OPTIMAL DESIGN AND EQUIVALENCY OF ACCELERATED LIFE TESTING PLANS

by

YADA ZHU

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And approved by

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ABSTRACT OF THE DISSERTATION

Optimal Design and Equivalency of Accelerated Life Testing Plans

By YADA ZHU

Dissertation Director:

Professor Elsayed A. Elsayed

Accelerated Life Testing (ALT) is an efficient approach to obtain failure observations by subjecting the test units to stresses severer than design stresses and utilize the test data to predict reliability at normal operating conditions.

ALT plans under multiple stresses needs to be designed to resemble the normal operating conditions and obtain useful failure observations for accurate reliability prediction. However, to date there is little research into the theory of planning ALT for reliability prediction with multiple stresses. Multiple stresses can result in a large number of stress-level combinations which presents a challenge for implementation. We propose an approach for the design of ALT plans with multiple stresses using Latin hypercube design (LHD) and demonstrate the proposed method with examples based on actual tests. The obtained optimal test plans are compared with those based on full factorial design. The comparison shows that ALT based on LHD not only increases the accuracy of reliability prediction significantly but also reduces the test duration dramatically.

ALT under Type-I and Type-II censoring has been extensively investigated. We generalize the one stage censoring to multi-stage progressive censoring, where the surviving test units are removed at intermediate stages other than the final termination of the test. This procedure further minimizes the test time and cost. We also combine the progressive censoring scheme with competing risk when test units experience different failure modes to investigate general, practical and optimal ALT plans.

ALT is usually conducted under constant-stresses which need a long time at low stress levels to yield sufficient failure data. Many stress loadings, such as step-stresses obtain failure times faster than constant-stresses but the accuracy of reliability predictions based on such loadings has not yet been investigated. We develop test plans under different stress applications such that the reliability prediction achieves equivalent statistical precision to that of the constant-stress. The research shows indeed there are such equivalent plans that reduce the test time, minimize the cost and result in the same accuracy of reliability predictions.

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CHAPTER 1

INTRODUCTION

1.1 Motivation of the Work

The significant increase in the introduction of new products and the significant reduction in time from product design to manufacturing, as well as the increasing customer's expectation for high reliability, have prompted industry to shorten its product test duration in order to assess the product's reliability before release. In many cases, Accelerated Life Testing (ALT) is one of the most common approaches that meet such requirements. The accuracy of the statistical inference obtained using ALT data has a profound effect on the reliability estimates and the subsequent decisions regarding system configuration, warranties and preventive maintenance schedules. Without an optimal test plan, it is likely that a sequence of expensive and time-consuming tests result in inaccurate reliability estimates and improper final product design requirements. This might also cause delays in product release or the termination of the entire product development (Elsayed, *et al*., 2007).

Most of the research on ALT planning is focused on single stress application. Optimal test plans in terms of stress applications, test unit allocation to stress levels and censoring time are usually formulated as nonlinear optimization problems. However, for products designed to operate without failure for years, it is difficult to obtain sufficient failure time data in a short time using only a single-stress. Therefore, reliability testing using multiple stresses is commonly used in practice to overcome such a difficulty (Escobar and Meeker, 1995). For instance, ceramic capacitors are tested simultaneously under higher than design temperature and voltage (Zelen, 1959, Minford, 1982, Mogilevsky and Shrin, 1988, Klinger, 1991), Semiconductor electronic components and outdoor optical products are tested at higher than operating humidity and temperature (Peck, 1986, Lam, 2007). Printed circuit boards are also tested at higher than operating voltage, humidity and temperature (LuValle, *et al*., 1986, Ghazikhanian, 2005).

In many situations, product's life depends on several stresses operating simultaneously. For outdoor products, multiple stresses represent a realistic field situation. For example, environments can cause rapid deterioration in the dielectric strength of lead zirconate titanates (PTZ) actuators (Pritchard, *et al*, 2001, Actuator, 2004). The deterioration rate increases by the use of high electrical field strengths required to achieve high mechanical output. However the PTZ actuators are deployed widely in devices that must work reliably for many years at multiple stresses, often in inaccessible locations (Lipscomb *et al.*, 2009). Lipscomb *et al.* (2009) evaluate the possible effect of accelerated stresses: temperature, humidity and electrical field on the reliability of PZT actuators. They place dry samples in an environmental chamber and fix two of the stresses while varying the third in a range of values, which is not a systematic and efficient approach. To date there is little research that deals with the theory of planning ALT for reliability prediction under multiple stresses.

Type-I and Type-II one-stage right censoring are the most common censoring schemes in reliability experiments and extensively studied by many researchers including Lawless (1982), Nelson (1990), Meeker and Escobar (1998). However, in many situations the surviving test units are removed at multiple stages before the final termination point of the experiment in order to reduce the test time and cost, save some of the surviving items for other tests, or to free up testing facilities for other experimentation. The conventional Type-I and Type-II censoring schemes do not provide these features. Progressive censoring could provide most of these features. However, existing research on progressive censoring is mainly based on an exponential failure time distribution or extreme value distribution and a single cause of failure. In many engineering situations, units may fail due to one of several possible failure modes (competing risk), e.g. the tensile strength of certain materials depends on two or more types of flows (Pascual, 2007), diodes may fail either open or short (Elsayed, 1996), cylinder liners present two dominant failure modes: wear degradation and thermal cracking (Bocchetti *et al*., 2009). These motivate us to investigate ALT under more general censoring schemes while considering the effects of competing risk with a practical field conditions, multiple stresses.

Current research on ALT plans has been focused on the design of optimum testing plans for a given stress loading. For instance, the constant-stress ALT plans have been investigated by Nelson and Kielpinski (1976), Maxim *et al*. (1977), Meeker and Hahn (1977), Nelson and Meeker (1978), Meeker (1984), Nelson (1990), Meeker (1994), and Yang (1994). The step-stress ALT plans have been studied by Miller and Nelson (1983), Bai *et al*. (1989), Bai and Chun (1991), Khamis and Higgins (1996), Xiong (1998), Xiong and Milliken (1999), and Xiong and Ji (2004), while the ramp-stress ALT plans have been considered by Bai *et al*. (1992), Bai and Chun (1993), Bai *et al*. (1993), and Park and Yum (1998). The wide range of stress applications, stress levels and corresponding test durations give rise to the investigation of the equivalency between testing plans.

Interestingly, because of wide range of polymer based advanced composite materials that are used in certified aircraft applications as well as the large variability of properties among these composites and within the same batch of a composite material the manufacturers and the Federal Aviation Administration develop a procedure to assess equivalence between different polymer based composites. The procedure utilizes essentially small data sets to generate test condition statistics such as population variability and corresponding basis values to pool results for a specific failure mode across all environments. The statistics from the test are compared and assessment of the "equivalency" is then made based on the mean and variance of the data, (Tomblin *et al*., 2002). Clearly, the term equivalency here refers to basic statistics about samples from populations but it does not provide information on reliability prediction or other time-dependent characteristics.

A brief literature review shows that fundamental research on the equivalency of test plans has not yet been investigated in the reliability engineering field. Without the understanding of such equivalency, it is difficult for a test engineer to determine the best experimental settings before conducting actual ALT. Meanwhile, accurate reliability prediction at normal operating conditions using the ALT results also requires appropriate ALT models. Complicated stress profiles create challenges in the development of regression analysis models that relate stress effects to the lifetime.

1.2 Problem Definition

Motivated by above discussion, we study three ALT planning related problems in this dissertation:

i. Planning ALT under multiple stresses

In many situations, product's life depends on several stresses operating simultaneously. For example, outdoor products usually operate under multiple stresses which represent a realistic field situation. However, the challenge of planning ALT with multiple stresses is the reduction of the number of stress-level combinations (experiments) in a test. When test units are subjected to two stresses and two levels for each stress, there are four combinations with Full Factorial Design (FFD). When the numbers of stresses and levels of each stress increase, FFD can lead to a large number of stress-level combinations, which makes it impractical to investigate or implement.

In this dissertation we present an approach for the design of ALT with multiple stresses using LHD. We review the literature on LHD and test plans for ALT under multiple stresses. We determine optimal ALT plans with respect to the variance-optimality, D-optimality and a multi-objective criterion which combines the D-optimality and a space-filling measure. We then compare the optimal ALT plans based on LHD with those based on FFD in terms of the variance-optimality and the D-optimality under different censoring situations. We develop an algorithm to efficiently obtain optimal solutions. We validate the performance of the algorithm by simulation.

ii. Design ALT under progressive censoring scheme

In order to develop ALT under a general scheme, we consider the progressive censoring and competing risk when test units are subject to multiple stresses. We assume each unit exhibits multiple independent failure modes. A unit fails when any of the potential failure modes occurs. The lifetime distribution of each failure mode follows an independent Weibull distribution with a common shape parameter. The observed failure time is the minimum of all the failure times. Under the progressive Type-I censoring scheme and the test condition of multiple stresses, we construct the likelihood function for MLE and develop the expression of Fisher information matrix. We determine optimum test plans under the following criteria:

1. Minimization of the asymptotic variance of the mean time to first failure in a group of units.
2. Minimization of asymptotic variance of the quantile failure at normal operating conditions.
3. D-optimality criterion that maximizes the determinant of the Fisher information matrix.

The first one is a new criterion that we firstly proposed for design of ALT plan. In addition to above three criteria, we also investigate the design of ALT plans under multi-objective optimization. We obtain optimal test plan subject to progressive censoring and competing risk under both single and multiple stresses for different objectives. We also conduct sensitivity study to indentify unknown parameters that should be initially estimated with special care.

iii. Design of equivalent ALT plans

In ALT, constant-stress is widely used due to the ease of conducting the test and the existence of acceptable reliability prediction models. However, constant-stress testing takes a long time at low stress levels to yield sufficient failures that can be used in providing accurate estimate of reliability characteristics. Due to time or cost constraint, there is an increasing interest in choosing time-varying stress loadings, e.g. step-stress (simple or multiple), ramp-stress, sinusoidal-cyclic stress or combinations. Each stress loading has some advantages and drawbacks. This has raised many practical questions such as: Can accelerating test plans involving different stress loadings be designed such that they are equivalent? What are the measures of equivalency? Can such test plans and their equivalency be developed for multiple stresses? Time-varying stresses also create challenges to relate the life of test units to the stress.

In this dissertation we propose an approach for the design of equivalent tests involving different stress applications. We define the measure of equivalency for reliability prediction. To quantify life-stress relationship under general time-varying stresses, we develop a model based on the well known cumulative exposure assumption. We formulate equivalent test plans under time-varying stresses to the baseline constant-stress based on the proposed measure of equivalency. We present examples of equivalent test plans under constant-stress, step-stress and ramp-stress. We conduct laboratory experiments using light bulbs to validate the equivalency of test plans.

1.3 Organization of the Dissertation

The remainder of the dissertation is organized as follows. Chapter 2 provides a review of the current literature of accelerated life testing with multiple stresses, and design of experiment with Latin hypercube. Then we present an approach for the design of ALT with multiple stresses using LHD. In chapter 3, we present a detailed review on the competing risk problem and planning ALT under different censoring schemes. Following that we present the design of ALT plans under progressive censoring for units subject to competing risk. In chapter 4, we present the design of equivalent ALT plans. In Chapter 5, we present the experimental set-up and results for the validation of the proposed model and equivalency of test plans. Chapter 6 concludes this dissertation and discusses the future research.

CHAPTER 2

ALT WITH MULTIPLE STRESSES

Accurate reliability prediction depends on both the ALT model and the test plan. In this chapter, we begin with a review of the widely used ALT models and the current research on test plans with single stress and multiple stresses. We also provide a review on the LHD. Following this, we present the assumption of this work and propose an approach to reduce the stress-level combinations using LHD. We then construct the likelihood function and develop the Fisher information matrix for Maximum Likelihood Estimate (MLE) of unknown parameters. We present and formulate three optimization criteria to determine optimal test plans. An algorithm is developed to evaluate the formulation which contains both continuous and discrete decision variables. Finally, examples based on a real test are given to demonstrate and validate the proposed method.

2.1 Literature Review

2.1.1 ALT Models

ALT models quantify the relationship between the failure time (hazard rate or reliability) and a set of explanatory variables (stresses in accelerated life testing area). We briefly present the most commonly used ALT models.

2.1.1.1 Accelerated Failure Time Models

Given a vector of covariates (stresses) **x**, *Accelerated Failure Time* (*AFT*) model represents the distribution of the lifetime T as a function of **x**. For example, the AFT model based on Weibull lifetime distribution with scale parameter α and shape parameter δ is obtained as:

$$f(t;\alpha,\delta) = \frac{\delta}{\alpha}\left(\frac{t}{\alpha}\right)^{\delta-1} \exp\left[-\left(\frac{t}{\alpha}\right)^{\delta}\right], \quad t \geq 0 \tag{5.1}$$

ALT models for which either α or δ depend on **x** may be considered. Since α or δ are positive-values, convenient specifications are

$$\alpha(\mathbf{x}) = \exp\left(\boldsymbol{\beta}'\mathbf{x}\right) \tag{5.2}$$

$$\delta(\mathbf{x}) = \exp\left(\boldsymbol{\gamma}'\mathbf{x}\right) \tag{5.3}$$

where $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are vectors of regression coefficients of the same length as **x**.

A Weibull model, that proves useful in many situations, has only α depending on **x** (Lawless, 1982), so that the reliability function of T is given by

$$R\left(t \mid \mathbf{x}\right) = \exp\left[-\left(\frac{t}{\alpha(\mathbf{x})}\right)^{\delta}\right] = \exp\left[-\left(\frac{t}{\exp\left(\boldsymbol{\beta}'\mathbf{x}\right)}\right)^{\delta}\right], \quad t \geq 0 \tag{5.4}$$

The factor $\exp(\boldsymbol{\beta}'\mathbf{x})$ is called an acceleration factor which relates the effect of a change in the covariate. The log-lifetime $Y = \log(T)$ in this case has the reliability function

$$R(y \mid \mathbf{x}) = \exp\left[-\exp\left(\frac{y-\mu(\mathbf{x})}{\sigma}\right)\right], \quad -\infty \le y \le \infty \tag{5.5}$$

where $\mu(\mathbf{x}) = \log \alpha(\mathbf{x})$ and $\sigma = \delta^{-1}$. This is also called an extreme value location-scale model. The extensions to other AFT models, such as exponential, log-normal, log-logistic, gamma and inverse Gaussian are also important and widely used (Lawless, 1982) for reliability prediction.

Advantages of the parametric models include simplicity, the availability of likelihood-based inference procedure, and ease of use for description, comparison, prediction, or decision (Lawless, 1982).

2.1.1.2 Cox's Proportional Hazards Model

The most widely used model describing the influence of covariates on the hazard rate function is the proportional hazards (PH) or Cox's model introduced by Cox (1972). The model is described as:

Let $\lambda(t;\mathbf{x})$ be the hazard rate at time t for a unit with a vector of stresses $\mathbf{x}$. The basic PH model is

$$\lambda(t;\mathbf{x}) = \lambda_0(t)c(\boldsymbol{\beta}'\mathbf{x}) \tag{5.6}$$

where

$\lambda_0(t)$	an arbitrary baseline hazard rate;
$c(\boldsymbol{\beta}'\mathbf{x})$	a known function

Because the hazard rate function $\lambda(t;\mathbf{x})$ must be positive, a common feasible function for $c(\boldsymbol{\beta}'\mathbf{x})$ is

$$c(\boldsymbol{\beta}'\mathbf{x}) = \exp(\boldsymbol{\beta}'\mathbf{x}) = \exp\left(\sum_{j=1}^{k}\beta_j x_j\right),$$

which results in

$$\lambda(t;\mathbf{x}) = \lambda_0(t)\exp(\boldsymbol{\beta}'\mathbf{x}) = \lambda_0(t)\exp\left(\sum_{j=1}^{k}\beta_j x_j\right) \tag{5.7}$$

The main assumption of the PH model is that the ratio of two hazard rates under two stress levels $\mathbf{x}_1$ and $\mathbf{x}_2$ is constant over time. In other words:

$$\frac{\lambda(t;\mathbf{x}_1)}{\lambda(t;\mathbf{x}_2)} = \frac{\lambda_0(t)\exp(\boldsymbol{\beta}'\mathbf{x}_1)}{\lambda_0(t)\exp(\boldsymbol{\beta}'\mathbf{x}_2)} = \exp[\sum_{j=1}^{k}\beta_j(x_{1j} - x_{2j})] \tag{5.8}$$

This implies that the hazard rates are proportional to the applied stress levels.

Without the specification of the form of baseline hazard rate function $\lambda_0(t)$, the coefficients of the covariates $\boldsymbol{\beta}$ could be obtained based on a partial or conditional likelihood rather than a full likelihood approach. The PH model usually produces "good" reliability estimation with failure data for which the proportional hazards assumption does not even hold exactly.

2.1.2 ALT Test Plans

In order to increase the accuracy of reliability prediction at normal operating conditions using accelerated life testing results, a carefully designed ALT plan is required. The test plan is designed to minimize a specified criterion, usually the variance of a reliability-related estimate, such as reliability function, mean time to failure and a percentile of failure time, under specific time and cost constraints. We review the work on the design of ALT plans for both cases when single or multiple stresses are used.

2.1.2.1 ALT Plans under Single Stress

Constant-stress test plans consisting of several stress levels are the most commonly used ALT plans due to ease of implementation and their acceptable reliability prediction models. Each stress level is allocated a proportion of the total number of test units. Earlier work by Nelson and Meeker (1978) propose optimal statistical plans for constant-stress ALTs which include only two stress levels. Such plans lack robustness since the assumed life-stress relationship is difficult, if not impossible, to validate. Meeker and Hahn (1985) propose a compromise plan with 4:2:1 allocation ratio for low, middle, and high stress levels and provide the optimal low level stress by assuming the middle stress to be the average of the high and the low stress levels. In recent years, by considering other test constraints and allowing non-constant shape parameter of the failure time distribution, Meeter and Meeker (1994) advocate the use of the compromised ALT plan for three stress levels without optimizing the middle stress level and allocation of test units. Yang (1994) proposes an optimum design of 4-level constant-stress ALT plans with various censoring times. The test plans derived are proven to be more robust than the 3-level best-compromise test plans.

On the other hand, a common criterion of interest in the existing work on constant-stress ALT plans is the estimate percentiles of the life distribution at specified design stress. Mann *et al.* (1974) consider linear estimation with order statistics to estimate a percentile of an extreme value (or Weibull) distribution at design stress and obtain optimal plans for failure data with censored observations. Nelson and Kielpinski (1976) obtain optimum plans and best traditional plans (traditional plans use equally spaced levels of stress with equal allocation of test units to each stress level) for the median of normal and lognormal distributions. Their model assumes that the normal distribution location parameter μ (also the mean) is a linear function of stress and the scale parameter σ (also the standard deviation) does not depend on stress. They also assume simultaneous testing of all test units and censoring at a pre-specified time. Nelson and Meeker (1978) provide similar optimum test plans to estimate percentiles of Weibull and smallest extreme-value distributions at a specified design stress when test units are overstressed. They assume that the smallest extreme-value location parameter μ (also the 0.632 percentile) is a linear function of stress and that the scale parameter σ is constant.

Other optimization criteria for constant-stress ALT plan are also widely investigated. Martz and Waterman (1977) use Bayesian methods for determining the optimal test stress for a single test unit to estimate the survival probability at a design stress. Meeker and Hahn (1985) consider the optimum allocation of test units to overstress conditions when it is desired to estimate the survival probability at a specified time and design conditions. The optimal criterion is to minimize the large sample variance under a logistic model assumption. Onar and Padget (2002) determine optimum accelerated test plans using the D-optimality criterion and assume an inverse Gaussian model. Ng *et al*. (2006) develop an optimal ALT plan based on the A-optimality criterion with complete data. In practice, the constant-stress ALTs need a long time at the low stress levels to obtain all or sufficient failure data. This has prompted the application of time-varying stresses in ALT such as step-stress and ramp-stress.

Under step-stress, the test units are first subjected to a lower stress level for some time; then the stress is increased to a higher level and held constant for another amount of time; the steps are repeated until all units fail or the predetermined test time has expired. To model the effects of time dependent stress on lifetime, Miller and Nelson (1983) present a cumulative exposure model which assumes that the remaining life of a test unit depends only on the “exposure” it has experienced and does not remember how the exposure was accumulated (this is a major drawback of the model). They obtain optimal test plans that minimize the asymptotic variance of MLE of the mean life at the design stress. Bai *et al.* (1989) extend the results to the case where a prescribed censoring time is involved. Bai and Chun (1991) obtain the optimal simple step-stress ALT with competing causes of failure. Khamis and Higgins (1998) present 3-step step-stress plans assuming a linear or quadratic relationship between the life and the stress. Xiong (1998) addresses the effect of the statistical inferences on the parameters of a simple step-stress ALT model with Type-II censoring. Xiong and Milliken (1999) study the statistical models in step-stress ALT when the stress change times are random. Xiong and Ji (2004) study the optimal design of a simple step-stress test plan involving grouped and censored data.

2.1.2.2 ALT Plans under Multiple Stresses

Most of the research on ALT planning is focused on single stress application. Optimal test plans in terms of stress applications, test unit allocation to stress levels and censoring time are usually formulated as nonlinear optimization problems. However, for products designed to operate without failure for years, it is difficult to obtain sufficient failure time data in a short time using only a single-stress. Meanwhile, in many situations product's life depends on several stresses operating simultaneously. To date there is little research into the theory of planning ALT for reliability prediction with multiple stresses.

Nelson (1990) describes a simulation-based method for planning ALT with two factors: voltage stress (volts/mm) and insulation thickness (mm). The voltage stress is the only acceleration factor and thickness is an ordinary experimental factor; the effect of the latter on the insulation life is of interest to engineers. Motivated by Nelson's work, Escobar and Meeker (1995) extend their previous work on the compromise plan method to design ALT with two types of stresses. More recently, Xu and Fei (2007) apply the compromise plan method to using two step-stresses. Though the approaches in Escobar and Meeker (1995) and Xu and Fei (2007) are interesting and practical for the design of test plans under two types of stresses, it is difficult to extend these methods to three or more stresses due to the difficulty of obtaining unique optimal solutions. Alternatively, Park and Yum (1996) and Elsayed and Zhang (2009) design ALT plans with factorial design arrangements assuming that the failure times follow exponential distribution and the proportional odds model, respectively. They consider the case when test units are subjected to two stresses and two levels for each stress. When the number of stresses and levels of each stress increases, complete factorial design can lead to a large number of stress-level combinations which makes it impractical to implement.

2.1.3 Latin Hypercube Design

A thorough literature review indicates that previous work on design of ALT plan is limited to single stress. The great challenge to performing ALT with multiple stresses is the reduction of stress-level combinations, i.e. the number of required experiments. To address this issue, we propose a new approach to design ALT experiments with multiple stresses based on LHD.

An LHD with *n* experiments and *k* factors, denoted by LHD (*n*, *k*), is an $n \times k$ matrix $\mathbf{X} = \left[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\right]^T$, where each row $\mathbf{x}_l^T = \left[x_{l,1}, x_{l,2}, \ldots, x_{l,k}\right]$ represents an experiment and each column represents a factor given by a permutation of its normalized levels

$\{1, 2, \ldots, n\}$. In this dissertation, stresses, factors and variables are used interchangeable; and runs, experiments and experiment points are also interchangeable. One of the main features of using LHD is that the stress-level combinations can be dramatically reduced as shown later. Consider a test with k factors and n levels for each factor. The Full–Factorial Design (FFD) requires n^k experiments, but an LHD needs only n experiments. Consequently, the overall testing time is significantly reduced. Though Fractional Factorial design (FFd) can facilitate the reduction of the number of experiments, the selection of appropriate fraction and allocation of test units present a challenge. On the other hand, a desirable property of an LHD is that when an n-experiment design is projected onto any factor, there are n different levels for that factor. For cases where one of the purposes of executing the experiment is to evaluate the effect of explanatory factors on reliability, the optimal LHD gives the best opportunity to investigate the true behavior of the response across the range of the factors (Zhao and Cui, 2007).

Table 2.1 gives two examples of LHD. In general, an n-experiment LHD can be generated using a random permutation of $\{1, \ldots, n\}$ for each factor. Each permutation leads to a different LHD. For k factors, there can be $(n!)^k$ LHDs.

Table 2.1 LHD (5, 3) and LHD (6, 4)

$n = 5, k = 3$	$n = 6, k = 4$
1 1 2 2 5 3 3 2 5 4 3 1 5 4 4	1 3 3 4 2 5 6 2 3 2 1 6 4 1 5 3 5 4 4 1 6 6 2 5

A random generated LHD may possess undesired properties and may act poorly in estimation and prediction. For example, consider an extreme case in Figure 2.1. For such an LHD, spurious correlation is introduced among the independent variables. As a result, it is impossible to distinguish between the effects of the two variables based on a test with such design. Specially, when the unexplored area is large, the effect of the ordinary experimental factor on reliability in the unexplored region cannot be assessed. Therefore, it is more appropriate to spread the design points as evenly as possible within the design space defined by the lowest and highest levels of each stress. There has been some work in the literature to improve the space-filling property.

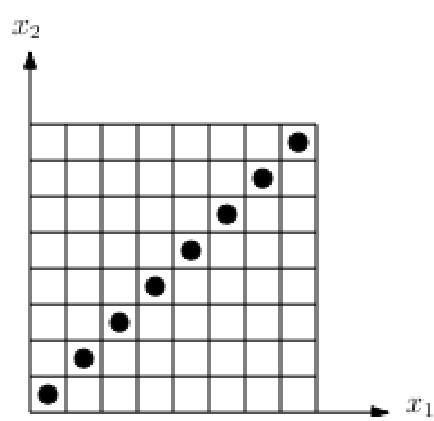


Figure 2.1 A two factor LHD-the extreme case

2.1.3.1 Maximin Distance Criterion and ϕ_p Criterion

A design is called a maximin distance design (John *et al.*, 1990) if it maximizes the minimum inter-site distance. For two experiment points (i.e. two rows in the design matrix $\boldsymbol{X}$) $\boldsymbol{s}$ and $\boldsymbol{t}$, the inter-site distance of order q (q-norm distance) is defined as:

$$d(\boldsymbol{s},\boldsymbol{t}) = \left\{ \sum_{j=1}^{k} \left| s_j - t_j \right|^q \right\}^{1/q} \tag{5.9}$$

where k is the number of factors, and $q \geq 1$. $q = 1$, $q = 2$, and $q = \infty$ corresponds to rectangular, Euclidean, and infinity distance, respectively. Morris and Mitchell (1995) proposed an intuitively appealing extension of the maximin distance criterion. For a given design, by sorting all the inter-sited distance, a distance list $(d_1, d_2, \ldots, d_s)$ and an index list $(J_1, J_2, \ldots, J_s)$ can be obtained, where d_i values are distinct distance values with $d_1 < d_2 < \ldots < d_s$, J_i is the number of pairs of sites in the design separated by d_i, s is the number of distinct distance values. A design is called a ϕ_p-optimal design if it minimizes:

$$\phi_p = \left(\sum_{i=1}^{s} J_i d_i^{-p} \right)^{1/p} \tag{5.10}$$

where p is a positive integer. The ϕ_p criterion is a variant of the maximin inter-site distance criterion.

2.1.3.2 Entropy Criterion

Shannon (1948) used entropy to quantify the "amount of information": the lower the entropy, the more precise the knowledge is. Minimizing the posterior entropy is equivalent to find a set of design points on which we have the least knowledge. It has been further shown that the entropy criterion is equivalent to minimizing the following (Koehler and Owen, 1996):

$$-\log|\mathbf{R}|,$$

where $\mathbf{R}$ is the correlation matrix of the experimental design matrix $\mathbf{X}=\left[\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n\right]^T$, whose elements are:

$$R_{ij}=\exp\left(\sum_{l=1}^{k}\theta_l\,|\,x_{li}-x_{lj}\,|^t\right),1\le i,\, j\le n;1\le t\le 2\,,$$

where θ_j (l = 1,…,k) are correlation coefficients.

2.1.3.3 Centered L_2 Discrepancy Criterion

The L_p discrepancy is a measure of the difference between the empirical cumulative distribution function of an experimental design and the uniform cumulative distribution function. In other words, the L_p discrepancy is a measure of non-uniformity of a design.

Among L_p discrepancy, L_2 discrepancy is used most frequently since it can be expressed analytically and is much easier to compute (John *et al*., 1990). Hickernell (1998) proposed three formulas of L_2 discrepancy, among which the centered L_2 –discrepancy (CL_2) seems the most interesting.

$$CL_2(\mathbf{X}) = \left(\frac{13}{12}\right)^2 - \frac{2}{n}\sum_{i=1}^{n}\prod_{l=1}^{k}\left(1+\frac{1}{2}|x_{il}-0.5|-\frac{1}{2}|x_{il}-0.5|^2\right)$$
$$+\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\prod_{l=1}^{k}\left(1+\frac{1}{2}|x_{il}-0.5|-\frac{1}{2}|x_{jl}-0.5|-\frac{1}{2}|x_{il}-x_{jl}|\right).$$

A design is called uniform design if it minimizes the centered L_2 discrepancy (Fang *et al*., 2000).

In this dissertation, we use the maximin inter-site distance criterion. Based on LHD, we develop optimal ALT plans under multiple stresses as follows.

2.2 The Lifetime Distribution

Assumptions

1. We consider $k\left(\in \mathbb{N}, k \geq 2\right)$ constant-stresses for ALT plans.
2. $N\pi_l$ units, randomly chosen from N, are allocated to stress-level combination $\mathbf{x}_l^T = \left[x_{l,1}, x_{l,2}, \ldots, x_{l,k}\right]$, where $\sum_{l=1}^{n}\pi_l = 1$ and $\pi_l \left(0 < \pi_l < 1\right)$ is the fraction of units allocated to stress-level combination *l*.

3. Under Type-I censoring, the test is continued until all test units fail or when a given censoring time η is reached. Under Type-II censoring, the test is continued until a specified number of failures is reached.
4. The lifetime T of a unit follows a Weibull distribution with scale parameter α and shape parameter λ with probability density function (*pdf*): $f\left(t;\alpha,\lambda\right)=\lambda\alpha^{-\lambda}t^{\lambda-1}\exp\left[-\left(t/\alpha\right)^{\lambda}\right]$, $t>0$ at both the normal operating conditions and the test conditions. We also assume that the shape parameter λ is constant while the scale parameter α depends on a vector of stresses through $\ln\alpha\left(\mathbf{x}\right)=\mu\left(\boldsymbol{\beta}\mathbf{x}\right)=\beta_0+\beta_1x_1+\beta_2x_2+\cdots+\beta_kx_k$, where $\boldsymbol{\beta}$ is a vector of regression coefficients. This assumption is a special case of the Cox's PH model.
5. The lifetimes of test units are statistically independent.

Due to the difficulty in extrapolation from a model with interactions among multiple factors, experimental factor definitions should be chosen such that the statistical interactions among the factors are minimized (Escobar and Meeker, 1995, 2006). In some cases, a sliding level technique can be used to avoid potential interactions. For example, Nelson (1990) uses the factors of insulation thickness and voltage stress (volts/mm) instead of thickness and volts which would result in a strong interaction. In other situations, physical or chemical characteristics might suggest ALT models with interaction terms such as the generalized Eyring model (Elsayed, 1996). In this dissertation, we limit our discussion to linear life-stress relationship without interactions. However, the methodology for finding optimal plans can be extended to nonlinear life-stress relationships.

Stress normalization

We choose the stress levels for each stress type using Eq. (2.11) as

$$x_l = x_L + (l-1)\frac{x_H - x_L}{n-1},\ \ l = 1, \dots, n, \tag{5.11}$$

where x_L and x_H are the lowest and highest testing levels of the stress, respectively. The x_l for $l = 1, \dots, n$ form an n-term arithmetic sequence which can always be transformed to $\{1, \dots, n\}$ using linear operation. The normalized model parameters ($\boldsymbol{\beta}$ and σ), and censoring time are then used in the computation.

On the other hand, the number of stress levels (experiments) n affects the prediction and estimation accuracy of the reliability characteristic. In planning single-stress ALT, it is sufficient to use three or four levels in general (Escobar and Meeker, 1995). For ALT with k ($k \geq 2$) stresses, we suggest using at least k+1 levels for each stress in order to ensure good properties of the Fisher information matrix. This will be interpreted in Sec. 2.3. In practices, simulation can be performed to evaluate the value of n.

2.3 Likelihood Estimate

Using the logarithmic transformation of T and inverse transformation $\sigma = \lambda^{-1}$, we obtain the distribution for $Y = \ln(T)$ with *pdf* as:

$$f_Y\left(y;\mu(\mathbf{\beta x}),\sigma\right)=\sigma^{-1}\exp\left[\left(y-\mu(\mathbf{\beta x})\right)\sigma^{-1}-\exp\left(\left(y-\mu(\mathbf{\beta x})\right)\sigma^{-1}\right)\right],\quad -\infty<y<\infty \tag{5.12}$$

Let $Y=\mu(\mathbf{\beta x})+\sigma Z$, Eq. (2.12) can be written with respect to Z as,

$$f_Z(z)=\exp\left(z-\exp(z)\right),\quad -\infty<z<\infty,$$

which is the smallest extreme-value distribution. The log likelihood of a single observation in the $l^{th}\left(l=1,\ldots n\right)$ experiment of an ALT is,

$$L(\mathbf{\beta},\sigma)=I(z_l)\left[z_l-\exp(z_l)-\ln(\sigma)\right]-\left(1-I(z_l)\right)\exp(\xi_l),$$

where $\xi_l=\left[\ln(\eta_l)-\mu\left(\mathbf{\beta}^T\mathbf{x}_l\right)\right]\sigma^{-1}$ is the standardized log censoring time of the l^{th} experiment, η_l is the censoring time, $\mathbf{x}_l$ is the l^{th} row of the LHD (n, k) that specifies the stress-level combination, and $I(z_l)$ is an indicator function defined by

$$I(z_l)=\begin{cases}1 & \text{if } z_l\le\xi_l\ \ (\text{failure})\\ 0 & \text{otherwise}\end{cases}$$

Under the regularly condition (Meeker and Escobar, 1998), the elements of the expected Fisher information matrix for an observation at ξ_l are the negative s-expectations of the second partial derivatives of the log likelihood with respect to the unknown model parameters:

$$E\left[-\frac{\partial^2 L}{\partial\beta_i \partial\beta_j}\right] = \left(\frac{x_{l,i}x_{l,j}}{\sigma^2}\right)\cdot A\left(\xi_l\right), i, j = 0,...,k \text{ ,}$$

$$E\left[-\frac{\partial^2 L}{\partial\beta_i \partial\sigma}\right] = \left(\frac{x_{l,i}}{\sigma^2}\right)\cdot B\left(\xi_l\right), i = 0,...,k \text{ ,}$$

$$E\left[-\frac{\partial^2 L}{\partial\sigma^2}\right] = \left(\frac{1}{\sigma^2}\right)\cdot C\left(\xi_l\right) \text{ ,}$$

where

$$x_{l,0} = 1,$$

$$A(\xi_l) = \Psi(\xi_l) = 1 - \exp(-\exp(\xi_l)) \text{ ,}$$

$$B(\xi_l) = \int_{-\infty}^{\xi_l} \exp\left(z_l - \exp\left(z_l\right)\right) z_l \exp\left(z_l\right) dz_l - \bar{\Psi}(\xi_l)\xi_l e^{\xi_l} \text{ ,}$$

$$C(\xi_l) = \Psi(\xi_l) + \int_{-\infty}^{\xi_l} \exp\left(z_l - \exp\left(z_l\right)\right) z_l^2 \exp\left(z_l\right) dz_l + \bar{\Psi}(\xi_l)\xi_l^2 e^{\xi_l} \text{ .}$$

Let F_l be the Fisher information matrix of an observation in the l^{th} experiment. The total Fisher information matrix for N s-independent observations is $F = N\sum_{l=1}^{n}\pi_l F_l$. Due to the invariant characteristic, we study $\left(\sigma^2/N\right)\cdot F$:

$$\begin{bmatrix} \sum_{l=1}^{n} A(\xi_l)\pi_l & \sum_{l=1}^{n} x_{1,l}A(\xi_l)\pi_l & \cdots & \sum_{l=1}^{n} x_{k,l}A(\xi_l)\pi_l & \sum_{l=1}^{n} B(\xi_l)\pi_l \\ & \sum_{l=1}^{n} x_{1,l}^2 A(\xi_l)\pi_l & \cdots & \sum_{l=1}^{n} x_{1,l}x_{k,l}A(\xi_l)\pi_l & \sum_{l=1}^{n} x_{1,l}B(\xi_l)\pi_l \\ & & \ddots & \vdots & \vdots \\ & & & \sum_{l=1}^{n} x_{k,l}^2 A(\xi_l)\pi_l & \sum_{l=1}^{n} x_{k,l}B(\xi_l)\pi_l \\ \text{symmetric} & & & & \sum_{l=1}^{n} C(\xi_l)\pi_l \end{bmatrix} \tag{5.13}$$

For the case without censoring, $\left(\sigma^2/N\right)\cdot F$ is given by (2.14). That is Eq. (2.13) with $A(\xi_l)$, $B(\xi_l)$, $C(\xi_l)$ replaced by their limits as $\xi \to \infty$, viz, 1, 1-γ, $\frac{\pi^2}{6}+\left(\gamma-1\right)^2$, respectively, where γ and π are the Euler's constant and circular constant, respectively. We observe from Eq. (2.14) that the Fisher information matrix is dependent of the model parameters. In addition, when $k = 3$ and $n = 2$, a test using three stresses and each with two levels, Eq. (2.14) becomes a 5×5 matrix, Eq. (2.15). The possible LHDs include $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{vmatrix}$, $\begin{vmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$, $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix}$ and $\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \end{vmatrix}$ which result in two identical rows (2&3, 2&4, or 3&4) in Eq. (2.15). Thus the Fisher information matrix is singular. If each stress has three levels, there is high chance that the Fisher information matrix is singular. Therefore, we recommend $k+1$ levels for a k factor design.

$$\begin{bmatrix} 1 & \sum_{l=1}^{n} x_{1,l}\pi_l & \cdots & \sum_{l=1}^{n} x_{k,l}\pi_l & (1-\gamma) \\ & \sum_{l=1}^{n} x_{1,l}^2\pi_l & \cdots & \sum_{l=1}^{n} x_{1,l}x_{k,l}\pi_l & (1-\gamma)\sum_{l=1}^{n} x_{1,l}\pi_l \\ & & \ddots & \vdots & \vdots \\ & & & \sum_{l=1}^{n} x_{k,l}^2\pi_l & (1-\gamma)\sum_{l=1}^{n} x_{k,l}\pi_l \\ \text{symmetric} & & & & \frac{\pi^2}{6}+(1-\gamma)^2 \end{bmatrix} \quad (5.14)$$

$$\begin{bmatrix} 1 & \sum_{l=1}^{2} x_{1,l}\pi_l & \sum_{l=1}^{2} x_{2,l}\pi_l & \sum_{l=1}^{2} x_{3,l}\pi_l & (1-\gamma) \\ \sum_{l=1}^{2} x_{1,l}\pi_l & \sum_{l=1}^{2} x_{1,l}^2\pi_l & \sum_{l=1}^{2} x_{1,l}x_{2,l}\pi_l & \sum_{l=1}^{2} x_{1,l}x_{3,l}\pi_l & (1-\gamma)\sum_{l=1}^{2} x_{1,l}\pi_l \\ \sum_{l=1}^{2} x_{2,l}\pi_l & \sum_{l=1}^{2} x_{1,l}x_{2,l}\pi_l & \sum_{l=1}^{2} x_{2,l}^2\pi_l & \sum_{l=1}^{2} x_{2,l}x_{3,l}\pi_l & (1-\gamma)\sum_{l=1}^{2} x_{2,l}\pi_l \\ \sum_{l=1}^{2} x_{3,l}\pi_l & \sum_{l=1}^{2} x_{1,l}x_{3,l}\pi_l & \sum_{l=1}^{2} x_{2,l}x_{3,l}\pi_l & \sum_{l=1}^{2} x_{3,l}^2\pi_l & (1-\gamma)\sum_{l=1}^{2} x_{3,l}\pi_l \\ (1-\gamma) & (1-\gamma)\sum_{l=1}^{2} x_{1,l}\pi_l & (1-\gamma)\sum_{l=1}^{2} x_{2,l}\pi_l & (1-\gamma)\sum_{l=1}^{2} x_{3,l}\pi_l & \frac{\pi^2}{6}+(1-\gamma)^2 \end{bmatrix} \quad (5.15)$$

Let $\boldsymbol{\theta} = (\beta_0, \beta_1, \cdots, \beta_k, \sigma)'$ be the vector of model parameters after normalization and $g(\boldsymbol{\theta})$ be a real-valued function, such as the quantile life, reliability function or hazard function at specified time and stress condition $\mathbf{x}$. Let $\hat{\boldsymbol{\theta}} = (\hat{\beta}_0, \hat{\beta}_1, \cdots, \hat{\beta}_k, \hat{\sigma},)'$ and $g(\hat{\boldsymbol{\theta}})$ be estimates of $\boldsymbol{\theta}$ and $g(\boldsymbol{\theta})$, respectively. Using the delta method (Meeker and Escobar, 1998), the asymptotic variance of $g(\hat{\boldsymbol{\theta}})$, denoted by $\text{Asvar}(g(\hat{\boldsymbol{\theta}}))$, can be approximated by

$$\text{Asvar}\left(g\left(\hat{\boldsymbol{\theta}}\right)\right) = \left[\frac{\partial g\left(\hat{\boldsymbol{\theta}}\right)}{\partial \hat{\boldsymbol{\theta}}}\right]^T \hat{\Sigma}_{\hat{\boldsymbol{\theta}}} \left[\frac{\partial g\left(\hat{\boldsymbol{\theta}}\right)}{\partial \hat{\boldsymbol{\theta}}}\right],$$

where $\hat{\Sigma}_{\hat{\boldsymbol{\theta}}}$ presents the asymptotic variance-covariance matrix which is the inverse of the Fisher information matrix evaluated at the MLE of $\hat{\boldsymbol{\theta}}$.

2.4 Optimal ALT Plan Formulation

We propose an optimal design of ALT plans based on the LHDs. This results in a significant reduction of the stress-level combinations for ALT with multiple stresses. Consider a test with three accelerated stresses with a minimum of three levels for each stress. As a result, the FFD requires 3^3=27 experiments but the LHD requires three experiments. To reduce the singularity of the Fisher information matrix, more levels for each stress are desired. When the stress levels are four for each stress, the FFD and LHD require 4^3 = 64 and four experiments, respectively. Clearly, as the number of stress level increases, the required experiments for FFD and LHD increase exponentially and linearly, respectively. If all the experiments have the same censoring time, the total test time based on LHD is significantly shorter than that based on FFD as illustrated by the examples in Sec. 2.6.2.

In practice, LHD can be randomly generated but such LHD may have undesired properties and may act poorly in estimation and prediction (Ye *et al*., 2000). In this section, we investigate three criteria for the design of the optimal plan in terms of stress-level combination matrix $\mathbf{X}^*$ and unit allocation $\pi_l \left(l = 1, \ldots, n-1 \right)$ under different censoring situations.

2.4.1 Variance Optimality

The main objective of an ALT experiment is to obtain accurate reliability estimates with minimum variance at normal operating conditions. In this section, we design an optimal test plan that minimizes the asymptotic variance of the logarithm of quantile failure at design stresses (normal operating conditions). The MLE of the log q^{th} quantile failure at design constant-stresses $\mathbf{x}_D$ is given by

$$\hat{y}_q = \hat{\beta}_0 + \hat{\beta}_1 x_{D,1} + \cdots + \hat{\beta}_k x_{D,k} + h\hat{\sigma} \,,$$

where $h = \ln\left[-\ln\left(1-q\right)\right]$ is the q^{th} quantile of the smallest extreme-value distribution. When h equals γ, $\hat{y}_q$ corresponds to the mean of the log lifetime distribution at $\mathbf{x}_D$.With the delta method, the asymptotic variance of $\hat{y}_q$ is given by,

$$\text{Asvar}\left(\hat{y}_q\right) = \left[1 \;\; x_{D,1} \cdots x_{D,k} \;\;\; h\right] \hat{\Sigma}_{\hat{\boldsymbol{\theta}}} \left[1 \;\; x_{D,1} \cdots x_{D,k} \;\;\; h\right]^T,$$

where $\hat{\Sigma}_{\hat{\boldsymbol{\theta}}}$ is the inverse of Eq. (2.13) evaluated at the MLE $\hat{\boldsymbol{\theta}} = \left(\hat{\beta}_0, \hat{\beta}_1, \cdots, \hat{\beta}_k, \hat{\sigma},\right)'$. We now formulate the optimization problem under different censoring situations. Let $\pi_l \left(\pi_l^* \le \pi_l < 1\right)$, for $l = 1, \ldots, n$ be the proportion of units allocated to each experiment and π_l^* be the specified lower bound of π_l.

Under Type-I censoring

Given the censoring time of each experiment $\eta_l \left(l = 1, \cdots, n\right)$, the optimization problem is formulated as,

$$\begin{aligned} &\text{Min} \quad \text{Asvar}\left(\hat{y}_q\right) \\ \text{s.t.} \quad &\pi_l^* \le \pi_l < 1,\ l = 1, \ldots, n \\ &\sum_{l=1}^{n} \pi_l = 1 \end{aligned}$$

The above formulation can be extended to the case without censoring when the censoring time is set to infinity ($\eta_l = \infty$), i.e. the test is terminated when all the units fail.

Under Type-II censoring

Suppose $n_l^* \left(l = 1, \cdots, n\right)$ be the minimum number of failures to be observed in the l^{th} experiment. Under Type-II censoring, the optimization problem is formulated as,

$$\text{Min} \quad \text{Asvar}\left(\hat{y}_q\right)$$

$$\text{s.t.} \quad \pi_l^* \le \pi_l < 1, \; l = 1, \ldots, n \;,$$

$$\sum_{l=1}^{n} \pi_l = 1$$

$$N\pi_l \Psi\left(\xi_l\right) = n_l^*, \; l = 1, \ldots, n \tag{5.16}$$

$$N\pi_l \ge n_l^*, \; l = 1, \ldots, n$$

where $\Psi\left(\bullet\right)$ is the cumulative failure distribution of *Z*. From (2.16), we obtain $\xi_l = \ln\left(-\ln\left(1 - n_l^* / N\pi_l\right)\right), \; l = 1, \ldots, n$ which is the upper integration limit of $B\left(\xi_l\right)$ and $C\left(\xi_l\right)$. This results in the Fisher information matrix given in Eq. (2.13).

The decision variables in above formulations are the $n \times k$ LHD matrix **X*** that specifies the stress-level combinations, and the proportion of test unit allocation $\pi_l \left(l = 1, \ldots, n-1\right)$. These formulations can be numerically evaluated by providing initial values for the model parameters.

Unit allocation

In above formulations, $\pi_l \left(l = 1, \ldots, n-1\right)$ are decision variables. To reduce the computational effort, we introduce an alternative method for unit allocation. Under single stress ALT, usually more test units are allocated to lower stress levels than higher stress levels, e.g. the widely used 4:2:1 rule. When the relative impact of each stress on the lifetime is known, then we use the following unit allocation:

$$\pi_l = \frac{w_1/x_{l,1} + w_2/x_{l,2} + \cdots + w_k/x_{l,k}}{w_1 \cdot \sum_{l=1}^{n} x_{l,1}^{-1} + w_2 \cdot \sum_{l=1}^{n} x_{l,2}^{-1} + \cdots + w_k \cdot \sum_{l=1}^{n} x_{l,k}^{-1}} \tag{5.17}$$

where $\sum_{i=1}^{k} w_i = 1$, $0 \le w_i \le 1$, for $i = 1, \ldots, k$ are weights reflecting the relative effect of each stress on lifetime. Eq. (2.17) is a monotonically decreasing function of stress levels of each factor. On the other hand, we use equal allocation of the test units to each experiment, i.e. $\pi_l = 1/n$ for $l = 1, \ldots, n$ when no information on the relative effect of each factor is known.

2.4.2 D-optimality

The volume of the asymptotic joint confidence region of the model parameters is proportional to the square root of the determinant of $\Sigma = F^{-1}$. A larger value of the determinant of the Fisher information matrix corresponds to a higher joint precision of the estimates of $\boldsymbol{\beta}$ and σ. Therefore, we choose D-optimality which maximizes the determinant of the Fisher information matrix as the second criterion. Due to the invariant characteristic, we investigate the determinant of $\left(\sigma^2/N\right) \cdot F$, which is given by Eq. (2.13). The optimization problems under different censoring situations are formulated as follows.

Under Type-I censoring

Given the censoring time of each experiment $\eta_l \left(l = 1, \cdots, n\right)$,

$$\text{Max} \quad \text{Det}\left(\left(\sigma^2/N\right)\cdot \hat{F}_{\hat{\boldsymbol{\beta}},\hat{\sigma}}\right)$$

$$\text{s.t.} \quad \pi_l^* \le \pi_l <, l = 1,\ldots,n$$

$$\sum_{l=1}^{n} \pi_l = 1$$

where $\text{Det}(\bullet)$ presents the determinant of the matrix. The above formulation also can be extended to the case without censoring when the censoring time $\eta_l = \infty$.

Under Type-II censoring

Given $n_l^*\left(l = 1,\cdots,n\right)$, the required number of failures for each experiment,

$$\text{Max} \quad \text{Det}\left(\left(\sigma^2/N\right)\cdot \hat{F}_{\hat{\beta},\hat{\sigma}}\right)$$

$$\text{s.t.} \quad \pi_l^* \le \pi_l < 1, l = 1,\ldots,n,$$

$$\sum_{l=1}^{n} \pi_l = 1,$$

$$N\pi_l \Psi\left(\xi_l\right) = n_l^*, l = 1,\ldots,n$$

$$N\pi_l \ge n_l^*, l = 1,\ldots,n$$

Initial estimate of the model parameters are required to evaluate the formulations under Type-I and Type-II censoring, whereas the formulation for the case without censoring does not depend on the model parameters.

2.4.3 Multi-objective Optimality

Although the reliability of a product might be affected by numerous factors, usually few are dominated. Oftentimes, the few important factors contain both accelerating variables and ordinary explanatory variables, such as the insulation thickness example (Nelson, 1990). ALTs are often designed not only to estimate unknown parameters but also investigate the effect of the ordinary experimental factors. Suppose that x_1 and x_2 are accelerating variables and x_3 is an ordinary experimental factor, or x_1 is an accelerating variable and x_2 and x_3 are ordinary experimental factors. Consider the model,

$$\mu(\mathbf{\beta x}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3,$$

where the β's are unknown. To reduce the variance of the estimate of unknown parameters, D-optimality without censoring is an appropriate choice as the determinant of the Fisher information matrix is independent of the model parameters. Meanwhile, to investigate the effect of ordinary experimental factors, the experimental points of a LHD should be spread as evenly as possible. The maximin inter-site distance criterion defined by Eq. (2.10) can be written as a scalar-valued function,

$$\phi_p = \left(\sum_{i=1}^{\binom{n}{2}} \frac{1}{d_i^p} \right)^{1/p}$$

where p is a positive number, d_i is defined by Eq. (2.9). The maximin inter-site design minimizes ϕ_p, so it is also called a ϕ_p-optimal design.

However, it is shown that LHDs with D-optimality may not achieve maximum minimum inter-site distances. For instance, consider 5-experiment and 3-stress LHDs. We plot the inter-site distance criterion ϕ_p^{-1} against the determinants of the Fisher information matrix in Figure 2.2. We observe that the data points are highly scattered indicating that optimization in terms of one criterion may not lead to optimization of the second criterion. The problem is worse with more stress-level combinations. Therefore, we propose combining the ϕ_p criterion and the D-optimality as a multi-objective optimization problem. In order to combine ϕ_p criterion with the D-optimality which maximizes the determinant of the Fisher information matrix, we use $\Phi_p = 1/\phi_p$ instead.

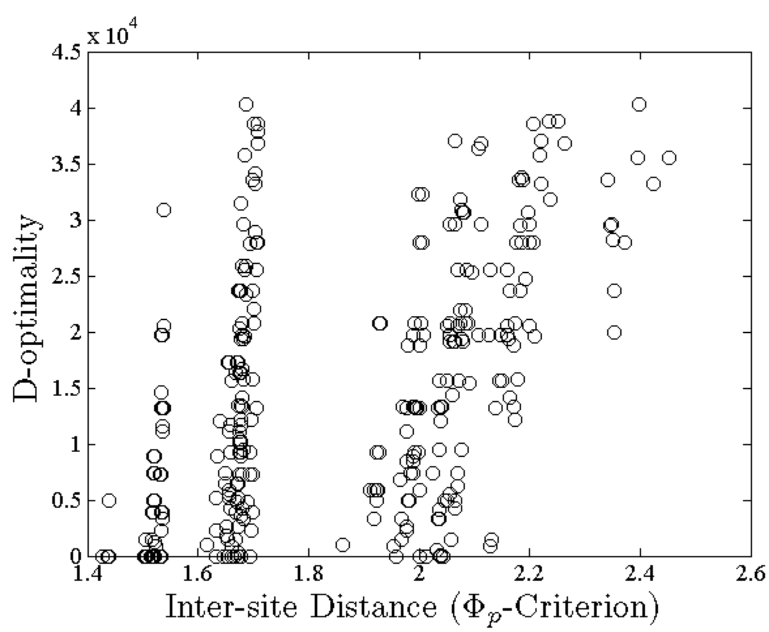


Figure 2.2 D-optimality vs. distance measure

2.4.3.1 Upper and Lower Bounds of Optimization Criterion

The D-optimality and Φ_p are two different criteria and their magnitudes are also different. In order to consider an optimization function that combines both criteria, normalization is required based on the upper and lower bounds of each. This is achieved as follows.

Since β's are unknown and investigation of the effect of the explanatory variables on lifetime is one of the purposes of the test, we assume that the test units are equally allocated to each stress-level combination ($\pi_l = n^{-1}$) for simplicity. Thus the information matrix $F_s = \left(n\sigma^2/N\right)\cdot F$ is given by Eq. (2.18). The associated upper and lower bounds are derived as discussed below.

$$\begin{bmatrix} n & \sum_{l=1}^{n} x_{1,l} & \sum_{l=1}^{n} x_{2,l} & \sum_{l=1}^{n} x_{3,l} & n(1-\gamma) \\ & \sum_{l=1}^{n} x_{1,l}^2 & \sum_{l=1}^{n} x_{1,l}x_{2,l} & \sum_{l=1}^{n} x_{1,l}x_{3,l} & (1-\gamma)\sum_{l=1}^{n} x_{1,l} \\ & & \sum_{l=1}^{n} x_{2,l}^2 & \sum_{l=1}^{n} x_{2,l}x_{3,l} & (1-\gamma)\sum_{l=1}^{n} x_{2,l} \\ & & & \sum_{l=1}^{n} x_{3,l}^2 & (1-\gamma)\sum_{l=1}^{n} x_{3,l} \\ & & & & n\left[\frac{\pi^2}{6}+(\gamma-1)^2\right] \end{bmatrix} \tag{5.18}$$

PROPOSITION 1. *For a three factor Latin hypercube design* $LHD(n,3)$*, the determinant of Fisher information matrix of MLE based on the smallest extreme-value model is,*

$$0 \le \mathrm{Det}\left(F(\mathbf{X})_s\right) \le \mathrm{Det}_U ,$$

where Det_U *denotes the upper bound and is given by*

$$\mathrm{Det}_{\mathrm{U}} = n^4 (n-1)^3 (n+1)^3 \left(n\pi^2 + 2n\gamma(1-\gamma)\right) / 10368,\ n > 1 .$$

The proof is given in Appendix 1.

For rectangular distance ($d(\boldsymbol{s},\boldsymbol{t}) = \sum_{j=1}^{k} \left| s_j - t_j \right|$) criterion, Joseph and Hung (2008) derive the upper and lower bounds of ϕ_p. In this dissertation, we derive upper and lower bounds of $\Phi_p = 1/\phi_p$ for general *q*-norm distance where $q \ge 1$.

Consider an LHD with *n* experiments and *k* factors, denoted by $\mathrm{LHD}(n,k)$. Let $d_1, d_2, \ldots < d_{\binom{n}{2}}$ be the inter-site distances among the *n* experiment points defined by Eq. (2.9).

LEMMA 1. *For an* $LHD(n,k)$*, the average of q's power of q-norm inter-site distance is a constant given by*

$$\bar{d}^{\,q} = \frac{2k\sum_{i=1}^{n-1}(n-i)\cdot i^q}{n(n-1)},\ q \geq 1.$$

For the case $q=2$ *(Euclidean distance),* $\bar{d}^{\,2} = kn(n+1)/6$.

PROPOSITION 2. *For an* $LHD(n,k)$, $\Phi_{p,L} \leq \Phi_p \leq \Phi_{p,U}$, *where* $\Phi_{p,U} = \left(\bar{d}^{\,q}\right)^{1/q} \Big/ \binom{n}{2}^{1/p}$,

and $\Phi_{p,L} = \left(\sum_{i=1}^{n-1} \frac{(n-i)}{k^{p/q}\cdot i^p}\right)^{(-1/p)}$.

When q =2, then $\Phi_{p,U} = \left[\sqrt{kn(n+1)/6} \Big/ \binom{n}{2}^{1/p}\right]$*, and* $\Phi_{p,L} = \left(\sum_{i=1}^{n-1} \frac{(n-i)}{k^{p/2}\cdot i^p}\right)^{(-1/p)}$.

Proofs of lemma 1 and proposition 2 are given in Appendix 2.

2.4.3.2 Multi-objective Formulation

Using the upper and lower bounds obtained above, the multi-objective criterion considering both the D-optimality and the inter-site distance is formulated as,

$$\text{Min } -w\frac{\det\left(F_s(\mathbf{X})\right)}{Det_U} - (1-w)\frac{\Phi_p(\mathbf{X}) - \Phi_{p,L}}{\Phi_{p,U} - \Phi_{p,L}},\ w \in [0,1] \tag{5.19}$$

where w is a pre-specified weight reflecting the test designers' preference for the criterion, and $\boldsymbol{X}$ is a $n \times 3$ LHD matrix.

2.5 Optimization of the Test Plan

The design matrix **X** contains permutations of integer values but the unit allocation π_l is continuous. Therefore, the optimization criteria presented in Sec. 2.4 are very difficult to evaluate by the classical analysis of function approach such as gradient search. In general, the generation of optimal LHD starts from a random LHD, then by swapping the order of two factor levels in a column of the matrix a new design is generated and evaluated. Since the generation of the design matrix **X** is a discrete problem, generic probabilistic metaheuristic might be utilized to obtain optimum solution in relatively short computational time. For instance, the threshold accepting heuristic is used to find optimal LHDs in terms of the *U*-type design (Winker and Fang, 1998), the stochastic evolutionary algorithm is used to evaluate the ϕ_p criterion, entropy criterion, and centered L_2 discrepancy criterion for optimal LHD, and the simulated annealing (SA) algorithm is used by Morris and Mitchell (1995) to find optimal LHDs according to the ϕ_p criterion and by Joseph and Hung (2008) to find orthogonal LHDs. However, searching for optimal unit allocation $\pi_l\left(l=1,\ldots,n-1\right)$ requires searching in a continuous space. Therefore, we propose a mixed algorithm to evaluate the criteria of variance optimality and D-optimality when optimal unit allocation is a decision variable. Since the convergence of a standard SA is already established (Lundy and Mees, 1986), we use SA to evaluate the objective functions and call a nonlinear optimization algorithm *fmincon* (see Matlab) to search the new best $\pi_l\left(l=1,\ldots,n-1\right)$ at each iteration. Regarding the multi-objective criterion, we propose a Modified SA algorithm which significantly improves the computational efficiency. All the MSA related parameters are set at the same values as those used in a standard SA.

Without loss of generality, the first column of a design $\boldsymbol{X}$ is fixed as $\{1, 2, \ldots, n\}'$. The mixed algorithm begins with a random permutation of the remaining columns and $\pi_l\ (l = 1, \ldots, n-1)$, and proceeds through examination of a sequence of designs and $\pi_l\ (l = 1, \ldots, n-1)$ values. Each new design is generated as a perturbation of the preceding one which is formed by interchanging two randomly chosen elements within a randomly chosen column (excluding the first column) of the design matrix. Given the perturbed matrix $\boldsymbol{X}_{try}$, SA calls *fmincon* to evaluate the same objective function and finds the best $\pi_l\ (l = 1, \ldots, n-1)$ values associated with $\boldsymbol{X}_{try}$. If the perturbed matrix $\boldsymbol{X}_{try}$ and the corresponding best $\pi_l\ (l = 1, \ldots, n-1)$ values lead to an improvement of the objective function, they are then adopted as the new current design and $\pi_l\ (l = 1, \ldots, n-1)$ values which are then used for the next perturbation and corresponding best π_l values. Otherwise, replacement of current design $\boldsymbol{X}$ and $\pi_l\ (l = 1, \ldots, n-1)$ values are made with probability $\exp\left\{-\left[\Omega\left(\mathbf{X}_{try}\right) - \Omega\left(\mathbf{X}\right)\right]/t\right\}$ (where t is the annealing temperature and $\Omega(\bullet)$ is the objective function). The step of using *fmincon* is not needed when the π_l values are given, e.g. by Eq. (2.17).

For the multi-objective optimality, we modify the SA algorithm such that the chosen elements to swap are based on defined probabilities. For an $n\times 3$ LHD, since the first column is fixed as $\{1,2,\ldots,n\}'$, the permutation of remaining two factors need to be determined. From the proof of proposition 1 we observe that the permutation of the second factor and third factor affects the elements $a_1=\sum_{l=1}^{n} x_{1,l}x_{2,l}$ and $a_2=\sum_{l=1}^{n} x_{1,l}x_{3,l}$ of F_s. The maximum $\det\left(F_s\left(\mathbf{X}\right)\right)$ is obtained when $a_1=a_2=n\left(n+1\right)^2/4$ for an n-experiment design. Suppose after some iterations, $a_1=\sum_{l=1}^{n} x_{1,l}x_{2,l}=n\left(n+1\right)^2/4$, then swapping elements of the second factor may not increase the determinant any more (Note the first factor is fixed). According to this observation, we propose to select the column to perturb base on the comparison of $\left|a_1-n\left(n+1\right)^2/4\right|$ and $\left|a_2-n\left(n+1\right)^2/4\right|$. That is, at each iteration, we compute $\left|a_1-n\left(n+1\right)^2/4\right|$, $\left|a_2-n\left(n+1\right)^2/4\right|$, and $\Phi_{p,i}=1\Big/\left(\sum_{j\neq i} d_{i,j}^{-p}\right)^{1/p}$ for each row $i=1,2,\ldots,n$, where $d_{i,j}$ is the q-norm inter-site distance between the points i and j. The second column is chosen to swap with probability p_2 given by $\left|a_1-n\left(n+1\right)^2/4\right|^{\delta}\Big/\sum_{i=1,2}\left|a_i-n\left(n+1\right)^2/4\right|^{\delta}$, $\delta\in\left[1,\infty\right)$. The third column is chosen with probability 1- p_2. Clearly, the selection of the column is not random and is based on the probability values calculated above. Within the selected column, element i is chosen with probability $\Phi_{p,i}^{p}\Big/\sum_{i=1}^{n}\Phi_{p,i}^{p}$ (Joseph and Hung, 2008) to swap with another randomly selected element in the same column. This generates a new design $\mathbf{X}_{try}$. If the objective function evaluated at the new design is smaller than that of the current design $\boldsymbol{X}$ then the new design replaces the current design; otherwise, it replaces the current design with probability $\exp\left\{-\left[\Omega\left(\mathbf{X}_{try}\right)-\Omega\left(\mathbf{X}\right)\right]/t\right\}$.

2.6 Examples

In this section, we demonstrate the application of LHD for finding optimal ALT plans based on an actual test. Then we compare the optimal ALT plans based on LHD with those obtained based on FFD and validate the performance of the MSA algorithm. Lipscomb *et al.* (2009) conduct tests to study the effect of Relative Humidity (RH), temperature (Temp) and electrical field (kilo Voltage per millimeter) on reliability of PTZ actuators by varying each accelerated stress independently. In their study, the range of Temp is 35-85 ℃, RH is 60-90%, and the applied electrical field is 0.31-2.2 kV/mm. In the following examples, we adopt the stresses and associated values that Lipscomb *et al*. (2009) use in their study. However, our objective is to develop optimal ALT plans based on LHD by simultaneously applying multiple stresses to predict reliability at normal operating conditions.

Suppose due to time constraint only five experiments can be conducted in an ALT. Using three stresses and five experiments, we construct an LHD (5, 3). Let the RH (%), Temp (℃), and electrical field strength (kV/mm) be the first, second and third stress, respectively. We use Eq. (2.11) to choose five equally spaced values (60, 67.5, 75, 82.5, and 90) from the range of RH (60-90%) and normalize them as (1, 2, 3, 4, and 5). Similarly, we choose 0.31, 0.7825, 1.255, 1.7275, and 2.2 from the range of the electrical field strength (0.31-2.2 kV/mm). The maximum and minimum Temp values are firstly converted to scaled Kelvin using $10000/(\text{Temp}+273.5)$. From the range of scaled Kelvin Temp (28-32), we use Eq. (2.11) to select five equally spaced values (28, 29, 30, 31, and 32) and normalize them as (1, 2, 3, 4, and 5). The results of stress normalization are summarized in Table 2.2. We assume the design stresses are 30%, 20 ℃, 0.2 kV/mm. Using the same linear operation to normalize each testing stress levels as $\{1,\ldots,5\}$, the corresponding design stress is normalized as -3, 7, 0.7672. According to the relative effects of each stress and the mean life of the PTZ actuators under different conditions provided by the tests in Lipscomb *et al*. (2009), we estimate the model parameters and normalize them as $[\boldsymbol{\beta},\sigma]^T = [5.23, -0.485, 0.427, -0.8, 0.8]^T$.

Table 2.2 Testing and normalized stress levels

Stress	LHD	1	2	3	4	5
I	Relative Humidity (%)	60	67.5	75	82.5	90
II	Temp (℃)	35	45	55	70	85
III	Electrical Field (kV/mm)	0.31	0.7825	1.255	1.7275	2.2

2.6.1 Optimal ALT Plans

Usually the lower quantile failure is of interest when one chooses the asymptotic variance optimality. Thus, in this example, we minimize the asymptotic variance of MLE of q = 0.1 log quantile failure at design stresses. The lower bound of the proportion of units allocated to each experiment is $\pi_l^* = 0.015$ of the total test units. We terminate the test when all units fail, censoring time $\eta_l = 10$ time-units is reached under Type-I censoring, or $n_l^* = 15$ failures are observed under Type-II censoring. With the formulation of the variance optimality given in Sec. 2.4, we search the optimal plans with the algorithm presented in Sec. 2.5. The results are shown in Table 2.3. For the case of no censoring, the η_l / n_l^* is represented by infinity.

Under each censoring situation, we study all three unit allocation methods given in Sec. 2.4. When π_l is determined by Eq. (2.17), the weights w_1, w_2, and w_3 corresponding to applied field (kV/mm), Temp, and RH are 5/9, 3/9, and 1/9, respectively. Under all cases, the minimum Asvar is achieved when $\pi_l \left(l = 1, \ldots, n-1\right)$ values are decision variables. Under Type-I censoring and no censoring, when the units are allocated according to Eq. (2.17), the achieved asymptotic variances are slightly smaller than those using equal allocation of test units. However, under Type-II censoring when the units are allocated equally, the asymptotic variance is smaller than that using Eq. (2.17).

The D-optimality is evaluated with the same constraints as the variance optimality. The obtained optimal ALT plans are presented in Table 2.4. The optimal plans under Type-II censoring and no censoring do not depend on the model parameters and design stresses. In other words, they are optimal plans regardless of the model parameters and stress conditions as long as they are subject to the same constrains. Table 2.4 shows that under all cases, the maximum determinant is achieved when $\pi_l \left(l = 1, \ldots, n-1\right)$ values are decision variables; the obtained determinants are comparable when the units are allocated equally to each experiment or based on Eq. (2.17).

Table 2.3 Optimal ALT plans based on variance optimality

Censoring	Experiment l	Normalized Factor Levels			η_l / n_l^*	π_l		Asvar
		RH	Temp	kV/mm		Values	Method	
Type-I	I	1	5	1	10	0.0150	Decision variables	39.43
	II	2	3	4	10	0.4704		
	III	3	4	3	10	0.0889		
	IV	4	2	2	10	0.0944		
	V	5	1	5	10	0.3313		
	I	1	3	4	10	0.3041	Eq. (2.17)	54.56
	II	2	5	1	10	0.1995		
	III	3	2	2	10	0.1784		
	IV	4	4	3	10	0.1135		
	V	5	1	5	10	0.2044		
	I	1	2	4	10	0.2	Equal allocation	59.50
	II	2	3	1	10	0.2		
	III	3	4	2	10	0.2		
	IV	4	5	3	10	0.2		
	V	5	1	5	10	0.2		
Type-II	I	1	5	2	15	0.0976	Decision variable2	50.77
	II	2	4	3	15	0.1266		
	III	3	1	4	15	0.0999		
	IV	4	3	1	15	0.2814		
	V	5	2	5	15	0.3945		
	I	1	3	1	15	0.3406	Eq. (2.17)	67.48
	II	2	4	4	15	0.1703		
	III	3	5	3	15	0.1265		
	IV	4	2	2	15	0.1582		
	V	5	1	5	15	0.2044		
	I	1	5	2	15	0.2	Equal allocation	62.50
	II	2	4	3	15	0.2		
	III	3	1	4	15	0.2		
	IV	4	3	1	15	0.2		
	V	5	2	5	15	0.2		
No	I	1	5	2	∞	0.6467	Decision variables	11.94
	II	2	1	1	∞	0.015		
	III	3	4	4	∞	0.015		
	IV	4	2	5	∞	0.0238		
	V	5	3	3	∞	0.2996		
	I	1	4	1	∞	0.3285	Eq. (2.17)	18.84
	II	2	3	4	∞	0.1825		
	III	3	5	3	∞	0.1265		
	IV	4	2	5	∞	0.1436		
	V	5	1	2	∞	0.2190		
	I	1	4	1	∞	0.2	Equal allocation	23.38
	II	2	3	5	∞	0.2		
	III	3	5	2	∞	0.2		
	IV	4	2	3	∞	0.2		
	V	5	1	4	∞	0.2		

Table 2.4 Optimal ALT plans based on D-optimality

Censoring	Experiment l	Normalized Factor levels			η_l / n_l^*	π_l		Det
		RH	Temp	kV/mm		Values	Method	
Type-I	I	1	2	5	10	0.2518	Decision variables	0.576
	II	2	4	1	10	0.0500		
	III	3	3	2	10	0.1889		
	IV	4	1	3	10	0.2664		
	V	5	5	4	10	0.2426		
	I	1	2	5	10	0.3260	Eq. (2.17)	0.252
	II	2	4	2	10	0.1825		
	III	3	3	1	10	0.1784		
	IV	4	1	3	10	0.2230		
	V	5	5	4	10	0.090		
	I	1	2	5	10	0.2	Equal allocation	0.318
	II	2	4	2	10	0.2		
	III	3	3	1	10	0.2		
	IV	4	1	3	10	0.2		
	V	5	5	4	10	0.2		
Type-II	I	1	3	5	15	0.0760	Decision variables	0.132
	II	2	5	1	15	0.0760		
	III	3	1	2	15	0.0852		
	IV	4	2	3	15	0.6811		
	V	5	4	4	15	0.0817		
	I	1	3	5	15	0.3017	Eq. (2.17)	0.103
	II	2	4	1	15	0.2068		
	III	3	2	2	15	0.1784		
	IV	4	1	3	15	0.2230		
	V	5	5	4	15	0.0900		
	I	1	4	5	15	0.2	Equal allocation	0.104
	II	2	1	2	15	0.2		
	III	3	5	1	15	0.2		
	IV	4	2	3	15	0.2		
	V	5	3	4	15	0.2		
No	I	1	5	4	∞	0.2462	Decision variables	22.106
	II	2	1	2	∞	0.2463		
	III	3	3	3	∞	0.0150		
	IV	4	4	1	∞	0.2462		
	V	5	2	5	∞	0.2463		
	I	1	5	4	∞	0.2847	Eq. (2.17)	13.825
	II	2	2	1	∞	0.2433		
	III	3	1	5	∞	0.2368		
	IV	4	3	3	∞	0.1257		
	V	5	4	2	∞	0.1095		
	I	1	3	5	∞	0.2	Equal allocation	12.896
	II	2	5	1	∞	0.2		
	III	3	1	2	∞	0.2		
	IV	4	2	3	∞	0.2		
	V	5	4	4	∞	0.2		

The multi-objective criterion considers both Φ_p criterion and D-optimality. For the $\phi_p\left(\Phi_p^{-1}\right)$ criterion, Morris and Mitchell (1995) show that the value of p has effects on the optimal solution. For a small problem, e.g. $n=5, k=3$, p as small as 5 is sufficient and a large problem (defined by large values of n and k) requires a much larger value of p. Therefore, in this example we let $p=5$. We consider both the rectangular and the Euclidean inter-site distance, i.e. $q=1, 2$. With the derived upper limit of the determinant of F_s, and the upper and lower limits of Φ_p, we have $\text{Det}_U = 4.3157e+004$; $\Phi_{p,U} = 3.7857$ and $\Phi_{p,L} = 2.2620$ when $q=1$; $\Phi_{p,U} = 2.4437$ and $\Phi_{p,L} = 1.3060$ when $q=2$. For the multi-objective, we set $w=0.5$; the tests are terminated when all units fail and the units are equally allocated to the five experiments. Using the MSA, we obtain optimal ALT plans as shown in Table 2.5.

Table 2.5 Optimal ALT plans based on multi-objective optimality

(p,q)	Experiment l	(5,1)			(5,2)			η_l / n_l^*	π_l
		RH	Temp	kV/mm	RH	Temp	kV/mm		
LHD (5,3)	I	1	4	2	1	4	2	∞	0.2
	II	2	3	5	2	1	3	∞	0.2
	III	3	2	1	3	5	4	∞	0.2
	IV	4	1	4	4	2	5	∞	0.2
	V	5	5	3	5	3	1	∞	0.2

2.6.2 ALT Plans based on LHD vs. FFD

We compare ALT plans based on FFD ($3^3 = 27$ experiments) with those based on LHD (5, 3) using the same optimality criteria. We use the test stress levels and corresponding normalized stress levels as shown in Table 2.6. As a result, the normalized model parameters, design stresses and stress ranges are the same as given in Sec. 2.6.1. We utilize the stress levels (1, 3, 5) of LHD in the FFD. We consider the cases when $\pi_l \left(l = 1, \ldots, n-1 \right)$ values are to be determined as well as when they are assigned equal values under Type-I censoring and no censoring.

The comparison based on the variance optimality is shown in Table 2.7. Under Type-I censoring, the asymptotic variance of log quantile failure at design stresses based on LHD is 18.35% lower than the asymptotic variance obtained from FFD when $\pi_l \left(l = 1, \ldots, n \right)$ values are decision variables. When $\pi_l \left(l = 1, \ldots, n \right)$ values are set equally, the achieved objective function value from LHD is 40.13% lower than the FFD's. Similar reduction in the objective function is obtained under no censoring. In addition, the total time and/or number of experiments required by LHD is 81.48% less than that required by FFD. Obviously, ALT with multiple stresses using LHD not only achieves higher precision of reliability prediction but also requires less time/experiments than that of the FFD.

Table 2.6 Test and normalized stress levels

LHD	**1**	**3**	**5**
Electrical Field Strength(kV/mm)	0.31	1.255	2.2
Temp (℃)	35	55	85
RH (%)	60	75	90

Table 2.7 Comparison based on the variance optimality

Min Asvar (q = 0.1) at design stresses		LHD (5 exp.)	FFD (27 exp.)	Asvar Red. (%)	Total time red. (%)	Number of exp. Red. (%)
Type-I	π_l : decision variables	39.43	48.29	18.35	81.48	81.48
	π_l : 1/n	59.50	99.38	40.13		
No censoring	π_l : decision variables	11.94	14.75	19.05	--	
	π_l : 1/n	23.38	26.71	12.47		

The comparison based on D-optimality is shown in Table 2.8. When $\pi_l\left(l=1,\ldots,n\right)$ values are decision variables, the obtained maximum determinants of the Fisher information matrix from LHD are larger than those obtained from FFD under both Type-I censoring and no censoring. The difference is significant. As a result, ALT with multiple stresses using LHD provides higher joint precision of parameters estimate than that of using FFD. In contrast, when test units are equally allocated to each experiment, the D-optimality values obtained from FFD are larger than those from LHD. However, the difference is insignificant.

Table 2.8 Comparison based on D-optimality

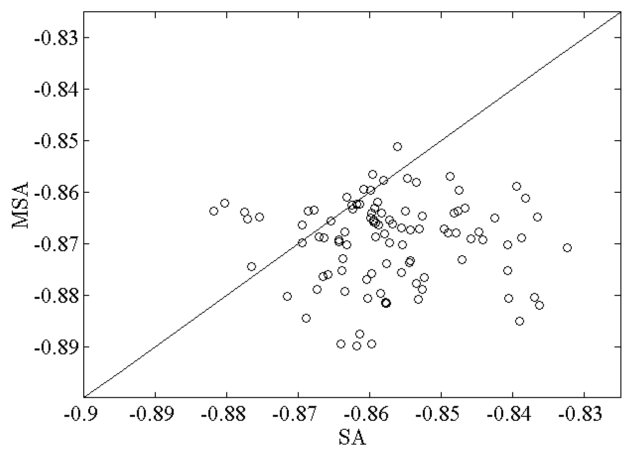
<table>
<tr><th colspan="2">Max Det. (Fisher)</th><th>LHD
(5 exp.)</th><th>FFD
(27 exp.)</th><th>Det.
Incr. (%)</th><th>Total time
red. (%)</th><th>Number of
exp. Red. (%)</th></tr>
<tr><td rowspan="2">Type-I</td><td>π_l : decision variables</td><td>0.576</td><td>0.0115</td><td>4909</td><td rowspan="2">81.48</td><td rowspan="4">81.48</td></tr>
<tr><td>π_l : 1/n</td><td>0.318</td><td>0.6399</td><td>-20.60</td></tr>
<tr><td rowspan="2">No censoring</td><td>π_l : decision variables</td><td>22.106</td><td>2.072</td><td>967</td><td rowspan="2">--</td></tr>
<tr><td>π_l : 1/n</td><td>12.90</td><td>31.19</td><td>-58.64</td></tr>
</table>

2.6.3 Performance of the MSA

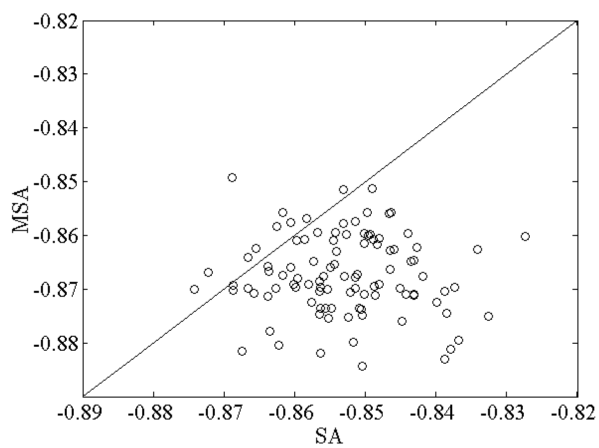
An MSA is developed in Sec. 2.5 to improve the efficiency of the search algorithm for the optimal LHD plans. In this section we validate the performance of the algorithm.

All the LHDs can be enumerated for an 5×3 design. Among the enumerations, the design that results in the minimum value of the multi-objective formulation when q = 1, 2 is the same as that obtained from the MSA. This shows that for a small problem, the solution from using MSA converges to the true optimal.

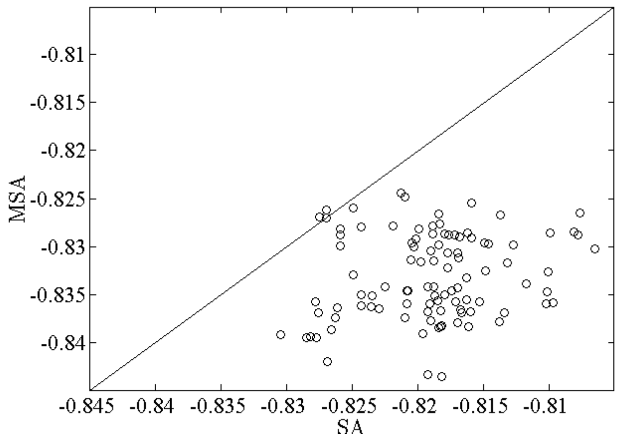
Now consider larger problems, LHD(10, 3) and LHD$(25, 3)$. Let $p = 15$, $w = 0.5$, and consider the two cases $q = 1, 2$. For the same initial LHD and setting of SA algorithm parameters, we evaluate the multi-objective formulation by MSA and SA. At the 500th iteration, we stop both algorithms and record their objective function values. We choose another initial LHD and run the two algorithms and record their objective function values at 500th iteration. This simulation is repeated 100 times. The results are plotted in Figure 2.3. The diagonal line is a collection of equivalent objective function values. We see that most of the data points are below the line, which verifies that MSA converges faster than SA algorithm.



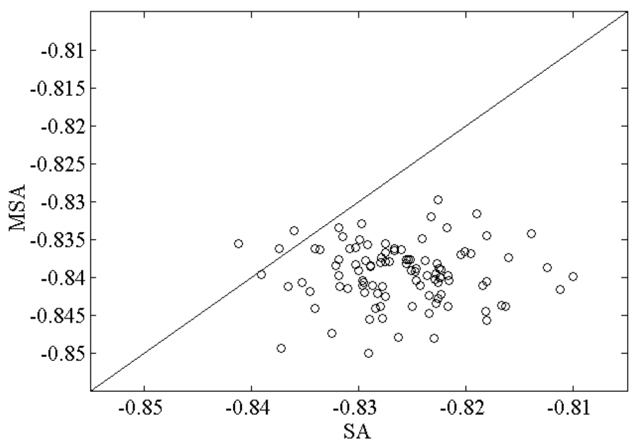
(a) q = 1, LHD(10, 3)



(b) q = 2, LHD(10, 3)



(c) q = 1, LHD(25, 3)



(d) q = 2, LHD(25, 3)

Figure 2.3 Objective function values from MSA vs. SA

The simulations are repeated for another 100 times and extended to the 1500th iterations and 3000th iterations. The MSA outperforms SA in more than 85 out of 100 times as shown in Table 2.9. Moreover, larger n values result in more significant improvements of MSA over SA. For instance, for q = 1, at the 500th iterations, when n = 10, MSA outperforms SA in 85.1% of the time. However, when n = 25, MSA outperforms SA in 98% of the time. In addition to the percentage of smaller objective function values from

MSA, we record the minimum, maximum and average achieved objective function values of the 200 times simulation from the two algorithms and summarize them in Tables 2.10 to 2.13. In each combination, the SMA always provides smaller minimum, maximum and average objective function values. These results validate the significant improvement obtained using the MSA.

Table 2.9 MSA outperforms SA

Iteration #	**LHD(10, 3)**		**LHD(25, 3)**	
	***q* = 1**	***q* = 2**	***q* = 1**	***q* = 2**
500	85.1%	90.2%	98%	95%
1500	88.3%	91%	98.5%	96.7%
3000	92%	92.5%	99.3%	98%

Table 2.10 LHD (10, 3), $(p, q) = (15, 1)$

Iteration #	Min		Max		Mean	
	MSA	SA	MSA	SA	MSA	SA
500	-.8899	-.8818	-.8511	-.8324	-.8697	-.8571
1500	-.8909	-.8823	-.8579	-.8480	-.8766	-.8645
3000	-.8909	-.8844	-.8667	-.8565	-.8796	-.8685

Table 2.11 LHD (10, 3), (p, q) = (15, 2)

Iteration #	Min		Max		Mean	
	MSA	SA	MSA	SA	MSA	SA
500	-.8843	-.8741	-.8493	-.8273	-.8673	-.8529
1500	-.8891	-.8794	-.8557	-.8375	-.8731	-.8606
3000	-.8950	-.8825	-.8654	-.8484	-.8756	-.8657

Table 2.12 LHD (25, 3), (p, q) = (15, 1)

Iteration #	Min		Max		Mean	
	MSA	SA	MSA	SA	MSA	SA
500	-.8436	-.8304	-.8244	-.8066	-.8331	-.8190
1500	-.8455	-.8304	-.8262	-.8098	-.8366	-.8215
3000	-.8455	-.8342	-.8278	-.8165	-.8380	-.8234

Table 2.13 LHD (25, 3), (p, q) = (15, 2)

Iteration #	Min		Max		Mean	
	MSA	SA	MSA	SA	MSA	SA
500	-.8500	-.8412	-.8298	-.8100	-.8395	-.8261
1500	-.8500	-.8423	-.8355	-.8112	-.8424	-.8287
3000	-.8519	-.8437	-.8360	-.8245	-.8433	-.8318

2.7 Summary

In this chapter, we present an approach for the design of ALT plans with multiple stresses using LHD. The lifetime of a test unit follows a Weibull distribution. The applied stresses affect only the scale parameter of the Weibull distribution through a logarithmic linear model. We develop the Fisher information matrix for MLE of the unknown parameters. We propose a multi-objective optimization criterion which maximizes the determinant of the Fisher information matrix of MLE as well as the maximum minimum inter-site distance between design points. We also formulate optimal test plans minimizing the asymptotic variance of log quantile failure and maximizing the determinant of the Fisher information matrix. The proposed approach results in efficient and practical ALT test plans. Moreover, these test plans are significantly better than those obtained based on FFD in terms of the asymptotic variance of reliability prediction and parameter estimates and total test time. We develop SA based algorithms to efficiently determine the optimal test plans. The effectiveness of the algorithm is validated by a simulation study.

CHAPTER 3

PROGRESSIVE CENSORING ALT PLANS

One of the purposes of conducting accelerated life testing is to obtain failure time data, in order to assess the reliability of products or material at normal operating conditions. A typical ALT can be terminated before the failure of all units under test. Moreover, units may experience different failure modes during the test as the case when testing circuit boards where failures occur due solder joints or device failure. Most of the previous research on ALT focuses on one failure mode.

In this chapter, we investigate ALT under a general censoring scheme considering multiple failure modes. We begin with the review of commonly used censoring schemes in accelerated life testing, likelihood inference with censored data and research on design of ALT plans under different censoring schemes. Then we briefly discuss failures under competing risk and related work. We then present the assumptions of this work and describe the censoring procedure (namely, progressive censoring), construct the likelihood function and develop the Fisher information matrix for MLE. Following this, we propose, discuss the motivation and develop a new test plan criterion; mean time of first failure. Then we formulate optimal ALT plans with respect to four optimization criteria: minimization of asymptotic variance of mean time of first failure and quantile failure, D-optimality, and multi-objective optimization. These formulations are applicable to tests under both single stress and multiple stresses. Finally, numerical examples based on parameters from real tests are presented. Sensitivity study is conducted to identify the parameters of the model that should be initially estimated with special care. We conclude this chapter with a summary.

3.1 Literature Review

3.1.1 Censoring

Censoring arises in a life test whenever the experimenter does not observe the lifetimes of all test units. In ALT, the most widely used censoring scheme is Type-I censoring, which is often called "time censoring". Under Type-I censoring, a test unit i is removed from test at a prespecified censoring time $C_i > 0$ if it does not fail at C_i. The failure time T_i can be observed if and only if $T_i \le C_i$.

Extensive work on planning ALT is based on Type-I censoring (Nelson, 1990, Meeker and Escobar, 1998, Lawless, 1982, 2003). Early discussions of asymptotic properties are given by Bartholomew (1957, 1963). Chernoff (1962) gives optimum plans for the estimation of the failure rate of exponentially distributed failure times at design stress level. The formulation of the likelihood function under censoring is investigated by Cox (1975), Kalbfleisch and MacKay (1978) and Kalbfleisch and Prentice (1980). Meeker (1984) presents a comparison of ALT plans for Weibull and lognormal distributions under Type-I censoring. Bai *et al*. (1992) obtain an optimal step-stress test plan that minimizes the asymptotic variance of the MLE of the mean life at the design stress under Type-I censoring.

Type-II censoring (or “failure censoring”) is commonly used in ALT. Under Type-II censoring, the test is terminated once *r* predetermined failures have been observed out of the *n* units under test. The total test time $t_{(r)}$ is random. This has limited the use of Type-II censoring in practice.

Early discussions of asymptotic properties for Type-II censoring are given in Halperin (1952). For the exponential distribution, Yum and Kim (1990) investigate reliability sampling plans based on the accelerated life testing. Type-II censoring is assumed at each overstressed level. Later, Hsieh (1994) extends their work such that the total number of failures is minimized. Solan (1968), Schneider (1989) and Kwon (1996) develop reliability sampling plans at the use condition assuming that the failure times follow Weibull distribution while Bai *et al*. (1993) develop reliability sampling plans based on accelerated test conditions. Menzefricke (1992) discusses sample size planning for ALTs when Type-II censoring is applied at each constant-stress level. More recently, Watkins and John (2008) consider ALT with Type-II censoring regime applied at one of the constant-stress levels.

However, in many situations, surviving test units may be removed before the termination time of the test in order to save some units for other tests or when the number of test units is limited and cost per unit is high. This is also desirable when a compromise between reduced time of experimentation and the observation of at least some extreme lifetimes is sought (Balakrishnan and Aggarwala, 2000). The traditional Type-I and Type-II censoring schemes do not provide such features. This leads to the investigation of progressive censoring which is a generalization of Type-I or Type-II censoring as explained below.

Suppose that *n* units are placed on a test simultaneously. At the first prespecified time point τ_1, r_1 surviving units are randomly removed from the test. Then at the second prespecified time point τ_2, r_2 surviving units are randomly removed from the test. This process continues until the prespecified time point τ_s is reached or when all the units fail. The r_i's are fixed with the provision that there are r_i surviving units at time $\tau_i, i = 1, 2, \ldots, s-1$. Sometimes r_i's are random, which is referred to as random progressive censoring. When $r_1 = r_2 = \ldots = r_{s-1} = 0$, the progressive censoring becomes the conventional Type-I censoring.

Likewise, the conventional Type-II censoring can be shown to a special case of progressive censoring. Under this censoring scheme, *n* units are placed on test at time zero. Immediately following the first failure, r_1 surviving units are removed from the test at random. Then immediately following the second failure, r_2 surviving units are removed from the test at random. The test is terminated and all of the remaining units are removed after the m^{th} failure is observed.

Statistical inference under progressive censoring is initiated by Cohen (1963). Then Srivastava (1967) develops the Fisher information matrix of the maximum likelihood estimate with changing failure rate under exponential distribution. Mann (1971) and

Thomas and Wilson (1972) discuss linear estimation of parameters of progressive censored data assuming a Weibull failure time distribution.

Current work on progressive censoring is focused on three areas: reliability sampling plan, ALT under step-stress and empirical methods. Detailed reviews are provided as follows.

Reliability sampling plan

Balasooriya *et al*. (2000) and Balasooriya and Balakrishnan (2000) study reliability sampling plans that to determine the acceptability of a product with respect to its lifetime under Weibull and lognormal distributions, respectively. They consider the progressive Type-II censoring with a predetermined number of removals $r_1, r_2, ..., r_s$ at each stage.

Ng *et al*. (2004) compute the expected Fisher information matrix based on progressively Type-II censored samples from a Weibull distribution and use EM algorithm to perform the calculation for the missing information. Given the number of units available for test and the number of failures allowed, they determine the optimal progressive Type-II censoring plan ($r_1, r_2, ..., r_s$) that results in optimal estimation of model parameters. Three optimality criteria are considered: minimizing the trace of the variance-covariance matrix of the MLEs, minimizing the determinant of the variance-covariance matrix of the MLEs, and maximizing the trace of the Fisher information matrix. The results are used to determine the minimum sample size *n* for reliability acceptance test.

In addition, Balasooriya and Low (2004) investigate reliability sampling plans for Weibull lifetimes under competing risk. They assume each cause of failure has a different set of parameters. The joint estimation of the parameters reduces to the estimation of parameters of a single lifetime distribution which simplifies the estimation of the parameters. They also construct the likelihood function and derive the expected Fisher information matrix.

The acceleration conditions of the test plan such as stress application, unit allocation and censoring time are not investigated in any of the work on reliability sampling plan.

ALT under step-stress

Conducting simple step-stress ALT in combination with progressive censoring has been investigated by several investigators. At each level of the step-stress, it is assumed that the failure time is exponentially distributed. The assumption of a constant hazard rate under exponential distribution is very restrictive, so the model's applicability is fairly limited (Lawless, 1982). In addition, investigators assume inspection is conducted intermittently $\left(\tau, 2\tau, ..., i\tau, ..., s\tau\right)$, i.e. only records of whether a test unit fails in an interval instead of the exact failure time are obtained. The grouped observation may dramatically impact the accuracy and precision of parameter estimate especially when the sample size is small. Gouno *et al*. (2004) and Wu *et al*. (2006) obtain optimal ALT plans based on predetermined $r_1, r_2, ..., r_s$. They determine the optimal inspection interval τ by minimizing the asymptotic variance of the MLE of log mean lifetime and D-optimality. Wu *et al*. (2008) treat the number of removals at each stage as a uniformly distributed random variable. They determine the optimal inspection interval τ by minimizing the asymptotic variance of the MLE of log mean lifetime and A-, D-, and E-optimality.

Empirical method

Researchers have utilized empirical cumulative distribution functions for reliability analysis and modeling. Obtaining such an empirical function under progressive censoring is not straight forward. Patel and Tsao (2009) derive closed-form expressions for the nonparametric estimate of failure probabilities under progressive Type-I censoring. They also develop a simple algorithm that not only produces these estimates but also provides a clear and intuitive justification for the estimates. However, nonparametric methods have limitations for time and stress level extrapolations.

3.1.2 Multiple Failure Modes

Oftentimes a unit can experience multiple failure modes and the failure time $T = \min_{i \in h} T_i$ is the minimum of the *h* latent failure times corresponding to *h* failure modes. Such scenario in statistical and reliability literature is referred to as "competing risk". In reality, there are many components and products that experience competing risk failures. For example, Nelson (1990) states that a Class-H insulation can fail due to turn, phase, and ground failures; assemblies of ball bearings can fail due to failure of the race or the ball. Cylinder liners present wear and thermal cracking failure modes (Bocchetti, *et al.*, 2009). Therefore, statistical inference and reliability analysis considering competing risk have been extensively studied.

The early work for ML analyses of competing risk data are given by Cox (1959), Moeschberger and David (1971), Herman and Tatell (1971), and Nelson (1982). David and Moeschberger (1971) present details of the competing risk theory and parametric estimate methods including the construction of general likelihood function and likelihood functions under specific distributions and the development of Fish information matrix. The recent work by Crowder (2001) provides a comprehensive review of the theory and methods of competing risk. Other work relevant to ALT plan and ALT data analysis when considering competing risk is described below.

Klein and Basu (1981, 1982) present the analysis of ALT data when more than one failure mode is present. The latent failure times are assumed to be independent and described as a series-system with Weibull component failure times having a common or different shape parameters. The authors obtain MLE estimates of the model parameters with data from Type-I, -II and progressive censoring. Similarly, Ishioka and Nonaka (1991) discuss the analysis of ALT data when test units are subject to two failure modes. More recently, Bunea and Mazzuchi (2006) present a Bayesian framework for the analysis of ALT data with possible multiple failure modes.

Bai and Chun (1991) present optimum ALT plans with single, simple step-stress for products subject to competing risk. The lifetime distribution of each failure mode is assumed to be independent exponential distribution with a mean expressed as a log-linear function of the stress. The optimum plans for time-step and failure-step ALTs are obtained in order to minimize the sum over all failure causes of asymptotic variances of the MLE of the log mean lives at design stress.

Bai and Bai (2002) model failure time in ALTs as a mixture of two Weibull distributions with the log of the scale parameter written as a linear function of stress. An EM algorithm for MLE is presented.

Zhao and Elsayed (2004) consider ALT under Type-II censoring with two competing failure modes: degradation and hard failure. The degradation process is described by a Brownian motion and the hard failure is modeled by a Weibull distribution. They construct the likelihood function for parameters estimate and conduct experiments to validate these estimates.

Pascual (2007) present a method for ALT planning when multiple failure modes are dependent on only one accelerated factor under Type-I censoring. The latent failure times are assumed to be independent Weibull distributions with known but common shape parameter. The optimal plans are achieved by minimizing the asymptotic variance of the MLE of failure quantile and hazard function at given conditions, and maximizing the determinant of the Fisher information matrix of MLE. They also perform sensitivity analysis of optimal plans to misspecification of the shape parameter.

In summary, the competing risk theory and methods have been investigated by many authors under traditional Type-I, -II censoring schemes. However, the application of multiple stresses under progressive censoring has not been studied.

3.2 Assumptions and Censoring Procedure

3.2.1 Assumptions

Suppose that each test unit experiences *h* statistically independent potential failure modes. A unit fails when any of the *h* failure modes occurs.

Let the random variable T_i be the lifetime of a unit when $C_i \left(i = 1, 2, \ldots h \right)$ were the only risk present. We assume that T_i follows an independent Weibull distribution with a common shape parameter λ. The *pdf* of T_i is:

$$f_i \left(t_i ; \lambda, \alpha_i \right) = \lambda \alpha_i^{-\lambda} t_i^{\lambda - 1} \exp\left[-\left(t_i / \alpha_i \right)^{\lambda} \right], \quad t_i > 0, \quad i = 1, 2, \ldots h \tag{6.1}$$

Suppose that the scale parameter α_i of the i^{th} failure mode is a log-linear function of stresses:

$$\ln\left(\alpha_i \right) = \mu_i \left(\boldsymbol{\beta} \mathbf{x} \right) = \beta_{i0} + \beta_{i1} x_1 + \ldots + \beta_{ik} x_k , \quad i = 1, \ldots h, \; k \geq 1 \tag{6.2}$$

where $\boldsymbol{\beta}_{i,\varpi}$ for $i = 1,\ldots h, \varpi = 1,\ldots k$ are unknown parameters associated with risk i and stress ϖ. This is a special case of the PH model. In the following derivations, μ_i is used to represent $\mu_i(\boldsymbol{\beta}\mathbf{x})$ for simplicity.

Let $\sigma = \lambda^{-1}$, then the log failure time $Y_i = \ln(T_i)$ is the smallest extreme-value distribution with *pdf*:

$$f_i(y_i;\mu_i,\sigma) = \frac{1}{\sigma}\exp\left[\left(\frac{y_i-\mu_i}{\sigma}\right) - \exp\left(\frac{y_i-\mu_i}{\sigma}\right)\right] \tag{6.3}$$

and cumulative density function (*cdf*):

$$F_i(y_i;\mu_i,\sigma) = 1 - \exp\left[-\exp\left(\frac{y_i-\mu_i}{\sigma}\right)\right] \tag{6.4}$$

where μ_i is the location parameter of the extreme-value distribution corresponding to failure mode i, and σ is the common scale parameter of the h extreme-value distributions.

Let $Z_i = \left(\frac{Y_i-\mu_i}{\sigma}\right)$, the *pdf* and *cdf* of Y_i can be rewritten in terms of Z_i as :

$$f_i(z_i;\mu_i,\sigma) = \exp\left[z_i - \exp(z_i)\right], -\infty < z_i < \infty \tag{6.5}$$

$$F_i(z_i;\mu_i,\sigma) = 1 - \exp\left[-\exp(z_i)\right] \tag{6.6}$$

Consider the situation when failure times and failure causes of test units are observed continuously. The failure time T of a test unit is the smallest of its h potential failure times: $T = \min\{T_1, \dots T_h\}$, i.e. $Y = \min_i Y_i$ and $Z = \min_i Z_i$. Therefore, the log failure time has cumulative *cdf* in terms of Y as:

$$F(y; \mu_1, \dots \mu_h, \sigma) = 1 - \prod_{i=1}^{h} \left[1 - F_i(y_i; \mu_i, \sigma)\right] = 1 - \exp\left[-\sum_{i=1}^{h} \exp\left(\frac{y_i - \mu_i}{\sigma}\right)\right] \tag{6.7}$$

and in terms of Z as:

$$F(z; \mu_1, \dots \mu_h, \sigma) = 1 - \prod_{i=1}^{h} \left[1 - F_i(z_i; \mu_i, \sigma)\right] = 1 - \exp\left[-\sum_{i=1}^{h} \exp(z_i)\right] \tag{6.8}$$

3.2.2 Stress Normalization

Without loss of generality, we first normalize the stress levels. Let S denote the single accelerated stress. Let S_D and S_H denote the normal operating stress (design stress) level and highest testing stress level that can be used in the ALT experiment, respectively. Then stress S is normalized as

$$x = \frac{S - S_D}{S_H - S_D} \tag{6.9}$$

As a result, the normal operating stress S_D and highest testing stress S_H become $x_D = 0$ and $x_H = 1$, respectively; and $0 \le x \le 1$. If multiple stresses are applied in an ALT, the *n* levels of each stress are calculated by Eq. (2.11) first and then linearized to form the *n*-term arithmetic sequence $\{1,...,n\}$.

3.2.3 Censoring Procedure

In this dissertation we study a progressive censoring procedure as shown in Figure 3.1. The procedure begins by placing *N* units under test at time zero and fractions of survival units $r_u\left(u=1,\ldots s-1\right)$ are removed at predetermined times $\tau_u\left(u=1,\ldots s-1\right)$ correspondingly. The test is terminated at a given time τ_s and the remaining surviving units are removed. Suppose I_{u-1} is the number of units on test at $\tau_{u-1}\left(u=1,\ldots s\right)$, N_u is the number of failures during (τ_{u-1}, τ_u) and R_u is the number of removed units at time τ_u. N_u, R_u, and I_u are random variables with values pending the outcome of the test (Note $R_u = r_u\left(I_u - N_u\right)$).

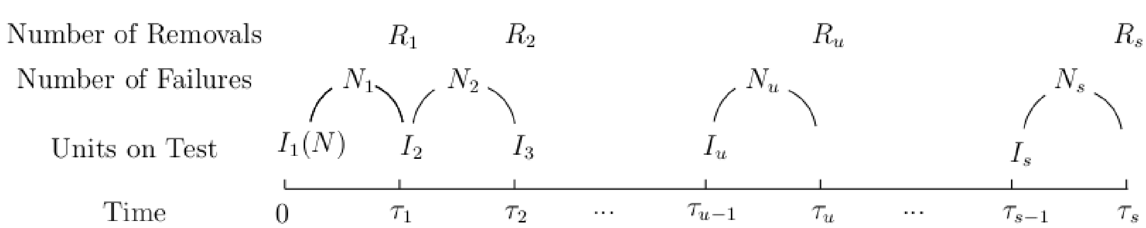


Figure 3.1 Progressive censoring

From the property of conditional expectation and by induction we have

$$E\left[I_u\right] = N\bar{F}\left(\eta_{u-1}\right)\prod_{\gamma=1}^{u-1}\left(1-r_\gamma\right) \tag{6.10}$$

$$E\left[N_u\right] = N\left[F\left(\eta_u\right) - F\left(\eta_{u-1}\right)\right]\prod_{\gamma=1}^{u-1}\left(1-r_\gamma\right) \tag{6.11}$$

$$E\left[R_u\right] = N\bar{F}\left(\eta_u\right)r_u\prod_{\gamma=1}^{u-1}\left(1-r_\gamma\right) \tag{6.12}$$

where $\eta_u = \ln\left(\tau_u\right)$.

3.3 Maximum Likelihood Estimate

Based on the assumptions and censoring procedure discussed in Sec. 3.2, we now develop the expected Fisher information matrix for the maximum likelihood estimate.

Suppose that a total of m_i units fail due to the i^{th} failure mode until time τ_s in a given experiment and we observe the following failure times

$$y_{i,j},\ \ i = 1, 2, \ldots, h, \text{ and } j = 1, 2, \ldots, m_i,$$

where $y_{i,j}$ denotes the j^{th} failure time due to the i^{th} failure mode. We construct the likelihood function under Type-I progressive censoring with competing failure modes in an analogous manner as that in Moeschberger and David (1971). Note that the last stage: *s*; before the termination of the test there are I_s, N_s units and there are R_s surviving units at the termination time:

$$L \propto \left[\prod_{i=1}^{h} \prod_{j=1}^{m_i} f_i\left(y_{i,j}\right) \prod_{\substack{l=1 \\ l \neq i}}^{h} \bar{F}_l\left(y_{i,j}\right) \right] \times \prod_{i=1}^{h} \prod_{u=1}^{s-1} \bar{F}_i\left(y_u\right)^{R_u} \times \prod_{i=1}^{h} \bar{F}_i\left(y_s\right)^{\left(I_s - N_s\right)} \tag{6.13}$$

The log likelihood function is,

$$\begin{aligned} \tilde{L} \propto & \sum_{i=1}^{h} \sum_{j=1}^{m_i} \left[-\ln\left(\sigma\right) + \left(\frac{y_{ij} - \mu_i}{\sigma} \right) - \sum_{w=1}^{h} \exp\left(\frac{y_{ij} - \mu_w}{\sigma} \right) \right] \\ & - \sum_{i=1}^{h} \sum_{u=1}^{s-1} R_u \exp\left(\frac{\eta_u - \mu_i}{\sigma} \right) - \sum_{i=1}^{h} \left(I_s - N_s\right) \exp\left(\frac{\eta_s - \mu_i}{\sigma} \right) \end{aligned} \tag{6.14}$$

The elements of the expected Fisher information matrix of MLE are obtained by taking the second order derivative of Eq. (3.14) with respect to the parameter of risk υ and the common scale parameter σ:

$$\frac{\partial^2 \tilde{L}}{\partial \mu_\upsilon^2} = -\frac{1}{\sigma} \frac{\partial \tilde{L}}{\partial \mu_\upsilon} - \frac{1}{\sigma^2} m_\upsilon \tag{6.15}$$

$$\begin{aligned} \frac{\partial^2 \tilde{L}}{\partial \mu_\upsilon \partial \sigma} = & -\frac{1}{\sigma} \frac{\partial \tilde{L}}{\partial \mu_\upsilon} - \frac{1}{\sigma^2} \sum_{i=1}^{h} \sum_{j=1}^{m_i} \frac{y_{ij} - \mu_\upsilon}{\sigma} \exp\left(\frac{y_{ij} - \mu_\upsilon}{\sigma} \right) \\ & - \frac{1}{\sigma^2} \sum_{u=1}^{s-1} R_u \frac{\eta_u - \mu_\upsilon}{\sigma} \exp\left(\frac{\eta_u - \mu_\upsilon}{\sigma} \right) - \frac{1}{\sigma^2} \left(I_s - N_s\right) \frac{\eta_s - \mu_\upsilon}{\sigma} \exp\left(\frac{\eta_s - \mu_\upsilon}{\sigma} \right) \end{aligned} \tag{6.16}$$

$$\frac{\partial^2 \tilde{L}}{\partial \sigma^2} = -\frac{2}{\sigma}\frac{\partial \tilde{L}}{\partial \sigma} - \frac{1}{\sigma^2}\sum_{i=1}^{h} m_i - \frac{1}{\sigma^2}\sum_{i=1}^{h}\sum_{j=1}^{m_i}\sum_{w=1}^{h}\left(\frac{y_{ij}-\mu_w}{\sigma}\right)^2 \exp\left(\frac{y_{ij}-\mu_w}{\sigma}\right)$$
$$-\frac{1}{\sigma^2}\sum_{i=1}^{h}\sum_{u=1}^{s-1} R_u \exp\left(\frac{\eta_u-\mu_i}{\sigma}\right)\cdot\left(\frac{\eta_u-\mu_i}{\sigma}\right)^2 - \frac{1}{\sigma^2}\sum_{i=1}^{h}\left(I_s-N_s\right)\left(\frac{\eta_s-\mu_i}{\sigma}\right)^2 \exp\left(\frac{\eta_s-\mu_i}{\sigma}\right) \tag{6.17}$$

Since the expectations of the first order derivatives in Eqs. (3.15)-(3.17) equal zeros at MLE, then the expectations of Eqs. (3.15)-(3.17) are

$$E\left[-\frac{\partial^2 \tilde{L}}{\partial \mu_\upsilon^2}\right] = \frac{1}{\sigma^2} E\left[m_\upsilon\right] \tag{6.18}$$

$$E\left[-\frac{\partial^2 \tilde{L}}{\partial \mu_\upsilon \partial \sigma}\right] = \frac{1}{\sigma^2}\sum_{i=1}^{h} E\left[\sum_{j=1}^{m_i}\frac{y_{ij}-\mu_\upsilon}{\sigma}\exp\left(\frac{y_{ij}-\mu_\upsilon}{\sigma}\right)\right]$$
$$+\frac{1}{\sigma^2}\sum_{u=1}^{s-1} E\left[R_u\right]\frac{\eta_u-\mu_\upsilon}{\sigma}\exp\left(\frac{\eta_u-\mu_\upsilon}{\sigma}\right) + \frac{1}{\sigma^2}E\left[I_s-N_s\right]\frac{\eta_s-\mu_\upsilon}{\sigma}\exp\left(\frac{\eta_s-\mu_\upsilon}{\sigma}\right) \tag{6.19}$$

$$E\left[-\frac{\partial^2 \tilde{L}}{\partial \sigma^2}\right] = \frac{1}{\sigma^2}\sum_{i=1}^{h} E\left[m_i\right] + \frac{1}{\sigma^2}\sum_{i=1}^{h} E\left[\sum_{j=1}^{m_i}\sum_{w=1}^{h}\left(\frac{y_{ij}-\mu_w}{\sigma}\right)^2 \exp\left(\frac{y_{ij}-\mu_w}{\sigma}\right)\right]$$
$$+\frac{1}{\sigma^2}\sum_{i=1}^{h}\sum_{u=1}^{s-1} E\left[R_u\right]\left(\frac{\eta_s-\mu_i}{\sigma}\right)^2 \cdot \exp\left(\frac{\eta_s-\mu_i}{\sigma}\right) \tag{6.20}$$
$$+\frac{1}{\sigma^2}\sum_{i=1}^{h} E\left[I_s-N_s\right]\left(\frac{\eta_s-\mu_i}{\sigma}\right)^2 \exp\left(\frac{\eta_s-\mu_i}{\sigma}\right)$$

Evaluation of Eqs. (3.18)-(3.20) requires the expectation of m_i, for $i = 1, \ldots, h$. Since the total number of failures with failure mode i equals the sum of the failures due to mode i over all the intervals, then

$$E[m_i] = \sum_{u=1}^{s} E[m_{iu}] = \sum_{u=1}^{s} E[I_u] \frac{\int_{-\infty}^{\eta_u} f_i(y_{ij};\mu_i,\sigma) \prod_{\substack{l=1 \\ l\neq i}}^{h} \bar{F}_l(y_{ij};\mu_l,\sigma) dy_{ij}}{\bar{F}(\tau_{u-1})}$$

by Eq. (3.10)

$$E[m_i] = \sum_{u=1}^{s} N \prod_{\gamma=1}^{u-1} (1-r_\gamma) \int_{-\infty}^{\eta_u} f_i(y_{ij};\mu_i,\sigma) \prod_{\substack{l=1 \\ l\neq i}}^{h} \bar{F}_l(y_{ij};\mu_l,\sigma) dy_{ij}$$

$$= \sum_{u=1}^{s} N \prod_{\gamma=1}^{u-1} (1-r_\gamma) \frac{1 - \dfrac{\exp\left[-(1+a)\exp\left(\dfrac{\eta_u-\mu_i}{\sigma}\right)\right]}{1+a}}{1+a} \tag{6.21}$$

where $a = \sum_{l=1}^{h} \exp\left(\dfrac{\mu_i-\mu_l}{\sigma}\right)$.

Evaluation of Eq. (3.19) requires the determination of $E\left[\sum_{j=1}^{m_i} \dfrac{y_{ij}-\mu_\upsilon}{\sigma} \exp\left(\dfrac{y_{ij}-\mu_\upsilon}{\sigma}\right)\right]$, $E[R_u]$ and $E[I_s - N_s]$. The last two can be obtained from Eqs. (3.10)-(3.12). Now we calculate the first expectation:

$$E\left[\sum_{j=1}^{m_i}\frac{y_{ij}-\mu_\upsilon}{\sigma}\exp\left(\frac{y_{ij}-\mu_\upsilon}{\sigma}\right)\right]=\sum_{u=1}^{s}E\left[\sum_{j=1}^{m_{iu}}\frac{y_{ij}-\mu_\upsilon}{\sigma}\exp\left(\frac{y_{ij}-\mu_\upsilon}{\sigma}\right)\right]$$

$$=\sum_{u=1}^{s}E\left[I_u\right]\frac{\int_{\eta_{u-1}}^{\eta_u}\left(\frac{y_{ij}-\mu_\upsilon}{\sigma}\right)\exp\left(\frac{y_{ij}-\mu_\upsilon}{\sigma}\right)\cdot f_i\left(y_{ij};\mu_i,\sigma\right)\prod_{\substack{l=1\\l\neq i}}^{h}\bar{F}_l\left(y_{ij};\mu_l,\sigma\right)dy_{ij}}{\bar{F}_z\left(\tau_{u-1}\right)} \tag{6.22}$$

$$=\sum_{u=1}^{s}N\prod_{r=1}^{u-1}\left(1-r_\gamma\right)\int_{\eta_{u-1}}^{\eta_u}\left(\frac{y_{ij}-\mu_\upsilon}{\sigma}\right)\exp\left(\frac{y_{ij}-\mu_\upsilon}{\sigma}\right)\cdot f_i\left(y_{ij};\mu_i,\sigma\right)\prod_{\substack{l=1\\l\neq i}}^{h}\bar{F}_l\left(y_{ij};\mu_l,\sigma\right)dy_{ij}$$

Similarly, to evaluate Eq. (3.20) we obtain $E\left[\sum_{j=1}^{m_i}\left(\frac{y_{ij}-\mu_\upsilon}{\sigma}\right)^2\exp\left(\frac{y_{ij}-\mu_\upsilon}{\sigma}\right)\right]$ which is calculated as follows

$$E\left[\sum_{j=1}^{m_i}\sum_{\upsilon=1}^{h}\left(\frac{y_{ij}-\mu_\upsilon}{\sigma}\right)^2\exp\left(\frac{y_{ij}-\mu_\upsilon}{\sigma}\right)\right]=\sum_{\upsilon=1}^{h}E\left[\sum_{j=1}^{m_i}\left(\frac{y_{ij}-\mu_\upsilon}{\sigma}\right)^2\exp\left(\frac{y_{ij}-\mu_\upsilon}{\sigma}\right)\right]$$

$$=\sum_{\upsilon=1}^{h}\sum_{u=1}^{s}N\prod_{r=1}^{u-1}\left(1-r_\gamma\right)\int_{\eta_{u-1}}^{\eta_u}\left(\frac{y_{ij}-\mu_\upsilon}{\sigma}\right)^2\exp\left(\frac{y_{ij}-\mu_\upsilon}{\sigma}\right)\cdot f_i\left(y_{ij};\mu_i,\sigma\right)\prod_{\substack{l=1\\l\neq i}}^{h}\bar{F}_l\left(y_{ij};\mu_l,\sigma\right)dy_{ij} \tag{6.23}$$

Consider a test plan Ξ: $\hat{N}$ units are available and in experiment ς, for $\varsigma=1,\ldots n$, p_ς proportion of test units with $p_1+\ldots+p_n=1$ are tested under $k(k\geq 1)$ stresses simultaneously,

$$\mathbf{x}_\varsigma^T=[x_{\varsigma,1},x_{\varsigma,2},\ldots.x_{\varsigma,k}],\ \varsigma=1,\ldots,n$$

where $\mathbf{x}_{\varsigma}^{T}$ represents one of the n stress-level combinations for a test with multiple stresses ($k \geq 1$) or one of the n stress levels for a test with single stress ($k = 1$). Also, $\mathbf{r}_{\varsigma} = \left[r_{\varsigma,1}, \ldots, r_{\varsigma,u}, \ldots, r_{\varsigma,s-1} \right]$ are the fractions of survival units removed at times $\boldsymbol{\tau}_{\varsigma} = \left[\tau_{\varsigma,1}, \ldots, \tau_{\varsigma,u}, \ldots, \tau_{\varsigma,s} \right]$ in experiment ς (under stress $\mathbf{x}_{\varsigma}^{T}$) for $\varsigma = 1, \ldots n$.

Replacing N, $\mathbf{x}^{T}$, $\mathbf{r} = \left[r_1, \ldots, r_u, \ldots, r_{s-1} \right]$ and $\boldsymbol{\tau} = \left[\tau_1, \ldots, \tau_u, \ldots, \tau_s \right]$ in the general expressions Eqs. (3.18)-(3.20) by $\mathbf{x}_{\varsigma}^{T}$, $\hat{N} p_{\varsigma}$, $\mathbf{r}_{\varsigma}$ and $\boldsymbol{\tau}_{\varsigma}$ for $\varsigma = 1, \ldots n$, we obtain Eqs. (3.18)-(3.20) under all the n experiments of test plan Ξ.

Let $\boldsymbol{\theta}$ represent the unknown parameters $\left(\beta_{10}, \beta_{11}, \ldots, \beta_{1k,} \ldots, \beta_{h0}, \beta_{h1}, \ldots, \beta_{hk,} \sigma \right)^{T}$. Since

$$\mu_i \left(\mathbf{x}_{\varsigma} \right) = \beta_{i0} + \beta_{i1} x_{\varsigma 1} + \ldots + \beta_{ik} x_{\varsigma k}$$

using the chain rule, we have

$$I_{\varsigma,i}^{\beta} \left(\boldsymbol{\theta}; \varsigma \right) = E \left[- \frac{\partial^2 \tilde{L} \left(\boldsymbol{\theta}; \varsigma \right)}{\partial \mu_i^2 \left(\mathbf{x}_{\varsigma} \right)} \right] \begin{bmatrix} 1 & x_{\varsigma,1} & \cdots & x_{\varsigma,k} \\ & x_{\varsigma,1}^2 & & x_{\varsigma,1} x_{\varsigma,k} \\ & & \ddots & \vdots \\ \text{Symmetric} & & & x_{\varsigma,k}^2 \end{bmatrix}$$

where $I_{\varsigma,i}^{\beta}$ represents the term of the Fisher information matrix associated with experiment ς with respect to risk *i*. Since the *h* failure modes are *s*-independent, the interaction terms with respect to risk *i* and $i^{'}$ ($i^{'} \neq i$) are zero.

Similarly, we have

$$I_{\varsigma,l}^{\beta\sigma}\left(\boldsymbol{\theta};\varsigma\right)=E\left[-\frac{\partial^{2}\tilde{L}\left(\boldsymbol{\theta};\varsigma\right)}{\partial\mu_{i}\left(\mathbf{x}_{\varsigma}\right)\partial\sigma}\right]\begin{bmatrix}1 & x_{\varsigma,1} & \cdots & x_{\varsigma,k}\end{bmatrix}^{T}$$

where $I_{\varsigma,l}^{\beta\sigma}$ represents the interaction term of the Fisher information matrix associated with experiment ς with respect to risk *i* and common parameter *σ*.

Let $I_{\varsigma}^{\sigma\sigma}\left(\boldsymbol{\theta};\varsigma\right)=E\left[-\frac{\partial^{2}\tilde{L}\left(\boldsymbol{\theta};\varsigma\right)}{\partial\sigma^{2}}\right]$. Now we have the Fisher information matrix associated with experiment ς of test plan Ξ:

$$I_{\varsigma}\left(\boldsymbol{\theta};\varsigma\right)=\begin{bmatrix}I_{\varsigma,1}^{\beta}\left(\boldsymbol{\theta};\varsigma\right) & 0 & 0 & I_{\varsigma,1}^{\beta\sigma}\left(\boldsymbol{\theta};\varsigma\right)\\ & \ddots & 0 & \vdots\\ & & I_{\varsigma,h}^{\beta}\left(\boldsymbol{\theta};\varsigma\right) & I_{\varsigma,h}^{\beta\sigma}\left(\boldsymbol{\theta};\varsigma\right)\\ \text{Symmetric} & & & I_{\varsigma}^{\sigma\sigma}\left(\boldsymbol{\theta};\varsigma\right)\end{bmatrix}$$

As the test units are *s*-independent, the total Fisher information matrix of test plan Ξ becomes

$$I(\boldsymbol{\theta};\Xi) = \hat{N}\sum_{\varsigma=1}^{n} p_{\varsigma} I_{\varsigma}(\boldsymbol{\theta};\varsigma) \tag{6.24}$$

If $G(\boldsymbol{\theta})$ is a function of $\boldsymbol{\theta}$, then $G(\widehat{\boldsymbol{\theta}})$ is the ML estimator of $G(\boldsymbol{\theta})$ with asymptotic variance

$$\text{Asvar}\left[G(\widehat{\boldsymbol{\theta}});\Xi\right] = \left[\frac{\partial G(\widehat{\boldsymbol{\theta}})}{\partial\boldsymbol{\theta}}\right]^{T}\left[I(\widehat{\boldsymbol{\theta}};\Xi)\right]^{-1}\left[\frac{\partial G(\widehat{\boldsymbol{\theta}})}{\partial\boldsymbol{\theta}}\right] \tag{6.25}$$

under test plan Ξ.

3.4 Optimal ALT Plan Criteria and Formulation

From the above Fisher information matrix we can now determine the optimal plans subject to different optimality criteria. In this section, we propose and develop a new test plan criterion, minimization of the asymptotic variance of mean time of first failure. We determine optimum test plans under the following criteria:

1. Variance optimality:
    a. Minimization of the asymptotic variance of the mean time of first failure in a group of units.
    b. Minimization of asymptotic variance of the quantile failure at normal operating conditions.
2. D-optimality that maximizes the determinant of the Fisher information matrix.
3. Multi-objective optimality

The motivation for implementing each criterion is discussed as well.

3.4.1 Variance Optimality

Early failures may significantly increase the warranty cost. Therefore accurate estimation of lower quantile failure at normal operating condition using accelerated life testing is important. However, the problem of early failures is exacerbated under certain situations such as units that are used in aerospace applications or devices implanted into humans because of safety and potential risk. The time of the first failure in a group of $\tilde{N}$ units represents the extreme case of lower life quantile. Therefore, we propose to determine the optimal test plan with respect to the minimum asymptotic variance of MLE of the mean time of the first failure in a group of $\tilde{N}$ units at normal operating conditions. The second criterion under variance optimality we investigate is the minimum asymptotic variance of the lower quantile failure at normal operating conditions.

3.4.1.1 Mean Time of First Failure

To determine the optimal test plan based on the criterion of minimum asymptotic variance of MLE of the mean time of first failure in a group of $\tilde{N}$ units at normal operating conditions, we derive the analytic expression of the time of first failure for $\tilde{N}$ units as follows.

As assumed in Sec. 3.2, the failure time of a single unit follows a Weibull competing risk model. The *cdf*, Eq. (3.7), at normal operating condition is given by

$$F\left(t \mid \mathbf{x}_D\right) = 1 - \exp\left\{-t^{1/\sigma} \sum_{i=1}^{h} \exp\left[-\frac{\boldsymbol{\beta}_i \mathbf{x}_D}{\sigma}\right]\right\} \tag{6.26}$$

From Eq. (3.26), we obtain the probability that a single unit fails in the time interval [*t*, *t*+*dt*]:

$$f\left(t \mid \mathbf{x}_D\right) = \frac{dF\left(t \mid \mathbf{x}_D\right)}{dt} = \frac{1}{\sigma} t^{\frac{1-\sigma}{\sigma}} \exp\left\{-t^{1/\sigma} \sum_{i=1}^{h} \exp\left[-\frac{\boldsymbol{\beta}_i \mathbf{x}_D}{\sigma}\right]\right\}\left[\sum_{i=1}^{h} \exp\left(-\frac{\boldsymbol{\beta}_i \mathbf{x}_D}{\sigma}\right)\right]$$

where $\mathbf{x}_D^T = [1, x_{D,1}, \ldots, x_{D,\varpi}, \ldots., x_{D,k}]$ represents the normal operating conditions. Now we introduce the probability $\frac{d\tilde{F}\left(t \mid \mathbf{x}_D\right)}{dt}$ that the first failure in a group of $\tilde{N}$ units occurs in [*t*, *t*+*dt*] at normal operating conditions:

$$\begin{aligned}\frac{d\tilde{F}\left(t \mid \mathbf{x}_D\right)}{dt} &= \tilde{N} f\left(t \mid \mathbf{x}_D\right)\left(\int_t^{\infty} f\left(t' \mid \mathbf{x}_D\right) dt'\right)^{\tilde{N}-1} \\ &= \frac{\tilde{N}}{\sigma} t^{\frac{1-\sigma}{\sigma}} \left[\sum_{i=1}^{h} \exp\left(-\frac{\boldsymbol{\beta}_i \mathbf{x}_D}{\sigma}\right)\right] \exp\left[-\tilde{N} t^{1/\sigma} \sum_{i=1}^{h} \exp\left(-\frac{\boldsymbol{\beta}_i \mathbf{x}_D}{\sigma}\right)\right]\end{aligned} \tag{6.27}$$

The term $\left(\int_t^{\infty} f\left(t' \mid \mathbf{x}_D\right) dt'\right)^{\tilde{N}-1}$ in Eq. (3.27) is the probability that $\tilde{N}-1$ units fail in $[t, \infty]$. $\tilde{N}$ is a combinatorial factor given the number of choices to choose the unit which fails in [*t*, *t*+*dt*].

To verify that Eq. (3.27) is the true failure time *pdf* of the first failure in a group of $\tilde{N}$ units, we perform simulation as follows. We choose $\tilde{N} = 20$, $\sigma = 0.5$, $h = 2$, and for simplicity $\boldsymbol{\beta}_i = \mathbf{0}$ which is equivalent to Weibull distribution with scale parameter $\alpha = 1$. We plot Eq. (3.27) from time zero to one time unit, shown as the dashed line in Figure 3.2.

To generate the first failure time for $\tilde{N} = 20$, we solve failure time *t* from Eq. (3.26),

$$t = \left[-0.5 \ln\left(1 - F\left(\cdot\right)\right) \right]^{\sigma}$$

where $F\left(\cdot\right)$ is the cumulative distribution function of the failure time.

$F\left(\cdot\right)$ can be simulated by generating random numbers from a unit uniform distribution. Since the group size is $\tilde{N} = 20$, we generate 20 $F\left(\cdot\right)$ values and solve the corresponding *t*. We choose the minimum *t* as the first failure time of the 20 units. This process is repeated 10^5 times to create the distribution. The distribution is normalized to the total number of generated failure times and is divided by the histogram bin width. The obtained histogram shows the normalized probability density function of the first failure time in a group of $\tilde{N} = 20$ units. As shown in Figure 3.2, the dashed line matches the histogram well. Hence, Eq. (3.27) correctly describes the failure time *pdf* of the first failure in a group of $\tilde{N}$ units.

Using Eq. (3.27), we calculate the mean time of first failure (t_1) as

$$E\left[t_1\right]=\int_0^{\infty} t\cdot\left[\frac{d\tilde{F}\left(t\,|\,\mathbf{x}_D\right)}{dt}\right]\cdot dt=\tilde{N}^{-\sigma}\Gamma(1+\sigma)\left[\sum_{i=1}^{h}\exp\left(-\frac{\beta_i\mathbf{x}_D}{\sigma}\right)\right]^{-\sigma} \tag{6.28}$$

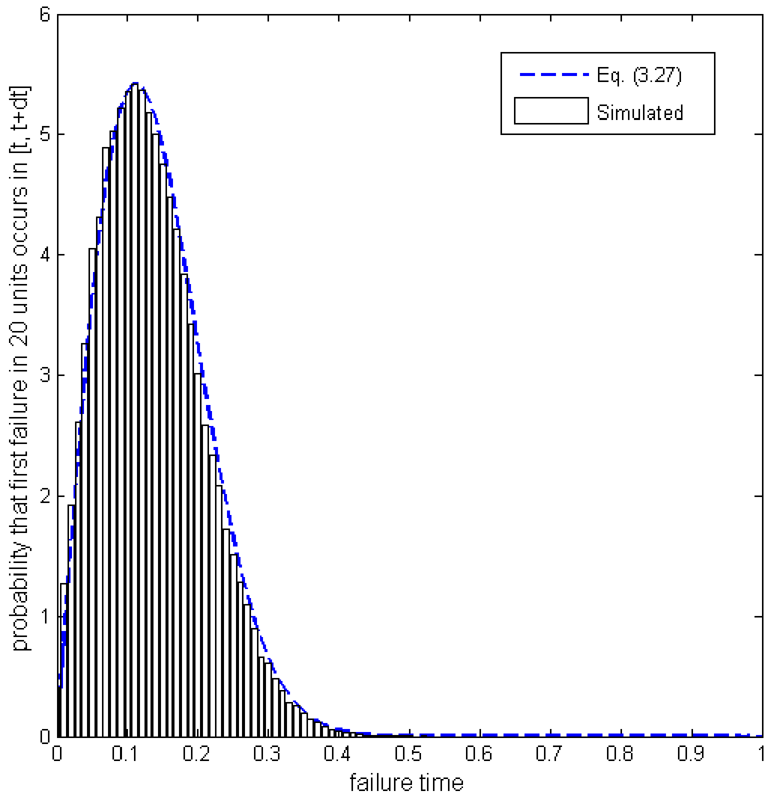


Figure 3.2 Verification of Eq. (3.27) by simulation

Eq. (3.28) is a function of unknown parameter $\boldsymbol{\theta}=\left(\beta_{10},\beta_{11},...,\beta_{1k},...,\beta_{h0},\beta_{h1},...,\beta_{hk},\sigma\right)^T$. We denote it by $G(\boldsymbol{\theta})=E\left[t_1\right]$, therefore we have

$$\frac{\partial G(\boldsymbol{\theta})}{\partial \beta_{i,\varpi}} = x_{D,\varpi}\tilde{N}^{-\sigma}\Gamma(1+\sigma)\exp\left(-\frac{\boldsymbol{\beta}_i\mathbf{x}_D}{\sigma}\right)\left[\sum_{i=1}^{h}\exp\left(-\frac{\boldsymbol{\beta}_i\mathbf{x}_D}{\sigma}\right)\right]^{-\sigma-1} \quad \varpi = 0,1,\ldots,k \tag{6.29}$$

and

$$\frac{\partial G(\boldsymbol{\theta})}{\partial \sigma} = \tilde{N}^{-\sigma}\Gamma(1+\sigma)\left[\sum_{i=1}^{h}\exp\left(-\frac{\boldsymbol{\beta}_i\mathbf{x}_D}{\sigma}\right)\right]^{-\sigma} \cdot\left[\ln\left(\sum_{i=1}^{h}\exp\left(-\frac{\boldsymbol{\beta}_i\mathbf{x}_D}{\sigma}\right)\right)+\ln\left(\tilde{N}\right)-\Psi\left(1+\sigma\right)\right] \tag{6.30}$$

Hence the asymptotic variance of the MLE of the mean time of first failure in a group of $\tilde{N}$ units at normal operating conditions is given by

$$\operatorname{Asvar}\left[G\left(\widehat{\boldsymbol{\theta}}\right);\Xi\right] = \left[\frac{\partial G\left(\widehat{\boldsymbol{\theta}}\right)}{\partial \beta_{10}},\ldots,\frac{\partial G\left(\widehat{\boldsymbol{\theta}}\right)}{\partial \beta_{ik}},\frac{\partial G\left(\widehat{\boldsymbol{\theta}}\right)}{\partial \sigma}\right] I\left(\widehat{\boldsymbol{\theta}};\Xi\right)^{-1}\left[\frac{\partial G\left(\widehat{\boldsymbol{\theta}}\right)}{\partial \beta_{10}},\ldots,\frac{\partial G\left(\widehat{\boldsymbol{\theta}}\right)}{\partial \beta_{ik}},\frac{\partial G\left(\widehat{\boldsymbol{\theta}}\right)}{\partial \sigma}\right]^{T} \tag{6.31}$$

Under progressive Type-I censoring and the given assumptions, the variance optimality of mean time of first failure for the test plan Ξ is formulated as

$$\text{Min} \quad \operatorname{Asvar}\left[G\left(\widehat{\boldsymbol{\theta}}\right);\Xi\right] \tag{6.32}$$

$$\text{s.t.} \qquad 0 < p_{\varsigma} < 1,\ \ \varsigma = 1,\ldots,n$$

$$\sum_{i=1}^{h} E\left[m_i;\mathbf{x}_{\varsigma}\right] \geq MENF_{\varsigma}$$

where $MENF_\varsigma$ is the specified Minimum Expected Number of Failures at test condition ς.

3.4.1.2 Quantile Failure

In this section we develop the expression of the quantile failure at normal operating conditions and the corresponding asymptotic variance and the formulation of optimal test plan based on the second optimality criterion which minimizes the asymptotic variance of MLE of lower quantile failure at normal operating conditions.

Let $t_q\left(\mathbf{x}_D\right)$ be the q^{th} quantile failure at normal operating conditions, then from Eq. (3.26) we solve

$$t_q\left(\mathbf{x}_D\right) = F^{-1}\left(q \mid \mathbf{x}_D\right) = \left\{\frac{-\ln\left(1-q\right)}{\sum_{i=1}^{k}\exp\left(-\frac{\boldsymbol{\beta}_i\mathbf{x}_D}{\sigma}\right)}\right\}^{\sigma} \tag{6.33}$$

where $\mathbf{x}_D^T = \left[1, x_{D,1}, \ldots, x_{D,\varpi}, \ldots, x_{D,k}\right]$ represents the normal operating conditions. $t_q\left(\mathbf{x}_D\right)$ is a function of parameters $\left(\beta_{10}, \beta_{11}, \ldots, \beta_{1k}, \ldots, \beta_{h0}, \beta_{h1}, \ldots, \beta_{hk}, \sigma\right)^T$. Now we take the derivatives of $t_q\left(\mathbf{x}_D\right)$ with respect to the parameters,

$$\frac{\partial t_q(\mathbf{x}_D)}{\partial \beta_{i\varpi}} = x_{D,\varpi} \frac{t_q(\mathbf{x}_D) \cdot \exp\left(-\frac{\boldsymbol{\beta}_i \mathbf{x}_D}{\sigma}\right)}{\sum_{i=1}^{h} \exp\left(-\frac{\boldsymbol{\beta}_i \mathbf{x}_D}{\sigma}\right)}, \quad \varpi = 0,1,\dots,k \tag{6.34}$$

$$\frac{\partial t_q(\mathbf{x}_D)}{\partial \sigma} = \left(\frac{-\ln(1-q)}{\sum_{i=1}^{h} \exp\left(-\frac{\boldsymbol{\beta}_i \mathbf{x}_D}{\sigma}\right)}\right)^{\sigma} \cdot \left\{ \ln\left[\frac{-\ln(1-q)}{\sum_{i=1}^{h} \exp\left(-\frac{\boldsymbol{\beta}_i \mathbf{x}_D}{\sigma}\right)}\right] - \frac{\sum_{i=1}^{h}\left(\frac{\boldsymbol{\beta}_i \mathbf{x}_D}{\sigma}\right) \cdot \exp\left(-\frac{\boldsymbol{\beta}_i \mathbf{x}_D}{\sigma}\right)}{\sum_{i=1}^{h} \exp\left(-\frac{\boldsymbol{\beta}_i \mathbf{x}_D}{\sigma}\right)} \right\} \tag{6.35}$$

The asymptotic variance of the MLE $\widehat{t}_q(\widehat{\boldsymbol{\theta}}; \mathbf{x}_D)$ at normal operating conditions is given by

$$\begin{aligned}\operatorname{Asvar}\left[t_q(\widehat{\boldsymbol{\theta}}; \mathbf{x}_D); \Xi\right] = & \left[\frac{\partial \widehat{t}_q(\widehat{\boldsymbol{\theta}}; \mathbf{x}_D)}{\partial \beta_{10}}, \dots, \frac{\partial \widehat{t}_q(\widehat{\boldsymbol{\theta}}; \mathbf{x}_D)}{\partial \beta_{ik}}, \frac{\partial \widehat{t}_q(\widehat{\boldsymbol{\theta}}; \mathbf{x}_D)}{\partial \sigma}\right] \\ & \times I(\widehat{\boldsymbol{\theta}}; \Xi)^{-1} \times \\ & \left[\frac{\partial \widehat{t}_q(\widehat{\boldsymbol{\theta}}; \mathbf{x}_D)}{\partial \beta_{10}}, \dots, \frac{\partial \widehat{t}_q(\widehat{\boldsymbol{\theta}}; \mathbf{x}_D)}{\partial \beta_{ik}}, \frac{\partial \widehat{t}_q(\widehat{\boldsymbol{\theta}}; \mathbf{x}_D)}{\partial \sigma}\right]^T\end{aligned}$$

Under progressive Type-I censoring, the variance optimality of quantile failure based on test plan Ξ is formulated as

$$\text{Min} \quad \operatorname{Asvar}\left[\widehat{t}_q(\widehat{\boldsymbol{\theta}}; \mathbf{x}_D); \Xi\right] \tag{6.36}$$

$$\text{s.t.} \quad 0 < p_\varsigma < 1, \ \varsigma = 1, \dots, n$$

$$\sum_{i=1}^{h} E\left[m_i; \mathbf{x}_\varsigma\right] \ge MENF_\varsigma$$

The optimal test plans in terms of stress levels (stress-level combination for multiple stresses) and unit allocation to each stress level can be determined by solving the formulation (3.32) and (3.36) for given initial values of the parameters, progressive censoring schedule $\mathbf{r}_\varsigma$ and $\boldsymbol{\tau}_\varsigma$ for $\varsigma = 1,...,n$ and the normal operating conditions.

3.4.2 D-Optimality

D-optimality maximizes the determinant of the Fisher information matrix which results in minimum volume of the Wald-type joint confidence region for the unknown parameters. D-optimality is a suitable criterion when the purpose of an ALT is to obtain more accurate estimates of the model parameters. Based on the Fisher information matrix developed in Sec. 3.3, the optimal test plan based on D-optimality is formulated as

$$\text{Max } \det\left[I\left(\widehat{\boldsymbol{\theta}};\Xi\right)\right] \tag{6.37}$$

$$\text{s.t.} \quad 0 < p_\varsigma < 1, \ \ \varsigma = 1,...,n$$

$$\sum_{i=1}^{h} E\left[m_i;\mathbf{x}_\varsigma\right] \geq MENF_\varsigma$$

Optimal test plans in terms of stress levels and unit allocation to each stress level are determined by solving (3.37) for given initial values of the parameters, progressive censoring schedule $\mathbf{r}_\varsigma$ and $\boldsymbol{\tau}_\varsigma$ for $\varsigma = 1,...,n$.

3.4.3 Multi-objective Optimization

Under progressive censoring, some of the surviving test units can be removed at different stages before the final termination time of the test. As a result, the test duration can be further reduced and the removed test units can be used for other tests or purposes. On the other hand, Tang and Yang (2002) revealed that, for multiple levels constant-stress ALT, there are many possible testing plans which are nearly statistically optimal. This motivates us to develop testing plans under multi-objective which not only obtain optimal statistical precision but also meet other practical constraints, e.g. time and cost. The total cost is affected by the fixed investment in facilities, overhead, stress applications, test duration and the number of test units, etc. The fixed investment has no effect on the optimization. The cost related to overhead, stress applications and test duration is difficult to estimate and may depend on the specific situation. Therefore, we investigate the total number of failures instead.

In this dissertation, we propose two formulations for multi-objective optimization under progressive censoring. In both formulations, we consider the objective of statistical precision $f_1(\cdot)$, which can be the asymptotic variance of mean time of first failure, mean time to failure and quantile failure, or the determinant of the Fisher information (variance-covariance) matrix. To simplify the presentation, we illustrate the formulation based on a 3-level single stress test. The extension to more stress levels and multiple stresses is straight forward. Under the given assumption and test plan Ξ, optimal multi-objective test plans are formulated as follows.

Formulation 1:

$$\text{Min} \quad \left\{ f_1\left[\hat{\boldsymbol{\theta}};\Xi(\Omega)\right],\, f_2\left[\hat{\boldsymbol{\theta}};\Xi(\Omega)\right] \right\} \tag{6.38}$$

$$\text{s.t.} \quad 0 < p_\varsigma < 1, \;\; \varsigma = 1,2,3$$

$$\boldsymbol{\tau}_\varsigma = \left[\tau_{\varsigma,1}, \ldots, \tau_{\varsigma,u}, \ldots, \tau_{\varsigma,s}\right], \;\; \varsigma = 1,2,3$$

$$x_3 = 1, \;\; x_2 = \frac{x_3 + x_1}{2}$$

where $f_2(\cdot)$ is the objective function of the number of failures. For example, $f_2(\cdot)$ can be the total number of failures, failures due to a particular failure mode, or failures under a specific stress level over the entire test duration or the test duration of a particular stress level. $\Omega = \left[x_1 \quad r_{\varsigma,1} \quad \ldots \quad r_{\varsigma,s-1}\right]'$ present the decision variables. $\boldsymbol{\tau}_\varsigma$ are the given times to remove surviving test units which can be equally, increasingly or decreasingly spaced.

Formulation 2:

$$\text{Min} \quad \left\{ f_1\left[\hat{\boldsymbol{\theta}};\Xi(\Omega)\right],\, f_2\left[\hat{\boldsymbol{\theta}};\Xi(\Omega)\right] \right\} \tag{6.39}$$

$$\text{s.t.} \quad 0 < p_\varsigma < 1, \;\; \varsigma = 1,2,3$$

$$\mathbf{r}_\varsigma = \left[r_{\varsigma,1}, \ldots, r_{\varsigma,u}, \ldots, r_{\varsigma,s-1}\right], \;\; \varsigma = 1,2,3$$

$$x_3 = 1, \;\; x_2 = \frac{x_3 + x_1}{2}$$

where $f_2(\cdot)$ is the objective function of test duration and $\Omega = \begin{bmatrix} x_1 & \tau_{\varsigma,1} & \dots & \tau_{\varsigma,s-1} \end{bmatrix}'$ are decision variables. $f_2(\cdot)$ can be the total test duration if the experiments under different stress levels are conducted sequentially, or the test duration under certain stress level. $\tau_{\varsigma,1} \quad \dots \quad \tau_{\varsigma,s-1}$ can follow certain relationship, for example,

$$\left(\tau_{\varsigma,u+1} - \tau_{\varsigma,u}\right) = \rho_{\varsigma}\left(\tau_{\varsigma,u} - \tau_{\varsigma,u-1}\right), \quad \rho > 0, \quad \varsigma = 1,2,3, \; u = 0,\dots,s-1$$

$\mathbf{r}_{\varsigma}$ is the given fraction of surviving units to remove, which can be a constant, decreasing or increasing function of the number of periods.

To evaluate above formulations, we can use multi-objective Genetic Algorithm to obtain the Pareto front.

3.5 Examples

In this section we present examples to demonstrate the application of the proposed approach for the design of ALT plans under competing risk and progressive Type-I censoring. The optimal test plans are obtained under both single stress and multiple stresses.

3.5.1 Optimal Test Plans under Single Stress

Parameters

Nelson (1990), page 393, presents data from ALTs of the Class-H insulation system of motorettes at temperature 190 ℃, 220 ℃, 240 ℃, and 260 ℃. Three potential failure modes, Turn, Phase, and Ground that occur on separated parts of the system are monitored. For each observation, the failure and/or censored time and corresponding failure mode are recorded. The objective of the life tests is to check if the median lifetime at normal operating condition of 180 ℃ is 20,000 hours under a lognormal Arrhenius model.

Pascual (2007) investigate ALT planning with independent Weibull competing risk with known shape parameter under single stress and Type-I censoring. Nelson's data (1990) from ALTs of the Class-H insulation system are used in his study, where only the Turn and Ground failure modes are considered. Planning values for the model parameters are obtained ($\beta_{10} = 2.6078$, $\beta_{11} = -2.1461$, $\beta_{20} = 3.1315$, and $\beta_{21} = -1.2796$) using ML methods, the *s*-independent Weibull-Arrhenius competing risk mode (same as Eq. (3.2)) with specified shape parameter $\sigma = 0.5$ and number of failure modes $h = 2$. In this dissertation, we use the same parameter values as Pascual (2007).

Single objective test plans

Using the optimization criteria presented in Sec. 3.4, we determine optimal single stress test plans with two and three stress levels. In all the cases, we specify $\mathbf{r}_{\varsigma} = [0.2, 0.2, 1]$, and $\boldsymbol{\tau}_{\varsigma} = [6, 8, 10], \varsigma = 1, 2, \ldots, n$ for simplicity. Due to the invariance property of the Fisher information matrix given in Eq. (3.24), we set the total number of test units $\hat{N} = 1$. With respect to the variance criterion of mean time of first failure, we set the group size

$\tilde{N}$ = 20 units. We investigate the lower life quantile $q = 0.01$ for the variance optimality criterion of quantile failure.

For the single stress test with two levels, we set the high stress level as $x_2 = 1$. Thus the search for the optimal test plan is limited to the determination of the low stress level $0 < x_1 < 1$ and the associated unit allocation $0 < p_1 < 1$. We require $MENF_1$ = 0.3, i.e. the minimum expected number of failure at the stress level x_1 is 30% of the units allocated to that condition.

For the test with three levels, we set the high stress level $x_3 = 1$ and the medium stress level $x_2 = \frac{x_1 + x_3}{2}$. The decision variable is the low stress level $0 < x_1 < 1$. Under such three equally-spaced stress levels the 4:2:1 rule is often used for unit allocation (Meeker and Hahn, 1985). Therefore, we set $p_1 = \frac{4}{7}$, $p_2 = \frac{2}{7}$, and $p_3 = \frac{1}{7}$. The $MENF_1$ at stress level x_1 is also 30% of the units allocated to the stress level.

The obtained optimal single stress plan with two levels and three levels are shown in Table 3.1 and Table 3.2, respectively. We observe that the obtained values of low-stress level (x_1) under all scenarios are close to the normal operating condition (x_D = 0). This is highly desirable as it reduces the extent of stress extrapolation. We also observe that the two-stress-level plans achieve better objective values relative to the corresponding three-stress-level plans. This implies that the prediction precision obtained from two-stress-level tests is relatively higher than that under three-stress-level tests for the given model and test plan parameters.

Table 3.1 Optimal test plans for two stress levels

Obj. Fun.	Stress Level	Unit Allocation	Obj. Fun. Value ($\hat{N}=1$)
Min Asvar$\left[G\left(\widehat{\boldsymbol{\theta}};\Xi\right)\right]$	x_1 = 0.0489	p_1 = 0.9334	18.3251
	x_2 = 1	p_2 = 0.0666	
Min Asvar$\left[\widehat{t}_{.01}\left(\widehat{\boldsymbol{\theta}};\mathbf{x}_D\right);\Xi\right]$	x_1 = 0.0489	p_1 = 0.4331	25.6451
	x_2 = 1	p_2 = 0.5669	
Max det$\left[I\left(\widehat{\boldsymbol{\theta}};\Xi\right)\right]$	x_1 = 0.1082	p_1 = 0.4538	1.5319e-04
	x_2 = 1	p_2 = 0.5462	

Table 3.2 Optimal 4:2:1 test plans

Obj. Fun.	Stress Level	Unit Allocation	Obj. Fun. Value ($\hat{N}=1$)
Min Asvar$\left[G\left(\widehat{\boldsymbol{\theta}};\Xi\right)\right]$	x_1 = 0.0489	p_1 = 4/7	23.1308
	x_2 = 0.5244	p_2 = 2/7	
	x_3 = 1	p_3 = 1/7	
Min Asvar$\left[\widehat{t}_{.01}\left(\widehat{\boldsymbol{\theta}};\mathbf{x}_D\right);\Xi\right]$	x_1 = 0.0610	p_1 = 4/7	27.0114
	x_2 = 0.5305	p_2 = 2/7	
	x_3 = 1	p_3 = 1/7	
Max det$\left[I\left(\widehat{\boldsymbol{\theta}};\Xi\right)\right]$	x_1 = 0.0707	p_1 = 4/7	5.4896e-05
	x_2 = 0.5353	p_2 = 2/7	
	x_3 = 1	p_3 = 1/7	

In design of ALT plans, initial estimates of unknown model parameters must be provided so as to derive a locally optimal test plan. Sometimes the optimality in terms of statistical precision cannot be achieved as planned due to the poor initial estimates of the parameters. We perform a sensitivity study to identify the parameters which must be estimated with special care. We increase and decrease the values of the parameters $\left[\hat{\beta}_{10}, \hat{\beta}_{11}, \hat{\beta}_{20}, \hat{\beta}_{21}, \hat{\sigma}\right]$ by 5% sequentially and investigate the corresponding effect on optimal test plans presented in Table 3.1 and Table 3.2. The results associated with the optimal 4:2:1 plans and 2-stress-level plans are summarized in Table 3.3 and Table 3.4, respectively, where "+" and "-" indicates 5% increase and 5% decrease, respectively.

We observe that $\hat{\sigma}$, inverse of the shape parameter of the Weibull distribution, is the most sensitive parameter. For the variance optimality, $\hat{\beta}_{10}$ is more sensitive relative to $\left[\hat{\beta}_{11}, \hat{\beta}_{20}, \hat{\beta}_{21}\right]$. On the contrary, for the D-optimality $\hat{\beta}_{10}$ is not so sensitive as it is for the variance optimality. In general, the sensitivity of $\boldsymbol{\beta}$ in terms of quantile failure is less than that of mean time of first failure. The reason is that mean time of first failure is the most extreme case. Also, the sensitivity of $\boldsymbol{\beta}$ in terms of mean time of first failure is less than that of D-optimality. In addition, parameters are less sensitive for 4:2:1 plans than those for two-stress-level plans. This implies that prediction obtained from three-stress-level test is more robust than that from two-stress-level test for the given parameters.

Table 3.3 Percent change in the objective function due to 5% change in the parameters--optimal 4:2:1 plans

Objective Function	Parameter and Objective Function Values (%)									
	$\hat{\beta}_{10}$ (2.6078)		$\hat{\beta}_{11}$ (-2.1461)		$\hat{\beta}_{20}$ (3.1315)		$\hat{\beta}_{21}$ (-1.2796)		$\hat{\sigma}$ (0.5)	
	+	-	+	-	+	-	+	-	+	-
Min Asvar$\left[G\left(\hat{\boldsymbol{\theta}};\Xi\right)\right]$	7.48	6.98	.14	.17	3.58	3.59	.28	.29	25.7	37.7
Min Asvar$\left[\hat{t}_{.01}\left(\hat{\boldsymbol{\theta}};x_D\right);\Xi\right]$	4.22	4.09	.33	.36	1.66	1.71	.11	.11	22.7	29.4
Max det$\left[I\left(\hat{\boldsymbol{\theta}};\Xi\right)\right]$	4.85	6.96	11.6	12.8	10.3	11.3	9.98	8.94	19.0	17.4

Table 3.4 Percent change in the objective function due to 5% change in the parameters--optimal 2-level plans

Objective Function	Parameter and Objective Function Values (%)									
	$\hat{\beta}_{10}$ (2.6078)		$\hat{\beta}_{11}$ (-2.1461)		$\hat{\beta}_{20}$ (3.1315)		$\hat{\beta}_{21}$ (-1.2796)		$\hat{\sigma}$ (0.5)	
	+	-	+	-	+	-	+	-	+	-
Min Asvar$\left[G\left(\hat{\boldsymbol{\theta}};\Xi\right)\right]$	7.93	7.41	.09	.04	4.33	4.31	.31	.34	30.9	48
Min Asvar$\left[\hat{t}_{.01}\left(\hat{\boldsymbol{\theta}};x_D\right);\Xi\right]$	4.24	4.11	.03	.02	1.71	1.77	.06	.07	23.1	30.3
Max det$\left[I\left(\hat{\boldsymbol{\theta}};\Xi\right)\right]$	8.49	9.67	16.4	19.1	10.4	11.5	13.2	11.7	21.2	18.9

Multi-objective optimal test plans

In this section, we present examples to illustrate the application of the multi-objective optimization to determine optimal test plans. For unit allocation to stress levels, we follow the 4:2:1 rule, that is, $p_1 = \frac{4}{7}$, $p_2 = \frac{2}{7}$, and $p_3 = \frac{1}{7}$ for equally spaced three-stress-level test. For simplicity, we set $\boldsymbol{\tau}_\varsigma = \boldsymbol{\tau}$ and $\mathbf{r}_\varsigma = \mathbf{r}$, for $\varsigma = 1,2,3$. In other words, we use the same progressive censoring schedule under different stress levels. The initial estimate of the unknown model parameters is the same as previous examples.

To develop examples based on the formulations (3.38) and (3.39), we use multi-objective Genetic Algorithm *gamultiobj* function which is built in Matlab and plot the Pareto front. In all the cases, we set the maximum number of iterations as 3000, population size as 100, and the Pareto fraction as 0.7. The larger the population size, the smoother the Pareto front, but the longer the time is needed for computation. We conducted some experimentation for the *gamultiobj* parameter setting and then chose these values. For other parameters of the algorithm, we use their default values.

For formulation (3.38), we consider three periods, i.e. $s = 3$. We investigate the objectives of the determinant of the Fisher information matrix and the total number of failure over all the periods and stress levels. We set $\hat{N} = 1$, so the total number of failures equals the total fraction of failure which can be calculated using Eq. (3.21). Therefore, we have formulation (3.38) written as

$$\text{Min } \left\{ -\det\left\{ I\left[\hat{\boldsymbol{\theta}};\Xi(\Omega)\right]\right\}, \sum_{i=1}^{2} E\left[m_i\right]\left[\hat{\boldsymbol{\theta}};\Xi(\Omega)\right]\right\}$$

$$\text{s.t.} \quad p_1 = \frac{4}{7},\; p_2 = \frac{2}{7},\; p_3 = \frac{1}{7}$$

$$\boldsymbol{\tau} = [6 \;\; 8 \;\; 10]$$

$$x_3 = 1,\;\; x_2 = \frac{x_3 + x_1}{2}$$

where $\Omega = \left[x_1 \quad r_1 \quad r_2\right]'$. As the test is terminated at the end of the 3rd period, $r_3 = 1$.

After evaluation of above formulation, we obtain the Pareto front as shown in Figure 3.3. We observe that the statistical precision, determinant of the Fisher information matrix, monotonically increases with the increase of the total fraction of failure. The increasing rate is very small when the total fraction of failure is more than 0.7. Each point on the Pareto front associates with not only the values of the two objective functions, but also the values of x_1, r_1 and r_2. For example, point *A* in Figure 3.3 represents a test plan that achieves 1st objective value 6.3438e-05 (D-optimality) and 2nd objective value 0.6931 (total ratio of failure). The corresponding three normalized stress levels from low to high are 0.0025, 0.5013 and 1. The fraction of test units allocated to low, medium and high stress levels is 4/7, 2/7 and 1/7, respectively. Under each stress level, 0.1216, 0.1016 and 1 fraction of surviving test units are removed at the 6th, 8th and 10th time unit, respectively. These parameters of optimal test plan associated with point *A* are summarized in Table 3.5.

In addition, with the value of x_1, r_1, r_2 and $\boldsymbol{\tau}$, we also can obtain the fraction of failure under different stress levels and the fraction of failure due to failure mode i based on Eq. (3.11) and Eq. (3.21), respectively.

Table 3.5 Optimal test plan (point A) on Pareto front

Det. (f_1)	**Total ratio of failure (f_2)**	**Stress levels**
6.3438e-05	0.6931	x = [0.0025, 0.5013, 1]
Unit allocation	**Censoring time**	**Ratio to remove**
p = [4/7, 2/7, 1/7]	τ = [6, 8, 10]	r =[0.1216, 0.1016, 1]

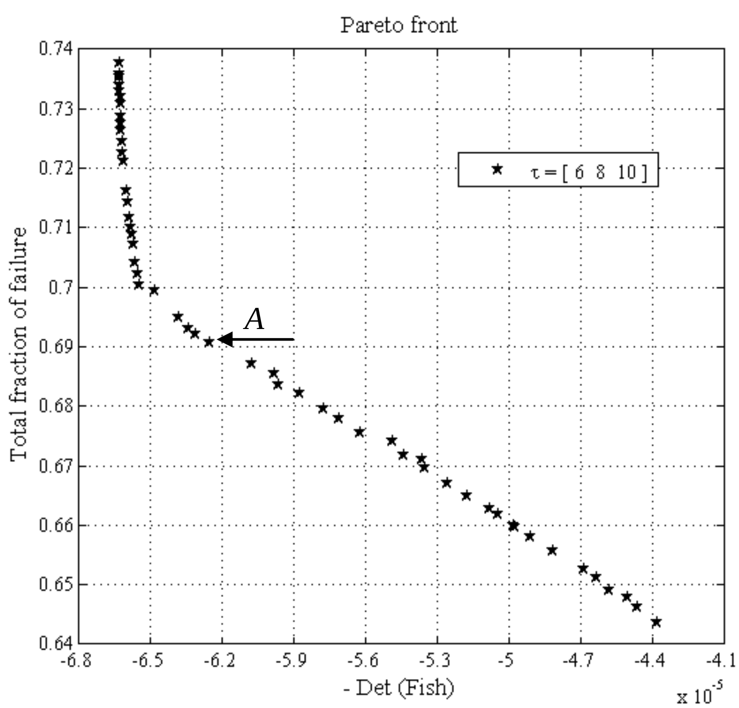


Figure 3.3 Pareto front for $\boldsymbol{\tau}$ = [6 8 10]

We keep all parameters the same as above formulation but $\boldsymbol{\tau}$ = [2 3 4] and compute another Pareto front, as shown in Figure 3.4. We observe that under shorter test duration, both the total fraction of failure and the statistical precision decrease. By plotting series of Pareto fronts under different $\boldsymbol{\tau}$, we can have clear idea about the relationship of statistical precision, test duration and total fraction of failure. This facilitates the choice of appropriate progressive censoring schedule, fraction of units to remove and test stress levels in order to obtain optimal parameter estimation as well as meet practical constraints on time and cost. Similarly, each point on the Pareto front represents an optimal test plan. The optimal test plan parameters associated with point *B* in Figure 3.4 is given in Table 3.6.

Table 3.6 Optimal test plan (point B) on Pareto front

Det. (f_1)	**Total T_L (f_2)**	**Stress levels**
1.8916e-06	0.4446	x = [0.2033, 0.6016, 1]
Unit allocation	**Censoring time**	**Ratio to remove**
p = [4/7, 2/7, 1/7]	τ =[2, 3, 4]	r =[0.1012, 0.1018, 1]

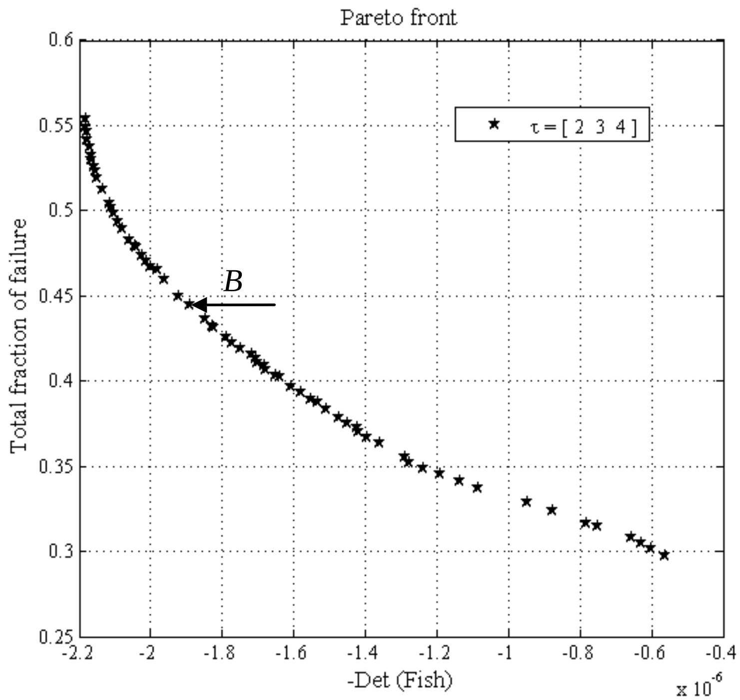


Figure 3.4 Pareto front for $\boldsymbol{\tau}$ = [2 3 4]

For formulation (3.39), we also consider three periods and set $\mathbf{r} = [0.4\ \ 0.2\ \ 1]$, $\left(\tau_{u+1} - \tau_u\right) = 0.5^u\left(\tau_u - \tau_{u-1}\right),\ u = 1, 2,\ \tau_0 = 0$, and $\hat{N} = 1$. For single stress ALT, the experiments under different stress levels can be conducted either sequentially or simultaneously. In either case, the test duration under the lowest stress level has the most crucial impact on the precision of statistical prediction. Therefore, we choose the test duration under the stress level x_1 as the objective as well as the determinant of the Fisher information matrix. The formulation (3.39) is written as

$$\text{Min} \quad \left\{-\det\left\{I\left[\hat{\boldsymbol{\theta}};\Xi(\Omega)\right]\right\}, \tau_3\left[\hat{\boldsymbol{\theta}};\Xi(\Omega)\right]\right\}$$

$$\text{s.t.} \quad p_1 = \frac{4}{7},\ p_2 = \frac{2}{7},\ p_3 = \frac{1}{7}$$

$$\mathbf{r} = [0.4 \ \ 0.2 \ \ 1]$$

$$\tau_1 = \frac{4}{7}\tau_3,\ \tau_2 = \frac{6}{7}\tau_3$$

$$x_3 = 1,\ \ x_2 = \frac{x_3 + x_1}{2}$$

where $\Omega = [x_1 \quad \tau_1 \quad \tau_2]'$.

After evaluation of above formulation, we obtain the Pareto front as shown in Figure 3.5. We observe that the statistical precision, determinant of the Fisher information matrix monotonically increases with the test duration. Each point on the Pareto front present not only the values of the two objective functions but also x_1, τ_1 and τ_2. Similarly, with the values of x_1, τ_1, τ_2 and $\boldsymbol{r}$, we can obtain the fraction of failure under different stress levels and the fraction of failure due to failure mode *i*.

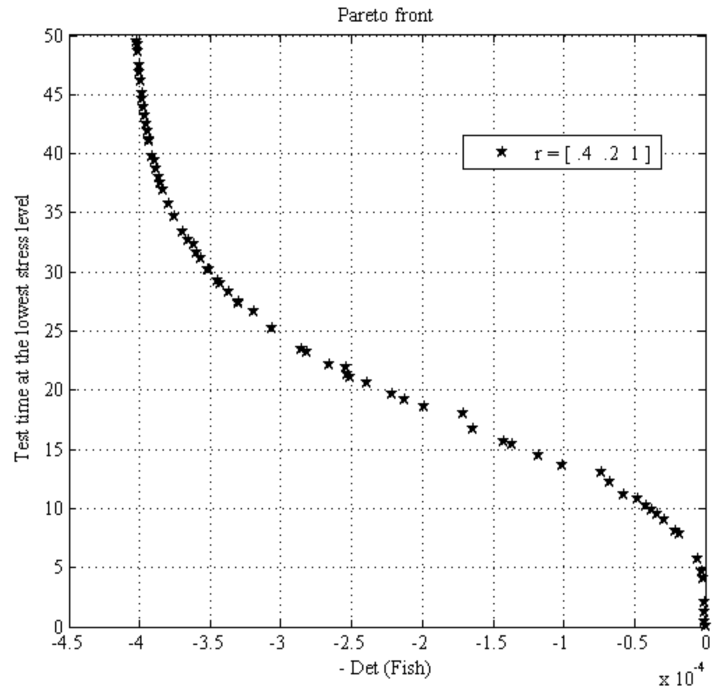


Figure 3.5 Pareto front for *r* = [.4 .2 1]

We keep all parameters the same as above formulation but change the fraction of removal to $\mathbf{r} = [0.2 \;\; 0.2 \;\; 1]$, $\mathbf{r} = [0.2 \;\; 0.4 \;\; 1]$ and $\mathbf{r} = [0 \;\; 0 \;\; 1]$, and compute the corresponding Pareto front, as shown in Figure 3.6. Under the same test duration, we obtain the best statistical precision when $\mathbf{r} = [0 \;\; 0 \;\; 1]$, which is the extreme case of progressive Type-I censoring, Type-I censoring. In other words, we do not remove any units until the final termination of the test. In this case, we have more information relative to other cases after the first period. Under the same time duration, we achieve the worst statistical precision when $\mathbf{r} = [0.4 \;\; 0.2 \;\; 1]$. This implies that the more the units are removed at the earlier stages, the worse the prediction is. The statistical precision under $\mathbf{r} = [0.2 \;\; 0.2 \;\; 1]$, $\mathbf{r} = [0.2 \;\; 0.4 \;\; 1]$ are very close. Thus the impact of the units removed at the later stages is insignificant under the given censoring time schedule. On the other hand, test duration for the same level of statistical precision is the shortest when $\mathbf{r} = [0 \;\; 0 \;\; 1]$ and longest when $\mathbf{r} = [0.4 \;\; 0.2 \;\; 1]$, respectively. Similarly, the series of Pareto front in Figure 3.6 can facilitate the choice of appropriate fraction of units to remove and censoring time at the lowest stress levels in order to obtain optimal parameter estimation as well as meet practical constraints on time and cost. The parameters of the optimal test plan associated with points *C* and *D* are given in Table 3.7 and Table 3.8, respectively.

Table 3.7 Optimal test plan (point *C*) on Pareto front

Det. (f_1)	**Total T_L (f_2)**	**Stress levels**
1.2640e-04	12.08	x = [0.0761, 0.5381, 1]
Unit allocation	**Censoring time**	**Ratio to remove**
p = [4/7, 2/7, 1/7]	τ = [6.9, 10.35, 12]	r =[0, 0, 1]

Table 3.8 Optimal test plan (point *D*) on Pareto front

Det. (f_1)	**Total T_L (f_2)**	**Stress levels**
3.2545e-04	27.16	x = [0.0023, 0.5012, 1]
Unit allocation	**Censoring time**	**Ratio to remove**
p = [4/7, 2/7, 1/7]	τ =[15.52, 23.28, 27.16]	r =[0.4, 0.2, 1]

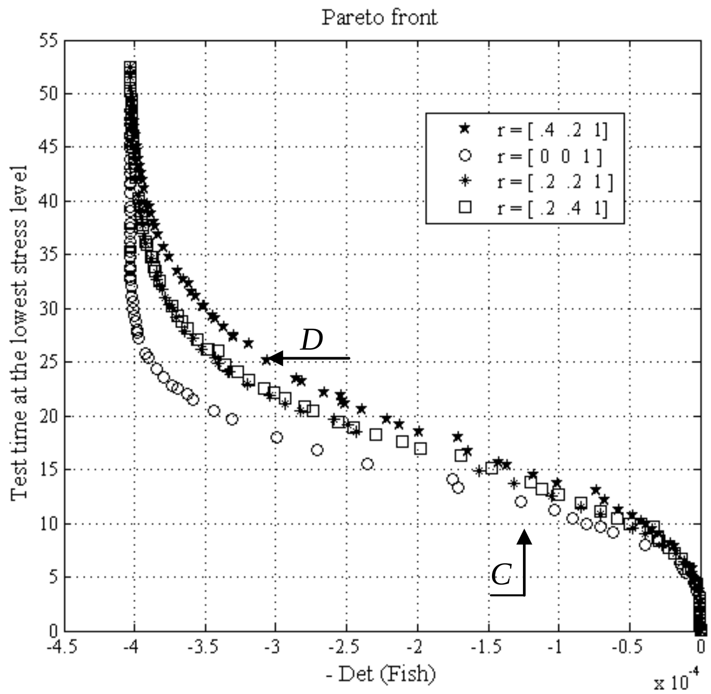


Figure 3.6 Pareto fronts for different ***r***

3.5.2 Optimal Test Plan under Multiple Stresses

We extend the example of optimal test plan with multiple stresses presented in Sec. 2.6 to multiple failure modes. We assume the parameters associated with the second failure mode are $\boldsymbol{\beta}_2 = [5.1, -0.469, 0.411, -0.65]$. We use the same values of $\hat{N}$, $\mathbf{r}$, $\boldsymbol{\tau}$, $\tilde{N}$ and q as in the example of the single stress test. The specified *MENF* at each stress-level combination is the fraction 0.015 times the number of units under test. We determine the optimal test plans based on the formulation given in Sec. 3.4 using the SA based algorithm presented in Sec. 2.5 and the results are shown in Table 3.9.

Table 3.9 shows that the achieved objective of the minimum asymptotic variance of mean time of first failure is 5.1408e+05. Correspondingly, 0.3868, 0.2150, 0.1049, 0.0150 and 0.2780 fractions of the test units are allocated to the stress-level combination of [60% RH, 70 ℃, 1.7275 kV/mm], [67.5%RH, 45 ℃, 0.7825 kV/mm], [75%RH, 85 ℃, 1.255 kV/mm], [82.5%RH, 55 ℃, 0.31 kV/mm] and [90%RH, 35 ℃, 2.2 kV/mm], respectively. The objective value of the optimal test plan for the minimum asymptotic variance of quantile failure at normal operating conditions is 8.4995e+06 and the corresponding test plan has 0.3423, 0.1852,0.0890, 0.0150 and 0.3685 fractions of the test units allocated to the stress-level combination of [60% RH, 70 ℃, 1.7275 kV/mm], [67.5%RH, 45 ℃, 0.7825 kV/mm], [75%RH, 85 ℃, 1.255 kV/mm], [82.5%RH, 55 ℃, 0.31 kV/mm] and [90%RH, 35 ℃, 2.2 kV/mm], respectively. The objective value of the optimal test plan that maximizes the determinant of the Fisher information matrix is 0.0034. Based on this test plan, 0.2505, 0.0150, 0.2393, 0.2304 and 0.2648 fractions of the test units are allocated to the stress-level combination of [60% RH, 55 ℃, 2.2 kV/mm], [67.5%RH, 70 ℃, 0.31 kV/mm], [75%RH, 35 ℃, 0.7825 kV/mm], [82.5%RH, 85 ℃, 1.255 kV/mm] and [90%RH, 45 ℃, 1.7275 kV/mm], respectively.

Table 3.9 Optimal test plans with multiple stresses

Obj. Fun.	**Stress-Level Combinations**			**Unit Allocation**	**Obj. Fun. Value (** $\hat{N}=1$ **)**
	x_1 (RH%)	x_2 (℃)	x_3 (kV/mm)		
Min Asvar$\left[G\left(\hat{\boldsymbol{\theta}};\Xi\right)\right]$	**1** (60%)	**4** (70 ℃)	**4** (1.7275 kV/mm)	0.3868	5.1408e+05
	2 (67.5%)	**2** (45 ℃)	**2** (0.7825 kV/mm)	0.2150	
	3	**5**	**3**	0.1049	

	(75%)	(85 ℃)	(1.255 kV/mm)		
	4 (82.5%)	**3** (55 ℃)	**1** (0.31 kV/mm)	0.0150	
	5 (90%)	**1** (35 ℃)	**5** (2.2 kV/mm)	0.2782	
Min $\text{Asvar}\left[\hat{t}_{.01}\left(\hat{\boldsymbol{\theta}}; x_D\right); \Xi\right]$	**1** (60%)	**4** (70 ℃)	**4** (1.7275 kV/mm)	0.3423	8.4995e+06
	2 (67.5%)	**2** (45 ℃)	**2** (0.7825 kV/mm)	0.1852	
	3 (75%)	**5** (85 ℃)	**3** (1.255 kV/mm)	0.0890	
	4 (82.5%)	**3** (55 ℃)	**1** (0.31 kV/mm)	0.0150	
	5 (90%)	**1** (35 ℃)	**5** (2.2 kV/mm)	0.3685	
Max $\det\left[I\left(\hat{\boldsymbol{\theta}}; \Xi\right)\right]$	**1** (60%)	**3** (55 ℃)	**5** (2.2 kV/mm)	0.2505	0.0034
	2 (67.5%)	**4** (70 ℃)	**1** (0.31 kV/mm)	0.0150	
	3 (75%)	**1** (35 ℃)	**2** (0.7825 kV/mm)	0.2393	
	4 (82.5%)	**5** (85 ℃)	**3** (1.255 kV/mm)	0.2304	
	5 (90%)	**2** (45 ℃)	**4** (1.7275 kV/mm)	0.2648	

3.6 Summary

This chapter presents approaches for planning ALT under progressive Type-I censoring and multiple failure modes. A unit is considered failed when any of the multiple failure modes occurs. Each failure mode is assumed to have an independent Weibull distribution with different unknown scale parameters and a common unknown shape parameter. Under progressive censoring, a proportion of the survival units is removed from the test at multiple stages before the final termination of the test under different stress levels.

We develop Fisher information matrix for the maximum likelihood estimate. We propose and develop a new optimization criterion for the design of test plans, that is, minimization of the asymptotic variance of MLE of mean time of first failure under normal operating conditions. This criterion is useful in circumstances where early failures are extremely crucial. In addition, we also develop optimal test plans in terms of the asymptotic variance of quantile failure, D-optimality and multi-objective optimization. The multi-objective criterion provides a practical guideline to seek test plan that not only achieves statistical optimality but also meet time and/or cost constraints.

This is also the first such work for the design of multiple stresses ALT plan under progressive censoring and competing risk. To illustrate the optimal test plan formulations, we present numerical examples based on the parameters from real tests under both single stress and multiple stresses. We also perform the sensitivity study to identify model parameters which should be initially estimated with special care.

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CHAPTER 4

DESIGN OF EQUIVALENT ALT PLANS

4.1 Introduction

Accelerated life testing is conducted under severer conditions than the normal operating conditions in order to obtain failure time data of test units in a much shorter time than testing at normal operating conditions. Typical ALT plans require the determination of stress types, stress levels, allocation of test units to the stress levels and duration of the test. ALT is usually conducted under constant-stresses during the entire test duration. In practice, the constant-stress tests need a long time at low stress levels to yield sufficient failure data. This has prompted industry to consider other stress loadings, such as step-stress (simple or multiple), ramp-stress, sinusoidal-cyclic stress or their combinations, as shown in Figure 4.1. These stress-loadings have been widely utilized in ALT experiments. For instance, static-fatigue tests and cyclic-fatigue tests (Matthewson and Yuce, 1994) have been frequently performed on optical fibers to study their reliability; dielectric-breakdown of thermal oxides (Elsayed, *et al*. 2006) have been studied under elevated constant electrical fields and temperatures; and the lifetime of ceramic components subject to slow crack growth due to stress corrosion have been investigated under cyclic stress by NASA (Choi and Salem, 1997).

Each stress-loading has both advantages and drawbacks. Complicated stress profiles may yield failures in a much shorter time than constant-stress tests but the statistical inference from the data might be more difficult to make. In other words, the accuracy of the reliability prediction might be affected. This has raised many practical questions such as: Can accelerated testing plans involving different stress loadings be designed such that they are equivalent? What are the measures of equivalency?

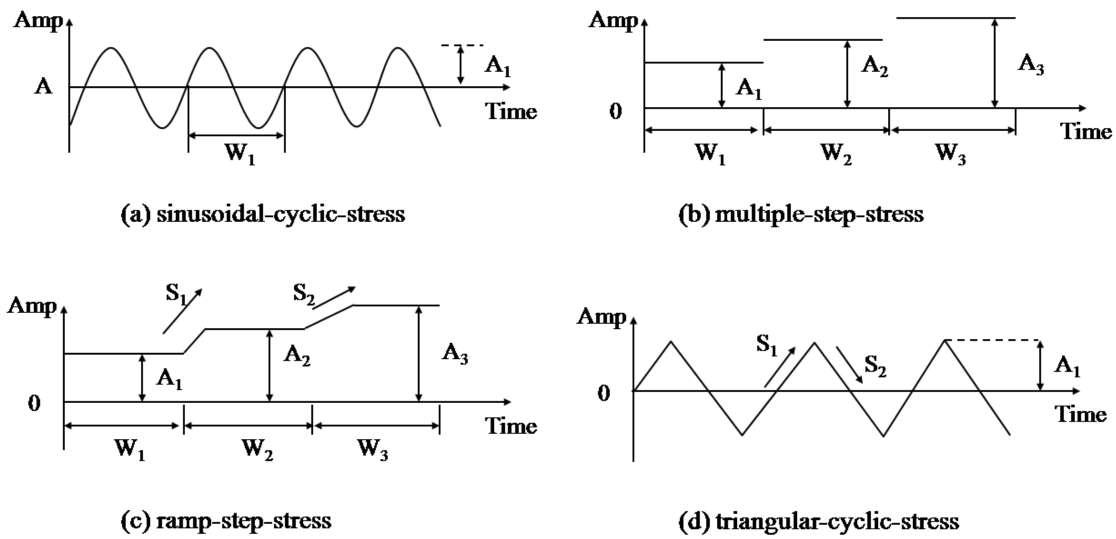

Figure 4.1 Various types of stress loadings

Literature review shows that current research on planning ALT has been focused on the design of optimum testing plans for given stress loading. For instance, the constant-stress ALT plans have been investigated by Nelson and Kielpinski (1976), Maxim *et al*. (1977), Meeker and Hahn (1977), Nelson and Meeker (1978), Meeker (1984), Nelson (1990) and Yang (1994). The step-stress ALT plans have been studied by Miller and Nelson (1983), Bai *et al*. (1989), Bai and Chun (1991), Khamis and Higgins (1996), Xiong (1998), Xiong and Milliken (1999), and Xiong and Ji (2004) while the ramp-stress ALT plans have been considered by Bai *et al*. (1992) and Park and Yum (1998).

The wide range of stress applications, stress levels and corresponding test durations give rise to the investigation of the equivalency between test plans. However, fundamental research on the equivalency of test plans has not yet been addressed in the reliability engineering field. Without understanding of such equivalency, it is difficult for practitioners to determine the best experimental settings before conducting actual ALT.

In this chapter, we present definitions of equivalent test plans, propose an approach for the design of equivalent ALT plans and apply the method to design of equivalent test plans under single constant-stress, step-stress and ramp-stress. The numerical results show that it is feasible to design equivalent and yet economical and efficient ALT plans having the same accuracy of reliability prediction. We also develop a model based on the well known cumulative exposure assumption to investigate the life-stress relationship under general time-varying stresses, e.g. ramp-stress.

4.2 Definition of Equivalent ALT Plans

In design of ALT plans, estimate of one or more reliability characteristics, such as the model parameters, hazard rate and the mean time to failure at certain conditions are common. Accordingly, different optimization criteria might be considered. For instance, if estimate of the model parameters is the main concern, D-optimality which maximizes the determinant of the Fisher information matrix is considered an appropriate criterion. When estimate of the quantile failure at normal operating conditions is the major task then the variance optimality that minimizes the asymptotic variance of quantile failure at normal operating conditions is commonly used. Meanwhile, different methods, e.g. MLE or Bayesian estimator can be used for estimation. However each method has its inherent statistical properties and efficiencies. In light of this, we discuss equivalent test plans with respect to the same reliability characteristics and optimization criterion and determine equivalent test plans using the same inference procedure. Four possible definitions of equivalency are presented as:

***Definition* 1**

Two or more test plans are equivalent if the absolute difference of the objectives for reliability prediction is less than $\delta\left(\delta \geq 0\right)$ while meeting given the same set of constraints on the number of test units, expected number of failures or total test time.

***Definition* 2**

Two or more test plans are equivalent if they achieve the same objective for reliability prediction while meeting the given constraints on the number of test units, expected number of failures or total test time within a margin $\delta\left(\delta \geq 0\right)$.

Definition 3

For the same reliability properties and inference procedure, two or more ALT plans are equivalent if they generate the same values of the same optimization criterion.

Definition 4

Two or more ALT plans are equivalent if the difference between the estimated times to failure and the respective confidence intervals by the plans at normal operating conditions are within $\delta(\delta \geq 0)$, where δ is an acceptable level of deviation.

4.3 Approach for Determining Optimal Equivalent ALT Plans

According to above definitions, the equivalent test plans are not unique. In this section, we discuss an approach for determining optimal equivalent ALT plans based on Definitions 1 and 2.

The first step of the approach is to obtain an optimal baseline test plan. Since constant-stress tests are the most commonly conducted accelerated life testing in industry and their statistical inference has been extensively investigated, we propose to use an optimal constant-stress plan as a baseline.

Suppose an optimal baseline test plan can be determined from the following general formulation,

$$\text{Min } f_B(x) \tag{7.1}$$

$$\text{s.t. } Lb \leq x \leq Ub$$

$$C(x) \leq 0$$

$$Ceq(x) = 0$$

where $f_B(x)$ is the objective function (e.g. the asymptotic variance of mean time to failure) and x is its decision variable which can be expressed as either a vector or a scalar, Lb and Ub are the corresponding lower and upper bounds of x. $C(x) \le 0$ and $Ceq(x) = 0$ are the possible inequality and equality constraints, respectively. .

The second step is to determine the equivalent test plan based on *Definitions* 1 and 2 using formulations (4.2) and (4.3), respectively. Formulation (4.2) is given as follows,

$$\text{Min } \Pi_i(y) \tag{7.2}$$

$$\text{s.t. } \left| f_B(x) - f_E(y) \right| \le \delta$$

$$\Pi_j(x) - \Pi_j(y) = 0$$

$$Lb' \le y \le Ub'$$

$$C'(y) \le 0$$

$$Ceq'(y) = 0$$

Where $f_B(x)$ and $f_E(y)$ are the base and equivalent objective functions respectively and y is the decision variable of the equivalent test plan, $\Pi(\cdot)$ represents the constraint of the total number of test units, expected number of failures or the test time. If $\Pi_j(y)$ is the total number of test units, $\Pi_i(y)$ can be the censoring time under Type-I censoring or expected number of failures under Type-II censoring and vice versa. The idea is to set the allowed *difference between objective values* as a constraint as well as seek other merits.

Similarly, based on *Definition* 2, the optimal equivalent test plan can be determined as,

$$\text{Min} \quad \Pi_i\left(y\right) \tag{7.3}$$

$$\text{s.t.} \;\; f_B\left(x\right) - f_E\left(y\right) = 0$$

$$\left|\Pi_j\left(x\right) - \Pi_j\left(y\right)\right| \le \delta$$

$$Lb' \le y \le Ub'$$

$$C'\left(y\right) \le 0$$

$$Ceq'\left(y\right) = 0$$

We now demonstrate these methods to determine an optimal equivalent step-stress test plan and an optimal equivalent ramp-stress test plan to the constant-stress test plan (baseline test plan).

4.4 Equivalent Test Plan Formulations

According to the definitions of equivalent ALT plans given in Sec. 4.3, we use the minimum asymptotic variance of quantile failure (e.g. q = 0.01) at normal operating condition as the objective for determining the optimal baseline test plan and equivalency of test plans. Without loss of generality, the stress is normalized using Eq. (3.9) to the range of $[0,1]$. In addition, the following assumptions are considered.

Assumptions

1. A single stress is used in the ALT plan.
2. The lifetimes of each test unit are statistically independent.
3. The failure time follows exponential distribution with a hazard rate function $h_0(t) = \lambda,\ \lambda > 0$.
4. The applied stress affects the lifetime of a test unit through PH model.

According to the proportional hazard assumption, the hazard function of the test units under test stress *z* is given by

$$h(t;z) = \lambda \exp(\beta z)$$

where β is the coefficient that reflects the effect of the stress.

Therefore we have reliability function under stress *z*

$$R(t;z) = \mathrm{e}^{-H(t;z)} = \exp\left[-\lambda t \exp(\beta z)\right] \tag{7.4}$$

and failure time distribution function

$$f(t;z) = \lambda \exp\left[\beta z - \lambda t \exp(\beta z)\right] \tag{7.5}$$

4.4.1 Optimal Baseline Constant-stress ALT Plan

The optimum baseline constant-stress ALT plan is designed under Type-I censoring with a predetermined censoring time τ. Three stress levels are used as shown in Figure 4.2. The high stress level is chosen to be the highest value $z_H = 1$. The medium level $z_M = \frac{(z_L + z_H)}{2}$ is the midway between the low level z_L and the high level z_H. The value of the low stress level is a decision variable. The allocation of test units to the low, medium and high stress levels follows the 4:2:1 rule. This unequal allocation is a compromise that extrapolates reasonably well and results in optimum design of test plans under constant-stress loading (Meeker and Hahn, 1985). The optimal test plan in terms of the low stress level $0 < z_L < 1$ is obtained such that the MLE of q = 0.01 quantile failure at the normal operating condition $z_D = 0$ is minimized. The total number of available test units is N_B. The expected number of failures at the low stress level is required to be greater than or equal to $N_B\, p_L$, where p_L is a fraction of the test units allocated to the low stress level.

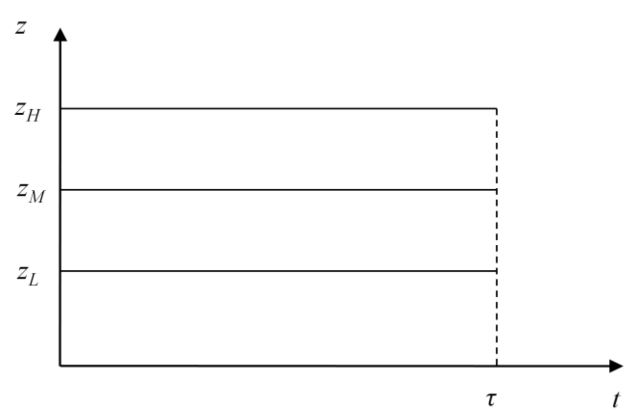

Figure 4.2 Constant-stress test

Fisher information matrix

Under Type-I censoring and the assumed failure time model under stress, the log likelihood function of an observation at stress level $z_k \left(k = L, M, H\right)$ is

$$L\left(\lambda, \beta; z_k\right) = I\left\{\ln\left(\lambda\right) + \beta z_k - \lambda t \exp\left(\beta z_k\right)\right\} - \left[1 - I\right]\lambda\tau\exp\left(\beta z_k\right) \tag{7.6}$$

where I is an indicator function defined by

$$I = \begin{cases} 1 & \text{if } t \le \tau \left(\text{failure}\right) \\ 0 & \text{otherwise} \end{cases}$$

By taking the second derivative of the log likelihood function with respect to the unknown parameters and taking the negative expectation, we can obtain the elements of the Fisher information matrix. Let F_k be the Fisher information matrix of observations corresponding to stress level $z_k \left(k = L, M, H\right)$ which is given by

$$F_k = N_B p_k \begin{bmatrix} E\left[-\dfrac{\partial L^2}{\partial \lambda^2}\right] & E\left[-\dfrac{\partial L^2}{\partial \lambda \partial \beta}\right] \\ E\left[-\dfrac{\partial L^2}{\partial \lambda \partial \beta}\right] & E\left[-\dfrac{\partial L^2}{\partial \beta^2}\right] \end{bmatrix}$$

where

$$E\left[-\frac{\partial L^2}{\partial \lambda^2}\right]=\frac{1}{\lambda^2}\left\{1-\exp\left[-\lambda\tau\exp\left(\beta z_k\right)\right]\right\}$$

$$E\left[-\frac{\partial L^2}{\partial \beta^2}\right]=z^2\left\{1-\exp\left[-\lambda\tau\exp\left(\beta z_k\right)\right]\right\}$$

$$E\left[-\frac{\partial L^2}{\partial \lambda \partial \beta}\right]=\frac{z}{\lambda}\left\{1-\exp\left[-\lambda\tau\exp\left(\beta \mathbf{z}_k\right)\right]\right\}$$

The total information matrix is given by $F_B = \sum_{k=L,M,H} F_k$.

Let $t_q\left(z_D\right)$ be the q^{th} quantile failure time at normal operating conditions z_D, then from Eq. (4.4) we solve

$$t_q\left(z_D\right)=\frac{\ln\left(1-q\right)}{-\lambda\exp\left(\beta z_D\right)}$$

The asymptotic variance of the MLE $\hat{t}_q\left(\hat{\lambda},\hat{\beta};z_D\right)$ at normal operating conditions z_D is given by

$$\text{Asvar}\left[\hat{t}_q\left(\hat{\lambda},\hat{\beta};z_D\right)\right]=\left[\frac{\partial\hat{t}_q\left(\hat{\lambda},\hat{\beta};z_D\right)}{\partial\lambda},\frac{\partial\hat{t}_q\left(\hat{\lambda},\hat{\beta};z_D\right)}{\partial\beta}\right]\times F_B^{-1}\times \left[\frac{\partial\hat{t}_q\left(\hat{\lambda},\hat{\beta};z_D\right)}{\partial\lambda},\ \frac{\partial\hat{t}_q\left(\hat{\lambda},\hat{\beta};z_D\right)}{\partial\beta}\right]^T \tag{7.7}$$

where

$$\frac{\partial\hat{t}_q\left(\hat{\lambda},\hat{\beta};z_D\right)}{\partial\lambda}=\frac{\ln\left(1-q\right)}{\hat{\lambda}^2\exp\left(\hat{\beta}z_D\right)} \tag{7.8}$$

and

$$\frac{\partial\hat{t}_q\left(\hat{\lambda},\hat{\beta};z_D\right)}{\partial\beta}=\frac{z_D\ln\left(1-q\right)}{\hat{\lambda}\exp\left(\hat{\beta}z_D\right)} \tag{7.9}$$

Optimal baseline test plan is obtained by solving the following optimization problem

$$\text{Min}\qquad f_B\left(x\right)=\text{Asvar}\left[\hat{t}_q\left(\hat{\lambda},\hat{\beta};z_D\right)\right] \tag{7.10}$$

$$\text{s.t.}\qquad 0<z_L<1$$

$$p_H=\frac{1}{7},\ \ p_M=\frac{2}{7},\ \ p_L=\frac{4}{7}$$

$$N_B p_L\left[1-R\left(\tau;z_L\right)\right]\geq \pi N_B p_L$$

where the decision variable is the low stress level $x=z_L$.

4.4.2 Equivalent Step-stress ALT Plan

Step-stress (shown in Figure 4.3) is often used in life testing in order to shorten the test duration. However, as the stress-level changes at a given time, the lifetime distribution under a step-stress needs to be related to that under a constant stress. This is accomplished using the cumulative exposure assumption described below.

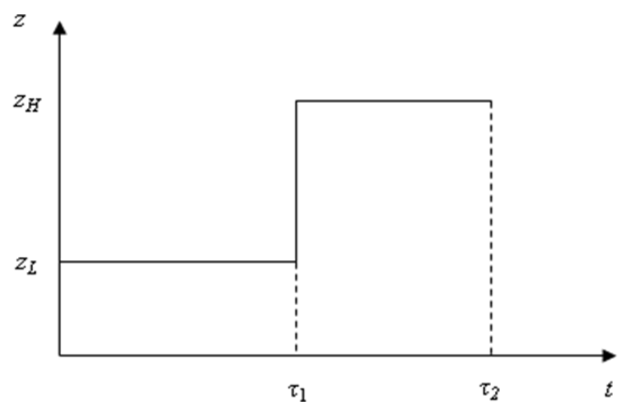

Figure 4.3 Simple step-stress test

4.4.2.1 Cumulative Exposure Assumption

According to the cumulative exposure assumption: 1) the remaining life of a test unit depends only on current cumulative fraction of damage and the current stress regardless how the fraction is accumulated. 2) If held at the current stress, survivors fail according to the cumulative distribution for that stress, but starting at the previously cumulative damage.

These assumptions form a joint cumulative damage function by horizontally shifting the individual cumulative damage function at the time that stress level changes. This can be explained by Figure 4.4. where $\Psi(t; z_s)$ denotes the *cdf* of damage time for units tested at constant-stress $z_s, s = L, H$.

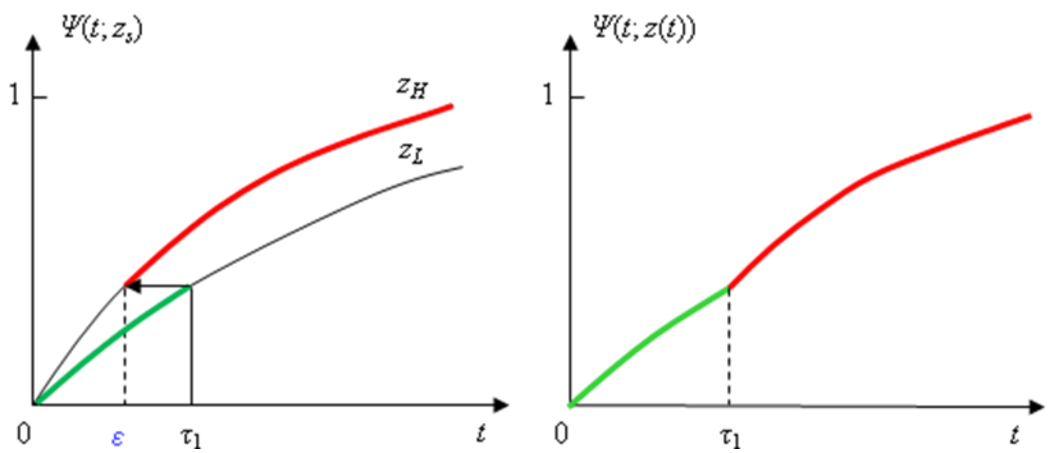

Figure 4.4 Cumulative exposure assumption

From Eq. (4.4) and Eq. (4.5) we have

$$\Psi(t; z_L) = 1 - \exp\left[-\lambda t \exp(\beta z_L)\right]$$

$$\Psi(t; z_H) = 1 - \exp\left[-\lambda t \exp(\beta z_H)\right]$$

Suppose from time zero the test runs under stress level z_L. At time τ_1 the stress level is increased to z_H. According to the cumulative exposure assumptions, at time τ_1, the test units have an equivalent starting time ε under stress level z_H which is the solution to the following equation

$$\Psi(\varepsilon; z_H) = \Psi(\tau_1; z_L) \tag{7.11}$$

That is,

$$\varepsilon = \tau_1 \exp\left[\beta_1 (x_L - x_H)\right]$$

Thereafter, the test units fail under stress level z_H following *cdf*

$$\Psi(t; z_H) = \Psi(t - \tau_1 + \varepsilon; z_H) = 1 - \exp\left[-\lambda(t - \tau_1 + \varepsilon)\exp(\beta z_H)\right], \; t \geq \tau_1$$

In summary, based on the cumulative exposure assumption the *cdf* of units subjected to simple step-stress as shown in Figure 4.3 is given by

$$\begin{aligned}\Psi(t, z(t)) &= \begin{cases}\Psi(t; z_L), & t < \tau_1 \\ \Psi(t - \tau_1 + \varepsilon; z_H), & t \geq \tau_1\end{cases} \\ &= \begin{cases}1 - \exp\left[-\lambda t \exp(\beta z_L)\right], & t < \tau_1 \\ 1 - \exp\left[-\lambda(t - \tau_1 + \varepsilon)\exp(\beta z_H)\right], & t \geq \tau_1\end{cases}\end{aligned} \tag{7.12}$$

The corresponding *pdf* is given by

$$f\left(t, z(t)\right) = \begin{cases} \lambda \exp\left[\beta z_L - \lambda t \exp\left(\beta z_L\right)\right] & t < \tau_1 \\ \lambda \exp\left[\beta z_H - \lambda\left(t - \tau_1 + \varepsilon\right)\exp\left(\beta z_H\right)\right] & t \geq \tau_1 \end{cases} \tag{7.13}$$

4.4.2.2 Fisher Information Matrix

Under simple step stress a test unit may either fail at stress level z_L before the stress changing time τ_1 or does not fail by time τ_1 and continues to run either to failure or to censoring time τ_2 at stress level z_H. Accordingly, we define following indicator functions to describe such a failure pattern:

$$I_1 = I_1(t \leq \tau_1) = \begin{cases} 1 & \text{if } t \leq \tau_1, \text{ failure observed before time } \tau_1 \\ 0 & \text{if } t > \tau_1, \text{otherwise} \end{cases}$$

$$I_2 = I_2(t \leq \tau_2) = \begin{cases} 1 & \text{if } t \leq \tau_2, \text{ failure observed before time } \tau_2 \\ 0 & \text{if } t > \tau_2, \text{ otherwise} \end{cases}$$

where $\tau_1 \leq \tau_2$.

Then the log likelihood of a single observation is given by

$$L(\lambda,\beta;z_L,z_H) = I_2\left\{I_1 \ln\left[f(t,z_L)\right] + (1-I_1)\ln\left[f(t,z_H)\right]\right\}$$

$$+(1-I_2)\ln\left[R(\tau_2-\tau_1+\varepsilon;z_H)\right]$$

$$= I_1 I_2\left[\ln(\lambda)+\beta z_L - \lambda t\exp(\beta z_L)\right]+$$

$$(1-I_1)I_2\left[\ln(\lambda)+\beta z_H-\lambda(t-\tau_1+\varepsilon)\exp(\beta z_H)\right]- \qquad (7.14)$$

$$(1-I_2)\lambda(\tau_2-\tau_1+\varepsilon)\exp(\beta x_H)$$

The Fisher information matrix of a single observation is given by

$$F_k = \begin{bmatrix} E\left[-\dfrac{\partial L^2}{\partial\lambda^2}\right] & E\left[-\dfrac{\partial L^2}{\partial\lambda\partial\beta}\right] \\ E\left[-\dfrac{\partial L^2}{\partial\lambda\partial\beta}\right] & E\left[-\dfrac{\partial L^2}{\partial\beta^2}\right] \end{bmatrix}$$

where the elements of the Fisher information matrix are the negative expectations of the second derivative of the log likelihood function with respect to unknown parameters,

$$E\left[-\frac{\partial L^2}{\partial\lambda^2}\right] = \frac{1}{\lambda^2}\left\{1-\exp\left[-\lambda(\tau_2-\tau_1+\varepsilon)\exp(\beta z_H)\right]\right\}$$

$$E\left[-\frac{\partial L^2}{\partial\beta^2}\right] = z_L^2 + \left(z_H^2 - z_L^2\right)\exp\left[-\lambda\tau_1\exp(\beta z_L)\right]-$$

$$x_H^2\exp\left[-\lambda(\tau_2-\tau_1+\varepsilon)\exp(\beta z_H)\right]$$

$$E\left[-\frac{\partial L^2}{\partial\lambda\partial\beta}\right]=\frac{z_L}{\lambda}\left\{1-\exp\left[-\lambda\tau_1\exp\left(\beta z_L\right)\right]\right\}+z_H\tau_1\exp\left(\beta z_L-\lambda\tau_1\exp\left(\beta z_L\right)\right)+$$

$$\frac{z_H\left\{\exp\left[-\lambda\tau_1\exp\left(\beta z_L\right)\right]-\exp\left[-\lambda\left(\tau_2-\tau_1+\varepsilon\right)\exp\left(\beta z_H\right)\right]\right\}}{\lambda}$$

The total information matrix is given by $F_s=\sum_{k=1}^{N_s}F_k$, where *Ns* is the total number of the test units.

4.4.2.3 Optimal Equivalent Step-stress Test Plan

According to *Definition* 1 and the approach for determining optimal equivalent test plan, we present two formulations for determining the optimal equivalent test plan under step-stress.

Formulation 1

The objective is to minimize the censoring time τ_2 under step-stress test using the same number of test units as that of the baseline test plan.

$$\text{Min} \quad \tau_2\left(y\right) \tag{7.15}$$

$$\text{s.t.} \quad \left|f_B\left(x\right)-f_s(y)\right|\le\delta$$

$$N_s-N_B=0$$

$$0<z_L<1,\ z_H=1,\ \tau_1<\tau_2$$

$$N_s\Psi\left(\tau_1;z_L\right)\geq \pi N_s$$

where $f_s(y)=\text{Asvar}\left[\widehat{t}_q\left(\widehat{\lambda},\widehat{\beta};z_D\right)\right]$, the decision variables are the low stress level z_L and time to change the stress level τ_1 represented by $y=\begin{bmatrix} z_L \\ \tau_1 \end{bmatrix}$. In all the following formulations for equivalent step-stress test plans the decision variables are expressed as $y=\begin{bmatrix} z_L \\ \tau_1 \end{bmatrix}$.

In formulation (4.15), the constraint $\left|f_B\left(x\right)-f_s(y)\right|\leq\delta$ maintains that the absolute difference between the values of the objective functions is less than or equal to $\delta\left(\delta\geq 0\right)$. The constraint $N_s-N_B=0$ ensures that the total number of test units under step-stress equals that of the baseline test. Likewise, the constraint $N_s\Psi\left(\tau_1;z_L\right)\geq\pi N_s$ ensures minimum expected number of failures at the low stress level under step-stress is greater than or equal to a fraction of the total test units. Such an optimal equivalent test plan intends to reduce the test time as well as obtain equivalent accuracy of reliability prediction as that of the constant-stress test plan.

Formulation 2

The objective is to minimize the total number of test units under step-stress test using the same censoring time as that of the baseline test.

$$\text{Min} \quad N_s\left(y\right) \tag{7.16}$$

$$\text{s.t.} \quad \left|f_B\left(x\right) - f_s(y)\right| \leq \delta$$

$$\tau_2 - \tau = 0\,, \tau_1 < \tau_2$$

$$0 < z_L < 1\,,\ z_H = 1$$

$$N_s \Psi\left(\tau_1; z_L\right) \geq \pi N_s$$

where τ is the censoring time of the baseline test plan.

According to *Definition* 2 we propose two formulations for determining optimal equivalent step-stress test plan as follows.

Formulation 1

$$\text{Min} \ \ \tau_2\left(y\right) \tag{7.17}$$

$$\text{s.t.} \ \ f_B\left(x\right) - f_s(y) = 0$$

$$\left|N_s - N_B\right| \leq \delta$$

$$0 < z_L < 1\,,\ z_H = 1, \tau_1 < \tau_2$$

$$N_s \Psi\left(\tau_1; z_L\right) \geq \pi N_s$$

Formulation 2

$$\text{Min} \ \ N_s\left(y\right) \tag{7.18}$$

$$\text{s.t.} \ \ f_B\left(x\right) - f_s(y) = 0$$

$$|\tau_s - \tau| \leq \delta$$

$$0 < z_L < 1,\ z_H = 1, \tau_1 < \tau_2$$

$$N_s \Psi(\tau_1; z_L) \geq \pi N_s$$

4.4.3 Equivalent Ramp-stress ALT Plan

Ramp-stress as shown in Figure 4.5 is a type of the stress loadings that can further reduce the test time than step-stress. However, the life-stress relationship is difficult to model, if not impossible. To obtain optimal equivalent ramp-stress test plan, generalized PH models for the lifetime of a test unit under time-varying stresses are developed in the next section.

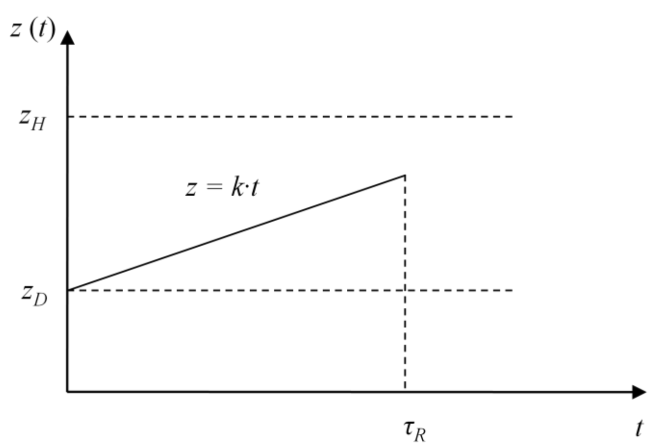

Figure 4.5 Ramp-stress loading (*k* is the rate of increase in stress per unit time)

4.4.3.1 Generalized PH Model

A ramp-stress can be approximated by a step-stress as shown in Figure 4.6.

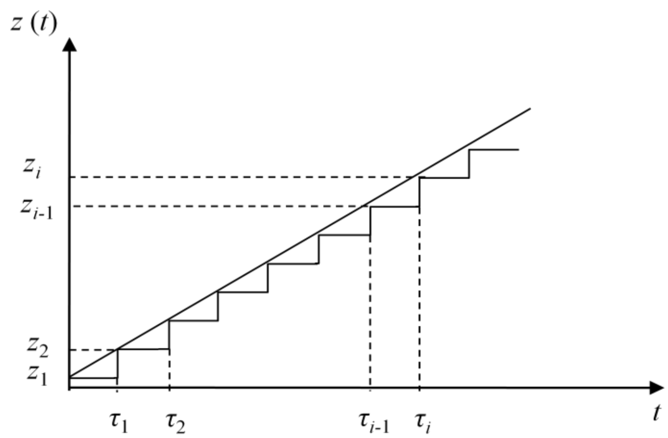

Figure 4.6 Approximation of ramp-stress using step-stress

Under step-stress, the cumulative exposure assumption can be applied to relate the cumulative failures under different stress levels as discussed in Sec.4.4.2.1. Since cumulative hazard rate is monotonically dependent on the cumulative failure, the cumulative exposure assumption then applies to the cumulative hazard rate.

Let $H(t; z_s)$ represent the cumulative hazard function at stress level z_s, $H_0(t)$ and $h_0(t)$ be the baseline cumulative hazard function and hazard function, respectively. Assume that the step-stress is applied at time zero as shown in Figure 4.6. Based on the cumulative exposure assumption, at the first stress-level changing time τ_1, we have

$$H(\tau_1; z_1) = H(\tau_1 + \Delta t_1; z_1 + \Delta z_1) = H(\tau_1 + \Delta t_1; z_2)$$

where $\Delta z_1 = z_2 - z_1$, Δt_1 is the time shift if from time zero the stress-level z_2 is applied.

At time τ_{i-1}, the stress level is increased from z_{i-1} to z_i, so we have

$$H\left(\tau_{i-1}; z_{i-1}\right) = H\left(\tau_{i-1} + \Delta t_1 + \ldots + \Delta t_{i-1}; z_{i-1} + \Delta z_{i-1}\right) = H\left(\tau_{i-1} + \Delta t_1 + \ldots + \Delta t_{i-1}; z_i\right) \qquad (7.19)$$

Using Taylor expansion of Eq. (4.19) we have

$$\left.\frac{\partial H(\zeta;\xi)}{\partial t}\right|_{\substack{\zeta=\sum_{j=1}^{i-1}\Delta t_j \\ \xi=z_{i-1}}} \cdot \Delta t_{i-1} + \left.\frac{\partial H(\zeta;\xi)}{\partial z}\right|_{\substack{\zeta=\sum_{j=1}^{i-1}\Delta t_j \\ \xi=z_{i-1}}} \cdot \Delta z_{i-1} = 0$$

Using the PH assumption we obtain

$$\frac{\Delta t_{i-1}}{\Delta z_{i-1}} = \left.\frac{\frac{\partial H(\zeta;\xi)}{\partial t}}{\frac{\partial H(\zeta;\xi)}{\partial z}}\right|_{\substack{\zeta=\sum_{j=1}^{i-1}\Delta t_j \\ \xi=z_{i-1}}} = -\frac{H_0\left(t+\sum_{j=1}^{i-1}\Delta t_j\right)\beta\exp(\beta z)}{h_0\left(t+\sum_{j=1}^{i-1}\Delta t_j\right)\exp(\beta z)} = -\beta\frac{H_0\left(t+\sum_{j=1}^{i-1}\Delta t_j\right)}{h_0\left(t+\sum_{j=1}^{i-1}\Delta t_j\right)}$$

Thus we have

$$\Delta t_{i-1} = -\beta\Delta z_{i-1}\frac{H_0\left(t+\sum_{j=1}^{i-1}\Delta t_j\right)}{h_0\left(t+\sum_{j=1}^{i-1}\Delta t_j\right)}$$

If stress $z(t)$ is differentiable, then the total time shift is

$$t^{*}(t)=\sum_{j=1}^{i-1}\Delta t_{j}=\int_{0}^{t}d\tau=-\int_{0}^{t}\beta z^{'}(\tau)\frac{H_{0}\left(\tau+t^{*}(\tau)\right)}{h_{0}\left(\tau+t^{*}(\tau)\right)}d\tau$$

Hence, the cumulative hazard function under time-varying stress $z(t)$ is given by

$$H\left(t,z(t)\right)=H_{0}\left(t+t^{*}(t)\right)\exp\left(\beta z(t)\right) \tag{7.20}$$

Under ramp-stress,

$$t^{*}(t)=-\int_{0}^{t}\beta k\frac{H_{0}\left(\tau+t^{*}(\tau)\right)}{h_{0}\left(\left(\tau+t^{*}(\tau)\right)\right)}d\tau \tag{7.21}$$

where k is the normalized ramp-rate.

If stress $z(t)$ is differentiable until time T where there is a jump Δz, then

$$H\left(T,z(T)\right)=H_{0}\left(T+t_{T}^{*}+\Delta t_{T}\right)\exp\beta\left[\left(z(T)+\Delta z\right)\right]$$

where

$$\Delta t_{T}=H_{0}^{-1}\left[H_{0}\left(T+t_{T}^{*}\right)\exp\left(-\beta\Delta z\right)-T-t_{T}^{*}\right]$$

If there are g jumps associated with $z(t)$ at times $T_{m}\left(m=1,\ldots,g\right)$, then

$$H\left(t, z(t)\right) = H_0\left(t + t_T^* + \sum_{m=1}^{g} \Delta t_{T_m}\right)\exp\left[\beta z(t)\right]$$

and

$$\Delta t_{T_m} = H_0^{-1}\left[H_0\left(T_m + t_{T_m}^*\right)\exp(-\beta\Delta z) - T_m - t_{T_m}^*\right]$$

Let the applied ramp-stress $z(t)$ be

$$z(t) = kt,\ \ k \geq 0 \tag{7.22}$$

where k is the normalized ramp-rate such that $0 = z_D \leq z(t) \leq z_H = 1$.

Then given the baseline hazard function $h_0(t) = \lambda,\ \lambda > 0$, using Eq. (4.21), we obtain

$$t^*(t) = \frac{1}{\beta k}\left[1 - \exp(-\beta kt)\right] - t$$

Therefore the cumulative hazard function under ramp-stress is

$$H\left[t, z(t)\right] = \frac{\lambda}{\beta k}\left[\exp(\beta kt) - 1\right] \tag{7.23}$$

From the cumulative hazard function Eq. (4.23), we obtain the cumulative failure function

$$\Psi\left[t, z(t)\right] = 1 - \exp\left\{\frac{\lambda}{\beta k}\left[1 - \exp(\beta k t)\right]\right\} \tag{7.24}$$

and the failure density function

$$f\left[t, z(t)\right] = \lambda \exp\left\{\beta k t + \frac{\lambda\left[1 - \exp(\beta k t)\right]}{\beta k}\right\} \tag{7.25}$$

4.4.3.2 The Fisher Information Matrix

The log likelihood function for an observation under a ramp-stress is given by

$$L\left(\lambda, \beta; z(t)\right) = I\left\{\ln(\lambda) + \beta k t + \frac{\lambda\left[1 - \exp(\beta k t)\right]}{\beta k}\right\} + (1 - I)\frac{\lambda}{\beta k}\left[1 - \exp(\beta k \tau_R)\right] \tag{7.26}$$

where

$$I = \begin{cases} 1 & \text{if } t \le \tau_R \ (\text{failure}) \\ 0 & \text{otherwise} \end{cases}$$

and τ_R is the censoring time of the ramp-test.

The Fisher information matrix of MLE corresponding to a single observation is given by

$$F_k = \begin{bmatrix} E\left[-\dfrac{\partial L^2}{\partial \lambda^2}\right] & E\left[-\dfrac{\partial L^2}{\partial \lambda \partial \beta}\right] \\ E\left[-\dfrac{\partial L^2}{\partial \lambda \partial \beta}\right] & E\left[-\dfrac{\partial L^2}{\partial \beta^2}\right] \end{bmatrix}$$

where the elements of the Fisher information matrix are the negative expectation of the second derivative of the log likelihood function with respect to unknown parameters.

$$E\left[-\frac{\partial L^2}{\partial \lambda^2}\right] = \frac{1}{\lambda^2}\left\{1-\exp\left[\frac{\lambda}{\beta k}\left(1-\exp\left(\beta k \tau\right)\right)\right]\right\}$$

$$E\left[-\frac{\partial L^2}{\partial \lambda \partial \beta}\right] = \int_0^{\tau} I\left\{\frac{1-\exp\left(\beta k t\right)}{\beta^2 k} + \frac{t\exp\left(\beta k t\right)}{\beta}\right\} dt +$$

$$\exp\left\{\frac{\lambda}{\beta k}\left[1-\exp\left(\beta k \tau\right)\right]\right\}\frac{\left[1-\exp\left(\beta k \tau\right)\right]+\beta k \tau \exp\left(\beta k \tau\right)}{\beta^2 k}$$

$$E\left[-\frac{\partial L^2}{\partial \beta^2}\right] = \int_0^{\tau} I\left\{\frac{\lambda k t^2 \exp\left(\beta k t\right)}{\beta} - \frac{2\lambda t \exp\left(\beta k t\right)}{\beta^2} - \frac{2\lambda\left[1-\exp\left(\beta k t\right)\right]}{\beta^3 k}\right\} dt +$$

$$\exp\left\{\frac{\lambda}{\beta k}\left[1-\exp\left(\beta k \tau\right)\right]\right\}\left\{\frac{\lambda k \tau^2 \exp\left(\beta k \tau\right)}{\beta} - \frac{2\lambda \tau \exp\left(\beta k \tau\right)}{\beta^2} - \frac{2\lambda\left[1-\exp\left(\beta k \tau\right)\right]}{\beta^3 k}\right\}$$

Suppose N_R units are available for the ramp-stress test. The total information matrix is given by $F_R = \sum_{k=1}^{N_R} F_k$.

4.4.3.3 Optimal Equivalent Ramp-stress Test Plan

We propose two formulations for equivalent ramp-stress test plan based on *Definition* 1.

Formulation 1

The objective is to minimize the censoring time τ_R under equivalent ramp-stress using the same number of test units as that of the baseline test.

$$\begin{aligned} &\text{Min } \tau_R(y) \\ &\text{s.t. } \left| f_B(x) - f_R(y) \right| \le \delta \\ &N_R - N = 0 \\ &k \le k_U,\ k\tau_R \le 1 \\ &N_R \Psi\left(\tau_R; z\left(\tau_R\right)\right) \ge \pi N_R \end{aligned} \tag{7.27}$$

where $f_R(y) = \text{Asvar}\left[\hat{t}_q\left(\hat{\lambda}, \hat{\beta}; z_D\right)\right]$, the decision variable is the normalized ramp-rate $y = k$. In the following formulations for equivalent ramp-stress test plan, all $y = k$.

The constraint $\left|f_B\left(x\right)-f_R(y)\right|\le\delta$ ensures that the absolute difference between the values of the objective functions is within $\delta\left(\delta\ge 0\right)$. The constraint $N_s-N_B=0$ ensures that the total number of test units equals that of the baseline test. The constraint $k\,\tau_R\le 1$ ensures that the highest test stress is not greater than or equal to the maximum allowed stress level z_H whose normalized value is 1. $N_R\Psi\left(\tau_R;z\left(\tau_R\right)\right)\ge\pi N_R$ ensures minimum expected number of failures under ramp-stress is greater than or equal to π fraction of the total test units. In addition, we specify an upper bound for the ramp-rate $k\le k_U$ in order to avoid different failure modes other than those occur at design stress.

Formulation 2 is similar to Formulation 1, but the objective is the minimum number of total test units $N_R\left(y\right)$ and the constraint of $N_R-N=0$ is replaced by $\tau_R-\tau=0$.

According to *Definition* 2 we propose two formulations for determining optimal equivalent ramp-stress test plans.

Formulation 1

$$\text{Min } \tau_R\left(y\right) \tag{7.28}$$

$$\text{s.t. } f_B\left(x\right)-f_R(y)=0$$

$$\left|N_R-N_B\right|\le\delta$$

$$k\le k_U,\ k\tau_R\le 1$$

$$N_R\Psi\left(\tau_R;z\left(\tau_R\right)\right)\ge\pi N_R$$

Formulation 2 is also similar to Formulation 1, but the objective is the minimum number of total test units $N_R\left(y\right)$ and the constraint of $\left|N_R - N_B\right| \le \delta$ is replaced by $\left|\tau_s - \tau\right| \le \delta$.

In the next section, we present examples to obtain equivalent test plans under single step-stress and ramp-stress.

4.5 Numerical Examples

Suppose the baseline accelerated life testing is to be carried out at three constant-voltage levels for MOS devices in order to estimate its 1% quantile failure at normal operating conditions z_D = 2V. The test needs to be completed in 300 hours. The total number of units available for testing is 200. To avoid the inducing of failure modes different from the expected at the design stress level, it has been determined, through engineering judgment, that the highest voltage level should not exceed z_H = 5V. The stress level is normalized to the range of [0, 1] using Eq. (3.9). The required minimum number of failures for the low stress level is 30% of test units allocated to that level.

Some experiments are conducted to obtain a set of initial values of the parameters for the PH model. These values are then normalized as $\widehat{\lambda} = 0.0015, \widehat{\beta} = 6.2$. The decision variable is the low stress level z_L. The optimal value of the decision variable is determined by solving the nonlinear optimization problem with nonlinear constrains as well as linear and boundary constraints, formulation (4.10). We use Matlab nonlinear constraint solver, *fmincon*, to solve this optimization problem. The optimum normalized low stress level is 0.1139 which is equivalent to z_L = 2.3417V. The corresponding asymptotic variance of 1% failure time at design stress is 0.8082, as shown in Table 4.1.

A simple step-voltage test, as shown in Figure 4.3, is conducted for the same MOS devices, using the same number of test units and censoring time (τ_2) as is used in the constant-voltage test. We investigate optimal test plan with respect to the same objective function as that of the constant-voltage test. The decision variables are the low voltage level z_L and voltage changing time τ_1. The optimum values of the decision variables are determined from following formulation.

$$\text{Min} \qquad f_s(x) = \text{Asvar}\left[\hat{t}_{.01}\left(\hat{\lambda},\hat{\beta};z_D=0\right)\right]$$

$$\text{s.t.} \qquad N_s = 200$$

$$\tau_2 = 300\,,\tau_1 < \tau_2$$

$$0 < z_L < 1,\ z_H = 1$$

$$N_s \Psi\left(\tau_1; z_L\right) \geq 0.1 N_s$$

where π = 0.1 is the required minimum fraction of failures of the test units before the stress level is increased. The obtained results are given in Table 4.1.

A ramp-voltage test as shown in Figure 4.5 is also conducted. The optimum ramp-rate is determined from following optimization problem.

$$\text{Min} \quad f_R(x) = \text{Asvar}\left[\hat{t}_{.01}\left(\hat{\lambda}, \hat{\beta}; z_D = 0\right)\right]$$

$$\text{s.t.} \quad N_R = 200$$

$$\tau_R = 300$$

$$k \le 0.01, \; 0 < k\tau_R \le z_H = 1$$

$$N_R \Psi(\tau_1; z_L) \ge 0.1 N_R$$

The optimum solution for above ramp-voltage test is presented in Table 4.1.

From Table 4.1 we observe that for the same objective function, initial estimate of unknown model parameters, censoring time and total number of test units, the ramp-test achieves significantly smaller asymptotic variance of the quantile failure prediction at normal conditions than that of the step-stress test, and the step-stress test achieves significantly smaller asymptotic variance of quantile failure prediction at normal conditions than that of the constant-stress test. This provides the possibility to investigate equivalent step-voltage and ramp-voltage test plans that achieve the same objective values as that of constant-voltage test but using less test duration or number of test units.

Since constant-stress tests are the most commonly conducted accelerated life tests in industry and their statistical inference has been extensively investigated, we set the constant-voltage test plan as the baseline. The first objective is to minimize the test duration under step-voltage and ramp-voltage tests while achieving an equivalent objective function values to that of the baseline. The efficiency of equivalent plan is measured by the percentage of reduction in the test time.

Table 4.1 Optimal test plans

$\hat{\lambda} = 0.0015$ $\hat{\beta} = 6.2$ $\tau = \tau_2 = \tau_R = 300$ hrs $N_B = N_s = N_R = 200$		
Test	Min Asvar$\left[\hat{t}_{.01}\left(\hat{\lambda}, \hat{\beta}; z_D\right)\right]$	Optimal decision value
Constant-voltage	0.8082	$Z_L = 0.1139$ (2.3417V)
Step-voltage	0.4826	$Z_L = 0.1472$ (2.4416V), $\tau_1 = 295$ hrs
Ramp-voltage	0.2245	$K = 0.0033$ (0.0099V/ hr)

To obtain the equivalent test plans for minimum censoring time we follow the formulations (4.15) and (4.27) for equivalent step-voltage test plan and ramp-voltage test plan, respectively. The allowed absolute difference between the objective function is less or equal to 0.01, i.e. $\delta = 0.01 = 1\%$. We use the same number of test units to that of the baseline test plan ($N_S = N_R = N_B = 200$). We set the upper bound of the ramp-rate as 0.01V/hr. Then by evaluation of the formulations (4.15) and (4.27) using nonlinear constraint solver, *fmincon*, built in Matlab, we obtain equivalent test plans parameters as shown in Table 4.2.

We observe that the step-voltage test significantly reduces the test time. The time reduction relative to the baseline test plan is 63.33% while the difference between the objective functions is less than 1%. The ramp-voltage test plan further reduces the test time. The time reduction relative to the baseline test plan is 85.6% while the difference between the objective functions is less than 0.5%. . This shows that it is feasible to design equivalent and yet efficient ALT plans having the same accuracy of reliability prediction.

Table 4.2 Equivalent test plans (minimum censoring time)

Test plan parameters	**Baseline constant-voltage test**	**Equivalent step-voltage test**	**Equivalent ramp-voltage test**
Obj. values	0.8082	0.8012 ($\delta = 0.0087 = 0.87\%$)	0.8044 $\delta = 0.0047 = 0.47\%$
Test time (hrs)	300	110	43.2
Test time reduction	--	63.33%	85.6%
Total number of units	200	200	200

When the cost per unit is high, it is extremely important to reduce the number of test units used in accelerated life testing. Therefore, the second objective is to minimize the total number of test units under step-voltage test and ramp-voltage test while achieving equivalent objective function values to that of the baseline test. The efficiency of equivalent plan is measured by the percentage of reduction in the number of test units.

To obtain the equivalent test plans for minimum number of test units we follow and evaluate the formulation (4.16) for step-voltage test plan. Similarly, we follow and evaluate formulation (4.27) with the objective function replaced by the minimum number of total test units $N_R(y)$ and the constraint of $N_R - N = 0$ replaced by $\tau_R - \tau = 0$ for the equivalent ramp-voltage test plan. The achieved equivalent test plan parameters are presented in Table 4.3.

We observe that the step-voltage test again significantly reduces the required number of test units. The reduction relative to the baseline test plan is 40.5% while the difference between the objective functions is less than 0.5%. The ramp-voltage test plan also further reduces the required number of test units. The reduction relative to the baseline test plan is 72% while the difference between the objective functions is less than 1%. This confirms that we can design equivalent and yet economical ALT plans having the same accuracy of reliability prediction.

Table 4.3 Equivalent test plans (minimum number of test units)

Test plan parameters	**Baseline constant-voltage test**	**Step-voltage test**	**Ramp-voltage test**
Obj. values	0.8082	0.8111 $\delta = 0.0036 = 0.36\%$	0.8017 $\delta = 0.0086 = 0.86\%$
Censoring time (hrs)	300	300	300
Total number of units	200	119	56
Number of test units reduction	--	40.5%	72%

4.6 Summary

In this chapter, we investigate the equivalency of ALT plans involving different stress loadings. We propose four definitions of equivalency in order to design equivalent ALT plan. Based on the definitions 1 and 2, we determine optimal equivalent ALT plans under the step-stress and the ramp-stress to the baseline constant-stress ALT plan. The objective is to shorten the test duration or reduce the number of test units without any significant errors in reliability predictions. Numerical examples demonstrate the feasibility of such equivalent ALT plans under different stress loadings. This has significant practical and economical impacts as it enables reliability practitioners to choose the appropriate ALT plan to accommodate restrictions of resource and duration of the test.

CHAPTER 5

EXPERIMENTAL VALIDATION

The objective of this chapter is to validate the general PH model for time-varying stress developed in Chapter 4 and the equivalent ALT plans by conducting accelerated life testing experiments in the Quality and Reliability Engineering Laboratory of the Industrial and Systems Engineering Department.

5.1 Experimental Samples

Each experimental set has a board that contains up to 32 miniature light bulbs as shown in Figure 5.1. The set is placed in a temperature and humidity chamber where humidity is held constant. The design working conditions of this light bulb are:

voltage: 2 Volts,

current: 0.06 amps.

The light bulbs may fail due to one of four modes: breakage of the glass bulb, sealing failure, thermal shock of the bulb filament and long term failure of the filament. The most common failure mode of the light bulbs is thermal shock. When the switch is turned on, full current suddenly flows to the filament at the speed of light. This sudden, massive vibration causes the filament to wildly bounce causing fatigue behavior of the filament which results in breakage of the filament. Long Term Failure occurs when the filament eventually becomes so fatigued that its electrical resistance increases to the point that current will not flow. We study and monitor the long term failure in this dissertation.

Figure 5.1 Samples of the miniature light bulbs (Zhang, 2006)

5.2 Experiments Setup

In order to continuously monitor the failure times of test units and to control the applied stresses, an automatic accelerated life testing environment is designed as shown in Figure 5.2. LabJack U3 is a connector block which interfaces directly to personal computer (PC) via USB. It retrieves the information of the current status of the test units and the testing environment. The SCB-68 is a shielded I/O connector block with 68 screw terminals for easy signal connection to LabJack U3. The SCB-68 features a general breadboard area for custom circuitry and sockets for interchanging electrical components. Each light bulb is connected in series with a resistor and the bulb-resistor sets are in parallel. To monitor the status of the light bulbs, Voltage across the resistor is measured.

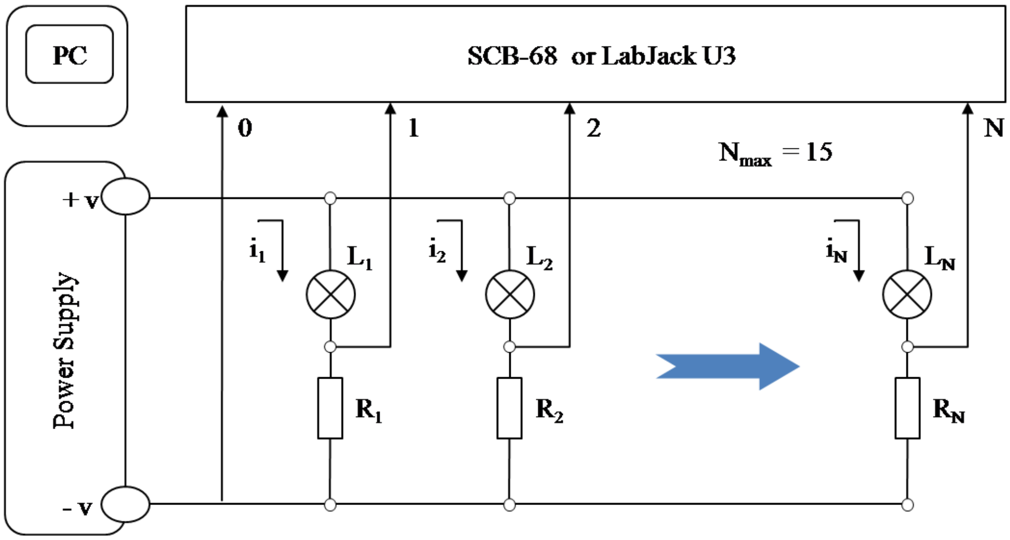

Figure 5.2 The layout of the accelerated life testing equipment

Figure 5.3 shows the programmable power supply, which is used to provide different voltage loadings. Under step-stress environment the BK Precision LabView GUI (Figure 5.4) controls the voltage of the power supply which increases by the step over the time interval. Under the ramp-stress environment, the power supply approximates the step-voltage to be close to a linear relationship.

Figure 5.3 The programmable power supply

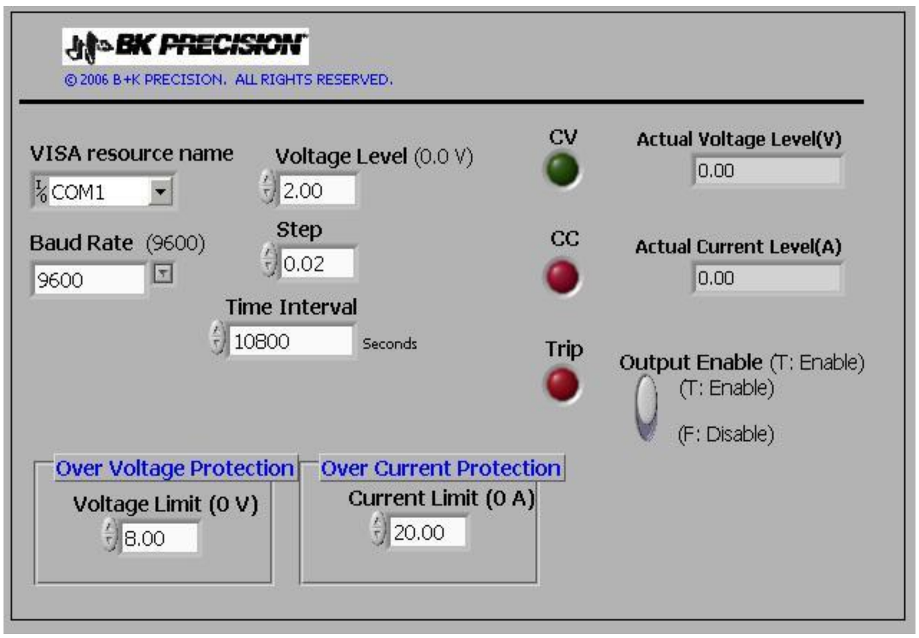

Figure 5.4 The LabView power supply control interface

National Instruments LabView software is used to develop the application for the continuous monitoring of the status of the test units. The failure time data are automatically saved as a spreadsheet file by LabView application. Figure 5.5 shows the graphical user interface of the programmed LabView application. It shows the start date and time of the record, and the current status of the units being monitored.

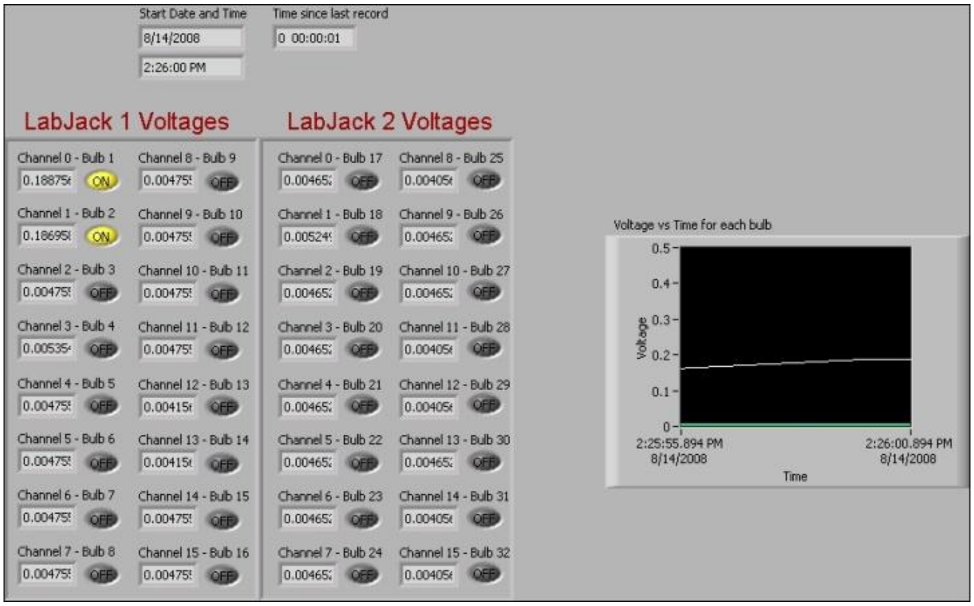

Figure 5.5 The LabView data collection interface

5.3 Test Conditions

To validate the general PH model for time-varying stresses, experiments on the light bulbs are conducted under two different voltage ramp-rates, 0.015V/hr and 0.01V/hr, starting from 2V at each condition. The working status of the light bulbs is continuously monitored and recorded by the LabView application.

To validate the equivalency of test plans under different stress loadings, experiments under constant-voltage, step-voltage and ramp-voltage are conducted. Under constant-voltage, we apply 2.2V and 2.46V for 160.13hr (47 samples) and 130.47hr (22 samples), respectively. Under step-voltage, we apply 2.25V on 64 samples for 96hr and then increase the voltage to 2.44V. We stop the step-voltage test at 140hr. As for the ramp-voltage test, we use the data from previous experiment under the condition: 0.015V/hr starting from 2V.

5.4 Analysis of the Experimental Results

The test begins with up to 32 miniature light bulbs being tested at each of the specified experimental conditions as stated in Sec. 5.3. Table 5.1 presents the lifetime (in hours) data from the ramp-voltage experiment. Each lifetime has a status indicator under the column labeled “Failure”. A value of 1 when failure occurs, otherwise the lifetime is a censoring time.

We maximize the likelihood function (4.26) and estimate the normalized model parameters $\hat{\tilde{\lambda}}$ and $\hat{\tilde{\beta}}$. The estimates under two different ramp-rates are close. The absolute difference is 3.24% and 6.24% for $\hat{\tilde{\lambda}}$ and $\hat{\tilde{\beta}}$, respectively. Then we calculate the estimation of the model parameters $\hat{\lambda}$ and $\hat{\beta}$ based on following normalization relationship:

$$\tilde{k} = \frac{k}{z_H - z_D} \tag{5.1}$$

$$\hat{\tilde{\beta}} = \hat{\beta}\left(z_H - z_D\right) = \frac{\hat{\beta}k}{\tilde{k}} \tag{5.2}$$

$$\hat{\hat{\lambda}} = \hat{\lambda}\exp\left(\hat{\beta} z_D\right) \tag{5.3}$$

where k is the stress level (in volts) increasing rate, $z_H = 3.5V$ and $z_D = 2V$ is the highest testing voltage and designed normal operating voltage, respectively.

Table 5.2 shows that the estimates of $\hat{\lambda}$ are significantly different, but $\hat{\beta}$ are close. This implies that $\hat{\lambda}$ might be a sensitive parameter. To obtain a better estimate, more samples are needed or more sets of experiment need be conducted. We also predict the mean time to failure (MTTF) under $z_D = 2V$. The prediction from ramp tests under two different ramp-rates is close. The difference is 3.32%. In general, the result validates the effectiveness of the general PH model.

The lifetime data from constant-voltage test and step-voltage tests are given in Table 5.3 and Table 5.4, respectively. To estimate the normalized model parameters $\hat{\hat{\lambda}}$ and $\hat{\hat{\beta}}$, we maximize the likelihood function (4.6) and (4.16) for the constant-voltage and step-voltage, respectively. We calculate the estimation of $\hat{\lambda}$ and $\hat{\beta}$ using Eqs. (5.2) and (5.3). Similarly, we also predict the MTTF under $z_D = 2V$ for tests under both the constant-voltage and the step-voltage. The results are presented in Tables 5.5 to 5.7.

Table 5.1 Lifetime data from ramp-voltage tests

Ramp-rate 1 (2V+0.015V/hr)						Ramp-rate 2 (2V+0.01V/hr)					
No.	Time	Failure	No.	Time	Failure	No.	Time	Failure	No.	Time	Failure
1	19.03	1	32	11.83	1	1	13.57	1	32	14.51	1
2	23.28	1	33	14.50	1	2	19.92	1	33	15.61	1
3	23.50	1	34	14.83	1	3	23.3	1	34	15.85	1
4	26.50	1	35	17.73	1	4	27.81	1	35	17.73	1
5	27.42	1	36	19.35	1	5	31.16	1	36	19.65	1
6	28.32	1	37	25.50	1	6	31.56	1	37	21.05	1
7	28.62	1	38	26.15	1	7	34.00	1	38	21.20	1
8	30.62	1	39	27.45	1	8	46.26	1	39	24.21	1
9	34.42	1	40	27.61	1	9	46.41	1	40	24.85	1
10	35.30	1	41	28.05	1	10	50.60	1	41	31.18	1
11	35.48	1	42	30.96	1	11	56.76	1	42	35.08	1
12	38.30	1	43	31.00	1	12	56.85	1	43	42.06	1
13	40.52	1	44	34.81	1	13	60.13	1	44	47.88	1
14	43.83	1	45	36.03	1	14	65.00	1	45	54.21	1
15	43.00	1	46	43.08	1	15	65.86	1	46	54.55	1
16	43.00	1	47	45.63	1	16	66.20	1	47	55.85	1
17	43.12	1	48	46.03	1	17	66.40	1	48	56.43	1
18	44.43	1	49	46.33	1	18	66.80	1	49	58.86	1
19	45.32	1	50	49.26	1	19	66.93	1	50	60.60	1
20	47.58	1	51	49.86	1	20	68.25	1	51	62.48	1
21	47.65	1	52	50.66	1	21	70.23	1	52	62.81	1
22	49.65	1	53	50.93	1	22	72.33	1	53	63.41	1
23	51.42	1	54	51.03	1	23	72.60	1	54	63.76	1
24	51.27	1	55	51.73	1	24	75.43	1	55	64.18	1
25	53.25	1	56	51.95	1	25	75.85	1	56	66.15	1
26	54.25	1	57	52.36	1	26	76.20	1	57	66.41	1
27	55.47	1	58	54.78	1	27	77.78	1	58	69.91	1
28	56.83	1	59	55.58	1	28	79.13	1	59	71.73	1
29	56.17	1	60	55.83	1	29	80.65	1	60	72.46	1
30	8.85	1	61	57.13	1	30	82.65	1	61	73.78	1
31	11.31	1				31	90.33	1	62	78.91	1

Table 5.2 Parameter estimation from ramp-voltage tests

Ramp-rates (k)	$\hat{\tilde{\lambda}}$	$\hat{\tilde{\beta}}$	$\hat{\lambda}$	$\hat{\beta}$	MTTF
2V+0.015V/hr ($\tilde{k} = 0.01$)	1.85e-03	8.6251	1.87e-08	5.75	541
2V+0.01V/hr ($\tilde{k} = 1/150$)	1.91e-03	8.0683	4.05e-08	5.38	523
Difference (%)	3.24	6.34	116	6.43	3.32

Table 5.3 Lifetime data from constant-voltage tests

Voltage: 2.2V						**Voltage: 2.46V**		
No.	Time	Failure	No.	Time	Failure	No.	Time	Failure
1	3.74	1	25	136.85	1	1	3.48	1
2	11.68	1	26	160.13	0	2	4.84	1
3	12.40	1	27	160.13	0	3	8.63	1
4	14.58	1	28	160.13	0	4	8.91	1
5	16.38	1	29	160.13	0	5	9.33	1
6	34.28	1	30	160.13	0	6	19.16	1
7	35.61	1	31	160.13	0	7	19.64	1
8	35.65	1	32	160.13	0	8	20.40	1
9	35.66	1	33	160.13	0	9	27.19	1
10	36.69	1	34	160.13	0	10	28.23	1
11	37.41	1	35	160.13	0	11	40.64	1
12	47.88	1	36	160.13	0	12	41.96	1
13	49.89	1	37	160.13	0	13	49.19	1
14	52.96	1	38	160.13	0	14	51.51	1
15	56.77	1	39	160.13	0	15	55.75	1
16	62.41	1	40	160.13	0	16	71.59	1
17	64.97	1	41	160.13	0	17	75.04	1
18	66.42	1	42	160.13	0	18	92.18	1
19	68.34	1	43	160.13	0	19	102.33	1
20	72.01	1	44	160.13	0	20	103.05	1
21	102.31	1	45	160.13	0	21	120.36	1

22	111.25	1	46	160.13	0	22	130.47	0
23	113.79	1	47	160.13	0			
24	114.37	1						

Table 5.4 Lifetime data from step-voltage test

Step-voltage test [2.25, 2.44] (V), [96, 140] (hr)											
No.	Time	Failure	No.	Time	Failure	No.	Time	Failure	No.	Time	Failure
1	12.07	1	17	91.22	1	33	14.00	1	49	94.38	1
2	19.50	1	18	102.10	1	34	17.95	1	50	97.71	2
3	22.10	1	19	105.10	2	35	24.00	1	51	101.53	2
4	23.11	1	20	109.20	2	36	26.46	1	52	105.11	2
5	24.00	1	21	114.40	2	37	26.58	1	53	112.11	2
6	25.10	1	22	117.90	2	38	28.06	1	54	119.58	2
7	26.90	1	23	121.90	2	39	34.00	1	55	120.20	2
8	36.64	1	24	122.50	2	40	36.13	1	56	126.95	2
9	44.10	1	25	123.60	2	41	40.85	1	57	129.25	2
10	46.30	1	26	126.50	2	42	41.11	1	58	136.31	2
11	54.00	1	27	130.10	2	43	42.63	1	59	140	0
12	58.09	1	28	140	0	44	52.51	1	60	140	0
13	64.17	1	29	140	0	45	62.68	1	61	140	0
14	72.25	1	30	140	0	46	73.13	1	62	140	0
15	86.90	1	31	140	0	47	83.63	1	63	140	0
16	90.09	1	32	140	0	48	91.56	1	64	140	0

Tables 5.5. to 5.7 show that the parameter estimates are very close under constant, step-voltage, and ramp-voltage, with the exception of $\hat{\lambda}$. The reliability prediction, MTTF, is also very close under different stress loadings. On the other hand, all of the test units under ramp-voltage fail within 91 hours. In other word, the ramp-voltage test requires

less than 91 hours. However the step-voltage test requires at least 140 hours to obtain similar accuracy for reliability prediction and parameter estimation. However, the constant-voltage test needs longer time, 161 hours. The time reduction between step-voltage and constant-voltage is about 12.57%. The time reduction between ramp-voltage and constant-voltage is even more, 43.59%. The time reduction between ramp-voltage and step-voltage is about 35.48%. These results confirm that it is feasible to design ALT under step-stress and ramp-stress to obtain approximately the same accuracy of reliability prediction as that of constant-stress while using shorter time.

Table 5.5 Parameter estimation from step-voltage and constant-voltage tests

Test	$\hat{\hat{\lambda}}$	$\hat{\hat{\beta}}$	$\hat{\lambda}$	$\hat{\beta}$	MTTF	Test duration
Step-voltage [2.25, 2.44] (V) [96, 140] (hr)	1.78e-03	8.6490	1.74e-08	5.77	561	140 (hr)
Constant-voltage	1.68e-03	8.1602	3.16e-08	5.44	597	160.13(hr)
Difference (%)	5.95	5.99	44.94	6.07	6.03	12.57

Table 5.6 Parameter estimation from ramp-voltage and constant-voltage tests

Test	$\hat{\hat{\lambda}}$	$\hat{\hat{\beta}}$	$\hat{\lambda}$	$\hat{\beta}$	MTTF	Test duration
Ramp-voltage (2V+0.015V/hr)	1.85e-03	8.6251	1.87e-08	5.75	541	57.13(hr)
Constant-voltage	1.68e-03	8.1602	3.16e-08	5.44	597	160.13(hr)
Difference (%)	10.12	5.70	40.82	5.70	9.38	64.32

Table 5.7 Parameter estimation from ram-voltage and step-voltage tests

Test	$\hat{\hat{\lambda}}$	$\hat{\hat{\beta}}$	$\hat{\lambda}$	$\hat{\beta}$	MTTF	Test duration
Ramp-voltage (2V+0.015V/hr)	1.85e-03	8.6251	1.87e-08	5.75	541	57.13(hr)
Step-voltage [2.25, 2.44] (V) [96, 140] (hr)	1.78e-03	8.649	1.74e-08	5.77	561	140 (hr)
Difference (%)	3.78	0.3	21.39	0.34	3.70	59.19

CHAPTER 6

CONCLUSIONS AND FUTURE WORK

In this dissertation we investigate several topics in design of accelerated life testing plans, including optimal ALT plans under multiple stresses, optimal ALT plans subject to competing risk and progressive censoring, and equivalent ALT plans. We summarize the main conclusions of these topics and future research areas.

6.1 Conclusions

In Chapter 2, we present an approach for the design of ALT plans with multiple stresses using LHD. The proposed approach results in efficient and practical ALT test plans which not only achieve significantly smaller asymptotic variance of reliability prediction but also require considerably less test time compared to those based on traditional FFD. We also develop an SA based algorithm to determine the optimal test plans based on the proposed approach. Simulation study shows that the algorithm is effective and efficient.

In Chapter 3, we present approaches for planning ALT under progressive censoring and competing risk for both single and multiple stresses in terms of different objectives. The proposed approaches result in optimal, general and practical test plans. The procedure further reduces the test time as well as saves surviving items for other purposes. We also propose and develop a new optimization criterion for the design of test plans, that is, minimization of the asymptotic variance of mean time of first failure. This is a valuable criterion for situations that early failure has significant safety and economic impacts.

In Chapter 4, we define and develop equivalent ALT plans involving different stress loadings. We obtain equivalent step-stress and ramp-stress ALT plans to the constant-stress plan, which considerably shorten the test duration and reduce the number of test units while maintaining the same accuracy of reliability predictions. This has noteworthy practical and economical impacts as it enables reliability practitioners to choose the appropriate test plan to accommodate restrictions of resources and test time.

In Chapter 5, we describe the experiment setup for validation of the work in Chapter 4. The results confirm our hypothesis and validate the feasibility of the development of equivalent ALT plans.

6.2 Future Work

The current work can be extended as follows. For the optimal ALT plan under multiple stresses, we assume that the scale parameter of the Weibull lifetime distribution depends on the stresses through a log linear relationship

$$\ln\alpha\left(\mathbf{x}\right)=\mu\left(\boldsymbol{\beta}\mathbf{x}\right)=\beta_0+\beta_1x_1+\beta_2x_2+\cdots+\beta_kx_k$$

This can be extended to a more general case to include higher order and interaction terms. Optimal ALT plans based on a nonlinear model could be designed to accommodate other applications.

Moreover, single, simple step-stress is widely utilized in ALT due to its shorter test duration than that of constant-stress. However, efficient and effective application of multiple, simple step-stresses is a challenging issue. Figure 6.1 shows two different applications of simple step-stress under multiple stresses. Methodology to design of optimal ALT plans under multiple step-stresses need be developed.

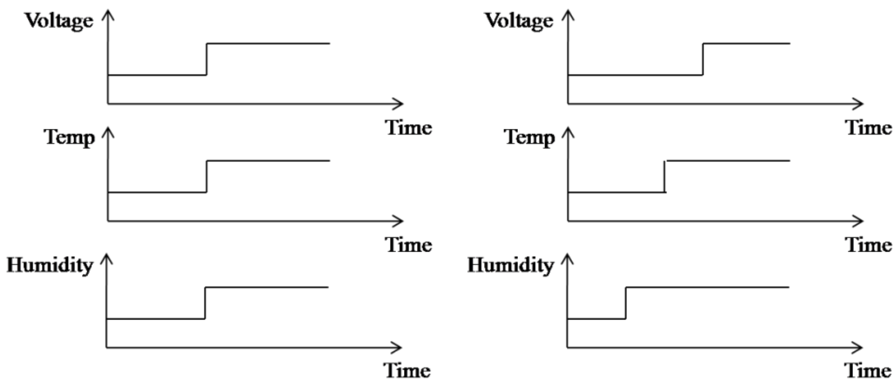

Figure 6.1 Simple multiple step-stresses

Under Type-I progressive censoring, surviving test units are removed at multiple stages before the final termination of the test. The multiple stages are time based. However in some situations it is more appropriate to determine the removal of surviving test units based on the number of observed failures. This is called Type-II progressive censoring. On the other hand, we assume the competing failure modes are independent. In many engineering application, the competing failure modes are dependent. These optimal ALT plans under Type-I progressive censoring and dependent failure modes are logical extension of the current work.

ALT can be used to obtain not only failure observations but also test unit's degradation information. Definition of equivalency can be extended to accelerated degradation tests. New approaches to determine optimal equivalent accelerated degradation test plans based on the definition of equivalency need be investigated.

In addition, equivalent test plans under a multi-stress multi-step test where the stress levels can be changed in different time and sequences is also an interesting and challenging problem. Figure 6.2 illustrates two experimental settings out of thousands of choices. Different stress change methods result in different cumulative damage to the test units. To obtain equivalent reliability prediction accuracy, each one may require different test duration and number of test units. Equivalent ALT plans under such a scenario can further refine the current work.

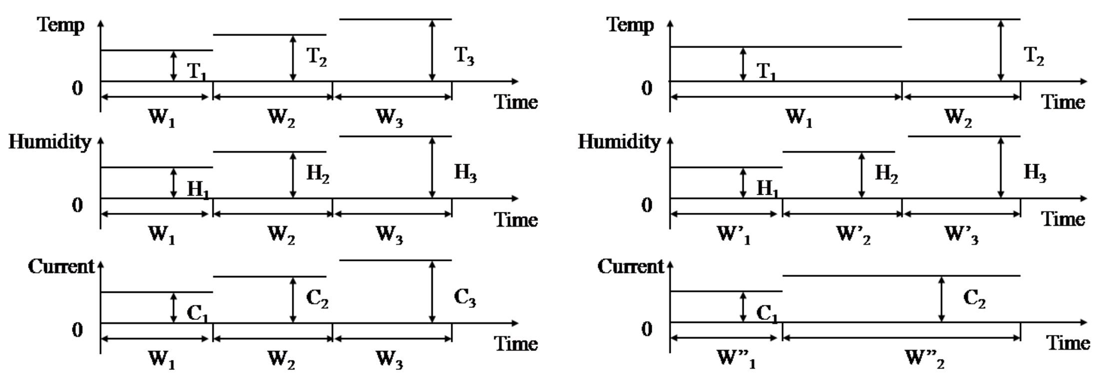

Figure 6.2 Multi-stress multi-step tests

APPENDIX 1

PROOF OF PROPOSITION 1

Fisher information matrix given by Eq. (2.18) is a positive semi-definite symmetric matrix, so its determinant is always larger than or equal to zero.

Now consider the upper bound of the determinant of Eq. (2.18). For a three factor $LHD(n,3)$, since each column (presents a factor) is a permutation of the normalized stress-values $\{1,2,\ldots,n\}$, the sum over all stress-values is given by

$$\sum_{l=1}^{n} x_{i,l} = n(n+1)/2, \; i = 1,2,3;$$

And the sum over all the quadric stress-values is given by

$$\sum_{l=1}^{n} x_{i,l}^{2} = n(n+1)(2n+1)/6, \; i = 1,2,3.$$

Let $a_1 = \sum_{l=1}^{n} x_{1,l} x_{2,l}$, $a_2 = \sum_{l=1}^{n} x_{1,l} x_{3,l}$, and $a_3 = \sum_{l=1}^{n} x_{2,l} x_{3,l}$ denote the sum of the cross product of stress-values between factor one and two, factor one and factor three, and factor two and factor three, respectively. Then matrix Eq. (2.18) can be written as

$$F_s = \begin{bmatrix} n & \frac{n(n+1)}{2} & \frac{n(n+1)}{2} & \frac{n(n+1)}{2} & n(1-\gamma) \\ \frac{n(n+1)}{2} & \frac{n(n+1)(2n+1)}{6} & a_1 & a_2 & (1-\gamma)\frac{n(n+1)}{2} \\ \frac{n(n+1)}{2} & a_1 & \frac{n(n+1)(2n+1)}{6} & a_3 & (1-\gamma)\frac{n(n+1)}{2} \\ \frac{n(n+1)}{2} & a_2 & a_3 & \frac{n(n+1)(2n+1)}{6} & (1-\gamma)\frac{n(n+1)}{2} \\ n(1-\gamma) & (1-\gamma)\frac{n(n+1)}{2} & (1-\gamma)\frac{n(n+1)}{2} & (1-\gamma)\frac{n(n+1)}{2} & n\left[\frac{\pi^2}{6}+(\gamma-1)^2\right] \end{bmatrix},$$

Since

$$F_s = -\begin{bmatrix} n & \frac{n(n+1)}{2} & \frac{n(n+1)}{2} & \frac{n(n+1)}{2} & n(1-\gamma) \\ \frac{n(n+1)}{2} & \frac{n(n+1)(2n+1)}{6} & a_1 & a_2 & (1-\gamma)\frac{n(n+1)}{2} \\ \frac{n(n+1)}{2} & a_2 & a_3 & \frac{n(n+1)(2n+1)}{6} & (1-\gamma)\frac{n(n+1)}{2} \\ \frac{n(n+1)}{2} & a_1 & \frac{n(n+1)(2n+1)}{6} & a_3 & (1-\gamma)\frac{n(n+1)}{2} \\ n(1-\gamma) & (1-\gamma)\frac{n(n+1)}{2} & (1-\gamma)\frac{n(n+1)}{2} & (1-\gamma)\frac{n(n+1)}{2} & n\left[\frac{\pi^2}{6}+(\gamma-1)^2\right] \end{bmatrix}.$$

By the property of matrix row operation we obtain

$$=\begin{bmatrix} n & \frac{n(n+1)}{2} & \frac{n(n+1)}{2} & \frac{n(n+1)}{2} & n(1-\gamma) \\ \frac{n(n+1)}{2} & \frac{n(n+1)(2n+1)}{6} & a_2 & a_1 & (1-\gamma)\frac{n(n+1)}{2} \\ \frac{n(n+1)}{2} & a_2 & \frac{n(n+1)(2n+1)}{6} & a_3 & (1-\gamma)\frac{n(n+1)}{2} \\ \frac{n(n+1)}{2} & a_1 & a_3 & \frac{n(n+1)(2n+1)}{6} & (1-\gamma)\frac{n(n+1)}{2} \\ n(1-\gamma) & (1-\gamma)\frac{n(n+1)}{2} & (1-\gamma)\frac{n(n+1)}{2} & (1-\gamma)\frac{n(n+1)}{2} & n\left[\frac{\pi^2}{6}+(\gamma-1)^2\right] \end{bmatrix}$$

This shows by exchanging a_1 and a_2, the determinant of the matrix does not change. The same results hold for exchanging a_2 and a_3, and exchanging a_1 and a_3. Therefore, a_2, a_2, and a_3 are pairwise symmetric.

By solving the partial differential equations $\partial \mathrm{Det}(F_s)/\partial \mathbf{a} = \mathbf{0}$, the solutions in terms of $\mathbf{a} = (a_1, a_2, a_3)$ that associate with the extreme values of $\mathrm{Det}(F_s)$ are obtained as shown,

$$\mathbf{a_1} = \begin{bmatrix} n(n+1)^2/4 \\ n(n+1)^2/4 \\ n(n+1)^2/4 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} n(n+1)(2n+1)/6 \\ n(n+1)(n+2)/6 \\ n(n+1)(n+2)/6 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} n(n+1)(n+2)/6 \\ n(n+1)(2n+1)/6 \\ n(n+1)(n+2)/6 \end{bmatrix},$$

$$\mathbf{a}_4 = \begin{bmatrix} n(n+1)(n+2)/6 \\ n(n+1)(n+2)/6 \\ n(n+1)(2n+1)/6 \end{bmatrix},\ \mathbf{a}_5 = \begin{bmatrix} n(n+1)(2n+1)/6 \\ n(n+1)(2n+1)/6 \\ n(n+1)(2n+1)/6 \end{bmatrix}.$$

The corresponding extreme values are,

$$\mathrm{Det}\left[F_s(\mathbf{a_1})\right] = n^4(n-1)^3(n+1)^3\left(n\pi^2 + 2n\gamma(1-\gamma)\right)\Big/10368 > 0 \qquad \text{(A1.1)}$$

$$\mathrm{Det}\left[F_s(\mathbf{a_2})\right] = \mathrm{Det}\left[F_s(\mathbf{a_3})\right] = \mathrm{Det}\left[F_s(\mathbf{a_4})\right] = \mathrm{Det}\left[F_s(\mathbf{a_5})\right] = 0 \qquad \text{(A1.2)}$$

In order to show that Eq. (A1.1) is the upper bound of $\mathrm{Det}\left(F_s(\mathbf{X})\right)$ we show it is larger than any other extreme determinant value and larger than the determinant values when a_1, a_2, and a_3 are at extreme values (boundary condition). The first condition is met according to Eq. (A1.2).

On the other hand, by definition, we know

$$\frac{n(n+1)(n+2)}{6} \le \mathbf{a} \le \frac{n(n+1)(2n+1)}{6}\ .$$

Consider the case

$$a_1 = \frac{n(n+1)(2n+1)}{6},$$

$$\frac{n(n+1)(n+2)}{6} \le a_2, a_3 \le \frac{n(n+1)(2n+1)}{6}.$$

The corresponding determinant is a function of a_2 and a_3 and we call it Eq. (A1.3). Let Eq. (A1.1) minus Eq. (A1.3), after some algebra, we obtain the following expression

$$n^2(n^2-1)\left[n^2(n^2-1)^2+144(a_3-a_2)^2\right]\left(n\pi^2+2n\gamma(1-\gamma)\right)\Big/10368 \qquad \text{(A1.4)}$$

Obviously, (A1.4) is strictly larger than zero. Similarly, consider the case

$$a_1 = n(n+1)(n+2)/6,$$

$$\frac{n(n+1)(n+2)}{6} \le a_2, a_3 \le \frac{n(n+1)(2n+1)}{6}.$$

After some algebra, we get exactly the same expression as Eq. (A1.4), which is strictly larger than zero.

Since a_1, a_2, and a_3 are pairwise symmetric, the above results hold for a_2 and a_3. This verifies the second condition for (A1.1) to be the upper bound of $\mathrm{Det}(F_s(\mathbf{X}))$.

APPENDIX 2

PROOF OF LEMMA 1 AND PROPOSITION 2

Proof of lemma 1

Consider the sum of d_i^q. By definition of *q*-norm inter-site distance, we have

$$\sum_{i=1}^{\binom{n}{2}} d_i^q = \sum_{i=2}^{n}\sum_{j=1}^{i-1} d\left(\mathbf{s}_i,\mathbf{s}_j\right)^q = \sum_{r=1}^{k}\sum_{i=2}^{n}\sum_{j=1}^{i-1}\left|s_{i,r}-s_{j,r}\right|^q$$

Since each column is a permutation of $\{1,2,\ldots,n\}$, we have

$$\sum_{i=1}^{\binom{n}{2}} d_i^q = \sum_{r=1}^{k}\sum_{i=2}^{n}\sum_{j=1}^{i-1}\left|s_{i,r}-s_{j,r}\right|^q = k\sum_{i=2}^{n}\sum_{j=1}^{i-1}\left|i-j\right|^q .$$

Since

$$\begin{aligned} & k\sum_{i=2}^{n}\sum_{j=1}^{i-1}\left|i-j\right|^q \\ &= k\left\{1^q+\left(1^q+2^q\right)+\left(1^q+2^q+3^q\right)+\ldots+\left(1^q+2^q+3^q+\ldots+\left(n-1\right)^q\right)\right\} \\ &= k\left[\left(n-1\right)\cdot 1^q+\left(n-2\right)\cdot 2^q+\left(n-3\right)\cdot 3^q+\ldots+1\cdot\left(n-1\right)^q\right] \\ &= k\sum_{i=1}^{n-1}\left(n-i\right)\cdot i^q \end{aligned} , \qquad \text{(A2.1)}$$

we obtain

$$\sum_{i=1}^{\binom{n}{2}} d_i^q = \sum_{r=1}^{k}\sum_{i=2}^{n}\sum_{j=1}^{i-1}\left|s_{i,r}-s_{j,r}\right|^q = k\sum_{i=2}^{n}\sum_{j=1}^{i-1}\left|i-j\right|^q = k\sum_{i=1}^{n-1}(n-i)\cdot i^q \ .$$

For the Euclidean distance,

$$\sum_{i=1}^{\binom{n}{2}} d_i^2 = k\sum_{i=2}^{n}\sum_{j=1}^{i-1}\left|i-j\right|^2 = kn^2\left(n^2-1\right)\Big/12 \ .$$

Therefore,

$$\bar{d}^{\,q} = \sum_{i=1}^{\binom{n}{2}} d_i^q \Bigg/ \binom{n}{2} = \frac{2k\sum_{i=1}^{n-1}(n-i)\cdot i^q}{n(n-1)},\ q \geq 1,$$

and

$$\bar{d}^{\,2} = kn^2\left(n^2-1\right)\Bigg/12\binom{n}{2} = kn(n+1)/6 \ .$$

Proof of proposition 2

To show $\Phi_{p,L} = \left(\sum_{i=1}^{n-1} \frac{(n-i)}{k^{p/q} \cdot i^p} \right)^{1/p}$ is the lower bound of Φ_p, a lemma from Joseph and Hung (2008) is needed:

For a set of positive values $\{d_{j1}, d_{j2}, \ldots d_{jm}\}$ *and its ordered sequence* $d_{j(1)} \le d_{j(2)} \le \ldots \le d_{j(m)}$ *for* $j = 1, 2, \ldots, k,$ *then* $\sum_{i=1}^{m} \left(\sum_{j=1}^{k} d_{ji} \right)^{-1} \le \sum_{i=1}^{m} \left(\sum_{j=1}^{k} d_{j(i)} \right)^{-1}$.

Since different LHDs have different inter-site distances values, by above lemma we obtain

$$\Phi_p = 1/\phi_p = \left(\sum_{i=1}^{\binom{n}{2}} d_i^{-p} \right)^{-1/p} = \left(\sum_{i=1}^{\binom{n}{2}} \left(\sum_{j=1}^{k} d_{j,i}^{q} \right)^{-p/q} \right)^{-1/p} \ge \left(\sum_{i=1}^{\binom{n}{2}} \left(\sum_{j=1}^{k} d_{j,(i)}^{q} \right)^{-p/q} \right)^{-1/p} .$$

When all the k factors are arranged in the same increasing sequence, from (A2.1), we know there are $(n-1)$ of the d_i^q's are k, $(n-2)$ of the d_i^q's are $2^q k$, …, and one is $(n-1)^q k$. Thus we have

$$\left(\sum_{i=1}^{\binom{n}{2}} \left(\sum_{j=1}^{k} d_{j,(i)}^{q} \right)^{-p/q} \right)^{-1/p} = \left(\sum_{i=1}^{n-1} \frac{(n-i)}{(k \cdot i^q)^{p/q}} \right)^{-1/p} = \left(\sum_{i=1}^{n-1} \frac{(n-i)}{k^{p/q} \cdot i^p} \right)^{(-1/p)} .$$

This proves

$$\Phi_{p,L} = \left(\sum_{i=1}^{n-1} \frac{(n-i)}{k^{p/q} \cdot i^p} \right)^{(-1/p)} .$$

Now consider the upper bound of $\Phi_p = 1/\phi_p$. This is equivalent to find the lower bound of ϕ_p. From lemma 1, we know that $\sum_{i=1}^{\binom{n}{2}} d_i^q = \binom{n}{2} \bar{d}^q$. Therefore, we can formulate the problem of finding lower bound of ϕ_p as a constraint optimization problem.

$$\text{Min} \qquad \phi_p = \left(\sum_{i=1}^{\binom{n}{2}} \frac{1}{d_i^p} \right)^{1/p} ,$$

$$\text{subject to} \qquad \sum_{i=1}^{\binom{n}{2}} d_i^q = \binom{n}{2} \bar{d}^q .$$

Using the Lagrange multiplier method, the constraint optimization problem can be rewritten as

$$\text{Min} \qquad \Lambda = \left(\sum_{i=1}^{\binom{n}{2}} \frac{1}{d_i^p} \right)^{1/p} + \lambda \left(\sum_{i=1}^{\binom{n}{2}} d_i^q - \binom{n}{2} \bar{d}^q \right).$$

From $\frac{\partial \Lambda}{\partial d_i} = 0$, we have $\frac{1}{\lambda q} = d_i^{p+q}$. As $q > 0$, and $p + q > 0$, then $\lambda > 0$, $d_i > 0$.

Since all the partial derivatives with respect to d_i are equal and $d_i > 0$, the optimal solution is obtained when

$$d_1 = d_2 = \ldots = d_{\binom{n}{2}} = \left(\overline{d}^{\,q}\right)^{1/q}.$$

Hence,

$$\min \phi_p = \left(\binom{n}{2} \Big/ \left(\overline{d}^{\,q}\right)^{p/q}\right)^{1/p} = \left(\overline{d}^{\,q}\right)^{1/q} \Big/ \binom{n}{2}^{1/p} = \Phi_{p,U}.$$

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Reliability of piezoelectric actuators at extreme operating conditions, *Actuator* 2004, Bremem, 2004.

Curriculum Vita

Yada Zhu

2002	B.S., Mechanical Engineering, Beihang University, Beijing, China
2002-2003	Design Engineer, Shanghai Aircraft Design and Research Institute, Shanghai, China
2005-2006	Process Engineer, Kulicke & Soffa, Singapore
2006	M.S., Mechanical Engineering, National University of Singapore, Singapore
2006-2010	Graduate Assistant, Department of Industrial & Systems Engineering, Rutgers, The State University of New Jersey, New Brunswick, NJ
2009	M.S., Statistics, Rutgers, The State University of New Jersey, New Brunswick, NJ