Minimize } L_1\left(w_1, w_2, T \middle| p, q\right)$$

$$= \frac{TC_1^W + c_{cpm} \cdot \left(p \sum_{j=1}^{\infty} q^{j-1} F_m\left(jT - w_1\right) \right) + c_{pm} \cdot \sum_{j=1}^{\infty} q^{j-1} R_m\left(jT - w_1\right)}{w_1 + \int_0^{T-w_1} R_m\left(t\right) dt + \sum_{j=2}^{\infty} q^{j-1} \int_{(j-1)T-w_1}^{jT-w_1} R_m\left(t\right) dt + y_1 \cdot \left(p \sum_{j=1}^{\infty} q^{j-1} F_m\left(jT - w_1\right) \right) + y_2 \cdot \sum_{j=1}^{\infty} q^{j-1} R_m\left(jT - w_1\right)}$$

subject to

$$w_1, w_2 \geq 0, T > 0, w_1 < T$$

where TC_1^W is given in eq. (7.3.7).

Similarly, under manufacturers' point of view, the function $L_2\left(w_1, w_2, T \middle| p, q\right)$ is also minimized with several constraints. Under manufacturers' point of view,

$$\text{Minimize } L_2\left(w_1, w_2, T \middle| p, q\right)$$
$$= \frac{TC_2^W}{w_1 + \int_0^{T-w_1} R_m(t)\,dt + \sum_{j=2}^{\infty} q^{j-1} \int_{(j-1)T-w_1}^{jT-w_1} R_m(t)\,dt + y_1 \cdot \left(p \sum_{j=1}^{\infty} q^{j-1} F_m\left(jT - w_1\right) \right) + y_2 \cdot \sum_{j=1}^{\infty} q^{j-1} R_m\left(jT - w_1\right)}$$

subject to

$$w_1, w_2 \geq 0, T > 0, w_1 < T$$

where TC_2^W is given in eq. (7.3.8).

The optimal maintenance policy can be determined from the optimization model using nonlinear function solving approaches. Also, the optimal warranty period could be obtained by computer software using various kinds of parameters and distributions for the failure rate.

7.5 An Application

In South Korea, there are four nuclear sites and, in 2010, there are 20 nuclear power plants in operation with a total licensed output amount to 17,716 MWe (MegaWatt electrical) and 8 nuclear power plants under construction, for a total of 28 units in operation by the end of 2016 [2]. We investigate the field data to check their dependency using a nonparametric method. We implement our proposed approaches to conduct warranty cost analysis using the field data.

7.5.1 Data Description

Among 20 nuclear power reactors in the four nuclear plants in South Korea, we pick one nuclear plant which has three nuclear power reactors. This nuclear plant is kind of 1-out-of-3 system because it is actually three parallel components system. If only one nuclear reactor is broken and fail to operate, then they would repair it immediately. So *m* is assumed to be 1. It is summarized that 10 failure data for nuclear power plants for relatively recent events or failures in Table 7.5.1.

Table 7.5.1 Failure times and repair times for nuclear power plants (failure times : days, repair times : hours)

No.	Reactor 1		Reactor 2		Reactor 3	
	Failure	Repair	Failure	Repair	Failure	Repair
1	465.43	197.83	34.85	276.92	218.85	29.25
2	717.26	202.50	383.85	85.13	12.04	278.87
3	7.00	641.87	188.86	310.25	110.94	5.25
4	39.11	372.58	666.26	316.00	278.84	35.67
5	174.62	3.79	666.06	447.79	633.10	622.08
6	362.15	225.83	115.29	102.96	505.41	207.50
7	196.84	108.08	1398.44	220.37	166.08	319.33
8	150.24	51.92	49.68	352.71	1178.80	162.79
9	132.55	842.25	230.30	355.63	368.25	41.71
10	520.24	851.13	75.02	126.33	62.35	28.54

From Operational Performance Information System for Nuclear Power Plant [107]

The proposed approach has warranty model in the warranty period and the maintenance model in the post warranty period. But the real application covers only the warranty model. From the website [107], we obtain the failure data and repair data but the maintenance related information is not available. The real application covers only the warranty period and the simulated data covers the post warranty period.

We investigate the warranty cost analysis using repair times and failure times of the nuclear power plants in the warranty period. To conduct cost analysis, we have to find out whether the failure times and the repair times are dependent or not. In next subsection, we are going to do the dependence test.

7.5.2 Dependency Test

In this section using a nonparametric method, Kendall's τ, we are going to test the hypothesis if the failure times and repair times are dependent. If failure times and repair times are independent then we can use the proposed model which is eq. (7.3.1). Otherwise, a bivariate function with the failure times and the repair times can be used in order to obtain the warranty cost. Kendall's rank correlation measures the strength of monotonic association between the failure times and repair times. It may also be noted that usual Pearson correlation is fairly robust and it usually agrees well in terms of statistical significance with results obtained using Kendall's rank correlation. The null and the alternative hypotheses are as follows:

$$\begin{cases} H_0: \text{The failure time and the repair time are independent.} \\ H_a: \text{The failure time and the repair time are dependent.} \end{cases}$$

or

$$\begin{cases} H_0: \tau = 0 \\ H_a: \tau \neq 0 \end{cases}$$

Based on the result of Kendall's τ method using *R software* [99], for Reactor 1, τ is -0.022 and the p value is 1. Therefore, at significant level $\alpha = 0.05$, we cannot accept the alternative hypothesis H_a. It is concluded that the failed times and repair times are independent. Similarly, for Reactor 2, τ is 0.067 and the p value is 0.8618. For Reactor 3, τ is 0.33 and the p value is 0.2105. At significant level $\alpha = 0.05$, because all Reactors' p values are larger then significant level $\alpha = 0.05$, it is concluded that the failed times and repair times for Reactor 2, and 3 are independent.

7.5.3 Best Fit Distributions

Given the field data from Table 7.5.1, we now want to figure out the best fit distributions for the repair times. Calculations are based on more than 10 distributions specified from computer software. Using the computer software, it shows that for Reactor 1's repair times, Weibull distribution, exponential and gamma distributions are best three well-fitted distribution by the order of log likelihood values. For Reactor 2 and 3, the three best well fitted distributions are described in Table 7.5.2. pdfs of each distributions are as follows.

Weibull distribution with three parameters :

$$f(x)=\frac{\beta}{x-\gamma}\left(\frac{x-\gamma}{\eta}\right)^{\beta} e^{-\left(\frac{x-\gamma}{\eta}\right)^{\beta}},\ x\geq 0,\ \beta,\eta>0 \tag{7.5.1}$$

Weibull distribution with two parameters :

$$f(x)=\frac{\beta}{x}\left(\frac{x}{\eta}\right)^{\beta} e^{-\left(\frac{x}{\eta}\right)^{\beta}},\ for\ x\geq 0,\ \beta,\eta>0 \tag{7.5.2}$$

Rayleigh distribution : $f(x)=\frac{2}{x}\left(\frac{x}{\eta}\right)^{2} e^{-\left(\frac{x}{\eta}\right)^{2}},\ for\ x\geq 0$ (7.5.3)

Exponential distribution with two parameters : $f(x)=\frac{1}{\lambda}e^{-\frac{(x-\eta)}{\lambda}},\ for\ x\geq 0\ \ \lambda>0$ (7.5.4)

Gamma distribution : $f(x)=\frac{1}{\Gamma(\alpha)\theta^{\alpha}}x^{\alpha-1}e^{-\frac{x}{\theta}},\ for\ x>0,\ \alpha,\theta>0$ (7.5.5)

Log normal distribution : $f(x)=\frac{1}{x\sqrt{2\pi\sigma^{2}}}e^{-\left(\frac{\ln x-\mu}{2\sigma^{2}}\right)^{2}},\ for\ x>0$ (7.5.6)

Table 7.5.2 Three best fitted distributions of repair times for 3 reactors

Fitted distribution	Reactor 1	Reactor 2	Reactor 3
Best	Weibull (3 Par.*) $\beta=0.72$, $\eta=303.98$, $\gamma=3.75$	Weibull (3 Par.) $\beta=3.1424$ $\eta=356.8643$ $\gamma=-58.7363$	Weibull (3 Par.) $\beta=0.6428$ $\eta=131.4138$ $\gamma=5.1975$
Second best	Exponential (2 Par.) $\lambda=0.0029$ $\eta=345.9875$	Weibull (2 Par.) $\beta=2.4628$ $\eta=293.05$	Exponential (2 Par.) $\lambda=0.0060$ $\eta=167.85$
Third best	Gamma $\alpha=0.8730$ $\theta=400.6667$	Rayleigh $\beta=2.0000$ $\eta=253.2770$	Weibull (2 Par.) $\beta=0.8496$ $\eta=159.1387$

* Par. : parameters

For Reactor 1, Weibull with 3 parameters, exponential with 2 parameters and gamma distributions are well-fitted distributions. Based on the output, we choose the Weibull distribution for the repair times' pdf then conduct warranty cost analysis. After we use the computer software, we figure out that for Reactor 2, the best fitting distributions for the repair times are listed by Weibull distribution with three parameters (β= 3.1424, η= 356.8643, γ= -58.7363), Weibull distribution with two parameters (β= 2.4628, η = 293.05) and Rayleigh distribution (β= 2, η= 253.2770). And the best fitting distributions for Reactor 3 are listed by Weibull distribution with three parameters (β = 0.6428, η = 131.4138, γ = 5.1975), exponential distribution with two parameters (β = 0.0060, η = 167.85) and Weibull distribution with two parameters (β= 0.8496, η= 159.2770).

For the failure rates of NHPP, many researchers [180] have investigated and studied the estimation and simulation of NHPP. However, in the study, we simplified the estimation for the failures rates of NHPP and used linear regression for cdf of failure times. Therefore, for each reactors, we obtain three different failure rates of NHPP which are given by $\lambda_1(t_1) = 214t_1 + 455$, $\lambda_2(t_1) = 475t_1 - 527$, and $\lambda_3(t_1) = 429t_1 - 713$. Using this failure rates, we can estimate the expected number of warranty services.

7.5.4 Long Run Expected Cost

Due to a complex nonlinear function, we use the Nelder-Mead downhill simplex method [92, 137] to obtain an optimum solution for the above optimization problem, with a hope to obtain the global solution. In this study, we also have used other non-linear optimization approaches such as random search method, differential evolution method and simulated annealing method. Among them, Nelder-Mead downhill simplex method seems to be the most popular direct search method for obtaining the optimum solution of a nonlinear function, which does not require the calculation of derivatives. Using the Nelder-Mead approach, we optimize the long run expected cost of the model. For the numerical example below, we apply our developed models. We wish to find the optimal periods w_1^*, w_2^* and the optimal maintenance cycle, T^*, which minimizes the long run expected costs. In summary, we can formulate the following optimization problem:

$$\text{Find } (\mathrm{w}_1^*, \mathrm{w}_2^*, T^*): \qquad \text{Minimize } L(\mathrm{w}_1, \mathrm{w}_2, T \mid p, q)$$

We consider that after PM service, a system is good as new with probability p and is bad as old with probability $1-p$. The PM service is considered imperfect, because it is more likely in real life.

A 1-out-of-3 system is considered under manufacturers' point of view and PRW policy and m is assumed to be 1. Suppose that a time to failure of each component follows the Weibull distribution eq. (7.5.2). Additionally, y_1 is the time to perform CPM and y_2 is the time to perform PM. c_{cpm} is cost of CPM and c_{pm} is cost of PM. Let

$$y_1 = 0.6, y_2 = 0.4, c_{cpm} = \$2000, c_{pm} = \$100$$

Under alternative PRW and under customers' point of view, total warranty cost from eq. (7.3.7) is given by

$$TC_2(w) = \left(1 - \frac{\frac{2w_1 + 12}{3}}{w_1}\right)\left(1 - \frac{\frac{w_2}{2}}{w_2}\right)\sum_{x_s=1}^{\infty} x_s \left(\left(1-R_1(x)\right)\left(1-R_2(x)\right)\left(1-R_3(x)\right)\right)^{x_s}$$

where $R_j(x) = 1 - \sum_{i=0}^{x-1} \frac{e^{-\iint_{\Psi} m_j(t_1)\cdot f_{R,j}(t_2)\,dt_2 dt_1} \left[\iint_{\Psi} m_j(t_1)\cdot f_{R,j}(t_2)\,dt_2 dt_1\right]^i}{i!}$, $j = 1,2,3$

Each nuclear reactor's reliability function, $R_j(x)$ can be obtained from previous Section 7.5.3 Best fitted distribution. Denote $E(T_1)$ and $E(T_2)$ as follows:

$$E(T_1)=\frac{2w_1+12}{3}$$
$$E(T_2)=\frac{w_2}{2} \tag{7.5.2}$$

Using eq. (7.3.9), the expected value of $g(t_{1i},t_{2i})$ is given by

$$E\left(g\left(t_{1i},t_{2i}\right)\right)=\begin{cases}\left(1-\dfrac{E(t_{1i})}{w_1}\right)\left(1-\dfrac{E(t_{2i})}{w_2}\right) & if\ (t_{1i},t_{2i})\in\Omega\\ 0 & if\ (t_{1i},t_{2i})\notin\Omega\end{cases} \tag{7.5.3}$$

We can rewrite the expected value of $g(t_{1i},t_{2i})$ as follows:

$$E\left(g\left(t_{1i},t_{2i}\right)\right)=\begin{cases}\left(1-\dfrac{\frac{2w_1+12}{3}}{w_1}\right)\left(1-\dfrac{\frac{w_2}{2}}{w_2}\right) & if\ (t_{1i},t_{2i})\in\Omega\\ 0 & if\ (t_{1i},t_{2i})\notin\Omega\end{cases} \tag{7.5.4}$$

Substituting the above data into eq. (7.3.28), the long run expected cost per unit time under the customers' point of view is given by

$$L_1\left(w_1,w_2,T\middle|p,q\right)$$
$$=\frac{\left(\left(1-\dfrac{\frac{2w_1+12}{3}}{w_1}\right)\left(1-\dfrac{\frac{w_2}{2}}{w_2}\right)\sum_{x_s=1}^{\infty}x_s\left(\prod_{j=1}^{3}\left(1-R_j(x)\right)\right)^{x_s}+2000\cdot\left(p\sum_{j=1}^{\infty}q^{j-1}\left(F_{m=1}(t)\right)\right)+100\cdot\sum_{j=1}^{\infty}q^{j-1}\left(R_{m=1}(t)\right)\right)}{\left(w_1+\int_0^{T-w_1}R_{m=1}(t)\,dt+\sum_{j=2}^{\infty}q^{j-1}\int_{(j-1)T-w_1}^{jT-w_1}R_{m=1}(t)\,dt+0.4\cdot\left(p\sum_{j=1}^{\infty}q^{j-1}\left(F_{m=1}(jT-w_1)\right)\right)+0.6\cdot\sum_{j=1}^{\infty}q^{j-1}\left(R_{m=1}(jT-w_1)\right)\right)}$$

subject to

$$w_1,w_2\geq 0,T>0,w_1<T$$

where $R_{m=1}\left(t\right)=1-\sum_{i=0}^{m=1}\binom{3}{i}\left(1-e^{\left(-\left(\frac{t}{200}\right)^{2}\right)}\right)^{i}\left(e^{\left(-\left(\frac{t}{200}\right)^{2}\right)}\right)^{3-i}$

We consider three optimum policies of the warranty period, the repair time limit, and the maintenance cycle. Under 60% for the perfect maintenance, we obtain the optimal solution as follows:

$$w_1^* = 297, w_2^* = 3.1,\ T^* = 518 \text{ and } L_1\left(w_1^*, w_2^*, T^* \middle| p, q\right) = 0.9639$$

Only minimal repair is performed for any component's failure in the warranty period. In the post warranty period, PM services would be repeated. The failed component will be subject to CM together with PM on the remaining deteriorating components when there are m numbers of failed components in the middle of maintenance cycles in the post warranty period. If less than m numbers of components fails until time T^*, PM is carried out at $T^* = 518$ and the optimized warranty period is 297.

7.6 Concluding Remarks

In this chapter, we showed the methodology for the warranty period, post warranty cost analyses and the long run expected cost per unit time subject to minimal repairs. In the warranty period, the warranty cost methodology is that a certain number of failures are considered in the censored area by the warrant period and the repair time limit. Obviously the longer warranty period is more helpful to increase the sales and the number of customers. We investigate the optimal warranty period when there is the limitation of the warranty cost reserve. Considering warranty conditions such as different perspective of views (customers vs manufacturers) and warranty policies (FRW, alternative PRW and CMW), we obtain the optimal warranty period and the repair time limit. We also proposed the two-dimensional NHPP and obtained the expected warranty cost.

For the post warranty period, we considered two types of warranty services such as CPM and PM. There are periodically interval T and when there are less than the threshold number of failures, manufacturers provide periodic PM services. However, when there are the threshold numbers of failures, then they provide CPM services. Also, we considered perfect PM services and imperfect PM services. We discussed the alternative PRW instead of the original PRW and obtained the long run expected cost per unit time and optimal policies for (w, T). We presented several numerical examples to demonstrate the applicability of the methodologies and results derived in the chapter. Eventually, we obtain two optimized local values for two numerical examples.

Chapter 8

A New Two-dimensional Warranty Policy with Repair Times and Failure Times using Field Data

8.1 Introduction

Some people insist that the most effective way to measure a manufacturer's performance is not the frequency of recalls but rather the amount manufacturers pay in warranty costs relative to their revenues [179]. Although Toyota has outshined its major American Competitors before, it is struggling by recent recalls for their popular models these days. Industry analyst estimates that it will cost some $250 million in warranty costs alone to address one of the two recalls in the United States in Feb. 2010 [48]. Reuters said that Toyota, reeling from its largest recall in history, is considering a new warranty program that at least matches Hyundai's market-leading 10-year, 100,000 mile power train warranty [86]. As we see, the warranty can be used as not only the important measurement for a manufacturer's performance but also big promotions to allure the customer.

Also, warranty policy is a guarantee for the seller to provide to the buyer with a specific service such as repair or replacement in the event of the product failure. Nowadays with a global competition amongst manufacturers, having a good quality product with an eye catching design is not sufficient enough to appeal to consumers to make a purchase. Other intangible features, warranty policy, after sales service (A/S), brand *et al.*, are becoming important factors in consumer behavior. Further as the economy contracts, consumers will want to extend the usage period by fixing or replacing parts to reduce cost. As a consequence, the consumer's demand for a more detailed warranty policy is increasing. This is why the warranty policy is becoming more important to companies, as they can strategically use warranty policy as a promotional tool to appeal to consumers.

Many different types of warranties may be defined based on the characteristics of warranty policies [20, 21]. Various common from a simple to complicates warranty policies are as follows. The one dimensional warranty is characterized by the warranty period and the two-dimensional warranty is characterized by a region in a two-dimensional plane with the usage and the age/time. In other words, single variable could be time, age or usage. In the case of two-dimensional warranties, one is representing time and the other representing item usage. Using two-dimensional warranty policy, we calculate the warranty cost and investigate the statistical properties of warranty models. Several researchers [32, 42, 43, 70, 71, 83, 107, 197] have proposed two-dimensional models under warranty. Yun and Kang [197] examine new warranty servicing strategy, considering imperfect repair with a two-dimensional warranty. Chukova and Johnston [43] consider that the warranty has options in choosing the degree of repair applied to an item that has failed within the warranty period and develop a particular warranty repair strategy, related to the degree of the warranty repair, for non-renewing, two-dimensional, free of charge to the consumer warranty policy. Chun and Tang [44] propose several decision models that estimate the expected total cost incurred under various types of two-attribute warranty policies. Kim and Rao [85] consider two-attribute warranty policies for non-repairable items and the item failures are described in terms of a bivariate exponential distribution. Iskandar *et al.* [71] investigate a new warranty servicing strategy for items sold with two-dimensional warranty where the failed item is replaced by a new one when it fails for the first time in a specified region of the warranty and all other failures are repaired minimally. In Chen and Popova's paper [32], they suggest a new maintenance policy which minimizes the total expected servicing cost for an item with two-dimensional warranty.

If the failure times and the repair times are independent, marked Poisson process could be used to develop the two-dimensional warranty model [121]. If they are dependent, bivariate distributions could be used to model the failure times and the repair times. From the proposed model, one can determine the number of warranty services under warranty. To test whether if they are independent or dependent, there are several ways might be considered. If the failure times and the repair times are following the bivariate normal distribution, t test could be used to check their dependence. On the contrary, if they are not following the bivariate normal distribution, nonparametric methods such as Spearman rank correlation coefficient test or Kendall's τ test could be used.

Using the field data, we can obtain the key information to evaluate product's reliability practically. Because of this reason, many researchers have investigated various topics from the field data. Among them, Several researchers [94, 95] have studied the warranty policy based on the field data from real applications. Majeske [94] proposes a general mixture model framework for automobile warranty data that includes parameters for product field performance, the manufacturing and assembly process and dealer preparation process. Oh and Bai [116] develop methods for estimating the lifetime distribution for situations where additional field data can be gathered after the warranty expires in a parametric time to failure distribution. Jung and Bai [83] consider a method of estimating lifetime distribution for products under two-dimensional warranty in which age and usage are used simultaneously to determine the eligibility of a warranty claim.

Field data provides important information which is useful to evaluate the reliability of the product, to investigate the weak point of the product and to compare the product's design, material and production methods. Additionally, instead of the usage and the age, we consider totally different two dimensions such as failure times and repair times. We study the bivariate distributions of product's failure times and repair times for the two-dimensional warranty policy using the field data.

The remainder of this chapter is organized as follows. In Section 8.2, the problem description is discussed. Section 8.3 focuses on the two-dimensional warranty models when failures times and repair times are dependent. Two-dimensional renewal function is obtained using BED for several types of warranty policies. An illustrative example is given in Section 8.4 to show the two-dimensional warranty models with BEDs using a field data. Finally, we discuss the strengths and the weakness of the proposed approach in Section 8.5 and concluding remarks are given in Section 8.6.

8.1.1 Nomenclature

w_1, w_2 : Warranty period and time limit of the repair service, respectively

$f(\cdot), F(\cdot), \bar{F}(\cdot), L(\cdot)$: pdf, cdf, reliability function and likelihood function, respectively

$f(x, y), F(x, y)$: Bivariate pdf and cdf, respectively for failure times X and repair times Y

$M(w_1, w_2)$: Expected number of warranty services by the warranty period w_1 and by the repair time limit w_2

c : a warranty cost per failure

$N_1(w_2), N_2(w_1, w_2), N_3(w_{11}, w_{12}, w_{21}, w_{22}), N_4(w_{11}, w_{12}, w_{21}, w_{22})$: number of repair services under warranty for Policy (1), for Policy (2), for Policy (3) and for Policy (4), respectively.

8.1.2 Assumptions

- Repair and replacement do not happen simultaneously. Although they tried to repair it for the time being in customer service center, but the failed product can't be repaired, the failed product would be replaced.
- When a product fails, the repair service would be provided first.
- Repair cost and replacement cost are constant.
- For the customer's satisfaction, the repair times are excluded in the warranty period.

8.2 Problem Description

The typical well-known two-dimensional warranty policy is to use usage and age/time as two-dimensions. In this chapter, totally different two-dimensions are used such as the failure times and the repair times. In this developed two-dimensional warranty policy, if a customer's failed product is delivered to the customer service center for repair services, the customer service center is supposed to return the fixed product back within the threshold time for the customer's satisfaction. Therefore, if the failed product can't be repaired after the time being, the replacement service is provided instead of continuing to repair. The failure times would be censored by the warranty period, on the other hand, the repair times would be censored by the limit of the repair time. We consider the warranty services which happen only within two censored limitations.

When we investigate two-dimensional warranty using failure times and repair times instead of usage and age/time, we separate the two cases: when repair times and failure times are dependent and when repair times and failure times are independent. If they are independent, Park and Pham [121] recently develop two-dimensional warranty model using non-homogeneous Poisson process and consider other conditions such as under different warranty policies (free repair/replacement, pro-rata and combination) and under different point of views (customer and manufacturer) including post warranty period. Based on the field data and since they are dependent, we first determine a bivariate distribution and then develop two-dimensional warranty models with repair times and failure times. The failure time is the interval between product's recovery time for previous failure and next failure time and the repair time is the interval between a failure time and its recovery time.

8.3 Model Formulation

Murthy *et al.* [107] derived expressions for the expected warranty cost for the four policies. We develop four different policies based on different shapes of the repair time limits and discuss the two-dimensional renewal functions to model warranty cost.

Four models with different time limit of repair services are described in Fig. 8.3.1. Let w_2 be a time limit for the repair service and w_1 be a warranty period. Also, T_1 axis has constant warranty period limitation for the failure times and on the other hand, T_2 axis has constant limit for the repair times. In Fig. 8.3.1 (a), we consider there is no warranty period. If we use this model in the customer service center, they can obtain the repair cost using the number of repair services without the warranty period. Then the customer service center can obtain the repair cost. They use repair time limit, as mentioned before, which indicates that if they can't repair the failed product within the time limit, then they provide the replacement service instead of continuing to repair. Using this model, the manufacturer could handle the repair cost and replacement cost using the repair time limit and find a way to maximize the customers' satisfaction and to minimize the company's cost.

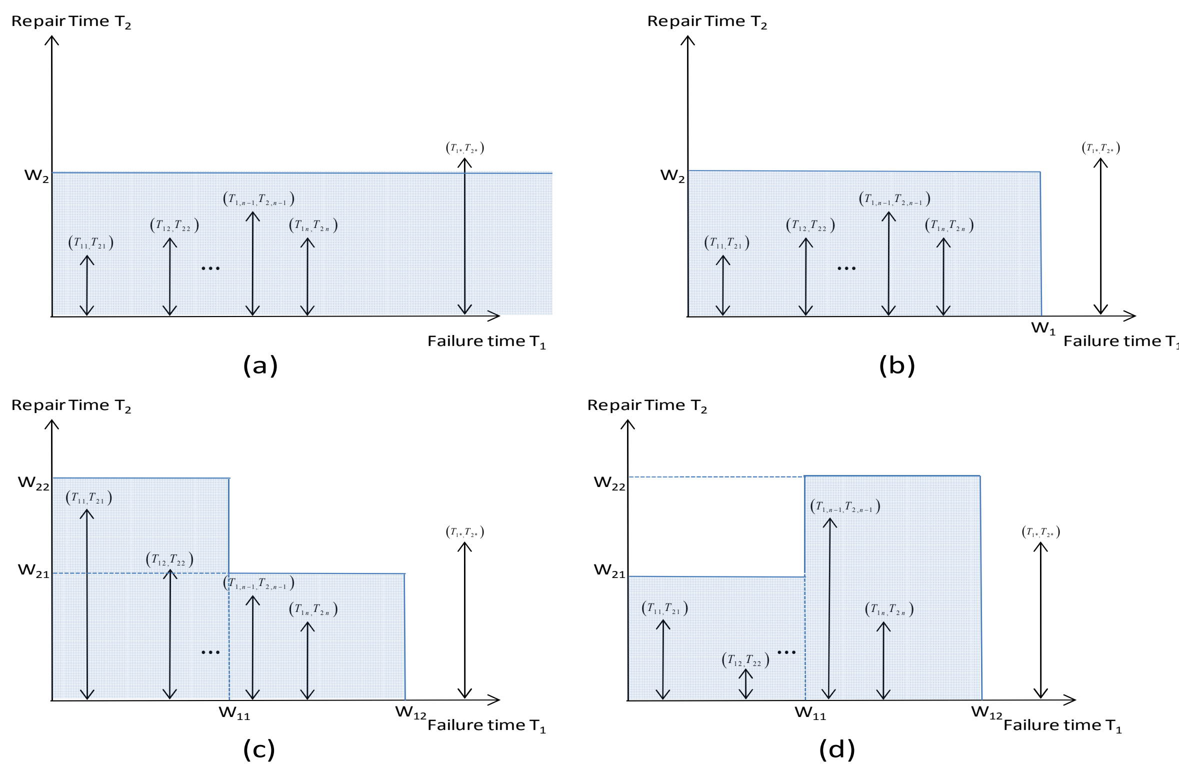


Fig. 8.3.1: Two-dimensional warranty policies with various time limits of the repair times

In Fig. 8.3.1 (b), the time limit is square-shaped which is a basic-shaped model using the repair time limit and the warranty period. Fig. 8.3.1 (c) and Fig. 8.3.1 (d) have ladder-shaped limits for the repair time. Fig. 8.3.1 (c) has the repair time limit which is higher level at the earlier period and then lower level at the later period. Lastly, Fig. 8.3.1 (d) has the repair time limit which is lower level at the earlier period and then higher level at the later period.

8.3.1 Two-dimensional Renewal Function

Let $M\left(w_1, w_2\right)$ be the bivariate renewal function. Two-dimensional renewal function plays an important role in the analysis of two-dimensional warranty policies. But it is difficult to obtain analytic expressions for $M\left(w_1, w_2\right)$ and computational procedures are generally required. Hunter [69] obtains the analytical expression for $M\left(w_1, w_2\right)$ using Downton's BED [52]. It is rare that the transform is invertible in closed form. For most of the bivariate models, closed Laplace transform inversions are not available [109]. Let $f^*\left(s_1, s_2\right)$ be bivariate Laplace transform of $f\left(x, y\right)$ and $F^*\left(s_1, s_2\right)$ be the bivariate Laplace transform of $F\left(x, y\right)$, the cumulative density function of $f\left(x, y\right)$. Then we know that

$$L\left\{F\left(x, y\right)\right\} = F^*\left(s_1, s_2\right) = \frac{f^*\left(s_1, s_2\right)}{s_1 s_2} \tag{8.3.1}$$

From the properties of Laplace transforms, we have

$$f^{*(n)}\left(s_1, s_2\right) = \left[f^*\left(s_1, s_2\right)\right]^n, \ F^{*(n)}\left(s_1, s_2\right) = \frac{\left[f^*\left(s_1, s_2\right)\right]^n}{s_1 s_2} \tag{8.3.2}$$

A requisite property of $f\left(x, y\right)$ is that the conditional expectations of X and Y must be increasing functions of the other variable. Clearly, the expected warranty cost as well as the variance increase linearly with y. Let $F^{(n)}\left(x, y\right)$ be *n-fold* convolution function of $F\left(x, y\right)$.

$$M\left(x, y\right) = \sum_{n=1}^{\infty} F^{(n)}\left(x, y\right) \tag{8.3.3}$$

We obtain the bivariate Laplace transform of $M_\rho(x,y)$ as:

$$M^*(s_1,s_2)=\frac{f^*(s_1,s_2)}{s_1 s_2\left[1-f^*(s_1,s_2)\right]} \tag{8.3.4}$$

If we use the Laplace transform, we obtain the reverse renewal function, eq. (8.3.4). However, it is difficult to obtain two-dimensional renewal function using the reverse Laplace transform.

Therefore, in the chapter, after Ross' algorithm [142] is extended, two-dimensional renewal function is obtained. Let $N(w_1,w_2)$ be the number of warranty services within the warranty period. Let (x,y) be the failure times and the repair times respectively. Later, their parameters could be calculated using the field data.

A bivariate extension of the exponential distribution is proposed as a model for certain problems in reliability engineering. The exponential distribution plays a fundamental role as a model in a variety of applications, typically connected with survival time, in some of its many forms of appearance. However, unfortunately, unlike the normal distribution, the exponential distribution does not have a natural extension to the bivariate or the multivariate case. Therefore, a large number of classes of bivariate distributions with exponential marginals have been proposed since 1960. Nadarajah and Kotz [113] derive the several distributions using bivariate exponential random variables. In BED, several researchers have developed their own type of bivariate exponential. Downton [52] derives his BED using a simple failure model in 1970. Hawkes [65] extends Downton's BED and proposes a more general distribution. Freund [58] proposes another bivariate extension of the exponential distribution as a model for certain problems in life testing for a two-component systems. Among them, the BEDs with the memoryless property are Marshall & Olkin's [98], Freund's [58] and Block &

Basu's [23] from [87], on the contrary, the BED without the memoryless property is Raftery's [135]. Also, if the marginal distributions of BED are exponential, then we can use the BED for the field data. Marshall & Olkin's BED [98] and Raftery's BED [135] have exponential marginals. Freund's BED [58] and Block & Basu's BED [23] have marginals which are mixture of exponential distributions. In the study, Marshall & Olkin's BED is chosen for the warranty cost analysis because it has memoryless property and exponential marginal. In Marshall & Olkin's BED, both the marginals have exponential distribution, and they can be equal with a positive probability. Because of that reason, if in a bivariate data set, for some cases two dimensions take values with positive probabilities, the Marshall & Olkin's BED can be used quite effectively to analyze such data set [91]. From [97, 98], the Marshall & Olkin's BED's joint probability density function is given by

$$f(x,y)=\theta_1(\theta_2+\theta_3)\exp\left(-\theta_1 x-(\theta_2+\theta_3)y\right) \tag{8.3.5}$$

for $0<x<y$ and

$$f(x,y)=\theta_2(\theta_1+\theta_3)\exp\left(-\theta_2 x-(\theta_1+\theta_3)y\right) \tag{8.3.6}$$

for $0<y<x$ and

$$f(x,y)=\theta_3\exp\left(-(\theta_1+\theta_2+\theta_3)y\right) \tag{8.3.7}$$

for $0<x=y,$ when $x>0,$ $y>0,$ $\theta_1>0,$ $\theta_2>0,$ $\theta_3>0.$

The marginal pdfs of X and Y are exponential with parameters $\theta_1+\theta_3$ and $\theta_2+\theta_3$, respectively; so, in particular,

$$E(X)=\frac{1}{\theta_1+\theta_3} \qquad E(Y)=\frac{1}{\theta_2+\theta_3} \tag{8.3.8}$$

The correlation coefficient $\rho=Cor(X,Y)$ is given by

$$\rho = \frac{\theta_3}{\theta_1 + \theta_2 + \theta_3}. \tag{8.3.9}$$

8.3.2 Expected Number of Warranty Service Modeling

We start by determining $E\left[N\left(w_1, w_2\right)\right]$, the expected number of renewals in the censored area of $\left(w_1, w_2\right)$. First, we condition on X_1 *and* Y_1, the times of the first failure renewal and the first repair renewal. Using the conditional probability, $E\left[N\left(w_1, w_2\right)\right]$ can be written as follows:

$$\begin{aligned} E\left[N\left(w_1, w_2\right)\right] &= E\left[E\left[N\left(w_1, w_2\right) \middle| X_1, Y_1\right]\right] \\ &= \int_0^\infty \int_0^\infty E\left[N\left(w_1, w_2\right) \middle| X_1 = x, Y_1 = y\right] f\left(x, y\right) dxdy \end{aligned} \tag{8.3.10}$$

where $f\left(x, y\right)$ is the joint inter-arrival density. To determine $E\left[N\left(w_1, w_2\right) \middle| X_1 = x, Y_1 = y\right]$, we now condition on whether or not the two constants $\left(w_1, w_2\right)$ exceed $\left(x, y\right)$, respectively. So we consider 4 cases as follows:

$$\begin{aligned} &1)\ w_1 < x \ and \ w_2 < y \\ &2)\ w_1 \geq x \ and \ w_2 < y \\ &3)\ w_1 < x \ and \ w_2 \geq y \\ &4)\ w_1 \geq x \ and \ w_2 \geq y \end{aligned} \tag{8.3.11}$$

If we are given that $w_1 \geq x$ *and* $w_2 \geq y$, then the number of renewals by time will equal 1 plus the number of additional renewals between w_1 and x and between w_1 and y. But if the inter-failure intervals follow a BED which has the bivariate lack of memory property, it follows that, given that $w_1 < x$ *and* $w_2 < y$, the amount by which they exceed x and y is also bivariate exponential, and so given that the number of renewals between w_1 and x and between w_2 and y will have the same distributions as $N(w_1, w_2)$ by the memoryless property of exponential random variables. On the other hand, for other cases, as the first renewal occurs by times x and y, it follows that the number of renewals by times (w_1, w_2) is equal to zero. Hence,

$$\begin{aligned}
&E\left[N(w_1,w_2)\middle| X_1 = x, Y_1 = y, w_1 < x, w_2 < y\right] = 0 \\
&E\left[N(w_1,w_2)\middle| X_1 = x, Y_1 = y, w_1 \geq x, w_2 < y\right] = 0 \\
&E\left[N(w_1,w_2)\middle| X_1 = x, Y_1 = y, w_1 < x, w_2 \geq y\right] = 0 \\
&E\left[N(w_1,w_2)\middle| X_1 = x, Y_1 = y, w_1 \geq x, w_2 \geq y\right] = 1 + E\left[N(w_1,w_2)\right]
\end{aligned} \tag{8.3.12}$$

And using eq. (8.3.12) if the first failure time is X_1 and its repair time is Y_1, the expected number of warranty services within repair service time limit w_2 and the warranty period w_1 is given by

$$\begin{aligned}
&E\left[N(w_1,w_2)\middle| X_1 = x, Y_1 = y\right] \\
&= E\left[N(w_1,w_2)\middle| X_1 = x, Y_1 = y, w_1 < x, w_2 < y\right] P\left\{w_1 < x, w_2 < y \middle| X_1 = x, Y_1 = y\right\} \\
&+ E\left[N(w_1,w_2)\middle| X_1 = x, Y_1 = y, w_1 \geq x, w_2 < y\right] P\left\{w_1 \geq x, w_2 < y \middle| X_1 = x, Y_1 = y\right\} \\
&+ E\left[N(w_1,w_2)\middle| X_1 = x, Y_1 = y, w_1 < x, w_2 \geq y\right] P\left\{w_1 < x, w_2 \geq y \middle| X_1 = x, Y_1 = y\right\} \\
&+ E\left[N(w_1,w_2)\middle| X_1 = x, Y_1 = y, w_1 \geq x, w_2 \geq y\right] P\left\{w_1 \geq x, w_2 \geq y \middle| X_1 = x, Y_1 = y\right\} \\
&= E\left[N(w_1,w_2)\middle| X_1 = x, Y_1 = y, w_1 \geq x, w_2 \geq y\right] P\left\{w_1 \geq x, w_2 \geq y\right\} \\
&= \left(1 + E\left[N(w_1,w_2)\right]\right) P\left\{w_1 \geq x, w_2 \geq y\right\}
\end{aligned} \tag{8.3.13}$$

Substituting this into eq. (8.3.10), we obtain

$$\begin{aligned}
E\left[N(w_1,w_2)\right] &= \int_0^\infty \int_0^\infty \left(1 + E\left[N(w_1,w_2)\right]\right) P\left\{w_1 \geq x, w_2 \geq y\right\} f(x,y)\,dxdy \\
&= \left(1 + E\left[N(w_1,w_2)\right]\right) \int_0^\infty \int_0^\infty P\left\{w_1 \geq x, w_2 \geq y\right\} f(x,y)\,dxdy
\end{aligned} \tag{8.3.14}$$

or

$$M(w_1,w_2)=E\left[N(w_1,w_2)\right]=\frac{\int_0^\infty\int_0^\infty P\{w_1\ge x,w_2\ge y\}f(x,y)\,dxdy}{1-\int_0^\infty\int_0^\infty P\{w_1\ge x,w_2\ge y\}f(x,y)\,dxdy} \tag{8.3.15}$$

To obtain the variance of the warranty system cost, we first need to calculate the second moment. Similarly to the first moment, we consider the first failure during the warranty period. We separate four cases such as eq. (8.3.12). Then, similarly to eq. (8.3.13),

$$E\left[N(w_1,w_2)^2\middle|X_1=x,Y_1=y,w_1\ge x,w_2\ge y\right]=E\left[\left(1+N(w_1,w_2)\right)^2\right] \tag{8.3.16}$$

and remaining three cases equal to zero. Therefore,

$$\begin{aligned}&E\left[\left(N(w_1,w_2)\right)^2|X_1=x,Y_1=y\right]\\&=\left(1+2E\left(N(w_1,w_2)\right)+E\left(N(w_1,w_2)^2\right)\right)P(w_1\ge x,w_2\ge y)\end{aligned} \tag{8.3.17}$$

Using eq. (8.3.17), we obtain the second moment as follows:

$$\begin{aligned}E\left[N(w_1,w_2)^2\right]&=E\left[E\left[\left(N(w_1,w_2)\right)^2|X_1=x,Y_1=y\right]\right]\\&=\int_0^\infty\int_0^\infty E\left[\left(N(w_1,w_2)\right)^2|X_1=x,Y_1=y\right]f(x,y)\,dxdy\\&=\int_0^\infty\int_0^\infty\left(1+2E\left(N(w_1,w_2)\right)+E\left(N(w_1,w_2)^2\right)\right)P(w_1\ge x,w_2\ge y)f(x,y)\,dxdy\\&=\left(1+2E\left(N(w_1,w_2)\right)+E\left[N(w_1,w_2)^2\right]\right)\int_0^\infty\int_0^\infty P\{w_1\ge x,w_2\ge y\}f(x,y)\,dxdy\end{aligned} \tag{8.3.18}$$

After simplifications, the second moment is given by

$$E\left[N(w_1,w_2)^2\right]=\frac{\left(1+2E\left(N(w_1,w_2)\right)\right)\int_0^\infty\int_0^\infty P\{w_1\ge x,w_2\ge y\}f(x,y)\,dxdy}{\left(1-\int_0^\infty\int_0^\infty P\{w_1\ge x,w_2\ge y\}f(x,y)\,dxdy\right)} \tag{8.3.19}$$

where $E\left[N(w_1,w_2)\right]$ is given as eq. (8.3.15).

Using the first moment and the second moment, we easily obtain the variance of the number of warranty services.

$$Var\left(N\left(w_1,w_2\right)\right)$$
$$=\frac{\left(1+2E\left[N\left(w_1,w_2\right)\right]\right)\int_0^\infty\int_0^\infty P\left\{w_1\ge x,w_2\ge y\right\}f\left(x,y\right)dxdy}{\left(1-\int_0^\infty\int_0^\infty P\left\{w_1\ge x,w_2\ge y\right\}f\left(x,y\right)dxdy\right)}-\left(E\left[N\left(w_1,w_2\right)\right]\right)^2 \quad (8.3.20)$$

where $E\left[N\left(w_1,w_2\right)\right]$ is given from eq. (8.3.15).

8.3.3 Various Policies - Expected Cost Models

In this section, based on the developed warranty policies in Fig. 8.3.1, we develop the number of warranty services under warranty and obtain the expected warranty cost.

Policy (a)

For Policy (a), we do not consider the warranty period. We only consider the repair time limit and it indicates that one-dimensional warranty policy is investigated. Park and Pham [120] obtain the renewal function for one-dimensional warranty model. If $N_1\left(w_2\right)$ denotes the number of repair services under warranty for Policy (a), then the expected number of repair service is given by

$$M\left(w_2\right)=E\left[N_1\left(w_2\right)\right]=\frac{\int_0^\infty P\left(w_2\ge y\right)f_Y\left(y\right)dy}{1-\int_0^\infty P\left(w_2\ge y\right)f_Y\left(y\right)dy} \quad (8.3.21)$$

and its variance is given by

$$Var\left(N_1\left(w_2\right)\right)=\frac{\left(1+2E\left[N\left(w_2\right)\right]\right)\int_0^\infty P\left(w_2\ge y\right)f_Y\left(y\right)dy}{\left(1-\int_0^\infty P\left(w_2\ge y\right)f_Y\left(y\right)dy\right)}-\left(E\left[N\left(w_2\right)\right]\right)^2 \quad (8.3.22)$$

where $E\left[N_1\left(w_2\right)\right]$ obtained from eq. (8.3.21).

Let c be a warranty cost per failure and $N_1(w_2)$ be the number of renewals over the warranty period w_2. The expected warranty cost per item sold is given by

$$\begin{aligned} E_1(C) &= c \cdot E\left(N_1(w_2)\right) \\ &= c \cdot M(w_2) \end{aligned} \tag{8.3.23}$$

where $M(w_2)$ obtained from eq. (8.3.21).

$$Var_2(C) = c^2 \cdot Var\left(N_1(w_2)\right) \tag{8.3.24}$$

where $Var\left(N_1(w_2)\right)$ obtained from eq. (8.3.22).

Policy (b)

If $N_2(w_1, w_2)$ denotes the number of repair services under warranty for Policy (b), then the expected warranty cost per item sold is given by

$$\begin{aligned} E_2(C) &= c \cdot E\left(N_2(w_1, w_2)\right) \\ &= c \cdot M(w_1, w_2) \end{aligned} \tag{8.3.25}$$

where $M(w_1, w_2)$ obtained from eq. (8.3.15).

$$Var_2(C) = c^2 \cdot Var\left(N(w_1, w_2)\right) \tag{8.3.26}$$

where $Var\left(N(w_1, w_2)\right)$ obtained from eq. (8.3.20).

Policy (c)

In case of Policy (c), there are two different levels of repair time limits. In the beginning, the time limit of repair services is a high level. Then, after a time being, the time limit is a low level. In other words, the time limits are changed by two different linear lines. It is described in Fig. 8.3.1 (c). If $N_3(w_{11}, w_{12}, w_{21}, w_{22})$ denotes the number of repair services under warranty for Policy (c), the expected warranty cost per item sold is given by

$$
\begin{aligned}
E_3(C) &= c \cdot E\left(N_3\left(w_{11}, w_{12}, w_{21}, w_{22}\right)\right) \\
&= c \cdot M\left(w_{11}, w_{12}, w_{21}, w_{22}\right) \\
&= c \cdot \left(M\left(w_{11}, w_{22}\right) + M\left(w_{12}, w_{21}\right) - M\left(w_{11}, w_{21}\right)\right)
\end{aligned}
\tag{8.3.27}
$$

where $M(\cdot,\cdot)$ obtained from (8.3.15)

$$
Var_3(C) = c^2 \cdot \left(Var\left(N\left(w_{11}, w_{22}\right)\right) + Var\left(N\left(w_{12}, w_{21}\right)\right) + Var\left(N\left(w_{11}, w_{21}\right)\right)\right) \tag{8.3.28}
$$

where $Var\left(N(\cdot,\cdot)\right)$ obtained from eq. (8.3.20).

Policy (d)

In case of Policy (d), there are two different levels of repair time limits. In the beginning, the time limit of repair services is a low level. Then, after a time being, the time limit is a high level. It is described in Fig. 8.3.1 (d). If $N_4\left(w_{11}, w_{12}, w_{21}, w_{22}\right)$ denotes the number of repair services under warranty for Policy (d), the expected warranty cost per item sold is given by

$$
\begin{aligned}
E_4(C) &= c \cdot E\left(N_4\left(w_{11}, w_{12}, w_{21}, w_{22}\right)\right) \\
&= c \cdot M\left(w_{11}, w_{12}, w_{21}, w_{22}\right) \\
&= c \cdot \left\{M\left(w_{12}, w_{22}\right) - \left(M\left(w_{11}, w_{22}\right) - M\left(w_{11}, w_{21}\right)\right)\right\}
\end{aligned}
\tag{8.3.29}
$$

where $M(\cdot,\cdot)$ obtained from (8.3.15).

$$
Var_4(C) = c^2 \cdot \left(Var\left(N\left(w_{12}, w_{22}\right)\right) + Var\left(N\left(w_{11}, w_{22}\right)\right) + Var\left(N\left(w_{11}, w_{21}\right)\right)\right) \tag{8.3.30}
$$

where $Var\left(N(\cdot,\cdot)\right)$ obtained from eq. (8.3.20).

8.4 Illustrative Example Using the Field Data

In South Korea, there are four nuclear sites and, in 2009, there are 20 nuclear power plants in operation with a total licensed output amount to 17,716 MWe (MegaWatt electrical) and 8 nuclear power plants under construction, for a total of 28 units in operation by the end of 2016 [2]. We investigate the field data to check their dependency using a nonparametric method. We implement our proposed approaches to conduct warranty cost analysis using the field data.

8.4.1 Data Description

Among 20 nuclear power plants in the four nuclear sites in South Korea, we summarize the 30 failure data for nuclear power plants for relatively recent events or failures in Table 1. It describes the failure data and the repair data.

Table 8.4.1: Failure times and repair times for nuclear power plants

No.	Failure (days)	Repair (hours)	No.	Failure (days)	Repair (hours)
1	353.04	104.88	16	30.27	87.12
2	334.72	45.84	17	117.37	65.52
3	80.04	48.96	18	126.27	61.2
4	6.49	41.28	19	56.45	17.28
5	1.34	6.96	20	45.28	88.56
6	467.19	46.32	21	267.31	8.64
7	74.18	405.36	22	615.64	255.12
8	398.86	42.48	23	115.37	269.76
9	1048.23	230.64	24	359.76	232.8
10	829.39	91.2	25	412.3	79.44
11	227.2	68.64	26	276.69	119.04
12	260.14	7.44	27	601.04	71.76
13	14	20.4	28	192.17	39.12
14	14.15	48.96	29	0.36	6.24
15	38.96	65.52	30	1021.01	56.64

From Operational Performance Information System for nuclear Power Plant (http://opis.kins.re.kr/index.jsp?Lan=US)

The exploratory data analysis on the failure times and the repair times is conducted to know their attribution. By the shape of each box plots and histograms in Fig. 8.4.1, we would know that the failure times and repair times are not normally distributed, respectively.

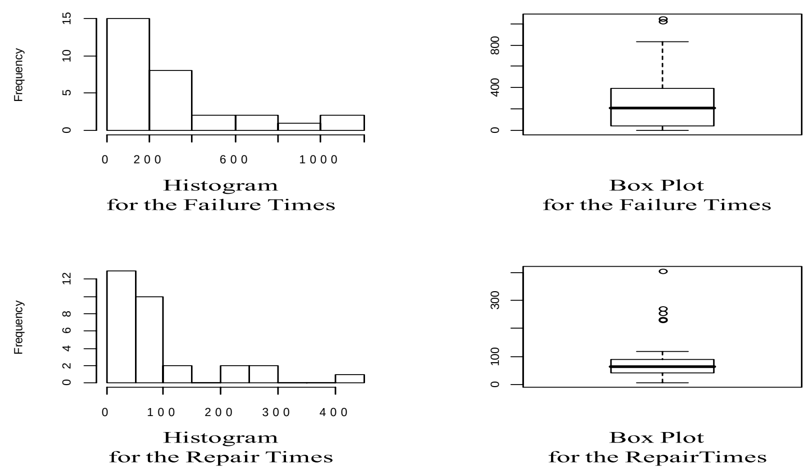


Fig. 8.4.1 Histograms and box plots for the failure time and the repair time before transformation

8.4.2 Nonparametric Method

In this section using a nonparametric method, Kendall's τ, we are going to test the hypothesis if the failure times and repair times are dependent. As mentioned before, if the two random variables, failure times and repair times, are independent then we can use NHPP model in Park and Pham [121]. Otherwise, a bivariate function with the failure times and the repair times is used in order to obtain the warranty cost. Kendall's rank correlation measures the strength of monotonic association between the vectors x and y. It may also be noted that usual Pearson correlation is fairly robust and it usually agrees well in terms of statistical significance with results obtained using Kendall's rank correlation. The null and the alternative hypotheses are as follows:

$$\begin{cases} H_0 : \text{The failure time and the repair time are independent.} \\ H_a : \text{The failure time and the repair time are dependent.} \end{cases}$$

Based on the result of Kendall's τ method using *R software* [99], τ is 0.277 and the *p* value is 0.012246. Therefore, at significant level $\alpha = 0.05$, we cannot accept the null hypothesis H_0. Therefore, it is concluded that the failed times and repair times are dependent.

Given the field failure data from Table 8.4.1, we now want to figure out the best fit distributions. Calculations are based on the distribution specified from *R software* [9]. Computer software calculated the β (the slope of the line), the η (the characteristic life, or the point at which 63.2% of the items in the data set have failed), ρ (a value between -1 and 1 that expresses how well the data fits the probability line) and residual of Y, $\hat{Y}_e$ (the error rate how far the data from the probability line). Using the computer software, it indicates that for the failure times, gamma, exponential and Weibull distributions are best three well-fitted distribution. For the repair times, exponential with 2 parameters and with 1 parameter, lognormal and gamma distributions are well-fitted distributions. Based on the output, we choose the BED then conduct warranty cost analysis. It is because the commonly fitting distribution for the repair time and the failure time is only exponential distribution with one parameter. We need to choose the commonly fitting distribution for both data because we want to get the bivariate distribution for dependable two-dimensional data. After we use the computer software, we figure out that the best fitting distributions for the failure time are listed by gamma distribution with two parameters (ρ^2= 0.9772, $\hat{Y}_e$=0.5173), exponential distribution with one parameter (ρ^2= 0.9787, $\hat{Y}_e$=0.7193) and Weibull with two parameters (ρ^2= 0.9721, $\hat{Y}_e$=1.1988). And the best fitting distributions for the repair time are listed by exponential distribution with two parameters (ρ^2= 0.9468, $\hat{Y}_e$= 1.3147) and exponential distribution with one parameter (ρ^2= 0.9468, $\hat{Y}_e$= 1.5303). Additionally, the BED is the most commonly used models for the joint distribution of failure times and repair times [113].

8.4.3 Expected Number of Warranty Services

To illustrate the proposed method, we assume that a two-dimensional warranty has been provided by the manufacturer to have the warranty period and the time limit of the repair services. Using the repair times and the failure times, we try to conduct warranty cost analysis by Marshall & Olkin's BED. From the nuclear power plant field data, we calculate their BED's parameters in Table 8.4.2 using eqs. (8.3.8) & (8.3.9).

Table 8.4.2 Estimated parameters in the Marshall & Olkin's BED

	θ_1	θ_2	θ_3	ρ
Estimated parameters	0.001016	0.008414	0.002562	0.213659

Based on the parameters in Table 8.4.2, we show the numerical example and the sensitivity analysis. For the numerical examples, Among four policies, Policy (b) and Policy (c) are considered in the numerical examples because Policy (a) is one dimensional warranty policy and Policy (d) is quite similar to Policy (c).

Policy (b) Analysis

Table 8.4.3 shows the expected number of failures under warranty for the limitation parameters for Policy (b). Using eqs (8.3.15) & (8.3.20), we investigate the repair cost. The repair time limit, w_2, starts from 5 to 50 by interval 5. The warranty period, w_1, is considered as 100, 200, 300 and 500. We obtain the expected number of warranty services and its variance. As a result of the sensitivity analysis, the expected number of warranty services and its variance are described in Table 8.4.3. If policy makers in companies can pre-determine the two constants, such as warranty period and repair time limit, they can use our model as a basic tool to make decisions for the company.

Table 8.4.3: Expected number of warranty services under warranty for Policy (b)

w_2	w_1=100		w_1=200		w_1=300		w_1=500	
	E(N)	Var(N)	E(N)	Var(N)	E(N)	Var(N)	E(N)	Var(N)
5	0.0001	0.0001	0.0002	0.0002	0.0003	0.0003	0.0003	0.0003
10	0.0004	0.0004	0.0008	0.0008	0.0010	0.0010	0.0012	0.0012
15	0.0009	0.0009	0.0018	0.0018	0.0023	0.0023	0.0027	0.0027
20	0.0015	0.0015	0.0032	0.0032	0.0041	0.0041	0.0046	0.0047
25	0.0024	0.0024	0.0049	0.0049	0.0062	0.0063	0.0072	0.0072
30	0.0034	0.0034	0.0069	0.0070	0.0088	0.0089	0.0102	0.0103
35	0.0045	0.0045	0.0093	0.0094	0.0119	0.0120	0.0136	0.0138
40	0.0058	0.0058	0.0119	0.0121	0.0153	0.0155	0.0176	0.0179
45	0.0072	0.0073	0.0149	0.0151	0.0191	0.0194	0.0219	0.0224
50	0.0088	0.0088	0.0181	0.0184	0.0232	0.0238	0.0267	0.0274

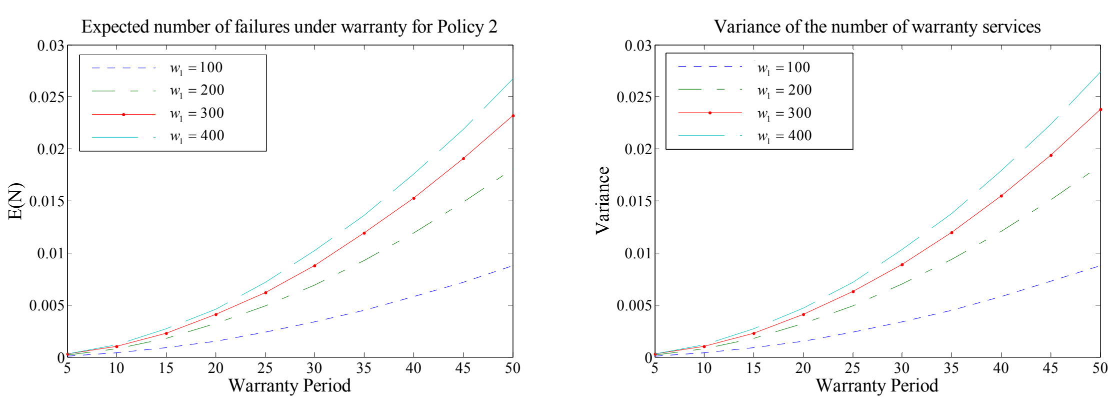


Fig. 8.4.2: Expected number of warranty services and its variance for Policy (b)

Policy (c) Analysis

Using Policy (c), the sensitivity analysis is conducted and Table 8.4.4 and Table 8.4.5 are obtained. Using the field data, the parameters of bivariate exponential distributions were calculated in Table 8.4.2. In a similar way to the Policy (b) Analysis, the expected number of warranty services and its variance are obtained in Table 8.4.4 and Table 8.4.5 using Policy 3. Four different constants such as w_{11}, w_{12}, w_{21} and w_{22} are used. For Table 8.4.4, $w_{11} = 300$ and $w_{12} = 500$ are fixed and other two constants, w_{21} and w_{22} are changed. And the expected number of warranty services and its variance are obtained using Policy (c). Similarly, in Table 8.4.5, $w_{11} = 500$ and $w_{12} = 1{,}000$ are fixed and other two constants, w_{21} and w_{22} are changed. Expected number of warranty services and its variance have similar pattern each other. And w_{21} does not affect the expected number of services severely, on the contrary, w_{11} and w_{12} have severe effect on the expected number and variance in Table 8.4.4 and Table 8.4.5.

Table 8.4.4: Expected number of warranty services when $w_{11} = 300, w_{12} = 500$ for Policy (c)

w_{21}	w_{22}=20		w_{22}=40		w_{22}=60		w_{22}=100	
	E(N)	Var(N)	E(N)	Var(N)	E(N)	Var(N)	E(N)	Var(N)
1	0.0042	0.0000	0.0166	0.0005	0.0366	0.0021	0.0971	0.0137
3	0.0043	0.0000	0.0166	0.0005	0.0366	0.0021	0.0972	0.0137
5	0.0043	0.0000	0.0166	0.0005	0.0366	0.0021	0.0972	0.0137
7	0.0043	0.0000	0.0167	0.0005	0.0366	0.0021	0.0972	0.0137
9	0.0044	0.0000	0.0167	0.0005	0.0367	0.0021	0.0973	0.0137
11	0.0044	0.0000	0.0168	0.0005	0.0367	0.0021	0.0973	0.0137
13	0.0045	0.0000	0.0169	0.0005	0.0368	0.0021	0.0974	0.0137
15	0.0046	0.0000	0.0169	0.0005	0.0369	0.0021	0.0975	0.0138
17	0.0047	0.0001	0.0170	0.0005	0.0370	0.0021	0.0976	0.0138
19	0.0048	0.0001	0.0171	0.0005	0.0371	0.0022	0.0977	0.0138

Table 8.4.5: Expected number of warranty services when $w_{11} = 500, w_{12} = 1{,}000$ for Policy (c)

w_{21}	w_{22}=50	w_{22}=100	w_{22}=150	w_{22}=200

	E(N)	Var(N)	E(N)	Var(N)	E(N)	Var(N)	E(N)	Var(N)
1	0.0308	0.0014	0.1226	0.0185	0.2752	0.0808	0.4894	0.2306
3	0.0308	0.0014	0.1226	0.0185	0.2752	0.0808	0.4894	0.2306
5	0.0308	0.0014	0.1226	0.0185	0.2752	0.0808	0.4895	0.2306
7	0.0308	0.0014	0.1226	0.0185	0.2752	0.0808	0.4895	0.2306
9	0.0308	0.0014	0.1226	0.0185	0.2752	0.0808	0.4895	0.2306
11	0.0308	0.0014	0.1226	0.0185	0.2752	0.0808	0.4895	0.2306
13	0.0309	0.0014	0.1226	0.0185	0.2752	0.0808	0.4895	0.2306
15	0.0309	0.0014	0.1226	0.0185	0.2753	0.0808	0.4895	0.2306
17	0.0309	0.0015	0.1227	0.0185	0.2753	0.0808	0.4895	0.2306
19	0.0309	0.0015	0.1227	0.0186	0.2753	0.0809	0.4896	0.2306

In Table 8.4.6 and Fig. 5, $w_{21} = 10$ and $w_{12} = 1,000$ are fixed and other two constants, w_{11} and w_{22} are changed.

Table 8.4.6: Expected number of warranty services when $w_{21} = 10, w_{12} = 1,000$ for Policy (c)

w_{22}	w_{11}=200		w_{11}=300		w_{11}=400		w_{11}=500	
	E(N)	Var(N)	E(N)	Var(N)	E(N)	Var(N)	E(N)	Var(N)
20	0.0037	0.0000	0.0045	0.0000	0.0048	0.0000	0.0050	0.0000
40	0.0133	0.0003	0.0171	0.0005	0.0189	0.0006	0.0198	0.0006
60	0.0287	0.0013	0.0377	0.0021	0.0423	0.0026	0.0443	0.0028
80	0.0493	0.0037	0.0661	0.0061	0.0746	0.0075	0.0786	0.0082
100	0.0748	0.0081	0.1019	0.0137	0.1160	0.0170	0.1226	0.0185
120	0.1046	0.0154	0.1448	0.0264	0.1662	0.0328	0.1763	0.0360
140	0.1383	0.0263	0.1945	0.0457	0.2251	0.0572	0.2398	0.0629
160	0.1755	0.0415	0.2508	0.0734	0.2927	0.0926	0.3131	0.1022
180	0.2157	0.0617	0.3132	0.1111	0.3689	0.1416	0.3963	0.1569
200	0.2586	0.0877	0.3815	0.1610	0.4535	0.2071	0.4895	0.2306

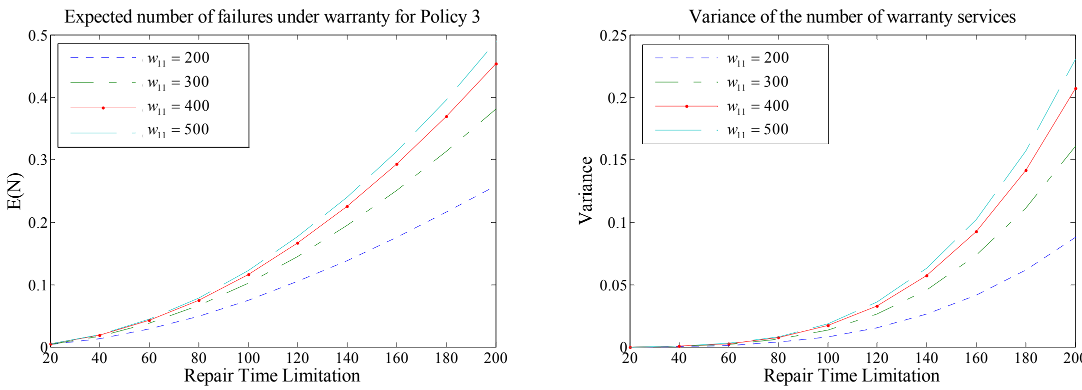


Fig. 8.4.3: Expected number of warranty services and its variance for Policy (c)

8.5 Discussion

Using BED and renewal theory, the two-dimensional renewal function is obtained. We consider the expected value and variance of the warranty cost together in the proposed approach. The expected warranty cost has been mainly investigated for warranty cost analysis. While expected warranty cost is a good measure on the overall cost of warranty, it provides little information of the risk contained in a warranty program. Therefore, we also obtain the variance of warranty cost with the expected value. The results provide more accurate and realistic cost analysis. Because we analyze the warranty cost based on the field data, the results of numerical example and the sensitivity analysis are more practical. It might give practitioners very useful tools to investigate the warranty cost as well as various warranty policies. Additionally, the repair service and the replacement service are considered at the same time. For the two dimensions, we consider the repair times and the failure times, not age and usage.

8.6 Concluding Remarks

We estimate the expected number of warranty services using the bivariate renewal functions based on the field data. We also perform the hypothesis to determine whether the failure times and repair times from the nuclear power plants field data are dependent to each other using nonparametric method. The BED is selected as the best fitting bivariate distribution using computer software. We obtain the number of warranty services under warranty and under the repair service limit. As a result of that, product's manufacturer can calculate the warranty cost using the developed approaches. This is very helpful for policy makers to make important decisions for their companies.

Chapter 9

Concluding Remarks & Future Research

9.1. Concluding Remarks

The contribution of this dissertation is to focus on the developments of warranty cost models with various maintenance policies as well as the warranty policy with post warranty periods for single-component and multi-component systems including parallel-series, series-parallel and *k-out-of-n* systems. Through various types of warranty cost models for each chapter, we want to distinguish this study from previous research in the following aspects:

- Present improved approaches on the cost analysis using both the expected value and the variance of the warranty cost for more complicated systems including *k-out-of-n* systems
- Consider the cost analysis in the post-warranty period under two perspectives (customers, manufacturers) to suggest accurate analyses to build warranty policies
- Suggest warranty cost structures based on two dimensional NHPP such as repair times and failure times

More specifically, in Chapter 4, based on the proposed alter- and mixed- quasi-renewal processes, we develop several cost models and also derive reliability measures for various systems.

In Chapter 5, warranty cost models are presented based on the quasi-renewal processes and exponential distribution. Cost analyses are conducted for various systems under the basic assumption that a repair service is imperfect. We develop warranty cost models, reliability, and other measures for several systems including multicomponent systems.

In Chapter 6, to minimize the expected total system cost, we develop a modified block replacement model for *k-out-of-n* systems and develop optimum policies of both a threshold level for the number of failed components to prevent the system's failures and the maintenance cycle. Additionally, we alleviate the existing block replacement policies' drawbacks which are rather wasteful if a preventive replacement happens just after a failure replacement. Our developed policy considers replacement service for a failure when *m* numbers of failed components occur. We also take into considerations the downtime period of each failed component using the order statistics for life time and age distributions for *k-out-of-n* systems.

In Chapter 7, we develop cost models by combining both warranty period and post warranty period and then derive the long run expected cost per unit time to find two decision variables including optimized maintenance cycle. The warranty services are separated into repair services and replacement services. Using the two-dimensional NHPP, we determine the threshold level for repair service time. In other words, as for the two kinds of warranty services, repair and replacement, if manufacturers can not finish the repair services within the threshold time, then they will have to provide replacement services instead of repair services to increase customers' satisfaction. So, we use two dimensional NHPP and obtain the expected warranty cost and the variance of the warranty cost.

In Chapter 8, a two-dimensional warranty policy is developed with repair times and failure times which are statistically correlated in bivariate distributions. Using the field data, the parameters are calculated and the warranty model is investigated using a bivariate exponential distribution which was the best fit distribution based on the field data.

9.2. Future Research

The following research problems that extend further research on these topics for a future study as follows:

Problem 1. Combine maintenance policies such as block replacement policy (BRP) and age replacement policy (ARP) and warranty policies such as FRW, PRW and CMW into the warranty cost modeling. The BRP and ARP can be compared based on two dimensional warranty policies with repair times and failure times in terms of the expected cost rates. Taking the product warranty into account, mathematical formulations for a product under ARP and BRP can be developed. For product with an increasing failure rate, the optimal replacement age can be obtained such that the long-run expected cost rate is minimized. The expected cost rate for all total number of failures during a warranty period can be calculated whereas the previous researchers [35, 192] have used a failure for determining the expected cost rate. In other words, we consider a certain number of failures in the testing period, and expected cost rates to obtain total cost of a certain number of failures, which is divided by their duration. In summary, warranty cost models may be developed considering two maintenance policies such as ARP and BRP under different warranty policies such as FRW, PRW and CMW.

Problem 2: With the renewable warranty policy, warranty cost models can be developed subject to different length of warranty periods and obtain the distribution of number of failures. The model would be suggested when the warranty period is assumed to be a random variable. For the globalized companies, they can sell their products with different warranty periods based on the locations and times. Additionally, customers can select the length of warranty according to their own needs if they may pay an additional fee. Therefore, the warranty period and the repair time limit can be changed and be considered as random variables. Under the non-renewable warranty policy, the expected number of warranty services and their variances within the warranty period can be obtained and maintenance policy be considered during post warranty period. In summary, warranty cost models considering the non-renewable warranty policies with different warranty periods would be an interesting research topic.

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