PROBLEMS ON THE GEOMETRIC FUNCTION THEORY IN SEVERAL COMPLEX VARIABLES AND COMPLEX GEOMETRY

BY YUAN YUAN

A dissertation submitted to the

Graduate School—New Brunswick

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

Prof. Xiaojun Huang(advisor) and Prof. Jian Song(co-advisor)

and approved by

New Brunswick, New Jersey

October, 2010

ABSTRACT OF THE DISSERTATION

Problems on the geometric function theory in several complex variables and complex geometry

by Yuan Yuan

Dissertation Director: Prof. Xiaojun Huang(advisor) and Prof. Jian Song(co-advisor)

The thesis consists of two parts. In the first part, we study the rigidity for the local holomorphic isometric embeddings. On the one hand, we prove the total geodesy for the local holomorphic conformal embedding from the unit ball of complex dimension at least 2 to the product of unit balls and hence the rigidity for the local holomorphic isometry is the natural corollary. Before obtaining the total geodesy, the algebraic extension theorem is derived following the idea in [MN] by considering the sphere bundle of the source and target domains. When conformal factors are not constant, we twist the sphere bundle to gain the pseudoconvexity. Then the algebraicity follows from the algebraicity theorem of Huang in the CR geometry. Different from the argument in the earlier works, the total geodesy of each factor does not directly follow from the properness because the codimension is arbitrary. By analyzing the real analytic subvariety carefully, we conclude that the factor is either a proper holomorphic rational map or a constant map. Lastly the total geodesy follows from a linearity criterion of Huang. On the other hand, we also derive the total geodesy for the local holomorphic isometries from the projective space to the product of projective spaces.

In the second part, we give a proof for the convergence of a modified Kähler-Ricci flow. The flow is defined by Zhang on Kähler manifolds while the Kähler class along the evolution is varying. When the limit cohomology class is semi-positive, big and integer, the convergence of the flow is conjectured by Zhang and we confirm it by using the monotonicity of some energy functional. When the limit class is Kähler, the convergence is proven by Zhang and we give an alternative proof by also using the energy functional. As a corollary, the convergence provides the solution to the degenerate Monge-Ampère equation on the Calabi-Yau manifold. Meanwhile we take the opportunity to describe the Kähler-Ricci flow on singular varieties.

Acknowledgements

I would like to express my most sincere gratitude to my advisor Professor Xiaojun Huang, for everything he has done to help me grow up in my academic life. I must mention that from the very beginning when I was totally lost being a junior graduate student, he has kept pointing out to me the right direction. During my Ph.D study, he provided me numerous interesting problems and shared with me his insight and new ideas to these problems without any reservation. He encouraged me when I was frustrated and gave me the pressure when I did not do my work well. Last but not least, the thesis would not be possible without all his help.

I also would like to express the sincere acknowledgements to my co-advisor Professor Jian Song. Without the numerous discussions and constant support, the second part of the thesis can not be accomplished.

I would like to thank Professor N. Mok for his very useful suggestions and constant support. I would like to thank Professor D.H. Phong for his very helpful discussions and constant encouragement. I benefit greatly from Professor Phong's seminar in Columbia University. I also would like to thank Professor G. Tian for the stimulating discussions.

I am indebted to Professor S. Berhanu, Professor S. Chanillo, Professor H. Jacobowitz for their constant support.

I am indebted to Professor S. Fu, Professor S. Greenfield, Professor Z. Han, Professor S. Li, Professor Y. Li, Professor F. Luo, Professor G. Mendoza, Professor X. Rong and Professor B. Shiffman for their encouragement and help during my graduate study. I am also indebted to C. Li, S. Ng, V. Tosatti, T. Yang, W. Yin, X. Yuan, Y. Zhang, Z. Zhang and many other people for their useful discussions.

I would like to thank all the faculty, staff and my fellow graduate students in the department of Mathematics at Rutgers for their care and help.

I am very grateful to my parents, grandparents and other family members for their love and support. I would like to express my gratitude to Lin Chen for everything she has brought to me.

Dedication

This thesis is dedicated to my family.

Table of Contents

Chapter 1

Introduction

1.1 Rigidity for local holomorphic isometric embeddings

Various rigidity problems in Several Complex Variables and Complex Geometry are among the basic problems in the subject. The understanding of these problems is not only important for the subject itself, but also plays a critical role in the application to many other research fields of mathematics.

The study of the topic will be focused on the rigidity problem for the local holomorphic isometric embeddings between Hermitian symmetric spaces. This type of problems is initialled in a celebrated paper of Calabi, who first studied the global extension and Borel type rigidity of a local holomorphic isometric embedding between Kähler manifolds with real analytic metrics [Ca]. Afterwards, there appeared quite a few papers along these lines of research (see [Um], for instance). In 2004, motivated from the problems in Arithmetic Algebraic Geometry, Clozel-Ullmo [CU] took up the problem again by considering the rigidity of local holomorphic isometries from a bounded symmetric domain into its product with respect to their Bergman metrics. More precisely, by reducing the problem to the rigidity problem for local holomorphic isometries, they proved the algebraic correspondence between such quotients of bounded symmetric domains preserving Bergman metric has to be modular correspondence in the case of (i) unit disc in the complex plane and (ii) bounded symmetric domains of rank at least 2. More recently, Mok carried out a systematic study of this problem in a very general setting. Many deep results have been obtained by Mok and Mok-Ng. (See [Mo1] [Mo3] [MN] [Ng1-3] and the references therein). For instance, Mok proved the total geodesy for the local holomorphic isometry between bounded symmetric domains D and Ω when either (i) $rank(D) \geq 2$ or (2) $D = \mathbb{B}^n$ and $\Omega = (\mathbb{B}^n)^p$ for $n \geq 2$.

In the second chapter, we prove the rigidity for local holomorphic conformal embeddings between the unit ball and the product of unit balls and as a corollary, obtain the rigidity for local holomorphic isometric embeddings between such domains. This theorem appears in [YZ]. (See chapter 2 for relevant notations)

Theorem 1.1.1 *Let*

$$F = (F_1, \ldots, F_m) : (U \subset \mathbb{B}^n, \lambda(z,\bar{z})ds_n^2) \to (\mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m}, \oplus_{j=1}^m \lambda_j(z,\bar{z})ds_{N_j}^2) \quad (1.1.1)$$

be a local holomorphic conformal embedding in the sense that

$$\lambda(z,\bar{z})ds_n^2 = \sum_{j=1}^m \lambda_j(z,\bar{z}) F_j^*(ds_{N_j}^2).$$

Assume that $n \geq 2$ and $\lambda(z,\bar{z}), \lambda_j(z,\bar{z})$ are positive Nash algebraic functions. We then have, for each j with $1 \leq j \leq m$, that either F_j is a constant map or F_j extends to a totally geodesic holomorphic embedding from $(\mathbb{B}^n, ds_n^2)$ into $(\mathbb{B}^{N_j}, ds_{N_j}^2)$.

When $\lambda(z,\bar{z}), \lambda_j(z,\bar{z})$ are constants, the result is due to Calabi when $m = 1$ [Ca], due to Mok [Mo1] [Mo3] when $N_1 = \cdots = N_m$, and due to Ng [Ng1] [Ng3] when $m = 2$ and $N_1, N_2 < 2n$. The novelty of our result is that the factor in the target manifold can be any $\mathbb{B}^N$.

Our proof of Theorem 1.1.1 is based on the algebraic extension theorem. In the case of the conformal embedding, it does not follow from [Mo2] directly. Following [MN], we try to use the unit sphere bundle of the corresponding domain and reduce the algebraicity to the mapping problem in CR geometry. However, since the pseudoconvexity is totally lost, we need to twist the sphere bundle to gain the positivity.

To obtain the total geodesy, different from the case considered in [Mo1] [Ng1], the properness of a factor of F does not imply the linearity of that factor, for the classical linearity theorem does not hold anymore for proper rational mappings from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N > 2n-2$. To make up this, we carried out a careful study on the precise boundary behavior of the Bergman metric pulled back by proper holomorphic maps. It turns out that the difference of the pull-back Bergman metric of the target space with the source Bergman metric is smooth up to the boundary and the values on the boundary are closely related to the boundary CR invariants inherited from the map. Another ingredient in our argument is a careful analysis for the (multiple-valued) holomorphic continuation of the local holomorphic map. The key for this part is to analyze the real analytic subvariety where the pull-back of the Bergman metric blows up. In our proof of Theorem 1.1.1, a major step is to prove that a non-constant component F_j of F must be proper from $\mathbb{B}^n$ into $\mathbb{B}^{N_j}$, using the multiple-valued holomorphic continuation technique. This then reduces the proof of Theorem 1.1.1 to the case when all components are proper. Unfortunately, due to the non-constancy for the conformal factors $\lambda_j(z,\bar{z})$ and $\lambda(z,\bar{z})$, it is not immediate that each component must also be conformal (and thus must have conformal factor constant) with respect to the normalized Bergman metric. However, we observe that the blowing-up rate for the Bergman metric of $\mathbb{B}^n$ with $n \geq 2$ in the complex normal direction is twice of that along the complex tangential direction, when approaching the boundary. From this, we will be able to derive an equation connecting the CR invariants associated from the map at the boundary of the ball. Lastly, a linearity criterion of Huang in [Hu1] can be applied to simultaneously conclude the linearity of all components.

Notice that Theorem 1.1.1 fails when $n = 1$ by Mok [Mo1] and later by Ng [Ng1]. By the previous work of Mok, we have a good understanding for the case when the rank of source domain is at least 2. Hence, our result provides a fairly complete picture for such a problem for the non-compact model in the rank one case.

When manifolds are compact Hermitian symmetric spaces, the rigidity property is unknown even in the case of projective spaces. The following question is the nature starting problem of the compact case:

Question 1.1.1 *Let*

$$F : (U \subset \mathbb{P}^n, \omega_n) \to (\mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_p}, \oplus_{l=1}^{p} \lambda_l \omega_{N_l})$$

be a local holomorphic isometric imbedding with respect to Fubini-Study metrics. How to classify F?

The global extension of the map F above has been established by Calabi and Mok [Ca] [Mo3] for some partial cases. Also, we know that the linearity for F fails even when

$p = 1$, for the Veronese imbedding from $(\mathbb{P}^n, 2\omega_n)$ into $(\mathbb{P}^{\frac{n(n+3)}{2}}, \omega_{\frac{n(n+3)}{2}})$ is an isometry and of degree 2. However, it is generally believed that a non-constant component of F is equivalent to the Veronese imbedding of some integer isometric constant.

In chapter 2, we study such problems. More precisely, we prove the following theorem. (See chapter 2 for relevant notations.)

Theorem 1.1.2 *Let $\lambda_l \in \mathbb{Q}$ and let*

$$F = (F_1, \cdots, F_p) : (U \subset \mathbb{P}^n, \omega_n) \to (\mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_p}, \oplus_{l=1}^p \lambda_l \omega_{N_l})$$

be a local holomorphic isometric imbedding with respect to Fubini-Study metrics, i.e.

$$\omega_n = \sum_l \lambda_l F_l^* \omega_{N_l}.$$

Let $\{\kappa_1, \cdots, \kappa_p\} \in (\mathbb{Z}^+)^p$ satisfy

(i) $N_l \geq M_{n,\kappa_l}$ *for* $1 \leq l \leq p$*,*

(ii) $\sum_{l=1}^p \kappa_l \lambda_l = 1$*.*

Then the possibility of $\{\kappa_1, \cdots, \kappa_p\}$ gives the classification of $F = (F_1, \cdots, F_p)$. In particular, F_l is equivalent to Veronese embedding V_{n,κ_l} with isometric constant κ_l modular Isom$(\mathbb{P}^{N_l}, \omega_{N_l})$ and F_l is a constant map if $\kappa_l = 0$.

1.2 The Kähler-Ricci flow

The Ricci flow was introduced by Richard Hamilton [H] on Riemannian manifolds to study the deformation of metrics. Its analogue in Kähler geometry, the Käher-Ricci flow, has been intensively studied in recent years. It turns out to be a powerful method for studying canonical metrics on Kähler manifolds. (See, for instance, the papers [Cao] [CT] [Pe] [PS] [PSSW] [ST1] [ST2] [TZhu] and the references therein.)

In a recent paper [Z3], a modified Kähler-Ricci flow was defined by Zhang by allowing the cohomology class to vary artificially. Consider the following Monge-Ampère flow (see Chapter 3 for relevant notations):

$$\begin{cases} \frac{\partial}{\partial t}\varphi = \log \frac{(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi)^n}{\Omega}, \\ \\ \varphi(0,\cdot) = 0. \end{cases} \tag{1.2.1}$$

Then the evolution for the corresponding Kähler metric is given by:

$$\begin{cases} \frac{\partial}{\partial t}\tilde{\omega}_t = -Ric(\tilde{\omega}_t) + Ric(\Omega) - e^{-t}\chi, \\ \\ \tilde{\omega}_t(0,\cdot) = \omega_0. \end{cases} \tag{1.2.2}$$

As pointed out by Zhang, the motivation is to apply the geometric flow technique to study the complex Monge-Ampère equation:

$$(\omega_\infty + \sqrt{-1}\partial\overline{\partial}\psi)^n = \Omega. \tag{1.2.3}$$

This equation has already been intensively studied very recently by using the pluripotential theory developed by Bedford-Taylor, Demailly, Kołodziej et al. When Ω is a smooth volume form and $[\omega_\infty]$ is Kähler, the equation is solved by Yau in his solution to the celebrated Calabi conjecture using the continuity method [Y1]. When Ω is L^p with respect to another smooth reference volume form and $[\omega_\infty]$ is Kähler, the continuous solution is obtained by Kołodziej [K1]. Later on, the bounded solution is obtained in [Z2] and [EGZ] independently, generalizing Kołodziej's theorem to the case when $[\omega_\infty]$ is big and semi-positive, and Ω is also L^p. On the other hand, as an interesting question, equation (1.2.3) is also studied on symplectic manifolds by Weinkove [We].

In the case of the unnormalized Kähler-Ricci flow, the evolution for the cohomology class of the metric is in the direction of the canonical class of the manifold. While in the case of (1.2.2), one can try to deform any initial metric class to an arbitrary desirable limit class. In particular, on Calabi-Yau manifolds, the flow (1.2.2) converges to a Ricci flat metric, if Ω is a Calabi-Yau volume form, with the initial metric also Ricci flat in a different cohomology class [Z3].

The existence and convergence of the solution are proved by Zhang [Z3] for the above flow in the case when $[\omega_\infty]$ is Kähler, which corresponds to the case considered by Cao in the classical Kähler-Ricci flow [Cao]. When $[\omega_\infty]$ is big, (1.2.2) may produce singularities at finite time $T < +\infty$. In this case, the local C^∞ convergence of the flow away from a proper analytic subvariety of X was obtained in [Z3] under the further assumption that $[\omega_T]$ is semi-positive. When $[\omega_\infty]$ is semi-positive and big, Zhang also obtained the long time existence of the solution as well as important estimates and conjectured the convergence even in this more general setting. In Chapter 3, we give a proof to this conjecture.

Theorem 1.2.1 *Let X be a Kähler manifold with a Kähler metric ω_0. Suppose that $[\omega_\infty] \in H^{1,1}(X,\mathbb{C}) \cap H^2(X,\mathbb{Z})$ is semi-positive and big. Then along the modified Kähler-Ricci flow (1.2.2), $\widetilde{\omega}_t$ converges weakly in the sense of currents and converges locally in C^∞-norm away from a proper analytic subvariety of X to the unique solution of the degenerate Monge-Ampère equation (1.2.3).*

Corollary 1.2.1 *When X is a Calabi-Yau manifold and Ω is a Calabi-Yau volume form, $\widetilde{\omega}_t$ converges to a singular Calabi-Yau metric.*

Singular Calabi-Yau metrics are already obtained in [EGZ] on normal Calabi-Yau Kähler spaces, and obtained by Song-Tian [ST2] and Tosatti [To] independently in the degenerate class on algebraic Calabi-Yau manifolds. The uniqueness of the solution to the equation (1.2.3) in the degenerate case when $[\omega_\infty]$ is semi-positive and big has been studied in [EGZ] [Z2] [DZ] etc. In particular, a stability theorem is proved in [DZ] which immediately implies the uniqueness. In [To], Tosatti studied the deformation for a family of Ricci flat Kähler metrics, whose cohomology classes are approaching a big and nef class. Here our deformation (equation (1.2.2)) gives different paths connecting non-singular and singular Calabi-Yau metrics.

The Kähler-Ricci flow on singular algebraic varieties is defined by Song-Tian [ST3] to study the singularities and surgeries. Starting with the singular and degenerate initial data on general algebraic varieties, the existence and uniqueness of the Kähler-Ricci flow is derived in [ST3]. We then prove the convergence of the Kähler-Ricci flow on projective Calabi-Yau varieties. As a corollary, we also obtain the singular Calabi-Yau metric on such varieties. This result appears in [SY].

Theorem 1.2.2 *Let $X \subset \mathbb{P}^N$ be a projective Calabi-Yau variety with log terminal singularities. Let ω_0 be the Kähler metric equivalent to the pull-back of the Fubini-Study metric on $\mathbb{P}^N$. Then the unnormalized weak Kähler-Ricci flow*

$$\begin{cases} \frac{\partial}{\partial t}\omega = -Ric(\omega), \\ \\ \omega(0,\cdot) = \omega_0 \quad \text{on} \quad X \end{cases} \tag{1.2.4}$$

has a unique weak solution $\omega(t,\cdot)$ on $[0,\infty) \times X$, where $\omega(t,\cdot) \in C^\infty([0,\infty) \times X_{reg})$ and $\omega(t,\cdot)$ admits a bounded local potenial. Furthermore, $\omega(t,\cdot)$ converges to the unique singular Ricci-flat Kähler metric $\omega_{CY} \in [\omega_0]$ in the sense of currents on X and in C^∞-topology on $X_{reg} = X \setminus X_{sing}$.

Chapter 2

Rigidity for local holomorphic isometric embeddings

2.1 Rigidity for local holomorphic conformal embeddings from the unit ball into the product of unit balls

2.1.1 Main results

Write $\mathbb{B}^n := \{z \in \mathbb{C}^n : |z| < 1\}$ for the unit ball in $\mathbb{C}^n$. Denote by ds_n^2 the normalized Bergman metric on $\mathbb{B}^n$ defined as follows:

$$ds_n^2 = \sum_{j,k\le n} \frac{1}{(1-|z|^2)^2}\big((1-|z|^2)\delta_{jk} + \bar{z}_j z_k\big) dz_j \otimes d\bar{z}_k. \tag{2.1.1}$$

Let $U \subset \mathbb{B}^n$ be a connected open subset and consider a holomorphic conformal embedding

$$F = (F_1, \ldots, F_m) : (U \subset \mathbb{B}^n, \lambda(z,\bar{z}) ds_n^2) \to (\mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m}, \oplus_{j=1}^m \lambda_j(z,\bar{z}) ds_{N_j}^2) \tag{2.1.2}$$

in the sense that $\lambda(z,\bar{z}) ds_n^2 = \sum_{j=1}^m \lambda_j(z,\bar{z}) F_j^*(ds_{N_j}^2)$. Here $\lambda_j(z,\bar{z}) > 0,\ \lambda(z,\bar{z}) > 0$ are assumed to be positive-valued smooth Nash algebraic functions over $\mathbb{C}^n$. Moreover, for each j with $1 \le j \le m$, $ds_{N_j}^2$ denotes the corresponding normalized Bergman metric of $\mathbb{B}^{N_j}$ and F_j is a holomorphic map from U to $\mathbb{B}^{N_j}$. We write $F_j = (f_{j,1}, \ldots, f_{j,l}, \ldots, f_{j,N_j})$, where $f_{j,l}$ is the l-th component of F_j. In this section, we prove the following rigidity theorem:

Theorem 2.1.1 *Suppose $n \ge 2$. Under the above notation and assumption, we then have, for each j with $1 \le j \le m$, that either F_j is a constant map or F_j extends to a totally geodesic holomorphic embedding from $(\mathbb{B}^n, ds_n^2)$ into $(\mathbb{B}^{N_j}, ds_{N_j}^2)$. Moreover, we have the following identity*

$$\sum_{F_j \text{ is not a constant}} \lambda_j(z,\bar{z}) = \lambda(z,\bar{z}).$$

In particular, when $\lambda_j(z,\bar{z}), \lambda(z,\bar{z})$ are positive constant functions, we have the following rigidity result for local isometric embeddings:

Corollary 2.1.1 *Let*

$$F = (F_1, \ldots, F_m) : (U \subset \mathbb{B}^n, \lambda ds_n^2) \to (\mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m}, \oplus_{j=1}^m \lambda_j ds_{N_j}^2) \tag{2.1.3}$$

be a local holomorphic isometric embedding in the sense that $\lambda ds_n^2 = \sum_{j=1}^m \lambda_j F_j^*(ds_{N_j}^2)$. *Assume that* $n \geq 2$ *and* λ, λ_j *are positive constants. We then have, for each* j *with* $1 \leq j \leq m$, *that either* F_j *is a constant map or* F_j *extends to a totally geodesic holomorphic embedding from* $(\mathbb{B}^n, ds_n^2)$ *into* $(\mathbb{B}^{N_j}, ds_{N_j}^2)$. *Moreover, we have the following identity*

$$\sum_{F_j \textit{ is not a constant}} \lambda_j = \lambda.$$

Recall that a function $h(z,\bar{z})$ is called a Nash algebraic function over $\mathbb{C}^n$ if it is either constant or if there is an irreducible polynomial $P(z,\xi,X)$ in $(z,\xi,X) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}$ with $P(z,\bar{z},h(z,\bar{z})) \equiv 0$ over $\mathbb{C}^n$. We mention that a holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ is a totally geodesic embedding with respect to the normalized Bergman metric if and only if there are a (holomorphic) automorphism $\sigma \in Aut(\mathbb{B}^n)$ and an automorphism $\tau \in Aut(\mathbb{B}^N)$ such that $\tau \circ F \circ \sigma(z) \equiv (z, 0)$.

2.1.2 Bergman metric and proper rational maps

Let $\mathbb{B}^n$ and ds_n^2 be the unit ball and its normalized Bergman metric, respectively, as defined before. Denote by $\mathbb{H}^n \subset \mathbb{C}^n$ the Siegel upper half space. Namely, $\mathbb{H}^n = \{(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \Im w - |z|^2 > 0\}$. Here, for m-tuples a, b, we write dot product $a \cdot b = \sum_{j=1}^m a_j b_j$ and $|z|^2 = z \cdot \bar{z}$. Recall the following Cayley transformation

$$\rho_n(z,w) = \left(\frac{2z}{1-iw}, \frac{1+iw}{1-iw} \right). \tag{2.1.4}$$

Then ρ_n biholomorphically maps $\mathbb{H}^n$ to $\mathbb{B}^n$, and biholomorphically maps $\partial\mathbb{H}^n$, the Heisenberg hypersurface, to $\partial\mathbb{B}^n \backslash \{(0,1)\}$. Applying the Cayley transformation, one can compute the normalized Bergman metric on $\mathbb{H}^n$ as follows:

$$\begin{aligned} ds^2_{\mathbb{H}^n} = &\sum_{j,k<n} \frac{\delta_{jk}(\Im w - |z|^2) + \bar{z}_j z_k}{(\Im w - |z|^2)^2} dz_j \otimes d\bar{z}_k + \frac{dw \otimes d\bar{w}}{4(\Im w - |z|^2)^2} \\ &+ \sum_{j<n} \frac{\bar{z}_j dz_j \otimes d\bar{w}}{2i(\Im w - |z|^2)^2} - \sum_{j<n} \frac{z_j dw \otimes d\bar{z}_j}{2i(\Im w - |z|^2)^2} \end{aligned} \tag{2.1.5}$$

One can easily check that

$$\begin{aligned} L_j &= \frac{\partial}{\partial z_j} + 2i\bar{z}_j \frac{\partial}{\partial w}, j = 1, \ldots, n-1. \\ \overline{L_j} &= \frac{\partial}{\partial \bar{z}_j} - 2iz_j \frac{\partial}{\partial \bar{w}}, j = 1, \ldots, n-1. \\ T &= 2(\frac{\partial}{\partial w} + \frac{\partial}{\partial \bar{w}}) \end{aligned}$$

span the complexified tangent vector bundle of $\partial\mathbb{H}^n$. (See, for instance, [BER], [Hu2], [Hu3] [HJX].)

Let F be a rational proper holomorphic map from $\mathbb{H}^n$ to $\mathbb{H}^N$. By a result of Cima-Suffridge [CS], F is holomorphic in a neighborhood of $\partial\mathbb{H}^n$. Assign the weight of w to be 2 and that of z 1. Denote by $o_{wt}(k)$ terms with weighted degree higher than k and by $P^{(k)}$ a function of weighted degree k. For $p_0 = (z_0, w_0) \in \partial\mathbb{H}^n$, write $\sigma^0_{p_0} : (z, w) \to (z + z_0, w + w_0 + 2iz \cdot \bar{z}_0)$ for the standard Heisenberg translation. The following normalization lemma will be used here:

Lemma 2.1.1 *[Hu2-3] For any $p \in \partial\mathbb{H}^n$, there is an element $\tau \in Aut(\mathbb{H}^{N+1})$ such that the map $F^{**}_p = ((f^{**}_p)_1(z), cdots, (f^{**}_p)_{n-1}(z), \phi^{**}_p, g^{**}_p) = (f^{**}_p, \phi^{**}_p, g^{**}_p) = \tau \circ F \circ \sigma^0_p$ takes the following normal form:*

$$\begin{aligned} f^{**}_p(z, w) &= z + \frac{i}{2} a^{(1)}(z) w + o_{wt}(3), \\ \phi^{**}_p(z, w) &= \phi^{(2)}(z) + o_{wt}(2), \\ g^{**}_p(z, w) &= w + o_{wt}(4) \end{aligned}$$

with

$$(\bar{z} \cdot a^{(1)}(z))|z|^2 = |\phi^{(2)}(z)|^2. \tag{2.1.6}$$

In particular, write $(f^{**}_p)_l(z) = z_j + \frac{i}{2}\sum_{k=1}^{n-1} a_{lk} z_k w + o_{wt}(3)$. Then, $(a_{lk})_{1 \le l,k \le n-1}$ is an $(n-1) \times (n-1)$ semi-positive Hermitian metrix. We next present the following key lemma for our proof of Theorem 2.1.1:

Lemma 2.1.2 *Let F be a proper rational map from $\mathbb{B}^n$ to $\mathbb{B}^N$. Then*

$$X := ds_n^2 - F^*(ds_N^2), \tag{2.1.7}$$

is a semi-positive real analytic symmetric (1,1)-tensor over $\mathbb{B}^n$, that extends also to a real analytic (1,1)-tensor in a small neighborhood of $\partial\mathbb{B}^n$ in $\mathbb{C}^n$.

Proof of lemma 2.1.2: Our original proof of Lemma 2.1 was largely simplified by Ng [Ng4] and Mok [Mo4] by considering the potential $-\log(1 - \|F(z)\|^2)$ of the pull-back metric $F^*(ds_N^2)$ as follows: Since $1 - \|F(z)\|^2$ vanishes identically on $\partial\mathbb{B}^n$ and since $1 - \|z\|^2$ is a defining equation for $\partial\mathbb{B}^n$, one obtains

$$1 - \|F(z)\|^2 = (1 - \|z\|^2)\varphi(z)$$

for a real analytic function $\varphi(z)$.

Since $\rho := \|F(z)\|^2 - 1$ is subharmonic over $\mathbb{B}^n$ and has maximum value 0 on the boundary, applying the classical Hopf lemma, we conclude that $\varphi(z)$ can not vanish at any boundary point of $\mathbb{B}^n$. Apparently, $\varphi(z)$ can not vanish inside $\mathbb{B}^n$. Therefore, $X = \sqrt{-1}\partial\bar{\partial}\log\varphi(z)$ is real analytic on an open neighborhood of $\overline{\mathbb{B}^n}$. The semi-positivity of X over $\mathbb{B}^n$ is an easy consequence of the Schwarz lemma. $\square$

Applying the Cayley transformation (and also a rotation transformation when handling the regularity near $(0,1)$), we have the following corollary:

Corollary 2.1.2 *Let F be a rational proper holomorphic map from $\mathbb{H}^n$ to $\mathbb{H}^N$. Then*

$$X := ds^2_{\mathbb{H}^n} - F^*(ds^2_{\mathbb{H}^N}), \tag{2.1.8}$$

is a semi-positive real analytic symmetric (1,1)-tensor over $\mathbb{H}^n$, that extends also to a real analytic (1,1)-tensor in a small neighborhood of $\partial\mathbb{H}^n$ in $\mathbb{C}^n$.

The boundary value of X is an intrinsic CR invariant associated with the equivalence class of the map F. Next, we compute X in the normal coordinates at the boundary point.

Write $t = \Im w - |z|^2$ and $H = \Im g - |\tilde{f}|^2$. Write $o(k)$ for terms whose degrees with respect to t are higher than k. For a real analytic function h in (z, w), we use h_z, h_w to denote the derivatives of h with respect to z, w. By replacing w by $u + i(t + |z|^2)$, H can also be regarded as an analytic function on $z, \bar{z}, u, t$. The following lemma gives an asymptotic behavior of H with respect to t:

Lemma 2.1.3 $H(z, \bar{z}, u, t) = (g_w - 2i\tilde{f}_w \cdot \bar{\tilde{f}})|_{t=0} t - (2|\tilde{f}_w|^2)|_{t=0} t^2 + \frac{1}{3}(-\frac{1}{2} g_{w^3} + 3i\tilde{f}_w \cdot \overline{\tilde{f}_{w^2}} + i\tilde{f}_{w^3} \cdot \bar{\tilde{f}})|_{t=0} t^3 + o(3)$.

Proof of Lemma 2.1.3: Notice that $H = H(z, \bar{z}, u + i(t + |z|^2), u - i(t + |z|^2))$. Since F is proper, H, as a function of t with parameters $\{z, u\}$, can be written as $P_1 t + P_2 t^2 + P_3 t^3 + o(3)$, where P_1, P_2, P_3 are analytic in $(z, \bar{z}, u)$. Then

$$
\begin{aligned}
P_1 &= \frac{\partial H(z, \bar{z}, u + i(t + |z|^2), u - i(t + |z|^2))}{\partial t}\Big|_{t=0} \\
&= iH_w - iH_{\bar{w}}\Big|_{t=0} \\
&= \frac{1}{2}(g_w + \overline{g_w}) + i(\tilde{f} \cdot \overline{\tilde{f}_w} - \bar{\tilde{f}} \cdot \tilde{f}_w)\Big|_{t=0},
\end{aligned}
\tag{2.1.9}
$$

$$
\begin{aligned}
P_2 &= \frac{1}{2}\frac{\partial^2 H(z, \bar{z}, u + i(t + |z|^2), u - i(t + |z|^2))}{\partial t^2}\Big|_{t=0} \\
&= \frac{1}{2}(-H_{w^2} + 2H_{w\bar{w}} - H_{\bar{w}\bar{w}})\Big|_{t=0} \\
&= \frac{1}{2}(\frac{i}{2} g_{w^2} - \frac{i}{2}\overline{g_{w^2}} - 2|\tilde{f}_w|^2 + \tilde{f}_{w^2} \cdot \bar{\tilde{f}} + \tilde{f} \cdot \overline{\tilde{f}_{w^2}})\Big|_{t=0},
\end{aligned}
\tag{2.1.10}
$$

and

$$
\begin{aligned}
P_3 &= \frac{1}{6}\frac{\partial^3 H(z, \bar{z}, u + i(t + |z|^2), u - i(t + |z|^2))}{\partial t^3}\Big|_{t=0} \\
&= \frac{1}{6}(-iH_{w^3} + 3iH_{w^2\bar{w}} - 3iH_{\bar{w}^2 w} + iH_{\bar{w}^3})\Big|_{t=0} \\
&= \frac{1}{6}(-\frac{1}{2} g_{w^3} - \frac{1}{2}\overline{g_{w^3}} + i\tilde{f}_{w^3} \cdot \bar{\tilde{f}} - i\tilde{f} \cdot \overline{\tilde{f}_{w^3}} - 3i\tilde{f}_{w^2} \cdot \overline{\tilde{f}_w} + 3i\tilde{f}_w \cdot \overline{\tilde{f}_{w^2}})\Big|_{t=0}.
\end{aligned}
\tag{2.1.11}
$$

On the other hand, applying T, T^2, T^3 to the defining equation $g - \bar{g} = 2i\tilde{f} \cdot \bar{\tilde{f}}$, we have

$$g_w - \overline{g_w} - 2i(\tilde{f}_w \cdot \bar{\tilde{f}} + \tilde{f} \cdot \overline{\tilde{f}_w}) = 0, \tag{2.1.12}$$

$$g_{w^2} - \overline{g_{w^2}} - 2i(\tilde{f}_{w^2} \cdot \bar{\tilde{f}} + \overline{\tilde{f}_{w^2}} \cdot \tilde{f} + 2|\tilde{f}_w|^2) = 0, \tag{2.1.13}$$

$$g_{w^3} - \overline{g_{w^3}} - 2i(\tilde{f}_{w^3} \cdot \bar{\tilde{f}} + \overline{\tilde{f}_{w^3}} \cdot \tilde{f} + 3\tilde{f}_{w^2} \cdot \overline{\tilde{f}_w} + 3\tilde{f}_w \cdot \overline{\tilde{f}_{w^2}}) = 0. \tag{2.1.14}$$

over $\Im w = |z|^2$.

Substituting (2.1.12), (2.1.13) and (2.1.14) into (2.1.9), (2.1.10) and (2.1.11), we get

$$\begin{aligned}
P_1 &= g_w - 2i\tilde{f}_w \cdot \bar{\tilde{f}}\Big|_{t=0}, \\
P_2 &= -2|\tilde{f}_w|^2\Big|_{t=0}, \\
P_3 &= \frac{1}{3}(-\frac{1}{2}g_{w^3} + 3i\tilde{f}_w \cdot \overline{\tilde{f}_{w^2}} + i\tilde{f}_{w^3} \cdot \bar{\tilde{f}})\Big|_{t=0}.
\end{aligned} \tag{2.1.15}$$

$\square$

We remark that by the Hopf Lemma, it follows easily that $P_1 \neq 0$ along $\partial\mathbb{H}^n$.

We next write $X = X_{jk}dz_j \otimes d\bar{z}_k + X_{jn}dz_j \otimes d\bar{w} + X_{nj}dw \otimes d\bar{z}_j + X_{nn}dw \otimes d\bar{w}$. By making use of Lemma 2.1.1, we shall compute in the next proposition the values of X at the origin. The proposition might be of independent interest, as the CR invariants in the study of proper holomorphic maps between Siegel upper half spaces are related to the CR geometry of the graph of the map.

Proposition 2.1.1 *Assume that $F = (\tilde{f}, g) = (f_1, \ldots, f_{N-1}, g) : \mathbb{H}^n \to \mathbb{H}^N$ is a proper rational holomorphic map, that satisfies the normalization (at the origin) stated in Lemma 2.1.1. Then*

$$\begin{aligned}
X_{jk}(0) &= -2i(f_k)_{z_j w}(0) = a_{kj}, \\
X_{jn}(0) &= \overline{X_{nj}}(0) = \frac{3i}{4}\overline{(f_j)_{w^2}}(0) + \frac{1}{8}g_{z_j w^2}(0), \\
X_{nn}(0) &= \frac{1}{6}g_{w^3}(0).
\end{aligned}$$

Proof of Proposition 2.1.1: Along the direction of $dz_j \otimes d\bar{z}_k$, collecting the coefficient of t^2 in the Taylor expansion of H^2X with respect to t, we get

$$\begin{aligned}
P_1^2X_{jk}(0) = \Big[&(2P_1P_2\delta_{jk} + (P_2^2 + 2P_1P_3)\bar{z}_j z_k) - \frac{1}{2}\{2iP_1(\tilde{f}_{wz_j} \cdot \overline{\tilde{f}_{z_k}} - \tilde{f}_{z_j} \cdot \overline{\tilde{f}_{wz_k}}) \\
&+ 2P_2\tilde{f}_{z_j} \cdot \overline{\tilde{f}_{z_k}} - (\overline{\tilde{f}_{w^2}} \cdot \tilde{f}_{z_j})(\tilde{f} \cdot \overline{\tilde{f}_{z_k}}) + 2(\overline{\tilde{f}_w} \cdot \tilde{f}_{z_j})(\tilde{f}_w \cdot \overline{\tilde{f}_{z_k}}) \\
&+ 2(\overline{\tilde{f}_w} \cdot \tilde{f}_{wz_j})(\tilde{f} \cdot \overline{\tilde{f}_{z_k}}) - 2(\overline{\tilde{f}_w} \cdot \tilde{f}_{z_j})(\tilde{f} \cdot \overline{\tilde{f}_{z_k w}}) - (\bar{\tilde{f}} \cdot \tilde{f}_{z_j})(\tilde{f}_{w^2} \cdot \overline{\tilde{f}_{z_k}}) \\
&- 2(\bar{\tilde{f}} \cdot \tilde{f}_{z_j w})(\tilde{f}_w \cdot \overline{\tilde{f}_{z_k}}) + 2(\bar{\tilde{f}} \cdot \tilde{f}_{z_j})(\tilde{f}_w \cdot \overline{\tilde{f}_{z_k w}}) - (\tilde{f}_{z_j w^2} \cdot \bar{\tilde{f}})(\tilde{f} \cdot \overline{\tilde{f}_{z_k}}) \\
&+ 2(\bar{\tilde{f}} \cdot \tilde{f}_{z_j w})(\tilde{f} \cdot \overline{\tilde{f}_{z_k w}}) - (\bar{\tilde{f}} \cdot \tilde{f}_{z_j})(\tilde{f} \cdot \overline{\tilde{f}_{z_k w^2}}) - \frac{1}{4}g_{z_j w^2}\overline{g_{z_k}} + \frac{1}{2}g_{z_j w}\overline{g_{z_k w}} \\
&- \frac{1}{4}g_{z_j}\overline{g_{z_k w^2}} + \frac{i}{2}(\overline{\tilde{f}_{w^2}} \cdot \tilde{f}_{z_j})\bar{g}_{z_k} - i(\overline{\tilde{f}_w} \cdot \tilde{f}_{wz_j})\overline{g_{z_k}} + i(\overline{\tilde{f}_w} \cdot \tilde{f}_{z_j})\overline{g_{z_k w}} \\
&+ \frac{i}{2}(\bar{\tilde{f}} \cdot \tilde{f}_{z_j w^2})\overline{g_{z_k}} - i(\bar{\tilde{f}} \cdot \tilde{f}_{z_j w})\overline{g_{z_k w}} + \frac{i}{2}(\bar{\tilde{f}} \cdot \tilde{f}_{z_j})\overline{g_{z_k w^2}} - \frac{i}{2}(\tilde{f}_{w^2} \cdot \overline{\tilde{f}_{z_k}})g_{z_j} \\
&- i(\tilde{f}_w \cdot \overline{\tilde{f}_{z_k}})g_{z_j w} + i(\tilde{f}_w \cdot \overline{\tilde{f}_{z_k w}})g_{z_j} - \frac{i}{2}(\tilde{f} \cdot \overline{\tilde{f}_{z_k}})g_{z_j w^2} + i(\tilde{f} \cdot \overline{\tilde{f}_{z_k w}})g_{z_j w} \\
&- \frac{i}{2}(\tilde{f} \cdot \overline{\tilde{f}_{z_k w^2}})g_{z_j}\}\Big]\Big|_{t=0}.
\end{aligned}$$

Letting $(z, w) = 0$ and applying the normalization condition as stated in Lemma 2.1.1, we have

$$X_{jk}(0) = \frac{\partial a_k^{(1)}(z)}{\partial z_j} = a_{kj}.$$

Similarly, considering the coefficient of t^2 along $dz_j \otimes d\bar{w}$ and $dw \otimes d\bar{w}$, respectively, we have

$$\begin{aligned}
P_1^2X_{jn}(0) = \Big[&(-iP_1P_3 - \frac{i}{2}P_2^2)\bar{z}_j - \frac{1}{2}\{2iP_1(\tilde{f}_{wz_j} \cdot \overline{\tilde{f}_w} - \tilde{f}_{z_j} \cdot \overline{\tilde{f}_{w^2}}) + 2P_2\tilde{f}_{z_j} \cdot \overline{\tilde{f}_w} \\
&- (\overline{\tilde{f}_{w^2}} \cdot \tilde{f}_{z_j})(\tilde{f} \cdot \overline{\tilde{f}_w}) + 2(\overline{\tilde{f}_w} \cdot \tilde{f}_{z_j})(\tilde{f}_w \cdot \overline{\tilde{f}_w}) + 2(\overline{\tilde{f}_w} \cdot \tilde{f}_{wz_j})(\tilde{f} \cdot \overline{\tilde{f}_w}) \\
&- 2(\overline{\tilde{f}_w} \cdot \tilde{f}_{z_j})(\tilde{f} \cdot \overline{\tilde{f}_{w^2}}) - (\bar{\tilde{f}} \cdot \tilde{f}_{z_j})(\tilde{f}_{w^2} \cdot \overline{\tilde{f}_w}) - 2(\bar{\tilde{f}} \cdot \tilde{f}_{z_j w})(\tilde{f}_w \cdot \overline{\tilde{f}_w})
\end{aligned}$$

$$\begin{aligned}
&+ 2(\bar{\tilde{f}} \cdot \tilde{f}_{z_j})(\tilde{f}_w \cdot \overline{\tilde{f}_{w^2}}) - (\tilde{f}_{z_j w^2} \cdot \bar{\tilde{f}})(\tilde{f} \cdot \overline{\tilde{f}_w}) + 2(\bar{\tilde{f}} \cdot \tilde{f}_{z_j w})(\tilde{f} \cdot \overline{\tilde{f}_{w^2}}) - (\bar{\tilde{f}} \cdot \tilde{f}_{z_j})(\tilde{f} \cdot \overline{\tilde{f}_{w^3}}) \\
&- \frac{1}{4}g_{z_j w^2}\overline{g_w} + \frac{1}{2}g_{z_j w}\overline{g_{w^2}} - \frac{1}{4}g_{z_j}\overline{g_{w^3}} + \frac{i}{2}(\overline{\tilde{f}_{w^2}} \cdot \tilde{f}_{z_j})\overline{g_w} - i(\overline{\tilde{f}_w} \cdot \tilde{f}_{wz_j})\overline{g_w} + i(\overline{\tilde{f}_w} \cdot \tilde{f}_{z_j})\overline{g_{w^2}} \\
&+ \frac{i}{2}(\bar{\tilde{f}} \cdot \tilde{f}_{z_j w^2})\overline{g_w} - i(\bar{\tilde{f}} \cdot \tilde{f}_{z_j w})\overline{g_{w^2}} + \frac{i}{2}(\bar{\tilde{f}} \cdot \tilde{f}_{z_j})\overline{g_{w^3}} - \frac{i}{2}(\tilde{f}_{w^2} \cdot \overline{\tilde{f}_w})g_{z_j} - i(\tilde{f}_w \cdot \overline{\tilde{f}_w})g_{z_j w} \\
&+ i(\tilde{f}_w \cdot \overline{\tilde{f}_{w^2}})g_{z_j} - \frac{i}{2}(\tilde{f} \cdot \overline{\tilde{f}_w})g_{z_j w^2} + i(\tilde{f} \cdot \overline{\tilde{f}_{w^2}})g_{z_j w} - \frac{i}{2}(\tilde{f} \cdot \overline{\tilde{f}_{w^3}})g_{z_j}\}\Big]\Big|_{t=0}
\end{aligned}$$

and

$$
\begin{aligned}
P_1^2 X_{nn}(0) = & \Big[\frac{1}{4}(2P_1P_3 + P_2^2) - \frac{1}{2}\{2iP_1(\tilde{f}_{w^2} \cdot \overline{\tilde{f}_w} - \tilde{f}_w \cdot \overline{\tilde{f}_{w^2}}) + 2P_2 \tilde{f}_w \cdot \overline{\tilde{f}_w} \\
& - (\overline{\tilde{f}_{w^2}} \cdot \tilde{f}_w)(\tilde{f} \cdot \overline{\tilde{f}_w}) + 2(\overline{\tilde{f}_w} \cdot \tilde{f}_w)(\tilde{f}_w \cdot \overline{\tilde{f}_w}) + 2(\overline{\tilde{f}_w} \cdot \tilde{f}_{w^2})(\tilde{f} \cdot \overline{\tilde{f}_w}) \\
& - 2(\overline{\tilde{f}_w} \cdot \tilde{f}_w)(\tilde{f} \cdot \overline{\tilde{f}_{w^2}}) - (\bar{\tilde{f}} \cdot \tilde{f}_w)(\tilde{f}_{w^2} \cdot \overline{\tilde{f}_w}) - 2(\bar{\tilde{f}} \cdot \tilde{f}_{w^2})(\tilde{f}_w \cdot \overline{\tilde{f}_w}) \\
& + 2(\bar{\tilde{f}} \cdot \tilde{f}_w)(\tilde{f}_w \cdot \overline{\tilde{f}_{w^2}}) - (\tilde{f}_{w^3} \cdot \bar{\tilde{f}})(\tilde{f} \cdot \overline{\tilde{f}_w}) + 2(\bar{\tilde{f}} \cdot \tilde{f}_{w^2})(\tilde{f} \cdot \overline{\tilde{f}_{w^2}}) \\
& - (\bar{\tilde{f}} \cdot \tilde{f}_w)(\tilde{f} \cdot \overline{\tilde{f}_{w^3}}) - \frac{1}{4} g_{w^3}\overline{g_w} + \frac{1}{2} g_{w^2}\overline{g_{w^2}} - \frac{1}{4} g_w \overline{g_{w^3}} \\
& + \frac{i}{2}(\overline{\tilde{f}_{w^2}} \cdot \tilde{f}_w)\bar{g}_w - i(\overline{\tilde{f}_w} \cdot \tilde{f}_{w^2})\overline{g_w} + i(\overline{\tilde{f}_w} \cdot \tilde{f}_w)\overline{g_{w^2}} + \frac{i}{2}(\bar{\tilde{f}} \cdot \tilde{f}_{w^3})\overline{g_w} \\
& - i(\bar{\tilde{f}} \cdot \tilde{f}_{w^2})\overline{g_{w^2}} + \frac{i}{2}(\bar{\tilde{f}} \cdot \tilde{f}_w)\overline{g_{w^3}} - \frac{i}{2}(\tilde{f}_{w^2} \cdot \overline{\tilde{f}_w}) g_w - i(\tilde{f}_w \cdot \overline{\tilde{f}_w}) g_{w^2} \\
& + i(\tilde{f}_w \cdot \overline{\tilde{f}_{w^2}}) g_w - \frac{i}{2}(\tilde{f} \cdot \overline{\tilde{f}_w}) g_{w^3} + i(\tilde{f} \cdot \overline{\tilde{f}_{w^2}}) g_{w^2} - \frac{i}{2}(\tilde{f} \cdot \overline{\tilde{f}_{w^3}}) g_w\}\Big]\Big|_{t=0}.
\end{aligned}
$$

Let $(z, w) = 0$. It follows that

$$
\begin{aligned}
X_{jn}(0) &= \frac{3i}{4}\overline{(f_j)_{w^2}}(0) + \frac{1}{8} g_{z_j w^2}(0), \\
X_{nn}(0) &= \frac{1}{6} g_{w^3}(0),
\end{aligned}
$$

for $g_{w^3}(0) = \overline{g_{w^3}(0)}$ by (2.1.14). $\square$

Making use of the computation in Proposition 2.5, we give a proof of Theorem 2.1.1 in the case when each component extends as a proper holomorphic map. Indeed, we prove a slightly more general result than what we need later as following:

Proposition 2.1.2 *Let*

$$F = (F_1, \ldots, F_m) : (\mathbb{B}^n, \lambda(z, \bar{z}) ds_n^2) \to (\mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m}, \oplus_{j=1}^m \lambda_j(z, \bar{z}) ds_{N_j}^2)$$

be a conformal embedding. Here for each j, $\lambda(z, \bar{z}), \lambda_j(z, \bar{z})$ are positive real-valued C^2-smooth functions over $\overline{\mathbb{B}^n}$, and F_j is a proper rational map from $\mathbb{B}^n$ into $\mathbb{B}^{N_j}$ for each j. Then $\lambda(z, \bar{z}) \equiv \sum_{j=1}^m \lambda_j(z, \bar{z})$ over $\overline{\mathbb{B}^n}$, and for any j, F_j is a totally geodesic embedding from $\mathbb{B}^n$ to $\mathbb{B}^{N_j}$.

Proof of Proposition 2.1.2: After applying the Cayley transformation and considering $\big((\rho_{N_1})^{-1}, \cdots, (\rho_{N_m})^{-1}\big) \circ F \circ \rho_n$ instead of F, we can assume, without loss of generality, that

$$F = (F_1, \ldots, F_m) : (\mathbb{H}^n, \lambda(Z, \bar{Z}) ds^2_{\mathbb{H}^n}) \to (\mathbb{H}^{N_1} \times \cdots \times \mathbb{H}^{N_m}, \oplus_{j=1}^m \lambda_j(Z, \bar{Z}) ds^2_{\mathbb{H}^{N_j}})$$

is a conformal map with each F_j a proper rational map from $\mathbb{H}^n$ into $\mathbb{H}^{N_j}$, respectively. Here we write $Z = (z, w)$. Moreover, we can assume, without loss of generality, that each component F_j of F satisfies the normalization condition as in Lemma 2.1. Since F is conformal, we have

$$\lambda(Z, \bar{Z}) ds^2_{\mathbb{H}^n} = \sum_{j=1}^m \lambda_j(Z, \bar{Z}) F_j^*(ds^2_{\mathbb{H}^{N_j}})$$

or

$$(\lambda(Z, \bar{Z}) - \sum_{j=1}^m \lambda_j(Z, \bar{Z})) ds^2_{\mathbb{H}^n} + \sum_{j=1}^m \lambda_j(Z, \bar{Z}) X(F_j) = 0. \tag{2.1.16}$$

Here, we write $X(F_j) = ds^2_{\mathbb{H}^n} - F_j^*(ds^2_{\mathbb{H}^{N_j}})$. Collecting the coefficient of $dw \otimes d\bar{w}$, one has

$$\frac{\lambda(Z, \bar{Z}) - \sum_{j=1}^m \lambda_j(Z, \bar{Z})}{4(\Im w - |z|^2)^2} + \sum_{j=1}^m \lambda_j(Z, \bar{Z})(X(F_j))_{nn} = 0. \tag{2.1.17}$$

Since $X(F_j)$ is smooth up to $\partial\mathbb{H}^n$, we see that $\lambda(Z, \bar{Z}) - \sum_{j=1}^m \lambda_j(Z, \bar{Z}) = O(t^2)$ as $Z = (z, w) \in \mathbb{H}^n \to 0$, where $t = \Im w - |z|^2$. However, since the $dz_l \otimes d\bar{z}_k$-component of $ds^2_{\mathbb{H}^n}$ blows up at the rate of $o(\frac{1}{t^2})$ as $(z, w)(\in \mathbb{H}^n) \to 0$, collecting the coefficients of the $dz_l \otimes d\bar{z}_k$-component in (2.1.16) and then letting $(z, w)(\in \mathbb{H}^n) \to 0$, we conclude that, for any $1 \leq l, k \leq n-1$,

$$\sum_{j=1}^m \lambda_j(0)(X(F_j))_{kl}(0) = 0.$$

By Proposition 2.5, we have $\sum_{j=1}^m \lambda_j(0) a^j_{lk}(0) = 0$, where a^j_{kl} is associated with F_j in the expansion of F_j at 0 as in Lemma 2.1. Since the matrix $(a^j_{kl})_{1 \leq l,k \leq n-1}$ is a semi-positive matrix and since $\lambda_j(0) > 0$, it follows immediately that $a^j_{kk} = 0$ for each $1 \leq k \leq n-1$. By the semi-positivity of $(a^j_{kl})_{1 \leq l,k \leq n-1}$, we conclude that $a^j_{lk} = 0$ for all j, k, l. Namely, $F_j = (z, w) + O_{wt}(3)$ for each j.

Next, for each $p \in \partial\mathbb{H}^n$, let $\tau_j \in Aut(\mathbb{H}^N)$ be such that $(F_j)^{**}_p = \tau_j \circ F_j \circ \sigma^0_p$ has the normalization as in Lemma 2.1.1. Let $\tau = (\tau_1, \cdots, \tau_m)$. Notice that $F^{**}_p :=$

$((F_1)_p^{**}, \cdots, (F_m)_p^{**}) = \tau \circ F \circ \sigma_p^0$ is still a conformal map satisfing the condition as in the proposition. Applying the just presented argument to F_p^{**}, we conclude that $(F_j)_p^{**} = (z, w) + O_{wt}(3)$. By Theorem 4.2 of [Hu2], this implies that $F_j = (Z, 0)$. Namely, F_j is a totally geodesic embedding. In particular, we have $X(F_j) \equiv 0$. This also implies that $\lambda(z, \bar{z}) \equiv \sum_{j=1}^m \lambda_j(z, \bar{z})$ over $\mathbb{B}^n$. The proof of Proposition 2.1.2 is complete. $\square$

2.1.3 Algebraic extension

In this section, we prove the algebraicity of the local holomorphic conformal embedding. As in the theorem, we let $U \subset \mathbb{B}^n$ be a connected open subset. Let

$$F = (F_1, \ldots, F_m) : (U \subset \mathbb{B}^n, \lambda(z, \bar{z}) ds_n^2) \to (\mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m}, \oplus_{j=1}^m \lambda_j(z, \bar{z}) ds_{N_j}^2)$$

be a holomorphic conformal embedding. Here $\lambda_j(z, \bar{z}), \lambda(z, \bar{z}) > 0$ are smooth Nash algebraic functions, and ds_n^2 and $ds_{N_j}^2$ are the Bergman metrics of $\mathbb{B}^n$ and $\mathbb{B}^{N_j}$, respectively. We further assume without loss of generality that none of the $F_j's$ is a constant map. Our proof uses the same method employed in the paper of Mok-Ng [MN]. Namely, we use the Grauert tube technique to reduce the problem to the algebraicity problem for CR mappings. However, different from the consideration in [MN], the Grauert tube constructed by using the unit sphere bundle over $\mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m}$ with respect to the metric $\oplus_{j=1}^m \lambda_j(z, \bar{z}) ds_{N_j}^2$ may have complicated geometry and may not even be pseudoconvex anymore in general. To overcome the difficulty, we bend the target hypersurface to make it sufficiently positively curved along the tangential direction of the source domain. For the convenience of the reader, we give the proof when $m = 2$ for the simplicity of our notation. The argument for general m is exactly the same as for $m = 2$ and is skipped.

Let $K > 0$ be a large constant to be determined. Consider $S_1 \subset TU$ and $S_2 \subset U \times T\mathbb{B}^{N_1} \times T\mathbb{B}^{N_2}$ as follows:

$$S_1 := \left\{(t, \zeta) \in TU : (1 + K\|t\|^2) \lambda(t, \bar{t}) ds_n^2(t)(\zeta, \zeta) = 1\right\}, \qquad (2.1.18)$$

$$S_2 := \{(t,z,\xi,w,\eta) \in U \times T\mathbb{B}^{N_1} \times T\mathbb{B}^{N_2} : \\ (1+K\|t\|^2)[\lambda_1(t,\bar t)ds^2_{N_1}(z)(\xi,\xi) + \lambda_2(t,\bar t)ds^2_{N_2}(w)(\eta,\eta)] = 1\}. \tag{2.1.19}$$

The defining functions ρ_1, ρ_2 of S_1, S_2 are, respectively, as follows:

$$\rho_1 = (1+K\|t\|^2)\lambda(t,\bar t)ds^2_n(t)(\zeta,\zeta) - 1,$$

$$\rho_2 = (1+K\|t\|^2)[\lambda_1(t,\bar t)ds^2_{N_1}(z)(\xi,\xi) + \lambda_2(t,\bar t)ds^2_{N_2}(w)(\eta,\eta)] - 1.$$

Then one can easily check that the map $(id, F_1, dF_1, F_2, dF_2)$ maps S_1 to S_2 according to the metric equation

$$\lambda(t,\bar t)ds^2_n = \lambda_1(t,\bar t)F_1^*(ds^2_{N_1}) + \lambda_2(t,\bar t)F_2^*(ds^2_{N_2}).$$

Lemma 2.1.4 *S_1, S_2 are both real algebraic hypersurfaces. Moreover for K sufficiently large, S_1 is smoothly strongly pseudoconvex. For any $\xi \neq 0, \eta \neq 0$, $(0,0,\xi,0,\eta) \in S_2$ is a smooth strongly pseudoconvex point when K is sufficiently large, where K depends on the choice of ξ and η.*

Proof It follows immediately from defining functions that S_1, S_2 are smooth real algebraic hypersurfaces. We show the strong pseudoconvexity of S_2 at $(0,0,\xi,0,\eta)$ as follows: (The strong pseudoconvexity of S_1 follows from the same computation.)

By applying $\partial\bar\partial$ to ρ_2 at $(0,0,\xi,0,\eta)$, we have the following Hessian matrix

$$\begin{bmatrix} A & 0 & D_1 & 0 & D_2 \\ 0 & B_1 & 0 & 0 & 0 \\ \bar D_1 & 0 & C_1 & 0 & 0 \\ 0 & 0 & 0 & B_2 & 0 \\ \bar D_2 & 0 & 0 & 0 & C_2 \end{bmatrix} \tag{2.1.20}$$

where

$$\begin{aligned}(A_1)_{t_i t_{\bar j}}(0,0,\xi,0,\eta) &= \partial_{t_i}\partial_{t_{\bar j}}\rho_2(0,0,\xi,0,\eta) \\ &= K(\lambda_1(0)|\xi|^2 + \lambda_2(0)|\eta|^2)\delta_{i\bar j} + \partial_{t_i}\partial_{t_{\bar j}}\lambda_1(0)|\xi|^2 + \partial_{t_i}\partial_{t_{\bar j}}\lambda_2(0)|\eta|^2 \\ &\geq \delta K(|\xi|^2 + |\eta|^2)\delta_{i\bar j},\end{aligned} \tag{2.1.21}$$

$$(B_1)_{z_k z_{\bar{l}}}(0,0,\xi,0,\eta) = \partial_{z_k}\partial_{z_{\bar{l}}}\rho_2(0,0,\xi,0,\eta) = -\lambda_1(0)R_{z_k z_{\bar{l}}\mu\bar{\nu}}(0)\xi^\mu\xi^{\bar{\nu}} \geq \delta|\xi|^2\delta_{k\bar{l}}, \tag{2.1.22}$$

$$(C_1)_{\xi_k\xi_{\bar{l}}} = \lambda_1(0)\delta_{k\bar{l}} \geq \delta\delta_{k\bar{l}}, \tag{2.1.23}$$

$$(D_1)_{t_i\xi_{\bar{l}}} = \partial_{t_i}\lambda_1(0)\xi^l \tag{2.1.24}$$

for some $\delta > 0$ and other matrices are similar.

Let

$$(e,r,r',s,s') = (e_1,\cdots,e_n,r_1,\cdots,r_{N_1},r'_1,\cdots,r'_{N_1},s_1,\cdots,s_{N_2},s'_1,\cdots,s'_{N_2}) \neq 0.$$

It holds that

$$\begin{aligned}
&\begin{bmatrix} e & r & r' & s & s' \end{bmatrix} \begin{bmatrix} A & 0 & D_1 & 0 & D_2 \\ 0 & B_1 & 0 & 0 & 0 \\ \bar{D}_1 & 0 & C_1 & 0 & 0 \\ 0 & 0 & 0 & B_2 & 0 \\ \bar{D}_2 & 0 & 0 & 0 & C_2 \end{bmatrix} \begin{bmatrix} \bar{e}^t \\ \bar{r}^t \\ \bar{r}'^t \\ \bar{s}^t \\ \bar{s}'^t \end{bmatrix} \\
&\geq \delta K(|\xi|^2+|\eta|^2)|e|^2 + \delta(|\xi|^2|r|^2+|\eta|^2|s|^2) + \delta(|r'|^2+|s'|^2) \\
&\quad - 2\left|\sum_{i,l} e_i\partial_{t_i}\lambda_1(0)\xi^l\bar{r}'_l\right| - 2\left|\sum_{i,l} e_i\partial_{t_i}\lambda_2(0)\eta^l\bar{s}'_l\right| \\
&\geq (\delta K - M)(|\eta|^2+|\xi|^2)|e|^2 + \delta(|\xi|^2|r|^2+|\eta|^2|s|^2) + (\delta-\epsilon)|r'|^2 + (\delta-\epsilon)|s'|^2 \\
&> 0.
\end{aligned} \tag{2.1.25}$$

Here the second inequality holds as

$$\left|\sum_{i,l} e_i\partial_{t_i}\lambda_1(0)\xi^l\bar{r}'_l\right| \leq M_1|e||\xi\cdot r| \leq M_1|e||\xi||r| \leq \frac{M}{2}|e|^2|\xi|^2 + \frac{\epsilon}{2}|r|^2$$

by the standard Cauchy-Schwarz inequality. Here $M = \frac{M_1^2}{\epsilon}$ and the last strict inequality holds as $\xi \neq 0$ and $\eta \neq 0$ by letting $\epsilon < \delta$ and raising K sufficiently large. $\square$

Theorem 2.1.2 *Under the assumption of Theorem 2.1.1, F is Nash algebraic.*

Proof Without loss of generality, one can arrange that $F(0) = 0$ by composing elements from $Aut(\mathbb{B}^n)$ and $Aut(\mathbb{B}^{N_1}) \times Aut(\mathbb{B}^{N_2})$. Furthermore, since F_1, F_2 are not constant maps, we can further assume that $dF_1|_0 \not\equiv 0$ and $dF_2|_0 \not\equiv 0$. Therefore, there exists $0 \neq \zeta \in T_0\mathbb{B}^n$, such that $dF_1(\zeta) \neq 0$ and $dF_2(\zeta) \neq 0$. After scaling, we assume that $(0, \zeta) \in S_1$. Notice that both the fiber of S_1 over $0 \in U$ and the fiber of S_2 over $(0, 0) \in U \times (\mathbb{B}^{N_1} \times \mathbb{B}^{N_2})$ are independent of the choice of K. Now the theorem follows from the algebracity theorem of Huang [Hu1] and Lemma 2.1.4 applied to the map $(id, F_1, dF_1, F_2, dF_2)$ from S_1 into S_2. $\square$

2.1.4 Proof of Theorem 2.1.1

In this section, we give a proof of Theorem 2.1.1. As in the theorem, we let $U \subset \mathbb{B}^n$ be a connected open subset. Let

$$F = (F_1, \ldots, F_m) : (U \subset \mathbb{B}^n, \lambda(z, \bar{z})ds_n^2) \to (\mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m}, \oplus_{j=1}^m \lambda_j(z, \bar{z})ds_{N_j}^2)$$

be a holomorphic conformal embedding. Here $\lambda_j(z, \bar{z}), \lambda(z, \bar{z}) > 0$ are smooth Nash algebraic functions, and ds_n^2 and $ds_{N_j}^2$ are the Bergman metrics of $\mathbb{B}^n$ and $\mathbb{B}^{N_j}$, respectively. F_j is a holomorphic map from U to $\mathbb{B}^{N_j}$ for each j. For the proof of Theorem 2.1.1, we can assume without loss of generality that none of the $F_j's$ is a constant map. Following the idea in [MN], we already showed that F extends to an algebraic map over $\mathbb{C}^n$. Namely, for each (non-constant) component $f_{j,l}$ of F_j, there is an irreducible polynomial $P_{j,l}(z, X) = a_{j,l}(z)X^{m_{jl}} + \ldots$ in $(z, X) \in \mathbb{C}^n \times \mathbb{C}$ of degree $m_{jl} \geq 1$ in X such that $P_{j,l}(z, f_{j,l}) \equiv 0$ for $z \in U$.

We will proceed to show that, for each j, F_j extends to a proper rational map from $\mathbb{B}^n$ into $\mathbb{B}^{N_j}$. For this purpose, we let $R_{j,l}(z)$ be the resultant of $P_{j,l}$ in X and let $E_{j,l} = \{R_{j,l} \equiv 0, a_{j,l} \equiv 0\}$, $E = \cup E_{j,l}$. Then E defines a proper complex analytic variety in $\mathbb{C}^n$. For any continuous curve $\gamma : [0, 1] \to \mathbb{C}^n \setminus E$ where $\gamma(0) \in U$, F can be continued holomorphically along γ to get a germ of holomorphic map at $\gamma(1)$. Also, if γ_1 is homotopic to γ_2 in $\mathbb{C}^n \setminus E$, $\gamma_1(0) = \gamma_2(0) \in U$ and $\gamma_1(1) = \gamma_2(1)$, then the continuations of F along γ_1 and γ_2 are the same at $\gamma_1(1) = \gamma_2(1)$. Now let $p_0 \in U$ and $p_1 \in \partial\mathbb{B}^n \setminus E$. Let $\gamma(t)$ be a smooth simple curve connecting p_0 to p_1 and $\gamma(t) \notin \partial\mathbb{B}^n$

for $t \in (0,1)$. Then each F_j defines a holomorphic map in a connected neighborhood V_γ of γ by continuing along γ the initial germ of F_j at p_0. (We can also assume that $V_\gamma \cap \mathbb{B}^n$ is connected.) Let

$$S_\gamma = \{p \in V_\gamma : \ \|F_j(p)\| = 1 \text{ for some } j\}.$$

Then S_γ is a real analytic (proper) subvarieties of V_γ. We first claim

Claim 1 *When V_γ is sufficiently close to γ, $dim(S_\gamma \cap \mathbb{B}^n) \leq 2n-2$.*

Proof of Claim 1: Seeking a contradiction, suppose not. Assume that $t_0 \in (0,1]$ is the first point such that for a certain j, and the local variety defined by $\|F_j(z)\|^2 = 1$ near $p^* = \gamma(t_0)$ has real dimension $2n-1$ at p^*. Since any real analytic subset of codimension two of a connected open set does not affect the connectivity, by slightly changing γ without changing its homotopy type and terminal points, we can assume that $\gamma(t) \notin S_\gamma$ for any $t < t_0$. Hence, p^* also lies on the boundary of the connected component $\hat{V}$ of $(V_\gamma \cap \mathbb{B}^n) \setminus S_\gamma$, that contains $\gamma(t)$ for $t < t_0$. Then a certain small open piece Σ of $S_\gamma \cap \mathbb{B}^n$ containing p^* lies in the boundary of $\hat{V}$. Now, for any $p \in \Sigma$, letting $q(\in \hat{V}) \to p$, we have along $\{q\}$,

$$\lambda(z,\bar{z}) ds_n^2 = \sum_j \lambda_j(z,\bar{z}) F_j^*(ds_{N_j}^2).$$

Suppose that $j^\sharp$ is such that $\|F_{j^\sharp}(p)\| = 1$ and $\|F_j(z)\| < 1$ for any j, $p \in \Sigma$ and $z \in \hat{V}$. Since $p \in \Sigma \subset \mathbb{B}^n$, $ds_n^2|_p < \infty$, we must have

$$\overline{\lim}_{q \to p} F_{j^\sharp}^*(ds_{N_{j^\sharp}}^2)|_q < \infty.$$

On the other hand,

$$F_{j^\sharp}^*(ds_{N_{j^\sharp}}^2) = \frac{\sum_{l,k}\{\delta_{lk}(1-\|F_{j^\sharp}\|^2) + \bar{f}_{j^\sharp,l} f_{j^\sharp,k}\} df_{j^\sharp,l} \otimes d\overline{f}_{j^\sharp,k}}{(1-\|F_{j^\sharp}\|^2)^2}.$$

For any vector $v \in \mathbb{C}^n$ with $\|v\| = 1$,

$$F_{j^\sharp}^*(ds_{N_{j^\sharp}}^2)(v,v)(q) = \frac{\|\sum_\xi \frac{\partial f_{j^\sharp,l}}{\partial z_\xi}(q) v_\xi\|^2}{(1-\|F_{j^\sharp}(q)\|^2)} + \frac{|\sum_{l,\xi} \overline{f_{j^\sharp,l}(q)} \frac{\partial f_{j^\sharp,l}}{\partial z_\xi}(q) v_\xi|^2}{(1-\|F_{j^\sharp}(q)\|^2)^2}. \qquad (2.1.26)$$

Letting $q \to p$, since $1 - \|F_{j^\sharp}(q)\|^2 \to 0^+$, we get

$$\|\sum_\xi \frac{\partial f_{j^\sharp,l}(p)}{\partial z_\xi} v_\xi\|^2 = 0$$

for any $v = (v_1, \ldots, v_\xi, \ldots, v_n)$ with $\|v\| = 1$. Thus

$$\frac{\partial f_{j^\sharp,l}(p)}{\partial z_\xi} = 0, \quad \text{for} \quad l = 1, \ldots, N_{j^\sharp}.$$

Hence, we see $dF_{j^\sharp} = 0$ in a certain open subset of Σ. Since any open subset of Σ is a uniqueness set for holomorphic functions, we see $F_{j^\sharp} \equiv const$. This is a contradiction. $\square$

Now, since $dim(S_\gamma \cap \mathbb{B}^n) \leq 2n-2$, we can always slightly change γ without changing the homotopy type of γ in $V_\gamma \setminus E$ and end points of γ so that $\gamma(t) \notin S_\gamma$ for any $t \in (0,1)$. Since $\lambda(z,\bar{z})ds_n^2 = \sum \lambda_j(z,\bar{z})F_j^*(ds_{N_j}^2)$ in $(V_\gamma \cap \mathbb{B}^n) \setminus S_\gamma$ and since ds_n^2 blows up when $q \in V_\gamma \cap \mathbb{B}^n$ approaches to $\partial\mathbb{B}^n$, we see that for each $q \in V_\gamma \cap \partial\mathbb{B}^n$, $\|F_{j_q}(q)\| = 1$ for some j_q. Hence, we can assume without loss of generality, that there is a $j_0 \geq 1$ such that each of the $F_1, \ldots, F_{j_0}$ maps a certain open piece of $\partial\mathbb{B}^n$ into $\partial\mathbb{B}^{N_1}, \ldots, \partial\mathbb{B}^{N_{j_0}}$, but for $j > j_0$,

$$dim\{q \in \partial\mathbb{B}^n \cap V_\gamma : \|F_j(q)\| = 1\} \leq 2n - 2.$$

By the Hopf lemma, we must have $N_j \geq n$ for $j \leq j_0$. By the result of Forstneric [Fo] and Cima-Suffridge [CS], F_j extends to a rational proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^{N_j}$ for $j \leq j_0$. Now, we must have

$$\lambda(z,\bar{z})ds_n^2 - \sum_{j=1}^{j_0} \lambda_j(z,\bar{z})F_j^*(ds_{N_j}^2) = \sum_{j=j_0+1}^{m} \lambda_j(z,\bar{z})F_j^*(ds_{N_j}^2)$$

in $(V_\gamma \cap \mathbb{B}^n) \setminus S_\gamma$, that is connected by Claim 1. Let $q \in (V_\gamma \cap \mathbb{B}^n) \setminus S_\gamma \to p \in \partial\mathbb{B}^n \cap V_\gamma$. Notice

$$(\lambda(z,\bar{z}) - \sum_{j=1}^{j_0} \lambda_j(z,\bar{z}))ds_n^2\Big|_q + \sum_{j=1}^{j_0} \lambda_j(z,\bar{z})(ds_n^2 - F_j^*(ds_{N_j}^2))\Big|_q = \sum_{j=j_0+1}^{m} \lambda_j(z,\bar{z})F_j^*(ds_{N_j}^2)\Big|_q.$$

By Lemma 2.2, $X_j := ds_n^2 - F_j^*(ds_{N_j}^2)$ is smooth up to $\partial\mathbb{B}^n$ for $j \leq j_0$. We also see, by the choice of j_0 and Claim 1, that for a generic point in $\partial\mathbb{B}^n \cap V_\gamma$, $F_j^*(ds_{N_j}^2)$ is real analytic in a small neighborhood of p for $j \geq j_0 + 1$. Thus by considering the normal component as before in the above equation, we see that $\lambda(z,\bar{z}) - \sum_{j=1}^{j_0} \lambda_j(z,\bar{z})$ has double vanishing order in an open set of the unit sphere. Since $\lambda(z,\bar{z}) - \sum_{j=1}^{j_0} \lambda_j(z,\bar{z})$ is real analytic over $\mathbb{C}^n$, we obtain

$$\lambda(z,\bar{z}) - \sum_{j=1}^{j_0} \lambda_j(z,\bar{z}) = (1-|z|^2)^2\psi(z,\bar{z}). \tag{2.1.27}$$

Here ψ is a certain real analytic function over $\mathbb{C}^n$. Next, write

$$Y = (\lambda(z,\bar{z}) - \sum_{j=1}^{j_0} \lambda_j(z,\bar{z}))ds_n^2.$$

Then Y extends real analytically to $\mathbb{C}^n$. Write $X = \sum_1^{j_0} \lambda_j(z,\bar{z})X_j$. From what we argued above, we easily see that there is a certain small neighborhood $\mathcal{O}$ of $q \in \partial\mathbb{B}^n$ in $\mathbb{C}^n$ such that (1): we can holomorphically continue the initial germ of F in U through a certain simple curve γ with $\gamma(t) \in \mathbb{B}^n$ for $t \in (0,1)$ to get a holomorpic map, still denoted by F, over $\mathcal{O}$; (2): $\|F_j\| < 1$ for $j > j_0$ and $\|F_j(z)\| > 1$ for $j \le j_0$ over $\mathcal{O} \setminus \mathbb{B}^n$; and (3):

$$X = \sum_{j=1}^{j_0} \lambda_j(z,\bar{z})(ds_n^2 - F_j^*(ds_{N_j}^2)) = \sum_{j=1}^{j_0} \lambda_j(z,\bar{z})X_j = \sum_{j=j_0+1}^{m} \lambda_j(z,\bar{z})F_j^*(ds_{N_j}^2) - Y. \tag{2.1.28}$$

We mention that we are able to make $|F_j| < 1$ for any $z \in \mathcal{O}$ and $j > j_0$ in the above due to the fact that $V_\gamma \cap \mathbb{B}^n \setminus S_\gamma$, as defined before, is connected.

Now, let $\mathcal{P}$ be the union of the poles of $F_1, \ldots, F_{j_0}$. Fix a certain $p^* \in \mathcal{O} \cap \partial\mathbb{B}^n$ and let $\widetilde{E} = E \cup \mathcal{P}$. Then for any $\gamma : [0,1] \to (\mathbb{C}^n \setminus \overline{\mathbb{B}^n}) \setminus \widetilde{E}$ with $\gamma(0) = p^*$, F_j extends holomorphically to a small neighborhood U_γ of γ that contracts to γ. Still denote the holomorphic continuation of F_j (from the initial germ of F_j at $p^* \in \mathcal{O}$) over U_γ by F_j. If for some $t \in (0,1)$, $\|F_j(\gamma(t))\| = 1$, then we similarly have

Claim 2 *Shrinking U_γ if necessary, we then have*

$$dim\{p \in U_\gamma : \|F_j(p)\| = 1 \text{ for some } j\} \le 2n-2.$$

Proof of the Claim 2: Still seek for a contradiction, if we suppose not. Define S_γ in a similar way. Without loss of generality, we assume that $t_0 \in (0,1)$ is the first point such that for a certain j_{t_0}, the local variety defined by $\|F_{j_{t_0}}(z)\|^2 = 1$ near $\gamma(t_0)$ has real dimension $2n-1$ at $\gamma(t_0)$. Then, as before, we have

$$X = \sum_{j=1}^{j_0} \lambda_j(z,\bar{z})(ds_n^2 - F_j^*(ds_{N_j}^2)) = \sum_{j=j_0+1}^{m} \lambda_j(z,\bar{z}) F_j^*(ds_{N_j}^2) - Y \tag{2.1.29}$$

in a connected component W of $U_\gamma \setminus S_\gamma$ that contains $\gamma(t)$ for $t << 1$ with $\gamma(t_0) \in \partial W$. Now, for any $q_j (\in W) \to p \in \partial W$ near $p_0 = \gamma(t_0)$ and $v \in \mathbb{C}^n$ with $\|v\| = 1$, we have the following:

$$\begin{aligned} &\sum_{j=1}^{j_0} \lambda_j(z,\bar{z}) \Big(\frac{\| \sum_\xi \frac{\partial f_{j,l}}{\partial z_\xi}(q) v_\xi \|^2}{(1-\|F_j(q)\|^2)} + \frac{| \sum_{l,\xi} \overline{f_{j,l}(q)} \frac{\partial f_{j,l}}{\partial z_\xi}(q) v_\xi |^2}{(1-\|F_j(q)\|^2)^2} \Big) \\ &= \sum_{j=j_0+1}^{m} \lambda_j(z,\bar{z}) \Big(\frac{\| \sum_\xi \frac{\partial f_{j,l}}{\partial z_\xi}(q) v_\xi \|^2}{(1-\|F_j(q)\|^2)} + \frac{| \sum_{l,\xi} \overline{f_{j,l}(q)} \frac{\partial f_{j,l}}{\partial z_\xi}(q) v_\xi |^2}{(1-\|F_j(q)\|^2)^2} \Big) - Y(v,v). \end{aligned} \tag{2.1.30}$$

Now, if p_0 is not a point of codimension one for any local variety defined by $\|F_j(z)\|^2 = 1$ near p_0 for $j \le j_0$, then it has to be a codimension one point for a certain local variety $S_{j'}$ defined by $\|F_{j'}(z)\|^2 = 1$ near p_0 for $j' > j_0$. Let J be the collection of all such j'. Let S^0 be a small open piece of the boundary of W near p_0. Then for a generic $p \in S^0$, the left hand side of (2.1.30) remains bounded as $q \to p \in S^0$. For a term in the right hand side with index $j \in J$, if $S^0 \cap S_j$ contains an open neigborhood of ∂W near p_0, then it approaches to $+\infty$ for a generic p unless $F_j = \mathit{constant}$ as argued before. The other terms on the right hand side remain bounded as $q \to p$ for a generic p. This is a contradiction to the assumption that none of the F_j for $j > j_0$ is constant. Hence, we can assume that p_0 is a point of codimension one for a local variety defined by $\|F_j(z)\|^2 = 1$ near p_0 for a certain $j \le j_0$. Let J be the set of indices such that for $j' \in J$, we have $j' \le j_0$ and $S_{j'} := \{\|F_{j'}\| = 1\}$ is a local real analytic variety of codimension one near p_0. For $j > j_0$, since $\|F_j(z)\| < 1$ for $z(\in U_\gamma) \approx p_0$ and since t_0 is the first point we have some j^* with $\|F_{j^*}\| = 1$ defining a variety of real codimension one, we see that $\|F_j(z)\| < 1$ for $z(\in W) \approx p_0$. Define S^0 similarly, as an open piece of ∂W. Hence, as $q(\in W) \to p \in S^0$, the right hand side of (2.1.30) remains to be non-negative. On the other hand, in the left hand side of (2.1.30) , for any $j' \in J$ with $S_{j'} \cap S^0$ containing an open piece of ∂W near p_0, if the numerator $| \sum_{l,\xi} \overline{f_{j',l}(q)} \frac{\partial f_{j',l}}{\partial z_\xi}(q) v_\xi |^2$ of the last term does not go to 0 for some vectors v, then the term with index j' on the left hand side would go to $-\infty$ for a generic $p \in S^0$. If this happens to such j', the left hand side would approach to $-\infty$. Notice that all other terms on the right hand side remain bounded as $q \to p \in S^0$ for a generic p. This is impossible. Therefore we must have for some $j' \in J$ that $|\sum_l \overline{f_{j',l}(q)} \frac{\partial f_{j',l}}{\partial z_\xi}(q)|^2 = \frac{\partial \sum_l |f_{j',l}|^2}{\partial z_\xi}(q) = \frac{\partial \|F_{j'}\|^2}{\partial z_\xi}(q) \to 0$ and thus $\frac{\partial \|F_{j'}\|^2}{\partial z_\xi}(p) = 0$ for all ξ and $p \in S_{j'}$. This immediately gives the equality $d(\|F_{j'}\|^2) = 0$ along $S_{j'}$. Assume, without loss of generality, that p_0 is also a smooth point of $S_{j'}$. If $S_{j'}$ has no complex hypersurface passing through p_0, by a result of Trepreau [Tr], the union of the image of local holomoprphic disks attached to $S_{j'}$ passing through p_0 fills in an open subset. Since $F_{j'}$ is not constant, there is a small holomorphic disk smooth up to the boundary $\phi(\tau) : \mathbb{B}^1 \to \mathbb{C}^n$ such that $\phi(\partial\mathbb{B}^1) \subset S_{j'}, \phi(1) = p_0$ and $F_{j'}$ is not constant along ϕ. Since $\partial\mathbb{B}^{N_{j'}}$ does not contain any non-trival complex curves, $r = (\|F_{j'}\|^2 - 1) \circ \phi \not\equiv 0$. Applying the maximum principle and then the Hopf lemma to the subharmonic function $r = (\|F_{j'}\|^2 - 1) \circ \phi$, we see that the outward normal derivative of r at $\tau = 1$ is positive. This contradicts the fact that $d(\|F_{j'}\|^2) = 0$ along $S_{j'}$. We can argue the same way for points $p \in S_{j'}$ near p_0 to conclude that for any $p \in S_{j'}$ near p_0, there is a complex hypersurface contained in $S_{j'}$ passing through p. Namely, $S_{j'}$ is Levi flat, foliated by a family of smooth complex hypersurfaces denoted by Y_η with real parameter η near p_0. Let Z be a holomorphic vector field along Y_η. We then easily see that $0 = \overline{Z}Z(\|F_{j'}\|^2 - 1) = \sum_{k=1}^{N_{j'}} |Z(f_{j',k})|^2$. Thus, we see that $F_{j'}$ is constant along each Y_η. Hence, $F_{j'}$ can not be a local embedding at each point of $S_{j'}$. On the other hand, noticing that $F_{j'}$ is a proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^{N'}$, $F_{j'}$ is a local embedding near $\partial\mathbb{B}^n$. Hence, the set of points where $F_{j'}$ is not a local embedding can be at most of complex codimension one (and thus real codimension two). This is a contradiction. This proves Claim 2.

Hence, we see that $\mathcal{E} = \{p \in (\mathbb{C}^n \backslash \overline{\mathbb{B}^n}) \backslash \widetilde{E}$: some branch, obtained by the holomorphic continuation through curves described before, of F_j for some j maps p to $\partial\mathbb{B}^{N_j}\}$ is a real analytic variety of real dimension at most $2n - 2$. Now, for any $p \in (\mathbb{C}^n \setminus \overline{\mathbb{B}^n}) \setminus \widetilde{E}$, any curve $\gamma : [0,1] \to (\mathbb{C}^n \setminus \overline{\mathbb{B}^n}) \setminus \widetilde{E}$ with $\gamma(0) = p^* \in \mathcal{O} \cap \partial\mathbb{B}^n$ and $\gamma(1) = p$, we can homotopically change γ in $(\mathbb{C}^n \setminus \overline{\mathbb{B}^n}) \setminus \widetilde{E}$ (but without changing the terminal points) such that $\gamma(t) \notin \mathcal{E}$ for $t \in (0,1)$. Now, the holomorphic continuation of the initial germ of F_j from p^* never cuts $\partial\mathbb{B}^{N_j}$ along $\gamma(t)$ $(0 < t < 1)$. We thus see that $\|F_j(p)\| \leq 1$ for $j > j_0$.

Let $\{(f_{j,l})_{k;p}\}_{k=1}^{n_{jl}}$ be all possible (distinct) germs of holomorphic functions that we can get at p by the holomorphic continuation, along curves described above in $(\mathbb{C}^n \setminus \overline{\mathbb{B}^n}) \setminus \tilde{E}$, of $f_{j,l}$. Let $\sigma_{jl,\tau}$ be the fundamental symmetric function of $\{(f_{j,l})_{k;p}\}_{k=1}^{n_{jl}}$ of degree τ. Then $\sigma_{jl,\tau}$ well defines a holomorphic function over $(\mathbb{C}^n \setminus \overline{\mathbb{B}^n})$. $\|\sigma_{jl,\tau}\|$ is bounded in $(\mathbb{C}^n \setminus \overline{\mathbb{B}^n}) \setminus \tilde{E}$. By the Riemann removable singularity theorem, $\sigma_{jl,\tau}$ is holomorphic over $(\mathbb{C}^n \setminus \overline{\mathbb{B}^n})$. By the Hartogs lemma, $\sigma_{jl,\tau}$ extends to a bounded holomorphic function over $\mathbb{C}^n$. Hence, by the Liouville theorem, $\sigma_{jl,\tau} \equiv const$. This forces $(f_{j,l})_k$ and thus F_j for $j > j_0$ to be constant. We obtain a contradiction. This proves that each F_j extends to a proper rational map from $\mathbb{B}^n$ into $\mathbb{B}^{N_j}$. Together with Proposition 2.1.2, we complete the proof of the main Theorem. $\square$

We conclude the section with a remark: The regularity of λ_j, λ can be reduced to be only real analytic in the complement of a certain real codimension two subset. This is obvious from our proof for Theorem 2.1.1.

2.2 Rigidity for local holomorphic isometric embeddings from the projective space into the product of projective spaces

In this section, we try to attack the classification problem 1.1.1. Let $\{[z_0, \cdots, z_n]\}$ be the homogeneous coordinate of $\mathbb{P}^n$ and ω_n be the Fubini-Study metric. Let $U_0 := \{z_0 \neq 0\}$ be one coordinate chart of $\mathbb{P}^n$, and the coordinate is given by $x_i = \frac{z_i}{z_0}$. Then on U_0, the Fubini-Study metric can be written as:

$$\omega_n = \sqrt{-1}\partial\overline{\partial} \log \big(1 + \sum_{i=1}^{n} |x_i|^2\big).$$

Calabi proved the local rigidity and global extension theorem for the local holomorphic embedding from a Kähler manifold into the projective space equipped with the Fubini-Study metric. We are going to apply them in the proof of our theorems.

Theorem 2.2.1 *[Ca] If M is a Kähler manifold and assume that an open set $G \subset M$ admits a holomorphic isometric embedding into $(\mathbb{P}^N, \omega_N)$, then the embedding is uniquely determined modular the group of isometries $Isom(\mathbb{P}^N, \omega_N)$.*

Theorem 2.2.2 *[Ca] Let M is a simply connected Kähler manifold and let $f : G \subset M \to \mathbb{P}^N$ be an holomorphic isometric embedding with respect to the Fubini-Study metric ω_N. Then there exists a global mapping $F : M \to \mathbb{P}^N$ extending f.*

Definition 2.2.1 *Let $f, g : M \to \mathbb{P}^n$ are holomorphic maps. f, g are called equivalent if f, g are equivalent up to the group of isometries of $\mathbb{P}^n$. More precisely, there exists $\sigma \in Isom(\mathbb{P}^n, \omega_n)$, such that $f = \sigma \circ g$.*

From now on, we consider our classification question 1.1.1. Let $\lambda_l > 0$ be a positive constant and let

$$F = (F_1, \cdots, F_p) : (U \subset \mathbb{P}^n, \omega_n) \to (\mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_p}, \oplus_{l=1}^{p} \lambda_l \omega_{N_l})$$

be a local holomorphic isometric imbedding with respect to Fubini-Study metrics, i.e.

$$\omega_n = \sum_l \lambda_l F_l^* \omega_{N_l}.$$

First, we prove the global extension theorem in a special case.

Theorem 2.2.3 *Under the above assumption and further assuming $\lambda_l \in \mathbb{Q}^+$, $\tilde{F}$ extends F to $\mathbb{P}^n$ as a holomorphic isometry, i.e. there exists a holomorphic map*

$$\tilde{F} = (\tilde{F}_1, \cdots, \tilde{F}_p) : \mathbb{P}^n \to \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_p}$$

such that

$$\tilde{F}\big|_U = F \quad and \quad \omega_n = \sum_l \lambda_l \tilde{F}_l^* \omega_{N_l}.$$

Proof For simplicity, we give the proof for $p = 2$. By multiplying the common denominators of λ_l, still denoted by λ_l, we assume that

$$F = (F_1, F_2) : U \subset \mathbb{P}^n \to \mathbb{P}^{N_1} \times \mathbb{P}^{N_2}$$

with

$$\lambda\omega_n = \lambda_1 F_1^* \omega_{N_1} + \lambda_2 F_2^* \omega_{N_2}$$

where λ, λ_i are positive integers. Then one has the following diagram:

$$\begin{tikzcd} U \subset \mathbb{P}^n \arrow[r, "F"] \arrow[dr, "G"'] & \mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \arrow[d, hook, "s"] \\ & \mathbb{P}^N \end{tikzcd}$$

where $G = s \circ F$ and s is the Segre embedding, which is a holomorphic isometry from the product of projective spaces into the bigger projective space.

$$\begin{array}{ccccccc} s: & \mathbb{P}^{N_1} & \times & \mathbb{P}^{N_2} & \longrightarrow & \mathbb{P}^N \\ & [Z] & , & [W] & \longmapsto & [Z^I W^J] \end{array}$$

where

$$[Z] = [Z_0, \cdots, Z_{N_1}], [W] = [W_0, \cdots, W_{N_2}], |I| = \lambda_1, |J| = \lambda_2,$$

and

$$N = (N_1 + 1)^{\lambda_1} (N_2 + 1)^{\lambda_2} - 1.$$

For s is an isometric embedding, i.e.

$$s^* \omega_N = \lambda_1 \omega_{N_1} \oplus \lambda_2 \omega_{N_2},$$

one obtains a local holomorphic isometric embedding G between projective spaces

$$G^* \omega_N = \lambda \omega_n.$$

By Theorem 2.2.2, G extends to the ambient space $\mathbb{P}^n$, denoted by $\tilde{G}$. Since the image of s is an algebraic subvariety in $\mathbb{P}^N$, $\tilde{G}(\mathbb{P}^n) \subset \text{Image}(s)$. Furthermore, s is invertible. Let $\tilde{F} = s^{-1} \circ \tilde{G}$. Then $\tilde{F}$ extends F. $\square$

Before classifying local holomorphic isometries from the projective space to the product of projective spaces, we give examples of holomorphic isometries between projective spaces.

Example 2.2.1 *We start with the Veronese embedding with isometric constant 2.*

$$v_{n,2}: \quad \mathbb{P}^n \longrightarrow \mathbb{P}^{\frac{(n+1)(n+2)}{2}-1}$$

$$[z] \hookrightarrow [\sqrt{2}\cos\big(\frac{\pi}{4}\delta_{ij}\big)z_i z_j]$$

One can check that

$$\sum_{i,j=0}^{n} |\sqrt{2}\cos\big(\frac{\pi}{4}\delta_{ij}\big)z_i z_j|^2 = \big(\sum_{i=0}^{n}|z_i|^2\big)^2,$$

yielding

$$v_{n,2}^*\left(\omega_{\frac{(n+1)(n+2)}{2}}\right) = 2\omega_n$$

by taking $\sqrt{-1}\partial\overline{\partial}\log$. *Let*

$$M_{n,2} = \frac{(n+1)(n+2)}{2} - 1.$$

One can check that $M_{n,2}$ *is the smallest dimension of the projective space admitting holomorphic isometry from* $\mathbb{P}^n$ *with isometric constant 2. Suppose that we have another holomorphic embedding* $f : \mathbb{P}^n \to \mathbb{P}^N$ *with*

$$f^*\omega_N = 2\omega_n \quad \text{and} \quad N \geq M_{n,2}.$$

Then f *is equivalent to* $(v_{n,2}, 0, \cdots, 0)$ *by Theorem 2.2.1.*

Similarly, one can also cook up the Veronese embedding $v_{n,\kappa}$ *with any integer isometric constant* κ. *Define* $M_{n,\kappa}$ *to be the smallest dimension of the projective space admitting holomorphic isometry from* $\mathbb{P}^n$ *with isometric constant* κ *and we have the same rigidity property as above.*

The following theorem gives the classification of the local holomorphic isometric embedding when $\lambda_l \in \mathbb{Q}$.

Theorem 2.2.4 *We suppose the assumption in Theorem 2.2.3. Let* $\{\kappa_1, \cdots, \kappa_p\} \in (\mathbb{Z}^+)^p$ *satisfy*

(i) $N_l \geq M_{n,\kappa_l}$ *for* $1 \leq l \leq q$,

(ii) $\sum_{l=1}^{p} \kappa_l \lambda_l = 1$.

Then the possibility of $\{\kappa_1, \cdots, \kappa_p\}$ *gives the classification of* $F = (F_1, \cdots, F_p)$. *In particular,* F_l *is equivalent to the Veronese embedding* V_{n,κ_l} *with isometric constant* κ_l *and* F_l *is a constant map if* $\kappa_l = 0$.

Proof Recall that $z = [z_0, \cdots, z_n], w = [w_0, \cdots, w_{N_l}]$ are the homogenous coordinates for $\mathbb{P}^n$ and $\mathbb{P}^{N_l}$ respectively. We use the coordinates $x_i = \frac{z_i}{z_0}$ on $U_0 \subset \mathbb{P}^n$ and let $V_{0,l} = \{w_0 \neq 0\}$ be a coordinate chart on $\mathbb{P}^{N_l}$.

By Theorem 2.2.3, we know that $F_l = (f_{l,0}, \cdots, f_{l,N_l})$ are all homogenous polynomials of z mapping $\mathbb{P}^n$ into $\mathbb{P}^{N_l}$. By composing with $\mathrm{Isom}(\mathbb{P}^{N_l}, \omega_{N_l})$, we can assume that $F_l([1, 0, \cdots, 0]) = [1, 0, \cdots, 0]$. Hence F_l maps some open subset of U_0 into $V_{0,l}$. Let

$$r_{l,q_l}(x) = \frac{f_{l,q_l}(z)}{f_{l,0}(z)}$$

be a rational function in x for $1 \leq q_l \leq N_l$ and write $F_l = (r_{l,1}(x), \cdots, r_{l,N_l}(x))$. As F_l is an isometric embedding, we have

$$\sqrt{-1}\partial\bar{\partial}\log(1 + \sum_{i=1}^{n} |x_i|^2) = \sum_l \lambda_l \sqrt{-1}\partial\bar{\partial}\log(1 + \sum_{q_l=1}^{N_l} |r_{l,q_l}(x)|^2).$$

Getting rid of $\sqrt{-1}\partial\bar{\partial}$, one has:

$$\log(1 + \sum_{i=1}^{n} |x_i|^2) = \sum_l \lambda_l \log(1 + \sum_{q_l=1}^{N_l} |r_{l,q_l}(x)|^2) + \mathrm{Re} h(x)$$

for some holomorphic function $h(x)$.

Comparing the Taylor expansion of two sides, when $r_{l,q_l}(0) = 0$, the other two quantities do not have the pure holomorphic and anti-holomorphic terms like $x^\alpha, \bar{x}^\alpha$ except for $\mathrm{Re}h$, implying that $h = 0$. Therefore,

$$1 + \sum_{i=1}^{n} |x_i|^2 = \prod_l (1 + \sum_{q_l=1}^{N_l} |r_{l,q_l}(x)|^2)^{\lambda_l}.$$

Note that the right hand side approaches ∞ when x approaches any pole of $r_{l,q_l}(x)$, while the left hand is bounded, implying that $r_{l,q_l}(x)$ is actually a polynomial. By polarization (replacing $\bar{x}_i$ by y_i), it follows that:

$$1+\sum_{i=1}^{n} x_i y_i = \prod_{l} (1+\sum_{q_l=1}^{N_l} r_{l,q_l}(x)\bar{r}_{l,q_l}(y))^{\lambda_l}.$$

Since $1+\sum_{i=1}^{n} x_i y_i$ is irreducible, and $\mathbb{C}[x,y]$ is a unique factorization domain, it follows that

$$1+\sum_{q_l=1}^{N_l} r_{l,q_l}(x)\bar{r}_{l,q_l}(y) = (1+\sum_{i=1}^{n} x_i y_i)^{\kappa_l}$$

for the possible integer κ_l satisfying condition (i) and (ii). This implies

$$1+\sum_{q_l=1}^{N_l} |r_{l,q_l}(x)|^2 = (1+\sum_{i=1}^{n} |x_i|^2)^{\kappa_l},$$

meaning that F_l is an isometry from $\mathbb{P}^n$ to $\mathbb{P}^{N_l}$ with isometric constant κ_l. By Theorem 2.2.1, F_l is equivalent to the Veronese embedding V_{n,κ_l}.

If $\kappa_l = 0$, i.e.

$$1+\sum_{q_l=1}^{N_l} |r_{l,q_l}(x)|^2 = 1,$$

it follows that $r_{l,q_l}(x) \equiv 0$, implying that F_l is a constant map. $\square$

Chapter 3

On the modified Kähler-Ricci flow

3.1 Preliminaries

Let X be a closed Kähler manifold of complex dimension n with a Kähler metric ω_0, and let ω_∞ be a real, smooth, closed $(1,1)$-form with $[\omega_\infty]^n = 1$. Let Ω be a smooth volume form on X such that $\int_X \Omega = 1$. Set $\chi = \omega_0 - \omega_\infty$, $\omega_t = \omega_\infty + e^{-t}\chi$. Let $\varphi : [0,\infty) \times X \to \mathbb{R}$ be a smooth function such that $\widetilde{\omega}_t = \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi > 0$. Consider the following Monge-Ampère flow:

$$\begin{cases} \frac{\partial}{\partial t}\varphi = \log \frac{(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi)^n}{\Omega}, \\ \\ \varphi(0,\cdot) = 0. \end{cases} \tag{3.1.1}$$

Then the evolution for the corresponding Kähler metric is given by:

$$\begin{cases} \frac{\partial}{\partial t}\tilde{\omega}_t = -Ric(\tilde{\omega}_t) + Ric(\Omega) - e^{-t}\chi, \\ \\ \tilde{\omega}_t(0,\cdot) = \omega_0. \end{cases} \tag{3.1.2}$$

We will give some definitions before stating our theorem.

Definition 3.1.1 $[\gamma] \in H^{1,1}(X,\mathbb{C})$ *is semi-positive if there exists* $\gamma' \in [\gamma]$ *such that* $\gamma' \geq 0$*, and is big if* $[\gamma]^n := \int_X \gamma^n > 0$.

Definition 3.1.2 *A closed, positive* $(1,1)$*-current* ω *is called a singular Calabi-Yau metric on* X *if* ω *is a smooth Kähler metric away from an analytic subvariety* $E \subset X$ *and satisfies* $Ric(\omega) = 0$ *away from* E.

Definition 3.1.3 *A volume form Ω is called a Calabi-Yau volume form if*

$$Ric(\Omega) := -\sqrt{-1}\partial\overline{\partial}\log\Omega = 0.$$

Theorem 3.1.1 *Let X be a Kähler manifold with a Kähler metric ω_0. Suppose that $[\omega_\infty] \in H^{1,1}(X,\mathbb{C})\cap H^2(X,\mathbb{Z})$ is semi-positive and big. Then along the modified Kähler-Ricci flow (3.1.2), $\widetilde{\omega}_t$ converges weakly in the sense of currents and converges locally in C^∞-norm away from a proper analytic subvariety of X to the unique solution of the degenerate Monge-Ampère equation $(\omega_\infty + \sqrt{-1}\partial\overline{\partial}\psi)^n = \Omega$.*

Corollary 3.1.1 *When X is a Calabi-Yau manifold and Ω is a Calabi-Yau volume form, $\widetilde{\omega}_t$ converges to a singular Calabi-Yau metric.*

As $[\omega_\infty] \in H^{1,1}(X,\mathbb{C}) \cap H^2(X,\mathbb{Z})$, there exists a line bundle L over X such that $\omega_\infty \in c_1(L)$. Moreover, L is big when $[\omega_\infty]$ is semi-positive and big [De]. Hence X is Moishezon. Furthermore, X is algebraic, for it is Kähler. Therefore, by applying Kodaira lemma, we see that for any small positive number $\epsilon \in \mathbb{Q}$, there exists an effective divisor E on X, such that $[L] - \epsilon[E] > 0$. Therefore, there exists a hermitian metric h_E on E such that $\omega_\infty - \epsilon Ric(h_E)$, denoted by ω_E, is strictly positive for any ϵ small. Let S be the defining section of the effective divisor E.

We now define some notations for the convenience of our later discussions. Let $\tilde{\nabla}$, $\tilde{\Delta}$ and $\frac{\partial}{\partial t} - \tilde{\Delta}$ be the gradient operator, the Laplacian and the heat operator with respect to the metric $\tilde{\omega}_t$ respectively. Let $V_t = [\tilde{\omega}_t]^n$ with V_t uniformly bounded and $V_0 = 1$. Write $\dot{\varphi} = \frac{\partial\varphi}{\partial t}$ for simplicity.

Before proving the theorem, we would like to sketch the standard uniform estimates of φ and recall some crucial estimates due to Zhang.

First of all, by the standard computation as in [Z3], the uniform upper bound of $\frac{\partial}{\partial t}\varphi$ is deduced from the maximum principle. Secondly, by the result of [Z2] [EGZ], generalizing the theorem of Kołodziej [K1] to the degenerate case, we have the C^0-estimate $\|u\|_{C^0(X)} \leq C$ independent of t where $u = \varphi - \int_X \varphi\Omega$. Then to estimate $\frac{\partial}{\partial t}\varphi$ locally, we will calculate $(\frac{\partial}{\partial t} - \tilde{\Delta})[\dot{\varphi} + A(u - \epsilon\log\|S\|^2_{h_E})]$, and then the maximum principle yields $\frac{\partial}{\partial t}\varphi \geq -C + \alpha\log\|S\|^2_{h_E}$ for $C, \alpha > 0$.

Next, we follow the standard second order estimate as in [Y1] [Cao] [Si] [Ts]. Calculating $(\frac{\partial}{\partial t} - \tilde{\Delta})[\log tr_{\omega_E + e^{-t}\chi}\tilde{\omega}_t - A(u - \epsilon \log \|S\|^2_{h_E})]$ and applying maximum principle, we have: $|\Delta_{\omega_E}\varphi| \leq C$. Then by using Schauder estimates and third order estimate as in [Z3], we can obtain the local uniform estimate: For any $k \geq 0, K \subset\subset X \setminus E$, there exists $C_{k,K} > 0$, such that:

$$\|u\|_{C^k([0,+\infty)\times K)} \leq C_{k,K}. \tag{3.1.3}$$

In [Z3], Zhang proved the following theorem by comparing (3.1.1) with the Kähler-Ricci flow (3.1.4). We include the detail of the proof here for the sake of completeness. This uniform lower bound appears to be crucial in the proof of the convergence.

Theorem 3.1.2 *([Z3]) There exists $C > 0$ such that $\frac{\partial}{\partial t}\varphi \geq -C$ holds uniformly along (3.1.1) or (3.1.2).*

We derive a calculus lemma here for later application.

Lemma 3.1.1 *Let $f(t) \in C^1([0, +\infty))$ be a non-negative function. If $\int_0^{+\infty} f(t)dt < +\infty$ and $\frac{\partial f}{\partial t}$ is uniformly bounded, then $f(t) \to 0$ as $t \to +\infty$.*

Proof We prove this calculus lemma by contradiction. Suppose that there exist a sequence $t_i \to +\infty$ and $\delta > 0$, such that $f(t_i) > \delta$. Since $\frac{\partial f}{\partial t}$ is uniformly bounded, there exist a sequence of connected, non-overlapping intervals I_i containing t_i with fixed length l, such that $f(t) \geq \frac{\delta}{2}$ over I_i. Then $\int_{\cup_i I_i} f(t)dt \geq \sum_i l\frac{\delta}{2} \to +\infty$, contradicting with $\int_0^{+\infty} f(t)dt < +\infty$. $\square$

Let $\hat{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}\phi$ and $\hat{\Delta}$ be the Laplacian operator with respect to the metric $\hat{\omega}_t$. Consider the Monge-Ampère flow, as well as its corresponding evolution of metrics:

$$\begin{cases} \frac{\partial}{\partial t}\phi = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\phi)^n}{\Omega} - \phi, \\ \\ \phi(0, \cdot) = 0. \end{cases} \tag{3.1.4}$$

$$\begin{cases} \frac{\partial}{\partial t}\hat{\omega}_t = -Ric(\hat{\omega}_t) + Ric(\Omega) - \hat{\omega}_t + \omega_\infty, \\ \\ \hat{\omega}_t(0,\cdot) = \omega_0. \end{cases} \tag{3.1.5}$$

The following theorem is proved in [Z1] and we sketch Zhang's argument here.

Theorem 3.1.3 *There exists $C > 0$, such that $\frac{\partial}{\partial t}\phi \geq -C$ uniformly along (3.1.4).*

Proof The standard computation shows:

$$(\frac{\partial}{\partial t} - \hat{\Delta})(\frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial t^2}) \leq -(\frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial t^2}).$$

Then the maximum principle yields:

$$\frac{\partial}{\partial t}(\phi + \frac{\partial \phi}{\partial t}) \leq C_1 e^{-t},$$

which further implies:

$$\frac{\partial}{\partial t}\phi \leq C_2 e^{-\frac{t}{2}}.$$

Hence $\phi + \frac{\partial \phi}{\partial t}$ and ϕ are essentially decreasing along the flow, for example: $\frac{\partial}{\partial t}(\phi + \dot{\phi} + C_1 e^{-t}) \leq 0$ and ϕ is uniformly bounded from above. As one also has the uniform estimate of ϕ that ϕ is uniformly bounded in $C^k([0,\infty) \times K)$-norm for any $k \geq 0$ and $K \subset\subset X \setminus E$, one can conclude that $\frac{\partial \phi}{\partial t} \to 0$ pointwisely away from E by applying Lemma 3.1.1 to $C_2 e^{-\frac{t}{2}} - \frac{\partial \phi}{\partial t}$. Suppose that ϕ_∞ is the limit of ϕ. Then $\phi_\infty \in PSH(X, \omega_\infty)$ and ϕ_∞ is also the pointwise limit of $\phi + \frac{\partial \phi}{\partial t}$ away from E. By the uniform estimate of ϕ once more, one knows that ϕ converges to ϕ_∞ in $C^\infty(K)$ for any $K \subset\subset X \setminus E$. Then the convergence of (3.1.4) is obtained [TZha] in the sense that

$$e^{\phi_\infty}\Omega = (\omega_\infty + \sqrt{-1}\partial\overline{\partial}\phi_\infty)^n \quad \leftarrow \quad e^{\phi + \dot{\phi}}\Omega = (\omega_t + \sqrt{-1}\partial\overline{\partial}\phi)^n$$

in $C^k(K)$-norm for any $k \geq 0, K \subset\subset X \setminus E$ as $t \to +\infty$. Furthermore, by [K1], $\phi(t,\cdot) \in L^\infty(X)$ for $0 \leq t < \infty$. In addition, $(\omega_\infty + \sqrt{-1}\partial\overline{\partial}\phi_\infty)^n$ does not charge any pluri-polar set, in particular, the effective divisor E, as ϕ_∞ is the decreasing limit of $\phi(t,\cdot) + 2C_2 e^{-\frac{t}{2}}$. Therefore,

$$e^{\phi_\infty}\Omega = (\omega_\infty + \sqrt{-1}\partial\overline{\partial}\phi_\infty)^n$$

holds in the sense of currents on X. On the other hand, from the pluri-potential theory [Z2] [EGZ], we know that $-C_3 \le \phi_\infty - \sup_X \phi_\infty \le C_3$ for $C_3 > 0$ as $\|e^{\phi_\infty}\|_{L^p(\Omega)} < C_4$ for any $p > 0$ and $\sup_X \phi_\infty \ne -\infty$. By the essentially decreasing property, it follows that away from E:

$$\phi + \dot{\phi} + C_1 e^{-t} \ge \phi_\infty \ge -C_3 + \sup_X \phi_\infty \ge -C_5.$$

Hence the uniform lower bound of $\dot{\phi}$ is obtained as ϕ is uniformly bounded from above and $\dot{\phi}$ is smooth on X.

□

We are now ready to give Zhang's proof to Theorem 3.1.2.

Proof of Theorem 3.1.2:

Notice that φ, ϕ are solutions to equation (3.1.1) and (3.1.4) respectively. Fix $T_0 > 0$. Let $\kappa(t,\cdot) = (1-e^{-T_0})\dot{\varphi} + u - \phi(t+T_0)$ with $\kappa(0,\cdot) \ge -C_1$. Then:

$$\begin{aligned}
\frac{\partial}{\partial t}\kappa(t,\cdot) &= \tilde{\Delta}((1-e^{-T_0})\dot{\varphi} + u) + \dot{u} - \dot{\phi}(t+T_0) - n + tr_{\tilde{\omega}_t}\omega_{t+T_0} \\
&= \tilde{\Delta}((1-e^{-T_0})\dot{\varphi} + u - \phi(t+T_0)) + \dot{u} - \dot{\phi}(t+T_0) - n + tr_{\tilde{\omega}_t}\hat{\omega}_{t+T_0} \\
&\ge \tilde{\Delta}((1-e^{-T_0})\dot{\varphi} + u - \phi(t+T_0)) + \dot{\varphi} - C_2 + n(\frac{C_3}{e^{\dot{\varphi}}})^{\frac{1}{n}},
\end{aligned}$$

where the fact: $\dot{u} \ge \dot{\varphi} - C, -C \le \frac{\partial}{\partial t}\phi \le C$ and $\hat{\omega}_t \ge C_3\Omega$ are used.

Suppose that $\kappa(t,\cdot)$ achieves minimum at (t_0,p_0) with $t_0 > 0$. Then the maximum principle yields $\dot{\varphi}(t_0,p_0) \ge -C_4$. Hence $\dot{\varphi}$ is bounded from below. Suppose that $\kappa(t,\cdot)$ achieves minimum at $t = 0$. Then the theorem follows trivially. □

3.2 Proof of Theorem 3.1.1

Inspired from the Mabuchi K-energy in the study on the convergence of the Kähler-Ricci flow on Fano manifolds, we similarly define an energy functional as follows:

$$\nu(\varphi) = \int_X \log \frac{(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi)^n}{\Omega}(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi)^n.$$

Next, we will give some properties of $\nu(\varphi)$ and then a key lemma for the proof of the main theorem.

Proposition 3.2.1 *$\nu(\varphi)$ is well-defined and there exists $C > 0$, such that $-C < \nu(\varphi) < C$ along (3.1.2).*

Proof It is easy to see that $\nu(\varphi)$ is well-defined. If we rewrite

$$\nu(\varphi) = \int_X \dot{\varphi}\tilde{\omega}_t^n,$$

then $\nu(\varphi)$ is uniformly bounded from above and below by the uniform upper and lower bound of $\dot{\varphi}$. Here, we derive the uniform lower bound by Jensen's inequality without using Theorem 3.1.2:

$$\nu(\varphi) = -V_t \int_X \log \frac{\Omega}{\tilde{\omega}_t^n} \frac{\tilde{\omega}_t^n}{V_t} \geq -V_t \log \int_X \frac{\Omega}{V_t} \geq -C,$$

as V_t is uniformly bounded. □

Proposition 3.2.2 *There exists constant $C > 0$ such that for all $t > 0$ along the flow (3.1.2):*

$$\frac{\partial}{\partial t}\nu(\varphi) \leq -\int_X \|\tilde{\nabla}\dot{\varphi}\|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n + Ce^{-t}. \tag{3.2.1}$$

Proof Along the flow (3.1.2), we have:

$$
\begin{aligned}
\frac{\partial}{\partial t}\nu(\varphi) &= \int_X \tilde{\Delta}\dot{\varphi}\tilde{\omega}_t^n - e^{-t}\int_X tr_{\tilde{\omega}_t}\chi\tilde{\omega}_t^n - ne^{-t}\int_X \dot{\varphi}\chi\wedge\tilde{\omega}_t^{n-1} + n\int_X \dot{\varphi}\sqrt{-1}\partial\overline{\partial}\dot{\varphi}\wedge\tilde{\omega}_t^{n-1} \\
&= -\int_X \|\tilde{\nabla}\dot{\varphi}\|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n - ne^{-t}\int_X \chi\wedge\tilde{\omega}_t^{n-1} - ne^{-t}\int_X \dot{\varphi}\chi\wedge\tilde{\omega}_t^{n-1} \\
&= -\int_X \|\tilde{\nabla}\dot{\varphi}\|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n - n[\chi][\omega_t]^{n-1}e^{-t} - ne^{-t}\int_X \dot{\varphi}(\omega_0-\omega_\infty)\wedge\tilde{\omega}_t^{n-1} \\
&\leq -\int_X \|\tilde{\nabla}\dot{\varphi}\|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n + n[\chi][\omega_t]^{n-1}e^{-t} + Ce^{-t}\int_X (\omega_0+\omega_\infty)\wedge\omega_t^{n-1} \\
&= -\int_X \|\tilde{\nabla}\dot{\varphi}\|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n + [n\chi + C\omega_0 + C\omega_\infty][\omega_t]^{n-1}e^{-t} \\
&\leq -\int_X \|\tilde{\nabla}\dot{\varphi}\|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n + C'e^{-t}.
\end{aligned}
$$

Notice that we used the evolution of $\dot{\varphi}$ and integration by parts in the first two equalities and the uniform bound of $\dot{\varphi}$ in the first inequality and the last inequality holds since ω_t is uniformly bounded. □

Lemma 3.2.1 *On each $K \subset\subset X\setminus E$, $\|\nabla_{\omega_\infty}\dot{\varphi}(t)\|^2_{\omega_\infty} \to 0$ uniformly as $t\to+\infty$.*

Proof Integrating (3.2.1) from 0 to T, we have:

$$-C \leq \nu(\varphi)(T) - \nu(\varphi)(0) \leq -\int_0^T\int_X \|\tilde{\nabla}\dot{\varphi}\|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n dt + C$$

for some constant $C>0$. It follows that

$$\int_0^{+\infty}\int_X \|\tilde{\nabla}\dot{\varphi}\|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n dt \leq 2C,$$

by letting $T\to+\infty$. Hence, (3.1.3) and Lemma 3.1.1 imply that for any compact set $K'\subset X\setminus E$,

$$\int_{K'} \|\tilde{\nabla}\dot{\varphi}(t,\cdot)\|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n \to 0. \tag{3.2.2}$$

Now, assume that there exists $\delta>0$, $z_j\in K$ and $t_j\to\infty$ such that $\|\nabla_{\omega_\infty}\dot{\varphi}(t_j,z_j)\|^2_{\omega_\infty} > \delta$. It follows from (3.1.3) that $\|\tilde{\nabla}\dot{\varphi}(t_j,z)\|^2_{\tilde{\omega}_{t_j}} > \frac{\delta}{2}$ for $z\in B(z_j,r)\subset K'$, $K\subset K'\subset\subset X\setminus E$ and $r>0$. This contradicts with (3.2.2).

□

Proof of Theorem 3.1.1:

First of all, we want to show that for any $K \subset\subset X\backslash E$, $u(t) \to \psi$ in $C^\infty(K)$. Exhaust $X\backslash E$ by compact sets K_i with $K_i \subset K_{i+1}$ and $\cup_i K_i = X\backslash E$. As $\|u\|_{C^k(K_i)} \leq C_{k,i}$, after passing to a subsequence t_{i_j}, we know $u(t_{i_j}) \to \psi$ in $C^\infty(K_i)$ topology. By picking up the diagonal subsequence of $u(t_{i_j})$, we know $\psi \in L^\infty(X) \cap C^\infty(X\backslash E) \cap PSH(X\backslash E, \omega_\infty)$. Furthermore, ψ can be extended over the pluri-polar set E as a bounded function in $PSH(X, \omega_\infty)$. Taking gradient of (3.1.1), by Lemma 3.2.1, we have on K_i as $t_{i_j} \to +\infty$,

$$\nabla_{\omega_\infty} \dot{\varphi} = \nabla_{\omega_\infty} \log \frac{(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi)^n}{\Omega} \to 0 = \nabla_{\omega_\infty} \log \frac{(\omega_\infty + \sqrt{-1}\partial\overline{\partial}\psi)^n}{\Omega}.$$

Hence, we know $\log \frac{(\omega_\infty + \sqrt{-1}\partial\overline{\partial}\psi)^n}{\Omega} = constant$ on $X \setminus E$. Then the constant can only be 0 as ψ is a bounded pluri-subharmonic function and $\int_X \omega_\infty^n = \int_X \Omega$, which means that ψ solves the degenerate Monge-Amperè equation (1.2.3) globally in the sense of currents and strongly on $X \setminus E$. Furthermore, we notice $\int_X \psi\Omega = 0$ as ψ is bounded.

Suppose $u(t) \nrightarrow \psi$ in $C^\infty(K)$ for some compact set $K \subset X \setminus E$, which means that there exist $\delta > 0$, $l \geq 0$, $K' \subset\subset X \setminus E$, and a subsequence $u(s_j)$ such that $\|u(s_j) - \psi\|_{C^l(K')} > \delta$. While $u(s_j)$ are bounded in $C^k(K)$ for any compact set K and $k \geq 0$, from the above argument, we know that by passing to a subsequence, $u(s_j)$ converges to ψ' in $C^\infty(K)$ for any $K \subset\subset X \setminus E$, where ψ' is also a solution to equation (1.2.3) under the normalization $\int_X \psi'\Omega = 0$, which has to be ψ by the uniqueness of the solution to (1.2.3). This is a contradiction. It thus follows that $u(t) \to \psi$ in $L^p(X)$ for any $p > 0$.

Notice that $u(t), \psi$ are uniformly bounded. Integrating by part, we easily deduce that $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\overline{\partial}u \to \omega_\infty + \sqrt{-1}\partial\overline{\partial}\psi$ weakly in the sense of currents. The proof of the theorem is complete.

□

Remark 3.2.1 Unlike the canonical Kähler-Ricci flow, it is not clear if the scalar curvature $s(\tilde{\omega}_t)$ is uniformly bounded from below along the flow (3.1.2). However applying $\sqrt{-1}\partial\overline{\partial}$ to (3.1.1), we have

$$s(\tilde{\omega}_t) = -\tilde{\Delta}\dot{\varphi} + tr_{\tilde{\omega}_t} Ric(\Omega).$$

It follows that there exists constant $C > 0$, such that:

$$-C \leq \int_X s(\tilde{\omega}_t)\tilde{\omega}_t^n = n\int_X Ric(\Omega) \wedge \omega_t^n = nc_1(X) \cdot [\omega_t]^{n-1} \leq C.$$

It would be very interesting to investigate the behavior of the scalar curvature along this modified Kähler-Ricci flow.

3.3 Remarks on non-degenerate case

In the case when $[\omega_\infty]$ is Kähler, the convergence of (3.1.2) has already been proven by Zhang in [Z3] by modifying Cao's argument in [Cao]. However, by using the functional $\nu(\varphi)$ defined in the previous section, we will have an alternative proof to the convergence, without using Li-Yau's Harnack inequality. We will sketch the proof in this section. We believe that this point of view is well-known to experts.

Firstly, under the same normalization $u = \varphi - \int_X \varphi\Omega$, we will have the following uniform estimates ([Z3]): for any integer $k \geq 0$, there exists $C_k > 0$, such that

$$\|u\|_{C^k([0,+\infty)\times X)} \leq C_k.$$

Secondly, following the convergence argument as in the previous section, we will obtain the C^∞ convergence of $\tilde{\omega}_t$ along (3.1.1). More precisely, $u \to \psi$ in C^∞-norm with ψ solving (1.2.3) as a strong solution. In particular, $\dot{\varphi} \to 0$ in C^∞-norm as $t \to +\infty$. Let $\tilde{\omega}_\infty = \omega_\infty + \sqrt{-1}\partial\overline{\partial}\psi$ be the limit metric. Furthermore, we have the bounded geometry along (3.1.2) for $0 \leq t \leq +\infty$:

$$\frac{1}{C}\tilde{\omega}_\infty \leq \tilde{\omega}_t \leq C\tilde{\omega}_\infty. \tag{3.3.1}$$

Finally, we need to prove the exponential convergence: $\|\tilde{\omega}_t - \tilde{\omega}_\infty\|_{C^k(X)} \leq C_k e^{-\alpha t}$ and $\|u(t) - \psi\|_{C^k(X)} \leq C_k e^{-\alpha t}$, for some $C_k, \alpha > 0$. Then it is sufficient to prove: for any integer $k \geq 0$, there exists $c_k > 0$, such that

$$\|D^k \dot{\varphi}\|^2_{\omega_0} \le c_k e^{-\alpha t}.$$

Essentially, by following the proof of Proposition 10.2 in the case of holomorphic vector fields $\eta(X) = 0$ in [CT], we can also prove the following proposition.

Proposition 3.3.1 *Let $c(t) = \int_X \frac{\partial \varphi}{\partial t} \tilde{\omega}_t^n$. There exists $\alpha > 0$ and $c'_k > 0$ for any integer $k \ge 0$, such that*

$$\int_X \|D^k(\frac{\partial \varphi}{\partial t} - c(t))\|^2_{\tilde{\omega}_t} \tilde{\omega}_t^n \le c'_k e^{-\alpha t}.$$

Immediately, we have this corollary

Corollary 3.3.1 *There exists $C > 0$, such that*

$$-Ce^{-t} \le c(t) \le Ce^{-\alpha t} + Ce^{-t} \qquad \forall\ t >> 1.$$

Proof Calculate

$$\begin{aligned} \dot{c}(t) &= \int_X \frac{\partial \dot{\varphi}}{\partial t} \tilde{\omega}_t^n + \int_X \dot{\varphi} \tilde{\Delta} \dot{\varphi} \tilde{\omega}_t^n - ne^{-t} \int_X \dot{\varphi} \chi \wedge \tilde{\omega}_t^{n-1} \\ &= -\int_X \|\nabla \dot{\varphi}\|^2_{\tilde{\omega}_t} \tilde{\omega}_t^n - ne^{-t} \int_X \chi \wedge \tilde{\omega}_t^{n-1} - ne^{-t} \int_X \dot{\varphi} \chi \wedge \tilde{\omega}_t^{n-1}. \end{aligned}$$

Hence,

$$-\int_X \|\nabla \dot{\varphi}\|^2_{\tilde{\omega}_t} \tilde{\omega}_t^n - Ce^{-t} \le \dot{c}(t) \le Ce^{-t}.$$

Integrating from t to $+\infty$ and using Proposition 3.3.1,

$$-Ce^{-t} \le c(t) \le Ce^{-\alpha t} + Ce^{-t}.$$

$\square$

Proof of Proposition 3.3.1:

Let $\kappa(t) = \int_X (\frac{\partial \varphi}{\partial t} - c(t))^2 \tilde{\omega}_t^n$. Then

$$\begin{aligned}
\dot{\kappa}(t) &= 2\int_X (\dot{\varphi} - c(t))(\frac{\partial \dot{\varphi}}{\partial t} - \dot{c}(t))\tilde{\omega}_t^n + \int_X (\dot{\varphi} - c(t))^2 \tilde{\Delta}\dot{\varphi}\tilde{\omega}_t^n \\
&\quad -ne^{-t}\int_X (\dot{\varphi} - c(t))^2 \chi \wedge \tilde{\omega}_t^{n-1} \\
&= 2\int_X (\dot{\varphi} - c(t))\tilde{\Delta}\dot{\varphi}\tilde{\omega}_t^n - 2\int_X (\dot{\varphi} - c(t))(e^{-t} tr_{\tilde{\omega}_t}\chi + \dot{c}(t))\tilde{\omega}_t^n \\
&\quad + \int_X (\dot{\varphi} - c(t))^2 \tilde{\Delta}\dot{\varphi}\tilde{\omega}_t^n - ne^{-t}\int_X \chi \wedge \tilde{\omega}_t^{n-1} \\
&\leq -2\int_X (1 + (\dot{\varphi} - c(t))(1 - V(t)))\|\nabla(\dot{\varphi} - c(t))\|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n + Ce^{-t}.
\end{aligned}$$

We make use of the expression of $\dot{c}(t)$ and the uniform estimate (3.1.3) in the last inequality. Since $\dot{\varphi}$ and $c(t)$ approach 0 uniformly as $t \to +\infty$, we have:

$$\dot{\kappa}(t) \leq -2(1-\epsilon)\int_X \|\nabla(\dot{\varphi} - c(t))\|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n + Ce^{-t} \leq -\alpha\kappa(t) + Ce^{-t}$$

with $0 < \epsilon << 1$ and $0 < \alpha < \inf\{1, 2(1-\epsilon)\lambda_\infty\}$, where λ_∞ is the first eigenvalue of metric $\tilde{\omega}_\infty$. Let $\tilde{\kappa}(t) = \kappa(t) + Ae^{-t}$. Therefore, by choosing A sufficiently large,

$$\begin{aligned}
\frac{\partial \tilde{\kappa}(t)}{\partial t} &= \dot{\kappa}(t) - Ae^{-t} \leq -\alpha\kappa(t) + (C - A)e^{-t} \\
&\leq -\alpha\tilde{\kappa}(t) + (-A + C + \alpha A)e^{-t} \\
&\leq -\alpha\tilde{\kappa}(t).
\end{aligned}$$

Solving this ODE, we get: $\tilde{\kappa}(t) \leq \tilde{\kappa}(0)e^{-\alpha t}$. Therefore, we have:

$$\kappa(t) \leq \tilde{\kappa}(t) \leq \tilde{\kappa}(0)e^{-\alpha t}.$$

Let $\kappa_k(t) = \int_X \|D^k(\frac{\partial \varphi}{\partial t} - c(t))\|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n$. Then

$$
\begin{aligned}
\frac{\partial}{\partial t}\kappa_k(t) &= \int_X \|D^k(\frac{\partial \varphi}{\partial t} - c(t))\|^2_{\tilde{\omega}_t}(\sqrt{-1}\partial\overline{\partial}\dot{\varphi} - e^{-t}\chi) \wedge \tilde{\omega}_t^{n-1} \\
&\quad + \int_X \frac{\partial}{\partial t}\|D^k(\frac{\partial \varphi}{\partial t} - c(t))\|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n \\
&\leq C(k)\int_X \|D^k(\frac{\partial \varphi}{\partial t} - c(t))\|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n + C'(k)e^{-t} \\
&\quad -2\int_X \|D^{k+1}(\frac{\partial \varphi}{\partial t} - c(t))\|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n \\
&\leq C(k)\Big[\epsilon\int_X \|D^{k+1}(\frac{\partial \varphi}{\partial t} - c(t))\|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n + C(\epsilon)\int_X (\frac{\partial \varphi}{\partial t} - c(t))^2\tilde{\omega}_t^n\Big] \\
&\quad -2\int_X \|D^{k+1}(\frac{\partial \varphi}{\partial t} - c(t))\|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n + C'(k)e^{-t} \\
&= C(k)C(\epsilon)\int_X (\frac{\partial \varphi}{\partial t} - c(t))^2\tilde{\omega}_t^n + C'(k)e^{-t} \\
&\quad -(2-\epsilon C(k))\int_X \|D^{k+1}(\frac{\partial \varphi}{\partial t} - c(t))\|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n \\
&\leq C(k)C(\epsilon)\int_X (\frac{\partial \varphi}{\partial t} - c(t))^2\tilde{\omega}_t^n + C'(k)e^{-t} \\
&\leq C''(k)e^{-\alpha t}.
\end{aligned}
$$

In the first inequality, we use integration by parts and the uniform estimate (3.1.3). In the second inequality, we use the interpolation inequality and choose ϵ sufficiently small to guarantee the third inequality. Integrating from t to ∞ and using $\lim_{t\to+\infty}\kappa_k(t) = 0$, we have:

$$\kappa_k(t) \leq C_k e^{-\alpha t}.$$

□

Now we are in position to prove the exponential convergence.

Theorem 3.3.1 *For any integer $k \geq 0$, $\frac{\partial \varphi}{\partial t}$ convergences exponentially to 0 in C^k-norm. More precisely, there exists $\alpha > 0$ and $c_k > 0$ such that for any $t >> 1$,*

$$\|D^k\dot{\varphi}\|^2_{\omega_0} \leq c_k e^{-\alpha t}.$$

Proof As $\frac{1}{C}\tilde{\omega}_\infty \leq \tilde{\omega}_t \leq C\tilde{\omega}_\infty$, the Sobolev constants of $\tilde{\omega}_t$ are uniformly bounded. Therefore it follows from Sobolev inequality that:

$$\|D^k(\dot{\varphi} - c(t))\|^2_{\tilde{\omega}_t} \le d_k \int_X \|D^{k'}(\frac{\partial \varphi}{\partial t} - c(t))\|^2_{\tilde{\omega}_t} \tilde{\omega}_t^n \le d_k C_{k'} e^{-\alpha t} = C'_k e^{-\alpha t}.$$

Hence when $k \neq 0$,

$$\|D^k \dot{\varphi}\|^2_{\omega_0} \le C \|D^k(\dot{\varphi} - c(t))\|^2_{\tilde{\omega}_t} \le c_k e^{-\alpha t}.$$

In the case of $k = 0$, Sobolev inequality yields:

$$\mid \frac{\partial \varphi}{\partial t} - c(t) \mid \le C \big(\int_X \|D^l(\frac{\partial \varphi}{\partial t} - c(t))\|^2 \tilde{\omega}_t^n \big)^{\frac{1}{2}} \le C' e^{-\alpha t},$$

for some $l > 0$. Hence

$$\mid \frac{\partial \varphi}{\partial t} \mid \le \mid \frac{\partial \varphi}{\partial t} - c(t) \mid + \mid c(t) \mid \le c e^{-\alpha t},$$

where we use Corollary (3.3.1). $\square$

References

[BER] Baouendi, S., Ebenfelt, P. and Rothschild, L. Real Submanifolds in Complex Space and Their Mappings, Princeton Mathematical Series, **47**, Princeton University Press, Princeton, NJ, 1999.

[Ca] Calabi, E. *Isometric imbedding of complex manifolds*, Ann. of Math. (2) 58, (1953). 1–23.

[Cao] Cao, H.-D. *Deformation of Kähler metrics to Kähler Einstein metrics on compact Kähler manifolds*, Invent. Math. 81 (1985), no. 2, 359–372.

[CT] Chen, X. X. and Tian, G. *Ricci flow on Kähler-Einstein surfaces*, Invent. Math. 147 (2002), no. 3, 487–544.

[CS] Cima, J. A. and Suffridge, T. J. *Boundary behavior of rational proper maps*, Duke Math. J. 60 (1990), no. 1, 135–138.

[CU] Clozel, L. and Ullmo, E. *Correspondances modulaires et mesures invariantes*, J. Reine Angew. Math. **558** (2003), 47–83.

[De] Demailly, J.-P. Complex analytic and algebraic geometry, online book available on the webpage of J.-P. Demailly.

[DZ] Dinew, S. and Zhang, Z. *Stability of Bounded Solutions for Degenerate Complex Monge-Ampère Equations*, ArXiv:0711.3643.

[Do] Donaldson, S. *Scalar curvature and projective embeddings I*, J. Differential Geom. 59 (2001), no. 3, 479–522.

[EGZ] Eyssidieux, P., Guedj, V. and Zeriahi, *A Singular Kähler-Einstein metrics*, J. Amer. Math. Soc. 22 (2009), no. 3, 607–639.

[Fo] Forstneric, F. *Extending proper holomorphic mappings of positive codimension*, Invent. Math.**95**, 31-62 (1989).

[H] Hamilton, R. *Three-manifolds with positive Ricci curvature*, J. Differential Geom. 17 (1982), no. 2, 255–306.

[Hu1] Huang, X. *On the mapping problem for algebraic real hypersurfaces in the complex spaces of different dimensions*, Ann. Inst. Fourier (Grenoble) **44** (1994), no. 2, 433–463.

[Hu2] Huang, X. *On a linearity problem of proper holomorphic mappings between balls in complex spaces of different dimensions*, J. Diff. Geom. **51**(1999), 13-33.

[Hu3] Huang, X. *On a semi-rigidity property for holomorphic maps*, Asian J. Math. **7**(2003), no. 4, 463–492. (A special issue dedicated to Y. T. Siu on the ocassion of his 60th birthday.)

[HJ] Huang, X. and Ji, S. *On some rigidity problems in Cauchy-Riemann analysis*, Proceedings of the International Conference on Complex Geometry and Related Fields, 89–107, AMS/IP Stud. Adv. Math., 39, Amer. Math. Soc., Providence, RI, 2007.

[HJX] Huang, X., Ji, S. and Xu, D. *A new gap phenomenon for proper holomorphic mappings from B^n into B^N*, Math. Res. Lett. **13**(2006), no. 4, 515–529.

[JX] Ji, S., and Xu, D. *Maps between B^n and B^N with geometric rank $k_0 \leq n-2$ and minimum N*, Asian J. Math. 8 (2004), no. 2, 233–257.

[K1] Kołodziej, S. *The complex Monge-Ampre equation*, Acta Math. 180 (1998), no. 1, 69–117.

[K2] Kołodziej, S. The complex Monge-Ampère equation and pluripotential theory, Mem. Amer. Math. Soc. 178 (2005), no. 840, x+64 pp.

[Mi] Mir, N. *Convergence of formal embeddings between real-analytic hypersurfaces in codimension one*, J. Differential Geom. 62 (2002), no. 1, 163–173.

[Mo1] Mok, N. *Local holomorphic isometric embeddings arising from correspondences in the rank-1 case*, Contemporary trends in algebraic geometry and algebraic topology (Tianjin, 2000), 155–165, Nankai Tracts Math., 5, World Sci. Publ., River Edge, NJ, 2002.

[Mo2] Mok, N. *Extension of germs of holomorphic isometries up to normalizing constants with respect to the Bergman metric*, http://www.hku.hk/math/imr/, Preprint.

[Mo3] Mok, N. Private communications.

[MN] Mok, N. and Ng, S. *Germs of measure-preserving holomorphic maps from bounded symmetric domains to their Cartesian products*, http://www.hku.hk/math/imr/, Preprint.

[Ng1] Ng, S. *On holomorphic isometric embeddings from the unit disk into polydisks and their generalizations*, Ph. D. thesis. 2008.

[Ng2] Ng, S. *On holomorphic isometric embeddings of the unit disk into polydisks*, Proc. Amer. Math. Soc. to appear.

[Ng3] Ng, S. *On holomorphic isometric embeddings of the unit n-ball into products of two unit m-balls*, Math. Z. to appear.

[Ng4] Ng, S. Private communications.

[Pe] Perelman, G. Unpublished notes on the Kähler-Ricci flow.

[PS] Phong, D. H. and Sturm, J. *On stability and the convergence of the Kähler-Ricci flow*, J. Differential Geom. 72 (2006), no. 1, 149–168.

[PSSW] Phong, D. H., Song, J., Sturm, J. and Weinkove, B. *The Kähler-Ricci flow and the $\bar{\partial}$ operator on vector fields*, J. Differential Geom. 81 (2009), no. 3, 631–647.

[Si] Siu, Y. T. Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics, DMV Seminar, 8. Birkhäuser Verlag, Basel, 1987. 171 pp.

[ST1] Song, J. and Tian, G. *The Kähler-Ricci flow on surfaces of positive Kodaira dimension*, Invent. Math. 170 (2007), no. 3, 609–653.

[ST2] Song, J. and Tian, G. *Canonical measures and Kähler-Ricci flow*, ArXiv:0802.2570.

[ST3] Song, J. and Tian, G. *The Kähler-Ricci flow through singularities*, ArXiv:0909.4898.

[SY] Song, J. and Yuan, Y. *The Kähler-Ricci flow on Calabi-Yau varieties*, Preprint.

[T1] Tian, G. *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. 130 (1997), no. 1, 1–37.

[T2] Tian, G. Canonical metrics in Kähler geometry, Lectures in Mathematics, ETH Zürich, Birkhauser Verlag, Basel 2000.

[TZha] Tian, G. and Zhang, Z. *On the Kähler-Ricci flow on projective manifolds of general type*, Chinese Ann. Math. Ser. B 27 (2006), no. 2, 179–192,

[TZhu] Tian, G. and Zhu, X. *Convergence of Kähler-Ricci flow*, J. Amer. Math. Soc. 20 (2007), no. 3, 675–699,

[To] Tosatti, V. *Limits of Calabi-Yau metrics when the Kahler class degenerates*, J. Eur. Math. Soc. (JEMS) 11 (2009), no. 4, 755–776.

[Tr] Trépreau, J.-M. *Sur le prolongement holomorphe des fonctions CR défines sur une hypersurface réelle de classe $\mathbb{C}^2$ dans $\mathbb{C}^n$*, Invent. Math. 83 (1986), no. 3, 583–592.

[Ts] Tsuji, H. *Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type*, Math. Ann. 281 (1988), 123-133.

[Um] Umehara, M. *Einstein Kähler submanifolds of a complex linear or hyperbolic space*, Tohoku Math. J. (2) 39 (1987), no. 3, 385–389.

[We] Weinkove, B. *The Calabi-Yau equation on almost-Kähler four-manifolds*, J. Differential Geom. 76 (2007), no. 2, 317–349.

[Y1] Yau, S.-T. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*, Comm. Pure Appl. Math. 31 (1978), 339-411.

[Y2] Yau, S.-T. *Problem section* Seminar on Differential Geometry, pp. 669–706, Ann. of Math. Stud., 102, Princeton Univ. Press, Princeton, N.J., 1982.

[Yu] Yuan, Y. *On the convergence of a modified Kähler-Ricci flow*, Math. Z., DOI:10.1007/s00209-010-0670-0.

[YZ] Yuan, Y. and Zhang, Y. *Rigidity for local holomorphic conformal embeddings from $\mathbb{B}^n$ into $\mathbb{B}^{N_1} \times ... \times \mathbb{B}^{N_m}$*, arXiv:1001.2941v1.

[Z1] Zhang, Z. *Degenerate Monge-Ampère equations over projective manifolds*, Ph.D. Thesis in MIT.

[Z2] Zhang, Z. *On degenerate Monge-Ampère equations over closed Kähler manifolds*, Int. Math. Res. Not. 2006, Art. ID 63640, 18 pp.

[Z3] Zhang, Z. *A Modified Kähler-Ricci Flow*, Math. Ann. 345 (2009), no. 3, 559–579.

Vita

Yuan Yuan

2010	Ph. D. in Mathematics, Rutgers University, New Brunswick, New Jersey, USA
2004	B. Sc. in Mathematics, Peking University, Beijing, China
2005-2010	Teaching assistant, Department of Mathematics, Rutgers University, New Brunswick, NJ, USA
2004-2005	Teaching assistant, Department of Mathematics, Indiana University, Bloomington, IN, USA