EVOLUTION EQUATIONS FOR MULTI-TIME WAVEFUNCTIONS

by

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ABSTRACT OF THE THESIS

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Multi-time wavefunctions are of particular interest in relativistic quantum mechanics. A multi-time wavefunction has separate time-variables for each particle; this makes it a manifestly Lorentz-invariant object. The time-evolution equations are systems of Schrödinger equations; one for each particle's time variable and each with a certain Hamiltonian. We derive conditions under which these systems of equations have a common solution. Also, we derive three main results about concrete multi-time models. First we show that a model proposed by Dürr and Tumulka in 2001 is inconsistent. The second result is a consistent model for a constant number of particles with a cutoff pair potential. The third result is a consistent theory for a simple quantum field theoretic model with creation and annihilation of particles. Existence and uniqueness of solutions is proven for both models.

Contents

Chapter 1

Introduction

In usual quantum mechanics the wavefunction (for N-particles) depends on $3N$ spatial variables and one time variable. This is not a Lorentz-invariant object (one has simultaneity for all particles). The idea of the multi-time formalism is to add a separate time-variable for each particle. One then has a multi-time wavefunction, i.e., a wavefunction that depends on $3N$ spatial and N time variables. The main reason for doing so is to get a manifestly Lorentz-invariant theory. A multi-time wavefunction can be a Lorentz-invariant object. Moreover, evolution laws for the multi-time wavefunction can be stated such that no additional structure on space-time is involved. This is believed to be one criterion for a fundamentally Lorentz-invariant theory (see for example [5]).

Multi-time wavefunctions were first introduced in 1932 by Dirac, Fock and Podolsky in [3]. After that they have widely been neglected in favour of a quantum field theoretic (QFT) description. Important work with the goal to make quantum mechanics more relativistic invariant has been done by Tomonaga and Schwinger in 1946 (see [7]). They defined wavefunctions on (spacelike) hypersurfaces and gave an evolution law from one hypersurface to another. The connection of our work with that of Tomonaga and Schwinger is that a multi-time wavefunction easily leads to a wavefunction on a hypersurface (with some restrictions).

The multi-time formalism which uses a wavefunction in the particle representation has the main advantage that the mathematical formalism is easier. For example, the fundamental equations in this work are just partial differential equations. So far, consistent multi-time theories have only been constructed for non-interacting particles. In this work we consider multi-time theories with interaction.

This work is structured as follows. In chapter 2 we state some general considerations about the multi-time formalism. We will see that there are two different ways of looking at the multi-time wavefunctions and its evolution laws. We also derive conditions for the existence of solutions to multi-time equations. Chapter 3 deals with a concrete proposal put forward by D. Dürr and R. Tumulka in 2001 for a multi-time model that is based on a simple quantum field theoretic model with creation and annihilation of particles. We show that this proposal is inconsistent, i.e., there is no solution to the proposed system of multi-time equations. Consistent multi-time models are presented in chapter 4. We construct two multi-time models and show that there is a unique solution to the multi-time equations. The first model describes N Dirac particles that interact via a pair potential if two particles are closer to each other than a certain radius δ. The second model is a consistent multi-time version of the simple quantum field theoretic model with creation and annihilation of particles. In both models we have to introduce a cutoff δ. (Note that because of this δ-cutoff our models are not actually fundamental Lorentz-invariant.) The main result of this work is Theorem 4.3.6 where the multi-time QFT model is defined and shown to be consistent.

Chapter 2

General Considerations about the Multi-Time Formalism

2.1 Wavefunctions on Hilbert Space and on Configuration Space-Time

Usual one-time quantum mechanics (in the Schrödinger picture) deals with a time-dependent wavefunction $\psi = \psi(t) = \psi_t$ that is the solution of a Schrödinger equation (throughout this work we will use units in which $\hbar = 1$)

$$i\frac{\partial}{\partial t}\psi(t) = H(t)\psi(t) \tag{2.1}$$

with a Hamiltonian $H(t)$. The wavefunction $\psi(t)$ is for each parameter $t \in \mathbb{R}$ usually considered to be an element of a (separable, complex) Hilbert space $\mathcal{H}$, so the Hamiltonian is for each t an operator from on $\mathcal{H}$. Examples of such Hamiltonians are the non-relativistic Schrödinger operator and the relativistic Dirac operator. $H(t)$ must be self-adjoint for each t in order to lead to a unitary time evolution. If $H(t)$ is self-adjoint for each t then there is a family of unitary operators $(U(t,s))_{t,s\in\mathbb{R},s<t}$ such that for every solution of (2.1) we have $\psi(t) = U(t,s)\psi(s)$, i.e., specifying initial conditions $\psi(0) \in \mathcal{H}$ determines the wavefunction $\psi(t) \in \mathcal{H}$. $U(t,s)$ is the solution of the equation $i\frac{\partial}{\partial t}U(t,s) = H(t)U(t,s)$. If the Hamiltonian $H(t)$ is time independent ($H(t) = H$) then U is given by $U(t,s) = e^{-iH[t-s]}$.

In this work another way of looking at the Schrödinger equation (2.1) will be useful. The wavefunction ψ can also be regarded as a function on configuration space and time in a suitable function space. For example, for N particles we could choose the set of all smooth functions, $C^\infty(\mathbb{R} \times \mathbb{R}^{3N}, \mathbb{C}^k)$ (for some k) as a suitable function space. (For $X = \mathbb{R}^d$ and $Y = \mathbb{C}^k$ $(d, k \in \mathbb{N})$, $C^\infty(X, Y)$ is the space of all infinitely often differentiable functions $f : X \to Y$.) The Hamiltonians are then operators that map smooth functions to smooth functions. We later show in Lemma 4.1 that for a certain class of operators H, specifying initial conditions $\psi_0 \in C^\infty(\{0\} \times X, Y)$ uniquely determines the wavefunction $\psi \in C^\infty(\mathbb{R} \times X, Y)$. Moreover, many practically relevant Hamiltonians are more or less local in configuration space (e.g. differential operators). This allows us to ask questions about what happens at a particular $x \in X$ that would not be meaningful for arbitrary operators (e.g., whether the commutator of two Hamiltonians vanishes at x). This will be particularly advantageous when dealing with multi-time wavefunctions. There we consider a configuration space-time. For example, in the case of N particles the configuration space-time is $\mathbb{R}^{4N}$. We consider wavefunctions $\psi : \mathbb{R}^{4N} \to \mathbb{C}^k$ in a suitable function space, e.g., $C^\infty(\mathbb{R}^{4N}, \mathbb{C}^k)$. We will also find that in the multi-time framework Hamiltonians on configuration space-time are given by simpler expressions than Hamiltonians on Hilbert space. Finally, to regard the Hamiltonians H in the multi-time framework as operators on configuration space-time is closer to the spirit of relativity since in this way space and time are considered on an equal footing.

2.2 Multi-Time Evolution Equations and Consistency Conditions

In this section we describe our idea of how a multi-time wavefunction could be defined by a system of equations. For simplicity we first regard a wavefunction for a system of two particles in $\mathbb{R}^3$. The multi-time wavefunction $\psi = \psi\left((t_1, \boldsymbol{x}_1), (t_2, \boldsymbol{x}_2)\right)$ then depends on two time and two position variables. The straightforward generalization of (2.1) is the following system of multi-time equations (where we did not write out the explicit variable dependence):

$$\begin{aligned} i\frac{\partial}{\partial t_1}\psi &= H_1\psi \\ i\frac{\partial}{\partial t_2}\psi &= H_2\psi. \end{aligned} \tag{2.2}$$

In order to coincide with the one-time theory the Hamiltonians H_1 and H_2 should be choosen in such a way that for equal times their sum is the one-time Hamiltonian. In the following sections 2.2.1 and 2.2.2 we regard this system of multi-time equations on Hilbert space and on configuration space-time. We find that a theorem about necessary conditions for solutions to a system of multi-time equations can be formulated in both settings. In the Hilbert space framework we have to impose very strong conditions on the Hamiltonians in order to prove the exact theorem though. However, we conjecture that the theorem also holds under more general conditions, like those we consider in section 3.3. In the configuration space-time framework we are able to prove a theorem that is immediately applicable to the equations we use in section 3.2. This is another advantage of using this framework.

2.2.1 Formulation on Hilbert Space

We first consider this system of equations for wavefunctions on Hilbert space. The multi-time wavefunction is $\psi(t_1, t_2) \in \mathcal{H}$ for all $t_1, t_2 \in \mathbb{R}$ and the Hamiltonians $H_i(t_1, t_2) : \mathcal{H} \to \mathcal{H}$ (for $i = 1, 2$) are self-adjoint operators for all t_1, t_2 that can possibly be time-dependent (i.e., dependent on t_1 and t_2). The system of multi-time equations then reads

$$\begin{aligned} i\frac{\partial}{\partial t_1}\psi(t_1,t_2) &= H_1(t_1,t_2)\psi(t_1,t_2) \\ i\frac{\partial}{\partial t_2}\psi(t_1,t_2) &= H_2(t_1,t_2)\psi(t_1,t_2). \end{aligned} \tag{2.3}$$

The multi-time Hamiltonians in (2.3) should be chosen such that they lead to the one-time theory (2.1) with Hamiltonian $H(t)$ if $t_1 = t_2 = t$, i.e.,

$$H_1(t,t) + H_2(t,t) = H(t). \tag{2.4}$$

Note that in this multi-time setting the bosonic (or fermionic) symmetry properties of the wavefunction are more complicated. To illustrate the difficulty, regard the standard example $\mathcal{H} = L^2(\mathbb{R}^6)$. Then for given t_1, t_2 we have a wavefunction $\psi_{t_1,t_2}(\boldsymbol{x}_1, \boldsymbol{x}_2)$. The correct bosonic symmetry is not $\psi_{t_1,t_2}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \psi_{t_1,t_2}(\boldsymbol{x}_2, \boldsymbol{x}_1)$ but rather $\psi_{t_1,t_2}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \psi_{t_2,t_1}(\boldsymbol{x}_2, \boldsymbol{x}_1)$. In section 3.3 this leads to an extra conditions on permissible wavefunctions.

We now derive a necessary and sufficient condition for the system of equations (2.3) to have a common solution $\psi(\cdot, \cdot) : \mathbb{R}^2 \to \mathcal{H}$. In the case of time independent Hamiltonians $H_i(t_1, t_2) = H_i$ (for $i = 1, 2$) we have $\psi(t_1, t_2) = e^{-iH_1 t_1}\psi(0, t_2)$ and $\psi(t_1, t_2) = e^{-iH_2 t_2}\psi(t_1, 0)$. Therefore we can write a solution of (2.3) as

$$\psi(t_1,t_2) = e^{-iH_1t_1}e^{-iH_2t_2}\psi(0,0) = e^{-iH_2t_2}e^{-iH_1t_1}\psi(0,0) \tag{2.5}$$

We conclude that (2.3) can have a solution only under certain conditions on the Hamiltonians. Equality in (2.5) holds for all $\psi(0,0) \in \mathcal{H}$ if and only if $e^{-iH_1t_1}e^{-iH_2t_2} = e^{-iH_2t_2}e^{-iH_1t_1}$, and thus if and only if $[H_1, H_2] = 0$. For time-dependent Hamiltonians we proceed in a similar way. The result for a necessary and sufficient condition on the Hamiltonians is stated in Assertion 2.1, which also generalizes the considerations so far to N instead of two multi-time equations. We call 2.1 an assertion since we do not rigorously prove it. (Also note, that for example for unbounded Hamiltonians it is not obvious how exactly the expression (2.7) is to be understood.)

Assertion 2.1 (Consistency Conditions on Hilbert Space)**.** *Let the operators $H_i(t_1, \ldots, t_N) : \mathcal{H} \to \mathcal{H}$ be selfadjoint for all $t_i \in \mathbb{R}$, $i = 1, \ldots, N$. Then a necessary and sufficient condition for the existence of a solution $\psi(t_1, \ldots, t_N)$, for all $\phi \in \mathcal{H}$, to the system of multi-time equations*

$$i\frac{\partial}{\partial t_i}\psi(t_1, \ldots, t_N) = H_i(t_1, \ldots, t_N)\psi(t_1, \ldots, t_N) \tag{2.6}$$

for all $i = 1, \ldots, N$, satisfying the initial condition $\psi(0, \ldots, 0) = \phi$, is

$$\Big[H_i(t_1, \ldots, t_N), H_j(t_1, \ldots, t_N)\Big] + i\frac{\partial H_i(t_1, \ldots, t_N)}{\partial t_j} - i\frac{\partial H_j(t_1, \ldots, t_N)}{\partial t_i} = 0 \tag{2.7}$$

for all $i, j = 1, \ldots, N$, $i \neq j$.

Proof. It suffices to consider an infinitesimal time evolution. In that case we know that for the one-time evolution operator we have $U(dt, 0) = 1 - iH(0)dt - \frac{1}{2}H(0)^2(dt)^2$ (to second order in dt). We now consider the i-th and j-th ($i \neq j$) equations of (2.6). For ease of notation we only explicitly write the t_i and t_j dependence of H_i, H_j and ψ. We regard the time evolution from initial times $(0, 0)$ to times (dt_i, dt_j) in two different ways (along two different infinitesimal paths in the (t_i, t_j)-plane),

$$\psi(0,0) \to \psi(0, dt_j) \to \psi(dt_i, dt_j) \quad \text{and} \quad \psi(0,0) \to \psi(dt_i, 0) \to \psi(dt_i, dt_j). \tag{2.8}$$

To second order in dt_i and dt_j we get two expressions for $\psi(dt_i, dt_j)$,

$$\begin{aligned}\psi(dt_i, dt_j) &= \Big(1 - iH_i(0, dt_j)dt_i - \frac{1}{2}H_i(0, dt_j)^2(dt_i)^2\Big)\psi(0, dt_j) \\ &= \Big(1 - iH_i(0, dt_j)dt_i - \frac{1}{2}H_i(0, dt_j)^2(dt_i)^2\Big)\Big(1 - iH_j(0,0)dt_j - \frac{1}{2}H_j(0,0)^2(dt_j)^2\Big)\psi(0,0)\end{aligned} \tag{2.9}$$

and

$$\psi(dt_i, dt_j) = \Big(1 - iH_j(dt_i, 0)dt_j - \frac{1}{2}H_j(dt_i, 0)^2(dt_j)^2\Big)\psi(dt_i, 0)$$
$$= \Big(1 - iH_j(dt_i, 0)dt_j - \frac{1}{2}H_j(dt_i, 0)^2(dt_j)^2\Big)\Big(1 - iH_i(0, 0)dt_i - \frac{1}{2}H_i(0, 0)^2(dt_i)^2\Big)\psi(0, 0). \tag{2.10}$$

Therefore a common solution to the i-th and j-th equation of (2.6) can only exist if

$$\Big(1 - iH_i(0, dt_j)dt_i - \frac{1}{2}H_i(0, dt_j)^2(dt_i)^2\Big)\Big(1 - iH_j(0, 0)dt_j - \frac{1}{2}H_j(0, 0)^2(dt_j)^2\Big)$$
$$= \Big(1 - iH_j(dt_i, 0)dt_j - \frac{1}{2}H_j(dt_i, 0)^2(dt_j)^2\Big)\Big(1 - iH_i(0, 0)dt_i - \frac{1}{2}H_i(0, 0)^2(dt_i)^2\Big). \tag{2.11}$$

We now evaluate this expression. We have (to first order in dt_i)

$$H_j(dt_i, 0) = H_j(0, 0) + \frac{\partial H_j(t_i, t_j)}{\partial t_i}|_{(0,0)}dt_i \tag{2.12}$$

and (to first order in dt_j)

$$H_i(0, dt_j) = H_i(0, 0) + \frac{\partial H_i(t_i, t_j)}{\partial t_j}|_{(0,0)}dt_j. \tag{2.13}$$

To simplify the notation we set $H_i(0, 0) = H_i$, $H_j(0, 0) = H_j$, $H_i(0, dt_j) = H_i(dt_j)$, $H_j(dt_i, 0) = H_j(dt_i)$, $\frac{\partial H_i(t_i,t_j)}{\partial t_j}|_{(0,0)} = \frac{\partial H_i}{\partial t_j}$ and $\frac{\partial H_j(t_i,t_j)}{\partial t_i}|_{(0,0)} = \frac{\partial H_j}{\partial t_i}$. With (2.12) and (2.13) the expression (2.11) becomes, neglecting terms in third and higher orders of dt_i and dt_j,

$$0 = \Big(1 - iH_i(dt_j)dt_i - \frac{1}{2}(H_i(dt_j))^2(dt_i)^2\Big)\Big(1 - iH_jdt_j - \frac{1}{2}H_j^2(dt_j)^2\Big)$$
$$- \Big(1 - iH_j(dt_i)dt_j - \frac{1}{2}(H_j(dt_i))^2(dt_j)^2\Big)\Big(1 - iH_idt_i - \frac{1}{2}H_i^2(dt_i)^2\Big)$$
$$= 1 - iH_i(dt_j)dt_i - iH_jdt_j - H_i(dt_j)H_jdt_idt_j - \frac{1}{2}(H_i(dt_j))^2(dt_i)^2 - \frac{1}{2}H_j^2(dt_j)^2$$
$$- 1 + iH_j(dt_i)dt_j + iH_idt_i + H_j(dt_i)H_idt_idt_j + \frac{1}{2}(H_j(dt_i))^2(dt_j)^2 + \frac{1}{2}H_i^2(dt_i)^2$$
$$= 1 - iH_idt_i - i\frac{\partial H_i}{\partial t_j}dt_idt_j - iH_jdt_j - H_iH_jdt_idt_j$$
$$- 1 + iH_jdt_j + i\frac{\partial H_j}{\partial t_i}dt_idt_j + iH_idt_i + H_jH_idt_j$$
$$= -i\frac{\partial H_i}{\partial t_j}dt_idt_j + i\frac{\partial H_j}{\partial t_i}dt_idt_j - [H_i, H_j]dt_idt_j. \tag{2.14}$$

In full notation the expression (2.14) is

$$[H_i(0,0), H_j(0,0)] + i\frac{\partial H_i(t_i,t_j)}{\partial t_j}|_{(0,0)} - i\frac{\partial H_j(t_i,t_j)}{\partial t_i}|_{(0,0)} = 0. \tag{2.15}$$

We conclude that an infinitesimal time evolution is path-independent, i.e., with condition (2.15) we get the same solution $\psi(dt_i, dt_j)$ from initial conditions $\psi(0,0)$ no matter if we first evolve in dt_i and then dt_j or the other way around. Let us now regard $\psi(t_i, t_j)$. We define $\psi(t_i, t_j)$ by first evolving $\psi(0,0)$ to $\psi(t_i, 0)$ and then to $\psi(t_i, t_j)$. According to the above reasoning this is the same as evolving $\psi(0,0)$ to $\psi(t_i - dt_i, 0)$, then to $\psi(t_i - dt_i, dt_j)$, then to $\psi(t_i, dt_j)$ and then to $\psi(t_i, t_j)$. This procedure of "cutting out" infinitesimal rectangles can repeated, so we finally find that we can also evolve $\psi(0,0)$ to $\psi(0, t_j)$ and then to $\psi(t_i, t_j)$ and arrive at the same solution as the other way around. Therefore, if the condition (2.15) holds for arbitrary initial times for any i and j, then we have a solution to (2.6) and if we have a solution to (2.6) then (2.15) holds for arbitrary initial times for any i and j. $\square$

Note that even though for unbounded Hamiltonians it is not clear how the expression (2.7) is to be understood, given concrete unbounded operators H_i and H_j we still might be able to calculate the expression (2.7) formally. This is the case for the (unbounded) Hamiltonians we consider in section 3.3. Therefore we think that Assertion 2.1 also holds for unbounded Hamiltonians. We also think that the assertion holds if we consider an infinite number of time variables.

2.2.2 Formulation on Configuration Space-Time

The system of equations (2.2) can also be considered in the framework of functions on configuration space-time. For N particles the configuration space-time is $\mathbb{R}^{4N}$. The operators in (2.2) then live on a suitable function space and act on functions on configurations space-time. This makes it possible to prove an exact consistency condition which can be applied to the model we consider in section 3.2. We denote $\mathbb{R}^4 \ni x = (x^0, x^1, x^2, x^3)$. For a two particle system the multi-time wavefunction is $\psi(x_1, x_2) : \mathbb{R}^8 \to \mathbb{C}^k$ (where $\mathbb{C}^k$, $k \in \mathbb{N}$ is a suitable spin space) and the Hamiltonians H_i (for $i = 1, 2$) are now operators on a suitable function space, e.g., on $C^\infty(\mathbb{R}^8, \mathbb{C}^k)$. Note that in this setting the Hamiltonians should not be regarded as explicitly time-dependent, since they are operators acting on the configuration space-time $\mathbb{R}^8$ (i.e., they act on those variables on which ψ depends). The system of multi-time equations reads

$$\begin{aligned} i\frac{\partial}{\partial x_1^0}\psi(x_1,x_2) &= (H_1\psi)(x_1,x_2) \\ i\frac{\partial}{\partial x_2^0}\psi(x_1,x_2) &= (H_2\psi)(x_1,x_2). \end{aligned} \tag{2.16}$$

Also note that in this setting a bosonic wavefunction satisfies $\psi(x_1,x_2)=\psi(x_2,x_1)$. This seems like a more straightforward generalization of the concept of bosonic symmetry to multi-time wavefunctions than the corresponding statement on Hilbert space.

The next Theorem 2.2 shows that the consistency condition is a necessary condition under which the system of multi-time equations (2.16), generalized to an arbitrary number of particles, can have solutions. We abbreviate $\mathbb{R}^{4m} \ni q^4 = (x_1,\ldots,x_m)$. For Theorem 2.2 we define the configuration space-time as $\mathscr{C} = \bigcup_{m=0}^{\infty}\mathbb{R}^{4m}$ and the spin space as $\mathcal{S} = \bigcup_{m=0}^{\infty}(\mathbb{C}^k)^{\otimes m}$ for a suitable $k \in \mathbb{N}$. The wavefunction evaluated at a certain configuration q^4 in the m-particle sector is an element of the m-particle spin space $\mathcal{S}_m = (\mathbb{C}^k)^{\otimes m}$. $C^2(\mathscr{C},\mathcal{S})$ denotes the space of all twice continuous differentiable functions from $\mathscr{C}$ to $\mathcal{S}$.

Theorem 2.2 (Consistency Conditions on Configuration Spacetime)**.** *Let $\psi : \mathscr{C} \to \mathcal{S}$ with $\psi(q^4) \in \mathcal{S}_m$ and $H_j : \mathscr{F} \to \mathscr{F}$, where $\mathscr{F}$ denotes a subspace of $C^2(\mathscr{C},\mathcal{S})$. Then every solution $\psi \in \mathscr{F}$ to the system of multi-time equations*

$$i\left(\frac{\partial}{\partial x_j^0}\psi\right)(q^4) = (H_j\psi)(q^4) \tag{2.17}$$

for all $j \in \mathbb{N}$, with $\frac{\partial \psi}{\partial x_j^0} \in \mathscr{F}$ for all $j \in \mathbb{N}$, must satisfy

$$\left[i\frac{\partial}{\partial x_i^0} - H_i, i\frac{\partial}{\partial x_j^0} - H_j\right]\psi(q^4) = 0, \tag{2.18}$$

or equivalently

$$\left([H_i,H_j]\,\psi\right)(q^4) + i\left(\frac{\partial H_i}{\partial x_j^0}\psi\right)(q^4) - i\left(\frac{\partial H_j}{\partial x_i^0}\psi\right)(q^4) = 0, \tag{2.19}$$

for all $i,j \in \mathbb{N}$, $i \neq j$ and all $q^4 \in \mathscr{C}$, where $\frac{\partial H_i}{\partial x_j^0} \overset{def}{=} \left[\frac{\partial}{\partial x_j^0}, H_i\right]$.

Proof. We consider the i-th and j-th ($i \neq j$) equations of (2.17). If ψ is a common solution to both equations, we have

$$\left(i\frac{\partial}{\partial x_i^0} - H_i\right)\psi(q^4) = 0 \quad \text{and} \quad \left(i\frac{\partial}{\partial x_j^0} - H_j\right)\psi(q^4) = 0. \tag{2.20}$$

Therefore also

$$\left[i\frac{\partial}{\partial x_i^0} - H_i, i\frac{\partial}{\partial x_j^0} - H_j\right]\psi(q^4) = 0. \tag{2.21}$$

This is the same as (2.19) for fixed $i, j \in \mathbb{N}$ as the following calculation shows. We use the product rule $\frac{\partial}{\partial x^0}(H\psi) = \frac{\partial H}{\partial x^0}\psi + H\frac{\partial \psi}{\partial x^0}$ and $\frac{\partial \psi}{\partial x_i^0 \partial x_j^0} = \frac{\partial \psi}{\partial x_j^0 \partial x_i^0}$.

$$\begin{aligned}
0 &= \left[i\frac{\partial}{\partial x_i^0} - H_i, i\frac{\partial}{\partial x_j^0} - H_j\right]\psi(q^4) \\
&= -\left(\frac{\partial}{\partial x_i^0}\frac{\partial \psi}{\partial x_j^0}\right)(q^4) - i\left(\frac{\partial}{\partial x_i^0}(H_j\psi)\right)(q^4) - i\left(H_i\frac{\partial \psi}{\partial x_j^0}\right)(q^4) + (H_iH_j\psi)(q^4) \\
&\quad + \left(\frac{\partial}{\partial x_j^0}\frac{\partial \psi}{\partial x_i^0}\right)(q^4) + i\left(\frac{\partial}{\partial x_j^0}(H_i\psi)\right)(q^4) + i\left(H_j\frac{\partial \psi}{\partial x_i^0}\right)(q^4) - (H_jH_i\psi)(q^4) \\
&= -i\left(\frac{\partial H_j}{\partial x_i^0}\psi\right)(q^4) - i\left(H_j\frac{\partial \psi}{\partial x_i^0}\right)(q^4) - i\left(H_i\frac{\partial \psi}{\partial x_j^0}\right)(q^4) + (H_iH_j\psi)(q^4) \\
&\quad + i\left(\frac{\partial H_i}{\partial x_j^0}\psi\right)(q^4) + i\left(H_i\frac{\partial \psi}{\partial x_j^0}\right)(q^4) + i\left(H_j\frac{\partial \psi}{\partial x_i^0}\right)(q^4) - (H_jH_i\psi)(q^4) \\
&= ([H_i, H_j]\psi)(q^4) + i\left(\frac{\partial H_i}{\partial x_j^0}\psi\right)(q^4) - i\left(\frac{\partial H_j}{\partial x_i^0}\psi\right)(q^4).
\end{aligned} \tag{2.22}$$

If ψ is a common solution to all equations (2.17) then the above condition has to hold for all $i, j \in \mathbb{N}$, $i \neq j$. □

Chapter 3

Inconsistency of a Proposed Multi-Time QFT Model on All Space-Time Configurations

This chapter deals with a conjecture made by D. Dürr and R. Tumulka in 2001 about a simple multi-time QFT model. The model describes a fixed number of electrons that can emit and absorb photons. First, we describe the one-time version in section 3.1. After that we state the conjecture about formulating this model in the multi-time formalism. The idea is to use the one-time Hamiltonian and to split it into several Hamiltonians, each associated with one particle. It is instructive to discuss this model for a wavefunction on Hilbert space and for a wavefunction on configuration space-time. On configuration space-time the model will have a simpler formulation, so the conjecture is first stated in this setting in section 3.2 and then on Hilbert space in section 3.3. We show that the multi-time equations in both settings do not have a solution, thus disproving the conjecture.

3.1 The One-Time QFT Model

We first discuss the one-time quantum field theoretic description of a model of free non-relativistic particles with emission and absorption (see [4] and originally [6, p. 339]). We consider two kinds of particles: electrons (or more generally: fermions) and photons (or bosons). We consider a constant electron number N and we allow the number of photons to change. Photons can be created in the vicinity of an electron (emission) and they get annihilated if they are close to an electron (absorption).

In the following we denote electron variables by $\boldsymbol{x}_i \in \mathbb{R}^3$. We abbreviate electron configurations by $\xi = \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ with constant N and photon configurations by $\eta = \boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ with $\boldsymbol{y}_k \in \mathbb{R}^3$.

The Hamiltonian for this theory is the sum of a free Hamiltonian and an interaction term which is the sum of a creation and an annihilation part:

$$H = H^{free} + H^{int} = H^{free} + H^c + H^a. \tag{3.1}$$

The time evolution equation is the usual Schrödinger equation:

$$i\frac{\partial}{\partial t}\psi = H\psi \tag{3.2}$$

for a wavefunction ψ on Fock space. The Fock space is $\mathcal{F} = \bigoplus_{m=0}^{\infty} \mathcal{H}^{(m)}$ with the (N,m)-particle sector $\mathcal{H}^{(m)} = AL^2(\mathbb{R}^{3N}) \otimes SL^2(\mathbb{R}^{3m})$ (anti-symmetrized in the electron and symmetrized in the photon variables). For the (N,m)-particle configuration space we denote $\mathcal{Q}^{(m)} = \mathbb{R}^{3N} \times \mathbb{R}^{3m}$. The full configuration space then is $\mathcal{Q} = \bigcup_{m=0}^{\infty} \mathcal{Q}^{(m)}$. We denote $\psi^{(N,m)} \in \mathcal{H}^{(m)}$. The scalar product on the (N,m)-sector in Fock space is given by

$$\langle\psi|\chi\rangle_{\mathcal{H}^{(m)}} = \int_{\mathbb{R}^{3N}} d\xi \int_{\mathbb{R}^{3m}} d\eta\, \psi^*(\xi,\eta)\chi(\xi,\eta) \tag{3.3}$$

and the scalar product on Fock space is

$$\langle\psi|\chi\rangle_{\mathcal{F}} = \sum_{m=0}^{\infty} \langle\psi|\chi\rangle_{\mathcal{H}^{(m)}} = \int_{\mathbb{R}^{3N}} d\xi \sum_{m=0}^{\infty} \int_{\mathbb{R}^{3m}} d\eta\, \psi^*(\xi,\eta)\chi(\xi,\eta). \tag{3.4}$$

The action of the Hamiltonian on the (N,m)-sector in Fock space is given by

$$(H^{free}\psi)(\xi,\eta) = \left(-\sum_{j=1}^{N} \frac{1}{2m_x}\Delta_{\boldsymbol{x}_j} - \sum_{k=1}^{m} \frac{1}{2m_y}\Delta_{\boldsymbol{y}_k}\right)\psi(\xi,\eta) \tag{3.5}$$

$$(H^c\psi)(\xi,\eta) = \frac{1}{\sqrt{m}} \sum_{j=1}^{N}\sum_{k=1}^{m} \varphi(\boldsymbol{y}_k - \boldsymbol{x}_j)\psi(\xi, \eta\backslash\boldsymbol{y}_k) \tag{3.6}$$

$$(H^a\psi)(\xi,\eta) = \sqrt{m+1}\sum_{j=1}^{N} \int_{\mathbb{R}^3} d\boldsymbol{y}\varphi^*(\boldsymbol{y} - \boldsymbol{x}_j)\psi(\xi, \eta \cup \boldsymbol{y}) \tag{3.7}$$

where $m_x > 0$ denotes the electron mass and $m_y > 0$ denotes the photon mass.[1] We use the notation $\eta\backslash\boldsymbol{y}_k = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{k-1}, \boldsymbol{y}_{k+1}, \ldots, \boldsymbol{y}_m)$ and $\eta \cup \boldsymbol{y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m, \boldsymbol{y})$. φ is a square-integrable

[1] Since this is a simple non-relativistic model the introduction of a photon mass makes sense.

cutoff function. It can be considered to be a form-factor that determines the effective range of the electron interaction. In terms of field operators the Hamiltonian can be written as follows:

$$H^{free} = -\sum_{j=1}^{N} \frac{1}{2m_x} \Delta_{\boldsymbol{x}_j} - \sum_{k=1}^{\infty} \frac{1}{2m_y} \Delta_{\boldsymbol{y}_k} \tag{3.8}$$

$$H^{int} = \sum_{j=1}^{N} \left(a_\varphi^\dagger(\boldsymbol{x}_j) + a_\varphi(\boldsymbol{x}_j) \right). \tag{3.9}$$

The definition of the smeared-out creation operators $a_\varphi^\dagger(\boldsymbol{x}_j)$ and annihilation operators $a_\varphi(\boldsymbol{x}_j)$ is:

$$a_\varphi^\dagger(\boldsymbol{x}_j) = \int_{\mathbb{R}^3} d\boldsymbol{y} \varphi(\boldsymbol{y} - \boldsymbol{x}_j) a^\dagger(\boldsymbol{y}) \tag{3.10}$$

$$a_\varphi(\boldsymbol{x}_j) = \int_{\mathbb{R}^3} d\boldsymbol{y} \varphi^*(\boldsymbol{y} - \boldsymbol{x}_j) a(\boldsymbol{y}) \tag{3.11}$$

with

$$(a^\dagger(\boldsymbol{y})\psi)(\xi, \eta) = \frac{1}{\sqrt{m}} \sum_{k=1}^{m} \delta(\boldsymbol{y}_k - \boldsymbol{y}) \psi(\xi, \eta \backslash \boldsymbol{y}_k) \tag{3.12}$$

$$(a(\boldsymbol{y})\psi)(\xi, \eta) = \sqrt{m+1}\, \psi(\xi, \eta \cup \boldsymbol{y}). \tag{3.13}$$

Their action on a certain sector in Fock space therefore is:

$$(a_\varphi^\dagger(\boldsymbol{x}_j)\psi)(\xi, \eta) = \frac{1}{\sqrt{m}} \sum_{k=1}^{m} \varphi(\boldsymbol{y}_k - \boldsymbol{x}_j) \psi(\xi, \eta \backslash \boldsymbol{y}_k) \tag{3.14}$$

$$(a_\varphi(\boldsymbol{x}_j)\psi)(\xi, \eta) = \sqrt{m+1} \int_{\mathbb{R}^3} d\boldsymbol{y} \varphi^*(\boldsymbol{y} - \boldsymbol{x}_j) \psi(\xi, \eta \cup \boldsymbol{y}). \tag{3.15}$$

Indeed $a^\dagger(\boldsymbol{y})$ is the adjoint of $a(\boldsymbol{y})$ (see Appendix A.1). Also $a_\varphi^\dagger(\boldsymbol{x}_j)$ is the adjoint of $a_\varphi(\boldsymbol{x}_j)$, i.e., $(a_\varphi(\boldsymbol{x}_j))^\dagger = a_\varphi^\dagger(\boldsymbol{x}_j)$, so H^{int} and therewith H is self-adjoint (see Appendix A.1).

In order to compare to subsequent calculations (in the multi-time model) it is worth calculating the commutation relations of the one-time creation and annihilation operators directly (with our notation). We find the following well-known results (see Appendix A.1):

$$\begin{aligned}
&[a_{\varphi_1}^\dagger(\boldsymbol{x}_{j_1}), a_{\varphi_2}^\dagger(\boldsymbol{x}_{j_2})] = 0 \\
&[a_{\varphi_1}(\boldsymbol{x}_{j_1}), a_{\varphi_2}(\boldsymbol{x}_{j_2})] = 0 \\
&[a_{\varphi_1}(\boldsymbol{x}_{j_1}), a_{\varphi_2}^\dagger(\boldsymbol{x}_{j_2})] = \int_{\mathbb{R}^3} d\boldsymbol{y} \varphi_1^*(\boldsymbol{y} - \boldsymbol{x}_{j_1}) \varphi_2(\boldsymbol{y} - \boldsymbol{x}_{j_2}).
\end{aligned} \tag{3.16}$$

For $\varphi_1(\boldsymbol{x}) = \varphi_2(\boldsymbol{x}) = \delta(\boldsymbol{x})$ (as in most physics textbooks) the last commutator corresponds to:

$$[a(\boldsymbol{x}_{j_1}), a^\dagger(\boldsymbol{x}_{j_2})] = \delta(\boldsymbol{x}_{j_1} - \boldsymbol{x}_{j_2}). \tag{3.17}$$

For $\boldsymbol{x}_{j_1} = \boldsymbol{x}_{j_2} = 0$ the last commutator of (3.16) corresponds to

$$[a_{\varphi_1}, a^\dagger_{\varphi_2}] = \int_{\mathbb{R}^3} d\boldsymbol{y}\ \varphi_1^*(\boldsymbol{y})\varphi_2(\boldsymbol{y}) = \langle\varphi_1|\varphi_2\rangle. \tag{3.18}$$

3.2 The Conjecture on Configuration Space-Time

We now state the conjecture about a multi-time version of the described model in the configuration space-time setting. We denote $\mathscr{C}^{(N,m)} = (\mathbb{R}^4)^N \times (\mathbb{R}^4)^m$, so the full configuration space-time is $\mathscr{C} = \bigcup_{m=0}^\infty \mathscr{C}^{(N,m)}$. Initial conditions will be specified on the subset of configuration space-time for which all time coordinates are zero. This set we denote by $\mathscr{C}_0 = (\{0\} \times \mathbb{R}^3)^N \times \bigcup_{m=0}^\infty (\{0\} \times \mathbb{R}^3)^m$. For a configuration $q^4 \in \mathscr{C}$ we use the notation $q^4 = (\xi^4, \eta^4) = (x_1, \ldots, x_N, y_1, \ldots, y_m)$ (with $x_j, y_k \in \mathbb{R}^4$). For single space-time points $x \in \mathbb{R}^4$ we denote $x = (x^0, \boldsymbol{x}) = (x^0, x^1, x^2, x^3)$. Let m_x and m_y be positive constants and $N, m \in \mathbb{N}$.

Conjecture 3.1 (R. Tumulka and D. Dürr, 2001 and 2009)**.** *The following equations consistently define a multi-time wave function $\psi : \mathscr{C} \to \mathbb{C}$ from initial data on $\mathscr{C}_0$ (i.e., the set where all time coordinates are zero): For any $(\xi^4, \eta^4) \in \mathscr{C}$, $m = |\eta^4|$, and any $i = 1, \ldots, N$ and $k = 1, \ldots, m$,*

$$\begin{aligned} i\frac{\partial\psi}{\partial x_i^0}(\xi^4, \eta^4) = &-\frac{1}{2m_x}\Delta_{\boldsymbol{x}_i}\psi(\xi^4, \eta^4) + \frac{1}{\sqrt{m}}\sum_{k=1}^m \tilde{\varphi}(y_k - x_i)\,\psi(\xi^4, \eta^4 \backslash y_k) \\ &+ \sqrt{m+1}\int_{\mathbb{R}^3} d\boldsymbol{y}\,\tilde{\varphi}^*\left((x_i^0, \boldsymbol{y}) - x_i\right)\ \psi\left(\xi^4, \eta^4 \cup (x_i^0, \boldsymbol{y})\right) \end{aligned} \tag{3.19}$$

$$i\frac{\partial\psi}{\partial y_k^0}(\xi^4, \eta^4) = -\frac{1}{2m_y}\Delta_{\boldsymbol{y}_k}\psi(\xi^4, \eta^4), \tag{3.20}$$

where $\tilde{\varphi} : \mathbb{R}^4 \to \mathbb{C}$ is a fixed function satisfying

$$i\frac{\partial\tilde{\varphi}}{\partial y^0}(y) = -\frac{1}{2m_y}\Delta_{\boldsymbol{y}}\tilde{\varphi}(y). \tag{3.21}$$

If this conjecture were correct, it would provide an example of consistent multi-time equations with interaction. However, the following Theorem 3.2 shows that the conjecture is not correct.

Theorem 3.2. *Suppose the function $\tilde{\varphi}(0,\cdot):\mathbb{R}^3\to\mathbb{C}$ is not identically zero and has compact support. Then the above system of equations* (3.19), (3.20) *has only one continuous solution $\psi:\mathscr{C}\to\mathbb{C}$, namely $\psi=0$.*

Proof. For every $i=1,\ldots,N$, we define the operator H_{x_i} on functions ψ on $\mathscr{C}$ by saying that $H_{x_i}\psi(\xi,\eta)$ is the right hand side of (3.19).[2] Now choose $i,j\in\{1,\ldots,N\}$ with $i\neq j$ and let[3]

$$K=\left[i\frac{\partial}{\partial x_i^0}-H_{x_i},i\frac{\partial}{\partial x_j^0}-H_{x_j}\right]. \tag{3.23}$$

Let Ψ be a solution of the above system of equations (3.19) and (3.20). We then have that ($\Psi\in\mathscr{S}_i'\cap\mathscr{S}_j'$ and)

$$\Big(i\frac{\partial}{\partial x_i^0}-H_{x_i}\Big)\Psi=0 \tag{3.24}$$

and

$$\Big(i\frac{\partial}{\partial x_j^0}-H_{x_j}\Big)\Psi=0. \tag{3.25}$$

It follows that ($\Psi\in\mathscr{S}_K$ and) $K\Psi=0$. We show in Appendix A.2 that K is a multiplication operator, multiplying by the function

$$f(\xi,\eta)=2i\,\mathrm{Im}\int_{\mathbb{R}^3}d\boldsymbol{y}\,\tilde{\varphi}^*(0,\boldsymbol{y}+\boldsymbol{x}_i-\boldsymbol{x}_j)\,\tilde{\varphi}(x_i^0-x_j^0,\boldsymbol{y}). \tag{3.26}$$

From this result about K the theorem follows in this way: We have that $f(q)\Psi(q)=0$ for every $q\in\mathscr{C}$ and will show that $f\neq0$ on a dense subset of $\mathscr{C}$; then Ψ vanishes on a dense subset, and by continuity vanishes everywhere. Let

$$g(x)=g(x^0,\boldsymbol{x})=\int_{\mathbb{R}^3}d\boldsymbol{y}\,\tilde{\varphi}^*(0,\boldsymbol{y}+\boldsymbol{x})\,\tilde{\varphi}(x^0,\boldsymbol{y})=\langle\varphi|T_{\boldsymbol{x}}U_{x^0}\varphi\rangle \tag{3.27}$$

with $\varphi=\tilde{\varphi}(0,\cdot)$, $T_{\boldsymbol{x}}$ the translation operator $T_{\boldsymbol{x}}\varphi(\boldsymbol{y})=\varphi(\boldsymbol{x}-\boldsymbol{y})$, and U_{x^0} the unitary time-evolution operator of the free Schrödinger equation. Since $f(\xi,\eta)=2i\,\mathrm{Im}\,g(x_i-x_j)$, it suffices to show that $\mathrm{Im}\,g(x)\neq0$ on a dense subset of $\mathbb{R}^4$. (The following argument we owe to Eric Carlen, Roderich Tumulka and Eugene Speer. Presumably, the hypothesis of compact support of φ can be weakened.)

[2] For the sake of completeness, we note that H_{x_i} is defined on the space $\mathscr{S}_i$ of those measurable functions $\psi:\mathscr{C}\to\mathbb{C}$ that are twice partially differentiable with respect to each of x_i^1,x_i^2, and x_i^3 and for which the integrand in (3.19),

$$\tilde{\varphi}^*\big((x_i^0,\boldsymbol{y})-x_i\big)\,\psi\big(\xi,\eta\cup(x_i^0,\boldsymbol{y})\big), \tag{3.22}$$

is an integrable function of $\boldsymbol{y}$ for every (ξ,η). For later use, let $\mathscr{S}_i'$ be the space of $\psi\in\mathscr{S}_i$ for which $\partial\psi/\partial x_i^0$ exists.

[3] For the sake of completeness, we note that K is defined on the space $\mathscr{S}_K$ of those $\psi:\mathscr{C}\to\mathbb{C}$ with $\psi\in\mathscr{S}_i'\cap\mathscr{S}_j'$ for which $i\partial\psi/\partial x_i^0-H_{x_i}\psi\in\mathscr{S}_j'$ and $i\partial\psi/\partial x_j^0-H_{x_j}\psi\in\mathscr{S}_i'$.

To this end, we will show that g possesses an analytic continuation on $\mathbb{C}^4 \setminus \{x^0 = 0\}$. It then follows that if $\operatorname{Im} g$ vanished on an open subset U of $\mathbb{R}^4 \setminus \{x^0 = 0\}$ then, by the Cauchy–Riemann partial differential equations,

$$\frac{\partial \operatorname{Re} g}{\partial \operatorname{Re} x^\mu} = \frac{\partial \operatorname{Im} g}{\partial \operatorname{Im} x^\mu}, \quad \frac{\partial \operatorname{Re} g}{\partial \operatorname{Im} x^\mu} = -\frac{\partial \operatorname{Im} g}{\partial \operatorname{Re} x^\mu} \tag{3.28}$$

g would have to be constant on U; again by analyticity, g would have to be constant everywhere on $\mathbb{R}^4 \setminus \{x^0 = 0\}$, which it is not: On the one hand, we have that

$$g(x) \to \|\varphi\|^2_{L^2} \text{ as } x \to 0 \tag{3.29}$$

because $g(0) = \|\varphi\|^2_{L^2}$ and g is continuous at 0, as $T_{\boldsymbol{x}}$ and U_t are strongly continuous in $\boldsymbol{x}$ and t. On the other hand,

$$|g(x)| < \|\varphi\|^2_{L^2} \text{ for } x^0 \neq 0 \tag{3.30}$$

because $g(t, \boldsymbol{x}) = \langle \varphi | T_{\boldsymbol{x}} U_t \varphi \rangle$ is the inner product between two functions of norm $\|\varphi\|$, thus must have absolute value no greater than $\|\varphi\|^2$, and can assume the maximum value only if $\varphi = e^{i\theta} T_{\boldsymbol{x}} U_t \varphi$ for some $\theta \in \mathbb{R}$, which occurs only if $\boldsymbol{x} = 0$ and $t = 0$. Thus, g is not constant.

To see that g is analytic, we use the Green function of the free Schrödinger equation,

$$U_t \varphi(\boldsymbol{x}) = \int_{\mathbb{R}^3} d\boldsymbol{y}\, G_t(\boldsymbol{x}, \boldsymbol{y})\, \varphi(\boldsymbol{y}), \tag{3.31}$$

$$G_t(\boldsymbol{x}, \boldsymbol{y}) = \Big(\frac{1}{2\pi i t}\Big)^{3/2} e^{-i(\boldsymbol{x}-\boldsymbol{y})^2/2t}, \tag{3.32}$$

and thus write

$$g(x) = \Big(\frac{1}{2\pi i x^0}\Big)^{3/2} \int_{\mathbb{R}^3} d\boldsymbol{y} \int_{\mathbb{R}^3} d\boldsymbol{z}\, \varphi^*(\boldsymbol{y})\, \varphi(\boldsymbol{z})\, e^{i(\boldsymbol{y}-\boldsymbol{z}-\boldsymbol{x})^2/2x^0}. \tag{3.33}$$

The pre-factor and the exponential term together form an analytic function in x on $\mathbb{C}^4 \setminus \{x^0 = 0\}$. From the Payley–Wiener theorem and standard arguments it follows that if a function $f(z, \lambda)$ is analytic in z for every λ and φ has compact support then $\int d\lambda\, \varphi(\lambda)\, f(z, \lambda)$ is analytic. Since our φ was assumed to have compact support, g is analytic. □

3.3 The Conjecture on Hilbert Space

We now switch to the Hilbert space setting and state the corresponding conjecture. ψ is now a wavefunction on Fock space. The Fock space is $\mathcal{F} = \bigoplus_{m=0}^{\infty} \mathcal{H}^{(m)}$ with the (N, m)-particle sector $\mathcal{H}^{(m)} = L^2(\mathcal{Q}^{(m)})$ (anti-symmetrized in the electron variables) where $\mathcal{Q}^{(m)} = \mathbb{R}^{3N} \times \mathbb{R}^{3m}$. Note that the Fock space is not symmetrized in the photon variables now. Instead, equation (3.36) below represents the bosonic symmetry.

We now "translate" the equations (3.19), (3.20) into equations on Hilbert space. The corresponding Hamiltonians (the right hand sides of (3.19), (3.20)) are

$$\begin{aligned} H_{x_i} &= H_{x_i}^{free} + H_{x_i}^{c} + H_{x_i}^{a}, \\ H_{y_k} &= H_{y_k}^{free}, \end{aligned} \tag{3.34}$$

with

$$\begin{aligned} (H_{x_i}^{free}\psi)(\xi,\eta) &= -\frac{1}{2m_x}\Delta_{x_i}\psi(\xi,\eta), \\ (H_{x_i}^{c}\psi)(\xi,\eta) &= \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\tilde{\varphi}(t_{y_k}-t_{x_i},\boldsymbol{y}_k-\boldsymbol{x}_i)\left[exp\left(\sum_{c=k}^{m-1}\frac{i}{2m_y}\Delta_c(t_{y_{c+1}}-t_{y_c})\right)\psi^{(m-1)}\right](\xi,\eta\backslash\boldsymbol{y}_k), \\ (H_{x_i}^{a}\psi)(\xi,\eta) &= \sqrt{m+1}\int_{\mathbb{R}^3} d\boldsymbol{y}_{m+1}\tilde{\varphi}^*(t_{y_{m+1}}-t_{x_i},\boldsymbol{y}_{m+1}-\boldsymbol{x}_i)\psi(\xi,\eta\cup\boldsymbol{y}_{m+1}), \\ (H_{y_k}^{free}\psi)(\xi,\eta) &= -\frac{1}{2m_y}\Delta_{y_k}\psi(\xi,\eta). \end{aligned} \tag{3.35}$$

The expression $\Delta_c\psi^{(m)}$ means the Laplace operator acting on the c-th variable of $\psi^{(m)}$. We have to insert the operator $exp\left(\sum_{c=k}^{m-1}\frac{i}{2m_y}\Delta_c(t_{y_{c+1}}-t_{y_c})\right)$ in the definition of $H_{x_i}^c$ in order to keep track of the corresponding time variables when we remove a variable from the η-configuration. In Appendix A.2 it is shown that $H_{x_i}^a$ is the adjoint of $H_{x_i}^c$, i.e., that both H_{x_i} and H_{y_k} are self-adjoint. Note that if all times in (3.35) are set equal we get the one-time Hamiltonians (3.5) (if we set $\tilde{\varphi}(\boldsymbol{y},0) = \varphi(\boldsymbol{y})$).

Keeping track of the corresponding time variables is also relevant to the condition for the bosonic symmetry. This condition on ψ is (we only write out the variables $\boldsymbol{y}_p$ and $\boldsymbol{y}_q$ in the η configuration)

$$\psi(\xi,\boldsymbol{y}_p,\boldsymbol{y}_q) = e^{-\frac{i}{2m_y}\Delta_{\boldsymbol{y}_p}(t_{y_q}-t_{y_p})}e^{-\frac{i}{2m_y}\Delta_{\boldsymbol{y}_q}(t_{y_p}-t_{y_q})}\psi(\xi,\boldsymbol{y}_q,\boldsymbol{y}_p) \tag{3.36}$$

for all $\boldsymbol{y}_q, \boldsymbol{y}_p \in \eta$. We denote with ψ_0 the wavefunction on Fock space with all parameters t_{x_j} and t_{y_k} set to zero. The conjecture about the multi-time model on Fock space is

Conjecture 3.3. *The following equations consistently define a multi-time wave function* ψ *on the Fock space* $\mathcal{F}$ *from initial data* $\psi_0 \in \mathcal{F}$ *that are symmetric in the y-variables: For the* (N, m)*-sector of Fock space,*

$$i \frac{\partial \psi}{\partial t_{x_i}}(\xi, \eta) = \left((H_{x_i}^{free} + H_{x_i}^{c} + H_{x_i}^{a})\psi \right)(\xi, \eta)$$
$$i \frac{\partial \psi}{\partial t_{y_k}}(\xi, \eta) = \left(H_{y_k}^{free}\psi \right)(\xi, \eta) \tag{3.37}$$

with the Hamiltonians (3.35) *and where* $\tilde{\varphi} : \mathbb{R}^4 \to \mathbb{C}$ *is a fixed function that is square integrable for each* t *and that satisfies*

$$i \frac{\partial \tilde{\varphi}}{\partial t}(t, \boldsymbol{y}) = -\frac{1}{2m_y} \Delta_{\boldsymbol{y}} \tilde{\varphi}(t, \boldsymbol{y}). \tag{3.38}$$

Indeed we find that the consistency conditions formulated in Assertion 2.1 are not satisfied by the Hamiltonians (3.35). We calculate in Appendix A.2 that

$$[H_{x_i}, H_{x_j}] + i \frac{\partial H_{x_i}}{\partial t_{x_j}} - i \frac{\partial H_{x_j}}{\partial t_{x_i}} = 2i \operatorname{Im} \int_{\mathbb{R}^3} d\boldsymbol{y}\, \tilde{\varphi}^*(y - x_i)\tilde{\varphi}(y - x_j),$$
$$[H_{y_k}, H_{y_j}] + i \frac{\partial H_{y_k}}{\partial t_{y_j}} - i \frac{\partial H_{y_j}}{\partial t_{y_k}} = 0,$$
$$[H_{x_i}, H_{y_k}] + i \frac{\partial H_{x_i}}{\partial t_{y_k}} - i \frac{\partial H_{y_k}}{\partial t_{x_i}} = 0. \tag{3.39}$$

Chapter 4

Two Consistent Multi-Time Models

As it does not seem possible to establish a consistent multi-time QFT model on all of space-time, we define in this chapter the multi-time wavefunction only on a subset of space-time. This subset is the set of all spacelike configurations with a δ cutoff taken into account.[1]

Our main result is a consistent multi-time QFT model with creation and annihilation of particles. This result is stated in Theorem 4.20. This chapter is organized as follows. In section 4.1 we first prove important statements about the existence of solutions for certain systems of equations. These results will be used in sections 4.2 and 4.3 to prove the existence of solutions for the multi-time evolutions. In section 4.2 we set up a multi-time model for N electrons that interact via a potential only if they are closer to each other than δ. This model will be particulary helpful in proving and understanding the next model in section 4.3. In this section we consider the QFT multi-time model with a constant number of electrons that can emit and absorb photons in a δ-ball around each electron.

The idea is to use the known one-time evolution with interaction for certain "groups" of particles. Those groups will be defined in such a way that particles within a group interact with each other, but particles in different groups do not. Our time-evolution equations will be "Dirac-type" equations and in the case of the QFT model "Dirac-type" equations with additional emission and absorption terms. We show that these equations have the property that solutions propagate with a finite speed.

[1] For N particles the set of all spacelike configurations is $\{(x_1, \dots, x_N) \subset \mathbb{R}^{4N} : \forall i \neq j : (x_i - x_j)^\mu (x_i - x_j)_\mu > 0\}$. For a four-vector x, $x^\mu x_\mu = -(x^0)^2 + \boldsymbol{x}^2$ denotes its Minkowski lenght. x is called spacelike if $x^\mu x_\mu > 0$, timelike if $x^\mu x_\mu < 0$ and lightlike if $x^\mu x_\mu = 0$.

This will be important in proving the Theorems about the multi-time evolution.

4.1 Existence of Solutions for Certain Systems of Equations

This section is due to Roderich Tumulka. If S is a closed subset of $\mathbb{R}^d$ then $C^\infty(S, \mathbb{C}^k)$ means the space of functions $f : S \to \mathbb{C}^k$ such that there is an open neighborhood U of S in $\mathbb{R}^d$ and a smooth function $g : U \to \mathbb{C}^k$ with $f = g$ on S.

If $\mathscr{B}$ is a vector bundle over the manifold $\mathscr{M}$ then $C^\infty(\mathscr{M}, \mathscr{B})$ denotes the space of smooth cross-sections of $\mathscr{B}$, i.e., the space of smooth functions $\psi : \mathscr{M} \to \mathscr{B}$ with $\psi(q) \in \mathscr{B}_q$ for all $q \in \mathscr{M}$. (If S is a closed subset of $\mathscr{M}$ then $C^\infty(S, \mathscr{B})$ means the space of functions $f : S \to \mathscr{B}$ such that there is an open neighborhood U of S in $\mathscr{M}$ and a smooth cross-section $g \in C^\infty(U, \mathscr{B}|_U)$ with $f = g$ on S.) By a *Hermitian* vector bundle we mean a complex vector bundle $\mathscr{B}$ equipped on each fiber space $\mathscr{B}_q$ with an inner product (i.e., a positive-definite, Hermitian sesqui-linear form) $\langle\cdot|\cdot\rangle_q$ that varies smoothly with q. By a *Dirac-type differential operator on $\mathscr{B}$* we mean an operator of the form

$$H\psi(q) = i \sum_{i=1}^{\dim T_q\mathscr{M}} A_i(q)\, \partial_i \psi(q) + B(q)\psi(q) \tag{4.1}$$

where the coefficients $A_i(q)$ and $B(q)$ are self-adjoint matrices on $\mathscr{B}_q$ (self-adjoint with respect to $\langle\cdot|\cdot\rangle_q$) and smooth functions of q. By an *emission-absorption-type integral operator* we mean an operator of the form

$$H\psi(q) = \sum_{i=1}^{n(q)} \int_{B_1^d} dy\, C_i(q,y)\, \psi(F_i(q,y)) + \sum_{i=1}^{n'(q)} C_i'(q)\, \psi(F_i'(q)), \tag{4.2}$$

where the functions $n, n' : \mathscr{M} \to \mathbb{N}$ are locally constant (i.e., constant on every connected component of $\mathscr{M}$), B_1^d is the closed unit ball in $\mathbb{R}^d$, $C_i(q,y) \in \mathscr{B}_q \otimes \mathscr{B}^*_{F_i(q,y)}$, $C_i'(q) \in \mathscr{B}_q \otimes \mathscr{B}_{F_i'(q)}$, the functions C_i, C_i', F_i, F_i' are smooth, and the $(\dim T_{F_i(q,y)}\mathscr{M} \times d)$ matrix $\partial F_i(q,y)/\partial y_j$ has rank d for all q, y, and $i \le n(q)$.

The following fact is known (see [2]):

Lemma 4.1. *Let $\mathscr{M}$ be a complete Riemannian manifold and $\mathscr{B}$ a Hermitian vector bundle over $\mathscr{M}$. When H is a Dirac-type differential operator then the PDE*

$$i\frac{\partial f(t,q)}{\partial t} = Hf(t,q), \tag{4.3}$$

where $t \in \mathbb{R}$, $q \in \mathscr{M}$, *and* f *is a cross-section of* $\mathscr{B}$, *possesses, for any smooth initial datum* $f_0 \in C^\infty(\mathscr{M}, \mathscr{B})$, *a unique smooth solution* $f \in C^\infty(\mathbb{R} \times \mathscr{M}, \mathscr{B})$ *with* $f(0, \cdot) = f_0(\cdot)$.

Lemma 4.2. *For each* $\alpha = 1, \ldots, L$, *let* $\mathscr{M}_\alpha$ *be a complete Riemannian manifold,* $\mathscr{B}_\alpha$ *a Hermitian vector bundle over* $\mathscr{M}_\alpha$, $\mathscr{G}_\alpha$ *a closed subset of* $\mathscr{M}_\alpha$, *and* $\mathscr{D}_\alpha$ *a closed subset of* $\mathbb{R} \times \mathscr{M}_\alpha$ *with* $\{0\} \times \mathscr{G}_\alpha \subseteq \mathscr{D}_\alpha$. *Let* $\mathscr{D}_\alpha(t) = \{q \in \mathscr{M}_\alpha : (t, q) \in \mathscr{D}_\alpha\}$. *Let* H_α *be a Dirac-type differential operator on* $\mathscr{B}_\alpha$. *Suppose that, for each* α, *the set* $\mathscr{D}_\alpha$ *is such that a solution* $\psi \in C^\infty(\mathscr{D}_\alpha, \mathscr{B}_\alpha)$ *of the equation*

$$i\frac{\partial \psi}{\partial t} = H_\alpha \psi \tag{4.4}$$

is uniquely determined by initial data on $\mathscr{G}_\alpha$, *i.e.,* $\psi(0, q) = \phi(q)$ *with* $\phi \in C^\infty(\mathscr{G}_\alpha, \mathscr{B}_\alpha)$. *Then also the (interaction-free) multi-time evolution equations*

$$i\frac{\partial \psi}{\partial t_\alpha} = H_\alpha \psi \tag{4.5}$$

possess, for every initial datum $\phi \in C^\infty(\prod_1^L \mathscr{G}_\alpha, \otimes_1^L \mathscr{B}_\alpha)$, *a unique solution* ψ, *and we have* $\psi \in C^\infty(\prod_1^L \mathscr{D}_\alpha, \otimes_1^L \mathscr{B}_\alpha)$.

Proof. We write $q = (q_1, \ldots, q_L)$ for a point in $\prod \mathscr{G}_\alpha$. First obtain $\chi_1(t_1, q)$ by solving (4.5) for $\alpha = 1$ with initial datum $\chi_1(0, q) = \phi(q)$. In detail, choose a smooth continuation $\tilde{\phi}$ of ϕ on $\mathscr{M} := \prod_1^L \mathscr{M}_\alpha$; by Lemma 4.1, there exists a smooth solution $\tilde{\chi}_1(t_1, q)$ of

$$i\frac{\partial f}{\partial t_1} = H_1 f \tag{4.6}$$

on $\mathbb{R} \times \mathscr{M}$. Since H acts only on q_1 and $\mathscr{B}_1$ but not on q_α and $\mathscr{B}_\alpha$ for $\alpha \geq 2$, different values of $q_2, \ldots, q_L$ decouple in (4.6), i.e., if we insert particular values $Q_2, \ldots, Q_L$ for the variables $q_2, \ldots, q_L$ in $\tilde{\chi}_1$ then the resulting function $\tilde{\chi}_1(t_1, q_1, Q_2, \ldots, Q_L)$ equals the function $\hat{\chi}(t_1, q_1)$ obtained by solving (4.6) from initial data $\hat{\phi}(q_1) = \tilde{\phi}(q_1, Q_2, \ldots, Q_L)$. By assumption, the solution $\hat{\chi}$ on $\mathscr{D}_1$ depends only on the initial data $\hat{\phi}$ on $\mathscr{G}_1$, so $\tilde{\chi}$ on $\mathscr{D}_1 \times \prod_2^L \mathscr{G}_\alpha$ depends only on the initial data $\tilde{\phi}$ on $\prod_1^L \mathscr{G}_\alpha$, which is ϕ; define $\chi_1 = \tilde{\chi}$ on $\mathscr{D}_1 \times \prod_2^L \mathscr{G}_\alpha$. Thus, we have that $\chi_1 \in C^\infty(\mathscr{D}_1 \times \prod_2^L \mathscr{G}_\alpha, \otimes_1^L \mathscr{B}_\alpha)$, solves (4.6), and agrees with ϕ at $t_1 = 0$; we also have that χ_1 is uniquely determined by (4.6) and the initial condition ϕ.

In the same way, obtain $\chi_2(t_1, t_2, q)$ by solving (4.5) for $\alpha = 2$ with initial datum $\chi_2(t_1, 0, q) = \chi_1(t_1, q)$. Continue this way to obtain $\chi_3, \ldots, \chi_L$, and set $\psi = \chi_L$. We then have that $\psi \in$

$C^\infty(\prod_1^L \mathscr{D}_\alpha, \otimes_1^L \mathscr{B}_\alpha)$, and we know that every solution of (4.5) for all $\alpha = 1, \dots, L$ with initial datum ϕ has to agree with ψ.

Now it is clear that ψ satisfies (4.5) for $\alpha = L$ but it remains to show this also for $\alpha = 1, \dots, L-1$. So consider some $\beta \in \{1, \dots, L-1\}$, and call ψ again χ_L. Since

$$\left[\frac{i\partial}{\partial t_\beta} - H_\beta, \frac{i\partial}{\partial t_L} - H_L\right] = 0, \tag{4.7}$$

we have that

$$\Big(\frac{i\partial}{\partial t_L} - H_L\Big)\Big(\frac{i\partial}{\partial t_\beta} - H_\beta\Big)\chi_L = \Big(\frac{i\partial}{\partial t_\beta} - H_\beta\Big)\Big(\frac{i\partial}{\partial t_L} - H_L\Big)\chi_L = 0 \tag{4.8}$$

because

$$\Big(\frac{i\partial}{\partial t_L} - H_L\Big)\chi_L = 0. \tag{4.9}$$

We need to show that the function

$$\chi'_{\beta,L} := \Big(\frac{i\partial}{\partial t_\beta} - H_\beta\Big)\chi_L \tag{4.10}$$

vanishes identically on $\prod_1^L \mathscr{D}_\alpha$. To this end, we note that $\chi'_{\beta,L}$ satisfies (4.5) for $\alpha = L$ with initial datum

$$\chi'_{\beta,L}(t_1, \dots, t_{L-1}, 0, q) = \Big(\frac{i\partial}{\partial t_\beta} - H_\beta\Big)\chi_{L-1}(t_1, \dots, t_{L-1}, q). \tag{4.11}$$

By the linearity of (4.5) and the uniqueness part of Lemma 4.1, it suffices to show that this initial datum vanishes identically; that is, it suffices to show that χ_{L-1} satisfies (4.5) with $\alpha = \beta$. If $\beta = L-1$, this is immediate from the construction of χ_{L-1}. If $\beta < L-1$, we repeat the above reasoning to find that it suffices to show that χ_{L-2} satisfies (4.5) with $\alpha = \beta$. After $L-\beta$ repetitions we are done. □

4.2 Model 1: Fixed Particle Number with δ-Range Interaction Potential

In this section a multi-time theory for a constant number of electrons N and an interaction pair potential with δ-range is described. We use the following notation: We abbreviate $q = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_N) \in \mathbb{R}^{3N}$ with each $\boldsymbol{x}_k \in \mathbb{R}^3$ and similarly $q^4 = (x_1, \dots, x_N) \in \mathbb{R}^{4N}$ with $x_k \in \mathbb{R}^4$.

After introducing the fundamental one-time-evolution equation in section 4.2.1, we prove the finite propagation speed of solutions in section 4.2.2. In section 4.2.3 we define the "groups" of particles and the subset of space-time on which the multi-time wavefunction will be defined. Finally, in section 4.2.4 these results and the results from section 4.1 are used to define the multi-time evolution for the wavefunction and prove the existence and uniqueness of solutions.

4.2.1 The Dirac Equation with Potential

For this multi-time model we use the Dirac equation with some interaction potential. The one-particle Dirac equation with potential for a wavefunction $\psi : \mathbb{R}^4 \to \mathbb{C}^4$ is

$$i\frac{\partial}{\partial t}\psi(t, \boldsymbol{x}) = \left(-ic\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta mc^2 + V(\boldsymbol{x})\right)\psi(t, \boldsymbol{x}) \tag{4.12}$$

with positive constants m and c, a Hermitian (4×4)-matrix valued function V and the (4×4) Dirac matrices $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$ and β. Note that we do not explicitly write out the summation over the spin indices. For example, with $V\psi$ we mean that the s-component of the spinor $V\psi$ is $\sum_{p=0}^{3} V^s{}_p \psi^p$.

The Dirac equation can be generalized to an N-particle equation. For a wavefunction $\psi : \mathbb{R}^{3N+1} \to (\mathbb{C}^4)^{\otimes N}$ the N-particle Dirac equation with potential is

$$i\frac{\partial}{\partial t}\psi(t, q) = \left(\sum_{k=1}^{N}\left(-ic\boldsymbol{\alpha}_k \cdot \boldsymbol{\nabla}_k + \beta_k mc^2\right) + V(q)\right)\psi(t, q) \tag{4.13}$$

with positive constants m and c and a Hermitian matrix valued function $V : \mathbb{R}^{3N} \to (\mathbb{C}^{4\times 4})^{\otimes N}$. The Dirac matrices $\boldsymbol{\alpha}_k$ and β_k act only on the k-th particle's spin indices and $\boldsymbol{\nabla}_k$ acts only on the k-th particle's position variables.

4.2.2 Finite Propagation Speed

For the Dirac equations (4.12) and (4.13) we now show that the wavefunction cannot propagate faster than at the speed of light. In Lemma 4.3 this statement is precisely formulated for the one-particle Dirac equation. Lemma 4.4 generalizes this statement to N particles.

These Lemmas also tell us about the domain of dependence of solutions of the Dirac equation. The domain of dependence of a wavefunction $\psi(t)$ for a certain region $M_0 \subset \mathbb{R}^3$ is the region on which initial conditions $\psi(0)$ have to be specified in order to determine the wavefunction $\psi(t)$ on M_0. In the following Lemma 4.3 we denote the ball around $\boldsymbol{x}$ with radius $r \geq 0$ as $B_r(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{R}^3 : ||\boldsymbol{y} - \boldsymbol{x}||_2 \leq r\}$. The ball around the origin is abbreviated $B_r := B_r(\boldsymbol{0})$. The ball around $\boldsymbol{x}$ can also be written as $B_r(\boldsymbol{x}) = \boldsymbol{x} + B_r$ with the summation defined as $a + B := \{a + b : b \in B\}$

for $a \in X$ and $B \subset X$. We also want to express that a certain region of space cannot grow faster than with the speed of light. For a subset $M_0 \subset \mathbb{R}^3$ we therefore define $M_t = M_0 + B_{ct}$ with the summation of sets defined as $A+B := \{a+b : a \in A, b \in B\}$. Pictorally speaking one obtains M_t by putting a ball with radius ct around each point of M_0. The next Lemma 4.3 states that the domain of dependence for $\boldsymbol{x}$ is given by $B_{ct}(\boldsymbol{x})$ and that a wavefunction with compact support M_0 has after time t compact support in M_t.

Lemma 4.3 (Finite Propagation Speed: One-particle Dirac Equation)**.** *Let ψ be a solution of the one-particle Dirac equation (4.12) with initial data $\psi(0,\cdot) \in L^2(\mathbb{R}^3, \mathbb{C}^4) \cap C^\infty(\mathbb{R}^3, \mathbb{C}^4)$ and with a Hermitian potential matrix $V \in C^\infty(\mathbb{R}^3, \mathbb{C}^{4\times 4})$. Let $t > 0$.*

(i) Then specifying initial conditions on $B_{ct}(\boldsymbol{x})$ uniquely determines $\psi(t, \boldsymbol{x})$. (I.e. the domain of dependence grows with the speed of light.)

(ii) Let $\psi(0,\cdot)$ have compact support, supp$(\psi(0,\cdot)) = M_0 \subset \mathbb{R}^3$. Then the support of $\psi(t,\cdot)$ is a subset of $M_t = M_0 + B_{ct}$. (I.e. the support of the wavefunction cannot grow faster than with the speed of light.)

Proof. For this proof we set $c = 1$. Let $(T, \boldsymbol{y}) \in \mathbb{R}^4$ with $T > 0$ and for $t \in [0, T]$ let

$$\Sigma_t = \left\{(t, \boldsymbol{x}) \in \mathbb{R}^4 : \boldsymbol{x} \in B_{T-t}(\boldsymbol{y})\right\} \tag{4.14}$$

denote the corresponding set of the equal-time hypersurface for time t (note that for $t = T$, $\Sigma_T = (T, \boldsymbol{y})$). We first prove the following statement $(*)$: If ψ vanishes on Σ_0 it also vanishes on Σ_t for all $t \in [0, T]$.

For any $t \in [0, T)$, let the lightcone between the surfaces Σ_0 and Σ_t be $C_t = \bigcup_{t' \in [0,t]} \Sigma_{t'}$. Let Σ^s denote the sides of the lightcone such that $\Sigma_0 \dot\cup \Sigma_t \dot\cup \Sigma^s = \partial C_t$ (the boundary of C_t) as shown in Figure 4.1. For ease of notation and since we consider a fixed time t we do not write an index t for the time-dependence of Σ^s and also define $C := C_t$. Let n be the outward-pointing unit vector field on ∂C orthogonal to ∂C in the Euclidean metric on $\mathbb{R}^4$, i.e., $||n||^2 = \sum_{i=1}^4 n^i n^i = 1$ and for any tangent vector s on ∂C, $n \cdot s = \sum_{i=1}^4 n^i s^i = 0$.[2] Let $\psi^\dagger$ denote the complex conjugate and transpose of the spinor ψ (so that $|\psi|^2 = \psi^\dagger \psi$). Let $j = (j^0, j^1, j^2, j^3) = (|\psi|^2, \psi^\dagger \alpha^1 \psi, \psi^\dagger \alpha^2 \psi, \psi^\dagger \alpha^3 \psi)$ denote

[2] We use a normal vector in the Euclidean metric here because we have to deal with the lightlike hypersurface Σ^s, so the normal vector in the Minkowski sense would be lightlike, too. Therefore the flux integrals (4.16) could not be written down in this simple form.

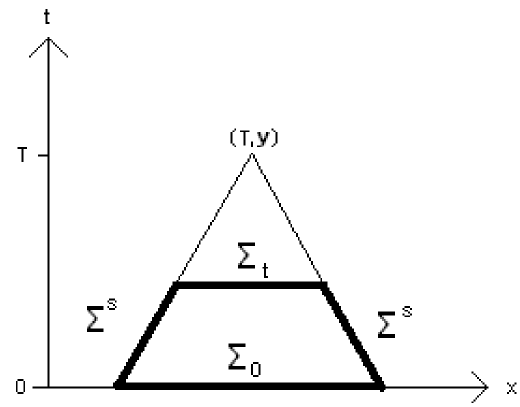

Figure 4.1: Σ_0 and Σ_t are parts of equal time hypersurfaces, Σ^s is lightlike. Σ_0, Σ_t and Σ^s enclose a volume in $\mathbb{R}^4$, the truncated lighcone of $(T, \boldsymbol{y})$.

the four-current. Then the continuity equation can be expressed as $div(j) = 0$.[3] According to Gauß' integral Theorem,

$$0 = \int_C div(j)\ d^4x = \int_{\partial C} j \cdot n\ d^3x = \int_{\Sigma_0} j \cdot n\ d^3x + \int_{\Sigma_t} j \cdot n\ d^3x + \int_{\Sigma^s} j \cdot n\ d^3x. \tag{4.16}$$

The differential d^3x denotes the volume on a 3-surface relative to the Euclidean metric and $j \cdot n = \sum_{k=1}^{3} j^k n^k$ denotes the Euclidean inner product. Now suppose that $\psi|_{\Sigma_0} = 0$. Since the (outward-pointing) normal vector on Σ_0 is $n = (-1, 0, 0, 0)$ we have $j \cdot n = -j^0 = -|\psi|^2 = 0$ on Σ_0. On Σ_t the normal vector is $n = (1, 0, 0, 0)$, so $j \cdot n = j^0 = |\psi|^2$ on Σ_t. Therefore (I denotes the identity on $\mathbb{C}^4$)

$$0 = \int_{\Sigma_t} j \cdot n\ d^3x + \int_{\Sigma_s} j \cdot n\ d^3x = \int_{\Sigma_t} |\psi|^2\ d^3x + \int_{\Sigma^s} \psi^\dagger (n^0 I + \boldsymbol{\alpha} \cdot \boldsymbol{n}) \psi\ d^3x. \tag{4.17}$$

Next we prove that the (4×4)-matrix $A := (n^0 I + \boldsymbol{\alpha} \cdot \boldsymbol{n})$ is positive semi-definite on Σ^s, i.e., that all eigenvalues are ≥ 0. For any unit vector $\boldsymbol{b} \in \mathbb{R}^3$ we have that $\boldsymbol{\alpha} \cdot \boldsymbol{b}$ has eigenvalues -1 and $+1$.[4] Therefore the matrix $||\boldsymbol{n}|| \boldsymbol{\alpha} \cdot \frac{\boldsymbol{n}}{||\boldsymbol{n}||}$ has eigenvalues $+||\boldsymbol{n}||$ and $-||\boldsymbol{n}||$, so the lowest eigenvalue of A is

[3] Note the difference between the four-divergence $div(j) = \partial_\mu j^\mu = \frac{\partial j^0}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j}$ and the three-divergence $div(\boldsymbol{j}) = \boldsymbol{\nabla} \cdot \boldsymbol{j}$. The continuity equation $div(j) = 0$ or $\frac{\partial |\psi|^2}{\partial t} = -div(\boldsymbol{j})$ is a consequence of the Dirac equation (4.12). Indeed, since V, β and α^i are Hermitian matrices and since for a Hermitian matrix A, $\psi^\dagger A \psi$ is real, we find:

$$\begin{aligned} \frac{\partial |\psi|^2}{\partial t} &= \frac{\partial \left(\psi^\dagger \psi\right)}{\partial t} = \frac{\partial \psi^\dagger}{\partial t} \psi + \psi^\dagger \frac{\partial \psi}{\partial t} = 2Re \left(\psi^\dagger \frac{\partial \psi}{\partial t}\right) \\ &= 2Re \left(\psi^\dagger \left(-\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} - i\beta m - iV\right) \psi\right) = -2Re \left(\psi^\dagger \boldsymbol{\alpha} \cdot \boldsymbol{\nabla} \psi\right) = -\left(\psi^\dagger \boldsymbol{\alpha} \cdot \boldsymbol{\nabla} \psi + \left(\boldsymbol{\nabla} \psi^\dagger\right) \cdot \boldsymbol{\alpha} \psi\right) \\ &= -\boldsymbol{\nabla} \cdot \left(\psi^\dagger \boldsymbol{\alpha} \psi\right) = -div(\boldsymbol{j}) \end{aligned} \tag{4.15}$$

[4] For the Dirac matrices the following relation holds: $\alpha^i \alpha^j + \alpha^j \alpha^i = 2\delta_{ij} I$. Therefore for any unit vector $\boldsymbol{b} \in \mathbb{R}^3$ we have $(\boldsymbol{\alpha} \cdot \boldsymbol{b})^2 = \sum_{i,j=1}^{3} b^i b^j \alpha^i \alpha^j = \sum_{i,j=1}^{3} b^i b^j (2\delta_{ij} I - \alpha^j \alpha^i) = -(\boldsymbol{\alpha} \cdot \boldsymbol{b})^2 + 2||\boldsymbol{b}||^2 I$, i.e., $(\boldsymbol{\alpha} \cdot \boldsymbol{b})^2 = ||\boldsymbol{b}||^2 I = I$, so $\boldsymbol{\alpha} \cdot \boldsymbol{b}$ has eigenvalues -1 and $+1$.

$e = n^0 - ||\boldsymbol{n}||$. On Σ^s the normal-vector is $n = (\frac{1}{\sqrt{2}}, \boldsymbol{n})$ with $||\boldsymbol{n}|| = \frac{1}{\sqrt{2}}$, so $e = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$. We then have

$$0 = \int_{\Sigma_t} |\psi|^2 \; d^3x + \int_{\Sigma^s} \underbrace{\psi^\dagger(n^0 I + \boldsymbol{\alpha} \cdot \boldsymbol{n})\psi}_{\geq 0} \; d^3x. \tag{4.18}$$

Since each integrand is ≥ 0, each integral is ≥ 0. Then according to (4.18) each integral has to vanish. In particular this means that the integrand $|\psi|^2$ has to vanish (almost everywhere) on Σ_t, i.e., $\psi|_{\Sigma_t} = 0$ (almost everywhere). From Lemma 4.1 we have that $\psi \in C^\infty(\mathbb{R}^4, \mathbb{C}^4)$, therefore ψ vanishes identically on Σ_t for all $t \in [0, T]$ (in particular also for $t = T$). We have proven the statement $(*)$: If ψ vanishes on Σ_0 it also vanishes on Σ_t for all $t \in [0, T]$. The two statements of the Lemma follow in this way:

(i) Suppose $\psi_1, \psi_2 \in C^\infty(\mathbb{R}^4, \mathbb{C}^4)$ are solutions of the Dirac equation (4.12) and are identical on Σ_0 and arbitrary on the rest of the $t = 0$ hypersurface. Then $\psi_1 - \psi_2$ is also a solution of the Dirac equation with $\psi_1 - \psi_2 = 0$ on Σ_0. According to statement $(*)$ also $\psi_1 - \psi_2 = 0$ in $(T, \boldsymbol{y})$, i.e., $\psi_1(T, \boldsymbol{y}) = \psi_2(T, \boldsymbol{y})$.

(ii) Suppose $\psi(0, \cdot)$ has compact support, $\mathit{supp}\,(\psi(0, \cdot)) = M_0 \subset \mathbb{R}^3$. Recall that $M_t = M_0 + B_t$. Now consider a point $(t, \boldsymbol{z}) \in \mathbb{R}^4$ for $t \geq 0$ and $(t, \boldsymbol{z}) \notin M_t$. This means that $B_t(\boldsymbol{z}) \cap M_0 = \emptyset$. Since $\psi(0, \cdot) = 0$ on $B_t(\boldsymbol{z})$ (outside M_0), according to $(*)$ also $\psi(t, \boldsymbol{z}) = 0$. Since this is true for any $(t, \boldsymbol{z}) \notin M_t$, $\psi(t, \cdot)$ vanishes outside M_t.

□

From Lemma 4.3 (i) it follows immediately that the wavefunction on any $M_0 \subset \mathbb{R}^3$ at time t is uniquely determined by specifying initial conditions on $M_t = M_0 + B_{ct}$ at time zero, i.e., the domain of dependence for M_0 is given by M_t.

Lemma 4.3 can be generalized for the N-particle Dirac equation. We first generalize our notion of sets growing with the speed of light from $\mathbb{R}^3$ to $\mathbb{R}^{3N}$. Consider a point $q \in \mathbb{R}^{3N}$. It denotes a certain configuration of N points in $\mathbb{R}^3$. Now let each of these points grow with the speed of light, i.e., put a ball of radius $ct \geq 0$ around each point. Then the corresponding set in configuration space is

$$B_{ct}^{(N)}(q) := B_{ct}(\boldsymbol{x}_1) \times \cdots \times B_{ct}(\boldsymbol{x}_N) = \prod_{i=1}^{N} B_{ct}(\boldsymbol{x}_i). \tag{4.19}$$

This definition can also be expressed by introducing the $||\cdot||_{2,\infty}$-norm. For any $q \in \mathbb{R}^{3N}$ we define

$$||q||_{2,\infty} := \max_{i=1,\ldots,N} ||\boldsymbol{x}_i||_2. \tag{4.20}$$

Then we find $B^{(N)}_{ct}(q) = \{p \in \mathbb{R}^{3N} : ||p - q||_{2,\infty} \leq ct\}$.[5] For the set around the origin we abbreviate $B^{(N)}_{ct}(0) = (B_{ct}(0))^N =: B^{(N)}_{ct}$. We then have $B^{(N)}_{ct}(q) = q + B^{(N)}_{ct}$. If an arbitrary subset of configuration space $M_0 \subset \mathbb{R}^{3N}$ grows with the speed of light one obtains $M_t = M_0 + B^{(N)}_{ct}$.

Lemma 4.4 (Finite Propagation Speed: N-particle Dirac Equation)**.** *Let ψ be a solution of the N-particle Dirac equation (4.13) with initial data $\psi(0,\cdot) \in L^2(\mathbb{R}^{3N}, (\mathbb{C}^4)^{\otimes N}) \cap C^\infty(\mathbb{R}^{3N}, (\mathbb{C}^4)^{\otimes N})$ and with a Hermitian potential matrix $V \in C^\infty(\mathbb{R}^{3N}, (\mathbb{C}^{4\times 4})^{\otimes N})$. Let $t > 0$.*

(i) Then specifying initial conditions on $B^{(N)}_{ct}(q)$ uniquely determines $\psi(t,q)$. (I.e., the domain of dependence grows with the speed of light.)

(ii) Let $\psi(0,\cdot)$ have compact support, supp($\psi(0,\cdot)) = M_0 \subset \mathbb{R}^{3N}$. Then the support of $\psi(t,\cdot)$ is a subset of $M_t = M_0 + B^{(N)}_{ct}$. (I.e., the support of the wavefunction cannot grow faster than with the speed of light.)

Proof. The proof goes along the same lines as in the one-particle case. Again for this proof we set $c = 1$. We first generalize the definition of the hypersurfaces to the N-particle case. Let $Q = (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N) \in \mathbb{R}^{3N}$ and $(T, Q) \in \mathbb{R}^{3N+1}$ with $T > 0$. The ball in configuration space (in the sense discussed above) around Q with radius t is $B^{(N)}_t(Q)$. For $t \in [0, T]$ let

$$\Sigma_t = \Big\{(t,q) \in \mathbb{R}^{3N+1} : q \in B^{(N)}_{T-t}(Q)\Big\} \tag{4.22}$$

denote the corresponding set of the equal-time hypersurface for time t. We first prove the following statement (∗∗): If ψ vanishes on Σ_0 it also vanishes on Σ_t for all $t \in [0, T]$.

We define $C_t = \bigcup_{t' \in [0,t]} \Sigma_{t'}$ as the "generalized lightcone" between the surfaces Σ_0 and Σ_t. Again, Σ^s denotes the sides of the generalized lightcone such that $\Sigma_0 \cup \Sigma_t \cup \Sigma^s = \partial C_t$. For ease of notation and since we consider a fixed time t we do not add an index t for the time-dependence of

[5]Both expressions for $B^{(N)}_{ct}(q)$ are equivalent (denote $p = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N)$):

$$\begin{aligned}\prod_{i=1}^N B_{ct}(\boldsymbol{x}_i) &= \{p \in \mathbb{R}^{3N} : \boldsymbol{p}_i \in B_{ct}(\boldsymbol{x}_i)\ \forall i = 1, \ldots, N\} \\ &= \{p \in \mathbb{R}^{3N} : ||\boldsymbol{p}_i - \boldsymbol{x}_i|| \leq ct\ \forall i = 1, \ldots, N\} \\ &= \{p \in \mathbb{R}^{3N} : \max_{i=1,\ldots,N} ||\boldsymbol{p}_i - \boldsymbol{x}_i|| \leq ct\} \\ &= \{p \in \mathbb{R}^{3N} : ||p - x||_{2,\infty} \leq ct\}.\end{aligned} \tag{4.21}$$

Σ^s and from now on we write C instead of C_t. Σ^s is composed of k faces in the following sense. We call

$$\Sigma^{s,k} = \bigcup_{t' \in (0,t)} \{(t', q) \in \Sigma^s : ||\boldsymbol{Q}_k - \boldsymbol{x}_k||_2 = T - t'\} \tag{4.23}$$

the k-th face of Σ^s. Then we have that $\Sigma^s = \bigcup_{k=1,\ldots,N} \Sigma^{s,k}$. Now let $n = (n^0, \boldsymbol{n}_1, \ldots, \boldsymbol{n}_N) \in \mathbb{R}^{3N+1}$ be the outward-poining unit vector field on ∂C orthogonal to ∂C in the Euclidean metric, i.e., $||n||^2 = \sum_{i=1}^{3N+1} n^i n^i = 1$ and for any tangent vector s on ∂C, $n \cdot s = \sum_{i=1}^{3N+1} n^i s^i = 0$. The current j for N particles is $j = (j^0, \boldsymbol{j}_1, \ldots, \boldsymbol{j}_N) = (|\psi|^2, \psi^\dagger \boldsymbol{\alpha}_1 \psi, \ldots, \psi^\dagger \boldsymbol{\alpha}_N \psi)$. The continuity equation for N particles is $div(j) = \frac{\partial |\psi|^2}{\partial t} + div(\boldsymbol{j}) = 0$ (with $div(\boldsymbol{j}) = \sum_{k=1}^N div(\boldsymbol{j}_k)$).[6] According to Gauß' integral Theorem,

$$0 = \int_C div(j)\ d^{3N+1}x = \int_{\partial C} j \cdot n\ d^{3N}x = \int_{\Sigma_0} j \cdot n\ d^{3N}x + \int_{\Sigma_t} j \cdot n\ d^{3N}x + \int_{\Sigma^s} j \cdot n\ d^{3N}x. \tag{4.25}$$

The differential $d^{3N}x$ denotes the $(3N)$-surface area relative to the Euclidean metric on $\mathbb{R}^{3N+1}$ and $j \cdot n = \sum_{k=1}^{3N+1} j^k n^k$ is the Euclidean inner product on $\mathbb{R}^{3N+1}$. We suppose $\psi|_{\Sigma_0} = 0$. As the normal vector on Σ_0 has components $n^0 = -1$ and $\boldsymbol{n}_k = \boldsymbol{0}$ (for all $k = 1, \ldots, N$) it follows $j \cdot n = -j^0 = -|\psi|^2 = 0$ on Σ_0. On Σ_t we have $n^0 = 1$ and $\boldsymbol{n}_k = \boldsymbol{0}$ (for all $k = 1, \ldots, N$), so $j \cdot n = j^0 = |\psi|^2$. Therefore ($I$ denotes the identity on $(\mathbb{C}^4)^{\otimes N}$)

$$0 = \int_{\Sigma_t} j \cdot n\ d^3x + \int_{\Sigma^s} j \cdot n\ d^3x = \int_{\Sigma_t} |\psi|^2\ d^3x + \int_{\Sigma^s} \psi^\dagger (n^0 I + \sum_{k=1}^N \boldsymbol{\alpha}_k \cdot \boldsymbol{n}_k) \psi\ d^3x. \tag{4.26}$$

The $(4^N \times 4^N)$-matrix $A := (n^0 I + \sum_{k=1}^N \boldsymbol{\alpha}_k \cdot \boldsymbol{n}_k)$ is positive semi-definite on Σ^s for the following reason. Since for any unit vector $\boldsymbol{b} \in \mathbb{R}^3$ we have that $\boldsymbol{\alpha} \cdot \boldsymbol{b}$ has eigenvalues -1 and $+1$ (see footnote 4) each matrix $||\boldsymbol{n}_k|| \boldsymbol{\alpha}_k \cdot \frac{\boldsymbol{n}_k}{||\boldsymbol{n}_k||}$ has eigenvalues $+||\boldsymbol{n}_k||$ and $-||\boldsymbol{n}_k||$. Then the lowest eigenvalue of A is $e = n^0 - \sum_{k=1}^N ||\boldsymbol{n}_k||$. On Σ^s the normal-vector has the component $n^0 = \frac{1}{\sqrt{2}}$. The spatial components depend on the face of Σ^s. At $\Sigma^{s,k}$ the spatial components have norm $||\boldsymbol{n}_j|| = \frac{1}{\sqrt{2}} \delta_{jk}$

[6] The continuity equation for the N-particle Dirac equation can be derived in the following way. Note that V, β and α_k^i are Hermitian matrices and that for a Hermitian matrix A, $\psi^\dagger A \psi$ is real. Then

$$\begin{aligned} \frac{\partial |\psi|^2}{\partial t} &= 2Re\left(\psi^\dagger \frac{\partial \psi}{\partial t}\right) = 2Re\left(\psi^\dagger \left(\sum_{k=1}^N (-\boldsymbol{\alpha}_k \cdot \boldsymbol{\nabla}_k - i\beta_k m) - iV\right)\psi\right) \\ &= 2Re\left(\psi^\dagger \sum_{k=1}^N (-\boldsymbol{\alpha}_k \cdot \boldsymbol{\nabla}_k)\psi\right) = -\sum_{k=1}^N \left(\psi^\dagger \boldsymbol{\alpha}_k \cdot \boldsymbol{\nabla}_k \psi + \left(\boldsymbol{\nabla}_k \psi^\dagger\right) \cdot \boldsymbol{\alpha}_k \psi\right) \\ &= -\sum_{k=1}^N \left(\boldsymbol{\nabla}_k \cdot \left(\psi^\dagger \boldsymbol{\alpha}_k \psi\right)\right) = -\sum_{k=1}^N div(\boldsymbol{j}_k) = -div(\boldsymbol{j}) \end{aligned} \tag{4.24}$$

(for all $j = 1, \dots, N$), so $e = \frac{1}{\sqrt{2}} - \sum_{j=1}^{N} \frac{1}{\sqrt{2}} \delta_{jk} = 0$. Thus

$$0 = \int_{\Sigma_t} |\psi|^2 \; d^3x + \int_{\Sigma^s} \underbrace{\psi^\dagger \left(n^0 I + \sum_{k=1}^{N} \boldsymbol{\alpha}_k \cdot \boldsymbol{n}_k \right) \psi}_{\geq 0} \; d^3x. \tag{4.27}$$

Since each integrand is ≥ 0, each integral is ≥ 0. Then according to (4.27) each integral has to vanish. This means in particular for the integral over Σ_t, that the integrand $|\psi|^2$ has to vanish (almost everywhere) and therefore $\psi = 0$ on Σ_t (almost everywhere). Since from Lemma 4.1 we know that $\psi \in C^\infty(\mathbb{R}^{3N+1}, (\mathbb{C}^4)^{\otimes N})$, ψ has to vanish identically on Σ_t. We have proven: If ψ vanishes on Σ_0 it also vanishes on Σ_t for all $t \in [0, T]$ (**). With that the two statements of Lemma 4.3 follow:

(i) As in the one-particle case, suppose $\psi_1, \psi_2 \in C^\infty(\mathbb{R}^{3N+1}, (\mathbb{C}^4)^{\otimes N})$ are both solutions of the Dirac equation (4.13) that are identical on Σ_0 and arbitrary on the rest of the $t = 0$ hypersurface. Then $\psi_1 - \psi_2$ is another solution of the Dirac equation (4.13) with $\psi_1 - \psi_2 = 0$ on Σ_0. From the statement (**) above it follows $\psi_1 - \psi_2 = 0$ in (T, Q), i.e., $\psi_1(T, Q) = \psi_2(T, Q)$.

(ii) Consider $\psi(0, \cdot)$ with compact support $\textit{supp}\,(\psi(0, \cdot)) = M_0 \subset \mathbb{R}^{3N}$. Recall that $M_t = M_0 + B_t^{(N)}$. Now for any $(t, q) \in \mathbb{R}^{3N+1}$ with $t \geq 0$ and $q \notin M_t$ we have $B_t^{(N)}(q) \cap M_0 = \emptyset$. Since $\psi(0, \cdot) = 0$ on $B_t^{(N)}(q)$ (outside M_0), also $\psi(t, q) = 0$ (according to (**)). This is true for any $(t, q) \notin M_t$, therefore $\psi(t, \cdot)$ vanishes outside M_t.

□

Lemma 4.4 (i) implies that the wavefunction at time t on any $M_0 \subset \mathbb{R}^{3N}$ is uniquely determined by the initial conditions on $M_t = M_0 + B_{ct}^{(N)}$ at time zero, i.e., the domain of dependence for M_0 is given by M_t.

4.2.3 The Configuration Space-Time and Partitions

Although the multi-time evolution that we define in Theorem 4.6 uses the Dirac equation, it is not fully relativistic but presumes a certain Lorentz frame. For the construction of a multi-time evolution we consider groups of particles that are too far away from each other to interact, i.e., every particle from one group does not interact with any particle from the other group. Then we have separate time-evolutions for each group (as they do not influence each other by interaction). The subset of space-time on which the multi-time wavefunction will be defined is

$$\mathcal{C}_\delta^{(N)} = \left\{q^4 \in \mathbb{R}^{4N} : \forall i \neq j : t_i = t_j \text{ or } ||\boldsymbol{x}_i - \boldsymbol{x}_j|| > c|t_i - t_j| + \delta\right\} \tag{4.28}$$

This is the set of all spacelike configurations in which additionally for each pair of particles either the times are equal or the spatial distance from the one's particle lightcone to the other particle is bigger than δ for equal times.

The subset of space-time $\mathcal{C}_\delta^{(N)}$ is divided into different regions in order to define the groups of particles that are too far away from each other (further than δ) to interact. For that we use a partition P into different groups of particles. A partition P of the set of the first N natural numbers is a set $P = \{S_1, \ldots, S_L\}$ with $\bigcup_{\alpha=1}^{L} S_\alpha = \mathcal{N}$ and $S_\alpha \cap S_\beta = \emptyset$ for $\alpha \neq \beta$. The set of space-time points with the property that for a given partition the particles in different groups do not interact with each other is

$$\begin{aligned}\mathcal{C}_{\delta,P}^{(N)} = \{q^4 \in \mathbb{R}^{4N} :&(1)\ \forall i,j \in S_k\ \forall k = 1,\ldots,L : t_i = t_j \\ &(2)\ \forall i \in S_k, j \in S_\ell\ \forall k \neq \ell : ||\boldsymbol{x}_i - \boldsymbol{x}_j||_2 > c|t_i - t_j| + \delta\}.\end{aligned} \tag{4.29}$$

An example is shown in Figure 4.2. We write $q^4 = (t_1, q_1; \ldots; t_L, q_L)$. With that notation we indicate to which partition q^4 belongs. We denote $\mathcal{P}_N = \{\text{partitions } P \text{ of } \mathcal{N}\}$. The set $\mathcal{C}_\delta^{(N)}$ on which the time evolution will be defined can therefore be written as $\mathcal{C}_\delta^{(N)} = \bigcup_{P \in \mathcal{P}_N} \mathcal{C}_{\delta,P}^{(N)}$.

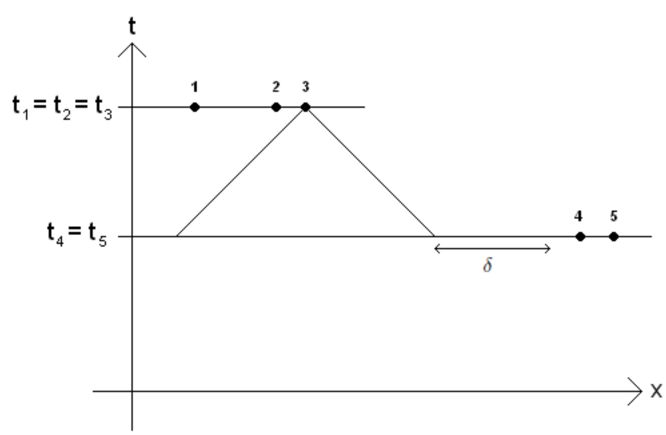

Figure 4.2: This space-time configuration is an element of $\mathcal{C}_\delta^{(N)}$, in fact of $\mathcal{C}_{\delta,\{S_1,S_2\}}^{(N)}$ with $S_1 = \{1,2,3\}$ and $S_2 = \{4,5\}$.

4.2.4 The Multi-Time Evolution

Definition 4.5. *For a pair potential V we write*

$$V = \sum_{\substack{i,j=1 \\ i \neq j}}^{N} W(\boldsymbol{x}_i - \boldsymbol{x}_j). \tag{4.30}$$

We say that a pair potential V has δ-range ($\delta > 0$) if for all $i, j \in \mathcal{N}$, $i \neq j$,

$$W(\boldsymbol{x}_i - \boldsymbol{x}_j) = 0 \quad \textit{for} \quad ||\boldsymbol{x}_i - \boldsymbol{x}_j|| > \delta. \tag{4.31}$$

The consistent multi-time model for N particles is defined in the following Theorem 4.6. We define $\mathcal{C}_{\delta,0}^{(N)} = (\{0\} \times \mathbb{R}^3)^N$. Given a partition $P = \{S_1, \dots, S_L\}$ we use the two notations

$$\mathcal{C}_{\delta,P}^{(N)} \ni q^4 = (t_1, q_1; \dots; t_L, q_L) = \big((x_1^0, \boldsymbol{x}_1), \dots, (x_N^0, \boldsymbol{x}_N)\big) \tag{4.32}$$

with $x_i^0 = t_\alpha$ for $i \in S_\alpha$.

Theorem 4.6. *Let $V \in C^\infty(\mathbb{R}^{3N}, (\mathbb{C}^{4\times 4})^{\otimes N})$ be a pair potential with δ-range. Then for all initial conditions $\phi : \mathcal{C}_{\delta,0}^{(N)} \to (\mathbb{C}^4)^{\otimes N}$, $\phi \in L^2 \cap C^\infty \left(\mathcal{C}_{\delta,0}^{(N)}, (\mathbb{C}^4)^{\otimes N}\right)$ there is a unique wavefunction $\psi : \mathcal{C}_{\delta}^{(N)} \to (\mathbb{C}^4)^{\otimes N}$ with $\psi|_{\mathcal{C}_{\delta,0}^{(N)}} = \phi$ and $\psi \in C^\infty\left(\mathcal{C}_{\delta}^{(N)}, (\mathbb{C}^4)^{\otimes N}\right)$, which satisfies on $\mathcal{C}_{\delta,P}^{(N)}$ for every partition $P = \{S_1, \dots, S_L\}$ the equations*

$$i\frac{\partial}{\partial t_\alpha}\psi(t_1, q_1; \dots; t_L, q_L) = \left(\sum_{j \in S_\alpha} \left(-ic\boldsymbol{\alpha}_j \cdot \boldsymbol{\nabla}_j + \beta_j mc^2\right) + \sum_{\substack{i,j \in S_\alpha \\ i \neq j}} W(\boldsymbol{x}_i - \boldsymbol{x}_j)\right)\psi(t_1, q_1; \dots; t_L, q_L) \tag{4.33}$$

for all $\alpha = 1, \dots, L$ and with $t_\alpha = x_i^0$ for all $i \in S_\alpha$.

Proof. The Theorem is proven in three steps: First, we define a function Ψ. Second, we show that any solution ψ of the evolution equations with initial conditions ϕ must agree with Ψ. Third, we show that Ψ satisfies the equations (4.33).

Step 1. We define Ψ on $\mathcal{C}_{\delta}^{(N)}$ by defining it on each $\mathcal{C}_{\delta,P}^{(N)}$ using induction on the number L of groups in a partition.

- We start the induction with $L = 1$. The corresponding partition is $P = \{S_1\}$ with $S_1 = \{1, \dots, N\}$. Note that with $P = \{S_1\}$ we have $q^4 = (t_1, q_1)$ with $q_1 = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_N)$. Let Ψ on $\mathcal{C}^{(N)}_{\delta,P}$ be the (unique) solution of

$$i\frac{\partial}{\partial t_1}\Psi(t_1,q_1) = \left(\sum_{j\in S_1}\left(-ic\boldsymbol{\alpha}_j\cdot\boldsymbol{\nabla}_j + \beta_j mc^2\right) + \sum_{\substack{i,j\in S_1\\ i\neq j}} W(\boldsymbol{x}_i - \boldsymbol{x}_j)\right)\Psi(t_1,q_1) \tag{4.34}$$

  with initial conditions given by ϕ on $\mathcal{C}^{(N)}_{\delta,0}$. This is just a one-time Dirac-type equation. (From Lemma 4.1 we know that there is a unique solution.)

- The induction assumption is that Ψ has been defined on $\mathcal{C}^{(N)}_{\delta,P'}$ for every partition P' with $L' = L - 1$ or fewer groups.

- Now we perform the induction step from $L - 1$ to L. Consider any P consisting of L groups. We construct $\Psi(Q^4)$ for an arbitrary $Q^4 = (T_1, Q_1; \dots; T_L, Q_L) \in \mathcal{C}^{(N)}_{\delta,P}$. (Note that according to our notation, $P = \{S_1, \dots, S_L\}$ and $T_1 < T_2 < \cdots < T_L$.) According to the induction assumption, Ψ is given for $\tilde{P} = \{S_1, \dots, S_{L-2}, S_{L-1} \cup S_L\}$, in particular, $\Psi(T_1, Q_1; \dots; T_{L-1}, Q_{L-1}; T_{L-1}, q_L)$ is given for any $q_L \in B^{(|S_L|)}_{T_L - T_{L-1}}(Q_L)$. Since the domain of dependence of Q_L is $B^{(|S_L|)}_{T_L - T_{L-1}}(Q_L)$ (see Lemma 4.4), we can solve the equation

$$\begin{aligned} &i\frac{\partial}{\partial t_L}\Psi(T_1,Q_1;\dots;T_{L-1},Q_{L-1};t_L,q_L) = \\ &\left(\sum_{j\in S_L}\left(-ic\boldsymbol{\alpha}_j\cdot\boldsymbol{\nabla}_j + \beta_j mc^2\right) + \sum_{\substack{i,j\in S_L\\ i\neq j}} W(\boldsymbol{x}_i - \boldsymbol{x}_j)\right)\Psi(T_1,Q_1;\dots;T_{L-1},Q_{L-1};t_L,q_L) \end{aligned} \tag{4.35}$$

  to obtain $\Psi(T_1, Q_1, \dots, T_L, Q_L)$. Since $Q^4 \in \mathcal{C}^{(N)}_{\delta,P}$ was arbitrary, we have defined Ψ on $\mathcal{C}^{(N)}_{\delta,P}$ for every partition P with L or fewer groups.

Step 2. Let ψ be a solution of the multi-time equations (4.33) with initial conditions ϕ. Then, for the induction start with $L = 1$, this ψ has to agree with Ψ on $\mathcal{C}^{(N)}_{\delta,\{\mathcal{N}\}}$, since both functions have the same initial conditions and are solutions of the same equation (4.34). The same holds for the induction step. Both ψ and Ψ have the same initial conditions and are solutions to the same equation (4.35). Therefore ψ agrees with Ψ.

Step 3. This part of the proof is due to Roderich Tumulka. For any space-time point $(T, \boldsymbol{Q})$ with

$T > 0$, let

$$D^{+}_{s,(T,\boldsymbol{Q})} = \bigcup_{0 \le t \le s} \{T - t\} \times B_{c|t|}(\boldsymbol{Q}) \tag{4.36}$$

be the "pyramid-shaped" region with tip at $(T, \boldsymbol{Q})$; it is the set of all space-time points whose domain of dependence (at time $T - s$) is contained in the domain of dependence of $(T, \boldsymbol{Q})$. Likewise, for a group of k particles, we define, for $T > 0$ and $Q_\alpha \in \mathbb{R}^{3k}$,

$$D^{+}_{s,(T,Q_\alpha)} = \bigcup_{0 \le t \le s} \{T - t\} \times B^{(k)}_{c|t|}(Q_\alpha). \tag{4.37}$$

Now, choose any $Q^4 = (T_1, Q_1; \ldots; T_L, Q_L)$. Choose $\varepsilon > 0$ in such a way that

$$2\varepsilon < T_{\alpha+1} - T_\alpha \tag{4.38}$$

for every $\alpha = 1, \ldots, L-1$ and

$$2\varepsilon < \|\boldsymbol{X}_i - \boldsymbol{X}_j\| - |X^0_i - X^0_j| - \delta \tag{4.39}$$

for every $i \in S_\alpha, j \in S_\beta$ with $\alpha \ne \beta$. We show that for any fixed $\beta = 1, \ldots, L$, fixed $\tilde{Q}^4_\alpha = (T_\alpha, Q_\alpha)$ for $\alpha \ne \beta$, and varying $\tilde{Q}^4_\beta \in D^{+}_{2\varepsilon,(\varepsilon + T_\beta, Q_\beta)}$, $\Psi(\tilde{Q}^4_1, \ldots, \tilde{Q}^4_L)$ satisfies

$$i\frac{\partial \Psi}{\partial t_\beta} = H_\beta \Psi = \left(\sum_{j \in S_\beta} \left(-ic\boldsymbol{\alpha}_j \cdot \boldsymbol{\nabla}_j + \beta_j mc^2 \right) + \sum_{\substack{i,j \in S_\beta \\ i \ne j}} W(\boldsymbol{x}_i - \boldsymbol{x}_j) \right) \Psi. \tag{4.40}$$

To this end, let $\psi(t_\beta, q_\beta, t_{>\beta}, q_{>\beta})$ be the solution, provided by Lemma 4.2, of the two-time evolution equations

$$i\frac{\partial \psi}{\partial t_\beta} = H_\beta \psi \tag{4.41}$$

and

$$i\frac{\partial \psi}{\partial t_{>\beta}} = H_{>\beta} \psi = \sum_{\alpha=\beta+1}^{L} H_\alpha \psi + \sum_{\substack{\alpha,\gamma=\beta+1 \\ \alpha \ne \gamma}}^{L} W_{\alpha\gamma} \psi \tag{4.42}$$

for $t_\beta \in [T_\beta - \varepsilon, T_\beta + \varepsilon]$ and $t_{>\beta} \in [T_\beta - \varepsilon, T_{\beta+1} + \varepsilon]$ with initial data

$$\psi(T_\beta - \varepsilon, q_\beta, T_\beta - \varepsilon, q_{\beta+1}, \ldots, q_L) = \Psi(T_1, Q_1, \ldots, T_{\beta-1}, Q_{\beta-1}, T_\beta - \varepsilon, q_\beta, T_\beta - \varepsilon, q_{\beta+1}, \ldots, q_L) \tag{4.43}$$

for $q_\alpha \in B^{(|S_\alpha|)}_{T_\alpha - T_\beta + 2\varepsilon}(Q_\alpha)$ for $\alpha \geq \beta$.

We now show that for $t_\beta = t_{>\beta}$ in the interval $[T_\beta - \varepsilon, T_\beta + \varepsilon]$,

$$\psi(t_\beta, q_\beta, t_\beta, q_{>\beta}) = \Psi(T_1, Q_1, \ldots, T_{\beta-1}, Q_{\beta-1}, t_\beta, q_\beta, \ldots, q_L). \tag{4.44}$$

The crucial point here is that the groups S_β and $S_{>\beta}$ do not interact in this time interval. Set

$$G_\beta = \prod_{\alpha=\beta+1}^{L} B^{(|S_\alpha|)}_{T_\alpha - T_\beta + 2\varepsilon}(Q_\alpha). \tag{4.45}$$

For any $(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots) \in B^{(|S_\alpha|)}_{2\varepsilon}(Q_\beta)$ and $(\boldsymbol{x}'_1, \boldsymbol{x}'_2, \ldots) \in G_\beta$ we have that

$$\|\boldsymbol{x}_i - \boldsymbol{x}'_j\| > \delta \tag{4.46}$$

for all i and j by virtue of the choice of ε. As a consequence, $W(\boldsymbol{x}_i - \boldsymbol{x}'_j) = 0$, that is, the equation for $S_{\geq\beta}$ contains no interaction term between S_β and $S_{>\beta}$ for $T_\beta - \varepsilon \leq t_\beta \leq T_\beta + \varepsilon$. That is,

$$H_{\geq\beta} = \sum_{\alpha=\beta}^{L} H_{0,\alpha} + \sum_{\substack{\alpha,\gamma=\beta \\ \alpha\neq\gamma}}^{L} W_{\alpha\gamma} = H_\beta + H_{>\beta}. \tag{4.47}$$

(Notabene: The operators called H are *not*, as usual, operators on the L^2 space over the configuration space; they are affine-linear combinations of the derivative operators ∂_i with matrix-valued coefficients. The difference is that the latter, but not the former, can be considered at a single configuration.) When both time variables are set equal in ψ, then its Hamiltonian is the sum of the partial Hamiltonians, $H_\beta + H_{>\beta}$; thus, it is the same as for Ψ with fixed $T_1, Q_1, \ldots, T_{\beta-1}, Q_{\beta-1}$ and the same time variable t_β for $q_\beta, \ldots, q_L$. From the uniqueness statement of Lemma 4.1, we thus obtain (4.44).

By definition of Ψ, the value of $\Psi(\tilde{Q})$ depends on

$$\Psi(T_1, Q_1, \ldots, T_{\beta-1}, Q_{\beta-1}, \tilde{T}_\beta, \tilde{Q}_\beta, T_{\beta+1}, q_{>\beta}) \tag{4.48}$$

with $q_{>\beta} \in \prod_{\alpha=\beta+1}^{L} B^{(|S_\alpha|)}_{T_\alpha - T_{\beta+1}}(Q_\alpha)$. The value of (4.48), in turn, is obtained from

$$\Psi(T_1, Q_1, \ldots, T_{\beta-1}, Q_{\beta-1}, \tilde{T}_\beta, \tilde{Q}_\beta, \tilde{T}_\beta, q_{>\beta}) \tag{4.49}$$

by solving the equation $i\partial\Psi/\partial t_{\beta+1} = H_{>\beta}\Psi$. It follows that (4.48) is equal to

$$\psi(\tilde{T}_\beta, \tilde{Q}_\beta, T_{\beta+1}, q_{>\beta}), \tag{4.50}$$

since the latter is obtained by solving $i\partial\psi/\partial t_{>\beta} = H_{>\beta}\psi$ (with the difference that q_β is not fixed but one of the variables of the function solved for in the PDE; but that difference does not affect the solution because the evolutions for different values of q_β decouple, so that it does not matter whether we first solve the PDE and then set $q_\beta = \tilde{Q}_\beta$ or the other way around) from the same initial data (4.44). Now (4.50) can be obtained from $\psi(T_\beta - \varepsilon, q_\beta, T_{\beta+1}, q_{>\beta})$ by solving (4.41). According to Lemma 4.2, the evolution of Ψ from (4.48) to $\Psi(\tilde{Q}^4)$ commutes with solving (4.41); therefore, $\Psi(\tilde{Q}^4)$ can be obtained from

$$\Psi(T_1, Q_1, \ldots, T_{\beta-1}, Q_{\beta-1}, T_\beta - \varepsilon, q_\beta, T_{\beta+1}, Q_{\beta+1}, \ldots, T_L, Q_L) \tag{4.51}$$

by solving (4.41), which is what we wanted to show.

Since β was arbitrary, we have shown that Ψ satisfies the multi-time evolution equations. Finally, to see that Ψ is smooth we note that on $\prod_{\alpha=1}^{L} D_\alpha^+$, Ψ simultaneously satisfies the multi-time evolution equations for all $\alpha = 1, \ldots, L$; Lemma 4.2 guarantees that the only solution is smooth. □

4.3 Model 2: A Multi-Time QFT Model with δ-Cutoff

Now the multi-time Dirac theory for a variable number of particles is formulated. Specifically a consistent multi-time theory for the QFT model with a constant electron number and emission and absorption of photons is defined. Therefore some of the definitions from above have to be generalized to a variable number of particles.

4.3.1 The Dirac Equation with Creation and Annihilation Terms

In the following emission-absorption model we use a wave equation for electrons and photons with creation and annihilation terms. For the electrons (massive spin-$\frac{1}{2}$ particles) we use the familiar N-particle free Dirac equation. For the photon-part we will use a Dirac-like equation for massless spin-1 particles (as described in [1]). For a single particle this equation reads

$$i\frac{\partial}{\partial t}\psi(t, \boldsymbol{y}) = -ic\,(\boldsymbol{S} \cdot \boldsymbol{\nabla})\,\psi(t, \boldsymbol{y}) \tag{4.52}$$

with $\boldsymbol{S} = (S_1, S_2, S_3)$ and a wavefunction $\psi : \mathbb{R}^3 \to \mathbb{C}^3$ with constraint $div(\psi) = \boldsymbol{\nabla} \cdot \psi = 0$. The Hermitian (3×3)-matrices S_j are the analog of the Pauli matrices for spin-1 particles. (They represent infinitesimal rotations for spin-1.) They can explicitly written down for example in the following representation as $(S_j)_{k\ell} = -i\varepsilon_{jk\ell}$, i.e.,

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, S_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, S_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.53}$$

They satisfy the relation

$$(S_i S_j + S_j S_i)_{ab} = 2\delta_{ij}\delta_{ab} - \delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi}. \tag{4.54}$$

Note that this relation implies that

$$[(\boldsymbol{S} \cdot \boldsymbol{\nabla})(\boldsymbol{S} \cdot \boldsymbol{\nabla})]_{ab} = \Delta\delta_{ab} - \nabla_a \nabla_b \tag{4.55}$$

which is the well-known identity from vector analysis

$$-\nabla \times (\nabla \times \psi) = \Delta\psi - \nabla(\nabla \cdot \psi). \tag{4.56}$$

Therefore for all ψ with vanishing divergence we have an expression for the square root of the Laplacian that leads to equation (4.52). Since $(S_j)_{k\ell} = -i\varepsilon_{jk\ell}$ the photon equation (4.52) can also be written as

$$i\frac{\partial}{\partial t}\psi(t, \boldsymbol{y}) = c\boldsymbol{\nabla} \times \psi(t, \boldsymbol{y}). \tag{4.57}$$

with constraint $div(\psi) = 0$. This equation is equivalent to the (vacuum) Maxwell equations by setting $\psi = \frac{1}{\sqrt{2}}\left(\frac{\boldsymbol{E}}{\sqrt{\epsilon_0}} + i\frac{\boldsymbol{B}}{\sqrt{\mu_0}}\right)$ with electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$. Further properties of the photon wavefunction are described in [1] and other papers by the same author.

With that the equation for massive spin-$\frac{1}{2}$ and massless spin-1 particles that includes creation and annihilation terms is the following equation for a wavefunction $\psi : \mathbb{R} \times \mathcal{Q} \to \mathcal{S}$. For the

(N, m)-particle sector it reads:

$$i\frac{\partial}{\partial t}\psi(t,\xi,\eta) = \sum_{j=1}^{N}\left(-ic\boldsymbol{\alpha}_j\cdot\boldsymbol{\nabla}_{x_j}+\beta_j mc^2\right)\psi(t,\xi,\eta)+\sum_{k=1}^{m}\left(-ic\boldsymbol{S}_k\cdot\boldsymbol{\nabla}_{y_k}\right)\psi(t,\xi,\eta)$$
$$+\frac{1}{\sqrt{m}}\sum_{j=1}^{N}\sum_{k=1}^{m}\varphi_\delta(\boldsymbol{y}_k-\boldsymbol{x}_j)\psi(t,\xi,\eta\backslash\boldsymbol{y}_k)+\sqrt{m+1}\sum_{j=1}^{N}\int_{\mathbb{R}^3}d\boldsymbol{y}\;\varphi_\delta^*(\boldsymbol{y}-\boldsymbol{x}_j)\psi(t,\xi,\eta\cup\boldsymbol{y}) \tag{4.58}$$

with positive constants m and c. The Dirac matrices $\boldsymbol{\alpha}_j$ and β_j act only on the j-th electron's spin indices and $\boldsymbol{\nabla}_{x_j}$ acts only on the j-th electron's position variables. The matrices $\boldsymbol{S}_k$ act only on the k-th photon's spin indices and $\boldsymbol{\nabla}_{y_k}$ acts only on the k-th photon's position variables. $\varphi_\delta(\boldsymbol{x}) : \mathbb{R}^3 \to \mathbb{C}^3$ is a square-integrable cut-off function with δ-range. (Again, note that for ease of notation we do not explicitly write out the summation over the spin indices.)

4.3.2 The Domain of Dependence

In the case of a constant particle number N we found that the wavefunction on $M_0 \subset \mathbb{R}^{3N}$ is determined by specifying initial conditions on $M_t = M_0 + B_{ct}^{(N)}$. A similar statement is true for a variable particle number. Suppose we have a certain point in configuration space $q \in \mathbb{R}^{3N} \times \mathbb{R}^{3m}$ and we want to determine its domain of dependence. A major difficulty is that the domain of dependence is a very complicated set because it has to be taken into account which photons can be absorbed and which not. Already for one electron and a small number of photons (both with finite propagation speed) it is a very difficult mathematical problem to determine which photons can possibly be "reached" in a certain time. Let us first define what kind of configuration jumps are allowed in this model. We define an allowed jump in the following way.

Definition 4.7. *For a configuration $(\xi, \eta) \in \mathbb{R}^{3N} \times \mathbb{R}^{3m}$ a jump $(\xi, \eta) \to (\xi', \eta')$ is called allowed if (ξ', η') is one of the following two configurations:*

1. *$\xi' = \xi$ and $\eta' = (\eta \cup \boldsymbol{y})$ with $\boldsymbol{y} \in B_\delta(\boldsymbol{x}_i)$ for some $i \in \mathcal{N}$.*

2. *$\xi' = \xi$ and $\eta' = (\eta\backslash\boldsymbol{y}_k)$ with $\boldsymbol{y}_k \in B_\delta(\boldsymbol{x}_i)$ for some $i \in \mathcal{N}$.*

With that, a very abstract definition for the domain of dependence of a configuration q can be given:

$$M_t(q) = \{\gamma(t) \text{ such that } \gamma : [0,t] \to \mathcal{Q}, \gamma(0) = q, \gamma \in \mathcal{E}\} \tag{4.59}$$

where $\mathcal{E}$ is the set of allowed histories. An allowed history γ fulfills two conditions: First, γ makes only allowed jumps and second, electrons and photons cannot move faster than with the speed of light between jumps, i.e., $||\frac{d\boldsymbol{x}_i}{dt}|| \leq c$ (for $i \in \mathcal{N}$) and $||\frac{d\boldsymbol{y}_k}{dt}|| \leq c$ (for $k \in \mathbb{N}$).

Since we want to avoid the difficulty of dealing with $M_t(\xi,\eta)$ we give an upper bound for the domain of dependence, $N_t(\xi,\eta) \supset M_t(\xi,\eta)$. We define

$$N_t(\xi,\eta) = \Big\{(\xi',\eta') \in \mathcal{Q} : \quad 1.\ \xi' \in B_{ct}^{(N)}(\xi) \quad 2.\ \eta' = \eta'' \cup \bigcup_{i=1}^{N} \eta_i''' \ \text{ with } \quad (i)\ \eta'' \in B_{ct}^{(\#\eta'''')}(\eta'''') \text{ for } \eta'''' \subset \eta \quad (ii)\ \eta_i''' \subset B_{ct+\delta}(\boldsymbol{x}_i)\Big\}. \tag{4.60}$$

This definition says that for a given configuration (ξ,η) the set $N_t(\xi,\eta)$ is constructed in the following way.

1. Electrons can maximally move with the speed of light.
2. The new photon configuration consists of two parts: (i) The photons that did not get absorbed by electrons. They can maximally move with the speed of light. We allow that an arbitrary number of photons did get absorbed. (ii) The photons that were created by any of the electrons in a ball with radius δ during the time interval $[0,t]$.

The approximation in the definition of $N_t(\xi,\eta)$ is that we allowed that an arbitrary number of photons can be absorbed, but as discussed above, in fact only a certain number of photons can be absorbed. We later need to consider subsets of configuration space that do not change if an allowed jump is performed. In the following we refer to unordered configurations.

Definition 4.8. $B \subset \Gamma^N(\mathbb{R}^3) \times \Gamma(\mathbb{R}^3)$ *is called closed under allowed jumps if for any* $(\xi,\eta) \in B$ *and for any allowed jump* $(\xi,\eta) \to (\xi',\eta')$*, also* $(\xi',\eta') \in B$*.*

Sets that are closed under allowed jumps can be further characterized. Since the ξ-configuration does not change under allowed jumps only the set of η-configuration is important in the question wheather a set is closed under allowed jumps. We define for $B \subset \Gamma^N(\mathbb{R}^3) \times \Gamma(\mathbb{R}^3)$ the set $B_\xi = \{\eta \in \Gamma(\mathbb{R}^3) : (\xi,\eta) \in B\}$ and abbreviate $U_\delta(\xi) = \bigcup_{i=1}^N B_\delta(\boldsymbol{x}_i)$ and $U_\delta(\xi)^c = \mathbb{R}^3 \backslash U_\delta(\xi)$. Then we have the following Lemma.

Lemma 4.9. *Let $B \subset \Gamma^N(\mathbb{R}^3) \times \Gamma(\mathbb{R}^3)$. Then the following two statements are equivalent.*

(i) B is closed under allowed jumps.

(ii) For every ξ, there is a $B'_\xi \subset \Gamma\left(U_\delta(\xi)^c\right)$ such that $B_\xi = B'_\xi \times \Gamma\left(U_\delta(\xi)\right)$.

Proof. First, note that $\Gamma(\mathbb{R}^3) = \Gamma\left(U_\delta(\xi)^c\right) \times \Gamma\left(U_\delta(\xi)\right)$.

- *(i)* $\Rightarrow$ *(ii)*: For a fixed ξ with $(\xi, \eta) \in B$, we can write $\eta = (\eta', \eta'')$ with $\eta' \in \Gamma\left(U_\delta(\xi)^c\right)$ and $\eta'' \in \Gamma\left(U_\delta(\xi)\right)$. Allowed jumps cannot change η' since η' is outside of the δ-neighborhood of the ξ-configuration. Now fix ξ and η'. Since B is closed under allowed jumps we have to add and remove an arbitrary number of photons within the δ-ball of the ξ-configuration. This is true for any ξ and η', therefore B can be written as in (ii).
- *(ii)* $\Rightarrow$ *(i)*: Any B for which $B_\xi = B'_\xi \times \Gamma\left(U_\delta(\xi)\right)$, is closed under allowed jumps, since for any configuration in B one can add or remove photons in $U_\delta(\xi)$ and still has a configuration in B.

□

Lemma 4.9 says that for a set that is closed under allowed jumps the ξ-configuration is arbitrary, the number of photons outside the δ-balls around each electron is arbitrary, but the set has to include all possible configurations in the union of the δ-balls around the electrons.

In the proof of Lemma 4.12 we need an important formula for integrals over the unordered configuration space. Let (S, μ_S) be a measure space. Then there is a corresponding measure on $\Gamma(S)$,

$$\mu_{\Gamma(S)}(B) = \sum_{m=0}^{\infty} \frac{1}{m!}\, \mu_S^{\otimes m}\left(\pi^{-1}\left(B^{(m)}\right)\right) \tag{4.61}$$

for $B \subset \Gamma(S)$, $B^{(m)} = B \cap \Gamma^{(m)}(S)$. $\mu_S^{\otimes m}$ is the product measure on S^m and π maps ordered configurations to unordered configurations, i.e.,

$$\pi\left((\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m)\right) = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m\}. \tag{4.62}$$

With that we have an explicit expression for integrals over the unordered configuration space:

$$\int_{\Gamma(S)} d\eta f(\eta) = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{S^m_{\neq}} d\boldsymbol{y}_1 \cdots d\boldsymbol{y}_m\, f(\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m\}) \tag{4.63}$$

with $S^m_{\neq} := \{\eta \in S^m : \boldsymbol{y}_i \neq \boldsymbol{y}_j \ \forall i \neq j\}$. For these integrals the following Lemma holds.

Lemma 4.10. *For any measure space (S, μ_S) one has*

$$\int_{\Gamma(S)} d\eta \sum_{\boldsymbol{y}\in\eta} f(\boldsymbol{y}, \eta\backslash\boldsymbol{y}) = \int_{\Gamma(S)} d\eta' \int_{S\backslash\eta'} d\boldsymbol{y}\, f(\boldsymbol{y}, \eta'). \tag{4.64}$$

Proof. With (4.63) we show that equation (4.64) holds. In the second step we define $\boldsymbol{z} := \boldsymbol{y}_k$ and $(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{k-1}) := (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{k-1})$ and $(\boldsymbol{z}_k, \ldots, \boldsymbol{z}_{m-1}) := (\boldsymbol{y}_{k+1}, \ldots, \boldsymbol{y}_m)$. Also $\eta' := \{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{m'}\}$ and $m' := m-1$. In the third step we use that the summands in the sum over k do not depend on k anymore.

$$\begin{aligned}
\int_{\Gamma(S)} d\eta \sum_{\boldsymbol{y}\in\eta} f(\boldsymbol{y}, \eta\backslash\boldsymbol{y}) &= \sum_{m=1}^{\infty} \frac{1}{m!} \int_{S^m_{\neq}} d\boldsymbol{y}_1 \cdots d\boldsymbol{y}_m \sum_{k=1}^{m} f(\boldsymbol{y}_k, \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m\}\backslash\boldsymbol{y}_k) \\
&= \sum_{m'=0}^{\infty} \frac{1}{(m'+1)!} \sum_{k=1}^{m'+1} \int_{S^{m'}_{\neq}} d\boldsymbol{z}_1 \cdots d\boldsymbol{z}_{m'} \int_{S\backslash\{\boldsymbol{z}_1,\ldots,\boldsymbol{z}_{m'}\}} d\boldsymbol{z}\, f(\boldsymbol{z}, \{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{m'}\}) \\
&= \sum_{m'=0}^{\infty} \frac{1}{m'!} \int_{S^{m'}_{\neq}} d\boldsymbol{z}_1 \cdots d\boldsymbol{z}_{m'} \int_{S\backslash\{\boldsymbol{z}_1,\ldots,\boldsymbol{z}_{m'}\}} d\boldsymbol{z}\, f(\boldsymbol{z}, \{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{m'}\}) \\
&= \int_{\Gamma(S)} d\eta' \int_{S\backslash\eta'} d\boldsymbol{y}\, f(\boldsymbol{z}, \eta').
\end{aligned} \tag{4.65}$$

□

For the set $N_t(\xi, \eta)$ we find:

Lemma 4.11. *$N_t(\xi, \eta)$ is closed under allowed jumps.*

Proof. We show that $N_t(\xi, \eta)$ is closed under the two possible allowed jumps:

1. Suppose $(\xi', \eta') \to (\xi', \eta'\backslash\boldsymbol{y}'_k)$ with $\boldsymbol{y}'_k \in B_\delta(\boldsymbol{x}'_i)$ for any $i \in \mathcal{N}$. This configuration is also in $N_t(\xi, \eta)$, since we allowed that an arbitrary number of photons could get absorbed (confer condition (2)(i) in definition (4.60)).

2. Suppose $(\xi', \eta') \to (\xi', \eta' \cup \boldsymbol{y}')$ with $\boldsymbol{y}' \in B_\delta(\boldsymbol{x}'_i)$ for any $i \in \mathcal{N}$. This configuration is also in $N_t(\xi, \eta)$, since $\boldsymbol{y}' \in B_\delta(\boldsymbol{x}'_i) \subset B_{ct+\delta}(\boldsymbol{x}_i)$ (confer condition (2)(ii) in definition (4.60)).

□

With that at hand we can prove the following Lemma 4.12 about the domain of dependence for the Dirac equation with photon creation and annihilation terms.

Lemma 4.12 (Domain of Dependence: Dirac Equation with Emission/Absorption)**.** *Let* ψ *be a solution of equation (4.58) with initial data* $\psi(0,\cdot) \in L^2(\mathcal{Q},\mathcal{S}) \cap C^\infty(\mathcal{Q},\mathcal{S})$. *Let* $t > 0$.

(i) Then specifying initial conditions on $N_t(\xi,\eta)$ *as defined in (4.60) uniquely determines* $\psi(t,\xi,\eta)$*. (I.e., the domain of dependence is contained in* $N_t(\xi,\eta)$*.)*

(ii) Let $\psi(0,\cdot)$ *have compact support, supp*$(\psi(0,\cdot)) = M_0 \subset \mathcal{Q}$*. Then the support of* $\psi(t,\cdot)$ *is a subset of* $N_t = \bigcup_{q\in M_0} N_t(q)$.

Proof. We set $c = 1$. Let $(\tilde{\xi},\tilde{\eta}) \in \mathcal{Q}^{(m)}$ and let $T > 0$. We define for $t \in [0,T]$

$$\Sigma_t = \left\{(t,q) \in \mathbb{R} \times \mathcal{Q} : q \in N_{T-t}(\tilde{\xi},\tilde{\eta})\right\}. \tag{4.66}$$

As in the case of a constant particle number we first prove $(***)$: If Ψ vanishes on Σ_0 it also vanishes on Σ_t for all $t \in [0,T]$.

We define $C_t = \bigcup_{t'\in[0,t]} \Sigma_{t'}$ and denote with Σ^s the sides of the region C_t such that $\Sigma_0 \dot\cup \Sigma_t \dot\cup \Sigma^s = \partial C_t$. Note that Σ^s is in fact dependent on t but for ease of notation we do not explicitly add an index t. C_t can be split into parts that belong to the (N,m)-particle sector for each m, i.e. $C_t^{(m)} = C_t \cap \left(\mathbb{R} \times \mathcal{Q}^{(m)}\right)$ such that $C_t = \bigcup_{m=0}^\infty C_t^{(m)}$. Let $n^{(m)} = (n^{(m),0}, \boldsymbol{n}_{x_1}^{(m)}, \dots, \boldsymbol{n}_{x_N}^{(m)}, \boldsymbol{n}_{y_1}^{(m)}, \dots, \boldsymbol{n}_{y_m}^{(m)})$ be the outward-pointing unit vector field on $\partial C_t^{(m)}$ orthogonal to $\partial C_t^{(m)}$ in the Euclidean metric on $\mathbb{R}^{1+3N+3m}$. We use the following notation for the current. For the $(N+m)$-particle sector we denote

$$j^{(m)} := \left(|\psi|^2, \psi^\dagger \boldsymbol{\alpha}_1 \psi, \dots, \psi^\dagger \boldsymbol{\alpha}_N \psi, \psi^\dagger \boldsymbol{S}_1 \psi, \dots, \psi^\dagger \boldsymbol{S}_m \psi\right) \tag{4.67}$$

$$j^{3,(m)} := \left(\psi^\dagger \boldsymbol{\alpha}_1 \psi, \dots, \psi^\dagger \boldsymbol{\alpha}_N \psi, \psi^\dagger \boldsymbol{S}_1 \psi, \dots, \psi^\dagger \boldsymbol{S}_m \psi\right). \tag{4.68}$$

The total "probability flux" through the "surface" ∂C_t is according to Gauß' integral Theorem given by

$$\sum_{m=0}^\infty \int_{C_t^{(m)}} ds\, d\xi\, d\eta\; div\left(j^{(m)}\right) = \sum_{m=0}^\infty \int_{\partial C_t^{(m)}} d\sigma^{(m)}\; j^{(m)} \cdot n^{(m)} \tag{4.69}$$

where $d\sigma^{(m)}$ denotes the $(3N+3m)$-surface area relative to the Euclidean metric. We now calculate $div\left(j^{(m)}\right)$. Using the results from the derivation of the continuity equation for a constant particle number (see equation (4.24)) we obtain for a solution ψ of equation (4.58):

$$\begin{aligned} div\left(j^{(m)}\right) &= \frac{\partial}{\partial t}|\psi|^2(t,\xi,\eta) + div\left(j^{3,(m)}\right) \\ &= 2\frac{1}{\sqrt{m}}\sum_{j=1}^{N}\sum_{k=1}^{m}\mathrm{Im}\left(\psi^\dagger(t,\xi,\eta)\varphi_\delta(\boldsymbol{y}_k-\boldsymbol{x}_j)\psi(t,\xi,\eta\backslash\boldsymbol{y}_k)\right) \\ &\quad + 2\sqrt{m+1}\sum_{j=1}^{N}\mathrm{Im}\left(\psi^\dagger(t,\xi,\eta)\int_{B_\delta(\boldsymbol{x}_j)} d^3y\ \varphi_\delta(\boldsymbol{y}-\boldsymbol{x}_j)\psi(t,\xi,\eta\cup\boldsymbol{y})\right) \end{aligned} \tag{4.70}$$

For every ξ, we define $N_{\xi,T-s} = \{\eta \in \bigcup_{m=0}^\infty \mathbb{R}^{3m} : (\xi,\eta) \in N_{T-s}(\tilde{\xi},\tilde{\eta})\}$ and $N^{(m)}_{\xi,T-s} = \mathbb{R}^{3m} \cap N_{\xi,T-s}$. We then have for the l.h.s. of (4.69):

$$\begin{aligned} &\sum_{m=0}^{\infty}\int_{C_t^{(m)}} ds\,d\xi\,d\eta\ div\left(j^{(m)}\right) = \int_0^t ds\sum_{m=0}^{\infty}\int_{\Sigma_s^{(m)}} d\xi\,d\eta\ div\left(j^{(m)}\right) \\ &= \int_0^t ds\int_{B^{(N)}_{T-s}(\tilde{\xi})} d\xi\ 2\sum_{j=1}^{N}\mathrm{Im}\Bigg[\underbrace{\sum_{m=1}^{\infty}\frac{1}{\sqrt{m}}\int_{N^{(m)}_{\xi,T-s}} d\eta\sum_{k=1}^{m}\psi^\dagger(t,\xi,\eta)\varphi_\delta(\boldsymbol{y}_k-\boldsymbol{x}_j)\psi(t,\xi,\eta\backslash\boldsymbol{y}_k)}_{A_1(N^{(m)}_{\xi,T-s})} \\ &\quad + \underbrace{\sum_{m=0}^{\infty}\sqrt{m+1}\int_{N^{(m)}_{\xi,T-s}} d\eta\,\psi^\dagger(t,\xi,\eta)\int_{B_\delta(\boldsymbol{x}_j)} d\boldsymbol{y}\,\varphi^*_\delta(\boldsymbol{y}-\boldsymbol{x}_j)\psi(t,\xi,\eta\cup\boldsymbol{y})}_{A_2(N^{(m)}_{\xi,T-s})}\Bigg] \end{aligned} \tag{4.71}$$

We now show that the expression (4.71) equals zero. We show for any (measurable) set $B \subset \bigcup_{m=0}^\infty \mathbb{R}^{3m}$ that is closed under allowed jumps that $A_1(B) = A_2^*(B)$. This holds in particular for $N_{\xi,T-s}$. To perform this calculation we switch to unordered configurations. We denote $\hat{B} = \pi(B_{\neq}) = \{\{\boldsymbol{y}_1,\dots,\boldsymbol{y}_m\} \in \Gamma(\mathbb{R}^3) : (\boldsymbol{y}_1,\dots,\boldsymbol{y}_m) \in B_{\neq}\}$ with $B_{\neq} = \{\eta \in B : \boldsymbol{y}_i \neq \boldsymbol{y}_j\ \forall i \neq j\}$. Also $B^{(m)} = \mathbb{R}^{3m} \cap B$ and $\hat{B}^{(m)} = \Gamma^{(m)} \cap \hat{B}$ and we denote $U_\delta := B_\delta(\boldsymbol{x}_j)$. According to Lemma 4.9, since B is closed under allowed jumps, so is $\hat{B}$, and we have $\hat{B} = \hat{B}' \times \Gamma(U_\delta)$ for a certain set $\hat{B}' \subset \Gamma(U_\delta^c)$. For ease of notation we do not explicitly write out the fixed variables ξ and t. We also use Lemma

4.10.

$$
\begin{aligned}
A_1(B) &= \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \int_{B^{(m)}} d\eta \sum_{k=1}^{m} \psi^\dagger(\eta)\varphi_\delta(\boldsymbol{y}_k - \boldsymbol{x}_j)\psi(\eta\backslash \boldsymbol{y}_k) \\
&= \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} m! \int_{\hat{B}^{(m)}} d\hat{\eta} \sum_{\boldsymbol{y}\in\hat{\eta}} \frac{1}{\sqrt{m!}} \hat{\psi}^\dagger(\hat{\eta})\varphi_\delta(\boldsymbol{y} - \boldsymbol{x}_j) \frac{1}{\sqrt{(m-1)!}} \hat{\psi}(\hat{\eta}\backslash \boldsymbol{y}) \\
&= \int_{\hat{B}} d\hat{\eta} \sum_{\boldsymbol{y}\in\hat{\eta}} \hat{\psi}^\dagger(\hat{\eta})\varphi_\delta(\boldsymbol{y} - \boldsymbol{x}_j)\hat{\psi}(\hat{\eta}\backslash \boldsymbol{y}) \\
&= \int_{\hat{B}'} d\hat{\eta}' \int_{\Gamma(U_\delta)} d\hat{\eta}'' \sum_{\boldsymbol{y}\in\hat{\eta}} \hat{\psi}^\dagger(\hat{\eta}'')\varphi_\delta(\boldsymbol{y} - \boldsymbol{x}_j)\hat{\psi}(\hat{\eta}''\backslash \boldsymbol{y}) \\
&= \int_{\hat{B}'} d\hat{\eta}' \int_{\Gamma(U_\delta)} d\hat{\eta}''' \int_{U_\delta} d\boldsymbol{y}\, \hat{\psi}^\dagger(\hat{\eta}''' \cup \boldsymbol{y})\varphi_\delta(\boldsymbol{y} - \boldsymbol{x}_j)\hat{\psi}(\hat{\eta}''') \\
&= \int_{\hat{B}'} d\hat{\eta}' \int_{\Gamma(U_\delta)} d\hat{\eta}''' \left(\hat{\psi}^\dagger(\hat{\eta}''') \int_{U_\delta} d\boldsymbol{y}\, \varphi^*_\delta(\boldsymbol{y} - \boldsymbol{x}_j)\hat{\psi}(\hat{\eta}''' \cup \boldsymbol{y}) \right)^* \\
&= \int_{\hat{B}} d\hat{\eta}'''' \left(\hat{\psi}^\dagger(\hat{\eta}'''') \int_{U_\delta} d\boldsymbol{y}\, \varphi^*_\delta(\boldsymbol{y} - \boldsymbol{x}_j)\hat{\psi}(\hat{\eta}'''' \cup \boldsymbol{y}) \right)^* \\
&= A_2^*(B)
\end{aligned} \tag{4.72}
$$

Since $A_1(B) = A_2^*(B)$ we have that $\mathrm{Im}\,(A_1(B) + A_2(B)) = \mathrm{Im}\,(A_1(B) + A_1^*(B)) = 0$ and therefore the flux-integral (4.71) is zero. Then it follows from (4.69) that

$$
\begin{aligned}
0 &= \sum_{m=0}^{\infty} \int_{\partial C_t^{(m)}} j^{(m)} \cdot n^{(m)}\; d\sigma^{(m)} \\
&= \sum_{m=0}^{\infty} \left[\int_{\Sigma_0^{(m)}} j^{(m)} \cdot n^{(m)}\; d\sigma^{(m)} + \int_{\Sigma_t^{(m)}} j^{(m)} \cdot n^{(m)}\; d\sigma^{(m)} + \int_{\Sigma^{s,(m)}} j^{(m)} \cdot n^{(m)}\; d\sigma^{(m)} \right].
\end{aligned} \tag{4.73}
$$

We now suppose that ψ vanishes on Σ_0, so ψ also vanishes on $\Sigma_0^{(m)}$ for all $m \in \mathbb{N}$. The normal vector on $\Sigma_0^{(m)}$ is $n^{(m)} = (-1, 0, \ldots, 0)$, so $j^{(m)} \cdot n^{(m)} = -|\psi|^2 = 0$ on $\Sigma_0^{(m)}$. The normal vector on $\Sigma_t^{(m)}$ is $n^{(m)} = (1, 0, \ldots, 0)$, so $j^{(m)} \cdot n^{(m)} = |\psi|^2$ on $\Sigma_t^{(m)}$. Let I denote the identity on $(\mathbb{C}^4)^{\otimes N} \times (\mathbb{C}^3)^{\otimes m}$. Then for $\Sigma^{s,(m)}$ we have

$$
\begin{aligned}
j^{(m)} \cdot n^{(m)}|_{\Sigma^{s,(m)}} &= \psi^\dagger \left(n^{(m),0} I + \sum_{j=1}^{N} \boldsymbol{\alpha}_i \cdot \boldsymbol{n}^{(m)}_{x_i} + \sum_{k=1}^{m} \boldsymbol{S}_k \cdot \boldsymbol{n}^{(m)}_{y_k} \right) \psi \\
&=: \psi^\dagger M \psi.
\end{aligned} \tag{4.74}
$$

Therefore from (4.73) it follows that

$$
0 = \sum_{m=0}^{\infty} \int_{\Sigma_t^{(m)}} |\psi|^2\; d\sigma^{(m)} + \sum_{m=0}^{\infty} \int_{\Sigma^{s,(m)}} \psi^\dagger M \psi\; d\sigma^{(m)}. \tag{4.75}
$$

We show that the matrix M is positive semi-definite. We know that for any unit vector $\boldsymbol{b} \in \mathbb{R}^3$ the matrices $(\boldsymbol{b} \cdot \boldsymbol{\alpha})$ and $(\boldsymbol{b} \cdot \boldsymbol{S})$ have the lowest eigenvalue $e = -1$.[7] Then the lowest eigenvalue of M is $e_M = n^{(m),0} - \sum_{j=1}^{N} ||\boldsymbol{n}_{x_i}^{(m)}|| - \sum_{k=1}^{m} ||\boldsymbol{n}_{y_k}^{(m)}||$. Note that there is now more than one "pyramid" in the m-photon sector because this sector has contributions from other sectors. Those pyramids can overlap so the surface $\Sigma_t^{(m)}$ looks more complicated than in the case of a constant particle number. However, since there are only finitely many pyramids the normal vector on $\Sigma^{s,(m)}$ has the component $n^{(m),0} = \frac{1}{\sqrt{2}}$ and for the spatial components we have $||\boldsymbol{n}^{(m),\ell}|| = \frac{1}{\sqrt{2}}\delta_{\ell j}$ (for $j = 1, \ldots, N+m$). Therefore $e_M = 0$. Then the rest of the argument goes along the same lines as the end of the proof of Lemma 4.4. □

4.3.3 On Existence and Uniqueness

Similar results as in section 4.1 should hold for equation (4.58). So far, we have not been able to prove the following statement though. It can therefore only be stated as a conjecture.

Conjecture 4.13. *Let $\mathscr{B}$ be a Hermitian vector bundle over the manifold $\mathscr{M}$. There is a suitable, large subspace $\mathscr{F}$ of $C^\infty(\mathcal{C}_{\delta,0} \times \mathscr{M}, \mathcal{S} \otimes \mathscr{B})$ such that for every initial condition $\phi \in \mathscr{F}$ there is a unique solution $\psi \in C^\infty(\mathbb{R} \times \mathcal{C}_{\delta,0} \times \mathscr{M}, \mathcal{S} \otimes \mathscr{B})$ of (4.58); for every time t, $\psi(t) \in \mathscr{F}$.*

With that we can state

Lemma 4.14. *Suppose that Conjecture 4.13 is true. For each $\alpha = 1, \ldots, L$, let H_α be the Hamiltonian in (4.58). The (interaction-free) multi-time evolution equations*

$$i\frac{\partial \psi}{\partial t_\alpha} = H_\alpha \psi \tag{4.78}$$

possess, for every initial datum $\phi \in \mathscr{F}(\mathcal{C}_{\delta,0}^L, \mathcal{S}^{\otimes L})$, a unique solution ψ, and $\psi \in C^\infty(\mathcal{C}_{\delta,0}^L, \mathcal{S}^{\otimes L})$.

This can be proved just like Lemma 4.2.

[7] For $(\boldsymbol{b} \cdot \boldsymbol{\alpha})$ this is shown in footnote 4. For $(\boldsymbol{b} \cdot \boldsymbol{S})$ this can be verified by direct calculation in the representation (4.53). We have

$$(\boldsymbol{b} \cdot \boldsymbol{S}) - \lambda = \begin{pmatrix} 0 & -ib_3 & ib_2 \\ ib_3 & 0 & -ib_1 \\ -ib_2 & ib_1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} -\lambda & -ib_3 & ib_2 \\ ib_3 & -\lambda & -ib_1 \\ -ib_2 & ib_1 & -\lambda \end{pmatrix}. \tag{4.76}$$

The characteristic polynomial is

$$det\,((\boldsymbol{b} \cdot \boldsymbol{S}) - \lambda) = -\lambda^3 + ib_1b_2b_3 - ib_1b_2b_3 + \lambda b_2^2 + \lambda b_1^2 + \lambda b_3^2 = -\lambda^3 + \lambda, \tag{4.77}$$

so the eigenvalues of $(\boldsymbol{b} \cdot \boldsymbol{S})$ are $-1, 0, +1$.

4.3.4 The Configuration Space-Time and Partitions

The multi-time evolution will be defined on

$$\mathcal{C}_\delta = \bigcup_{m=0}^{\infty} \mathcal{C}_\delta^{(N,m)} \tag{4.79}$$

with the following generalization of the subset of space-time (for $N+m$ particles):

$$\begin{aligned}\mathcal{C}_\delta^{(N,m)} = \Big\{(\xi^4, \eta^4) \in \mathbb{R}^{4N} \times \mathbb{R}^{4m} :\\ &(1)\ \forall i,j \in \mathcal{N}^2, i \neq j :\ t_{x_i} = t_{x_j} \text{ or } ||\boldsymbol{x}_i - \boldsymbol{x}_j|| > c|t_{x_i} - t_{x_j}| + 2\delta \\ &(2)\ \forall i \in \mathcal{N}\ \forall k \in \mathcal{M} :\ t_{x_i} = t_{y_k} \text{ or } ||\boldsymbol{x}_i - \boldsymbol{y}_k|| > c|t_{x_i} - t_{y_k}| + \delta\Big\}.\end{aligned} \tag{4.80}$$

where we abbreviate the set of the first N natural numbers as $\mathcal{N} := \{1, \ldots, N\}$ and likewise $\mathcal{M} := \{1, \ldots, m\}$. For a given partition $P = \{S_1, \ldots, S_L\} = \{(S_1^\xi, S_1^\eta), \ldots, (S_L^\xi, S_L^\eta)\}$ we define

$$\begin{aligned}\mathcal{C}_{\delta,P}^{(N,m)} = \Big\{(\xi^4, \eta^4) \in \mathbb{R}^{4N} \times \mathbb{R}^{4m} :\\ &(1)\ \forall i,j \in S_v^\xi,\ \forall k,\ell \in S_v^\eta,\ \forall v = 1, \ldots, L :\ t_{x_i} = t_{x_j} = t_{y_k} = t_{y_\ell} \\ &(2)\ \forall i \in S_v^\xi,\ \forall j \in S_w^\xi,\ \forall v,w = 1, \ldots, L,\ v \neq w : \boldsymbol{x}_i \notin B_{c|t_{x_i} - t_{x_j}| + 2\delta}(\boldsymbol{x}_j) \\ &(3)\ \forall i \in S_v^\xi,\ \forall k \in S_w^\eta,\ \forall v,w = 1, \ldots, L,\ v \neq w : \boldsymbol{x}_i \notin B_{c|t_{x_i} - t_{y_k}| + \delta}(\boldsymbol{y}_k)\Big\}.\end{aligned} \tag{4.81}$$

Again, we have that

$$\mathcal{C}_\delta^{(N,m)} = \bigcup_{P \in \mathcal{P}} \mathcal{C}_{\delta,P}^{(N,m)}. \tag{4.82}$$

For an element $q^4 \in \mathcal{C}_{\delta,P}^{(N,m)}$ and for a partition $P = \{(S_1^\xi, S_1^\eta), \ldots, (S_L^\xi, S_L^\eta)\}$ we write:

$$\begin{aligned}q^4 &= (t_1, q_1; \ldots; t_L, q_L) = (t_1, q_1^\xi, q_1^\eta; \ldots; t_L, q_L^\xi, q_L^\eta) \\ &= \big((x_1^0, \boldsymbol{x}_1), \ldots, (x_N^0, \boldsymbol{x}_N), (y_1^0, \boldsymbol{y}_1), \ldots, (y_m^0, \boldsymbol{y}_m)\big)\end{aligned} \tag{4.83}$$

with $x_j^0 = y_k^0 = t_\alpha$ for $j \in S_\alpha^\xi$, $k \in S_\alpha^\eta$.

4.3.5 Properties of $\mathcal{C}_{\delta,P}^{(N,m)}$ and N_t

We collect some Lemmas which will be used in the proof of Theorem 4.20.

Lemma 4.15. *Let $\tau_2 \geq \tau_1$ and $(Q_{S_1}, \tau_1; Q_{S_2}, \tau_2) \in \mathcal{C}_{\delta,P}^{(N,m_1+m_2)}$. Then for all $q_{S_2'} \in N_{\tau_2-\tau_1}(Q_{S_2})$ the configuration $(Q_{S_1}, \tau_1; q_{S_2'}, \tau_1) \in \mathcal{C}_{\delta,P}^{(N,m_1+m_2')}$.*

Proof. We denote $\Delta t := \tau_2 - \tau_1$ and set $c = 1$. Then according to the definition (4.81) of $\mathcal{C}_{\delta,P}^{(N,m)}$, $(Q_{S_1}, \tau_1; Q_{S_2}, \tau_2) \in \mathcal{C}_{\delta,P}^{(N,m)}$ means that

$$\text{For all } \boldsymbol{X} \in Q_{S_1^\xi},\ \tilde{\boldsymbol{X}} \in Q_{S_2^\xi}:\ \boldsymbol{X} \notin B_{\Delta t+2\delta}(\tilde{\boldsymbol{X}}). \tag{4.84}$$

$$\text{For all } \boldsymbol{X} \in Q_{S_1^\xi},\ \tilde{\boldsymbol{Y}} \in Q_{S_2^\eta}:\ \boldsymbol{X} \notin B_{\Delta t+\delta}(\tilde{\boldsymbol{Y}}). \tag{4.85}$$

$$\text{For all } \boldsymbol{Y} \in Q_{S_1^\eta},\ \tilde{\boldsymbol{X}} \in Q_{S_2^\xi}:\ \boldsymbol{Y} \notin B_{\Delta t+\delta}(\tilde{\boldsymbol{X}}). \tag{4.86}$$

Now choose an arbitrary $q_{S_2'} \in N_{\Delta t}(Q_{S_2})$. We denote $q_{S_2'} = (q_{S_2^\xi}, q_{S_2'^\eta})$. (Here we use the notation $\boldsymbol{x}_{\beta_j} \in q_{S_2^\xi}$ and $\tilde{\boldsymbol{X}}_{\beta_j} \in Q_{S_2^\xi}$.) According to the definition of $N_{\Delta t}(Q_{S_2})$, see (4.60), this means:

$$\text{For all } j = 1, \ldots, N_2:\ \boldsymbol{x}_{\beta_j} \in B_{\Delta t}(\tilde{\boldsymbol{X}}_{\beta_j}). \tag{4.87}$$

$$\text{For all } \boldsymbol{y} \in q_{S_2'^\eta} \text{ and some } \tilde{\boldsymbol{X}} \in Q_{S_2^\xi},\ \tilde{\boldsymbol{Y}} \in Q_{S_2^\eta}:\ \boldsymbol{y} \in B_{\Delta t}(\tilde{\boldsymbol{Y}}) \quad \text{or} \quad \boldsymbol{y} \in B_{\Delta t+\delta}(\tilde{\boldsymbol{X}}). \tag{4.88}$$

Now we show that $(Q_{S_1}, \tau_1; q_{S_2'}, \tau_1) \in \mathcal{C}_{\delta,P}^{(N,m)}$. The next statements follow from the simple fact that if $\boldsymbol{a} \notin B_{r+\varepsilon}(\boldsymbol{c})$ and $\boldsymbol{b} \in B_r(\boldsymbol{c})$ then $\boldsymbol{a} \notin B_\varepsilon(\boldsymbol{b})$.

- From (4.84) and (4.87) it follows immediately that

$$\text{for all } \boldsymbol{X} \in Q_{S_1^\xi},\ \boldsymbol{x} \in q_{S_2^\xi}:\ \boldsymbol{X} \notin B_{2\delta}(\boldsymbol{x}). \tag{4.89}$$

- If $\boldsymbol{y} \in B_{\Delta t}(\tilde{\boldsymbol{Y}})$ in (4.88), then with (4.85) the next statement (4.90) follows immediately. If $\boldsymbol{y} \in B_{\Delta t+\delta}(\tilde{\boldsymbol{X}})$ in (4.88), then it follows with (4.84).

$$\text{For all } \boldsymbol{X} \in Q_{S_1^\xi},\ \boldsymbol{y} \in q_{S_2'^\eta}: \boldsymbol{X} \notin B_\delta(\boldsymbol{y}). \tag{4.90}$$

- From (4.86) and (4.87) it follows immediately that

$$\text{for all } \boldsymbol{Y} \in Q_{S_1^\eta},\ \boldsymbol{x} \in q_{S_2^\xi}: \boldsymbol{Y} \notin B_\delta(\boldsymbol{x}). \tag{4.91}$$

Statements (4.89), (4.90) and (4.91) together are equivalent to $(Q_{S_1}, \tau_1; q_{S_2'}, \tau_1) \in \mathcal{C}_{\delta,P}^{(N,m)}$. Since $q_{S_2'} \in N_{\Delta t}(Q_{S_2})$ was arbitrary, the Lemma is proven. □

Lemma 4.16. *Let* $(\xi_1, \eta_1, \tau; \xi_2, \eta_2, \tau) \in \mathcal{C}_{\delta,\{S_1,S_2\}}^{(N,m)}$. *Let* $(H\psi)(\xi, \eta)$ *be the emission-absorption Dirac-type operator, the right hand side of* (4.58)*. Then*

$$(H\psi)(\xi, \eta) = (H_1\psi)(\xi, \eta) + (H_2\psi)(\xi, \eta), \tag{4.92}$$

where H_1 *acts only on the variables* ξ_1, η_1 *and* H_2 *acts only on* ξ_2, η_2.

Proof. The Lemma follows from a simple calculation. Note that $\varphi_\delta(\boldsymbol{x}) = 0$ for $||\boldsymbol{x}|| > \delta$ and that

$||\boldsymbol{x}-\boldsymbol{y}|| > \delta$ for $\boldsymbol{x} \notin B_\delta(\boldsymbol{y})$.

$$\begin{aligned}
(H\psi)(\xi,\eta) &= \sum_{j=1}^{N} \left(-ic\boldsymbol{\alpha}_j \cdot \boldsymbol{\nabla}_{x_j} + \beta_j mc^2\right)\psi(\xi,\eta) + \sum_{\boldsymbol{y}\in\eta} \left(-ic\boldsymbol{S}_y \cdot \boldsymbol{\nabla}_y\right)\psi(\xi,\eta) \\
&\quad + \sum_{j=1}^{N}\sum_{\boldsymbol{y}\in\eta} \varphi_\delta(\boldsymbol{y}-\boldsymbol{x}_j)\psi(\xi,\eta\backslash\boldsymbol{y}) + \sum_{j=1}^{N}\int_{\mathbb{R}^3} d\boldsymbol{y}\ \varphi^*_\delta(\boldsymbol{y}-\boldsymbol{x}_j)\psi(\xi,\eta\cup\boldsymbol{y}) \\
&= \underbrace{\sum_{j\in S_1^\xi} \left(-ic\boldsymbol{\alpha}_j \cdot \boldsymbol{\nabla}_{x_j} + \beta_j mc^2\right)\psi(\xi,\eta)}_{(H_1^{x,free}\psi)(\xi,\eta)} + \underbrace{\sum_{j\in S_2^\xi} \left(-ic\boldsymbol{\alpha}_j \cdot \boldsymbol{\nabla}_{x_j} + \beta_j mc^2\right)\psi(\xi,\eta)}_{(H_2^{x,free}\psi)(\xi,\eta)} \\
&\quad + \underbrace{\sum_{\boldsymbol{y}\in\eta_1} \left(-ic\boldsymbol{S}_y \cdot \boldsymbol{\nabla}_y\right)\psi(\xi,\eta)}_{(H_1^{y,free}\psi)(\xi,\eta)} + \underbrace{\sum_{\boldsymbol{y}\in\eta_2} \left(-ic\boldsymbol{S}_y \cdot \boldsymbol{\nabla}_y\right)\psi(\xi,\eta)}_{(H_2^{y,free}\psi)(\xi,\eta)} \\
&\quad + \underbrace{\sum_{j\in S_1^\xi}\sum_{\boldsymbol{y}\in\eta_1} \varphi_\delta(\boldsymbol{y}-\boldsymbol{x}_j)\psi(\xi,\eta\backslash\boldsymbol{y})}_{(H_1^c\psi)(\xi,\eta)} + \sum_{j\in S_1^\xi}\sum_{\boldsymbol{y}\in\eta_2} \underbrace{\varphi_\delta(\boldsymbol{y}-\boldsymbol{x}_j)}_{=0 \text{ as } \boldsymbol{x}_j\notin B_\delta(\boldsymbol{y})} \psi(\xi,\eta\backslash\boldsymbol{y}) \\
&\quad + \sum_{j\in S_2^\xi}\sum_{\boldsymbol{y}\in\eta_1} \underbrace{\varphi_\delta(\boldsymbol{y}-\boldsymbol{x}_j)}_{=0 \text{ as } \boldsymbol{x}_j\notin B_\delta(\boldsymbol{y})} \psi(\xi,\eta\backslash\boldsymbol{y}) + \underbrace{\sum_{j\in S_2^\xi}\sum_{\boldsymbol{y}\in\eta_2} \varphi_\delta(\boldsymbol{y}-\boldsymbol{x}_j)\psi(\xi,\eta\backslash\boldsymbol{y})}_{(H_2^c\psi)(\xi,\eta)} \\
&\quad + \underbrace{\sum_{j\in S_1^\xi}\int_{B_\delta(\boldsymbol{x}_j)} d\boldsymbol{y}\ \varphi^*_\delta(\boldsymbol{y}-\boldsymbol{x}_j)\psi(\xi,\eta\cup\boldsymbol{y})}_{(H_1^a\psi)(\xi,\eta)} + \underbrace{\sum_{j\in S_2^\xi}\int_{B_\delta(\boldsymbol{x}_j)} d\boldsymbol{y}\ \varphi^*_\delta(\boldsymbol{y}-\boldsymbol{x}_j)\psi(\xi,\eta\cup\boldsymbol{y})}_{(H_2^a\psi)(\xi,\eta)} \\
&= \left(\underbrace{\left(H_1^{x,free} + H_1^{y,free} + H_1^c + H_1^a\right)}_{H_1}\psi\right)(\xi,\eta) \\
&\quad + \left(\underbrace{\left(H_2^{x,free} + H_2^{y,free} + H_2^c + H_2^a\right)}_{H_2}\psi\right)(\xi,\eta). \qquad (4.93)
\end{aligned}$$

□

If the photons variables are unordered configurations we have to adjust the definition of a cartesian product of two configurations. We define

$$A_1 \hat{\times} A_2 := \{(\xi_1,\xi_2,\eta_1\cup\eta_2) : (\xi_i,\eta_i)\in A_i, \eta_1\cap\eta_2=\emptyset\}. \qquad (4.94)$$

Then we get the following results that follow directly from the definition of $N_t(q)$, see (4.60).

Lemma 4.17. *For any configuration* $Q_1 \hat{\times} Q_2 = (Q_1, Q_2)$ *we have*

$$N_t(Q_1, Q_2) = N_t(Q_1) \hat{\times} N_t(Q_2). \tag{4.95}$$

Lemma 4.18. *For any configuration* Q *we have*

$$N_s(N_t(Q)) = N_{s+t}(Q). \tag{4.96}$$

Lemma 4.19. *For any configuration* Q_1 *and* Q_2 *we have*

$$N_{t_1}(\{Q_1\} \hat{\times} N_{t_2 - t_1}(Q_2)) = N_{t_1}(Q_1) \hat{\times} N_{t_2}(Q_2). \tag{4.97}$$

Proof. This is an immediate consequence of Lemmas 4.17 and 4.18. □

4.3.6 The Multi-Time Evolution

We now have everything together to prove the main result of this work. Theorem 4.20 is a consistent multi-time model with creation and annihilation of particles. We define $\mathcal{C}_{\delta,0} = (\{0\} \times \mathbb{R}^3)^N \times \bigcup_{m=0}^{\infty} (\{0\} \times \mathbb{R}^3)^m$, i.e., $\mathcal{C}_{\delta,0}$ is the subset of $\mathcal{C}_\delta$ where all times are zero. The spin space is $\mathcal{S} = (\mathcal{C}^4)^{\otimes N} \otimes \bigcup_{m=0}^{\infty} (\mathcal{C}^3)^{\otimes m}$. If $\mathscr{B}$ is a vector bundle over the manifold $\mathscr{M}$ then $C^\infty(\mathscr{M}, \mathscr{B})$ means the smooth cross-section of $\mathscr{B}$.

Theorem 4.20. *Let* $\delta > 0$. *Suppose that Conjecture 4.13 is true and that* $\varphi_\delta \in C^\infty(\mathbb{R}^3, \mathbb{C}^3)$ *vanishes outside the* δ*-ball around the origin. Then for every* $\phi \in C^\infty(\mathcal{C}_{\delta,0}, \mathcal{S})$ *with* $\phi \in \mathcal{F}(\mathcal{C}_{\delta,0}, \mathcal{S})$ *as in Conjecture 4.13, there is a unique multi-time wavefunction* $\psi \in C^\infty(\mathcal{C}_\delta, \mathcal{S})$ *with* $\psi|_{\mathcal{C}_{\delta,0}} = \phi$, *solving at every* $q^4 \in \mathcal{C}_\delta$ *(writing* $q^4 = (\xi^4, \eta^4) = (t_1, q_1; \ldots; t_L, q_L)$*) the equations*

$$\begin{aligned} i\frac{\partial}{\partial t_\alpha}\psi(\xi^4, \eta^4) = &\sum_{j \in S_\alpha^\xi} \left(-ic\boldsymbol{\alpha}_j \cdot \boldsymbol{\nabla}_{x_j} + \beta_j mc^2\right) \psi(\xi^4, \eta^4) + \sum_{k \in S_\alpha^\eta} (-ic\boldsymbol{S}_k \cdot \boldsymbol{\nabla}_{y_k}) \psi(\xi^4, \eta^4) \\ &+ \frac{1}{\sqrt{m}} \sum_{j \in S_\alpha^\xi} \sum_{k \in S_\alpha^\eta} \varphi_\delta(\boldsymbol{y}_k - \boldsymbol{x}_j) \psi(\xi^4, \eta^4 \backslash y_k) \\ &+ \sqrt{m+1} \sum_{j \in S_\alpha^\xi} \int_{\mathbb{R}^3} d\boldsymbol{y}\ \varphi_\delta^*(\boldsymbol{y} - \boldsymbol{x}_j) \psi(\xi^4, \eta^4 \cup (t_\alpha, \boldsymbol{y})) \end{aligned} \tag{4.98}$$

for all $\alpha = 1, \ldots, L$.

Proof. The Theorem is proven along the same line of arguments as Theorem 4.6: First, we define a function Ψ. Second, we show that any solution ψ of the evolution equations with initial conditions ϕ must agree with Ψ. Third, we show that Ψ satisfies the equations (4.98).

Step 1. We define Ψ on $\mathcal{C}_\delta$ by defining it on each $\mathcal{C}_{\delta,P}$ using induction on the number L of groups in a partition.

- We start the induction with $L = 1$. For any $m = 0, 1, 2, \ldots$ the only partition with $L = 1$ is $P = \{(S_1^\xi, S_1^\eta)\}$ with $S_1^\xi = \{1, \ldots, N\}$, $S_1^\eta = \{1, \ldots, m\}$, and we can write $q^4 \in \mathcal{C}_{\delta,P}^{(N,m)}$ as $q^4 = (t_1, q_1)$. Let $\Psi(q^4) = \psi_{t_1}(q_1)$, where ψ is the solution of (4.58) with initial conditions given by ϕ on $\mathcal{C}_{\delta,0}$ which, by Conjecture 4.13, exists, is unique and satisfies $\psi_{t_1} \in \mathcal{F}(\mathcal{C}_{\delta,0}, \mathcal{S})$ for every t_1.

- The induction assumption is that Ψ has been defined on $\mathcal{C}_{\delta,P'}^{(N,m)}$ for every m and every partition P' with $L' = L - 1$ or fewer groups, and $\Psi(t_1, q_1; \ldots; t_{L'}, q_{L'})$ as a function of $q_{L'}$ always lies in $\mathcal{F}(\mathcal{C}_{\delta,0}, \mathcal{S} \otimes \mathcal{C}^k)$ with $k = 3^{(\sum_{\alpha=1}^{L'-1} |S_\alpha^\eta|)} 4^{(\sum_{\alpha=1}^{L'-1} |S_\alpha^\xi|)}$, with $\mathcal{F}$ as in Conjecture 4.13.

- Now we perform the induction step from $L - 1$ to L. We construct $\Psi(Q^4)$ for an arbitrary $Q^4 = (T_1, Q_1; \ldots; T_L, Q_L) \in \mathcal{C}_{\delta,P}^{(N,m)}$. (Note that according to our notation, $P = \{(S_1^\xi, S_1^\eta), \ldots, (S_L^\xi, S_L^\eta)\}$ and $T_1 < T_2 < \cdots < T_L$.) According to the induction assumption, $\Psi(T_1, Q_1; \ldots; T_{L-1}, Q_{L-1}; T_{L-1}, q_L)$ is given for any $q_L = (\xi_L, \eta_L) \in N_{T_L - T_{L-1}}(Q_L)$. Since the domain of dependence of Q_L is $N_{T_L - T_{L-1}}(Q_L)$ (see Lemma 4.12), we can solve the equation

$$
\begin{aligned}
&i\frac{\partial}{\partial t_L}\Psi(T_1, Q_1; \ldots; T_{L-1}, Q_{L-1}; t_L, q_L) = \\
&\left(\sum_{j \in S_L^\xi} \left(-ic\boldsymbol{\alpha}_j \cdot \boldsymbol{\nabla}_{x_j} + \beta_j mc^2\right) + \sum_{k \in S_L^\eta} \left(-ic\boldsymbol{S}_k \cdot \boldsymbol{\nabla}_{y_k}\right)\right) \Psi(T_1, Q_1; \ldots; T_{L-1}, Q_{L-1}; t_L, q_L) \\
&+ \frac{1}{\sqrt{|S_L^\eta|}} \sum_{j \in S_L^\xi} \sum_{k \in S_L^\eta} \varphi_\delta(\boldsymbol{y}_k - \boldsymbol{x}_j) \Psi(T_1, Q_1; \ldots; T_{L-1}, Q_{L-1}; t_L, \xi_L, \eta_L \backslash \boldsymbol{y}_k) \\
&+ \sqrt{|S_L^\eta| + 1} \sum_{j \in S_L^\xi} \int_{\mathbb{R}^3} d\boldsymbol{y}\; \varphi_\delta^*(\boldsymbol{y} - \boldsymbol{x}_j) \Psi(T_1, Q_1; \ldots; T_{L-1}, Q_{L-1}; t_L, \xi_L, \eta_L \cup \boldsymbol{y})
\end{aligned}
\tag{4.99}
$$

  to obtain $\Psi(T_1, Q_1, \ldots, T_L, Q_L)$. By Conjecture 4.13, there is a unique solution, and we have that $\Psi(T_1, Q_1; \ldots; T_{L-1}, Q_{L-1}; t_L, q_L)$ as a function of q_L lies, for every t_L in $\mathcal{F}$. Since $Q^4 \in \mathcal{C}_{\delta,P}^{(N,m)}$ was arbitrary, we have defined Ψ for every Q^4 lying in some $\mathcal{C}_{\delta,P}^{(N,m)}$ with a partition P into L or fewer groups.

Step 2. Let ψ be a solution of the multi-time equations (4.98) with initial conditions ϕ. Then, for the induction start with $L = 1$, this ψ has to agree with Ψ on the equal-time configurations by Conjecture 4.13, since both functions have the same initial conditions and are solutions of the same equation (4.58). The same holds for the induction step. Both ψ and Ψ have the same initial conditions and are solutions to the same equation (4.99). Therefore ψ agrees with Ψ.

Step 3. This part of the proof is due to Roderich Tumulka. Choose any $Q^4 = (T_1, Q_1; \ldots; T_L, Q_L) \in \mathcal{C}_{\delta,P}^{(N,m)}$. Choose $\varepsilon > 0$ in such a way that

$$2\varepsilon < T_{\alpha+1} - T_\alpha \tag{4.100}$$

for every $\alpha = 1, \ldots, L-1$ and

$$\begin{aligned} 2\varepsilon &< \|\boldsymbol{X}_i - \boldsymbol{X}_j\| - |X_i^0 - X_j^0| - 2\delta, \\ 2\varepsilon &< \|\boldsymbol{X}_i - \mathbf{Y}_k\| - |X_i^0 - Y_k^0| - \delta \end{aligned} \tag{4.101}$$

for every $i \in S_\alpha^\xi$, $j \in S_\beta^\xi$ and $k \in S_\beta^\eta$ with $\alpha \neq \beta$. For this proof we define

$$D^+_{s,(T,Q_\alpha)} = \bigcup_{0 \le t \le s} \{T - t\} \times N_{ct}(Q_\alpha). \tag{4.102}$$

We show that for any fixed $\beta = 1, \ldots, L$, fixed $\tilde{Q}_\alpha^4 = (T_\alpha, Q_\alpha)$ for $\alpha \neq \beta$, and varying $\tilde{Q}_\beta^4 \in D^+_{2\varepsilon,(\varepsilon+T_\beta,Q_\beta)}$, $\Psi(\tilde{Q}_1^4, \ldots, \tilde{Q}_L^4)$ satisfies

$$i\frac{\partial \Psi}{\partial t_\beta} = H_\beta \Psi, \tag{4.103}$$

with H_β defined as the right-hand side of (4.99) with L replaced by β (and $\Psi(\ldots)$ replaced by $\Psi(\tilde{Q}_1^4, \ldots, \tilde{Q}_L^4)$ and the corresponding addition or removal of photon variables). To this end, let $\psi(t_\beta, q_\beta, t_{>\beta}, q_{>\beta})$ be the solution, provided by Lemma 4.12 and 4.14, of the two-time evolution equations

$$i\frac{\partial \psi}{\partial t_\beta} = H_\beta \psi \tag{4.104}$$

and

$$i\frac{\partial \psi}{\partial t_{>\beta}} = H_{>\beta} \psi \tag{4.105}$$

(with $H_{>\beta}$ defined as the right-hand side of (4.99) with L replaced by $> \beta$ and Ψ replaced by ψ)

for $t_\beta \in [T_\beta - \varepsilon, T_\beta + \varepsilon]$ and $t_{>\beta} \in [T_\beta - \varepsilon, T_{\beta+1} + \varepsilon]$ with initial data

$$\psi(T_\beta - \varepsilon, q_\beta, T_\beta - \varepsilon, q_{\beta+1}, \ldots, q_L) \\ = \Psi(T_1, Q_1, \ldots, T_{\beta-1}, Q_{\beta-1}, T_\beta - \varepsilon, q_\beta, T_\beta - \varepsilon, q_{\beta+1}, \ldots, q_L) \quad (4.106)$$

for $q_\alpha \in N_{T_\alpha - T_\beta + 2\varepsilon}(Q_\alpha)$ for $\alpha \geq \beta$.

We now show that for $t_\beta = t_{>\beta}$ in the interval $[T_\beta - \varepsilon, T_\beta + \varepsilon]$,

$$\psi(t_\beta, q_\beta, t_\beta, q_{>\beta}) = \Psi(T_1, Q_1, \ldots, T_{\beta-1}, Q_{\beta-1}, t_\beta, q_\beta, \ldots, q_L). \quad (4.107)$$

The crucial point here is that the groups S_β and $S_{>\beta}$ do not interact in this time interval. Set

$$G_\beta = \prod_{\alpha=\beta+1}^{L} N_{T_\alpha - T_\beta + 2\varepsilon}(Q_\alpha). \quad (4.108)$$

For any $q \in N_{2\varepsilon}(Q_\beta)$ and $q' \in G_\beta$ it follows from Lemmas 4.15 and 4.19 and by virtue of the choice of ε, that $(q, T_\beta - \varepsilon; q', T_\beta - \varepsilon) \in \mathcal{C}_\delta^{(N,m)}$. By Lemma 4.16 the equation for $S_{\geq\beta}$ contains no interaction term between S_β and $S_{>\beta}$ for $T_\beta - \varepsilon \leq t_\beta \leq T_\beta + \varepsilon$. That is,

$$H_{\geq\beta} = H_\beta + H_{>\beta}. \quad (4.109)$$

When both time variables are set equal in ψ, then its Hamiltonian is the sum of the partial Hamiltonians, $H_\beta + H_{>\beta}$; thus, it is the same as for Ψ with fixed $T_1, Q_1, \ldots, T_{\beta-1}, Q_{\beta-1}$ and the same time variable t_β for $q_\beta, \ldots, q_L$. From the uniqueness statement of Lemma 4.14, we thus obtain (4.107).

By definition of Ψ, the value of $\Psi(\tilde{Q})$ depends on

$$\Psi(T_1, Q_1, \ldots, T_{\beta-1}, Q_{\beta-1}, \tilde{T}_\beta, \tilde{Q}_\beta, T_{\beta+1}, q_{>\beta}) \quad (4.110)$$

with $q_{>\beta} \in \prod_{\alpha=\beta+1}^{L} N_{T_\alpha - T_{\beta+1}}(Q_\alpha)$. The value of (4.110), in turn, is obtained from

$$\Psi(T_1, Q_1, \ldots, T_{\beta-1}, Q_{\beta-1}, \tilde{T}_\beta, \tilde{Q}_\beta, \tilde{T}_\beta, q_{>\beta}) \quad (4.111)$$

by solving the equation $i\partial\Psi/\partial t_{\beta+1} = H_{>\beta}\Psi$. It follows that (4.110) is equal to

$$\psi(\tilde{T}_\beta, \tilde{Q}_\beta, T_{\beta+1}, q_{>\beta}), \quad (4.112)$$

since the latter is obtained by solving $i\partial\psi/\partial t_{>\beta} = H_{>\beta}\psi$ from the same initial data (4.107). Now (4.112) can be obtained from $\psi(T_\beta - \varepsilon, q_\beta, T_{\beta+1}, q_{>\beta})$ by solving (4.104). According to Lemma 4.14, the evolution of Ψ from (4.110) to $\Psi(\tilde{Q}^4)$ commutes with solving (4.104); therefore, $\Psi(\tilde{Q}^4)$ can be obtained from

$$\Psi(T_1, Q_1, \ldots, T_{\beta-1}, Q_{\beta-1}, T_\beta - \varepsilon, q_\beta, T_{\beta+1}, Q_{\beta+1}, \ldots, T_L, Q_L) \tag{4.113}$$

by solving (4.104), which is what we wanted to show.

Since β was arbitrary, we have shown that Ψ satisfies the multi-time evolution equations at Q^4. Finally, to see that Ψ is smooth we note that on $\prod_{\alpha=1}^{L} D_\alpha^+$, Ψ simultaneously satisfies the multi-time evolution equations for all $\alpha = 1, \ldots, L$; Lemma 4.14 guarantees that the only solution is smooth. $\square$

Appendix A

Appendix

A.1 Calculations for the One-Time QFT Model

Adjointness of $a(\boldsymbol{y})$ and $a^\dagger(\boldsymbol{y})$. We regard only one sector in Fock space (the m-photon sector):

$$\begin{aligned}
\langle\psi^{(N,m)}|a^\dagger(\boldsymbol{y})\chi^{(N,m)}\rangle &= \int_{\mathbb{R}^{3N}} d\xi \int_{\mathbb{R}^{3m}} d\eta\ \psi^*(\xi,\eta)\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\delta(\boldsymbol{y}_k-\boldsymbol{y})\chi(\xi,\eta\backslash\boldsymbol{y}_k) \\
&= \int_{\mathbb{R}^{3N}} d\xi\ \sum_{k=1}^{m}\int_{\mathbb{R}^{3(m-1)}} d\eta' \int_{\mathbb{R}^3} d\boldsymbol{y}_k\ \psi^*(\xi,\eta'\cup\boldsymbol{y}_k)\frac{1}{\sqrt{m}}\delta(\boldsymbol{y}_k-\boldsymbol{y})\chi(\xi,\eta') \\
&= \int_{\mathbb{R}^{3N}} d\xi\ \sum_{k=1}^{m'+1}\int_{\mathbb{R}^{3m'}} d\eta'\ \chi(\xi,\eta')\frac{1}{\sqrt{m'+1}}\psi^*(\xi,\eta'\cup\boldsymbol{y}) \\
&= \int_{\mathbb{R}^{3N}} d\xi \int_{\mathbb{R}^{3m'}} d\eta'\ \chi(\xi,\eta')\sqrt{m'+1}\psi^*(\xi,\eta'\cup\boldsymbol{y}) \\
&= \langle a(\boldsymbol{y})\psi^{(N,m)}|\chi^{(N,m)}\rangle. \qquad \text{(A.1)}
\end{aligned}$$

Adjointness of $a_\varphi(\boldsymbol{x}_i)$ and $a^\dagger_\varphi(\boldsymbol{x}_i)$.

$$\begin{aligned}
\langle\psi|a^\dagger_\varphi(\boldsymbol{x}_i)\chi\rangle &= \int_{\mathbb{R}^{3N}} d\xi \ \sum_{m=1}^{\infty}\int_{\mathbb{R}^{3m}} d\eta \ \psi^*(\xi,\eta)\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\varphi(\boldsymbol{y}_k-\boldsymbol{x}_i)\chi(\xi,\eta\backslash\boldsymbol{y}_k)\\
&= \int_{\mathbb{R}^{3N}} d\xi \ \sum_{m=1}^{\infty}\sum_{k=1}^{m}\int_{\mathbb{R}^{3(m-1)}} d\eta' \int_{\mathbb{R}^3} d\boldsymbol{y}_k \ \frac{1}{\sqrt{m}}\psi^*(\xi,\eta'\cup\boldsymbol{y}_k)\varphi(\boldsymbol{y}_k-\boldsymbol{x}_i)\chi(\xi,\eta')\\
&= \int_{\mathbb{R}^{3N}} d\xi \ \sum_{m'=0}^{\infty}\sum_{k=1}^{m'+1}\int_{\mathbb{R}^{3m'}} d\eta' \ \frac{1}{\sqrt{m'+1}}\int_{\mathbb{R}^3} d\boldsymbol{y}_k\varphi(\boldsymbol{y}_k-\boldsymbol{x}_i)\psi^*(\xi,\eta'\cup\boldsymbol{y}_k)\chi(\xi,\eta')\\
&= \int_{\mathbb{R}^{3N}} d\xi \ \sum_{m'=0}^{\infty}\int_{\mathbb{R}^{3m'}} d\eta' \ \left(\sqrt{m'+1}\int_{\mathbb{R}^3} d\boldsymbol{y}\varphi^*(\boldsymbol{y}-\boldsymbol{x}_i)\psi(\xi,\eta'\cup\boldsymbol{y})\right)^*\chi(\xi,\eta')\\
&= \langle a_\varphi(\boldsymbol{x}_i)\psi|\chi\rangle. \qquad\text{(A.2)}
\end{aligned}$$

Commutation relations of the one-time creation and annihilation operators.

- Creation operators:

$$\begin{aligned}
a^\dagger_{\varphi_1}(\boldsymbol{x}_{i_1})(a^\dagger_{\varphi_2}(\boldsymbol{x}_{i_2})\psi)(\xi,\eta) &= \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\varphi_1(\boldsymbol{y}_k-\boldsymbol{x}_{i_1})(a^\dagger_{\varphi_2}(\boldsymbol{x}_{i_2})\psi)(\xi,\eta\backslash\boldsymbol{y}_k)\\
&= \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\varphi_1(\boldsymbol{y}_k-\boldsymbol{x}_{i_1})\frac{1}{\sqrt{m-1}}\sum_{\substack{k'=1\\k'\neq k}}^{m}\varphi_2(\boldsymbol{y}_{k'}-\boldsymbol{x}_{i_2})\psi(\xi,\eta\backslash\boldsymbol{y}_k\backslash\boldsymbol{y}_{k'})\\
&= \frac{1}{\sqrt{m(m-1)}}\sum_{\substack{k'=1\\k'\neq k}}^{m}\varphi_1(\boldsymbol{y}_k-\boldsymbol{x}_{i_1})\varphi_2(\boldsymbol{y}_{k'}-\boldsymbol{x}_{i_2})\psi(\xi,\eta\backslash\boldsymbol{y}_k\backslash\boldsymbol{y}_{k'}).
\end{aligned} \qquad\text{(A.3)}$$

$$a^\dagger_{\varphi_2}(\boldsymbol{x}_{i_2})(a^\dagger_{\varphi_1}(\boldsymbol{x}_{i_1})\psi)(\xi,\eta) = \frac{1}{\sqrt{m(m-1)}}\sum_{\substack{k'=1\\k'\neq k}}^{m}\varphi_2(\boldsymbol{y}_k-\boldsymbol{x}_{i_2})\varphi_1(\boldsymbol{y}_{k'}-\boldsymbol{x}_{i_1})\psi(\xi,\eta\backslash\boldsymbol{y}_k\backslash\boldsymbol{y}_{k'}). \qquad\text{(A.4)}$$

$$\Rightarrow [a^\dagger_{\varphi_1}(\boldsymbol{x}_{i_1}), a^\dagger_{\varphi_2}(\boldsymbol{x}_{i_2})] = 0. \qquad\text{(A.5)}$$

- Annihilation operators:

$$
\begin{aligned}
a_{\varphi_1}(\boldsymbol{x}_{i_1})(a_{\varphi_2}(\boldsymbol{x}_{i_2})\psi)(\xi,\eta) &= \sqrt{m+1}\int_{\mathbb{R}^3} d\boldsymbol{y}\varphi_1^*(\boldsymbol{y}-\boldsymbol{x}_{i_1})(a_{\varphi_2^*}(\boldsymbol{x}_{i_2})\psi)(\xi,\eta\cup\boldsymbol{y}) \\
&= \sqrt{m+1}\int_{\mathbb{R}^3} d\boldsymbol{y}\varphi_1^*(\boldsymbol{y}-\boldsymbol{x}_{i_1})\sqrt{m+2}\int_{\mathbb{R}^3} d\boldsymbol{y}'\varphi_2^*(\boldsymbol{y}'-\boldsymbol{x}_{i_2})\cdot \\
&\quad \cdot\psi(\xi,\eta\cup\boldsymbol{y}\cup\boldsymbol{y}') \\
&= \sqrt{(m+1)(m+2)}\int_{\mathbb{R}^3} d\boldsymbol{y}\int_{\mathbb{R}^3} d\boldsymbol{y}'\varphi_1^*(\boldsymbol{y}-\boldsymbol{x}_{i_1})\varphi_2^*(\boldsymbol{y}'-\boldsymbol{x}_{i_2})\cdot \\
&\quad \cdot\psi(\xi,\eta\cup\boldsymbol{y}\cup\boldsymbol{y}').
\end{aligned} \tag{A.6}
$$

$$
\begin{aligned}
a_{\varphi_2}(\boldsymbol{x}_{i_2})(a_{\varphi_1}(\boldsymbol{x}_{i_1})\psi)(\xi,\eta) &= \sqrt{(m+1)(m+2)}\int_{\mathbb{R}^3} d\boldsymbol{y}\int_{\mathbb{R}^3} d\boldsymbol{y}'\varphi_2^*(\boldsymbol{y}-\boldsymbol{x}_{i_2})\varphi_1^*(\boldsymbol{y}'-\boldsymbol{x}_{i_1})\cdot \\
&\quad \cdot\psi(\xi,\eta\cup\boldsymbol{y}\cup\boldsymbol{y}').
\end{aligned} \tag{A.7}
$$

$$
\Rightarrow [a_{\varphi_1}(\boldsymbol{x}_{i_1}), a_{\varphi_2}(\boldsymbol{x}_{i_2})] = 0. \tag{A.8}
$$

- Creation and annihilation operators:

$$
\begin{aligned}
a^\dagger_{\varphi_2}(\boldsymbol{x}_{i_2})(a_{\varphi_1}(\boldsymbol{x}_{i_1})\psi)(\xi,\eta) &= \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\varphi_2(\boldsymbol{y}_k-\boldsymbol{x}_{i_2})(a_{\varphi_1^*}(\boldsymbol{x}_{i_1})\psi)(\xi,\eta\backslash\boldsymbol{y}_k) \\
&= \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\varphi_2(\boldsymbol{y}_k-\boldsymbol{x}_{i_2})\sqrt{m}\int_{\mathbb{R}^3} d\boldsymbol{y}\varphi_1^*(\boldsymbol{y}-\boldsymbol{x}_{i_1})\psi(\xi,\eta\backslash\boldsymbol{y}_k\cup\boldsymbol{y}) \\
&= \sum_{k=1}^{m}\int_{\mathbb{R}^3} d\boldsymbol{y}\varphi_2(\boldsymbol{y}_k-\boldsymbol{x}_{i_2})\varphi_1^*(\boldsymbol{y}-\boldsymbol{x}_{i_1})\psi(\xi,\eta\backslash\boldsymbol{y}_k\cup\boldsymbol{y}).
\end{aligned} \tag{A.9}
$$

$$
\begin{aligned}
a_{\varphi_1}(\boldsymbol{x}_{i_1})(a^\dagger_{\varphi_2}(\boldsymbol{x}_{i_2})\psi)(\xi,\eta) &= \sqrt{m+1}\int_{\mathbb{R}^3} d\boldsymbol{y}\varphi_1^*(\boldsymbol{y}-\boldsymbol{x}_{i_1})(a^\dagger_{\varphi_2}(\boldsymbol{x}_{i_2})\psi)(\xi,\eta\cup\boldsymbol{y}) \\
\text{(notation: } \boldsymbol{y} := \boldsymbol{y}_{m+1}) \quad &= \sqrt{m+1}\int_{\mathbb{R}^3} d\boldsymbol{y}\varphi_1^*(\boldsymbol{y}-\boldsymbol{x}_{i_1})\frac{1}{\sqrt{m+1}}\sum_{k=1}^{m+1}\varphi_2(\boldsymbol{y}_k-\boldsymbol{x}_{i_2})\psi(\xi,\eta\cup\boldsymbol{y}\backslash\boldsymbol{y}_k) \\
&= \sum_{k=1}^{m+1}\int_{\mathbb{R}^3} d\boldsymbol{y}\varphi_2(\boldsymbol{y}_k-\boldsymbol{x}_{i_2})\varphi_1^*(\boldsymbol{y}-\boldsymbol{x}_{i_1})\psi(\xi,\eta\cup\boldsymbol{y}\backslash\boldsymbol{y}_k).
\end{aligned} \tag{A.10}
$$

$$
\Rightarrow ([a_{\varphi_1}(\boldsymbol{x}_{i_1}), a^\dagger_{\varphi_2}(\boldsymbol{x}_{i_2})]\psi)(\xi,\eta) = \int_{\mathbb{R}^3} d\boldsymbol{y}\varphi_1^*(\boldsymbol{y}-\boldsymbol{x}_{i_1})\varphi_2(\boldsymbol{y}-\boldsymbol{x}_{i_2})\psi(\xi,\eta). \tag{A.11}
$$

$$
\Rightarrow [a_{\varphi_1}(\boldsymbol{x}_{i_1}), a^\dagger_{\varphi_2}(\boldsymbol{x}_{i_2})] = \int_{\mathbb{R}^3} d\boldsymbol{y}\varphi_1^*(\boldsymbol{y}-\boldsymbol{x}_{i_1})\varphi_2(\boldsymbol{y}-\boldsymbol{x}_{i_2}). \tag{A.12}
$$

A.2 Calculations for the Multi-Time QFT Model on Space-Time

Calculation of K**.** In the following, in order to keep the notation as simple as possible, we work with unordered configurations η, i.e., the combinatorial factors $\frac{1}{m}$ and $\sqrt{m+1}$ vanish. Also, we set $\frac{1}{2m_x} = 1$. We abbreviate $H_i = H_{x_i}$, $H_j = H_{x_j}$. With Theorem 2.2 we have

$$\begin{aligned} K\psi(\xi^4, \eta^4) &= \left[i\frac{\partial}{\partial x_i^0} - H_{x_i}, i\frac{\partial}{\partial x_j^0} - H_{x_j} \right] \psi(\xi^4, \eta^4) \\ &= [H_i, H_j]\, \psi(\xi^4, \eta^4) + i\frac{\partial H_i}{\partial x_j^0}\psi(\xi^4, \eta^4) - i\frac{\partial H_j}{\partial x_i^0}\psi(\xi^4, \eta^4) \\ &= [H_i, H_j]\, \psi(\xi^4, \eta^4) \\ &= H_i H_j \psi(\xi^4, \eta^4) - H_j H_i \psi(\xi^4, \eta^4). \end{aligned} \tag{A.13}$$

We find

$$
\begin{aligned}
H_i H_j \psi(\xi^4,\eta^4) = H_i \Bigg(& -\Delta_{\boldsymbol{x}_j} \psi(\xi^4,\eta^4) + \sum_{y\in\eta^4} \tilde{\varphi}(y-x_j)\, \psi(\xi^4,\eta^4\backslash y) \\
& + \int_{\mathbb{R}^3} d\boldsymbol{y}\, \tilde{\varphi}^* \left((x_j^0,\boldsymbol{y})-x_j\right)\, \psi\left(\xi^4,\eta^4\cup(x_j^0,\boldsymbol{y})\right) \Bigg) \\
= \Bigg(& \Delta_{\boldsymbol{x}_i}\Delta_{\boldsymbol{x}_j} \psi(\xi^4,\eta^4) - \sum_{y\in\eta^4} \tilde{\varphi}(y-x_j)\, \Delta_{\boldsymbol{x}_i}\psi(\xi^4,\eta^4\backslash y) \\
& - \int_{\mathbb{R}^3} d\boldsymbol{y}\, \tilde{\varphi}^* \left((x_j^0,\boldsymbol{y})-x_j\right)\, \Delta_{\boldsymbol{x}_i}\psi\left(\xi^4,\eta^4\cup(x_j^0,\boldsymbol{y})\right) \Bigg) \\
& + \Bigg(-\sum_{y'\in\eta^4} \tilde{\varphi}(y'-x_i)\Delta_{\boldsymbol{x}_j}\psi(\xi^4,\eta^4\backslash y') \\
& + \sum_{y'\in\eta^4\backslash y} \tilde{\varphi}(y'-x_i) \sum_{y\in\eta^4} \tilde{\varphi}(y-x_j)\, \psi(\xi^4,\eta^4\backslash y\backslash y') \\
& + \sum_{y'\in\eta^4\cup(x_j^0,\boldsymbol{y})} \tilde{\varphi}(y'-x_i) \int_{\mathbb{R}^3} d\boldsymbol{y}\, \tilde{\varphi}^* \left((x_j^0,\boldsymbol{y})-x_j\right)\, \psi\left(\xi^4,\eta^4\cup(x_j^0,\boldsymbol{y})\backslash y'\right) \Bigg) \\
& + \Bigg(-\int_{\mathbb{R}^3} d\boldsymbol{y'}\, \tilde{\varphi}^* \left((x_i^0,\boldsymbol{y'})-x_i\right) \Delta_{\boldsymbol{x}_j}\psi(\xi^4,\eta^4\cup(x_i^0,\boldsymbol{y'})) \\
& + \int_{\mathbb{R}^3} d\boldsymbol{y'}\, \tilde{\varphi}^* \left((x_i^0,\boldsymbol{y'})-x_i\right) \sum_{y\in\eta^4} \tilde{\varphi}(y-x_j)\, \psi(\xi^4,\eta^4\backslash y\cup(x_i^0,\boldsymbol{y'})) \\
& + \int_{\mathbb{R}^3} d\boldsymbol{y'}\tilde{\varphi}^* \left((x_i^0,\boldsymbol{y'})-x_i\right) \int_{\mathbb{R}^3} d\boldsymbol{y}\tilde{\varphi}^* \left((x_j^0,\boldsymbol{y})-x_j\right) \psi\left(\xi^4,\eta^4\cup(x_j^0,\boldsymbol{y})\cup(x_i^0,\boldsymbol{y'})\right) \Bigg).
\end{aligned}
\tag{A.14}
$$

Therefore

$$
\begin{aligned}
&H_i H_j \psi(\xi^4,\eta^4) - H_j H_i \psi(\xi^4,\eta^4) \\
&= \sum_{y' \in \eta^4 \cup (x_j^0,\boldsymbol{y})} \int_{\mathbb{R}^3} d\boldsymbol{y}\, \tilde{\varphi}(y'-x_i)\tilde{\varphi}^*\left((x_j^0,\boldsymbol{y})-x_j\right)\, \psi\left(\xi^4, \eta^4 \cup (x_j^0,\boldsymbol{y})\backslash y'\right) \\
&\quad + \sum_{y\in\eta^4} \int_{\mathbb{R}^3} d\boldsymbol{y'}\, \tilde{\varphi}^*\left((x_i^0,\boldsymbol{y'})-x_i\right) \tilde{\varphi}(y-x_j)\, \psi(\xi^4, \eta^4\backslash y \cup (x_i^0,\boldsymbol{y'})) \\
&\quad - \sum_{y \in \eta^4 \cup (x_i^0,\boldsymbol{y'})} \int_{\mathbb{R}^3} d\boldsymbol{y'}\, \tilde{\varphi}(y-x_j)\tilde{\varphi}^*\left((x_i^0,\boldsymbol{y'})-x_i\right)\, \psi\left(\xi^4, \eta^4 \cup (x_i^0,\boldsymbol{y'})\backslash y\right) \\
&\quad - \sum_{y'\in\eta^4} \int_{\mathbb{R}^3} d\boldsymbol{y}\, \tilde{\varphi}^*\left((x_j^0,\boldsymbol{y})-x_j\right) \tilde{\varphi}(y'-x_i)\, \psi(\xi^4, \eta^4\backslash y' \cup (x_j^0,\boldsymbol{y})) \\
&= \int_{\mathbb{R}^3} d\boldsymbol{y}\, \tilde{\varphi}((x_j^0,\boldsymbol{y})-x_i)\tilde{\varphi}^*\left((x_j^0,\boldsymbol{y})-x_j\right)\, \psi\left(\xi^4,\eta^4\right) \\
&\quad - \int_{\mathbb{R}^3} d\boldsymbol{y}\, \tilde{\varphi}((x_i^0,\boldsymbol{y})-x_j)\tilde{\varphi}^*\left((x_i^0,\boldsymbol{y})-x_i\right)\, \psi\left(\xi^4,\eta^4\right) \\
&= \int_{\mathbb{R}^3} d\boldsymbol{y} \Bigg(\tilde{\varphi}((x_j^0,\boldsymbol{y})-x_i)\varphi^*\left(\boldsymbol{y}-\boldsymbol{x}_j\right) - \tilde{\varphi}((x_i^0,\boldsymbol{y})-x_j)\varphi^*\left(\boldsymbol{y}-\boldsymbol{x}_i\right) \Bigg) \psi\left(\xi^4,\eta^4\right) \\
&= \Bigg(\langle T_{\boldsymbol{x}_j}\varphi | U_{x_j^0-x_i^0} T_{\boldsymbol{x}_i}\varphi\rangle - \langle T_{\boldsymbol{x}_i}\varphi | U_{x_i^0-x_j^0} T_{\boldsymbol{x}_j}\varphi\rangle \Bigg) \psi\left(\xi^4,\eta^4\right) \\
&= \Bigg(\langle T_{\boldsymbol{x}_j-\boldsymbol{x}_i}\varphi | U_{x_j^0-x_i^0}\varphi\rangle - \langle U_{x_j^0-x_i^0}\varphi | T_{\boldsymbol{x}_j-\boldsymbol{x}_i}\varphi\rangle \Bigg) \psi\left(\xi^4,\eta^4\right) \\
&= \Bigg(\langle T_{\boldsymbol{x}_j-\boldsymbol{x}_i}\varphi | U_{x_j^0-x_i^0}\varphi\rangle - \left(\langle T_{\boldsymbol{x}_j-\boldsymbol{x}_i}\varphi | U_{x_j^0-x_i^0}\varphi\rangle \right)^* \Bigg) \psi\left(\xi^4,\eta^4\right) \\
&= 2i\,\mathrm{Im} \Bigg(\langle T_{\boldsymbol{x}_j-\boldsymbol{x}_i}\varphi | U_{x_j^0-x_i^0}\varphi\rangle \Bigg) \psi\left(\xi^4,\eta^4\right) \\
&= 2i\,\mathrm{Im} \int_{\mathbb{R}^3} d\boldsymbol{y}\, \tilde{\varphi}^*(0,\boldsymbol{y}+\boldsymbol{x}_i-\boldsymbol{x}_j)\, \tilde{\varphi}(x_j^0-x_i^0,\boldsymbol{y})\psi(\xi^4,\eta^4) \qquad \text{(A.15)}
\end{aligned}
$$

where $T_{\boldsymbol{x}}$ is the translation operator $T_{\boldsymbol{x}}\varphi(\boldsymbol{y}) = \varphi(\boldsymbol{y}-\boldsymbol{x})$, and U_{x^0} the unitary time-evolution operator of the free Schrödinger equation.

Adjointness of $H^a_{x_i}$ and $H^c_{x_i}$. We set $m' = m - 1$.

$$
\begin{aligned}
&\langle \psi | H^c_{x_i} \chi \rangle_{\mathcal{F}} \\
&= \int_{\mathbb{R}^{3N}} d\xi \sum_{m=0}^{\infty} \int_{\mathbb{R}^{3m}} d\eta \; \psi^*(\xi,\eta)(H^c_{x_i}\chi)(\xi,\eta) \\
&= \int_{\mathbb{R}^{3N}} d\xi \sum_{m=1}^{\infty} \int_{\mathbb{R}^{3m}} d\eta \; \psi^*(\xi,\eta) \frac{1}{\sqrt{m}} \sum_{k=1}^{m} \tilde{\varphi}(t_{y_k} - t_{x_i}, \boldsymbol{y}_k - \boldsymbol{x}_i) \cdot \\
&\quad \cdot \left[exp\left(\sum_{c=k}^{m-1} \frac{i}{2m_y} \Delta_c (t_{y_{c+1}} - t_{y_c}) \right) \chi^{(m-1)} \right] (\xi, \eta \backslash \boldsymbol{y}_k) \\
&= \int_{\mathbb{R}^{3N}} d\xi \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \sum_{k=1}^{m} \int_{\mathbb{R}^{3(m-1)}} d(\eta \backslash \boldsymbol{y}_k) \; \int_{\mathbb{R}^3} d\boldsymbol{y}_k \; \psi^*(\xi,\eta) \tilde{\varphi}(t_{y_k} - t_{x_i}, \boldsymbol{y}_k - \boldsymbol{x}_i) \cdot \\
&\quad \cdot \left[exp\left(\sum_{c=k}^{m-1} \frac{i}{2m_y} \Delta_c (t_{y_{c+1}} - t_{y_c}) \right) \chi^{(m-1)} \right] (\xi, \eta \backslash \boldsymbol{y}_k) \\
&= \int_{\mathbb{R}^{3N}} d\xi \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \sum_{k=1}^{m} \int_{\mathbb{R}^{3(m-1)}} d\eta' \int_{\mathbb{R}^3} d\boldsymbol{y}_k \left[exp\left(\sum_{c=k}^{m-1} \frac{-i}{2m_y} \Delta_c (t_{y_{c+1}} - t_{y_c}) \right) \psi^{*(m)} \right] (\xi, \eta' \cup \boldsymbol{y}_k) \cdot \\
&\quad \cdot \tilde{\varphi}(t_{y_k} - t_{x_i}, \boldsymbol{y}_k - \boldsymbol{x}_i) \left[exp\left(\sum_{c=k}^{m-1} \frac{i}{2m_y} \Delta_c (t_{y_{c+1}} - t_{y_c}) \right) \chi^{(m-1)} \right] (\xi, \eta') \\
&= \int_{\mathbb{R}^{3N}} d\xi \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \sum_{k=1}^{m} \int_{\mathbb{R}^{3(m-1)}} d\eta' \; \int_{\mathbb{R}^3} d\boldsymbol{y}_{m+1} \; \psi^*(\xi, \eta' \cup \boldsymbol{y}_{m+1}) \tilde{\varphi}(t_{y_{m+1}} - t_{x_i}, \boldsymbol{y}_{m+1} - \boldsymbol{x}_i) \chi(\xi, \eta') \\
&= \int_{\mathbb{R}^{3N}} d\xi \sum_{m'=0}^{\infty} \int_{\mathbb{R}^{3m'}} d\eta' \; \left[\frac{m'+1}{\sqrt{m'+1}} \int_{\mathbb{R}^3} d\boldsymbol{y}_{m'+1} \; \tilde{\varphi}^*(t_{y_{m'+1}} - t_{x_i}, \boldsymbol{y}_{m'+1} - \boldsymbol{x}_i) \psi(\xi, \eta' \cup \boldsymbol{y}_{m'+1}) \right]^* \cdot \\
&\quad \cdot \chi(\xi, \eta') \\
&= \langle H^a_{x_i} \psi | \chi \rangle_{\mathcal{F}}
\end{aligned}
\tag{A.16}
$$

Commutation relations of the multi-time Hamiltonians.

$$
\begin{aligned}
\left[H^{free}_{x_i}, H^{free}_{x_j} \right] &= \left[\frac{1}{2m_x} \Delta_{x_i}, \frac{1}{2m_x} \Delta_{x_j} \right] = 0 \\
\left[H^{free}_{y_k}, H^{free}_{y_j} \right] &= \left[\frac{1}{2m_y} \Delta_{y_k}, \frac{1}{2m_y} \Delta_{y_j} \right] = 0 \\
\left[H^{free}_{x_i}, H^{free}_{y_k} \right] &= \left[\frac{1}{2m_x} \Delta_{x_i}, \frac{1}{2m_y} \Delta_{y_k} \right] = 0
\end{aligned}
\tag{A.17}
$$

$$
\begin{aligned}
&\left(\left[H^{free}_{x_i}, H^{c}_{x_j}\right]\psi\right)(\xi,\eta)\\
&= -\frac{1}{2m_x}\frac{1}{\sqrt{m}}\Delta_{\boldsymbol{x}_i}\left(\sum_{k'=1}^{m}\tilde{\varphi}(y_{k'}-x_j)\left[exp\left(\sum_{c=k}^{m-1}\frac{i}{2m_y}\Delta_c(t_{y_{c+1}}-t_{y_c})\right)\psi^{(m-1)}\right](\xi,\eta\backslash\boldsymbol{y}_k)\right)\\
&\quad+\frac{1}{2m_x}\frac{1}{\sqrt{m}}\sum_{k'=1}^{m}\tilde{\varphi}(y_{k'}-x_j)\Delta_{x_i}\left[exp\left(\sum_{c=k}^{m-1}\frac{i}{2m_y}\Delta_c(t_{y_{c+1}}-t_{y_c})\right)\psi^{(m-1)}\right](\xi,\eta\backslash\boldsymbol{y}_k)\\
&= 0 \quad (\text{for } i\neq j)
\end{aligned}
\tag{A.18}
$$

$$
\begin{aligned}
\left(\left[H^{free}_{x_i}, H^{a}_{x_j}\right]\psi\right)(\xi,\eta) &= -\frac{1}{2m_x}\Delta_{\boldsymbol{x}_i}\left(\sqrt{m+1}\int_{\mathbb{R}^3}d\boldsymbol{y}_{m+1}\tilde{\varphi}^*(y_{m+1}-x_j)\psi(\xi,\eta\cup\boldsymbol{y}_{m+1})\right)\\
&\quad-\sqrt{m+1}\int_{\mathbb{R}^3}d\boldsymbol{y}_{m+1}\tilde{\varphi}^*(y_{m+1}-x_j)\left(-\frac{1}{2m_x}\Delta_{\boldsymbol{x}_i}\psi\right)(\xi,\eta\cup\boldsymbol{y}_{m+1})\\
&= 0 \quad (\text{for } i\neq j)
\end{aligned}
\tag{A.19}
$$

$$
\begin{aligned}
&\left(\left[H^{c}_{x_i}, H^{c}_{x_j}\right]\psi\right)(\xi,\eta)\\
&= \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\tilde{\varphi}(t_{y_k}-t_{x_i},\boldsymbol{y}_k-\boldsymbol{x}_i)\frac{1}{\sqrt{m-1}}\sum_{k'=1}^{m-1}\tilde{\varphi}(t_{y_{k'}}-t_{x_j},\boldsymbol{y}_{k'}-\boldsymbol{x}_j)\cdot\\
&\quad\cdot\left[exp\left(\sum_{c'=k'}^{m-2}\frac{i}{2m_y}\Delta_{c'}(t_{y_{c'+1}}-t_{y_{c'}})\right)exp\left(\sum_{c=k}^{m-1}\frac{i}{2m_y}\Delta_c(t_{y_{c+1}}-t_{y_c})\right)\psi^{(m-2)}\right](\xi,\eta\backslash\boldsymbol{y}_k\backslash\boldsymbol{y}_{k'})\\
&\quad-(i\leftrightarrow j)\\
&= 0
\end{aligned}
\tag{A.20}
$$

$$
\begin{aligned}
&\left(\left[H^{a}_{x_i}, H^{a}_{x_j}\right]\psi\right)(\xi,\eta)\\
&= \sqrt{m+1}\int_{\mathbb{R}^3}d\boldsymbol{y}_{m+1}\tilde{\varphi}^*(y_{m+1}-x_i)\left(H^{a}_{x_j}\psi\right)(\xi,\eta\cup\boldsymbol{y}_{m+1})\\
&\quad-\sqrt{m+1}\int_{\mathbb{R}^3}d\boldsymbol{y}_{m+1}\tilde{\varphi}^*(y_{m+1}-x_j)\left(H^{a}_{x_i}\psi\right)(\xi,\eta\cup\boldsymbol{y}_{m+1})\\
&= \sqrt{m+1}\int_{\mathbb{R}^3}d\boldsymbol{y}_{m+1}\tilde{\varphi}^*(y_{m+1}-x_i)\sqrt{m+2}\int_{\mathbb{R}^3}d\boldsymbol{y}_{m+2}\tilde{\varphi}^*(y_{m+2}-x_j)\psi(\xi,\eta\cup\boldsymbol{y}_{m+1}\cup\boldsymbol{y}_{m+2})\\
&\quad-\sqrt{m+1}\int_{\mathbb{R}^3}d\boldsymbol{y}_{m+1}\tilde{\varphi}^*(y_{m+1}-x_j)\sqrt{m+2}\int_{\mathbb{R}^3}d\boldsymbol{y}_{m+2}\tilde{\varphi}^*(y_{m+2}-x_i)\psi(\xi,\eta\cup\boldsymbol{y}_{m+1}\cup\boldsymbol{y}_{m+2})\\
&= 0
\end{aligned}
\tag{A.21}
$$

$$
\begin{aligned}
&\left(\left[H^a_{x_i}, H^c_{x_j}\right]\psi\right)(\xi,\eta)\\
&= \sqrt{m+1}\int_{\mathbb{R}^3} d\boldsymbol{y}_{m+1}\tilde{\varphi}^*(y_{m+1}-x_i)\left(H^c_{x_j}\psi\right)(\xi,\eta\cup\boldsymbol{y}_{m+1})\\
&\quad -\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\tilde{\varphi}(y_k-x_j)\left(H^a_{x_i}\left[exp\left(\sum_{c=k}^{m-1}\frac{i}{2m_y}\Delta_c(t_{y_{c+1}}-t_{y_c})\right)\psi^{(m-1)}\right]\right)(\xi,\eta\backslash\boldsymbol{y}_k)\\
&= \sqrt{m+1}\int_{\mathbb{R}^3} d\boldsymbol{y}_{m+1}\tilde{\varphi}^*(y_{m+1}-x_i)\frac{1}{\sqrt{m+1}}\sum_{k=1}^{m+1}\tilde{\varphi}(y_k-x_j)\cdot\\
&\qquad\cdot\left[exp\left(\sum_{c=k}^{m}\frac{i}{2m_y}\Delta_c(t_{y_{c+1}}-t_{y_c})\right)\psi^{(m)}\right](\xi,\eta\cup\boldsymbol{y}_{m+1}\backslash\boldsymbol{y}_k)\\
&\quad -\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\tilde{\varphi}(y_k-x_j)\sqrt{m}\int_{\mathbb{R}^3} d\boldsymbol{y}_{m+1}\tilde{\varphi}^*(y_{m+1}-x_i)\cdot\\
&\qquad\cdot\left[exp\left(\sum_{c=k}^{m-1}\frac{i}{2m_y}\Delta_c(t_{y_{c+1}}-t_{y_c})\right)\psi^{(m-1)}\right](\xi,\eta\backslash\boldsymbol{y}_k\cup\boldsymbol{y}_{m+1})\\
&= \int_{\mathbb{R}^3} d\boldsymbol{y}\tilde{\varphi}^*(y-x_i)\tilde{\varphi}(y-x_j)\psi(\xi,\eta)
\end{aligned}
\tag{A.22}
$$

$$
\begin{aligned}
&\left(\left[H^{free}_{y_k}, H^c_{x_i}\right]\psi\right)(\xi,\eta)\\
&= -\frac{1}{2m_y}\frac{1}{\sqrt{m}}\Delta_{y_k}\left(\sum_{k'=1}^{m}\tilde{\varphi}(y_{k'}-x_i)\left[exp\left(\sum_{c=k}^{m-1}\frac{i}{2m_y}\Delta_c(t_{y_{c+1}}-t_{y_c})\right)\psi^{(m-1)}\right](\xi,\eta\backslash\boldsymbol{y}_{k'})\right)\\
&\quad +\frac{1}{2m_y}\frac{1}{\sqrt{m}}\sum_{k'=1}^{m}\tilde{\varphi}(y_{k'}-x_i)\left[exp\left(\sum_{c=k}^{m-1}\frac{i}{2m_y}\Delta_c(t_{y_{c+1}}-t_{y_c})\right)\Delta_k\psi^{(m-1)}\right](\xi,\eta\backslash\boldsymbol{y}_{k'})\\
&= -\frac{1}{2m_y}\frac{1}{\sqrt{m}}\sum_{k'=1}^{k-1}\tilde{\varphi}(y_{k'}-x_i)\left[exp\left(\sum_{c=k}^{m-1}\frac{i}{2m_y}\Delta_c(t_{y_{c+1}}-t_{y_c})\right)(\Delta_{k-1}-\Delta_k)\psi^{(m-1)}\right](\xi,\eta\backslash\boldsymbol{y}_{k'})\\
&\quad +\frac{1}{2m_y}\frac{1}{\sqrt{m}}\tilde{\varphi}(y_k-x_i)\left[exp\left(\sum_{c=k}^{m-1}\frac{i}{2m_y}\Delta_c(t_{y_{c+1}}-t_{y_c})\right)\Delta_k\psi^{(m-1)}\right](\xi,\eta\backslash\boldsymbol{y}_k)\\
&\quad + -\frac{1}{2m_y}\frac{1}{\sqrt{m}}\left(\Delta_{y_k}\tilde{\varphi}(y_k-x_i)\right)\left[exp\left(\sum_{c=k}^{m-1}\frac{i}{2m_y}\Delta_c(t_{y_{c+1}}-t_{y_c})\right)\psi^{(m-1)}\right](\xi,\eta\backslash\boldsymbol{y}_k)
\end{aligned}
\tag{A.23}
$$

$$
\begin{aligned}
&\left(\left[H^{free}_{y_k}, H^a_{x_i}\right]\psi\right)(\xi,\eta)\\
&= -\frac{1}{2m_y}\sqrt{m+1}\Delta_{\boldsymbol{y}_k}\left(\int_{\mathbb{R}^3} d\boldsymbol{y}_{m+1}\tilde{\varphi}^*(y_{m+1}-x_i)\psi(\xi,\eta\cup\boldsymbol{y}_{m+1})\right)\\
&\quad +\frac{1}{2m_y}\sqrt{m+1}\int_{\mathbb{R}^3} d\boldsymbol{y}_{m+1}\tilde{\varphi}^*(y_{m+1}-x_i)\left(\Delta_{\boldsymbol{y}_k}\psi\right)(\xi,\eta\cup\boldsymbol{y}_{m+1})\\
&= 0 \qquad \text{for } k\neq m+1
\end{aligned}
\tag{A.24}
$$

We also find

$$
\begin{aligned}
&\left(\left(i\frac{\partial H_{x_i}}{\partial t_{y_k}}\right)\psi\right)(\xi,\eta)\\
&=\left(\left(i\frac{\partial H^{free}_{x_i}}{\partial t_{y_k}}+i\frac{\partial H^{c}_{x_i}}{\partial t_{y_k}}+i\frac{\partial H^{a}_{x_i}}{\partial t_{y_k}}\right)\psi\right)(\xi,\eta)\\
&=i\frac{1}{\sqrt{m}}\sum_{k'=1}^{m}\frac{\partial}{\partial t_{y_k}}\left[\tilde{\varphi}(t_{y_{k'}}-t_{x_i},\boldsymbol{y}_{k'}-\boldsymbol{x}_i)\left[exp\left(\sum_{c=k}^{m-1}\frac{i}{2m_y}\Delta_c(t_{y_{c+1}}-t_{y_c})\right)\psi^{(m-1)}\right]\right](\xi,\eta\backslash\boldsymbol{y}_{k'})\\
&=i\frac{1}{2m_y}\frac{1}{\sqrt{m}}\sum_{k'=1}^{k-1}\tilde{\varphi}(y_{k'}-x_i)\left[exp\left(\sum_{c=k}^{m-1}\frac{i}{2m_y}\Delta_c(t_{y_{c+1}}-t_{y_c})\right)(i\Delta_{k-1}-i\Delta_k)\psi^{(m-1)}\right](\xi,\eta\backslash\boldsymbol{y}_{k'})\\
&\quad+i\frac{1}{2m_y}\frac{1}{\sqrt{m}}\tilde{\varphi}(y_k-x_i)\left[exp\left(\sum_{c=k}^{m-1}\frac{i}{2m_y}\Delta_c(t_{y_{c+1}}-t_{y_c})\right)(-i\Delta_k)\psi^{(m-1)}\right](\xi,\eta\backslash\boldsymbol{y}_k)\\
&\quad+i\frac{1}{\sqrt{m}}\Big(\frac{\partial}{\partial t_{y_k}}\tilde{\varphi}(y_k-x_i)\Big)\left[exp\left(\sum_{c=k}^{m-1}\frac{i}{2m_y}\Delta_c(t_{y_{c+1}}-t_{y_c})\right)\psi^{(m-1)}\right](\xi,\eta\backslash\boldsymbol{y}_k)
\end{aligned}
\tag{A.25}
$$

From that and the fact that $\tilde{\varphi}$ satisfies the free Schrödinger equation, the results (3.39) follow.

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