A CASE STUDY: THE DEVELOPMENT OF STEPHANIE'S ALGEBRAIC REASONING

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ABSTRACT OF THE DISSERTATION:

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This research provides an analysis of the mathematical growth and development of one student, Stephanie, as she worked on early algebra tasks during her eighth-grade year as part of a teaching experiment. Stephanie was among the original participants in a longitudinal study which investigated how students develop mathematical ideas under conditions that fostered independent exploration, reasoning, and justification of ideas (Maher, 2005). A qualitative approach based on the analytical model described by Powell, Francisco, and Maher (2003), was taken in analyzing videotape data from the Robert B. Davis Institute of Learning archive, along with student work. Seven task-based interview sessions were analyzed, spanning a six month period, beginning from November 8, 1995 to April 17, 1996. The research focused on Stephanie's algebraic reasoning; in particular, how she built an understanding of the binomial theorem and related it to Pascal's triangle. Stephanie's representations, her explanations and justifications, and her methods of dealing with obstacles to understanding, were all examined and provided the basis for this research.

The analysis shows that Stephanie built her mathematical understanding through the development of multiple representations of concepts and moved fluidly between and among the representations that she organized into 'symbolic' and 'visual' representations. Symbolic representations included algebraic expressions, combinatorics notation, and

Pascal's triangle while visual representations included drawings, tables, models formed by algebra blocks and other manipulatives, and towers built with unifix cubes. Furthermore, through Stephanie's explanations and justification of her representations and reasoning in general, she invented strategies to convince herself as well as the researchers that she had fulfilled the requirements of the problem task. When dealing with obstacles to her understanding such as lack of information, or calculating obstacles, Stephanie acquired the use of several heuristic methods in order to overcome them. These included the use of substituting in numbers in order to test a conjecture; returning to basic meaning; drawing diagrams; building models; and considering a simpler problem. Throughout the task-based interviews, Stephanie retrieved knowledge from her earlier problem solving and extended this knowledge to build new ideas, while tackling more challenging problems. In particular, Stephanie mapped the coefficients in the binomial expansion to particular rows in Pascal's Triangle; she connected these ideas to her problem solving from earlier work in the elementary grades. The findings are relevant to the timing and method of early algebraic instruction in schools.

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CHAPTER 1 INTRODUCTION

1.1 Statement of the Problem

The importance of algebra within mathematics instruction, how it is taught, and when it is taught, has been and still is the subject of debate and research. The late Robert B. Davis (1927 – 1997) attributed much of the difference in opinion to two differing views of mathematics in general and algebra in particular (1985). In a report on "Algebraic thinking in the early grades" based on the sessions of the Fifth International Congress on Mathematics Education held in Australia in 1984, he described the first view of algebra as "learning to manipulate meaningless symbols by following rules learned by rote"; the second view maintains that students build up mental representations based on experience (1985, p. 203). According to Davis, researchers that hold the first view do not see a need to teach algebra to younger students, while those who hold the second opinion favor the idea of developing mathematical thinking among younger students (1985).

A second part of the problem facing mathematics educators is when to teach algebra. Should it be taught in the elementary grades in conjunction with arithmetic or should it wait until high school – typically eighth or ninth grade? Traditionally, algebra was first introduced at the ninth-grade level (Usiskin, 1999; Davis, 1995). However, the National Council of Teachers of Mathematics (NCTM) recommends that algebra should be viewed as a strand in the curriculum beginning from pre-kindergarten through high school. They suggest that teachers can then help students build a solid foundation of understanding and experience as a preparation for more sophisticated work in algebra in the middle grades and high school (NCTM, 2000).

A third issue is how algebra should be taught. The traditional approach to teaching algebraic concepts stresses students' memorization of rules and procedures and manipulation of symbols (Davis, 1985). How well a student acquired knowledge was directly related to the teacher's ability to clearly explain the concepts and demonstrate how to solve problems (Davis, 1985). Researchers now make the claim that students develop mathematical understanding as they invent and examine methods for solving mathematical problems (Hiebert et al., 1997). They suggest that this can be accomplished by engaging in tasks that allow students to build mathematical understanding by individual reflection and communication with others.

1.2 Purpose of the Study

This study will investigate the aforementioned issues by analyzing data from a twenty-year longitudinal study[1] designed to study children's mathematical thinking. In particular, I will be examining and analyzing the early algebra work of Stephanie, an eighth-grade student who participated in the longitudinal study since the first grade. The longitudinal study developed from partnerships between Rutgers University and three New Jersey school districts: Kenilworth, New Brunswick, and Colts Neck. These partnerships included professional development for teachers and research about children's mathematical thinking. A primary goal of each was to gain a deeper understanding of how students develop mathematical ideas under particular conditions (Maher: 2002, 2005; Maher & Martino, 1996). These conditions included acknowledging the importance of various instructional settings, providing opportunities

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for children to explore ideas, providing flexibility of content, and allowing extended periods of time for children to pursue their own mathematical thinking (Maher, Davis, & Alston, 1991). The goal of the researchers who facilitated the sessions was to create learning environments where the personal attributes of each student were valued. The role of the facilitator was modified from dispenser of knowledge to a questioner or coach (Maher & Martino, 1996). Children were given opportunities to express their thinking about mathematics by participating in a variety of problem solving tasks. These tasks were based on five strands: Combinatorics, Counting, Probability, Algebra, and Precalculus. Prior to her eighth-grade year, Stephanie participated as a member of the class in a number of problem solving tasks that have been analyzed by other researchers within the Robert B. Davis Institute for Learning (RBDIL). For instance, Muter analyzed the Pizza Problem and the Towers Problems from the Counting Strand (Muter, 1999). The Tower of Hanoi, a problem solving task within the algebra strand, was analyzed by Mayansky (2007). Giordano (2008) and Bellisio (1999) analyzed Guess my Rule tasks dealing with the concept of function. A number of these tasks, along with Stephanie's involvement in them, are discussed in more detail in the Literature Review of this dissertation in chapter 3.

Stephanie participated in the longitudinal study as a part of the class through seventh grade. After seventh grade, she moved to a different school and was enrolled in an eighth-grade algebra course. In order for her to continue in the longitudinal study, eight individual task-based interviews were carried out in which she was given the opportunity to question and think about early algebraic ideas and principles and build new ones (Maher & Speiser, 1997). In prior years, she had initiated new ideas, formed representations, and justified those representations to her classmates. At times, when collaborating with her peers, she modified aspects of her earlier work (Maher & Martino, 1996). Examples are discussed in detail in the literature review of this paper.

The purpose of this study is to investigate how Stephanie built meaning for particular algebraic ideas during her eighth-grade year. In this series of interviews, she investigated ideas related to perimeter, expansion of a binomial, and mathematical properties such as the Distributive Property, the Binomial Theorem, and Pascal's Triangle. She had opportunities to make conjectures, test them out, and draw conclusions. She was invited to think seriously about topics that she mentioned she was studying in school. Through these interviews, she was able to draw upon some of her earlier experiences with tasks she worked on during the longitudinal study. The following research questions will guide this study.

1.3 Research Questions

This study is designed to describe how Stephanie explored and built algebraic ideas. It will focus on how Stephanie built meaning of the Binomial Theorem and related the meaning to Pascal's Triangle.

In particular, I will investigate:

1) What representations does Stephanie use to construct, develop, and present her responses to the tasks, problems, and/or questions posed?

2) What explanations and justifications does she give for her solutions and/or the representations that she constructs?

3) What, if any, obstacles to understanding does she encounter?

4) How, if at all, does she overcome these obstacles?

CHAPTER 2 THEORETICAL FRAMEWORK

The late Robert B. Davis described mathematics as a way of thinking that involves creating mental representations of problem situations, and then selecting and/or developing relevant knowledge to deal with these mental representations, and finally, using appropriate heuristic methods to solve a given problem. This way of thinking may make use of written symbols (or even physical representations with manipulatable materials), but the real essence is something that takes place within the student's mind (Davis, 1992).

Mathematicians use these mental and physical representations to help them analyze problems and create algorithms to solve them (Davis & Maher, 1990). Numerous studies by Davis, Maher and others show that children are capable of doing the same creative types of things that mathematicians do (Davis, Maher & Martino, 1992; Maher & Martino, 1996; Warren, Cooper & Lamb, 2006; Davis & Maher, 1997; Maher & Martino, 1992; Maher & Speiser, 1997; Davis & Maher, 1990). Many of these are discussed within the literature review.

One question that arises is whether the typical K-12 mathematics curriculum encourages and supports this belief or does it still rely on memorizing routine algorithmic procedures? Davis and Maher (1997) describe a "transmission" theory of sharing knowledge, in which teachers focus on presenting mathematical ideas to students, whose job it is to remember them. Within the past three decades, a number of alternatives to this procedural approach to learning and teaching mathematics have developed. Among them is an approach referred to as "constructivist" teaching and learning (Davis & Maher, 1997). The central idea, according to Davis and Maher, is that the teacher is concerned with the mental representations that a student is building. The teacher attempts to recognize the meaning within the student's representations to the best of his/her ability. In addition, the teacher tries to provide experiences for that student that will be useful for further revision of the mental structures that are being built (Davis & Maher, 1997).

The proper environment needs to exist so that constructivist learning can take place. According to Burns, "Children's classroom experiences need to lead them to make predictions, formulate generalizations, justify their thinking, consider how ideas can be expanded or shifted, look for alternative approaches and search for those insights that, rather than converging toward an answer, open up new areas to investigate" (1985, p. 17). Maher and Martino (1996) describe the following conditions that facilitate such learning: (a) opportunities for students to work in a variety of social settings, (b) flexibility in the curriculum for students to continue working on a problem or to pursue a new idea, (c) teacher restraint from telling students what to do, and (d) teaching guided by student thinking. The data for this study comes from a longitudinal study that strove to create these conditions, thereby providing an atmosphere of sense making wherein researchers could study children's mathematical thinking.

According to Davis, when these conditions are present, the learner cycles through a series of steps in order to think carefully about a mathematical situation. These steps include building a representation for the input data, carrying out a memory search for relevant knowledge that can be used in solving the problem or moving further with the task, constructing a mapping between the data representation and the knowledge representation, checking the mapping, and, finally, applying the knowledge representation in order to solve the problem. At times, the learner must cycle through one of these steps several times before moving forward (Davis & Maher, 1990).

The first step in this series, building representations, is based on cognitive building blocks. Davis, Maher, and Martino (1992) argue that effective cognitive building blocks initially result from experience, usually "concrete" experience. Often, these are mental representations based on ideas and concepts that the children are familiar with and can envision and relate to. In order to build accurate mental representations for mathematical situations and tasks, Davis and Maher (1997) suggest that a student needs to be able to draw on a large collection of such fundamental blocks.

One way to do this is to use the "paradigm teaching strategy" (Davis & Maher, 1997, p.99). This requires that the teacher provide a carefully designed experience for a student that is similar in structure to the relevant mathematics. These experiences are then referred to as assimilation paradigms (i.e., a conceptual framework for viewing the mathematical topic) and can be used to build more abstract representations in the future (Davis & Maher, 1997). Examples are described in detail in chapter 3 of this paper. These assimilation paradigms allow the mathematics learner to refer back to prior experiences that may help them with their current problem. This allows the student to form a knowledge representation.

By working with their peers to develop and justify representations and solutions to particular tasks, students are able to construct a mapping between the given data and their knowledge representation (Davis & Maher, 1990). They can check their mappings by using concrete numbers and comparing their ideas with those of their classmates. This will initiate discussion and sometimes disagreement, whereby students can go back and modify and improve upon their ideas and representations. They may have to go through this process several times before being able to solve a particular problem (Davis & Maher, 1990). The end result, however, is that they will have established meaning for themselves through the process of solving the problem. This theoretical framework will provide the context and inform the methodology for this study of Stephanie's mathematical activity as she builds and justifies representations for the expansion of a binomial, the binomial coefficients, Pascal's Triangle, and the use of various mathematical properties.

CHAPTER 3 - LITERATURE REVIEW

3.1 Algebra

According to Motz and Weaver (1993), algebra is important to learn because it can be viewed as a mathematical bridge that connects all of the branches of mathematics to each other. Many school districts are taking heed of the importance of learning algebra and hence making algebra a graduation requirement for all high school students (Choike, 2000; NCTM, 2000). For many of these students, algebra becomes a stumbling block to the study of higher-level mathematics (Bellisio & Maher, 1998). One reason is that the students do not make a smooth transition from concrete arithmetic to the more abstract ideas in algebra (Spang, 2009). Another reason is that traditional algebra, because of its terminology and use of symbols, has been likened to a foreign language and therefore can be more difficult to learn than arithmetic (Usiskin, 1999).

The importance of algebra is minimized when algebraic ideas are presented as separate independent facts or as "a collection of tricks" (Thorpe, 1999). Students frequently fail to see their relevance and connection (Thorpe, 1999). Spang's research indicates that students learning algebra need to learn that the ideas are not independent from each other, rather they should be shown how the ideas fit together within the bigger picture in mathematics. This can be done by presenting algebraic concepts within the context of real life situations that they can relate to. They will then see that algebra can help to solve other types of mathematical problems in other disciplines (Spang, 2009). According to Usiskin, traditionally, algebra has been formally studied by students from grade levels as early as seventh grade and as late as college. He recommends that formal algebra be taught at the eighth-grade level with a strong preparation course as a precursor

(Usiskin, 1999). NCTM (2000) recommends that the teaching of algebraic ideas be included for all grades K-12, with special emphasis on the more abstract ideas beginning in middle school (2000). "Students in the middle grades should learn algebra both as a set of concepts and competencies tied to the representation of quantitative relationships and as a style of mathematical thinking for formalizing patterns, functions and generalizations" (NCTM, 2000, p. 223). In order to achieve these goals and make algebra more accessible to students, further research and discussion are needed (Davis, 1995).

3.1.1 How Algebra is Taught

Kieran (1992) reports that algebra has traditionally been taught as a cycle of procedural-structural steps. This traditional approach to teaching algebraic concepts focuses on memorization of rules and procedures, as well as the manipulation of symbols. Without a proper understanding of "why," the rules may seem meaningless to many of the students. Furthermore, when the rules get mixed up or forgotten, students have no prior building blocks of understanding to refer back to (Kieran, 1992). According to Davis, these rules are taught in bits and pieces in isolation from each other (Davis, 1994). As a result, the students are not able to assemble these small bits and pieces of ideas into a meaningful larger whole that will give mathematical power to their thinking (Davis & Maher, 1996). In order to achieve this goal, teaching algebra should move away from teaching by rote methods with a focus on mindless manipulation and move toward inspired teaching that will convey the true power, the utility and the beauty of algebra (Taylor, 1990).

One of the earliest advocates for reform in teaching algebra was the late Robert B. Davis. He emphasized teaching algebra for understanding, rather than relying on rote learning where the students were told what to do and how to do it with a substantial amount of drill and practice (Davis, 1990). Davis provided examples of environments that enabled students to challenge themselves and successfully solve non-routine problems,[2] enabling discussion, exploration, and discovery of important ideas (Maher, Davis, & Alston, 1991). He was able to do this through a method of discovery, in which students would independently build up their own mental representations of key algebraic concepts. This method of instruction emphasized how students think mathematically in the classroom and how students explain, argue for, justify, and represent their mathematical ideas (Ginsburg, Maher, & Speiser, 1997). Davis also recommends shifting the responsibility of discovering or inventing methods for dealing with problems to the learner. According to Davis, students gain more mathematical power from discovery learning than they do from the more passive approach of rote learning (Davis: 1992, 1997).

Davis also believed that this was not only possible, but also necessary for children in elementary grades (Davis, 1993). He claimed that children have a greater mental processing ability with mathematical ideas at a younger age (Davis, 1994). This claim emphasizes that younger children are quite capable of learning algebra in their elementary years.

[2] Non-routine problems are problems that are not the standard textbook tasks that require drill and practice to solve.

3.1.2 Algebra in the Earlier Grades

Davis is not the only strong advocate of teaching algebraic ideas in the elementary grades rather than waiting until eighth or ninth grade. Since the advent of the NCTM (2000) curriculum standards, which indicate that algebraic ideas be taught in all grades, there has been a trend toward developing algebraic concepts, such as patterns, functions, variables, etc., in instructional programs for students at the elementary school level. According to Wickett, Kjaras, and Burns (2002), the algebra taught in the elementary grades should not be a watered down version of standard high school algebra. The goal in grades 3 through 5 is to develop students' algebraic thinking, building a foundation of understanding and skills so that they can be successful in their later, more formal study of algebra (Spang, 2009).

Kaput (2008) stated, "the underlying goal of early algebra is for children to learn to see and express generality in mathematics." He emphasized that the goal of elementary mathematics should be to capitalize on young children's innate powers of generalization to guide them to "engage in generalizing as a *mathematical activity*."

Wickett, Kharas, and Burns (2002) emphasize that the understanding of patterns is an essential component to help develop a young child's algebraic thinking. They contend that patterns provide a useful foundation on which teachers can develop student's algebraic thinking because patterns draw on experiences and contexts that are familiar to students. The students should have experience creating, recognizing, and extending patterns while describing them verbally and symbolically in several ways.

Davis developed, as part of the Madison Project,[3] two approaches to give students an understanding of algebraic concepts such as integers, variables, and functions. He introduced integers through "Pebbles in a Bag" and functions through "Guess my Rule" (Davis & Maher, 1997). According to Davis and Maher (1997), the "Pebbles in a Bag" idea can be used to introduce the idea that 5 – 6 = -1 (Davis & Maher, 1997). A bag containing a reasonably large number of pebbles is required, as well as a pile of loose pebbles on the table. A student gives a signal, whereupon the teacher takes 5 pebbles from the pile on the table and places them in the bag. The students are asked if there are more pebbles in the bag now than there were prior to the signal. They would usually answer "more." The teacher asks, "How many more?" The students would reply, "5." The teacher then asks "How can we write this?" which opens up the opportunity to discuss notation. Now, 6 pebbles can be taken out of the bag. It can be written as 5 – 6. When asked, "Are there more pebbles in the bag now than there were at the signal or are there fewer," the students typically reply that there are fewer pebbles now. If asked, "How many less?" they reply that there is now one pebble less than at the signal. They are then asked, "How can we write this?" This provides an opportunity to invent and discuss the notation: $^{-}1$. If repeated often enough, then the children will now develop new basic building blocks in their minds, from which they can construct mental representations for similar problems (Davis & Maher, 1997).

To introduce the concept of functions, Davis used a game known as Guess My Rule (Giordano, 2008). Giordano describes it as an activity in which the teacher creates a rule, or asks one or more students to create a rule using a box and a triangle. The other

[3] The project, created during the mid-1950s and running through the early 1960s, emphasized basic algebraic tasks that were ordinarily omitted from the curriculum (Davis, 1992).

students then provide a number for the box, and the creator(s) respond with the appropriate value for the triangle. When students think they know the rule, they guess and the rule's creator(s) tells them if they were right or not. As they play the game, students are expected to develop their own theories and understandings. These, in turn, help to build their understanding of the concept of a function. From this activity, they move on to explore multiple representations of functions, including equations, tables, and graphs. The next step is to explore how these representations relate to each other.

Another example of research on early algebraic reasoning at the elementary level took place in Brisbane, Australia. Warren, Cooper, and Lamb (2006) examine the development of student functional thinking during a teaching experiment that was conducted in two classrooms with a total of 45 children whose average age was nine years and six months. The teaching experiment was part of a longitudinal study that tracked the same students for three years. According to the researchers, the longitudinal study was based on "a reconceptualization of content and pedagogy for algebra in the elementary school" (p. 210). The specific goals of the investigation were to (a) document the development and implementation of the lessons; (b) identify examples of students' algebraic and functional thinking; and (c) determine teacher actions and students' material use and classroom activities that facilitate functional thinking.

The intervention included four lessons taught by a researcher, with a second researcher and classroom teacher acting as observers. The data included videotapes of the sessions, field notes, student worksheets, and pre- and post-tests. Each of the four lessons was given to two classes. The researchers describe four representations that were developed throughout the lesson series: a model of the function, table of values generated by the function machine, specific language to assist student description of this action, and symbols used to summarize the change process. The lessons included practice activities, for example: the Guess my Rule games, as well as technology, specifically, the use of spreadsheets to represent differing change rules.

Each lesson was described in detail along with the results and then summarized. To give a brief overview, the lessons developed student understanding of addition and subtraction as functional relationships and as inverses of each other. Two formal notation systems were introduced. Problem solving was used to draw out the relationship between generalized arithmetic and functional thinking. Each lesson was presented in the context of a different 'real world' situation.

The results were described in terms of the pre-test, the lessons, and the post-test. The researchers note that the results suggest that the four lessons had a significant impact on students' understanding of the notion of a function when there were two operations, although none of the lessons had instructions on two-operation change. Based on their results, the researchers formed the following conclusions: (1) Young students are capable of thinking functionally and the approach taken in the four lessons helped them in doing so; (2) The order in which numbers appear in the table of values appears to help with the search for functional relationships; (3) The sequencing of the lessons assisted students' learning and understanding of functional relationships; (4) Students experienced less difficulty with addition change rules than with subtraction change rules, particularly when having to reverse them; (5) Use of materials has to focus at the heart of the mathematics inherent in the task to be effective. The researchers found that the function machine material was effective in identifying a relationship or pattern in a table of changes that are presented in random order; (6) Students' inability to work mentally with larger numbers affected their ability to find generalizations.

According to the researchers, these findings support the premise that young children do have the ability to exhibit functional thinking. They suggest that the development of functional thinking along with arithmetic could serve as the foundation for algebra.

3.2 Representations

Representations are an inherent part of mathematics. Students not only use representations to help them understand abstract concepts but also to explain their ideas to others. Pape (2001) uses the term representation(s) to refer to both the internal and external manifestations of mathematical concepts. He describes the internal representation(s) as "abstractions of mathematical ideas or cognitive schemata that are developed by a learner through experience" (p. 119). He also designates representations such as numerals, algebraic equations, graphs, tables, diagrams, and charts as external manifestations of mathematical concepts. Additionally, Pape refers to representation(s) as the act of externalizing an internal, mental abstraction. Furthermore, he argues that there is a mutual influence between the two forms of representations: the external will have an effect on the internal and vice versa (Pape, 2001).

Other mathematics educators who have contributed to the literature concerning representations include Gerald Goldin from Rutgers University and Claude Janvier from the University of Quebec. They co-chaired the initial Working Group on Representations; a subset of the International Group for the Psychology of Mathematics

Education (PME) from 1989 to 1993. In the following year, Janvier co-chaired the group with Gerard Vergnaud. Goldin and Janvier described the following interpretations of the terms "representation" and "system of representation" in connection with mathematics learning, teaching, and development that evolved from those five years:

> 1) An external, structured physical situation, or structured set of situations in the physical environment, that can be described mathematically or seen as embodying mathematical ideas;
>
> 2) A linguistic embodiment, or a system of language, where a problem is posed or mathematics is discussed, with emphasis on syntactic and semantic structural characteristics;
>
> 3) A formal mathematical construct, or a system of constructs, that can represent situations through symbols or through a system of symbols, usually obeying certain axioms or conforming to precise definition – including mathematical constructs that may represent aspects of other mathematical constructs;
>
> 4) An internal, individual cognitive configuration, or a complex system of such configurations, inferred from behavior or introspection, describing some aspects of the processes of mathematical thinking and problem solving (Goldin & Janvier, 1998, p.1).

Goldin and Janvier note the emergence of three theoretical perspectives.

First, Brian Greer and Guershon Harel bring up the idea of isomorphism and its role in mathematical cognition. Goldin and Janvier argue that these researchers view isomorphisms as "components of mental representations constructed by individuals" (1998, p. 2). They cite two examples: the insight of Poincare and the reasoning of the fourth-grade children studied by Maher, Martino, and Alston.

Secondly, Goldin and Janvier note the research of Norma Presmeg, whose research focuses on the visualization process in mathematical thinking. She "develops the importance to individuals' representation of mathematical constructs based on metaphor and metonymy" (Goldin & Janvier, 1998, p. 2). She claims that these representations, along with their corresponding imagery and symbolism, “help individuals resolve ambiguities inherent in cognitive representation.”

Thirdly, Neil Hall is mentioned as the initiator of the “procedural analogy theory” as a basis for understanding learning through concrete representations. According to Goldin and Janvier, Hall addresses the type of knowledge that includes strategies, tactics, and heuristic planning. He emphasizes that in order to understand how to make abstract mathematical ideas accessible to students, it is important to characterize and establish measures of isomorphisms between procedures (Goldin & Janvier, 1998, p. 2).

Goldin (1998), similarly to Pape (2001), views a representational system as having both intrinsic structure (i.e., within itself), and extrinsic symbolic relations (i.e., with other systems of representation). This is similar to Pape’s ideas, as mentioned earlier (2001). Goldin asserts that we lack a unified psychological model based on several different types of representational systems and their stages of development that serves the growing need of the educational community and offers a framework for discussion of how representational systems develop. Among other topics, he considers the issue of ambiguity inherent in representation, and he considers the issue of human affect as a representational system (Goldin & Janvier, 1998; Goldin, 1998).

According to Goldin and Janvier, Vergnaud considers the role of action in representation. They argue that he views representation as a dynamic process rather than a static entity (Goldin & Janvier, 1998). Everyone experiences representation “as a stream of internal images, gestures, and words” (Vergnaud, 1998, p. 167). Vergnaud further elaborates that words and symbols used in communication do not refer directly to reality but to represented entities. He cites the following examples: objects, properties, relationships, processes, actions, and constructs; thereby claiming that there is no assumed agreement between two people (Vergnaud, 1998). He defines a scheme "to be the invariant organization of behavior in a class of situations, with theorems-in-action and concepts-in-action as components of schemes" (Goldin & Janvier, 1998, p. 3). Hence, Vergnaud's alternative view of representation is based on both action and language and stems from the aforementioned ideas.

In 2000, the National Council of Teachers of Mathematics (NCTM) introduced representation as a process standard, based on its growing importance in the literature. Both the NCTM standards documents (2000) and the National Research Council's science standards (NRC, 1996) call for students to be able to use various forms of representations flexibly to investigate and communicate about real-world phenomena. Standards (2000) refers to representation as both process and product. The new process standard calls for all students to be able to:

1) Create and use representations to organize, record, and communicate mathematical ideas;
2) Select, apply, and translate among mathematical representations to solve problems; and
3) Use representation(s) to model and interpret physical, social, and mathematical phenomena. (NCTM, 2000, p. 67)

The remaining question is how to accomplish these goals. According to Davis and Maher, in order to build accurate mental representations for mathematical tasks, a student needs to be able to draw on a large collection of fundamental cognitive building blocks. It then becomes the job of the teacher to help students "build larger, and accurate, repertoires of 'basic' ideas" (1997, p. 99). This can be accomplished by drawing on common experiences taken from daily life, as well as from an accumulation of classroom experiences (Davis & Maher, 1997). According to the *paradigm teaching* *strategy* described by Davis, the teacher should provide a carefully designed experience for a student that is essentially isomorphic to the relevant mathematics. Davis further elaborates, "When any of us is confronted by some form of data that requires processing, our first attempt is to see if it can be made to match something that we already know. If so, the 'something we already know' is an *assimilation paradigm*" (Davis, 1996).

One example offered by Davis and Maher is Brandon, a fourth-grade boy from Colts Neck who participated in the longitudinal study. He invents a notation that he is then able to use (Davis & Maher, 1997). Brandon was presented with the Pizza Problem as follows:

> How many different pizzas can be made if every pizza has cheese, but to this you can add whichever of the following toppings you wish and in any combination you wish: green peppers, sausage, mushrooms, and pepperoni?

On his own, Brandon, invented a notation where the numeral "1" meant the presence of a particular topping, and "0" meant the absence of this particular topping. Previously he had solved the Towers Problem:

> How many different towers can you build, if each tower is 4 cubes high, and you have as many red cubes, and as many yellow cubes, as you want?

Davis and Maher claim that because of this representation, Brandon was able to demonstrate that the pizza problem was actually the same as the tower problem. They further explain that if you make the towers lie down horizontally then you could match the 0 with a red cube and the 1 with a yellow cube. They affirm that Brandon had demonstrated a true isomorphism between two tasks that, at first, seemed completely different. The researchers suggest that Brandon was able to make these connections because he had earlier been exposed to the Tower Problem, which allowed him to refer back to mental representations when dealing with the Pizza Problem (Davis & Maher, 1997).

Another important aspect of representations involves the ability to translate between representations.   This concept is explored in a study that included 195 students from the Department of Education of the University of Cyprus (Gagatsis & Shiakalli, 2004). The researchers attempted to identify the translation ability of university students within the context of functions. They refer to translation ability as the psychological processes involved in going from one mode of representation to another. The study attempts to clarify the "relationship between translation ability and problem solving by identifying direct translation tasks using different representations of the concept of function – the verbal, the graphical, and the algebraic" (Gagatsis & Shiakalli, 2004, p. 646).

The researchers administered two tests consisting of six direct translation tasks from one representation of the concept of function to another, a table, and two word problems.. These tests were designated Test A and Test B. Test A required the students to pass from a verbal representation of the six tasks to the corresponding graphical and algebraic representation. Test B, alternately, presented the six tasks in graphical form and required the students to express them in their corresponding verbal and algebraic form of representation. The students are told at the beginning of each test that a piece of information in mathematics can be expressed in three ways: verbally, graphically, and algebraically, and are given an example to illustrate. The given task is considered by the researchers to be the "source representation" (Gagatsis & Shiakalli, 2004, p. 650). The representation that the students need to translate into is considered the "target representation." The direct translation tasks were given a score of 1 if the representation produced by the students was the one corresponding to the source representation while a score of 0 was given to wrong answers or no answer. The other tasks were scored similarly.

The data was analyzed using SPSS statistical analysis. The researchers found no relationship between the verbal and graphical representations of function. They suggest that the students conceive the two representations of the same concept as two different tasks and not as different means of representing the same idea. This would indicate that "they are unable to recognize an idea embedded in a variety of qualitatively different representational systems and as a result they do not understand this idea" (Gagatsis & Shiakalli, 2004, p. 652). Furthermore, the researchers found that the percentages of success were lower whenever the graphical representation was involved in the translation task. The researchers concluded that instruction should include all types of representation in translation tasks because each representation has its own characteristics and poses different challenges for students (Gagatsis & Shiakalli, 2004).

3.3 Background on Stephanie

3.3.1 The Rutgers- Kenilworth Partnership

A partnership between Rutgers University and Harding Elementary School in Kenilworth, New Jersey, was developed by Professor Carolyn Maher (Maher: 2002, 2005; Maher & Martino, 1996). The goal of this partnership was to create classroom environments in which children would be actively engaged in building mathematical models, and in which the curriculum would be based on students' construction of meaning (Maher, 2005; Maher and Alston, 1990). According to the researchers, the collaboration has been highly successful. Maher, Davis, and Alston write:

> From its beginning, the Rutgers-Kenilworth partnership has emphasized the establishment of meaning, questioning, and thoughtful analysis – despite the fact that the pervasive approach in most mathematics classrooms across the nation is a routine sequence of often mindless activities. Kenilworth has dared to be different. At Kenilworth, children continue to grow in their understanding of mathematical ideas, have become better problem solvers, and rank mathematics as one of their favorite subjects (1991, p. 222).

The main goal of the longitudinal study has been to gain a deeper understanding of how students develop mathematical ideas under particular conditions. The conditions were created such that the children were given the opportunity to express their thinking about mathematics by building mathematical ideas, creating models, and inventing notations in order to justify and generalize their ideas (Maher, 2002). Students were presented with a variety of problem solving tasks based in five mathematical strands: counting, combinatorics, algebra, probability, and precalculus. The algebra strand reflected the ideas of Robert B. Davis.

The longitudinal study was initiated at Harding School in 1989 with a class of 18 first graders (Martino, 1992; Maher & Martino, 1996). The study has followed some of the students from grade one through college. The students benefited from being in a school where building meaning in doing mathematics is a serious goal and one that encourages reform in teaching mathematics (Maher, Davis, & Alston, 1991). One of these students is the subject of this study: Stephanie.

Stephanie was one of a class of first graders, at a public school in a blue-collar district (Maher & Speiser, 1997). She and her classmates were presented with problems in which they were challenged to build solutions and construct models for their solutions. According to Maher and Speiser, this setting encouraged differences in thinking that were discussed and negotiated. Stephanie continued in this setting through grade 7. The following year, in the fall of 1995, Stephanie moved to another community and was transferred to a parochial girl's school. At that time, her mathematics program for grade 8 was a conventional algebra course. However, she continued to participate in the longitudinal study by taking part in a series of individual task-based interviews (Maher & Speiser, 1997). These interviews provide the data for this dissertation.

3.3.2 Shirts and Pants

Stephanie has been involved in a thoughtful approach to doing mathematics problems since she was in grade 1. During a problem-solving session designed by the Rutgers researchers, she was observed working freely with three other boys. She was willing to share her ideas and challenged them when their ideas did not make sense to her (Maher & Martino, 1996).

In the spring of grade 2, she and her classmates were presented with the Shirts and Pants problem in one of a series of open-ended problem-solving sessions planned and facilitated by the Rutgers research team in collaboration with the children's classroom teacher. The problem was presented as follows:

> Stephen has a white shirt, a blue shirt, and a yellow shirt. He has a pair of blue jeans and a pair of white jeans. How many different outfits can he make?

The class worked in groups of three and was not told in advance any method for solution (Davis, Maher, & Martino, 1992). Following the group working sessions, the children were asked to share their ideas with the entire class. Stephanie was one of six children who were videotaped during the class and after the third-grade implementation during individual interviews regarding the problem task (Davis, Maher, & Martino, 1992).

According to Davis, Maher, and Martino (1992), their individual ways of representing the problem, and the methods for solution which they invented, were their own, and usually different from those of others in the group. Although the children listened and argued with one another, they did not usually give up their idea in order to accept another student's point of view (Davis, Maher, & Martino, 1992).

For example, the students decided that differences in the kinds of outfits are relevant. Stephanie used a diagram to develop a coding strategy to form her combinations (Davis, Maher, & Martino, 1992). She illustrated each distinct outfit with a pair of letters, the first for the shirt and second for the jeans. She recorded each outfit and kept track of them by numbering each combination (Davis, Maher, & Martino, 1992). She concluded that there were a total of five combinations. Stephanie took a leading role when she pointed out to one of her group members, Dana, that the outfits do not have to match. She also told another group member, Michael, that "you can make it in different ways too," referring to the possible outfits (Davis, Maher, & Martino, 1992). All three students seemed satisfied with their results and were happy to share them with the class, but there was no agreed upon answer (Davis, Maher, & Martino, 1992).

Five months later, in October, 1990, the same problem was revisited, although this time the children were in the third grade (Maher & Martino, 1996). In the interim, there was no class consideration of the problem (Davis, Maher, & Martino, 1992). This time the students worked in pairs. Stephanie worked with Dana. As Dana read the problem aloud, Stephanie suggested drawing a picture (Davis, Maher, & Martino, 1992). The girls began by coloring in the shirts and pants with the appropriate colors but then Stephanie suggested identifying the different shirts and pants by using the first letter of the color. For example, a blue shirt was labeled "B." The girls worked together to find possible outfits. Stephanie began drawing lines from different shirts to different pants (Davis, Maher, & Martino, 1992). Later, the girls were asked by the researcher why they used connecting lines. Stephanie replied, "…so that we didn't do that again," i.e., repeating a combination. According to Davis, Maher, and Martino (1992), Stephanie's justification of the use of the line strategy indicated a shift from working with the representation of the problem data to working instead with a representation of the process by which she solved the problem.

Furthermore, she had now invented notation to monitor her own behavior. She kept track of how many combinations she found and this time the girls were able to come up with six combinations for the original three shirts and two pairs of jeans problem. According to Maher and Martino (1996), Stephanie was able to extend her solution to three shirts and three pairs of jeans by systematically considering each pair of jeans with the blue shirt, each pair of jeans with the white shirt, and each pair of jeans with the yellow shirt.

3.3.3 Towers

In grade 3, in a second problem-solving session following the Shirts and Pants activity on October 11, 1990, Stephanie was introduced to investigations with block towers. According to Maher and Martino (1996), Stephanie and her classmates were asked to build towers four cubes tall selecting from red and blue cubes. Working with her partner, Dana, Stephanie initially used trial-and-error and guess-and-check strategies to create new towers and to search for duplicates. However, they quickly moved to searching for patterns within and among towers. Along the way, Stephanie invented descriptive names to identify particular towers. Stephanie also began to notice relationships between pairs of towers and referred to some of the pairs as '*opposites*' or '*cousins*' (Maher & Martino, 1996). According to Maher and Speiser, this early introduction enabled her to build visual patterns that allowed her to represent her ideas. Furthermore, these working theories triggered the development of arguments to support a component of a solution and the extension of arguments to build more complete solutions later (Maher & Speiser, 1997).

The Towers problem was revisited on February 6, 1992, when Stephanie was in the fourth grade. At that time, the students were asked to build all possible towers five cubes tall while still using two colors of Unifix cubes. Stephanie and her partner Dana utilized the strategy of '*opposites*' to generate new combinations (Maher & Martino, 1992). Stephanie explained how she and her partner would build a tower, then build the '*opposite*' of that tower, then build the '*cousin*' of the tower, then finally the opposite of the '*cousin*' of the tower (Maher & Martino, 1992). They used this "upside down and opposite" pattern to group towers into sets (Maher &Martino, 1992). Their work and explorations in grade 3 provided the building blocks to build more complicated towers, and enabled them to invent ways of doing it more efficiently and justify their results.

After each of the sessions in grades 3 and 4, the following day, individual interviews with the children were conducted. In these interviews, Stephanie worked on further extending her organizations of groups of towers according to certain color categories in an effort to avoid producing duplicate towers (Maher & Martino, 1996). She responded to the interviewer's suggestion to continue working on it at home and to further explore the cases by considering towers six cubes tall. According to Maher &

Martino (1996), Stephanie's early use of patterns and local organizations suggested to her that these methods were not reliable either for conducting an exhaustive search or for monitoring the presence of duplicates.

In addition to working on the towers tasks with her classmates in grades 3 and 4, Stephanie was one of four children who took part in a small-group assessment in which they were asked to find all possible towers three cubes tall when selecting from red and blue cubes, as well as provide a convincing argument that every possible tower had been found (Maher & Martino, 1996). They were later referred to as the "Gang of Four" (Maher & Martino, 1992). Sessions included small-group discussions with the Gang of Four, as well as individual interviews with each of the four children. These interviews continued and built upon the discussions from the earlier interviews. These began about a month after the classroom session in grade 4 (Maher & Martino, 2000).

It was within these sessions that Stephanie acknowledged a 'doubling pattern' for the total number of towers of different heights and where she produced an argument by cases to account for all possible towers (Maher & Martino, 2000). According to Maher and Martino (1996), Stephanie used the letter-grid notation she developed earlier to present her argument. She organized the towers as follows: towers with no blue cubes, towers with exactly one blue cube, towers with exactly two blue cubes next to each other, towers with exactly three cubes, and towers with exactly two blue cubes separated by a red cube. Her classmates pointed out that she could classify the two categories with the two blue cubes into one. Stephanie admitted that this was a possibility.

Seven months later on October 25, 1992, when Stephanie was in grade 5, the towers problem was again revisited. Stephanie and her classmates were asked to find all possible towers three cubes tall and to write a convincing argument for having found all possible arrangements for a person who was unfamiliar with the problem. This took the form of a letter to a student who was not present during the towers task sessions (Maher & Martino, 2000). Maher and Martino (1996) relate that her earlier argument had now become an elegant written version of ‘proof by cases’ to justify her solution to the Tower problem. Furthermore, they found that Stephanie’s written justification utilized the suggestions made by her classmates earlier so that she now had four cases instead of five. Stephanie also made use of the ‘doubling pattern’ to monitor her justifications (Maher & Martino, 2000).

Later that same year on February 26, 1993, Stephanie, still in grade 5, had another opportunity to think about the Tower problem and its variations. Maher and Martino (2000) describe a new problem, Guess My Tower, which called for finding all possible arrangements of towers three and four cubes tall when selecting from red and yellow cubes within the context of a probability problem. Stephanie, along with her partner Matt, predicted a total of 16 towers four cubes tall, again referring to the ‘doubling method’ but without being able to explain why the method worked. According to Maher and Martino (2000), the prediction did not match the number of cases Stephanie actually found, leading her to question and modify her theory. When asked to explain and justify her theory to another student, Stephanie found it difficult to do so, since she and Matt were still unable to build all 16 cases. At that point, the interviewer suggested that Stephanie try to explain “how the towers grow in height” (p. 265). Stephanie attempted to do so, inventing notation such as "parents" to refer to the original generation of towers, and "children" to refer to the new generation of towers built by adding a red or yellow cube. Matt joined the discussion and presented the "tree organization" initiated by another student, Milin. Maher and Martino (2000) relate that, as Stephanie listened, she began building higher towers based on the "tree organization," eventually making the connection between the 'doubling pattern' and the generation of towers by use of a "tree of towers" (p. 253, 266). They document that by the end of the session, Stephanie confidently demonstrated the 16 cases and explained the "doubling pattern" to the entire class.

3.3.4 Tower of Hanoi

Another problem that Stephanie took part in was the Tower of Hanoi problem in the fall of 1993 during her sixth-grade year. Stephanie was one of eleven students who took part in the four problem-solving sessions led by Robert B. Davis discussing the Tower of Hanoi problem.

The Tower of Hanoi problem (sometimes referred to as Tower of Brahma or the End of the World Puzzle) was invented in 1883 by French number theorist and recreational mathematician Francois-Edouardo-Anatole Lucas and was sold as a toy. The problem was posed by Lucas in the following manner: The player is given a tower of eight disks, initially stacked in decreasing size on one of three pegs. The objective is to transfer the entire tower to one of the other pegs, moving only one disk at a time and never moving a larger disk onto a smaller disk. Although not an official requirement, an implied condition is that the transfer of all the disks be accomplished in the least number of steps (in an efficient manner), and all three pegs must be used in working the problem

(Mayansky, 2007). The mathematical question becomes: How many moves are necessary and sufficient to perform the task?

Mayansky (2007) relates that the game is always possible and solvable with a simple recursive algorithm. According to Mayansky, the problem lends itself to more than one solution: a recursive solution and an open-ended, closed form solution. The students were able to see the recursive nature of the solution which created a possibility for them to also explore the closed form (Mayansky, 2007). This gave them the opportunity to explore other mathematics that they had not yet seen. According to Mayansky, since they had the opportunity earlier to learn algebraic reasoning and "Guess my Rule," the students were able to build on their earlier learning and learn about exponents and powers of 10. In addition, it allowed the students to work with patterns. The students were split up into 3 groups. Stephanie worked with three other students. At the beginning of the session, Dr. Davis reviewed "Guess my Rule" and reminded the children about the concept of functions. The children found solutions to the problem with one disk, two disks, and three disks. Stephanie was given credit for finding the solution to the two disk problem (Mayansky, 2007). She also took an active role in participating, by attempting to find the answers with more disks, and by explaining herself to her group and to the class. In the third session that took place in November 1993, Stephanie was able to find the pattern in the table: the number plus the number plus one more. Later on, Davis brought up the towers problem that many of them had worked on earlier in fourth- grade. He asked them how many towers 100 cubes tall they could make. Stephanie went to the board and wrote 2^{100}. The students adjusted this answer to $2^{100} - 1$ in order to fit the Tower of Hanoi problem. The children then explored how to get the numerical answer.

3.3.5 Advanced Guess my Rule

After students worked on tasks such as Guess my Rule and Tower of Hanoi, a task was developed by Robert Davis, in order to continue with the development of algebraic thinking of the students. This task was labeled Advanced Guess my Rule and focused on students working with inverse functions. In her seventh-grade year, Stephanie was one of thirteen students who took part in a three-day inquiry in which students were engaged in Guess my Rule activities (Bellisio & Maher, 1998). The first session was held on May 16, 1995. The sessions were videotaped using one video camera (Bellisio & Maher, 1998). The data for this study consisted of videotapes, student papers, and researcher notes (Bellisio & Maher, 1998).

The researcher presented the students with a two-column table of values with a box heading the left column and a triangle heading the right column. According to Bellisio and Maher (1998), the students were asked to write the rule that demonstrated the relationship between the box and triangle. They were then asked to show how to get the box value if the triangle value was known, thereby writing the inverse rule. Bellisio and Maher relate that the students worked in groups and then shared their rules with the class. The researcher, who also was the classroom teacher, also shared a rule. The students then tested the various rules to see if they worked with the given table. They were then challenged to justify and explain why these rules, which appeared different, could both work (Bellisio & Maher, 1998).

3.3.6 Interviews with Stephanie

As a precursor to this research, Maher and Speiser (1997) conducted a teaching experiment that allowed them to investigate the process by which Stephanie related more concrete representations to abstract symbolic notation. In the eighth grade, Stephanie's mathematics program was a conventional algebra course. Her continued participation in the longitudinal study included a series of eight individual task-based interviews. The data for this study comes from two of the eight interviews.

During the first of the two interviews conducted on March 13, 1996, Stephanie, on her own, made a connection to towers in examining her symbolic representation of the expansion of $(a + b)^2$ and $(a + b)^3$. She asserted that each 3-high tower with a choice of two colors can represent a monomial of degree 3 in two variables. The researchers inferred that Stephanie was visualizing the towers by referring to mental models since she did not have the physical ones in front of her in order to organize her monomials (Maher and Speiser, 1997).

In the second interview conducted on March 27, 1996, Stephanie again began with towers and then introduced the binomial coefficient notation as $C(n, r)$ based on organizing towers. Here, Stephanie explained that "*r* is a variable" which can range from 0 to *n*. According to the researchers, this observation shifts from the level of concrete towers to the more abstract patterns of formal symbols. Furthermore, it views the index *n*, the height of a tower, also as a variable. Stephanie then proceeded to explain that she could use Pascal's triangle to predict the terms of $(a + b)^n$ for binomial expansions with larger exponents. The researchers suggest that by revisiting explorations with block towers, Stephanie was able to trigger previous mental representations of towers that enabled her to shift to a more abstract level of symbolic notation for the binomial expansion.

3.3.7 Pizza Problem

Stephanie was first presented with the Pizza problem in the fifth-grade (Maher, Sran, & Yankelewitz, 2010). Maher, Sran, and Yankelewitz (2010) relate that the students were asked to find the number of different pizzas that can be made when selecting from two toppings: pepperoni and sausage. They were also given the option of putting a topping on only half of the pizza. In the first session, Stephanie participated as part of a group where the students in the group debated and discussed the merits of different ways of organizing the possible pizzas, eventually deciding on an answer. The next day, in the next session, the students were asked to find a way to convince the others that their answer was correct. According to Maher, Sran, and Yankelewitz (2010), Stephanie's group developed a notation referring to three possibilities: one topping, two toppings, and three toppings. The plain cheese combination was considered to be one topping. Stephanie proceeded to use this notation to explain to the facilitator how her group formed their pizzas. Throughout her work, she explained why and justified her reasoning.

One month later, the same group of students met for an extended session. This time they were presented with an easier version of the Pizza problem. They were asked to find the number of pizzas that can be made when selecting from four toppings. This time, the option of putting a topping on half of a pizza was not available. The students were given time to formulate a solution. Afterwards, they were asked to convince the facilitator of each of their solutions. Stephanie immediately volunteered and then proceeded to use an extension of her earlier notation of one topping, two toppings, and three toppings to explain her reasoning. Later within the same session, the students were presented with a more advanced version allowing for a regular, thin crust, or Sicilian crust, and allowing the option of putting a topping on half of a pizza. The students had to learn to extend their ideas from the earlier tasks in order to construct a solution; strategies such as trial and error were no longer efficient (Maher, Sran, & Yankelewitz, 2010).

In the eleventh grade, Stephanie was presented with another version of the Pizza problem. She, along with three other students from the longitudinal study, considered alternative approaches in order to solve the problem (Robert B. Davis Institute for Learning 1999a; 1999b; 1999c; 1999d; 1999e; 1999f). The problem was presented as follows:

> A local pizza shop has asked us to help design a form to keep track of certain pizza choices. They offer a plain pizza that is cheese and tomato sauce. A customer can then select from the following toppings: pepper, sausage, mushrooms, and pepperoni. How many different choices for pizza does a customer have? List all the choices. Find a way to convince each other that you have accounted for all possible choices. Suppose a fifth topping, anchovies, were available. How many different choices for pizza does a customer now have? Why?

Stephanie suggested that previous problems, such as Shirts and Pants and Towers, and previous strategies such as tree diagrams, might be helpful. The students proceeded to generate an exhaustive list of pizza options while choosing from four toppings. They accomplished this by constructing modified tree diagrams to represent systematic lists of pizza choices. They noticed patterns and categorized their list into cases. Furthermore, they wrote what they remembered to be the first several rows of Pascal's triangle in order to see whether the numbers in each row matched the corresponding number of toppings for pizza choices with fewer than four toppings. Finally, they recognized that the 16 choices correspond to the fourth row of Pascal's triangle and used that discovery to predict 32 choices when choosing from five toppings in the problem.

Stephanie and the three students then delved deeper into the task by considering how the addition rule of Pascal's triangle works to generate each successive level of the Pizza problem. They were able to recognize the isomorphism between the structures of adding an additional pizza topping choice and the addition rule for successive rows of Pascal's triangle. Furthermore, they recognized exponential growth for the problem. In the course of explaining their solutions, the students connected the Pizza problem with the Towers problems. They discussed how this problem appeared to be similar to building Unifix towers when selecting from different colored cubes. They concluded that the total number of pizza choices for a particular number of toppings is 2 to the power of that number of toppings.

CHAPTER 4 - METHODOLOGY

4.1 Design of the Study

This study utilizes a qualitative approach based on selected videotape data from the longitudinal study. The longitudinal study of students from Kenilworth began in the spring of 1989 and is currently in its twenty-fourth year. The present study addresses the issue of Stephanie's development of mathematical ideas within an early algebra strand during her eighth-grade year.

4.2 Data Collection

The data for this study comes from two primary sources. The first includes a collection of twenty-four CDs comprising the eight individual task- based interviews with Stephanie. The videotaping was done from two cameras: a work view as well as a people view. The work view zoomed in on Stephanie's written work while the people view was a little farther away but allowed the viewer to observe the interactions between Stephanie and the researcher as well as gestures and facial expressions. It also allows the viewer to see any observers watching the session. Each session was recorded on one to two CDs lasting from one to two hours. The second major source of data comes from the transcripts of these eight videotaped sessions. These transcripts have been verified and retyped with updated formatting and a line numbering system.

The eight sessions with Stephanie occur during her eighth-grade year. At that time she was enrolled in a conventional algebra class in a parochial girl's school. The interviews were conducted over a six-month interval from November 8, 1995 to May 1, 1996. Each interview typically began with questions regarding the mathematics that Stephanie was currently studying in school. She was given an opportunity to talk about it and ask questions. This in turn allowed her to pursue her thinking about fundamental ideas in greater detail.

Maher and Speiser (1997), the two main researchers in these interviews, describe the interview structure as their way of *working with* the children. They describe the following process, a working theory, which is reflected in the research interview structure. First, the interviewer engages the child in an exploration that attempts to reveal the child's thinking. Later in the same interview or in a subsequent interview, the child pursues these ideas by initiating and accepting responsibility for the direction of the discourse. The researchers refer to this process as "folding back" (p. 126). They further describe the following step as the "teaching phase" which strives to investigate deeper connections. They note that often children make surprising connections on their own initiative. When this occurs, the researchers sometimes form new hypotheses. They question the child about the structural similarity that is being visualized or built and then invite the child to explain further. The child is then able to construct explanations of her own (Maher & Speiser, 1997).

The following table outlines the dates and topics of the eight interview sessions.

***Table 4-1:* Summary of Interview Sessions with Stephanie**

Date	Topic(s)	Researcher(s)
11/08/95	Perimeter; Distributive Property; Square of a binomial;	Carolyn A. Maher
01/29/96	Exploration of Concept of area of square units; Continue with the square of a binomial;	Carolyn A. Maher; Alice S. Alston
02/07/96	Stephanie builds an understanding of the square of a binomial using square units; Begin discussion of a cube of a binomial;	Carolyn A. Maher
02/21/96	Stephanie builds the cube of a binomial;	Carolyn A. Maher

03/13/96	Combinatorics notation is introduced using trains of 4 cubes long; Stephanie relates work from previous session to Pascal's triangle;	Carolyn A. Maher
03/27/96	Stephanie explains content of previous session to another researcher; She builds an understanding of the Binomial Theorem;	Carolyn A. Maher; Robert Speiser;
04/17/96	Stephanie rebuilds towers 2, 3, and 4 high with unifix cubes; Pascal's triangle; Combinatorics notation;	Carolyn A. Maher
05/01/96	Exploration of a Tetrahedron; Recap of Towers;	Robert Speiser

4.3 Analysis of Interview Data

Analysis of the video data is based on the analytical model described in Powell, Francisco, & Maher (2003) which employs "interacting, non-linear phases" (p. 413). This model comes from a methodology developed from the video data of research at the Robert B. Davis Institute for Learning (RBDIL) at Rutgers University and is based on over two decades of research on the development of mathematical ideas of a focus group of students (Davis & Maher, 1990, 1997; Maher & Martino, 1996; Maher & Speiser, 1997). Analysis begins by repeatedly viewing the video data. The intent is for the researcher to gain an overall picture of the session. The video data is then transcribed and verified, and critical events are flagged. Once critical events are established, they are grouped into various categories according to emerging, common themes. These categories are then assigned codes so that a narrative can be composed that connects the critical events within a story that addresses the research question(s) (Powell, Francisco, & Maher, 2003). These steps are now described in more detail.

4.3.1 Viewing the Video Data

In order to become familiar with the video data, the researcher must watch and listen to the video several times. According to Powell, Francisco, and Maher, he/she should watch and listen without imposing a "specific analytic lens on their viewing." This allows the researcher to gain a general view of what is happening and perhaps allow him/her to identify parts that may require further scrutiny (Powell, Francisco, & Maher).

4.3.2 Transcribing and Verifying the Interviews

The reasons for transcribing data vary. Powell, Francisco, and Maher (2003), claim transcripts allow the researcher to provide evidence of students' assertions. This is true in this case where line numbers are referenced to justify claims. They further assert that for researchers analyzing participants' discursive practices, transcription is useful in order to allow them to "view the printed, sequential rendering of speech to see what it reveals about the mathematical meanings and understandings participants construct" (p. 422). In this study, the transcript includes little pertaining to body movements, but does include some inscriptions of Stephanie's work. Furthermore, they note that transcripts can reveal important categories that are not immediately identified by viewing video data, even if viewed repeatedly. In the case of this research, much of the organization of the coding scheme developed throughout the transcription process as emergent themes unfolded.

Data for this study that was transcribed earlier is verified and updated. The line numbering begins with the first line in the first of the eight CDs. Each of the sessions begins with the number one and is assigned a letter. For instance, any reference to a line number in the first session would look like (A: 117). The second session is designated B and so on. The transcript is separated into five minute time intervals with the time labeled as hour:minute. Researchers are referred to as R1, R2, etc. Speakers' remarks are precisely recorded.

4.3.3 Critical Events

A critical event demonstrates a significant or contrasting change from a previous understanding (Powell, Francisco & Maher, 2003; Maher, 2002; Maher & Martino,1996). Significant contrasting moments can be events that either confirm research hypotheses or provide evidence to the contrary; they can be instances of cognitive victories, conflicting schemes, or generalizations (Powell, Francisco & Maher, 2003). In short, a critical event can be any event that is somehow significant to the researcher's agenda. Critical events can emerge from repeated viewing of video data, transcripts, or student work. Powell, Francisco, and Maher describe a critical event as contextual: its importance is directly related to the particular research questions being studied. In this case, where Stephanie's development of mathematical ideas or the growth of her mathematical understanding is under observation, a critical event can be associated with a time line that allows different events to be categorized into different strands within a narrative (Powell, Francisco, & Maher, 2003). This is accomplished by developing a coding scheme that categorizes the critical events.

4.3.4 Coding Scheme

Powell, Francisco, and Maher (2003) describe this step in the analysis of video data as an activity aimed at identifying themes that help a researcher interpret data. At this stage, the researcher focuses attention on the content of the critical events. They advise the researcher studying the growth of mathematical understanding to code for learners' mathematical ideas, mathematical explanations or arguments, mathematical presentations, and features and functions of discourse.

Example of Coding Scheme

The following is an explanation of the coding scheme that emerged in the course of analyzing seven of the eight sessions of Stephanie with Carolyn Maher as the primary researcher/teacher.

Building meaning
BCA – Uses a generic form of reasoning to support solutions. This is defined by Rowland as "Reasoning about a paradigmatic example whose properties can be applied to the set and lends insight into a more general truth, which in turn verifies the claim made about the particular example" (2002). More specifically, I am looking for instances where Stephanie extends meaning from the concrete or simple to more general, or abstract.

BMP – Using mathematical properties and/or concepts.
BDI – Discovering new ideas or having insight into some topic.

BEJ – Stephanie's explanations/justifications for solutions/representations.

BR – Representations Stephanie uses to present her responses.
BR – S Symbolic representations: algebraic expressions, notation
BR – V Visual representations: diagrams, models, tables

Dealing with problems (obstacles)
PPK – Referring back to prior experience.
PNE – Using concrete examples/numbers to prove/disprove an idea.
PAH – Asking for help and/or seeking clarification.

Obstacles **-**
OBS – misconceptions, confusion, lack of information.

Most of these codes correspond to at least one of the steps in Davis' cycle of thinking about mathematical knowledge, mentioned earlier in this paper (Davis & Maher, 1990). The noted instances using these codes constitute the critical events pertaining to the research questions. Upon reviewing the transcript and video data several times, I was able to categorize them into the aforementioned codes and elaborate on how they fit within Davis' five steps.

4.3.5 Composing a Narrative

Once a coding scheme is established and critical events are grouped according to their respective themes, a narrative naturally unfolds that tells the story of Stephanie's mathematical growth and understanding. One must keep in mind that ideas for this narrative could begin at the beginning of the research (Powell, Francisco, & Maher, 2003). Furthermore, the composition of the narrative takes place amid constant revisions to the code as well as the choice of critical events. The following is an excerpt from the analysis of the session that took place on 11/08/95.

> R1 presents Stephanie with the expression *a(x + y)* and asks her if she's done anything like that. Stephanie says no. She then asks Stephanie what it could possibly mean. Stephanie conjectures that it is equal to *ax + ay.* This forms another of Stephanie's symbolic representations. R1 then suggests that Stephanie put in numbers for *a*, *x*, and *y* in order to test her conjecture. Stephanie puts in 2 for *a*, 3 for *x*, and 4 for *y* and finds that both expressions are equal. She is applying Davis' step (4) *Check this mapping (and these constructions) to see if they seem to be correct*. She will frequently use this technique in later episodes to prove/disprove an idea or representation.

Table 4-2: **Excerpt of transcript of first session – 11/8/95**

15:00-19:59	248	R1	Okay. Have you done anything like this yet? Okay – as we do these examples. Did you do anything like this? *[a(x + y)]*	
	249	Stephanie	Um hm. Not that I can recall. No.	
	250	R1	No, what do you think that could possibly mean?	
	251	Stephanie	It's any number times two other variables that could also stand for any number – so – can you get a number that's like *ax + ay*?	BMP
	252	R1	Let's think about that? Why don't you write –	
	253	Stephanie	'Cause that's what it's telling you to do. It's telling you.	BMP
	254	R1	So you think that's going to be *[Stephanie writes ax + ay]*.	
	255	Stephanie	That's what it's telling you.	
	256	R1	That's an *a*, right? *[corrects Stephanie's handwriting]*	
	257	Stephanie	Yeah.	
	258	R1	Okay – So your conjecture is that – why don't you test it? Why don't you try some numbers for *a*, *x*, and *y*? And see if it works?	
	259	Stephanie	Alright *[2(3 + 4)]* is six plus eight is fourteen.	PNE

	260	R1	Does that work?	
	261	Stephanie	Well, actually I have to try one number at it – well…	PNE
	262	R1	Well.	
	263	Stephanie	I have to just plug in one number and then – what I really have is go like equals fourteen and then plug in – if I just plugged in like the two.	PNE
	264	R1	Okay. So what you're saying here – two – is…	
	265	Stephanie	It comes out to be the same – I mean, I guess you could put a variable with another variable and multiply it.	BCA
	266	R1	Right.	
	267	Stephanie	We just never did it before.	

CHAPTER 5 - RESULTS

5.1 INTRODUCTION

Through analysis of video data taken from Stephanie's eighth-grade year, this study focuses on Stephanie's development of mathematical ideas within the context of early algebra and combinatorics. The research examines Stephanie's mathematical behavior as the subject of eight task-based interviews, conducted over a six- month interval from 11/8/95 to 5/1/96. These are summarized in the following table.

Table 5-1: **Summary of Interview Sessions with Stephanie**

Date	Topic(s)	Researcher(s)
11/08/95	Perimeter; Distributive Property; Square of a binomial	Carolyn A. Maher
01/29/96	Exploration of Concept of area of square units; Continue with the square of a binomial;	Carolyn A. Maher; Alice S. Alston
02/07/96	Stephanie builds an understanding of the square of a binomial using square units; Begin discussion of a cube of a binomial;	Carolyn A. Maher
02/21/96	Stephanie builds the cube of a binomial;	Carolyn A. Maher
03/13/96	Combinatorics notation is introduced using trains of 4 cubes long; Stephanie relates work from previous session to Pascal's triangle;	Carolyn A. Maher
03/27/96	Stephanie explains content of previous session to another researcher; She builds the Binomial Theorem;	Carolyn A. Maher; Robert Speiser;
04/17/96	Stephanie rebuilds towers 2, 3, and 4 high with unifix cubes; Pascal's triangle; Combinatorics notation;	Carolyn A. Maher
05/01/96	Exploration of a Tetrahedron; Recap of Towers;	Robert Speiser

This study is designed to describe how Stephanie explored and built algebraic ideas in the context of problem solving. It will focus on how Stephanie built meaning of the Binomial Theorem and related the meaning to a row in Pascal's triangle. The research questions guiding this study are: 1) What representations does Stephanie use to present her problem solving in the tasks, problems, and/or questions posed? 2) What explanations and justifications does she give for her solutions and/or the representations that she uses? 3) What, if any, obstacles to understanding does she encounter? 4) How, if at all, does she overcome these obstacles?

In searching for answers to these questions, I examined Stephanie's problem solving using the framework posed by Robert B. Davis, in which he describes five steps that a learner uses in order to build mathematical meaning and understanding. According to Davis, when the proper conditions are present, in order to think carefully and deeply about a mathematical situation, the learner cycles through a series of steps, as follows: (1) building a representation for the input data; (2) carrying out a memory search for relevant knowledge that can be used in solving the problem or moving further with the task; (3) constructing a mapping between the data representation and the knowledge representation; (4) checking the mapping; and (5) applying the knowledge representation in order to solve the problem. According to Davis and Maher (1990), at times, one must cycle several times through one of these steps before moving forward.

Results are divided into chapters corresponding to each interview session. Each of the interview sessions was analyzed individually and divided into subtasks, usually according to topic in a chronological manner. The breakdown of the sections varied according to content. Data was analyzed chronologically in order to preserve the continuity and the connections between the sessions as well as within a particular session.

5.2 SETTING

The eight task-based interviews took place after school hours. Generally, there was one main researcher, although in some sessions there were two to three. Carolyn A. Maher served as the primary researcher/facilitator and is referred to as R1 in all the sessions. Each interview lasted approximately one and a half hours. The format of the interviews was generally the same, beginning with inquiries about the mathematics that Stephanie was currently studying in her eighth-grade algebra course, although naturally the content changed.

Two camera views were combined and transcribed for analysis. The "People View" camera focused on the people sitting around Stephanie. It allowed the observer to catch expressions, gestures, and general movement. The "Work View" camera zoomed in on the writing occurring at the table as well as any representations built with manipulatives such as the Unifix cubes. A transcript combining audio and visual information from both views corresponding to sessions one through seven is included in this research as appendices. For instance, the transcript for the first session occurring on November 8, 1995 is in Appendix A. When referencing the transcript, line numbers are preceded by the appendix letter, i.e. A: 15 would refer to line number 15 in Appendix A. The eighth session will not be analyzed as the content does not pertain to the focus of this research.

5.3 SESSION 1- November 8, 1995
Perimeter, Distributive Property, Square of a Binomial

5.3.1 Background

The video begins with the primary researcher, Carolyn Maher, designated R1 in the transcripts, asking Stephanie what she is doing in school. Stephanie pulls out her algebra book and homework papers and mentions, “Now we're . . .um, consecutive int - problem solving, perimeter and angle measure with algebra. That's the last thing we did.” (A: 8) R1 replies, “Oh - so you've done some perimeter stuff” (A: 9)? Stephanie shows her homework to the researcher, then comments that “red problems are hard” (A: 14, 16). Stephanie shows her an example of a “red problem” found in her homework that has to do with perimeter. R1 then presents Stephanie with a problem having to do with perimeter. This problem originated from one of the college level classes of another observing researcher, Elena Steencken, designated R2. It is as follows:

> I need to construct a dog kennel for my very large Akita. I want to enclose a space fifteen feet wide and twenty-five feet long. The fence company needs to know how much fencing is needed, I need, so that the company can give me an estimate of the cost. Can you help me decide how much fence I will need to order? (A: 27)

Stephanie immediately responds with “It’s just um, twenty-five plus twenty-five plus fifteen plus fifteen.” (A: 31) R1 then modifies the problem and asks Stephanie to come up with a general way of expressing how much space is needed if the length is l and the width is w. Stephanie responds with $2l + 2w = s$ for space. R1 then presents Stephanie with the expression $2(w + l) = s$ and states that this is the space. Stephanie replies, “It’s the same thing.” (A: 65)

This discussion forms the first segment of the video data, lasting approximately 18 minutes. In the second segment, R1 presents Stephanie with the expression $a(x + y)$ and asks her if she’s done anything like that. Stephanie says no. She then asks her what it could possibly mean. A discussion ensues lasting approximately 10 minutes. From there, R1 asks Stephanie to think about the expression $x \cdot x$. Stephanie responds with “the variable x x amount of times”. R1 then suggests that Stephanie use numbers to help her write a general expression for $x \cdot x$. This portion of the video, lasting approximately

3 minutes, serves as the third segment. The aforementioned subtasks culminate in the fourth segment of the video which lasts approximately 25 minutes. In this final part, Stephanie is asked to think about the square of a binomial $(x+y)(x+y)$ how to simplify it and what it means. The previous subtasks now serve as assimilation paradigms or building blocks in order to complete the last task.

5.3.2 Subtask 1: Perimeter, Distributive Property

This subtask revolves around the relationship between the two expressions representing the perimeter of a space: $s = 2l + 2w \text{ and } s = 2(l + w)$. The first expression forms the first of many subsequent representations that Stephanie produces. This one is a symbolic representation and serves as the first of Davis' five steps of thinking about a mathematical situation: (1) *Build a representation for the input data*. Stephanie recognizes that the two expressions are equal due to the distributive property (A: 63, 65, 67). She is aptly able to describe the mechanics of the distributive property as follows: "Well, distribute – you have to – if you have a number outside the brackets and there's no plus sign you can't remove the parentheses until you um multiply it by each number inside the parentheses." (A: 69) However, when asked whether she has thought about why it works, Stephanie admits to just using it without considering its meaning (A: 93). She mentions, "…we just were told that it was because you um you put parentheses for a certain reason and it means that you have to do whatever is in the parentheses first" (A: 97).

The researcher then asks Stephanie about $2x$ and $x + x$ and whether they are related. Stephanie responds that they are the same and explains why. The following conversation is an excerpt of the transcript:

Table 5-2: **Stephanie explains the equality of the expressions 2*x* and *x* + *x*.**

108	R1	Yeah – If I write an x and I tell you that…	
109	Stephanie	x stands for any number.	BMP
110	R1	Any number.	
111	Stephanie	Any number.	
112	R1	So if I write two x.	
113	Stephanie	It stands for any number times two.	BMP
114	R1	Any number times two. Okay could it stand for – for this? *[writes x + x]*	
115	Stephanie	Um. Yeah, because with um with two you're do – well yeah because you're just adding another one of those.	BMP
116	R1	Okay so. The two x could mean twice that number I'm thinking about?	
117	Stephanie	Yeah. Twice the number.	BMP
118	R1	Or it could mean that number plus that number?	
119	Stephanie	Yes.	BMP

In this instance, Stephanie is applying Davis' step (2) *carry out memory searches to retrieve or construct a representation of relevant knowledge that can be used in solving the problem or otherwise going further with the task*; and (3) *construct a mapping between the data representation and the knowledge representation*. Stephanie is able to use her knowledge of mathematical properties to connect the representations.

R1 then refers back to the expression $2(w+l)$ and asks Stephanie to think about $(w+l)$ in the same way as she thought about the x. Stephanie replies, "So you – in other words, you can also write it. It's also the same as *w* plus *w* plus *1* plus *1*. And you can put that in parentheses, because it's doubling each one." (A: 131) This is an example of Stephanie's generic reasoning. She is able to extend the concepts discussed when considering $2x$ and $x + x$ and is able to apply them to the representation $2(w+l)$. The researcher then asks Stephanie to think about $5(w+l)$ and what it means. Stephanie responds with, "You're only saying that you're multiplying – you're taking any number

'w' and you're um I guess – if you're going to do it like how the $2x$ was – um – any number twice, you can do the $5w$ – any number five times. And the $5l$. " (A: 185) Stephanie realizes that the expression is equivalent to $5w + 5l$ but skips over the step that it is the same as $(w + l)$ five times (see lines 198, 199, 215, 217). They briefly discuss the intermediary step.

5.3.3 Subtask 2: Distributive Property: $a(x + y)$

In the second part of the interview, R1 presents Stephanie with the expression $a(x+y)$ and asks her if she's done anything like that. Stephanie says no. She then asks Stephanie what it could possibly mean. Stephanie conjectures that it is equal to $ax + ay$. This forms another of Stephanie's symbolic representations. R1 then suggests that Stephanie put in numbers for a, x, and y in order to test her conjecture. Stephanie puts in 2 for a, 3 for x, and 4 for y and finds that both expressions are equal. She is applying Davis' step (4) *Check this mapping (and these constructions) to see if they seem to be correct*. She frequently uses this technique in later episodes to prove/disprove an idea or representation. They discuss whether the representation will always hold true. In the course of their discussion, Stephanie questions the conditions under which the distributive property should be used and attempts to deal with an obstacle to her understanding. Stephanie: "Um, I don't know. If it works every time, I don't understand why they make us um distribute in the first place – if it works every time. So I don't think – I think there's going to be a problem *(inaudible)* I mean 'cause – it's pretty dumb then if we always have to distribute – you know." (A: 293) She goes on to further question when to use the distributive property as follows:

***Table 5-3:* Stephanie discusses conditions of using the distributive property.**

305	Stephanie	If I had a variable. Like if it was *[writes 2(x + 4)]*. $2(x + 4)$ right. I have to distribute first 'cause I can't add four to x.	BEJ; BMP
306	R1	Okay, so what would that look like?	
307	Stephanie	So that would have to be $2x + 8 = 14$.	
308	R1	Where did you get the fourteen from?	
309	Stephanie	Well, fourteen was my answer up here. I'm just doing – using	
310	R1	That's if you don't know what x is?	
311	Stephanie	Yeah.	
312	R1	Okay.	
313	Stephanie	Eight minus eight. *[writes 2x + 8 – 8 = 14 – 8]* equals *(inaudible) [continues "figuring*"] x equals three. It worked.	PNE; BMP
314	R1	Interesting.	
315	Stephanie	This problem's working out.	
316	R1	What's the it that worked? What were you thinking when you did it?	
317	Stephanie	Well, um I was just try – 'cause like here I didn't have to distribute, but if I had a problem where I had a variable in the inside of the parentheses I would have to distribute.	BDI
318	R1	Um hm.	
319	Stephanie	Because I can't combine like terms if they're not the same -- so	BMP
320	R1	Um hm.	
321	Stephanie	I was just saying that you know if you have a variable, you have to distribute first.	BMP

Notice that Stephanie is going back to her basic understanding of mathematical properties as well as trying out different possibilities with numbers to convince herself (Davis' step (4) *Check this mapping (and these constructions) to see if they seem to be correct*;).

R1 then goes back to the earlier representations: $2x$, $2(l + w)$, $5(l + w)$, and adds another one $8(l + w)$. She and Stephanie revisit the concept that for instance: 5 times $(l + w)$ just means $(l + w)$ five times which is the same as 5 of l and 5 of w. They become convinced that it will be true for any number times $(l + w)$ (A: 332 – 341). R1 then brings up

$a(x+y)$ again. Stephanie is now able to describe what it means in her own words as follows:

***Table 5-4:* Stephanie explains the meaning of the expression** $a(x+y)$**.**

350	R1	If you have $(x + y)$ a times? How would how would you reason it in your head? How would you think about it?	
351	Stephanie	That you're taking any number.	BCA
352	R1	Um hm.	
353	Stephanie	And you're adding it with itself.	BCA
354	R1	Um hm.	
355	Stephanie	as many times as a is.	BCA

Again, Stephanie is able to apply generic reasoning in order to extend the concept with concrete numbers to a general "*a*." Before she can get to the final step however, Stephanie forms some intermediary visual representations of what this expression $a(x+y)$ actually means. She imagines "rows of x's and rows of y's a amount of times." (A: 395, 397, 399, 401) Upon being encouraged to write it down, Stephanie is able to express $a(x+y)$ as $ax+ay$, which is what she initially conjectured.

5.3.4 Subtask 3: Exponents: $x \cdot x$

In this segment, R1 then presents Stephanie with the expression $x \cdot x$ and asks her what she thinks it means. Stephanie responds with "the variable x x amount of times," thereby again using generic reasoning to apply the previous concept to this situation (A: 424). R1 then suggests that Stephanie use numbers to help her write a general expression for $x \cdot x$. Stephanie suggests $x = 2$ and gets $2 \cdot 2$ and then $x = 3$ and gets $3 \cdot 3$. When asked if there was another way to write these expressions, Stephanie suggests using exponents, thereby retrieving previous knowledge to help her with the task at hand (Davis' step (2): *carry out memory searches to retrieve or construct a representation of* *relevant knowledge that can be used in solving the problem or otherwise going further with the task*). She then builds on her knowledge of exponents to write *2 · 2* as 2 to the second power and *3 · 3* as 3 to the second power, Davis' step (5): *When the constructions and the mapping appear satisfactory, use technical devices (or other information) associated with the knowledge representation in order to solve the problem*. In this case, Stephanie is using mathematical properties. Stephanie, at this point, is unsure whether $x \cdot x$ is x to the x power or x to the second power. She then conjectures that it is x to the second power. When asked why, Stephanie responds with "cause x to the x power would mean – say x is – x is one thousand one hundred and fifteen. That would mean one thousand one hundred and fifteen one thousand one hundred and fifteen times and that's-" (A: 460). Stephanie is using an example with concrete numbers to disprove her other conjecture that $x \cdot x$ is equal to x to the x power.

5.3.5 Subtask 4: Square of a Binomial: $(x + y)(x + y)$

Finally, in the last part of the interview, R1 presents Stephanie with the expression $(x + y)(x + y)$ and asks her what she thinks it means. Stephanie initially explains that you have to multiply them since you can't combine the terms in the parentheses because "they're not the same variable" (A: 488, 492). Stephanie is drawing upon her procedural knowledge of algebra to justify and explain her responses. She mentions that she "can't figure out how to get around it" and then conjectures that the expression would be equal to $x^2 + y^2$ (A: 498). She is building her initial representation (Davis' steps (1)-(3)) then proceeding to test some numbers (Davis' step (4)) and finding that it doesn't work (A: 502).

Upon reaching another obstacle in understanding, Stephanie, with R1's encouragement, attempts to explore the meaning of the expression, hoping to discover something insightful. When asked how she thought about this expression, Stephanie responds, "Oh. That it's um x times x plus y or x plus and y plus y a amount of times. And since I didn't know a, it was just like rows and rows and rows of numbers." (A: 514) Stephanie clearly has a visual representation in mind. However, she is having difficulty limiting it since a is unknown. The following excerpt illustrates this:

***Table 5-5:* Stephanie has difficulty representing $(x + y)$ *a* amount of times.**

515	R1	Okay. How many times did you get those rows of x's and those rows of y's?	
516	Stephanie	A lot. 'Cause I didn't have any stopping point and that –	BR-V; OBS
517	R1	You did have a stopping point.	
518	Stephanie	Well, it was a, but I didn't –	
519	R1	It was a.	
520	Stephanie	But I didn't know what a was.	
521	R1	Right. Exactly. Okay. Show – Now we're thinking of – Remember that. Remember. a could be anything.	
522	Stephanie	a could be anything.	

The next step is to think of a as $(x + y)$. After grasping this concept (A: 527-531), Stephanie is able to explain what it means to her although she is still a bit unsure. She states, "I have x plus y times x plus y, so I have it x plus y amount of times, but I don't know." (A: 556) She is using generic reasoning to apply the concepts established earlier with numbers to extend them to a more complicated expression. This gives her a way to think about it but she still is not sure how to simplify it. After discussing with R1, Stephanie then decides she can break it down by thinking about it x amount of times and *y* amount of times (A: 572). She tests out her conjecture by putting in numbers. Her first set works with no problems.

Table 5-6: **Stephanie tests her conjecture that** $(x+y)(x+y) = x(x+y) + y(x+y)$**.**

591	R1	That's interesting. *[Stephanie writes: 2(2 + 3) + 3(2 + 3)* *4 + 6 + 6 + 9* *10 + 15* *25]*	PNE
592	Stephanie	…six plus nine equals ten plus (inaudible) twenty-five.	
593	R1	Is that what you were supposed to get before?	
594	Stephanie	Yep.	

Stephanie is ready to move on but R1 causes her to question whether one attempt is enough to show that it will 'always work'. She, therefore, substitutes a different set of numbers:

> *[Stephanie tries 4 and 5:*
> *4(4 + 5) + 5(4 + 5)*
> *16 + 20 + 20 + 25]*

However, she seems to think that her result is incorrect and is unsure how to interpret it.

Table 5-7: **Stephanie questions whether she is testing her conjecture correctly.**

609	Stephanie	Now it didn't work.	OBS
610	R1	It didn't work?	
611	Stephanie	No.	
612	R1	Let's see what you had happen here.	
613	Stephanie	Well, now I got a higher number. - - - But I'm using higher numbers.	OBS
614	R1	Right. So?	
615	Stephanie	So – it's okay?	OBS
616	R1	Did you test it on both sides?	
617	Stephanie	Not yet.	OBS
618	R1	Remember what you're testing that works. Remember what you're – See why did the twenty-five work here? Remember. Look back and see what you did here.	
619	Stephanie	Oh.	

Stephanie has perhaps lost sight of what exactly she is trying to test. This represents one of the obstacles that she deals with in working on these tasks. She eventually finds that it works: *(x + y)(x + y)* can be thought of as *x(x + y) + y(x + y)*.

Now that Stephanie has a visual as well as a symbolic representation, it remains for her to simplify it. She uses the Distributive Property to simplify and comes up with $x^2 + x \cdot y + y \cdot x + y^2$. Stephanie is now able to see why her initial conjecture of $(x+y)(x+y) = x^2 + y^2$ was incorrect (A: 687-688). However, she is not quite sure how to simplify the middle terms. This confusion regarding algebraic simplification serves as another of Stephanie's obstacles to understanding. She initially conjectures the following:

Table 5-8: **Stephanie conjectures that (*x* + *y*)(*x* + *y*) =** $(x^2 + x + x) + (y + y + y^2)$**.**

696	Stephanie	Yeah. I can get – Could I – Now if I added another *x* there, it could be *x* to the third, right? Could I do –	OBS
697	R1	Now I'm confused. Let's think what you're doing here. So –	
698	Stephanie	Alright. Because then – alright – it would be *x* plus *x* plus *x* plus – just so that it's easier for me - - *y* plus *y* plus *y*-squared. *[Stephanie writes:* $(x^2 + x + x) + (y + y + y^2)$ *]*	OBS

In order to test her conjecture, Stephanie draws upon a heuristic method used earlier when overcoming obstacles: putting in numbers on both sides. She finds that it doesn't work (A: 706). Stephanie keeps trying to manipulate the two terms. She describes her struggle as follows: "Yes. *[pause]* Oh, but I can't move them. I have to keep them – Can I just – 'cause when I tried to do do just *y* plus *y* last time it was like…" (A: 741)

She then proceeds to put in the numbers $x = 2$ and $y = 3$ into the terms $x \cdot y$ *and* $y \cdot x$. She recognizes that they both give her six and that they'll always be the same (A: 755). R1 asks her what would happen if she used the numbers five and six: would the two terms still remain the same? Stephanie responds that yes they would be the same and initiates an explanation as follows:

***Table 5-9:* Stephanie explains that *xy* and *yx* are the same.**

756	R1	Suppose you use five times six and six times five?	
757	Stephanie	Yeah – 'cause it's the communative-	BDI, PPK
758	R1	So *xy* is that always the same as *yx*?	
759	Stephanie	Yeah – No – Wait – Yeah, 'cause it's the same thing.	BDI
760	R1	You just – you just –used a big word. What was that word? You just used?	
761	Stephanie	Oh. Communative.	
762	R1	Commun – Commutative?	
763	Stephanie	Commu – Yeah that one.	
764	R1	Yeah. What's that mean?	
765	Stephanie	It means for addition and multiplication, it doesn't matter the order.	BEJ

Here, Stephanie is drawing upon her knowledge of mathematical properties in order to simplify her representations. She is applying Davis step (5) *When the constructions and the mapping appear satisfactory, use technical devices (or other information) associated with the knowledge representation in order to solve the problem.*

Stephanie then attempts to combine them. She rewrites $y \cdot x$ *as* $x \cdot y$. Using the idea that there are two of them, she expresses their sum as $2(x \cdot y)$, thereupon applying the concept discussed earlier in the video where the sum of two identical quantities is equivalent to doubling that quantity (A: 785 – 789). At this point, Stephanie faces another obstacle in that she is unsure how to simplify $2(x \cdot y)$. Stephanie initially vetoes the idea that it is equal to two times x times two times y, but conjectures that it may equal $(x \cdot y)$ squared (A: 790 – 791). Stephanie is unsure where to go from here and admits to being confused (A: 800 – 803). R1 suggests that Stephanie put in some numbers to test both conjectures. Stephanie is learning the heuristic method of testing out her conjectures by putting in numbers and using this technique to overcome confusion and obstacles to understanding. Upon putting numbers into both expressions, Stephanie concludes that they are both incorrect (A: 821). Part of the difficulty is that Stephanie is attempting to use the distributive property incorrectly.

Table 5-10: **Stephanie misuses the distributive property.**

835	Stephanie	I was just – 'cause I always do it like that 'cause I'm used to having like a variable and when you have a variable in there you can't – you have to distribute first. So I'm used to distributing first.	BEJ
836	R1	So this isn't – Is this the Distributive Law? Two *xy*? I mean – do you need that dot? - *x* times *y*. Can you write that as *xy*? Have you had that yet?	

The other issue is the notation. Once Stephanie realizes she can drop the parentheses and the dot, she is able to express $2(x \cdot y)$ *as* $2xy$. In order to make sure, Stephanie puts in numbers in both sides of the expression $(x+y)^2 = x^2 + 2xy + y^2$ and finds that they satisfy the equation. The end result is that Stephanie was able to prove that that the square of a binomial is $(x+y)^2 = x^2 + 2xy + y^2$ based on meaning and understanding.

CHAPTER 6 - SESSION 2 - January 29, 1996

Square of a Binomial, The Area Problem

6.1 Background

The video begins with Carolyn Maher, designated as R1 throughout the transcript, and Stephanie, discussing the merits of various Catholic high schools that Stephanie is considering attending. The conversation then shifts to what was covered in the last session. Upon being asked if she remembered the contents of the last session, Stephanie responded, "Something with x and y. It was like grouping them. It was something like x to the y or something. I don't remember like exactly." And, "But I know it had to do with changing around like the way it was placed. It was x to the y plus like x to the y. It could be just be like – x y like parentheses or something. I don't remember – like exactly." (B: 35, 37) Since Stephanie clearly does not remember the contents of the last session (occurring over two months ago), R1 suggests they rebuild what they did last time and presents Stephanie with the expression $(a+b)^2$. She then asks Stephanie what she thinks it means. Stephanie answers that they distributed and it would be $a^2 + b^2$. With regard to meaning, Stephanie states that the initial expression means $a \cdot a + b \cdot b$. R1 suggests that Stephanie tests this conjecture with numbers. At first, Stephanie is unsure how exactly to do this, but then it comes back to her with the help of R1 and she concludes that the two expressions are not equal to each other (B: 47 – 79). R1 points out that she has just proved that $(a+b)^2 \neq a^2 + b^2$ by counter-example. This portion of the video lasted approximately six minutes.

For the next six minutes, R1 and Stephanie return to the expression $(a+b)^2$ and continue to explore its meaning. They establish that the 'squared' means to multiply something by itself (B: 102 – 109). Stephanie is retrieving prior knowledge of mathematical terminology in order to build meaning (Davis' step (2) *Carry out memory searches to retrieve or construct a representation of (hopefully) relevant knowledge that can be used in solving the problem or otherwise going further with the task.*) Stephanie then applies that definition to $(a+b)^2$ and is able to express it as $(a+b)(a+b)$. After writing it down, Stephanie realizes that this is what they did in the last session (B: 129 – 131). R1 and Stephanie then discuss the various reasons why students commonly express $(a+b)^2$ as a^2+b^2. From there, R1 asks Stephanie to come up with any special cases where $(a+b)^2 = a^2+b^2$. Stephanie immediately responds with zero. They test it out and come up with zero equals zero. She has just cycled through Davis' steps (2) *Construct a mapping between the data representation and the knowledge representation*; and (3) *Check this mapping (and these constructions) to see if they seem to be correct*. R1 then suggests $a = b = 1$, but upon testing it on both sides, they find that it does not satisfy the equation.

In the next part, lasting approximately forty minutes, R1 returns to the expression $(a+b)(a+b)$ and asks Stephanie how she could express it not as a product. Stephanie is having difficulty understanding what is required of her. R1 decides to illustrate by considering it as an area problem. At first, Stephanie seems to be having difficulty shifting from symbolic representations such as a^2 to visual representations such as a square (B: 264 – 267). R1 gives her a concrete example of a square with side five units long. Stephanie realizes that the area would be side squared, which in this case would be twenty-five square units. R1 then asks her to think about why that works. In order to illustrate it, she draws a square with side three units long and marks off three intervals on each side (B: 300). She and Stephanie discuss the concept of square units and use the drawing to illustrate the area (B: 306 – 324). R1 then brings the discussion back to a square with side *a* and asks Stephanie to draw it. Stephanie is dumbfounded (B: 331 – 337). She explains how, if it was a concrete number, she would have no problem completing the task but because she doesn't know what *a* is, she is unable to draw the square (B: 369). R1, Stephanie, and another researcher Ethel Muter, designated R2, discuss different approaches to representing a square with side *a* visually. R1 then suggests that Stephanie draw some more squares with sides with concrete lengths, sectioning them off into square units as done earlier. She also asks Stephanie to think about why the area of a unit square is one. They again consider a square of side three units. R1 uses this example to question and correct Stephanie regarding the difference between a length of three units and three square units (B: 534 – 538). Once the proper terminology is established, R1 revisits the reason the area of a unit square is one. Stephanie now understands that it is because each side has length of one unit and that one unit times one unit is one square unit (B: 569 – 577). However, when finding the area of a square with given side, she opts to multiply side by side (B: 622). R1 encourages her to think about it from another lens, that of counting up the square units (B: 623 – 636). Stephanie is now able to envision the number of square units for a square with side *a* and states that "there'd be *a* squared number of one square units" (B: 644). R1 eventually returns to the original task and again asks Stephanie to draw a picture representing a square with side *a*. Using one of the earlier suggested representations, Stephanie is able to draw a visual representation of a square with side *a* and clearly labels that there are a^2 square units inside (B: 790, 792).

The third part of the session lasts approximately twenty minutes. R1 now asks Stephanie to show her a square that has one side *a* plus *b* and another side *a* plus *b* (B: 823). After a couple of starts, Stephanie eventually gets a drawing that she is satisfied with: one that is sectioned off with clear divisions between *a* and *b* on the sides and a total of four pieces in the inside of the square (B: 875 – 880). Her next task is to find the area of each of the four pieces. Stephanie recognizes that the total area of the square is $(a+b)\cdot(a+b)=(a+b)^2$ (B: 920, 932). After some discussion, she also realizes that the total area of the square would be equal to the sum of the four inside pieces (B: 944, 946). Stephanie is able to express that sum as follows: $a\cdot a+a\cdot b+b\cdot b+a\cdot b$ (B: 955). R1 then asks Stephanie to simplify the expression. Stephanie ends up with the following result: $a^2+2ab+b^2$ (B: 976). R1 suggests that she test it with numbers and compare it with what they started out with: $(a+b)^2$. Stephanie tests the expression out with two different sets of numbers and finds that they satisfy the equation. However, when asked if this is enough to prove that it is always true, Stephanie replies that it would have to be true for all numbers (B: 1024 – 1029). Since it would be impossible to test out an infinite set of numbers, R1 and Stephanie set out to convince themselves that the two expressions are equal based on meaning.

The last part consists of Stephanie interacting with another researcher, Alice Alston, designated R3. This part lasts approximately twenty minutes and requires Stephanie to dictate to R3 exactly what she drew earlier so that R3 can replicate it without seeing Stephanie's paper. Stephanie describes to R3 a specific example of a rectangle with length five units and width two units (B: 1163 – 1165). Stephanie explains how to section it off into square units. This is a challenging task because Stephanie is unable to see R3's paper and vice versa. At some point, before they continue any further, they want to make sure that they are envisioning the same figure so they decide that Stephanie will also draw it with R3 dictating back to her what Stephanie had dictated to her earlier (B: 1400 – 1406). Eventually they end up with the same picture and use it as a reference point for discussion.

The remaining five minutes of the video data is spent organizing papers and discussing the importance of being able to share this work. A potential idea arises as to whether they should share this work with the students from the longitudinal study at Harding.

6.2 Subtask 1: Building Meaning of $(a+b)^2$

In the first segment of this session, Stephanie comes up with an algebraic representation for the expression $(a+b)^2$ that is, in fact, incorrect.

Table 6-1: **Stephanie forms a conjecture that** $(a+b)^2 = a^2 + b^2$**.**

40	R1	Okay. – Um. – Let's see. Maybe you can rebuild it. Okay? Um. [*takes paper and pen. Writes* $(a+b)^2$] Do you remember what that means?	
41	Stephanie	Um. I – this is yeah and didn't we distribute it so that it was like [*writes* $a^2 + b^2$]?	BR-S; OBS

Stephanie doesn't realize it at the moment, but she came up with the same incorrect representation in the first session, over two months ago, where she was able to show that it was incorrect. R1 suggests that Stephanie substitute numbers in both sides of the equation in order to test her conjecture (B: 42). Stephanie is learning the heuristic method of substituting concrete numbers in her representations in order to test them. She chooses two for *a* and three for *b*. In the process, she makes a couple of computation errors which she discovers and corrects, eventually concluding that the left side does not equal the right side (B: 77).

Stephanie is still faced with the original task at hand: what does $(a + b)^2$ mean? She draws upon her understanding of what 'squared' means and describes it as follows: "It means that you're multiplying it by itself" (B: 105, 107). She then expresses $(a+b)^2$ *as* $(a+b)(a+b)$ forming a symbolic representation for $(a + b)^2$ by using generic reasoning to apply the definition of 'squared' to $(a + b)^2$ (B: 125). Upon writing it down, Stephanie immediately realizes that they had done this before, but with x and y instead of a and b.

The conversation moves to potential special cases where $(a + b)^2$ does equal $a^2 + b^2$. Stephanie immediately suggests zero. When asked to elaborate, Stephanie explains, "Oh. Well then a would have to be equal to b." (B: 203) She continues explaining that a and b would be equal to zero and that zero squared would mean zero times zero which is zero (B: 204 – 209). Notice that Stephanie developed her own representation of $a = b = 0$ and using her knowledge of mathematical properties was able to prove her conjecture (Davis' steps (3) *Construct a mapping between the data representation and the knowledge representation*; and (4) *Check this mapping (and these constructions) to see if they seem to be correct;*). R1 then asks Stephanie, "What about one?" (B: 224) Stephanie is able to explain that it wouldn't work because $(a + b)^2$ would equal four while $a^2 + b^2$ would equal two, this time proving the conjecture incorrect (B: 225 – 234). Stephanie is again applying the heuristic method of substituting in numbers to prove or disprove a conjecture.

6.3 Subtask 2: Geometric Approach - The Area Problem

Obstacle to Understanding

Stephanie is now being asked to express $(a+b)(a+b)$ "not as a product" (B: 238). She is unclear as to what exactly is being asked (B: 255, 257). R1 suggests they think of it as an area problem and asks Stephanie to represent *a* squared. Stephanie is still unclear as to what is expected of her. She deals with this obstacle to understanding by seeking clarification by asking specific questions.

Table 6-2: **Stephanie asks questions to clarify task.**

260	R1	You know, if I asked you to represent *a* squared.	
261	Stephanie	With the - you mean with the box that we did last time?	PAH
262	R1	Yeah. How would you represent *a* squared? Let's get another piece of paper. Can you draw me a picture of what *a* squared could be?	
263	Stephanie	Um. Do you want it to represent like one side of the – 'cause that, I'm trying to think how we did it?	PAH
264	R1	Does anything come to your mind when you say *a* squared?	
265	Stephanie	Just well *a* times *a*.	BMP
266	R1	All right. That's true. But can you think of in geometry, what that might represent? [*pause*]	
267	Stephanie	Not like – I don't know like what you mean.	PAH

Stephanie is honest about her confusion. R1 breaks it down in more concrete terms using a square with side equal to three units. She draws it using three equal interval marks. She then asks Stephanie what the area is. Stephanie is able to reply, "nine square units," using the proper terminology. She is able to accept that there are nine of the "one square unit" enclosed in the square (B: 322-323). But when R1 brings up *a* squared again and wants Stephanie to represent it in a picture, Stephanie still has difficulty extending the representation. This is illustrated in the following excerpt:

Table 6-3: **Stephanie has difficulty representing a square with side *a* pictorially.**

327	Stephanie	You want me to show you *a* squared? Or?	
328	R1	Yeah.	
329	Stephanie	But you have it, like here.	
330	R1	Yeah. What would it look like in the picture? [*pause*]	
331	Stephanie	[*noise*] Um. [*pause*] I	
332	R1	It's a big leap, isn't it?	
333	Stephanie	I don't know, 'cause there's no like number to work.	OBS
334	R1	Yeah. Right. So.	
335	Stephanie	I can't draw anything 'cause there's no no number to like separate anything with or to like square it off in like little	OBS
336	R1	Hm.	
337	Stephanie	sections, you know?	OBS

Stephanie is having difficulty extending the concept from the concrete (a = 3) to the abstract a. She defines a as a variable "that represents like like all or like any number 'cause it doesn't have one. – a number." (B: 373, 375) Since Stephanie is unable to assign a mental quantity to a, she is not able to move forward with a visual representation. Stephanie states: "I mean I can show, I can, I can show you if you give me a number. But I can't just like show you what a is." (B: 386)

In order to get Stephanie to be able to imagine it, R2 uses a square of side a = 4 units as an example. Stephanie has no problem drawing it and sectioning off each side of the square into four equal parts (B: 473). Once she extends those lines, she ends up with sixteen equal parts enclosed in the square (B: 475). Since Stephanie is still unable to extend this type of representation to a drawing of a, R1 suggests that Stephanie draw a couple more concrete examples choosing side length herself. Stephanie draws a square sectioned off into six equal parts on each side thereby forming thirty-six equal parts. When asked what a 'square unit' is, Stephanie points to the upper right unit square of the six by six square (B: 493). She knows that the area of one of those squares is one but when asked why, she responds, "Oh! Because it just is. It's – it's one because um I don't know." (B: 499) A discussion ensues regarding the difference in terminology and meaning between a 'unit' and a 'square unit' using the square with side three units as a model. One conclusion that is established is that whether there are sixteen square units or thirty-six square units, each square unit is the same and has an area of one (B: 617 – 618). The other idea that results from the discussion is that the area of the square can be found by counting the number of square units enclosed in it instead of just finding side squared (B: 623 - 626). These ideas enable Stephanie to develop a mental representation for a square with side *a*.

Table 6-4: **Stephanie forms a mental representation for a square with side *a*.**

	637	R1	I want you to try to think of that. Now that should help you to figure out how to do *a* times *a*. If you think about that. 'Cause what's the difference now? You did it for three. You did it for four. You did it for six. What would it be for *a*?	
	638	Stephanie	Oh. It would be like	
	639	R1	What would be different in the *a* as compared to the three?	
	640	Stephanie	What would be different?	
	641	R1	Four, six.	
	642	Stephanie	The fact that you don't have a number?	OBS
	643	R1	Yes.	
35:00-39:59	644	Stephanie	But, I mean, the same, it would be like, there'd be *a* squared number of one square units.	BDI; BCA
	645	R1	All right. That's what's gonna be inside. So we're gonna	
	646	Stephanie	Yes.	
	647	R1	have all these little squares in here, you're telling me, right?	
	648	Stephanie	And that's gonna be *a* squared number.	
	649	R1	And when you add them all up, right, you're gonna have	
	650	Stephanie	It's gonna equal	

	651	R1	*a* squared of them.	
	652	Stephanie	Yes.	

From here, Stephanie is in a better position to envision a drawing of a square with side *a*. R1 suggests that Stephanie draw it and leave it open with slash marks according to a suggestion that R2 had made earlier (B: 402, 777, 779, and 781). Stephanie does so and writes that the area is a^2 square units.

6.4 Subtask 3: Area of $(a+b)(a+b)$

Visual Representation leads to Symbolic Representation

In this segment, the main task is to find a representation for a square with side $a + b$. Stephanie draws a square, labeling each side $a + b$. She sections each side off into two parts, calling the longer segment *a* and the shorter segment *b*. Upon finding that her square resembles a rectangle and that it is sectioned off incorrectly, Stephanie decides to start over (B: 852 -853). The final picture is that of a square partitioned into four pieces.

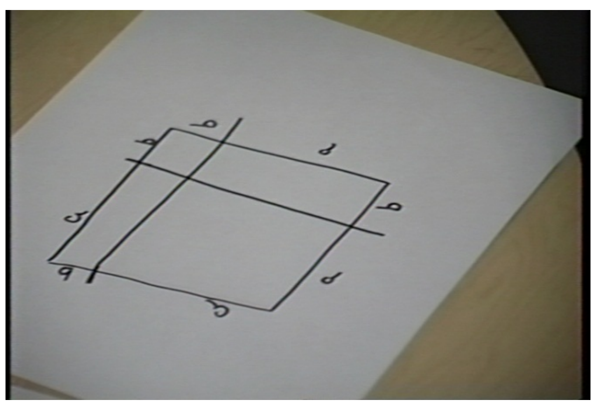

Figure 6-1: **Stephanie's diagram of a square with side *a* + *b*.**

Stephanie has just completed Davis' step (1) of his steps required in thinking about a mathematical situation: *Build a representation for the input data*. At this point, she uses her representation to apply Davis' step (2) *From this data representation, carry out* *memory searches to retrieve or construct a representation of (hopefully) relevant knowledge that can be used in solving the problem or otherwise going further with the task*. In this case, Stephanie uses her knowledge of area to find the area of each of the four pieces (B: 883 – 886). Stephanie writes *ab* in the upper left rectangle, *bb* in the upper right square, *ab* in the lower right rectangle and *aa* in the lower left square. R1 then asks Stephanie what is the area of the whole square, the original one. Stephanie initially responds, "Um. *ab* times *ab*." (B: 890) R1 then asks her about the area of some of the earlier concrete examples, such as a square with side four units (area = 16 square units) and a square with side six units (area = 36 square units) and a square with side *a* units (area = a^2 square units). Stephanie is able to explain that she is multiplying the sides together (B: 904, 910, 912). Now, when she asks Stephanie the area of a square with side $a + b$, Stephanie is able to respond, using generic reasoning to extend the concept of area, that it is *a* plus *b* times *a* plus *b* (B: 920). Stephanie writes that $(a+b)(a+b)=(a+b)^2$.

At this point, Stephanie must work on Davis' step (3) *Construct a mapping between the data representation and the knowledge representation*. In order to do this, she must make a connection between the equation $(a+b)(a+b)=(a+b)^2$ and her visual representation of the square with side $a + b$. This is illustrated in the following excerpt:

***Table 6-5:* Stephanie connects her visual representation of a square with side *a* + *b* to her symbolic representation** $(a+b)(a+b)=(a+b)^2$ **.**

937	R1	All right. But now in this picture, what part of the picture represents this $[(a+b)^2]$ piece? I know what part is *a* plus *b*. You told me that it's this side.	
938	Stephanie	Like the whole thing?	
939	R1	The whole thing.	
940	Stephanie	Yeah. The whole thing.	

941	R1	Okay. So this whole area is what this is equals. Let's write it out. What is the whole thing? You have pieces of it.	
942	Stephanie	Um hm.	
943	R1	So it's the whole thing. That means, this piece [*the* $a \cdot a$]	
944	Stephanie	and this piece [*the top left* $a \cdot b$] and this piece [*the* $b \cdot b$] and this piece [*the bottom right* $a \cdot b$]	BR-V
945	R1	Okay. So	
946	Stephanie	All together.	
947	R1	All together, when you	
948	Stephanie	Yes.	
949	R1	talk about things all together, what do you do?	
950	Stephanie	You add them.	BMP

Stephanie has realized that the area of a square with side ($a + b$) is equal to ($a + b$)($a + b$) which represents the sum of the areas of each of the four pieces enclosed in the square. Stephanie is now able to represent $(a + b)^2$ symbolically as $a \cdot a + a \cdot b + b \cdot b + a \cdot b$. She attempts to simplify this by changing the order of the terms and expressing $a \cdot a$ as a^2 and $b \cdot b$ as b^2 in order to get $a^2 + a \cdot b + a \cdot b + b^2$. In trying to combine $a \cdot b + a \cdot b$, Stephanie initially conjectures *ab* squared, but then corrects herself by realizing that since there are two of them, then their sum would be equal to 2*ab* (B: 966, 972, 976). When asked what her final expression $a^2 + 2ab + b^2$ represented, Stephanie replied, "This" and put her hand over the ($a + b$) square (B: 980) thereby solidifying the connection between the visual representation and the symbolic representation.

The next step is to check the representation, Davis' step (4) *Check this mapping (and these constructions) to see if they seem to be correct*. Stephanie does this by using the heuristic method of substituting numbers in both sides of the equation and checking to see if the left side is equal to the right side. She initially chooses two for *a* and three for *b*. She finds that it satisfies both sides of the equation (B: 1021). However, Stephanie realizes that one correct example is not sufficient to prove the equation as illustrated by the following conversation:

Table 6-6: **Stephanie and R1 discuss the conditions of proof.**

1022	R1	It worked for that example.	
1023	Stephanie	Yeah.	
1024	R1	But when you claim it's true, how many does it have to work for?	
1025	Stephanie	All of them?	BMP
1026	R1	All of them. Yeah.	
1027	Stephanie	(inaudible)	
1028	R1	Could you possibly test all of them?	
1029	Stephanie	No-o! [*laughs*] There's too many numbers. Um. Do you want me to try again?	

In order to convince herself, Stephanie decides to test another example, this time using four for *a* and five for *b* and finds that it works (B: 1061). R1 then again questions Stephanie about the meaning behind the area, bringing up concepts established earlier: the difference between a unit and a square unit and the origin of the representations *ab* and *ba* (B: 1098 – 1102; 1106 – 1130). In order to further establish the meaning behind the square, R1 supposes that *a* is five and *b* is two and encourages Stephanie to be able to imagine how the area of ten unit squares gets generated (B: 1142). This example leads to the following subtask.

6.5 Subtask 4: Recounting - An Example

Explanations

This subtask forms the final segment of this session. R1 gives the other researchers/observers an opportunity to ask questions or comment. One of them, Alice Alston, designated R3, was unable to see the drawings that were being discussed. R3 asks Stephanie if it would be possible for Stephanie to dictate instructions on how to draw the square, so that R3 would end up with the same drawing without having seen it (B: 1150 – 1153, 1157). The researchers present suggest that the example of the rectangle with sides five and two would be a good one to dictate. This refers back to the *ab* rectangle mentioned earlier. Stephanie instructs R3 to draw a line segment five units in length and another line segment perpendicular to it two units in width (B: 1183 – 1184; 1189 - 1196). Stephanie is to get R3 to draw a five by two rectangle, which she does (B: 1231 – 1237). R3 then claims that she has fourteen units (B: 1238). Stephanie is unsure of what she means so she goes over to where R3 is writing so that she can see her drawing. When R3 begins counting up units on the sides of the rectangles, Stephanie realizes that she is referring to perimeter (B: 1254). R3 sends Stephanie back to her seat and asks her to explain where the "ten thing" came from that Stephanie and R1 had discussed earlier (B: 1257).

Stephanie begins by attempting to give R3 instructions on how to section off the rectangle by drawing parallel lines straight "through where each unit would stop." She further elaborates, "Section off one unit at a time but section them all the way across through the rectangle." (B: 1260) R3 does this vertically and states that she has five sections (B: 1317). Stephanie then instructs her to do it horizontally into two sections (B: 1318). Before she does that though, R3 wants to know how long each of the horizontal lines is. Stephanie answers, "Five units." (B: 1348) However, there seems to be some confusion regarding whether R3 is asking about the length of a section or the area of a section. The following conversation illustrates this:

***Table 6-7:* There is confusion regarding whether R3 is asking about the length of a section or the area of a section.**

	1388	R1	I don't understand the question.	

	1389	R3	Oh! Well, maybe we've lost each other. Um	
	1390	R1	We have a different understanding of the question here.	
	1391	R3	Uh my uh I just well I'm not even sure	
	1392	R1	When you're talking about a section, do you mean the region inside?	
	1393	R3	I have a section across. A while ago I was wanting to know how long the how long – my original question to Stephanie was how long	
	1394	Stephanie	Well, how do I know that she has what I'm thinking?	
	1395	R3	how long the line is that divides each of the sections?	
	1396	R1	Oh. So that is a different question than I thought.	
	1397	R3	Yeah, but then it moved into how much space was (inaudible)	
	1398	Stephanie	But	
	1399	R3	Which is a slightly different question.	
	1400	Stephanie	but how do I know she has what I'm thinking?	

To alleviate the confusion, R2 suggests that they construct it simultaneously so that they can make sure that they are referring to the same picture. R3 then proceeds to dictate to Stephanie the exact steps that Stephanie had dictated to her. Once Stephanie has a matching picture, R3 repeats her question regarding the length of the lines used to "make off those sections to go" (B: 1456). This time, Stephanie replies, "Two units long." (B: 1459) R3 now wants to know how big each of those sections is. Stephanie is not sure what "big" means so she questions, "Perimeter or area?" Stephanie frequently seeks clarification by asking specific questions. R3 replies, "Just how big is it?" Stephanie replies that the perimeter of each of them is six but the area is two square units. R3 then asks her how many sections she has. Stephanie replies, "Five" and that each one is two square units (1472 – 1482).

R1 then asks Stephanie to draw the two square units. She and R3 both draw as Stephanie gives instruction to R3. Stephanie is able to show her two square units at the bottom of her rectangle (B: 1515 – 1517). Furthermore, when asked "why" is each unit a square, she immediately responds, "Oh! Because each side of it is one." (B: 1527) She has clearly grasped the concept of a square unit. The following excerpt from the transcript shows how Stephanie found and explained to R3 how to find the remaining sections and how to prove that the area was equal to ten square units.

Table 6-8: **Stephanie shows R3 that the area of a five by two rectangle is ten square units.**

1530	R3	What about the other sections?	
1531	Stephanie	Well, draw your line, keep drawing your line.	
1532	R3	Okay. So in my section two?	
1533	Stephanie	And that has two more square units.	
1534	R3	Okay.	
1535	Stephanie	And if you keep drawing it,	
1536	R3	My section three?	
1537	Stephanie	two more.	
1538	R3	Oh. Uh huh.	
1539	Stephanie	And that has two more. And that has two more.	
1540	R3	Huh. And so that means I have five sections, like you just told me?	
1541	Stephanie	Yes.	
1542	R3	Each with two	
1543	Stephanie	Um hm.	
1544	R3	square units?	
1545	Stephanie	Yes.	
1546	R3	How much is that?	
1547	Stephanie	Ten. [*laughs*]	BMP; BR-V

In the remaining few minutes, R1, R2, R3, and Stephanie briefly discuss if this would still work for a square. Stephanie explains that there's no difference but "it's gonna be the same" referring to the sides of the square (B: 1604 – 1607). They organize the papers and discuss sharing this information with the students at Harding as well as R2's daughter.

CHAPTER 7 SESSION 3 - February 7, 1996
Square of a Binomial; Cube of a Binomial

7.1 Background

The session begins with Carolyn Maher, the primary researcher, designated R1, asking Stephanie to explain what they worked on in the last session. Stephanie has prepared a detailed write-up of the work from the previous session. R1 begins reading it aloud and then hands it off to Stephanie to continue reading the rest of it. In it, Stephanie summarizes the work done in representing $(a + b)^2$ as a square, her struggles in understanding, the discussions regarding the difference between a square unit and a unit, and the attempts made to test the symbolic representations by substituting in numbers. This portion of the video lasts approximately the first four minutes.

Next, R1 pulls out a bag full of different kinds of manipulatives. She then asks Stephanie to try to use them to explain to her younger sister Susie, who at that time was in sixth grade, some of the things they discussed in the session. R1 will role play Susie. Stephanie begins by explaining what a square unit and a unit are using a 'flat', a ten by ten square with height one unit and a 'cube' with length, width, and height all equal to one unit. She also defines the concept of area and explains it in more detail using the flat and the cube. Then, R1 (as Susie), asks her to explain the *a*'s and the *b*'s. She does so using an unmarked blue shape that resembles a square. This portion of the video lasted approximately six minutes.

In the next segment, lasting approximately ten minutes, R1 asks Stephanie to illustrate a model either with the 'flat' or with something else, where *a* is three units and *b* is seven units (C: 199). Stephanie uses three cubes to represent *a* and seven cubes to represent *b*, then puts them all together to get a side equal to ten units. She and R1 then use these values for *a* and *b* to test their earlier representation from the previous session that $(a+b)^2 = a^2 + 2ab + b^2$. Using the fact that the area of a square of side $a + b$ with $a = 3$ and $b = 7$ would be equal to ten times ten or 100 square units, they test both sides of the equation by substituting $a = 3$ and $b = 7$, also getting one hundred (C: 250 – 251). In order to create a model of $(a + b)^2$ with $a = 3$ and $b = 7$, Stephanie uses the ten by ten 'flat' to mark off four sections: a section representing a^2, one representing *ab*, one representing *ba*, and one representing b^2.

Stephanie is now presented with another subtask: to consider the cube of a binomial $(a + b)^3$ (C: 385 – 386). For approximately the first eight minutes, Stephanie and R1 establish the meaning behind the cube of a binomial. Stephanie is attempting to use a cube with length, width, and depth equal to ten units. Stephanie and R1 discuss the fact that this particular cube has a volume of 1000 cubic units. R1 then brings up the more general case of $(a + b)^3$ and asks Stephanie what she thinks it means. Stephanie responds, "*a* plus *b* times *a* plus *b* times *a* plus *b*" then writes, "$(a+b)\cdot(a+b)\cdot(a+b)$" (C: 504 – 506). Before dealing with this representation, R1 suggests that it might be helpful to think of it as $(3 + 7)^3 = (3 + 7)(3 + 7)(3 + 7)$. R1 then writes = $(3+7)(3^2 + 2\cdot 3\cdot 7 + 7^2)$.

For the next seven minutes, R1 and Stephanie focus on this particular example as a basis for discussion. They discuss the possibility of simplifying without actually adding up the terms in the parentheses. R1 suggests using the distributive property, which Stephanie is familiar with (C: 573 – 574). Stephanie distributes one way while R1 distributes a different way. R1 then suggests that they test out both ways and see if they actually equal one thousand (C: 591). They find that both approaches work.

In the next twenty minutes, R1 and Stephanie return to the symbolic representation of $(a+b)^3 = (a+b)(a^2+2ab+b^2)$. R1 wants Stephanie to apply what they just did with the distributive property to the expression $(a+b)(a^2+2ab+b^2)$ (C: 668). Stephanie is able to apply the distributive property in the following manner: $a^2(a+b)+2ab(a+b)+b^2(a+b)$ (C: 718 – 720). She and R1 discuss the multiplication of the terms by considering how many times each variable is a factor, i.e., for $a^2 \cdot a$, *a* is a factor three times so it is equal to a^3. After multiplying all of the terms to get $a^3+a^2b+2a^2b+2b^2a+b^2a+b^3$, R1 and Stephanie discuss the possibility of combining any of the terms. R1 suggests that it may help to put the variables in alphabetical order. After rearranging the variables and combining like terms, Stephanie gets $a^3+3a^2b+3ab^2+b^3$. R1 then suggests that they test out the expression by putting in numbers for *a* and *b*. They use the previous numbers of three for *a* and seven for *b*, knowing that the result should be one thousand (C: 909). After verifying the expression for $a = 3$ and $b = 7$, Stephanie finds that she gets one thousand. She and R1 then discuss the possibility of it working for $a = 2$ and $b = 10$ and other combinations of numbers for *a* and *b* that add up to ten. In the remaining four minutes, they end the session with a discussion of potential ways they could model $(a + b)^3$ three dimensionally.

7.2 Recap: Stephanie's Write-up; Explaining to Susie

In this first segment of the video, Stephanie shares with R1 the write-up of the work she has done on the square of a binomial $(a + b)^2$ from the two previous sessions.

She describes her initial conjecture of $(a+b)^2 = a^2 + b^2$ and how they proved that it was incorrect by substituting in numbers. She further recounts the special case when $a = b = 0$ that does make $(a+b)^2 = a^2 + b^2$ true. She describes her feelings when she was first asked to represent $(a + b)^2$ as a square and refers to the discussion regarding a unit square and a unit. The following excerpt illustrates this:

Table 7-1: **Stephanie reads her write-up from the last session aloud.**

26	Stephanie	"Disregarding that answer I was asked what *a* plus *b* quantity squared really meant. My answer was easy. *a* plus *b* quantity squared equals *a* plus *b* times *a* plus *b*. Then for a moment, we got slightly off the subject. I was asked the question 'Of any circumstance when *a* plus *b* quantity squared equals *a* squared plus *b* squared'. I said "Yes, there was one circumstance. When *a* equals *b* equals zero, then *a* plus *b* quantity squared equals *a* squared plus *b* squared. With this question answered, we came back to the original problem of *a* plus *b* quantity squared. Now a new concept was brought into the picture. I was asked if I could explain and display *a* squared on a square. I was so dumbfounded. I really had no idea how to show them. Many squares were drawn. The subject of area was discussed. The area of a square is length times width or *a* squared. Still this didn't help me. Around this time we started to discuss the difference between a unit and a square unit. This is a unit in length." This is a unit, or this is a unit, or this – you know? [*Stephanie points to different parts on the paper.*]	BR-V

Stephanie goes on to explain how she used her representation of the square sectioned into four parts to help her come up with a symbolic representation for $(a + b)^2$: "So we found out that *a* plus *b* times *a* plus *b* equaled *a* plus *b* quantity squared equals *a* plus *aa* plus *ab* plus *bb* plus *aa,* -which also equals *a* squared plus *ab* plus *ab* plus *b* squared, which equals *a* squared plus 2*ab* squared plus *b* squared." (C: 32) Stephanie then describes how they simplified the expression to $a^2 + 2ab + b^2$, then tested it by substituting in numbers in both sides of the equation. Finally, Stephanie mentions how she had to explain "why one unit by one unit equaled one square unit by having Dr. Alston draw a picture of a square without having me see it" (C: 34). R1 comments that it is a lovely write-up. Stephanie is learning to explain and document the different steps in a problem solving task, an important step in building meaning and understanding of mathematical ideas.

Furthermore, Stephanie is asked to explain in detail many of the concepts that were discussed or introduced in the previous session. This is done by having Stephanie explain these concepts to her younger sister Susie; she does this with a bag of manipulatives placed at her disposal. R1 role plays Susie. Stephanie chooses to begin with 'unit' and 'square unit'. She is also encouraged by R1 to explain the whole concept of area. Stephanie explains it as follows: "Okay. Well, area is like um the amount of space inside like a sp…an object. Um. So and to find the area of a square it's like length times width or if – especially when you're dealing with a square 'cause like the sides are all equal it's like one side squared. So if this is ten, it would be ten squared." (C: 77) Stephanie is referring to a 'flat' a square shaped model with side equal to ten units and a height equal to one unit. R1, as Susie, questions Stephanie regarding what a unit is and where the 'ten' comes from. The following excerpt from the transcript illustrates how Stephanie explained 'unit' and 'square unit':

Table 7-2: **Stephanie explains a 'unit' and a 'square unit'.**

82	R1	What do you mean ten? Where did you get ten?	
83	Stephanie	Oh. Well, there's ten – you see, it's ten units long. This is like one unit.	BR-V
84	R1	Can you show me what's a unit?	
85	Stephanie	See this *[Stephanie puts a cube over the 'square' in the top left corner of the 'flat'*.]	

86	R1	This square is one unit?	
87	Stephanie	Yeah. Like this square, [*a cube*] this is like a littler piece like that's how big that is.	BR-V
88	R1	And you're calling this a unit?	
89	Stephanie	Yes.	
90	R1	Okay.	
91	Stephanie	Oh. One square unit.	OBS; BR-V
92	R1	Oh. This is one square unit. But I don't know what a unit is.	
93	Stephanie	This is a unit. You see this like side right here. [*Stephanie points out the length of one unit on the side of the 'flat'.*]	BR-V
94	R1	Can you show me here too? [*R1 holds up a cube.*]	
95	Stephanie	Like this. [*Stephanie shows R1 the length of the edge on the cube.*]	BR-V
96	R1	Oh. Okay.	

At first, Stephanie is referring to a square unit as a unit. She then corrects herself (C: 91). After that, she is clearly able to point out the difference between them on the manipulatives. Stephanie then explains that in order to get the area of the cube, it would be one unit by one unit, which is equal to one square unit (C: 99, 101). She then moves on to finding the area of the 'flat', explaining it in two ways.

R1, as Susie, then moves on to ask about the *a* and the *b* and what they represent. Stephanie searches in the bag of manipulatives, pulling out a blue shape resembling a square. She uses it because it's unmarked and as she explains to R1, *a* can be any number. She explains how to get the area as follows: "So like if this wasn't marked it would be *a* length by *a* length and to find the area of an object that's like *a* length long it would be *a* length squared or *a* length times *a* length." (C: 153 – 155) R1 then asks her to explain the *a* plus *b*. Stephanie describes *a* and *b* as "not the same" but "they stand for any number" and shows R1 on the blue figure how each side would be equal to *a* plus *b*.

R1 then asks Stephanie to demonstrate it on one of the models, using three for *a* and seven for *b*. This forms the first subtask.

7.3 Subtask 1: Building a model of (a + b)²

After attempting to use a couple of models, Stephanie decides to use the cubes to form a row of ten cubes. She then separates them into two groups – she calls one group of three cubes *a* and she calls one group of seven cubes *b*. R1 then asks Stephanie to recall the symbolic representation of $(a + b)^2$ which she does stating that it was equal to $a^2 + 2ab + b^2$ (C: 228 – 231). R1 reminds her that it represents a square with side $a + b$ equal to ten where *a* is three and *b* is seven. They discuss the fact that they already know that the area is one hundred. Furthermore, since they know that *a* is three and *b* is seven, they can substitute into $a^2 + 2ab + b^2$ and should also be able to get one hundred. Stephanie calculates the numbers and finds that they do add up to one hundred. R1 tells Stephanie that they just showed that the area is equal to $a^2 + 2ab + b^2$ but reminds her that it came from $a^2 + ab + ba + b^2$ (C: 258). R1 then asks Stephanie to build her a model showing all four pieces for the special case of $a = 3$ and $b = 7$.

Stephanie begins by using the 'flat' as a basis for her model. She then tries out different ways of sectioning off *a* and *b* by using the cubes and/or using a row of ten attached square units, which is referred to as a 'long' in the transcript. Eventually, she uses the 'long' in addition to the cubes to mark off the rows dividing *a* and *b*.

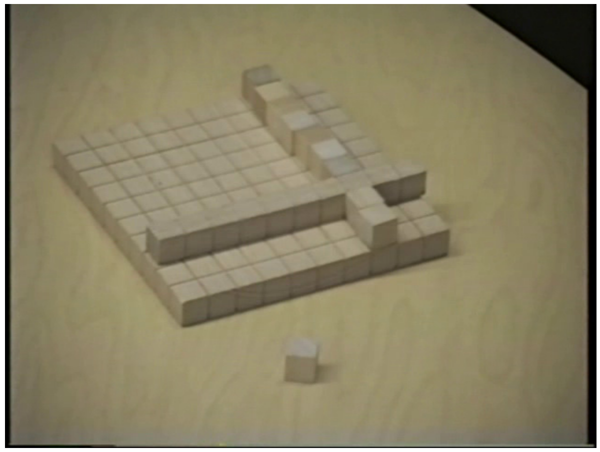

Figure 7-1: **Stephanie's model of a square with side $a + b$ when $a = 3$ and $b = 7$.**

Stephanie is now able to point out to R1 each of the four pieces: a^2, ab, ba, and b^2 on her model, thereby connecting her symbolic representation of $(a + b)^2$ with her visual representation of a square with side a plus b (C: 324, 342 – 346, 362 – 366). Furthermore, since her model is concrete rather than abstract, i.e., it represents a square with side ten units sectioned off into three units and seven units, Stephanie is physically able to see the number of square units for each section. For instance, for the b^2 section, the area is forty-nine square units, and this could always be confirmed by counting. Therefore, Stephanie is forming a mental representation that she can use in future when dealing with a more complicated or abstract model.

7.4 Subtask 2: Cube of a Binomial $(a + b)^3$: From the Specific to the General

7.4.1 Building Meaning

R1 begins this next segment by suggesting to Stephanie that they consider an explicit example and find the volume. She asks Stephanie to explain volume. Stephanie describes it as follows: "It would be length, width, times depth." (C: 408) When asked what that means, she replies, "That means this way, times this way, times this way" as she traces the edges of the cube and "- it's like three dimensional"(C: 410, 414). At this stage, Stephanie is retrieving prior knowledge in order to "*construct a representation of (hopefully) relevant knowledge that can be used in solving the problem or going further with the task*" (Davis' step (2)). R1 then asks her what the volume is of a cube that she is holding with each side equal to ten units. Stephanie conjectures "a thousand units cubed" then goes on to explain, "cause um squared is like two-dimensional, so cubed is like three-dimensional" (C: 456, 460, 462) She convinces R1 that there are one thousand cubic units by showing her that if you take ten of the 'flats,' which have an area of one hundred square units, and stack them, then ten times one hundred is one thousand (C: 482, 484).

R1 now brings the conversation to $(a + b)^3$. Stephanie's first instinct is to build it before coming up with a theoretical representation (C: 500). She and R1, however, begin by discussing meaning first. When asked what it means, Stephanie responds, "*a* plus *b* times *a* plus *b* times *a* plus *b*" and writes, "$(a+b)\cdot(a+b)\cdot(a+b)$" (C: 504, 506). R1 then asks Stephanie how she could express any two of the binomials $(a + b)(a + b)$. Initially, Stephanie responds, "$(a + b)^2$" but when R1 tells her to actually square it, Stephanie is able to recall that $(a + b)^2 = a^2 + 2ab + b^2$ (C: 522). They now have the expression $(a+b)(a+b)(a+b) = (a+b)(a^2 + 2ab + b^2)$. The task remains to simplify it. R1 suggests that they think of it as $(3 + 7)^3$ in order to make the problem easier.

7.4.2 The Specific: $(3 + 7)^3$

R1 begins by expressing $(3 + 7)^3$ as $(3 + 7)(3 + 7)(3 + 7)$ and stating that they know the answer (C: 549, 551). She then equates it to $(3+7)(3^2 + 2\cdot 3\cdot 7 + 7^2)$ (C: 553). Stephanie's first instinct is to "do everything in the parentheses" first (C: 558). R1 however does not want her to do everything in the parentheses (C: 559). She describes what she wants Stephanie to do as follows:

***Table 7-3:* R1 describes task of simplifying** $(3+7)(3^2+2\cdot3\cdot7+7^2)$.

30:00-34:59	567	R1	I mean you could add these up and add these up but we know it's a thousand. So. But suppose rather than do everything in the parentheses – is there anything that you've learned about arithmetic that you could stop this from being a multiplication problem. Does any of that look familiar to you? [*pause*]	
	568	Stephanie	I don't know. I've usually – 'cause if you just have numbers like that you just like	
	569	R1	But suppose they were letters?	
	570	Stephanie	Well, if they were letters I'd probably like – get help or something to figure it out. I don't know. I don't – um – to stop it from being a multiplication problem?	PAH; OBS
	571	R1	Um hm.	
	572	Stephanie	I don't know.	

As Stephanie is still a bit unclear regarding what is expected of her, R1 mentions the distributive property and asks Stephanie to apply it as if they were letters instead of numbers (C: 573 – 579). Stephanie responds, "Well, it would be nine – like parentheses – three plus seven plus forty-two parentheses three plus seven plus forty-nine parentheses, you know, three plus seven. Like, you know." (C: 580) Stephanie is distributing the three plus seven times the nine, the three plus seven times the forty-two, and the three plus seven times the forty-nine thereby applying her knowledge of the distributive property to simplify (Davis' step (5) *When the constructions and the mapping appear satisfactory, use technical devices (or other information) associated with the knowledge representation in order to solve the problem*). It turns out that R1 would have distributed the three times each of the terms then the seven times each of the terms (C:

587 – 589). They decide to continue by simplifying it using both approaches and checking to see if they are able to get one thousand. Stephanie is learning one way to test an idea. After completing the calculations with the aid of a calculator, Stephanie finds that both of their approaches lead to the same conclusion: an answer equal to one thousand.

7.4.3 The General: $(a + b)^3$

Stephanie is now asked whether the same approach using the distributive property can be applied to $(a + b)^3$. Stephanie uses generic reasoning to reply, "Yeah. I guess. Except they're like *a*'s and *b*'s, but yeah." (C: 667) Stephanie begins simplifying using what she did earlier with the $(3 + 7)^3$ example as an assimilation paradigm: *a* plus *b* times *a* squared plus *a* plus *b* times 2*ab* plus *a* plus *b* times *b* squared (C: 689 – 693). She has doubts that she can complete the rest of the simplification correctly, so R1 suggests that she treat each part of it as a separate problem and consider the meaning behind it (C: 696 – 700).

Stephanie begins by describing $(a+b)\cdot a^2$ as *a* squared times *a* plus *a* squared times *b* (C: 701). She continues with the rest of the simplification expressing $(a+b)\cdot 2ab+(a+b)\cdot b^2$ as two *ab* times *a* plus two *ab* times *b* plus *b* squared times *a* plus *b* squared times *b* correctly applying the distributive property again (C: 718, 720). She ends up with the expression $a^2\cdot a+a^2\cdot b+2ab\cdot a+2ab\cdot b+b^2\cdot a+b^2\cdot b$.

In order to simplify the first term, Stephanie suggests making it a^3 because, "Well, it's another *a*. She sees that it is *a* three times (C: 731 – 732). R1 likens it to three times three squared becoming twenty-seven, which is just three cubed (C:735 – 739). Stephanie moves on to simplify the third term $2ab\cdot a$. She conjectures, "Could it be –

um – there'd be another *a*, right? So could I make it like three *a* times two *b*?" (C: 752) In order to assess this conjecture, R1 suggests that they go back to *a* squared times *a* and think about why it is *a* cubed. Stephanie explains, "Because you're multiplying it by itself again." (C: 762) They further discuss the idea of *a* as a factor:

***Table 7-4:* Stephanie and R1 discuss the meaning of a^3.**

763	R1	Okay. Um. So – another way I think about it is – here you have – when there's no exponent – that means you have one of them.	
764	Stephanie	Yeah.	
765	R1	Right?	
766	Stephanie	Um hm.	
767	R1	Okay. That means you have one factor *a*.	
768	Stephanie	Um hm.	
769	R1	And here you have two factors of *a*.	
770	Stephanie	Yes.	
771	R1	So that means you have three	
772	Stephanie	Three.	
773	R1	So *a* cubed.	
774	Stephanie	Um hm.	

R1 and Stephanie now return to the term $2ab \cdot a$. They discuss the fact that there is one factor of *a*, one factor of *b*, and one factor of *a* again (C: 831 – 834). R1 then asks Stephanie if it can be simplified. Stephanie responds, " – Oh! I can I can make it *a* squared" using the concept that two factors of *a* is equivalent to *a* squared (C: 836). She continues simplifying as follows:

***Table 7-5:* Stephanie simplifies the terms resulting from the expansion of $(a+b)^3$.**

845	R1	So this term can be written – the second term – as	
846	Stephanie	Two *a* squared *b*	BR-S
847	R1	Good.	
848	Stephanie	plus and then it again, right? Oh. No. Now this time it's two *b* squared *a*.	BR-S
849	R1	Or two *a b* squared. If you're keeping them alphabetically.	

850	Stephanie	Okay. Plus you know that one is *b* squared times *a*. You can't do anything with that one.	BR-S

R1 suggests that Stephanie write them alphabetically in order to aid in simplification. They then go about combining like terms. The following excerpt illustrates their discussion:

***Table 7-6:* Stephanie combines the terms resulting from the expansion of $(a+b)^3$.**

881	R1	Now here we have an *a* squared *b*. Right? We have one of those.	
882	Stephanie	Um hm.	
883	R1	When we don't have a number, that means one of them, isn't that right?	
884	Stephanie	Yes.	
885	R1	We have one *a* squared *b*.	
886	Stephanie	Oh. Well here you have two *a* squared.	BR-S
887	R1	Oh. We have two of them. Okay. So we have one of them and two of them. How many of them will that give us?	
888	Stephanie	Three of them. Three *a* squared.	BMP

They use the same idea to simplify the remaining terms in order to get the expression $a^3+3a^2b+3ab^2+b^3$ (C: 897 – 908). R1 suggests that they test the expression for *a* equal to three and *b* equal to seven and see if it will give them an answer equal to one thousand. They are applying Davis' step (4) *Check this mapping (and these constructions) to see if they seem to be correct* by using the heuristic method of substituting in numbers to test the validity of an expression. They find that the expression with $a = 3$ and $b = 7$ does indeed give them one thousand.

In the final few minutes of the video, R1 discusses with Stephanie the idea of building a model of $(a + b)^3$. She points out that it would have to be three-dimensional and would have to include the four pieces that would correspond to the four terms in the expression they formed earlier. Furthermore, she could use the example where $a = 3$ and $b = 7$ as a special case. R1 gives to Stephanie various manipulatives to use, one of which R2 has used in the past to create a similar model. Stephanie is charged with the task of working on it in preparation for the next session. They organize Stephanie's papers and the session ends.

CHAPTER 8 - Session 4 - February 21, 1996
Cube of a Binomial

8.1 Background

The session begins with Stephanie relating her anguish to Carolyn Maher, the primary researcher, designated R1, when she lost the papers from the previous session two weeks ago. She describes how she went to her teacher Miss Colosimo for help and they redid the problem of finding $(a + b)^3$. Stephanie offers to retrieve the papers showing that work but instead R1 suggests that Stephanie explain to her what she did from the beginning. For the next four minutes, Stephanie summarizes her experiences simplifying and modeling the square of a binomial. She finishes up with her attempts to draw a model of the cube of a binomial $(a + b)^3$.

R1 then asks Stephanie to conjecture what $(a + b)^3$ looks like. For the next four minutes, Stephanie attempts to simplify $(a + b)^3$ beginning by expressing it as $(a+b)(a+b)(a+b)$. She then replaces $(a+b)(a+b)$ with $a^2 + 2ab + b^2$. Applying the distributive property and then combining like terms, Stephanie is able to simplify $(a + b)^3$ to $a^3 + 3a^2b + 3ab^2 + b^3$ (D: 113).

In the next segment of the video, R1 asks Stephanie if she can represent *a* plus *b* quantity squared *a* plus *b* times (D: 138 – 149). Stephanie suggests possibly showing it on a cube. For the next eleven minutes, R1 and Stephanie explore the meaning behind the cube of a binomial $(a + b)^3$ using the manipulatives, the drawing of a square with side $a + b$, as well as the different 'pieces' of the symbolic representations, as a point of discussion. They debate whether the 'flat' (*the 10x10x1 box)* is really two dimensional or three dimensional. They discuss the difference between area and volume. Stephanie uses the ‘cube’ (*the 10x10x10 box*) and the ‘flat’ to convince R1 that the volume of the cube is one thousand and that this means that you can “fill it up with one thousand square units” (D: 249).

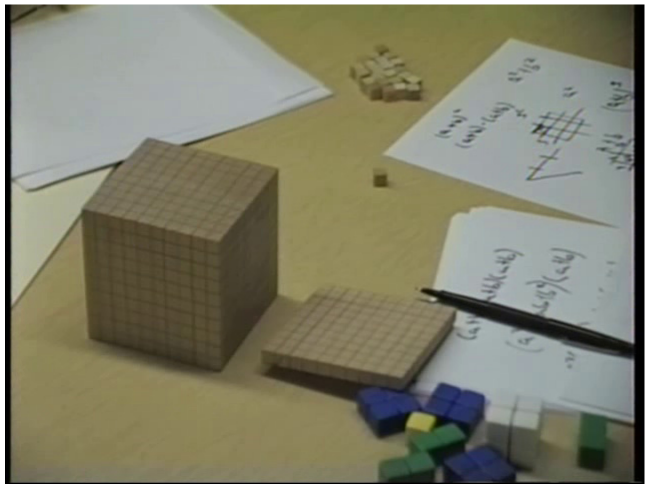

Figure 8-1: **The ‘cube’ – the 10x10x10 box and the ‘flat’ – the 10x10x1 box.**

She also formulates a “peel back” argument that she uses to further illustrate the volume of the cube (D: 263 – 277). R1 then asks her to use the same argument to represent $(a + b)(a^2 + 2ab + b^2)$ which Stephanie does.

In the next segment of the video, lasting approximately twenty minutes, R1 and Stephanie use a set of eight algebra blocks that Ethel Muter, designated R2, had provided, in order to build a physical model of $(a + b)^3$. These blocks consisted of four different colors and a variety of shapes. R1 begins by arranging four of the blocks in a square resembling the square with side $(a + b)$ drawn earlier by Stephanie.

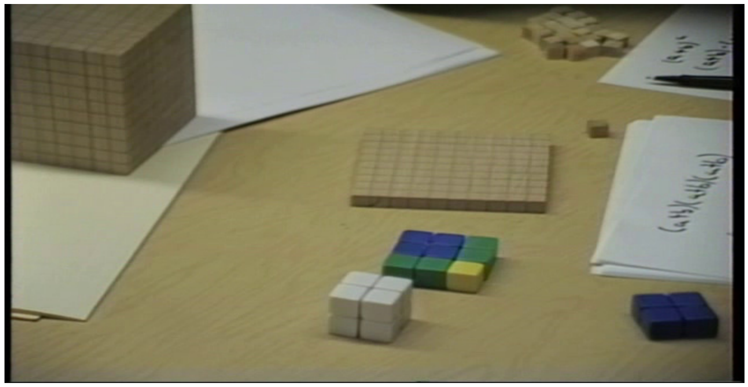

Figure 8-2: **R1 uses algebra blocks to represent** $(a+b)^2$**.**

Stephanie recognizes the isomorphism to her earlier picture and so rearranges them slightly to match her picture exactly (D: 337 – 341). She assigns symbolic representations to each piece, maintaining a two-dimensional lens (D: 345 – 351).

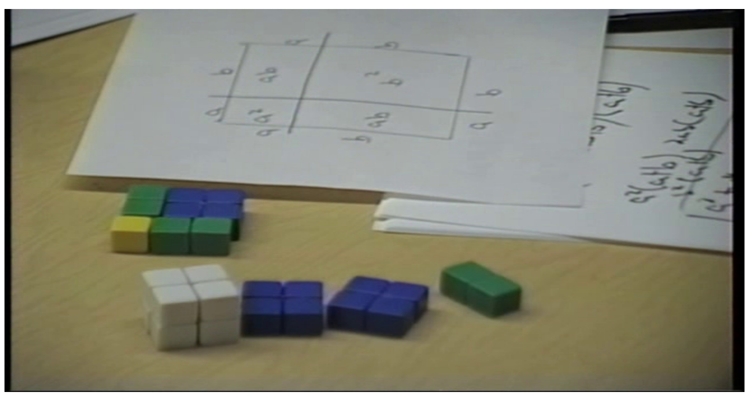

Figure 8-3: **Stephanie rearranges the algebra blocks to correspond to her drawing of a square with side *a* + *b*.**

R1 then asks her how they could show $(a + b)^3$. Stephanie struggles with the transition to the three-dimensional, so R1 suggests a way of "going up *a* plus *b*" (D: 378, 380). However, Stephanie is still unsure how to continue. In dealing with this obstacle to understanding, R1 and Stephanie discuss the dimensionality of the model they currently have for $(a + b)^2$. They question whether it in fact is two dimensional, since it does have a height unlike its corresponding picture. Once Stephanie sees the model as three dimensional, she is able to modify her representations of the model. For example, since the height is a, the a^2 piece became a^3, the ab piece became a^2b, and so on (D: 457, 466, 474).

At this point, R1 reminds Stephanie that they are supposed to go up a plus b. She suggests that Stephanie think about how to finish it at some later point. She then asks Stephanie to use the remaining algebra blocks to build a cube, which she does. R1 and Stephanie then compare the cube with the symbolic representation that they formed earlier: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ (D: 519). R1 suggests considering the symbolic representation before simplification and checking to see if the 'pieces' from the model correspond to the 'pieces' in the symbolic representation (D: 546, 548). Stephanie proceeds to assign representations to each of the 'pieces' in the model as follows: i.e., *Blue piece-* ab^2, until she has accounted for all the 'pieces' (D: 581).

In the final segment of the video, the other researcher/observers present are given the opportunity to ask Stephanie questions regarding the model. First, she explains her work representing $(a + b)^2$ as the area of a square with side $(a + b)$ resulting in the expression $(a^2 + 2ab + b^2)$. She then continues by explaining her simplification of $(a + b)(a^2 + 2ab + b^2) = a^3 + 3a^2b + 3ab^2 + b^3$. She then describes how she "built up" the model of $(a + b)^3$ (D: 679). One of the researcher/observers, Terry Pearl, designated R3, questions Stephanie on the significance of the colors. Stephanie explains to her that the colors have no importance in the model, but just serve as a convenient way of allowing her to label her 'pieces' (D: 708 – 712). Another issue that arises is the concept of dimensionality. In order to alleviate confusion, Stephanie returns to her two-dimensional picture of the square with side $(a + b)$ and explains in more detail how the four basic algebra blocks that fill the square correspond to their symbolic representations (D: 752 – 774). This segment lasted approximately twenty minutes.

In the last five minutes, the researchers and Stephanie discuss the benefits and limitations of different manipulatives, sharing their various experiences. In addition, Stephanie is asked to think about $(a + b)^4$ in preparation for the next session.

8.2 Recap: Symbolic Representaton of $(a + b)^3$

In this segment of the video, Stephanie explains what she did in the last session. She begins by describing her feelings upon losing her work from the previous session: "Cause that was, like, the information you gave me, but I lost the sheet! I went home and I was - I went nuts looking for the folder with the papers." (D: 5) She then explains that she rebuilt $(a + b)^3$ with her teacher and offers to get the work. Instead, R1 asks that she describe what she remembers having done. Stephanie begins by recounting her experiences with $(a + b)^2$: how she initially expressed it as $(a + b)(a + b)$ then was asked to represent it as a square with side a plus b (D: 25). The following is an excerpt of her description:

Table 8-1: **Stephanie describes the work she did in the previous session.**

27	Stephanie	But, so, then we got into, like, if the square was three parts [*writing*] what this was- and that that was a unit, and that that was like one square unit.	PPK; BR-V
28	R1	Mhm.	
29	Stephanie	And um, that it would be nine, and because it was like three by three, three squared. And we did a couple of those. And then, um, [*pause*], we- you asked me if it was um, if one side was [*writing*] a plus b [*writing*]	PPK; BR-V; BR-S; BMP
30	R1	Oh yes, I remember that one.	
31	Stephanie	Then what it would be.	BCA; BR-V
32	R1	Yeah.	
33	Stephanie	And um, if the small part's a and the big part's b [*draws square divided into parts representing* $(a+b)^2$]	PPK; BCA;

			BR-V
34	R1	Mhm. [*pause, Stephanie writes*] did you figure out what all those pieces were?	
35	Stephanie	Yeah. It was *a* squared, *ab*, ahem, *b* squared, *ab*, and it would be *a* squared plus 2*ab* plus *b* squared, and that's what we figured out then. [*pause, writes*] *a* plus *b* squared equals.	PPK; BR-S; BR-V; BCA; BMP
36	R1	Oh, okay, right. And the original conjecture what *a* plus *b* squared equaled you were testing.	

Stephanie then concludes by sharing her experience in school trying to draw a three-dimensional figure to represent $(a + b)^3$ and being unable to do so (D: 61).

R1 then asks Stephanie to form a conjecture for $(a + b)^3$. Stephanie begins by expressing it as $(a+b)(a+b)(a+b)$ since, as she puts it, "That's, like, what it's saying to do." (D: 73) Stephanie has just *built a representation for the input data* (Davis' step (1)). She explains that if you multiply two of the binomials, it comes out to $a^2+2ab+b^2(a+b)$ initially leaving off the parentheses but then realizing they were needed to get $(a^2+2ab+b^2)(a+b)$ (D: 75, 79, 81). Stephanie is now retrieving prior knowledge that can be used in helping her to progress further with the task (Davis' step (2) *From this data representation, carry out memory searches to retrieve or construct a representation of (hopefully) relevant knowledge that can be used in solving the problem or otherwise going further with the task*). Knowing that $(a + b)^2 = (a + b)(a + b)$, which is equivalent to $a^2 + 2ab + b^2$, has provided Stephanie with a building block that can now be used in working on other tasks. Stephanie now applies Davis' step (5) *When the constructions and the mapping appear satisfactory, use technical devices (or other information) associated with the knowledge representation in order to solve the problem*. In this case, she uses the distributive property and properties of exponents to distribute as follows:

$a^2(a+b)+2ab(a+b)+b^2(a+b)=a^3+a^2b+2a^2b+2ab^2+ab^2+b^3$ (D: 87 – 91). At this point, Stephanie recognizes that there are 'like terms' that can be further simplified, so she proceeds to simplify the expression to $a^3+3a^2b+3ab^2+b^3$ (D: 98 – 103).

R1 then points the discussion back to the expression $a^2+2ab+b^2$ by asking Stephanie how she would "say in words how you were thinking about this piece" (D: 120 – 124). Stephanie replies, "Well, it's *a* plus *b* quantity squared." (D: 127) Referring to her picture, she describes it as a square with side *a* plus *b*. R1 now asks her to think about what $a^2+2ab+b^2$ would look like *a* plus *b* times as illustrated in the following excerpt:

Table 8-2: **R1 asks Stephanie think of a way to represent** $(a^2+2ab+b^2)$ ***a* plus *b* times.**

136	R1	So you've made a square with side length *a* plus *b* [*Stephanie nods*] and this piece represents the area of that square.	
137	Stephanie	Mhm.	
138	R1	Right? Okay, and what you've said here- so we know we have this piece, but we have it *a* plus *b* times, don't we?	
139	Stephanie	Mhm.	
140	R1	And this piece *a* plus *b* times, [*points at paper*] cause we're finding the product.	
141	Stephanie	Yeah.	
142	R1	Can you conjecture what that might look like?	
143	Stephanie	What that might look like… [*pauses, thinking*]	
144	R1	We're going back to this piece [*points to* $a^2+2ab+b^2$ *on paper*] *a* plus *b* times. Now remember, when you can't make sense of something with letters, try to imagine-	
145	Stephanie	With numbers?	PNE
146	R1	-if you're doing it with numbers. Sometimes that's a useful way to think about it. So you might not want to think about it as *a* plus *b*.	
147	Stephanie	Alright. [*nods*]	
148	R1	But you might. But you know what this piece [*points at paper- speaking inaudible*]	
149	Stephanie	But you want me to show like, you how that would look if it was *a* plus *b* times? Like, how, *a* plus *b* quantity squared would look *a* plus *b* times?	PAH

Stephanie suggests possibly showing it on a cube leading to the first subtask.

8.3 Subtask 1: Building meaning of $(a + b)^3$

R1 and Stephanie begin by experimenting with various manipulatives. Stephanie suggests using a *10x10x10* box referred to as a 'cube' as well as a *10x10x1* box referred to as a 'flat'. They decide that they are going to think of the 'flat' as two dimensional (D: 177 – 178). R1 uses the 'flat' to trace out a square box on paper. Stephanie then redraws a diagram of the square with side *a* plus *b* on the new square drawing. However, when it comes to drawing $(a + b)^3$, Stephanie is not sure how to proceed (D: 209, 211). Previously, when Stephanie was facing an obstacle to understanding, she would use the heuristic method of considering a simpler problem or using concrete numbers instead of abstract variables. In this instance, R1 suggests they view the 'flat' and the 'cube' in terms of the number of actual units, i.e., a length of ten units instead of *a* plus *b*.

8.3.1 The Specific

Stephanie describes the 'flat' as having an area of one hundred square units. She justifies this as follows: "Cause it's ten units here, and ten square units here and ten times ten is a hundred." (D: 223) When asked about the 'cube', Stephanie responds that the volume would be a thousand (D: 241, 243). She describes volume as "length times width times height" or "the three dimensions of the cube" (D: 245, 247). R1 then asks Stephanie what "the thousand" means. Stephanie replies, "There's a thousand little, like [*picks up little one-unit cube*] units. Square units in there. Like, you could fill it up with a thousand square units." (D: 249) Notice that Stephanie is still referring to the units two dimensionally. In order to convince R1, Stephanie formulates a 'peel back' argument as illustrated in the following excerpt:

***Table 8-3:* Stephanie explains why the volume of the 'cube' is one thousand.**

260	R1	And you're telling me there are a thousand.	
261	Stephanie	Yes.	
262	R1	So why?	
263	Stephanie	Okay. Well it's 10 high, right? [*picks up cube, compares to box*] If it was just th- one of these, [*indicating box*] is like the same as like, it's one. Like one part.	BEJ; BR-V
264	R1	A hundred.	
265	Stephanie	A hundred. And you know that like [*touching side of cube*] this, is the same as that [*indicating box*].	BEJ; BR-V
266	R1	So that's a hundred, okay. So what-	
267	Stephanie	It's the same thing, if I took this off it would be one hundred [*indicating side of cube*], and you know that there's ten of them [*pointing to each layer of the cube along the edge*]. So you see that ten of them would make this- ten high? You know?	BEJ; BR-V
268	R1	So if I took one of them off, I would get one hundred.	
269	Stephanie	Yes.	
270	R1	If I peeled another off…	
271	Stephanie	It would be two hundred.	BEJ
272	R1	[*tapping top of cube*] 300, 400, 500 [*trails off*]. That's the way you think about getting a thousand.	
273	Stephanie	Yeah.	
274	R1	I peel- How many times would I peel them off?	
275	Stephanie	Ten.	BEJ
276	R1	Ten times? Why ten?	
277	Stephanie	[*coughs*] 'Cause that's how high it is. That's how many fill it up.	BEJ
278	R1	Oh. How wide is it?	
279	Stephanie	Ten.	BEJ
280	R1	And how long is it?	
281	Stephanie	Ten.	BEJ

Stephanie clearly has a mental representation of the volume of the 'cube' and is able to explain it and justify her answer using a well-formed argument when dealing with a concrete example. She now attempts to revisit the abstract version $(a + b)^3$.

8.3.2 The General

R1 now asks Stephanie to use the same argument to show what $(a+b)(a^2+2ab+b^2)$ represents. She describes it initially as "it's *a* plus *b* high, it's *a* plus *b* long, and it's *a* plus *b* wide-" (D: 283). She continues explaining, "So, um [*picks up 10x10x1 flat*] if this is *a* plus *b* squared, you see that like, [*pointing at 10x10x10 cube*], this, if you t- took this off, it would be *a* plus *b* squared, and you need to take *a* plus *b* amount of these off to get *a* plus *b* cubed." (D: 285) She is now using generic reasoning to apply her mental representation for the concrete example of a *10x10x10* cube to the more abstract example of an *a* plus *b* times *a* plus *b* times *a* plus *b* cube. This is illustrated by the following conversation:

***Table 8-4:* Stephanie uses generic reasoning to extend her earlier explanation to justify the volume of a cube with side *a + b*.**

299	Stephanie	Alright, well, if this is *a* plus *b*, like this side is *a* plus *b* [*uses box*] and this side is *a* plus *b*, then there are *a* plus *b* squared number of pieces in here. Do you believe that?	BEJ; BR-V
300	R1	I believe that. And I even believe that it is *a* squared plus 2*ab* plus *b* squared.	
301	Stephanie	Yes. So-	
302	R1	You've convinced me of that.	
303	Stephanie	So, if I were- there's *a* plus *b*, like, rows of these. If I took *a* plus *b* number of, like, this [*indicates box*], it would make that- it would fill that up [*indicates cube*].	BEJ; BR-V
304	R1	Okay.	
305	Stephanie	If I took off one of these [*indicates box*], you see if I took this first row off, right here, I'd have *a* plus *b* squared, of- *a* plus *b* squared number-	BEJ; BR-V
306	R1	*a* plus *b* quantity squared	
307	Stephanie	Yeah, and so I'd have to take up *a* plus *b* number of those, to like, fill it up [*indicates cube*], or something?	BEJ; BR-V
308	R1	Okay, so,	
309	Stephanie	Yeah.	

R1 is now convinced and returns the conversation to the symbolic representation of $(a+b)(a+b)(a+b) = (a^2 + 2ab + b^2)(a+b)$ (D: 310). This leads to the following subtask.

8.4 Subtask 2: Algebra Blocks - the model of (a + b)^2

R1 pulls out a set of eight algebra blocks that come in four colors.

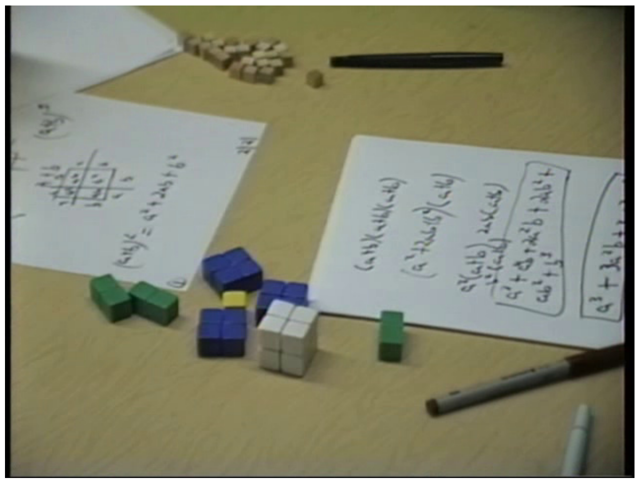

Figure 8-4: **Set of 8 Algebra blocks in four colors.**

Meanwhile she is considering the symbolic representation of $(a+b)(a+b)(a+b) = (a^2 + 2ab + b^2)(a+b)$. She encourages Stephanie to think about the 'pieces' before simplification and points out that in the model of $(a + b)^2$, the two 'pieces' *ab* and *ba* were two separate regions, although in the symbolic representation they simplified to 2*ab* (D: 318, 320). As Stephanie is unclear on how to use the algebra blocks, R1 rearranges some of them to resemble the model of $(a + b)^2$ in the drawing of the square with side *a* plus *b*. Stephanie then rearranges it to match the drawing exactly, now able to see the isomorphism between the two representations (D: 341 – 343). R1 then asks her how it works. She explains as follows:

Table 8-5: **Stephanie explains the isomorphism between the algebra blocks and her earlier drawing of a square with side *a* + *b*.**

345	Stephanie	[*Points to pieces in model*] *a* squared.	BEJ; BR-V
346	R1	What's *a* and what's *b*?	
347	Stephanie	This is *a* and this is *b*.	BEJ; BR-V
348	R1	That's *a* and that's *b*? Oh, okay, this is *a* squared…	
349	Stephanie	[*Points to pieces in model*] *a* squared, *a* plus *b*, err- *ab*	BEJ; BR-V
350	R1	Okay-	
351	Stephanie	*b* squared-	BEJ; BR-V
352	R1	Okay-	
353	Stephanie	*ab*.	BEJ; BR-V
354	R1	Oh, okay, that's neat. Now, I'll buy that.	

Figure 8-5: **Model of ($a + b)^2$ using algebra blocks.**

Stephanie now has a model of $(a + b)^2$ which she views as two-dimensional although in reality it is three-dimensional.

8.5 Subtask 3: Algebra Blocks - the model of $(a + b)^3$

R1 now asks Stephanie to create a model for $(a + b)^3$. Stephanie realizes that it would have to be three-dimensional, but is not sure where to go from there (D: 357, 359). In order to clarify the task, R1 explains that they have a square with area $a^2 + 2ab + b^2$

and now that they want to make a cube, they have to "go up *a* plus *b*." (D: 370, 372) At first, Stephanie responds that there aren't enough pieces, but when R1 takes one of the algebra blocks and places it vertically on top of the model and states, "But that's up *a* plus *b*", Stephanie is now able to see it (D: 379 – 385).

The next obstacle to understanding is the dimensionality of the model. The model used to represent $(a + b)^2$ is actually three-dimensional because it has a height of *a*. However, if viewed two dimensionally, then it has the four 'pieces': a^2, *ab*, *ba*, and b^2. In order to help Stephanie make the transition to a three-dimensional form, R1 asks Stephanie to look for a^3, a^2b, etc. and see if they exist in the present model. The following conversation illustrates this:

***Table 8-6:* Stephanie maps blocks from model to terms in the expansion of $(a+b)^3$.**

25:00-29:59	390	R1	Okay, so now when we have a cube, we know [*picking up blue piece*] right? What do we know about all these? Any- all- of these components? [*pauses*] Okay, [*points at paper*] is there an *a* cubed any place?	
	391	Stephanie	[*pauses*] I don't- [*sighs*]	OBS
	392	R1	Is there an *a* squared *b* any place?	
	393	Stephanie	I- guess-	OBS
	394	R1	Where's there an *ab*?	
	395	Stephanie	An *ab*? Is right here [*points at set of green cubes*], well, no. An *ab* is like, is this piece right here? Or this piece?	PAH; BR-V
	396	R1	Okay, so it's *a* [*pointing to one side of piece*] *b* [*pointing to other side*]. So this piece is *a* and this piece is *b*.	
	397	Stephanie	Yes.	
	398	R1	So where would a, *ab* squared be? I wonder…	
	399	Stephanie	*ab* squared? Is that what you said?	PAH
	400	R1	Yeah. [*pause*] This is *b*. [*points to side green piece on model*] Think about this, it's so easy …	
	401	Stephanie	[*Sighs*] Um, I guess…	OBS

Stephanie still seems to be struggling, so R1 returns to the $(a + b)^2$ model and suggests they "pull it apart" and mark off each component (D: 412 – 415). Stephanie labels each component of the $(a + b)^2$ model as follows:

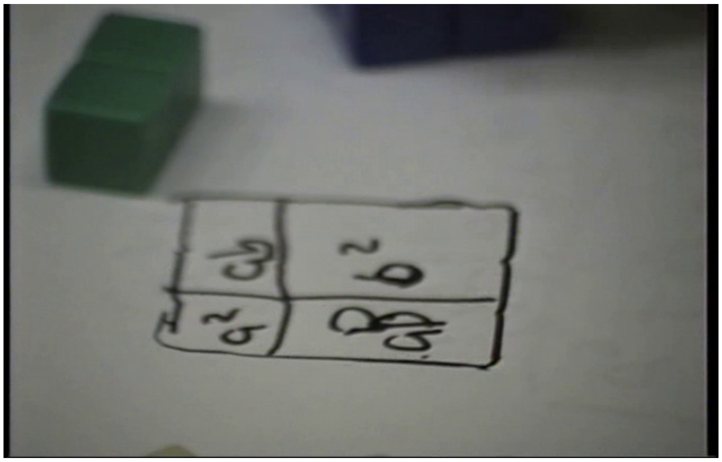

***Figure 8-6:* Stephanie labels the 'pieces' of her square of side *a* + *b* picture.**

R1 then begins with the a^2 piece and asks Stephanie "How many times have you gone up now?" (D: 435) Stephanie responds that "you went up, like, *a*" and so wouldn't the piece be *a* cubed? (D: 436, 438). R1 and Stephanie then go back and consider each piece from a three-dimensional perspective with height *a*. The following excerpt illustrates the process:

***Table 8-7:* R1 and Stephanie transform the two-dimensional model into a three-dimensional model by giving each piece a height of *a*.**

459	R1	How much did you go up here? [*pointing to green piece*]	
460	Stephanie	You went up- you went up *a*.	BEJ
461	R1	You went up *a* here, okay. So you went up *a*-	
462	Stephanie	Yeah.	
463	R1	And how much were you down? [*pointing to tracing on paper*] What's the area of this little piece?	
464	Stephanie	The area of that little piece was *ab*.	PPK; BMP
465	R1	But you went up- You did *ab*, *a* times.	
466	Stephanie	So it would be *a* squared *b*?	BDI
467	R1	Does that make sense?	

468	Stephanie	Yeah. So, can I like write that on the side?	BR-S
469	R1	Whatever you want. 'Cause they're getting interesting. [*Stephanie labels a^3, a^2b around edge of traced diagram]* Okay, so this is *a* cubed, and we're saying this piece is *a* squared *b*. What about this piece? [*blue piece]*	
470	Stephanie	Hm- You still went up *a*.	BEJ
471	R1	Okay.	
472	Stephanie	So it would be *ab* squared?	BCA; BR-V
473	R1	Does that make sense?	
474	Stephanie	Yes. And this one you went up *a*, [*indicating other green piece]*, so it would be *a* squared *b*? I guess? 'Cause it's the same as that one [*indicating other first green piece]*.	BEJ; BCA; BR-S

Stephanie now has the base of a model of $(a + b)^3$. She can clearly see some of the 'pieces' of the symbolic representation of $(a + b)^3$. However, R1 reminds her that they have only gone up '*a*' yet they still have to go up '*a* + *b*' (D: 479). Stephanie picks up the green algebra block and suggests that this is '*b*' and sets it vertically on top of the model (D: 484, 486).

At this point, R1 asks Stephanie if she can build a cube with the rest of the algebra blocks. Stephanie hesitates at first, rearranging pieces, but then successfully builds it (D: 494). R1 points out that she has accounted for all the components of the "first layer of the cube" and then "went up *b*" (D: 497, 499).

***Table 8-8:* Stephanie assigns a symbolic representation to each of her 'pieces' in the base layer of the model.**

514	Stephanie	We have *a* cubed [*writes terms on paper*], we have *a* cubed- squared *b*, we have *ab* squared, and we have another *a* squared *b*. And I guess, on the base level [*pulling apart a piece of the cube*], does that count? [*Drops some pieces, reassembles cube*]	BR-S/V; PAH
515	R1	That was all those pieces- you-	
516	Stephanie	Yeah, so it doesn't. So like, we have these four [*pointing to paper*] pieces… With just this layer.	BR-S
517	R1	Hmm. Just the bottom layer.	
518	Stephanie	Yeah.	

519	R1	Mhm. And [*returning to previous work on paper, before simplified*], according to this thing we needed three *a* squared *b*, you only had one. You need *3ab* squared, you only had one. Right?	
520	Stephanie	Well we have two *a* squared *b*. [*pause*] Don't we?	BMP; PAH
521	R1	Hmm. I guess we do. Right.	

Stephanie has now assigned a symbolic representation to each of the pieces in the bottom layer of the model. She writes them down as follows: $a^3 \quad a^2b \quad ab^2 \quad a^2b$. Stephanie conjectures that she could account for each of the terms in the symbolic representation of $(a + b)^3 = a^3 + a^2b + 2a^2b + 2ab^2 + ab^2 + b^3$ by finding their corresponding representation within the model (D: 546 – 547). She proceeds to do this by labeling each piece of the model according to color, i.e., blue piece – ab^2 and writing down the corresponding symbolic term for each piece as she identifies it. She then simplifies the terms until she has shown that she has $a^3 + 3a^2b + 3ab^2 + b^3$. The following excerpt illustrates the process:

***Table 8-9:* Stephanie accounts for all of the 'pieces' in the three-dimensional model and uses it to simplify the expansion of $(a+b)^3$.**

565	Stephanie	Okay [*continues writing*]. White is *b* cubed. Yellow is *a* squared [*pauses, corrects "2" with "3" on paper*] cubed.	BR-S
566	R1	Why did you change it?	
567	Stephanie	Because I was talking about the paper, instead of the yellow.	BEJ
568	R1	Okay, good. So when you think about the paper, it's the two dimensions, and when you think of the actual block-	
569	Stephanie	Mhm-	
570	R1	You have to think of three dimensions.	
571	Stephanie	Mhm. And the green was [*writes*] *a* squared *b*.	BR-S
572	R1	Okay. So. [*pauses*] The green one is *a* squared *b* [*gathers green pieces*], how many of those do you have?	
573	Stephanie	Three. Three *a* squared *b*.	BEJ
574	R1	And what's the blue one [*picking up blue piece*]	
575	Stephanie	Oh, so we have 3*a* squared *b* [*pointing to original paper*	BDI;

		with simplified work]	BMP
576	R1	Oh.	
577	Stephanie	[*Crosses out two a^2b terms on newer paper, rewrites "$3a^2b$" instead*] And we have *a* cubed [*writes*] and we have *b* cubed, and we have *ab* cubed- squared- [*looks at pieces*] we have 3*ab* squared [*writes*]	BR-S; BMP
578	R1	So, why are these *ab* squared [*picks up blue piece*]	
579	Stephanie	Because, it's like, *a* up, *b* over [*pointing to edges of piece*]	BEJ
580	R1	Believe that, absolutely. Okay.	
581	Stephanie	So that's it, we have all the pieces.	BDI

Stephanie has convinced herself, as well as R1, that the model aptly represents the symbolic representation of $(a + b)^3$. She was able to construct a mapping between her visual representation and symbolic representation (Davis' step (3) *Construct a mapping between the data representation and the knowledge representation*) thereby acquiring another assimilation paradigm to use in future tasks.

8.6 Subtask 4: Recounting - Explanation

After Stephanie has successfully built her model of $(a + b)^3$ and connected it to the symbolic representation, R1 invites the other researchers and/or observers to ask Stephanie questions. She particularly introduces her friend Terry Pearl, designated R3 in the transcript. R1 then asks Stephanie to explain to them what she's done beginning with *a* plus *b* squared.

Stephanie begins by expressing $(a + b)^2$ as $(a + b)(a + b)$. Underneath it, she recreates her drawing of a square with side *a* plus *b*, sectioning it off into four 'pieces': a^2, ab, b^2, ab and finally ending with $a^2 + 2ab + b^2$ (D: 639, 641).

In order to explain $(a + b)^3$, Stephanie uses the same approach by expressing it as $(a+b)(a+b)(a+b)$. She describes how she now knows that $(a+b)(a+b)$ is equal to $a^2+2ab+b^2$. Stephanie then proceeds to explain to the researcher/observers how she multiplied $a^2+2ab+b^2$ by $(a + b)$ using the distributive property. Stephanie continues in this fashion with Carmella Colosimo, designated R4, asking questions along the way until she gets $(a+b)^3=a^3+3a^2b+3ab^2+b^3$ (D: 659).

In order to introduce the model, Stephanie mentions that they should begin with a two-dimensional model: the drawing of the square on the paper. From there, you would "build it up" (D: 667). She then describes to them her representations of each of the four 'pieces' in the bottom layer of the model of a cube. For example:

***Table 8-10:* Stephanie describes how she formed three dimensional-representations.**

674	R4	How's that- How did you determine that that was *a* squared *b*?	
675	Stephanie	Oh, because, um, [*removes* ab^2 *piece to show drawing*] this is *b* squared, and you built it up *a*, like, 'cause, it's- this [*indicating height*] is *a*.	BEJ; BR-V
676	R4	Okay.	
677	Stephanie	Like this piece and this piece, so you built it up *a*, so it would be *a* squared- *b* squared.	BEJ; BR-V
678	R4	Oh, okay. Alright.	
679	Stephanie	But. Here, this piece [*moving* a^2b *piece*] is *a*- *ab*, [*moving* ab^2 *piece*] this is *b* squared. So this piece [ab^2 *piece*] would be- Um- Like this piece here [*picks up* a^2b *piece*] 'Cause it's *ab*, if you built it up *a*, it would be *a* squared *b*. And this piece [*picks up* a^3 *piece*] 'cause this piece is *a* squared, and you build it up *a*, it would be *a* cubed.	BEJ; BR-V
680	R4	Oh, building it up, okay.	
681	Stephanie	Yeah, you built it up.	BEJ

One issue that arises is that of color. R3 questions the effect of the color on the representations (D: 707). Stephanie responds that "the color itself has, like, nothing to do with it. It could be purple- and- it doesn't make a difference." (D: 710) She explains that she wrote the representations down by color only to help her remember each piece (D: 712). Another point of confusion for R3 is that she is viewing the cube that Stephanie refers to as a^3 and is thinking that it is one cubic unit. Stephanie addresses her concern as follows:

Table 8-11: **Stephanie explains why the little 'cube' is represented by a^3.**

752	Stephanie	[*moves pieces, gets paper with drawing on it*] This [*emphasizing a portion of side length on square*] like, is a un- this is *a* long. This piece right here is *a* long. Okay? And this [*emphasizing corresponding side length on other side*] is *a* long. And so it's *a* squared. We're saying that this is *a* cubed [*indicating a^3 piece*]. We're saying that this is *a* long [*pointing to edge of cube*], by *a* long [*pointing to other edge of cube*] y-you know? Length, width, and height; they're all *a*. [*pause*] Okay? [*pause*] This [*referring back to the drawing*] is *b* like, um, this is *b* long by *b* long [*redrawing segments on sides of b x b square in $(a+b)^2$ model*]. Okay? So we're saying- and this is *b* cubed- s- or well [*mumbles to self; picks between algebra block pieces*] this is *b* cubed [*choosing b^3 piece*] and they're saying that this is *b* u- they're all *b*. We're not saying that like [*pauses, picks up a^3 piece again*]- this isn't *a* [*indicating whole cube*] this is *a* [*indicating side length of cube*] this little piece, this unit is *a*. Okay? [*pauses*] Okay. So-	BEJ; BR-V

Stephanie then takes each of the pieces in the bottom layer of the cubic model and explains beginning with the two-dimensional representation how she built up to get the three-dimensional representation that corresponds to the color. For example, she takes the 'a^2' that represents one region of the two-dimensional square and explains that if you build it up *a* then you have a three-dimensional representation of the yellow cube labeled as a^3 (D: 762, 764). After explaining the bottom layer, Stephanie moves on to show how she formed representations for the other pieces in the model. In addition, she demonstrates how there are three of the a^2b piece and three of the ab^2 piece and shows that they correspond to the symbolic representation. The following excerpt illustrates this:

Table 8-12: **Stephanie demonstrates the isomorphism between the 'pieces' in the model and the symbolic representations of the terms in the expansion of $(a + b)^3$.**

777	Stephanie	[*Stephanie reaches for paper from earlier with $(a+b)^3$ expanded and simplified*] –oops- [*knocks into table*] – was- find out if we had, like, all the pieces that were here, and so if you build, um, and then [*reaches for Algebra blocks, drops one*]–oops- if we build this up, like if you keep building like that, like this is *ab* cubed [*placed ab^2 piece on diagram*], *a* cubed *b* [*places a^2b piece on diagram, then a^3*]-um *a* squared *b*, *a* cubed [*places a^2b piece*], *a* squared *b*, [*places b^3 piece on top of ab^2*] and you build it up. If you built [*removes b^3, ab^2 pieces, points to b^2 part of diagram, holding b^3 piece*] *b* squared up *b* times- *b* units, it would become *b* to the third. So this piece is *b* cubed. So you have every piece here [*referring back to the paper with $(a+b)^3$ work on it*]. You have *a* cubed [*picks up a^3 piece, places it down; picks up a^2b piece*], you have, um [*pauses*], what is that? *a* squared *b* [*places piece down, picks up ab^2 piece*] you have *ab* squared [*places piece down, picks up b^3 piece*] and you have, um, *b* cubed [*places piece down, gathers all a^2b pieces*]. And you have three of these, so that becomes 3*a* squared *b* [*gathers ab^2 pieces*], and you have three of these, so it becomes 3*b*- 3*ab* squared, and you have your *a* cubed and your *b* cubed. And that makes up the problem. And you can build that into like [*pauses, assembles pieces into cube*].	BEJ; BR-V
778	R4	And it doesn't matter which way you put the colors?	
779	Stephanie	No, because the colors don't matter. It's the [*points to edge of cube*] units.	BEJ

After separating the pieces into groups by color, Stephanie reassembles them into the cubic model for $(a + b)^3$.

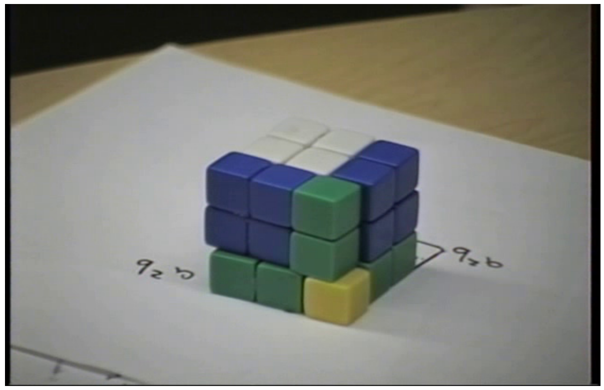

Figure 8-7: **Stephanie's model of $(a + b)^3$.**

The conversation then turns to the concrete. The question arises as to what the volume would be if the little yellow cube was in fact just one cubic unit. After initially responding 9 then 81, R1 asks Stephanie to explain why. In trying to explain herself, Stephanie realizes those answers do not represent the volume of a cube of side three units (D: 804 – 811). She represents the volume as three cubed then as 27. In order to confirm her conjecture, Stephanie decides to physically count up the units contained in the model and gets 27 cubic units (D: 821 – 825). She now has a mental picture of volume (D: 826 – 827).

The session concludes with the researcher/observers discussing potential ideas and experiences with manipulatives that can be used in modeling. Stephanie is asked, in preparation for next time, to think about $(a + b)^4$. R1 suggests that she use the results of $(a + b)^3$ to conjecture or predict what $(a + b)^4$ will look like. Stephanie responds that there will be an a^4 and a b^4 and some other terms in the middle. They organize papers and the session ends.

CHAPTER 9 - Session 5 - March 13, 1996
Combinatorics: Trains/Towers, Pascal's Triangle

9.1 Background

The session begins with Carolyn Maher, designated R1 throughout the transcript, relating to Stephanie how she and other researchers are telling the story of the students from the longitudinal study, focusing primarily on their work with the unifix cubes. R1 then asks Stephanie if she remembers anything regarding her work with the unifix cubes. Stephanie replies, "That we finally had to go up to ten or something. Like you, but we figured out that it was just, like, you just multiply the last number's amount by two to get the next number's amount. So that's, like, what I remember." And, "we were building like families with it" (E: 10, 12, 14). R1 then suggests that they do "something different" and "put the algebra stuff on hold" (E: 19, 21). She introduces the term 'combinatorics' and tells Stephanie that they've been doing it all along. Instead of towers, however, they are going to use trains and keep them 'flat' (E: 25, 27).

R1 explains combinatorics as the process of selecting and introduces two forms of notation to represent combinations: C_r^n *and* $\binom{n}{r}$ where the top number represents the number of cubes and the bottom tells you "how many of a certain kind you're picking" (E: 61, 63). She is using a train four cubes long with two possible colors of red and yellow to illustrate. R1 and Stephanie use this scenario to discuss the possibilities of selecting one red, then two red, then three red, and four red. Since there are no unifix cubes available, Stephanie draws a diagram of the cubes in order to form her representations (E: 90, 92). She uses a strategy of separating by none, by one, and by two to find all possible cases for selecting two red cubes. For selecting three red cubes, she also uses a separation strategy (E: 126 – 134). Stephanie then forms a diagram of the case where she is selecting one red cube as follows:

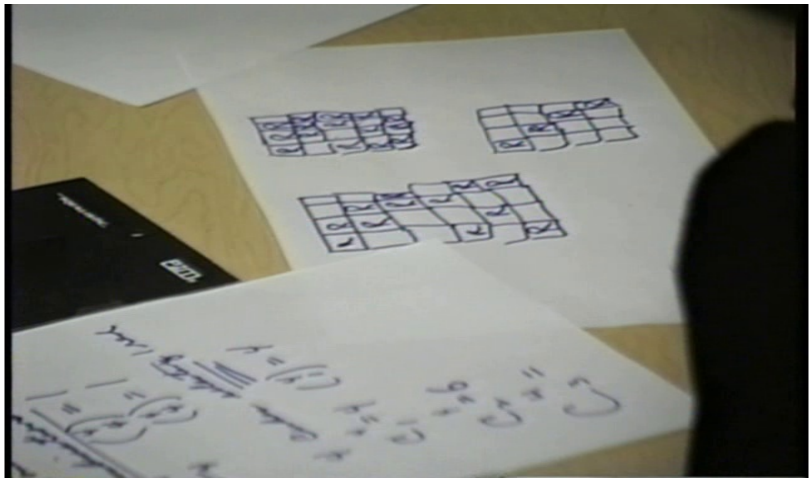

Figure 9-1: **Stephanie's diagram of trains with 2 red, 3 red, and 1 red using '*separation*' strategy.**

Meanwhile she uses the notation to represent the different combinations as follows:

$C_4^4 = 1, \quad C_3^4 = 4, \quad C_2^4 = 6, \quad C_1^4 = 4, \quad C_0^4 = 1$

R1 and Stephanie also discuss whether there is any relationship between selecting three reds versus the case of selecting one red. Stephanie recognizes that they both have four combinations and are the same if "flipped over" (E: 144, 152, 154). R1 and Stephanie then add up all of the possible ways of selecting red to get sixteen. Stephanie recognizes the isomorphism to the towers problems that she did in the past (E: 212 – 218). This segment of the video lasted approximately fifteen minutes.

In the next segment, lasting approximately eight minutes, R1 reminds Stephanie about a strategy she used in the past, where Stephanie grouped them into 'families' (E: 225 – 227). In order to "build them up," she used a tree diagram (E: 227 – 236, 243).

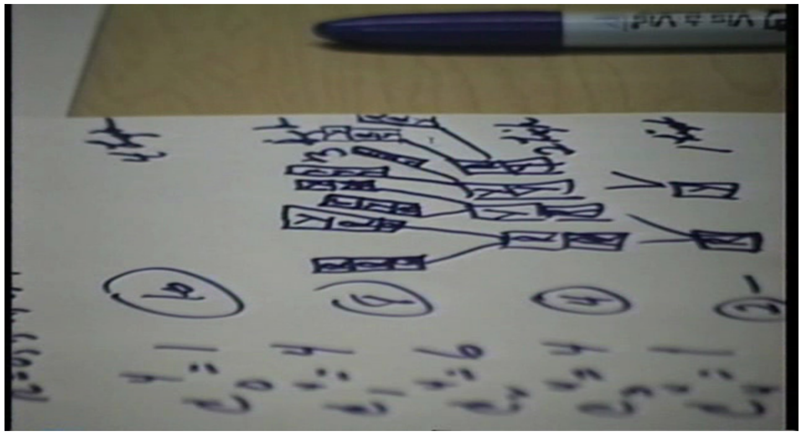

Figure 9-2: **Stephanie's tree diagram.**

She finds a total of sixteen combinations. R1 points out that when they initially added up the combinations found earlier, they were only focusing on selecting red. If they were to select yellow, they would find another sixteen combinations. Stephanie realizes that selecting yellow is essentially forming the same combinations as when selecting red; she refers to it as 'opposite'. (E: 304, 316) Therefore, she and R1 are able to reconcile the different answers (sixteen and thirty-two) that they arrived at by using two different strategies.

The next segment of this session lasts approximately twenty minutes. R1 returns to the C_r^n notation introduced earlier and discusses with Stephanie the representations when n = 1, 2, 3, 4, 5. For instance, for n = 3, they write down

$C_0^3 = 1, \quad C_1^3 = 3, \quad C_2^3 = 3, \quad C_3^3 = 1$ (E: 363 – 368, 388 – 396). Stephanie is able to predict the values for n = 4. R1 then expresses the values in a horizontal fashion.

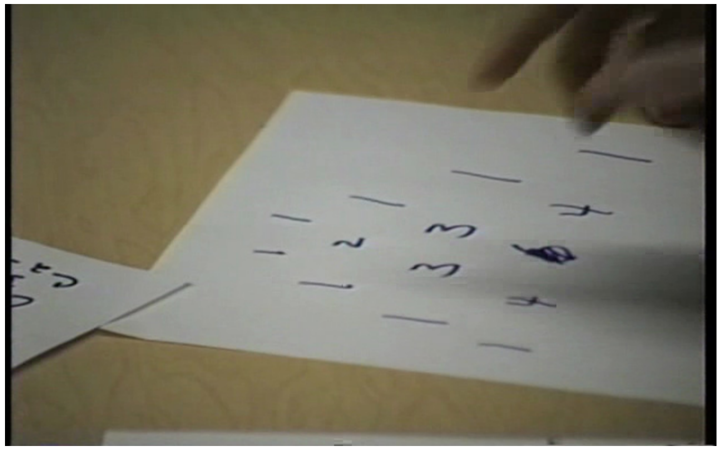

Figure 9-3: **Pascal's triangle up to *n* = 4.**

Stephanie is then able to discover the relationship of the numbers in Pascal's triangle and uses it to predict the next row when $n = 5$, mapping each number back to its corresponding tower representation (E: 404, 423 – 438, 445 – 447, 457 – 460, 468). R1 eventually introduces the name Pascal and discusses the importance of having a visual representation instead of just using the relationship of the numbers.

R1 then suggests forming Pascal's triangle using the notation instead of the numbers (E: 506). She forms two different triangles for each of the notations: C_r^n *and* $\binom{n}{r}$. R1 then begins listing parts of the triangle such as $\binom{1}{0}+\binom{1}{1}=\binom{2}{1}$, and $\binom{2}{0}+\binom{2}{1}=\binom{3}{1}$ (E: 552 – 554). Meanwhile, she encourages Stephanie to look for patterns and relationships that would allow her to form a hypothesis for a general form (E: 578 – 598).

The next segment of the video, lasting approximately five minutes, relates to a task that was assigned to Stephanie at the end of the previous session. She was asked at that time to consider $(a+b)^4$ and $(a+b)^5$. R1 asks her about it and Stephanie replies that she "worked it out" and goes to retrieve her papers. She returns with the following:

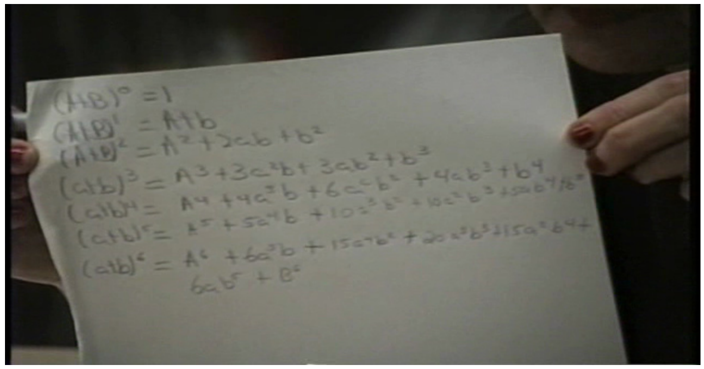

Figure 9-4: **Stephanie's write-up from the previous session.**

R1 then begins to write all of the coefficients of the binomial expansions that Stephanie has found. For instance, $(a+b)^0$ is 1, $(a+b)^1$ is $a + b$, which has coefficients one and one, $(a+b)^2$ is equal to $a^2+2ab+b^2$ which has coefficients one, two, and one, etc. R1 writes them out in the form of Pascal's triangle. As she writes them, Stephanie conjectures that it is the same as the towers (E: 644, 646). Stephanie is then able to predict the next couple of rows in the triangle and then compare with the coefficients in her binomial expansions (E: 649 – 658). R1 and Stephanie then discuss the a^2. At first they represent it as two factors of a, then R1 suggests that they view a as red. Stephanie is able to shift to the new representation and identifies a^2 as one way of representing a tower two high with two reds (E: 673 – 674). They continue to represent the terms in the binomial expansions as towers, then R1 asks Stephanie to write up their results for the next session.

The next five minutes consist of the researcher/observers experimenting with different calculators in order to show Stephanie how to use the calculator to find the combinations. They discuss the similarities and differences between the various calculators and share their experiences using them with Stephanie.

In the last ten minutes of the video, R1 discusses the beauty of the mathematics they just did: the symmetry and how things fit together. R1 and Stephanie discuss $(a + b)^7$ and what the coefficients would look like. As they predict the coefficients, R1 asks Stephanie to think about why the numbers repeat. Stephanie comments that it's just like with the cubes, "isn't it just 'cause it's the opposite" (E: 827). They choose an example of towers four high as a point of discussion, bringing up opposites within the same category, then applying it to the $(a + b)^7$ example. The session closes with the researcher/observers making general conversation and Stephanie volunteering to make copies of her papers.

9.2 Subtask 1: Separation Strategy

R1 begins this segment of the video by suggesting that she and Stephanie do something different and "put the algebra stuff on hold" (E: 19, 21). R1 then introduces the term 'combinatorics' to Stephanie and decides to explain it to her using a train of four cubes with two possible colors: red and yellow. She mentions that unlike towers, "we can keep them flat" (E: 27). R1 asks Stephanie how many trains of four cubes can be made with one color. Stephanie immediately responds, "One color? Two!" (E: 34) When asked to explain herself, she replies, "Well, if there's only- you can only- you only have, um, two colors, so you can only- and you can only use one color each, so there'd only be two colors." (E: 36) R1 then uses this example to introduce notations used to express combinations. She describes combinations as 'selections' and introduces two ways of writing them: C_4^4 *and* $\binom{4}{4}$ (E: 61). R1 explains to Stephanie that the number on top tells you the number of cubes you have and the number on the bottom tells you "how many of a certain kind you're picking" (E: 63). Therefore, in this case, Stephanie is selecting four from four.

The subtask begins with R1 asking Stephanie: given a train four cubes long, how many ways could she select one red. R1 proceeds to write it down as C_1^4 *and* $\binom{4}{1}$ (E: 73). Stephanie already knows the answer without writing anything down. She responds, "There's four, if you're saying that I could put a red here, and three yellows and a red here. That would be four." (E: 74) R1 continues by supposing they were selecting two red cubes. Stephanie initially conjectures two ways, then immediately changes her mind to three ways then to "a lot" of ways (E: 84 – 88). Since there are no unifix cubes available to demonstrate, Stephanie begins drawing diagrams of the possible trains with two red cubes. At first, Stephanie draws three horizontal trains with the two red cubes next to each other. Then, underneath, she draws two trains where the red cubes are separated by one cube.

R	R		
	R	R	
		R	R
R		R	
	R		R

Stephanie describes her strategy as follows:

***Table 9-1:* Stephanie explains her representation of trains with two red cubes.**

91	R1	Okay, show me what you did.	
92	Stephanie	Red, red, like, two reds. Two reds. They're all just two reds.	BEJ
93	R1	So that's two reds. You put them this way here and you put them another way here. What's the difference?	
94	Stephanie	Oh, well, I just separated them. Here, they were together and here they're not. But they're still two reds.	BEJ
95	R1	And so these are all the ways they can be together and these are all possible ways they can be separated?	
96	Stephanie	Yes.	

Upon being asked if she was sure, Stephanie takes a closer look then proceeds to add another row to her drawing where the two red cubes are separated by two cubes.

R			R

When asked to justify her results and prove that these are the only possibilities, Stephanie explains her reasoning as follows:

***Table 9-2:* Stephanie explains her *'separation'* strategy.**

107	R1	So you separated them by none and how do you know you can't do any more of that, that you can separate by none?	
108	Stephanie	Because I filled up all the spaces.	BEJ
109	R1	Okay, and how do you know that there are no more that you can separate by one? Because you filled up all the spaces.	
110	Stephanie	Yeah. Ok.	
111	R1	How do you know that there are no more you can separate by two? Now what about separate by three?	
112	Stephanie	Because there's not enough space.	BEJ

In order to account for all possible cases with one red cube, Stephanie uses a '*separation*' strategy where she lists all the possible ways of separating the two red cubes: separating by none, separating by one, and separating by two. She acknowledges that she can't separate by any more than that because "there's not enough space" (E: 112). She has just proved that there are six possible horizontal trains with two red cubes.

Stephanie then uses the same strategy of 'separating' to figure out the number of possibilities for three red cubes. She draws a diagram as follows:

R	R	R	
	R	R	R
R		R	R
R	R		R

Using the notation, Stephanie represents it symbolically as $C_3^4 = 4$. When R1 asks Stephanie to convince her that all possibilities are accounted for, she describes her strategy as follows:

***Table 9-3:* Stephanie uses her '*separation*' strategy to convince R1 that she has found all the cases of selecting three red cubes.**

131	R1	Okay, how can you convince me that you have them all?	
132	Stephanie	All right, well, here they're not separated by any, so there's only two ways you can do that. There's not enough space, to, like, move them again.	BEJ
133	R1	Okay.	
134	Stephanie	And here, they're separated by one, so you have one standing by itself over here and then two over here with a space in between, and then you switch it. But like, you can't.	BEJ

She has proven that there are four horizontal combinations. R1 then asks her to visually represent the case with one red cube. Stephanie draws the following diagram:

R			
	R		
		R	
			R

R1 asks her what would be in the blanks. Stephanie responds that those would be yellow cubes (E: 139 – 142). She then asks her if there is any relationship between the diagrams of the cases of three red and one red. Stephanie recognizes that there are four combinations for each case. R1 then suggests moving the top row of the four cases representing three reds to the bottom and claims that it "makes it easier for me to see" (E: 163). They end up with the following figure:

	R	R	R
R		R	R
R	R		R
R	R	R	

Stephanie now is able to point out the symmetry along the diagonal as in the case with one red cube (E: 164). She and R1 write out all the combinations including the case of ‘no red’ which corresponds to $C_0^4 = 1$ (E: 183 – 186). They find that the sum of all the combinations selecting red is sixteen (E: 203 – 206). Stephanie is able to see the isomorphism of this task with the trains to the towers task she worked on years ago (E: 212 – 216). She is able to apply Davis’ step (2) *From this data representation, carry out memory searches to retrieve or construct a representation of (hopefully) relevant knowledge that can be used in solving the problem or otherwise going further with the task.* However, she acknowledges that her approach has changed. In the present task, she utilizes a more systematic approach. The following excerpt illustrates her perspective:

Table 9-4: **Stephanie discusses the change in her approach to organizing towers.**

217	R1	How did you do it differently with the towers?	
218	Stephanie	Well, with the towers, I just didn’t have this, to, like, say “All right, now I’m going to try it with three.” I just, like, did all these different things until we couldn’t do them any more.	PPK; BEJ
219	R1	Mm-hmm.	
220	Stephanie	So, it was like, more just like guessing. You know?	BEJ

R1 points out that in her later work on towers she did use a strategy rather than just guessing albeit a different one. This leads to the following subtask.

9.3 Subtask 2: Family and Opposites Strategy

R1 recalls a strategy that Stephanie used in the past when working on the towers task. She refers to it as “some family thing” where Stephanie “built up” (E: 221 – 225). R1 describes how Stephanie would build all combinations one high then in order to build two high she “would build on top of” (E: 229, 231). The following excerpt illustrates the process:

***Table 9-5:* R1 and Stephanie refer to a '*family*' and '*opposites*' strategy used in Stephanie's earlier work in elementary school with the towers.**

239	R1	And then, you talked about, "Ok, now I want to move from one to two high."	
240	Stephanie	Mm-hmm.	
241	R1	So you said, "Ok, if I start with the red, what could I do to make two high?"	
242	Stephanie	Well, I could have um, red-red.	BR-V
243	R1	You did something like this, right? [*draws a tree diagram showing how the towers build by adding a red and yellow to each previous tower.]*	
244	Stephanie	Yeah. Or I could have yellow-yellow. Oh if you want to use the red, you can have red-yellow.	BR-V; BEJ
245	R1	If you start with red on the bottom?	
246	Stephanie	Well, yellow-red.	BR-V
247	R1	Is that right?	
248	Stephanie	Yeah.	

R1 reminds Stephanie of a tree diagram that they used in the past to represent the different combinations. She begins drawing one and then has Stephanie fill in the rest of the combinations for towers two high and three high. They discuss how many combinations there would be for towers four high without drawing that case and decide that there would be sixteen (E: 287 – 289). R1 then points out that for all of these combinations, they were selecting red. She mentions that if they were selecting yellow instead, wouldn't they end up with another sixteen combinations for a total of thirty-two combinations? Stephanie affirms this fact. R1 points out that that is not the same result as the sixteen combinations they would get using the tree diagram and building up (E: 303). Stephanie reconciles the two answers by explaining that the yellow combinations are the same as the red combinations but are opposites of each other, thereby forming another strategy that can be helpful in forming combinations. This is illustrated in the following excerpt:

***Table 9-6:* R1 and Stephanie discuss the *'opposites'* strategy.**

304	Stephanie	But wouldn't it be the same thing? Like, only the opposite way? 'Cause, wait, if there's two red, then there's two yellow. [*writing*] And if there's three red, then there's one yellow. And if there's one red, then there's three yellow, so isn't it the same thing?	BDI; BEJ; BR-V
305	R1	Is it?	
306	Stephanie	Yeah.	
307	R1	Ok, you're sure of that?	
308	Stephanie	Yeah.	
309	R1	And-and that's why if you think about that as a strategy, if you've already figured out exactly one, do you know exactly three?	
310	Stephanie	Um?	
311	R1	See this was the exactly one here, right?	
312	Stephanie	Mm-hmm.	
313	R1	Right?	
314	Stephanie	Yes.	
315	R1	That was exactly one red. And when you did exactly three red, I asked you to move one, you also got four.	
316	Stephanie	Yeah, well, I guess it's just the opposite.	BEJ; BCA
317	R1	Isn't that interesting?	
318	Stephanie	Yeah.	

9.4 Subtask 3: Pascal's Triangle

9.4.1 The Specific

In this next segment, R1 begins listing all of the different combinations using the C_r^n notation for *n* = 1, 2, 3. For example, for *n* = 1, she writes $C_0^1 = 1$ *and* $C_1^1 = 1$; for *n* = 2, she writes $C_0^2 = 1$, $C_1^2 = 2$, *and* $C_2^2 = 1$ etc. Stephanie is able to come up with these values using the context of selecting red for towers *n* high. She is able to use her visual and mental representations in order to formulate symbolic representations for the number of combinations, thereby applying Davis' step (3) *Construct a mapping between the data* *representation and the knowledge representation*. This process is illustrated in the following excerpt:

Table 9-7: **Stephanie forms symbolic representations for the case of selecting red for towers 2 high.**

351	R1	So I thought we'd do something else that might. . . . now two. Right? So if we're doing two now, again, what do you want to think of red or yellow? Does it matter? You told me it doesn't matter.	
352	Stephanie	Yeah, it would be one.	BR-S
353	R1	There's one way. You saw that right away. What made you see that right away?	
354	Stephanie	Well, because there's always going to- if there's- you can't do none of one, and there's another color, it's obviously going to be all the other color.	BEJ
355	R1	Good, that's great. Ok, so now, if we're gonna do – I'm going to pick one out of two.	
356	Stephanie	Um, two ways, I guess. One on top or one on bottom.	BR-V/S
357	R1	Mm-hmm. Can you see that?	
358	Stephanie	Yes.	
359	R1	And if it's two out of two?	
360	Stephanie	It would be one.	BR-S
361	R1	Okay. So, when I have $n = 2$, here I had one, right, that's no reds or one, that was one red, which was one high. Now, if I'm talking two high, I could have one red, I could have two reds, or I could have one red. No reds. One red or two reds. So this one is this piece, this one is this piece, this one is . . . let me just put the numbers in now.	
362	Stephanie	Okay.	

For $n = 3$, Stephanie finds some of the combinations by drawing a diagram and then finds the remaining ones by using the '*opposite*' strategy (E: 388 – 394).

R1 then begins to write the number of combinations in a horizontal fashion with row under row forming a triangle.

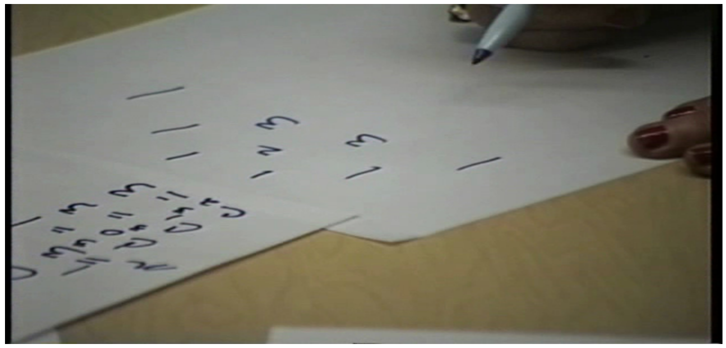

Figure 9-5. **The first few rows of Pascal's triangle.**

R1 then invites Stephanie to predict the number of combinations for $n = 4$. Stephanie is able to predict all but one of the numbers, but then goes on to say that she knows it would be six. She is also able to recognize the relationship between the numbers in each row and the previous row. This is illustrated by the following excerpt:

***Table 9-8:* Stephanie predicts the next row of Pascal's triangle using the towers and discovers the relationship of the numbers between the rows.**

399	R1	Now, I'm going to write for *n* equals three here, look, put a one, three, three, one. Now do you notice something happening here. I have a one-one, for these two. I have a one-two-one, a one-two-one for none, one and two. I have a one-three-three-one, one-three-three-one for the case of three. Do you want to predict what it's going to be like for four?	
400	Stephanie	It's going to be, like, one-four and then there's another number. And then, four-one.	BCA
401	R1	Okay, now that's the interesting. . . .	
402	Stephanie	Well, I know that that one's six though.	BR-S
403	R1	Oh, but notice something, no?	
404	Stephanie	Oh, is it, cause like, the 1 and 2- 1 and 1 are 2, 1 and 2 are 3, 1 and 2 are 3, 1 and 3 are 4, 1 and 3 are 4, 3 and 3 is 6?	BDI; BEJ

They then use the pattern to formulate the number of combinations for $n = 0$: $C_0^0 = 1$, selecting none from none, in order to complete the top of the triangle (E: 409 – 420). R1 now asks Stephanie if she can predict the numbers that would correspond to towers five.

Stephanie states that they would be one, five, ten, ten, five, one (E: 424). They write out the corresponding symbolic notation as follows:

$C_0^5 = 1, C_1^5 = 5, C_2^5 = 10, C_3^5 = 10, C_4^5 = 5, C_5^5 = 1$. Meanwhile, they discuss the corresponding tower representations for the first three cases as follows:

Table 9-9. **R1 and Stephanie discuss the mapping between the symbolic representations and the visual towers representations.**

429	R1	So, see if you can tell me what that one is? We're selecting . . .	
430	Stephanie	One from five.	BR-S
431	R1	Ok and you're telling me that this is the case that should be one.	
432	Stephanie	Mm-hmm.	
433	R1	And what's the five?	
434	Stephanie	Oh, no, that . . .	
435	R1	Is this one from five?	
436	Stephanie	Yeah, I thought, wasn't the five one from five. That would be zero.	BEJ; BR-S
437	R1	Okay, so you're going to make this, oh ok. So the five would be one from five, you're saying?	
438	Stephanie	Yeah.	BR-S
439	R1	And you believe that? You can see that in your mind?	
440	Stephanie	Yes.	
441	R1	What are you seeing? I'm curious.	
442	Stephanie	It would be like this, only longer.	BR-V; BEJ
443	R1	How long?	
444	Stephanie	Well, five.	
445	R1	Okay, just checking. Just checking. Ok, so the next one is going to be…	
446	Stephanie	Um, two from five. And that equals two.	BR-S
447	R1	And that's ten cases. You wouldn't want to write those out. You kinda wish this is gonna be true, don't you?	

Stephanie then uses the concept of '*opposites*' to figure out the remaining cases. For instance: $C_3^5 = C_2^5 = 10$ and $C_1^5 = C_4^5 = 5$. In other words, a tower five high with three reds is essentially the same as a tower five high with two yellows. R1 and Stephanie then discuss the total number of cases for $n = 5$. Stephanie predicts that there are 32 combinations. She then predicts that there will be 64 for the next row. In order to check, Stephanie predicts the next row to be 1, 6, 15, 20, 15, 6, 1 and then adds them up to get 64 (E: 466 – 470). They discuss the fact that not only are they able to get the next row but also the number of combinations for each row (E: 473 – 476). R1 shares with Stephanie the information that Blaise Pascal is responsible for this relationship called Pascal's Triangle and discusses the importance of having a visual representation instead of just using the triangle.

9.4.2 The General

R1 suggests building Pascal's triangle by using the more general combinatorics notation to generate the numbers. She initially begins using the C_r^n notation then switches over to the $\binom{n}{r}$ notation. She and Stephanie form a triangle that looks like the following:

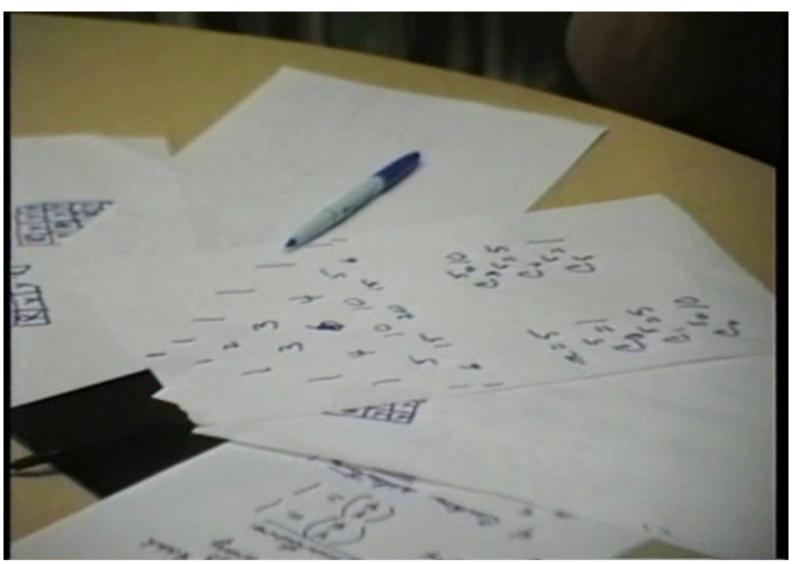

Figure 9-6: **Pascal's triangle and the corresponding symbolic representations.**

After forming four rows of Pascal's triangle, R1 begins writing down relationships derived from the triangle and suggests to Stephanie that they can hypothesize their value. For instance, she adds up $\binom{1}{0}+\binom{1}{1}=\binom{2}{1}$. They are forming these sums based on their relationship within Pascal's triangle. The following excerpt illustrates this:

Table 9-10: **R1 and Stephanie are using the symbolic representations and Pascal's triangle to hypothesize a general representation.**

576	Stephanie	Yeah, it would just keep going.	BCA
577	R1	So let's go- let's write it one more time. This is the row of…	
578	Stephanie	Zero from three, hmm- plus one from three equals one from four.	BR-S; BCA
579	R1	Ok, so in general, can you propose a rule?	
580	Stephanie	Well, that . . .	
581	R1	In other words, instead of um, having threes and fours,	
582	Stephanie	Oh, you want me to put in like, letters? Like, *m* from zero plus . . .	PAH; BCA
583	R1	If I'm taking- okay- taking zero from *n*.	
584	Stephanie	Oh yeah. And one from-	BCA
585	R1	-*n* should give me	
586	Stephanie	Um, one from, I guess, just a different letter.	BCA
587	R1	Well, you can tell me that letter because this is- this was a two, this was a three. This was a three, this is a four.	
588	Stephanie	Well, if that's n, then it would be *m*.	BCA
589	R1	Well, you can tell me more about *m*. What- how much- what's the relationship between *m* and *n*? We're not talking about candies. [*laughing*] In other words, what's the relationship between these two and this? These are two, this is three. If these are three, this is four. Maybe you need to write some more out. I propose it's one bigger.	
590	Stephanie	Ok.	

R1 is encouraging Stephanie to use generic reasoning by using the 'specific' values in the triangle to formulate a 'general' formula for a combination in the *n*th row.

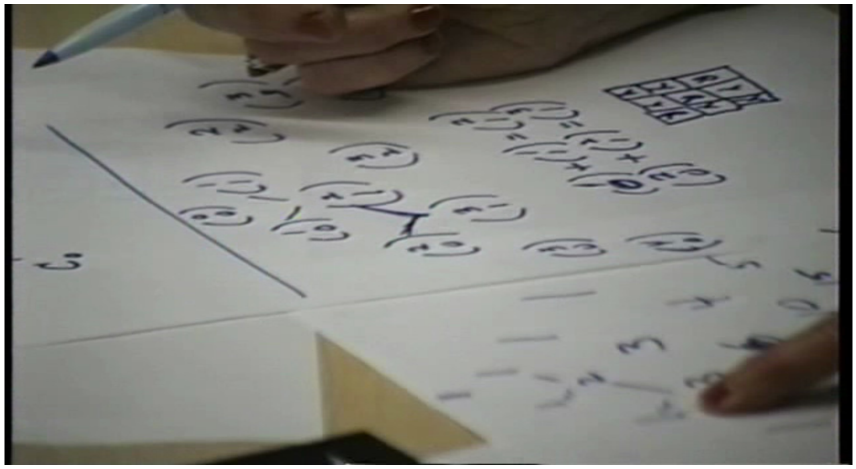

Figure 9-7: **R1 writes down specific sums extracted from Pascal's triangle.**

Stephanie is left to ponder this proposition later and form her own hypotheses. They then move on to the next subtask.

9.5 Subtask 4: Binomial Expansions: Isomorphisms

This segment of the video begins with R1 reminding Stephanie that she asked her to do something from the last session. Stephanie recalls that the task was to think about $(a+b)^4$ *and* $(a+b)^5$. She has worked it out and goes to retrieve her papers.

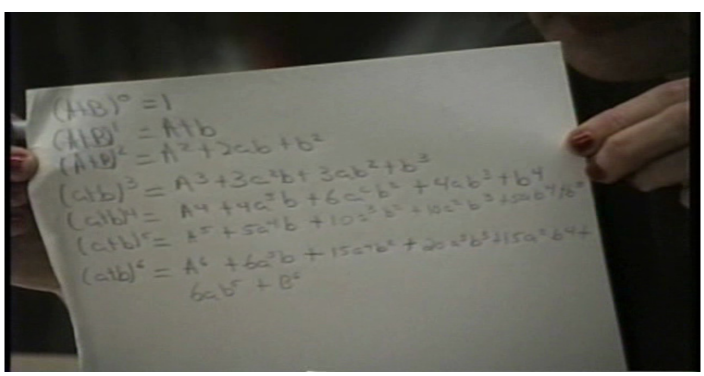

Figure 9-8. **Stephanie's write-up from the previous session.**

R1 begins the discussion by asking Stephanie to think about the coefficients in front of the terms (E: 626 – 628). They discuss the convention that when there is no number in front of a single, variable term, the coefficient is one. R1 then begins to make a diagram of the coefficients row by row as she maps it to the binomials in Stephanie's work. For example, for $(a + b)$, R1 writes the line 1 1 corresponding to the coefficient of a and the coefficient of b. As they do this, Stephanie realizes that the numbers correspond to Pascal's triangle (E: 641 – 648). She is able to recognize the isomorphism of the two representations and likens them to the towers claiming that "it's the same thing" (E: 644). This claim is supported by the next two rows corresponding to the coefficients of the simplified forms of $(a + b)^4$ and $(a + b)^5$.

R1 and Stephanie then discuss the meaning of a squared. They refer to it as a times a or two factors of a (E: 661 – 663). They then decide to represent it as red. Stephanie makes the transition and represents two factors of a or a^2 as a tower two high with two reds (E: 673 – 674). She is able to represent the remaining terms in terms of towers as follows:

***Table 9-11:* Stephanie represents the terms in her binomial expansions in the context of towers.**

675	R1	And here I'm talking about two things.	
676	Stephanie	Mm-hmm.	
677	R1	One is-	
678	Stephanie	-red-	BR-V
679	Stephanie/ R1	-and one is yellow.	BR-V
680	R1	Is that possible in two high?	
681	Stephanie	Yeah.	
682	R1	To have the one red and one yellow? There are two of them.	
683	Stephanie	Yeah. 'Cause one is- the red can be on top or on the bottom. And the yellow -same thing.	BEJ; BR-V
684	R1	And what about b squared?	
685	Stephanie	Um- two yellow.	BR-V

Stephanie continues by representing the simplified coefficients of $(a + b)^3$ as different combinations of towers three tall (E: 693 – 719). For instance, when R1 points to the term $3a^2b$ in the expansion $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, Stephanie is able to use her mental representation of the corresponding towers to recite the different combinations and to affirm that there are three of them just like the coefficient of the term.

Table 9-12: **Stephanie maps the terms in her binomial expansion of $(a+b)^3$ to towers three high.**

698	R1	And here I have…	
699	Stephanie	Um . . . towers . . . um . . . of red and yellow, three high, I guess? Since there's three of them?	BR-S/V
700	R1	Right, and how many are reds and how many of them are yellow?	
701	Stephanie	Two are red and one is yellow. And . . .	BR-S/V
702	R1	And there are three of those.	
703	Stephanie	Yes. And the next. . .	
704	R1	Do you really believe that?	
705	Stephanie	Yes.	
706	R1	Two are reds and one are yellow? Can you see them? The three? The yellow, the yellow, the yellow?	
707	Stephanie	Yeah uh yeah. I mean, you could have, um, the red, the red, the yellow. The red, the yellow, the red. The yellow, red, red.	BR-V

Stephanie has just applied Davis' step (3) *Construct a mapping between the data representation and the knowledge representation*. In future, Stephanie can use this mapping between the towers representation and the binomial expansions to aid her in working on other tasks and forming new conjectures. R1 then asks her to write up these results for the next session.

9.6 Subtask 5: Symmetry

The next subtask begins with R1 and the other researcher/observers demonstrating to Stephanie how to calculate the number of selections/combinations using a calculator.

She and R1 then begin discussing the case $(a + b)^7$. They begin predicting the next row of numbers in Pascal's triangle, as follows: 1, 7, 21, 35, 21… When they reach 21 again, R1 points out that there is symmetry. Stephanie responds, "Oh, like with the cubes, isn't it just 'cause it's the opposite?" (E: 827) She elaborates, "Cause, like, if I have two – if I have towers of four, and it's two red, then there's going to be two yellow." (E: 829, 831) R1 and Stephanie look at the particular example of towers four high and the corresponding row in Pascal's triangle: 1, 4, 6, 4, 1. Stephanie is identifying the symmetry in the coefficients while R1 is identifying the symmetry within the actual terms. This is illustrated in the following excerpt:

***Table 9-13:* R1 and Stephanie discuss the symmetry between the coefficients within a row and the terms themselves.**

834	R1	Ok, so let's look at a particular line. You said towers – how high did you say?	
835	Stephanie	Of four.	
836	R1	So towers of four is which line here?	
837	Stephanie	Yeah, that one. And if I have two- I have two red on it –	BEJ
838	R1	So two red would be this.	
839	Stephanie	Yeah.	
840	R1	So this would contain two red and two yellow?	
841	Stephanie	Well, I mean, wouldn't it just be – Yeah, see, right here. See *a* squared and see *b* squared.	BEJ; BR-S
842	R1	I was thinking of the symmetry here, like, 4 *a cubed b* and *4 a b cubed.*	

Stephanie's response is that it's the same thing (E: 843). R1 describes it as "opposites in the same categories" (E: 846 – 847). R1 and Stephanie then use both views of '*opposites*' to predict the binomial expansion of $(a + b)^7$. They have already predicted the coefficients of each term but now need to formulate the actual terms. Stephanie is confident that a^7 is the first one and the next one would be a^6b (E: 850 – 859). However, when continuing, she recognizes that the next term would contain a^5 but is not sure what the b would be raised to (E: 861). In order to get past this obstacle to understanding, R1 suggests they look at the case $(a + b)^6$. She encourages Stephanie to study it and look for patterns. Stephanie is being encouraged to use the heuristic method of looking at a simpler or more concrete problem in order to gain insight into the task at hand. Stephanie realizes that the sum of the exponents in all of the terms within the expansion adds up to six (E: 874 – 877). She then uses generic reasoning to deduce that the exponents in each of the terms in the expansion of $(a + b)^7$ must therefore add up to seven, explaining that it's because "you're building it seven high" (E: 879 – 881). She now returns to the term with a^5 and realizes that the b would be squared to get a^5b^2 (E: 883). Stephanie then conjectures that the next term would be a^4b^3 (E: 893). She then realizes that the remaining terms are '*opposites*' and states, "Mm-hmm. Um, the next one would be the opposite, a to the third b to the fourth and then it would just keep going the opposite." (E: 897)

In the remaining couple of minutes, R1 encourages Stephanie to search for meaning and to imagine the towers when studying these relationships and patterns. They organize papers and casual conversation follows.

CHAPTER 10 Session 6 - March 27, 1996 Building Towers

10.1 Background

This session contains two main parts. The first part consists of Stephanie explaining what was discussed in previous sessions, while in the second part, Stephanie works on a new subtask. The session begins with Carolyn Maher, designated R1 throughout the transcript, advising Stephanie on her interaction with Robert Speiser, another researcher/observer, who is designated R2 throughout the transcript. R1 tells Stephanie that he is capable of understanding anything she explains, but to start from the beginning and not to assume that he already knows the information. Stephanie will be explaining to R2 the content of the previous session. R1 adds that she will help if she can. This time there are blue and green unifix cubes available for Stephanie to use.

Stephanie recalls that they began with trains four cubes long with a choice of two colors. Stephanie mentions that R1 asked how many different combinations could be made when selecting one color. Stephanie demonstrates by building four different trains, each with three blue cubes and one green cube. Furthermore, Stephanie represents this situation symbolically by writing down C_1^4 *and* $\binom{4}{1}$. She explains them by saying, "that means that you're selecting one out of four choices" and that there are a total of four (F: 53, 55, 61, 63). R2 indicates that he is convinced (F: 66). Stephanie then discusses the case in which one is selecting two green from four. She builds six possible combinations with two green and claims that there are no more. R2 then asks her how she knows that. She begins by saying, "Cause I tried all the combinations possible" then proceeds to explain to him her *'separation'* strategy: separating by none, by one, and by two (F: 81 –

88). Stephanie has now switched to towers (i.e., unifix cubes standing up instead of flat). Stephanie convinces R2 of her solution and then writes down $C_2^4 = 6$ (F: 124, 130). She then moves to the case of selecting three green from towers four high. As she builds, she comments that it's the *'opposite'* of the case of selecting one green, so therefore there will also be four combinations (F: 133, 137, 143). She then writes $C_3^4 = 4$. She then discusses the cases of selecting four blues out of four and no blues out of four, expressing them as $C_4^4 = 1 \textit{ and } C_0^4 = 1$ and using the *'opposite'* strategy to explain herself. This segment of the video lasted approximately ten minutes.

In the next seven minutes, Stephanie refers back to the approach used in building combinations in her earlier work with the towers. In the previous session, she used a tree diagram to illustrate the concept of 'building up' but in the present session, Stephanie uses the unifix cubes beginning with one blue and one green, then building up the next case for towers two high, then for towers three high, and finally for towers four high. She demonstrates two combinations for towers one high, four combinations for towers two high, eight combinations for towers three high, and sixteen combinations for towers four high. Stephanie then describes how she and R1 defined $C_0^0 = 1, C_0^1 = 1, \textit{and } C_1^1 = 1$.

Stephanie then goes on to describe how she and R1 arrived at Pascal's triangle. Stephanie begins by writing out the rows of Pascal's triangle as R2 asks questions regarding where the numbers originated. For example, for the two in the third row of Pascal's triangle, Stephanie explains that the row corresponding to $n = 2$ maps to towers two high. Furthermore, the 2 represents the number of possible combinations when selecting one color for towers two high. She proceeds to demonstrate the relationship between the towers she had built and the numbers found in Pascal's triangle. She explains the relationship between the numbers in a given row and the row immediately above it: you add in order to get the number in the next row. Stephanie and R2 discuss different ways of organizing the towers.

The question then arises of "why the adding works" in Pascal's triangle (F: 324). R1 and R2 both ask Stephanie to explain it for towers three high. Stephanie tells them how she arrived at each of the towers three high. However, R2 is still unsure of the relationship between the 'building up' and the succession of rows in Pascal's triangle. Stephanie rearranges the towers in a way that better illustrates the 'building up'. R1, R2, and Stephanie then discuss various organizations of the towers, realizing that they can reach the same goal but from different perspectives, all of which work. This segment of the video lasted approximately ten minutes.

The next question posed to Stephanie has to do with the relationship, if any, of the binomials to Pascal's triangle and/or to the towers representations. This segment lasts approximately fifteen minutes. Stephanie begins by relating to R2 how she came up with $(a+b)^2 = a^2 + 2ab + b^2$. In order to answer the question, she first describes the relationship of the coefficients and the exponents to the towers. She explains that if *a* represents green and *b* represents blue then there are two combinations of a two high tower with one blue cube and one green cube (F: 495 – 500). She then explains that "you have one that's all *a* [*indicates* $\begin{bmatrix} G \\ G \end{bmatrix}$] and one that's all *b* [*indicates* $\begin{bmatrix} B \\ B \end{bmatrix}$], corresponding to the coefficients of a^2 and b^2 (F: 501 – 507). Furthermore, she describes the a^2 as "two factors of *a*", the b^2 as "two factors of *b*", and the *ab* and the *ba* as one factor of each (F: 517 – 521). Stephanie then points out the correspondence of the coefficients to the third row in Pascal's triangle: 1, 2, 1.

R1 then asks Stephanie if she could show the relationships between $(a + b)^3$, the towers, and the triangle. Stephanie proceeds to demonstrate these relationships by first simplifying $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ then searching for the towers that correspond to each term, expecting to find a total of eight towers. She then matches the coefficients to the fourth row in Pascal's triangle: 1, 3, 3, 1.

When demonstrating $(a + b)^4$, Stephanie uses a different approach as she does not remember the simplification of $(a + b)^4$ completely. She begins with a^4 recognizing that it corresponds to a tower four high with all green plus $4a^3b$ (F: 553 – 557). She then uses the triangle as well as the towers previously built to find the remaining terms in the expansion, eventually getting $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, after changing the order of the terms (F: 558 – 563).

In the next subtask, R1 and R2 shift to take an in depth look at the fourth row in Pascal's triangle: 1, 3, 3, 1. This portion of the video lasts approximately thirty-two minutes. R2 asks Stephanie how you get from one number in a row to the next number in the same row. He poses it in the context of the following problem: *Given a three high tower with one green, suppose we were to trade one of the blues for a new green, how many different ways could we do that*? (F: 644 – 652). Stephanie experiments and finds that each tower three high with one green produces two towers as soon as one of the blues is traded for a green. After considering this for three different towers, Stephanie realizes that since each one produced two towers, there are a total of six towers. However, some of them are *'duplicates'*. (F: 694 – 709) R1, R2, and Stephanie then discuss the concept of *'duplicates'* and how many combinations would result. They then decide to test Stephanie's idea and "build it".

R1 and R2 suggest saving the originals and then building, by taking one tower at a time as the base. Stephanie places $\begin{bmatrix} G \\ B \\ B \end{bmatrix}$ and $\begin{bmatrix} B \\ B \\ G \end{bmatrix}$ on the sides of $\begin{bmatrix} B \\ G \\ B \end{bmatrix}$ (F: 750).

Beginning with the $\begin{bmatrix} G \\ B \\ B \end{bmatrix}$ tower, Stephanie builds the two that would come about by trading in one of the blues for a green: $\begin{bmatrix} G \\ G \\ B \end{bmatrix}$ and $\begin{bmatrix} G \\ B \\ G \end{bmatrix}$ meanwhile keeping the top one constant. She does the same thing for the other two originals. R1, R2, and Stephanie then discuss the idea that there are duplicates and that they come in pairs (F: 767 – 774). R1 and R2 then question Stephanie regarding whether towers four high with one green would work in the same way. Stephanie conjectures that since there are four combinations of towers four high with one green, if she replaced one of the blues with green, she would end up with eight towers (F: 799, 801). Since half of them would be duplicates, she would really end up with four (F: 805). Stephanie begins to build them. She finds that each of the four 'originals' generates three other towers, giving her a total of twelve (F: 827, 829, 832). Again, Stephanie finds that the duplicates come in pairs, therefore, she really has six towers (F: 1014 – 1015, 1026 – 1029).

In the next subtask, R2 asks Stephanie to build a generation of new four-tall towers that have three green using the six four-tall towers with two green and two blue as the 'originals' (F: 1101). Stephanie begins by conjecturing that there would be towers from each 'original' thereby getting a total of twelve towers (F: 1114, 1116). She also predicts that the towers may be duplicates in triples, therefore, giving a total of four towers (F: 1137, 1141, 1143). Stephanie proceeds to build them, but then decides to look for them among the ones previously built. She locates three of them and builds the fourth. R2 then wants to continue the process, where now the 'originals' would be four four-tall towers with three green. Stephanie responds that there would be four duplicates with four greens (F: 1176 – 1178). If you divided by four you would get one.

In the last segment, lasting approximately fifteen minutes, R1, R2, and Stephanie decide to take stock of what they have done so far and review by writing it down. They document exactly what was done in order to get the 'new generation' after disposing of the duplicates. For example, R2 writes that they started with four towers four high with one green and three blues. They then multiplied by three since each tower produced three choices and got twelve. Since the duplicates came in pairs for that case, they divided by two in order to get six. They write out the remaining cases and then look for patterns. Stephanie is able to use the pattern to predict the case of producing new towers from towers four high with three green (F: 1413 – 1416).

In the last couple of minutes, R1 brings up the possibility of exploring another row in Pascal's triangle, that of towers five high. Stephanie builds one tower and uses it as a point of discussion. Since she is working backwards, she concludes that in the last case when dividing by duplicates, she would divide by five (F: 1473 – 1477). The session ends with R1 describing how this topic originated from a dinner conversation. R1 and R2 explain the history of the problem as they organize papers.

10.2 Recap: Building Towers

Stephanie is asked to relate the content of the previous session to R2. She is to assume that he is not familiar with the material but is capable of understanding anything she presents to him. The problem revolves around the number of combinations possible when selecting from four objects.

10.2.1 Separation Strategy

Stephanie began explaining to R2 what she had done by building four trains four cubes long with one green cube. She likens the visual representation to the symbolic representation C_1^4 *and* $\binom{4}{1}$ and explains that they mean that “you’re selecting one out of four choices” (F: 53, 55).

Stephanie then moves on to the case of selecting two green for trains four cubes long. She builds six trains with two green cubes and two blue cubes. She then reorganizes them into three groups by standing them upright, explaining her strategy as ‘separating by no greens’, ‘separating by one green’, and ‘separating by two greens’ (F: 95 – 103). She convinces R2 that there can’t possibly be any more “because you can’t move them anymore” (F: 107). She further elaborates that “there’s only four spaces for you to move them” (F: 107).

Figure 10-1: **Towers with 2 green cubes grouped by ‘*separation*’ strategy.**

Stephanie then expresses the symbolic representation as $C_2^4 = 6$.

Stephanie proceeds to demonstrate the case of selecting three green by building four towers four high with three green cubes and organizing them in a 'staircase' pattern.

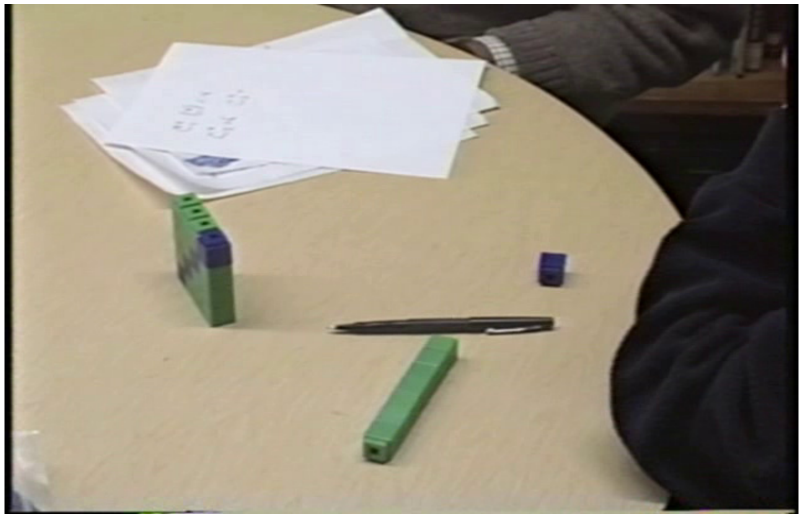

Figure 10-2: **Stephanie arranges towers with three green cubes in 'staircase' pattern.**

She then points to the four towers with one green cube and claims that they are just 'opposites' of each other, explaining that the blue is in the spot where the green was (F: 133, 137 - 140). Stephanie represents this case symbolically as $C_3^4 = 4$.

For the last case, Stephanie claims that there are two possibilities. If you are selecting four blue out of four then you have one tower with four blue cubes, while if you are selecting four green out of four then you have one tower with four green cubes. From the perspective of selecting green, the first possibility can be expressed as $C_0^4 = 1$ since the selection is of zero green. The second possibility can be expressed as $C_4^4 = 1$ since the selection is of four green. Again, Stephanie mentions that they are '*opposites*' of each other (F: 155).

R2 comments that Stephanie organized her towers differently when she was initially building them and when she was explaining to him why there were no more possibilities and questions the reasoning behind it. She responds as follows:

***Table 10-1:* Stephanie explains her organization of towers to R2.**

171	Stephanie	Oh. Well. – Because it's easier for me to look at them as opposites when I'm building them.	BEJ
172	R2	Um hm.	
173	Stephanie	Then – 'cause I know – 'cause it's like pairing them up – like – if there's one separated on top, there's one – you know -	BEJ
174	R2	Yeah -	
175	Stephanie	But, it's easier for you to look at them when they're done if they're like this. So you can see the pattern that they make. That you can't build down any more.	BEJ
176	R2	Um hm.	
177	Stephanie	Or you can't build up any more, 'cause there's no more to – do it.	BEJ

In other words, the pattern she formed when 'explaining' or justifying her reasoning made it easier to see that there were no more possibilities. Stephanie is not only learning to solve the problem task but also the more challenging task of proving her results.

10.2.2 Family Strategy

Stephanie relates to R2 how she and R1 went back to her earlier work with the towers where they used the strategy of 'building up'. In the previous session, she and R1 used a tree diagram, but now that unifix cubes are available, Stephanie offers to show R2 their representations by building them. She begins with towers one high: one blue and one green. She then moves to towers two high by generating two from the first one and two from the second one. Again, she is accomplishing this by 'building up'.

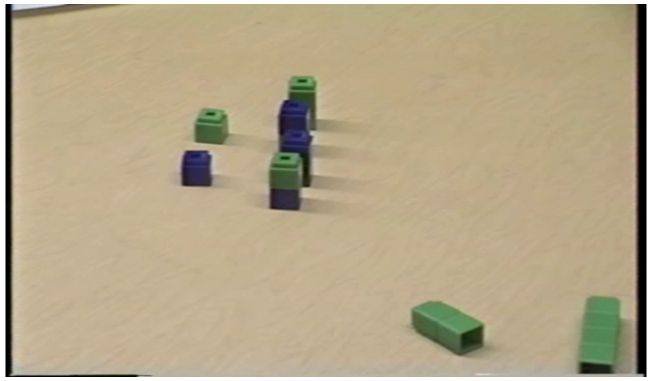

Figure 10-3: **Stephanie 'builds up' towers two high from towers one high.**

To formulate towers three high, Stephanie generates them by adding on either a green cube or a blue cube on top of all of the towers two high, resulting in a total of eight towers (F: 212). Stephanie continues to 'build up' by adding either a blue or green cube to each of the eight towers three high in order to get a total of sixteen towers (F: 238).

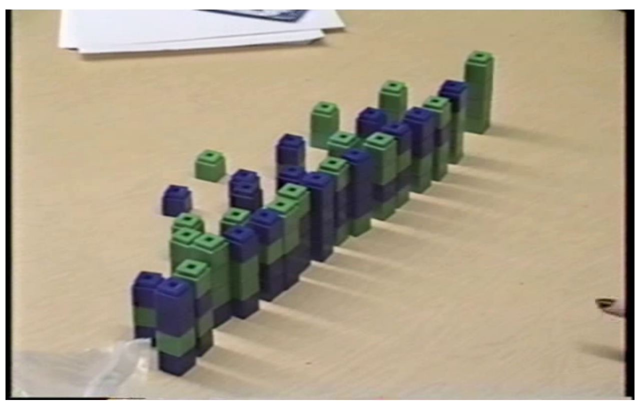

Figure 10-4: **Stephanie 'builds up' towers up to four high.**

10.2.3 Pascal's Triangle

Stephanie then recalls how she and R1 defined $C_0^0 = 1, C_0^1 = 1, \textit{and } C_1^1 = 1$ and how she used them in 'building the triangle'. She describes the first one as "how many you can get if you take zero from zero" and the other two as "zero out of one or one out of one" as she writes the first three rows of Pascal's triangle and points to the three ones at the top of the triangle (F: 253, 257). She describes the other numbers in the triangle in the same way, while matching each number to its corresponding tower, thereby mapping the visual representation of towers to the symbolic representation of the triangle. Stephanie has just applied Davis' step (3) *Construct a mapping between the data representation and the knowledge representation.*

Stephanie also explains to R2 how she got each row by adding up the numbers in the previous row (F: 314 – 319). He questions "why the adding works" (F: 324 – 325). He and R1 ask Stephanie to demonstrate it for the fourth row: 1, 3, 3, 1. Stephanie begins by explaining to them how she formed her towers three high. However, R2 wants to know how the towers get added to each other in order to get the combinations in the next row in Pascal's triangle (F: 387, 389). Stephanie tries to illustrate by reorganizing the different combinations of towers three high. One organization is by putting each one by its 'opposite'. Another way is to use the resulting organization from 'building up'. R1 prefers the organization where you can see the patterns (i.e. the 'staircase' pattern) . They conclude that although they are all different organizations, "they all work" (F: 466 – 468).

10.2.4 Isomorphism

It is interesting to note that Stephanie voluntarily decides to share with R2 her work with the binomials (F: 478 – 479). She begins by writing down the square of a binomial, expressing it as $a^2 + 2ab + b^2$. She then adds rows five and six to her previous diagram of Pascal's triangle, meanwhile explaining to R2 how she went up to $(a + b)^6$. As Stephanie struggles to recall the relationship between the exponents and the numbers in Pascal's triangle, R1 asks her if $(a+b)^2 = a^2 + 2ab + b^2$ can be related to the triangle and/or to the towers (F: 485, 488 – 490). In order to answer the question, at first, she describes the relationship of the coefficients and the exponents to the towers. She explains that if a is green and b is blue then there are two combinations of a two high tower with one blue cube and one green cube (F: 495 – 500). She then explains that "you have one that's all a [*indicates* $\begin{bmatrix} G \\ G \end{bmatrix}$] and one that's all b [*indicates* $\begin{bmatrix} B \\ B \end{bmatrix}$], corresponding to the coefficients of a^2 and b^2 (F: 501 – 507). Furthermore, she describes the a^2 as "two factors of a", the b^2 as "two factors of b", and the ab and the ba as one factor of each (F: 517 – 521). When asked how many there are of each, Stephanie mentions that there are one of the a^2, two of the ab, and one of the b^2 and writes in the one before the a^2 and the b^2 (F: 517, 519, 521). Stephanie then points out the correspondence of the coefficients to the third row in Pascal's triangle: 1, 2, 1. She has clearly delineated the isomorphism between the binomials, the triangle, and the towers thereby applying Davis' step (3) *Construct a mapping between the data representation and the knowledge representation*.

R1 then asks Stephanie to represent the quantity $(a + b)^3$ without simplifying, but by using the towers and the triangle to figure it out. Stephanie begins by writing an a^3 and a b^3. When asked why, Stephanie replies, "Because there's a one and a one", meanwhile pointing to the one and one at the ends of the fourth row of the triangle (F: 529). She is able to identify the corresponding towers: $\begin{bmatrix} G \\ G \\ G \end{bmatrix}$ for a^3 and $\begin{bmatrix} B \\ B \\ B \end{bmatrix}$ for b^3 (F: 531, 533). Stephanie then decides that there will be a $3a^2b$ and a $3ab^2$ (F: 535).

***Table 10-2:* Stephanie forms the terms in the expansion of $(a + b)^3$ by using the towers and Pascal's triangle.**

546	Stephanie	I have three with two factors of a and one factor of b.	BR-S/V

		[*Stephanie indicates* $\begin{bmatrix} G \\ G \\ B \end{bmatrix} \begin{bmatrix} G \\ B \\ G \end{bmatrix} \begin{bmatrix} B \\ G \\ G \end{bmatrix}$]	
547	R1	Okay.	
548	Stephanie	And I have three with two factors of *b* and one factor of *a* [*indicates* $\begin{bmatrix} G \\ B \\ B \end{bmatrix} \begin{bmatrix} B \\ G \\ B \end{bmatrix} \begin{bmatrix} B \\ B \\ G \end{bmatrix}$] so I guess it would be *a* cubed plus three *a* squared *b* plus three *ab* squared plus *b* cubed. [*inserts plus signs so that her paper now reads:* $a^3 + 3a^2b + 3ab^2 + b^3$]	BR-S/V

Stephanie now chooses to represent $(a + b)^4$ at R1's request to describe another one of the binomial expansions. This time, Stephanie doesn't know all of the terms in the expansion of $(a + b)^4$. She recalls that it will contain the terms $a^4 + 4a^3b + 4ab^3 + \ldots$ but is not sure about the rest (F: 556 – 560). In order to find the remaining terms, she uses Pascal's triangle and the towers to formulate them as illustrated in the following excerpt:

Table 10-3: **Stephanie forms the terms in the expansion of $(a + b)^4$ by using the towers and Pascal's triangle.**

574	Stephanie	*a* to the fourth [*writes* a^4] so you have all *a*. [indicates $\begin{bmatrix} G \\ G \\ G \\ G \end{bmatrix}$]	BR-S/V
575	R2	Um hm.	
576	Stephanie	Um plus four *a* cubed *b*. You have four with three *a*. [*indicates* $\begin{bmatrix} B \\ G \\ G \\ G \end{bmatrix} \begin{bmatrix} G \\ B \\ G \\ G \end{bmatrix} \begin{bmatrix} G \\ G \\ B \\ G \end{bmatrix} \begin{bmatrix} G \\ G \\ G \\ B \end{bmatrix}$]	BR-S/V
577	R2	Um hm.	
578	Stephanie	You have four with three *a* and one *b* and you - plus if you're following this the next one's the six. [*indicates towers four high with two blues and two greens*]	BR-S/V

579	R2	Ah.	
580	Stephanie	You have six with two *a* and two *b* [*writes* $6a^2b^2$] with two factors of *a* and two factors of *b* and then you have another one where it's four except it has three factors of um yeah three factors of *b* [*writes* $4ab^3$]and then you have one where it's just factors of …	BR-S/V
581	R1	I can't see what you wrote. There's three factors of *b*. [*Stephanie moves the towers away*.]	
582	Stephanie	And one where it's just um four factors of *b*. [*writes* + b^4]	BR-S/V

Stephanie then rearranges them to get the expression: $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

Stephanie has just used the isomorphism between the binomials, the triangle, and the towers to expand $(a + b)^4$ successfully, thereby applying Davis' step (5) *When the constructions and the mapping appear satisfactory, use technical devices (or other information) associated with the knowledge representation in order to solve the problem*.

10.3 Subtask 1: An in depth look at the fourth row in Pascal's Triangle

10.3.1 Exploration

In this subtask, R1 and R2 invite Stephanie to explore the relationship between the numbers in the fourth row of Pascal's triangle: 1, 3, 3, 1. They pose this within the context of the following problem: *Given a three-high tower with one green, suppose we were to trade one of the blues for a new green, how many different ways could we do that*? (F: 644 – 652) Stephanie begins by experimenting with the following tower: $\begin{bmatrix} B \\ G \\ B \end{bmatrix}$.

She decides that there are two ways of replacing a blue with a green: $\begin{bmatrix} G \\ G \\ B \end{bmatrix}$ and $\begin{bmatrix} B \\ G \\ G \end{bmatrix}$ (F: 673 – 677). She then considers the other two possible towers three high with one green:

$\begin{bmatrix} G \\ B \\ B \end{bmatrix}$ and $\begin{bmatrix} B \\ B \\ G \end{bmatrix}$. Again she finds that each tower produces two possible cases. She also discovers that there are duplicates and that they come in pairs. Therefore, not counting duplicates, there are really three possible towers. This is illustrated in the following excerpt:

***Table 10-4:* Stephanie realizes that in 'building up' towers three high, duplicates come in pairs so there are only three resulting towers.**

710	Stephanie	Um, hum but they're both duplicates.	BEJ
711	R2	But they're both duplicates	
712	Stephanie	Yes.	
713	R2	Okay. Very good. Okay.	
714	R1	How's that?	
715	Stephanie	Well if you put one on the top, you have this one with the one on the top, and one on the bottom. [*points to* $\begin{bmatrix} G \\ B \\ B \end{bmatrix}$] If you put one there, you have this one with the one there, and the one there. [*points to* $\begin{bmatrix} B \\ G \\ B \end{bmatrix}$] So that's just that doesn't really do anything.	BEJ
716	R2	Okay. Good. Okay. So if so we built six. We imagined building six towers but we noticed that they came in pairs.	
717	Stephanie	Yeah.	
718	R2	Is that right?	
719	Stephanie	Um hum.	
720	R2	Okay, so um so what's the real number of towers?	
721	Stephanie	Three.	BR-V

They then decide they want to build them up.

10.3.2 Building it

Stephanie proceeds to build up the six cases discussed and arranges them as follows:

$$\begin{bmatrix} G \\ B \\ B \end{bmatrix} \qquad \begin{bmatrix} B \\ G \\ B \end{bmatrix} \qquad \begin{bmatrix} B \\ B \\ G \end{bmatrix}$$

$$\begin{bmatrix} G \\ G \\ B \end{bmatrix} \quad \begin{bmatrix} G \\ B \\ G \end{bmatrix} \quad \begin{bmatrix} G \\ G \\ B \end{bmatrix} \quad \begin{bmatrix} B \\ G \\ G \end{bmatrix} \quad \begin{bmatrix} G \\ B \\ G \end{bmatrix} \quad \begin{bmatrix} B \\ G \\ G \end{bmatrix}$$

They are able to clearly see that there are three sets of duplicates that do indeed "come in pairs" (F: 769 – 770). Therefore, the original ones really only produced three cases. At this time, R2 and Stephanie begin to use terms like "parents" to refer to the ones they started out with before replacing a blue cube and "children" to refer to the resulting ones after replacing one of the blue cubes (F: 776 – 780).

R1 now questions whether the same thing would happen for towers four high with one green. Stephanie conjectures that if they replaced a blue cube with a green, they would result in eight towers with duplicates again coming in pairs; therefore there would actually be four possibilities (F: 800 – 806). Again, Stephanie proceeds to build this case.

She starts with the original four towers four high with one green: $\begin{bmatrix} B \\ B \\ G \\ B \end{bmatrix} \begin{bmatrix} B \\ G \\ B \\ B \end{bmatrix} \begin{bmatrix} G \\ B \\ B \\ B \end{bmatrix} \begin{bmatrix} B \\ B \\ B \\ G \end{bmatrix}$.

She builds $\begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix} \begin{bmatrix} G \\ B \\ G \\ B \end{bmatrix} \begin{bmatrix} G \\ B \\ B \\ G \end{bmatrix}$ from $\begin{bmatrix} G \\ B \\ B \\ B \end{bmatrix}$. When asked if she's sure that these are the only ones and why this time the original (the ‘parent’) results in three ‘children’, Stephanie justifies her results as follows:

***Table 10-5:* Stephanie explains how in ‘building up’ towers four high, each one produces three duplicates.**

830	Stephanie	Um. All right. Well, I moved it down one, I moved it down, yeah, yeah, that’s it.	BEJ
831	R2	Okay.	
832	R1	So from that one you got how many now?	
833	Stephanie	Three.	BR-V
834	R1	So when you did it for three high.	
835	Stephanie	You got two for each.	BEJ
836	R1	I wonder why you get two of them?	
837	Stephanie	I don’t know. Maybe cause it’s bigger.	BEJ; OBS
838	R1	What would that have to do with it?	
839	Stephanie	I don’t cause you have more room to build on.	BEJ; OBS
840	R1	Tell me can you explain to me?	
841	Stephanie	Oh, well maybe it’s because like you have you already have one that’s taking up space so you only have three places to move it. [*indicates* $\begin{bmatrix} G \\ B \\ B \\ B \end{bmatrix}$]	BEJ
842	R1	Oh, three places.	
843	R2	Three places.	
844	Stephanie	Where before you had one that’s taking up space and you only had two spaces to move it. [*indicates* $\begin{bmatrix} G \\ B \\ G \end{bmatrix}$]	BEJ

Stephanie continues to use this approach in building the remaining ‘children’. She ends up with three corresponding to each ‘parent’ tower for a total of twelve. Along the way, Stephanie and the researcher/observers rearrange into alternative organizations.

Figure 10-5: **Stephanie builds a 'new generation' of towers four high with two green cubes from towers four high with one green cube.**

Stephanie now searches for duplicates as had appeared within the towers three high cases. She finds that, again, they come in pairs and that there are six sets of them giving a total of six resulting towers (F: 1003 – 1010). R1 points out that the resulting six cases are twice the three cases that they found earlier with towers three high (F: 1018).

10.3.3 Building a New Generation

Stephanie is now given a challenging extension of the original task. R2 asks her to use the six resulting towers with two green and two blue and build a "generation of new ones that have three green" (F: 1102). Stephanie begins by predicting that each tower will produce two because "there's only two places for you to move" (F: 1107, 1109). Regarding the possibility of duplicates, she initially predicts that the towers will be in pairs but after considering that each tower originated from three possible towers, Stephanie changes her conjecture from duplicates to triplicates (F: 1123, 1135). She explains herself as follows:

***Table 10-6:* Stephanie explains why duplicates emerge in triples for the case of towers four high with three green cubes.**

1135	Stephanie	Oh. Okay. So, maybe they'll be, um, groups of three?	BDI
1136	R2	So, you think they might be in groups of three.	
1137	R1	Okay. Now explain to me how that happened.	
1138	Stephanie	Mmm, because, here I could either, I could have one here [*points to* $\begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix}$]. Which would make this one [*the* $\begin{bmatrix} G \\ G \\ B \\ G \end{bmatrix}$ *tower*]. I could have here, I could have one here, another green here [*points to* $\begin{bmatrix} G \\ B \\ B \\ G \end{bmatrix}$], which would make this one. And here, I could put another green here [points to $\begin{bmatrix} B \\ G \\ B \\ G \end{bmatrix}$], which would make this one. So, there's three of them that can make that one [*the* $\begin{bmatrix} G \\ G \\ B \\ G \end{bmatrix}$ *tower*]. [*pause*] So, um, I guess they'll be groups of three, maybe?	BEJ

Stephanie is connecting between the number of 'parent' representations and the number of duplicates thereby applying Davis' step (3) *Construct a mapping between the data representation and the knowledge representation*. She continues, stating that since the duplicates come in groups of three, then there would actually be four representations after multiplying by two and then dividing by three (F: 1142, 1144). Stephanie then searches the already built towers for the four towers with three green and one blue. They are as follows:

$$\begin{bmatrix} B \\ G \\ G \\ G \end{bmatrix} \begin{bmatrix} G \\ B \\ G \\ G \end{bmatrix} \begin{bmatrix} G \\ G \\ B \\ G \end{bmatrix} \begin{bmatrix} G \\ G \\ G \\ B \end{bmatrix}$$

R2 then asks Stephanie to consider the case of creating another generation from these four. Stephanie immediately responds, "There's only one way to do that one." (F: 1169) She explains herself as follows:

***Table 10-7:* Stephanie explains why there is only one possible tower when moving from three green cubes to four green cubes for towers four high.**

1175	Stephanie	You're only gonna get one from each, because there's only one place you can put the green.	BEJ
1176	R2	Green.	
1177	Stephanie	And you're gonna get four all green, and so you're going to come up with one 'cause	BEJ
1178	R2	Oh.	
1179	Stephanie	they're all the same.	BDI

When asked how many duplicates, Stephanie answers that there would be four and that to "undo the duplicates", she would have to divide by four (F: 1183, 1189 – 1191).

10.3.4 Review

At this point, R1 and R2 decide to review what they've done by recording the results of each stage. They note that at each stage, they multiplied by one number, then divided by another (F: 1198, 1200). Stephanie suggests that they begin with the last one they discussed and work backwards. Stephanie clearly recalls that they divided by four because of the four duplicates but is unsure what it would be multiplied by (F: 1207, 1209). R1 suggests thinking of it as the tower with one blue and when switching the blue to green there are four of them, so one times four, then you divide by the four duplicates to get one (F: 1210, 1212). She further suggests using this case as a way of generalizing a rule or of identifying patterns. They then attempt to go back to the stage before this one but find that it's difficult to go backwards, so they decide to start from the beginning and meet in the middle (F: 1249, 1251). This time, R2 does the writing instead of Stephanie.

They begin with the case of four towers with one green and three blues. From each one, they built three, which gave them a total of twelve towers with two greens and two blues (F: 1299 – 1302). Since the duplicates came in pairs, they divided by two in order to get six towers. They discuss the fact that each one forms three because "there are three spaces to move it" and the duplicates come in pairs of two because they originated from two 'parents' (F: 1366, 1374 – 1379).

They move on to the next case where they take each of the six towers with two green and two blue and move to three green with one blue. Stephanie describes how each of the six towers produces two possible towers to give a total of twelve, but that you have to divide by three because the duplicates come in triples. When asked how, Stephanie replies, "You'll get two from each, but you have to divide it by three 'cause there's three green?" (F: 1403).

Stephanie now predicts the next case. In going from three green and one blue to four green and no blue, Stephanie conjectures that "you multiply by one and you divide by four" (F: 1413). When asked why, she explains herself as follows, "Um, like I guess, the numerator, decreased. And the denominator, increased." (F: 1421) Stephanie is using generic reasoning to identify the patterns in the concrete examples they already did and generalize them. She is then using that generalization to predict the following example. She further explains her reasoning as follows:

Table 10-8: **Stephanie explains how to predict the number of towers for the next generation.**

1438	R2	How do you see them multiplying by one and dividing by four when we make the next generation?	
1439	Stephanie	Well. Each one gives off one new one, one with four green, 'cause there's only one place for you to put the green.	BEJ; BR-V
1440	R2	Excellent.	
1441	Stephanie	And because there's four greens, you divided by four. Like the new generation has four greens. You divided by four.	BEJ; BR-V

Stephanie has convinced R2 (F: 1444). The session ends with a brief discussion by R1, R2, and Stephanie in which Stephanie speculates on the types of cases that would arise if they were to represent the next line (the fifth row) in Pascal's triangle. R1 and R2 also share with Stephanie the history of the problem and how it came to be the topic of that day's session.

CHAPTER 11 Session 7 - April 17, 1996
Towers, The Formula, Pascal's Triangle

11.1 Background

The session begins with Carolyn Maher, the main researcher/observer, designated R1 in the transcript, encouraging Donna Weir, another researcher/observer designated R2, to listen in on the session since she was not present at the last one. R1 suggests to Stephanie that it would be easier to explain to R2 as if she is not familiar with towers or Pascal's triangle. Stephanie agrees and requests the Unifix cubes. This time they are red and yellow.

Stephanie begins by explaining that she was given the task of finding the number of towers with two reds given four towers with one red cube and three yellows but without moving the red cube that was originally there. As she is talking, she builds the four possible towers with one red and three yellow as follows:

$$\begin{bmatrix} R \\ Y \\ Y \\ Y \end{bmatrix} \begin{bmatrix} Y \\ R \\ Y \\ Y \end{bmatrix} \begin{bmatrix} Y \\ Y \\ R \\ Y \end{bmatrix} \begin{bmatrix} Y \\ Y \\ Y \\ R \end{bmatrix}$$

She then builds three possibilities for each one of them, resulting in a total of twelve towers. Stephanie then points out that half of them are duplicates and since they come in pairs, she can divide by two to get six possible ways (G: 70 – 78).

Stephanie describes how the next step was to find the number of towers with three red and one yellow from the six towers with two red and two yellow. Stephanie predicts that each of the six will produce two since there are only two possible places to put them (G: 94 – 96). She recognizes that there will be duplicates, but is unsure exactly how many (G: 98). She predicts that there will be three duplicates giving a total of four towers. Stephanie begins to build new towers, creating a total of twelve towers with three red and one yellow. She then proceeds to group the duplicate towers together, finding that her prediction that the duplicates came as triplicates was correct and that there are indeed four towers.

R1 then reminds Stephanie that she skipped the case of no red or all yellow cubes. Stephanie predicts that there will be four but as they are all duplicates, in essence there will be one. A discussion then ensues regarding the number to divide by. Stephanie suggests dividing by the number of red cubes but then realizes that wouldn't work in this case because there are zero red cubes (G: 152). This portion of the video lasted approximately fifteen minutes.

R1 then suggests recording the cases that work. She refers back to Stephanie's written work from the previous session and notes that Stephanie had written a formula down to represent the case with two blue and two green: $\frac{4\ x\ 3}{2} = 6$ (G: 192). They discuss the formula and its relevance to the case of four towers with one red going to towers with two red and two yellow. They discuss what each number represents and then begin to generalize using *n* (G: 267 – 268). Stephanie represents $\binom{n}{1}$ as $n \cdot (n-1)$ where *n* can be the total number of towers or the total number of positions (G: 327 – 336). They then discuss finding a way of generalizing the number of duplicates as they would still have to divide by that number. They continue to examine the next case of moving from two red to three red. Moving on to the other cases of no red and four red, they explore how those would correspond to the formula. This segment of the video lasted approximately twenty-five minutes.

In the next segment, lasting approximately five minutes, R1 suggests that they explore the situation of towers five high. They begin with a tower five high with no red. Stephanie records that it would generate five towers with one red or $\frac{1 \cdot 5}{1}$ =5 and $\binom{5}{1}$(G: 643 – 652). For the next case of moving from one red to two reds, Stephanie is unsure about the number of duplicates. She forms a conjecture after building a couple of towers while R1 suggests using Pascal's triangle to check (G: 698 – 700). Stephanie confirms her conjecture then uses that pattern to predict the rest of the cases, meanwhile confirming them with Pascal's triangle (G: 712 – 720).

In the next segment, lasting approximately three minutes, R1 and Stephanie move on to consider the case of towers six high. Stephanie is able to quickly use her formula to represent each of the combinations, then check them with the coefficients in the seventh row of Pascal's triangle (G: 730). For instance, for towers six high with two red, Stephanie writes $\frac{6 \cdot 5}{2} = 15$. She is using the same pattern that she used in the previous case.

The question now arises as to how one could predict the grouping of the duplicates without having to build them. R1 suggests they think about why the numbers match the pattern. She then suggests putting it aside for later.

R1 then changes the focus of the conversation to another aspect of Pascal's triangle. Instead of just considering the horizontal relationship within one row, she wants

Stephanie to consider the relationship from row to row. This segment of the video lasts approximately seven minutes. In order to discuss this relationship, R1 takes a specific section of the triangle $\begin{array}{ccc} 1 & & 3 \\ & \backslash \; / & \\ & 4 & \end{array}$ and asks Stephanie what the towers that correspond to these numbers look like and how they relate to the row below. After building towers and discussing this case, they conclude that from the one that is three tall with no reds and the three three-tall towers with one red, the result is four four- tall towers with one red (G: 843 – 850).

They decide to do another example and this time choose $\begin{array}{ccc} 6 & & 4 \\ & \backslash \; / & \\ & 10 & \end{array}$. Again, Stephanie represents the numbers using the towers, but this time R1 also asks her to represent the number of choices symbolically using the combinatorics notation. Stephanie expresses them as $C_2^4 + C_3^4 = C_3^5$ (G: 888 – 892). They continue to discuss the meaning behind the notation. R1 then charges Stephanie with the task of examining specific examples as many times as necessary for her to offer a generalization of the relationship using *n* and *r* and predict a rule (G: 933 -941).

In the remainder of the video, R1, Stephanie, and Steve, another researcher/observer, designated R3, discuss various topics. They discuss the field of combinatorics and other related fields. This leads to discussion of a report on a mathematician that Stephanie was investigating as a school project. R1 and Stephanie then revisit Stephanie's earlier explorations with Dr. Davis on The Tower of Hanoi. The session ends with their discussion of how certain definitions arise; they complete the session by organizing and labeling papers.

11.2 Recap: Building Towers

Stephanie is asked to explain to R2 the content of the previous session. R1 suggests that she approach it as if R2 is unfamiliar with the material. Stephanie begins by requesting the Unifix cubes, now available in red and yellow. She explains that she was given the task of finding the number of towers with two reds given four towers with one red cube and three yellow cubes but without moving the red cube that was originally there. Stephanie quickly builds the following towers: $\begin{bmatrix} R \\ Y \\ Y \\ Y \end{bmatrix}\begin{bmatrix} Y \\ R \\ Y \\ Y \end{bmatrix}\begin{bmatrix} Y \\ Y \\ R \\ Y \end{bmatrix}\begin{bmatrix} Y \\ Y \\ Y \\ R \end{bmatrix}$. From the first one $\begin{bmatrix} R \\ Y \\ Y \\ Y \end{bmatrix}$, she produces $\begin{bmatrix} R \\ R \\ Y \\ Y \end{bmatrix}\begin{bmatrix} R \\ Y \\ R \\ Y \end{bmatrix}\begin{bmatrix} R \\ Y \\ Y \\ R \end{bmatrix}$. From the second one $\begin{bmatrix} Y \\ R \\ Y \\ Y \end{bmatrix}$, she produces $\begin{bmatrix} R \\ R \\ Y \\ Y \end{bmatrix}\begin{bmatrix} Y \\ R \\ R \\ Y \end{bmatrix}$ $\begin{bmatrix} Y \\ R \\ Y \\ R \end{bmatrix}$. Stephanie continues to produce three for the each of the last two, getting a total of twelve towers with two red and two yellow (G: 36 – 40). Stephanie then relates how there are duplicates of each one that come in pairs, so therefore, there are really six (G: 44). She clarifies that each one produces three because there are three possible positions in each tower to place the new red in moving from four things taken one at a time to four things taken two at a time (G: 55 – 64).

Stephanie then decides to explain the case of moving to four things taken three at a time or a tower four high with three red and one yellow. Before she builds anything, Stephanie predicts that each of the six towers will produce two each, explaining that there are now only two available spots to place a third red thereby giving a total of twelve towers (G: 94). She acknowledges that there will be duplicates, but is not quite sure of the number (G: 98). She predicts that there will be three duplicates, resulting in a total of four towers with three red and one yellow (G: 102 – 104). She bases her conjecture on a strategy used in the past: that of opposites. She explains as follows:

***Table 11-1:* Stephanie justifies her conjecture using the *'opposites'* strategy.**

105	R1	You think there would be three duplicates and then four of them.	
106	Stephanie	Yeah. Because it's just the opposite of that.	BEJ
107	R1	What do you mean?	
108	Stephanie	Well, like this is one red and three yellow [*Stephanie picks up a tower* $\begin{bmatrix} Y \\ Y \\ R \\ Y \end{bmatrix}$]. It'll be three red and one yellow. So it'll be just the opposite.	BEJ; BDI

Stephanie has just applied Davis' step (2) *From this data representation, carry out memory searches to retrieve or construct a representation of (hopefully) relevant knowledge that can be used in solving the problem or otherwise going further with the task;* in order to form her conjecture. She then builds this case finding that the duplicates did indeed come in groups of three and that once twelve is divided by three, the result is four.

Stephanie now returns to the case with no red or all yellows. She predicts that there will be one and explains herself as follows, "Because you're going to get yellow – all yellow from all four. 'Cause there's only one space you can put a yellow. So you're going to get one all yellow from here, one all yellow from here, one all yellow from here and one all yellow from here. So there's going to be one." (G: 144) She suggests that since there are four, but all duplicates, that you can divide by four. R1 encourages her to come with a general way of deciding the number to divide by. Initially Stephanie suggests dividing by the number of red, but finds that although it works for the rest of the cases, it wouldn't work for the case of all yellow because the number of red is zero (G: 152 – 154). Stephanie is facing an obstacle in understanding at this point (G: 174, 176). In this case, it is lack of information. She is correct in stating that there is one way of selecting four yellow or zero red. However, she is unaware that zero is a special case. R1 suggests writing out the cases that do work. Stephanie is being taught the heuristic method of reviewing and evaluating what is known when facing an obstacle to understanding, in the hopes of perhaps gaining some insight or discovering something overlooked. In reviewing papers from the previous session, R1 comes across a formula resulting from a pattern that Stephanie had formed and written down (G: 192).

11.3 Stephanie's Formula

11.3.1 Building Meaning

R1 reads off a formula that Stephanie had formed in the previous session for the case with towers four high with one red going to towers four high with two red: $\frac{4\ x\ 3}{2} = 6$ (G: 192). Stephanie realizes that this formula corresponded to the row of six towers with two red cubes. R1 asks her if she would be able to predict by looking at the row before it (four towers with one red cube) that each one would yield three more towers when moving from one red to two red. Stephanie replies affirmatively, because "there's three places where you can put the second one" (G: 225). They then discuss whether the four in the formula represents the number of towers or the height of the towers (G: 246 – 249). Upon exploring the case of towers three high and two high they find that the height and the number of towers are equal when considering towers with one red: i.e., there are three towers three high with one red (G: 255 – 264). They then generalize, claiming that if the tower is *n* high with one red, there would be *n* possible towers. R1 then asks Stephanie how many for towers *n* high, "how many positions are there to place that second one?" (G: 281) Stephanie uses generic reasoning to extend the concrete example discussed to the more general form of $n - 1$ (G: 282). Stephanie then writes, "For towers *n* high…" representing the case of moving from *n* things taken one at a time to *n* things taken two at a time (G: 288 – 289). R1 encourages Stephanie to use the combinatorics notation resulting in Stephanie's response: "…moving from towers $\binom{n}{1}$ to $\binom{n}{2}$, it would be $n \cdot (n-1)$." (G: 324) R1 then makes the point that *n* could also represent the number of positions. Stephanie replies, "Because it – the number of towers is useful, but the number of positions is useful when you're talking about *n* minus one." R1 then questions exactly what *n* minus one means when moving from towers with one red to towers with two reds. Stephanie explains it as follows:

Table 11-2: **Stephanie explains her perspective of *n* minus one in moving from towers with one red cube to towers with two red cubes.**

364	Stephanie	That means that you're taking away – like – well, we're talking about – like – aren't you talking about like *n* minus one being like yellow minus space like yellow being like – replaced by red?	BEJ; PAH
365	R1	Yeah. Right. That's exactly what I'm thinking about.	
366	Stephanie	So like *n* minus one would be like this [*Stephanie points to the tower of all yellow.*] being replaced by this [*Stephanie indicates* $\begin{bmatrix} R \\ Y \\ Y \\ Y \end{bmatrix}$.] Right? 'Cause like – it's not	BEJ; PAH

		taking away - -	

They use this idea to refer back to the case of moving from one red to two red. They summarize by noting that they have *n* positions, one red, and *n* minus one yellow (G: 403 – 405). Furthermore, this can occur four times or generally *n* times (G: 409 – 411).

R1 and Stephanie then return to the original formula that Stephanie had written down: $\frac{4 \; x \; 3}{2} = 6$. She records that the four is the number of positions or the height of the tower, the three is "the number of spaces you can put a red", and the two is because the duplicates came in pairs (G: 459 – 463). Stephanie attempts to apply this relationship for towers with one red to the case of towers with two red. Her reasoning is as follows:

***Table 11-3:* Stephanie applies formula to case of towers with two red cubes.**

479	Stephanie	Well. It's six, because there's six towers.	BEJ
480	R1	Um hm.	
481	Stephanie	And it's going to be six times um, three because – wait – no it was six times two because there's two places to put it. Because it was four times three. Yeah.	BEJ; BR-S/V
482	R1	It's getting a little trickier. Huh?	
483	Stephanie	It's it's six times two because there's two places to put it.	BEJ

R1 notes that the height of the tower is no longer an option in this case (G: 502). They decide to just use the number of towers in order to stay consistent. Regarding the number to divide by, Stephanie recognizes that the number represents the number of duplicates but she is not sure how to predict that number without building the towers. R1 and R2 encourage Stephanie to go back and look at the previous cases and see if she can identify a pattern. The following excerpt illustrates their discussion:

***Table 11-4:* Stephanie attempts to identify a pattern by looking at previous cases.**

589	R2	Maybe if we think about how you grouped things when	

		you were finished. If they're related.	
590	R1	Here you divided by two. To make this work – what would you have to divide by here?	
591	Stephanie	Oh! These are groups of one!	BDI
592	R1	Okay. So here you divided by two. Here you divided by three. Here you divided by four.	
593	Stephanie	Oh!	BDI
594	R1	To make this work – what would you have to divide by	
595	Stephanie	Yeah. But – oh – 'cause here we divided by the groups. 'Cause here there were groups of two. Here there were groups of three. Here there's groups of one.	BEJ; BDI
596	R1	I don't understand. Help me.	
597	Stephanie	All right. For this one. For like the second one, where there were four times three. There were groups of two. Like they came in pairs. There were two of these. Right?	BEJ
598	R1	Um hm.	
599	Stephanie	So they came in groups of two. So we divided by two.	BEJ
600	R1	Um hm.	
601	Stephanie	And here for the six, they came in groups of three. So you divided by three.	BEJ
602	R1	Um hm.	
603	Stephanie	But here	
604	R1	How would you know what they come in groups of? – Unless you were all done?	
605	Stephanie	Because there were three duplicates. Here's the two duplicates.	BEJ
606	R1	How would you know before you start how many duplicates there would be?	
607	Stephanie	You mean -	
608	R1	I mean here you divided by one; here you divided by two; here you divided by	
609	Stephanie	The number of reds in it?	PAH
610	R1	But isn't that nice? It goes one, two, three, four.	

Once Stephanie decides that the first group of towers with zero red comes in "groups of one", she is able to identify a pattern for the duplicates: zero red comes in groups of one, one red comes in pairs, two red comes in groups of three, and three red comes in groups of four. Stephanie has now accounted for every number in her formula.

11.3.2 Towers five high

The next task is to apply this formula to towers five high and check to see if the numbers correspond to the numbers in the triangle. R1 and Stephanie begin with a tower five high with zero red or all yellow. Stephanie describes the first case as "one times five because there's five positions divided by one because they come in groups of one" as she writes $\frac{1 \cdot 5}{1} = 5$ (G: 643 – 649). She then begins describing the next case of moving from one red to two red as five times four, but she's not sure how many groups of duplicates there will be so she is not sure what number she should divide by (G: 655). Stephanie predicts groups of two but because she is unsure, R1 suggests they build that case. They build one tower with five yellow, and from that one they build five towers with one red. From there, Stephanie is able to see that she was correct in predicting that each of the five towers produces four towers with two red for a total of twenty towers with two red. If she applies her conjecture that the towers come in pairs, then she would divide the twenty by two to get a total of ten towers with two red. Meanwhile R1 draws Pascal's triangle so that Stephanie can confirm her conjecture by checking the number that corresponds to towers five high with two red. Stephanie finds that she was correct and continues writing $\frac{5 \cdot 4}{2} = 10$. She is able to write out the next case with three reds without building them as $\frac{10 \cdot 3}{3} = 10$, checking it against the triangle and finding that it works (G: 718). Stephanie writes out the remaining cases as follows: for four red, she writes $\frac{10 \cdot 2}{4} = 5$ and for five red, she writes $\frac{5 \cdot 1}{5} = 1$. Again, she checks them against the triangle and finds that they match the numbers in the sixth row of the triangle. Stephanie uses her mapping between the triangle and the towers to construct a mapping to a symbolic representation such as her ‘formula’, thereby applying Davis’ steps (3) *Construct a mapping between the data representation and the knowledge representation*; and (4) *Check this mapping (and these constructions) to see if they seem to be correct*.

11.3.3 Towers six high

R1 then suggests that they look at the next row, corresponding to towers six high. Stephanie is able to describe and write down each representation quite quickly. The following excerpt illustrates this:

Table 11-5: **Stephanie forms the terms in Pascal’s triangle for *n* = 6 by using her formula.**

729	R1	You think that makes sense? One – six – fifteen – twenty – fifteen – six – one. So we know the one and six. That’s easy, right?	
730	Stephanie	Times six divided by one – six – [*Stephanie writes.*] The next one is six times five divided by two. That’s fifteen. The next one is fifteen times four divided by three. Gosh. Fifteen times four – sixty divided by three – twenty. The next one is twenty times three divided by four. Oops. Sixty. Fifteen. Next is fifteen times oh and there’s two spaces. That’s thirty um divided by five. That’s um six – six – [*Stephanie is writing very quickly as she is speaking.*] is one. Yeah. That works.	BR-S/V

The topic of the duplicates again gets raised. R1 questions how one would know in advance the number of duplicates. Stephanie responds that it is related to the number of towers that it originated from (G: 760). For instance, in moving from two reds to three reds, three potential towers could have produced the one with the three reds, therefore the duplicates would come in groups of three (G: 761 – 767). Stephanie admits that she would need to see it in order to identify the originating towers (G: 778 – 780).

11.4 SUBTASK 2 - Pascal's Triangle: Moving vertically

R1 decides to "switch gears" and consider the vertical relationship between the rows in Pascal's triangle (G: 741 – 743). She decides to do this by considering sections of the triangle and discussing their representations with Stephanie. R1 selects $\begin{array}{ccc} 1 & & 3 \\ & \backslash\ / & \\ & 4 & \end{array}$ to consider first. She asks Stephanie what the one and the three would look like as towers. Stephanie proceeds to build one tower with no red: $\begin{bmatrix} Y \\ Y \\ Y \end{bmatrix}$ and three towers with one red: $\begin{bmatrix} R \\ Y \\ Y \end{bmatrix}\begin{bmatrix} Y \\ R \\ Y \end{bmatrix}\begin{bmatrix} Y \\ Y \\ R \end{bmatrix}$(G: 786 – 792). R1 then asks Stephanie to describe any differences between the top row with the one and three and the bottom row with the four. Stephanie notes that the row with the four represents towers four high rather than three high. She further claims that there will be one red and three yellow (G: 810 – 820). R1 then asks Stephanie to explain to her exactly what she would do to the three-high towers in order to get a four-high tower with one red. Stephanie explains as follows:

***Table 11-6:* Stephanie explains how she would form a tower four high with one red cube from the three high towers with one red cube and the three high tower with no red cubes.**

838	Stephanie	I'm going to put a yellow here [*points to* $\begin{bmatrix} R \\ Y \\ Y \end{bmatrix}$]	BEJ; BR-S/V
839	R1	Okay.	
840	Stephanie	I'm gonna put a yellow there. [*points to* $\begin{bmatrix} Y \\ R \\ Y \end{bmatrix}$]	BEJ; BR-S/V
841	R1	Right.	

842	Stephanie	I'm going to put a yellow there. [*points to* $\begin{bmatrix} Y \\ Y \\ R \end{bmatrix}$] and I'm gonna put a red there. [*points to* $\begin{bmatrix} Y \\ Y \\ Y \end{bmatrix}$]	BEJ; BR-S/V
843	R1	Okay. So how many ways – how many do you end up with?	
844	Stephanie	Four	
845	R1	Four. – So from the one three tall with no reds	
846	Stephanie	Um hm.	
847	R1	And the three three tall with one red, right?	
848	Stephanie	Yes.	
849	R1	You end up with four four tall with one red.	
850	Stephanie	Um hm.	

R1 then selects another example from the triangle to peruse: $\begin{array}{ccc} 6 & & 4 \\ \backslash & & / \\ & 10 & \end{array}$. Stephanie describes the six as six towers four high with two red and two yellow and the four as four towers four high with three red and one yellow (G: 858 – 860; 872). Stephanie then builds some of them and locates others from the previously built towers. R1 rearranges them and together they get the following towers: $\begin{bmatrix} Y \\ R \\ R \\ R \end{bmatrix} \begin{bmatrix} R \\ Y \\ R \\ R \end{bmatrix} \begin{bmatrix} R \\ R \\ Y \\ R \end{bmatrix} \begin{bmatrix} R \\ R \\ R \\ Y \end{bmatrix} \quad \begin{bmatrix} R \\ Y \\ Y \\ R \end{bmatrix} \begin{bmatrix} Y \\ R \\ Y \\ R \end{bmatrix} \begin{bmatrix} R \\ Y \\ R \\ Y \end{bmatrix}$

$\begin{bmatrix} Y \\ Y \\ R \\ R \end{bmatrix} \begin{bmatrix} Y \\ R \\ R \\ Y \end{bmatrix} \begin{bmatrix} R \\ R \\ Y \\ Y \end{bmatrix}$. They have formed a mapping between the relationships between rows within the triangle to the towers. It remains to form a symbolic mapping using combinatorics notation. Stephanie represents the six as C_2^4 and the four as C_3^4. She then adds them together to get C_3^5 (G: 888 – 892). R1 and Stephanie then discuss the meaning of the symbolic relationship as follows:

Table 11-7: **R1 and Stephanie discuss the meaning of the symbolic relationship $C_2^4 + C_3^4 = C_3^5$ within the context of their corresponding towers representations.**

897	R1	It is three. Okay. So. Um. What does that mean? What is -	
898	Stephanie	That means you have four and you're selecting two. You're taking – well you're taking two red	BEJ
899	R1	Okay. Exactly two red.	
900	Stephanie	and	
901	R1	And then you have exactly three red.	
902	Stephanie	Yes.	
903	R1	And now you're making them – how tall?	
904	Stephanie	Five tall.	BR-S/V
905	R1	Five tall. And how many reds are there going to be?	
906	Stephanie	Three.	BR-S/V
907	R1	So how can you make them five tall with three reds?	
908	Stephanie	Red there. Red there. Red there.	BEJ
909	R1	So here you get three ways, right?	
910	Stephanie	A red there. A red there. A red there. A yellow there. A yellow there, a yellow there and a yellow there.	BEJ
911	R1	There's your ten.	

For a visual picture of their corresponding towers representations, see Figure 11-1:

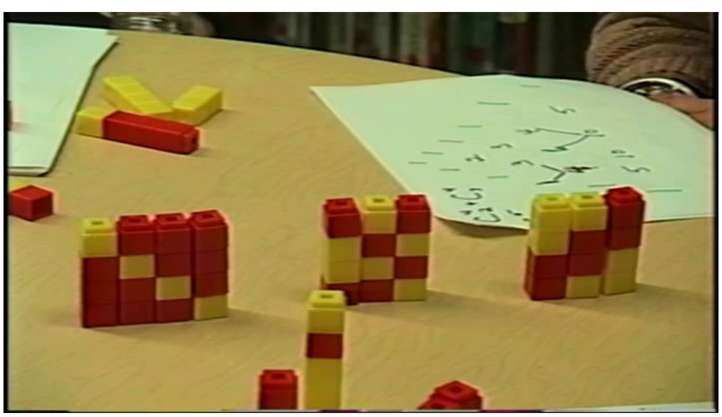

Figure 11-1: **Towers representations of the four high towers with two red cubes and three red cubes that would be used to build towers five high with three red cubes.**

Stephanie has succeeded in creating a mapping between Pascal's Triangle, the visual representation of the towers, and the symbolic representation again applying Davis' step (3) *Construct a mapping between the data representation and the knowledge representation*. R1 now wants her to repeat this process as many times as necessary in order to formulate a general rule using either *n* and *n* minus one or *n* and *r* (G: 935).

For the remainder of the session, R1, R2, R3, and Stephanie discuss various topics. R1 discusses the field of Combinatorics and its applicability to the work force. Stephanie shares that she had to do a report on a mathematician, which sparks a reference to Fermat, the originator of this problem (G: 963). She and R1 then recall her work on the Tower of Hanoi with Dr. Davis years earlier. Finally, they discuss the way definitions sometimes arise out of convenience. Stephanie's mother arrives and the session ends.

CHAPTER 12 – CONCLUSIONS

12.1 INTRODUCTION

Educators, researchers, administrators, students, and parents hold a variety of views of education. Some of these views are influenced by time and place, or society, while other perspectives of education transcend time, place, and circumstances. These perspectives of education have a direct impact on standards, policies, and curricula formulated then dictated to schools and teachers and finally, to the student. According to John Dewey, a philosopher of education, "any social arrangement that remains vitally social, or vitally shared, is educative to those who participate in it. Only when it becomes cast in a mold and runs in a routine way does it lose its educative power" (Dewey, 1944, p. 6). He further warns against the possibility of formal instruction becoming "remote and dead – abstract and bookish" (p. 8). Several generations later, we hear Robert Davis echoing the same call with regard to mathematics education. According to Davis, the controversy centers around whether mathematics can be told to children, or whether most children need to build up mathematical ideas themselves, in their own minds, perhaps with the aid of a knowledgeable adult (Davis, 1997). He describes mathematics as a way of thinking about and analyzing situations, as a matter of inventing strategies for attacking problems (Davis, 1992). In fact, he forms a theory that, in order to think about a mathematical situation, one must cycle (perhaps many times) through the following steps (Davis & Maher, 1990, p. 65):

1) Build a representation for the input data.
2) From this data representation, carry out memory searches to retrieve or construct a representation of (hopefully) relevant knowledge that can be used in solving the problem or otherwise going further with the task.

3) Construct a mapping between the data representation and the knowledge representation.
4) Check this mapping (and these constructions) to see if they seem to be correct.
5) When the constructions and the mapping appear satisfactory, use technical devices (or other information) associated with the knowledge representation in order to solve the problem.

This study demonstrates Stephanie's mathematical development and growth as she did, indeed, think about and analyze situations, construct representation, and invent strategies for completing the tasks presented to her. Furthermore, she learned to explain and justify her ideas to others, sometimes modifying them along the way. Finally, when faced with cognitive obstacles to understanding, she learned to employ various heuristic methods as a means of overcoming the confusion.

Stephanie participated in a longitudinal study from her first-grade year; her mathematical understanding was studied, encouraged, and given room to develop. By her eighth-grade year, she no longer had the opportunity to engage with the classmates she had worked with for the previous seven years because of a move to another town and a transfer to a parochial school. The transition was not an easy one for her. In her eighth-grade year, Stephanie was placed in a traditional algebra course. Early on, she expressed concern that the rules she was learning did not make sense. Her mother approached Professor Maher and asked if it were possible for Stephanie to continue to participate in the Rutgers study. Professor Maher, with the support of Stephanie's math teacher, arranged for after-school sessions in which Stephanie could explore some of the ideas she was learning in greater detail. Stephanie was asked if she would like another classmate to join her, but she indicated a preference to work alone. Thus, the activities took the form of individual task-based interviews. These sessions were initially designed to provide an opportunity for Stephanie to build meaning to the algebra she was currently studying in her algebra class. As a follow-up to these sessions, Stephanie's teacher invited her to share her ideas with her new classmates as the occasion to do so presented itself.

As I studied Stephanie's mathematical behavior during those interviews, I found that Stephanie frequently cycled through the steps in Davis' model when engaging in the problems; she answered questions, formed conjectures, switched between representations, and overcame obstacles to understanding. In analyzing the data, I searched for ways in which Stephanie explored and built algebraic ideas. In particular, I focused on how Stephanie built meaning for the binomial theorem and related this meaning to Pascal's triangle. The following research questions guided my analysis:

1) What representations does Stephanie use to construct, develop, and present her responses to the tasks, problems, and/or questions posed?

2) What explanations and justifications does she give for her solutions and/or the representations that she constructs?

3) What, if any, obstacles to understanding does she encounter?

4) How, if at all, does she overcome these obstacles?

I found that one of the important tools that Stephanie used in solving problems and working on given tasks was to try and build meaning for herself. In the following sections, examples from the analysis illustrate the different conditions under which Stephanie returns to building meaning. These include the following: difficulty recognizing the structural isomorphism between representations or developing new representations, explaining or justifying her reasoning, and overcoming obstacles to understanding. In discussing conclusions, I hope to demonstrate how Stephanie's search for meaning formed the umbrella of other heuristic methods that developed when forming representations, explaining and justifying, and overcoming obstacles to understanding.

12.2 REPRESENTATIONS

This section seeks to highlight some of the many representations that Stephanie used in building mathematical understanding. Representations are an inherent part of mathematics. They can provide a window into a learner's ideas and mathematical understanding. As noted in the literature review, Pape refers to representation(s) as the act of externalizing an internal, mental abstraction (2001). Furthermore, he designates representations such as numerals, algebraic equations, graphs, tables, diagrams, and charts as external manifestations of mathematical concepts. Similarly, Goldin (1998) views a representational system as having both intrinsic structure (i.e., within itself), and extrinsic symbolic relations (i.e., with other systems of representation).

Throughout the video data, we see evidence of Stephanie using multiple representations of her ideas. She simplified algebraic expressions, drew diagrams, built area models using algebra blocks, built trains and towers using unifix cubes, and finally used combinatorics notation to express various selections. Some of these representations fall under what I refer to as 'symbolic' representations: algebraic expressions, combinatorics notation, and Pascal's triangle. The remaining representations would be considered 'visual' representations: drawings, tables, models formed by algebra blocks and other manipulatives, and towers built with unifix cubes. By the end of the sessions, Stephanie showed a fluid movement between and among her symbolic representation of a binomial, Pascal's triangle, the combinatorics notation, and her visual representation of the towers; and was able to lucidly explain their interrelation, thereby demonstrating her mathematical understanding of the binomial theorem. In the following two sections, I will highlight some of the more relevant symbolic and visual representations that Stephanie used not only to build meaning and understanding, but also to demonstrate it.

12.2.1 Symbolic

Session 1 – November 8, 1995

One of the first representations Stephanie forms is an expression for perimeter. She is asked to come up with a general way of expressing how much space is needed if the length is l and the width is w. She forms the symbolic representation $2l + 2w = s$. She is then asked about the representation $2(l + w) = s$. Stephanie recognizes the equivalence of the representations and through her explanation of why they are equal, she is able to demonstrate her knowledge of the distributive property.

Another expression $x \cdot x$ is presented to Stephanie. In attempting to form a representation for it, Stephanie had to return to the meaning of the expression. She described it as "the variable x x amount of times" and by searching for meaning, she was able to form the representation x^2 by using her knowledge of exponents. Both of these representations, along with the discussions over their meaning, provided cognitive building blocks for Stephanie to use when dealing with the expressions $a(x+y)$ *and* $(x+y)(x+y)$, thereby providing her with the tools to form new representations. In particular, these two representations allowed her to form the representation for a square of a binomial $(x+y)^2 = x^2 + 2xy + y^2$.

Session 3 – February 7, 1996

In the third session, on February 7, 1996, Stephanie is called upon to form a representation for the cube of a binomial $(a+b)^3$. She is able to use her earlier representation for $(a+b)^2 = a^2 + 2ab + b^2$ as a building block in the simplification of $(a+b)^3 = (a+b)(a^2 + 2ab + b^2)$. Using mathematical properties such as the distributive property, Stephanie was able to form a new representation for the cube of a binomial equal to $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Session 5 – March 13, 1996

This session provides Stephanie's first introduction to combinatorics notation: $C_r^n \textit{ and } \binom{n}{r}$. Stephanie learns to use this symbolic notation to describe visual representations of the towers. For example, to represent a tower with three reds and one yellow, Stephanie writes $C_3^4 = 4 \textit{ or } \binom{4}{3} = 4$. This means that there are four ways to select three from four.

Later in the same session, Stephanie has formed symbolic representations for $(a+b)^4 \textit{ and } (a+b)^5$. She expresses $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ and $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$. The researcher then forms the first few rows of Pascal's triangle. Stephanie immediately identifies the pattern between the numbers and is able to predict the next few rows. Furthermore, she conjectures that it is the same as the towers, thereby identifying an isomorphism between the symbolic representation, Pascal's triangle, and the towers.

Sessions 6 and 7 – March 27, 1996 and April 14, 1996

Stephanie gains confidence using the combinatorics notation along with Pascal's triangle as is evidenced by the analyses chapters of those sessions. Furthermore, she shows flexibility in moving between the different types of representations. In recording her work, she writes a formula for finding the coefficients in Pascal's triangle: $\frac{4\ x\ 3}{2} = 6$ that corresponds to the case of four towers with one red becoming towers with two red and two yellow (G: 192). In these sessions, Stephanie and the researcher discuss the meaning and significance of each of the numbers, improving Stephanie's mathematical understanding of the concept. In addition, in considering the relationship between the rows in Pascal's triangle, Stephanie learns to express it symbolically using the combinatorics notation. For example, for the row containing 6 and 4 and the next row containing 10, Stephanie was able to represent them symbolically as $C_2^4 + C_3^4 = C_3^5$ (G: 888 – 892). By doing more examples, Stephanie could identify patterns. By the end of the last session, Stephanie had found a general representation for the binomial coefficients in Pascal's triangle.

12.2.2 Visual

Session 2 – January 29, 1996

Stephanie makes a big leap from the symbolic representation of $(a+b)^2 = a^2 + 2ab + b^2$ to a visual representation of $(a+b)^2$ as a square. The researcher is using the concept of area as another way of representing $(a+b)^2$. Stephanie initially forms simpler representations, such as drawing a square with side three units and finding its area, then a square with five units and finding its area, then extending it to a square with *a* units and learning to represent its area as a^2 square units. These visual representations then formed cognitive building blocks for the more difficult representation of $(a+b)^2$ as a square. By building the square and finding the area of each section within the square, Stephanie was able to form another representation for $(a+b)^2$ that turned out to be equivalent to her earlier symbolic representation. This, in turn, allowed Stephanie to see the isomorphism between the two representations and concepts, thereby building her mathematical understanding of the square of a binomial.

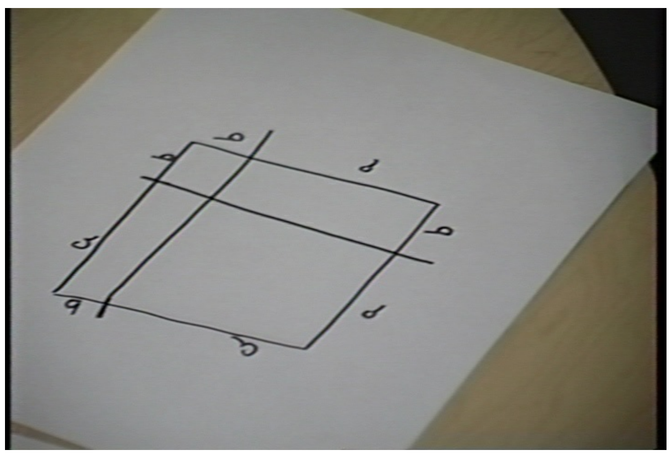

Figure 12-1: **Stephanie's diagram of a square with side *a* + *b*.**

Session 3 - February 7, 1996

Stephanie is provided with manipulatives and is asked to build a physical model of $(a+b)^2$ but using *a* = 3 and *b* = 7. She does so using a ten by ten by one 'flat' and a one by one by one 'cube' to demonstrate.

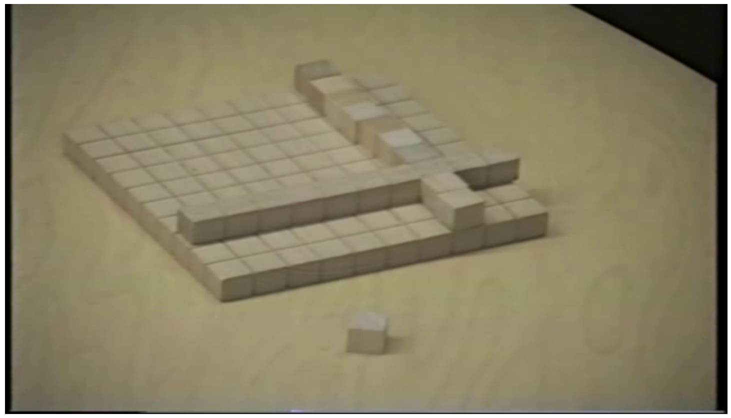

Figure 12-2: **Stephanie's model of (a + b)^2 for a = 3 and b = 7 using the 'flat' and the 'cube'.**

Notice that although both shapes are three dimensional, in Stephanie's representation, they function as two dimensional. This physical representation forms a building block for a later subtask arising within the fourth session.

Session 4 – February 21, 1996

Stephanie is asked to build a model of $(a+b)^3$ using algebra blocks. In order to accomplish this task, Stephanie needs to consider the concept of volume. She does this by using a larger 'cube' that is ten by ten by ten cubic units and the 'flat' to form a representation of the volume of the cube. She describes the volume as one thousand and explains that this means that you can "fill it up with one thousand square units" (D: 249). Clearly, Stephanie has a mental representation corresponding to her physical representation of volume. However, when transitioning to the algebra blocks, Stephanie is initially unsure how to go about using them to build a model. As she, with the researcher, experiments with the blocks, Stephanie creates the base for her three-dimensional model by fitting in four of the blocks into her earlier representation of the square with side $a + b$. She assigns symbolic representations to each piece using a two-dimensional lens.

Figure 12-3: **Algebra blocks corresponding to 2-dimensional diagram of square with side a + b.**

Stephanie recognizes the isomorphism between the two representations, eventually allowing her to extend the idea and 'build up' $a + b$ in order to complete the model. Once Stephanie has physically built a cube using the algebra blocks, she is able to use her earlier symbolic representation to map the 'pieces' from her expression to the 'pieces' from her model. Again, Stephanie recognizes an isomorphism between the two representations, thereby expanding her mathematical understanding of the cube of a binomial.

Session 5 – March 13, 1996

Most of Stephanie's visual representations in this session relate to trains and towers represented by diagrams with R for red and Y for yellow. Stephanie is initially asked to consider the possibilities of selecting one red, two red, three red, and four red for trains four cubes long. She makes diagrams such as the following:

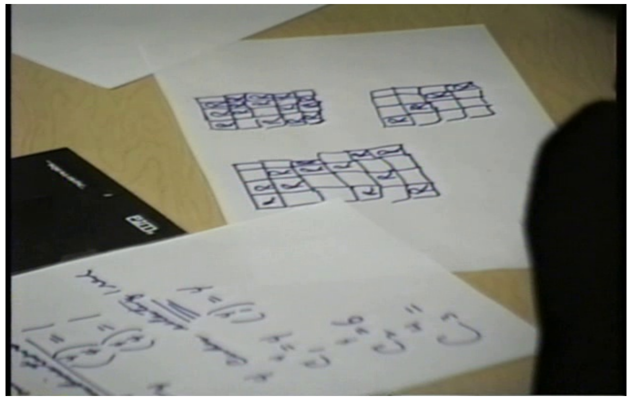

Figure 12-4: **Stephanie's drawings of trains four cubes long.**

In order to organize her representations, she uses several strategies. Among these are: the separation strategy, opposites, families, and 'building up' using a tree diagram. Many of these strategies are ideas that she used in the past and is now revisiting and expanding on in order to build meaning for herself.

Sessions 6 and 7 – March 27, 1996 and April 14, 1996

Stephanie uses Unifix cubes in two colors to model her representations of the towers. In the sixth session, Stephanie uses the coefficients in Pascal's triangle as well as the towers to expand $(a+b)^3, (a+b)^4, and\ (a+b)^5$. Her flexibility in moving from various representations of the same concept allows her to accomplish this task without having to do all of the algebra. Later on in the same session, Stephanie and the researchers take an in-depth look at the fourth row in Pascal's triangle using the following problem as a context for exploration: *Given a three high tower with one green, suppose we were to trade one of the blues for a new green, how many different ways could we do that*? (F: 644 – 652) As Stephanie builds the possible towers using the unifix cubes, she discovers that she inevitably gets 'duplicates'. The number of ‘duplicates’ and the conditions that produce them form the basis for a discussion that leads to the discovery of the binomial coefficients in the Binomial Theorem.

12.3 EXPLANATIONS/JUSTIFICATIONS

One of the ways in which an observer/researcher can gain insight into Stephanie’s mathematical understanding is to examine her representations, conjectures, and ideas. She is consistently invited to explain and justify her reasoning and her representations in a way that the observer/researcher can understand. At times, she is called upon to clarify something that may be unclear. Throughout this process, Stephanie learns to question and assess the validity of her own representations and hypotheses. Stephanie has ample opportunities to explain her thinking and reasoning. These occur either at the beginning of the session when Stephanie is invited to review and relate what she remembers from the previous session or at the end when other observer/researchers are invited to ask her questions. Furthermore, throughout the sessions, R1 repeatedly asks her “why” providing her with further opportunity to explain and justify. I will highlight some of the more prominent situations where Stephanie was asked to explain and justify her reasoning and representations.

Session 2 – January 29, 1996

One of the most challenging explanations required of Stephanie occurred towards the end of the second session. Stephanie had just completed her diagram of a square with side $a + b$ and partitioned it into four parts. She added up the area of the four parts and after simplifying her terms obtains $(a+b)^2 = a^2 + 2ab + b^2$. One of the observer/researchers, Alice Alston, designated R3 in that session, was unable to see

Stephanie's paper as Stephanie created her area model. She asked Stephanie to describe (without seeing what R3 was writing) exactly how to create the model again. As Stephanie did so she needed to be very precise in giving R3 directions. The example used to illustrate was a five by two rectangle. The distinction between a 'unit' and a 'square unit' arises as well as the concept of area and perimeter. When questioned by R3 as to "why" each unit is a square, Stephanie immediately responds, "Oh! Because each side of it is one." (B: 1527) Her explanation as well as her instructions to R3 demonstrated her mathematical understanding of the concepts under discussion.

Session 3 – February 7, 1996

Another example of Stephanie's explanations and justifications occurs when she is asked to explain her write-up of the previous session to her younger sister Susie. Although Susie was not present, R1 role plays Susie. Through her questions, Stephanie is motivated to think of the concepts in very simplistic ways and to really consider the meaning behind it. Her explanation of the concept of area gives insight into the mental representation she has of it: "Okay. Well, area is like um the amount of space inside like a sp…an object." (C: 77) Initially, Stephanie refers to a square unit as a unit, then in the course of explaining, she corrects herself (C: 91). When demonstrating using the manipulatives, she clearly points out the difference between them, thereby reflecting her mathematical understanding of the terminology. When explaining how to find the area of the 'flat' (the 10x10x1 box), Stephanie describes it in two ways. First, she mentions that there are ten of the cubes along each side of the 'flat' each of which has an area of one square unit. And since there are ten rows in the flat, there would be a hundred square units, which Susie could confirm by counting. She also mentions alternatively that you could get area equals one hundred by squaring ten (C: 103 – 115). Stephanie's multiple mental representations of area clearly demonstrate her understanding of the concept. She was able to explain it in a way that related back to her mental representation of "filling up space" that could be understood even by a younger child who had never been exposed to the formula for area. Finally, when asked about the *a* plus *b*, Stephanie described *a* and *b* as "not the same" but "they stand for any number" and shows R1 on the blue figure manipulative how each side would be equal to *a* plus *b*. Through her explanations and justifications, Stephanie has reinforced her mental representations and was prepared to use them as cognitive building blocks for the next task.

Session 4 – February 21, 1996

At the end of the fourth session, the other observer/researchers are invited to ask Stephanie questions. She begins by summarizing what has been done beginning with $(a + b)^2$ and finishing with her three-dimensional model of $(a + b)^3$. Among the topics that arise are those of color and dimensionality. Stephanie explained to R3 that the color has nothing to do with the actual representations: "the color itself has, like, nothing to do with it. It could be purple- and- it doesn't make a difference." (D: 710) She indicated that she only uses the colors as an aid in labeling her representations. When R3 questions the dimensionality of Stephanie's model, Stephanie clearly explains how she formed each of her representations forming the model. She begins with the base of the model which she treats as two dimensional and explains how she 'built up' each of the four 'pieces'. For example, she takes the 'a^2' that represents one region of the two dimensional square and explains that if you build it up *a* then you have a three dimensional representation of the yellow cube labeled as a^3 (D: 762, 764). After explaining the bottom layer, Stephanie moves on to show how she formed representations for the other pieces in the model.

Session 5 – March 13, 1996

In this session, Stephanie is initially asked to consider combinations using horizontal trains. For example, when considering the problem: *given a train four cubes long, how many ways could she select one red*; Stephanie formed a diagram to represent the different combinations. One of the ways she does this is by using a 'separation' strategy.

R	R		
	R	R	
		R	R
R		R	
	R		R

When asked to justify her combinations and prove that these are the only possibilities, Stephanie explains her reasoning as follows: "Oh, well, I just separated them. Here, they were together and here they're not. But they're still two reds." (E: 94) When asked if she was sure she had all possible cases, Stephanie takes a closer look then adds another row where the two red cubes are separated by two cubes:

R			R

When asked if there was any other way to separate, she acknowledges that she can't separate by any more than that because "there's not enough space" (E: 112). She has justified that there are six possible trains with two red cubes. Stephanie is then able to use this strategy to consider the situation with three red cubes as well as more difficult tasks.

Session 6 – March 27, 1996

One of the most relevant questions posed to Stephanie has to do with the relationship, if any, of the binomials to Pascal's triangle and/or to the towers representations. Stephanie first described the relationship of the coefficients of $(a+b)^2 = a^2 + 2ab + b^2$ and exponents to the towers. She explained that if a is green and b is blue then there are two combinations of a two high tower with one blue cube and one green cube (F: 495 – 500). She then explained that "you have one that's all a [*indicates* $\begin{bmatrix} G \\ G \end{bmatrix}$] and one that's all b [*indicates* $\begin{bmatrix} B \\ B \end{bmatrix}$], corresponding to the coefficients of a^2 and b^2 (F: 501 – 507). Furthermore, she described the a^2 as "two factors of a", the b^2 as "two factors of b", and the ab and the ba as one factor of each (F: 517 – 521). Stephanie then pointed out the correspondence of the coefficients to the third row in Pascal's triangle: 1, 2, 1. Through her explanation, Stephanie, in her problem solving, demonstrated that she recognized the structural isomorphism between all three representations: $a^2 + 2ab + b^2$, the towers, and Pascal's triangle. Furthermore, when applying these representations to the cube of a binomial and a binomial to the fourth power, she successfully built upon her earlier ideas and representations in order to extend the concepts to more complicated tasks.

Session 7 – April 17, 1996

Stephanie was called upon to explain her towers from the previous session to another observer/researcher, designated R2 in this session who was not present at the last session. She explained that initially she was given the task of finding the number of towers with two reds given four towers with one red cube and three yellows but without moving the red cube that was originally there. Stephanie quickly built the following towers: $\begin{bmatrix} R \\ Y \\ Y \\ Y \end{bmatrix} \begin{bmatrix} Y \\ R \\ Y \\ Y \end{bmatrix} \begin{bmatrix} Y \\ Y \\ R \\ Y \end{bmatrix} \begin{bmatrix} Y \\ Y \\ Y \\ R \end{bmatrix}$. From each of these, she produced three others with two red and two yellow. Stephanie then related how there are duplicates of each one that came in pairs so therefore, there were actually six (G: 44). She clarified that each tower produced three, because there are three possible positions in each tower to place the new red in moving from four things taken one at a time to four things taken two at a time (G: 55 – 64).

Stephanie continued to explain the next case of moving to four things taken three at a time or a tower four high with three red and one yellow. Before she began building models, Stephanie predicted that each of the six towers will produce two each, explaining that there are now only two available spots to place a third red thereby giving a total of twelve towers (G: 94). She predicted that there would be three duplicates resulting in a total of four towers with three red and one yellow (G: 102 – 104). She based her conjecture on a strategy used in the past: that of opposites. She explained that a tower with one red and three yellow is just the opposite of a tower with three red and one yellow (G: 108). Stephanie then showed the cases and confirmed her predictions. In explaining to the researcher, Stephanie was in effect justifying what she worked on in the previous session. This time around, she built more quickly and was more confident in justifying her organization of towers.

12.4 OBSTACLES TO UNDERSTANDING

According to Davis, "understanding" occurs when a new idea can be fitted into a larger framework of previously-assembled ideas (1992, p. 229). An obstacle to understanding therefore would be something that prevents the learner from achieving this fit. It is inevitable that mathematical learners will encounter obstacles to understanding as they solve problems or work on a task, as it is inherent to the problem solving process. Obstacles can occur due to lack of knowledge, misconceptions, computational errors, or plain miscommunication. The learner's challenge is to develop and use heuristics that allow him/her to overcome the obstacle by perhaps gaining insight that clears the way for solving the problem or completing the required task.

Undoubtedly, Stephanie faced various obstacles to understanding in her attempts to answer questions posed to her, complete problem-- solving tasks, and explain her thinking to others. Along the way, with the help and support of the researcher, she learned to use the following heuristics to overcome these obstacles. In the next four sections, I will explain them as well as highlight some examples to illustrate Stephanie's use of them. These examples are documented in the analysis in greater detail.

12.4.1 Substituting in Numbers

In many of the tasks presented to Stephanie, she was required to form a symbolic representation, or simplify another representation. Many times, Stephanie formed conjectures for a particular representation. One technique that R1 introduced Stephanie to was the heuristic of putting in numbers and checking the result. For instance, if an expression was being simplified, then Stephanie could substitute the same numbers in the original expression as well as the simplified expression and compare them. She learned to use this heuristic to disprove a conjecture by counter-example. Furthermore, its use allowed her to catch computational errors along the way and build confidence in an expression when she found that both sides to be equal. The following examples from the analysis highlight the different ways this heuristic was used.

Session 1- November 8, 1995

Example 1: The first time this heuristic was suggested to Stephanie was when she was presented with the expression $a(x + y)$. When asked what it meant, Stephanie conjectured that it was equal to $ax + ay$. R1 suggested putting in numbers for a, x, and y in order to test her conjecture. Stephanie substituted in 2 for a, 3 for x, and 4 for y and found that both expressions are equal. She applied Davis' step (4) *Check this mapping (and these constructions) to see if they seem to be correct*.

Example 2: R1 has asked Stephanie to form a general expression for $x \cdot x$. Again, she encourages Stephanie to put in numbers to help her. Stephanie does so for $x = 2$ and gets $2 \cdot 2$ and $x = 3$ and gets $3 \cdot 3$. Now when Stephanie again tries to find an alternative way of expressing, she suggests using exponents. She expresses $2 \cdot 2$ as 2^2 and $3 \cdot 3$ as 3^2, eventually leading her to conclude that $x \cdot x$ is equal to x^2.

Example 3: Stephanie is presented with the expression $(x + y)(x + y)$. She conjectures that it is equal to $x^2 + y^2$. In order to test her conjecture, she again substitutes 2 for x and 3 for y and finds that it does not work. She has just learned to prove by counter example. As she is still not sure what the expression is equal to, she must use a different approach to overcome this obstacle to understanding. She and the researcher R1 consider a simpler problem as discussed in section 6.4.2.

Session 3 – February 7, 1996

Example 1: Stephanie has just created a concrete model of a square with side $a + b$ with $a = 3$ and $b = 7$. From her model, she is able to see that the area would be equal to one hundred square units. However, she is still in the process of grasping the isomorphism between the area model and the symbolic expression for $(a+b)^2 = a^2 + 2ab + b^2$. R1 suggests she substitute in the numbers 3 and 7 for a and b into both sides of the symbolic representation. Stephanie does so and finds that she also gets an answer of one hundred on both sides, thereby allowing her to see the relationship between the two representations.

Example 2: In simplifying the cube of a binomial, Stephanie was able to recognize that $(a+b)^3 = (a+b)(a^2 + 2ab + b^2)$. However, after applying the Distributive Property and combining like terms, Stephanie wasn't quite sure that she did it without making any mistakes. Her simplification resulted in the expression $a^3 + 3a^2b + 3ab^2 + b^3$. In order to check this mapping, Stephanie substitutes the numbers 3 and 7 for *a* and *b* expecting to get a thousand, which she does. She uses the heuristic of substituting in numbers to test the validity of an expression.

12.4.2 Considering a simpler problem

Considering a simpler problem was a heuristic initially suggested by the researcher and subsequently applied by Stephanie. At times, when Stephanie was presented with a question or a task, she simply had no idea where to begin. A technique suggested by the researcher was to consider a simpler problem. When this occurred, generally Stephanie was able to make the connection between the simpler problem and the original problem and overcome her initial obstacle to understanding. This occurred throughout the sessions. The following examples from the analysis demonstrate the application of this heuristic.

Session 1 – November 8, 1995

In dealing with the expression $(x+y)(x+y)$, Stephanie and R1 return to a simpler representation they had previously discussed: $a(x+y)$. When asked how she thought about this expression, Stephanie responds, "Oh. That it's um *x* times *x* plus *y* or *x* plus and *y* plus *y* *a* amount of times. And since I didn't know *a*, it was just like rows and rows and rows of numbers." (A: 514) Stephanie clearly has a visual representation in mind. However, she is having difficulty limiting it since *a* is unknown. She does however know that $a(x+y) = ax + ay$. With further discussion, Stephanie applied generic reasoning and replaced *a* with $(x + y)$. Stephanie remarked, "I have *x* plus *y* times *x* plus *y*, so I have it *x* plus *y* amount of times, but I don't know." (A: 556) From there, Stephanie distributed the binomial $(x + y)$, first to x and then to y, as she did with the a earlier. By considering a simpler problem, Stephanie overcame the difficulty she was having in building a representation for $(x + y)(x + y)$. She then used her knowledge of algebraic properties as well as other heuristics to simplify the expression.

Session 2 – January 29, 1996

Another example that occurred in the second session required Stephanie's representation of the binomial $(a + b)$ squared. She was asked to represent $(a+b)^2 = (a+b)(a+b)$ as an area problem. Initially, she struggled with shifting from the symbolic representation to a visual representation. R1 suggested that she begin with a^2. When Stephanie was still unsure, R1 offered a concrete example of a square with side of measure three units. Stephanie realized that the area would be side squared which in this case would be nine, and produced a drawing with three equal intervals marked off on each side. This drawing served as the point of discussion for why the area of the square was nine square units and what nine square units meant. Stephanie then successfully transitioned to a square with side a with area equal to a^2 square units. From there, Stephanie generalized even further by representing a square with side a plus b. This led Stephanie to the simplification of $(a + b)^2$ by representing it as a square and finding its area as $(a+b)^2$ square units.

Session 3 – February 7, 1996

In this session, Stephanie was presented with the cube of a binomial $(a + b)^3$. She recognized that it meant $(a+b)\cdot(a+b)\cdot(a+b)$ but was unsure how to proceed. R1 suggested that it might be helpful to think of it as $(3 + 7)^3 = (3 + 7)(3 + 7)(3 + 7)$. She then wrote = $(3+7)(3^2+2\cdot 3\cdot 7+7^2)$. By putting in concrete numbers, R1 has just created a simpler problem that Stephanie could consider. They discussed using the distributive property instead of just simplifying the numbers. After doing so, Stephanie again used generic reasoning to apply the same idea to the more abstract representation $(a+b)\cdot(a+b)\cdot(a+b)$ and successfully simplified the expression.

12.4.3 Building Meaning

Throughout these sessions, Stephanie was encouraged to build meaning and understanding for herself. At times throughout the sessions, building meaning was used as a heuristic in overcoming an obstacle to understanding. By going through the process of making sense, Stephanie was generally able to gain insight and develop an idea/approach/representation for the task at hand. The following example demonstrates the use of this heuristic.

Session 4 – February 21, 1996

At one point, Stephanie was called upon to build a model of $(a+b)^3$ using a set of eight algebra blocks. She built the base of the model recognizing the isomorphism between the pieces and her earlier drawing of a square with side $a + b$. However, when it came to 'building up' $a + b$, Stephanie faced an obstacle to understanding. R1 and Stephanie dealt with this obstacle by returning to basic meaning. They discuss the dimensionality of the base of the model. One of the roadblocks was that Stephanie was trying to represent the problem with a two dimensional model when in fact it called for a three dimensional representation, to include height. Once Stephanie recognized the obstacle, she was successful in representing each of the 'pieces' in the $(a+b)^2$ model: a^2, ab, ba, and b^2, as a three dimensional entity, i.e. a^2 became a^3 and ab became a^2b since the height was a. This enabled her to see that she could then 'build up' $a + b$ and complete the model.

12.4.4 Writing things down

At the end of each of the sessions, Stephanie was asked to organize and label her work. Several times, she was asked to write up the work done in a particular session in preparation for the following session. Stephanie became accustomed to describing and documenting her ideas, her notation, and her representations. Furthermore, she had to accomplish this in a way that was clear and understandable to others. In the later sessions when topics became more complicated and abstract, Stephanie learned to use the 'reviewing' or 'writing down' process as a heuristic when dealing with an obstacle to understanding. By pausing to write down and organize what she did know, often she was able to gain insight into what she didn't know. The following examples illustrates the use of this heuristic.

Session 6 – March 27, 1996

Stephanie was engaged in a task of building a new generation of towers from the previous one as she sought to solve the problem posed as follows: *Given a four high tower with two green cubes, suppose we were to trade one of the blues for a new green, how many different ways could we do that*? She was asked to begin with the six resulting towers with two green and two blue and build a "generation of new ones that have three green" (F: 1102). She conjectured that the number of duplicates would come in triples and predicted that overall there would be a total of four towers with three green cubes.

She was asked about forming the next generation and predicted that "there's only one way to do that" (F: 1169).

At this time during the interview, Stephanie was called upon to consider movement within a row of Pascal's triangle. More detail is provided within the Analysis chapters of this paper in chapters 10 and 11 and in a chapter in a book on Combinatorial Reasoning (Speiser, 2010). Robert Speiser, one of the researchers in session 6, describes in detail the mathematics relating to the movement within and between the rows of Pascal's triangle and documents the way in which Stephanie's earlier background with the Towers tasks and other counting tasks related to her work in the eighth-grade (Speiser, 2010, p. 73-74, 77-79).

The researchers and Stephanie decided to review what they had built by recording the results of each stage of construction. As they returned to the first case, they documented exactly what they did to find the numbers of towers. They continued to break things down case by case until Stephanie noticed patterns and, more confidently, conjectured what the next case would look like without physically building the towers. For example, Stephanie indicated that the solution for the case of towers with two green cubes and two blue cubes was $\frac{4 \; x \; 3}{2} = 6$.

Session 7 – April 17, 1996

In recounting to another researcher what she did with towers in the previous session, the issue of the number to divide by resurfaced. Stephanie knew that the divisor represented the number of duplicates but still seemed unable to explain how it was related to finding the total number of towers. She was also unsure of how to predict the divisor without actually building the towers. Researcher R1 suggested that they record their calculations, case by case. They returned back to the earlier finding that $\frac{4 \ x \ 3}{2} = 6$ from the previous session and decided to use it as a model for recording other cases. This process helped Stephanie to work on developing a general way of describing the process using *n,* thereby bringing her closer to finding the coefficients in the Binomial Theorem. Stephanie's written work not only helped her to continue the thought process developed in the previous session, but also made it easier for her to recognize and identify patterns in the different cases and form conclusions.

12.5 LIMITATIONS

Limitations are inherent to any study. The present study is a case study, one that focuses on the mathematical development and learning of one student, Stephanie. The study focused on her work during seven individual task- based problem- solving sessions taking place in her eighth grade year. The findings apply to this case and, as with any case study design, should not be generalized. More importantly, this research focused on how Stephanie built ideas and responded to posed questions and tasks.

Stephanie had been a participant in the Kenilworth longitudinal study since the first grade. Therefore, she was accustomed to talking about her ideas and sharing them with others. Maher and Martino (1996), in a five-year case study describing the process of how Stephanie invents the idea of proof, indicate how Stephanie's argument by cases evolved while investigating building towers tasks. According to Maher and Martino, she progressed in coordinating multiple bits of information by moving from local arrangements of ideas to more global ones. How was this accomplished? They cite that Stephanie had multiple opportunities to modify and extend her original ideas and to reflect on them. Furthermore, she considered the input of others in revising and expanding her ideas (Maher and Martino, 1996). Speiser also discussed how Stephanie's earlier work with towers and other counting tasks related to her representations when working with Pascal's triangle in the eighth-grade. In addition, Speiser notes that Stephanie's initial discoveries informed and dictated the direction of the interview task (Speiser, 2010). One can see a parallel between the opportunities presented to Stephanie in her earlier work within the longitudinal study and her participation in the individual task based interviews in her eighth grade year. Between the interviews, she had an average of two to three weeks to reflect upon, review, and write up her work from the previous session. One can see that her ideas grew and matured across the span of the interviews, a period of about six months, through the refinement and extension of multiple representations of mathematical ideas. Also, during this period, it is apparent that Stephanie switched easily between and among her representations. It has been documented that Stephanie's earlier work within the longitudinal study provided certain cognitive building blocks that provided a strong foundation for her to deal with more difficult tasks presented in the eighth grade sessions (Speiser, 2010; Davis & Maher, 1997; Maher, Martino, & Alston, 1993; Maher & Davis, 1990).

The design of the longitudinal study provided a setting that ensured that certain conditions were present that promoted an invitation for sense making and collaboration of ideas. The conditions called for an informal environment with sufficient time to explore and revisit problems, limited researcher intervention, and access to resources such as manipulatives and calculators.

Furthermore, it should be noted that this study focused on how Stephanie built her personal understanding. It did not analyze the interventions of the researchers in terms of how their moves were followed by certain actions by Stephanie. It may be important to trace these moves and Stephanie's behavior to gain insight into the complexity of the teaching experiment.

12.6 IMPLICATIONS

Numerous views, opinions, and theories regarding education bombard educators and parents, and perhaps even students on a daily basis. With regards to math education, the debate centers on traditional mathematics and reform mathematics philosophy and curricula. One aspect of the debate revolves around how explicitly children must be taught skills based on formulas or algorithms versus a more inquiry based approach in which students are exposed to real-world problems (Preliminary Report, 2007). Traditional mathematics relies on fixed, step-by-step procedures for solving math problems while reform mathematics places more importance on conceptual understanding. The National Council of Teachers of Mathematics responded to this debate through the revision of its standards. In its algebra strand, the NCTM calls upon educators to enable all students to understand patterns, relations, and functions; represent and analyze mathematical situations and structures using algebraic symbols; use mathematical models to represent and understand quantitative relationships; and analyze change in various contexts (NCTM, 2000). These requirements have a direct impact on the dissemination of algebraic ideas in schools.

A consequence of these requirements is the recommendation to expose students at an earlier age to many of these topics. Advocates of teaching algebra in the earlier grades include: Davis, 1992; Kaput, 2008; Wicket, Kharas and Burns, 2002. For example, Kaput (2008) stated: “the underlying goal of early algebra is for children to learn to see and express generality in mathematics.” He describes the goal of elementary mathematics is to guide children to engage in generalizing as a mathematical activity. Wickett, Kharas and Burns (2002) stress the importance of patterns in developing a young child’s algebraic thinking because patterns draw on experiences and contexts that are familiar to students. They assert that students should have experience creating, recognizing and extending patterns while describing them verbally and symbolically in several ways.

Stephanie, as a participant in a longitudinal study since the first grade, was exposed to early algebra ideas in grade six (Spang, 2009; Giordano, 2008; Mayansky, 2007; DePaolo, in progress). These included working with linear, quadratic, exponential functions and inverse functions. These were presented within the context of various tasks. Giordano (2008) documents that Davis introduced a group of students from the longitudinal study that included Stephanie, to the concept of functions using an activity called Guess My Rule using boxes and triangles. Spang, (2009) in her dissertation, describes Davis’s approach to quadratic equations when again working with a group of students from the longitudinal study that included Stephanie (p. 28). Through her participation in the Tower of Hanoi task, Stephanie learned to search for patterns and interact with exponential functions (Mayansky, 2007). In grade seven, Bellisio and Maher document the participation of thirteen students from the longitudinal study in a task called Advanced Guess My Rule where the students were introduced to inverse functions. Throughout these tasks, references were made to earlier activities such as Guess My Rule and the towers tasks (Giordano, 2008; Mayansky, 2007). Furthermore, as is documented in the Literature Review of this paper, Stephanie had ample opportunity to describe her findings verbally and symbolically when working within small groups as well as when explaining her findings to the entire group. Through her earlier work on the towers tasks, she learned to justify and generalize her strategies and ideas (Maher & Martino, 1996). Therefore, when she began the interviews in the eighth grade, her earlier experiences provided a foundation for mathematical growth and development. A resulting implication is that learners who have been exposed to opportunities such as those provided by the longitudinal study including: developing the ability to generalize mathematically, make conjectures and prove them, explain themselves verbally and symbolically, are better able to mathematically grow and develop when faced with more abstract topics in later years. The conditions of the longitudinal study as well as the applicability of its components to the larger classroom require further study.

One of the important aspects of the "reform approach" to teaching is a shifting of focus from the class as a whole group to individuals (Davis, 1992). It is recognized that individuals learn in different ways and at different speeds. If this idea is considered, then teaching would move toward students developing their own "personal" curriculum (Davis, 1992). Stephanie's developing understanding enabled her to build more abstract ideas. In another case study focusing on a student named Robert, it was found that he, as well as other students, made similar discoveries in grade seven (Kalsi, 2010). At that time, he was working in small groups. Future case studies could further illustrate the learner's development of similar understandings but in different ways, at different speeds, and within the context of different problems.

The results of this study suggest some optimism about what students can learn. Stephanie, as an eighth grader, learned to build meaning for the coefficients of the binomial theorem, and map that meaning to the respective rows of Pascal's triangle. Also, she explored the relationship between the coefficients within a particular row. She shared her ideas with multiple representations, visual as well as symbolic, and traveled comfortably between them. She justified her reasoning and provided explanations of her ideas in a lucid manner. Finally, she learned to overcome obstacles to understanding in order to advance working on her tasks. These are all impressive accomplishments, particularly for an eighth grader. Many of these ideas and skills are difficult for even college level students as I have personally witnessed in the classroom as an educator at a community college. Until this study, I, as a graduate student, have never had the opportunity to think deeply about the binomial theorem. My experience with Stephanie will hopefully provide me with a better approach and clearer insight the next time I am called upon to introduce these ideas to my own students.

Furthermore, this study offers suggestions for the teaching of early algebra. As previously mentioned in the literature review of this paper, algebra is a requirement for many other courses in high school and college. Unfortunately, many students find it extremely challenging. The techniques, heuristics, and approaches to topics such as the distributive property, simplification in general, the square of a binomial, the cube of a binomial, area, and volume that formed the content of the earlier sessions deserve attention. A challenge for mathematics educators and researchers is to develop methods of introducing these ideas to the high school and college level classrooms. Perhaps some conditions in the environment for learning as well as teacher expectations need to change. Mathematics educators have been offered guidance by the work of Robert Davis. He has advocated for:

> (1)A greater commitment to listening to students; (2) a greater realization of the intellectual potential of children; (3) greater recognition of the many "thinking" processes that must take place when anyone attempts to deal with a mathematical problem; and (4) the deliberate construction of assimilation paradigms. (Maher, 1999, p. 89)

The idea of constructing an assimilation paradigm requires comment. When viewing the interview sessions as a whole, one can see that many of the representations developed by Stephanie in the earlier sessions functioned as cognitive building blocks for the representations she used in the later sessions. Perhaps, the selection and sequencing of tasks explored by Stephanie enabled her to build an assimilation paradigm that facilitated her recognition of the isomorphism between the multiple representations developed. Perhaps, too, they were helpful to her in posing conjectures. The importance of choosing a sequence of topics conducive to developing an assimilation paradigm has direct implications for traditional algebra classes. According to Davis (1994), algebraic rules are taught in bits and pieces in isolation from each other. As a result, the students will not be able to assemble these small bits and pieces of ideas into a meaningful larger whole that will give mathematical power to their thinking (Davis & Maher, 1996). Educators, administrators, and researchers therefore are called upon to give much thought and consideration to the sequence of topics within an algebra course or textbook.

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APPENDIX A: TRANSCRIPT – SESSION 1

INTERVIEW WITH STEPHANIE Time: 58 minutes (2 CDs)
November 8, 1995

R1: Dr. Carolyn Maher Stephanie: Stephanie R2: Dr. Elena Steencken

[D r . Maher gives Stephanie a gift from South Africa . The observers can be heard chatting in the background.]

Time	Line	Speaker	Transcript	Code
00:00 –04:59	1	R1	So, you're doing algebra now?	
	2	Stephanie	Yeah.	
	3	R1	Do you have your book (inaudible) 'cause I ' d love to . . . *[Stephanie gets her algebra book out from her backpack.]*	
	4	Stephanie	This is my Algebra book.	
	5	R1	We haven't told Dr. Davis - that we might be working with your school. I know he's going to be very excited to hear about that.	
	6	Stephanie	And this is a lot of the work - like this is all the homework (inaudible) in there.	
	7	R1	So what are you doing now?	
	8	Stephanie	Uh. - Let me look at the notes. 'cause it says -we we're just interpreting stuff the other day. Now we're . . .um, consecutive int -problem solving, perimeter and angle measure with algebra. That's the last thing we did. *[R1 looks at Stephanie's work.]*	
	9	R1	Oh. so you've done some perimeter stuff? - - So you did some story problems like this - or word problems ?	
	10	Stephanie	Yeah. A lot of those - those are really- They get really confusing.	
	11	R1	They what?	
	12	Stephanie	Because then they just - like they're not really confusing, but then it just gets like, um, there was one. It was like - Horace's mother is three years older than her oldest - than her only child. So, obviously, Horace's mother, Horace is the only kid. Then it was like Horace's grandmother is thirty years older than Horace's mother. So you know why -you know it's Horace's age, Horace's mother is three age, Horace's grandmother is three	

Time	Line	Speaker	Transcript	Code
			age plus thirty. If Horace's mother is seven years younger she would be five times older than Horace's grandmother. Then they ask you for something like what's the grandmother's maiden name? They didn't ask that, but they're like asking you for stuff that you don't know.	
	13	R1	*[chuckles]*	
	14	Stephanie	And you're like, oh, yeah. So it gets harder. 'Cause like at the bottom - like all the problems will be easy and they they 'll have two or three red problems.	
	15	R1	Red problems? What are red problems?	
	16	Stephanie	Which are hard.	
	17	R1	Show me red problems.	
	18	Stephanie	*[gets out book]* Like these. *[shows R1 an example of a red problem.]* But these are back further in the book. That's like..	
	19	R1	So where are you - about? *[Stephanie flips the pages.]*	
	20	R1	Oh wow. (inaudible)	
	21	Stephanie	I think, in fact, I have like one of these (inaudible) Yeah. We're right here. This is my homework for tonight.	
	22	R1	Perimeter? Interesting.	
	23	Stephanie	Um hm.	
	24	R1	The reason I say it's interesting, because um see this lady here. She's teaching algebra also. I suspect some of those others are too. And uh she's just – is teaching at the university - people have come back and never had it. And so she and Dr. Davis are teaching a class together. And they just gave a midterm. Is that right? And so I was sorta asking- in fact I have - I took the class a couple of days when she was sick and so I know the class a little bit and I asked about the midterm and so she wrote out a problem. Is this a midterm problem?	
	25	R2	No.	
	26	R1	Was it a problem that you did recently? Is that it? (inaudible) Is that anything like you do? *[R1 gives the paper to Stephanie.]* You could read it.	
	27	Stephanie	Alright. "I need to construct a dog kennel for my very large Akita. I want to enclose a space fifteen feet wide and twenty-five feet long. The fence company needs to know how much fencing is needed; I need, so that the company can give me	

Time	Line	Speaker	Transcript	Code
			an estimate of the cost. Can you help me decide how much fence I will need to order?" No we haven't done this this year. But we did a lot of it last year.	
	28	R1	What do you think it is?	
	29	Stephanie	We did...	
	30	R1	What's involved here?	
	31	Stephanie	A lot of it last year. It's perimeter. It's just um, twenty-five plus twenty-five plus fifteen plus fifteen.	PPK, BMP
	32	R1	Um um.	
	33	Stephanie	So... I mean... this isn't . . . unless there's any other... no, that's about it, except you don't know - oh, no. You don't need to know the price. So that's all.	BEJ
	34	R1	So it seems like – a bit – Suppose I changed it. Suppose I, um, changed the problem a little bit. She really does have a very large Akita.	
	35	Stephanie	*[Chuckles]*	
05:00 –09:59	36	R1	I went there recently and I couldn't believe her friendly dog greeted me – and I ran! She really wasn't kidding. But suppose, suppose we wanted to make a whole bunch of dog kennels. And I wasn't sure what the space was. Suppose, for instance, I said that I could have lots of spaces – different width and different long, so the space could be, let's say "w" wide and "l" long. Right? So if I wanted to have a general way of talking about um how much space I need, how would you write that? *[instructions from video taper] [Stephanie switches pens]*	
	37	Stephanie	It would be *w* plus *w* plus *1* plus *l* equals *s* for space.	BCA
	38	R1	Sure, okay.	
	39	Stephanie	Then it would be two *w* plus two *l* equals *s* and then what you could do – is – I think what we did was we 'cause we did a problem like this , where you had to come up with a uh uh in fact – I'm trying to find (inaudible) it was in here somewhere. And we had to do a problem just recently – like the other night for homework.	BMP; PPK
	40	R1	Uh hm.	
	41	Stephanie	Where what you had to find, I think it was this	

Time	Line	Speaker	Transcript	Code
			one. It was this one. Where you had to prove that for all integers *n*, it *n* plus that equals.	
	42	R1	Um hm.	
	43	Stephanie	And so that's – it was something like that one where you had to move it over to the other side. Like one of these over.	
	44	R1	Okay but so – can you write this another way?	
	45	Stephanie	Can I write…?	
	46	R1	Is there another way of writing that sum besides two *w* plus two *l*?	
	47	Stephanie	Well. Yeah. You could. If you moved one of these over…	
	48	R1	Okay. But suppose you kept everything…	
	49	Stephanie	You can't.	
	50	R1	The same side. Is there another way…	
	51	Stephanie	I don't know.	
	52	R1	…you could write that? *[pause]* Is there another…	
	53	Stephanie	I don't - - I don't know if you could write it like four *w l* – like if you're – if you're allowed to put two variables on one number.	BMP; OBS
	54	R1	Um hm.	
	55	Stephanie	I don't know. Not that I – I mean, not that we learned.	
	56	R1	Do you think that's – Okay. So. So if you were going to be this manufacturer of these little spaces for dogs –	
	57	Stephanie	Um hm.	
	58	R1	Um and I wanted to know how much space – um – to manufacture each one – um – what would I need to tell you before you could tell me how much space?	
	59	Stephanie	Oh! You'd have to tell me the length and the width.	BCA
	60	R1	Okay. So. So if I told you the length and the width you would tell me how much space.	
	61	Stephanie	Yeah.	
	62	R1	Okay. And you would figure it how?	
	63	Stephanie	Well, I'd just multiply whatever the width was by two and the length by two and then add them together.	BMP
	64	R1	Okay. Okay. Now let me change this a little bit. Suppose um I said to you um *[R1 writes 2(w +l) = s]* this was the space.	

Time	Line	Speaker	Transcript	Code
	65	Stephanie	It's the same thing.	BDI; BR-S
	66	R1	Why is it the same?	
	67	Stephanie	Because if you distribute it's just going to be two *w* plus two *l* equals *s*.	BEJ; BMP
	68	R1	Okay. So what do you mean distribute?	
	69	Stephanie	Well, distribute – you have to – if you have a number outside the brackets and there's no plus sign you can't remove the parentheses until you um multiply it by each number inside the parentheses.	BEJ
	70	R1	Does that always work?	
	71	Stephanie	Does that al – well what do you mean? With like distributing?	PAH
	72	R1	Uh huh.	
	73	Stephanie	Yeah.	BMP
	74	R1	Yeah.	
	75	Stephanie	That I know of. You always have to …	
	76	R1	Why does it work? Why does it work?	
	77	Stephanie	I don't – It's just because – hm – 'cause that's what it's telling you to do. It's telling you to um multiply it by – alright – well with parentheses, you're supposed to do whatever is in the parentheses first. But you can't add two different variables together.	BEJ
	78	R1	Um hm.	
	79	Stephanie	And so you're like – you can't combine any like terms. So what you have to do is um the next step which would be multiply everything inside the parentheses. 'cause otherwise, the parentheses, they just wouldn't put them there.	BEJ
	80	R1	Um hm.	
	81	Stephanie	And that would screw the problem up. Because it would be two *w* plus two *l*.	BEJ
	82	R1	Um hm.	
	83	Stephanie	And then you'd have like a different shape.	
	84	R1	It would be a different shape.	
	85	Stephanie	Yeah.	
	86	R1	Okay. Now. So. You've sorta said to me that this is what you're supposed to do, but that you could rewrite this: Two times the expression *w* plus *1*.	
	87	Stephanie	Um hm.	
	88	R1	You said was two *w* plus…	
	89	Stephanie	Two *l*.	

Time	Line	Speaker	Transcript	Code
	90	R1	Two *l*. You said they were the same. And you said that's the Distributive Law.	
	91	Stephanie	Yeah.	
	92	R1	That's what I heard you say. Right? Um. Now. You're sorta telling me it's like a rule. Or a – or you can just use the Distributive Law. Um. Did you ever think about why that worked? That someone someone made this rule? Do you think…	
	93	Stephanie	No, not really. I just figure it worked. I think – we talked about it a little when we got into the negative – the minus sign in front of it.	PPK
	94	R1	Um hm.	
10:00-14:59	95	Stephanie	And you had to distribute the minus sign. 'Cause then it got harder.	
	96	R1	Um hm.	
	97	Stephanie	But um we just were told that it was because you um you put parentheses for a certain reason and it means that you have to do whatever is in the parentheses first.	PPK; BMP
	98	R1	Um hm.	
	99	Stephanie	Otherwise you'll mess the problem up. And then…	
	100	R1	Um hm.	
	101	Stephanie	It's just what we were doing.	
	102	R1	Okay. – um – okay. Now. Let's – let's think about this for a minute we have two times *w* plus *l* *[2(w + l)]* okay let's say that. If I wrote down here um something like – if I wrote an *x*. Do you know what that means?	
	103	Stephanie	No.	
	104	R1	If I just write an *x*.	
	105	Stephanie	Oh it's a variable.	BMP
	106	R1	Give me an example. If you were to explain this to someone who never had algebra, what would you say?	
	107	Stephanie	Alright Um – Well – explain what a variable is? Is that what you're saying?	PAH
	108	R1	Yeah – If I write an *x* and I tell you that…	
	109	Stephanie	*x* stands for any number.	BMP
	110	R1	Any number.	
	111	Stephanie	Any number.	
	112	R1	So if I write two *x*.	
	113	Stephanie	It stands for any number times two.	BMP

Time	Line	Speaker	Transcript	Code
	114	R1	Any number times two. Okay could it stand for – for this? *[writes $x + x$]*	
	115	Stephanie	Um. Yeah, because with um with two you're do – well yeah because you're just adding another one of those.	BMP
	116	R1	Okay so. The two x could mean twice that number I'm thinking about?	
	117	Stephanie	Yeah. Twice the number.	BMP
	118	R1	Or it could mean that number plus that number?	
	119	Stephanie	Yes.	BMP
	120	R1	Okay. Is that consistent with what you knew about arithmetic?	
	121	Stephanie	Um hm.	BMP
	122	R1	If I write five. Right? If I tell you what do you mean by two times five. Is that – could I say that's the same as five plus five?	
	123	Stephanie	Yeah.	PNE
	124	R1	Okay. That sorta works the same way?	
	125	Stephanie	Um hm.	
	126	R1	Okay now let's go back here. *[$2(w + l)$]*	
	127	Stephanie	Okay.	
	128	R1	Okay. Now. I'd like you to think about *[w plus 1]* like you thought about the x.	
	129	Stephanie	Oh. Okay.	
	130	R1	What do you think?	
	131	Stephanie	So you – in other words, you can also write it. It's also the same as w plus w plus *1* plus *1*. And you can put that in parentheses, because it's doubling each one.	BCA
	132	R1	Oh. That's interesting. This step –	
	133	Stephanie	You don't have to put in parentheses, but that's just how our teacher likes us to do it. When we're doing two different steps – so I'm just like used to doing it – like there's no need for parentheses (inaudible).	BEJ
	134	R1	Okay. Sorta like you skipped a step for me.	
	135	Stephanie	Oh. Okay.	
	136	R1	You know – what do you think the step is you skipped for me?	
	137	Stephanie	Um.	
	138	R1	When I wrote $2(w + l)$.	
	139	Stephanie	I have no idea…Um, Did you want…	PAH
	140	R1	No, I…	
	141	Stephanie	me to just write $2w + 2l$ and then …	PAH

Time	Line	Speaker	Transcript	Code
	142	R1	No, no. That would be skipping two steps for me. *[Chuckles]* For me, when when you go – I'm this little kid. Right?	
	143	Stephanie	Um hm.	
	144	R1	And I'm trying to think, if you're telling me – if you're telling me $2x$ is the same as $x + x$, I'm this little child whose thinking $2 (w + l)$ is the same as $(w + l) + (w + l)$. That you have it twice.	
	145	Stephanie	Okay.	
	146	R1	Do you see what I'm saying?	
	147	Stephanie	Yeah.	
	148	R1	But then you went and skipped it. You know, you put the w's and the l's.	
	149	Stephanie	Oh. Okay.	
	150	R1	But I don't know how you were thinking about it. And I'm kind of interested –	
	151	Stephanie	I was just…	
	152	R1	Did you think about this step that I was thinking of here? That $2(w + l)$ is $(w + l) + (w + l)$?	
	153	Stephanie	No, actually I didn't.	
	154	R1	You didn't even think about it?	
	155	Stephanie	I didn't even think about it. I just went right to…	
	156	R1	Does that make any sense?	
	157	Stephanie	Yeah. It's the same thing.	BMP
	158	R1	Um hm.	
	159	Stephanie	Cause it – I just – all of a sudden put the w's and the l's together.	BEJ
	160	R1	Um hm. But it gets you the same place? The two w –	
	161	Stephanie	Yeah, you can do it either…	BEJ
	162	R1	You see I wasn't sure if you were thinking backwards from knowing that it was $2w + 2l$ because it was a rule.	
	163	Stephanie	Um hm.	
	164	R1	And then thinking well two w's – I didn't know if you were working from this side and thinking well the two w's is $w + w$ and the 2 l's is l plus l.	
	165	Stephanie	Oh.	
	166	R1	Do you see what I'm saying?	
	167	Stephanie	Yeah.	
	168	R1	I wasn't sure that you were thinking from here, because when I asked I was pretending like I didn't know what this was. I'm this little kid	

Time	Line	Speaker	Transcript	Code
			who's trying to figure out what this is. So I'm thinking if you told me 2*(w + l)* – two times the expression *w* plus *l* it means you have it twice. Do you see what I'm saying?	
	169	Stephanie	Yeah. I understand what you're saying.	
	170	R1	So I don't know. It's just interesting how you were thinking about that. But when you mentioned that some of the perimeter stuff and some of the Distributive Law, I started thinking about this. Did you do anything like this? Question – yet –	
	171	Stephanie	Okay.	
	172	R1	Okay so you did 2(*w* + *l*) and that's kind of perimeter stuff. Could you do 3 times the expression (*w* + *l*) and 5 times (*w* + *l*)?	
	173	Stephanie	I think we did a couple three – like towards the end of the problems. Like with the red problems.	PPK
	174	R1	But if we just did something like this for a minute.	
	175	Stephanie	Um hm.	
	176	R1	You can do that? *[5(w + l)]*	
	177	Stephanie	Yeah.	
	178	R1	Right.	
	179	Stephanie	Yeah.	
	180	R1	And that's?	
	181	Stephanie	It would be 5*w* + 5*l*.	BMP
	182	R1	Right? You could actually imagine why that works.	
	183	Stephanie	Um hm. Yes.	
	184	R1	If I said – you know – to convince yourself that why this rule sorta – this rule works?	
15:00-19:59	185	Stephanie	You're only saying that you're multiplying – you're taking any number "*w*" and you're um I guess – if you're going to do it like how the 2*x* was – um – any number twice, you can do the 5*w* – any number five times. And the 5*l*.	BEJ; BCA
	186	R1	Um hm.	
	187	Stephanie	Um – any number 5 um *l* five times but – and the distributive part is just because that's all the parentheses (inaudible) mess the problem up. I guess that's how you…	BEJ
	188	R1	So I think of if I was this little kid, if you tell me I have five of something, right?	
	189	Stephanie	Um hm.	
	190	R1	5*(w + l)*. I'm this little kid and I say well you have one of them, you have two of them.	

Time	Line	Speaker	Transcript	Code
	191	Stephanie	(inaudible) whatever you think you…	
	192	R1	You have three of them. I'm just trying to think it in the most elementary way. Right?	
	193	Stephanie	And that's the same thing.	
	194	R1	And I'm adding all this right.	
	195	Stephanie	I guess that's what…	
	196	R1	Which gives you this. That's the way I was thinking about it.	
	197	Stephanie	That's the same thing.	
	198	R1	So you're saying that's all the – and that's always going to work.	
	199	Stephanie	Yeah. I just always put the varia –that's how –but I understand –	
	200	R1	You just skip that step it seems.	
	201	Stephanie	I skip that step.	
	202	R1	But you see that – does that?	
	203	Stephanie	Yeah. But I understand.	
	204	R1	Have you ever thought about it that way?	
	205	Stephanie	I – When – I guess way when we first…	
	206	R1	A long time ago.	
	207	Stephanie	'Cause I think we first took uh little steps in sixth grade with Mr. Poe.	
	208	R1	Okay.	
	209	Stephanie	And then it really screwed me up. I was like really bad in that class. I had no idea what we were doing at all.	
	210	R1	Why do you think?	
	211	Stephanie	I don't know. It was just – Matt – It was Matt 'cause I'd understand what Mr. Poe was saying and then Matt would get up in front of the class and describe how he got the answer and it would totally mess me up 'cause I was doing it totally differently.	
	212	R1	Then Matt…	
	213	Stephanie	And then Matt would go and do it like this totally weird complex way and I was just going like (w + w) is 2w he was like well you know and I was like – so I was really bad at that.	
	214	R1	So it was hard to understand what Matt was thinking?	
	215	Stephanie	It was just like…	
	216	R1	Yeah.	
	217	Stephanie	But now it's just like after you've done it, like, we did it like last year and we're doing it like this year	BEJ

Time	Line	Speaker	Transcript	Code
			you cancel it out. Like I don't even – like when you first said it I couldn't even recall having to do that. I just always go you put the same variables together.	
	218	R1	Um hm.	
	219	Stephanie	Instead of – I just like skip that entirely now.	
	220	R1	Um hm. Um hm. I see. I remember, um a long time ago in Kenilworth, actually, Harding School.	
	221	Stephanie	Um hm.	
	222	R1	That um you used to do some of this without *x*'s. You used to use boxes. Do you remember that?	
	223	Stephanie	And triangles and stuff.	PPK
	224	R1	Do you remember that?	
	225	Stephanie	Yes.	
	226	R1	Okay.	
	227	Stephanie	That, that I remember.	
	228	R1	What, what would I have to do with something like this? I mean would you think about this if you had to explain it with boxes and triangles?	
	229	Stephanie	Well, you… a box is good 'cause it's always like a blank.	PPK
	230	R1	Um hm.	
	231	Stephanie	You know and you can put any number in the box.	PPK
	232	R1	Um hm.	
	233	Stephanie	And multiply it by two and you're going to get two of those boxes.	PPK
	234	R1	Okay.	
	235	Stephanie	You know and the same thing with like the five.	
	236	R1	Um hm.	
	237	Stephanie	Big um.	
	238	R1	So does that help you to think about it that way? About the meaning of the *x*'s and the *a*'s and *w*'s and *r*'s and *p*'s?	
	239	Stephanie	I guess.	
	240	R1	Um hm.	
	241	Stephanie	I mean, I haven't thought about the boxes.	
	242	R1	You haven't thought about the boxes?	
	243	Stephanie	I never thought about the boxes. But I guess it – it's basically the same thing – no matter what variable you use – a triangle or an *x* or …	BR-S/V
	244	R1	Um hm.	
	245	Stephanie	It's still saying that you have any number.	BMP
	246	R1	Um hm.	
	247	Stephanie	So…	

Time	Line	Speaker	Transcript	Code
	248	R1	Okay. Have you done anything like this yet? Okay – as we do these examples. Did you do anything like this? *[a(x + y)]*	
	249	Stephanie	Um hm. Not that I can recall. No.	
	250	R1	No, what do you think that could possibly mean?	
	251	Stephanie	It's any number times two other variables that could also stand for any number – so – can you get a number that's like *ax* + *ay*?	BMP
	252	R1	Let's think about that? Why don't you write –	
	253	Stephanie	'Cause that's what it's telling you to do. It's telling you.	BMP
	254	R1	So you think that's going to be *[Stephanie writes ax + ay]*.	
	255	Stephanie	That's what it's telling you.	
	256	R1	That's an *a*, right? *[corrects Stephanie's handwriting]*	
	257	Stephanie	Yeah.	
	258	R1	Okay – So your conjecture is that – why don't you test it? Why don't you try some numbers for *a*, *x*, and *y*? And see if it works?	
	259	Stephanie	Alright *[2(3 + 4)]* is six plus eight is fourteen.	PNE
	260	R1	Does that work?	
	261	Stephanie	Well, actually I have to try one number at it – well…	PNE
	262	R1	Well.	
	263	Stephanie	I have to just plug in one number and then – what I really have is go like equals fourteen and then plug in – if I just plugged in like the two.	PNE
	264	R1	Okay. So what you're saying here – two – is…	
	265	Stephanie	It comes out to be the same – I mean, I guess you could put a variable with another variable and multiply it.	BCA
	266	R1	Right.	
	267	Stephanie	We just never did it before.	
	268	R1	So, Let's think about this. What did you use for *a*? What did you use for *x*? And what did you use for *y*?	
20:00-24:59	269	Stephanie	Um two for *a*.	PNE
	270	R1	And…	
	271	Stephanie	Three for *x*.	PNE
	272	R1	For *x*.	
	273	Stephanie	And four for *y*.	PNE
	274	R1	Okay, so um your conjecturing is that is, it *ax* plus	

Time	Line	Speaker	Transcript	Code
			ay and so *ax* would be six and *ay* would be eight.	
	275	Stephanie	Oh, I see.	
	276	R1	Okay, so is it-	
	277	Stephanie	What I should have done is – *[writes 2·3 + 2·4 between her steps]* 'Cause I just leave that step out. Like sometimes too.	BEJ
	278	R1	That's okay. That doesn't bother me. Um but that's if you distribute, right?	
	279	Stephanie	Yes.	
	280	R1	If you don't distribute, what do you get?	
	281	Stephanie	If you don't distribute?	
	282	R1	If you did it without distributing.	
	283	Stephanie	You get twelve plus two – fourteen.	BMP
	284	R1	You get two times seven, right?	
	285	Stephanie	Yeah?	
	286	R1	So it still worked for two, three, and four?	
	287	Stephanie	Yeah, it worked the same way.	BDI
	288	R1	Okay, now do you think it's always going to work? *[pause]* For any choice of…	
	289	Stephanie	I don't know.	OBS
	290	R1	Of *a, x* and *y*.	
	291	Stephanie	I think we might of – we went over something like this, but it was with um exponents and I don't remember if it'll work every time. Should I try it with different variables?	PPK
	292	R1	Okay. What's your intuition on it?	
	293	Stephanie	Um, I don't know. If it works every time, I don't understand why they make us um distribute in the first place – if it works every time. So I don't think – I think there's going to be a problem *(inaudible)* I mean 'cause – it's pretty dumb then if we always have to distribute – you know.	OBS
	294	R1	Um hm. Do you think you always have to distribute?	
	295	Stephanie	Well, obviously not in this problem.	
	296	R1	Um hm.	
	297	Stephanie	So I mean…	
	298	R1	You know you could've gotten the answer without distributing, if that were true.	
	299	Stephanie	Yeah, I could've just…	
	300	R1	If that were true. If they were equivalent, you didn't have to, did you?	
	301	Stephanie	Well actually I shouldn't have – I should've just	

Time	Line	Speaker	Transcript	Code
			distributed after I added those two.	
	302	R1	Well no. I don't know that you should've…	
	303	Stephanie	I mean – it doesn't matter. Like it I had…	
	304	R1	Let me ask you a question? Does it matter?	
	305	Stephanie	If I had a variable. Like if it was *(writes) [2(x + 4)]*. 2(x + 4) right. I have to distribute first 'cause I can't add four to x.	BEJ; BMP
	306	R1	Okay, so what would that look like?	
	307	Stephanie	So that would have to be $2x + 8 = 14$.	
	308	R1	Where did you get the fourteen from?	
	309	Stephanie	Well, fourteen was my answer up here. I'm just doing – using	
	310	R1	That's if you don't know what x is?	
	311	Stephanie	Yeah.	
	312	R1	Okay.	
	313	Stephanie	Eight minus eight. *[writes 2x + 8 – 8 = 14 – 8]* equals *(inaudible) [continues "figuring"]* x equals three. It worked.	PNE; BMP
	314	R1	Interesting.	
	315	Stephanie	This problem's working out.	
	316	R1	What's the it that worked? What were you thinking when you did it?	
	317	Stephanie	Well, um I was just try – 'cause like here I didn't have to distribute, but if I had a problem where I had a variable in the inside of the parentheses I would have to distribute.	BDI
	318	R1	Um hm.	
	319	Stephanie	Because I can't combine like terms if they're not the same -- so	BMP
	320	R1	Um hm.	
	321	Stephanie	I was just saying that you know if you have a variable, you have to distribute first.	BMP
	322	R1	Okay. Now. Do you see – Do you see how I convinced myself that that would always work?	
	323	Stephanie	Um hm.	
	324	R1	Do you see how I was trying to think if I was this little kid – you know how we would persuade a little kid that why if I doubled x, you get two x. What does doubling x mean?	
	325	Stephanie	Um hm. Right.	
	326	R1	Right? So why does this work? Multiplying w plus l times five?	
	327	Stephanie	Right?	
	328	R1	Are you convinced that will always work? That I	

Time	Line	Speaker	Transcript	Code
			– did I convince you that this is always going to work when I multiply by five.	
	329	Stephanie	Well, yeah. ‘Cause I – uh, I mean I always thought, you know.	
	330	R1	I mean someone saying that’s not a rule but – I’m saying that what *w* plus *l* means five times – it means that you have it five times.	
	331	Stephanie	Yeah.	
	332	R1	Alright, does that make sense?	
	333	Stephanie	Yeah.	
	334	R1	So suppose I asked you to convince me that why the Distributive Law works for eight times *w* plus the expression *w* plus *l*.	
	335	Stephanie	It’s the same reason that it works for two.	
	336	R1	Right. What’s that reason?	
	337	Stephanie	That you’re simply taking that number and adding it with the same number the amount of times it’s telling you.	BEJ
	338	R1	Okay. So the same time as two and the same time as five.	
	339	Stephanie	Um hm.	
	340	R1	And you would do eight the way – okay so that’s not too bad if you know the number here: two or five or eight. So you think it will work for any number? Two, five, eight, eleven? Will it work the same way?	
	341	Stephanie	I think so.	
	342	R1	Sixteen? – One million?	
	343	Stephanie	Yeah.	
	344	R1	Okay. So if I say any number…	
	345	Stephanie	Okay.	
	346	R1	Say “a”.	
	347	Stephanie	Well that was – what you had here.	
	348	R1	So how would you convince me that it would work for say any number “a”?	
	349	Stephanie	That…	
	350	R1	If you have $(x + y)$ a times? How would how would you reason it in your head? How would you think about it?	
	351	Stephanie	That you’re taking any number.	BCA
	352	R1	Um hm.	
	353	Stephanie	And you’re adding it with itself.	BCA
	354	R1	Um hm.	
	355	Stephanie	as many times as a is.	BCA

Time	Line	Speaker	Transcript	Code
	356	R1	Okay.	
	357	Stephanie	Like…	
	358	R1	Yeah.	
	359	Stephanie	I'm trying.	
25:00-29:59	360	R1	Okay. That's interesting. That's pretty neat. So um. Why don't you write that down? That's really kinda nice Stephanie what you just said. I just want to be sure you get a chance to think about it. You're going to try now to tell me what this is. *[a(x + y) = ?]* Okay? Why the distributive law works here. I'm interested in…	
	361	Stephanie	Um.	
	362	R1	Think about that for a minute and write it and I'll have a glass of water.	
	363	Stephanie	I – it just…	
	364	R1	Yeah just say it, just write what you've said.	
	365	Stephanie	Oh what I just said before.	
	366	R1	Sure, yeah.	
	367	Stephanie	That *[Stephanie writes "Your taking any number and adding it of itself the amount of times that the variable a equals"]* repeats what she wrote.	BEJ
	368	R1	Okay. So what do you end up with when you've done that? When you've added *(x + y)* to itself *a* times?	
	369	Stephanie	Um.	
	370	R1	What do you end up with when you add *(x + y)*?	
	371	Stephanie	That –	
	372	R1	Do you end up with *a* times *(x + y)*?	
	373	Stephanie	Well –	
	374	R1	Is there another way you could say that?	
	375	Stephanie	Well – I would – I don't – *(x + y)*.	
	376	R1	See, I don't want to end up where I started. If I'm adding *(w + l*) to itself five times, I didn't end up with this *[5(w + l)]*. Do you see?	
	377	Stephanie	Um hm yeah. I understand. I just – I don't know how many times *a* is.	OBS
	378	R1	But can you write an expression that says how many times in general? Without knowing it? - See when you knew it was two times, you knew how to write it. When you knew it was five times you knew how to write it.	
	379	Stephanie	We just –	
	380	R1	And we conjectured when it was eight times how to write it.	

Time	Line	Speaker	Transcript	Code
	381	Stephanie	It would just be like *(x + y) = (x + y)* – I don't know – cause like here I don't have the –	
	382	R1	Okay. Well, how many times are you going to do this now?	
	383	Stephanie	As many times as *a* is.	
	384	R1	Okay. So write that down. *(x + y)* as many times as *a*. Okay now.	
	385	Stephanie	Oh, do you want me to say just *(x + y)* as many times as *a*?	PAH
	386	R1	Sure. *[Stephanie writes.]*	
	387	Stephanie	Okay.	
	388	R1	Okay. Can you imagine this in your head?	
	389	Stephanie	Yeah.	
	390	R1	You got *(x + y)* as many times as *a*.	
	391	Stephanie	Okay.	
	392	R1	Can you imagine that?	
	393	Stephanie	Well yeah. But I'm just imagining like any number.	
	394	R1	Tell me what's in your head when you see this.	
	395	Stephanie	Just like rows of *x*'s and –	BR-V
	396	R1	Rows of *x*'s – how many *x*'s would you end up with when you're all done?	
	397	Stephanie	Um. A lot.	
	398	R1	How many? If you're doing it – if you have *(x + y) a* times?	
	399	Stephanie	*a* amount of *x*'s.	BR-S; BCA
	400	R1	Right. And how many *y*'s?	
	401	Stephanie	*a* amount – like	BR-S; BCA
	402	R1	So why don't you write that down? Isn't that what it's going to be? How would you write *a* amount of *x*'s and *a* amount of *y*'s?	
	403	Stephanie	Um. Could I just write you would end up with *a* amount of *x*'s?	
	404	R1	Yeah. I'd like to see how you would write that. *[Stephanie writes: "a amount of x's + a amount of y's".]*	
	405	Stephanie	It's just like – like – It looks like this, only it's –	
	406	R1	Okay. So can you write it in a simple form? The *a* amount of *x*'s and *a* amount of *y*'s? What does *a* times the expression *(x + y)* now equal? If we want to replace this question mark, how could you write *a* amount of *x*'s and *a* amount of *y*'s?	

Time	Line	Speaker	Transcript	Code
	407	Stephanie	$ax + ay$?	BR-S; BCA
	408	R1	Doesn't that make sense?	
	409	Stephanie	Um hm.	
	410	R1	Which is what you conjectured before.	
	411	Stephanie	Yeah.	
	412	R1	Based on – Does that make sense?	
	413	Stephanie	Yeah.	
	414	R1	You really believe it, right?	
	415	Stephanie	Yeah.	
	416	R1	Good. Now we'll get to why I wanted you here today. *[Stephanie chuckles.]*	
	417	R1	'Cause you why they're all wondering over there? What am I up to? I have no clue. That's what makes it so much fun. 'Cause I want to give you something to think about. 'Cause I thought it would be fun to do this. This is great. Um. Since you have this, can we make it a little bit harder?	
	418	Stephanie	Okay.	
	419	R1	Did you ever have anything like this? *[Dr. Maher writes $x \cdot x$.]*	
	420	Stephanie	x times x?	PAH
	421	R1	We had x plus x. You told me that was two x's. What about x times x?	
	422	Stephanie	I think I might've. I don't think I've done anything like this this year.	
	423	R1	Okay. What do – what do you think that means?	
	424	Stephanie	It means x, the variable x, x amount of times.	BCA
30:00-34:59	425	R1	Okay. That's that's interesting. The variable x x amount of times. Really neat. I'll buy that.	
	426	Stephanie	Um hm.	
	427	R1	Okay. Suppose you had numbers.	
	428	Stephanie	Okay.	
	429	R1	Um 'cause you said it could be anything. Suppose we had x and we had x dot x. Right? And if x were two…	
	430	Stephanie	It would be two times two.	PNE
	431	R1	Two times two. Right?	
	432	Stephanie	Um hm.	
	433	R1	And if x were three –	
	434	Stephanie	Three times three.	BMP
	435	R1	Is there another way two times two?	
	436	Stephanie	Um, you could write it two plus two?	

Time	Line	Speaker	Transcript	Code
	437	R1	Okay. And three times three? Does that work the same way?	
	438	Stephanie	No.	
	439	R1	Is there a way of writing, you know –	
	440	Stephanie	everything the same?	
	441	R1	Write everything the same? Right. All the way up to …if you had four. This is four times four.	
	442	Stephanie	Oh! Well, you could use um you could use exponents.	BDI
	443	R1	How would you do that?	
	444	Stephanie	Well. Two to the second and three to the second.	BMP
	445	R1	Okay. So you could – why don't you write that?	
	446	Stephanie	Okay. Like do you want me to make another chart?	PAH
	447	R1	Well – well –sure.	
	448	Stephanie	Okay.	
	449	R1	And then tell me how you would write x times x then. *[Stephanie writes.]*	
	450	Stephanie	How far do you want me to go?	
	451	R1	Um until you can give me a writing x times x in general.	
	452	Stephanie	Um. x to the x power?	OBS
	453	R1	Okay.	
	454	Stephanie	Can you do it like that?	
	455	R1	Well. Check it out.	
	456	Stephanie	Or – x to the second! Oh no! x to the second power?	BDI
	457	R1	What do you think? Which do you think it is? x to the x or x to the second?	
	458	Stephanie	x to the second.	BCA
	459	R1	Why?	
	460	Stephanie	'cause x to the x power would mean – say x is – x is one thousand one hundred and fifteen. That would mean one thousand one hundred and fifteen one thousand one hundred and fifteen times and that's –	BEJ; PNE
	461	R1	Pretty big.	
	462	Stephanie	Really long.	
	463	R1	Okay.	
	464	Stephanie	So two x – uh – x to the second.	
	465	R1	Okay. Do you know how you read that?	
	466	Stephanie	What?	
	467	R1	Another way people read the x to the second	

Time	Line	Speaker	Transcript	Code
			power? Sometimes that's called x-squared.	
	468	Stephanie	Oh. Yeah	
	469	R1	You knew that. Okay. So is that familiar to you?	
	470	Stephanie	Yes.	
	471	R1	Okay. So. You had some of this at Harding, didn't you?	
	472	Stephanie	Yes.	
	473	R1	Some exponents – but you're not doing that here yet?	
	474	Stephanie	We did some exponents.	
	475	R1	A little bit.	
	476	Stephanie	Here, I think as a review-	
	477	R1	Um hm.	
	478	Stephanie	-in the beginning, but right now we're not working with them at all.	
	479	R1	Okay. Now that we've done this, let's move on to something else. Now. You – you already came up with a times x plus y. Another way to write that is $ax + ay$. And you believe that that's always true. And you sorta gave me a nice little argument.	
	480	Stephanie	Okay.	
	481	R1	Okay, I'll buy that.	
	482	Stephanie	Okay.	
	483	R1	Okay. Um. Now – could I do this? *[Dr. Maher writes $(x + y)(x + y)$.]*	
	484	Stephanie	You probably could. I don't know how, but you probably um	
	485	R1	What do you think it means?	
	486	Stephanie	It means that – um – x plus y times – OH! Could you just do it x squared times y squared?	OBS
	487	R1	What do you think this means?	
	488	Stephanie	It means that you're multiplying – 'cause you can't combine these terms, right?	BMP
	489	R1	I'll buy that.	
	490	Stephanie	So…	
	491	R1	Why can't you, by the way?	
	492	Stephanie	'Cause they're not the same variable.	BMP
	493	R1	Okay.	
	494	Stephanie	Uh. Because you can't combine them, um you have to multiply them by – okay – you're supposed to multiply these. But you can't combine these either.	BMP
	495	R1	Um hm.	
	496	Stephanie	So – but you can't exactly take this (inaudible)	

Time	Line	Speaker	Transcript	Code
	497	R1	Um. It's interesting, isn't it?	
	498	Stephanie	I can't figure out how to get around it. But I'm pretty sure that if I could, the answer would be *x*-squared plus *y* – squared.	OBS
	499	R1	Why don't you put a question mark here and let's test it.	
	500	Stephanie	Okay.	
	501	R1	Okay. Your conjecture – Stephanie's conjecture – this is *x* – squared plus *y* – squared. Test it. Try some numbers and see.	
	502	Stephanie	Alright. *[She tries 2 and 3.]* Two plus three. Two plus three. Two squared plus three squared. Nine. Four. Equals five – oops, that's not right. *[She throws down her pen.]*	PNE
	503	R1	Didn't work, huh?	
	504	Stephanie	No, it didn't work.	
	505	R1	Hm.	
	506	Stephanie	I don't know what it is then.	OBS
	507	R1	Hm. So – It was a reasonable guess. *[makes a noise]* These things are tricky, aren't they?	
	508	Stephanie	I don't know, because I can't – I can't figure out how to get rid of the – the um.	OBS
	509	R1	Hm. Let's try to think about meaning. Let's try to think about meaning.	
	510	Stephanie	Okay.	
	511	R1	Okay. Remember how you made an argument that this is the same as this?	
	512	Stephanie	Um hm.	
	513	R1	What did you imagine in your head? Tell me about what you saw in your head.	
35:00-39:59	514	Stephanie	Oh. That it's um *x* times *x* plus *y* or *x* plus and *y* plus *y a* amount of times. And since I didn't know *a*, it was just like rows and rows and rows of numbers.	BR-S/V
	515	R1	Okay. How many times did you get those rows of *x*'s and those rows of *y*'s?	
	516	Stephanie	A lot. 'Cause I didn't have any stopping point and that –	BR-V; OBS
	517	R1	You did have a stopping point.	
	518	Stephanie	Well, it was *a*, but I didn't –	
	519	R1	It was *a*.	
	520	Stephanie	But I didn't know what *a* was.	
	521	R1	Right. Exactly. Okay. Show – Now we're thinking of – Remember that. Remember. *a* could	

Time	Line	Speaker	Transcript	Code
			be anything.	
	522	Stephanie	*a* could be anything.	
	523	R1	Could *a* be *x* plus *y*?	
	524	Stephanie	Oh.	BDI
	525	R1	You said *a* could be anything. That's what you're telling me.	
	526	Stephanie	(inaudible)	
	527	R1	Could *a* be *x* plus *y*? Now does it help you now to think of what this means if you think of *a* as *x* plus *y*?	
	528	Stephanie	I don't see why it couldn't.	
	529	R1	Okay.	
	530	Stephanie	I mean-	
	531	R1	So tell me what you're imagining in your head.	
	532	Stephanie	Well, now I just see *a* times *a*. 'Cause you told me…	BDI
	533	R1	You told me – Oh, 'cause that's – that's really neat. That's very nice. *a* times *a*. I could buy that. That's nice.	
	534	Stephanie	Um.	
	535	R1	Okay. That's true. But that's not going to get you out of figuring –	
	536	Stephanie	Yeah.	
	537	R1	-out what *x* plus *y* times *x* plus *y* is.	
	538	Stephanie	I (inaudible)	
	539	R1	But that's absolutely reasonable. I like that. Nice and simple.	
	540	Stephanie	I – (inaudible) I'd have to find a way to make – I can't just say it's *a* though. I can't just go *x* plus *y* is *a*. You know.	BEJ
	541	R1	Okay. Well. So. Now. But – you're thinking of *a* times *x* plus *y*, right?	
	542	Stephanie	I can't even add *x* plus *y*, though. Which is my problem. Like I can't add *x* plus *y* together. 'Cause they're different.	BEJ; BMP
	543	R1	Okay. But if this were *a* times, you would imagine *x* plus *y* *a* times in your head?	
	544	Stephanie	Um hm.	
	545	R1	*x* plus *y*, *x* plus *y*, *x* plus y. But now, this is not *a*, right?	
	546	Stephanie	Right.	
	547	R1	This is *x* plus *y*. So how many times are you imagining *x* plus *y* in your head?	
	548	Stephanie	Once. Right now, just because there's not an *a*	BCA

Time	Line	Speaker	Transcript	Code
			amount of times. And it's x plus y, x plus y amount of times.	
	549	R1	Is it x plus y x plus y amount of times? Okay. I'm asking how many? - This is your x plus y.	
	550	Stephanie	Okay.	
	551	R1	Alright?	
	552	Stephanie	Yeah.	
	553	R1	You have a bunch of them.	
	554	Stephanie	Yeah.	
	555	R1	How many of them do you have?	
	556	Stephanie	I have x plus y times x plus y, so I have it x plus y amount of times, but I don't know.	BCA
	557	R1	Okay. Don't lose that idea.	
	558	Stephanie	Okay.	
	559	R1	Why don't you just get that idea? Make sure of it. Write it down, 'cause that's a that's a good thing to hold on to. - - You have x plus y x plus y amount of times. That's pretty good. - - Do you really believe that?	
	560	Stephanie	That's what I'm getting.	
	561	R1	Or a.	
	562	Stephanie	Or a plus a – a times a. I just – 'cause…	
	563	R1	So you've got x plus y x plus y amount of times.	
	564	Stephanie	That's what it is.	BDI
	565	R1	Okay.	
	566	Stephanie	That's what it is.	
	567	R1	Alright. Now. What makes this kind of messy, 'cause you're thinking about this x plus y amount of times. It was nice when you – you thought a amount of times was bad enough – but that was sure easier than –	
	568	Stephanie	Yeah.	
	569	R1	x plus y. Right?	
	570	Stephanie	Yeah.	
	571	R1	So can you break down the way you think about it in terms of x plus y amount of times? Do you have to think about it x plus y amount of times?	
	572	Stephanie	No. If I break down, I can think of it x amount of times and y amount of times.	BMP
	573	R1	Okay. Very interesting. So – so you can think of x plus y – you could have x amount of times, right?	
	574	Stephanie	Um hm.	
	575	R1	And you could have it y amount of times. Isn't that right?	

Time	Line	Speaker	Transcript	Code
	576	Stephanie	Yes.	
	577	R1	Is that a way to think about it?	
	578	Stephanie	Oh! Yeah.	BDI
	579	R1	Does that make sense?	
	580	Stephanie	Yeah. You could do it like that.	BDI
	581	R1	If you did, does it make it simpler to now…	
	582	Stephanie	Yeah.	
	583	R1	rewrite these – these – we know it didn't work to be *x*-squared plus *y*-squared. Why don't you play with that and see what you can do with that?	
	584	Stephanie	Alright. Should I put in like some numbers?	PNE
	585	R1	Well – See what you –	
	586	Stephanie	Oh, well otherwise –	
	587	R1	Yeah, that's – Yeah. Put in some numbers. Sure. That's a great idea.	
	588	Stephanie	Alright. I'll just put in like a number for *x*. So I'll make *x* two.	PNE
	589	R1	Well, put in a number for *y* and *x*.	
	590	Stephanie	Alright.	
	591	R1	That's interesting. *[Stephanie writes: 2(2 + 3) + 3(2 + 3)* *4 + 6 + 6 + 9* *10 + 15* *25]*	PNE
	592	Stephanie	…six plus nine equals ten plus (inaudible) twenty-five.	
	593	R1	Is that what you were supposed to get before?	
	594	Stephanie	Yep.	
	595	R1	You like that, huh?	
	596	Stephanie	Um hm.	
	597	R1	So it worked at least for two numbers? Does that mean it's always going to work?	
	598	Stephanie	It might. I –	
	599	R1	But does that mean it always is gonna work?	
	600	Stephanie	I think so. I think that's allowed. To do it like that.	BDI
	601	R1	Does that make sense? What you did?	
	602	Stephanie	Yeah.	
	603	R1	Alright. Now. Okay. So you sorta think you're on the right track. You don't want to test any more just to be sure? Another one or two? *[Stephanie inaudible]*	

Time	Line	Speaker	Transcript	Code
	604	R1	It's up to you. I really – I'm not trying to persuade you.	
	605	Stephanie	Well.	
	606	R1	If you're satisfied, we can go on.	
	607	Stephanie	I'll do it.	
	608	R1	It's just that my students back there might be saying 'You're letting her be convinced on one try!' I'll bet they've got to tell me things I'm doing wrong later. *[Stephanie tries 4 and 5: 4(4 + 5) + 5(4 + 5) 16 + 20 + 20 + 25]*	PNE
	609	Stephanie	Now it didn't work.	OBS
	610	R1	It didn't work?	
	611	Stephanie	No.	
	612	R1	Let's see what you had happen here.	
	613	Stephanie	Well, now I got a higher number. - - - But I'm using higher numbers.	OBS
	614	R1	Right. So?	
	615	Stephanie	So – it's okay?	OBS
	616	R1	Did you test it on both sides?	
	617	Stephanie	Not yet.	OBS
	618	R1	Remember what you're testing that works. Remember what you're – See why did the twenty-five work here? Remember. Look back and see what you did here.	
40:00-44:59	619	Stephanie	Oh.	
	620	R1	You were testing?	
	621	Stephanie	Because I did use two plus three here.	
	622	R1	Let's find…	
	623	Stephanie	Oh. I did use two plus three here.	
	624	R1	Right.	
	625	Stephanie	That's why it worked.	BDI
	626	R1	Right. So now when you use four plus five –	
	627	Stephanie	So now I would have to use four plus five there.	BDI
	628	R1	Would you would you expect to get something different?	
	629	Stephanie	Let me …(inaudible)	
	630	R1	Yes. (inaudible) This is going to be very confusing.	
	631	Stephanie	I got eighty-one.	
	632	R1	Is that what you should get?	

Time	Line	Speaker	Transcript	Code
	633	Stephanie	Um – well four plus five…	
	634	R1	If you use four plus five.	
	635	Stephanie	…is twenty.	OBS
	636	R1	Four plus five times four plus five?	
	637	Stephanie	Plus (inaudible) that would be twenty plus twenty. I get –	
	638	R1	Four plus five.	
	639	Stephanie	Oh. Nine!	BDI
	640	R1	Nine times.	
	641	Stephanie	I'm doing four times five! Nine times nine. That's eighty-one. That works.	BDI
	642	R1	Getting more confident in this rule?	
	643	Stephanie	Yeah. Yeah.	
	644	R1	You think it's going to work now? Or do you need to try any more?	
	645	Stephanie	Um no. But should I?	
	646	R1	Uh, it's up to you. – Alright. If you're confident with the reason of breaking it down, you should feel pretty good about why it should work. You know?	
	647	Stephanie	Yeah.	
	648	R1	Okay. But do you think this makes sense? That you have x plus y x plus y amount of times?	
	649	Stephanie	It makes it easier to do, because…	
	650	R1	Alright.	
	651	Stephanie	looking at that it's a lot harder.	
	652	R1	Let's see if we can make this simple. So we said so far that possibly x plus y times x plus y, right?	
	653	Stephanie	Um hm.	
	654	R1	Could be thought of as x x plus y's right? Plus y x plus y's.	
	655	Stephanie	Yeah.	
	656	R1	You like that?	
	657	Stephanie	Yes.	
	658	R1	Now let's – um – could you make this simple with your Distributive Law?	
	659	Stephanie	Yes.	
	660	R1	Do you think you can – do you know enough – what does it mean to write x times x plus y?	
	661	Stephanie	Oh. Can I - - ?	
	662	R1	What does that mean: x times the quantity x plus y?	
	663	Stephanie	Well, x times – no. Wait. That's – It – See if it was just x times x I could do an x-squared.	BMP

Time	Line	Speaker	Transcript	Code
	664	R1	Well, it is. You have *x* times *x*.	
	665	Stephanie	Yeah, but I can't do it with *y*, 'cause *y*-squared is different than *x*-squared.	
	666	R1	Okay. But this piece you think is *x*-squared?	
	667	Stephanie	I can do it.	
	668	R1	*x* times *x*.	
	669	Stephanie	Yeah.	
	670	R1	Well, do that.	
	671	Stephanie	It would just be – do you want me to write *x* times *x* or *x*-squared?	PAH
	672	R1	*x*-squared.	
	673	Stephanie	*x*-squared, okay.	
	674	R1	Okay.	
	675	Stephanie	But here it would be *x* to the *y* power.	OBS
	676	R1	Let's think about that. What are you saying here? You're trying to guess what *x* times *y* is, right?	
	677	Stephanie	Yeah.	
	678	R1	So let's get a paper to conjecture. You can conjecture here.	
	679	Stephanie	Okay.	
	680	R1	How do you think you would write – what do you think it means '*x* times *y*'?	
	681	Stephanie	Well, it's an um *x* amount *y* number of times or *y* amount *x* number of times. It can go either way.	BMP
	682	R1	So. Well. Look at what you just wrote.	
	683	Stephanie	Um hm.	
	684	R1	Do you think that's a way to write it?	
	685	Stephanie	Well, yeah. You can write it like that. I'm just saying –	
	686	R1	Yeah. That's fine. I like it that way. Okay. *[Stephanie writes: $x^2 + x \cdot y + y \cdot x + y^2$]*	BMP
	687	R1	Okay. So you see why your other guess didn't work before? If what you're doing is right – there's your *x*-squared, there's your *y*-squared, but there's something else.	
	688	Stephanie	Yeah. I understand.	BDI
	689	R1	See that. What is that something else?	
	690	Stephanie	It's the *x* times the *y*.	BDI
	691	R1	Or – what's next?	
	692	Stephanie	Or the *y* times the *x*. Or –	BMP
	693	R1	Okay. So you have this *xy* and you have this *yx*, right?	
	694	Stephanie	Um hm.	
	695	R1	Can you simplify that?	

Time	Line	Speaker	Transcript	Code
	696	Stephanie	Yeah. I can get – Could I – Now if I added another *x* there, it could be *x* to the third, right? Could I do –	OBS
	697	R1	Now I'm confused. Let's think what you're doing here. So –	
	698	Stephanie	Alright. Because then – alright – it would be *x* plus *x* plus *x* plus – just so that it's easier for me - - *y* plus *y* plus *y*-squared. *[Stephanie writes:* $(x^2 + x + x) + (y + y + y^2)$ *]*	OBS
	699	R1	So you're conjecturing that this is the same as this?	
	700	Stephanie	Yeah. Because you're just putting all the –	
	701	R1	Let's try it with numbers and see if that makes sense – what you're conjecturing.	
	702	Stephanie	Alright.	
	703	R1	What does that mean?	
	704	Stephanie	That means like –	
	705	R1	Try some numbers. Try easy numbers. *[Stephanie writes:* $(2^2 + 2 + 2) + (3 + 3 + 3^2)$ *]*	PNE
	706	Stephanie	And that's two squared, that's four, plus two, six, eight, plus three, plus three, that's six, plus nine is fifteen. That works! *[she writes: 8 + 15]* No. It doesn't. That's twenty-three.	BMP
	707	R1	That gives you twenty-three	
	708	Stephanie	Yeah.	
	709	R1	So something isn't working here, huh?	
	710	Stephanie	No.	
	711	R1	So that might not be a valid step.	
	712	Stephanie	No.	
	713	R1	Okay. So. I'm kind of curious. What did you want to do with this thing here?	
	714	Stephanie	Well, because – well – when we add the um-	
	715	R1	You have *x*-squared plus *xy* plus *yx* plus *y*-squared.	
	716	Stephanie	It was just putting the terms together.	
	717	R1	What terms were you putting together?	
	718	Stephanie	Well, the *x* and the –oh. Is it that maybe I can't put the *x*'s with the *x*-squared, 'cause they're two different terms? Would that make a difference?	BDI; BMP
	719	R1	Okay. Where's the *x*?	
	720	Stephanie	Right here and here. [*points to the xy and yx*]	
	721	R1	But is this an *x*?	
	722	Stephanie	No. It's *x* times *y*, actually. (inaudible)	BDI

Time	Line	Speaker	Transcript	Code
	723	R1	(inaudible) Sure.	
	724	Stephanie	So this is (inaudible).	
	725	R1	(inaudible) change your mind in that one, huh? Okay. So this is *x*-squared plus, this is an *x*.	
45:00-47:47	726	Stephanie	Yes.	
	727	R1	And this is – tell me what this is?	
	728	Stephanie	*x* times *y* plus *x* times *y* – or – hm?	
	729	R1	What are you thinking?	
	730	Stephanie	I don't know.	
	731	R1	Tell me what you're thinking.	
	732	Stephanie	Well. Because I can't multiply *x* times *y*.	BMP
	733	R1	Okay. So right now, where we are –	
	734	Stephanie	Um hm.	
	735	R1	Get another piece of paper. We're going to have fun keeping track of (inaudible). Um – How much time do you have? Are you getting tired?	
	736	Stephanie	Oh. No. I'm fine.	
	737	R1	This is so much fun. *[Stephanie chuckles.]*	
	738	R1	Okay. Here we go. Um. This is what we're dealing with: *x* times *y*, right?	
	739	Stephanie	Um hm.	
	740	R1	Plus *y* times *x*. Right?	
	741	Stephanie	Yes. *[pause]* Oh, but I can't move them. I have to keep them – Can I just – 'cause when I tried to do do just *y* plus *y* last time it was like…	
	742	R1	Why don't why don't you try some numbers here – and write numbers here?	
	743	Stephanie	Easy numbers. *[writes 2·3 + 3·2]* six plus six. That's twelve.	PNE
	744	R1	So you were able to do them with numbers two and three.	
	745	Stephanie	Yeah.	
	746	R1	You were able to finally add them – numbers. I guess that's not helping too much. So this is –	
	747	Stephanie	Well, yeah, but there were parentheses around them here.	
	748	R1	Okay, but let's look at the two times three and three times two.	
	749	Stephanie	Okay.	
	750	R1	Look at those two terms: two times three and three times two.	

Time	Line	Speaker	Transcript	Code
	751	Stephanie	Um hm.	
	752	R1	Are they always going to be the same?	
	753	Stephanie	Two times three.	
	754	R1	And three times two.	
	755	Stephanie	And three times two. – Yeah.	BDI
	756	R1	Suppose you use five times six and six times five?	
	757	Stephanie	Yeah – 'cause it's the communative-	BDI, PPK
	758	R1	So *xy* is that always the same as *yx*?	
	759	Stephanie	Yeah – No – Wait – Yeah, 'cause it's the same thing.	BDI
	760	R1	You just – you just –used a big word. What was that word? You just used?	
	761	Stephanie	Oh. Communative.	
	762	R1	Commun – Commutative?	
	763	Stephanie	Commu – Yeah that one.	
	764	R1	Yeah. What's that mean?	
	765	Stephanie	It means for addition and multiplication, it doesn't matter the order.	BEJ
	766	R1	Okay so what are you doing here? This is *xy* and this is *yx*?	
	767	Stephanie	Well I just switched "em" around. I just switched the two numbers around.	BMP
	768	R1	And what op – what operation is involved with *xy* and *yx*?	
	769	Stephanie	Multiplication?	BMP
	770	R1	Multiplication. So you should be able to write *xy* as *yx*, or *yx* as *xy*.	
	771	Stephanie	Yeah.	
	772	R1	It's the same.	
	773	Stephanie	Yeah.	
	774	R1	If that's true and you're telling me that's true whenever you use a four and a five or an eight or a nine, or a million and a ten million.	
	775	Stephanie	Yeah now that's true.	
	776	R1	Okay, so if *xy* is the same as *yx*…	
	777	Stephanie	Um hm.	
	778	R1	Is there a way you can rewrite any one of these so you can write that as a simpler term?	
	779	Stephanie	*xy* – I don't – I mean I don't – I don't	OBS
	780	R1	Okay. You wrote *xy* plus *xy*.	
	781	Stephanie	Plus *xy*.	
	782	R1	Can you write that in a simpler way?	
	783	Stephanie	Two *x*.	

Time	Line	Speaker	Transcript	Code
	784	R1	Two *xy*.	
	785	Stephanie	Two *xy*?	BMP
	786	R1	Sure.	
	787	Stephanie	*[Stephanie writes.]* Or is it – is that	
	788	R1	Does that make sense?	
CD 2 00:00-04:59	789	Stephanie	-What we're doing? 'Cause – Yes! Two *x* plus times two *y*. Right? That would be the same thing.	BDI
	790	R1	Can we say that this is two *x* times two *y*? $2(x \cdot y)$	
	791	Stephanie	No. No. So would it be like $(x \cdot y)$ squared.	OBS
	792	R1	But - - - Let's go back to this and see what (inaudible). There are three interesting things here. So let's go back and think.	
	793	Stephanie	Okay.	
	794	R1	Alright. You're saying *xy*.	
	795	Stephanie	Um hm plus *xy*.	
	796	R1	Plus *xy*. You said that's a valid thing to do.	
	797	Stephanie	Yes.	
	798	R1	You can rewrite *yx*. Okay. And you said you can write that two parentheses *x* dot *y*.	
	799	Stephanie	Um hm.	
	800	R1	Right? Now. This is confusing you a little bit – this expression?	
	801	Stephanie	Yes.	
	802	R1	Right?	
	803	Stephanie	Well it – it's saying two *x* times two *y*, though.	OBS
	804	R1	Well try it. Is two *xy* the same as two *x* times two *y*? -Try some numbers –	
	805	Stephanie	Oh. Well okay.	
	806	R1	Do you do that when you're not sure if something's allowed? Do you try some numbers?	
	807	Stephanie	Sometimes. Most of the time we're just we're not dealing with like problems where we know – we don't know (inaudible).	
	808	R1	You will. You will. That's yet to come. *[Stephanie writes:* $2 \cdot 2 \cdot 2 \cdot 3$ $4 \cdot 6$ *24]*	PNE
	809	Stephanie	Two times three, that's six times four. That's twenty-four.	PNE
	810	R1	Hm.	
	811	Stephanie	And the other way, it would just be be um two times two times three times three, but that's	BMP

Time	Line	Speaker	Transcript	Code
			different. That's not (inaudible). That's nine times four and that's thirty-six.	
	812	R1	Um, what is - ?	
	813	Stephanie	Something is wrong.	BDI
	814	R1	Okay. What is it this way?	
	815	Stephanie	Oh.	
	816	R1	You know this way is right.	
	817	Stephanie	Well, yeah. So (inaudible) twelve.	
	818	R1	Okay. So this is tw-	
	819	Stephanie	Oh. Boy.	
	820	R1	So what's happening?	
	821	Stephanie	Well, neither of them are right then! – 'cause I mean – this is –	OBS
	822	R1	Well, this one isn't *[points to 2x · 2y]* Right?	
	823	Stephanie	No.	
	824	R1	And this one isn't *[can't see which one R1 points to]*. But what about this one? *[Stephanie writes: 2(2·3) 4·6]*	
	825	Stephanie	Twenty-four.	PNE
	826	R1	Well. I'm trying to see how you got twenty-four.	
	827	Stephanie	Oh. Wait.	
	828	R1	I'm doing the two and the three first.	
	829	Stephanie	Oh.	
	830	R1	'Cause it's in the parentheses.	
	831	Stephanie	Yeah. Yeah. That works. Okay.	
	832	R1	Do you see what you were just doing?	
	833	Stephanie	Yeah.	
	834	R1	What you were just doing? It's common – lots of people –	
	835	Stephanie	I was just – 'cause I always do it like that 'cause I'm used to having like a variable and when you have a variable in there you can't – you have to distribute first. So I'm used to distributing first.	BEJ
	836	R1	So this isn't – Is this the Distributive Law? Two *xy*? I mean – do you need that dot? - *x* times *y*. Can you write that as *xy*? Have you had that yet?	
	837	Stephanie	No.	
	838	R1	If you have *x* dot *y*. You have dots. Have you used dots?	
	839	Stephanie	Yes. We used to use dots.	PPK
	840	R1	So like if I write two dot *x* – you would know that's two *x*, right?	

Time	Line	Speaker	Transcript	Code
	841	Stephanie	Yeah.	
	842	R1	But if I write *x* dot y would you know that's *xy*?	
	843	Stephanie	*x* times – yeah.	
	844	R1	So I didn't need that dot, maybe, huh?	
	845	Stephanie	Um. Because –	
	846	R1	Do I need the parentheses?	
	847	Stephanie	I don't – Maybe not – I don't know – Because –	OBS
	848	R1	See I always go back to basic meaning. What helps me – you see you have an *xy* and an *xy*. So I think you have an *xy*, you have another *xy*. You have one of them here and one of them here. So you have –	
	849	Stephanie	That's two *xy*.	BDI; BMP
	850	R1	That's two of them.	
	851	Stephanie	Okay. Yeah.	
	852	R1	I like to think of basic meaning and simplicity. That's what helps me. Do you see what I'm saying?	
	853	Stephanie	Yeah.	
	854	R1	So when I'm confused, I think, 'Now. Wait a minute. All this language is just getting too confusing. Let's try to make it simple. What does it mean?' I keep saying – 'What does it mean?' Go back to basic meaning. So you see in this lovely thing you did here – this thing here looks very complicated. *x* times *y* plus *y* times *x*. Right? But as soon as you recognize that's *xy* and that's *yx* – as soon as you recognize commutative, (that was very nice), that you can think of that as *xy* and *xy*. One of them and another one. And that's a nice way to think about it.	
	855	Stephanie	Yeah.	
	856	R1	Okay. So what did we end up – What's the sim - What's another way of rewriting this in its simplest way? Can you – that's what we started with.	
	857	Stephanie	Well. Okay. Now we didn't eliminate these.	
	858	R1	See what you believe any more.	
	859	Stephanie	*x* to the second.	
	860	R1	You believe that still. Okay.	
	861	Stephanie	But what we got was plus 2 *xy* plus *y* to the second.	BMP

Time	Line	Speaker	Transcript	Code
	862	R1	Okay.	
	863	Stephanie	That's what we have now.	BDI
	864	R1	You believe that? Try some numbers and test it.	
05:00-09:59	865	Stephanie	Okay. Two to the second plus two (inaudible) four plus twelve plus nine. It worked.	PNE
	866	R1	You like that, huh? Isn't that wonderful?	
	867	Stephanie	It looks a lot easier than this.	
	868	R1	Um. Okay. So you believe that this is the same as this? *[indicates:* $(x+y)^2 = x^2 + 2xy + y^2$ *]*	
	869	Stephanie	Yes.	
	870	R1	And we could keep testing lots of numbers?	
	871	Stephanie	Um hm.	
	872	R1	Actually, you've proved it. What you've just done is gone through a proof. What you've done here is is your proof is based upon you know the meaning of these things. If you think about what you've done. I'd really like you to go back and think about this when you go home. What the meaning is. When you talk about your Distributive Law. What does it mean with numbers? And then what does it mean if you don't have numbers.	
	873	Stephanie	Um hm.	
	874	R1	And if you think about the evolution of it. And now you've even made it more complicated. I could make this more complicated. The one I could leave you with is this. Suppose you had *[(x + y)(x + y + z)]* *x* plus *y*, *x* plus *y* plus *z* times. Could you come up with a general expression to show me?	
	875	Stephanie	Well, it's this *[* $x^2 + 2xy + y^2$ *]* with an extra –	
	876	R1	So you see here – look what you did here? You made it simple. *x* plus *y* times *x* plus *y*. You said was – you have *x* is your distributive law *x* plus *y* plus *y* *x* plus *y*.	
	877	Stephanie	Um hm.	
	878	R1	And then you simplified it a little bit more. But now I'm saying you'll have *x* plus *y* times *x* plus *y* plus *z*. Can you just tell me what you think this means?	
	879	Stephanie	What I think it means? -Boy. I think it means *x* plus *y* *x* plus *y* plus *z* amount of times. Or –	BMP; BCA
	880	R1	Or.	
	881	Stephanie	Or *x* plus *y* plus *z* *x* plus *y* amount of times?	BMP; BCA

Time	Line	Speaker	Transcript	Code
	882	R1	Um hm right. Now suppose I didn't want to think of something x plus y plus z amount of times? Can I think about it in simpler ways?	
	883	Stephanie	*[Stephanie inaudible]*	
	884	R1	I don't want thinking about something x plus y plus z amount of times is hard for me.	
	885	Stephanie	Um.	
	886	R1	I like to break it up into simple ways of thinking about it.	
	887	Stephanie	Alright. I guess you could make it x times – well – it's basically – it's still the same – I think it still is.	BEJ
	888	R1	Go ahead.	
	889	Stephanie	x times x plus y plus z <u>plus</u> y times x plus y plus z.	BMP; BCA
	890	R1	Um hm, yeah. And if I put an "r" and "w" and a "t" –	
	891	Stephanie	Yeah, it would just be the same thing.	BCA
	892	R1	Okay, really neat. So um what I was trying to do and now it's four o'clock and we probably have to stop soon, don't we? Um. I wanted, you know, to see how you thought about these ideas. But what I would really like to do, if you're interested, is some of the things you've been doing at Harding School for the last few years.	
	893	Stephanie	Um hm.	
	894	R1	Um. That you've been thinking about deeply – um – as problem solving. What I've tried to work with you with to see what you think about these things algebraically. I'd like to see if you could take some of those ideas that you've thought a lot about –	
	895	Stephanie	And?	
	896	R1	and deal with them.	
	897	Stephanie	Alright.	
	898	R1	And in order to do that we need some of this notation.	
	899	Stephanie	Alright.	
	900	R1	You see um –	
	901	Stephanie	Okay.	
	902	R1	So I think we probably skipped a lot of stuff here. (I don't know how this quite works.) But I think you have a really good understanding of what you're doing. If you think about it. And test it. Does anyone in the audience have anything to ask Stephanie?	

Time	Line	Speaker	Transcript	Code

APPENDIX B: TRANSCRIPT – SESSION 2

INTERVIEW WITH STEPHANIE Time: 91 minutes (2 CDs)
January 29, 1996

R1: Dr. Carolyn Maher Stephanie: Stephanie R2: Dr. Ethel Muter
R3: Dr. Alice Alston R4: Dr. Elena Steencken

Time	Line	Speaker	Transcript	Code
00:00 –04:59	1	Stephanie	Well. I don't know. I just have to like register on February 3rd – like at my high school. At – if I choose	
	2	R1	Oh. Really.	
	3	Stephanie	a Catholic high school. They register so early. So if I want to go to um Union Catholic or Roselle Catholic or Mother Seton or Benedictine Academy…	
	4	R1	Oh. It's Mother Seton where	
	5	Stephanie	But I'm not go-	
	6	R1	Amy's sister teaches, right?	
	7	Stephanie	Yeah. I don't think I'm going to Mother Seton.	
	8	R1	That's not your choice.	
	9	Stephanie	Un uh. I don't want to go to an all girls school.	
	10	R1	Oh, that's an all girl school. And Union Catholic's coed.	
	11	Stephanie	Union Catholic and Roselle Catholic are coed. Benedictine and um	
	12	R1	That's all girls.	
	13	Stephanie	And Mother Seton are all girls. So I don't think I'm	
	14	R1	So you like Union Catholic?	
	15	Stephanie	Um. I haven't been there like on a- I haven't gone there yet for like 'Student for a Day'. I have gone there	
	16	R1	You need to do that.	
	17	Stephanie	twice to take tests. I've gone there twice to take tests and everyone seems – they're all like – the kids I've seen –	
	18	R1	Um hm.	
	19	Stephanie	and stuff. They're really nice.	
	20	R1	You might want to go for a day.	
	21	Stephanie	Yeah. I think we're going to do it this week,	

Time	Line	Speaker	Transcript	Code
			like Thursday or something.	
	22	R1	Right.	
	23	Stephanie	Because I mean	
	24	R1	If you could sit in some of the classes that would help.	
	25	Stephanie	Yeah. I also know some kids that have like that go there, but they're like juniors and stuff and they say that-	
	26	R1	Do they like it?	
	27	Stephanie	Oh, yeah. They like it 'cause it was like their alternative choice to from instead of Burly. I mean they – they in- *[camera person conversation]*	
	28	R2	Do they all have geometry for freshman?	
	29	Stephanie	Yeah. They have geometry, geometry honors – like Benedictine didn't have that for freshmen.	
	30	R1	That's your paper, Stephanie. [*Stephanie chuckles.*] Um. Ethel's daughter is going to be a freshman next year. Is that right?	
	31	R2	Yeah. They're also registering now.	
	32	R1	They are?	
	33	R2	For courses at the high school. She had to sign up for um the classes that she wanted and she had to apply for honors courses.	
	34	R1	Well, interesting. Well. Do you remember what we did last time?	
	35	Stephanie	Something with x and y. It was like grouping them. It was something like x to the y or something. I don't remember like exactly.	
	36	R1	Um hm.	
	37	Stephanie	But I know it had to with changing around like the way it was placed. It was x to the y plus like x to the y. It could be just be like – x y like parentheses or something. I don't remember – like exactly.	PPK
	38	R1	Um hm.	
	39	Stephanie	But it was something like that.	
	40	R1	Okay. – Um. – Let's see. Maybe you can rebuild it. Okay? Um. [*takes paper and pen. Writes* $(a + b)^2$] Do you remember what that means?	
	41	Stephanie	Um. I – this is yeah and didn't we distribute it so that it was like [*writes* $a^2 + b^2$]?	BR-S; OBS

Time	Line	Speaker	Transcript	Code
	42	R1	Okay. Do you want to test it? [*Stephanie makes a noise*.] Tell me what it means and test it.	
	43	Stephanie	[*Stephanie writes* $a \cdot a + b \cdot b$ *; puts down pen*] or like two *a* plus two *b*.	OBS
	44	R1	Well. Let's let's try some things. Um. Pick something for *a* and pick something for *b* and	
	45	Stephanie	Okay.	
	46	R1	test it.	
	47	Stephanie	[*Stephanie writes* $2 \cdot 2 + 3 \cdot 3$; *under* $3 \cdot 3$ *she writes 9, brings down the + and under the* $2 \cdot 2$ *she writes 4. She follows the 4 + 9 expression with + 13*] Now do you want me to …?	PNE
	48	R1	Okay. So tell me what you did.	
	49	Stephanie	Well	
	50	R1	What were you testing?	
	51	Stephanie	This. [*points the pen at* $a^2 + b^2$] Like, oh. Wait – should I do it this way too? That would be [*writes 2 above the a in* $(a + b)^2$ *and 3 above the b*] – six. Seven. That's twelve. – that's one less. *[writes 12 to the right of* $(a + b)^2$*]*	OBS
	52	R1	Now tell me what you just did.	
	53	Stephanie	Well. Um. Like from the start? Or what I was testing?	PAH
	54	R1	Well. Anything you think you want to tell me.	
	55	Stephanie	All right. Well. Um. I put distributed – well you gave me that and I distributed the um *z*, I guess, to um *a* and *b*.	BEJ
	56	R1	This is a two. [*points to the square of* $(a + b)^2$]	
	57	Stephanie	Oh. That's a two. The two to *a* and *b* and then um you told me to like work it out, so it would be a times *a* plus *b* times *b*. And then it was, you told me to put in numbers. Two times two plus three times three.	BEJ; BMP
	58	R1	Okay. I'm confused now. What number is that? [*points to the 12*]	
	59	Stephanie	Twelve.	
	60	R1	And what number's that? [*points to the 13*]	
	61	Stephanie	Oh! Wait! That's five. [*crosses out the 12 and writes 5*]	BDI
	62	R1	And how did you get five?	
	63	Stephanie	Well, because two plus three is five. – And then it's five times five makes twenty-five. [*writes 25 below the crossed out 12*]	BEJ; BMP

Time	Line	Speaker	Transcript	Code
	64	R1	So what's twenty-five?	
	65	Stephanie	This. [*draws a line around* $(a + b)^2$] Like if you distribute um if you put two and three in here.	BEJ
	66	R1	So – you're putting, why are you putting the two and the three in there? Tell me again.	
	67	Stephanie	'Cause you asked me to put numbers in-	BEJ
	68	R1	So	
	69	Stephanie	-in place of the letters	BEJ
	70	R1	So so what so the two is being used for	
	71	Stephanie	*a* and the three is *b*.	BEJ
	72	R1	Three is for *b*. And when you did that you have	
	73	Stephanie	Um. Well, this	
	74	R1	This to be twenty-five.	
	75	Stephanie	Turns out to be five and then five squared is	BMP
	76	R1	Okay. And when you did it, when you, you said, what is this? [*points to* $a \cdot a + b \cdot b$ *that Stephanie wrote earlier*]	
	77	Stephanie	Oh. You told me um well you said 'what is this?' [*the* $a^2 + b^2$] and I said that it would be like *a* squared plus *b* squared. Obviously, it's not.	BDI
	78	R1	Ah ha.	
	79	Stephanie	Because it doesn't work out.	
	80	R1	Okay. So. So then in in your testing it	
	81	Stephanie	[*Stephanie chuckles.*]	
05:00 –09:59	82	R1	Your conjecture	
	83	Stephanie	Yeah.	
	84	R1	that *a* plus *b* in parentheses	
	85	Stephanie	Um hm.	
	86	R1	that quantity squared is not the same as *a* squared plus *b* squared. You've just proved it's not.	
	87	Stephanie	Yes.	
	88	R1	By counter-example, haven't you? That's sort of a proof.	
	89	Stephanie	Yeah.	
	90	R1	So, so why don't you write down what you just said- that *a* that this [$(a + b)^2$] is not equal to this [*points in the vicinity of* $a^2 + b^2$] is something you just found. Why don't you write out what you just discovered?	

Time	Line	Speaker	Transcript	Code
	91	Stephanie	So like [*pause*] is not equal to um [*writing*]	BR-S
	92	R1	Would you have to test something else to prove it's not equal? If if you show it doesn't work once is that- is that okay?	
	93	Stephanie	Well, yeah. Because if it doesn't work once then it can't like be true.	BCA; BEJ
	94	R1	Okay. So so you proved in essence then that this is not true. So the question was, I go back to my original question.	
	95	Stephanie	[*chuckling*] What is that?	
	96	R1	What is it, right?	
	97	Stephanie	Yeah.	
	98	R1	Okay. So I'll let you struggle a little bit and think about that.	
	99	Stephanie	Um.	
	100	R1	That about- you know- what it means. Think about meaning.	
	101	Stephanie	[*Stephanie inaudible*]	
	102	R1	And maybe maybe what might help you – think about what what you know about meaning in the simplest way, to think about what this could be in meaning. What does *a* plus *b*, that quantity squared, mean?	
	103	Stephanie	It means that you [*chuckles*] it means like – well – I	
	104	R1	What does something squared mean?	
	105	Stephanie	It means that	
	106	R1	Try something.	
	107	Stephanie	you're multiplying it by itself.	BMP
	108	R1	Oh. Okay. So what is being	
	109	Stephanie	*a* plus *b*.	
	110	R1	So so tell me what you just – let's number these pages. Because I know what will happen. This is number one and today's date is the twenty-ninth.	
	111	Stephanie	Twenty-ninth.	
	112	R1	Okay. This is for my benefit.	
	113	Stephanie	Um hm.	
	114	R1	'Cause I – This is what we know. So this - you can be numbering them now. Um. So so you know what *a* plus *b* quantity squared means.	
	115	Stephanie	Yeah.	
	116	R1	So moving from meaning	
	117	Stephanie	Oh. What does it like	

Time	Line	Speaker	Transcript	Code
	118	R1	So write down what you think it means. You know what *a* squared means. You clearly know what *a* squared means.	
	119	Stephanie	Well, yeah.	
	120	R1	You believe that *a* squared, if *a* is two, is the same as two times two?	
	121	Stephanie	Yes.	
	122	R1	You know that. Right? And *b* squared here is the same as three times three. That you believe?	
	123	Stephanie	Yes.	
	124	R1	Okay. So what does *a* plus *b*, that quantity squared, what does that mean?	
	125	Stephanie	*a* plus *b* times *a* plus *b*?	BCA; BMP
	126	R1	So why don't you write that down? What that means: *a* plus *b* quantity squared. [*pause*] Okay.	
	127	Stephanie	Oh! Okay.	BR-S
	128	R1	Right?	
	129	Stephanie	This is this is what we did last time (inaudible).	PPK
	130	R1	I don't know. Does it look familiar to you?	
	131	Stephanie	Yeah, but we used *x* and *y*.	
	132	R1	Oh! Does it matter?	
	133	Stephanie	No.	BMP
	134	R1	Okay. Could we use *w* and *r*?	
	135	Stephanie	Yeah.	
	136	R1	Do you prefer to use *x* and *y*?	
	137	Stephanie	No. This is fine. [*chuckling]*	
	138	R1	Is *a* and *b* okay? Okay. I didn't really do that deliberately to throw you off.	
	139	Stephanie	No. I just – that's what I remembered.	PPK
	140	R1	Okay. So. Uh. It might be useful, um, Stephanie - - to write down that this *[*$(a + b)(a + b)$*]* equals this thing [*it appears that the researcher is pointing to the* $(a + b)^2$] or you know – not to lose sight of what this is supposed to represent.	
	141	Stephanie	Oh.	
	142	R1	You know what I'm saying. As a as a whole sentence. Because that you absolutely believe, right?	
	143	Stephanie	Um hm.	

Time	Line	Speaker	Transcript	Code
	144	R1	You believe that?	
	145	Stephanie	Yes.	
	146	R1	And why do you believe that? Why is that true?	
	147	Stephanie	Because um when you square something it's like multiplying it by like itself? And so it would be like *a* plus *b* times *a* plus *b*.	BMP; BEJ
	148	R1	Okay. So. Um. Here you have squared.	
	149	Stephanie	Um hm.	
	150	R1	And you have two factors of what you're squaring. You have *a* plus *b* as a factor two times. Right?	
	151	Stephanie	Um hm.	
	152	R1	'Cause it's squared.	
	153	Stephanie	Yes.	
	154	R1	And if I had a three here? [*indicates the exponent*]	
	155	Stephanie	You'd do it three times.	BMP
	156	R1	What would you do three times?	
	157	Stephanie	*a* plus *b* times *a* plus *b* times *a* plus *b*.	BMP; BEJ
	158	R1	times *a* plus *b* [*simultaneously with Stephanie's last 'a plus b'*] Okay. And you get twenty-five times?	
	159	Stephanie	It would be *a* plus *b* twenty-five times. Like times *a* plus *b*.	BMP
	160	R1	We we're going to get there – twenty-five times. That's sort of our vision.	
	161	Stephanie	Okay.	
	162	R1	All right. But I want you to build this for yourself so that um if we get to come together next time and I ask you what *a* plus *b* quantity squared, you may not remember what that is, but you're going to tell me what it's not.	
	163	Stephanie	Okay.	
	164	R1	Because I know your teacher's sitting there. I can't tell you – every year she must have students over and over and over again, even if she stands on her head and does dances and all kinds of things – they're going to tell her it's *a* squared plus *b* squared. Isn't that right? Over and over again. So what is it that entices students to want to do this?	
	165	Stephanie	Distributive Property. And even if it doesn't say to distribute, that's the first thing that	BEJ

Time	Line	Speaker	Transcript	Code
			everybody thinks.	
	166	R1	Okay. So intuitively it looks like	
	167	Stephanie	Yeah. I mean 'cause if you just look at it and you're not putting thought into it, you're just going	BEJ
	168	R1	If you look at it and not put thought in it	
	169	Stephanie	say if there's something outside the parentheses, you distribute.	BEJ
	170	R1	Okay. So you're thinking that um	
	171	Stephanie	that's just (inaudible)	
	172	R1	raising to a power is is but we're finding out	
	173	Stephanie	Yeah.	
	174	R1	that's a counter example of	
	175	Stephanie	Yeah.	
	176	R1	of that. So	
	177	Stephanie	It's like – it's just like if you're just looking at it that's what most people	
	178	R1	Um hm.	
	179	Stephanie	that don't know would just	
	180	R1	Um hm.	
	181	Stephanie	that's what you think – 'cause	
	182	R1	Right.	
	183	Stephanie	you just -	
	184	R1	Right.	
	185	Stephanie	think that.	
10:00 –14:59	186	R1	Hm. That's interesting. Um what what I would sort of like you to think about for next time is: Does this ever work? Is ever *a* plus *b* quantity squared equal to *a* squared plus *b* squared? Is there any special case when that could possibly be true? I don't want you to do that now. But if you could write yourself a little note of things to think about. You you've shown shown me that, by a counter example, this [*indicates* $(a+b)^2 \neq a^2 + b^2$] is not a true statement.	
	187	Stephanie	Um hm.	
	188	R1	Is that right?	
	189	Stephanie	Yes.	
	190	R1	And you can't expect to generalize this.	
	191	Stephanie	But you want to know is it always	PAH
	192	R1	Is it – could you ever?	
	193	Stephanie	not true.	PAH
	194	R1	Can you ever think of a situation when it might	

Time	Line	Speaker	Transcript	Code
			be true in some special case? And then you want to think about that.	
	195	Stephanie	Okay.	
	196	R1	What what makes that special case, if it exists?	
	197	Stephanie	Well, what about zero?	BDI
	198	R1	What about zero? Would it work for zero?	
	199	Stephanie	Yeah. You can't square zero.	BEJ
	200	R1	Well, what happens, what what do you mean by zero? What are you thinking of as zero?	
	201	Stephanie	Well –	
	202	R1	What are you making zero?	
	203	Stephanie	Oh. Well then *a* would have to be equal to *b*.	BEJ; BR-S
	204	R1	And and what would *a* and *b* be equal to?	
	205	Stephanie	Zero.	BR-S
	206	R1	Zero. Okay. And and if that were the case, what would you get? Why can't you square zero? What does zero squared mean?	
	207	Stephanie	It means like zero times zero.	BMP
	208	R1	Which is?	
	209	Stephanie	Zero.	BMP
	210	R1	Okay.	
	211	Stephanie	So – if *a* were equal to *b*	
	212	R1	Which would equal zero.	
	213	Stephanie	Yes.	
	214	R1	So write that down: 'If *a* equaled *b* equaled zero' [*Stephanie writes: $a = b = 0$*] Okay. Okay. So	
	215	Stephanie	Then like should I write?	
	216	R1	Then write 'then'. If, write the word 'if' so you can follow your reasoning.	
	217	Stephanie	Oh. Like 'if' here?	
	218	R1	'If this is true', right? Then what will happen?	
	219	Stephanie	Then *a* plus *b* [*writes* $(a+b)^2 = a^2 + b^2$]	BR-S
	220	R1	And *a* plus *b* quantity squared would equal it. And what would they both equal?	
	221	Stephanie	Zero.	BMP
	222	R1	Zero.	
	223	Stephanie	So. I can't fit that in there.	
	224	R1	So. Isn't that interesting? What about one?	
	225	Stephanie	Well it would equal one, wouldn't it?	OBS
	226	R1	Does it work if *a* and *b* were both equal to one?	
	227	Stephanie	Yeah. Wouldn't – no!	BDI

Time	Line	Speaker	Transcript	Code
	228	R1	Why not?	
	229	Stephanie	Well – yeah. Yeah. Because – that would be two – no it wouldn't.	BEJ
	230	R1	Why not?	
	231	Stephanie	Because this [*the* $(a + b)^2$] would equal four	PNE
	232	R1	Um hm.	
	233	Stephanie	and this [*the* $a^2 + b^2$] would equal two.	PNE; BDI
	234	R1	And four doesn't equal two. Okay. So it's interesting, isn't it?	
	235	Stephanie	Yeah.	
	236	R1	So it's something to think about. But to go back here you by by this isn't by definition of what it means to raise something to a power. You know this is true. [*indicates* $(a+b)\cdot(a+b)=(a+b)^2$]	
	237	Stephanie	Um hm.	
	238	R1	Right? But is – how can you express this not as a product – right? We have a product.	
	239	Stephanie	Um hm.	
	240	R1	Right? By the way, if you – you may have run across this term in school and you may not have. But if you have two terms that are being added together, do you know what you call that?	
	241	Stephanie	Um. Oh. Gosh. Not (inaudible). We just did this.	
	242	R1	This is Greek. This really is Greek. When students say you know algebra is Greek to them, some of the terms come from –	
	243	Stephanie	Um hm.	
	244	R1	-come from some Greek prefixes. Uh. The Greek prefix 'bi' means two.	
	245	Stephanie	Um hm.	
	246	R1	So it's a binomial. There are two of them. Did you ever hear of that?	
	247	Stephanie	I've heard of it – (inaudible).	
	248	R1	Okay. If you if you have one 'mo' monomial. If you had just one like x or	
	249	Stephanie	or um	
	250	R1	y	
	251	Stephanie	we're doing	
	252	R1	if you have two it's 'bi' so you have a binomial times a binomial. That's what you're	

Time	Line	Speaker	Transcript	Code
			doing here and that, or a binomial squared, which is not the sum of the squares. We know it's not. You've just shown me that. Okay? That was your, what you wrote here.	
	253	Stephanie	Um hm.	
	254	R1	So the question is – think of meaning again: What does it mean to multiply *a* plus *b* times (this is an *a* plus *b*) times *a* plus *b*? What does that mean?	
	255	Stephanie	But it means to like square it.	BMP
	256	R1	Right.	
	257	Stephanie	But I don't – like I can't think of another way to put it.	OBS
	258	R1	Um hm. So. Um. Well, there are a couple of ways directions to go. One direction we went last time was to think of of this as um an area problem.	
	259	Stephanie	Um hm.	
	260	R1	You know, if I asked you to represent *a* squared.	
	261	Stephanie	With the you mean with the box that we did last time?	PPK
	262	R1	Yeah. How would you represent *a* squared? Let's get another piece of paper. Can you draw me a picture of what *a* squared could be?	
	263	Stephanie	Um. Do you want it to represent like one side of the – 'cause that, I'm trying to think how we did it?	PAH
	264	R1	Does anything come to your mind when you say *a* squared?	
	265	Stephanie	Just well *a* times *a*.	BMP
	266	R1	All right. That's true. But can you think of in geometry, what that might represent? [*pause*]	
	267	Stephanie	Not like – I don't know like what you mean.	PAH
	268	R1	What I'm what I'm fishing for? Let me be more direct than that. Okay?	
	269	Stephanie	Yeah.	
	270	R1	If that were a square,	
15:00-19:59	271	Stephanie	Yeah.	
	272	R1	Right? And this side had length *a*.	
	273	Stephanie	Um hm.	
	274	R1	And this side had length *a*.	
	275	Stephanie	Um hm.	

Time	Line	Speaker	Transcript	Code
	276	R1	If you were finding the area of a square? Remember?	
	277	Stephanie	Um.	
	278	R1	How do you find area of a square?	
	279	Stephanie	Multiply the two sides.	BMP
	280	R1	Length times width. Right?	
	281	Stephanie	Um hm.	
	282	R1	In this case or side squared? So if one side is *a*, right?	
	283	Stephanie	So it would be	
	284	R1	And the other side is *a*, so the area is?	
	285	Stephanie	*a* squared.	BMP
	286	R1	*a* squared, right? Remember that?	
	287	Stephanie	Um hm.	
	288	R1	So when you were in lower grades, you'd be finding area where you had, find the area of square of side, when the length of a side maybe is 5 units.	
	289	Stephanie	Um hm.	
	290	R1	So what would the area of that square be?	
	291	Stephanie	Twenty-five.	BMP
	292	R1	Twenty-five square units.	
	293	Stephanie	Um hm.	
	294	R1	All right? Does that make sense?	
	295	Stephanie	Yeah.	
	296	R1	Uh. I wonder why that works? What that what that means?	
	297	Stephanie	Like why *a* like length times width works? Or?	PAH
	298	R1	Well, I wonder if um if I didn't have an *a*. Suppose I made a three, right?	
	299	Stephanie	Um hm.	
	300	R1	Okay. One, two, three. [*marks off three intervals on the sides of a square*] This is – can you imagine these being the same size?	
	301	Stephanie	Okay, so all	
	302	R1	So this length of this side is three units.	
	303	Stephanie	All the little sections are	
	304	R1	This is three units, right?	
	305	Stephanie	In each one is one? Like each of the little sections is one?	BR-V
	306	R1	Yeah. Can you tell me what I mean when I talk about the area? What's the area of that square?	

Time	Line	Speaker	Transcript	Code
	307	Stephanie	Um. Isn't that-	
	308	R1	If this side is three units and this side is three units?	
	309	Stephanie	Nine?	BMP
	310	R1	Nine what?	
	311	Stephanie	Nine square.	
	312	R1	Can you draw me a picture of that? To show that? Nine, you told me, nine square units. So show me those nine square units.	
	313	Stephanie	Um. Like if each one of these – oh! You want me to [*draws two verticals and then the two horizontal lines which divide the square into nine square units*]	BR-V
	314	R1	So what's the area?	
	315	Stephanie	Nine square units.	
	316	R1	What's a square unit?	
	317	Stephanie	One of these little squares.	
	318	R1	Okay. And that little square, right? See that little square there? [*colors the top left unit square blue*]	
	319	Stephanie	Um hm.	
	320	R1	What is the length of one of its sides?	
	321	Stephanie	One?	
	322	R1	One. So you see, this is really a square unit. It has one, one. It's a one by one square and look how many of them are in here. There are nine of them.	
	323	Stephanie	Um hm.	
	324	R1	Right? So that square has area nine square units. So – if we were thinking about *a* squared,	
	325	Stephanie	Um.	
	326	R1	How does – what does that have to do with this? It looks like a nine. [*indicates the a label on the left side of the square*] Maybe an *x* would have been better.	
	327	Stephanie	You want me to show you *a* squared? Or?	
	328	R1	Yeah.	
	329	Stephanie	But you have it, like here.	
	330	R1	Yeah. What would it look like in the picture? [*pause*]	
	331	Stephanie	[*noise*] Um. [*pause*] I	
	332	R1	It's a big leap, isn't it?	
	333	Stephanie	I don't know, 'cause there's no like number to	OBS

Time	Line	Speaker	Transcript	Code
			work.	
	334	R1	Yeah. Right. So.	
	335	Stephanie	I can't draw anything 'cause there's no no number to like separate any thing with or to like square it off in like little	OBS
	336	R1	Hm.	
	337	Stephanie	sections, you know?	OBS
	338	R1	So if I gave you a number would you be able to do it? Pick a number. And do it.	
	339	Stephanie	Well, if it was like four, right?	PNE
	340	R1	Hm.	
	341	Stephanie	And I could divide it each into four parts,	BEJ
	342	R1	Um hm. Um hm.	
	343	Stephanie	then I could show you	BEJ
	344	R1	Um hm.	
	345	Stephanie	like what four squared looked like.	BEJ
	346	R1	Um hm.	
	347	Stephanie	But because *a* has no number	BEJ
	348	R1	Um hm.	
	349	Stephanie	I can't just like make *a*, like you, 'cause you're asking me what *a* is.	OBS; BEJ
	350	R1	Um hm.	
	351	Stephanie	You're not asking me what like four is.	BEJ
	352	R1	Um hm.	
	353	Stephanie	And I can't just like materialize like *a* is this	OBS; BEJ
	354	R1	Hm.	
	355	Stephanie	is like this	BEJ
	356	R1	Yeah.	
	357	Stephanie	extra number or something.	BEJ
	358	R1	That's the same problem here, isn't it?	
	359	Stephanie	It has parts.	
	360	R1	It's sort of the same problem. You're dealing with these these letters here. Right? In the sense, when you have an *a*, it's not a two. Or it's not a three.	
	361	Stephanie	Um hm.	
	362	R1	Or it's not a five or a seven or a half or a third or whatever? Right?	
	363	Stephanie	Um hm.	
	364	R1	It's gotta stand for whatever you want it to be. Isn't that right?	
	365	Stephanie	Yeah.	
	366	R1	So, um, so what are you supposed to be	

Time	Line	Speaker	Transcript	Code
			thinking about to help you make the transition from making this a particular number and thinking about it as an *a*?	
	367	Stephanie	I could – I – I –I understand what you want me to do.	
	368	R1	I don't know. I don't even know what I want you to do, so	
	369	Stephanie	You asked me to show you – like you said could you show me like like cubed like if there were three parts and and I can. But because *a* isn't a number, I can't show you like what *a* squared would be?	
	370	R1	What is the *a*, if it's not a number?	
	371	Stephanie	It's a variable.	BMP
	372	R1	What does that mean?	
	373	Stephanie	It means like a letter that repre- that represents like like all or like any number 'cause it doesn't have one.	BEJ; BMP
	374	R1	Um hm.	
	375	Stephanie	a number.	
	376	R1	Um hm.	
	377	Stephanie	So I can't like show you.	
	378	R1	Hm. How do we handle that one, Ethel?	
	379	R2	That's a tough one. Can you maybe use like three dots or something to imagine that there's space in between a beginning point and an ending point?	
	380	Stephanie	But I don't like- if you want me to uh- but how far apart do you want me to make the dots? How many sections do you want me to make?	PAH
	381	R1	What do you think?	
	382	Stephanie	I don't know. 'Cause *a* isn't a number.	OBS
	383	R1	Is there a way you can try to draw a picture that kinda would work for some numbers that people might be thinking of in this room? That will work if I'm thinking about *a* to be maybe five; and Ethel thinking of *a* to be three; and Mrs. Colosimo is thinking of *a* to be two? And	
	384	Stephanie	Well	
	385	R1	and Dr. Alston's thinking of a to be twenty-seven? She'll always do that. Or a half?	
	386	Stephanie	I mean I can show, I can, I can show you if you give me a number. But I can't just like show you what *a* is.	OBS

Time	Line	Speaker	Transcript	Code
20:00-24:59	387	R1	Um hm.	
	388	Stephanie	'Cause I don't know.	
	389	R1	(inaudible) I see all these pictures in these books and they have here's a square of length *a* and here's a rectangle: this piece is *a* and this piece is *b*.	
	390	Stephanie	Um hm.	
	391	R1	And I'm trying to imagine what am I supposed to	
	392	Stephanie	I mean if you just say that way	
	393	R1	What am I supposed to keep in my head when I see these things?	
	394	Stephanie	I can't section *a* off. If this whole thing is *a*, then there's four parts and that's it. I can't like section *a* off.	OBS
	395	R1	Hm.	
	396	Stephanie	You know?	
	397	R1	Hm. Well, the whole thing is *a*.	
	398	Stephanie	I know. I can't like make it into little	
	399	R1	Um hm.	
	400	Stephanie	I I can make it into little parts, but it's not what *a* is.	
	401	R1	What do you do with your students when you're trying to show them a diagram of *a*? [*turns to R2*]	
	402	R2	Sometimes I try to draw with like little slash marks in between to show that it's a break in the thing. Like here. This is a slash like that to show that there's more space in there.	
	403	R1	That's interesting.	
	404	R2	And then you can kind of think of it as expanding out or shrinking in as much as you need it to.	
	405	Stephanie	Um hm.	
	406	R1	But, but you see	
	407	R2	That's tough to do.	
	408	R1	What's tough to do, though, is what I want you to think about here, which is a little bit of a shift, Stephanie, is that what is the size of this square here?	
	409	Stephanie	One unit. One square unit.	BR-V
	410	R1	It's one square unit.	
	411	Stephanie	Um hm.	

Time	Line	Speaker	Transcript	Code
	412	R1	And we know the area is nine square units.	
	413	Stephanie	Yeah.	
	414	R1	Because we added up how many of them.	
	415	Stephanie	There's nine of them in there.	BR-V
	416	R1	'Cause we added up nine of them.	
	417	Stephanie	Yeah.	
	418	R1	Now when you did your um other example of a square, um of, you were doing four you said. Isn't that what you did?	
	419	Stephanie	Um.	
	420	R1	Four units? Is that the example you said earlier?	
	421	Stephanie	Yeah. Well you could use four.	
	422	R1	So, so if you were doing four, what would your picture look like if if you were	
	423	Stephanie	Well, it would have an extra it would	BCA?
	424	R1	would have an extra one. Right?	
	425	Stephanie	Yeah.	
	426	R1	Okay. And so how many would there be inside?	
	427	Stephanie	There'd be um sixteen.	
	428	R1	Right. But what would the size of each one still be?	
	429	Stephanie	One square unit.	BCA?
	430	R1	One square unit.	
	431	Stephanie	Um hm.	
	432	R1	So if I were doing five by five?	
	433	Stephanie	It would be twenty-five.	
	434	R1	Okay. But what's common in all of these?	
	435	Stephanie	They'd all still be one square unit.	BCA
	436	R1	Right.	
	437	Stephanie	Like the little things that make up inside.	
	438	R1	Whatever you choose to – suppose I did a square where the side is a half of a unit?	
	439	Stephanie	But th that still doesn't tell me what *a* is.	
	440	R1	Well	
	441	Stephanie	'Cause I'm like making up numbers.	
	442	R1	Right. But. Okay. But, but what do you what's gonna be whether whether *a* is a three or whether whether *a* is a five or whether *a* is a twenty-seven or an eight, there's gonna be something common about what your picture looks like.	
	443	Stephanie	It's going to be made up of square units?	BCA

Time	Line	Speaker	Transcript	Code
	444	R1	Isn't that right?	
	445	Stephanie	Yeah.	
	446	R1	It's and so whatever it's going to be it's going to be so many of them.	
	447	Stephanie	Yeah.	
	448	R1	Do you buy that?	
	449	Stephanie	Yeah.	
	450	R1	Okay. So. The point is how many of them are you going to have in here?	
	451	Stephanie	Well, can I just a pick a number? 'Cause	
	452	R1	Well, if this is *a*, if this is three, the side is three.	
	453	Stephanie	You're gonna have *a*, you're gonna have *a* time, *a* squared number of square units or something?	BCA
	454	R1	Why is that? How is why is that gonna be?	
	455	Stephanie	'Cause if it was four	BEJ
	456	R1	Um hm.	
	457	Stephanie	it would be four times four and that would be sixteen and it would be sixteen square units.	BEJ
	458	R1	So how do you get the sixteen?	
	459	Stephanie	You multiply four times four. So if it was *a* times *a* you'd have	BEJ; BMP
	460	R1	But I did I didn't multiply when I did this. I just counted them.	
	461	Stephanie	Well, yeah. But you you can multiply. Like length times width.	BEJ; BMP
	462	R1	Um hm.	
	463	Stephanie	So.	
	464	R1	'Cause that doesn't help me imagine what's inside. Do you know what I'm saying?	
	465	Stephanie	Yes. Um.	
	466	R1	I'm trying in my head imagining	
	467	Stephanie	All right. You want me to draw a…	
	468	R1	No, that's okay.	
	469	Stephanie	But it's like if I had a square [*draws a square*] that was four by four, [*writes 4 above the top of the square and 4 beside the left side of the square*] right? And you divide it into four p-, then it would have like four here, right?	BR-V
	470	R1	Hm.	
	471	Stephanie	[*sections the top edges of the square into four equal segments*] And they'd all be even. [*draws lines all the way through the square,*	BR-V

Time	Line	Speaker	Transcript	Code
			intersecting the side closest to her (the bottom)]	
	472	R1	Um hm.	
25:00-29:59	473	Stephanie	And you'd have four here. [*divides the left side of the square into four segments, then extends the lines all the way across the square*]	BR-V
	474	R1	Um hm.	
	475	Stephanie	And that would – there'd be sixteen. [*places a dot in each of the sixteen square units inside the four by four square*]	BR-V
	476	R1	Um hm.	
	477	Stephanie	I can't like	
	478	R1	Okay.	
	479	Stephanie	And	
	480	R1	So so tell me what this would look like then. If you could draw it?	
	481	Stephanie	It would have like *a* squared number of, I can't tell you what it would look like.	OBS
	482	R1	Sure you can. I bet you can. You're doing it right now. Um. Make a couple of more of these and then I bet you can describe mine. If you talk to me about these others. Force yourself to talk to me and describe them. I bet you can tell me about this one.	
	483	Stephanie	Okay. [*draws a square*] How many like units?	
	484	R1	You decide.	
	485	Stephanie	Oh. All right. If this is like six.	
	486	R1	Um hm.	
	487	Stephanie	All right. [*divides the square vertically into six strips, then divides the square horizontally*] It would be thirty-six. 'Cause it was like six by six.	BR-V
	488	R1	Um hm.	
	489	Stephanie	But 'cause there's like thirty-six like square units inside.	
	490	R1	Um hm.	
	491	Stephanie	There's	
	492	R1	One more time. Tell me what a square unit is.	
	493	Stephanie	It's like one of these. [*traces the upper right unit square of the six by six square*]	BR-V
	494	R1	And what's the area of one of those squares?	
	495	Stephanie	One.	
	496	R1	One. Okay. Why is the area of that one?	

Time	Line	Speaker	Transcript	Code
	497	Stephanie	So it would be filled up with – [*pause*] wait. Why is the area one?	
	498	R1	Right.	
	499	Stephanie	Oh! Because it just is. It's – it's one because um I don't know.	OBS
	500	R1	See. Um. It sounded like such an easy question, didn't it?	
	501	Stephanie	Yeah.	
	502	R1	But when you have to think deeply about it, it comes out that- let's now let's go back and try to figure out. You're saying that the area of this little square is one. [*colors the top right unit square of the six by six square*]	
	503	Stephanie	Yes.	
	504	R1	Right? And all the other little ones.	
	505	Stephanie	Um hm.	
	506	R1	And here you're saying the area of this is one- [*colors the top left unit square of the four by four square*]	
	507	Stephanie	-is one.	
	508	R1	And this is one. [*indicates the top left unit square in the three by three square*]	
	509	Stephanie	One.	
	510	R1	And let's analyze it to figure out how we got this to have an area one.	
	511	Stephanie	Um.	
	512	R1	That is a square.	
	513	Stephanie	Yeah.	
	514	R1	and it has an area of one. [*pause*] One square unit.	
	515	Stephanie	Yeah. I don't know. I never thought about it.	
	516	R1	Yeah. I know. Isn't that interesting? There's a professor at Rutgers called Professor Gelfand. He's supposed to be the world's greatest living mathematician.	
	517	Stephanie	Really.	
	518	R1	I don't know. He's about eighty-seven or something. Eighty-six. Even has a daughter your age. [*chuckles*] This is true. Um. And he has seminars in the mathematics department all the time. And this is what he does to mathematicians. He tries to get them to think very deeply about very fundamental ideas that they never really thought about before. And he, every great mathematician who came out	

Time	Line	Speaker	Transcript	Code
			of the Soviet Union worked with him. And uh, so, there are these great research professors who go to these seminars, he has them on Mondays, and uh they just come out of there sort of ‘I never thought about that before. I never thought about that before.’ And so what he really believes is that what you should be doing is thinking very deeply about basic ideas.	
	519	Stephanie	Um hm.	
	520	R1	And, uh, that sort of, what it is that you’ll be doing as good mathematics. So that’s good. That’s good that now you’re thinking about it. Otherwise, I’d be wasting your time.	
	521	Stephanie	Um hm.	
	522	R1	Right? But but this is exciting. Let’s try to understand. What does the three units mean on this side? [*indicates the left side of the three by three square*]	
	523	Stephanie	It means that it’s made up of three one square units. Like each, it’s made up of three squares that are like one square unit each in that, you know, one, two, three (inaudible).	BEJ
	524	R1	Okay. Great. Um. That isn’t what it means.	
	525	Stephanie	So what does it mean?	
	526	R1	(inaudible) okay. Um. See, I said three units.	
	527	Stephanie	Yeah.	
	528	R1	And I said this is a square unit.	
	529	Stephanie	Um hm.	
	530	R1	When I use the- this terminology is very important. The length of the side of the square is three units. Not three square units. Three units. [*emphasizing each word*]	
	531	Stephanie	Okay.	
	532	R1	Now is there a difference? Am I, the words are different, but do they in your head trigger anything different?	
	533	Stephanie	Well, they didn’t at first. But I guess three square units is	
	534	R1	I didn’t say that this is three	
	535	Stephanie	I	
	536	R1	square units. I said this side is three units. [*the left side*]	
	537	Stephanie	Yeah.	
	538	R1	And this piece is a square unit. [*indicates the*	

Time	Line	Speaker	Transcript	Code
			lower left one square unit] But this side of the square is made up of one, two, three [*traces the segments on the left side one at a time beginning with the segment closest to Stephanie and working toward the edge farthest from Stephanie*] units.	
	539	Stephanie	Um hm.	
	540	R1	Right?	
	541	Stephanie	Yes.	
	542	R1	Units. So let's let's worry about one of them for a minute. Let's not worry about one, two, three of them, right? Let's worry about one of them.	
	543	Stephanie	So this square [*the lower left square unit*] is like a square unit?	PAH
	544	R1	Why?	
30:00-34:59	545	Stephanie	Well, no. I was just asking.	
	546	R1	Okay. Let's, let's, it's sort like we're getting a, you know, it's sort of a telescope into this.	
	547	Stephanie	Um hm.	
	548	R1	I think we're getting somewhere.	
	549	Stephanie	Ok.	
	550	R1	This is exciting. Okay. Let's make another picture. [*draws a square; marks off three equal segments on each side*] Let's hope they don't fall asleep there because I got this excited. [*Stephanie chuckles.*]	
	551	R1	[*extends the lines across the square both vertically and horizontally*] I'm saying if I took [*traces the vertical side of the lower left unit square*] a measuring, whatever you want to measure.	
	552	Stephanie	Yes.	
	553	R1	Whether it's an inch, yard.	
	554	Stephanie	So this is one unit. [*the same unit segment that R1 traced*]	PAH
	555	R1	This is one.	
	556	Stephanie	Like this line.	
	557	R1	Right.	
	558	Stephanie	Okay.	
	559	R1	Think of it as an inch or whatever you want.	
	560	Stephanie	But this square [*the lower left one square unit*] is one square unit.	
	561	R1	Now why is that? What what's the length of	

Time	Line	Speaker	Transcript	Code
			this line? [*the horizontal line closest to Stephanie of the lower left one square unit*] We know that-	
	562	Stephanie	It's – one.	BR-V
	563	R1	-this is also one?	
	564	Stephanie	Yeah.	
	565	R1	Right? So this side has length one. This side has length one.	
	566	Stephanie	Yeah.	
	567	R1	One unit.	
	568	Stephanie	Yeah.	
	569	R1	All right. So here we have a square	
	570	Stephanie	And	
	571	R1	with side	
	572	Stephanie	Yeah.	
	573	R1	one side is one unit. Right? The other side is one unit. Right? How do you find the area of a square?	
	574	Stephanie	Um. Length times width. Which would be one times one and that's one squared. So that makes	BMP
	575	R1	One unit.	
	576	Stephanie	One squared unit.	BMP
	577	R1	times one unit. [*writes as she is speaking = 1 unit x 1 unit*] One times one is one square unit. [*writes = 1 square unit*] Sometimes they'll write it like this. [*writes 1 unit*^{2}] one unit; put a square here or something.	
	578	Stephanie	Um hm.	
	579	R1	If it's inches, they put inches squared.	
	580	Stephanie	Yeah.	
	581	R1	Okay. So remember the length of the side is three. The length of this square is one square, do you see the difference? This is really kind of tricky because they both have a one in it. You'd see it more easily-	
	582	Stephanie	So it's like the area	
	583	R1	-if I made it two.	
	584	Stephanie	of that little unit. Oh wait.	
	585	R1	Yeah. We're talking about two different things. Here we're talking about	
	586	Stephanie	so the lines are the unit. Okay.	BDI
	587	R1	sort of like perimeter.	
	588	Stephanie	Um hm. Yeah.	

Time	Line	Speaker	Transcript	Code
	589	R1	Right. But this is one. You see, I could've changed it. I said if this were two [*indicates the two edges labeled one unit previously*] and this were two, then the area of the square would be	
	590	Stephanie	Four square units.	BMP
	591	R1	four square units. Right? But I chose it to be one. You did, actually. So this is one.	
	592	Stephanie	Okay.	
	593	R1	Okay. And that's how we got one square unit. Oops, I'm off the paper. And I got another one square unit, another, and so forth. Right? And that's how we got nine.	
	594	Stephanie	Um hm.	
	595	R1	So let's go back to the initial idea here. Three units.	
	596	Stephanie	So that's one unit, two units, three units. [*counts up the left side of the original three by three square units*] What was I supposed to like - ?	PAH
	597	R1	Well. But I'm trying to think is think in smaller terms. You you found the ar-	
	598	Stephanie	Yes.	
	599	R1	Because this is one and this is one, this area is one square unit, right?	
	600	Stephanie	Um hm.	
	601	R1	Could I figure out the area of this square? [*the unit square below the one colored in the original three by three square*]	
	602	Stephanie	Yeah.	
	603	R1	Just this little one, right here?	
	604	Stephanie	Well, yeah, I guess. If you knew that that was that length.	
	605	R1	Well, do you? You know this is one, right? [*indicates the left vertical edge of the unit square under consideration*]	
	606	Stephanie	Well, yeah. And if the sides are the same, then yeah.	
	607	R1	So. That's why that's	
	608	Stephanie	Yeah. So you could figure it out if you had both sides.	BEJ
	609	R1	Right. But the fundamental idea here is that this is a unit and this is a square unit.	
	610	Stephanie	Um hm.	
	611	R1	That's why they call it a unit square sometimes	

Time	Line	Speaker	Transcript	Code
			or a square unit. Now notice. If I have three of them, you marked out three units.	
	612	Stephanie	Um hm.	
	613	R1	If you had six, you marked out six of these.	
	614	Stephanie	Um hm.	
	615	R1	Right.	
	616	Stephanie	Um hm.	
	617	R1	Okay. But what's this little one going to be even if you marked out six? If you took one of these little boxes?	
	618	Stephanie	It's going to be one square unit.	
	619	R1	It's going to be one. How many of them are going to be in there?	
	620	Stephanie	Thirty-six.	
	621	R1	Do you buy that? Do you really believe that? I mean, you really, fully, deeply, if you had to go and explain this to your class.	
	622	Stephanie	Well, yeah. But I mean I'm not going to sit there and figure it out like that. I believe it because six times six is thirty-six.	BMP
	623	R1	All right. But what if you um, some little child, you have a younger sister, but suppose she were really much younger and didn't know multiplication.	
	624	Stephanie	Um hm.	
	625	R1	Right? How could you convince her that the area of this is thirty-six and all we know is the length of this little one is one and you have six of them. What she could do is count. That's all she could do. I mean, does she have to? Could she still understand what you're doing?	
	626	Stephanie	I guess. But she'd, I mean, you'd still have to kinda explain it to her like squared.	
	627	R1	Yeah. I suppose.	
	628	Stephanie	Ya know. I mean she might understand that picture, but	
	629	R1	Okay. What what I'm what I'm pushing you to think about is not just think about four times four is sixteen.	
	630	Stephanie	Um hm.	
	631	R1	I want you to imagine in your head that when you find the area of a square that has four units on one side	
	632	Stephanie	Um hm.	
	633	R1	Which means four units on every side. I want	

Time	Line	Speaker	Transcript	Code
			you to think of each of these [*the four line segments*] and I want you to imagine inside you have all these little squares. Right?	
	634	Stephanie	Um hm.	
	635	R1	All one unit and totally you have sixteen of them. That's where you get your sixteen square units.	
	636	Stephanie	Okay.	
	637	R1	I want you to try to think of that. Now that should help you to figure out how to do *a* times *a*. If you think about that. 'Cause what's the difference now? You did it for three. You did it for four. You did it for six. What would it be for *a*?	
	638	Stephanie	Oh. It would be like	
	639	R1	What would be different in the *a* as compared to the three?	
	640	Stephanie	What would be different?	
	641	R1	Four, six.	
	642	Stephanie	The fact that you don't have a number?	OBS
	643	R1	Yes.	
35:00-39:59	644	Stephanie	But, I mean, the same, it would be like, there'd be *a* squared number of one square units.	BDI; BCA
	645	R1	All right. That's what's gonna be inside. So we're gonna	
	646	Stephanie	Yes.	
	647	R1	have all these little squares in here, you're telling me, right?	
	648	Stephanie	And that's gonna be *a* squared number.	
	649	R1	And when you add them all up, right, you're gonna have	
	650	Stephanie	It's gonna equal	
	651	R1	*a* squared of them.	
	652	Stephanie	Yes.	
	653	R1	Okay, h—Why? How come?	
	654	Stephanie	[*chuckling*] Because – um – [*sighs*] because – if *a* was, can I make *a* a number?	PNE
	655	R1	Sure.	
	656	Stephanie	(inaudible) If *a* was like four	BEJ
	657	R1	Um hm.	
	658	Stephanie	and – can – and it was divided into that many sections and you did each little area?	BEJ
	659	R1	Um.	
	660	Stephanie	It would equal one square unit.	BEJ

Time	Line	Speaker	Transcript	Code
	661	R1	Okay. But if *a* were four, how would you how would you start marking them off?	
	662	Stephanie	Well, if *a* was four, [*draws a square; can't see as she is working; the final version is a square sectioned off into sixteen square units*] then [*pause*]	BR-V
	663	R1	But even though *a* is four	
	664	Stephanie	Okay.	
	665	R1	What's this one? [*points to the line segment in the upper left corner of Stephanie's drawing*] What's this length here?	
	666	Stephanie	One.	
	667	R1	One. Right? Now, if *a* were five	
	668	Stephanie	And you know they're gonna be one square units because one times one	BMP
	669	R1	Okay.	
	670	Stephanie	is one.	BMP
	671	R1	If *a* were five?	
	672	Stephanie	There'd be five of these.	
	673	R1	But what would still one of these little pieces be?	
	674	Stephanie	One square unit.	
	675	R1	It would still be one, right?	
	676	Stephanie	Oh! Well, it's still, yeah.	BDI
	677	R1	But you would have how many of them?	
	678	Stephanie	Five. Oh! You mean total?	PAH
	679	R1	Well, you would have the five of these markings.	
	680	Stephanie	Yeah.	
	681	R1	And then total you would have?	
	682	Stephanie	Twenty-five.	BMP
	683	R1	Twenty-five. If *a* were seven?	
	684	Stephanie	It would be seven.	
	685	R1	What would one little one be?	
	686	Stephanie	One.	BR-V
	687	R1	If *a* were *a*?	
	688	Stephanie	One little one would still be one.	BCA
	689	R1	It would still be one.	
	690	Stephanie	It would always be one.	BDI; BCA
	691	R1	Okay. Now that's that's very important, isn't it?	
	692	Stephanie	Yes.	
	693	R1	You'd you'd have but but you would stop at *a*.	

Time	Line	Speaker	Transcript	Code
	694	Stephanie	Yes.	
	695	R1	Just like you stopped at marking them at four or stopped marking them at five.	
	696	Stephanie	So *a* – well	
	697	R1	You know you have these little one squares in there.	
	698	Stephanie	Yeah.	
	699	R1	Right? But when you added them up all up	
	700	Stephanie	So, so you have like *a* number of like units?	
	701	R1	Go ahead: *a* number of units.	
	702	Stephanie	And um *a* number of squares, I don't	
	703	R1	Let's try-	
	704	Stephanie	like	
	705	R1	-try to get down your idea. This is, think about all you've done here for a minute. Just think about all these things and try to write that down for *a* squared.	
	706	Stephanie	[*pause*] Um. [*pause*] I don't, I mean, you'd have *a* number of units no matter what. Like no matter what number it is, you're always gonna have that number of units. Yeah.	BEJ
	707	R1	What's the length of one of those units?	
	708	Stephanie	One.	
	709	R1	One. Okay.	
	710	Stephanie	So [*pause*] so no matter how many number of units you're always going to have um that many units with a length of one. Like each unit has a length of one.	BEJ
	711	R1	Um hm.	
	712	Stephanie	And each square unit has a length of - has an area of one.	BEJ
	713	R1	Of one what?	
	714	Stephanie	Square units.	BEJ
	715	R1	Square units.	
	716	Stephanie	Okay. Is that it?	
	717	R1	Okay. Um. So if if I were calling the length one	
	718	Stephanie	Um hm.	
	719	R1	Right? As you did. Is is do you see that the important difference between labeling something a unit and labeling something a square unit?	
	720	Stephanie	Yeah.	
	721	R1	What's that difference?	

Time	Line	Speaker	Transcript	Code
	722	Stephanie	One's like a square and one's just a little piece of the thing.	BEJ
	723	R1	Okay. Okay. So the difference is one is like we call a linear measure.	
	724	Stephanie	Um hm.	
	725	R1	Right. It's a length.	
	726	Stephanie	Okay.	
	727	R1	The other is a square measure. Which is a square. Literally a square. Right?	
	728	Stephanie	Yes.	
	729	R1	Which has an area of whatever that unit is.	
	730	Stephanie	Um hm.	
	731	R1	Okay. That that's great. So let's go back to the sketch here. So if if I really pushed you on Ethel's diagram, imagine that you were going to represent *a*, what would the length of this unit be?	
	732	Stephanie	Wait. One?	
	733	R1	And this one would be?	
	734	Stephanie	One.	
	735	R1	Okay. All the way up to what?	
	736	Stephanie	They'd all be one.	
	737	R1	But altogether, you'd have how many of them?	
	738	Stephanie	*a*	BR-V
	739	R1	See if you can	
	740	Stephanie	Do you want me to write that down?	
	741	R1	Yeah. See if you can make a picture that	
	742	Stephanie	Like in words or in a picture?	
	743	R1	In a picture. I think. Try that. Any way you want.	
	744	Stephanie	Um. I don't know.	OBS
	745	R1	Wh what what	
	746	Stephanie	It's still gonna	
	747	R1	Ethel did this you know, by the way, so that you if put one, one, one, one that this isn't four. What that means is there's one, one, a whole lot of others maybe.	
	748	Stephanie	Okay.	
	749	R1	Right?	
40:00-44:59	750	Stephanie	Yeah.	
	751	R1	Or maybe not. And then you could	
	752	Stephanie	So I could just mark them one then	BR-V
	753	R1	Yeah	

Time	Line	Speaker	Transcript	Code
	754	Stephanie	And like it doesn't matter if *a* equals one.	
	755	R1	Because this tells you there could be more in here, but the point is how many of these ones do you have?	
	756	Stephanie	*a* number.	
	757	R1	All right. Write that on top. [*Stephanie writes 'A number of ones' above the diagram.*]	BR-V
	758	Stephanie	Oops. Ones.	
	759	R1	Does that make sense? If you add the number one *a* times, you get *a*?	
	760	Stephanie	Yes.	
	761	R1	You see. Fundamental ideas. Huh? It's not so trivial. Okay. But that gives me a piece of it. Finish, finish this story.	
	762	Stephanie	Finish? You want - ?	
	763	R1	All I know is about the length of one side of this square.	
	764	Stephanie	Well, they're all gonna have the same number of ones. Well, if it's *a*	BEJ
	765	R1	Um hm.	
	766	Stephanie	and there's an *a* over here,	BEJ
	767	R1	Okay.	
	768	Stephanie	then they're all gonna have the same number of ones on each side.	BEJ
	769	R1	Okay. So finish your sketch. [*Stephanie draws.*]	
	770	Stephanie	Do you want me to write a number of ones on each side?	PAH
	771	R1	Any way you want, but – so show me what this inside is going to start to look like and what it's going to resemble.	
	772	Stephanie	Oh. All right. Well. If I'm just using four ones, 'cause	BEJ
	773	R1	You're not using four. You're using *a* of them.	
	774	Stephanie	Well, I know. But I can't, how am I supposed to draw a square	OBS
	775	R1	Well	
	776	Stephanie	if it's like expanding?	OBS
	777	R1	What you might do, what I think Ethel is suggesting is, don't finish it. Leave it sort of open in the middle. [*pause*] So it's sorta, see what she did here?	
	778	Stephanie	Yeah. I understand, but like	

Time	Line	Speaker	Transcript	Code
	779	R1	So what you might do is something like start showing it. Right?	
	780	Stephanie	Um hm.	
	781	R1	But don't finish it. You know what I'm saying? 'Cause you don't know what the inside is.	
	782	Stephanie	Oh, so just leave the inside.	
	783	R1	You see what I'm saying? That's a way.	
	784	Stephanie	Okay.	
	785	R1	That's a possibility, right?	
	786	Stephanie	Yeah.	
	787	R1	You you get the sorta general picture of what this is looking like, but the important the importance is can you tell me how many of these	
	788	Stephanie	*a*	
	789	R1	are in there?	
	790	Stephanie	There are *a* squared.	BCA
	791	R1	Okay. And so label that. *a* squared.	
	792	Stephanie	Do you want me to just put *a* squared?	
	793	R1	Um hm. [*Stephanie writes* A^2 *in the middle of the picture.*] But you really know what that means now. I mean there's no	
	794	Stephanie	Yeah.	BR
	795	R1	question in your mind. You know what that means.	
	796	Stephanie	Yes.	
	797	R1	That that's great. I mean there are students in college who don't know what that means.	
	798	Stephanie	Um.	
	799	R1	Does that surprise you? That students in college don't know what that means? Because uh um Mrs. Steencken is teaching a college course in Algebra right now for students at Rutgers. And you could give them a quiz and see if they know what it means. Students in high school don't know what that means. Students in county college- they don't know what that means. And so this is not baby stuff. I mean, this is really fundamental, important ideas. Right, Dr. Alston?	
	800	R3	(inaudible)	
	801	R1	And and it's, if you can't make the leap to understanding this, you're just gonna be doing things to these letters that you remember	

Time	Line	Speaker	Transcript	Code
			worked for your arithmetic.	
	802	Stephanie	Uh huh.	
	803	R1	Like distributing as you said	
	804	Stephanie	Yes.	
	805	R1	the exponent. And you're going to get yourself in big trouble, unless you begin to think about the meaning of these ideas, deeply. And um what I'm going to ask you to do is to sorta write up what you did today.	
	806	Stephanie	Okay.	
	807	R1	I don't know what our time schedule is here. We've been here about an hour, um, but I think if we move on to the next piece of it.	
	808	Stephanie	Okay.	
	809	R1	*a* plus *b*.	
	810	R2	To the third power?	
	811	R1	No, squared. How can we talk about a picture of *a* plus *b*?	
	812	R2	I I was thinking maybe, maybe um you still want a square, right?	
	813	R1	Could we have a rectangle? *a* plus *b* *a* plus *b*. We want a square.	
	814	R2	*a* plus *b* *a* plus *b*.	
	815	R1	Let's keep it a square.	
	816	Stephanie	Okay.	
	817	R2	Right? Okay? So we're going to need a pretty big square maybe.	
	818	Stephanie	Okay.	
	819	R1	How would the square *a* with the side *a* plus *b* look different from a square with side *a*?	
	820	Stephanie	Well. It would have two different parts.	BR-V
	821	R1	Okay! That's very good. You make the picture.	
	822	R2	Yeah.	
	823	R1	Show me a square that has one side *a* plus *b* and another side *a* plus *b*.	
	824	Stephanie	It's not gonna be even.	
	825	R1	Oh. I I hope it's not. 'Cause then I'll feel good. I mean mine are never even. [*pause; Stephanie draws a square; labels the top A + B and the left side A + B*] So already I'm confused because you wrote *a* plus *b* and you skipped a step and I'm this really slow kid. You know this younger sister of yours and	BR-V

Time	Line	Speaker	Transcript	Code
			you're already up to *a* plus *b* and I don't even know how you got that. I want to know where *a* is and I want to know where *b* is. To know where you got	
	826	Stephanie	Oh. So you want me to section it off.	PAH
	827	R1	*a* plus *b*. Thank you.	
	828	Stephanie	Okay.	
	829	R1	It would help me a lot. I want to know which part is *a* and which part is *b*, where you got your *a* plus *b*.	
	830	Stephanie	All right. This is *b*. [*marks off a small section at the right side of the top edge of the square; labels it b*]	BR-V
	831	R1	That little piece?	
	832	Stephanie	And this is *a*. [*labels the longer segment of the top edge of the square a*] But it's gonna be really confusing, 'cause there's like *a* – should I just turn it over?	BR-V
	833	R1	You can do it again.	
	834	Stephanie	Yeah, because (inaudible)	
	835	R1	Here you go. Take another piece of paper. [*Stephanie redraws the square.*]	
45:00-49:59	836	Stephanie	It looks like a rectangle. Oh well. [*marks the top edge with a and b as before*] That's *a*. And like, should I write *a* plus *b* up top? That way it	
	837	R1	Sure. [*Stephanie writes A + B above the top edge of the square; marks off a length similar to the previous b's length down from the top of the left edge; labels it b; labels the longer segment on that edge A. She marks off 'b' length on the left side of the bottom edge; labels it b and the longer side A. She then marks off b at the bottom edge of the right side; labels it b and labels the longer segment A. The segments representing lengths a and b are at opposite ends of the top and bottom sides of the square and are similarly misplaced for the left and right edges.]*	BR-V
	838	Stephanie	Okay.	
	839	R1	Hmmm. [*pause*] So tell me what you did.	
	840	Stephanie	I sectioned it off.	BEJ
	841	R1	Ok.	
	842	Stephanie	Um. I represented *a* plus *b*. I don't [*pause*] I labeled it.	BEJ;BR-V

Time	Line	Speaker	Transcript	Code
	843	R1	Okay. [*pause*] So [*pause*]	
	844	Stephanie	Um. Do you want me to draw it all the way down and ?	
	845	R1	See what happens. [*Stephanie draws line segments which intersect parallel sides of the square at each of the b markings that she made; pause*]	
	846	Stephanie	They're all different sizes.	BEJ;BR-V
	847	R1	Hmm. What do you mean?	
	848	Stephanie	Well, like the squares – they're not even squares. They're like rectangles. But they're all different sizes. [*pause*] Um. Was it sectioned wrong? Or was it…?	BEJ;BR-V
	849	R1	Well [*pause; sighs*] I guess, um, maybe maybe you'd find it easier, remember when we did this, the *a* squared, when we did one side and then another. Maybe you should just do two sides.	
	850	Stephanie	Okay.	
	851	R1	For now. Uh. Rather than	
	852	Stephanie	But was it sectioned correctly? Or?	PAH
	853	R1	Well, no. Do do two sides again. Let's go through this again. Let me see what you're doing. Try to make it a square this time. Okay? Maybe that'll help. [*Stephanie draws.*] I mean you could measure them that way, but sometimes the orientation of it might make it easier for you to work with, Stephanie.	
	854	Stephanie	Yeah. [*finishes drawing a square*]	
	855	R1	So m- mark on one of the sides your *a*. [*Stephanie writes a at the left side of the top segment of her square.*] And put a line like you did. That's a good idea. And the *b*.	
	856	Stephanie	Okay.	
	857	R1	Why don't you try your section right now? [*Stephanie draws a vertical line through the square at her marking.*] 'Cause I think where you got in trouble before, maybe you can tell me. 'Cause what you've just done you've already decided what the length of those other sides are, haven't you?	
	858	Stephanie	Oh. I guess this is *a* and this is *b*. [*marks the appropriate segments on the bottom side of the square.*]	BR-V

Time	Line	Speaker	Transcript	Code
	859	R1	Now notice what you did in your original drawing that made it tough for you to deal with. See see you're *a* up here?	
	860	Stephanie	Um hm.	
	861	R1	Look what you made where you put your *a* on the bottom.	
	862	Stephanie	Ooh!	BDI
	863	R1	You see what I'm saying. You measured this to be *a*	
	864	Stephanie	Um hm.	
	865	R1	and then when you went down there, you want to be *a* here rather than to keep them the same.	
	866	Stephanie	Okay.	
	867	R1	You see what I'm saying?	
	868	Stephanie	Yes.	
	869	R1	I mean it's not that it's wrong. It just may not be helpful.	
	870	Stephanie	Yeah.	
	871	R1	You know what I'm saying? So. Now do another side. The same way.	
	872	Stephanie	Does it - ?	
	873	R1	If you do one side, the other automatically gets determined. You don't have to	
	874	Stephanie	Does it matter if I section it *ab* or *ba*? Or or like?	PAH
	875	R1	We'll try it both ways. And then you can tell me if it matters. [*Stephanie marks off a from the bottom of the left side of the square*.]	
	876	Stephanie	Okay.	
	877	R1	What you have to kinda watch is that the lengths of *a* and *b* that you picked are pretty close. [*Stephanie draws a horizontal line through the square at her marking between a and b*.] Okay. So now you know this. What do you know on this side now that you've done that?	
	878	Stephanie	That this is *b* and this is *a*. [*labels the appropriate line segments on the opposite side; (the right side)*]	BR-V
	879	R1	That's neat.	
	880	Stephanie	Um hm.	
	881	R1	Okay, so you have here, if we've done this carefully, you have partitioned this square into four pieces.	
	882	Stephanie	Yes.	

Time	Line	Speaker	Transcript	Code
	883	R1	Isn't that right? And, um, you know how to find the area of a square. How do you find the area of a square?	
	884	Stephanie	Multiply the two, the length and the width?	
	885	R1	Yeah. Or a rectangle, you know how to do that, right? So you should be able to find the area of each of these four pieces.	
	886	Stephanie	Yeah.	
	887	R1	Go for it! [*Stephanie writes ab in the upper left rectangle, bb in the upper right square, ab in the lower right rectangle and aa in the lower left square.*] Okay. So. What's the area of the square? The big one?	
	888	Stephanie	[*Stephanie grunts.*]	
	889	R1	The one you started with?	
	890	Stephanie	Um. *ab* times *ab*.	OBS
	891	R1	No. What's the	
	892	Stephanie	Oh. *a*	
	893	R1	You've done four pieces.	
	894	Stephanie	plus *b* times *a* plus *b*. Or	OBS
	895	R1	I'm going to try my question again.	
	896	Stephanie	Okay.	
50:00-54:59	897	R1	Let's go back to some of these other things. [*sorts through some of the papers on the desk*] Okay. When this was six and this was six	
	898	Stephanie	Um hm.	
	899	R1	You found the area inside, right?	
	900	Stephanie	Um hm.	
	901	R1	Which was what?	
	902	Stephanie	Um. Thirty-six. Or	BMP
	903	R1	How did you get that?	
	904	Stephanie	How did I get that? I multiplied six times six.	BMP; BEJ
	905	R1	You you really did. You didn't count them. I know you multiplied. You said six is the length of this side	
	906	Stephanie	Yeah.	
	907	R1	times the length of this side. And for this one you said the area was	
	908	Stephanie	Um. Sixteen.	
	909	R1	Because you took	
	910	Stephanie	I multiplied	
	911	R1	(inaudible) And this one you said the area was *a* squared. Because you took	

Time	Line	Speaker	Transcript	Code
	912	Stephanie	*a* and I multiplied it by *a*.	BEJ
	913	R1	Right? So. What's this side here? [*can't tell which*]	
	914	Stephanie	Um. *ab* or *a* plus *b*.	
	915	R1	That's what you told me up in the other	
	916	Stephanie	Yeah. *a* plus *b*.	BR-V
	917	R1	Okay. Why don't you write *a* plus *b* on top of it, lest not we lose that idea. And what's the side here? [*the left side*]	
	918	Stephanie	*a* plus *b*.	
	919	R1	Okay. Okay. So	
	920	Stephanie	So it would be *a* plus *b* times *a* plus *b*?	BCA
	921	R1	Why don't you write that down? *a* plus *b* times *a* plus *b*. [*Stephanie writes* $a+b \cdot a+b$] Don't you need some parentheses in there? [*Stephanie inserts parentheses so it now reads* $(a+b)\cdot(a+b)$] Does it matter?	
	922	Stephanie	Mm. I don't know. Um. I guess it just tells you to do that first.	BMP
	923	R1	Okay. We'll get back to 'do you need them?'	
	924	Stephanie	Yeah.	
	925	R1	in a minute. But it's you said *a* plus *b* times *a* plus *b*, right? Equals	
	926	Stephanie	Um hm.	
	927	R1	Put an equal. [*Stephanie does.*] Equals what? If I know the length of this side and I know the length of this side, what part will give me the area?	
	928	Stephanie	What part will give you the area?	
	929	R1	Um hm. What's the area of that square?	
	930	Stephanie	In other words than *a* plus *b* times *a* plus *b*.	PAH
	931	R1	Um hm.	
	932	Stephanie	Well, doesn't that go back to that? Then it becomes like, if *a* plus, wouldn't it, wouldn't it just be um *a* plus *b* squared?	BR-S
	933	R1	Write that down. [*Stephanie completes the algebra sentence:* $(a+b)\cdot(a+b)=(a+b)^2$] And why is it?	
	934	Stephanie	Because that's what it was before? Because it's um two *a*'s and two *b*? Like there's two of each?	BEJ; OBS
	935	R1	Okay. So. *a* plus *b*. I'm not sure – you're not telling me *a* squared plus *b* squared. You're saying that this [*points to (a + b)*] and this	

Time	Line	Speaker	Transcript	Code
			[*points to (a + b)*] twice.	
	936	Stephanie	Yes.	
	937	R1	All right. But now in this picture, what part of the picture represents this [$(a+b)^2$] piece? I know what part is *a* plus *b*. You told me that it's this side.	
	938	Stephanie	Like the whole thing?	
	939	R1	The whole thing.	
	940	Stephanie	Yeah. The whole thing.	
	941	R1	Okay. So this whole area is what this is equals. Let's write it out. What is the whole thing? You have pieces of it.	
	942	Stephanie	Um hm.	
	943	R1	So it's the whole thing. That means, this piece [*the* $a \cdot a$]	
	944	Stephanie	and this piece [*the top left* $a \cdot b$] and this piece [*the* $b \cdot b$] and this piece [*the bottom right* $a \cdot b$]	BR-V
	945	R1	Okay. So	
	946	Stephanie	All together.	
	947	R1	All together, when you	
	948	Stephanie	Yes.	
	949	R1	talk about things all together, what do you do?	
	950	Stephanie	You add them.	BMP
	951	R1	You add them. So it's this piece, plus this piece, plus this piece, plus this piece. [*indicates the pieces in the same order as before]*	
	952	Stephanie	You want me to add them.	
	953	R1	I want you to write this piece, plus this piece, plus	
	954	Stephanie	Okay.	
	955	R1	this piece, plus this piece, and not skip any steps. [*Stephanie writes* $a \cdot a + a \cdot b + b \cdot b + a \cdot b$.] You have four terms?	
	956	Stephanie	Yes.	
	957	R1	Okay. Let's simplify them. Equal	
	958	Stephanie	Just put it like back down here?	
	959	R1	Just put the equal underneath that and let's simplify.	
	960	Stephanie	All right.	
	961	R1	Is there another way you can write *a* times *a*?	
	962	Stephanie	*a* squared. [*writes* a^2]	BR-S

Time	Line	Speaker	Transcript	Code
	963	R1	Okay.	
	964	Stephanie	Plus it could be *b* squared, 'cause there's a	BR-S
	965	R1	Put that at the end.	
	966	Stephanie	Okay. So *a* squared plus *ab* (inaudible) plus *b* squared. [*writes* $a^2 + a \cdot b + a \cdot b + b^2$] And you can simplify that. Couldn't it be *ab* squared?	BR-S; BMP
	967	R1	Okay. So what you have here: *a* squared plus *ab*	
	968	Stephanie	Yeah.	
	969	R1	plus *ab*	
	970	Stephanie	Um hm.	
	971	R1	plus *b* squared.	
	972	Stephanie	Yes. That would be two *ab* or	BMP
	973	R1	You have *ab* and you have another *ab*	
	974	Stephanie	Yes.	
	975	R1	so you have two *ab*, so write that down.	
	976	Stephanie	*a* squared plus two *ab* plus *b* squared. [*writes:* $a^2 + 2ab + b^2$ *while speaking; pause*]	BR-S; BMP
	977	R1	Hm. What did you just do?	
	978	Stephanie	I um simplified it?	
	979	R1	Okay. So what is this *a* squared plus two *ab* plus *b* squared represent?	
	980	Stephanie	This. [*puts her hand over the (a + b) square*]	BR-V
	981	R1	The area of the square?	
	982	Stephanie	Yes.	
	983	R1	With what side? What length side? [*pause*]	
	984	Stephanie	Well, it represents like the area of the square.	BR-V
	985	R1	This, what particular square? What is the length of the side of that square?	
	986	Stephanie	Oh. *a* plus *b*.	BR-V
	987	R1	*a* plus *b*. Now. *a* plus *b* is the length of the side.	
	988	Stephanie	Um hm.	
	989	R1	The area you told me in simplified form – you said the area is *a* plus *b* quantity squared.	
	990	Stephanie	Um hm.	
	991	R1	But didn't we start this whole visit here	
	992	Stephanie	With (inaudible)	
	993	R1	to try and figure out what *a* plus *b* quantity squared meant?	
	994	Stephanie	Yes.	
55:00-55:16	995	R1	And now you're telling me it's *a* squared plus two *ab* plus *b* squared.	

Time	Line	Speaker	Transcript	Code
	996	Stephanie	[*hesitantly*] Yeah.	
	997	R1	Why don't we test these with the numbers you tested before as a start for some numbers. We're getting so organized here, Lynda. I know you're disappointed in me with numbering pages. [*chuckles*]	
CD 2 00:00-04:59	998	R1	Um. Is this what we started with? You said *a* plus *b* quantity squared does not equal *a* squared plus *b* squared.	
	999	Stephanie	Yes.	
	1000	R1	Okay. Now you, using some geometry and things about area of a square	
	1001	Stephanie	Um hm.	
	1002	R1	you told me that *a* plus *b* quantity square equals *a* squared plus two *ab* plus *b* squared.	
	1003	Stephanie	Yes.	
	1004	R1	That's what you, I believe, were working on for this last hour and fifteen minutes.	
	1005	Stephanie	Yes.	
	1006	R1	Okay. So if your arithmetic work is correct, I , you should be able to test some numbers – at least to see if you don't get a counter example right away.	
	1007	Stephanie	So you want me to test numbers?	
	1008	R1	What do you think? Wouldn't you be inclined to test	
	1009	Stephanie	Oh. Well, yeah	
	1010	R1	some numbers.	
	1011	Stephanie	I didn't know	
	1012	R1	for *a*'s and *b*'s and see what happens?	
	1013	Stephanie	All right. So let me do some really easy numbers. Um. If	PNE
	1014	R1	Try a very try a easy number. That's a good idea.	
	1015	Stephanie	Yeah. So	
	1016	R1	Especially this time of day.	
	1017	Stephanie	*a* is two and *b* is three.	PNE
	1018	R1	That's what you did before.	
	1019	Stephanie	Yeah. So it would be	
	1020	R1	You've got half of it done already.	
	1021	Stephanie	[*talking under her breath as she writes*] Two is four, plus two times two time three plus three squared, that's a nine (inaudible) [*Stephanie has written:* $2^2 + (2 \cdot 2 \cdot 3) + 3^2$*;*	PNE

Time	Line	Speaker	Transcript	Code
			beneath that she wrote: 4 + 12 + 9; beside her work she added 16 + 9 and got 25] [*pause*] Twenty-five. It works.	
	1022	R1	It worked for that example.	
	1023	Stephanie	Yeah.	
	1024	R1	But when you claim it's true, how many does it have to work for?	
	1025	Stephanie	All of them?	BMP
	1026	R1	All of them. Yeah.	
	1027	Stephanie	(inaudible)	
	1028	R1	Could you possibly test all of them?	
	1029	Stephanie	No-o! [*laughs*] There's too many numbers. Um. Do you want me to try again?	
	1030	R1	Well, you might want to convince yourself with something else.	
	1031	Stephanie	All right.	
	1032	R1	Does it work for zero?	
	1033	Stephanie	Well, zero you'd just get zero.	BMP
	1034	R1	Maybe that will give you some insight why zero worked here and why it	
	1035	Stephanie	Well, zero would work anywhere 'cause it's always gonna be zero.	BMP; BCA
	1036	R1	Um hm. Okay. Now, do you believe this? What you just built? That *a* plus *b* quantity squared is *a* squared plus two *ab* plus *b* squared, by that geometry you've just done? You've just done some geometry.	
	1037	Stephanie	Yeah.	
	1038	R1	Now the question is: How can we take what we know about arithmetic and algebra to convince us that's true, because we can't test every number to prove it. Right? You just said that there are infinitely many of them.	
	1039	Stephanie	Um hm.	
	1040	R1	Isn't that true?	
	1041	Stephanie	Yes.	
	1042	R1	And we impossibly can't – you you tried one. You might want to try a few more.	
	1043	Stephanie	Um hm.	
	1044	R1	The problem is with when students try a couple and they make a mistake in computation,	
	1045	Stephanie	Um hm.	
	1046	R1	they they might discard something that they	

Time	Line	Speaker	Transcript	Code
			worked real hard to build because they've made a computation mistake. So you've got to be real careful with your computation. It might not be a bad idea to try another one.	
	1047	Stephanie	Okay.	
	1048	R1	(inaudible) another piece of paper. Just to convince yourself and then	
	1049	Stephanie	And what should I use? Four and five?	PNE
	1050	R1	Whatever you think.	
	1051	Stephanie	Okay. Four squared	PNE
	1052	R1	It depends on how much you want to do arithmetic.	
	1053	Stephanie	[*laughs*] plus four times four times five plus five squared. [*writes:* $4^2 + (4 \cdot 4 \cdot 5) + 5^2$] Twenty-five. [*writes 25 under the 5^2; brings down the + to the left of 25; writes 80 below the* $(4 \cdot 4 \cdot 5)$*; brings down the + to the left of 80; writes 16 under the 4^2. To the right of this, Stephanie adds 96 + 25 and obtains 121*] And what was the original? *a* plus *b* squared?	PNE: BMP
	1054	R1	Tell me what you're doing. [*taps near the 4^2*]	
	1055	Stephanie	Four times four.	
	1056	R1	No. What's what's this first sentence?	
	1057	Stephanie	Oh. Four squared plus four times four times five	BEJ
	1058	R1	Where did that four come from? I don't see the four	
	1059	Stephanie	Oh! It's two!	BDI
	1060	R1	Okay.	
	1061	Stephanie	Okay. [*corrects her work; writes 2 over the first 4 of the middle term; scribbles out the 80 and writes 40 in its place*] Forty. [*crosses out the previous addition; adds 56 + 25 and gets 81*] Um. (inaudible) [*writes:* $(4+5)^2$] Nine squared. That's eighty-one. Yeah. It works.	PNE; BMP
	1062	R1	Just a lucky two numbers. [*Stephanie laughs*] We're gonna try again. If you don't make a computation mistake.	
	1063	Stephanie	Yeah. If I don't make a mistake. Yeah.	
	1064	R1	You sort of inclined to believe this?	
	1065	Stephanie	Yeah.	
	1066	R1	Does this make sense to you? What you did here?	
	1067	Stephanie	Well, after I kinda knew what I was like doing,	

Time	Line	Speaker	Transcript	Code
			yeah.	
	1068	R1	See, see I really think I think um we may have to stop here 'cause of the time, but but what I would like you to be thinking to yourself – you wrote *aa a* times *a* you mean	
	1069	Stephanie	Um hm.	
	1070	R1	*a* squared. In your head, do you know what you're imagining here? In this piece? [*indicates the lower left corner of the* $(a+b)^2$ *model Stephanie drew*]	
	1071	Stephanie	Well, it's just this square would be *a.*	BEJ
	1072	R1	Right.	
	1073	Stephanie	It would just be like this piece right here. [*Stephanie traces the* a^2 *section of the model.*]	BEJ
	1074	R1	Tell me about this piece. What does this *a* squared mean for this piece?	
	1075	Stephanie	It means that that's like the area of the piece.	BEJ; BR-V
	1076	R1	Right. But what am I supposed to imagine in my piece.	
	1077	Stephanie	There's *a* squared number of units uh square units.	BEJ; BR-V
	1078	R1	Okay.	
	1079	Stephanie	In there.	
	1080	R1	And what's the length of one?	
	1081	Stephanie	One.	BR-V
	1082	R1	Okay. So you have two, three, four, five, dot, dot, dot [*marks off intervals along the left side of the* a^2 *section*].	
	1083	Stephanie	Um hm.	
	1084	R1	Each of these is one.	
05:00-09:59	1085	Stephanie	Yes.	
	1086	R1	And you have that many and (inaudible)	
	1087	Stephanie	And the squares would be one square unit.	BR-V
	1088	R1	Okay. What about this? This is not a square. [*indicates the* $a \cdot b$ *rectangle in the upper left corner of the model*]	
	1089	Stephanie	It would – [*makes a noise*] – um so wouldn't it be there'd be *ab* number of square units and - would each one still be one?	PAH
	1090	R1	That's an interesting question. I want you to think about – now this is *b*. [*traces the line segment labeled b on the upper left side of the*	

Time	Line	Speaker	Transcript	Code
			model] the way you made the picture, do you have more of these [*the b's*] than you have of these [*traces the a edge of the* $a \cdot b$ *rectangle in the upper left corner of the drawing*]	
	1091	Stephanie	Um. No. I have more – well *a* is larger.	
	1092	R1	*a* is larger than *b*. Okay.	
	1093	Stephanie	So there's more	
	1094	R1	Okay. But what's, if I have one of these?	
	1095	Stephanie	Like what do you mean? Like if there's one divider?	PAH
	1096	R1	Well, I have *b* of how ma- I have *b* of something here.	
	1097	Stephanie	Yes.	
	1098	R1	What what are these things I have *b* of?	
	1099	Stephanie	Units.	BR-V
	1100	R1	Units. And so I have a unit here. I keep going until I have *b* of them. Right?	
	1101	Stephanie	Yes.	
	1102	R1	And here, I keep going until I have *a* of them. Right?	
	1103	Stephanie	Yes.	
	1104	R1	So if you can imagine these. [*marking off intervals on both sides of the* $a \cdot b$ *rectangle as she is speaking. Then she extends the lines to give the impression of square units.]*	
	1105	Stephanie	Um hm.	
	1106	R1	So what does what does the the *ab* have to do with it? How do I get *ab*?	
	1107	Stephanie	Well, that's how many units there are.	
	1108	R1	What does that (inaudible)	
	1109	Stephanie	Square units there are	
	1110	R1	Why?	
	1111	Stephanie	Because um [*pause*] Oh! Um! Because there's *a* number of units here [*along the top side of the* $a \cdot b$ *rectangle*]	BEJ; BR-V
	1112	R1	Um hm.	
	1113	Stephanie	And okay. There's like if this is what it is, right?	
	1114	R1	Um hm.	
	1115	Stephanie	Like if this is that piece [*redraws the* $a \cdot b$ *rectangle on the upper left side of the paper*] this is *a* and this is *b*. [*labels the longer (horizontal) side of the rectangle A and the shorter (vertical) side of the rectangle B*]	BEJ; BR-V

Time	Line	Speaker	Transcript	Code
	1116	R1	Um hm.	
	1117	Stephanie	There's *a* number of units here, like this part [*traces the air over the side she labeled A*]	BEJ; BR-V
	1118	R1	Hm.	
	1119	Stephanie	and there's *b* number of units here [*vertically*],	BEJ; BR-V
	1120	R1	Um hm.	
	1121	Stephanie	so if you mult and you want to get like this square. [*sectioned off what looks like one square unit at the left side of the rectangle she drew*]	BEJ
	1122	R1	Um hm.	
	1123	Stephanie	And that's *a* [*touches the top of the square she sectioned off*] times *b* [*touches the left side of the square*] and there's like that many number [*marks off 2 more squares by drawing vertical lines through the rectangle*] and that would be *a* times *b*, so you'd have	BEJ
	1124	R1	But this is not *a*. This is one and this is one.	
	1125	Stephanie	Well, yeah, but	
	1126	R1	It's just that you have uh *a* of these ones [*indicates horizontally*] and *b* of these ones. [*indicates vertically*]	
	1127	Stephanie	Yeah.	
	1128	R1	So I'm trying to understand, how do you get *ab*?	
	1129	Stephanie	*ab* what? Like?	OBS
	1130	R1	As a total number of square units in that section.	
	1131	Stephanie	In this whole	
	1132	R1	Yeah.	
	1133	Stephanie	thing?	
	1134	R1	Yeah. [*pause*] Well, suppose you thought of *a* and *b* being particular numbers.	
	1135	Stephanie	Um hm.	
	1136	R1	Suppose *a* were five and *b* were two.	
	1137	Stephanie	Okay.	
	1138	R1	You know ahead of time	
	1139	Stephanie	(inaudible)	
	1140	R1	without thinking that you're going to get	
	1141	Stephanie	Ten.	BMP
	1142	R1	How many of those little squares? Ten. But I want you to be able to imagine how those ten get generated when *b* is two	

Time	Line	Speaker	Transcript	Code
	1143	Stephanie	Um hm.	
	1144	R1	and *a* is five. I want you to really in your mind to try to think of how they come about. 'Cause because this is the kind of power that's going to help you in mathematics as you move along. Not just to say that there are *ab*. Let's not worry think about that. That's a fast way to get an answer, but how are they coming? That's that's the real way you're going to develop this ability to do higher level mathematics.	
	1145	Stephanie	Okay.	
	1146	R1	And this is like what Gelfand would say. These are the fundamental idea. And Dr. Alston, who's been so good, and hasn't said a word. And she's not going to be happy until I let her say or ask you a question.	
	1147	R3	No. I've been sitting over here spellbound. This has been an interesting thing for me, 'cause I've been sitting over here and can't see what you guys are doing. And	
	1148	R1	You should have moved over by a monitor.	
	1149	R3	Uh. And I but I but I was trying trying to imagine is if I didn't know at all what you were talking about and I was sitting over here and I was just listening to you, all of that um (inaudible) uh and and you were going to try to explain it so I would draw what you're thinking. How would you do that? If it were even if it were one of those squares that were numbered. Could you tell me what to draw (inaudible)?	
	1150	Stephanie	Oh! You mean like dictate to you like the square that you're gonna draw	PAH
	1151	R3	Yeah.	
	1152	Stephanie	without seeing it?	PAH
	1153	R3	Yeah. Yeah. Uh. All that stuff about the things on the insides and the things on the outside.	
	1154	Stephanie	Ooh.	
	1155	R3	And if, I didn't know it all, uh I don't understand that three times three bit. Why why does that work?	
	1156	Stephanie	Um. Well – do you want me to explain to you how to draw this…?	

Time	Line	Speaker	Transcript	Code
	1157	R3	Uh uh hang on. I want you I want you to tell me so it will come out that I have what you're talking about.	
	1158	R1	You give her instructions, and she's going to draw it there	
	1159	R3	Yeah.	
	1160	R1	and Mrs. Cosmo is gonna see if what	
	1161	R3	Anything with real numbers (inaudible)	
	1162	Stephanie	Oh!	
	1163	R1	So just give her a specific example.	
	1164	R2	The two and the five was a good one.	
	1165	R3	Yeah. The two and the five (inaudible)	
	1166	Stephanie	Okay. So, well, so draw a line.	
	1167	R3	Uh hum. A line?	
	1168	Stephanie	Okay. Yeah. And you can	
	1169	R3	Okay. What kind of a line?	
	1170	Stephanie	A straight line.	
	1171	R3	Okay. Forever? Lines go on forever.	
	1172	Stephanie	Well – draw a line segment.	
	1173	R3	Uh huh.	
	1174	Stephanie	A straight line segment.	
	1175	R3	Oh. I can do that. [*Stephanie laughs.*]	
	1176	R3	And then?	
	1177	Stephanie	Label and when you're done, you can make it any length and well, do you have graph paper, or is that just …?	
	1178	R3	I can do it. Just tell me how long. I wanna do	
	1179	Stephanie	Okay. Five.	
	1180	R3	a specific example. Five?	
	1181	Stephanie	Yes.	
	1182	R3	Okay. Five what?	
	1183	Stephanie	Five units.	
10:00-14:59	1184	R3	Okay. [*pause*] Okay [*pause*] Okay. I've drawn a line segment that's five units long.	
	1185	Stephanie	Yeah. Now on one of the points on the line segment.	
	1186	R3	Um hm.	
	1187	Stephanie	Um- you want to draw a line either like going up or going down like um – did you draw the line? [*softly: 'I'm trying to think of how to get her to draw a rectangle.'*] Did you draw the line vertically or horizontally?	
	1188	R3	I drew it vertically.	
	1189	Stephanie	Okay. So you want to draw this line	

Time	Line	Speaker	Transcript	Code
			horizontally.	
	1190	R3	Uh huh.	
	1191	Stephanie	And you can draw it either going up or going down, starting from that- a point.	
	1192	R3	Which point?	
	1193	Stephanie	Um, the beginning point.	
	1194	R3	Okay.	
	1195	Stephanie	Um. And draw it two units.	
	1196	R3	Okay. Units being the same (inaudible)	
	1197	Stephanie	Yeah.	
	1198	R3	Or (inaudible)	
	1199	Stephanie	What?	
	1200	R3	The same or different from the units before or what?	
	1201	Stephanie	Oh. Well. You drew five units for the first one, right?	
	1202	R3	Yeah.	
	1203	Stephanie	Well, the same, each unit's the same length, but you're only gonna draw two of these.	
	1204	R3	Okay. And now?	
	1205	Stephanie	And now draw vertically.	
	1206	R3	Okay. The first one went vertically?	
	1207	Stephanie	So now this one's gonna go vertically, too. And go – all right – is the um the first one	
	1208	R3	Okay. I think you mean vertically and horizontally being [*her hands move out of the viewing screen in some sort of motion*]	
	1209	Stephanie	Yes.	
	1210	R3	Ninety – ninety degrees.	
	1211	Stephanie	Yes. Um. Now is the first one – like, is it facing up or is it facing down? The your vertical line? [*pause*]	
	1212	R3	I think up.	
	1213	Stephanie	Okay. So you want to draw – and is the horizontal line on top of the vertical line or below it?	
	1214	R3	Does it matter?	
	1215	Stephanie	Well, yeah. 'Cause you have to connect it.	
	1216	R3	Uh. It's vertical and it's at the bottom.	
	1217	Stephanie	Okay. So you want to draw up from the um like end point of the horizontal line that you just drew.	
	1218	R3	Um hm.	
	1219	Stephanie	You want to draw up um a straight line	

Time	Line	Speaker	Transcript	Code
			segment that's um five units long.	
	1220	R3	What am I drawing?	
	1221	Stephanie	A rectangle.	
	1222	R3	Oh. [*pause*] Okay.	
	1223	Stephanie	And now you, can you connect the space that's missing? Oh, well, um, now from that point	
	1224	R3	Yeah.	
	1225	Stephanie	you want to go horizontally towards the other	
	1226	R3	Yeah.	
	1227	Stephanie	Do you know what I'm like?	
	1228	R3	How many units?	
	1229	Stephanie	Two.	
	1230	R3	Okay. And so now I have	
	1231	Stephanie	Do you have a rectangle?	
	1232	R3	Yeah, I do.	
	1233	Stephanie	Okay. Good.	
	1234	R3	So now I have a rectangle.	
	1235	Stephanie	Yeah.	
	1236	R3	Okay. That's fine. I have five units	
	1237	Stephanie	Yeah. Five by two.	
	1238	R3	And five units and two units. I have two units. Sounds to me like now I have five, ten, fourteen units?	
	1239	Stephanie	What?	
	1240	R3	I have five, ten, fourteen units, and you were talking about ten. You got ten a little while ago, and now we've got fourteen. I have one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen units.	
	1241	Stephanie	But it's a rectangle?	
	1242	R3	Yeah.	
	1243	Stephanie	And it's connecting?	
	1244	R3	I just drew one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen units. [*counting the perimeter of the rectangle*]	
	1245	R1	You can go over there and help her if you want to, Stephanie.	
	1246	Stephanie	Oh. Okay.	
	1247	R3	I have one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen units in that entire thing.	
	1248	Stephanie	Five by two?	
	1249	R3	One, two, three, four, five [*counts one side*	

Time	Line	Speaker	Transcript	Code
			while Stephanie watches]	
	1250	Stephanie	Yeah.	
	1251	R3	And you take another two.	
	1252	Stephanie	Two and then five.	
	1253	R3	One, two, three, four, five. One, two. That's fourteen.	
	1254	Stephanie	Oh! But that's the perimeter.	BMP; BR-V
	1255	R3	Huh?	
	1256	Stephanie	But that's the perimeter.	
	1257	R3	Okay. Then I want you back over there. I want you to tell me how it is that you come out with this ten thing from here.	
	1258	Stephanie	Oh! All right. You know where you sectioned each unit off?	
	1259	R3	Yeah.	
	1260	Stephanie	Can you draw lines like straight acr- like draw lines straight through where each unit would stop? Wait, I'm trying to think of how to-- All right. Section off one unit at a time but section them all the way across through the rectangle.	BR-V
	1261	R3	You mean draw another line?	
	1262	Stephanie	Well,	
	1263	R1	Parallel	
	1264	Stephanie	For, yeah, like for each unit?	
	1265	R3	Um hm.	
	1266	Stephanie	Like where you sectioned each one?	
	1267	R3	Yeah.	
	1268	Stephanie	Draw it all the way through the rectangle so that you section off a part.	BR-V
	1269	R3	Okay. I did it for one.	
	1270	Stephanie	Okay. Now do it for	
	1271	R3	And what does that mean?	
	1272	Stephanie	Well, do it for all of them.	
	1273	R3	Okay. I'm going up vertically. Is that okay?	
	1274	Stephanie	Yeah. Well.	
	1275	R3	How many am I going to have?	
	1276	Stephanie	Well, you should have two lines – wait	
	1277	R3	(inaudible) I'm just (inaudible)	
	1278	Stephanie	Is the vertical line - ?	
	1279	R3	The vertical was my five one.	
	1280	Stephanie	Okay, then you should have	
	1281	R3	The vertical line is my five one.	
	1282	Stephanie	you should have five lines going up then.	

Time	Line	Speaker	Transcript	Code
	1283	R3	Okay. I should have five lines going up. I only have four.	
15:00-19:59	1284	Stephanie	[*pause*] Oh! That's fine.	
	1285	R3	(inaudible)	
	1286	Stephanie	No no That's okay.	
	1287	R3	(inaudible) but I only have (inaudible)	
	1288	Stephanie	Yeah, but you're supposed to have five parts; four lines	
	1289	R3	I have five parts.	
	1290	Stephanie	Would make five parts.	
	1291	R3	What are those parts?	
	1292	Stephanie	They're sections. But they're not	
	1293	R3	They're sections?	
	1294	Stephanie	Yeah.	
	1295	R3	Five sections?	
	1296	Stephanie	Do you have five sections?	
	1297	R3	Oh. Okay. So I have five sections in my (inaudible)	
	1298	Stephanie	But four lines.	
	1299	R3	Yeah.	
	1300	Stephanie	And now just	
	1301	R3	And a top line and a bottom line. So (inaudible)	
	1302	Stephanie	Now now now section them like	
	1303	R3	Yeah.	
	1304	Stephanie	Horizontally. You should have	
	1305	R3	Okay. And so, I'm gonna I've got I've got to keep that in my head. I have five sections going upward. Is that right?	
	1306	Stephanie	Yeah.	
	1307	R3	Yeah. I have five	
	1308	Stephanie	Yes.	
	1309	R3	sections – (inaudible) Five sections going vertically. I actually had one, two, three, four, five, six lines going across to make those sections, don't I? - can you imagine that in your head? 'Cause that's what matches what I got here. I have a bottom line [*pause*]	
	1310	Stephanie	All right, what?	
	1311	R3	Okay. If you told me to draw, to section off and to use four lines and I did that.	
	1312	Stephanie	Okay.	
	1313	R3	But then I said first I had five lines and you	

Time	Line	Speaker	Transcript	Code
			said that was not (inaudible) but I really had six lines.	
	1314	Stephanie	Why?	
	1315	R3	I had the bottom.	
	1316	Stephanie	Oh. Well. [*laughs*]	
	1317	R3	Okay. That's fine. I now have five sections.	
	1318	Stephanie	Okay. Now section horizontally – like – okay – horizontally you have two, two units, right?	
	1319	R3	You mean each one of those sections? [*pause*] Is each one of those sections.	
	1320	Stephanie	Yes.	
	1321	R3	lines	
	1322	Stephanie	Yes.	
	1323	R3	should be how long did you say?	
	1324	Stephanie	Five?	
	1325	R3	(inaudible)	
	1326	Stephanie	Okay. Five going up, two going	
	1327	R3	Okay. So each of my section lines that you just had me draw	
	1328	Stephanie	Um hm.	
	1329	R3	should be how many units long?	
	1330	Stephanie	Oh. No. No. No. That doesn't	
	1331	R3	Yeah!	
	1332	Stephanie	Well, does it?	
	1333	R3	I think.	
	1334	Stephanie	Oh well.	
	1335	R3	You just told me uh I had the two five five unit long sides.	
	1336	Stephanie	Um hm.	
	1337	R3	And you told, I think, to go up and section it off.	
	1338	Stephanie	Yes.	
	1339	R3	With horizontal lines, okay?	
	1340	Stephanie	Yes.	
	1341	R3	And you told me I would end up with five sections.	
	1342	Stephanie	And you got five sections?	
	1343	R3	And I do have five sections.	
	1344	Stephanie	Okay.	
	1345	R3	Okay. But then I was asking you uh then you started talking about how big each of those sections was.	
	1346	Stephanie	Well, -can you draw the other line and then I can tell you how big they are?	

Time	Line	Speaker	Transcript	Code
	1347	R3	Well, I'd first like to know how long is each of those uh how long is each of my	
	1348	Stephanie	Five units.	
	1349	R3	Horizontal lines.	
	1350	Stephanie	Five units.	
	1351	R3	(inaudible) there are five sections.	
	1352	Stephanie	Five units long.	
	1353	R3	Yeah.	
	1354	Stephanie	Right?	
	1355	R3	Yeah. Five linear units long.	
	1356	Stephanie	Yeah.	
	1357	R3	Okay. Okay. But now I don't have but now I'm not thinking about – I've got these five sections	
	1358	Stephanie	Yeah. You have five sections.	
	1359	R3	Um hm.	
	1360	Stephanie	And they should be five units long.	
	1361	R3	Each section	
	1362	Stephanie	Oh. Well, each section separately?	
	1363	R3	Um hm. What does a section look like	
	1364	Stephanie	Like	
	1365	R3	in your head?	
	1366	Stephanie	Well, you haven't crossed it off vertically so, they're just like this, right? With like – oh well, if this was one section, it would look [*can't see what Stephanie is describing*]	BR-V
	1367	R3	That's what I thought.	
	1368	Stephanie	Yeah, but – wait – I'm trying to think of how it would look. Like if this was one – and you didn't cross it off vertically [*pause*] then it would look like that, right?	
	1369	R1	I think what um you're being asked is, how big is that section?	
	1370	Stephanie	Five units.	OBS
	1371	R1	No. She you said	
	1372	Stephanie	Oh. Um.	
	1373	R1	That this is five units.	
	1374	R3	The section is the	
	1375	R1	The section is	
	1376	R3	little slice	
	1377	R1	How much is the	
	1378	R3	that goes across	
	1379	R1	inside?	
	1380	Stephanie	How much is this?	

Time	Line	Speaker	Transcript	Code
	1381	R1	She's asking for the area of the section.	
	1382	R3	How big is this?	
	1383	Stephanie	Um what's it on top. Oh! Five square units?	OBS
	1384	R3	One of these little – there are five of those stripes across. Five sections.	
	1385	Stephanie	Um hm.	
	1386	R3	Aren't there? How big is each one of them? [*whispering in the background*]	
	1387	Stephanie	Um.	
	1388	R1	I don't understand the question.	
	1389	R3	Oh! Well, maybe we've lost each other. Um	
	1390	R1	We have a different understanding of the question here.	
	1391	R3	Uh my uh I just well I'm not even sure	
	1392	R1	When you're talking about a section, do you mean the region inside?	
	1393	R3	I have a section across. A while ago I was wanting to know how long the how long – my original question to Stephanie was how long	
	1394	Stephanie	Well, how do I know that she has what I'm thinking?	
	1395	R3	how long the line is that divides each of the sections?	
	1396	R1	Oh. So that is a different question than I thought.	
	1397	R3	Yeah, but then it moved into how much space was (inaudible)	
	1398	Stephanie	But	
	1399	R3	Which is a slightly different question.	
	1400	Stephanie	but how do I know she has what I'm thinking?	
	1401	R2	Maybe you should construct it at the same time as she's constructing it.	
	1402	R3	Yeah. You want me to talk you back through it. [*laughs*]	
20:00-24:59	1403	R2	Here.	
	1404	Stephanie	Okay. [*takes a blank sheet of paper*]	
	1405	R1	Now she's going to tell you to do it.	
	1406	R3	Yeah. I'm going to tell you to do it, because I'm not finished yet. I want to make sure we're matching. Um. Uh. You told me the first thing to do was to draw a line.	
	1407	Stephanie	Just a line.	
	1408	R3	Yeah.	

Time	Line	Speaker	Transcript	Code
	1409	Stephanie	Okay.	
	1410	R3	That was five units long.	
	1411	Stephanie	Okay. *[pause]* Okay.	
	1412	R3	Okay. And have you done that?	
	1413	Stephanie	Yes.	
	1414	R3	Uh and you you've marked off the five units so you	
	1415	Stephanie	I'm doing it right now.	BR-V
	1416	R3	So that you really believe it. Okay. Now you asked me, well you told me at the beginning it didn't matter, but then you asked me whether it was vertical or horizontal.	
	1417	Stephanie	Um hm.	
	1418	R3	And I told you mine was vertical.	
	1419	Stephanie	Yeah.	
	1420	R3	So let's make yours be vertical too.	
	1421	Stephanie	Okay.	
	1422	R3	And then uh on either end of it	
	1423	Stephanie	Um hm.	
	1424	R3	You said it didn't matter, but I did mine from the bottom.	
	1425	Stephanie	Okay.	
	1426	R3	Okay. Uh. Then you asked me to draw a horizontal line and you told me that meant it was ninety degrees. Uh. Uh. That was two units.	
	1427	Stephanie	Okay.	
	1428	R3	And you told me that the units for that had to be the same length as the	
	1429	Stephanie	Yes.	
	1430	R3	(inaudible) isn't that what you said?	
	1431	Stephanie	Um hm.	
	1432	R3	Okay. So now you have a side of a rectangle and a bottom of a rect-. Well about that time you told me that we (inaudible) a rectangle. And so then you said to do another line that was vertical on the other side of the bottom.	
	1433	Stephanie	Um hm.	
	1434	R3	That was also five units. [*pause*]	
	1435	Stephanie	Okay.	
	1436	R3	And then you said, [*chuckles*] 'can't you see? There's got to be a top' or something like that. And so you did uh a two unit (inaudible)	
	1437	Stephanie	Okay.	

Time	Line	Speaker	Transcript	Code
	1438	R3	Okay. That was when I said I got fourteen units.	
	1439	Stephanie	Mmm.	
	1440	R3	Remember?	
	1441	Stephanie	Yeah, but that was the perimeter.	
	1442	R3	Yeah. Do you have fourteen units also?	
	1443	Stephanie	Yeah.	
	1444	R3	Okay. But then you said 'Well that's the perimeter. You got to deal with something else.' And you told to go back and to to go from the bottom of of you know one that five units side. And you told me to draw uh uh four more horizontal lines	
	1445	Stephanie	Like	
	1446	R3	to make that's what you called the sections.	
	1447	Stephanie	Yes.	
	1448	R3	You said you said four horizontal lines that are going to section it off.	
	1449	Stephanie	Okay.	
	1450	R3	And then I said (inaudible). You said that I should have five sections.	
	1451	Stephanie	Um hm.	
	1452	R3	Do you?	
	1453	Stephanie	Yeah.	
	1454	R3	You have five sections.	
	1455	Stephanie	Of the, yes.	
	1456	R3	Yeah. Okay. And so that's fine. I said I have five sections that are going this way up the vertical side. Then my question to you was uh the lines that that that I used to make off those sections to go, I asked you how long they were. That was really all.	
	1457	Stephanie	Um hm.	
	1458	R3	How long are they?	
	1459	Stephanie	Um. Two units long.	
	1460	R3	I think. Mine are.	
	1461	Stephanie	Yes.	
	1462	R3	And so we agree. Okay. Uh. And then then we we all got talking and then I said well how how big is each of those sections?	
	1463	Stephanie	How big is each of those sections?	
	1464	R3	Um hm.	
	1465	Stephanie	Um. Of like perimeter? Or like area?	PAH
	1466	R3	Just how big it it? I mean, how big?	

Time	Line	Speaker	Transcript	Code
	1467	Stephanie	Well, do you want to know like 'cause the perimeter is um six.	BMP
	1468	R3	It sure is. So the perimeter of each of them is six.	
	1469	Stephanie	But the area is um two.	BMP
	1470	R3	Two what?	
	1471	Stephanie	Square units.	
	1472	R3	Okay. So the area of each of my sections is two square	
	1473	Stephanie	Yeah.	
	1474	R3	And I have how many?	
	1475	Stephanie	Total?	
	1476	R3	No, how many sections? I mean counting them up, how many of these sections do I have?	
	1477	Stephanie	Oh! Five.	BR-V
	1478	R3	Yeah. And each one is?	
	1479	Stephanie	Um. What do you want to know?	
	1480	R3	Each of those sections is? You just told me.	
	1481	Stephanie	Oh. Two square units?	
	1482	R3	Yeah.	
	1483	R1	Why don't you draw those two square units for me? [*pause*]	
	1484	R3	I'm going to do it (inaudible)	
	1485	R1	Show me the section that's two square units, Stephanie. Show me a section that's two square units. [*You can hear the marker coloring, but what is being colored can not be seen.*] Um hm.	BR-V
	1486	R3	I did it for my bottom. Did you do it for your bottom one?	
	1487	R1	She could make it for her bottom one easily. [*laughs*]	
	1488	R3	Um. I have I have my bottom section sectioned off into – how do I know it's two? Where does the line go to make it two?	
	1489	Stephanie	Okay. You you know that section line that you made?	
	1490	R3	Um hmm.	
	1491	Stephanie	For um the two side?	
	1492	R3	You mean this is the first section line up?	
	1493	Stephanie	Yeah.	
	1494	R3	(inaudible)	
	1495	Stephanie	You know that line that you made to divide it	

Time	Line	Speaker	Transcript	Code
			into two parts?	
	1496	R3	Um.	
	1497	Stephanie	Well, just draw it all the way through your ss – rectangle.	
25:00-29:59	1498	R3	I'm mostly only concerned in the bottom section right now.	
	1499	Stephanie	Oh. Well, then just draw it through to the section line.	
	1500	R3	From the middle or what?	
	1501	Stephanie	Oh. No. From the bottom.	
	1502	R3	Oh, from the bottom?	
	1503	Stephanie	From like um the divider for the two sections.	
	1504	R3	Yeah.	
	1505	Stephanie	Right?	
	1506	R3	Uh huh.	
	1507	Stephanie	You, do you know what I'm talking about? On the two side. Not on the five side.	
	1508	R3	Yeah. Yeah. Oh, I see.	
	1509	Stephanie	Okay.	
	1510	R3	Yeah.	
	1511	Stephanie	Well, draw that from the bottom line to the first marking of the section.	
	1512	R3	The first the first rung on the ladder more or less?	
	1513	Stephanie	Yeah.	
	1514	R3	Okay. And so?	
	1515	Stephanie	And that shows you your um	BR-V
	1516	R3	Two square units.	
	1517	Stephanie	Two parts. Yeah.	BR-V
	1518	R3	One and then the other.	
	1519	Stephanie	Um hm.	
	1520	R3	Okay. And so, in that bottom section I've got	
	1521	Stephanie	Two square units.	
	1522	R3	Two square units. Why again?	
	1523	Stephanie	Um because um	
	1524	R3	Why is each one a square unit?	
	1525	Stephanie	What?	
	1526	R3	Why is each one a square	
	1527	Stephanie	Oh! Because each side of it is one.	BEJ
	1528	R3	Ooh. Okay. So in that bottom section, I have one section there on the bottom. It's got two square units.	
	1529	Stephanie	Yes.	
	1530	R3	What about the other sections?	

Time	Line	Speaker	Transcript	Code
	1531	Stephanie	Well, draw your line, keep drawing your line.	
	1532	R3	Okay. So in my section two?	
	1533	Stephanie	And that has two more square units.	
	1534	R3	Okay.	
	1535	Stephanie	And if you keep drawing it,	
	1536	R3	My section three?	
	1537	Stephanie	two more.	
	1538	R3	Oh. Uh huh.	
	1539	Stephanie	And that has two more. And that has two more.	
	1540	R3	Huh. And so that means I have five sections, like you just told me?	
	1541	Stephanie	Yes.	
	1542	R3	Each with two	
	1543	Stephanie	Um hm.	
	1544	R3	square units?	
	1545	Stephanie	Yes.	
	1546	R3	How much is that?	
	1547	Stephanie	Ten. [*laughs*]	BMP; BR-V
	1548	R3	That's nice. That really helped me to think about it because I've been over here trying to say if I didn't know-	
	1549	Stephanie	Um hm.	
	1550	R3	-how could I know that there weren't just ten little square units floating around in there?	
	1551	Stephanie	Um hm.	
	1552	R3	You know what I mean?	
	1553	Stephanie	Yes.	
	1554	R3	Thank you.	
	1555	Stephanie	Thank you.	
	1556	R1	Well. Tired?	
	1557	R3	Does that for squa- hey Steph, would that work for squares too?	
	1558	Stephanie	Yeah.	
	1559	R3	Or it's just because it was a rectangle?	
	1560	Stephanie	No. It would work for squares. Wouldn't it? Yeah.	
	1561	R3	Okay.	
	1562	Stephanie	Yeah, yeah it would work for squares.	BCA
	1563	R3	That same kind of description?	
	1564	Stephanie	Um hm. Well, like the lines would have to be like the same length.	BEJ; BR-V
	1565	R3	Um hm.	

Time	Line	Speaker	Transcript	Code
	1566	Stephanie	But it would work. Like I could tell you to draw the sections and mark them off and it would still come out the same.	BEJ; BR-V
	1567	R3	Okay. If it was a square, so three by three?	
	1568	Stephanie	Um hm.	
	1569	R3	What would a section be?	
	1570	Stephanie	A section like?	
	1571	R3	Like we've been talking about here?	
	1572	Stephanie	Like if a square was [*pause*] it would be, do you mean one section by itself? It would just be one square unit.	BR-V
	1573	R3	My section for this rectangle was two.	
	1574	Stephanie	Well, you're using um, what?	
	1575	R3	You just told me a while ago	
	1576	Stephanie	Oh. You mean if you didn't section it off totally. You just sectioned	
	1577	R3	(inaudible)	
	1578	Stephanie	it one way.	
	1579	R3	A while ago, we drew this rectangle. And after I got messed up with the fourteen, you told me to split it off. And then I had five sections going across here.	
	1580	Stephanie	Oh. But you on- you only sectioned it one way.	
	1581	R3	But that was a section.	
	1582	Stephanie	Yeah. I, I	
	1583	R3	And then in each of those sections you told me I had two square units.	
	1584	Stephanie	Well, then it would be three. No.	BR-V
	1585	R3	Yeah. Help me to think about that.	
	1586	Stephanie	Okay. Because what I did was I had sectioned it off both ways right away. But if you only sectioned the three, if the square is three by three	BEJ; BR-V
	1587	R3	Uh huh.	
	1588	Stephanie	and you section it off one way.	BR-V
	1589	R3	Uh huh.	
	1590	Stephanie	One section is gonna be three	BR-V
	1591	R3	Square units.	
	1592	Stephanie	Yes.	
	1593	R3	And then the second section up?	
	1594	Stephanie	Is gonna be the same.	BR-V
	1595	R3	And the third section?	
	1596	Stephanie	Is gonna be the same.	

Time	Line	Speaker	Transcript	Code
	1597	R3	And each one is gonna be (inaudible)	
	1598	Stephanie	Three square units.	BR-V
	1599	R3	And you're gonna have three sections	
	1600	Stephanie	Yes.	
	1601	R3	and each is gonna have	
	1602	Stephanie	Three square units.	BR-V
	1603	R3	Three square units, yes.	
	1604	R1	What's the difference in the when you do a rectangle as compared to doing a square? For finding the area of the square?	
	1605	Stephanie	Nothing, except it's gonna be the same.	BR-V
	1606	R1	What's gonna be the same?	
	1607	Stephanie	The sides. And that's it.	BR-V
	1608	R1	Are you tired? [*Stephanie laughs*] You've worked so hard, Stephanie. You really have. Um. Does this make any sense?	
	1609	Stephanie	Well, yeah, but –I just, I mean, I knew how to do it beforehand. I just never sat down and went like	
	1610	R1	Well, there's a difference between knowing how to do math and get a right answer, and thinking about the meaning of it.	
	1611	Stephanie	Um hm.	
	1612	R1	That's where it separates the clerks from the mathematicians. Do you know what I'm saying? If I just tell you do this, do this, do this, and you do it and you don't think about it, um it's hard for you to build in your head the things to think more deeply about. So – if it's useful for you and if if you find it interesting, that's what I want to spend our time doing.	
	1613	Stephanie	Um hm.	
30:00-35:32	1614	R1	Think deeply about the things that you're doing. You know, I think that's um where maybe we're still doing algebra, and sometimes thinking about it – thinking about some ideas in geometry help you to understand the ideas in algebra. But also thinking about some of the arithmetic that makes sense to you, to see some of that arithmetic works in general. And that sorta connects the arithmetic to the algebra. Um. I think I think what maybe, I don't know what you think um, do you need to think about this a little bit more maybe? And see if when all these papers get pushed aside if	

Time	Line	Speaker	Transcript	Code
			you if you kept a journal and wrote about what we did today as best as you can remember and	
	1615	Stephanie	Okay.	
	1616		reconstruct it? I think now it would be nice to sort of keep a portfolio, uh, because I think that it helps to um if we could make copies of these and give you copies of everything. So we could a set, having a copy might help you. So if you had to like write a report of someone – pretend that we were um you were writing to a friend at Harding. And they wanted to know what you did.	
	1617	Stephanie	Okay.	
	1618	R1	You know. Imagine, just pretend for a minute, you were writing to someone like uh Michelle or Jeff or some some of that crew. And or Ankur. He's not, he's still there. Right? Ankur's still there. Milan's not there. And um reading what you did, they'd have a better sense.	
	1619	Stephanie	Okay.	
	1620	R1	Dr. Alston goes there a lot.	
	1621	R3	Yeah.	
	1622	R1	She might even um start a dialog going which might be useful.	
	1623	R3	(inaudible) that going (inaudible)	
	1624	R1	I don't know. It's extra work, but I I think before we move on it would be useful to try to think about this as if you were explaining it to someone who hasn't thought about it. 'Cause I think there is going to some point – you're not doing binomials yet, and binomial expansions, but there's going to be some point in your class that's it's gonna be helpful to share, to talk about these ideas. You know, at some point in your class, after they've had a chance to think about it.	
	1625	Stephanie	Okay.	
	1626	R1	You know what I'm saying?	
	1627	Stephanie	Yes.	
	1628	R1	So I think this is kind of uh useful.	
	1629	Stephanie	Um hm.	
	1630	R1	And I'm not kidding. Where I hope to go by the end of this year is to *a* plus *b* to the twenty-fifth or something or thirtieth or fiftieth.	

Time	Line	Speaker	Transcript	Code
			Right? It's sorta where I want to go. And what it means and uh how we can begin to think about this. And you really thought about these ideas in different forms in the past. Which is how you can pull them in and connect them.	
	1631	Stephanie	Okay.	
	1632	R1	Okay. So I thank you all. When I tell you about time, this is what tends to happen. Do you have anything, uh, Elena?	
	1633	R4	No.	
	1634	R1	Ethel?	
	1635	R2	No. I was just thinking I'm gonna go home and ask Christine to think about this, too. And maybe Christine can write to you what she's thinking and	
	1636	R1	Would you like that? Where does Christine go to school?	
	1637	R2	Bridgewater Raritan Middle School. She's in the eighth grade, taking algebra also. So …I'll ask her.	
	1638	R1	Would that be interesting? To have a pen pal so you can make it real. It also would be nice to bring in our friends at Harding.	
	1639	R2	Yeah.	
	1640	R1	I'm going to push them. They can't get away without thinking about this. Dr. Alston, let's put them to work.	
	1641	R3	I think we should.	
	1642	Stephanie	Yeah!	
	1643	R3	Yeah. As I said I'm I'm I'm working with a group and (inaudible)	
	1644	R1	are doing the same thing.	
	1645	R3	(inaudible)	
	1646	R1	So now what's the easiest? I really have lost track of this in terms of the order. Maybe, Stephanie, you could look at these and see what order makes sense to you. And we'll renumber them.	
	1647	Stephanie	Not number other than just the order that would	
	1648	R1	Well, that you think and then maybe we can make copies for you and then I could have a set. You know, make sure you put them in some sort of order.	

Time	Line	Speaker	Transcript	Code
	1649	Stephanie	Well, it might make sense if you put the um squares first.	
	1650	R1	This was this was number two. So we know numbers one and two.	
	1651	Stephanie	And this was three.	
	1652	R1	So why don't you put them one, two, three.	
	1653	Stephanie	It might make more sense if you put the squares first.	
	1654	R1	Okay.	
	1655	Stephanie	'Cause that *a* squared and then you get into the *a* plus *b* thing.	
	1656	R1	Okay. So the *a* squared (inaudible)	
	1657	Stephanie	You see how this was like the first paper we did when	
	1658	R1	So that'd be four then. Let's make that a four.	
	1659	Stephanie	All right.	
	1660	R1	This is one, two, three.	
	1661	Stephanie	Yeah.	
	1662	R1	Why don't you do it? Okay.	
	1663	Stephanie	Okay.	
	1664	R1	And after this did we have any divergence here? Um.	
	1665	Stephanie	With that was this.	
	1666	R1	Okay, that was, this was a suggestion from Ethel. Why don't we make that five?	
	1667	Stephanie	Okay.	
	1668	R1	Then you think we did this one? Okay.	
	1669	Stephanie	This is six?	
	1670	R1	This is six. Okay.	
	1671	Stephanie	Okay.	
	1672	R1	Um. Then I guess, I think – where's that other one? I think you ought to try to take a good look at that and then maybe explain to me why when you looked at that it wasn't helpful.	
	1673	Stephanie	Okay. So you want me to	
	1674	R1	Yeah. Let's make that seven.	
	1675	Stephanie	Okay.	
	1676	R1	Okay.	
	1677	Stephanie	This is eight?	
	1678	R1	Right. So that if you were discussing this with some of your classmates who might do this, you know, where do we get in trouble?	
	1679	Stephanie	Okay.	
	1680	R1	You know this is *a*, right?	

Time	Line	Speaker	Transcript	Code
	1681	Stephanie	Um hm.	
	1682	R1	Is it possible that this piece could be *a*? Would you call this *a*? *[points out some inconsistencies in Stephanie's first attempt at sectioning a square into (a + b) segments*] You got to think about that. That's a very interesting problem. And this is eight and that's nine.	
	1683	Stephanie	Okay.	
	1684	R1	You call this one nine. 'Cause you tested it here.	
	1685	Stephanie	Okay.	
	1686	R1	And then maybe we should schedule another time to come back. I was thinking of every other week.	
	1687	Stephanie	Okay.	
	1688	R1	Is that too much?	
	1689	Stephanie	No. That's fine.	
	1690	R2	I don't have a problem with that.	
	1691	R1	Is that okay, Ethel? Okay. So uh we'll talk to your teacher and try to schedule a convenient time for you.	
	1692	Stephanie	Okay.	

APPENDIX C: TRANSCRIPT – SESSION 3

INTERVIEW WITH STEPHANIE Time: 56 minutes (1 CD)
February 7, 1996

R1: Dr. Carolyn Maher Stephanie: Stephanie R2: Dr. Ethel Muter
R3: Dr. Alice Alston R4: Dr. Elena Steencken

Time	Line	Speaker	Transcript	Code
00:00 –04:59	1	R1	Okay. What I'm going to ask you to do is tell me what you did.	
	2	Stephanie	Alright. Well –	
	3	R1	You can go through this with me.	
	4	Stephanie	'Cause like I didn't know what you wanted, so I basically – it just says like what we did – like in the papers.	
	5	R1	You want me to …[*R1reads from the paper.*] "The problem *a* plus *b* quantity squared is the problem I have worked on the last two times. Rutgers" [*pause*]	
	6	Stephanie	(inaudible) Oh. My handwriting's a little sloppy.	
	7	R1	"My first answer" – It's better than mine. – "My first answer was very simply to distribute the square. *a* squared plus *b* squared was proved wrong, though, when I was asked to use numbers in place of variables – two plus three quantity squared to test two squared plus three squared." You got twenty-five and thirteen –	
	8	Stephanie	Um hm.	
	9	R1	They're not the same.	
	10	Stephanie	Yeah.	
	11	R1	You know what we sometimes do? This might be helpful to you. You have two plus three quantity squared – you can put a question mark – does it equal two squared plus three squared? [*R1 writes* $((2+3)^2 = 2^2 + 3^2$ *with a ? on top of the* =.]	
	12	Stephanie	Okay?	
	13	R1	Do you see what I'm doing here?	
	14	Stephanie	Um hm.	
	15	R1	And so what you do next, then you have five squared. You still don't know yet, right? Does it equal	

Time	Line	Speaker	Transcript	Code
	16	Stephanie	Um hm.	
	17	R1	four plus nine. Now you have twenty-five and thirteen and now [*R1 writes* $5^2 = 4 + 9, \quad 25 \neq 13$]	
	18	Stephanie	Not equal.	
	19	R1	you can say 'not equal'. So that might help you with notation a little bit.	
	20	Stephanie	Okay.	
	21	R1	Okay.	
	22	Stephanie	(inaudible)	
	23	R1	(inaudible) answer.	
	24	Stephanie	'Cause it's really hard to	
	25	R1	Why don't you read?	
	26	Stephanie	"Disregarding that answer I was asked what *a* plus *b* quantity squared really meant. My answer was easy. *a* plus *b* quantity squared equals *a* plus *b* times *a* plus *b*. Then for a moment, we got slightly off the subject. I was asked the question 'Of any circumstance when *a* plus *b* quantity squared equals *a* squared plus *b* squared'. I said "Yes, there was one circumstance. When *a* equals *b* equals zero, then *a* plus *b* quantity squared equals *a* squared plus *b* squared. With this question answered, we came back to the original problem of *a* plus *b* quantity squared. Now a new concept was brought into the picture. I was asked if I could explain and display *a* squared on a square. I was so dumbfounded. I really had no idea how to show them. Many squares were drawn. The subject of area was discussed. The area of a square is length times width or *a* squared. Still this didn't help me. Around this time we started to discuss the difference between a unit and a square unit. This is a unit in length." This is a unit, or this is a unit, or this – you know? [*Stephanie points to different parts on the paper.*]	BR-V
	27	R1	Okay.	
	28	Stephanie	And this is a square unit, you know?	BR-V
	29	R1	Okay.	
	30	Stephanie	"Next we talked about a square having – wait – we talked about a square with each side equaling *a* plus *b*. After a failed attempt at drawing the square, we came up with this-"	BR-V

Time	Line	Speaker	Transcript	Code
			[*Stephanie looks for the picture she drew.*] You know, the *a* plus *b*, which was, I don't know, one of these papers. [*continues shuffling the papers on the table*] This one?	
	31	R1	Okay.	
	32	Stephanie	And then [*shuffles more of the paper*] Oh. I must've – Okay. [*Stephanie returns to reading her summary of the last session.*] "So we found out that *a* plus *b* times *a* plus *b* equaled *a* plus *b* quantity squared equals *a* plus *aa* plus *ab* plus *bb* plus *aa* which also equals *a* squared plus *ab* plus *ab* plus *b* squared, which equals *a* squared plus *2ab* squared plus *b* squared.	BR-S
	33	R1	*2ab*	
	34	Stephanie	Yeah. "Which like went down to *a* plus *b* quantity squared equals *a* squared plus *2ab* plus *b* squared. And then we tested it. After conducting this and concluding that it worked, we decided that I should try and explain why one unit by one unit equaled one square unit by having Dr. Alston draw a picture of a square without having me see it. And then we did the same thing with a rectangle."	BMP
	35	R1	Um hm.	
	36	Stephanie	And that was	
	37	R1	Where we left off.	
	38	Stephanie	basically what we did.	
	39	R1	Okay. Does that make sense?	
	40	Stephanie	Yeah. I understand what we did. But it's still – I'd probably like take a minute to try to explain it again. Like if I had to explain like the whole thing, it'd probably take me a minute once we got down to like explaining the square-	
	41	R1	Um hm.	
	42	Stephanie	-like if I had to explain a square again, I'd be (inaudible)	
	43	R1	That's the hard part?	
	44	Stephanie	Well – 'cause you don't know what the person's drawing. So I could be like 'Draw a line' and they could be like – you know?	
	45	R1	Slanting?	
	46	Stephanie	So I'd – it's really easier if you can see what you're doing.	
	47	R1	Um hm. Right. I think so. Neat! Um. Okay. Um. Just to – that's actually very nice,	

Time	Line	Speaker	Transcript	Code
			Stephanie. That's a very lovely write up. Um. How about – you have a younger sister? Susie?	
	48	Stephanie	Yes.	
	49	R1	Is that her name?	
	50	Stephanie	Um hm.	
	51	R1	Okay. I talked to her briefly.	
	52	Stephanie	[*whispers*] Oh God.	
	53	R1	And um. She's very friendly. [*Stephanie chuckles.*] Now um Susie's in what grade?	
	54	Stephanie	Sixth.	
	55	R1	Sixth grade. Okay, that's good. Um. Now suppose Susie wanted to understand what you were doing. But she's not studying algebra, right?	
	56	Stephanie	Yeah.	
	57	R1	Okay. We have some things here. Okay. [*R1 removes several items from the bag beside her. These include squared materials and a bag of odd shaped plastic pieces.*] We have these things –these things. Right?	
05:00-09:59	58	Stephanie	Um hm.	
	59	R1	Um. We have some things in the bag here. And we have some of these things, which, by the way, I have never used before. Um. So when I tell you, I really haven't – [*Stephanie chuckles.*]	
	60	R1	Um. Dr. Alston threw some of these things in – thinking maybe you want to use any of these.	
	61	Stephanie	Okay.	
	62	R1	Now. Can you kinda maybe think for a minute and see how you could use some of these things to explain to Susie what you were doing here?	
	63	Stephanie	Oh. Um.	
	64	R1	That might be – that might be appropriate for her in the sixth grade.	
	65	Stephanie	Do you want me to explain *a* squared or do you want me to explain like *a* plus *b* quantity squared?	PAH
	66	R1	Well, you know Susie.	
	67	Stephanie	Yeah.	
	68	R1	(inaudible)	
	69	Stephanie	We'd have to start out with 'unit' and 'square unit'.	BR-V
	70	R1	Okay. You start out where you think Susie is	

Time	Line	Speaker	Transcript	Code
			and kind of try to think what you might do.	
	71	Stephanie	Like what's this? Ten by ten? [*Stephanie picks up a 'flat' and counts the intervals on one side*.] Yeah. It's ten by ten. And like [*Stephanie takes more of the squared materials from the bag*.] if she knew it was – she knows how to get um - I'm sure she knows how to get area. And it would be – you know	BR-V
	72	R1	Why don't you pretend she doesn't. Don't take anything for granted. 'Cause she just might know a formula.	
	73	Stephanie	Um hm.	
	74	R1	But she might not know what it means. You know? So suppose you even – you wanted her to really understand what she's doing.	
	75	Stephanie	So I'd have to explain what area was	
	76	R1	So you might even want to come back and try to introduce the whole idea of area to her.	
	77	Stephanie	Okay. Well, area is like um the amount of space inside like a sp…an object. Um. So and to find the area of a square it's like length times width or if – especially when you're dealing with a square 'cause like the sides are all equal it's like one side squared. So if this is ten, it would be ten squared.	BR-V; BMP
	78	R1	So, I'm going to be Susie.	
	79	Stephanie	Or ten times ten.	BMP
	80	R1	Can I role play?	
	81	Stephanie	Yeah.	
	82	R1	What do you mean ten? Where did you get ten?	
	83	Stephanie	Oh. Well, there's ten – you see, it's ten units long. This is like one unit.	BR-V
	84	R1	Can you show me what's a unit?	
	85	Stephanie	See this *[Stephanie puts a cube over the 'square' in the top left corner of the 'flat'*.]	
	86	R1	This square is one unit?	
	87	Stephanie	Yeah. Like this square, [*a cube*] this is like a littler piece like that's how big that is.	BR-V
	88	R1	And you're calling this a unit?	
	89	Stephanie	Yes.	
	90	R1	Okay.	
	91	Stephanie	Oh. One square unit.	OBS; BR-V
	92	R1	Oh. This is one square unit. But I don't know what a unit is.	

Time	Line	Speaker	Transcript	Code
	93	Stephanie	This is a unit. You see this like side right here. [*Stephanie points out the length of one unit on the side of the 'flat'.*]	BR-V
	94	R1	Can you show me here too? [*R1 holds up a cube*.]	
	95	Stephanie	Like this. [*Stephanie shows R1 the length of the edge on the cube.*]	BR-V
	96	R1	Oh. Okay.	
	97	Stephanie	Or like that. Or any – that's a unit	BR-V
	98	R1	Okay.	
	99	Stephanie	And so this – If you were going to get the area of this [*the cube*] it would be one unit by one unit	BR-V
	100	R1	Um hm.	
	101	Stephanie	and so it would be one square unit.	BMP
	102	R1	Okay.	
	103	Stephanie	So to get the area of this [*the flat*] – there are ten units – ten square units going this way and ten – like length and width	BR-V
	104	R1	Um hm.	
	105	Stephanie	And so it would be ten times ten	BMP
	106	R1	Um hm.	
	107	Stephanie	and you'd get a hundred.	BMP
	108	R1	Um hm. And how can I be sure there's a hundred?	
	109	Stephanie	Well, you could count them if you wanted to.	PNE
	110	R1	I don't want to do that, okay.	
	111	Stephanie	But…	
	112	R1	I believe you.	
	113	Stephanie	Yeah. And so – and another way you could get like um a hundred like if you're multiplying any number by itself you can also say like ten squared or nine squared or eight squared or you know eleven squared.	BR-S
	114	R1	Um hm.	
	115	Stephanie	And it just means that number multiplied once by itself. So ten times ten.	BMP
	116	R1	Okay. But now you were doing algebra. *a*'s and *b*'s. and I'm a very curious little sister. And I really want to sorta know what you're doing with *a*'s and *b*'s.	
	117	Stephanie	Um hm.	
	118	R1	What does this have to do with *a*'s and *b*'s?	
	119	Stephanie	Well.	

Time	Line	Speaker	Transcript	Code
	120	R1	Can you make me something that looks like …	
	121	Stephanie	*a* is any length.	BR-S
	122	R1	Okay.	
	123	Stephanie	So *a* stands for any number.	BR-S
	124	R1	Um hm.	
	125	Stephanie	And we're gonna – like if this side was *a* units long	BR-V
	126	R1	Um hm	
	127	Stephanie	Like you didn't – I'm trying to think if there's anything in there that's not marked – [*Stephanie looks through the materials on the table for an example.*] Well - like	BR-V
	128	R1	I don't know what these are.	
	129	Stephanie	Yeah.	
	130	R1	You might want to take a look. I've never seen them.	
	131	Stephanie	I think I – we used them last year to build like weird shapes. Oh, here's [*Stephanie pulls a blue square of the bag of shapes.*] like if this was a square.	BR-V
	132	R1	Square?	
	133	Stephanie	Oh, well this is kinda – [*Stephanie puts aside the blue shape and picks up the flat.*] We'll just use this. It's easier.	
	134	R1	Well, no. I'm just curious.	
	135	Stephanie	Well like if this was a square?	BR-V
	136	R1	So, what am I supposed to imagine, that this is a straight line?	
	137	Stephanie	Yeah, that those are all straight lines.	BR-V
	138	R1	Um hm.	
	139	Stephanie	But this isn't marked, so you don't know how many units long it is.	BR-V
	140	R1	Um hm.	
	141	Stephanie	And you don't know how many units wide it is.	BR-V
	142	R1	Um hm.	
	143	Stephanie	And then that would – uh – *a* length long, *a* length long, *a* length – 'cause you don't know - *a* can stand for any number. And… [*Stephanie indicates that each side of the blue shape is 'a' length long.*]	BR-V
	144	R1	Why can't you do the same thing here? Why can't I pretend…	
	145	Stephanie	'Cause it's marked.	OBS
	146	R1	I don't know.	

Time	Line	Speaker	Transcript	Code
	147	Stephanie	It's marked so it's harder you know.	OBS
	148	R1	Um hm.	
	149	Stephanie	Like it would, but it would be easier to imagine if you	
	150	R1	I see.	
	151	Stephanie	had something that wasn't marked.	
	152	R1	I see.	
	153	Stephanie	So like if this wasn't marked it would be *a* length by *a* length and to find the area	BR-V
	154	R1	Um hm.	
	155	Stephanie	of an object that's like *a* length long it would be *a* length squared or *a* length times *a* length.	BR-V; BMP
	156	R1	Um hm.	
	157	Stephanie	And you'd get area.	BMP
	158	R1	Um hm.	
	159	Stephanie	And so that's where *a* comes into it.	
	160	R1	Um hm. Okay.	
10:00-14:59	161	R1	What always confuses me about these blocks is that it also has a height. [*R1 and Stephanie chuckle.*] You know. And this kinda [*R1 indicates the blue square Stephanie had selected earlier.*] does, too, but it sorta doesn't look like it does.	
	162	Stephanie	Yeah.	
	163	R1	You know. Um. Okay. Interesting. So, um, you said, you'd make this *a*, but what about the *a* plus *b*? How would you handle that?	
	164	Stephanie	Oh. Okay. Well, if you were to say *a* and *b* are both numbers that – they're not the same, but they stand for any number.	BR-S
	165	R1	Okay.	
	166	Stephanie	So *a* doesn't equal *b*.	BR-S
	167	R1	Okay.	
	168	Stephanie	And	
	169	R1	But could it?	
	170	Stephanie	It could, but like in this problem, it's not.	
	171	R1	Okay.	
	172	Stephanie	Well, it could, but either way – um	
	173	R1	Okay. So it works even when they're not equal.	
	174	Stephanie	Yeah.	
	175	R1	You're saying	
	176	Stephanie	But… so if this side was *a* plus *b* long.	BR-S
	177	R1	Um hm.	
	178	Stephanie	[*Stephanie uses the blue 'square' from earlier.*]	BR-V

Time	Line	Speaker	Transcript	Code
			We'll say that this corner from here over is *b*.	
	179	R1	Oh. Okay.	
	180	Stephanie	And this is *a*.	BR-V
	181	R1	Okay.	
	182	Stephanie	And that this is *b* and this is *a*	BR-V
	183	R1	Okay.	
	184	Stephanie	And this is *b* and this is *a* – No. This is *a* and this is *b*. [*Stephanie indicates these lengths on the edges of the blue 'square'.*]	BR-V
	185	R1	Okay.	
	186	Stephanie	And this this is *a* and this is *b*.	BR-V
	187	R1	And so you called this piece *b*.	
	188	Stephanie	Yeah. The shorter piece.	BR-V
	189	R1	So what's this piece here?	
	190	Stephanie	*b*	BR-V
	191	R1	And this is?	
	192	Stephanie	*a*	BR-V
	193	R1	*a*	
	194	Stephanie	And that's *b*.	BR-V
	195	R1	Um. Wait a minute.	
	196	Stephanie	Oh. – this little piece is *b*, but this piece is *a*.	BR-V
	197	R1	Okay. Alright. Interesting. Now – now I'm going to be your little sister again. Suppose I wanted *a* to be three and *b* to be seven.	
	198	Stephanie	Okay. Well. You could mark it off like *a*.	BR-V
	199	R1	Can you show me with this particular model, or anything else, with *a* being three and *b* being seven, that'll work here, right? Because *a* plus *b* is ten, right? How that works?	
	200	Stephanie	Well	
	201	R1	Just – you can show me – Yeah, any way you want to.	
	202	Stephanie	You want me to	
	203	R1	Yeah.	
	204	Stephanie	'cause I can build it with like…	BR-V
	205	R1	Okay.	
	206	Stephanie	[*Stephanie takes out more of the squared materials.*] Oh. Well. [*She tries to use three 'longs', unsuccessfully.*] Actually… *a* is that long, right? [*Stephanie tries to use three cubes.*]	BR-V
	207	R1	Okay.	
	208	Stephanie	You're saying that you want *a* to be that long.	
	209	R1	*a* to be three	

Time	Line	Speaker	Transcript	Code
	210	Stephanie	and *b*	
	211	R1	to be seven.	
	212	Stephanie	to be seven.	
	213	R1	Um hm. [*Stephanie builds.*]	
	214	R1	If we're not going to use these, so let's get them out of the way of here. Let's get (inaudible)	
	215	Stephanie	[*Stephanie continues to build- she makes a row of ten cubes, and she then separates them into two groups – one of three cubes and one of seven cubes.*] There. *a* is these these three and *b* is those seven. [*She pushes the cubes back together.*]	BR-V
	216	R1	Um hm.	
	217	Stephanie	And that makes ten.	
	218	R1	Okay. But how can you – can you tell me that *a* plus *b* quantity squared – three plus seven quantity squared would be *a* squared – three squared, right? How does that work? Let's actually write it out. Let's take this piece of paper. You told me that *a* plus *b* quantity squared does not equal *a* squared plus *b* squared when *a*	
	219	Stephanie	*a* and *b* are not equal to zero.	
	220	R1	Right, *a* is not equal to *b* which is not – let's just say *a* equals *b*, but they're not equal to zero.	
	221	Stephanie	Yes.	
	222	R1	Alright. Agreed. But we're not – we're saying let *a* equal – What did we make *a*? Three? Is that what we said?	
	223	Stephanie	Um hm.	
	224	R1	And *b* equals seven. Right?	
	225	Stephanie	Um hm.	
	226	R1	Now you said that *a* plus *b* quantity squared was not *a* squared plus *b* squared, but there was some other thing. Right? Do you remember what that was?	
	227	Stephanie	I said that *a* plus *b* was not *a* squared plus *b* squared.	
	228	R1	What was it?	
	229	Stephanie	Oh! That it was *a* squared plus two *ab* plus	BR-S; PPK
	230	R1	plus *b* squared	
	231	Stephanie	*b* squared.	BR-S; PPK

Time	Line	Speaker	Transcript	Code
	232	R1	Right. So if *a* is three and *b* is seven, right?	
	233	Stephanie	Um hm.	
	234	R1	Okay. So the … so we're talking about a square. Right?	
	235	Stephanie	Um hm.	
	236	R1	Okay. With side equal to ten. Right?	
	237	Stephanie	Yes.	
	238	R1	*a* plus *b* equals ten. Right? And *a* is three and *b* is seven.	
	239	Stephanie	Um hm.	
	240	R1	So. So. We know what the area of that square is going to be, right?	
	241	Stephanie	Um hm.	
	242	R1	But what you're telling me if *a* is, if we were to use what it equals, we would say it's three squared plus two times	
	243	Stephanie	times three times seven?	PNE
	244	R1	plus?	
	245	Stephanie	seven squared.	PNE
	246	R1	Which equals nine plus	
	247	Stephanie	Um – six – forty-two	PNE
	248	R1	plus	
	249	Stephanie	forty-nine.	PNE
	250	R1	And if we add all those up together, let's hope that we get one hundred. Do we? [*pause*]	
	251	Stephanie	Can I have a pen? I can't do it in my head. Thank you. [*Stephanie calculates.*] Yes!	PNE; BMP
	252	R1	We would hope that. Right?	
	253	Stephanie	Yes.	
	254	R1	Because if we know that the side	
	255	Stephanie	Well, that's what you get	
	256	R1	If we knew how – you told me that if the side were ten it would be ten squared or a hundred	
15:00-19:59	257	Stephanie	Um hm.	
	258	R1	but it's three plus seven. Um. But what you did is – um – you showed me that the area is *a* squared plus two *ab* plus *b* squared, but you also showed me that the area was *a* squared plus *ab* plus *ba* plus *b* squared before you showed me two *ab*. Isn't that right?	
	259	Stephanie	Yes, but	
	260	R1	Isn't that right?	
	261	Stephanie	Yes.	

Time	Line	Speaker	Transcript	Code
	262	R1	Can you show me that with the three and seven?	
	263	Stephanie	Well. Okay.	
	264	R1	Can you build me a model? 'Cause I'm this little sixth grader.	
	265	Stephanie	Um hm.	
	266	R1	Do you understand what I'm saying? That shows this special case when *a* is three and *b* is seven – to show me all the little pieces and convince me that they do work because you'd have a piece that has an area of nine, right? For three squared?	
	267	Stephanie	Okay.	
	268	R1	You'd have a piece that has an area of twenty-one, another one twenty-one and another one forty-nine. I want to see all these little pieces – nine, twenty-one, and forty-nine. Do you understand what I'm asking you?	
	269	Stephanie	Yes.	
	270	R1	Great.	
	271	Stephanie	Um. Okay. If this is your whole thing – [*Stephanie picks up the flats.*]	BR-V
	272	R1	Right.	
	273	Stephanie	Can I maybe square it off with these? [*Stephanie picks up a 'ten'.*]	BR-V
	274	R1	Absolutely.	
	275	Stephanie	Okay. This is your three. You see it, one, two, and counting this – three. [*Stephanie places a long on the flat. It lies on the third interval from the left edge of the flat.*]	BR-V
	276	R1	I can imagine that.	
	277	Stephanie	This is another three. [*Stephanie places a long on the third interval from the front edge of the flat.*]	BR-V
	278	R1	Um.	
	279	Stephanie	Well, actually.	
	280	R1	You could use the little ones, you know, to go across.	
	281	Stephanie	Yeah.	
	282	R1	You know what I'm saying?	
	283	Stephanie	Yeah. Here's another one. Here's another three. Here's your other three. And here – I'm trying to make it even that way. [*Stephanie builds a cross hatched pattern.*] Here's your other three. Do you see, like do you understand	BR-V

Time	Line	Speaker	Transcript	Code
			that?	
	284	R1	No. I really – I've got to go very slow. [*Stephanie chuckles.*] I'm not	
	285	Stephanie	Let me just make sure. Okay. I don't want to like build it wrong.	
	286	R1	You could take, you could take your (inaudible) off	
	287	Stephanie	'Cause otherwise I'd build it wrong, like I did the last time. [*Stephanie refers to her papers to refresh her memory.*] Alright. Um. You understand that – you want me to mark off three and seven. Well this is your three. One, two, three.	BR-V
	288	R1	Okay.	
	289	Stephanie	So I'm marking that off.	BR-V
	290	R1	I follow that.	
	291	Stephanie	You follow that.	
	292	R1	Um hm.	
	293	Stephanie	Now here is another three. [*Stephanie places a long along row eight of the flat. She now has longs placed on row three and row eight.*] Oh. Dar! Wait – here – [*She removes the long placed over row eight.*]	BR-V
	294	R1	So this is *a* [*R1 indicates rows one through three*] and this is *b*. [*R1 indicates rows four through ten of the flat.*]	
	295	Stephanie	Yes.	
	296	R1	Okay.	
	297	Stephanie	That's *b*. That's *a*. [*Stephanie points to the appropriate locations on her model.*]	BR-V
	298	R1	And so this is *a* and this is *b*. [*R1 indicates the same lengths on the opposite edge of the model.*]	
	299	Stephanie	Yes.	
	300	R1	'Cause they're the same on both sides. – Now we got to do them on these sides. [*R1 indicates the vertical edges of the model.*] What's *a* and what's going to be *b*?	
	301	Stephanie	And so on these sides it would be – wait- [*Stephanie tries to use the longs again to complete the model.*]	
	302	R1	If you don't want to go over you can use the little ones to go across.	
	303	Stephanie	Yes.	
	304	R1	You know what I'm saying. I don't know how	

Time	Line	Speaker	Transcript	Code
			you're going to do 'em.	
	305	Stephanie	So that's *a* and this would be *b*.	BR-V
	306	R1	Well, mark them off – the line for me so that I can understand it. [*Stephanie builds a line of unit cubes across the eighth row from the top of the flat.*]	
	307	Stephanie	Okay.	
	308	R1	So tell me what you did again.	
	309	Stephanie	I marked off *a* and *b*. This is *a*. Yeah. This is - no – we said *a* was three, right?	BR-V
	310	R1	*a* is three.	
	311	Stephanie	This is *a*, counting this. So it's one, two, three [*Stephanie counts up from the bottom of the model along its left side.*]	BR-V
	312	R1	Um hm.	
	313	Stephanie	And this is b. One, two, three, four, five, six, seven.	BR-V
	314	R1	Okay. *a*. *b*.	
	315	Stephanie	Yes. This is *a* [*Stephanie counts along the bottom of the model.*] one, two, three.	BR-V
	316	R1	Um hm.	
	317	Stephanie	And this is *b*. One, two, three, four, five, six, seven.	BR-V
	318	R1	Um hm.	
	319	Stephanie	This is *a*. One, two, three. [*Stephanie counts along the right side of the model.]*	BR-V
	320	R1	Um hm.	
	321	Stephanie	And this is *b*. One, two, three, four, five, six, seven. And this is *a*. [*Stephanie is now counting the appropriate segments on the top of the model.*] And this is *b*.	BR-V
	322	R1	Okay. So now you've partitioned it into four pieces.	
	323	Stephanie	Yes. Now	
	324	R1	Can you show me the *a* squared piece, the *ab* piece, the *ba* piece, and the *b* squared piece?	
	325	Stephanie	This piece is the *b* squared piece. [*Stephanie points to the large square area in the upper right corner of her model.*]	BR-V
	326	R1	Okay. Let's see. This is *b* [*R1 indicates the top of the model.*]	
	327	Stephanie	If you…if you…	
	328	R1	One, two, three, four, five, six – wait – I'm going	

Time	Line	Speaker	Transcript	Code
	329	Stephanie	From here [*Stephanie points to the location.*]	
	330	R1	One, two, three, four, five, six, seven by seven.	
	331	Stephanie	Yes. Now you're	
	332	R1	And that's going to be forty-nine of these squares.	
	333	Stephanie	And that's – Yes.	
	334	R1	And the way to find out – suppose I didn't have this on the bottom. Can I fill these in here?	
	335	Stephanie	Yeah.	
	336	R1	And I should – if I filled them in? Right?	
	337	Stephanie	Um hm.	
	338	R1	I should have how many? And that's a way I could figure it out.	
	339	Stephanie	Yes.	
	340	R1	Fill them all in and I should have forty-nine. Okay. I'll buy that. Forty-nine square units.	
	342	Stephanie	Okay. Now here	
	343	R1	That's *b* squared. That's a square. That's neat.	
	344	Stephanie	Yeah. That's *b* squared.	BR-V
	345	R1	Um hm.	
	346	Stephanie	[*rotates the flat.*] This little corner is *a* squared.	BR-V
	347	R1	Okay. Can you fill in the whole corner for me? Because it doesn't look like a square. Oh. I see. I've got to look at this line here, too.	
	348	Stephanie	Yeah.	
	349	R1	Gotcha.	
	350	Stephanie	That counts, too. Oops. [*She accidentally knocks off one of the cubes.*]	BR-V
	351	R1	Cool. This helps me a lot.	
	352	Stephanie	Yeah.	
	353	R1	If I were your little sister, I'd like this.	
	354	Stephanie	That's *a* squared.	BR-V
	355	R1	Okay. So that's	
	356	Stephanie	Three by three	BR-V
	357	R1	Three by three square. And this has what area? That's the	
	358	Stephanie	Nine	
	359	R1	So we've done this	
	360	Stephanie	Forty-nine. And we've done nine.	
	361	R1	And we've done this. Great!	
	362	Stephanie	Now this [*Stephanie indicates the area which had originally been on the upper left side*] is your *a* times *b*.	BR-V
	363	R1	Okay.	

Time	Line	Speaker	Transcript	Code
	364	Stephanie	And – you understand that?	
	365	R1	Um hm.	
	366	Stephanie	And this is your other *a* times *b*.	BR-V
	367	R1	So. What am I talking about? I'm talking about – length is	
	368	Stephanie	Length is – Well, if you're going like this [*Stephanie moves the model to its original orientation.*] the length is three and the width is seven?	BR-V
20:00-24:59	369	R1	Does it matter?	
	370	Stephanie	No, but … I mean it all depends on which way you look at it, but still	BR-V
	371	R1	Okay. So as long as you turn it around, you're okay.	
	372	Stephanie	Um hm.	
	373	R1	It depends on which way	
	374	Stephanie	(inaudible) like three and you know	
	375	R1	Okay. Gotcha. And the other one? So which is *ab* and which is *ba*? Does it matter?	
	376	Stephanie	No. It doesn't matter.	BMP
	377	R1	Okay.	
	378	Stephanie	'Cause you're using the same numbers.	BMP
	379	R1	Okay. Alright. So does that make sense to you?	
	380	Stephanie	Yes.	
	381	R1	Okay. So what do you think I'm going to ask you next?	
	382	Stephanie	I don't know. What? I don't…	
	383	R1	What do you think?	
	384	Stephanie	I don't	
	385	R1	Now if you were in my place, what do you think would be a logical thing to ask?	
	386	Stephanie	I don't know. Do you want me to like do it cubed or something?	
	387	R1	Yeah. I think that's a very good thing. I think a cube would be a great thing to ask.	
	388	Stephanie	Okay. [*Stephanie takes a cube from the bag on the floor and sets it on the table.*]	
	389	R1	Okay. So we did *a* plus *b* quantity squared. And I really think, uh, you have a good mental model of that.	
	390	Stephanie	Um hm.	
	391	R1	Don't you think?	

Time	Line	Speaker	Transcript	Code
	392	Stephanie	Yes.	
	393	R1	Don’t you feel good about that? I betcha you could really take these blocks home and explain it to your little sister.	
	394	Stephanie	Yes.	
	395	R1	Do you think she’d be interested?	
	396	Stephanie	No. [*They both chuckle.]*	
	397	R1	That’s what happens sometimes.	
	398	Stephanie	Yeah.	
	399	R1	I used to come home and think I’d want to explain this to my son. Do you think he’d be interested? No. Okay. But let’s suppose you have this real interested younger sister.	
	400	Stephanie	Okay.	
	401	R1	Okay. Um. So. What should we do first? Should we start with something we know – that has a very explicit length	
	402	Stephanie	Um for	
	403	R1	and find the	
	404	Stephanie	for cubed?	
	405	R1	and find the volume? What’s volume?	
	406	Stephanie	Volume is like	
	407	R1	Is this volume? [*R1 taps the large cube which is sitting on the table between Stephanie and herself.*]	
	408	Stephanie	Yes. It would be length, width, times depth.	BMP
	409	R1	What does that mean?	
	410	Stephanie	That means this way, times this way, times this way. [*Stephanie traces the edges of the cube as she speaks.*]	BEJ
	411	R1	Okay.	
	412	Stephanie	‘Cause there’s – it – not – oh – it’s length, width, height.	
	413	R1	Um hm.	
	414	Stephanie	‘Cause it’s not only – it’s like three dimensional.	BMP
	415	R1	Alright. Now. You sorta convinced me that this square [*the flat used in the model previously*] has area, right? A hundred square units.	
	416	Stephanie	Yes.	
	417	R1	Even with the three and the seven and you could’ve done it with six and four or one and nine and all of this would work, right?	

Time	Line	Speaker	Transcript	Code
	418	Stephanie	Um hm.	
	419	R1	If we did *a* is one and *b* is nine and you took one plus nine quantity squared.	
	420	Stephanie	Um hm.	
	421	R1	and you apply this, that would work? Would it?	
	422	Stephanie	Um. Yeah. I thought – didn't – did we do that last time?	
	423	R1	You see – I mean – we have ten, right?	
	424	Stephanie	Yes.	
	425	R1	We have a length of ten to play with. If we wanted *a* plus *b* to be ten, we could have a lot of fun with this. And if we really wanted to, we could have *a* to be two and a half, right? [*Stephanie chuckles.*] Right? And what could we have *b* to be?	
	426	Stephanie	*b* would be seven and a half.	BMP
	427	R1	Seven and a half? You know. The question is if you had a calculator with you now. The question is "Is two and a half the quantity squared, right?	
	428	Stephanie	Um hm.	
	429	R1	- plus two and a half times seven and a half plus seven and a half times two and a half, right? plus seven and a half squared?"	
	430	Stephanie	Squared.	
	431	R1	Right. What would you expect that to be?	
	432	Stephanie	One hundred.	BMP
	433	R1	One hundred and so forth. You can think of lots of things. How many things add up to ten? – How many numbers	
	434	Stephanie	Five?	
	435	R1	add up to ten?	
	436	Stephanie	Right? Is there five sets? - One plus nine. Eight plus um eight plus two, uh, seven plus three, six plus four, and five plus five.	OBS
	437	R1	Two and a half, seven and a half?	
	438	Stephanie	Oh. Well then there's like – isn't there like an infinite amount?	BCA; BDI
	439	R1	Hm. Why do you say infinite amount?	
	440	Stephanie	Because you could have like one point nine nine nine nine nine like forever, 'cause like numbers don't stop.	BMP
	441	R1	So it's really interesting. Even with this special case, right?	

Time	Line	Speaker	Transcript	Code
	442	Stephanie	Yeah.	
	443	R1	We can keep trying this. But- we even have our shortcut of doing it. You know. Whatever you want to choose for *a* and *b*, you know, two and a half quantity squared plus twice two and a half times seven and a half	
	444	Stephanie	Um hm.	
	445	R1	plus seven and a half squared, but we always know that no matter what we do	
	446	Stephanie	it's going to be a hundred.	BMP
	447	R1	Isn't that really powerful? Isn't that exciting?	
	448	Stephanie	Yeah.	
	449	R1	Now you can – so if you took this example, you could've just taken it and sliced it any place in here. Arbitrarily, you can pick *a*.	
	450	Stephanie	Um hm.	
	451	R1	Right?	
	452	Stephanie	Yeah.	
	453	R1	But once you've picked *a* and this is ten, you know what *b* is going to be.	
	454	Stephanie	Yeah.	
	455	R1	Okay. So that can be great fun. So you've convinced me of this. How do you then begin to convince someone to move from here to volume? What is the volume of this by the way?	
	456	Stephanie	A thousand?	
	457	R1	A thousand what?	
	458	Stephanie	A thousand units cubed.	BR-V
	459	R1	What do you mean by that?	
	460	Stephanie	Well – [*chuckles*] 'cause um squared is like two dimensional,	BEJ; BR-V
	461	R1	Um hm.	
	462	Stephanie	so cubed is like three dimensional.	BEJ; BR-V
	463	R1	Yeah.	
	464	Stephanie	(inaudible)	
	465	R1	That's what makes these hard for me. I have a lot of trouble with these. You know why? Because this really is a cubic unit, isn't it?	
	466	Stephanie	Um hm.	
	467	R1	It's not a square unit. If I put this on an overhead	
	468	Stephanie	Yeah.	

Time	Line	Speaker	Transcript	Code
	469	R1	it'll look like a square unit, but it really is a cubic unit. So what does that mean in terms of this little – this is a cubic unit – only one side of it is a square unit, right?	
	470	Stephanie	Um hm.	
	471	R1	'Cause we're supposed to imagine there's no depth here and that's really hard for for a lot of students to do. Because that's one of the criticisms of using these blocks – you're supposed to imagine this is only two dimensions, but it really is three. And any time you have anything, I mean, this really is three dimensions. [*R1 holds up a piece of paper*.] This has a thickness. Isn't that right?	
25:00-29:59	472	Stephanie	Yeah.	
	473	R1	You'll be dealing with these ideas in geometry next year. But you're supposed to imagine it doesn't	
	474	Stephanie	Um hm.	
	475	R1	Even when we write something on the board. The thickness is the chalk, but you're supposed to imagine it's not there. Do you see where students get confused?	
	476	Stephanie	Yes.	
	477	R1	"Boy, this teacher, boy, is really losing it!" Right? [*Stephanie chuckles.*] But anyway, so this really is a cubic unit. So that's what's nice about this. It has – so how many of these are in here – if you (inaudible)	
	478	Stephanie	A thousand.	
	479	R1	A thousand. And you really you really believe that.	
	480	Stephanie	Yes.	
	481	R1	You really imagine building it up. How can you – you know there are a hundred here? How can you quickly show there are a thousand?	
	482	Stephanie	Well. It would take ten of these [*indicates the flat*]	BR-V
	483	R1	Um hm.	
	484	Stephanie	to build all the way up and ten times a hundred is a thousand.	BR-V, BMP
	485	R1	Ten times a hundred?	
	486	Stephanie	Yeah. 'Cause there's a hundred here.	BR-V
	487	R1	Okay.	

Time	Line	Speaker	Transcript	Code
	488	Stephanie	So	
	489	R1	So, a hundred, a hundred, a hundred…ten times.	
	490	Stephanie	Um hm.	
	491	R1	Okay. Alright. Neat. So my next question is: We know how this works for a cube with a very explicit length	
	492	Stephanie	Um hm.	
	493	R1	of one side, right?	
	494	Stephanie	Yes.	
	495	R1	Now suppose we wanted to do *a* plus *b* quantity cubed. That's our next question. [*R1 writes:* $(a + b)^3 =$] Right?	
	496	Stephanie	Um hm.	
	497	R1	Now. What do you want to do? Do you want to come up with the theoretical answer for this?	
	498	Stephanie	Mmm.	
	499	R1	Before you build it? Or do you want to build it?	
	500	Stephanie	It would probably be easier to build it than to come up with it.	BR-V
	501	R1	Oh. That's interesting. Okay. Sure.	
	502	Stephanie	'Cause …I guess, 'cause	
	503	R1	What does this mean, though? Can you tell me what this means?	
	504	Stephanie	*a* plus *b* times *a* plus *b*	BMP
	505	R1	Why don't you write that?	
	506	Stephanie	Okay. [*Stephanie writes:* $(a+b)\cdot(a+b)\cdot(a+b)$]	
	507	R1	Okay. Do you know any – you have a binomial times a binomial, right? And then you have another binomial, right? That's what this means.	
	508	Stephanie	Okay.	
	509	R1	Now if I said to you? - [*pause*] Sometimes it's convenient to go from right to left or left to right. I don't know which way you're comfortable doing it, but suppose we multiplied the last two. Do you know what that product is?	
	510	Stephanie	That product?	
	511	R1	*a* plus *b* times *a* plus *b*	
	512	Stephanie	Well. – Do I know what it is or could I just say like *a* plus *b* squared?	

Time	Line	Speaker	Transcript	Code
	513	R1	You're telling me *a* plus *b* times *a* plus *b* is *a* plus *b* quantity squared?	
	514	Stephanie	Yes.	BR-S
	515	R1	But if you are to actually square it, do you know what that would be? The quantity – *a* plus *b* times the *a* plus *b*?	
	516	Stephanie	Without numbers? Like just…	PAH
	517	R1	Without numbers – the general way. Whenever I multiply *a* plus *b* times another *a* plus *b*. We know what it's not. We know that it's not *a* squared plus *b* squared.	
	518	Stephanie	Um hm.	
	519	R1	Do we know what it is? 'Cause you proved it last time.	
	520	Stephanie	I did?	
	521	R1	The quantity *a* plus *b* times *a* plus *b*.	
	522	Stephanie	Oh. Well, I proved that it was *a* squared plus *ab* plus uh two *ab* plus *b* squared.	PPK
	523	R1	Right. Isn't that right?	
	524	Stephanie	Yeah.	
	525	R1	Okay. So you know what this piece is [*the last two factors*].	
	526	Stephanie	Um hm.	
	527	R1	Let's put – let's actually do it. Right? *a* plus *b* times *a* plus *b* you told me is *a* squared plus two *ab* plus *b* squared. Isn't that right?	
	528	Stephanie	Yes.	
	529	R1	Now I didn't have to do these two. I could've done these two.	
	530	Stephanie	Yeah. You could've	BMP
	531	R1	You understand that?	
	532	Stephanie	Yeah. I understand.	
	533	R1	I could've done the first and the last, but I just chose to do that.	
	534	Stephanie	Um hm.	
	535	R1	Right. So. So, we've done part of it already. Isn't that right?	
	536	Stephanie	Um hm.	
	537	R1	Well. We haven't finished it. 'Cause what part didn't we do?	
	538	Stephanie	We have *a* plus *b* left.	
	539	R1	We have to multiply *a* plus *b* times this. [*indicates* $(a^2+2ab+b^2)$]	
	540	Stephanie	Yes.	

Time	Line	Speaker	Transcript	Code
	541	R1	Okay.	
	542	Stephanie	Okay.	
	543	R1	Now. If you were to multiply *a* plus *b* times this that's going to be you know a bit of algebraic work here, right?	
	544	Stephanie	Yeah.	
	545	R1	You know. I don't know if you – what that means – if you have *a* plus *b* of these – we can talk about what does that mean – to have *a* plus *b* of these.	
	546	Stephanie	Um hm [*pause*] I – um – I don't	OBS
	547	R1	But we can think of it this way. Three plus seven cubed. [*R1 writes:* $(3 + 7)^3$]	
	548	Stephanie	Okay.	
	549	R1	That might help. Is three plus seven times three plus seven times three plus seven. Right? Isn't that right? [*R1 writes* $= (3 + 7)(3 + 7)(3 + 7)$]	
	550	Stephanie	Um hm.	
	551	R1	We know the answer to that.	
	552	Stephanie	Um hm.	
	553	R1	Or, what I just did here is this, didn't I? I said it's the same as three squared. Right? You told me plus two times three times seven plus seven squared. Right? [*R1 writes* = $(3+7)(3^2+2\cdot 3\cdot 7+7^2)$]	
	554	Stephanie	Um.	
	555	R1	Now you know enough arithmetic- how to finish this.	
	556	Stephanie	Um. Yes.	
	557	R1	What would you do next?	
	558	Stephanie	Well. First, I'd um do everything in the parentheses.	BMP
	559	R1	What do you mean? Well, okay. But I don't want you to do everything in the parentheses.	
	560	Stephanie	Well, I can	
	561	R1	All I want you to do is this. Okay. – I want – come with – suppose you said that was nine and you said this was what? Forty-two?	
	562	Stephanie	Yeah.	
	563	R1	And this is forty-nine. Okay?	
	564	Stephanie	Um hm.	
	565	R1	That's as much as I want you to do in the parentheses. Right?	
	566	Stephanie	So you want	

Time	Line	Speaker	Transcript	Code
30:00-34:59	567	R1	I mean you could add these up and add these up but we know it's a thousand. So. But suppose rather than do everything in the parentheses – is there anything that you've learned about arithmetic that you could stop this from being a multiplication problem. Does any of that look familiar to you? [*pause*]	
	568	Stephanie	I don't know. I've usually – 'cause if you just have numbers like that you just like	
	569	R1	But suppose they were letters?	
	570	Stephanie	Well, if they were letters I'd probably like – get help or something to figure it out. I don't know. I don't – um – to stop it from being a multiplication problem?	PAH; OBS
	571	R1	Um hm.	
	572	Stephanie	I don't know.	
	573	R1	Have you heard of the Distributive Property?	
	574	Stephanie	Well. Yeah. But I mean if I was really going to distribute, I'd just	
	575	R1	What would you distribute?	
	576	Stephanie	I'd add them first. And then I'd distribute that.	BMP
	577	R1	But sometimes you can't add them. Like if they're *a* and *ab*.	
	578	Stephanie	Well. Yeah.	
	579	R1	So if you can't add them first, how would you distribute them?	
	580	Stephanie	Well, it would be nine – like parentheses – three plus seven plus forty-two parentheses three plus seven plus forty-nine parentheses, you know, three plus seven. Like, you know.	BMP
	581	R1	Okay. That's a way. That's a absolutely perfectly good way to do that. So you would distribute the three plus seven times the nine, the three plus seven times the forty-two, and the three plus seven times the forty-nine.	
	582	Stephanie	Yeah.	
	583	R1	I would do it a little differently, but that's – your way is as good as mine.	
	584	Stephanie	Well, how else – how can you do it?	PAH
	585	R1	Um. We can test to see if it matters. Your way is interesting. You're distributing three plus seven times each of these.	
	586	Stephanie	Yeah. 'Cause	
	587	R1	I would distribute just three times each of them.	
	588	Stephanie	Oh. Okay.	

Time	Line	Speaker	Transcript	Code
	589	R1	And then the seven times each of them. But we can test to see if that works.	
	590	Stephanie	Um hm.	
	591	R1	Why don't we do it? Why don't we do it your way and my way and see if this really comes out to a thousand. We know the answer. Right?	
	592	Stephanie	Um hm.	
	593	R1	Why don't we try the three plus seven times the nine, the three plus seven times the forty-two and the three plus seven times the forty-nine? Do you understand what we're doing here?	
	594	Stephanie	Yeah. But still	
	595	R1	We're testing an idea.	
	596	Stephanie	Then can I add them? Like after I distribute	PNE
	597	R1	Yeah.	
	598	Stephanie	Three plus seven to each – can I add them?	
	599	R1	Absolutely.	
	600	Stephanie	Okay.	
	601	R1	Three plus seven times the nine plus three plus seven you told me times the forty-two, right?	
	602	Stephanie	Um hm	
	603	R1	Plus three plus seven times the forty-nine.	
	604	Stephanie	Yes.	
	605	R1	That's what you told me?	
	606	Stephanie	Um hm.	
	607	R1	Right?	
	608	Stephanie	And that would be ninety.	
	609	R1	And I'm going to put my – the way I would've done it here. I would've done the three	
	610	Stephanie	Okay.	
	611	R1	times the nine plus the three times the forty-two plus the three times the forty-nine and then I would've done the seven times the nine	
	612	Stephanie	Um hm	
	613	R1	plus the seven times the forty-two plus the seven – I don't know why? And the question is – let's do the arithmetic	
	614	Stephanie	Okay.	
	615	R1	and see if it works.	
	616	Stephanie	So that would come	PNE
	617	R1	We know what the	
	618	Stephanie	out to ninety.	
	619	R1	Alright. Write that down. Ninety.	

Time	Line	Speaker	Transcript	Code
	620	Stephanie	Ninety plus four twenty	PNE
	621	R1	Um hm.	
	622	Stephanie	plus four ninety. And your answer is – is one thousand.	PNE
	623	R1	Um hm. Check my way.	
	624	Stephanie	Oh God.	
	625	R1	You did that really fast, Stephanie.	
	626	Stephanie	Oh. Yeah. Well, they're the same numbers as before except with the zeros on the end, so…	BMP
	627	R1	Um hm.	
	628	Stephanie	Okay. Three times nine plus three? Okay. So. Three times nine is twenty-seven plus three.	
	629	R1	Is that twenty-seven plus three or plus three times forty-two?	
	630	Stephanie	Oh. Okay. So those are like each squared – like they're each – like that?	BDI
	631	R1	So – yeah. Maybe we should put parentheses. [*They do.*]	
	632	Stephanie	Okay. That's three times nine, twenty-seven, plus three times forty-two, that's – twelve – plus three times forty-nine – (inaudible) twelve, fourteen, plus seven times nine is sixty-three plus seven times (inaudible) that's (inaudible) twenty-eight, twenty-nine, plus (inaudible), three, six, (inaudible). Okay.	
	633	R1	Your's was less work than mine.	
	634	Stephanie	[*Stephanie sighs.*]	
	635	R1	So – gee	
	636	Stephanie	Um.	
	637	R1	No one has a calculator.	
	638	Stephanie	[*chuckles*] Okay. I 'm just gonna do it up here.	
	639	R1	Here take another piece of paper.	
	640	Stephanie	Okay.	
	641	R1	… a calculator? (inaudible) Oh. There's a math teacher. Travels with a calculator.	
	642	Stephanie	[*Stephanie chuckles.*]	
	643	R1	You could show her how (inaudible) you know how to do it.	
	644	Stephanie	Oh. We used those last year.	
	645	R1	Okay. Great.	
	646	Stephanie	Yeah.	
	647	R1	Texas Instruments.	
	648	Stephanie	Um hm – [*Stephanie works with the calculator.*] Oh. There's stuff on here.	

Time	Line	Speaker	Transcript	Code
	649	R2	That's okay. Clear it.	
	650	Stephanie	Okay. Um. If I remember how I can. Okay. Okay. Twenty-seven.	
	651	R1	You want me to read them to you?	
	652	Stephanie	Yeah.	
	653	R1	Twenty-seven. Okay. Plus one twenty-six	
	654	Stephanie	Um hm.	
	655	R1	plus one forty-seven	
	656	Stephanie	Um hm.	
	657	R1	plus sixty-three, plus two ninety-four, plus three forty-three.	
	658	Stephanie	That's it?	
	659	R1	Um hm. Equals?	
	660	Stephanie	One thousand. They both work.	
	661	R1	They both worked.	
	662	Stephanie	Um hm.	
	663	R1	Okay. Hm. Now. Why don't you leave it here? We may need this again. Um.	
35:00-39:59	664	R1	My next question to you then is: Is there – what you did here – right –distributed the three plus seven to all of these,	
	665	Stephanie	Um hm.	
	666	R1	can that be applied here? [*to the abstract case*]	
	667	Stephanie	Yeah. I guess. Except they're like *a*'s and *b*'s, but yeah.	BCA
	668	R1	Okay. So if you were to write – let's – let's start all over again then. [*R1 takes a new sheet of paper.*] Why don't we write *a* plus *b* cubed. What it means and try to take it to the step and distribute it and see what all that comes out to be.	
	669	Stephanie	*a* plus *b* cubed. And so you want me to write that equals that and then	BR-S
	670	R1	Yeah. Right. Um hm.	
	671	Stephanie	[*Stephanie writes.*] Equals. You want me to write that one, too? Oh, sorry.	
	672	R1	Yeah.	
	673	Stephanie	Okay.	
	674	R1	I would – you know what – you might for format don't put the equals here.	
	675	Stephanie	Put it here?	
	676	R1	Yeah. [*Stephanie writes.*]	
	677	Stephanie	Oh.	
	678	R1	And you know where this came from.	

Time	Line	Speaker	Transcript	Code
	679	Stephanie	Yes.	
	680	R1	You really know this. You believe this. You're absolutely convinced?	
	681	Stephanie	Yes.	
	682	R1	No doubt in your mind?	
	683	Stephanie	We worked it out.	
	684	R1	That's always true?	
	685	Stephanie	Yes.	
	686	R1	You've proved it. Okay.	
	687	Stephanie	And then it would be	
	688	R1	You're going to need more space. You may want to start here rewriting it.	
	689	Stephanie	Oh. Yeah. Okay. Well – equals – *a*	
	690	R1	Do it your way.	
	691	Stephanie	*a* squared *b*. Okay.	BMP
	692	R1	That's fine. You're doing well.	
	693	Stephanie	And it's – yeah – *a* plus – two *ab* [*Stephanie continues working.*] Okay.	BMP
	694	R1	Okay. Do you know enough algebra to to actually do each of these little problems?	
	695	Stephanie	Um.	
	696	R1	Have you learned how to multiply *a* squared times *a* plus *b*?	
	697	Stephanie	We might've. I'm just – I doubt I could do it like correctly. Um.	
	698	R1	I mean 'cause – what I'm what I'm suggesting that you think about here is think of this as one special problem. Just this little piece.	
	699	Stephanie	Um hm.	
	700	R1	What does that mean?	
	701	Stephanie	That means like – *a* squared times *a* plus *a* squared times *b*.	BMP
	702	R1	Why don't you write that down?	
	703	Stephanie	Oh. Okay.	
	704	R1	Underneath. [*Stephanie writes those values on the paper.*]	
	705	R1	Now you said *a* squared times *a*.	
	706	Stephanie	Oh. Yeah.	
	707	R1	And you didn't write *a* squared times *a*. [*Stephanie makes the correction.*]	
	708	Stephanie	Yeah.	
	709	R1	Put a dot. *a* squared times *a*. That might help you.	
	710	Stephanie	*a* squared times *a* and then	

Time	Line	Speaker	Transcript	Code
	711	R1	Um hm.	
	712	Stephanie	Plus *a* squared times *b*.	
	713	R1	Neat. Okay. So we did this piece. Why don't you put an equal?	
	714	Stephanie	Okay.	
	715	R1	The reason I'm covering it now – we have plus	
	716	Stephanie	Plus	
	717	R1	Now can you do this piece?	
	718	Stephanie	Oh, God. Two *ab*	BMP
	719	R1	You gotta write small.	
	720	Stephanie	times *a* plus two *ab* times *b* plus *b* squared times *a* plus *b* squared times *b*.	BMP
	721	R1	Cool.	
	722	Stephanie	Okay.	
	723	R1	Are any of these you can simplify?	
	724	Stephanie	Can I simplify that?	
	725	R1	(inaudible)	
	726	Stephanie	Can't I make that *a*	BMP
	727	R1	Equal	
	728	Stephanie	to the *a* cubed?	BMP
	729	R1	You believe it's *a* cubed?	
	730	Stephanie	Well, it's another *a*.	BEJ
	731	R1	Okay. So that means you have *a* three times.	
	732	Stephanie	Yeah.	
	733	R1	You believe that? Right?	
	734	Stephanie	Yeah.	
	735	R1	See in a sense um that was like my three times three squared	
	736	Stephanie	Um hm.	
	737	R1	became a three cubed or twenty-seven. Isn't that right?	
	738	Stephanie	Um hm.	
	739	R1	You can think of twenty-seven as three cubed.	
	740	Stephanie	Yes.	
	741	R1	Okay.	
	742	Stephanie	Okay. This I can't – well	
	743	R1	Okay, so you can leave it.	
	744	Stephanie	[*Stephanie continues working.*]	
	745	R1	Now what's this piece say?	
	746	Stephanie	It says two *ab* like two *a* times *b*	
	747	R1	It says two	
	748	Stephanie	times *a* times *b*	
	749	R1	*a* times *b* times	
	750	Stephanie	*a*.	

Time	Line	Speaker	Transcript	Code
	751	R1	Can that be simplified?	
	752	Stephanie	*[pause]* Could it be – um – there'd be another *a*, right? So could I make it like three *a* times two *b*?	
	753	R1	Okay. Let's look at this piece. Okay. Let's try to think of what you did. I want to go back to this *a* squared times *a*.	
	754	Stephanie	Okay.	
	755	R1	I better use this pen.	
	756	Stephanie	Oh.	
	757	R1	This *a* squared times *a*, right?	
	758	Stephanie	Um hm.	
	759	R1	You said could be *a* cubed.	
	760	Stephanie	Yes.	
	761	R1	Why?	
	762	Stephanie	Because you're multiplying it by itself again.	BEJ; BMP
	763	R1	Okay. Um. So – another way I think about it is – here you have – when there's no exponent – that means you have one of them.	
	764	Stephanie	Yeah.	
	765	R1	Right?	
	766	Stephanie	Um hm.	
	767	R1	Okay. That means you have one factor *a*.	
	768	Stephanie	Um hm.	
	769	R1	And here you have two factors of *a*.	
	770	Stephanie	Yes.	
	771	R1	So that means you have three	
	772	Stephanie	Three.	
	773	R1	So *a* cubed.	
	774	Stephanie	Um hm.	
	775	R1	So that was sorta like my – if I go back to my three story down here – three times nine could be thought of as three times three squared.	
	776	Stephanie	Um hm.	
	777	R1	Right. Or three squared times three if we write it the squared term first.	
	778	Stephanie	Um hm.	
	779	R1	And that tells you we have three factors of three.	
	780	Stephanie	Yeah.	
40:00-44:59	781	R1	Isn't that right?	
	782	Stephanie	Yes.	

Time	Line	Speaker	Transcript	Code
	783	R1	Two factors and that gives you your twenty-seven which I got down here. Isn't that what that means?	
	784	Stephanie	Um hm.	
	785	R1	Okay. Now. Now. If you think about that and look at the second piece – um – here you have two factors of *a* and one of *b*. That's what you told me, right? *a* squared times *b* meant you had two factors of *a* and one factor of *b*. Nothing you can do with that.	
	786	Stephanie	Um hm.	
	787	R1	Right. That's sorta like – What is it sorta like? [*R1 goes back to an earlier example.*] Do we have that little piece represented here?	
	788	Stephanie	Um. Well	
	789	R1	Do you see any two factors of *a* and one of *b* any place here?	
	790	Stephanie	Two of *a* and one of *b*? Um – *a* was three, right?	
	791	R1	*a* was three.	
	792	Stephanie	Okay.	
	793	R1	So we had two factors of *a* and *b* was seven. Do any of these terms represent that?	
	794	Stephanie	I don't know. Um. This one?	
	795	R1	Um.	
	796	Stephanie	Well that that can be divided by three –	
	797	R1	Let's test it. A little diversion here, but this is interesting. You have seven times forty-two.	
	798	Stephanie	And forty-two can be divided (inaudible)	PNE
	799	R1	Or seven times seven	
	800	Stephanie	No. What I meant was	
	801	R1	Remember we want *a*'s and *b*'s. So we only want three's and seven's.	
	802	Stephanie	Okay.	
	803	R1	Remember *a* was three and *b* was seven.	
	804	Stephanie	Yes.	
	805	R1	Right? So we only want three's and seven's.	
	806	Stephanie	Um hm.	
	807	R1	Alright. So forty-two is	
	808	Stephanie	Fourteen.	
	809	R1	Seven times six.	
	810	Stephanie	Yeah.	
	811	R1	Or seven times seven times three times two. I'm having a little trouble here.	

Time	Line	Speaker	Transcript	Code
	812	Stephanie	Now – what – I don't – you want me to find one that has one seven and two three's?	PAH
	813	R1	One one -we wanted one that has two factors of *a*	
	814	Stephanie	Um hm.	
	815	R1	which means two factors of three and one factor of *b* – one factor of seven.	
	816	Stephanie	Um hm.	
	817	R1	Right? Isn't that what that means?	
	818	Stephanie	Yes.	
	819	R1	Three squared times seven means – Is there something that has	
	820	Stephanie	Okay. So you want nine and seven?	BDI
	821	R1	Does that make sense?	
	822	Stephanie	Oh. Yeah. Right here.	BDI
	823	R1	So that's sorta what you're talking about here.	
	824	Stephanie	Yeah.	
	825	R1	Right. Okay. Now this one is two *ab* times *a*. Now what does this mean? You have two times – how many factors of *a* do you have here?	
	826	Stephanie	[*Stephanie sneezes.*]	
	827	R1	God bless you. God bless you.	
	828	Stephanie	I have	
	829	R1	Do you need a tissue?	
	830	Stephanie	No. I always get that feeling that I have to sneeze and I never do.	
	831	R1	So here you have one factor of *a*. Right?	
	832	Stephanie	Um hm.	
	833	R1	One factor of *b* and one factor of *a* again.	
	834	Stephanie	Yes.	
	835	R1	Can you simplify that?	
	836	Stephanie	Um. Can I simplify that like – Oh! I can I can make it *a* squared.	BMP; BDI
	837	R1	So you could make it two	
	838	Stephanie	*a* squared	
	839	R1	*a* squared	
	840	Stephanie	*b.*	
	841	R1	*b.*	
	842	Stephanie	Okay.	
	843	R1	You got that?	
	844	Stephanie	Yes.	
	845	R1	So this term can be written – the second term – as	
	846	Stephanie	Two *a* squared *b*	BR-S

Time	Line	Speaker	Transcript	Code
	847	R1	Good.	
	848	Stephanie	plus and then it again, right? Oh. No. Now this time it's two *b* squared *a*.	BR-S
	849	R1	Or two *ab* squared. If you're keeping them alphabetically.	
	850	Stephanie	Okay. Plus you know that one is *b* squared times *a*. You can't do anything with that one.	BR-S
	851	R1	You could put them alphabetically.	
	852	Stephanie	Do you want me to?	
	853	R1	You might want to put them alphabetically.	
	854	Stephanie	Okay.	
	855	R1	'Cause it may be you can simplify them. Maybe you can't. Do you know what I'm saying?	
	856	Stephanie	Yes. And plus and this one can become *b* to the third.	BR-S
	857	R1	Third. Okay. So let's take a look at this. *a* plus *b* quantity cubed. Wasn't that the problem?	
	858	Stephanie	Um hm.	
	859	R1	Oh. You wrote squared here. Don't you mean	
	860	Stephanie	Oh.	
	861	R1	Cubed? Right? We know this has an *a* cubed and we know it has a *b* cubed.	
	862	Stephanie	Um hm.	
	863	R1	Just like in *a* plus *b* quantity squared has an *a* squared and a *b* squared.	
	864	Stephanie	Yes.	
	865	R1	Agreed? But it has all this stuff in between.	
	866	Stephanie	Um hm.	
	867	R1	Can we simplify that? Are any of them alike?	
	868	Stephanie	Yeah. Probably.	
	869	R1	Here's an *a* squared *b*. Are there any other *a* squared *b*'s here?	
	870	Stephanie	*a* squared *b*. Is this one of those, too, or is this a whole new thing?	
	871	R1	Well. We know we have *a* cubed.	
	872	Stephanie	No. But is this part of this problem?	
	873	R1	And we know we have *ab* cubed. – Yeah, this is the second line.	
	874	Stephanie	Whoa!	
	875	R1	You just simplified this line to this line. That's what you did.	
	876	Stephanie	Okay. So this is just this.	

Time	Line	Speaker	Transcript	Code
	877	R1	Right.	
	878	Stephanie	In the same (inaudible)	
	879	R1	So we have the *a* cubed and the *b* cubed. Right. We have one of those. Right?	
	880	Stephanie	Um hm.	
	881	R1	Now here we have an *a* squared *b*. Right? We have one of those.	
	882	Stephanie	Um hm.	
	883	R1	When we don't have a number, that means one of them, isn't that right?	
	884	Stephanie	Yes.	
	885	R1	We have one *a* squared *b*.	
	886	Stephanie	Oh. Well here you have two *a* squared.	BR-S
	887	R1	Oh. We have two of them. Okay. So we have one of them and two of them. How many of them will that give us?	
	888	Stephanie	Three of them. Three *a* squared.	BMP
	889	R1	Okay. So these two together	
	890	Stephanie	Yes.	
	891	R1	Right? Why don't you write – rewrite the line again? The *a* cubed we have –	
	892	Stephanie	Oh. Okay. – *a* cubed and then it's three *a* squared – what about the *b*?	
	893	R1	*b* – it's still there.	
	894	Stephanie	Okay.	
	895	R1	It's plussed. Don't lose your plus.	
	896	Stephanie	Yeah.	
	897	R1	Okay. So we took care of this. We took care of these two. Right?	
	898	Stephanie	Um hm.	
	899	R1	Now what? This is two.	
	900	Stephanie	A two *ab* squared.	BR-S
	901	R1	And this is?	
	902	Stephanie	*ab* squared.	BR-S
	903	R1	How many of them are here?	
	904	Stephanie	Three *ab* squared.	BMP
	905	R1	Okay. So you have three *ab* squared. Okay.	
45:00-49:59	906	Stephanie	Three *ab* squared.	
	907	R1	And then the last one is?	
	908	Stephanie	Plus *b* cubed. Okay. Oh. Can this be like – oh – it's different.	
	909	R1	Why don't we test it? If *a* is three and *b* is seven, see if this rule will give us a thousand.	

Time	Line	Speaker	Transcript	Code
	910	Stephanie	Okay. So that's three times three – twenty-seven plus – um – would I do in the parentheses first? Like 'cause this would be like – oh – well, it would be three times twenty-seven times seven? So three times	PNE
	911	R1	Could you just tell me what you're doing?	
	912	Stephanie	Oh. Well. This would become three. Do you want me to write it out like?	PNE
	913	R1	This is *a* squared.	
	914	Stephanie	Yeah. So it would be three. Oh. Squared. So that's nine.	PNE
	915	R1	Um hm.	
	916	Stephanie	So that would be three times nine – that's twenty-seven	PNE
	917	R1	Um hm.	
	918	Stephanie	times seven.	PNE
	919	R1	Um hm. I think that's what you said. Why don't you use scrap paper for that? Or the calculator?	
	920	Stephanie	Oh. Yeah. [*She continues working on the verification.*] One hundred eighty-nine plus – um – *b* squared – so that's – okay – uh – what am I doing? Oh, *b* squared – so that's forty-nine times three times three times three plus seven to the third – oops! – where is the –forget it – is- and then add them [*Stephanie is writing the products as she determines them using the calculator*] – would be twenty-seven plus - Yeah. One thousand.	PNE
	921	R1	It worked.	
	922	Stephanie	Um hm.	
	923	R1	Interesting. So if *a* were three and *b* were seven – do you think it would work if *a* were two and *b* were eight?	
	924	Stephanie	Um hm.	BCA
	925	R1	And *a* were two and – so we have here something that you're saying is *a* plus *b* quantity third.	
	926	Stephanie	Um hm.	
	927	R1	Right? Okay. And what might be interesting for you to do is to test it with different *a*'s and *b*'s to convince yourself if you've done all the arith – all the algebra right. You know.	
	928	Stephanie	Um hm.	
	929	R1	The interesting thing is: Could you reproduce	

Time	Line	Speaker	Transcript	Code
			this without me here? Could you go through this on your own? Could you think about - [*Stephanie makes a noise.*] You see what I'm saying?	
	930	Stephanie	Yeah.	
	931	R1	There's a lot of detail, but you know that's really to pay attention to all those details is incredible.	
	932	Stephanie	Yeah.	
	933	R1	It's really neat that you're doing that. Um. But what you're – you had developed what *a* plus *b* quantity squared – and this is theoretical – that's not what you said you wanted to do, by the way. You said that you wanted to build it first.	
	934	Stephanie	[*Stephanie chuckles.*]	
	935	R1	I know. I heard that.	
	936	Stephanie	Yeah.	
	937	R1	But I'm kinda curious now that um if you built it, right? There's going to have to be pieces to what you build.	
	938	Stephanie	Yeah. I'd have to have an *a* cubed piece; a – like a three *a* squared *b*.	BCA
	939	R1	Yeah. So you're going to have to have something in there with *a*, *a*, *a*. Three dimensional, right?	
	940	Stephanie	Yes.	
	941	R1	You're going to have to have a three dimensional *b* cubed piece, like you had your two dimensional. Then you're going to have to have another three dimensional. What would a three *a* squared *b* – kind of	
	942	Stephanie	Oh my gosh! I don't know – I –I mean – 'cause *a* cubed would have to have three parts. It would have to have length, width and height?	OBS
	943	R1	What you might try to do – this is what I'd like you to maybe think about. Think about *a* being three.	
	944	Stephanie	Um hm.	
	945	R1	Think about *b* being seven. Right? Now you went through the two dimensional model, right? Of the pieces where *a* is three and b is seven. Right?	
	946	Stephanie	Um hm.	
	947	R1	Okay. Now see if you can think about building	

Time	Line	Speaker	Transcript	Code
			it up. And what would the little cube pieces be and what will the pieces that, you know – we call this a cube – um – you know, because the length and the width and the depth are the same.	
	948	Stephanie	Um hm.	
	949	R1	Okay. but if it – if I can take that one – if you can help me grab that. Imagine these glued together. [*R1 has placed two of the large cubes side by side.*]	
	950	Stephanie	Okay.	
	951	R1	Right? Okay.	
	952	Stephanie	Yes.	
	953	R1	You can find the volume of this.	
	954	Stephanie	Um hm.	
	955	R1	But it's not a cube any more.	
	956	Stephanie	No.	
	957	R1	But you know how you would do it, right?	
	958	Stephanie	Yes.	
	959	R1	How?	
	960	Stephanie	It would be twenty times ten times ten.	BMP
	961	R1	Okay. So we would call this a rectangular solid. Does it have another name?	
	962	??????	I always call it a rectangular solid.	
	963	R1	A rectangular solid. We call this a cube. We call this a rect…- 'cause it's a solid. It's three dimensional and you can see this piece looks like a rectangle, even though this piece does look like a square. So you have to imagine, as you did with your *a* and *b*, you had two pieces that were squares, the *a* squares and the *b* squares. But your other two pieces were rectangles, weren't they?	
50:00-55:59	964	Stephanie	Yes.	
	965	R1	Remember that one had length *a*,	
	966	Stephanie	Um hm.	
	967	R1	the other had length *b*.	
	968	Stephanie	And they were the same.	
	969	R1	Right. The other had length *b* and the other one had width *a*.	
	970	Stephanie	Um hm.	
	971	R1	But then you had those sides were *a* and sides were *b*. So you got to imagine within what's inside of this thing, you're going to have your	

Time	Line	Speaker	Transcript	Code
			cube – your perfect cube pieces and your other pieces. And that's what I want you to work on – imagining how you would build that. And you could think of the special case: three and seven	
	972	Stephanie	Okay.	
	973	R1	and if you want to, you can take some of these things.	
	974	Stephanie	Okay.	
	975	R1	Would you like to work on that?	
	976	Stephanie	Yeah. I'll work on that.	
	977	R1	In a couple of weeks – I'm going away for a week - I can come back and see what you think.	
	978	Stephanie	Okay. Yeah. I can work on it.	
	979	R1	Isn't that an interesting problem?	
	980	Stephanie	Yeah.	
	981	R1	Um. Now I don't know if these are going to help you or not. But you can take these too.	
	982	Stephanie	Okay.	
	983	R1	I have not looked to see how these work. [*R1 gives Stephanie some items in a paper bag.*]	
	984	Stephanie	Okay.	
	985	R1	Um. But. Um. Mrs. Muter has – was able to build a model using these with *a*'s and *b*'s and maybe you could figure out what she had in her head and then explain it to me.	
	986	Stephanie	Okay.	
	987	R1	Okay. – You could explain it to your teacher, Mrs. Pelosi. Pardon? Do you want to see them? Go ahead.	
	988	Stephanie	Okay.	
	989	R1	You can show them.	
	990	Stephanie	[*Stephanie takes them out of the bag.*] They're like littler cubes. And like (inaudible) piece	
	991	R1	I'm kinda curious to know what you find more helpful, if at all.	
	992	Stephanie	I don't know. I'm sure – you could probably build like – 'cause they're basically – these are just smaller.	
	993	R1	Um hm.	
	994	Stephanie	So that you could probably – you could use either one and like come up with it.	
	995	R1	A model.	

Time	Line	Speaker	Transcript	Code
	996	Stephanie	Yeah.	
	997	R1	This is – boy, she has four colors here. That's interesting. Four colors. And we have four pieces here. [*R1 points to each of the terms of* $a^3 + 3a^2b + 3ab^2 + b^3$] Right? We have one, two, three, four pieces, don't we? We have the *a* cubed piece.	
	998	Stephanie	Yep.	
	999	R1	We have the three *a* squared *b* piece, we have the three *ab* squared piece and the *b* cubed piece. I wonder if that is accidental?	
	1000	Stephanie	[*Stephanie chuckles.*]	
	1001	R1	That's the only hint I'm going to give you.	
	1002	Stephanie	Okay.	
	1003	R1	Fair enough?	
	1004	Stephanie	Yes.	
	1005	R1	Okay. So we need copies of this. And then set up another time, if we can. 'Cause I can't wait to come back.	
	1006	Stephanie	Okay.	
	1007	R1	This is great. So now, let's see. You're up to probably Algebra 2. Wouldn't you think she's into Algebra 2 by now? And um probably even beyond.	
	1008	?????	(inaudible)	
	1009	R1	And you're also doing geometry.	
	1010	Stephanie	Okay.	
	1011	R1	You're also doing some geometry. So don't worry about your colleagues in the other school. You're doing some really good geometry. Because what you're trying to do, Stephanie, here, is: you're trying to think of algebra, right? In a general expression – what does it mean in very explicit terms? It's kind of a helpful thing. If I don't know what this means – you know if your teacher writes something on the board with *a*'s and *b*'s, always look for 'what does that mean?' If I were to put numbers in there, what would that be telling me? Always try to think of meaning. And then, always, if you can, think of 'can there be a model?' something I can build that's concrete that could go with that. So you want to do this mapping three ways. You want to map it to something you already can imagine with	

Time	Line	Speaker	Transcript	Code
			numbers and all you know about numbers to see if this works. To the general and then back to a model and then back and forth.	
	1012	Stephanie	Okay.	
	1013	R1	Just like you did the *a* plus *b* squared. And try not to lose that. *a* plus *b* squared – what that means – the *a* squared plus the two *ab* plus the *b* squared hides it a little bit. 'Cause remember the two *ab* came from an *ab* and a *ba*. Right?	
	1014	Stephanie	Yes.	
	1015	R1	The rectangle *ab* and the rectangle *ba*. So the two *ab* could – so be careful. Maybe there are things hidden in here. By the fact that we collapsed it. I don't know. Right? 'Cause she really ended up with one, two, three, four, five, six, seven, eight. By the way, I wonder how many pieces there are here. [*R1 points to an equation near the middle of Stephanie's development of* $a^3+3a^2b+3ab^2+b^3$.] One, two, three, … I don't know if that helps. One, two, three, four. Interesting. You can play with that.	
	1016	Stephanie	*[Stephanie chuckles.]*	
	1017	R1	You'd like to play with that. [*R1 responds to one of the researchers observing.*]	
	1018	R1	So, you could recopy and write it. Can I have your other papers?	
	1019	Stephanie	Oh. Yeah. You want these? [*Stephanie picks up a folder.*]	
	1020	R1	I'd like to keep a portfolio of of your work. And um we can make copies for you also to keep but I like to keep the originals, if you don't mind, okay?	
	1021	Stephanie	Here is the (inaudible).	
	1022	R1	Wonderful.	
	1023	Stephanie	And	
	1024	R1	We'll make copies of this. Why don't you try to organize them in a way	
	1025	Stephanie	Okay.	
	1026	R1	and decide what's scrap paper and what's really useful for you.	
	1027	Stephanie	Okay.	
	1028	R1	Also, if you put your name and date. I am really very sloppy about dates.	
	1029	Stephanie	Okay.	

Time	Line	Speaker	Transcript	Code
	1030	R1	Stephanie, when were you born? When's your birthday?	
	1031	Stephanie	June 25th.	
	1032	R1	June 25th?	
	1033	Stephanie	1982.	
	1034	R1	So you're how old now?	
	1035	Stephanie	Thirteen.	
	1036	R1	Thirteen. Well, so you're born – you're one of the younger people in your class?	
	1037	Stephanie	Um, I I yeah I guess.	
	1038	R1	Right, so you get to choose one or the other. June 25th. Sort of like my son. He was June 13th. Right? Andrew was even younger, right Linda, he was August?	
	1039	Linda	July.	
	1040	R1	July? That's very interesting.	
	1041	R1	'Cause I wrote up something about you and I made you older than you were when you did it. I said you were ten and you were really nine. So I have that wrong. We all know the truth here. [*Stephanie is shuffling papers.*]	
	1042	R1	So why don't you look at that	
	1043	Stephanie	Okay.	
	1044	R1	and if you have any questions, you can ask anybody here.	
	1045	Stephanie	Okay. Let me see.	
	1046	R1	So name and number and if we could get copies again that would be wonderful. [*R1 looks at watch.*] We're really on task today.	
	1047	R1	What do you think, Ethel?	
	1048	R2	(inaudible)	
	1049	R1	[*Dr. Maher gets up.*] Why don't you take of this what you need.	
	1050	Stephanie	Okay.	
	1051	R1	I don't think you want to take all of them. So you can take what you need.	
	1052	Stephanie	Okay.	
	1053	R1	I think it's okay – that Stephanie can borrow them? [*R1 is speaking to other researchers.*] [*The other researchers are talking in the background* (inaudible) *as Stephanie is organizing her papers. R4 comes over to Stephanie and demonstrates one of the manipulatives. She then tells Stephanie to take*	

Time	Line	Speaker	Transcript	Code
			the whole bag if she wants to take them. Stephanie agrees and then resumes organizing papers.]	

APPENDIX D: TRANSCRIPT – SESSION 4

INTERVIEW WITH STEPHANIE — Time: 69 minutes (1 CD)
February 21, 1996

R1: Dr. Carolyn Maher	Stephanie: Stephanie	R2: Ethel Muter
R3: Dr. Terry Pearl	R4: Mrs. Carmela Colosimo	R5: Dr. Maher's son

Time	Line	Speaker	Transcript	Code
0:00-4:59	1	Stephanie	We just- I went to Miss Colosimo the other day to make sure I- I redid the problem.	PAH
	2	R1	Yeah. Which problem did you do?	
	3	Stephanie	*a* plus *b* s-cubed.	
	4	R1	Oh, you worked on that one. Let me think.	
	5	Stephanie	'Cause that was, like, the information you gave me, but I lost the sheet! I went home and I was- I went nuts looking for the folder with the papers.	
	6	R1	Alright, I might've not have brought the right one either. This is one twenty nine, the February one, but we can rebuild it.	
	7	Stephanie	Okay, I have the-I have paper in my- that I rebuilt it with Miss Colosimo.	
	8	R1	Um-	
	9	Stephanie	Do you want me to get it?	
	10	R1	If you want to, but I'd rather you tell me what you did Friday.	
	11	Stephanie	Okay.	
	12	R1	Bring me up to date first, it would sort of help me remember. It's been a couple weeks for me.	
	13	Stephanie	Oh you mean like the *a* plus *b* square-- cubed.	PPK
	14	R1	Tell me- start from the beginning.	
	15	Stephanie	Squared?	
	16	R1	Start from whatever you remember so far, from when we started, what would it be?	
	17	Stephanie	I remember the-	
	18	R1	Well, pretend you know this- see this young man here? He's gonna-that's my son-	
	19	Stephanie	Mhm.	
	20	R1	And, so you can sort of let him know what- what you did.	
	21	Stephanie	Alright, so it was like- I don't know- we did *a* plus *b* squared, and you asked me to explain	PPK

Time	Line	Speaker	Transcript	Code
			what *a* squared was-	
	22	R1	Mhm.	
	23	Stephanie	With like, a square.	PPK; BR-V
	24	R1	So tell me, help me remember what you did.	
	25	Stephanie	Oh, so [*reaches for pen, writes*], and then you asked me what that was, and it was [*more writing*] it was *a* plus *b* times *a* plus *b*. And um, ahem, then you asked me what, like, to show *a* squared on a square [*more writing*] and that was like, confusing 'cause I didn't know like how you wanted me to show it-	PPK; BR-S; BR-V
	26	R1	Mhm.	
	27	Stephanie	But, so, then we got into, like, if the square was three parts [*writing*] what this was- and that that was a unit, and that that was like one square unit.	PPK; BR-V
	28	R1	Mhm.	
	29	Stephanie	And um, that it would be nine, and because it was like three by three, three squared. And we did a couple of those. And then, um, [*pause*], we- you asked me if it was um, if one side was [*writing*] *a* plus *b* [*writing*]	PPK; BR-V; BR-S; BMP
	30	R1	Oh yes, I remember that one.	
	31	Stephanie	Then what it would be.	BCA; BR-V
	32	R1	Yeah.	
	33	Stephanie	And um, if the small part's *a* and the big part's *b* [*draws square divided into parts representing* $(a+b)^2$]	PPK; BCA; BR-V
	34	R1	Mhm. [*pause, Stephanie writes*] did you figure out what all those pieces were?	
	35	Stephanie	Yeah. It was *a* squared, *ab*, ahem, *b* squared, *ab*, and it would be *a* squared plus 2*ab* plus *b* squared, and that's what we figured out then. [*pause, writes*] *a* plus *b* squared equals.	PPK; BR-S; BR-V; BCA; BMP
	36	R1	Oh, okay, right. And the original conjecture what *a* plus *b* squared equaled you were testing.	
	37	Stephanie	Yes.	PPK
	38	R1	And originally, what did you conjecture?	
	39	Stephanie	Um-	
	40	R1	What most people-	
	41	Stephanie	I think it was *a* squared plus *b* squared.	PPK

Time	Line	Speaker	Transcript	Code
	42	R1	Yeah, lots of students	
	43	Stephanie	And that was wrong.	PPK
	44	R1	conjecture that, right, so-	
	45	Stephanie	Yeah.	
	46	R1	Does this help you-	
	47	Stephanie	Yes.	
	48	R1	-see that its-	
	49	Stephanie	'Cause I tried to, today, um, 'cause when I finally, I went to Miss Colosimo to figure out what- to make sure I knew what I was doing, 'cause I was going to reconstruct *a* plus *b* cubed.	PAH
	50	R1	Mhm.	
	51	Stephanie	But I wasn't sure if I had figured the problem out correctly, so Miss- I went to Miss Colosimo to make sure that I did it right, and- I really didn't have a lot done, she helped me make, like, sure, like, make sure that I did it right. 'Cause I wasn't going to like do the project over and then, like, have a wrong number and the whole thing was like, wrong.	PAH
	52	R1	Mhm.	
	53	Stephanie	So, um, ahem, so then, we had religion class and we didn't do anything so I sat down with the graph paper	
	54	R1	Mhm.	
	55	Stephanie	that you gave me, and um, I started um, trying to figure out that but I measured wrong.	BEJ; BR-V; OBS
	56	R1	Mhm.	
	57	Stephanie	Because, I mean it's hard to draw a three-dimensional figure-	
	58	R1	Hard for me too, yes.	
	59	Stephanie	-on graph paper, so like, instead of like, 'cause when you're drawing diagonally it's not the same as- like that's three [*points to paper*] you know,	BEJ; BR-V
	60	R1	-Mhm.	
	61	Stephanie	So when like, on graph paper, but diagonally it's different, so I just measured and it was like three and- three centimeters, so you measured like that, but I measured off, by like a half a centimeter, or something, and it threw the whole thing off, so… I had it, like, all done	BEJ; BR-V

Time	Line	Speaker	Transcript	Code
			and then Melanie, who sits next to me, was like, what are you doing? And I showed her and she- I'm like, she starts to measure it and she's like, you're wrong. And I'm like, what? And she's like, you're off by a half a centimeter, so I just like, stopped doing it 'cause it was the end of the period, so um, that's like what we did last time.	
	62	R1	Okay, so, let's put the date and the number on the bottom so we don't (inaudible)	
	63	Stephanie	What is it, the twenty first?	
	64	R1	That sounds good.	
	65	Stephanie	Okay.	
	66	R1	Okay, so this is page one, right there so we can keep track. Alright, now, um, so you did this really fast, this *a* plus *b* … *a* plus *b*	
	67	Stephanie	'Cause we did it, like, a couple of times-	
	68	R1	So you really- you really feel comfortable	
	69	Stephanie	Yes.	
	70	R1	with thinking about what *a* plus *b* quantity squared is. What would you conjecture, ah, what *a* plus *b* quantity cubed looks like?	
5:00-9:59	71	Stephanie	Oh, okay.	
	72	R1	What would you conjecture?	
	73	Stephanie	Well, if you wanna, it starts out, it could be [*writes* $(a+b)$ $(a+b)$ $(a+b)$]. That's, like, what it's saying to do. And if you do, like, two of these-	BMP; BR-S
	74	R1	Mhm.	
	75	Stephanie	It comes out to [*writes* $a^2+2ab+b^2(a+b)$] *a* squared plus 2*ab* plus *b* squared times *a* plus *b*.	BMP; PPK; BR-S
	76	R1	Mhm.	
	77	Stephanie	And you can multiply each one of those by that-	BMP
	78	R1	Do you need parentheses around that, this part here [*points to paper*]?	
	79	Stephanie	Well you could, but-[*writes*]	
	80	R1	But what if you didn't?	
	81	Stephanie	Well, then it wouldn't- yeah you do because then it would just be that times that.	BMP; BEJ
	82	R1	Okay.	
	83	Stephanie	So [*pauses*] then you can multiply this by	BMP;

Time	Line	Speaker	Transcript	Code
			everything.	BEJ
	84	R1	What's the "this?"	
	85	Stephanie	*a* plus *b*	BEJ
	86	R1	Okay.	
	87	Stephanie	It would be [*pauses, writes* $a^2(a+b)$, *simplifies to* a^3+a^2b]. Which- *a* to the third, plus *ab*- *a* squared, *b*?	BMP; BR-S
	88	R1	Mhm.	
	89	Stephanie	Um. And then it'd be 2*ab* times *a* plus *b* [*whispers; inaudible, writes* $2ab(a+b)$, *simplifies to* $2a^2b+2ab^2$] and that would be- plus 2*a* squared *b* plus 2*ab* squared? I think?	BMP; BR-S
	90	R1	Mhm.	
	91	Stephanie	And then it would be *a* plus *b* times *b* squared and that would be *ab* squared plus *b* to the third [*writes* $b^2(a+b)$, *simplifies to* ab^2+b^3]	BMP; BR-S
	92	R1	Okay, and can that be simplified?	
	93	Stephanie	Yeah, I think so. [*pauses*]	
	94	R1	Well, even before you simplify it, why don't you put a box around it 'cause you went into two lines.	
	95	Stephanie	Okay.	
	96	R1	Alright. Okay, so you have here 1, 2, 3, 4, 5, 6 terms, right?	
	97	Stephanie	Yeah.	
	98	R1	I'm wondering if any of that can be simplified.	
	99	Stephanie	[*writes*] *a* cubed plus *ab* plus 2*ab* can be simplified, so it would be, um-	BMP; BR-S
	100	R1	[*points to paper*] *a* squared *b* plus 2*a* squared *b*.	
	101	Stephanie	Yeah, um, what would it be? 3*a* squared *b*? [*writes down* a^3+3a^2b]	BR-S
	102	R1	Yeah.	
	103	Stephanie	Plus- and these two can be simplified, it would be 3*ab* squared plus *b* to the third [*adds* $+3ab^2+b^3$ *to the expression*]	BMP; BR-S
	104	R1	Okay. So why don't you put a box- put a different color [*hands Stephanie a different pen*]	
	105	Stephanie	[*draws box, which crosses out some writing*] Whoops.	BR-V
	106	R1	The color's not going to show a little bit, but okay. Okay. So, um, *a* plus *b* quantity cubed, you said, means	

Time	Line	Speaker	Transcript	Code
	107	Stephanie /R1	*a* plus *b* times *a* plus *b* times *a* plus *b*	BMP; BR-S
	108	R1	Okay, so you used it three times as a factor. And then when you actually used your distributive property	
	109	Stephanie	Mhm.	
	110	R1	and then simplified	
	111	Stephanie	Mhm.	
	112	R1	you ended up with	
	113	Stephanie	*a* cubed plus 3 *a* squared *b* plus 3 *ab* squared plus *b* cubed.	BR-S
	114	R1	Um, so that's, um, that's very interesting. Now, you have, this you- you first said has these two pieces.	
	115	Stephanie	Mhm.	
	116	R1	Mhm. [*points to line with* $(a^2+2ab+b^2)(a+b)$ *on paper while speaking*] This piece, and this piece. Right? [*Stephanie nods.*] Now, you have a way of thinking about this piece [*R1 points to* $(a^2+2ab+b^2)$] , don't you?	
	117	Stephanie	Oh, you mean like this? Like what we did here? [*Pulls paper from beginning and points to box representation of square with side a + b*]	BR-V
	118	R1	Is that a way of thinking about it?	
	119	Stephanie	Yeah.	
	120	R1	Okay, so if you had to say in words how you were thinking about this piece [*points to paper*], right?	
	121	Stephanie	Yeah…	
	122	R1	Which you happen to have made a picture, and you've actually shown me-	
	123	Stephanie	Mhm.	BR-V
	124	R1	How would you say, in words?	
	125	Stephanie	Like what this piece is? [*Points at* $(a^2+2ab+b^2)$]	PAH
	126	R1	Yes.	
	127	Stephanie	Well, it's *a* plus *b* quantity squared.	BR-S
	128	R1	Right, but if you were to talk about this picture, what did you make a picture of, what's it showing?	
	129	Stephanie	Oh, I made a picture of a square.	BEJ; BR-V

Time	Line	Speaker	Transcript	Code
	130	R1	Of a square, tell me more about that square.	
	131	Stephanie	Oh, a square and-with sides measuring *ab*- *a* plus *b*.	BEJ; BR-V
	132	R1	Okay.	
	133	Stephanie	And, um, [*pauses*]	
	134	R1	That's good enough, right?	
	135	Stephanie	Yeah.	
	136	R1	So you've made a square with side length *a* plus *b* [*Stephanie nods*] and this piece represents the area of that square.	
	137	Stephanie	Mhm.	
	138	R1	Right? Okay, and what you've said here- so we know we have this piece, but we have it *a* plus *b* times, don't we?	
	139	Stephanie	Mhm.	
	140	R1	And this piece *a* plus *b* times, [*points at paper*] cause we're finding the product.	
	141	Stephanie	Yeah.	
	142	R1	Can you conjecture what that might look like?	
	143	Stephanie	What that might look like… [*pauses, thinking*]	
	144	R1	We're going back to this piece [*points to* $a^2+2ab+b^2$ *on paper*] *a* plus *b* times. Now remember, when you can't make sense of something with letters, try to imagine-	
	145	Stephanie	With numbers?	PNE
	146	R1	-if you're doing it with numbers. Sometimes that's a useful way to think about it. So you might not want to think about it as *a* plus *b*.	
	147	Stephanie	Alright. [*nods*]	
	148	R1	But you might. But you know what this piece [*points at paper- speaking inaudible*]	
	149	Stephanie	But you want me to show like, you how that would look if it was *a* plus *b* times? Like, how, *a* plus *b* quantity squared would look *a* plus *b* times?	PAH
	150	R1	Do you have any idea?	
	151	Stephanie	[*pauses, thinking*] Do you want me to show it on a cube?	BDI; BR-V
	152	R1	I don't know-	
	153	Stephanie	Is that what you want me to do?	PAH
	154	R1	I don't know, can you do it on a cube?	
	155	Stephanie	I don't know, I don't-	OBS
	156	R1	Can you show it on a cube, do you think? Did	

Time	Line	Speaker	Transcript	Code
			you play with any of that stuff?	
	157	Stephanie	You could show it. [*pauses*]	
10:00-14:59	158	R1	That's an interesting thing- [*gets bag and pours contents, Algebra blocks, on table*] Alright. [*rearranges blocks on table with Stephanie*] I'm interested if you could show it with this.	
	159	Stephanie	Alright.	
	160	R1	I don't know if you can or not, I haven't tried it, but I'm not really- I honestly haven't done it myself. This is something that Ethel brought, and um, Ethel seemed to think these are useful, and, so the question is, um, can you [*rearranges Algebra blocks, counts them*] 1, 2, 3, 4, 5, 6, 7, 8 pieces [*points to count terms written on paper*] Hm. That's interesting.	
	161	Stephanie	Um, I mean I guess the side would be like *ab* by *ab* by *ab*- *a* plus *b*, by *a* plus *b*, by *a* plus *b*, but…	BEJ; BR-V
	162	R1	That's reasonable.	
	163	Stephanie	And that's really what this is. [*points at paper, pauses*]	BEJ; BR-S
	164	R1	And you could make your own, you could use your own things to make- you don't have to-	
	165	Stephanie	Well, I'm just like-	
	166	R1	I don't have a clue where to start, frankly.	
	167	Stephanie	It's-	
	168	R1	We could work on this together.	
	169	Stephanie	-it's the same thing. I mean, you could use, this is just smaller than like, one of these… [*reaches to get something from below table, puts 10x10x10 cube on table*] You know? But like…	BEJ; BR-V
	170	R1	[*Gets something from below table, places a 'flat' a 10x10x1 box on table, rearranges pieces on table*] Okay. So we have this piece, and we have that piece.	
	171	Stephanie	Mhm.	
	172	R1	I like to do this, in case you want to- [*gets paper, places 'flat' on table, traces perimeter on paper*] label it in any way. I'm going to do this for you. Probably not gonna have it as neat as you're gonna have it- already I have corners that are rounder than they're supposed	

Time	Line	Speaker	Transcript	Code
			to be, right? Is that good enough?	
	173	Stephanie	Yeah that's fine.	
	174	R1	Alright?	
	175	Stephanie	Mhm.	
	176	R1	I wonder what you think about it-	
	177	Stephanie	Well, you see, the thing is, that [*coughs*] that's like, two-dimensional.	BR-V
	178	R1	Okay, so we can think of this as two-dimensional [*picks up flat*]	
	179	Stephanie	Yeah, but-	
	180	R1	So what part would that be? [*Pulls out paper with work on it*]	
	181	Stephanie	I guess that would be this part [*points at paper*] *a* squared plus *a* plus- Oh! Okay, it would be this [*points at paper*]	BDI; BR-V; BR-S
	182	R1	Oh, Okay, well can you, kind of draw it [*pulls out paper with traced square*]	
	183	Stephanie	Yeah [*draws* $(a+b)^2$ *diagram on new square drawing*]	BR-V
	184	R1	Okay, great. So we can think of this [*places 10x10x1 box on picture*] that way.	
	185	Stephanie	Mhm.	
	186	R1	Okay, [*removes box*] so we know that piece.	
	187	Stephanie	It's *a* plus *b* number of times, and this is *a* plus *b*, like right here [*points at side of drawn square*], like this, side-	BEJ
	188	R1	This length is-	
	189	Stephanie	Yeah- is *a* plus *b* [*pauses*]. So I don't know what that means…	BR-V; OBS
	190	R1	I guess, okay, this side is *a* plus *b* [*points at paper*]	
	191	Stephanie	Yes.	
	192	R1	This side is *a* plus *b* [*points*]	
	193	Stephanie	Mhm.	
	194	R1	These are different, [*pulls 10x10x10 cube and 10x10x1 box together*] in what way?	
	195	Stephanie	[*picks up cube*] This, like, this is a cube.	BEJ
	196	R1	What's the difference?	
	197	Stephanie	It's got three dimensions.	BEJ
	198	R1	Okay, what are they?	
	199	Stephanie	Length, width, and depth.	BMP
	200	R1	Okay, what is the length?	
	201	Stephanie	Length [*points along edge of cube*]	BR-V
	202	R1	Remember, remember you called it-in terms	

Time	Line	Speaker	Transcript	Code
			of *a* plus *b*.	
	203	Stephanie	Well then it's just like, oh! [*laughing*] It's *a* plus *b*. The length is like, *a* plus *b*, the width is *a* plus *b*, the height is *a* plus *b*.	BDI; BEJ; BR-V
	204	R1	So.	
	205	Stephanie	So. It's just…	
	206	R1	[*picks up 10x10x1 box, turns it on side to show it is same height as 10x10x10 cube*] So , I think that's true, the height is *a* plus *b*.	
	207	Stephanie	Yeah, it's all *a* plus *b*.	BEJ
	208	R1	So, is that helpful?	
	209	Stephanie	I don't know, I mean I already… I just don't know how to show this [*paper*] that *a* plus *b*, length times.	OBS
	210	R1	Well, [*points at paper*] you showed this.	
	211	Stephanie	Yeah, I mean that's just *a* plus *b*, I don't know- I- I mean, that's like…	OBS
	212	R1	Okay, let me ask you a question; forget *a* plus *b* for a minute.	
	213	Stephanie	Okay.	
	214	R1	What is this really [*points at 10x10x1 box*]? What do we consider this?	
	215	Stephanie	Oh, it's a hundred units. Like, the area? Or like-	BMP; BR-V
	216	R1	It's a hundred, well, h- how- where did a hundred come from?	
	217	Stephanie	Well,	
	218	R1	Is it the area? The area is what? The area is a hundred?	
	219	Stephanie	Mhm.	
	220	R1	A hundred what?	
	221	Stephanie	Square units.	BR-V
	222	R1	So we can think of this as a hu- where did that come from?	
	223	Stephanie	[*Runs pen along sides of box*] 'Cause it's ten units here, and ten square units here and ten times ten is a hundred.	BEJ; BMP
	224	R1	Do you believe it? Do you believe there are a hundred square units here?	
	225	Stephanie	Yeah.	
	226	R1	You absolutely-	
	227	Stephanie	[*laughing*] Yes.	
	228	R1	-believe it?	
	229	Stephanie	Yes [*nodding*]	

Time	Line	Speaker	Transcript	Code
	230	R1	I couldn't convince you that, that that's not true [*Stephanie shakes head*]. Okay, so you know that.	
	231	Stephanie	Yes.	
	232	R1	Okay. Now, [*points at box's edges*] this has length ten, width ten, right-	
	233	Stephanie	Mhm.	
	234	R1	It has area, a hundred square units, I could have said it had [*runs finger along edge of box*] *a* is 4, *b* is 6…	
15:00-19:59	235	Stephanie	Mhm [*nods*] It wouldn't make-	
	236	R1	*a* is 2…	
	237	Stephanie	Yeah, it wouldn't make a difference.	BR-V
	238	R1	*a* is two and a half… [*Stephanie laughs*] Now. What about this one [*points to cube*]	
	239	Stephanie	[*Pause*] In numbers, or like…	PAH
	240	R1	Yeah, numbers, if you're dealing with…	
	241	Stephanie	Well, um. The area would be, um, one thousand? The area would be a thousand.	BMP; BR-V; OBS
	242	R1	Area?	
	243	Stephanie	Well, oh, the volume.	
	244	R1	What do you mean by volume?	
	245	Stephanie	Um, length times width times height.	BMP; PPK
	246	R1	But I don't know what that means.	
	247	Stephanie	Well it's the three dimensions of the cube.	BEJ
	248	R1	What does the thousand mean. I know what the hundred means [*points to box*], I can count-	
	249	Stephanie	There's a thousand little, like [*picks up little one-unit cube*] units. Square units in there. Like, you could fill it up with a thousand square units.	BEJ; BR-V
	250	R1	How do I know that? Can you- do you see that? I only see [*points to cube's faces*] 10, 20, 30 40, 50, [*picks up, points at bottom*] 60.	
	251	Stephanie	You only see 60?	
	252	R1	I'm sorry, 600.	
	253	Stephanie	Okay.	
	254	R1	[*touches top face*] 100.	
	255	Stephanie	Yes.	
	256	R1	[*points to side*] And I see another hundred.	

Time	Line	Speaker	Transcript	Code
	257	Stephanie	Mhm.	
	258	R1	[*points to other sides*] And this would be…	
	259	Stephanie	Okay.	
	260	R1	And you're telling me there are a thousand.	
	261	Stephanie	Yes.	
	262	R1	So why?	
	263	Stephanie	Okay. Well it's 10 high, right? [*picks up cube, compares to box*] If it was just th- one of these, [*indicating box*] is like the same as like, it's one. Like one part.	BEJ; BR-V
	264	R1	A hundred.	
	265	Stephanie	A hundred. And you know that like [*touching side of cube*] this, is the same as that [*indicating box*].	BEJ; BR-V
	266	R1	So that's a hundred, okay. So what-	
	267	Stephanie	It's the same thing, if I took this off it would be one hundred [*indicating side of cube*], and you know that there's ten of them [*pointing to each layer of cube along the edge*]. So you see that ten of them would make this- ten high? You know?	BEJ; BR-V
	268	R1	So if I took one of them off, I would get one hundred.	
	269	Stephanie	Yes.	
	270	R1	If I peeled another off…	
	271	Stephanie	It would be two hundred.	BEJ
	272	R1	[*tapping top of cube*] 300, 400, 500 [*trails off*]. That's the way you think about getting a thousand.	
	273	Stephanie	Yeah.	
	274	R1	I peel- How many times would I peel them off?	
	275	Stephanie	Ten.	BEJ
	276	R1	Ten times? Why ten?	
	277	Stephanie	[*coughs*] 'Cause that's how high it is. That's how many fill it up.	BEJ
	278	R1	Oh. How wide is it?	
	279	Stephanie	Ten.	BEJ
	280	R1	And how long is it?	
	281	Stephanie	Ten.	BEJ
	282	R1	Can you use that same argument to show me what *a* plus *b* is times that [*points at paper*]? Only now we're calling this *a* plus *b*.	

Time	Line	Speaker	Transcript	Code
	283	Stephanie	Well it's *a* plus *b* high, it's *a* plus *b* long, and it's *a* plus *b* wide-	BR-V; BCA
	284	R1	Mhm.	
	285	Stephanie	So, um [*picks up 10x10x1 box*] if this is *a* plus *b* squared, you see that like, [*pointing at 10x10x10 cube*], this, if you t- took this off, it would be *a* plus *b* squared, and you need to take *a* plus *b* amount of these off to get *a* plus *b* cubed.	BEJ; BCA; BR-V
	286	R1	Does that make sense?	
	287	Stephanie	A little bit.	
	288	R1	A little bit, not quite [*Stephanie laughs*], a little fuzzy yet?	
	289	Stephanie	Yeah, 'cause it's harder using letters-	OBS
	290	R1	You bet.	
	291	Stephanie	Especially when there's like, *a* plus *b*. It's not just like *a* number of…	OBS
	292	R1	Yeah, right. So it's easy to think of it as we're thinking of ten. [*points to edge of cube*] Peeling it off ten times. But if I said I'm peeling it off six and then four times to make my ten. Or eight then two times to make my ten, or seven then three times to make my ten, *a* and then *b* times to make my ten, is that easier to see?	
	293	Stephanie	Yeah. It's easier to like-	
	294	R1	Harder to think in those abstract-	
	295	Stephanie	Yeah-	
	296	R1	-terms, in those symbols. It really is. [*Stephanie nods*] A lot of people don't think - they just do things. They don't try to think and imagine what it means. But, um, every now and then you oughta try to think about- 'cause it's elusive, it's gonna come, and it's gonna go, and it's gonna come and it's gonna go. Which is very interesting. Okay well that's something to think about some more, and- so, I can think of the- Tell me again how I can think of *a* plus *b* quantity cubed, one more time.	
	297	Stephanie	You want like this [*points to paper*] or you want me to show you like here [*points to cube*].	PAH
	298	R1	Well, what- show me here first [*points to cube*], then we're going to try to break it	

Time	Line	Speaker	Transcript	Code
			down [*rearranges papers*] to pieces.	
	299	Stephanie	Alright, well, if this is *a* plus *b*, like this side is *a* plus *b* [*uses box*] and this side is *a* plus *b*, then there are *a* plus *b* squared number of pieces in here. Do you believe that?	BEJ; BR-V
	300	R1	I believe that. And I even believe that it is *a* squared plus 2*ab* plus *b* squared.	
	301	Stephanie	Yes. So-	
	302	R1	You've convinced me of that.	
	303	Stephanie	So, if I were- there's *a* plus *b*, like, rows of these. If I took *a* plus *b* number of, like, this [*indicates box*], it would make that- it would fill that up [*indicates cube*].	BEJ; BR-V
	304	R1	Okay.	
	305	Stephanie	If I took off one of these [*indicates box*], you see if I took this first row off, right here, I'd have *a* plus *b* squared, of- *a* plus *b* squared number-	BEJ; BR-V
	306	R1	*a* plus *b* quantity squared	
	307	Stephanie	Yeah, and so I'd have to take up *a* plus *b* number of those, to like, fill it up [*indicates cube*], or something?	BEJ; BR-V
	308	R1	Okay, so,	
	309	Stephanie	Yeah.	
	310	R1	Alright, now. This is real interesting, how you told me this was *a* plus *b*, and this was *a* plus *b* [*points to sides of box*], and you said the area here was *a* plus *b* quantity squared. But then when you actually showed me the *a* plus *b* you actually showed me the *a* squared, and the *ab*, and the *ab* and the *b* squared. And you got the *a* squared plus two *ab* plus *b* squared.	
20:00-24:59	311	Stephanie	Yeah.	
	312	R1	No, this is how we're talking and you're following me. Isn't that amazing?	
	313	Stephanie	Yeah.	
	314	R1	It really is great. You're doing great. Okay, I'm curious, now, and I've never had to do this as a student before- I'm curious now- you also said that *a* plus *b* times *a* plus *b* times *a* plus *b* is *a* squared plus 2*ab* plus *b* squared, that quantity, times *a* plus *b*. And then when you simplified it, you got *a* cubed, plus 3*a* squared *b*, plus 3*ab* squared, plus *b* cubed.	

Time	Line	Speaker	Transcript	Code
	315	Stephanie	Yeah.	
	316	R1	Right, and before that you got all these terms before you simplified.	
	317	Stephanie	Mhm.	
	318	R1	I think it's important to think about what you did before you simplified them too, because you see here [*pointing at paper with square representation of* $(a + b)^2$] you had an *ab* and a *ba*, and they each had different regions.	
	319	Stephanie	Mhm.	
	320	R1	Even though it simplified to 2*ab* when you actually built your model, they had different regions representing each of these components before you simplified them. So, it may very well be these 1, 2, 3, 4, 5, 6. It may be six pieces. It may not be, I don't- I- I've never done this before. I haven't done it with these pieces. [*Rearranges Algebra blocks from earlier*] And I don't if, um, Ethel did this to us to distract us, give us more pieces than you need. I don't know what she did. She's a teacher, teachers do sneaky things.	
	321	Stephanie	Uh huh.	
	322	R1	And- Or she expects us to use all of them? [*Stephanie nods*] Or, um, she thinks we can model it that way [*Stephanie nods*]. Um, you can ask her anything you want, but I'm kind of curious to see if we can see these components [*pointing at paper*] in building the model. Um, she's here. I don't think she's gonna tell us too much, 'cause she's not allowed to, but she might tell us the basics.	
	323	Stephanie	I don't- I don't know.	
	324	R1	W- What can we start with? [*Picks up some of the Algebra blocks*] I don't know.	
	325	Stephanie	[*Picks up blue piece*] Well if that's *a* plus *b* by *a* plus *b*, if you're- if you're saying that this is *a* plus *b* squared, and that this is *a* plus *b* high, [*picks up white piece*] it's the s- I can- I can just explain it the same way.	BR-V
	326	R1	Mhm.	
	327	Stephanie	That I explained it with this [*10x10x10 cube*] and this [*10x10x1 flat*].	BR-V
	328	R1	Mhm.	

Time	Line	Speaker	Transcript	Code
	329	Stephanie	You know?	
	330	R1	Mhm.	
	331	Stephanie	I don't know…	
	332	R1	But suppose if you wanted to… [*rearranges Algebra blocks to resemble* $(a+b)^2$ *model in drawing, sighs*] funny little one in there… Um.	
	333	Stephanie	[*coughs*] Um.	
	334	R1	You like that funny little one in there?	
	335	R2	[*off screen*] I like that 'cause it matches up with what she's shown us.	
	336	R1	'Cause I'm looking at what you did here, [*points to drawing*] in terms of *a* plus *b*.	
	337	Stephanie	Mhm. Oh…	
	338	R1	Is that-	
	339	Stephanie	Oh-	BDI
	340	R1	I don't know, does that do it? Is that the way?	
	341	Stephanie	Oh [*mumbling*], if you wanted to- [*rearranges to model drawing exactly*] that's how it's drawn.	BDI; BEJ; BR-V
	342	R1	Is that like what you drew?	
	343	Stephanie	Yeah.	BEJ
	344	R1	How does that work?	
	345	Stephanie	[*Points to pieces in model*] *a* squared.	BEJ; BR-V
	346	R1	What's *a* and what's *b*?	
	347	Stephanie	This is *a* and this is *b*.	BEJ; BR-V
	348	R1	That's *a* and that's *b*? Oh, okay, this is *a* squared…	
	349	Stephanie	[*Points to pieces in model*] *a* squared, *a* plus *b*, err- *ab*	BEJ; BR-V
	350	R1	Okay-	
	351	Stephanie	*b* squared-	BEJ; BR-V
	352	R1	Okay-	
	353	Stephanie	*ab*.	BEJ; BR-V
	354	R1	Oh, okay, that's neat. Now, I'll buy that.	
	355	Stephanie	Okay.	
	356	R1	Now, how would we show *a* plus *b* quantity cubed?	
	357	Stephanie	Oh, it'd have to be, like, more, it'd have to be three dimensional. I couldn't-	OBS

Time	Line	Speaker	Transcript	Code
	358	R1	Okay-	
	359	Stephanie	‘Cause it doesn’t have three parts, I couldn’t, like, say, well I… [*pauses, picking up small cube*]	BEJ
	360	R1	Okay, let’s leave this.	
	361	Stephanie	I guess if I…	
	362	R1	That’s interesting. We have all these pieces here. If I were doing it I’d give you more than- I don’t know what she had in mind, but we… We need to show [*pauses, points to parts of paper*] this is *a* [*pointing to small cube*], and this is *b* [*pointing to cubes*]. We need to show this [*pauses*] right?	
	363	Stephanie	Yeah.	
	364	R1	Up now [*indicating height*].	
	365	Stephanie	Yeah.	
	366	R1	*a* plus *b*. Is that right? [*Stephanie looks off screen*] Don’t look at her, she’s not going to tell us. [*Stephanie laughs*]	
	367	Stephanie	Um.	
	368	R1	Right?	
	369	Stephanie	I don’t-like, if you want me to show you, like…	OBS
	370	R1	How can we make a cube? Now, we have the- we have a square, right? [*Stephanie nods*] With area *a* squared plus 2*ab* plus *b* squared.	
	371	Stephanie	Mhm.	
	372	R1	Okay, that’s an interesting puzzle. Now we wanna make a cube, so we have to go up *a* plus *b*.	
	373	Stephanie	Yes.	
	374	R1	What- what’s *a* plus *b*?	
	375	Stephanie	[*points to model already assembled*] That right there.	BR-V
	376	R1	[*Picks up the pieces modeling the side length*] This is *a* plus *b*, right?	
	377	Stephanie	Yes.	
	378	R1	So we wanna go up *a* plus *b*.	
	379	Stephanie	[*coughs*] But there’s not enough pieces.	OBS
	380	R1	Oh, I don’t know. [*Places a block vertically*] But that’s up *a* plus *b*.	
	381	Stephanie	Oh. Oh. Alright.	
	382	R1	Isn’t it?	
	383	Stephanie	Well, yeah.	

Time	Line	Speaker	Transcript	Code
	384	R1	No?	
	385	Stephanie	Well, yeah. I just- I didn't think of it like that. So, do you like-	BDI; BR-V
	386	R1	So we know *a* plus *b* up-	
	387	Stephanie	Yeah.	
	388	R1	Okay.	
	389	Stephanie	So… What do you want me to show? Like…	PAH
25:00-29:59	390	R1	Okay, so now when we have a cube, we know [*picking up blue piece*] right? What do we know about all these? Any- all- of these components? [*pauses*] Okay, [*points at paper*] is there an *a* cubed any place?	
	391	Stephanie	[*pauses*] I don't- [*sighs*]	OBS
	392	R1	Is there an *a* squared *b* any place?	
	393	Stephanie	I- guess-	OBS
	394	R1	Where's there an *ab*?	
	395	Stephanie	An *ab*? Is right here [*points at set of green cubes*], well, no. An *ab* is like, is this piece right here? Or this piece?	PAH; BR-V
	396	R1	Okay, so it's *a* [*pointing to one side of piece*] *b* [*pointing to other side*]. So this piece is *a* and this piece is *b*.	
	397	Stephanie	Yes.	
	398	R1	So where would a, *ab* squared be? I wonder…	
	399	Stephanie	*ab* squared? Is that what you said?	PAH
	400	R1	Yeah. [*pause*] This is *b*. [*points to side green piece on model*] Think about this, it's so easy …	
	401	Stephanie	[*Sighs*] Um, I guess…	OBS
	402	R1	Here, maybe we can make a picture with this, like we did here [*collects papers*]-	
	403	Stephanie	Can we go like-	PAH
	404	R1	That might help.	
	405	Stephanie	Alright.	
	406	R1	If we trace it, right? I'll let you do it this time. We're up to page what… This was such a nice one. [*referring to paper from earlier, picks up blank sheet*] This is two, why don't you label this is two, this three [*shuffles paper*] and then we'll make that one four. [*Stephanie traces around blocks in* $(a+b)^2$ *model*] You know what confuses me in this? Um, I don't know if it bothers you, Stephanie, I'm gonna tell you where I get confused…	
	407	Stephanie	Where?	

Time	Line	Speaker	Transcript	Code
	408	R1	I'll tell you after you draw it.	
	409	Stephanie	I can't trace…	
	410	R1	It's not going to be any worse that what I would have done, I promise you that.	
	411	Stephanie	Okay. [*finishes tracing*]	
	412	R1	See, you know what might help? To mark off on- see- [*indicating on paper*], because this already has the height, okay, so let's pull it apart and-	
	413	Stephanie	And mark off where- [*pointing to edge of Algebra block*]	BR-V
	414	R1	Yeah-	
	415	Stephanie	Yeah where each thing is. [*marks on paper where each block comes together*]	BR-V
	416	R1	Make it two dimensional, right. So where's your *a*? That's right. Put a line, like a line there. Okay-	
	417	Stephanie	Okay.	
	418	R1	Okay. So let's mark off the components [*Stephanie marks off next piece*] you can do that, you know what *a* is…	
	419	Stephanie	[*rearranging pieces, tracing*] Alright. [*pulls each component apart*]	BR-V
	420	R1	So, let's mark it exactly [*places Algebra block back on tracing as guide*]	
	430	Stephanie	Oh, you want me to like, label it?	PAH
	431	R1	Yeah.	
	432	Stephanie	Okay. [*labels each component of* $(a+b)^2$ *model*] Okay.	BR-V
	433	R1	Okay, now, we wanna really be fussy about this. This is *a* squared and this is *ab*…	
	434	Stephanie	Mhm.	
	435	R1	Alright, but that's this and this [*indicating side lengths*] now we're going up [*indicating height*]. How many times have you gone up now?	
	436	Stephanie	Here? [*Pointing at yellow piece on model*] This piece? You went up, like, *a*.	BEJ; PAH; BR-V
	437	R1	Mhm.	
	438	Stephanie	So like, this piece here, wouldn't it be *a* cubed?	BDI
	439	R1	Hmm. Okay, that piece is *a* cubed.	
	440	Stephanie	And this piece, what was this [*moving green piece of model*], *a* plus *b*, *ab*? So… I don't	OBS

Time	Line	Speaker	Transcript	Code
			know if this is like…	
	441	R1	So if you went up *a*, this is *a* cubed [*indicating yellow piece*]	
	442	Stephanie	Yeah.	
	443	R1	Okay. Now how much did you go up over here?	
	444	Stephanie	You went up *ab*.	BEJ
	445	R1	How-how much did you go- Tell me how you decided you went up *a* here [*indicating yellow piece*]	
	446	Stephanie	Well, 'cause this is an *a* piece, this is an *a*.	BEJ; BR-V
	447	R1	What's the *a*?	
	448	Stephanie	This yellow piece [*points at yellow piece*].	BEJ
	449	R1	No, the piece isn't an *a*.	
	450	Stephanie	Oh, well, like…	
	451	R1	What's the *a*?	
	452	Stephanie	This is *a* [*indicating side length*], like the unit.	BEJ; BR-V
	453	R1	Okay, the length-the side of this is an *a*.	
	454	Stephanie	Yes.	
	455	R1	Okay, 'cause this thing [*picking up* a^3 *piece*] is not an *a* squared, it's –	
	456	Stephanie	Going up-	BEJ
	457	R1	Going up, it's an *a* cubed. So you went up *a*.	
	458	Stephanie	Yeah.	
	459	R1	How much did you go up here? [*pointing to green piece*]	
	460	Stephanie	You went up- you went up *a*.	BEJ
	461	R1	You went up *a* here, okay. So you went up *a*-	
	462	Stephanie	Yeah.	
	463	R1	And how much were you down? [*pointing to tracing on paper*] What's the area of this little piece?	
	464	Stephanie	The area of that little piece was *ab*.	PPK; BMP
30:00-34:59	465	R1	But you went up- You did *ab*, *a* times.	
	466	Stephanie	So it would be *a* squared *b*?	BDI
	467	R1	Does that make sense?	
	468	Stephanie	Yeah. So, can I like write that on the side?	BR-S
	469	R1	Whatever you want. 'Cause they're getting interesting. [*Stephanie labels* a^3, a^2b *around edge of traced diagram]* Okay, so this is *a*	

Time	Line	Speaker	Transcript	Code
			cubed, and we're saying this piece is *a* squared *b*. What about this piece? [*blue piece]*	
	470	Stephanie	Hm- You still went up *a*.	BEJ
	471	R1	Okay.	
	472	Stephanie	So it would be *ab* squared?	BCA; BR-V
	473	R1	Does that make sense?	
	474	Stephanie	Yes. And this one you went up *a*, [*indicating other green piece]*, so it would be *a* squared *b*? I guess? 'Cause it's the same as that one [*indicating other first green piece]*.	BEJ; BCA; BR-S
	475	R1	Okay. But we only went up *a*, remember we're supposed to go up *a* plus *b*. [*laughs]*	
	476	Stephanie	Okay, so-	
	477	R1	Isn't that interesting?	
	478	Stephanie	So now we have to go up two more?	PAH
	479	R1	I don't know, I'm gonna let you think about that. I'm not gonna- I think maybe this is something to think about some more, right? 'Cause we've only gone up *a*.	
	480	Stephanie	Mhm.	
	481	R1	Remember like here, um, when we went up ten, [*indicating 10x10x10 cube from earlier]* we could've gone up four and six? We wanna do *a*, this is *a* [*indicating yellow piece's height]*, now we wanna go up *b*. Do we know what *b* is? Do we know the length of *b* any place?	
	482	Stephanie	*b*?	
	483	R1	We wanna go up *a* plus *b*.	
	484	Stephanie	Is like this [*picking up green block vertically]* So I guess we'd have to go up this much more [*placing green block on top of yellow block vertically]*.	BEJ; BR-V
	485	R1	That's interesting. [*pause]*	
	486	Stephanie	So *a* cubed would be- I don't know, *a* squared- *a* cubed *b*?	PAH; OBS
	487	R1	Well, I don't think it's fair to have you think about this right now, but I think this is something you could be thinking about.	
	488	Stephanie	Okay.	
	489	R1	Does it give you a direction to think?	
	490	Stephanie	Yeah.	
	491	R1	'Cause I'm thinking here, see, [*replaces* a^2b	

Time	Line	Speaker	Transcript	Code
			piece horizontally in original model; adds new a squared b piece on top of a cubed piece vertically, then moves it vertically on top of a^2b piece] can you make a cube with these pieces? Can you build a cube?	
	492	Stephanie	That, like, all that length?	PAH
	493	R1	With this as a base [*indicating model]*? Just build it, without worrying about what they are. Can you just make it, can you put the puzzle together? [*Stephanie attempts to put pieces together to create cube*]	
	494	Stephanie	I don't know if there's enough, like [*hesitates*], no. Not a – well [*resumes rearranging pieces, succeeds at assembling cube*]. Oh. There.	BR-V
	495	R1	My goodness. That's pretty neat. Now.	
	496	Stephanie	Oh boy…	
	497	R1	What kind of question might you be asking? You've done a really nice job, saying what all those pieces are, and what it was coming up what, one layer of it, you know?	
	498	Stephanie	Mhm.	
	499	R1	You did all those components of the first layer, that's very lovely. And then you went up *b*, right?	
	500	Stephanie	Mhm.	
	501	R1	So I'm kind of interested in [*pause*] you know, you had- you ended up with an *a* squared *b*, and an *a* squared *b*.	
	502	Stephanie	Yeah.	
	503	R1	An *ab* squared, but you ended up with this [*pointing to paper with work from before*] before that, with this [*showing work from previous; work before simplifying; accidentally knocking over cube*] whoops. What did I do, I destroyed it. I don't wanna put it together the way you didn't have it? Do you remember what you did? Was it like this? [*reassembling cube*]	
	504	Stephanie	Yes.	
	505	R1	I don't know if they belong in those places or not [*reassembling cube*] That's something we can think about, maybe they do, maybe they don't, I haven't thought about it. But, we know where the *a* cubed is.	

Time	Line	Speaker	Transcript	Code
	506	Stephanie	Yes.	
	507	R1	That's this little piece.	
	508	Stephanie	Yes.	
	509	R1	I mean, are all of these pieces there? [*Indicating terms on paper and pieces of cube*]	
	510	Stephanie	Probably.	
	511	R1	This is *a* plus *b*, here [*indicating cube*]. Should we- How can we figure that out?	
	512	Stephanie	Well, we already have, we have this piece [*going to write on paper*]	BR-S
	513	R1	Let's get another piece of paper [*gets another sheet of paper*]. We already have the *a* cubed piece.	
	514	Stephanie	We have *a* cubed [*writes terms on paper*], we have *a* cubed- squared *b*, we have *ab* squared, and we have another *a* squared *b*. And I guess, on the base level [*pulling apart a piece of the cube*], does that count? [*Drops some pieces, reassembles cube*]	BR-S/V; PAH
	515	R1	That was all those pieces- you-	
	516	Stephanie	Yeah, so it doesn't. So like, we have these four [*pointing to paper*] pieces… With just this layer.	BR-S
	517	R1	Hmm. Just the bottom layer.	
	518	Stephanie	Yeah.	
	519	R1	Mhm. And [*returning to previous work on paper, before simplified*], according to this thing we needed three *a* squared *b*, you only had one. You need *3ab* squared, you only had one. Right?	
	520	Stephanie	Well we have two *a* squared *b*. [*pause*] Don't we?	BMP; PAH
	521	R1	Hmm. I guess we do. Right.	
35:00-39:59	522	R1	We have an *a* squared *b*, we have two *a* squared *b*. [*places old and new work next to each other*] I don't know, is this the right way to think about this? It's interesting. [*pause*] What's a *b* cubed?	
	523	Stephanie	*b* cubed? Um… [*deconstructs cube, picks up* ab^2 *piece from bottom layer*] That's *b* squared [*puts cube back together*]. And that's gonna be… [*pauses*]	BR-V
	524	R1	You said this was *b* squared? Over here, right? [*removes piece, pointing to bottom*	

Time	Line	Speaker	Transcript	Code
			layer of cube]	
	525	Stephanie	Yeah, that was *b* squared.	BR-V
	526	R1	What was *b*? Show me *b*. What was the length *b*?	
	527	Stephanie	*b* is like this, [*running finger along edge of* ab^2 *piece*] or this [*running finger along* b^3 *piece*], so I guess it's going up another *b*, so… But it's already *ab* squared, but there's no *ab* cubed.	BEJ; BR-V
	528	R1	Well, [*pulling out* ab^2 *piece, pointing to tracing on paper*] this was *b* squared, right? And then when you went up one it became *ab* squared. That was this piece [*replacing* ab^2 *piece*].	
	529	Stephanie	Yes.	
	530	R1	Right? Isn't that right?	
	531	Stephanie	Yes.	
	532	R1	[*moving piece in and out of place*] 'Cause you went up *a*. So you went to *ab* squared.	
	533	Stephanie	Mhm.	
	534	R1	So, what's this [*places* b^3 *piece on tracing on paper*]?	
	535	Stephanie	Well that's *b*. That's going up *b*. Like, that would be going up *b* [*pointing along edge of* b^3 *piece*].	BR-V
	536	R1	So.	
	537	Stephanie	So I guess that would be *b* cubed.	BDI; BR-V
	538	R1	So tell me why that's *b* cubed.	
	539	Stephanie	'Cause you're going up, like, you already have *b* squared and you're going up another *b*.	BEJ
	540	R1	Okay, so this piece is *b* cubed [*picks up piece*]	
	541	Stephanie	Okay.	
	542	R1	Yeah, I think so. 'Cause if you're telling me this is *b*, and this is *b*, and this is *b* [*points to edges*], does that look like a cube?	
	543	Stephanie	Yeah.	
	544	R1	That looks like a *b* cubed, and that looks like an *a* cubed [*pointing at pieces*] *a-a-a*. So we know the *a* and the *b* cubed. That's pretty good.	
	545	Stephanie	So we have *b* cubed [*writes on paper*].	BR-S
	546	R1	Now do you believe you can find all these pieces in here [*pointing at previous paper with terms before simplifying*]? What's your	

Time	Line	Speaker	Transcript	Code
			conjecture at this point?	
	547	Stephanie	I don't- Probably.	
	548	R1	Okay, you kind of think that's a reasonable thing to pursue. That's why I think we should stop.	
	549	Stephanie	Okay…	
	550	R1	I think you know enough. W- If you think about it, [*picks up pieces from cube*] you know, you can give names to some of these, right? Right?	
	551	Stephanie	Yes.	
	552	R1	K. So what did you call this one again [*holding up* ab^2]?	
	553	Stephanie	*b* squared [*pauses*]. Didn't I? It was- yeah that was *b* squared.	BEJ;OBS
	554	R1	Which part is *b* squared? The whole piece?	
	555	Stephanie	Well this is *b*, and this is *b* [*pointing to edges*]	BEJ; OBS
	556	R1	Is this solid *b* squared, or-	
	557	Stephanie	Oh, it's flat *b* squared.	BEJ; BR-V
	558	R1	Flat is *b* squared, but when you-	
	559	Stephanie	*a*- *ab*?	PAH
	560	R1	So it's *ab* squared.	
	561	Stephanie	Okay.	
	562	R1	Does that make sense? This is *ab* squared. Isn't that interesting? We can think of this piece as *ab* squared.	
	563	Stephanie	Okay.	
	564	R1	Okay. So, it might help you to write this down, or draw pictures, anything you need to remind yourself of what pieces you know and that you believe. Because remember, you're the one who gave them all these names here [*pointing at tracings on paper*], should I move this for a minute [*slides cube off of tracing, Stephanie writes on new paper "Blue piece-* ab^2]	
	565	Stephanie	Okay [*continues writing*]. White is *b* cubed. Yellow is *a* squared [*pauses, corrects "2" with "3" on paper*] cubed.	BR-S
	566	R1	Why did you change it?	
	567	Stephanie	Because I was talking about the paper, instead of the yellow.	BEJ
	568	R1	Okay, good. So when you think about the	

Time	Line	Speaker	Transcript	Code
			paper, it's the two dimensions, and when you think of the actual block-	
	569	Stephanie	Mhm-	
	570	R1	You have to think of three dimensions.	
	571	Stephanie	Mhm. And the green was [*writes*] *a* squared *b*.	BR-S
	572	R1	Okay. So. [*pauses*] The green one is *a* squared *b* [*gathers green pieces*], how many of those do you have?	
	573	Stephanie	Three. Three *a* squared *b.*	BEJ
	574	R1	And what's the blue one [*picking up blue piece*]	
	575	Stephanie	Oh, so we have 3*a* squared *b* [*pointing to original paper with simplified work*]	BDI; BMP
	576	R1	Oh.	
	577	Stephanie	[*Crosses out two* a^2b *terms on newer paper, rewrites "*$3a^2b$*" instead*] And we have *a* cubed [*writes*] and we have *b* cubed, and we have *ab* cubed- squared- [*looks at pieces*] we have 3*ab* squared [*writes*]	BR-S; BMP
	578	R1	So, why are these *ab* squared [*picks up blue piece*]	
	579	Stephanie	Because, it's like, *a* up, *b* over [*pointing to edges of piece*]	BEJ
	580	R1	Believe that, absolutely. Okay.	
	581	Stephanie	So that's it, we have all the pieces.	BDI
	582	R1	So you believe…	
	583	Stephanie	Yeah.	
	584	R1	You're absolutely convinced?	
	585	Stephanie	Yes.	
	586	R1	You can explain that to your teacher?	
	587	Stephanie	Yeah, kind of.	
	588	R1	And to Melanie?	
	589	Stephanie	Yes.	
	590	R1	Kind of? Or- If you think about- this is really cool. This was a nice problem, Ethel. But you should've given us more pieces. To throw us off.	
	591	R2	Should've made them all the same color, too.	
40:00-44:59	592	R1	Should've made them all the same color? That would have been very hard [*laughs*]. It's nicer to have them different colors, don't you think? So next time you can make them a little harder.	

Time	Line	Speaker	Transcript	Code
			Okay, so you believe that the quantity *a* plus *b* squared means *a* plus *b* three times [*points at paper*]. You'd have to think about this a lot until you have the *a* cubed piece, you have the *a* squared *b* piece three times-	
	593	Stephanie	Mhm.	
	594	R1	You have the *a b* squared piece	
	595	Together	three times.	
	596	R1	And you have the *b* cubed piece three times.	
	597	Stephanie	Yes.	
	598	R1	And when you build it all up, I'm going to ask you to do it one more time…	
	599	Stephanie	Okay.	
	600	R1	Okay [*Stephanie builds*]. You're gonna have a cube?	
	601	Stephanie	Um, yeah.	
	602	R1	And what are the length, width, and height of that cube?	
	603	Stephanie	[*pause; building*] The length is *ab*, the width is *ab*, and the height is *ab*.	BEJ; OBS
	604	R1	*ab*?	
	605	Stephanie	What?	
	606	R1	*ab*? Show me.	
	607	Stephanie	*a* plus *b*.	BR-V
	608	R1	*a* plus *b*. And what's the *a*, and what's the *b*?	
	609	Stephanie	Well, this can, this is the *a* [*pointing to top of* a^2b *piece*] and this is the *b* [*running finger along remaining edge of cube*].	BEJ
	610	R1	Okay, and so you have to keep separate the linear measure and then the two dimension and then three? It's easy to-	
	611	Stephanie	Okay.	
	612	R1	It's easy to… That's interesting. So, um, you have a way of doing *a* plus *b* quantity cubed, you have a model that I think I want you to think about a little bit more.	
	613	Stephanie	Okay-	
	614	R1	Okay, and then, what do you think I'd ask [*scene cut*]	
	615	Stephanie	*a* plus *b* to the fourth?	BR-S; BCA
	616	R1	Yeah.	
	617	Stephanie	Okay.	
	618	R1	And you could even anticipate what I would	

Time	Line	Speaker	Transcript	Code
			ask you after that.	
	619	Stephanie	Yeah.	
	620	R1	Uh huh. You might work them out, and look at them, and study them a little bit.	
	621	Stephanie	Okay, [*scene skips*]	
	622	R1	Was this horrible? Was it fun a little bit? My son is gonna say I'm torturing you. [*laughing*] …that it's fun 'cause I'm torturing him.	
	623	R1	Does anyone have any questions? Anyone back there? Did you all…? 'Cause you all can come close and I think she'll show you now.	
	624	Stephanie	Okay.	
	625	R1	Come on, 'cause I know you're far away. Don't worry, you've gotta go slow.	
	626	R1	*[inaudible]* Come on Terry. Did you meet my friend?	
	627	Stephanie	No.	
	628	R1	This is Dr. Pearl, who's visiting from California.	
	629	Stephanie	Hi.	
	630	R1	She's a very good friend of mine.	
	631	Stephanie	Okay.	
	632	R1	And, um, I'll tell you about it some other time. I won't embarrass… Why don't you pull up a chair [*to R3*] and I'll let Stephanie tell you. You can ask her all the questions (inaudible) I feel very comfortable that she's going to be able to answer any question you ask her.	
	633	Stephanie	Do I have to start with *a* plus *b*? Squared?	
	634	R1	You've gotta start with where they are-	
	635	Stephanie	Do I have to start with *a* plus *b* quantity squared?	
	636	R1	You may have to start with the very basic-	
	637	Stephanie	Alright.	
	638	R1	Feel free to ask Stephanie questions.	
	639	Stephanie	Alright [*begins writing*]. *a* plus *b*, quantity, squared, is *a* plus *b*, times *a* plus *b*. Right? Okay. So, if I were to like, draw it as a square, like [*begins to use 10x10x1 box*], if this were- this is a square, and say that, well [*draws a square*] if that was a square, and that piece is *a* [*divides square in drawing*] and that piece is *b* [*labels drawing*]. Okay? [*Divides in other*	PPK; BEJ; BR-V

Time	Line	Speaker	Transcript	Code
			direction, labels] That piece is *a*, and that piece is *b*. Okay, so, each, like, little section, like, has its own area. And it would be [*labels drawing*] *a* squared [*trails off*]. So, you understand that?	
	640	R3	Yes.	
	641	Stephanie	Okay. So then *a* plus *b* squared would be *a* squared, plus *ab*, plus *ab*, plus *b* squared [*points to diagram*]. Or, *a* squared plus two *ab*, plus *b* squared. Okay?	BMP; BEJ
	642	R3	Mhm.	
	643	Stephanie	So then, um, [*begins to write on new paper*]	BR-S
	644	R3	What is that *ab*? The *a* squared was a square, and the *b* squared was a square, (inaudible), what was the *ab*?	
	645	Stephanie	Oh, it's a rectangle.	BEJ
	646	R3	Oh, okay.	
	647	Stephanie	So [*resumes writing*] *a* plus *b* cubed. *a* plus *b* quantity cubed, which is the same thing as [*writes*] *a* plus *b*, quantity *a* plus *b*, *a* plus *b*. But we already know that quantity *a* plus *b* times *a* plus *b* is *a* plus *b* squared, or [*writes*] *a* squared, plus 2*ab*, plus *b* squared. Right?	PPK; BEJ; BR-S
45:00-49:59	648	R3	Right.	
	649	Stephanie	So… You'd have to multiply that times [*writes*] the other *a* plus *b*. Right?	BEJ; BMP
	650	R3	Okay.	
	651	Stephanie	So… It would be *a* squared [*writes*] times *a* plus *b*, which is- *a* times *a* squared is *a* to the third- plus *a* squared times *b*, which is *a* squared *b*.	BEJ; BMP; BR-S
	652	R4	How did you get that? How did you get from one step to the other? How'd you go- Where'd you get that *a* squared from?	
	653	Stephanie	Oh-This *a* sq- Oh-	
	654	R4	Yeah.	
	655	Stephanie	'Cause you're multiplying it by *a* squared.	BEJ
	656	R4	Okay. Let me see. [*turns paper around to see*] So you have *a* squared plus 2*ab* plus *b* squared, oh okay, that's *a* squared, and then you're multiplying it by that *ab*, quantity *ab*.	
	657	Stephanie	Yes.	
	658	R4	Oh, okay.	
	659	Stephanie	Okay. So then it would be [*resumes writing*]	BEJ;

Time	Line	Speaker	Transcript	Code
			2*ab* times *a* plus *b*, which is, *a* times 2*ab* is 2 *a* squared *b*. And *b* times 2*ab* is 2*a b* squared. Ahem. Plus… um… *b* squared times *a* plus *b*, which would be *a* times *b* squared is *ab* squared, plus *b* times *b* squared, which is *b* to the third. And that can be simplified. [*pause*] That can be [*writes*] *a* s- cubed plus you can – ahem- *a* squared *b* plus 2*a* squared *b* is [*writes*] 3*a* squared *b*. Plus 2*a* squared- 2*ab* squared plus *ab* squared is 3*ab* squared plus *b* to the third. And that's [*turns paper to show work*]- can't be simplified anymore, so that's the same thing as- um- [*writes*] *a* plus *b* quantity cubed. And then- ahem- we- [*pause, flips through papers*] So then if you were gonna use these [*places Algebra blocks on table*] to show this, um, we'd start out with the two dimensional figure, which was [*retrieves paper*] *a* plus *b* quantity squared.	BMP; BR-S
	660	R4	So *a* plus *b* quantity squared is a two dimensional?	
	661	Stephanie	Yes.	
	662	R4	Even though you showed that, right there? [*indicating 10x10x1 block used earlier*]	
	663	Stephanie	Yeah.	
	664	R4	That's two dimensional?	
	665	Stephanie	Well, no-	BEJ
	666	R4	Okay.	
	667	Stephanie	But I was just… Cause you know, there's nothing else to use, to show it… so *that*, *a* squared [*pointing at drawing*], *ab*, *b* squared, *ab*, makes up *a* plus *b* quantity squared. So, um, if you took like, if this was *ab* [*places* a^2b *piece on picture*], if this fit there and that fit there [*places base layer of Algebra cubes on drawing*]. There you built it up-	BEJ; BR-S/V
	668	R4	How-How is that-	
	669	Stephanie	Like-	
	670	R4	How is that *a* squared *b*? That, the green with the- *a* squared *b*? How is that *a* squared *b*? And the other one is *ab* squared? How do they differ? 'Cause I was back there, I couldn't really see what you were doing with Dr. Maher…	
	671	Stephanie	Wait, which one's *ab*?	PAH

Time	Line	Speaker	Transcript	Code
	672	R4	This one's- you said this was *a* squared *b*?	
	673	Stephanie	Oh.	
	674	R4	How's that- How did you determine that that was *a* squared *b*?	
	675	Stephanie	Oh, because, um, [*removes* ab^2 *piece to show drawing*] this is *b* squared, and you built it up *a*, like, 'cause, it's- this [*indicating height*] is *a*.	BEJ; BR-V
	676	R4	Okay.	
	677	Stephanie	Like this piece and this piece, so you built it up *a*, so it would be *a* squared- *b* squared.	BEJ; BR-V
	678	R4	Oh, okay. Alright.	
	679	Stephanie	But. Here, this piece [*moving* a^2b *piece*] is *a*- *ab*, [*moving* ab^2 *piece*] this is *b* squared. So this piece [ab^2 *piece*] would be- Um- Like this piece here [*picks up* a^2 *b piece*] 'Cause it's *ab*, if you built it up *a*, it would be *a* squared *b*. And this piece [*picks up* a^3 *piece*] 'cause this piece is *a* squared, and you build it up *a*, it would be *a* cubed.	BEJ; BR-V
	680	R4	Oh, building it up, okay.	
	681	Stephanie	Yeah, you built it up.	BEJ
	682	R4	Now I understand what you mean by build it up, okay. I wasn't sure.	
	683	Stephanie	Yeah, that's why. And so, [*rearranging pieces*] like, you know that this piece is *a* squared *b*.	BEJ; BR-V
	684	R4	Why is that *a* squared *b*, show me-	
	685	Stephanie	Okay-	
	686	R4	So if I were to-	
	687	R3	*a* doesn't have a color, *a* has just the dimension, the height.	
	688	Stephanie	Yeah…	
	689	R3	That's the problem with it. That's why it's so hard to visualize.	
	690	Stephanie	Yeah.	
	691	R4	'Cause if I were to show my class I want to be able to explain it to them.	
	692	Stephanie	So like, this is *a*- 'cause on like, one- two dimensional, if this is.. this would be *ab* [*referring back to* $(a+b)^2$ *drawing*].	BEJ; BR-V
50:00-54:59	693	R4	Okay.	
	694	Stephanie	And it you build it up *a*, it would become *a* squared *b* [*placing* a^2b *piece on drawing,*	BEJ; BR-V

Time	Line	Speaker	Transcript	Code
			indicating dimensions]. Okay? And this piece [*picks up piece*], so we know that this piece is *a* squared *b* [*places piece back on paper on the side, picks up* a^3 *piece, points to* a^2 *region of tracing*]. This piece is *a* squared. If you build it *a* [*places* a^3 *piece on tracing*], it's *a* cubed.	
	695	R4	Alright, okay.	
	696	Stephanie	So this piece-	
	697	R3	It's *a* squared *b* like that, isn't it? [*places* a^2b *piece on table vertically*] This is the square [*pointing at base of piece*], and it's *b* high. Isn't that it?	
	698	Stephanie	Well…	
	699	R3	'Cause *a* is- has no color, it's just this centimeter [*indicating base of piece*]. This is the square. [*places piece on table*] I mean, how are you gonna call this *a* squared *b*? [*pauses*] Unless [*picks piece up*], that's the square, and *b*'s the height. [*Stephanie picks up piece*]	
	700	R4	You're building it up *a*…	
	701	Stephanie	Yeah, because…	
	702	R1	But Terry wants to build it up a different way.	
	703	R3	No, I want to see how this is- I see when you sort of come together, but with the- the square one, the *a*'s and *b*'s each had color, and you could clearly see that the square of *a*- it was a square of *a*- you had an edge that was *a* plus *b*, and then you ended up with a square that was the color of *a*, and a square that was the color of *b*, it was very clear. Then suddenly something here is happening. I've only got *a* and *b*, and I'm cubing them, so I know I'd have a cube, and where is that other color- you've suddenly got all these colors. [*pause*] Something has happened to the v-	
	704	Stephanie	I don't understand what you're saying though. Like-	PAH
	705	R3	I'm saying that I am very- I mean I see where you're making the model, and I see you've got something [*rearranging Algebra blocks*] that's got a 4 by 4 by 4 cube, and you've got a 1 by 1 by 1 cube-	
	706	Stephanie	Mhm.	

Time	Line	Speaker	Transcript	Code
	707	R3	And then you've got three of these things [*a^2b pieces*], and three of these things [*ab^2 pieces*], you can build it all up into a cube. But the colors are confusing me. They're not helping me.	
	708	Stephanie	But the colors don't, like-	BEJ
	709	R3	But they did before	
	710	Stephanie	No, that was just because it helped me remember, that the green piece- but the color itself has, like, nothing to do with it. It could be purple- and- it doesn't make a difference [*pause*]	BEJ; BDI
	711	R3	Right.	
	712	Stephanie	But I just wrote it down in colors, that way it helped me remember that this piece was *a* squared *b*. But why is it- do you wanna know, like…	BEJ
	713	R3	[*hesitates, rearranges papers, pulls out paper with $(a+b)^2$ drawing*] Over here, you had *a*'s and *b*'s. Okay, two things. And you modeled it with two colors [*pause*]. Right?	
	714	Stephanie	I mo- I didn't model it with-	BEJ
	715	R3	Well actually, you didn't. I guess I (inaudible)	
	716	Stephanie	I didn't model it with colors.	BEJ
	717	R3	Well…	
	718	R1	See I guess that, um…	
	719	R3	There's something wrong with the model.	
	720	R1	The color can get in the way. Because, um, if you take the 10 by 10 by 10 cube, right	
	721	Stephanie	This one?	PAH
	722	R1	The reason it ends up being a 10 by 10 by 10.	
	723	R3	It's really a 10 by 10 by 1.	
	724	R1	Exactly. Well that one is a 10 by 10 by 1.	
	725	R3	We're treating it like a flat, but it's really a-	
	726	R1	But in what Stephanie's building, she didn't call that one, she called that *a*.	
	727	R3	Okay. Alright.	
	728	R1	See you referred to- you referred to the little yellow cube as a unit cube, but Stephanie's referring to the little yellow cube as an *a* by *a* by *a*.	
	729	R3	So-	
	730	R1	See the difference?	
	731	R3	-is this 4 of it? [*referring to b^3 piece*]	

Time	Line	Speaker	Transcript	Code
	732	R1	No-	
	733	R3	Or is it not 4 of it?	
	734	R1	-it doesn't matter.	
	735	R3	Well, yeah, but, you can't do it both ways. I don't think- it's confusing-	
	736	R1	Let's ask Stephanie the question. That- I think she called that *a* by *a* by *a*, the yellow. Is that right?	
	737	Stephanie	Yeah. The yellow is *a* cubed.	BEJ; BR-V
	738	R1	*a* cubed?	
	739	Stephanie	Yes.	
	740	R1	So the length, width, and height are *a*, which is what (inaudible) used in her model. So she's thinking in terms of length *a*, which happens to be the length, width, and height of that yellow cube.	
	741	Stephanie	Mhm.	
	742	R1	And she's thinking of *b*, which happens to be…	
	743	Stephanie	*b* is like, like [*picks up* a^2b *piece*] this high- it's almost like 2*a*. 'Cause, like, [*picks up* a^3 *piece*] the thing is they're all-	BEJ; BR-V
	744	R1	*b* is like 2*a*, exactly.	
	745	Stephanie	*b* is like 2*a.*	BEJ; BR-V
	746	R1	In this case, right.	
	747	R3	You mean 2*a* cubed.	
	748	Stephanie	'Cause like, [*looks at* a^3 *piece and* a^2b *piece*]	
	749	R1	No, *b* is linear. *a* and *b* are linear. [*pause*] It's length *a*, and length *b*-	
	750	R3	But if this [*picks up* a^3 *piece*] is linear, then this is a square. Then suddenly-	
	751	R1	How can you help Dr. Pearl with that? She's seeing that as a square. How can you help her with that? You may have to go to the picture, you know, 'cause that's exactly where, what's helped before.	
55:00-59:59	752	Stephanie	[*moves pieces, gets paper with drawing on it*] This [*emphasizing a portion of side length on square*] like, is a un- this is *a* long. This piece right here is *a* long. Okay? And this [*emphasizing corresponding side length on other side*] is *a* long. And so it's *a* squared. We're saying that this is *a* cubed [*indicating*	BEJ; BR-V

Time	Line	Speaker	Transcript	Code
			a^3 *piece*]. We're saying that this is *a* long [*pointing to edge of cube*], by *a* long [*pointing to other edge of cube*] y-you know? Length, width, and height; they're all *a*. [*pause*] Okay? [*pause*] This [*referring back to the drawing*] is *b* like, um, this is *b* long by *b* long [*redrawing segments on sides of b x b square in* $(a+b)^2$ *model*]. Okay? So we're saying- and this is *b* cubed- s- or well [*mumbles to self; picks between Algebra block pieces*] this is *b* cubed [*choosing* b^3 *piece*] and they're saying that this is *b* u- they're all *b*. We're not saying that like [*pauses, picks up* a^3 *piece again*]- this isn't *a* [*indicating whole cube*] this is *a* [*indicating side length of cube*] this little piece, this unit is *a*. Okay? [*pauses*] Okay. So-	
	753	R4	And the white, that's all *b*? The-	
	754	Stephanie	The white is-	
	755	R4	'Cause I think we're looking at the whole–	
	756	Stephanie	-*b* by *b* by-	
	757	R4	-cubes, and that's	
	758	Stephanie	-yeah-	
	759	R4	throwing us off	
	760	Stephanie	yeah.	
	761	R4	Okay.	
	762	Stephanie	So, this is *a* squared [*looking at* a^2 *box in drawing, placing* a^3 *piece on top*] you build it up *a* units, and it would be *a* cubed.	BEJ; BR-V
	763	R4	Okay.	
	764	Stephanie	So this piece is *a* cubed [*picks up* a^3 *piece*]. Okay.	BEJ; B-V
	765	R4	Mhm.	
	766	Stephanie	This is *ab* [*pointing to ab rectangle in picture, holding* a^2b *piece in hand*], and you're- and if you build it up *a*, it's *a* units, it's *a* squared *b*. So this is *a* squared *b* [*picking up piece again*]. K?	BEJ; BR-V
	767	R1	Or *a* squared *b* times. See how you have the *a* squared *b* times? You have another *a* squared and another *a* squared.	
	768	Stephanie	Yes.	
	769	R4	Mhm.	
	770	R1	Or *a* squared *b* times.	
	771	Stephanie	This is [*cough*] *b* squared [*points at* b^2 *part of diagram on paper*]. If you build it up *a* units,	BEJ; BR-V

Time	Line	Speaker	Transcript	Code
			it's *ab* squared. Okay? So this piece is *ab* squared. [*pause*] And, um-	
	772	???	Where's the other piece?	
	773	Stephanie	Oh, this is like the same thing [*places second* a^2b *piece on diagram*]. It's *a* squared *b*. So you know that this piece is *a* squared *b* [*picks up* a^2b *piece*], this piece is- no [*picking up* ab^2 *piece*]- yeah. This piece is *a* squared *b* [*picks up* a^2b *piece*], this piece is *ab* squared [*picks up* ab^2], and this piece is *a* cubed [*picks up* a^3 *piece*].	BEJ; BR-V
	774	R1	Right.	
	775	Stephanie	Alright [*places all pieces on table*]. And so then what we-	BEJ
	776	???	(inaudible)	
	777	Stephanie	[*Stephanie reaches for paper from earlier with* $(a+b)^3$ *expanded and simplified*] –oops- [*knocks into table*] –was- find out if we had, like, all the pieces that were here, and so if you build, um, and then [*reaches for Algebra blocks, drops one*]–oops- if we build this up, like if you keep building like that, like this is *ab* cubed [*placed* ab^2 *piece on diagram*], *a* cubed *b* [*places* a^2b *piece on diagram, then* a^3]-um *a* squared *b*, *a* cubed [*places* a^2b *piece*], *a* squared *b*, [*places* b^3 *piece on top of* ab^2] and you build it up. If you built [*removes* b^3, ab^2 *pieces, points to* b^2 *part of diagram, holding* b^3 *piece*] *b* squared up *b* times- *b* units, it would become *b* to the third. So this piece is *b* cubed. So you have every piece here [*referring back to the paper with* $(a+b)^3$ *work on it*]. You have *a* cubed [*picks up* a^3 *piece, places it down; picks up* a^2b *piece*], you have, um [*pauses*], what is that? *a* squared *b* [*places piece down, picks up* ab^2 *piece*] you have *ab* squared [*places piece down, picks up* b^3 *piece*] and you have, um, *b* cubed [*places piece down, gathers all* a^2b *pieces*]. And you have three of these, so that becomes 3*a* squared *b* [*gathers* ab^2 *pieces*], and you have three of these, so it becomes 3*b*- 3*ab* squared, and you have your *a* cubed and your *b* cubed. And that makes up the problem. And you can build that into like	BEJ; BR-V

Time	Line	Speaker	Transcript	Code
			[*pauses, assembles pieces into cube*].	
	778	R4	And it doesn't matter which way you put the colors?	
	779	Stephanie	No, because the colors don't matter. It's the [*points to edge of cube*] units.	BEJ
	780	R4	I have to tell you, I find that very interesting, because I- I know what *a* plus *b* quantity cubed, uh, raised to the third power is, but I never saw it like that.	
	781	Stephanie	Yeah.	
	782	R4	And why is it 3*a* squared *b*, and 3*ab* squared- that- th- I- I find that totally interesting.	
	783	R1	You know what? I think this -this is sort of difficult for me. [*pause*] Sort of- when you take that little yellow one-	
	784	Stephanie	Yeah.	
	785	R1	We think- we've been taught to think about that as a unit cube- as length, width, and height being one unit. And so we're not thinking in term of algebraic or general terms, we're thinking of something very specific. This is a cube with volume one. Right? And, we- if we think about that yellow cube as a cube of volume one, we've now made, um-	
1:00:00-1:04:59	786	R3	Well then this is, is 8 [b^3 *piece*]-	
	787	R1	That's 8.	
	788	R3	-and this is 4 [ab^2 *piece*], and this is 2 [a^2b *piece*], and so on.	
	789	R1	So what does it all become? Wh-what-	
	790	R3	It all becomes-	
	791	R1	-if we think of the yellow cube as a cube of volume one, if we think of the unit as one unit, what kind of-what kind of model are we doing with, um , it's not, wh-what are the values of *a* and *b*?	
	792	Stephanie	Oh, well then *a* would be 1-	BEJ
	793	R1	And *b*?	
	794	Stephanie	2.	BEJ
	795	R1	So okay, so th-the cube you constructed has what volume?	
	796	Stephanie	The cube I constructed? Is a- if *a* is 1 and *b* is 2?	PAH;
	797	R1	Mhm.	
	798	Stephanie	It would be, um, [*muttering*] 1 plus *b* 2 is… is	BMP;

Time	Line	Speaker	Transcript	Code
			9.	PNE
	799	R1	Cubed? You put a square.	
	800	Stephanie	Oh. 3 squared is 9.	BMP
	801	R3	You can sort of count them [*gathers Algebra blocks, constructs cube*]	
	802	Stephanie	Yeah, you could.	
	803	R3	Count them.	
	804	R1	What's the cube? What's the volume of the cube with side-	
	805	Stephanie	What? Oh with side-	
	806	R1	3.	
	807	Stephanie	-um 1+2? [*muttering*] 3 plus… 9 times… 9 um, yeah, 81?	OBS
	808	R1	How'd you get that?	
	809	Stephanie	Wait.	
	810	R1	81 will get you –	
	811	Stephanie	Forget it. It would be [*reaching for cube*] now, [*deconstructs cube, reconstructs cube*] well it would just be, um, [*writes on paper*] 3 cubed.	BR-S
	812	R1	Or?	
	813	Stephanie	Oh.	
	814	R1	What is 3 cubed?	
	815	Stephanie	3 cubed is 3 times 3, and that's 9. Then it would be 9 times 3, and that's 27.	BMP
	816	R1	So is that true, are there 27 little cubes there?	
	817	Stephanie	Yeah, I guess.	
	818	R1	You check 'em? Didn't look like it. [*Stephanie deconstructs cube, counts unit cubes*]	
	819	Stephanie	1 [*moves a^3 piece*] 2, 3 [*moves a^2b piece*], 4, 5 [*moves a^2b piece*], 6, 7, 8, 9 [*moves ab^2 piece*] –	PNE
	820	R1	I'm beginning to believe you.	
	821	Stephanie	10, 11 [*moves a^2b piece*], 12, 13, 14, 15 [*moves ab^2piece*], 16, 17, 18, 19 [*moves ab^2piece*], 20, 21, 22, 23, 24, 25, 26, 27 [*moves b^3 piece*].	PNE
	822	R1	Is that neat?	
	823	Stephanie	Yeah.	
	824	R1	So if *a* is 1 and *b* is 2…	
	825	Stephanie	Then, it's 27. The volume is 27.	BMP
	826	R1	You have a mental picture of volume, you have 27 of those-	

Time	Line	Speaker	Transcript	Code
	827	Stephanie	Yes.	
	828	R1	-little unit cubes now.	
	829	Stephanie	Mhm.	
	830	R1	But here now, when I say the, um, yellow is *a*, right-	
	831	Stephanie	Mhm.	
	832	R1	How many [*pause*] what's your unit cube now? It's not volume 1, the unit cube, what is the volume, what is the volume, what is the size of *a*, the yellow one	
	833	Stephanie	The-	
	834	R1	with side *a*?	
	835	Stephanie	It would- I- what? Like, you wanna know the volume of the yellow one if it's *a*?	PAH
	836	R1	Mhm.	
	837	Stephanie	*a* cubed.	BMP; BCA
	838	R1	*a* cubed.	
	839	Stephanie	Yeah.	
	840	R1	And so it's moving in that kind of thinking, something very specific to something general, that- that's hard because it is specific, isn't it. Once you built your model it's very specific, and y- you're forcing yourself to think in somewhat of an artificial way, you know? And that could be very difficult to do. Don't you think? I mean I could still- I could be a student saying, but wait a minute, what are you calling that *a*, *a*, *a*.	
	841	R3	You should build a model of cubes that are all the same color that are, uh, have Velcro on the edges so you could sort of do a true *a* plus *b* and sort of build all the parts.	
	842	R1	What do you think Stephanie? Do you know what Dr. Pearl's saying?	
	843	Stephanie	Yeah.	
	844	R1	That would be a great class project.	
	845	R3	It's a great…	
	846	R1	What do you think?	
	847	R3	It's probably a new manipulative [*laughing*]	
	848	R1	-with sugar cubes and glue.	
	849	R4	Good idea.	
	850	R3	No, no, you've got to be able to open them and close them, you do need the Velcro, I admit that makes it complicated.	

Time	Line	Speaker	Transcript	Code
	851	R1	Yeah that does make it com-	
	852	R3	You do want to be able to change it, you don't want to get locked into any particular set.	
	853	R2	Actually, when I was trying to do this originally I was trying to go for something with Velcro, and I couldn't- I need a, a saw, 'cause I was trying to do it with, um, Styrofoam, and I couldn't cut the Styrofoam straight-	
	854	R3	Well you could do it with these, [*picking up Algebra block*] with the Velcro.	
	855	R5	You could do that with an electric saw.	
	856	R3	-Velcro dots or something.	
	857	R2	Yes, well if I had an electric saw I could do it, but I was trying to cut it with a knife, an exacto knife, and I couldn't, it just, (inaudible)	
	858	R5	Diane, you have, uh, um, Lego cubes.	
	859	R2	Yeah	
	860	R3	No.	
	861	R5	Lego cubes might be…	
	862	R3	What're the ones-that snap (inaudible) that fix, snap, there's something that snaps in lots of- all directions. What are those that snap in all directions?	
	863	R1	What's in the bag	
	864	R5	The big legos they have for the little kids that snap in any direction you want.	
1:05:00-1:09:23	865	R3	No, they're cubes that snap in different, that have connectors, no, they're not legos.	
	866	R2	I know what you're talking about, I saw them in the Creative Publications, um…	
	867	R5	My friend's little daughter has it.	
	868	R3	Yeah.	
	869	R1	It's um-	
	870	R3	They've been around a long time. [*pause*] They can snap in all directions, they can sort of make, uh, 'cause I found this [*referring to Algebra blocks*] confusing. The color part. When you clarified, you know, and since you threw the color out, and just made it one and two…	
	871	R1	Well that's what Stephanie did, Stephanie said "I don't care about the color, this is length *a* and this is length *b*."	

Time	Line	Speaker	Transcript	Code
	872	R3	Mhm.	
	873	R1	It's sort of a different thing, very interesting. Yeah. Well thank you Stephanie. So what are you going to think about for next time?	
	874	Stephanie	Oh. Did you want me to think about the, the um, 4? The-	PAH
	875	R1	Yeah.	
	876	Stephanie	-to the fourth power?	PAH
	877	R1	Right.	
	878	Stephanie	Okay.	
	879	R1	And let's see if you can conjecture what it would be to the fifth after you've done the fourth.	
	880	Stephanie	Okay.	
	881	R1	And you know the first. What's *a* plus *b* raised to the first power?	
	882	Stephanie	*a* plus *b*.	PPK
	883	R1	Okay, so you know *a* plus *b* -what is *a* plus *b* raised to the zero power?	
	884	Stephanie	Oh, one.	BMP
	885	R1	Okay. By definition, right?	
	886	Stephanie	Yes.	
	887	R1	You can get into that with your- we've done that already.	
	889	R4	Yes. She has.	
	890	R1	*a* plus *b* to the zero is one. *a* plus *b* to the first is *a* plus *b*. *a* plus *b* to the second is	
	891	Stephanie	*a* squared plus 2*ab* plus *b* squared.	PPK; BMP
	892	R1	*a* plus *b* to the third is	
	893	Stephanie	*a* cubed plus 3*a* cubed- *a* squared *b* plus 3*ab* squared plus *b* cubed.	PPK; BMP
	894	R1	That's interesting. So you might want to hypothesize once you've written these down before you do anything, before you do any work.	
	895	Stephanie	Okay.	
	896	R1	That-that's what I would suggest you do, write these down and you might hypothesize as you study those, what you think *a* plus *b* to the fourth might be.	
	897	Stephanie	Okay.	
	898	R1	D-can you think of any terms that will be in *a* plus *b* to the fourth, for sure?	

Time	Line	Speaker	Transcript	Code
	899	Stephanie	Okay. Um, for the first one – the first one was *a* plus-what was the first? *a* squared plus- and the second one- it'll probably be *a* to the fourth and *b* to the fourth will probably definitely be there.	BCA; BDI
	900	R1	Okay, that's pretty good, you think there's anything in the middle or the answer's just *a* to the fourth-	
	901	Stephanie	No, there's stuff in the middle. [*laughing*]	
	902	R1	That's very good. Some of our college students have difficulty with that one, they'll say, our teachers here, they all think this should be *a* to the fourth plus *b* to the fourth and not want to think about it much anymore, farther than just doing this. So how can we avoid that thinking?	
	903	R2	At least that there isn't anything in the middle.	
	904	R1	Okay, I want you to do something else. If you look at *a* plus *b* quantity squared, right?	
	905	Stephanie	Mhm.	
	906	R1	Okay? You have *a* squared plus 2*ab* plus *b* squared. Right?	
	907	Stephanie	Yes.	
	908	R1	Okay, so you have two factors of *a*, right?	
	909	Stephanie	Yes.	
	910	R1	And then you have two factors again, but one is *a* and one is *b*, but you have two factors, right?	
	911	Stephanie	Okay.	
	912	R1	Then you have two factors of *b*.	
	913	Stephanie	Okay.	
	914	R1	Okay. Now if you look at the *a* plus *b* cubed you have one fac- you have three factors of *a*, that's your *a* cubed-	
	915	Stephanie	Mhm.	
	916	R1	And then you have an *a* squared *b*.	
	917	Stephanie	Yes.	
	918	R1	Right? You have two factors of *a* and one of *b*. Right? That sort of adds up to three if you think about it.	
	919	Stephanie	Mm yeah.	
	920	R1	And then the *ab* squared also adds up to three. And then the *b* cubed. Just think about that.	
	921	Stephanie	Okay.	

Time	Line	Speaker	Transcript	Code
	922	R1	So, if you think about those things, maybe you start to look for some patterns you might be able to see. And your teacher is not allowed- she's absolutely, positively not allowed to give you any hints.	
	923	R4	Sorry Steph.	
	924	Stephanie	Okay. [*all laugh*] Okay.	
	925	R1	Okay? 'Cause then we have a real hard problem next time.	
	926	Stephanie	Okay.	
	927	R1	This was great, thank you. I wanna set up another date-	
	928	Stephanie	Okay.	
	929	R1	-if we can. Can we do that?	
	930	R4	Sure.	

APPENDIX E: TRANSCRIPT – SESSION 5

INTERVIEW WITH STEPHANIE Time: 66 minutes (1 CD)
March 13, 1996

R1: Dr. Carolyn Maher Stephanie: Stephanie R2: Steve
R3: Ethel Muter

Time	Line	Speaker	Transcript	Code
00:00-04:59	1	R1	This is really neat. Stephanie probably doesn't know this. Someday we'll have to show her this, but a lot of . . . you may not want to read that; it may be very boring to you, right? Some of the teachers do read it because they try to look to see the way students think about some of these ideas but, um, all of the story we're telling about you and your classmates, well, we're telling the story mostly through you, is when you were working with those unifix cubes.	
	2	Stephanie	Mm-hmm.	
	3	R1	Remember that?	
	4	Stephanie	Yes.	
	5	R1	Um, and so, you know at first, you were making these towers.	
	6	Stephanie	Mm-hmm.	
	7	R1	But do you remember that the problems were getting more complicated?	
	8	Stephanie	Yes.	
	9	R1	Do you remember why, at all? What do you remember? I'm curious.	
	10	Stephanie	That we finally had to go up to ten or something. Like you, but we figured out that it was just, like, you just multiply the last number's amount by two	PPK
	11	R1	Mm-hmm.	
	12	Stephanie	to get the next number's amount. So that's, like, what I remember.	PPK
	13	R1	That's what you remember about it, mm-hmm.	
	14	Stephanie	when we were in the home ec room one time.	PPK
	15	R1	Mm-hmm.	
	16	Stephanie	Built these, that's like it, just like stuff like that.	
	17	R1	And we were just looking at those videos of your building families, weren't we?	

Time	Line	Speaker	Transcript	Code
	18	R3	Yes.	
	19	R1	Just recently. And, um, so we're working on looking at that piece of it. And I was thinking maybe we should do something different today; you and I could do something different.	
	20	Stephanie	Ok.	
	21	R1	Um, it really is up to you. Um, we're going to kinda put the algebra stuff a little bit on hold for a moment, 'cause your teacher's not here anyway, and deal with something else. Have you ever heard of, um, combinatorics?	
	22	Stephanie	No.	
	23	R1	But you've been doing combinatorics problems for a few years in our project without even realizing it 'cause if, um, if you can remember these unifix cubes, remember, if you can imagine you have two piles. I wish I had brought some but I forgot. But if you can imagine we have two piles of them in two different colors.	
	24	Stephanie	Mm-hmm.	
	25	R1	Ok, and for the sake of argument, we can build towers, let's make trains.	
	26	Stephanie	Ok.	
	27	R1	So we can keep them flat. Okay. Um, and if I said to you we're making – let's say for the sake of argument – make them a train of four.	
	28	Stephanie	Okay.	
	29	R1	You can imagine that?	
	30	Stephanie	Yes.	
	31	R1	And if I said to you, making them of one color	
	32	Stephanie	Okay.	
	33	R1	-how many trains of four that you can make?	
	34	Stephanie	One color? Two!	BR-V
	35	R1	That's one color. Two? How did you get that?	
	36	Stephanie	Well, if there's only- you can only- you only have, um, two colors, so you can only- and you can only use one color each, so there'd only be two colors.	BEJ
	37	R1	Ok, so, for now, let's pretend they're red and yellow.	
	38	Stephanie	Ok.	
	39	R1	Do you want to do this a little bit?	
	40	Stephanie	Ok.	
	41	R1	I'll show you what this has to do with	

Time	Line	Speaker	Transcript	Code
			combinatorics.	
	42	Stephanie	Okay.	
	43	R1	And what it has to do with some of the notation we've been using. You might like to think about this a little differently.	
	44	Stephanie	Alright.	
	45	R1	So, you can imagine, right, we're making these . . .	
	46	Stephanie	Uh huh.	
	47	R1	Ok, you just told me that there's one way you can make them-	
	48	Stephanie	Yeah.	
	49	R1	-with red and there's	
	50	R1/Steph	One way you can make them with yellow.	
	51	R1	Um, so in a sense, what we're saying is that you have four cubes, right?	
	52	Stephanie	Mm-hmm.	
	53	R1	Ok and out of the four cubes, you're selecting four red-	
	54	Stephanie	Mm-hmm.	
	55	R1	-and you can do that, right-	
	56	Stephanie	Mm-hmm.	
	57	R1	-in one way. Ok?	
	58	Stephanie	Mm-hmm.	
	59	R1	So we have a fancy way of writing that. Do you want to know what that is?	
	60	Stephanie	Ok.	
	61	R1	There are a couple of fancy ways. I really am curious, both, help me, Elena and Ethel and Steve, if you know of another way to write it 'cause there are two ways of notation that I'm familiar with, ok? Now, when we talk about selecting, we call those combinatorics, combinations. And sort of, if you think of, when someone says combinations, you think of selections, that's kind of, combinations means selections. So we can say that, well, one notation is combinations – isn't this kind of weird looking? You put a C and a four on top and a four on the bottom. [*writes* C_4^4] Have you ever seen those? Go looking through some high school or books on probability or something. You might see a notation like that. Now, this number on top tells you the number	

Time	Line	Speaker	Transcript	Code
			of cubes you have, or the number of objects. In this case, you have four.	
	62	Stephanie	Um-hm.	
	63	R1	And this tells you how many of a certain kind you're picking. So you're selecting four from four. Another way they do this is like an elongated parenthesis [*writes* $\binom{4}{4}$] that says you're selecting four let's say red cubes from-	
	64	Stephanie	Okay.	
	65	R1	-your cubes. So if I'm selecting 4 things from four, you can do that in only one way. Towers of four tall.	
05:00 – 09:59	66	Stephanie	Ok.	
	67	R1	Now, that's when you selected four reds. But if you selected four yellows, so that's how you got one.	
	68	Stephanie	Mm-hmm.	
	69	R1	But if- so this meant reds, that was one, and if it meant, of the four, in your train, you were selecting yellow, that would be one. And your one and one gave you	
	70	Stephanie /R1	Two.	
	71	R1	Ok.	
	72	Stephanie	Okay.	
	73	R1	Now, there are certain, um, let's change it a little bit. Now I have four cubes and I want to know how many ways I could select exactly one red. Four cubes. Selecting, again, one red. Now, because I said selecting, we're talking about, I'm selecting one from four. The combinatorics is selecting. Or I could have written it this way. Now, you know the answer to that. You don't know any formulas or anything. But you can figure that out, I'm sure.	
	74	Stephanie	There's four, if you're saying that I could put a red here, and three yellows and a red here. That would be four.	BR-V
	75	R1	Ok, so you know the answer to this is four, right?	
	76	Stephanie	Mm-hmm.	
	77	R1	Right?	
	78	Stephanie	Yes.	

Time	Line	Speaker	Transcript	Code
	79	R1	Let's just focus on red for awhile. So, if I'm selecting all red, four red, from my cubes, right-	
	80	Stephanie	Mm-hmm.	
	81	R1	- my pile which has yellow and red, I can do that in one way. If I'm selecting one red, I could do that in four ways. Shall we continue?	
	82	Stephanie	Ok.	
	83	R1	Suppose I was selecting two reds?	
	84	Stephanie	We could do it, um, two ways. Right? No.	BR-V
	85	R1	Well here, why don't you-	
	86	Stephanie	Wait…	
	87	R1	-play with it? I wish we had the cubes, but you can play with it any way you want. Doesn't this look a little bit familiar? We did stuff like this, didn't we?	
	88	Stephanie	You could do it three ways. No, you could do it a lot.	BR-V
	89	R1	Well think about it a little bit. I wish we had chips or cubes.	
	90	Stephanie	You could do red, red. [*draws diagram of possible arrangements*] You could do red, red. You could do red, red. [*pauses, continues drawing*] And that's it.	BR-V
	91	R1	Okay, show me what you did.	
	92	Stephanie	Red, red, like, two reds. Two reds. They're all just two reds.	BEJ
	93	R1	So that's two reds. You put them this way here and you put them another way here. What's the difference?	
	94	Stephanie	Oh, well, I just separated them. Here, they were together and here they're not. But they're still two reds.	BEJ
	95	R1	And so these are all the ways they can be together and these are all possible ways they can be separated?	
	96	Stephanie	Yes.	
	97	R1	And you're sure of that?	
	98	Stephanie	Yes.	
	99	R1	You're saying there are five ways? Now, I can see that these are the only ways they could be together, but I'm not convinced that those are the only ways you can separate them.	
	100	Stephanie	Oh, well, oh! [*draws more*]	BDI; BR-V

Time	Line	Speaker	Transcript	Code
	101	R1	What did you forget?	
	102	Stephanie	I forgot that one.	
	103	R1	Now, your strategy, it seems to me, is here you separated them by one and you forgot the case of separating them by two.	
	104	Stephanie	Yeah.	
	105	R1	Right?	
	106	Stephanie	Mm-hmm.	
	107	R1	So you separated them by none and how do you know you can't do any more of that, that you can separate by none?	
	108	Stephanie	Because I filled up all the spaces.	BEJ
	109	R1	Okay, and how do you know that there are no more that you can separate by one? Because you filled up all the spaces.	
	110	Stephanie	Yeah. Ok.	
	111	R1	How do you know that there are no more you can separate by two? Now what about separate by three?	
	112	Stephanie	Because there's not enough space.	BEJ
	113	R1	Okay, do you see what I'm saying? Now you're sure. It's not like trial and error anymore. You've accounted for all possible ways of separating and there's nothing else possible. You've really thought that out. Okay?	
	114	Stephanie	Mm-hmm.	
	115	R1	So what did you come up with? Why don't you start writing this? So, if you're selecting two red from all of these cubes.	
	116	Stephanie	There's six.	BR-V
	117	R1	Right?	
	118	Stephanie	Mm-hmm.	
	119	R1	Okay, what am I going to ask you next?	
	120	Stephanie	3 red, I guess?	BCA
	121	R1	How about that?	
	122	Stephanie	Ok, um, do you want me to draw it out, like, here?	PAH
	123	R1	You can do it any way you want.	
	124	Stephanie	Okay.	
	125	R1	Write- Actually, start using the notation, so that you can be using some of the new notation.	
10:00 –	126	Stephanie	[*writes for about a minute*] I don't know	

Time	Line	Speaker	Transcript	Code
14:59			[*pause*] um, and that's it?	
	127	R1	What do you think?	
	128	Stephanie	Well, I can't do any more like this. And I can only separate them like- there's not enough space to separate them, like, into threes.	BEJ
	129	R1	So you think you have them all?	
	130	Stephanie	I don't know, I guess. Um. [*pause*] Yeah.	OBS
	131	R1	Okay, how can you convince me that you have them all?	
	132	Stephanie	All right, well, here they're not separated by any, so there's only two ways you can do that. There's not enough space, to, like, move them again.	BEJ
	133	R1	Okay.	
	134	Stephanie	And here, they're separated by one, so you have one standing by itself over here and then two over here with a space in between, and then you switch it. But like, you can't.	BEJ
	135	R1	Can- can you draw me a picture to show me this case because you talked about it, but you didn't draw me a picture because it was so obvious to you.	
	136	Stephanie	What? The one with the one red? [*draws*]	PAH
	137	R1	Yeah. [*Pause as Stephanie draws*] Right?	
	138	Stephanie	Mm-hmm.	
	139	R1	And what goes in these other ones?	
	140	Stephanie	Yellow.	BR-V
	141	R1	Yellow, and what goes in these other ones?	
	142	Stephanie	Yellow.	BR-V
	143	R1	Is there any relation between these two?	
	144	Stephanie	I don't know. I mean, they're using red and yellow and there's – oh – they both have four combinations.	BDI; BR-V
	145	R1	Right. That's true.	
	146	Stephanie	And, I don't know, they both, I don't know. [*laughs*]	
	147	R1	Ok let's do a little geometry since you're going to be doing geometry next. Let's- let's try to imagine that these are really the unifix cubes and these are the reds and that's the yellow and just- you know what I'm saying?	
	148	Stephanie	Yes.	
	149	R1	Can I take this one-	
	150	Stephanie	Mm-hmm.	

Time	Line	Speaker	Transcript	Code
	151	R1	-and flip it like that?	
	152	Stephanie	Yeah but you'd get that one.	BR-V
	153	R1	Hm.	
	154	Stephanie	Right? If you were to flip it over?	PAH
	155	R1	Could I take this one and move it down here?	
	156	Stephanie	If you wanted to, I guess. It wouldn't make a- I mean-just like, take it, and instead of having it up here, having it down here? [*moves row of four to the bottom of that series of cases to create a symmetrical pattern*]	PAH
	157	R1	See, I'm not so sure that these two are different. If I can take this and flip it, is it really different?	
	158	Stephanie	Well, but you can do it here though too.	BEJ
	159	R1	It's true. It's easier with the towers when you know what's the top and what's the bottom. Isn't it?	
	160	Stephanie	Yeah.	
	161	R1	You have a point [*pauses*]. Hmm. That's interesting. So, let's see, somehow, when I take this one and move it to the bottom.	
	162	Stephanie	Mm-hmm. Like you want me to put it here?	PAH
	163	R1	Yeah, just move it for a minute. Right. That's easier for me to see all possibilities. I don't have to work as hard in my head.	
	164	Stephanie	Oh, you mean, 'cause it like- like that? [*outlines how the reds form a symmetrical pattern along a diagonal]*	BR-V
	165	R1	Yeah, right. 'Cause like-	
	166	Stephanie	Oh, okay.	
	167	R1	You see what I'm saying?	
	168	Stephanie	Yes.	
	169	R1	So you can take this and move it. It's true. As towers, you can't flip them.	
	170	Stephanie	Mm-hmm.	
	171	R1	But theoretically, that's what makes this a little bit different. That's why towers are nice, they have a chimney. Remember that?	
	172	Stephanie	Yes, that's how they fit.	
	173	R1	Alright. So we have four here.	
	174	Stephanie	Mm-hmm.	
	175	R1	And so we have- do we have all cases?	
	176	Stephanie	Yeah, we have four, four…	
	177	R1	Exactly one. Exactly two. Exactly three.	

Time	Line	Speaker	Transcript	Code
			Exactly four.	
	178	Stephanie	Yeah.	
	179	R1	Exactly none.	
	180	Stephanie	None?	PAH
	181	R1	Exactly no reds.	
	182	Stephanie	Oh.	
	183	R1	Can you make one with exactly no reds?	
	184	Stephanie	Yeah. You can make one with exactly no reds.	BR-V
	185	R1	Okay, so why don't you write that down?	
	186	Stephanie	That would be zero, on the bottom I guess?	BR-S
15:00 – 19:59	187	R1	Okay, so we've looked at selecting, right?	
	188	Stephanie	Mm-hmm.	
	189	R1	Well we're going to do a little algebra here. We have four and we're selecting *r* and *r* could go- be zero, one, two, three or four. Isn't that right?	
	190	Stephanie	Yeah.	
	191	R1	When *r* is zero, we have this, and you told me that's one. Right?	
	192	Stephanie	Ok.	
	193	R1	When *r* is one, you told me that was . . . [*writing*]	
	194	Stephanie	Um, with one red, four.	BR-S
	195	R1	And this was . . . [*writing*]	
	196	Stephanie	Six.	BR-S
	197	R1	And this was . . . [*writing*]	
	198	Stephanie	Four.	BR-S
	199	R1	And this was . . . one, two, three. [*writing*]	
	200	Stephanie	Four out of four, you'd have one.	BR-S
	201	R1	One. Right?	
	202	Stephanie	Yeah.	
	203	R1	So, if I wanted to know the total number-	
	204	Stephanie	Mm-hmm.	
	205	R1	-where you could have no reds, exactly one, exactly two, exactly three, exactly four. What does it turn out to be?	
	206	Stephanie	Sixteen.	BR-S
	207	R1	Does that surprise you?	
	208	Stephanie	Not really. I-I mean, I wasn't thinking about it like that-	
	209	R1	I know.	
	210	Stephanie	-but I mean, no.	
	211	R1	Isn't that interesting?	

Time	Line	Speaker	Transcript	Code
	212	Stephanie	Yeah, it's the same thing.	BDI
	213	R1	What do you mean?	
	214	Stephanie	Like with just the towers-	PPK;BEJ
	215	R1	Mm-hmm.	
	216	Stephanie	-except that I just did it different.	BEJ
	217	R1	How did you do it differently with the towers?	
	218	Stephanie	Well, with the towers, I just didn't have this, to, like, say "All right, now I'm going to try it with three." I just, like, did all these different things until we couldn't do them any more.	PPK; BEJ
	219	R1	Mm-hmm.	
	220	Stephanie	So, it was like, more just like guessing. You know?	BEJ
	221	R1	Well, but I noticed in the towers later on you did something different. Um, something I just looked at recently. Um, you didn't start, y-you- in order to figure out how many you can build, let's say, four high-	
	222	Stephanie	Mm-hmm.	
	223	R1	-you started building one high. Like, you said this is one high. You said it could be a red or a yellow-	
	224	Stephanie	Mm-hmm.	
	225	R1	-you did some family thing.	
	226	Stephanie	Yeah, and we had them, I think, when we first showed it we had them all lined up. Like, and their opposites. We did, like, one red, all red, all yellow. And stuff like that.	PPK; BEJ; BR-V
	227	R1	Do you remember how you built up the family? This was for one high, right?	
	228	Stephanie	Oh, okay.	
	229	R1	Then, when you went for two high, right-	
	230	Stephanie	Mm-hmm.	
	231	R1	-you built on top of. You all were talking about a way of doing that. Um, you said that, something like, I remember you starting something like, someone asked you how many can you build one high when they could be red or yellow.	
	232	Stephanie	Mm-hmm. And, there could be two.	BR-V
	233	R1	There could be red.	
	234	R1/Steph	Or yellow.	
	235	R1	And then you built those.	

Time	Line	Speaker	Transcript	Code
	236	Stephanie	Yes.	
	237	R1	And you see them standing in front of the camera. Beautiful shots of red or yellow.	
	238	Stephanie	Yeah.	
	239	R1	And then, you talked about, "Ok, now I want to move from one to two high."	
	240	Stephanie	Mm-hmm.	
	241	R1	So you said, "Ok, if I start with the red, what could I do to make two high?"	
	242	Stephanie	Well, I could have um, red-red.	BR-V
	243	R1	You did something like this, right? [*draws a tree diagram showing how the towers build by adding a red and yellow to each previous tower.]*	
	244	Stephanie	Yeah. Or I could have yellow-yellow. Oh if you want to use the red, you can have red-yellow.	BR-V; BEJ
	245	R1	If you start with red on the bottom?	
	246	Stephanie	Well, yellow-red.	BR-V
	247	R1	Is that right?	
	248	Stephanie	Yeah.	
	249	R1	Millan did something like this. Do you remember that?	
	250	Stephanie	Mm-hmm.	
	251	R1	So you got two, the family grew.	
	252	Stephanie	Yeah.	
	253	R1	You did something like that. Do you remember that?	
	254	Stephanie	Yes.	
	255	R1	And then you used the same argument here.	
	256	Stephanie	That'd be yellow-yellow and red-yellow.	BR-V
	257	R1	And you could put, ok, you could put yellow on the top or you could put red on the top of that yellow.	
	258	Stephanie	Mm-hmm.	
	259	R1	And so, two high you ended up–for one high you ended up with a total of two, and for two high, you ended up with a total of-	
	260	Stephanie	Four.	PPK
	261	R1	And then you predicted for three high, there'd be how many?	
	262	Stephanie	Um, Eight.	PPK
	263	R1	And you predicted for four high, there'd be	

Time	Line	Speaker	Transcript	Code
	264	Stephanie	Sixteen.	PPK
	265	R1	Sixteen, and?	
	266	Stephanie	Thirty-two.	PPK
	267	R1	And so, yeah, but how did you get the eight from these four?	
	268	Stephanie	Um, well, you could do red-red-red or you could do red-yellow-red or red-red-yellow.	BR-V; BEJ
	269	R1	I'm having trouble following you if you're making a family.	
	270	Stephanie	Oh ok, if you're doing- ok. You could do it. And I have to have two red on the bottom?	PAH
	271	R1	Well, I don't know, you tell me, I don't…	
	272	Stephanie	Well, here, I have to have-I can have [*writing*] red-red-red or I can have red-red-yellow or I can have . . .	BR-V
	273	R1	That goes from that one?	
20:00 – 24:59	274	Stephanie	Yeah, that goes from the red-red. Or I can have, [*writing*] like, red-yellow-red. Or I can have – whoops – red-yellow-yellow. You can't see that. Or I can have, um, yellow-yellow-yellow. Or I can have yellow-yellow-red. Or I can have, um, yellow-red-yellow. Or I can have yellow-red-red. Yeah.	BR-V
	275	R1	So where did the eight come from, from the four?	
	276	Stephanie	From the four? Well, like, red-red-red or yellow-red-red.	BR-V; BEJ
	277	R1	How did that happen that you got two from that one? Did you always get two from the one?	
	278	Stephanie	Um…	
	279	R1	As you build up from one, you got two here, didn't you?	
	280	Stephanie	Mm-hmm.	
	281	R1	From this one, you got two here, right?	
	282	Stephanie	Yeah, probably. Yeah.	
	283	R1	Why?	
	284	Stephanie	'Cause, I guess, there's always going to be two combinations with whatever you have on the bottom-	BEJ
	285	R1	Mm-hmm.	
	286	Stephanie	-like, 'cause if you're building it from here, it's got to have three reds on the bottom, and there's only two other things 'cause you only have two colors. So you can only do two other	BEJ

Time	Line	Speaker	Transcript	Code
			things with that. You can either put a red on top or a yellow.	
	287	R1	So, so that means four high, you would get?	
	288	Stephanie	You would get sixteen.	BR-V
	289	R1	You would get sixteen, so, in this, I'm not gonna ask you to do that, you just told me what it would look like and I can follow what you're saying. So you do get sixteen four-high.	
	290	Stephanie	Mm-hmm.	
	291	R1	Right?	
	292	Stephanie	Yes.	
	293	R1	And, um, in all of these, I focused on red. Talked about the positions for red, right?	
	294	Stephanie	Mm-hmm.	
	295	R1	For these four high, you can imagine these sixteen there. And, of these sixteen, I could say, of these sixteen, there'll be no reds and there's going to be one of those. And there's going to be exactly one red-	
	296	Stephanie	And there'd be four of those.	BR-V
	297	R1	And so forth, right? Um, what about yellows? Don't we have to do the same thing for yellows? So wouldn't that give us 32?	
	298	Stephanie	Yeah.	
	299	R1	But this thing only produces sixteen. If I were to do the same thing here for yellow, right-	
	300	Stephanie	Mm-hmm.	
	301	R1	-and if I said, let's now find out how many exactly no yellows, let's find out exactly one yellow out of the four, exactly two yellows out of the four, three yellows out of the four, don't you agree that you'd get another sixteen?	
	302	Stephanie	Yeah.	
	303	R1	But then 16 and 16 gives you 32, not 16.	
	304	Stephanie	But wouldn't it be the same thing? Like, only the opposite way? 'Cause, wait, if there's two red, then there's two yellow. [*writing*] And if there's three red, then there's one yellow. And if there's one red, then there's three yellow, so isn't it the same thing?	BDI; BEJ; BR-V
	305	R1	Is it?	
	306	Stephanie	Yeah.	
	307	R1	Ok, you're sure of that?	
	308	Stephanie	Yeah.	
	309	R1	And-and that's why if you think about that as a	

Time	Line	Speaker	Transcript	Code
			strategy, if you've already figured out exactly one, do you know exactly three?	
	310	Stephanie	Um?	
	311	R1	See this was the exactly one here, right?	
	312	Stephanie	Mm-hmm.	
	313	R1	Right?	
	314	Stephanie	Yes.	
	315	R1	That was exactly one red. And when you did exactly three red, I asked you to move one, you also got four.	
	316	Stephanie	Yeah, well, I guess it's just the opposite.	BEJ; BCA
	317	R1	Isn't that interesting?	
	318	Stephanie	Yeah.	
	319	R1	So, it saves you some work.	
	320	Stephanie	Yeah.	
	321	R1	And that's kind of important to realize. If you know exactly none, right, do you know exactly all?	
	322	Stephanie	Yeah, but I mean, I wouldn't have thought of that. Like-	
	323	R1	Yeah, well, that kind of pulls some of the ideas together.	
	324	Stephanie	Yeah.	
	325	R1	I think also if you think about that, it might help you. So if we went, to towers five, it might be interesting to look at some of this, now that you're looking at it from another point of view – combinations or selections – which, by the way, um, is a field of math that's called counting, and counting, um, is a field of math that you study as sort of a prelude to studying things like probability	
	326	Stephanie	Mm-hmm.	
	327	R1	and statistics. So it's a very important field, and, um, if you start to pick up a book at the college level or advanced high school, and you see all these formulas and you see all this notation, and with the notation, there's formulas.	
25:00 – 29:59	328	Stephanie	Mm-hmm.	
	329	R1	There are students who work with this and have no sense of what it means. See, the advantage you're going to have when you get	

Time	Line	Speaker	Transcript	Code
			to work with this is if you could think about what this means, you say “Oh, selection, towers.”	
	330	Stephanie	Yeah.	
	331	R1	You know what I’m saying?	
	332	Stephanie	Yeah.	
	333	R1	That’s like exactly one out of the four being this. See what helps is if you can, all the work- all the hard work you’ve done for years, if you can, in your mind, try to say, “This is like this” or “This is almost like this”, then you can build on these ideas and then when you get the formulas, you know, they don’t always apply directly. It’s like, sort of, the problem you had yesterday with the factoring.	
	334	Stephanie	Yeah.	
	335	R1	It really was the same problem. You know, sort of tricky, wasn’t it? Once you saw it a certain way, you realized it was the same problem. Well that’s part of what you have to do. You have to be able to see it, you know, to be able to visualize it, which is part of the strength. But if we were to go through the same thing again, and do this for combinations- let’s do this. If I picked none, exactly none, out of one.	
	336	Stephanie	Out of one?	PAH
	337	R1	Does that make any sense? Okay, I have one high. I have this one high, if I have no red. I still have my yellow-	
	338	Stephanie	But- oh- but you have the yellow though.	PAH
	339	R1	See notice that it didn’t make any sense, but once you started thinking about-	
	340	Stephanie	Oh, well then there’s one.	BR-S
	341	R1	Oh, isn’t that right? And if I said to you, “Exactly one out of one.” See this is no reds. You said there’s one, right?	
	342	Stephanie	Yeah.	
	343	R1	Exactly one red.	
	344	Stephanie	That would be one.	BR-S
	345	R1	That would be one. See, now it has meaning.	
	346	Stephanie	Yeah.	
	347	R1	But you look at this notation and say, “What does this mean?” But see, this will help you think of selections. Ok, so if we were to think	

Time	Line	Speaker	Transcript	Code
			about this, um, if we're thinking of for towers for n = 1, that's one high towers, right?	
	348	Stephanie	Mm-hmm.	
	349	R1	So, we can think about this as [*writing*] this and this, right? Or we can think about this as one and one. Isn't that cool?	
	350	Stephanie	Mm-hmm.	
	351	R1	So I thought we'd do something else that might. . . . now two. Right? So if we're doing two now, again, what do you want to think of red or yellow? Does it matter? You told me it doesn't matter.	
	352	Stephanie	Yeah, it would be one.	BR-S
	353	R1	There's one way. You saw that right away. What made you see that right away?	
	354	Stephanie	Well, because there's always going to- if there's- you can't do none of one, and there's another color, it's obviously going to be all the other color.	BEJ
	355	R1	Good, that's great. Ok, so now, if we're gonna do – I'm going to pick one out of two.	
	356	Stephanie	Um, two ways, I guess. One on top or one on bottom.	BR-V/S
	357	R1	Mm-hmm. Can you see that?	
	358	Stephanie	Yes.	
	359	R1	And if it's two out of two?	
	360	Stephanie	It would be one.	BR-S
	361	R1	Okay. So, when I have n = 2, here I had one, right, that's no reds or one, that was one red, which was one high. Now, if I'm talking two high, I could have one red, I could have two reds, or I could have one red. No reds. One red or two reds. So this one is this piece, this one is this piece, this one is . . . let me just put the numbers in now.	
	362	Stephanie	Okay.	
	363	R1	See if you notice what's happening here. $n = 3$.	
	364	Stephanie	Ok, so, for , like, there's one.	BR-S
	365	R1	Okay.	
	366	Stephanie	Um, I don't know, maybe there's two?	PAH
	367	R1	Want to think about that? (inaudible) yeah-	
	368	Stephanie	Yeah, I think there's more than . . . . I don't know.	OBS
	369	R1	Think about it.	
	370	Stephanie	Um, I need a few…	

Time	Line	Speaker	Transcript	Code
	371	R1	Yeah, that's fair enough. It's always good to take your time to think about it.	
	372	Stephanie	There's one choice, I'm gonna do them, like, as towers this time. When there's three it could be, um, you have red and yellow, it could be red-yellow-yellow and there's gonna be three. It could be red and it could be like that. There's three.	BR-V; BEJ
	373	R1	You absolutely sure of that? What was-um, what was- combinations were you selecting one from?	
	374	Stephanie	Two.	BR-V
	375	R1	Ok. Um, what do you think it would be when selecting one from four? Exactly one from four?	
	376	Stephanie	Four?	BR-V
	377	R1	What would you think it would be if I could select one from *n*?	
	378	Stephanie	*n*?	BCA
	379	R1	See that? Can you imagine that?	
	380	Stephanie	Yes.	
	381	R1	If it's five, can you see them all up there? If it's six, can you see them? You can make it as tall as you want, you can just see them exactly-	
	382	Stephanie	Yes.	
	383	R1	Isn't that helpful?	
	384	Stephanie	Yeah.	
	385	R1	To have that visual kind of thing?	
	386	Stephanie	Yes.	
	387	R1	You didn't even have any unifix cubes, that's great. Okay, so-	
	388	Stephanie	So, there would be three-	BR-S
	389	R1	You know that, do you know exactly two? Do you know that? Do you have to think a lot?	
30:00-34:59	390	Stephanie	I don't know. There's- oh- wouldn't it be the same thing?	BDI
	391	R1	Why?	
	392	Stephanie	Because it's just the opposite, right?	BEJ
	393	R1	Isn't that right?	
	394	Stephanie	So that would be three. And then, three, three, is one.	BR-S
	395	R1	Right?	
	396	Stephanie	Yeah.	
	397	R1	See how fast you got those?	

Time	Line	Speaker	Transcript	Code
	398	Stephanie	Yeah.	
	399	R1	Now, I'm going to write for *n* equals three here, look, put a one, three, three, one. Now do you notice something happening here. I have a one-one, for these two. I have a one-two-one, a one-two-one for none, one and two. I have a one-three-three-one, one-three-three-one for the case of three. Do you want to predict what it's going to be like for four?	
	400	Stephanie	It's going to be, like, one-four and then there's another number. And then, four-one.	BCA
	401	R1	Okay, now that's the interesting. . . .	
	402	Stephanie	Well, I know that that one's six though.	BR-S
	403	R1	Oh, but notice something, no?	
	404	Stephanie	Oh, is it, cause like, the 1 and 2- 1 and 1 are 2, 1 and 2 are 3, 1 and 2 are 3, 1 and 3 are 4, 1 and 3 are 4, 3 and 3 is 6?	BDI; BEJ
	405	R1	Isn't that exciting? Now, I'd like to have this case in here [*writes*].	
	406	Stephanie	Okay.	
	407	R1	It looks pretty, doesn't it? So, what would that be? Gosh. This was $n = 1$.	
	408	Stephanie	Mm-hmm.	
	409	R1	This would have to be $n = 0$. Right? Right?	
	410	Stephanie	Mm-hmm.	
	411	R1	So, what would you have to make selecting none from none, by definition, to make this all look pretty?	
	412	Stephanie	Selecting none from none?	PAH
	413	R1	See it makes almost no sense to think about.	
	414	Stephanie	Yeah, cause like . . .	
	415	R1	But remember you told me, like, if I took a number to the zero power, that doesn't make any sense?	
	416	Stephanie	Yeah.	
	417	R1	Remember we had that conversation in the car?	
	418	Stephanie	Yes.	
	419	R1	Well, this is almost like that. It doesn't make any sense, but if you want this picture to be so nice and symmetry and all, and if you want it to turn out to be that way, what would you want it to be?	
	420	Stephanie	I guess it would have to equal one.	BCA
	421	R1	Yeah. So people find it convenient to make	

Time	Line	Speaker	Transcript	Code
			that one. That's how definitions sometimes arise. There's- motivated by some symmetry or beauty. Is there another reason to make that one? I don't know of any. Do you? Taking no things from nothing? One way? [*to researchers*]	
	422	R2	Well, (inaudible)	
	423	R1	See, it just works out nicely. Can you guess five high, what these numbers would be?	
	424	Stephanie	All right. It would be 1. Um, and then it would be 1 + 3, oh, 5. And then it would be 10, 10, 5, 1.	BR-S
	425	R1	I put the one there. So this would be towers- this is no high.	
	426	R1/ Stephanie	One high. Two high. Three high. Four high. Five high.	BR-S/V
	427	R1	So now, I'm going to tell you what those numbers mean. Let's go backwards again. You know this is for n = five high.	
	428	Stephanie	Mm-hmm.	
	429	R1	So, see if you can tell me what that one is? We're selecting . . .	
	430	Stephanie	One from five.	BR-S
	431	R1	Ok and you're telling me that this is the case that should be one.	
	432	Stephanie	Mm-hmm.	
	433	R1	And what's the five?	
	434	Stephanie	Oh, no, that . . .	
	435	R1	Is this one from five?	
	436	Stephanie	Yeah, I thought, wasn't the five one from five. That would be zero.	BEJ; BR-S
	437	R1	Okay, so you're going to make this, oh ok. So the five would be one from five, you're saying?	
	438	Stephanie	Yeah.	BR-S
	439	R1	And you believe that? You can see that in your mind?	
	440	Stephanie	Yes.	
	441	R1	What are you seeing? I'm curious.	
	442	Stephanie	It would be like this, only longer.	BR-V; BEJ
	443	R1	How long?	
	444	Stephanie	Well, five.	
	445	R1	Okay, just checking. Just checking. Ok, so the	

Time	Line	Speaker	Transcript	Code
			next one is going to be…	
	446	Stephanie	Um, two from five. And that equals two.	BR-S
	447	R1	And that's ten cases. You wouldn't want to write those out. You kinda wish this is gonna be true, don't you?	
	448	Stephanie	Yeah.	
	449	R1	Actually, you did write that out when you were in the fourth grade.	
	450	Stephanie	Oh yeah.	
	451	R1	Right, you really did. We have a video to show it. Ok, and this ten, would that surprise you that it would be-if this is two, this would be three?	
	452	Stephanie	No. I mean-	
	453	R1	You would expect that wouldn't you?	
	454	Stephanie	Yeah.	
	455	R1	Because if you've done one, you've done half your work.	
	456	Stephanie	Mm-hmm.	
	457	R1	See this nice symmetry here. And the next one will be . . .	
	458	Stephanie	Four.	BR-S
	459	R1	And that doesn't surprise you, does it? That that's like this?	
	460	Stephanie	Nope and the last one will be five. One.	BR-S
	461	R1	So if I asked you, I'm now building these six, could you tell me how many that are exactly no red-	
	462	Stephanie	Yeah. Yes.	BCA
	463	R1	-exactly one, exactly two, exactly three, exactly four? Now, you expect this should all add up to what if it's five high? If you total them, you should get a total of?	
	464	Stephanie	Um, 32?	BCA; BR-S
	465	R1	And does it? 6? 11? 21? Wait a minute, something's wrong here. Oh, I shouldn't be adding the 5- 6, 16, 26, 31, 32. So if this thing works, what should it add- what should this next row add up to?	
35:00-39:59	466	Stephanie	Um, 64?	BCA
	467	R1	Let's try it. Let's predict what this is going to be.	
	468	Stephanie	It's going to be 1, 6, 15, 20, 15, 6, 1.	BCA

Time	Line	Speaker	Transcript	Code
	469	R1	And does that add up to 64?	
	470	Stephanie	Um, 30, 50 , um, 12, Yeah.	BMP
	471	R1	You like that?	
	472	Stephanie	Yes.	
	473	R1	So not only do you know how many towers you're going to get by adding, what else do you know?	
	474	Stephanie	I know the next row.	BCA
	475	R1	You know the next row.	
	476	Stephanie	And, I don't know, I know how many combinations I get for each row.	BCA; BR-S/V
	477	R1	Mh-hmm.	
	478	Stephanie	Um.	
	479	R1	Wasn't it clever, the person who found this out? Do you know who that was, would you like to know?	
	480	Stephanie	Yes.	
	481	R1	I don't know the guy's first name, but the last name is Pascal. Does anybody know his first name?	
	482	R3	Blaise. B-l-a-i-s-e.	
	483	R1	B-l-a-i-s-e. How do you say that? "Blaze" Pascal?	
	484	R3	(inaudible) I'm not French.	
	485	R1	And this thing is called Pascal's Triangle. And so, I don't think you realize, when you read this paper now, and see how hard you worked, you were really working pieces of Pascal's Triangle.	
	486	Stephanie	Hmm. It makes it easier.	
	487	R1	It makes it easier?	
	488	Stephanie	A lot easier.	
	489	R1	You know something, Stephanie? I hate to get preachy, 'cause my son will tell me "Ma, you're getting preachy", but if you hadn't done all that hard work all those years	
	490	Stephanie	Yeah.	
	491	R1	this would make no sense to you now, I don't think. Because I taught college and Mrs. Muter teaches college and Mrs. Steencken teaches college and the students work with this and they don't see it. You know what I mean by see it?	
	492	Stephanie	Yeah.	

Time	Line	Speaker	Transcript	Code
	493	R1	You see those cubes. You worked so hard at those.	
	494	Stephanie	Yeah.	
	495	R1	You know what I'm saying?	
	496	Stephanie	Mh-hmm.	
	497	R1	I mean, I don't know. But it's hard to visualize and see 'cause they only deal with the numbers. They just learned this rule that you add these numbers you get this and you add these numbers, you get this.	
	498	Stephanie	Mm-hmm.	
	499	R1	And if someone asks me what is the combinations of selecting exactly one of a color from five. You know, they'll give you the answer to that, but they have no picture of what they are giving you the answer to. They just are picking it out as a formula.	
	500	Stephanie	Yeah.	
	501	R1	You see that difference?	
	502	Stephanie	Yeah.	
	503	R1	So I don't know. Um, so, why- what are some interesting things about this? Things you might, I might leave you with to look at.	
	504	R1	Let me have another piece of paper. Let me give you (inaudible) Instead of making this triangle with these numbers as we did it-	
	505	Stephanie	Mm-hmm.	
	506	R1	-it might be interesting to make the triangle using the notation. I'll show you what I mean.	
	507	Stephanie	You mean, like C zero zero?	BR-S
	508	R1	Right. Exactly.	
	509	Stephanie	Ok.	
	510	R1	So this would be C zero zero . This is one by this. So this would be C one zero and this would be	
	511	R1/ Stephanie	C one one.	BR-S
	512	R1	Now this, sometimes this notation looks better. One zero. One one. Right? [*writes in both combination notations*]	
	513	Stephanie	Mh-hmm.	
	514	R1	And then this would be two zero, right?	
	515	Stephanie	Mh-hmm.	
	516	R1	On the end, and this would be three zero on the end and these are all ones, right?	

Time	Line	Speaker	Transcript	Code
	517	Stephanie	Mm-hmm.	
	518	R1	You could take a big piece of paper and write this out. Same thing here 1-1, 2-2, 3-3, 4-4, 5-5. And rather than write out, we know, when these are numbers, we add them and we get this, but now we're going to write this as . . . two . . .	
	519	Stephanie	Oh, um, two one. [*combination notation*]	BR-S
	520	R1	Right. And this two two. See if I wanted to invent a formula, 'cause the problem with this is I can tell you the next line but I have to know the line before it.	
	521	Stephanie	Oh.	
	522	R1	What if I wanted to know the sixtieth line? And it's tedious to know the line before it. Isn't it?	
	523	Stephanie	Yeah.	
	524	R1	You get the picture?	
	525	Stephanie	Mm-hmm.	
	526	R1	So wouldn't it be nice if we could figure out how that might begin to work, the relationship of one line to the other so that we don't necessarily-	
	527	Stephanie	Mm-hmm.	
	528	R1	-have to know the line in front of it.	
	529	Stephanie	Ok.	
	530	R1	So this is not a simple thing but it's something to think about. And this last one's going to be four four right?	
	531	Stephanie	Mm-hmm.	
	532	R1	So in essence you're saying, right, that this plus this is this.	
	533	Stephanie	Yeah.	
	534	R1	Right?	
	535	Stephanie	Mm-hmm…	
	536	R1	Okay, now you know what's going to go here and you know what's going to go here, right?	
	537	Stephanie	Yes.	
	538	R1	There's going to be, because there's three terms in here, there's going to be how many?	
40:00-44:59	539	Stephanie	Um. There's going to be what? Like number-wise? Or-	PAH
	540	R1	Right, with the three column, you have one-three- three- one.	

Time	Line	Speaker	Transcript	Code
	541	Stephanie	Yeah.	
	542	R1	Is that right? This is n=0, n=1, n=2, n=3.	
	543	Stephanie	Yes.	
	544	R1	So you've got to write this out yourself for it to make sense. So this was three...	
	545	Stephanie	Three zero.	BR-S
	546	R1	You could say from three things we're selecting zero of them. From three things, we're selecting one. From three things, we're selecting two. From three things, we're selecting exactly three. Right?	
	547	Stephanie	Mm-hmm.	
	548	R1	Ok, so we can see some interesting things here if you believe that this rule, that there's some pattern here. Right?	
	549	Stephanie	Mm-hmm.	
	550	R1	You know these are always ones, but you can get this by adding these two. Right, isn't that what we did here?	
	551	Stephanie	Yes.	
	552	R1	We know that's true, right? We know when we add these two, right to get this one?	
	553	Stephanie	Mm-hmm.	
	554	R1	So-w-we could begin to hypothesize one thing zero plus one thing one gives you two things one.	
	555	Stephanie	Okay.	
	556	R1	Right?	
	557	Stephanie	Yes. Um.	
	558	R1	Um, I'm not going to ask you to do this now; I'm going to ask you to think about it. I'm going to ask you to make your own, first of all. This is only going to make sense if you can-	
	559	Stephanie	Okay…	
	560	R1	-reproduce all this yourself. That's how it's going to make sense.	
	562	Stephanie	Okay.	
	563	R1	It's sort of like you're following me and saying "Yeah, this is interesting. I see this relationship", but you have to sit down and see if you can make the triangle with the numbers yourself and if you can think of these cases out.	
	564	Stephanie	Ok.	
	565	R1	And then put it in with the notation and say,	

Time	Line	Speaker	Transcript	Code
			"Ok, does that work?" And I'm also saying that I'm selecting zero from two, plus if I'm selecting one from two, my answer is…	
	566	Stephanie	Um...	
	567	R1	One from three. Is that right? One plus three is four. Did I do that wrong? I think I did something wrong. Yeah, here it is; it's this one. Right? If I'm selecting none from two, exactly one from two.	
	568	Stephanie	You get three.	BMP
	569	R1	I add those two together-	
	570	Stephanie	Yes.	
	571	R1	-it's the same as selecting one from three.	
	572	Stephanie	Mm-hmm.	
	573	R1	Right?	
	574	Stephanie	Yes.	
	575	R1	We're saying something like this here too. Let's just go down the side.	
	576	Stephanie	Yeah, it would just keep going.	BCA
	577	R1	So let's go- let's write it one more time. This is the row of…	
	578	Stephanie	Zero from three, hmm- plus one from three equals one from four.	BR-S; BCA
	579	R1	Ok, so in general, can you propose a rule?	
	580	Stephanie	Well, that . . .	
	581	R1	In other words, instead of um, having threes and fours,	
	582	Stephanie	Oh, you want me to put in like, letters? Like, *m* from zero plus . . .	PAH; BCA
	583	R1	If I'm taking- okay- taking zero from *n*.	
	584	Stephanie	Oh yeah. And one from-	BCA
	585	R1	-*n* should give me	
	586	Stephanie	Um, one from, I guess, just a different letter.	BCA
	587	R1	Well, you can tell me that letter because this is- this was a two, this was a three. This was a three, this is a four.	
	588	Stephanie	Well, if that's n, then it would be *m*.	BCA
	589	R1	Well, you can tell me more about *m*. What- how much- what's the relationship between *m* and *n*? We're not talking about candies. [*laughing*] In other words, what's the relationship between these two and this? These are two, this is three. If these are three, this is four. Maybe you need to write some more out.	

Time	Line	Speaker	Transcript	Code
			I propose it's one bigger.	
	590	Stephanie	Ok.	
	591	R1	Fair enough?	
	592	Stephanie	Yes.	
	593	R1	Is it always one bigger? Right? Isn't it-	
	594	Stephanie	Yes.	
	595	R1	I mean, that's something to propose. So the question is, if you could study that chart and see what relationships you might conjecture-	
	596	Stephanie	Ok.	
	597	R1	-you might be able to come up with ways of thinking about this in general that could be helpful to you in about three years.	
	598	Stephanie	Ok. Alright.	
	599	R1	Did you find this interesting or?	
	600	Stephanie	Yeah. I mean, it made it a lot easier than, like, when I did it, like, when we were in fourth grade or something.	
	601	R1	Well, I asked you to think about something the last time. Do you remember that?	
	602	Stephanie	Yes.	
	603	R1	Remember what it was?	
	604	Stephanie	It was, um, like, a plus b quantity to the fourth and then to the fifth. I worked it out on paper.	
	605	R1	Do you have it?	
	606	Stephanie	Yes.	
	607	R1	Can you show it to me?	
	608	Stephanie	All right. [*leaves table*]	
	609	R2	[*off camera; whispering*] The standard with combinatorics is two (inaudible) subscripts on each side…	
	610	R1	[*off camera*] Well that's another way.	
45:00-49:59	611	R2	[*off camera; whispering*]-that's (inaudible)	
	612	R1	[*off camera*] I thought that was permutations.	
	613	R2	[*off camera; whispering*] One's a *P* and one's a *C*.	
	614	R1	Oh, okay, thank you.	
	615	Stephanie	[*returns to table*] There's, um, the fourth.	
	616	R1	I just want to look at something. Do you mind if I do something? I don't want to mess up yours.	
	617	Stephanie	No, go ahead.	
	618	R1	I want the camera to take a picture of this	

Time	Line	Speaker	Transcript	Code
			before I mess it up. Who's taking a picture of this?	
	619	R3	Me- one second.	
	620	R1	I need another piece of paper.	
	621	R3	Lined? (inaudible)	
	622	R1	I'll take lined paper. I want you to study this for a minute. This is really very nice. I can see you worked hard at this. Now, for a moment, you see your *a*' s and *b*'s?	
	623	Stephanie	Mm-hmm.	
	624	R1	Right? *a* plus *b* to the zero equals one, right?	
	625	Stephanie	Mm-hmm.	
	626	R1	I'm not going to worry about the *a* and *b*, I'm going to worry about the number in front of it. What's the number in front of *a* here, when you don't put a number- when you have just *a* and you don't have any number written, there's a number that it's understood. Did you know that? So if I write *a*-	
	627	Stephanie	Oh one.	BMP
	628	R1	It's one. So that's one *a* plus one *b*.	
	629	Stephanie	Yeah.	
	630	R1	Right, so if I'm going to write this is one you wrote. And then *a* plus *b*. That's one *a* and one *b*. That's my one.	
	631	Stephanie	Yes.	
	632	R1	And *a* squared plus 2*ab* plus *b* squared. That's how many *a* squared?	
	633	Stephanie	Uh-huh. [*pause*] One?	BR-S
	634	R1	There's one *a* squared.	
	635	Stephanie	And one *b* squared.	BR-S
	636	R1	And two *ab*.	
	637	Stephanie	Yeah.	
	638	Stephanie /R1	And one *b* squared.	BR-S
	639	R1	One- two- one- those are my coefficients. The coefficients here, even though you don't see them, are ones.	
	640	Stephanie	Yeah.	
	641	R1	Right? Now, read off my next set of coefficients.	
	642	Stephanie	There's *a* cubed.	BR-S
	643	R1	One.	
	644	Stephanie	And there's three *a* squared *b* and there's three	BR-S;

Time	Line	Speaker	Transcript	Code
			a b squared and there's *b* cubed. Isn't that the same thing?	BDI
	645	R1	What do you mean?	
	646	Stephanie	As the towers?	BDI
	647	R1	Why?	
	648	Stephanie	It just is.	BEJ
	649	R1	Ok so, what do you have for the next one? Let's continue and compare this.	
	650	Stephanie	Um, *a* to the fourth.	BCA
	651	R1	One of them?	
	652	Stephanie	Yeah. And you have	
	653	R1	4-	
	654	Stephanie	four *a* cubed *b*.	BCA
	655	R1	And 6 – 4 – 1 .	
	656	Stephanie	Oh, okay.	BDI
	657	R1	And the next one?	
	658	Stephanie	1 –5 – 10 – 10 – 5 – 1 .	BR-S
	659	R1	You did all that hard work but I'm going to tell you what those coefficients are- 1 – 6- 15 – 20 – 15 – 6 and 1 . Let's see if I'm right.	
	660	Stephanie	Yeah.	
	661	R1	Hmmm. So, the only difference is here you have an *a* squared and what does *a* squared mean?	
	662	Stephanie	*a* times *a*.	BMP
	663	R1	Or two factors of *a*.	
	664	Stephanie	Yes.	
	665	R1	Right?	
	666	Stephanie	Ok.	
	667	R1	So you have two factors of *a*, right?	
	668	Stephanie	Mm-hmm.	
	669	R1	You have one of those. One thing with two factors of *a*, one thing with two *a*'s in it.	
	670	Stephanie	Mm-hmm.	
	671	R1	I don't want to think of *a*'s; I want to think of red.	
	672	Stephanie	Ok.	
	673	R1	Can you switch that a minute? So now I have one thing with two reds. What thing could I be thinking of if I have two reds?	
	674	Stephanie	A tower that's two high?	BR-V; BCA
	675	R1	And here I'm talking about two things.	
	676	Stephanie	Mm-hmm.	

Time	Line	Speaker	Transcript	Code
	677	R1	One is-	
	678	Stephanie	-red-	BR-V
	679	Stephanie /R1	-and one is yellow.	BR-V
	680	R1	Is that possible in two high?	
	681	Stephanie	Yeah.	
	682	R1	To have the one red and one yellow? There are two of them.	
	683	Stephanie	Yeah. 'Cause one is- the red can be on top or on the bottom. And the yellow -same thing.	BEJ; BR-V
	684	R1	And what about b squared?	
	685	Stephanie	Um- two yellow.	BR-V
	686	R1	Ok, so I could think about this as these coefficients tell me how many of combinations of them and these tell me which ones – exactly two red, right?	
	687	Stephanie	Mm-hmm.	
	688	R1	-exactly one red and a yellow-	
	689	Stephanie	Mm-hmm.	
	690	R1	-exactly two yellow.	
	691	Stephanie	Yeah.	
	692	R1	Does that work here?	
	693	Stephanie	It's- yeah, I guess, there's three red.	BR-S/V
	694	R1	So I'm talking about towers of three red. How many of those? Exactly three red?	
	695	Stephanie	Mm-hmm.	BR-S/V
	696	R1	There's one?	
	697	Stephanie	Yes.	
	698	R1	And here I have…	
	699	Stephanie	Um . . . towers . . . um . . . of red and yellow, three high, I guess? Since there's three of them?	BR-S/V
	700	R1	Right, and how many are reds and how many of them are yellow?	
	701	Stephanie	Two are red and one is yellow. And . . .	BR-S/V
	702	R1	And there are three of those.	
	703	Stephanie	Yes. And the next. . .	
	704	R1	Do you really believe that?	
	705	Stephanie	Yes.	
	706	R1	Two are reds and one are yellow? Can you see	

Time	Line	Speaker	Transcript	Code
			them? The three? The yellow, the yellow, the yellow?	
50:00-54:59	707	Stephanie	Yeah uh yeah. I mean, you could have, um, the red, the red, the yellow. The red, the yellow, the red. The yellow, red, red.	BR-V
	708	R1	Say that again. That was too fast for me. I was trying to concentrate.	
	709	Stephanie	The red, the red, the yellow. The red, the yellow, the red. Or the, um, yellow, yellow, red- uh- yellow, red, red.	BR-V
	710	R1	I'm going to believe what you said is true, but somehow I'm having trouble focusing. Um. One more time.	
	711	Stephanie	Red-	BR-V
	712	R1	On the bottom?	
	713	Stephanie	On the top.	BR-V
	714	R1	Ok, that's why I'm having trouble. Red on the top.	
	715	Stephanie	-red, yellow.	BR-V
	716	R1	Red, red, yellow's on the bottom.	
	717	Stephanie	Red, on the top, yellow, red.	BR-V
	718	R1	Mm-hmm.	
	719	Stephanie	Or yellow on the top, red, red.	BR-V
	720	R1	Alright, I think I see it. We'll listen to the tape. I'm getting tired. But you see the relationship here between towers?	
	721	Stephanie	Yes	
	722	R1	Good. You can write that up for me for next time.	
	723	Stephanie	Ok.	
	724	R1	So you see, all this hard work, when you get a test, you know, or take a college board-	
	725	Stephanie	Mm-hmm.	
	726	R1	-and they say expand *a* plus- you know, *a* plus *b* to the sixth,	
	727	Stephanie	Mm-hmm.	
	728	R1	Think how fast you can do that.	
	729	Stephanie	Yeah.	
	730	R1	That's very nice. Maybe this is where we should stop unless Ethel has a question, or Steve? Or Elena? Oh, Steve had said- said- whispered something to me when you went over there, that, he reminded me of something that I had not remembered, that you might also	

Time	Line	Speaker	Transcript	Code
			see this notation in books. You might see the five here and the zero here [*writes* ${}_5C_0$].	
	731	Stephanie	Ok, but it's the same thing?	PAH
	732	R1	It's the same thing. I was thinking, um, well, let's stay with selections and combinations. And um, yeah, come on, show her on the computer.	
	733	R3	(inaudible)	
	734	R1	Yeah, Ethel's going to show you how you could use the uh-	
	735	Stephanie	Calculator?	
	736	R1	-computer. Maybe next time you can bring yours, Steve.	
	737	R2	Actually, I have one in my car.	
	738	R1	You do? That'd be neat. Do you mind going to get it? The calculator?	
	739	R3	And then you come over here to the probability section.	
	740	Stephanie	Mm-hmm.	
	741	R3	See where it is- those entries are. Just what- just what Dr. Maher just wrote, where the *n* would be the five and the *r* would be the zero.	
	742	R1	Let me see. I can't see over there. So what did you do here? Oh, but we didn't tell her about the *P*.	
	743	R3	No, we didn't talk about random generators or factorials or	
	744	R1	That's going to be another time we're gonna talk about all that. But now, that's what-	
	745	R3	You're just looking at number three right now-	
	746	R1	Let's- let's let Stephanie put one of these in and see if it works. Suppose we wanted to know-	
	747	Stephanie /R3	You've got to do-	
	748	R1	-let's do something like this, suppose I wanted to know- I'm building towers eight and I wanted to find all with exactly three red. Isn't that what I'm asking?	
	749	Stephanie	Oh, okay.	
	750	R3	Ok, so you put in 8, and then you go to Math.	
	751	Stephanie	(inaudible)	
	752	R3	Third button down on the left hand side.	
	753	R1	The black one.	

Time	Line	Speaker	Transcript	Code
	754	Stephanie	Ok.	
	755	R3	Then go across with your arrow to probability and then select- no – oops.	
	756	Stephanie	Sorry.	
	757	R1	That's ok. Do it again. I don't know how to do this either, Stephanie. So I'm watching too.	
	758	R3	Back up one space and hit delete. Ok, delete, now hit Math again. And then go across to Probability. Now, you want to go down and select the third one 'cause that's where we have. . .	
	759	Stephanie	Yeah, ok.	
	760	R3	And then, how many are you taking from that group of eight?	
	761	Stephanie	Three.	BR-S
	762	R3	Ok, taking three, and then tell it to do it.	
	763	Stephanie	56.	BR-S
	764	R1	See, I don't know if that's true. I'm going to have to do all of this to figure it out. Right, you can double check. Let's do one we know. Ok, which one is this?	
	765	Stephanie	Um, that would be [*pauses*] two out of six. I think?	BR-S
	766	R1	I think so too.	
	767	Stephanie	So it would be six [*presses buttons on calculator to do six choose two*] . . . fifteen.	BR-S
	768	R1	Hey!	
	769	Stephanie	Yeah.	
	770	R1	Your calculator's a little different, Steve? This is a TI-82. What do you have?	
	771	R2	Same thing, basically.	
	772	R1	Is it a TI?	
	773	R2	No, it's Hewlett-Packard.	
	774	R1	Why don't you show Stephanie how to work that? They're all a little bit different.	
	775	R3	They all are.	
	776	R2	You just um-	
	777	R1	Show me too.	
	778	R2	-hit six, two, then you hit com.	
	779	R1	Oh that's different. Wait, do that again. I didn't – that's too complicated for me. Go slow.	
	780	R2	Um.	
	781	R1	You're doing six things taking two at a time, so what mode are you in? Did you get fifteen?	

Time	Line	Speaker	Transcript	Code
			I don't know what you did.	
	782	R2	Okay. Um, you want six things.	
	783	R1	Enter.	
	784	R2	Right. Taken two at a time.	
	785	R1	Enter again.	
	786	R2	And you hit this button – see, how it says com – combinations.	
	787	R1	Where does it say combinations? I can't see that.	
	788	R2	Right there. See the –	
	789	R1	Oh oh oh, ok. I can hardly see that. So this has a lot more things in it, you know. But, see, if you worry about in the future once you understand it and you want to do some of the harder problems like 560 things. Ok?	
55:00-59:59	790	Stephanie	Yeah.	
	791	R1	So, um, you've done some really nice pulling together of stuff here. So what I guess, um, do you have a calculator like this?	
	792	Stephanie	Like this? No. I have, um, I don't have a TI-82. I think, do I?	
	793	R3	What kind of calculator do you have, Stephanie?	
	794	Stephanie	I'm trying to think. I think we all have, um, the Sony ones.	
	795	R3	Does it say scientific on it?	
	796	Stephanie	Yes	
	797	R3	Does it do logs and sines and cosines?	
	798	Stephanie	Yes, I know it's a scientific calculator 'cause they required them.	
	799	R3	Then you should be able to find a key on it that says nCr somewhere. You'll probably have to use your shift or your inverse key to access it. But almost every scientific calculator has that capability on it.	
	800	R1	So why don't you explore, you know, checking out some of these that you know with the calculator so that you get to know how to use the calculator because you know when you take college boards and all in the future you're going to be allowed to use a calculator.	
	801	Stephanie	Mm-hmm.	
	802	R1	But the question is to know when to use it because you have to know what's in your head	

Time	Line	Speaker	Transcript	Code
			when to do this. Um, and I'd like you to kind of write up how all of this fits together, like a little essay.	
	803	Stephanie	Ok.	
	804	R1	You know, starting with the cubes. Like, you can even write a little short story. The cubes. The cubes. The cubes. I know you're all getting sick of them, right? But a lot of that, um, powerful mathematical ideas can be developed, I think, from it, even the probability. You're going to play with that a little bit. But do you see how the algebra fits?	
	805	Stephanie	Yes.	
	806	R1	But you couldn't do this stuff until you had some algebra.	
	807	Stephanie	Yeah.	
	808	R1	You see? And the exponents and I want you to think real hard when you look at all of these terms and you know these coefficients are important. You see, they can be mapped right into Pascal's Triangle.	
	809	Stephanie	Yeah.	
	810	R1	Right? But not only do you – what's nice about it when you think of these terms, once you know the coefficients and how many of them there are – let's look at this – this is the sixth, right? And we know they have to be 1, 2, 3, 4, 5, 6, 7 terms. 1, 2, 3, 4, 5, 6, 7, terms, right?	
	811	Stephanie	Mm-hmm.	
	812	R1	And I know you probably worked hard to do this. Now, so now, if you asked me to do, do it what I'd say, the seventh one, I'd say, well, it's going to be a one, right? That's the first term.	
	813	Stephanie	Mmhmm.	
	814	R1	Next one's gonna have a seven, right? The next one's gonna have a 21, right?	
	815	Stephanie	Mm-hmm.	
	816	R1	The next one's gonna have	
	817	Stephanie	35…	BMP
	818	R1	And then –	
	819	Stephanie	21. Oh, that's another one. 35. And then 21.	BMP
	820	R1	There's a little symmetry here. 21.	
	821	Stephanie	Yeah. Oh and then seven and one.	BMP
	822	R1	Ok, and then you ought to think about why that's symmetry.	

Time	Line	Speaker	Transcript	Code
	823	Stephanie	Ok.	
	824	R1	Ok, now the key –	
	825	Stephanie	[*interrupting*] Oh, well,	
	826	R1	Go ahead.	
	827	Stephanie	Oh, like with the cubes, isn't it just 'cause it's the opposite?	PPK; BDI;BCA
	828	R1	All right, say more.	
	829	Stephanie	'Cause, like, if I have two – if I have towers of four, and it's two red.	BEJ
	830	R1	[*interrupting*] let's say towers of four	
	831	Stephanie	Right, then there's going to be two yellow.	BEJ
	832	R1	Ok is that why you think so?	
	833	Stephanie	So it's just like the opposite.	BEJ
	834	R1	Ok, so let's look at a particular line. You said towers – how high did you say?	
	835	Stephanie	Of four.	
	836	R1	So towers of four is which line here?	
	837	Stephanie	Yeah, that one. And if I have two- I have two red on it –	BEJ
	838	R1	So two red would be this.	
	839	Stephanie	Yeah.	
	840	R1	So this would contain two red and two yellow?	
	841	Stephanie	Well, I mean, wouldn't it just be – Yeah, see, right here. See *a* squared and see *b* squared.	BEJ; BR-S
	842	R1	I was thinking of the symmetry here, like, 4 *a cubed b* and *4 a b cubed.*	
	843	Stephanie	But isn't that like the same thing?	BCA
	844	R1	Okay, tell me why it's the same.	
	845	Stephanie	Well, 'cause here it's just there's two but here it's three.	BEJ
	846	R1	Okay. So you have those opposites in those same categories is what you're saying?	
	847	Stephanie	Yeah.	
	848	R1	You once said that in an interview I had with you when you were in fourth grade. You said the opposites are in the same categories, but you were thinking of cubes then. Now, I know there are how many – 1, 2, 3, 4 , 5, 6, 7, 8, - terms here-	
	849	Stephanie	Mm-hmm	
	850	R1	-and if I'm doing this to the –	
	851	Stephanie	Seventh.	
	852	R1	Seventh, I know this is going to be *a* to the	

Time	Line	Speaker	Transcript	Code
			what?	
	853	Stephanie	Um, seventh.	BR-S
	854	R1	Seventh. Ok, that means all of them are going to be red.	
	855	Stephanie	Mm-hmm.	
	856	R1	Right? Now, I'm going to have seven of them, of which-	
	857	Stephanie	Um, *a* is to the sixth and *b*	BR-S
	858	R1	Six red and one yellow. Right?	
	859	Stephanie	Yeah.	
	860	R1	And this is going to be –	
1:00:00-1:04:59	861	Stephanie	*a*, um, fifth, *b* to the third. *b* to the, um, fourth. I don't know.	BR-S; OBS
	862	R1	Ok, now that's a question, right?	
	863	Stephanie	Mm-hmm.	
	864	R1	Now, let's see if there's anything in here that can help you. Know that. Let's look at something you know. Now in the sixth, you said this was *a* to the sixth.	
	865	Stephanie	Mmhmm. Ok.	
	866	R1	This is *a* to the fifth *b.* What's the exponent of *b* here?	
	867	Stephanie	Oh it's- is it gonna be squared? It's- uh- gonna be squared?	BR-S; PAH
	868	R1	Why do you think squared?	
	869	Stephanie	Well, because they're all squared. Like, all, like that one, every one that looks like that is all squared. But, I don't know.	BEJ; OBS
	870	R1	So the question is the exponents. What do you think these exponents need to be? 'Cause if you knew that, gosh-	
	871	Stephanie	Yeah.	
	872	R1	-you'd have it all, right? You'd be able to write the next one out-	
	873	Stephanie	Mm-hmm.	
	874	R1	-without- so the question is studying this and seeing if there are any patterns that might-	
	875	Stephanie	Well-	
	876	R1	-but think of the towers, because, remember, you're building your towers how tall?	
	877	Stephanie	Um, in this one? Six.	BR-V
	878	R1	Alright, so you're building them six tall. What does this five mean?	
	879	Stephanie	Oh, would it- it would have to add up to seven.	BR-S

Time	Line	Speaker	Transcript	Code
	880	R1	Why?	
	881	Stephanie	Well, because you're building it seven high.	BEJ
	882	R1	Right, so what does this –	
	883	Stephanie	So five and two, you could do that.	BEJ
	884	R1	So the five means what?	
	885	R1/ Stephanie	Five reds and two yellows.	BEJ
	886	R1	So the next would be . . .	
	887	Stephanie	Um, *a* to the - I don't know 'cause the, see here, it's like an . .	OBS
	888	R1	Well, think of what case this is. Here, all your seven are red.	
	889	Stephanie	Yeah.	
	890	R1	Right? Here, six are red and one is yellow. Here, five are red-	
	891	Stephanie	Mm-hmm. Four-	
	892	R1	-and two are yellow.	
	893	Stephanie	Four *a*. Three *b?* [a^4b^3]	BR-S; BCA
	894	R1	Doesn't that make sense?	
	895	Stephanie	Yeah.	
	896	R1	So, uh, notice something. These are seven tall. They can't be more than seven tall. They could be distributed.	
	897	Stephanie	Mm-hmm. Um, the next one would be the opposite, *a* to the third *b* to the fourth and then it would just keep going the opposite.	BR-S; BCA
	898	R1	Ok. So you need to study that. Those numbers and those relationships, but always look for meaning, Stephanie.	
	899	Stephanie	Ok.	
	900	R1	Try to imagine these towers and what does this mean? This means, this is the part of the, you know what these mean. These mean seven are exactly red. This means, ok.	
	901	Stephanie	Yeah.	
	902	R1	Oh this was none of them exactly red and this was all of them exactly red.	
	903	Stephanie	Yes.	
	904	R1	I'm sorry, I didn't want to confuse you. I think I said that wrong. So, some interesting things to think about. Um, how do we do this? I really do want a copy of what you've done here, but how do we get copies? Now, we don't . . .	

Time	Line	Speaker	Transcript	Code
	905	Stephanie	I can go down to the office and see if I can get them there.	
	906	R1	So can you do the same thing again and make copies of these? I'd like you to put your name on them and a date on them and if you can remember to order and number them, that would be absolutely phenomenal.	
	907	Stephanie	Alright. Do you know what the date is?	
	908	R1	Today is the thirteenth. March thirteenth.	
	909	Stephanie	Ok.	
	910	R1	Since we're so unorganized. Ok. Anything you can write me about, your whole, you know, thinking about these towers and this notation and whatever.	
	911	Stephanie	Ok.	
	912	R1	You're just probably done the first, what, she's done some Algebra II. Is some of this in Algebra II, Steve?	
	913	R2	Um, combinatorics?	
	914	R1	Yeah.	
	915	R2	Um, I don't think so.	
	916	R1	What would it be in? Pre-calculus?	
	917	R2	Well, no, okay now- um there's some binomial expansion in Algebra II.	
	918	R1	So binomial expansion is in Calculus and Algebra II. Ok, that would also be in finite math. It would also be in statistics.	
	919	R2	In a probability class you're just gonna-	
	920	R1	A lot of this in probability.	
	921	R2	They do lots of cool stuff like (inaudible) card games.	
	922	R1	Now, Stephanie, you're going to be in ninth grade next year. My son is in ninth grade; he took probability.	
	923	Stephanie	Really?	
	924	R1	With a satellite course. His high school didn't have it; isn't that right? He dabbled with a little of these ideas. But he didn't build towers, so he was at a direct disadvantage. Any other questions, Elena or Ethel?	
	925	R2	(inaudible)	
	926	R1	If they're not exactly right, the camera, Ethel will number them correctly for us when she looks at the tape, right? So it'll be real important to get as much of what you see here	

Time	Line	Speaker	Transcript	Code
			in your ideas pulled together in writing to get ready as if you were going to make a presentation and maybe I'd like to invite you to our Summer Institute and present this to the teachers, ok?	
	927	Stephanie	Ok, uh-huh. Um. [*laughs*]	
	928	R1	Steve, you can be the cameraperson that day. You'll help us. I should also say in all the institutes we've run, the teachers never got this far. It's true.	
1:05:00-1:05:56	929	R2	(inaudible)	
	930	Stephanie	Alright, you want me to go down and see if they'll make copies?	
	931	R1	That would be wonderful.	
	932	Stephanie	Alright, how many copies do you want?	
	933	R1	I like this piece of mathematics a lot. I think this is one of the prettiest things- the way these different things come together.	
	934	Stephanie	Alright.	
	935	R1	And I have to tell you, Stephanie, it didn't come together to me until I was in college.	
	936	Stephanie	Alright. [*exits*]	
	937	R1	Pardon? [*to off camera]*	
	938	R3	Same for me.	

APPENDIX F: TRANSCRIPT – SESSION 6

INTERVIEW WITH STEPHANIE Time: 92 minutes (2 CDs)
March 27, 1996

R1: Dr. Carolyn Maher Stephanie: Stephanie R2: Dr. Robert Speiser

Time	Line	Speaker	Transcript	Code
00:00-04:59	1	R1	I have to take my glasses off to see up close.	
	2	R2	(inaudible)	
	3	R1	Stephanie. What I would suggest to you is, I know him	
	4	Stephanie	Um hm.	
	5	R1	and we've been friends for a few years – is – don't assume he knows anything. [*Stephanie smiles.*]	
	6	R2	Good.	
	7	R1	And then it'll work better. And don't, don't pretend [*the camera person moves the microphone*] he even knows the towers.	
	8	Stephanie	Okay	
	9	R1	And start with the assumption that this is a person who is certainly capable of understanding anything you explain – honestly-	
	10	R2	Maybe.	
	11	R1	But don't assume –	
	12	Stephanie	All right.	
	13	R1	Okay.	
	14	R2	Okay.	
	15	Stephanie	(inaudible)	
	16	R1	So as much as you can remember what last time was about um – and I have paper here and pens and things if you need them. Any way you can be helpful.	
	17	Stephanie	All right.	
	18	R1	And if you need to come closer, I'll just move back. How we even started this discussion – which I can't even remember. I'll help – if I can be helpful.	
	19	R2	What was it about?	
	20	Stephanie	I think. Did you – you started with um	PPK;

Time	Line	Speaker	Transcript	Code
			explaining that if you had like – four – like a towers of four –	BR-V
	21	R1	I'm going to let you move up.	
	22	Stephanie	Or	
	22	R1	So you can	
	23	Stephanie	trains of four	BR-V
	24	R1	(inaudible)	
	25	R2	Okay.	
	26	R1	switch positions, Bob.	
	27	Stephanie	that um	
	28	R2	Thank you.	
	29	Stephanie	and you have two different choices.	PPK
	30	R2	Two different colors?	
	31	Stephanie	Yeah. Like	
	32	R2	Um hm.	
	33	Stephanie	if it was [*Stephanie gets some Unifix cubes*] like that	BR-V
	34	R2	Okay.	
	35	Stephanie	Um. I think it started with her explaining that um if you took one of the four colors [*Stephanie pauses, she rolls her eyes, and appears to be thinking – recalling the last interview.*] Yeah. One of the fours. Oh. One color.	PPK
	36	R2	Um hm.	
	37	Stephanie	How many different combinations you could make – out of four high. Like you could have	PPK
	38	R2	You mean they'd be four high with one green?	
	39	Stephanie	Yeah.	
	40	R2	Somewhere.	
	41	Stephanie	Yes.	
	42	R2	Okay.	
	43	Stephanie	So. Well, it would – we did trains so it would be four like this.	BR-V
	44	R2	Okay.	
	45	Stephanie	Or um. Oh. Four like this [*Stephanie builds a train.*] or four like um – [*she builds another train*] this or four like [*continues building*] this or four [*builds a fourth train*] like that.	BR-V
	46	R2	(inaudible)	
	47	Stephanie	And that taking one out of four, like one out of four choices was the same as um –	BR-S

Time	Line	Speaker	Transcript	Code
			[*Stephanie writes* C_1^4 *and* $\binom{4}{1}$ *on the paper before her*]- or – I think – that's how we started. [*Four trains are now visible on the table. They are arranged in a row on the table in front of Stephanie from Stephanie's right to left.*]	
	48	R2	Okay. So this is the way you would write –	
	49	Stephanie	Yeah.	
	50	R2	What?	
	51	Stephanie	Um that –	
	52	R2	What do those symbols stand for?	
	53	Stephanie	That – well – that means that you're selecting one out of four	BEJ
	54	R2	four	
	55	Stephanie	choices.	BEJ
	56	R2	Out of four choices.	
	57	Stephanie	Uh hm. Yeah. I think that's how we started. And then what happened was	PPK
	58	R2	Um hm.	
	59	Stephanie	she asked like two out of - if I had two green?	PPK
	60	R2	Um hm.	
	61	Stephanie	What would it be? And it – how many choices would there be? And for one, there was four. [*Stephanie writes on the paper in front of her.*]	BR-S
	62	R2	And they're the four that you've shown?	
	63	Stephanie	Yeah. There's no more.	BR-V
	64	R2	And there are no more.	
	65	Stephanie	Yeah. You can't make any more.	BEJ
	66	R2	Okay. I'm ready to believe that.	
	67	Stephanie	Okay.	
	68	R2	Uh. When you start – when you work with two, though, the question might be interesting.	
	69	Stephanie	Yeah.	
	70	R2	So let's see what happens.	
	71	Stephanie	Well, for two there's um…this one. [*Stephanie builds [G G B B]*]	BR-V
	72	R2	Um hm.	
	73	Stephanie	And there's –this one. [*She builds [B B G G]*] And there's –	BR-V
	74	R2	Um hm.	

Time	Line	Speaker	Transcript	Code
	75	Stephanie	This one. [*She builds [G B G B]*] and there's –[*builds [B G B G]*]	BR-V
	76	R2	Um hm.	
	77	Stephanie	[*Stephanie builds [B G G B] and [G B B G]*] That's it.	BR-V
	78	R2	Six?	
	79	Stephanie	Um hm. There's no more.	
	80	R2	How do you know that?	
	81	Stephanie	Um. 'Cause I tried all the combinations-	BEJ
	82	R2	Um.	
	83	Stephanie	-possible – like – um alright. If you start out with – um – two blue on top, [*Stephanie picks up that tower*] there, you can have – if you start out with two blue together, you can	BEJ
	84	R2	Yes.	
	85	Stephanie	put them on top. You can put them in the middle – you can move them one down-	BEJ
	86	R2	Um hm.	
	87	Stephanie	-or you can put them on the bottom. [*Stephanie rearranges the towers, lining them up in the following order:* $\begin{bmatrix} B \\ B \\ G \\ G \end{bmatrix} \begin{bmatrix} G \\ B \\ B \\ G \end{bmatrix} \begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix}$]	BR-V
	88	R2	Yes.	
	89	Stephanie	If you start with them separated by a green	BR-V
	90	R2	Um hm.	
	91	Stephanie	There'd be one – on top- or like this. [*Stephanie shows the two towers:* $\begin{bmatrix} B \\ G \\ B \\ G \end{bmatrix} \begin{bmatrix} G \\ B \\ G \\ B \end{bmatrix}$]. You can't move it anymore, because you only have four spaces to move it.	BR-V
	92	R2	Um hm.	
05:00-09:59	93	Stephanie	And there's only one like that. {*Stephanie indicates the* $\begin{bmatrix} B \\ G \\ G \\ B \end{bmatrix}$ *tower.*}	BR-V
	94	R2	How would you describe this one?	

Time	Line	Speaker	Transcript	Code
	95	Stephanie	It's separated by two green.	BEJ
	96	R2	So here it's like they're separated by no greens?	
	97	Stephanie	Yeah.	
	98	R2	And here separated	
	99	Stephanie	By one.	BEJ
	100	R2	The two blues are separated by	
	101	Stephanie	one green.	BEJ
	102	R2	one green. And here, they're separated by two.	
	103	Stephanie	Um hm.	
	104	R2	Um. – Is it possible that there could be another tower that you haven't built yet?	
	105	Stephanie	Oh. Yeah. No. No. Un uh.	
	106	R2	How would you explain that?	
	107	Stephanie	All right. Wait. Let me think. *[Stephanie writes something on the paper in front of her.]* Yeah, because you can't move them any more. –There's only four spaces for you to move them.	BEJ
	108	R2	Um hm.	
	109	Stephanie	And – like with this one – if they're separated by none, you can have them here	BEJ
	110	R2	Yes.	
	111	Stephanie	up top. You could move them down one and have them here.	BEJ
	112	R2	Right.	
	113	Stephanie	And you can down them move them down another – You can't move them down any more. There's no more	BEJ
	114	R2	Because	
	115	Stephanie	spaces for you to move them.	BEJ
	116	R2	That's true.	
	117	Stephanie	Here, you have	
	118	R2	separated by one	
	119	Stephanie	separated by one green, you can have them here. You can move them down one and have them here. You can't move them down any more.	BEJ
	120	R2	That's true.	
	121	Stephanie	Because there's only four. If they're separated by two. You can't move them – you have one on the top and one of the bottom and that's it. You can't do anything	BEJ

Time	Line	Speaker	Transcript	Code
			else to it.	
	122	R2	Okay.	
	123	Stephanie	So there's six.	
	124	R2	I think – I think I'm convinced.	
	125	Stephanie	Okay.	
	126	R2	Okay. Good.	
	127	Stephanie	So then it was -	
	128	R2	Did you do this last week?	
	129	Stephanie	Yeah.	
	130	R2	Yeah. Okay. So you found six.	
	131	Stephanie	Um hm. And then with three – [*Stephanie begins building towers with three green and one blue. She first builds* $\begin{bmatrix}B\\G\\G\\G\end{bmatrix}$] three, there's only – you can have one at the top – Oh. No. – You can have one at the bottom. [*builds* $\begin{bmatrix}G\\B\\G\\G\end{bmatrix}$] You can have – there. [*builds* $\begin{bmatrix}G\\G\\B\\G\end{bmatrix}$] And one there. [*builds* $\begin{bmatrix}G\\G\\G\\B\end{bmatrix}$] *[Stephanie looks at R2.]* And that's it. *{Stephanie has built her 'traditional' "staircase".* $\begin{bmatrix}B\\G\\G\\G\end{bmatrix}$ $\begin{bmatrix}G\\B\\G\\G\end{bmatrix}\begin{bmatrix}G\\G\\B\\G\end{bmatrix}\begin{bmatrix}G\\G\\G\\B\end{bmatrix}$*}*	BR-V
	132	R2	That's it?	
	133	Stephanie	Um hm. 'Cause if there's – well, what it is, is it's the opposite of this one. *[Stephanie indicates the towers with three blues and one*	BDI

Time	Line	Speaker	Transcript	Code
			green.]	
	134	R2	Ah! – Are you – are you saying that three greens is the same as one blue?	
	135	Stephanie	Yeah.	
	136	R2	Ah?	
	137	Stephanie	They're the opposite. Because here it's blue separated by one green and here's it's green and one blue.	BEJ
	138	R2	So by opposite, you mean – wherever there's a green on this side, you put a blue on this side.	
	139	Stephanie	Yeah.	
	140	R2	And wherever there's blue on this side, you put a green on that side.	
	141	Stephanie	Yeah.	
	142	R2	And then - - Yeahh - -	
	143	Stephanie	And so you have four.	
	144	R2	And so four.	
	145	Stephanie	*[writing]* And there's only two ways to do this one. Oops -	BR-S
	146	R1	Let's leave those.	
	147	Stephanie	No. There's only one way to do this one.	BR-V
	148	R1	Why don't we leave these?	
	149	Stephanie	If it's blue, you can only do it like this. *[builds* $\begin{bmatrix} B \\ B \\ B \\ B \end{bmatrix}$*]*	BR-V
	150	R2	I see.	
	151	Stephanie	You're selecting four blue out of four.	BR-S
	152	R2	Four.	
	153	Stephanie	So you can't do any thing else.	
	154	R2	Um hm.	
	155	Stephanie	And zero is the opposite. You're selecting no blue, so you're selecting all green. *[builds* $\begin{bmatrix} G \\ G \\ G \\ G \end{bmatrix}$*]*	BR-S/V
	156	R2	Yeah.	
	157	Stephanie	That's it.	
	158	R2	Those seem so much simpler.	

Time	Line	Speaker	Transcript	Code
	159	Stephanie	Yeah.	
	160	R2	But again, I've seen you're using the opposite in order to	
	161	Stephanie	Um hm.	
	162	R2	connect them together. Uh. Let me just ask you one more question	
	163	Stephanie	Okay.	
	164	R2	about when you um I think it was when you built these six towers. Uh. It looked to me like you were making pairs of opposites	
	165	Stephanie	Yeah.	
	166	R2	at the beginning, when you were constructing them,	
	167	Stephanie	Um hm.	
	168	R2	but then when you were explaining to me how many there were you organized them differently.	
	169	Stephanie	Um hm.	
	170	R2	Um. Could you say a little more about that?	
	171	Stephanie	Oh. Well. – Because it's easier for me to look at them as opposites when I'm building them.	BEJ
	172	R2	Um hm.	
	173	Stephanie	Then – 'cause I know – 'cause it's like pairing them up –like – if there's one separated on top, there's one – you know -	BEJ
	174	R2	Yeah -	
	175	Stephanie	But, it's easier for you to look at them when they're done if they're like this. So you can see the pattern that they make. That you can't build down any more.	BEJ
	176	R2	Um hm.	
	177	Stephanie	Or you can't build up any more, 'cause there's no more to – do it.	BEJ
	178	R2	So it was more for your explanation that	
	179	Stephanie	Um hm.	
		R2	you rearranged them.	
10:00-14:59	180	Stephanie	Like you could see it better like this, than if I said – I mean – 'cause when we first did the towers problems, we went through – I mean there were tons of Unifix cubes and all it was was those two are opposites. "Well, how do you know?"	BEJ
	181	R2	Um hm.	
	182	Stephanie	And I don't know. I didn't know how to	BEJ

Time	Line	Speaker	Transcript	Code
			explain it. So it's easier for you to see that – there's the – you know – because it goes down – you can't build anymore. That's why.	
	183	R2	Thank you.	
	184	Stephanie	[*chuckles*] And then	
	185	R2	What did you do next?	
	186	Stephanie	All right. – Next – we um – hm, what did we do next? - - - I think - - [*Stephanie looks through her paper*.] We – oh – we um- said if there were um – if it was 'C' and you still had four – um – four cubes, but you didn't how many of them you were taking, what would it be? Um – like what could 'r' be?	PPK
	187	R2	'r' is the lower number?	
	188	Stephanie	Yeah. – so 'r' would be like how many green you were selecting.	PPK
	189	R2	Um hm.	
	190	Stephanie	But you didn't know 'cause it was a variable and then it was – it could either be um zero, one, two, three, or four. 'Cause those were how many selections you could make. And then – do you remember what we did next? I think -	PPK
	191	R2	Could 'r' be five?	
	192	Stephanie	No.	
	193	R2	Why?	
	194	Stephanie	'Cause you're selecting four.	BEJ
	195	R2	Okay	
	196	Stephanie	So 'r' couldn't be anything more than four.	BEJ
	197	R2	Okay.	
	198	Stephanie	And um then – All right – We went back to um the beginning with the towers. And we went way back to when we were building towers like a long time ago. And we built – and started with like the first tower. And you could have towers of – either – towers one high	PPK
	199	R2	Um hm.	
	200	Stephanie	in two colors. So you could have	
	201	R2	Okay.	
	202	Stephanie	- Well, actually, I could just show you. You could either have blue or green. That's it.	BR-V
	203	R2	I'm convinced.	
	204	Stephanie	Now for towers two tall, you could have – from there you could have a two green or –	BR-V

Time	Line	Speaker	Transcript	Code
			[*builds* $\begin{bmatrix} B \\ G \end{bmatrix}$]	
	205	R2	Okay. Why did you choose these particular two? That you placed next to that green one?	
	206	Stephanie	Because the green was on the bottom here	BEJ
	207	R2	The bottom	
	208	Stephanie	So you keep building up from it. Like for the next one, there'll be either three green or green, blue, um, green.	BEJ
	209	R2	Okay. Continue.	
	210	Stephanie	And over there you can have that. [*Builds* $\begin{bmatrix} B \\ B \end{bmatrix}$ *and* $\begin{bmatrix} G \\ B \end{bmatrix}$] Those are the two you can get from that.	BR-V
	211	R2	Um hm.	
	212	Stephanie	You'll always get two, like, from each of them. And then for three it'll be like that [*builds* $\begin{bmatrix} G \\ G \\ G \end{bmatrix}$] or that [*builds* $\begin{bmatrix} G \\ B \\ G \end{bmatrix}$] or that [*builds* $\begin{bmatrix} B \\ B \\ B \end{bmatrix}$] or that [*builds* $\begin{bmatrix} B \\ G \\ B \end{bmatrix}$] or um [*builds* $\begin{bmatrix} G \\ B \\ B \end{bmatrix}$] I'm sorry. That or that. [*Stephanie rearranges the towers she has built.*] or that [*builds* $\begin{bmatrix} G \\ B \\ B \end{bmatrix}$*, Stephanie then places a* $\begin{bmatrix} B \\ G \\ G \end{bmatrix}$ *beside the* $\begin{bmatrix} G \\ G \\ G \end{bmatrix}$] Or that. [*builds* $\begin{bmatrix} B \\ B \\ G \end{bmatrix}$]	BR-V
	213	R2	Okay. Can I just ask you about the last one	

Time	Line	Speaker	Transcript	Code
			you built?	
	214	Stephanie	Yeah.	
	215	R2	Okay. Um. If I'm not mistaken, you thought for a moment and then decided that something – that this one needed to be there.	
	216	Stephanie	Yeah.	
	217	R2	Okay. Um. If I'm not mistaken, you thought for a moment and then decided that something – that this one needed to be there.	
	218	Stephanie	Yeah.	
	219	R2	Is that right?	
	220	Stephanie	Yeah.	
	221	R2	Why did you decide that it needed to be there?	
	222	Stephanie	Because, -well, what happened was – I started building and I forgot that each one had two.	BEJ
	223	R2	Um hm.	
	224	Stephanie	So I just built like one for each of 'em and then I had to go back an rebuild it with the other one. And I wasn't sure if I had already built one for this one, like two, but I noticed that there was none – see- for this – since this is green-blue.	BEJ
	225	R2	Yes.	
	226	Stephanie	Its choices can be green – you build on to it – it can either have a green on top of it	BEJ
	227	R2	Um hm.	
	228	Stephanie	or a blue on top of it.	BEJ
	229	R2	or a blue on top of it.	
	230	Stephanie	and there was no one	BEJ
	231	R2	I see.	
	232	Stephanie	with green-blue-blue. That's why.	BEJ
	233	R2	Good.	
	234	Stephanie	Oops	
	235	R2	It looks to me like the others work the same way…	
	236	Stephanie	Yeah.	
	237	R2	Yeah.	
15:00-19:59	238	Stephanie	You can – you can just keep building on. And then for four it was – [*Stephanie builds towers four high in the following order:*	BR-V

Time	Line	Speaker	Transcript	Code
			$\begin{bmatrix}G\\G\\G\\G\end{bmatrix}\begin{bmatrix}B\\G\\G\\G\end{bmatrix}\begin{bmatrix}G\\B\\G\\G\end{bmatrix}\begin{bmatrix}B\\B\\G\\G\end{bmatrix}\begin{bmatrix}B\\G\\B\\G\end{bmatrix}\begin{bmatrix}G\\G\\B\\G\end{bmatrix}\begin{bmatrix}B\\B\\B\\G\end{bmatrix}\begin{bmatrix}G\\B\\B\\G\end{bmatrix}\begin{bmatrix}B\\B\\B\\B\end{bmatrix}\begin{bmatrix}G\\B\\B\\B\end{bmatrix}$ $\begin{bmatrix}G\\B\\G\\B\end{bmatrix}\begin{bmatrix}B\\B\\G\\B\end{bmatrix}$] There. *[Stephanie counts the number of towers she has built.]* Okay.	
	239	R2	I was counting, too.	
	240	Stephanie	That's all there are. – And that's it for four. But we did it on paper.	BEJ
	241	R2	Um hm.	
	242	Stephanie	And then – um – then I think we started to build – um – we figured out all of them like from like from this.	PPK
	243	R2	All of?	
	244	Stephanie	Like if you started out with you see from one from zero	BR-S
	245	R2	Oh.	
	246	Stephanie	And we said that would equal one, 'cause	BR-S
	247	R2	Um hm.	
	248	Stephanie	And then – um –one – and you figured that out all the way up to um	
	249	R2	Um hm	
	250	Stephanie	four. And then, she showed me how to build the triangle – one -	PPK
	251	R2	Okay. Um –Tell me a little more about the triangle. Um. What is this number?	
	251	Stephanie	That's	
	252	R2	What does that count?	
	253	Stephanie	That's how many you can get if you take zero from zero.	BEJ
	254	R2	So that's the zero-zero.	
	255	Stephanie	Yes.	
	256	R2	And then these two ones?	
	257	Stephanie	That's –um –zero out of one or one out of one.	BEJ
	258	R2	Um hm.	
	259	Stephanie	That's zero out of two.	BR-S
	260	R2	Uh huh.	

Time	Line	Speaker	Transcript	Code
	261	Stephanie	One out of two, two out of two.	BEJ
	262	R2	Two out of two?	
	263	Stephanie	Um hm.	
	264	R2	So this counts the ways – so which towers – okay – does this have to do with towers?	
	265	Stephanie	Yeah. I – it wou-	
	266	R2	Show me.	
	267	Stephanie	It would be – *[Stephanie grabs the towers two tall.]*	BR-V
20:00-24:59	268	R2	Okay.	
	269	Stephanie	And this one - those	
	270	R2	So these are the towers that are two high.	
	271	Stephanie	Yeah.	
	272	R2	-two blocks high and then um how do find the one, the two, and the one?	
	273	Stephanie	It would be –um – if you're selecting green, it would be – one well if you're selecting blue, it would be one with no selections of blue.	BEJ
	274	R2	Right.	
	275	Stephanie	Two with one selection of blue and one with um one's with	BEJ
	276	R2	Okay.	
	277	Stephanie	all selections of blue.	BEJ
	278	R2	Oh. Okay. Okay. So this has more to do	
	279	Stephanie	It's like the towers	BCA; BDI
	280	R2	It's like the way you'd organized	
	281	Stephanie	Um hm.	
	282	R2	the towers before.	
	283	Stephanie	Yeah.	
	284	R2	Uh – I was interested in how – Do you remember if you had – if these were the original order in which you arranged them – you know when you had them here – or whether you had rearranged them?	
	285	Stephanie	When we were there, I think it was here, here, here, and there. [*the towers are arranged from her left to her right:* $\begin{bmatrix} B \\ B \end{bmatrix} \begin{bmatrix} G \\ B \end{bmatrix} \begin{bmatrix} B \\ G \end{bmatrix} \begin{bmatrix} G \\ G \end{bmatrix}$]	
	286	R2	and there.	
	287	Stephanie	Yeah. Because this has all the blue. And	
	288	R2	Um hm.	
	289	Stephanie	on the bottom. *[Pause]*	

Time	Line	Speaker	Transcript	Code
	290	R2	Oh. Very fine. Yeah.	
	291	Stephanie	Actually [*Stephanie reverses the pattern:* $\begin{bmatrix}G\\B\end{bmatrix}\begin{bmatrix}B\\B\end{bmatrix}\begin{bmatrix}B\\G\end{bmatrix}\begin{bmatrix}G\\G\end{bmatrix}$] There. I think I messed this row up. *[She is talking about the towers three cubes high.]* When I moved I think I moved these around.	BEJ
	292	R2	Okay. I'm still understanding this one. Um. These two came from this green one by putting	
	293	Stephanie	Yes.	
	294	R2	different tops on it. And similarly *[R2 indicates the two with blue bottom cubes.]*	
	295	Stephanie	Um hm.	
	296	R2	those two. *[pause]* Ah ha. It's interesting because the way you arranged them to show the one, two, and the one, switched these two.	
	297	Stephanie	I think I just messed them up when I was making these. I couldn't see and I had to move those.	BEJ
	298	R2	Oh. But they're a different	
	299	Stephanie	Yeah.	
	300	R2	But they're different choices anyway, but it is interesting, it was interesting to me	
	301	Stephanie	Um, but	
	302	R2	how you see the -	
	303	Stephanie	It's just	
	304	R2	Yeah. How would you organize the next row, so that it makes more sense?	
	305	R1	(inaudible)	
	306	R2	So it makes the most sense for you?	
	307	Stephanie	Oh.	
	308	R1	It works for the chart.	
	309	R2	Could it work for the chart? Yeah. You want to try that?	
	310	Stephanie	For the chart?	
	311	R1	You can come around here.	
	312	R2	Yeah.	
	313	Stephanie	Well for the chart it would be um *[Stephanie writes]* wait – *[writes some more]* So	BR-S
	314	R2	How did you know to write those numbers?	
	315	Stephanie	'Cause - - one goes to one and one and then one goes here. One plus one is two.	BEJ
	316	R2	Oh.	

Time	Line	Speaker	Transcript	Code
	317	Stephanie	One goes here. One.	BR-S
	318	R2	So you do it by adding.	
	319	Stephanie	Yeah. One plus two is three. One plus two is three. And one goes there. It…it that's how you figure it out.	BEJ
	320	R2	Ahh so that so that's how you got this row.	
	321	Stephanie	Yes.	
	322	R2	Okay.	
	323	Stephanie	That's how I got it.	
	324	R2	Did you explore why the adding works?	
	325	Stephanie	Um, I don't know, I, I mean we, um, we worked it out like this on paper, but -	
	326	R1	What, what's – that's a good question. You just took these four *[points to the row of towers two high:* $\begin{bmatrix} B \\ B \end{bmatrix}\begin{bmatrix} G \\ B \end{bmatrix}\begin{bmatrix} B \\ G \end{bmatrix}\begin{bmatrix} G \\ G \end{bmatrix}$]	
	327	Stephanie	Um hm.	
	328	R1	and you explained how the adding works for this, for the row one-two-one.	
	329	Stephanie	Um hm.	
	330	R1	Isn't that what you just did?	
	331	Stephanie	Yeah.	
	332	R1	Any you had them like this [*rearranges the row so that the towers are in* $\begin{bmatrix} G \\ B \end{bmatrix}\begin{bmatrix} B \\ B \end{bmatrix}\begin{bmatrix} B \\ G \end{bmatrix}\begin{bmatrix} G \\ G \end{bmatrix}$ *order*] Right?	
	333	Stephanie	Um hm.	
	334	R1	But you said it could also go like this [*rearranges the row so that the towers are back in* $\begin{bmatrix} B \\ B \end{bmatrix}\begin{bmatrix} G \\ B \end{bmatrix}\begin{bmatrix} B \\ G \end{bmatrix}\begin{bmatrix} G \\ G \end{bmatrix}$ *order*]. It didn't matter	
	335	Stephanie	Not really.	
	336	R1	with this design.	
	337	Stephanie	No.	
	338	R1	It still works.	
	339	Stephanie	Yeah, it still works.	
	340	R1	Why?	
	341	Stephanie	Because you're taking there's two choices you can do from each. There are two blue	BEJ

Time	Line	Speaker	Transcript	Code
			[*picks up the* $\begin{bmatrix} B \\ B \end{bmatrix}$ *tower.*] If you're building up, you have a blue on the bottom and a blue on the top or a blue on the bottom and a green on the top. [*Stephanie indicates towers* $\begin{bmatrix} B \\ B \end{bmatrix}\begin{bmatrix} G \\ B \end{bmatrix}$.]	
	342	R1	Okay. So it works for here. [*indicates* $\begin{bmatrix} B \\ B \end{bmatrix}\begin{bmatrix} G \\ B \end{bmatrix}$] It also works [*indicates* $\begin{bmatrix} B \\ G \end{bmatrix}\begin{bmatrix} G \\ G \end{bmatrix}$]	
	343	Stephanie	Yes.	
	344	R1	I thought I heard you say that here you have, this would be the one [*lifts* $\begin{bmatrix} G \\ G \end{bmatrix}$]	
	345	Stephanie	Right.	
	346	R1	because that says from the two you have no blue and here you have one with one blue [*indicates* $\begin{bmatrix} B \\ G \end{bmatrix}$] and one with one blue [*indicates* $\begin{bmatrix} G \\ B \end{bmatrix}$] So together you could, that gives you the two. [*pushes the* $\begin{bmatrix} G \\ B \end{bmatrix}$ *and* $\begin{bmatrix} B \\ G \end{bmatrix}$ *closer together.*]	
	353	Stephanie	Um hm.	
	354	R1	And then, that's the one [*indicates* $\begin{bmatrix} B \\ B \end{bmatrix}$]. And I think you were asking the question, Bob, can – how does that work for the next row? [*indicates the row of towers three high*]	
	355	Stephanie	Like	
	356	R1	How how do you get the one-three-three-one out of the next row?	
	357	Stephanie	Oh, all right. [*Stephanie sweeps away the rows of towers four high.*]	
	358	R1	I think that was (inaudible) this one here. [*points to the third row of towers still remaining*]	

Time	Line	Speaker	Transcript	Code
	359	R2	Yes.	
	360	Stephanie	'Cause this, oops	
	361	R2	Oops *[Some of the towers three high fall over.]*	
	362	Stephanie	There's millions of these	
	363	R2	This was part of the row	
	364	Stephanie	All right.	
	365	R2	that we wanted, right? [*lifts the* $\begin{bmatrix} B \\ B \\ B \end{bmatrix}$ *that was knocked over*]	
	366	R1	Why why don't we move them away? [*moves away the four high fallen towers*]	
	367	Stephanie	There's the one.	
	368	R2	Okay.	
	369	Stephanie	Here's one [*indicates* $\begin{bmatrix} B \\ B \\ B \end{bmatrix}$] if you've selected none, uh, no greens out of towers of three you have all blue.	BEJ; BR-V
	370	R2	That's right.	
	371	Stephanie	Then if you're selecting one green out of the towers it can be um [*pause*]	BEJ; BR-V
	372	R2	Um hm.	
25:00-29:59	373	Stephanie	It could be these three. [*takes* $\begin{bmatrix} G \\ B \\ G \end{bmatrix}$ $\begin{bmatrix} B \\ G \\ B \end{bmatrix}$ $\begin{bmatrix} G \\ B \\ B \end{bmatrix}$]	BEJ; BR-V
	374	R2	Um hm.	
	375	Stephanie	No these three. [*exchanges* $\begin{bmatrix} G \\ B \\ G \end{bmatrix}$ *for* $\begin{bmatrix} B \\ B \\ G \end{bmatrix}$]	BEJ; BR-V
	376	R2	One green.	
	377	Stephanie	With one green.	BEJ; BR-V
	378	R2	Good.	
	379	Stephanie	And then if you're selecting two green it	BEJ;

Time	Line	Speaker	Transcript	Code
			would be these [*takes* $\begin{bmatrix} G \\ G \\ B \end{bmatrix} \begin{bmatrix} B \\ G \\ G \end{bmatrix} \begin{bmatrix} G \\ B \\ G \end{bmatrix}$]	BR-V
	380	R2	Those three. Okay.	
	381	Stephanie	And then if you're selecting all green, there'd be one way to do it. [*takes* $\begin{bmatrix} G \\ G \\ G \end{bmatrix}$]	BEJ; BR-V
	382	R2	Um hm.	
	383	Stephanie	So I guess.	
	384	R1	So, so I guess – Let's go back, I think you were asking this question but I'm not sure you were – um – can these work also this pattern- can you both patterns work at the same time?	
	385	Stephanie	Yeah. Uh, I mean	
	386	R1	Where would you place these so that they fit the pattern you're building here as well as looking like that. Is it possible, is that, was that your question. I don't know if that's it.	
	387	R2	This was- it was really curious to me. I wanted to understand the addition. For example, we're adding these two [*indicates* $\begin{bmatrix} G \\ B \end{bmatrix}$ *and* $\begin{bmatrix} B \\ G \end{bmatrix}$] and this one [*indicates* $\begin{bmatrix} G \\ G \end{bmatrix}$] to get- isn't it these three? [*indicates* $\begin{bmatrix} G \\ G \\ B \end{bmatrix}$ $\begin{bmatrix} B \\ G \\ G \end{bmatrix} \begin{bmatrix} G \\ B \\ G \end{bmatrix}$]	
	388	Stephanie	Um hm.	
	389	R2	How did that, how does that happen?	
	390	Stephanie	Oh. All right. There's the one with the two on the bottom. [*indicates* $\begin{bmatrix} B \\ G \\ G \end{bmatrix}$] There's the one with the blue in the middle. [*indicates*	BEJ

Time	Line	Speaker	Transcript	Code
			$\begin{bmatrix} G \\ B \\ G \end{bmatrix}$] And there's the one with the blue on the bottom, see how you're adding like on. [*indicates* $\begin{bmatrix} G \\ G \\ B \end{bmatrix}$]	
	391	R2	Okay. Let me see if I see it. So these two both had a green placed on top which keeps the one blue, right?	
	392	Stephanie	Um hm.	
	393	R2	And then these two greens had a had to have a blue on top in order to get	
	394	Stephanie	Yes.	
	395	R2	one with one blue.	
	396	Stephanie	And those would be the three	BEJ
	397	R2	And the (inaudible)	
	398	Stephanie	Yeah.	
	399	R1	Is there any other way you can do it though how do I know there's not another way you can do it.	
	400	Stephanie	Because.	
	401	R1	Do you understand my question? I, I, I believe you can do these to get	
	402	R2	Yeah.	
	403	R1	to keep the one blue. But how do I know there's one we haven't missed in our counting? Do you know my question?	
	404	R2	Ahhhh!	
	405	Stephanie	Yeah. Um, oh, well I think you can – can't I just do this again? Like, cause there's one blue - I can put the one blue on the top.	BEJ
	406	R2	Oops.	
	407	Stephanie	I can put the one blue on the top. I can move it down one to the middle. I can move it down one to the bottom. I can't move it up or down anymore. [*rearranges the towers to:* $\begin{bmatrix} B \\ G \\ G \end{bmatrix} \begin{bmatrix} G \\ B \\ G \end{bmatrix} \begin{bmatrix} G \\ G \\ B \end{bmatrix}$]	BEJ
	408	R2	Right.	

Time	Line	Speaker	Transcript	Code
	409	Stephanie	There's no more blocks.	BEJ
	410	R1	So you're using the position argument.	
	411	Stephanie	Yeah, I can't, there's nothing else.	BEJ
	412	R1	Is that ok with you?	
	413	R2	It's ok with me. This, yeah, the position is fine. I'm convinced that these are the only ones that have one blue, no doubt about it. I was convinced before. I'm convinced again. Um. What I was interested in was that uh was where these came from	
	414	Stephanie	Oh. Well	
	415	R2	You know it's like a family tree.	
	416	Stephanie	Well, they keep building up. That's the whole	BEJ
	417	R2	Yeah.	
	418	Stephanie	thing.	
	419	R2	Yeah.	
	420	Stephanie	Like if I placed them there, there, and there it's just building up [*replaces* $\begin{bmatrix} G \\ G \\ B \end{bmatrix} \begin{bmatrix} G \\ B \\ G \end{bmatrix} \begin{bmatrix} B \\ G \\ G \end{bmatrix}$ *in the triangle of towers*].	BEJ
	421	R2	Um hm.	
	422	Stephanie	And I also place um this here	BEJ
	423	R2	Uh huh	
	424	Stephanie	and this here and this here. So now the row is [*replaces towers:* $\begin{bmatrix} B \\ B \\ B \end{bmatrix} \begin{bmatrix} B \\ G \\ B \end{bmatrix} \begin{bmatrix} G \\ G \\ B \end{bmatrix} \begin{bmatrix} B \\ B \\ G \end{bmatrix} \begin{bmatrix} G \\ B \\ G \end{bmatrix} \begin{bmatrix} B \\ G \\ G \end{bmatrix}$]	BEJ; BR-V
	425	R2	Because of the way they built up.	
	426	R1	That bothers me because it's messed up the three you wanted to keep together. Is there any way you keep my three together and not mess up your pattern? Because this bothers me a lot.	
	427	Stephanie	Um.	
	428	R1	See what I'm saying?	
	429	Stephanie	You mean like	

Time	Line	Speaker	Transcript	Code
	430	R1	I like, I like patterns.	
	431	Stephanie	separate the ones with two and the ones with one. [*pulls* $\begin{bmatrix} G \\ G \\ B \end{bmatrix}$ *and* $\begin{bmatrix} B \\ B \\ G \end{bmatrix}$ *in front*]	PAH
	432	R1	Yeah. Is there a way of doing it and still keep the pattern and keeping, in other words, keeping both at the same time? I don't know. Is it possible?	
	433	R2	Is it possible?	
	434	Stephanie	They mix in the middle though. I mean they're gonna	BEJ
	435	R1	Do they have to mix in the middle? There's no way of avoiding it?	
	436	Stephanie	Oh. I have to put one here and one like, [*pause*] like if you're talking; about how they build up.	BEJ
	437	R1	Yeah.	
	438	Stephanie	They go together. *[replaces the towers she had put in front so they are grouped:* $\begin{bmatrix} G \\ B \\ B \end{bmatrix}$ - $\begin{bmatrix} G \\ G \\ B \end{bmatrix}\begin{bmatrix} B \\ G \\ B \end{bmatrix}$ - $\begin{bmatrix} B \\ B \\ G \end{bmatrix}\begin{bmatrix} G \\ B \\ G \end{bmatrix}$ - $\begin{bmatrix} B \\ G \\ G \end{bmatrix}$]	BEJ
	439	R1	So they're always gonna have to	
	440	Stephanie	Even if I – there's	
	441	R2	(inaudible)	
	442	Stephanie	There's no matter what you do they're gonna be	BEJ
	443	R1	No matter what you do there's gonna be	
	444	Stephanie	They're gonna touch.	BEJ
	445	R2	Okay. Yeah. I think I see it. This one has one blue and one green [*indicates* $\begin{bmatrix} B \\ G \end{bmatrix}$].	
	446	Stephanie	Um hm.	
	447	R2	So what can happen to the number of blues and greens when we build on top of it?	
	448	Stephanie	They (inaudible) two blues or two greens	BEJ
	449	R2	and so the two cases get shuffled	
	450	Stephanie	Yes.	

Time	Line	Speaker	Transcript	Code
	451	R2	That way they - it seems that they have to.	
	452	Stephanie	Um hm.	
	453	R2	Yeah.	
	454	Stephanie	Oh, so	
	455	R2	Yeah.	
	456	Stephanie	you want to keep these	
	457	R2	We wanted to keep those	
	458	Stephanie	Over here and these over here? *[arranges towers:* $\begin{bmatrix} G \\ B \\ B \end{bmatrix} \quad \begin{bmatrix} G \\ B \\ G \end{bmatrix}\begin{bmatrix} B \\ G \\ B \end{bmatrix} \quad \begin{bmatrix} B \\ B \\ G \end{bmatrix}\begin{bmatrix} G \\ G \\ B \end{bmatrix} \quad \begin{bmatrix} B \\ G \\ G \end{bmatrix};$ *exchanges towers to make this arrangement]* Or um. Yeah, those would have to go the other way.	BR-V
	459	R1	This one bothers me now [*indicates* $\begin{bmatrix} G \\ G \\ B \end{bmatrix}$] 'cause there's a blue on the bottom and it's next to the green. That really bothers me.	
	460	Stephanie	Yeah, but then I'd have to move them again and	
	461	R1	Exactly.	
	462	Stephanie	it would still- [*Stephanie rearranges the towers back to the first groupings.*]	
	463	R2	So it looks like there's different…	
	464	R1	organizations	
	465	R2	organizations	
	466	Stephanie	They, they all work but	BDI
	467	R2	They all work.	
30:00-34:59	468	R1	But, but they're different, aren't they?	
	469	Stephanie	Yeah.	
	470	R2	But they seem to do something different, okay, but that looks like a kind of a victory in its own way (inaudible)	
	471	Stephanie	And then um [*places the* $\begin{bmatrix} B \\ B \\ B \end{bmatrix}$ *and* $\begin{bmatrix} G \\ G \\ G \end{bmatrix}$ *at the ends of the row of towers.*]	
	472	R2	(inaudible) they had to be different	

Time	Line	Speaker	Transcript	Code
			(inaudible) good	
	473	Stephanie	And that's how you can get (inaudible) Should I keep going with that?	
	474	R2	Did you do that last night?	
	475	Stephanie	Last	
	476	R2	Last time	
	477	Stephanie	Um	
	478	R2	Did you carry it further?	
	479	Stephanie	Yeah, I think we went a little bit. I think 'cause what happened was we were doing this problem like before that, like way before we started this, on 'a' plus 'b' quantity squared [*writes* $(a + b)^2$]	PPK
	480	R2	Um hm.	
	481	Stephanie	And at first, I did um [*writes* $a^2 + b^2$] but that was proved wrong, and it was *a* squared plus two *ab*.	PPK
	482	R2	Two *ab*!	
	483	Stephanie	Plus *b* squared [*writes* $a^2 + 2ab + b^2$] and um we kept going like I think I got up to like six like *a* plus *b* quantity squared, quantity like to the sixth power.	PPK
	484	R2	Ah.	
	485	Stephanie	And I think see this is where I forgot and um, I think with the numbers let's see (inaudible) [*draws Pascal's triangle until the sixth row*] There. I think that's one…zero, one, two, three, four, five, six. [*Stephanie points to each row as she counts.*] All right. That's six, and um, I think, using that see this is where I forget, I think she figured out the exponents or something to some of the numbers or like you know that there's going to be an *a* but I think she figured out like what the numbers were going to be up here [*indicates the position of the exponents*]. The exponents, is that what you did? I don't	BR-S; PPK
	486	R1	I don't know. I don't remember it myself and I didn't look at the tape, but I have a question now. You just wrote down what *a* plus *b* quantity squared was. Why don't you write it on the top of this paper? [*gives Stephanie a new piece of paper and she writes* $(a+b)^2 = a^2 + 2ab + b^2$*]*	

Time	Line	Speaker	Transcript	Code
	487	Stephanie	Okay.	
	488	R1	And I guess my question now is can that at all be related to the triangle or what you built with your	
	489	Stephanie	I'm sorry. [*Stephanie moves a tower that was in the way.*]	
	490	R1	With, with your tow- with your cubes, can you take each of those terms in that expansion *a* squared, 2*ab*, *b* squared and see any relationship to the towers or any of those lines of the triangle or any part of the triangle – column, line, diagonal, anything.	
	491	Stephanie	I guess like here [*takes the towers two high*] there's, I don't, I don't, I mean, not with the exponents. Like I don't see how *a* squared	OBS
	492	R1	Tell us what you do see.	
	493	Stephanie	Well, I guess cause like there's two with an *a* and a *b*. [*indicates* $\begin{bmatrix}G\\B\end{bmatrix}$ *and* $\begin{bmatrix}B\\G\end{bmatrix}$] Like	BEJ
	494	R1	What's an *a* and a *b*?	
	495	Stephanie	If green was *a*. And	BR-S/V
	496	R1	Okay. Lets call green *a* and lets call blue *b*.	
	497	Stephanie	[*lifts* $\begin{bmatrix}G\\B\end{bmatrix}\begin{bmatrix}B\\G\end{bmatrix}$] you have two with green is *a*, and blue is *b*.	BR-S/V
	498	R1	Okay.	
	499	Stephanie	You know like one of each.	BR-S/V
	500	R1	Okay, so you have an *ab* and a *ba* or 2*ab*. [*points to the towers* $\begin{bmatrix}G\\B\end{bmatrix}$ *and* $\begin{bmatrix}B\\G\end{bmatrix}$ *that Stephanie put aside*]	
	501	Stephanie	You have one that's all *a* [*indicates* $\begin{bmatrix}G\\G\end{bmatrix}$] and one that's all *b*. [*indicates* $\begin{bmatrix}B\\B\end{bmatrix}$]	BR-S/V
	502	R1	Ok but this says what do you mean by all *a*? This is an *a* and a *b* [*indicates* $\begin{bmatrix}G\\B\end{bmatrix}$] and an *a* and a *b* [*indicate* $\begin{bmatrix}B\\G\end{bmatrix}$] .	
	503	Stephanie	Yeah, well	

Time	Line	Speaker	Transcript	Code
	504	R1	*aa* [*points to* $\begin{bmatrix} G \\ G \end{bmatrix}$] *bb* [*points to* $\begin{bmatrix} B \\ B \end{bmatrix}$]	
	505	Stephanie	Yes.	
	506	R1	So what do you mean *aa*? What could these *aa* and *ab* mean? Is that *a*	
	507	Stephanie	Oh. I get to, oh, well if you're saying that this is *a* [*takes one green cube*] and two of them would like *aa* would be like *a* squared. [*lifts* $\begin{bmatrix} G \\ G \end{bmatrix}$]	BR-S/V; BDI
	508	R1	Could be (inaudible) how many of those do you have?	
	509	Stephanie	Well one of *a*	BR-S/V
	510	R1	So where's the one I don't see the one in this.	
	511	Stephanie	Well, the one's just there. [*points in front of* a^2 *on her paper*]	BR-S
	512	R1	So imagine there's a one	
	513	Stephanie	Yeah.	
	514	R1	in front of that *a* squared.	
	515	Stephanie	I mean I could put it	
	516	R1	Yeah. Put it somewhere okay? [*Stephanie writes ones on the paper in front of* a^2 *and* b^2.] So now, now help me see what that might mean.	
	517	Stephanie	Okay, there's one with two *a*'s with like *aa* or *a* squared. [*lifts* $\begin{bmatrix} G \\ G \end{bmatrix}$]	BEJ; BR-S/V
	518	R1	Two factors of *a*.	
	519	Stephanie	Yeah, and there's two with *ab*, with *a* and *b*. [*indicates* $\begin{bmatrix} G \\ B \end{bmatrix}$ *and* $\begin{bmatrix} B \\ G \end{bmatrix}$]	BEJ; BR-S/V
	520	R1	One factor of *a* and one factor of *b*.	
	521	Stephanie	One factor of *b*. And there's one with two factors of *b*.	BR-S/V
	522	R1	So, so that relates to the *a* plus *b* quantity squared. What about the triangle?	
	523	Stephanie	One, two, one. [*points to the third row of the triangle*]	BR-S
35:00-39:59	524	R1	Okay, tell me what you think *a* plus *b* quantity cubed will be. Without having to work out all the details of it now. Using your cubes and using what you just told me.	

Time	Line	Speaker	Transcript	Code
	525	Stephanie	I guess it would be	
	526	R1	'Cause you didn't like multiplying those out all the time. That was a lot of hard work.	
	527	Stephanie	I know there'll be an *a* cubed and a *b* cubed. [*writes* a^3 *and* b^3 *on the paper leaving a large space between them.*]	
	528	R1	How do you know that?	
	529	Stephanie	Because there's a one and a one [*points to the fourth row of the triangle*] and besides I mean	BEJ
	530	R1	What's the *a* cubed? Which cube, which tower is this? Don't make new ones. You have them made, I think.	
	531	Stephanie	That would be that. [*indicates* $\begin{bmatrix} G \\ G \\ G \end{bmatrix}$]	BR-S/V
	532	R1	Oh, okay.	
	533	Stephanie	And the *b* would be that. [*indicates* $\begin{bmatrix} B \\ B \\ B \end{bmatrix}$]	BR-S/V
	534	R1	That was easy.	
	535	Stephanie	And there's gonna be, I guess, three *a* squared *b* cubed and three *ab* squared.	BR-S
	536	R1	Ok. Why don't you write that down and then see if we can find them. [*Stephanie writes:* $3ab^2$] Tell me why you think that.	
	537	Stephanie	All right. Here. The *a* is the green. So here's the [*5 second pause; then picks up* $\begin{bmatrix} G \\ G \\ B \end{bmatrix}$ *and* $\begin{bmatrix} B \\ G \\ G \end{bmatrix}$] Am I missing one?	BR-S/V
	538	R1	How many do you want? How many towers three high should you have and let's, let's find them. How many should you have altogether?	
	539	Stephanie	I should have eight.	
	540	R1	Okay. I see eight. There's four here and then you have four up there. [*indicates towers three high*] Let's get these out of the way. [*pushes away the towers two high*] Right,	

Time	Line	Speaker	Transcript	Code
			here's eight of them. Right? [*R2 uprights four towers that have fallen.*]	
	541	Stephanie	Zero, one, two, three, yeah, that's three high. Oh here [*takes* $\begin{bmatrix} G \\ B \\ G \end{bmatrix}$] um okay so.	BR-S/V
	542	R1	Tell me what's *a* and what's *b* again. I keep forgetting.	
	543	Stephanie	Green is *a*.	BR-S/V
	545	R1	Why don't you write that down what *a* is. I get [*Stephanie writes: Green – A, Blue – B*] Okay, green is *a*, blue is *b*.	
	546	Stephanie	I have three with two factors of *a* and one factor of *b*. [*Stephanie indicates* $\begin{bmatrix} G \\ G \\ B \end{bmatrix}\begin{bmatrix} G \\ B \\ G \end{bmatrix}$ $\begin{bmatrix} B \\ G \\ G \end{bmatrix}$]	BR-S/V
	547	R1	Okay.	
	548	Stephanie	And I have three with two factors of *b* and one factor of *a* [*indicates* $\begin{bmatrix} G \\ B \\ B \end{bmatrix}\begin{bmatrix} B \\ G \\ B \end{bmatrix}\begin{bmatrix} B \\ B \\ G \end{bmatrix}$] so I guess it would be *a* cubed plus three *a* squared *b* plus three *ab* squared plus *b* cubed. [*inserts plus signs so that her paper now reads:* $a^3 + 3a^2b + 3ab^2 + b^3$]	BR-S/V
	549	R1	So how do you know there can't be a *c* in here?	
	550	Stephanie	Because I only have two colors.	BEJ
	551	R1	Oh.	
	552	Stephanie	If I had a third color there could be a *c*, but	BEJ
	553	R1	That's interesting. That's something to explore later. [*Stephanie writes* $(a + b)^3$ *before the expansion she has written previously.*] We could look into that. Okay so now could you tell me about another one of those binomials raised to a power?	
	554	Stephanie	*a* plus *b* to the fourth. [*writes* $(a + b)^4$]	

Time	Line	Speaker	Transcript	Code
	555	R1	Sure why not [*Stephanie writes:* $a^4 + 4a^3$] just think of all those students working out *a* plus *b* to the fourth out and there you're going on your piece of paper and doing it so fast.	
	556	Stephanie	Four *a* to third – oh. There's got to be more than that.	BR-S; BDI
	557	R1	Well, how many are there?	
	558	Stephanie	Um, well there's four with um um, three quantities of *a* and one quantity of *b*.	BEJ
	559	R1	Okay, write that down. [*After the* $a^4 + 4a^3$ *Stephanie has already written, Stephanie continues so it reads* $a^4 + 4a^3b + 4ab^3 +$]	BR-S
	560	Stephanie	And there's four with one quantity of *a* and three quantities of *b* but there's um there's other ones too.	OBS
	561	R1	Remember you have your chart. Use it.	
	562	Stephanie	There's six…..oh…..um…	OBS
	563	R1	Don't hesitate to use what you have.	
	564	Stephanie	Okay. The ones with two, I know those are like [*Stephanie takes* $\begin{bmatrix} G \\ B \\ G \\ B \end{bmatrix} \begin{bmatrix} B \\ G \\ G \\ B \end{bmatrix} \begin{bmatrix} B \\ B \\ G \\ G \end{bmatrix} \begin{bmatrix} G \\ B \\ B \\ G \end{bmatrix} \begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix} \begin{bmatrix} B \\ G \\ B \\ G \end{bmatrix}$] Oh um, all right. Here's my six. I have [*pause*] um [*pause*] I have to get six with – oh [*writes* $6a^2b^2 + b^4$] plus *b* to the fourth I guess that's it.	BR-S/V
	565	R2	Wait. I'm not sure what you wrote. Um, could you read this to me exactly as you like it? [*He gives Stephanie another piece of paper.*]	
	566	R1	Why don't you write it again the long way so you can write with the long paper you might have more space	
	567	Stephanie	Oh. You mean	
	568	R1	Yeah horizontally.	
	569	Stephanie	Do you want me to copy the whole thing all over or just *a* to the fourth *ab* to the	
	570	R1	Whatever helps (inaudible)	

Time	Line	Speaker	Transcript	Code
	571	Stephanie	Quantity *ab* to the fourth plus *b* to the fourth. [*writes* $(a + b)^4$]	BR-S
	572	R1	Well	
	573	R2	Sure why not? Now get it exactly so that it says what you mean.	
	574	Stephanie	*a* to the fourth [*writes* a^4] so you have all *a*. [indicates $\begin{bmatrix} G \\ G \\ G \\ G \end{bmatrix}$]	BR-S/V
	575	R2	Um hm.	
	576	Stephanie	Um plus four *a* cubed *b*. You have four with three *a*. [*indicates* $\begin{bmatrix} B \\ G \\ G \\ G \end{bmatrix}\begin{bmatrix} G \\ B \\ G \\ G \end{bmatrix}\begin{bmatrix} G \\ G \\ B \\ G \end{bmatrix}\begin{bmatrix} G \\ G \\ G \\ B \end{bmatrix}$]	BR-S/V
	577	R2	Um hm.	
40:00-44:59	578	Stephanie	You have four with three *a* and one *b* and you have - plus if you're following this the next one's the six. [*indicates towers four high with two blues and two greens*]	BR-S/V
	579	R2	Ah.	
	580	Stephanie	You have six with two *a* and two *b* [*writes* $6a^2b^2$] with two factors of *a* and two factors of *b* and then you have another one where it's four except it has three factors of um yeah three factors of *b* [*writes* $4ab^3$]and then you have one where it's just factors of …	BR-S/V
	581	R1	I can't see what you wrote. There's three factors of *b*. [*Stephanie moves the towers away*.]	
	582	Stephanie	And one where it's just um four factors of *b*. [*writes* $+ b^4$]	BR-S/V
	583	R1	Oh, but you wrote down one factor of *a*. Okay. I didn't hear that.	
	584	R2	Oh, okay, so you arranged this in a different order from this one.	
	585	Stephanie	Yeah, because I forgot the six	BEJ
	586	R2	Because	
	587	Stephanie	before [*Stephanie underlines the 6 in the 6 in the* $6a^2b^2$ *she had written the first time after the* $4a^3b + 4ab^3$] . I couldn't figure out the six	BEJ

Time	Line	Speaker	Transcript	Code
			right away.	
	588	R2	Oh, I see. This is the way you prefer to think about it.	
	589	Stephanie	Oh I mean that that goes in order	BEJ
	590	R2	Um hm.	
	591	Stephanie	so	
	592	R2	in increasing numbers (inaudible)	
	593	Stephanie	Yes	
	594	R2	Okay, good.	
	595	R1	Now I have a question. Um look at the, take the six for a minute.	
	596	Stephanie	Okay.	
	597	R1	If you had to write that as a combinatoric [*uses her finger to draw parentheses in the air*] right, where you said *a* represents green, *b* represents blue, right. That number six came about how? Which combinatoric is it? What is the number on top, the number on bottom? What is the number?	
	598	Stephanie	The number on top is four.	BR-S
	599	R1	The number on top is four. [*Stephanie writes* $C_.^4$]	
	600	Stephanie	The number on the bottom is um two.. [*writes* $C_2^4 = 3$] It equals three. Right? Yeah, because it's it's combining both it's it's backwards like both ways 'cause um with the six um two of each. I guess you could just – it's six [*corrects* $C_2^4 = 3$ *so that it reads* $C_2^4 = 6$] I guess you could just like that um yeah there's	BR-S
	601	R1	It's interesting if you're thinking about it in your head about the three	
	602	Stephanie	Yeah.	
	603	R1	and the two together. I found that interesting – that the early ways of thinking about it still stay with you when you start moving to towers. [*Stephanie has grouped the towers into this arrangement:* $\begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix} \begin{bmatrix} B \\ G \\ G \\ B \end{bmatrix} \begin{bmatrix} B \\ B \\ G \\ G \end{bmatrix}$ *and*	

Time	Line	Speaker	Transcript	Code
			$\begin{bmatrix} G \\ B \\ G \\ B \end{bmatrix} \begin{bmatrix} B \\ G \\ B \\ G \end{bmatrix} \begin{bmatrix} G \\ B \\ B \\ G \end{bmatrix}$]	
	604	Stephanie	Yeah that's	
	605	R1	Um hm.	
	606	R2	This is some something that goes back to the earlier interview	
	607	R1	Right. One of the things	
	608	R2	because I'm not sure that I followed it.	
	609	R1	One of the things that you've done over the years is you've you've found that staircase kind of idea to be a way of making one case and then you look at them separated you look at them together and separated	
	610	Stephanie	Um hm.	
	611	R1	and you sort of naturally go to that.	
	612	Stephanie	Yeah.	
	613	R1	Which I find interesting and um but just now you just said. Well let me think it all as six but you were originally thinking of it, I think, were you thinking of it as three and three?	
	614	Stephanie	Yeah, I was thinking of it as separated.	BEJ
	615	R1	Or maybe you weren't…I don't want to	
	616	R2	Are these the three and the three?	
	617	Stephanie	Yeah, 'cause those are all together.	BEJ
	618	R2	And those are separated.	
	619	Stephanie	Yeah.	
	620	R2	Okay. So this is where the greens are together, I guess, and this is where the greens are separated.	
	621	Stephanie	Greens are separated. Or you could just- [*rearranges towers to:* $\begin{bmatrix} B \\ B \\ G \\ G \end{bmatrix} \begin{bmatrix} G \\ B \\ B \\ G \end{bmatrix} \begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix} \begin{bmatrix} G \\ B \\ G \\ B \end{bmatrix} \begin{bmatrix} B \\ G \\ B \\ G \end{bmatrix} \begin{bmatrix} B \\ G \\ G \\ B \end{bmatrix}$]	
	622	R2	You can – oh – so they rearrange if you're	

Time	Line	Speaker	Transcript	Code
			looking	
	623	Stephanie	Doing blue.	
	624	R2	at blue instead.	
	625	Stephanie	It's just the opposite.	BEJ
	626	R2	Yeah, that's that's the part I didn't understand yet. Now I see. Thank you.	
	627	Stephanie	Um hm. Oops. [*Stephanie moves the towers back.*] There's – do you want me to do another one or	
	628	R1	Bob, what do you think? Do you want to see another one or do you want to go in a different direction?	
	629	R2	Oh, I'm sure Stephanie can produce the rest of the triangle.	
	630	R1	I think so too. I think she can too. I guess um what we're to explore from here. There's lots of ways. But I'm sort of interested in maybe exploring this way [*indicates with her finger moving horizontally across a row in the triangle*]. At least on one of the rows or so you want to look at that exploration?	
45:00-49:59	631	R2	Uh, I'd be delighted, uh, let me think about, let me think for a moment about how I'd start – what I'm curious about and maybe you can help me is um let's look at the at the towers that are three high where you have one, three, three, and one here in the different cases. [*indicated the fourth row of the triangle*] Um, now, uh, let's see, um, this is the case where there are no blues. [*points to the left one in the one, three, three, one row*]	
	632	Stephanie	Um hm.	
	633	R2	This is the case where there's one blue, okay? [*points to the left three in the one, three, three, one row*]	
	634	Stephanie	Yes.	
	635	R2	Now, what I'm interested in is reading this this row of numbers from the left to the right. How do we get from one number to the next?	
	636	Stephanie	Like, like	
	637	R2	I'm looking for a new idea.	
	638	Stephanie	Okay.	
	639	R2	Okay. In other words, suppose we know that – okay. Suppose we start with what we do know - that if there are towers three high with	

Time	Line	Speaker	Transcript	Code
			one green, there are exactly three of them. If I remember your explanation, it was because there were only three places where we can put that one green.	
	640	Stephanie	Yes.	
	641	R2	So we're very sure of that. Now suppose we wanted to start with that knowledge. In other words, not just that there were three towers but also remembering what the towers were.	
	642	Stephanie	Okay.	
	643	R2	Okay, and then um the next question is, okay, imagine one of those towers, okay?	
	644	Stephanie	Um hm.	
	645	R2	Um. The next number in this row, if we didn't know it, we know what it's supposed to count. It's supposed to count the towers with two greens. So now we've got a tower with one green.	
	646	Stephanie	Oh, okay.	
	647	R2	Okay. Now let's imagine trading	
	648	Stephanie	Switching this	
	649	R2	one of the blues for a new green.	
	650	Stephanie	Okay.	
	651	R2	Okay, how many different ways could we do that? How many new towers could we…	
	652	Stephanie	Well, I know there's three but like all you if you're saying these with one green. [*takes* $\begin{bmatrix} B \\ G \\ B \end{bmatrix}$]	BEJ
	653	R2	Yeah, take one of the towers with one green.	
	654	Stephanie	If I could picture this one okay, all I'd have to really do is picture it back like if instead of green have this be blue [*indicates the green cube*] and have these two be green [*indicates the blue cubes*]. Oh you want with two green.	BEJ
	655	R2	Yeah.	
	656	Stephanie	Oh.	
	657	R2	But starting with the one green and then taking one of the blues and trading it for a green.	
	658	Stephanie	All right let me start.	
	659	R2	How many different towers could we make that way?	

Time	Line	Speaker	Transcript	Code
	660	R1	From the one you have in your hand…just worry about the one you have in your hand.	
	661	Stephanie	All right, um.	
	662	R2	Just that one.	
	663	Stephanie	Well, I just couldn't I just imagine it the opposite like if I imagined it as um two green and one blue 'cause there's they all have an opposite one.	BEJ: BR-V
	664	R1	Okay, let's stop for a minute. I think I understand what the problem is. That one in your hand	
	665	Stephanie	Okay.	
	666	R1	has one green. That one green can't move.	
	667	Stephanie	Okay.	
	668	R1	'Cause you've picked it and you've picked this one with the one green. That one green can't move. Right?	
	669	Stephanie	Okay.	
	670	R1	But now you know what to change this tower to have two greens.	
	671	Stephanie	Okay.	
	672	R1	So obviously that one.	
	673	Stephanie	Well, you can put a green on the top or a green on the bottom.	BEJ
	674	R1	Okay.	
	675	R2	Good so there are two ways.	
	676	Stephanie	Yes.	
	677	R1	So there are two ways that you can change that one to have exactly two greens from a one green.	
	678	Stephanie	Yes.	
	679	R1	Okay, now is that the only one green tower that you can make two greens?	
	680	Stephanie	No.	
	681	R1	What are the others?	
	682	Stephanie	Um-	
	683	R1	You don't have to show me if you can tell me without showing me and then you can go to the towers.	
	684	Stephanie	Um, the one let me see. [*takes* $\begin{bmatrix} G \\ B \\ B \end{bmatrix}$]	
	685	R1	Okay, that's one.	

Time	Line	Speaker	Transcript	Code
	686	Stephanie	Yeah you cause you can make put a green here. [*indicates blue on the bottom*]	BR-V
	687	R1	Right, that'll make it two greens.	
	688	Stephanie	You can put it here too [*indicates blue in the middle*] but you already did that.	BR-V
	689	R1	Well, forget about that one for a minute.	
	690	R2	Don't worry about that.	
	691	Stephanie	Well, here you have two ways too. They all have two ways.	BEJ
	692	R2	Good.	
	693	R1	I agree with you that we've already done that. That that's good that you remember that's wonderful, but from that one you could do it two ways, right?	
	694	Stephanie	Yeah.	
	695	R2	So let's see where we are now.	
	696	Stephanie	Okay.	
	697	R2	From this one you made two new ones. [*indicates* $\begin{bmatrix} B \\ G \\ B \end{bmatrix}$]	
	698	Stephanie	Yes.	
	699	R2	And from this one you made also two new ones. [*indicates* $\begin{bmatrix} G \\ B \\ B \end{bmatrix}$] And you also noticed	
	700	R1	You have a	
	701	R2	that there is a duplicate.	
	702	R1	Very good.	
	703	R2	Okay. Good. Okay. This is all strong. Okay there's one tower left.	
	704	Stephanie	Okay.	
	705	R2	It's this one. [*takes* $\begin{bmatrix} B \\ B \\ G \end{bmatrix}$]	
	706	Stephanie	Yes.	
	707	R2	Um, how many ways can you?	
	708	Stephanie	Two.	BR-V
	709	R2	Two.	
	710	Stephanie	Um, hum but they're both duplicates.	BEJ
	711	R2	But they're both duplicates	

Time	Line	Speaker	Transcript	Code
	712	Stephanie	Yes.	
	713	R2	Okay. Very good. Okay.	
	714	R1	How's that?	
	715	Stephanie	Well if you put one on the top, you have this one with the one on the top, and one on the bottom. [*points to* $\begin{bmatrix} G \\ B \\ B \end{bmatrix}$] If you put one there, you have this one with the one there, and the one there. [*points to* $\begin{bmatrix} B \\ G \\ B \end{bmatrix}$] So that's just that doesn't really do anything.	BEJ
	716	R2	Okay. Good. Okay. So if so we built six. We imagined building six towers but we noticed that they came in pairs.	
	717	Stephanie	Yeah.	
	718	R2	Is that right?	
	719	Stephanie	Um hum.	
	720	R2	Okay, so um so what's the real number of towers?	
	721	Stephanie	Three.	BR-V
	722	R2	There's three because…	
	723	Stephanie	Because you can have one with two green.	BEJ
	724	R2	Um hm.	
	725	Stephanie	Here.	
	726	R2	Yes.	
	727	Stephanie	One with two green, one here, and one here [*lifts* $\begin{bmatrix} G \\ B \\ B \end{bmatrix}$]. One with one green here and one green here. [*lifts* $\begin{bmatrix} B \\ G \\ B \end{bmatrix}$]	BEJ
	728	R2	Okay.	
	729	Stephanie	That's it without having any like duplication.	
50:00-54:59	730	R2	Okay, so the duplicates seem to come up two at a time.	
	731	Stephanie	Um hm.	
	732	R2	Right?	

Time	Line	Speaker	Transcript	Code
	733	Stephanie	They come like	BEJ
	734	R2	Is that right?	
	735	Stephanie	Oh. I wouldn't like two at a time you mean like two from the same one or cause this one they're they're just	BEJ
	736	R1	You know what? Let's do it.	
	737	R2	Okay.	
	738	R1	Let's do it.	
	739	Stephanie	Oh you mean like let's build a tower.	
	740	R1	Let's start all over again cause this is too confusing for me. [*brushes aside all towers*]	
	741	R2	Yeah. This is getting lovely. Let's build it, okay?	
	742	R1	Let's do it.	
	743	R2	Okay. Good.	
	744	Stephanie	Start with this one. [*indicates* $\begin{bmatrix} B \\ G \\ B \end{bmatrix}$]	
	745	R2	Okay.	
	746	Stephanie	You can have one like that. [*takes* $\begin{bmatrix} B \\ G \\ G \end{bmatrix}$]	
	747	R1	No. Let's leave these alone. [*pushes away all towers except* $\begin{bmatrix} B \\ G \\ B \end{bmatrix}$] Let's start with the three with one	
	748	R2	Let's save all the originals.	
	749	R1	Let's leave all the originals by themselves and let's just start with the three with one.	
	750	Stephanie	Okay.	
	751	R1	Let's pull those aside and let's start all over again. [*Stephanie places* $\begin{bmatrix} G \\ B \\ B \end{bmatrix}$ *and* $\begin{bmatrix} B \\ B \\ G \end{bmatrix}$ *on the sides of* $\begin{bmatrix} B \\ G \\ B \end{bmatrix}$.] We could take apart (excuse us) okay. Now.	

Time	Line	Speaker	Transcript	Code
	752	Stephanie	From this one [*indicates* $\begin{bmatrix} G \\ B \\ B \end{bmatrix}$], you could have one like that [*indicates* $\begin{bmatrix} G \\ G \\ B \end{bmatrix}$]. Or one like that [*indicates* $\begin{bmatrix} G \\ B \\ G \end{bmatrix}$]. That's without moving the top one.	BR-V
	753	R1	Replacing…	
	754	R2	Replacing…	
	755	R1	each blue	
	756	R2	one of the blues	
	757	R1	one or the other ones?	
	758	R2	Good.	
	759	R1	I believe that…	
	760	R2	Now let's go on to the next one. [$\begin{bmatrix} B \\ G \\ B \end{bmatrix}$ *is the next one.*]	
	761	Stephanie	You can have one like that. [*indicates* $\begin{bmatrix} G \\ G \\ B \end{bmatrix}$]	BR-V
	762	R2	Um hm.	
	763	Stephanie	Or (inaudible) um (inaudible)…one like that. [*indicates* $\begin{bmatrix} B \\ G \\ G \end{bmatrix}$]	BR-V
	764	R2	Okay	
	765	Stephanie	And then the next one – you can either – one like that [*indicates* $\begin{bmatrix} G \\ B \\ G \end{bmatrix}$] or one like that	BR-V

Time	Line	Speaker	Transcript	Code
			[*indicates* $\begin{bmatrix} B \\ G \\ G \end{bmatrix}$]	
	766	R2	Okay.	
	767	Stephanie	and that's it.	
	768	R2	Okay, now which are duplicates in this row? You know in this new row that you constructed?	
	769	Stephanie	These two, [$\begin{bmatrix} G \\ B \\ G \end{bmatrix}\begin{bmatrix} G \\ B \\ G \end{bmatrix}$] these two, [$\begin{bmatrix} G \\ G \\ B \end{bmatrix}\begin{bmatrix} G \\ G \\ B \end{bmatrix}$] and these two [$\begin{bmatrix} B \\ G \\ G \end{bmatrix}\begin{bmatrix} B \\ G \\ G \end{bmatrix}$].	BR-V
	770	R2	Aha, so they do come in pairs.	
	771	Stephanie	Yes, oh okay like that.	
	772	R2	Yeah, that's what we meant.	
	773	Stephanie	Okay.	
	774	R2	I think that's what Carolyn meant.	
	775	R1	Right.	
	776	R2	Right. Okay, now let's put them back with the parents. It's okay to call these the parents	
	777	Stephanie	Yeah.	
	778	R2	and the new ones the children?	
	779	R1	Different kind of parents. I'm getting very mixed up.	
	780	R2	Oh, I'm sorry. Okay, would you call them step one and step two or something like that. But um…	
	781	Stephanie	Um. [*Stephanie replaces duplicates. The towers are now arranged:* $\begin{bmatrix} G \\ B \\ B \end{bmatrix}$ $\begin{bmatrix} B \\ G \\ B \end{bmatrix}$ $\begin{bmatrix} B \\ B \\ G \end{bmatrix}$	BR-V

Time	Line	Speaker	Transcript	Code
			$\begin{bmatrix} G \\ G \\ B \end{bmatrix}$ $\begin{bmatrix} G \\ B \\ G \end{bmatrix}$ $\begin{bmatrix} G \\ G \\ B \end{bmatrix}$ $\begin{bmatrix} B \\ G \\ G \end{bmatrix}$ $\begin{bmatrix} G \\ B \\ G \end{bmatrix}$ $\begin{bmatrix} B \\ G \\ G \end{bmatrix}$	
	782	R2	Let me see. Good, okay so…	
	783	R1	Our next question is okay so you predicted that when you went to exactly three with one green - is that right?	
	784	Stephanie	Um, hm.	
	785	R1	When you were replacing a blue with a green with two greens you ended up with three. That's what you told me.	
	786	Stephanie	Um, hm.	
	787	R1	You ended up with six.	
	788	Stephanie	But…	
	789	R1	But you had only half	
	790	Stephanie	Yeah.	
	791	R1	because each one had a duplicate. That's what you just told me. Is that right? I guess there's two ways. I'm curious to know if that also works with four.	
	792	Stephanie	Probably.	
	793	R1	Or is this one case?	
	794	Stephanie	Yeah.	
	795	R1	Would it work the same way? What would you predict with if you had towers four high and you had exactly one green and now you want to replace a blue?	
	796	Stephanie	Should I show you?	
	797	R1	Well, what do you think is going to happen before you show me and why. Then show me.	
	798	Stephanie	Well, if four high with one green is um, can I check them to see how many there are so I can…	PAH
	799	R1	How many are there?	
	800	Stephanie	Well, if for four high, for one green, there's four	BR-V
	801	R1	Okay.	
	802	Stephanie	then you'd probably get eight.	BR-V

Time	Line	Speaker	Transcript	Code
	803	R1	So you think there'd be eight?	
	804	Stephanie	Yeah, you'd probably get eight.	
	805	R1	And what about duplicates?	
	806	Stephanie	Well, there they'd be in pairs. It would come out to four.	BEJ
	807	R1	So what would you have to do if they came out in pairs?	
	808	Stephanie	You'd just have to use one like…	BEJ
	809	R1	All right, so you'd have to take half of them then.	
	810	Stephanie	Yeah.	
	811	R1	All right. So you'd have to take eight divided by two, I guess.	
	812	Stephanie	Um hm.	
	813	R1	To undo those pairs that's a way of thinking about it or subtract out four, divide by two… (inaudible).	
	814	Stephanie	Um hm.	
	815	R2	Could you build this?	
	816	Stephanie	Yeah.	
	817	R2	Let's start with the ones that have one green.	
	818	R1	Let let me give you the ones with one green so we… [*the interviewer and Stephanie pull out of the stack towers four high with one green:* $\begin{bmatrix} B \\ B \\ G \\ B \end{bmatrix} \begin{bmatrix} B \\ G \\ B \\ B \end{bmatrix} \begin{bmatrix} B \\ B \\ G \\ B \end{bmatrix} \begin{bmatrix} B \\ B \\ B \\ G \end{bmatrix}$]	
	819	R2	You're looking for four towers with one green (inaudible).	
	820	Stephanie	Wait.	
55:00-55:38	821	R2	We have a duplicate. Aha.	
	822	Stephanie	Let me see [*pulls aside* $\begin{bmatrix} B \\ B \\ G \\ B \end{bmatrix}$].	
	823	R2	Which one are we missing?	

Time	Line	Speaker	Transcript	Code
	824	Stephanie	The one with the one on top. [*R2 places* $\begin{bmatrix} B \\ B \\ B \\ G \end{bmatrix}$ *along side.*] No, we already had that one.	BR-V
	825	R1	How about one [*indicates the pile*]?	
	826	Stephanie	Here it is. [*takes* $\begin{bmatrix} G \\ B \\ B \\ B \end{bmatrix}$]	BR-V
	827	R2	How many can you build from that?	
CD 2: 00:00-04:59	828	Stephanie	I can have one like that. [$\begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix}$] I can have one like that. [$\begin{bmatrix} G \\ B \\ G \\ B \end{bmatrix}$] And I can have one like that. [$\begin{bmatrix} G \\ B \\ B \\ G \end{bmatrix}$] Is that it?	BR-V
	829	R2	Well, how would you tell?	
	830	Stephanie	Um. All right. Well, I moved it down one, I moved it down, yeah, yeah, that's it.	BEJ
	831	R2	Okay.	
	832	R1	So from that one you got how many now?	
	833	Stephanie	Three.	BR-V
	834	R1	So when you did it for three high.	
	835	Stephanie	You got two for each.	BEJ
	836	R1	I wonder why you get two of them?	
	837	Stephanie	I don't know. Maybe cause it's bigger.	BEJ; OBS
	838	R1	What would that have to do with it?	
	839	Stephanie	I don't… cause you have more room to build on.	BEJ; OBS
	840	R1	Tell me can you explain to me?	
	841	Stephanie	Oh, well maybe it's because like you have	BEJ

Time	Line	Speaker	Transcript	Code
			you already have one that's taking up space so you only have three places to move it. [*indicates* $\begin{bmatrix} G \\ B \\ B \\ B \end{bmatrix}$]	
	842	R1	Oh, three places.	
	843	R2	Three places.	
	844	Stephanie	Where before you had one that's taking up space and you only had two spaces to move it. [*indicates* $\begin{bmatrix} G \\ B \\ G \end{bmatrix}$]	BEJ
	845	R1	I got you okay.	
	846	R2	Okay.	
	847	R1	So what would you predict if you were building towers five high.	
	848	Stephanie	You'd have four.	BEJ
	849	R1	And you went from one to two. You would have four places to move it.	
	850	Stephanie	You'd have four.	BEJ
	851	R2	Good.	
	852	Stephanie	Um hm.	
	853	R1	I see that. It's neat.	
	854	Stephanie	All right. So for this one… [indicates $\begin{bmatrix} B \\ G \\ B \\ B \end{bmatrix}$]	BR-V
	855	R1	You gonna get duplicates, you think?	
	856	Stephanie	Um hm.	
	857	R1	Let's think about this too because…	
	858	Stephanie	Um, this one [$\begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix}$] that's already a duplicate um now this one [$\begin{bmatrix} B \\ G \\ G \\ B \end{bmatrix}$] that's a duplicate or	BR-V

Time	Line	Speaker	Transcript	Code
			you could have…	
	859	R2	Which one is a duplicate?	
	860	Stephanie	Oh, no it doesn't, forget it.	
	861	R2	Okay.	
	862	Stephanie	It doesn't.	
	863	R2	So it's new?	
	864	Stephanie	Yeah, it's new.	BR-V
	865	R2	Okay. So, we found one duplicate and another one that's new.	
	866	R2	And again a three.	
	867	Stephanie	Yeah, but this one's a duplicate. [$\begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix}$]	BR-V
	868	R2	Okay.	
	869	Stephanie	Some yeah, except for that one [*lifts* $\begin{bmatrix} B \\ B \\ G \\ B \end{bmatrix}$] the next one…could I have that blue one please?	BEJ; BR-V
	870	R1	Yes, sure. [*hands Stephanie a long stack of blue Unifix cubes*]	
	871	Stephanie	Thank you. Two like that [$\begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix}$] or two like that [$\begin{bmatrix} B \\ G \\ G \\ B \end{bmatrix}$]…like that [$\begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix}$] and this one's a duplicate [$\begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix}$] and this one is [$\begin{bmatrix} B \\ G \\ G \\ B \end{bmatrix}$] next one all three of them will probably be duplicates.	BR-V

Time	Line	Speaker	Transcript	Code
	872	R1	Okay. Let's [*lifts* $\begin{bmatrix}B\\B\\B\\G\end{bmatrix}$]	
	873	Stephanie	Oh. Wait. [*R2 pushes the* $\begin{bmatrix}B\\B\\B\\G\end{bmatrix}$ *more into line.*] Let me just (inaudible) all right um [*places* $\begin{bmatrix}B\\B\\G\\G\end{bmatrix}\begin{bmatrix}B\\G\\B\\G\end{bmatrix}\begin{bmatrix}G\\B\\G\\B\end{bmatrix}$] like that and…	BR-V
	874	R1	How many do we have? Three, six, nine, twelve?	
	875	Stephanie	Um hm, yeah, um only um, how many? There's twelve. All right, um, this one's fine. [*Stephanie indicates the* $\begin{bmatrix}G\\G\\B\\B\end{bmatrix}$ *tower.*] One, two, alright there's three of this one.	BEJ; BR-V
	876	R1	So we have three duplicates.	
	877	Stephanie	Yes.	
	878	R1	Three of the same.	
	879	Stephanie	Um hm.	
	880	R1	Okay. Wait, let's put that back, let me go through this again.	
	881	Stephanie	Okay.	
	882	R1	It's kind of fast for me. Um. Can we put those	
	883	R2	It might be helpful if we put the whole row back together	
	884	R1	Yeah, let's put the whole	
	885	R2	and started from the beginning.	
	886	Stephanie	Okay.	
05:00–09:59	887	R2	And then let's be, um – let's see if we're	
	888	Stephanie	Oops. [*Stephanie knocks over a tower.*]	
	889	R2	quite sure	

Time	Line	Speaker	Transcript	Code
	890	R1	Let, let's go back. Here you got this one [*indicates the* $\begin{bmatrix} G \\ B \\ B \\ B \end{bmatrix}$ *tower*] because you replaced this [*indicates the second position from the top in the* $\begin{bmatrix} G \\ B \\ B \\ B \end{bmatrix}$ *tower*] with green?	
	891	Stephanie	Yeah. Um, I got that one from	BEJ
	892	R1	This one, making this one [*indicates the second position from the top in the tower again*] a green.	
	893	Stephanie	Yeah.	
	894	R1	Okay. And – or you made, um, the bottom one a green – I'm confused here - or the middle one a green.	
	895	Stephanie	I can't move this one. [*indicates the top green in the* $\begin{bmatrix} G \\ B \\ B \\ B \end{bmatrix}$ *tower*]	BEJ
	896	R1	Right.	
	897	Stephanie	I can move – I can put one under it, I can put one	BEJ
	898	R1	Oh that helps.	
	899	Stephanie	separated by one	BEJ
	900	R1	Oh, I like that.	
	901	R2	That does	
	902	R1	And one on the bottom.	
	903	R2	Okay.	
	904	R1	Oh, that helps me if you organize it that way. [*R1 is referring to Stephanie's set up of the towers in this manner:* $\begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix}\begin{bmatrix} G \\ B \\ G \\ B \end{bmatrix}\begin{bmatrix} G \\ B \\ B \\ G \end{bmatrix}$.]	
	905	R2	Okay. Now how did you	
	906	R1	Okay. I follow you.	

Time	Line	Speaker	Transcript	Code
	907	R2	organize these three? [*He refers to the three from the* $\begin{bmatrix} B \\ G \\ B \\ B \end{bmatrix}$ *tower.*]	
	908	Stephanie	Well…	
	909	R1	Now what about this? Help me with this.	
	910	Stephanie	I can go like that. [*Stephanie changes the blue on the top of the tower to green, forming the* $\begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix}$ *tower.*]	BEJ; BR-V
	911	R1	On the top.	
	912	R2	So the tops are the same.	
	913	Stephanie	Well-	
	914	R1	The second one is the same and the tops …	
	915	R2	Oh, the second one is the same.	
	916	R1	But now you made the top different.	
	917	Stephanie	Yeah, the second one is the same.	
	918	R2	Oh. I see.	
	919	R1	Here you	
	920	Stephanie	Or you want me to put it like that?	PAH
	921	R1	Oh, but that helps me. And you made underneath it different. Then you made	
	922	Stephanie	Um hm.	
	923	R2	That way	
	924	Stephanie	I separated by one-	BEJ; BR-V
	925	R1	On the bottom different. I follow that.	
	926	R2	Okay. Now what happens with these? [*He refers to the three from the* $\begin{bmatrix} B \\ B \\ G \\ B \end{bmatrix}$ *tower.*]	
	927	Stephanie	Mm, you can have	
	928	R1	This is the one that's the same. [*She points to*	

Time	Line	Speaker	Transcript	Code
			the green block in the $\begin{bmatrix} B \\ B \\ G \\ B \end{bmatrix}$ *tower.*] I don't see it the same on all of them.	
	929	Stephanie	What is this. This doesn't go there. [*Stephanie says this quietly, as if to herself. She points to the* $\begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix}$ *tower.*]	BEJ; BR-V
	930	R2	Ahhh.	
	931	R1	Ah. Okay, but let's see what does go there.	
	932	R2	Okay. – Yeah. What should?	
	933	Stephanie	This goes here. [*The* $\begin{bmatrix} B \\ B \\ G \\ G \end{bmatrix}$ *tower.*]	BEJ; BR-V
	934	R1	That's the bottom.	
	935	Stephanie	This goes here. [*The* $\begin{bmatrix} B \\ G \\ G \\ B \end{bmatrix}$ *tower.*]	BEJ; BR-V
	936	R1	That's when you put it here. [*She points to the green in the second from the top position.*] Where should the next one go?	
	937	Stephanie	You know what? I think this one [*The* $\begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix}$ *tower*] goes here and um…	BEJ; BR-V
	938	R2	Wait.	
	939	Stephanie	Maybe this one goes there.	
	940	R1	Wait, wait, wait, but we needed that one there.	
	941	R2	Wait, wait, wait, wait, wait, …	
	942	R1	We checked these, I thought. Right? Stephanie this is	
	943	Stephanie	Yeah, I, yeah.	

Time	Line	Speaker	Transcript	Code
	944	R1	on the top, this is here, and this is on the bottom. Maybe this one is, let's fix this one.	
	945	Stephanie	No, I think I just might have messed them up when I moved them.	
	946	R1	Okay, but…	
	947	R2	But wait, let's, let's check them through.	
	948	R1	But, what should it be? Okay, if this is the one that's one up [*she points to the* $\begin{bmatrix} B \\ G \\ G \\ B \end{bmatrix}$ *tower*] and one down [*she points to the* $\begin{bmatrix} B \\ B \\ G \\ B \end{bmatrix}$ *tower*], what should this one look like?	
	949	Stephanie	It should be one like this. Um [*Stephanie builds it.*]	BEJ; BR-V
	950	R1	We're going to check them all, so…	
	951	Stephanie	[*Stephanie continues to build.*]	
	952	R1	Okay. Let's double check that.	
	953	Stephanie	Like that. [*Stephanie has built the* $\begin{bmatrix} G \\ B \\ G \\ B \end{bmatrix}$ *tower.*]	BR-V
	954	R1	It's the top.	
	955	Stephanie	Or do you want me to put it there? Wherever you want me to put it.	
	956	R2	Let me see if I can understand. [*Stephanie is engaged with R1 now and does not acknowledge this comment.*]	
	957	Stephanie	You like it down there?	
	958	R1	I like it the other way.	
	959	Stephanie	Okay, there. [*Stephanie and Dr. Maher laugh together.*]	
	960	R2	Okay, so – I'm looking at the new – okay, this row is the green that was there first.	
	961	Stephanie	Yes.	
	962	R2	Right? And then the new green goes into the different places, and you're starting at the top	

Time	Line	Speaker	Transcript	Code
			and then working your way down	
	963	Stephanie	Um hm.	
	964	R2	through three places. Where the blues were on the other one. Does that look right to you?	
	965	Stephanie	Yes.	
	966	R2	Okay. Let's go on to the next one. [*R2 refers to the three from the* $\begin{bmatrix} B \\ B \\ B \\ G \end{bmatrix}$ *tower.*]	
	967	R1	How do we check this?	
	968	R2	We've checked it all the way to here. How would you…	
	969	Stephanie	Um. You have…	
	970	R1	What's, what's the same about all of these?	
	971	Stephanie	Well, they have one on the bottom.	
	972	R1	One on the bottom. Okay.	
	973	Stephanie	And, um, I guess you probably want them like that.	
	974	R2	Okay.	
	975	Stephanie	Oops. [*Stephanie knocks over some towers.*]	
	976	R2	So, you're following exactly the same organization.	
	977	R1	Green on top	
	978	R2	Every single case.	
	979	R1	green next, green next.	
	980	R2	Good.	
	981	R1	Okay. I see that. So one of them wasn't right.	
	982	Stephanie	Um hm.	
	983	R1	Okay. Now we do still have twelve, though.	
	984	Stephanie	Yes.	
	985	R1	And the question's we're looking for duplicates.	
	986	Stephanie	Okay.	
	987	R1	So let's do that.	
	988	Stephanie	This is one. And this goes with this. And, um, okay there was two.	BEJ; BR-V
	989	R2	(inaudible)	
	990	Stephanie	Yeah, that's it. So, one of those is a duplicate. So, I'm gonna take one out?	BR-V
	991	R1	No, let's just leave it organized.	
	992	R2	No, leave the two together.	

Time	Line	Speaker	Transcript	Code
	993	R1	Let's leave them together.	
	994	R2	Let's just keep the organization.	
	995	R1	Let's find them all. Let's find all the …	
	996	R2	Are there others that are in duplicate pairs?	
	997	Stephanie	Yeah.	
	998	R2	Or maybe, more than two, or whatever? If you could just put like with like.	
	999	Stephanie	Yeah. [*Stephanie works.*]	
	1000	R2	Yeah.	
	1001	R1	Stephanie, we've never done this before. So, we're, we're doing it with these cubes for the first time ourselves.	
	1002	R2	That's true.	
	1003	Stephanie	…and that goes with that… [*Stephanie continues to work, talking to herself under her breath.*]	BR-V
	1004	R2	Okay. So they all come in pairs.	
	1005	Stephanie	Yeah.	
	1006	R2	Okay.	
	1007	R1	So we found six.	
	1008	Stephanie	Yes.	
	1009	R1	We found six.	
	1010	Stephanie	Um hm.	
	1011	R1	Right? We were building them four tall – that when we moved – let me try to understand. When we moved from one green exactly. Right? To two green exactly. Right? From each of the three positions you got two.	
	1012	Stephanie	Um hm.	
	1013	R1	But you found out you had two when you were all done.	
	1014	Stephanie	Yeah.	
	1015	R1	You got a pair, so it wasn't twelve again you had this…	
	1016	Stephanie	It was six.	
	1017	R2	It was six.	
	1018	R1	It was six. It was twice the three, right?	
	1019	Stephanie	Um hm.	
	1020	R2	Right.	
	1021	Stephanie	Yeah.	
	1022	R1	(inaudible) in an interesting way.	
	1023	R2	I have a question now.	
	1024	Stephanie	Yeah.	
	1025	R2	Okay. If these are the old ones, and these are	

Time	Line	Speaker	Transcript	Code
			the new ones, then we had a method for building new ones from the old ones.	
	1026	Stephanie	Um hm.	
	1027	R2	And then when we went from the old ones to the new ones we found that we got twelve, but they weren't all different.	
	1028	Stephanie	Um hm.	
	1029	R2	They came in…	
	1030	Stephanie	Pairs.	BR-V
10:00-14:59	1031	R2	They came in pairs. Okay. I'm interested in why they came in pairs, instead of triplets or quadruplets, or whatever. And, uh, so let me ask you a question maybe that would point the other way. Okay. Here's a tower that's four high with two greens. [*He picks up the* $\begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix}$ *tower.*]	
	1032	Stephanie	Um hm.	
	1033	R2	Let's suppose it's a new one.	
	1034	Stephanie	Okay.	
	1035	R2	Where did it come from?	
	1036	Stephanie	Umm.	
	1037	R1	That second green you mean?	
	1038	R2	In other words	
	1039	R1	Where did the second green come from?	
	1040	R2	Yeah. No, no, no	
	1041	Stephanie	Well, it could come from	
	1042	R1	I'm not so sure I understand the question.	
	1043	R2	I'm just seeing two greens here, and I don't know which is first and which is second. Right?	
	1044	Stephanie	Okay.	
	1045	R2	But, let's just say it came up in this process and it was one of the new ones.	
	1046	Stephanie	Oh, you mean, where would I put it?	PAH
	1047	R2	No. Where did it come from?	
	1048	Stephanie	Like, which one of these? [*Stephanie points to the towers with one green and three blue.*]	PAH
	1049	R2	Yeah.	
	1050	Stephanie	It came from one of these. [*She points to*	BEJ

Time	Line	Speaker	Transcript	Code
			$\begin{bmatrix} G \\ B \\ B \\ B \end{bmatrix}$ *and* $\begin{bmatrix} B \\ G \\ B \\ B \end{bmatrix}$.]	
	1051	R2	Which one?	
	1052	Stephanie	It probably came from both. You probably had two, like if you had two. I don't know which one exactly it came from.	BEJ
	1053	R2	Why two and not three? Why two ancestors instead of three?	
	1054	Stephanie	Because, there's two of them with, um, with the, um, that can move, that like that can have a green on the bottom of it and on the top, to get that. Like I can have, I have one with a green on top, so I can put a green under it.	BEJ
	1055	R2	Um hm.	
	1056	Stephanie	I have one with a green on the bottom, so I can put a green on top.	BEJ
	1057	R2	Okay. Your, your fingers were, uh, focusing on these two positions in the tower. [*He points to the two green in the* $\begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix}$ *tower.*]	
	1058	Stephanie	Yeah.	
	1059	R2	Why did you focus on those two in the old towers, to relate to this new tower?	
	1060	Stephanie	Because, um, oh those were the two positions where the greens were.	BEJ
	1061	R2	Excellent. Thank you. I understand you. Okay. So it seems like, if you took, okay, so let's take another one. Let's take one of these two. [*The duplicate* $\begin{bmatrix} B \\ G \\ B \\ G \end{bmatrix}$ *towers.*] Which one, which ancestors could it have had? Which of the old ones could have produced it?	

Time	Line	Speaker	Transcript	Code
	1062	Stephanie	Ummm – this one [*the* $\begin{bmatrix} B \\ G \\ B \\ B \end{bmatrix}$ *tower*]	BEJ
	1063	R2	Okay.	
	1064	Stephanie	or this one. [*the* $\begin{bmatrix} B \\ B \\ B \\ G \end{bmatrix}$ *tower*]	BEJ
	1065	R2	This one.	
	1066	Stephanie	Because this one has the green there and that one has the green there.	BEJ
	1067	R2	Excellent.	
	1068	Stephanie	So, these were the two places.	BEJ
	1069	R2	Now, can you tell me why they came in pairs, instead of, say triples?	
	1070	Stephanie	Because, there are two like, I guess, parents that have a green in that position.	BEJ
	1071	R2	And why two?	
	1072	Stephanie	Because, I guess, maybe before that, I don't, because they came from, I don't know, just	OBS
	1073	R2	Well, remember when we were looking at this one? [*the* $\begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix}$ *tower*]	
	1074	Stephanie	Yeah.	
	1075	R2	Your fingers were touching the two greens.	
	1076	Stephanie	Well, because there were probably two before them that had two in that position.	
	1077	R2	Okay, but it was	
	1078	Stephanie	Er, um.	
	1079	R2	Well, let's imagine it this way.	
	1080	Stephanie	Okay.	
	1081	R2	How does, how does a new one come from, how do you get the old one starting from a new one? Um. Let's say this is the new one.	

Time	Line	Speaker	Transcript	Code
			[*the* $\begin{bmatrix} B \\ B \\ G \\ G \end{bmatrix}$ *tower*]	
	1082	Stephanie	Okay.	
	1083	R2	How do we find, how would we actually take the blocks and then, make one of the old ones that it came from?	
	1084	Stephanie	Well, if you're using green, um, you would have to take away either this one or this one, and make it a blue.	BEJ
	1085	R2	Ah, so the, the old ones depend on the, that there are two greens	
	1086	Stephanie	Yes.	
	1087	R2	over here. Okay. Very fine. Okay. Should we try the big step?	
	1088	R1	Why not? Are you very tired?	
	1089	Stephanie	Umm.	
	1090	R2	How are you doing?	
	1091	Stephanie	I'm fine.	
	1092	R2	Okay. How are you doing? Okay? So, now, look what we've gone from.	
	1093	R1	So we just woke – um – Professor Spieser up. Notice how animated he is. [*All laugh.*]	
	1094	R2	Yeah. I'm starting to get lively. I'm very excited by the way you're moving through this. Um, so we started, let's remember what we did. We started with the towers that had one green.	
	1095	Stephanie	Yeah.	
	1096	R2	And then by putting in a second green, we built the towers that had two greens. So, they came up in pairs.	
	1097	Stephanie	Yes.	
	1098	R2	That's the way it happened. That's what we found.	
	1099	Stephanie	Um hm.	
	1100	R2	Okay. So now we've got, but now I think we've got a hold on, the six different towers with the two greens. Or the twelve when they came up in pairs. Okay. So now let's take those six. Okay, so, let's forget that, we can use these duplicates as extra blocks if we find	

Time	Line	Speaker	Transcript	Code
			we need them.	
	1101	Stephanie	Okay.	
	1102	R2	Okay. Now, these are the old ones [*the six towers with two green and two blue*], and we're gonna build a generation of new ones that have three green.	
	1103	Stephanie	Okay.	
	1104	R2	Okay. And we're expecting, maybe, that something interesting will happen. Okay, so first of all, how many do we produce?	
	1105	Stephanie	Um, with three greens?	PAH
	1106	R2	Um hm. Well, how many would you produce from each of the towers first and	
	1107	Stephanie	Well, you'd produce two from each.	BR-V
	1108	R2	then how much from…two from each! Why?	
	1109	Stephanie	Well, because there's only two places for you to move. [*Stephanie points to the blue blocks in the* $\begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix}$ *tower.*]	BEJ
	1110	R2	Excellent.	
	1111	Stephanie	You can't	
	1112	R2	And you're looking, you're pointing at the blues.	
	1113	Stephanie	Yeah. You can only put a green here or here for that one.	BEJ
	1114	R2	So, how many new towers would that produce?	
	1115	Stephanie	Well, you'd get two from each.	BEJ
	1116	R2	Two from each. So…	
	1117	Stephanie	So. And we have six, so you'd get twelve.	BEJ
15:00-19:59	1118	R2	So, we'd get twelve. Okay. I'm correcting you. [*R2 moves two towers in line with the others.*] Right. But, yeah. Looks like twelve to me.	
	1119	Stephanie	Um hm.	
	1120	R2	Good. Uh. Now the question is, um, are they all, are these twelve new towers all different?	
	1121	Stephanie	I don't think so. But, um	
	1122	R2	Okay. What do you think?	
	1123	Stephanie	Um. There'll probably be pairs again.	BEJ
	1124	R2	You think there will be pairs again. Okay.	

Time	Line	Speaker	Transcript	Code
	1125	Stephanie	Uh maybe. ‘Cause I mean – [*Stephanie sighs heavily.*]	
	1126	R2	Okay. So here’s a tower with three greens. [*the* $\begin{bmatrix} G \\ G \\ B \\ G \end{bmatrix}$ *tower*] So, it’s one of the new ones. Right?	
	1127	Stephanie	Um hm.	
	1128	R2	Which would be the old ones that gave it?	
	1129	Stephanie	Um, this one [*the* $\begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix}$ *tower*]	BEJ; BR-V
	1130	R2	Um hm.	
	1131	Stephanie	or, this one [*the* $\begin{bmatrix} B \\ G \\ B \\ G \end{bmatrix}$ *tower*]	BEJ; BR-V
	1132	R2	Um hm.	
	1133	Stephanie	or this one. [*the* $\begin{bmatrix} G \\ B \\ B \\ G \end{bmatrix}$ *tower*]	BEV; BR-V
	1134	R2	You found three.	
	1135	Stephanie	Oh. Okay. So, maybe they’ll be, um, groups of three?	BDI
	1136	R2	So, you think they might be in groups of three.	
	1137	R1	Okay. Now explain to me how that happened.	
	1138	Stephanie	Mmm, because, here I could either, I could have one here [*points to* $\begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix}$]. Which would	BEJ

Time	Line	Speaker	Transcript	Code
			make this one [*the* $\begin{bmatrix} G \\ G \\ B \\ G \end{bmatrix}$ *tower*]. I could have here, I could have one here, another green here [*points to* $\begin{bmatrix} G \\ B \\ B \\ G \end{bmatrix}$], which would make this one. And here, I could put another green here [points to $\begin{bmatrix} B \\ G \\ B \\ G \end{bmatrix}$], which would make this one. So, there's three of them that can make that one [*the* $\begin{bmatrix} G \\ G \\ B \\ G \end{bmatrix}$ *tower*]. [*pause*] So, um, I guess there'll be groups of three, maybe?	
	1139	R2	Ah. You're guessing.	
	1140	Stephanie	So, I guess	
	1141	R2	Would you like to	
	1142	Stephanie	it would come up, the answer would be like four. There would be four without duplicates, if there's groups of three. 'Cause	BEJ
	1143	R2	Four, ah.	
	1144	Stephanie	you'll come up with twelve, and then, if there's groups of three, you'll get four. 'Cause four divided by three is twelve.	BEJ
	1145	R1	In triples?	
	1146	R2	So we're multiplying by two and dividing by three.	
	1147	Stephanie	Yes.	
	1148	R2	And, you're saying there, there are four. Well, we started with four and then we found six, and then we found four. Okay.	
	1149	Stephanie	There's only, if you wanted, (inaudible)	
	1150	R2	Okay. So now we've got- Okay, let's. Do we have the four?	

Time	Line	Speaker	Transcript	Code
	1151	Stephanie	Um, the four would be	
	1152	R2	Can you build them?	
	1153	Stephanie	(inaudible) over here	
	1154	R2	Have we got them already? Um.	
	1155	Stephanie	With three green. Here's one. Um. [*She picks up the* $\begin{bmatrix} G \\ G \\ B \\ G \end{bmatrix}$ *tower.*]	BEJ; BR-V
	1156	R2	You're looking for three greens?	
	1157	Stephanie	Yeah.	
	1158	R2	Here's one. [*The* $\begin{bmatrix} B \\ G \\ G \\ G \end{bmatrix}$ *tower*]	
	1159	Stephanie	Thank you.	
	1160	R2	Here's another. [*The* $\begin{bmatrix} G \\ B \\ G \\ G \end{bmatrix}$ *tower*]	
	1161	Stephanie	That one we already have.	
	1162	R2	Are they all different?	
	1163	Stephanie	No, I'll just make one.	
	1164	R2	Oh. Okay.	
	1165	Stephanie	There. [*Stephanie has made the* $\begin{bmatrix} G \\ G \\ G \\ B \end{bmatrix}$ *tower.*]	
	1166	R2	Okay.	
	1167	Stephanie	There's your four. Now	BEJ; BR-V
	1168	R2	Okay. So suppose we play another round.	
	1169	Stephanie	There's only one way to do that one.	BEJ
	1170	R1	But, suppose we're getting it from there [*The row of towers four tall containing three greens and one blue*] though. We know the answer	
	1171	R2	But, we're gonna start with these old ones.	
	1172	R1	We're making it from there.	
	1173	Stephanie	Well, there's only	

Time	Line	Speaker	Transcript	Code
	1174	R2	How do we produce it?	
	1175	Stephanie	You're only gonna get one from each, because there's only one place you can put the green.	BEJ
	1176	R2	Green.	
	1177	Stephanie	And you're gonna get four all green, and so you're going to come up with one 'cause	BEJ
	1178	R2	Oh.	
	1179	Stephanie	they're all the same.	BDI
	1180	R1	But, you got really four.	
	1181	Stephanie	Well, yeah.	
	1182	R1	How many duplicates did you get?	
	1183	Stephanie	You got four dup - well	
	1184	R1	So you have four	
	1185	Stephanie	I guess you	
	1186	R1	but you have four duplicates	
	1187	Stephanie	Yeah.	
	1188	R1	So you have to, you got four times and you have to undo the duplicates, what do you do to undo the duplicates?	
	1189	Stephanie	You get rid of 'em?	
	1190	R1	By dividing by what?	
	1191	Stephanie	Oh, by four.	
	1192	R1	By four.	
	1193	R2	Terrific.	
	1194	R1	You divided, you got rid of the duplicates the last time by dividing by three. You got rid of the duplicates the last time by dividing by two. Isn't that right?	
	1195	Stephanie	Yeah. Okay. So that would give you the, um, all green ones.	BEJ
	1196	R2	Okay. So let's just review a little bit.	
	1197	R1	Maybe we should write down what we say.	
	1198	R2	In each round we multiplied by one number	
	1199	Stephanie	Okay.	
	1200	R2	and we divided by another number.	
	1201	Stephanie	Yes.	
	1202	R2	Okay. So why don't we write, just to remember what happened, why don't we write down at each stage the number that we multiplied by and the number we divided by.	
	1203	Stephanie	All right. For – should I start with like from here and work backwards?	PAH
	1204	R2	If that's easiest for you, that's fine.	

Time	Line	Speaker	Transcript	Code
	1205	Stephanie	All right.	
	1206	R2	Okay. As long as we remember exactly what, you know, which round of the, the process we were in.	
	1207	Stephanie	From, I guess from, oops [*Stephanie knocks over a tower*] – three green to four green. [*Stephanie writes this on the paper.*] We multiplied by, um, well we multiplied, I guess we divided by four, and I guess we multiplied it by, we only got one of each. So, we came up with four.	BEJ
	1208	R1	So, you had four.	
20:00-24:59	1209	Stephanie	Yeah, so it was…um. Found, I guess four and divided by four? Um, should, I guess, go on to the next one.	BEJ; OBS
	1210	R1	Does it help to think about that as having the one tower, go to that one tower with blue. There are four of them. So from one you have four.	
	1211	Stephanie	Um hm. All right.	
	1212	R1	One times four, and then you have the duplicates and you divide by four.	
	1213	Stephanie	Okay. Um.	
	1214	R1	Does that help Bob? To think of it that way, or to represent it that way?	
	1215	R2	It helps me.	
	1216	R1	So, you might think of that, from that one, I don't know, you might, if you're looking for a way of generalizing a rule or a pattern. You might think of that one tower with the blue, any one you want. You can think of any one you want.	
	1217	Stephanie	Um hm.	
	1218	R1	But, from that one, right?	
	1219	Stephanie	There were	
	1220	R1	You can generate	
	1221	Stephanie	Like, well	
	1222	R1	The green, all green.	
	1223	Stephanie	Yeah.	
	1224	R1	But you can do that from the one, four times.	
	1225	Stephanie	Four, okay.	
	1226	R1	But, then you get the duplicates. I don't know –if that's helpful or not. See this is, see it's just as easy. I mean what Stephanie was doing right now, that's intriguing to me is that	

Time	Line	Speaker	Transcript	Code
			she's	
	1227	R2	It looks like	
	1228	R1	She's	
	1229	R2	Yeah.	
	1230	R1	Yeah. It looks like she's counting.	
	1231	R2	It looks like she's multiplying by one, 'cause each	
	1232	Stephanie	Yeah, see that's	
	1233	R1	Exactly.	
	1234	R2	(inaudible) old tower produced one new tower. And then we needed to divide by four in order to take care of the duplication.	
	1235	Stephanie	Um hm.	
	1236	R2	Okay. So. What about, okay, so why don't we just go back to one step before that?	
	1237	Stephanie	One step before that was from, um	
	1238	R2	Let me move this a little bit so you can actually see what you wrote on the paper.	
	1239	Stephanie	From two green…two green…[*Stephanie writes.*] Um, we found four and we divided – well, no we found more than four. We found twelve, and we divided by three.	BEJ; BR-V; BMP
	1240	R2	Excellent.	
	1241	Stephanie	…found twelve… [*Stephanie writes more on the paper.*]	BEJ; BR-V
	1242	R2	So, how did we find the twelve? Did we find, didn't we find the twelve by multiplying?	
	1243	Stephanie	Well, what happened was, they were duplicates. They were groups of three.	BEJ; BR-V
	1244	R2	Yes. That's right.	
	1245	Stephanie	So…	
	1246	R1	Where did the twelve come from?	
	1247	Stephanie	They came from, um, from these. [*Stephanie points to a group of towers with two green and two blue.*] Oh, well, they didn't come from all of these, they came from like (inaudible)	BEJ
	1248	R1	I think it's hard to go backwards.	
	1249	R2	I think it's hard to go backwards. Let's, maybe we can, you want to try going forwards	
	1250	Stephanie	Okay.	
	1251	R2	and then see if we can meet in the middle and then put all our information together. Okay.	

Time	Line	Speaker	Transcript	Code
			We started with four towers that had one green	
	1252	R1	Let's get another piece of paper.	
	1253	Stephanie	Okay.	
	1254	R2	And, then, um – one green and three blues. Ready?	
	1255	Stephanie	All right.	
	1256	R2	We started with four towers	
	1257	Stephanie	Yes.	
	1258	R2	that had one green and three blues. [*R2 points to the '4' in Pascal's Triangle.*]	
	1259	Stephanie	All right.	
	1260	R2	Right? Now those were the old ones in that round.	
	1261	Stephanie	Um hm.	
	1262	R2	To produce new ones, what did we do?	
	1263	Stephanie	We, um, added another green. To any other part of the tower.	
	1264	R2	So how many choices?	
	1265	Stephanie	Three.	
	1266	R2	Three. So we multiplied by three, the four towers we had by three, we multiplied by three choices.	
	1267	Stephanie	Um hm.	
	1268	R2	And then we found	
	1269	Stephanie	That we had duplicates and we divided it by, um. You divided it by three, right? Or did we, we divided it by four?	BEJ; PAH
	1270	R2	I think you found that	
	1271	R1	Well, why don't we	
	1272	R2	the number of duplicates was the number of greens.	
	1273	R1	Let's, let's um, maybe it would help Bob	
	1274	R2	If I remember it.	
	1275	R1	if you did the writing and Stephanie did the thinking.	
	1276	R2	Okay. So. Well, let me swing around so that we're actually sort of sitting straight up. [*R2 moves his chair next to Stephanie's chair.*]	
	1277	R1	So, you could write down what Stephanie's saying. Right.	
	1278	R2	Okay. So, we started – can you read my writing?	
	1279	Stephanie	Yes.	

Time	Line	Speaker	Transcript	Code
	1280	R2	Good. Okay. We started with four towers with one green [*R2 writes this on the paper.*] and three blues. And then	
	1281	R1	Here they are.	
	1282	R2	Okay. So that was the, that was the first one. And then, from each.	
	1283	R1	From each. Here's one. [*the* $\begin{bmatrix}B\\B\\B\\G\end{bmatrix}$ *tower*]	
	1284	R2	We built	
	1285	R1	How many Stephanie?	
	1286	Stephanie	From each, we built three.	
	1287	R1	Okay, this one you built three [*R1 points to the* $\begin{bmatrix}B\\B\\B\\G\end{bmatrix}$ *tower.*]	
	1288	R2	We built three.	
	1289	R1	This one you built three [*the* $\begin{bmatrix}B\\B\\G\\B\end{bmatrix}$ *tower*], this one you built three [*the* $\begin{bmatrix}B\\G\\B\\B\end{bmatrix}$ *tower*], this one you built three [*the* $\begin{bmatrix}G\\B\\B\\B\end{bmatrix}$ *tower*]	
	1290	Stephanie	Well-	
	1291	R1	Right?	
	1292	Stephanie	How many green, we're adding how many greens on though?	
	1293	R1	Exactly one green.	
	1294	Stephanie	Like? Yeah.	
	1295	R1	Okay. So- right? So from	
	1296	Stephanie	'Cause I have three spaces to put it.	BEJ

Time	Line	Speaker	Transcript	Code
	1297	R1	'Cause you have three spaces to put it.	
	1298	Stephanie	Yeah.	
	1299	R1	So, from this you got	
	1300	Stephanie	Um, three.	BEJ; BR-V
	1301	R1	three.	
	1302	Stephanie	Yeah. I got three from all of them. So, I got twelve.	BEJ; BR-V
	1303	R1	Three from the blue spaces, three from the blue spaces… So from the four	
	1304	Stephanie	Um hm.	
	1305	R1	you tripled it.	
	1306	Stephanie	(inaudible)	
	1307	R1	You started with the four, you tripled it. Right?	
	1308	Stephanie	Yeah.	
	1309	R1	And you got twelve?	
	1310	Stephanie	Yes.	
25:00-29:59	1311	R1	But, you know that, there aren't	
	1312	Stephanie	Um hm. There's three of each kind.	BEJ
	1313	R1	exactly two green. You know there aren't twelve.	
	1314	Stephanie	Yes.	
	1315	R1	Well, how many are there?	
	1316	Stephanie	There's four, so you divided it by	BEJ
	1317	R1	No, think a minute, think a minute. When you have exactly two green	
	1318	Stephanie	Um hm.	
	1319	R1	how many are there?	
	1320	Stephanie	When I have exactly two green?	PAH
	1321	R1	Towers four high.	
	1322	Stephanie	Towers four high, and exactly two green, I'm building on or that's how many I have?	PAH
	1323	R1	Just, just tell me what, you know the result of that. When you have exactly two green.	
	1324	Stephanie	I have four. No, wait, no. I have six.	BR-V
	1325	R1	It should be six, right?	
	1326	R2	That's right.	
	1327	Stephanie	Yeah.	
	1328	R1	But when you, you started with the four, you ended up with four times three, or twelve. You're supposed to have six…	
	1329	Stephanie	Um hm.	

Time	Line	Speaker	Transcript	Code
	1330	R1	So, how many duplicates, did you have?	
	1331	Stephanie	Two.	BMP; BR-V
	1332	R2	Two. So they came in	
	1333	R1	That first time you did it, there were only, there was only one duplicate for each one. I couldn't remember all of this. But, doesn't that make sense? Here's the six. Right? [*R1 points to the '6' on Pascal's Triangle.*]	
	1334	Stephanie	Yeah.	
	1335	R1	But you didn't get six, you got twelve. So, and then if you pulled them out and you did… We weren't recording as we went along, and that's what's hard.	
	1336	R2	Let me check out what I'm writing and see if it makes sense to you.	
	1337	Stephanie	Okay.	
	1338	R2	And, then, um, what I'd like to do is, is correct it if I need to so that it begins to look like what you're really thinking.	
	1339	Stephanie	All right.	
	1340	R2	Okay, because what I'm thinking may be different from what you're thinking. And I really want to understand your thinking.	
	1341	Stephanie	Um hm. Okay.	
	1342	R2	Okay? Okay. Ah. We built three towers with two greens – from each, okay, we started with four towers with one green and from each of those four, we built three towers with two greens. This gave four times three, which was twelve towers, but they came in pairs of two. [*R2 reads this from the paper on which he has been writing.*]	
	1343	Stephanie	Okay.	
	1344	R2	These are the duplicates. So, there are really, it seems like, four times three is twice as many as we should have had. So, that's four times three, is twice as many.	
	1345	Stephanie	Yeah.	
	1346	R2	So, we had to divide it by two,	
	1347	Stephanie	Um, hm.	
	1348	R2	and that gives	
	1349	Stephanie	Six.	
	1350	R2	Six.	
	1351	R1	Okay. I'd like you to look at this [*R1 points*	

Time	Line	Speaker	Transcript	Code
			to the towers.] again Stephanie. Because, it helps me when I see four. Right?	
	1352	Stephanie	Um hm.	
	1353	R1	Three. Right? Three blue?	
	1354	Stephanie	Um hm.	
	1355	R1	Imagining twelve. And then when you looked at them and pulled them together, you saw the duplicates.	
	1356	Stephanie	Um hm. Okay.	
	1357	R1	But, it may be hard to remember, because each of these were chunked separately.	
	1358	Stephanie	All right.	
	1359	R2	Where do you think	
	1360	R1	Do you think you'll remember that? You're not sure, you're not really sure where you got how many duplicates each time, I think.	
	1361	Stephanie	It, ehh. I understand that like, from these you're going to get three.	BEJ
	1362	R2	Right.	
	1363	R1	Right.	
	1364	Stephanie	Like, 'cause, oops. [*Stephanie knocks over some towers and sighs.*]	
	1365	R1	Um hm.	
	1366	Stephanie	'Cause, there's only three places for you to move them.	BEJ
	1367	R1	Um hm.	
	1368	Stephanie	And then, I think what messed me up was how many duplicates you were going to get from each of them.	BEJ
	1369	R2	Okay.	
	1370	R1	Um hm. You know there have to be a total of six when you're done.	
	1371	Stephanie	Yeah. But, I just	
	1372	R1	Right. Sure.	
	1373	Stephanie	I	
	1374	R2	Yeah. There was a step that we talked about at that point. And, um, that was if we took any one of these with the two greens, how many of the old ones did it come from?	
	1375	Stephanie	Two.	
	1376	R2	And it was because there were…	
	1377	Stephanie	Oh. 'Cause there were two, two of, that would have the possibility, like two parents,	BEJ; BR-V

Time	Line	Speaker	Transcript	Code
			or two, like these two [*the* $\begin{bmatrix} G \\ B \\ B \\ B \end{bmatrix}$ *and* $\begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix}$ *towers*].	
	1378	R2	Yeah. And how do you count, okay, what is it about this tower [*the* $\begin{bmatrix} G \\ G \\ B \\ B \end{bmatrix}$ *tower*] that counts the number of parents?	
	1379	Stephanie	It has a green in two places where	BEJ
	1380	R2	Excellent. So, it's that two [*the two green blocks*] which counts the parents.	
	1381	Stephanie	Okay.	
	1382	R2	And then it's this two [*the two blue blocks*] that count the next one.	
	1383	Stephanie	Yeah. So, like here I divide by three, because there's three green?	PAH
	1384	R2	Excellent. Excellent.	
	1385	Stephanie	Okay.	
	1386	R2	Okay. In the next step – uh- we took each of the six towers with two greens. [*R2 writes this on the paper.*] Right?	
	1387	Stephanie	Um hm.	
	1388	R2	And produced how many new ones?	
	1389	Stephanie	Um. We produced, from the six with two greens?	PAH
	1390	R2	Yeah. How many new ones would you get from this one [*the* $\begin{bmatrix} B \\ G \\ B \\ G \end{bmatrix}$ *tower*], for example.	
	1391	Stephanie	Two.	BR-V
	1392	R2	Two. Any different from the others?	
	1393	Stephanie	No. So you produce twelve again.	
	1394	R2	Okay. What were you counting when you got the two?	
	1395	Stephanie	The spaces left over. There were two blue spaces.	BEJ; BR-V
	1396	R2	The blues that were left over.	

Time	Line	Speaker	Transcript	Code
	1397	Stephanie	Yeah.	
	1398	R2	Okay. Now, you just	
	1399	R1	Maybe you ought to put in parentheses blue, parentheses green, when it's appropriate. No?	
	1400	R2	Come and write. [*R2 laughs.*] Okay. Um. Okay, so we. Uh. So, each of the six produce two which gives, six times two equals twelve. But, you just told me that, uh, that these twelve, uh, came up in	
	1401	Stephanie	Are duplicates?	
	1402	R2	And how many came up at a time?	
	1403	Stephanie	Two. Um. Two at a time. Well, like. You'll get two from each, but you have to divide it by three 'cause there's three green?	BEJ
	1404	R2	Aha. There's three green in the next generation.	
30:00-34:59	1405	Stephanie	Yes.	
	1406	R2	Okay. [*R2 writes.*] Okay. So this gives, uh, six times two divided by three actual towers, with, now it's three greens, right?	
	1407	Stephanie	Yes.	
	1408	R2	Okay. So, the first time we multiplied by three, the second time we multiplied by two.	
	1409	Stephanie	Um hm.	
	1410	R2	The first time we divided by two, and then, the second time we multiplied by	
	1411	Stephanie	By three.	
	1412	R2	three. Can you guess what will happen?	
	1413	Stephanie	You'll multiply by um four and divide by one. Oh, wait, no! The opposite. You multiply by one and you divide by four.	BDI
	1414	R2	Okay. So in the next step… [*R2 writes.*] Prediction. This is by you. [*Stephanie laughs.*] Okay. We, we'd multiply by	
	1415	Stephanie	By one	
	1416	R2	one and	
	1417	Stephanie	and divide by four.	
	1418	R2	and divide by four. Okay. How did you guess one and how did you guess four?	
	1419	Stephanie	'Cause it decreased on	BEJ
	1420	R2	Or how did you predict one?	
	1421	Stephanie	Um, like I guess, the numerator, decreased. And the denominator, increased.	BEJ; BDI

Time	Line	Speaker	Transcript	Code
	1422	R2	Increased. Terrific. Suppose we do that	
	1423	Stephanie	Okay.	
	1424	R2	and let's just see what turns up. So, the actual number of towers here is six times two over three, which is?	
	1425	Stephanie	Um, six, oh, three, four. [*Stephanie laughs and covers her face.*]	
	1426	R2	Four. So that's, so what happens when we multiply by one and divide by four?	
	1427	Stephanie	(inaudible)	
	1428	R2	Which is?	
	1429	Stephanie	One.	BMP
	1430	R2	Is that what you found?	
	1431	Stephanie	Yes.	
	1432	R2	So, the prediction is that there are this many towers.	
	1433	Stephanie	Um hm.	
	1434	R2	with four greens.	
	1435	Stephanie	Yes.	
	1436	R2	Which is an old story. Okay. But, um. Now the next, the final question is this. Okay. Um, here are the actual four towers with the three greens. Right?	
	1437	Stephanie	Um hm.	
	1438	R2	How do you see them multiplying by one and dividing by four when we make the next generation?	
	1439	Stephanie	Well. Each one gives off one new one, one with four green, 'cause there's only one place for you to put the green.	BEJ; BR-V
	1440	R2	Excellent.	
	1441	Stephanie	And because there's four greens, you divided by four. Like the new generation has four greens. You divided by four.	BEJ; BR-V
	1442	R2	You know what?	
	1443	Stephanie	What?	
	1444	R2	I'm convinced.	
	1445	Stephanie	Oh, good.	
	1446	R2	Do you have questions?	
	1447	R1	Well, of course the next, the next thing I would want to know is, um, we took this row. Okay?	
	1448	Stephanie	Um hm.	
	1449	R1	And we showed from exactly one green,	

Time	Line	Speaker	Transcript	Code
			right?	
	1450	Stephanie	Um hm.	
	1451	R1	And then, if we were to replace a blue, and now had exactly two greens, and so forth. Whatever. Which row was this? Four high?	
	1452	Stephanie	Um hm.	
	1453	R1	Um. We were able to go through this, this process, um, how would it work for the next line? This goes across four lines. Of how does it go for the line of five?	
	1454	R2	Can you see the numbers through the towers?	
	1455	Stephanie	Yeah.	
	1456	R2	If not, you're welcome to move.	
	1457	Stephanie	Um. I'm sure it would probably work the same way, I guess. I mean, like, um, one would be, um, like for five, one would be no greens. And all right… [*Stephanie builds a tower five high of all blues.*] so this would be one. Umm, one like this, well, alright. [*Stephanie finds a tower of five with one green on the bottom and four blue above that.*] You get that from one like this. Or from…	BEJ; BR-V
	1458	R1	So, you're going backwards now.	
	1459	Stephanie	Um. Yeah. I'm kinda	
	1460	R1	But, it, okay	
	1461	R2	Well, we have to start somewhere.	
	1462	R1	That's fine.	
	1463	Stephanie	Yeah.	
	1464	R2	Yeah.	
	1465	Stephanie	Um, or one like this. [*the* $\begin{bmatrix} B \\ B \\ B \\ G \\ B \end{bmatrix}$ *tower*] If you're, if you're building with blue this time.	BEJ; BR-V
	1466	R1	All right. You can just tell us, if you want to, right.	
	1467	Stephanie	Yeah, well, you know, the other ones. Like one with a green here [*She indicates the*	BEJ; BR-V

Time	Line	Speaker	Transcript	Code
			position in the tower $\begin{bmatrix} B \\ B \\ B \leftarrow \\ G \\ B \end{bmatrix}$.], one with a green here [*She indicates the position in the tower* $\begin{bmatrix} B \\ B \leftarrow \\ B \\ G \\ B \end{bmatrix}$.], or one with a green there. [*She indicates the position in the tower* $\begin{bmatrix} B \leftarrow \\ B \\ B \\ G \\ B \end{bmatrix}$.] Um, so you have – you have one way to do it. Like you have one space left. Wait, I have to think because I'm working	
35:00-36:23	1468	R1	In that one you have one space left. In that particular one. [*the* $\begin{bmatrix} B \\ B \\ B \\ G \\ B \end{bmatrix}$ *tower*]	
	1469	Stephanie	I have one	
	1470	R1	In that tower.	
	1471	Stephanie	space to put a ca, um. [*Stephanie sighs.*] A blue tower, a blue cube, so you're multiplying by one. Or, yeah. And, I guess this would kinda be like, um, [*Stephanie sighs again.*] the last one. Not, not five over zero, but five over five, [*She is referring to C(5, 0) and C(5, 5).*] like it would be this one, not the other one.	
	1472	R1	You want to go at the other end now.	
	1473	R2	Oh good. Okay. Okay.	
	1474	Stephanie	Yeah, because otherwise, um, so. Um, and	BEJ

Time	Line	Speaker	Transcript	Code
			because there's five, you divide by five. And you get one.	
	1475	R2	Ah. So you're looking at the numbers we divide by, first.	
	1476	Stephanie	Yes.	
	1477	R2	Okay. so instead of dividing by four at the last step, you divide by five.	
	1478	Stephanie	Well, if you were building with five.	
	1479	R1	And where did the five come from, one more time, Stephanie?	
	1480	Stephanie	Well, there's these five. [*The five blues in the all blue tower five high.*] Like there's five blue. If there were four blue, it would be, in your final (inaudible)	BEJ
	1481	R2	Okay.	
	1482	R1	Um hm. You know what would be interesting to me, I know it's late and you've worked very, very hard, and um, this, um, problem came out of a dinner conversation we had the night before last with Professor Davis.	
	1483	Stephanie	Um hm.	
	1484	R1	And, um, just, I thought you would be interested in the conversation	
	1485	Stephanie	Yeah.	
	1486	R1	which is why I brought the cubes, lest anyone question why. Of course, Dr. Spieser didn't really know we were going to do this, but since he started it with his conversation	
	1487	R2	Yeah, I was expecting to be a silent observer. You know, I'm just as surprised as you.	
	1488	R1	He was telling us that…	
			R1 and R2 explain the history of the problem. Then R1 asks Stephanie to write up her work as if explaining it to her friends back at Harding school. Stephanie does not do any more work in the tape.	

APPENDIX G: TRANSCRIPT – SESSION 7

INTERVIEW WITH STEPHANIE Time: 71 minutes (1 CD)
April 17, 1996

R1: Dr. Carolyn Maher Stephanie: Stephanie R2: Donna Weir R3: Steve

Time	Line	Speaker	Transcript	Code
00:00-04:59	1	R1	He's just very excited. I'll be visiting him in Utah next week to see what's going on there. So, uh… But um Donna wasn't here last time. You weren't here last time, were you?	
	2	R2	No. I wasn't.	
	3	R1	You were away. Yeah. So you might want to listen in on what went on last time.	
	4	R2	Are you starting?	
	5	R1	Pretty much. Uh. Because um it isn't what I was going to do with you. But since he came, uh, and we had been thinking about that idea, because he's changed it. 'Cause he was (inaudible) – we were going to do something else that was somehow related, but um, why don't you tell Donna [*R1 speaks to R2.*]. You can come up - if you want to. [*R2 joins Stephanie and R1 at the table. R1 speaks to Stephanie.*] I don't know how you want to do this. I think it's easier to explain it to somebody who really doesn't know it (inaudible due to background noise). You know what I'm saying?	
	6	Stephanie	Yeah.	
	7	R1	Than to somebody who does. And she's a quick study.	
	8	R2	Thank you.	
	9	Stephanie	Alright.	
	10	R1	As you know	
	11	Stephanie	Um	
	12	R1	She – she um is very interested in this.	
	13	Stephanie	Can I have the Unifix cubes?	
	14	R1	Okay.	
	15	Stephanie	Alright.	
	16	R1	Do you want this color? Or blue?	
	17	Stephanie	Uh.	
	18	R1	Does it matter?	

Time	Line	Speaker	Transcript	Code
	19	Stephanie	It doesn't matter. Okay. [*Stephanie spills the Unifix cubes out onto the table. She seems eager to begin.*]	
	20	R1	(inaudible)	
	21	Stephanie	So like what we did was – we were building towers of four.	PPK
	22	R1	Um hm.	
	22	Stephanie	And we started out with towers of four made of red and green – uh – well – at the time it was blue and green, but now it's red and yellow, um, with one red.	PPK
	23	R2	Okay.	
	24	Stephanie	And there's four ways to do that – [*Stephanie builds* $\begin{bmatrix} R \\ Y \\ Y \\ Y \end{bmatrix}\begin{bmatrix} Y \\ R \\ Y \\ Y \end{bmatrix}\begin{bmatrix} Y \\ Y \\ R \\ Y \end{bmatrix}\begin{bmatrix} Y \\ Y \\ Y \\ R \end{bmatrix}$.] There's four of them.	PPK; BR-V
	25	R2	Um hm. Okay.	
	26	Stephanie	And then I was asked: For each of them, without moving the one that's red	PPK
	27	R2	Okay.	
	28	Stephanie	how many I could build with two reds. So like from this one – [*Stephanie chooses* $\begin{bmatrix} R \\ Y \\ Y \\ Y \end{bmatrix}$ *and moves the other towers to the side.*] like how many I could build with two red, but one of them has to be on top.	PPK; BR-V
	29	R2	Okay.	
	30	Stephanie	So – [*Stephanie builds.*] I built them like this. [$\begin{bmatrix} R \\ R \\ Y \\ Y \end{bmatrix}\begin{bmatrix} R \\ Y \\ R \\ Y \end{bmatrix}\begin{bmatrix} R \\ Y \\ Y \\ R \end{bmatrix}$] and like that.	
	31	R2	Okay. That's all?	
	32	Stephanie	Yeah. That's all you can build. And the same	

Time	Line	Speaker	Transcript	Code
			with that one. [*She chooses* $\begin{bmatrix} Y \\ R \\ Y \\ Y \end{bmatrix}$.] You can make one like that, [*She builds* $\begin{bmatrix} R \\ R \\ Y \\ Y \end{bmatrix}$.] one like that [*builds* $\begin{bmatrix} Y \\ R \\ R \\ Y \end{bmatrix}$]	
	33	R2	Stephanie, what – now you've changed what you're doing when you came here?	
	34	Stephanie	Oh. [*builds* $\begin{bmatrix} R \\ Y \\ Y \\ R \end{bmatrix}$] Wait a minute. [*Stephanie changes the tower to* $\begin{bmatrix} Y \\ R \\ Y \\ R \end{bmatrix}$] No, I'm still – uh – this time I have to build them all with the red, the two red, but one has to be in the second spot.	BEJ; BR-V
	35	R2	Oh, okay.	
	36	Stephanie	And for this one, it'll be the same thing, only one has to be in the third spot.	BEJ
	37	R2	Okay. Now I've got what you're doing.	
	38	Stephanie	[*Stephanie builds* $\begin{bmatrix} R \\ Y \\ R \\ Y \end{bmatrix}\begin{bmatrix} Y \\ R \\ R \\ Y \end{bmatrix}\begin{bmatrix} Y \\ Y \\ R \\ R \end{bmatrix}$.] These three.	BR-V
	39	R2	Okay.	
05:00-09:59	40	Stephanie	And then the fourth one. [*Stephanie moves the trios that she has built to the back of the table and moves the fourth tower into the front. She*	BR-V

Time	Line	Speaker	Transcript	Code
			builds $\begin{bmatrix}R\\Y\\Y\\R\end{bmatrix}\begin{bmatrix}Y\\R\\Y\\R\end{bmatrix}\begin{bmatrix}Y\\Y\\R\\R\end{bmatrix}$.] And one like that [*as she places the last tower onto the table*].	
	41	R2	Okay.	
	42	Stephanie	And that's it. But, the problem is, we made um three for each one.	BEJ
	43	R2	Um hm.	
	44	Stephanie	But the thing is that there there's like duplicates of each – like – this one [*Stephanie selects* $\begin{bmatrix}R\\R\\Y\\Y\end{bmatrix}$ *from the first group of three*] and this one [*the tower with the same pattern from the second group of three. Pause.*] This one [*Stephanie selects* $\begin{bmatrix}R\\Y\\R\\Y\end{bmatrix}$ *from group one and then* $\begin{bmatrix}Y\\R\\Y\\R\end{bmatrix}$ *from group two.] and that one. [She continues to sort the towers into pairs. The result is:* $\begin{bmatrix}R\\R\\Y\\Y\end{bmatrix}\quad\begin{bmatrix}R\\R\\Y\\Y\end{bmatrix}\begin{bmatrix}R\\Y\\R\\Y\end{bmatrix}\quad\begin{bmatrix}Y\\R\\Y\\R\end{bmatrix}\begin{bmatrix}R\\Y\\Y\\R\end{bmatrix}\quad\begin{bmatrix}R\\Y\\Y\\R\end{bmatrix}\begin{bmatrix}Y\\R\\R\\Y\end{bmatrix}\quad\begin{bmatrix}Y\\R\\R\\Y\end{bmatrix}\begin{bmatrix}R\\Y\\R\\Y\end{bmatrix}$ $\begin{bmatrix}Y\\R\\Y\\R\end{bmatrix}\begin{bmatrix}Y\\Y\\R\\R\end{bmatrix}\begin{bmatrix}Y\\Y\\R\\R\end{bmatrix}$ *Stephanie and the interviewers do not notice that she has made an error in groups two and*	BEJ; BR-V

Time	Line	Speaker	Transcript	Code
			five at this point.] So really we made six. [*pause*] Okay.	
	45	R2	Okay.	
	46	Stephanie	So. Then the next question – [*She grabs some more Unifix cubes.*]	
	47	R1	Could we stay here for a minute? Before the next one?	
	48	Stephanie	Yeah.	
	49	R1	Um. So you started with towers of exactly one red.	
	50	Stephanie	Um hm.	
	51	R1	Okay. And you moved to make towers four tall with exactly two reds and you worked with each of these [*R1 points to each of the original four towers Stephanie had built.*].	
	52	Stephanie	Um hm.	
	53	R1	Okay. And you said something to Donna – you said: When you add another red, this red [*R1 touches the red cube at the top of the first tower.*] position stays the same.	
	54	Stephanie	Um hm.	
	55	R1	And so you can only add a red in how many places?	
	56	Stephanie	Here. [*Stephanie points to the cube just below the top (position two).*]	BEJ
	57	R1	[*reiterating Stephanie's statement and gesture*] Here.	
	58	Stephanie	Here [*Stephanie points to the cube two below the top (position three)*] or here [*Stephanie points to the bottom cube.*]	BEJ
	59	R1	Okay. And here [*R1 indicates the second tower.*] you can add a red	
	60	Stephanie	Here.	
	61	R1	Here, here, or here. [*R1 points to the top, third and bottom positions.*]	
	62	Stephanie	Um hm. And that's why you'll have three, like three	BEJ
	63	R1	Okay. So you'll get	
	64	Stephanie	from each.	
	65	R1	From each of these four you get three	
	66	Stephanie	Right.	
	67	R1	but that gives you twelve.	
	68	Stephanie	Twelve [*simultaneously with R1*]	BR-V
	69	R1	Two, four, six, eight, ten [*R1 counts the pairs*	

Time	Line	Speaker	Transcript	Code
			of towers.]	
	70	Stephanie	But they come in pairs of two	BEJ; BR-V
	71	R1	Um. They come in pairs of two.	
	72	Stephanie	Um hm.	
	73	R1	Um. So – you divide by	
	74	Stephanie	By two.	BEJ
	75	R1	You divide by two.	
	76	Stephanie	to get	
	77	R1	to get six	
	78	Stephanie	Yeah.	
	79	R1	Because of the two duplicates. Okay. So that's in moving from	
	80	Stephanie	Um hm.	
	81	R1	four things taken one at a time to four things taken two at a time.	
	82	Stephanie	Um hm.	
	83	R1	Okay.	
	84	R2	Okay.	
	85	R1	So you were going to ask another question – but you were going to do something?	
	86	Stephanie	No. I was just going to keep building.	
	87	R1	So what would you do – be building next?	
	88	Stephanie	Um. Towers with three reds?	
	89	R1	Four taken three – can you tell us what you think is going to happen and why before you do it?	
	90	Stephanie	Well, it would be six [*Stephanie points to the row of six pairs.*] times two. Because	BEJ
	91	R1	Hold on a minute.	
	92	Stephanie	Oh. Well, here it was	
	93	R1	Tell me – explain what you're doing.	
	94	Stephanie	'Cause here it was four times three to get the twelve, because you could have red here [*She picks up* $\begin{bmatrix} Y \\ Y \\ Y \\ R \end{bmatrix}$ *and points to the three yellow positions on the tower one at a time as she speaks.*], a red here or a red here. So that's three – it'll produce three. And here [*This time*	BEJ; BR-V

Time	Line	Speaker	Transcript	Code
			she picks up $\begin{bmatrix} Y \\ R \\ Y \\ R \end{bmatrix}$] you can only put a red here [*the top position*] or here [*the third position*]. So it'll produce two. And so	
	95	R1	And so from the six you could produce	
	96	Stephanie	You could produce two.	BEJ
	97	R1	two. Any duplicates?	
	98	Stephanie	Yeah. There'll be duplicates. There'll be um two duplicates for each. So you divide by two – No. – Will there be two for each? I forget how many there were.	BEJ; OBS
	99	R1	You see. That that's the question.	
	100	Stephanie	I forgot how many there were. So you have to like build it.	OBS
	101	R1	Okay. So could you think about it for a minute without doing it and predict?	
	102	Stephanie	Oh. Four.	BR-V
	103	R1	You predict there would be	
	104	Stephanie	Four. Oh. Three duplicates and then there'll be four.	BEJ; BR-V
	105	R1	You think there would be three duplicates and then four of them.	
	106	Stephanie	Yeah. Because it's just the opposite of that.	BEJ
	107	R1	What do you mean?	
	108	Stephanie	Well, like this is one red and three yellow [*Stephanie picks up a tower* $\begin{bmatrix} Y \\ Y \\ R \\ Y \end{bmatrix}$]. It'll be three red and one yellow. So it'll be just the opposite.	BEJ; BDI
	109	R1	Should we try it? [*R2 says something inaudible.*] [*Stephanie immediately begins to build towers.*] What do you think? [*Pause*] So tell us what you're doing while you're doing it.	
	110	Stephanie	Alright. Well, for this one	
	111	R1	Um hm	
10:00-14:59	112	Stephanie	It has to have two red on top. So I put a red down there [*third position*] and now it's going	BEJ; BR-V

Time	Line	Speaker	Transcript	Code
			to go [$\begin{bmatrix}R\\R\\Y\\R\end{bmatrix}$] so there are two for that one. And for the second one – that one there [$\begin{bmatrix}R\\R\\R\\Y\end{bmatrix}$] and [$\begin{bmatrix}R\\Y\\R\\R\end{bmatrix}$] that one there. The third one	
	113	R1	Now let me see. The second one – help me understand this. You have to have a red -	
	114	Stephanie	[*The mistakenly placed tower is corrected.*] Oooh. – This is -	BDI
	115	R1	Does that belong there?	
	116	Stephanie	No. [*Stephanie switches* $\begin{bmatrix}Y\\R\\Y\\R\end{bmatrix}$ *with* $\begin{bmatrix}R\\Y\\R\\Y\end{bmatrix}$.]	BDI
	117	R1	Okay. – Okay. So, let's see. Let's go through this again. So you have the red there and there [*R1 indicates the top and third positions.*]	
	118	Stephanie	Um hm. [*She builds* $\begin{bmatrix}R\\R\\Y\\R\end{bmatrix}$ *while R1 is speaking.*]	
	119	R1	That's still the same. You put it in the middle and here and here. Then you put it on the bottom.	
	120	Stephanie	Um hm. [*Then she builds* $\begin{bmatrix}R\\Y\\R\\R\end{bmatrix}$. *She continues building towers very confidently and quite*	BR-V

Time	Line	Speaker	Transcript	Code
			quickly. She produces $\begin{bmatrix}R\\R\\R\\Y\end{bmatrix}\begin{bmatrix}Y\\R\\R\\R\end{bmatrix}\begin{bmatrix}R\\R\\Y\\R\end{bmatrix}\begin{bmatrix}Y\\R\\R\\R\end{bmatrix}$.] And then, um, this one [*begins to collect the duplicates from among the towers* $\begin{bmatrix}R\\R\\R\\Y\end{bmatrix}\begin{bmatrix}R\\R\\R\\Y\end{bmatrix}$ $\begin{bmatrix}R\\R\\R\\Y\end{bmatrix}$*, for example*] this one and this one are the same. This one [*she picks up* $\begin{bmatrix}R\\R\\Y\\R\end{bmatrix}$]	
	121	R1	So you tripled them.	
	122	Stephanie	This one	
	123	R1	Triplicates.	
	124	Stephanie	Yeah. And that one. That one, that one, that one, and these three. So you have four. [*She continues to sort the towers into triples. The result is:* $\begin{bmatrix}R\\R\\R\\Y\end{bmatrix}\begin{bmatrix}R\\R\\R\\Y\end{bmatrix}$ $\begin{bmatrix}R\\R\\R\\Y\end{bmatrix}\begin{bmatrix}R\\R\\Y\\R\end{bmatrix}\begin{bmatrix}R\\R\\Y\\R\end{bmatrix}$ $\begin{bmatrix}R\\R\\Y\\R\end{bmatrix}\begin{bmatrix}R\\Y\\R\\R\end{bmatrix}\begin{bmatrix}R\\Y\\R\\R\end{bmatrix}$ $\begin{bmatrix}R\\Y\\R\\R\end{bmatrix}\begin{bmatrix}Y\\R\\R\\R\end{bmatrix}\begin{bmatrix}Y\\R\\R\\R\end{bmatrix}$ $\begin{bmatrix}Y\\R\\R\\R\end{bmatrix}$	BR-V
	125	R1	So. – Is that what you predicted?	
	126	Stephanie	Yes.	
	127	R1	You predicted you would get triplicates.	
	128	Stephanie	I said there would be four, so it would be groups of three.	BEJ
	129	R1	So what would you be dividing by?	

Time	Line	Speaker	Transcript	Code
	130	Stephanie	Three	BEJ
	131	R1	By three. And before that you were dividing by?	
	132	Stephanie	Two.	BEJ
	133	R1	And before that you were dividing by? We didn't do before that.	
	134	Stephanie	We didn't do that.	
	135	R1	We could've. We could've started with all yellow.	
	136	Stephanie	Um. Well. Yeah, but, we're gonna	
	137	R1	No red. – Let's see how that works with all yellow. Sorta I like (inaudible) explain (inaudible) from the beginning. Do you know what I'm saying?	
	138	Stephanie	Um hm.	
	139	R1	We might as well do them all. Is this the first one? We started with this? [*Stephanie sneezes.*] God bless you.	
	140	Stephanie	Thank you.	
	141	R1	You need some tissues? [*Stephanie goes on building towers. This time every tower (four of them) are entirely yellow.*] Before you do it, why don't you predict what will happen?	
	142	Stephanie	Oh. There'll be one.	BR-V
	143	R1	Why?	
	144	Stephanie	Because you're going to get yellow – all yellow from all four. 'Cause there's only one space you can put a yellow. So you're going to get one all yellow from here, one all yellow from here, one all yellow from here and one all yellow from here. So there's going to be one. [*pause; Stephanie lines up four towers each built using four yellow Unifix cubes.*]	BEJ; BR-V
	145	R1	Um. But there are four of them there. So…	
	146	Stephanie	You divide by four.	BEJ
	147	R1	Okay. So – If you were trying to help me know what to divide by, is there anything that helps you?	
	148	Stephanie	You – um – (inaudible) [*Stephanie repeats the question.*] What – is there anything that helps you	PAH
	149	R1	Um hm.	
	150	Stephanie	like to know what to divide by?	PAH
	151	R1	Um hm.	
	152	Stephanie	Um – the number of red, I guess? 'Cause here	BEJ

Time	Line	Speaker	Transcript	Code
			you divided by one. [*Stephanie points to the towers with one red*.] Here you divided by two. [*Now she indicates the towers with two reds*.] And here you divided by three. [*She indicates the towers built with three reds*.]	
	153	R1	Here the number of red is zero. [*R1 points to the four all yellow towers.*]	
	154	Stephanie	No. But I'm saying here [*pointing to the area between the all yellow towers and the towers with one red.*] you divided by one. Like to get that -	BEJ
	155	R1	Um hm	
	156	Stephanie	I don't know.	OBS
	157	R1	Does that work? For all of them?	
	158	Stephanie	I guess not for this one, but like – the number of --[*long pause*]	OBS
	159	R1	Well, how does it work from here to here? [*R1 points from the towers with three reds to the non-existent towers of four red.*]	
	160	Stephanie	Well. The same way it works there. It's just all red.	BEJ
	161	R1	All red. So--	
	162	Stephanie	And it would be four times one divided by one, 'cause there's only one spot.	BEJ
	163	R1	So it works here. So - -	
	164	Stephanie	Yeah.	
	165	R1	It's one spot here. One spot - - [*R1 points to the one yellow cube in each of the towers with three reds and one yellow.*]	
	166	Stephanie	Um hm.	
	167	R1	Um hm. So it works there.	
	168	Stephanie	Yeah.	
	169	R1	It's just that this situation here is – a little bit different. If you wanted it to be nice and consistent, you would –sometimes that forces	
	170	Stephanie	Um hm.	
	171	R1	people to make definitions particular ways.	
	172	Stephanie	Um	
15:00-19:59	173	R1	'Cause you know you can't divide by zero. – Um - - so -	
	174	Stephanie	D – um – the number of spots you're pu – filling in – like – to get these [*She indicates going from three yellow and one red to all yellow.*] You put a yellow in one spot. Like – um – you know – I don't -	BEJ; OBS

Time	Line	Speaker	Transcript	Code
	175	R1	Yeah.	
	176	Stephanie	Like to get these [*pointing to the towers with two reds and two yellows*] you put – there's – like – [*Stephanie sounds frustrated.*] I don't -	BEJ; OBS
	177	R1	No. You can't force it. It's very interesting. What do you think about it? [*to R2*]	
	178	R2	It's interesting.	
	179	R1	It is. Do you have a question?	
	180	R2	I feel like it's something missing though (inaudible) – with the four yellows. Is this the end here?	
	181	Stephanie	No. You can do four reds.	
	182	R1	Now. Maybe what we should do is write out the cases that work. We wrote out one of them really very clearly here. [*R1 takes out some papers.*] Right? You did the one here um where you started with um exactly one when it was green and blue in this case, right?	
	183	Stephanie	Um.	
	184	R1	And all those that had one green, and you said there were four. And then you said the next step was to build all possible towers with two green. All right. And that's what you did here. And you showed how you had to keep it.	
	185	Stephanie	Um hm.	
	186	R1	The one green in the same place so either of these could be green.	
	187	Stephanie	Um hm.	
	188	R1	I like that. Okay. So then you said so you multiply by four. What's this? Let's see if I'm reading this right. You say multiply four by one and divide by four.	
	189	Stephanie	There's stuff on the back. [*R1 turns over each of the papers she is holding.*] No, on the other back.	
	190	R1	On this. Oh. That helps. Now let's see which one am I doing on this? After doing this with all four of the towers we had twelve towers with two green. That you built here. [*R1 indicates the towers with two red cubes and two yellow cubes.*]	
	191	Stephanie	Um hm.	
	192	R1	[*reading*] 'There were some duplications. Each new tower came in a pair so there was only really six new towers. We took the six	

Time	Line	Speaker	Transcript	Code
			new towers and from each one, using the same method, created towers of four high with three green.' And that's this stack. *[R1 indicates towers on the table.]* 'We created twelve towers, but like before, there were duplicates. This time they came in groups of three.' Here's your groups of three. [*R1 indicates the appropriate towers on the table. She continues reading Stephanie's synopsis.*] 'four new towers and we started to see a pattern.' And this is where you wrote 'the formula looked like this: $\frac{4\ x\ 3}{2} = 6$ Right? Towers four high'	
	193	Stephanie	Hm.	
	194	R1	Four and now this is where – maybe it would be helpful to go over again. Let's talk about the formula. Okay? Explain that to us. – The four	
	195	Stephanie	Wait.	
	196	R1	Times three over two equals six. – That must be	
	197	Stephanie	For this one, I guess?	PAH
	198	R1	for	
	199	Stephanie	I – uh – for this one? With um – these. And it would be	BEJ
	200	R1	(inaudible) a number one, two, three, four, five, six. So - - How would that work?	
	201	Stephanie	That was wrong.	BDI
	202	R1	Is it? Why do you say it's wrong?	
	203	Stephanie	Wouldn't it be um six times two? Because um	BEJ
	204	R1	Well. Think about what you did. Remember you reorganized them. When you first had this one here [*R1 points to* $\begin{bmatrix} R \\ Y \\ Y \\ Y \end{bmatrix}$.] right?	
	205	Stephanie	Um hm.	
	206	R1	Okay.	
	207	Stephanie	Um hm.	
	208	R1	When you were making two's out of them	
	209	Stephanie	Um hm.	
	210	R1	Right? - You had – you kept this [*points to the top red cube of the tower*] the same so you had	

Time	Line	Speaker	Transcript	Code
			here and here [*indicates the top and second positions*]. That was one of them, right?	
	211	Stephanie	Yeah.	
	212	R1	Then you had here and here [*the top and third positions*]. That was another one. Then you had here and here. [*top and bottom positions*] That was another one. So you got three, didn't you?	
	213	Stephanie	Yes.	
	214	R1	You got three.	
	215	Stephanie	Oh. Okay.	BDI
	216	R1	Right?	
	217	Stephanie	Yeah.	
	218	R1	Isn't that right? And the same thing here. This was the same. [*points to the red cube in the* $\begin{bmatrix} Y \\ R \\ Y \\ Y \end{bmatrix}$ *tower*] But you went here and here. [*points to the top and second positions*] One. Here and here. [*second and third positions*] Two. Here and here. [*second and bottom positions*] Three.	
	219	Stephanie	Yeah.	
	220	R1	So for each of these four, you got three.	
	221	Stephanie	Yeah.	
	222	R1	Now, can you predict by looking at this row [*pointing to the row of towers four high with one red*] that there would be three that would come when you go from one to two?	
	223	Stephanie	Yes.	
	224	R1	What helps you predict?	
	225	Stephanie	Because there's three places where you can put the second one.	BEJ
	226	R1	Three places. Okay. So -	
	227	Stephanie	Yes. I think it should -	BEJ
	228	R1	Well. Let's write this down, so we can – so we don't – 'cause it's easy -	
	229	Stephanie	Alright.	
	230	R1	It's easy for me when you disassemble it to remember, you know what I'm saying,	
	231	Stephanie	Um hm.	
	232	R1	what happened before it was assembled.	

Time	Line	Speaker	Transcript	Code
	233	Stephanie	So what do you want me to write?	
	234	R1	So let's think. Well, let's think – what makes sense to you? Um. You started with – right? – four towers	
	235	Stephanie	Four towers.	
	236	R1	with exactly one red, if you want to say there, right?	
	237	Stephanie	Okay.	
	238	R1	And you told from each of these, right?	
	239	Stephanie	Um hm.	
	240	R1	from each one of these	
	241	Stephanie	You can get three.	BEJ
	242	R1	Three. Because there are three positions	
	243	Stephanie	Um hm.	
	244	R1	to place a red.	
	245	Stephanie	Yes.	
20:00-24:59	246	R1	Isn't that right? So it's - - so you have three positions by four towers. So four is the number of towers. Just happens to be that they're height four. Or is that…?	
	247	Stephanie	Oh. No no no. I think what it should've been is four towers.	BEJ; BDI
	248	R1	Four towers?	
	249	Stephanie	That's what I should've wrote down.	BDI
	250	R1	But – but – well. If the – if the towers are height three, would there be four towers?	
	251	Stephanie	No-o.	
	251	R1	How many would there've been?	
	252	Stephanie	Two.	
	253	R1	If it's height three, can you tell me why?	
	254	Stephanie	Because you only have two other places where you can put a red. If it's height	BEJ
	255	R1	Yes, but how many would you start up with if it were height three, where you had one in each position? One red in each position?	
	256	Stephanie	Three.	
	257	R1	You'd start with three. – If it was height	
	258	Stephanie	Yeah. It would start with – Oh. Like that it would work, but I think it – it – that what I should've wrote was four towers.	BEJ; BDI
	259	R1	Right, but but if they're four high -	
	260	Stephanie	Yeah. It – it really	
	261	R1	it would be four. If they're three high	
	262	Stephanie	Um hm	

Time	Line	Speaker	Transcript	Code
	263	R1	If they're two high	
	264	Stephanie	Yeah. It would be two.	
	265	R1	Does that make sense?	
	266	Stephanie	Yeah.	
	267	R1	So if they're *n* high?	
	268	Stephanie	It would be *n*.	BR-S
	269	R1	Okay. So. So this would be	
	270	Stephanie	It – it – it still works.	BEJ
	271	R1	Actually it works that it's the number of towers, doesn't it?	
	272	Stephanie	Yeah.	
	273	R1	It's a way to think about it. So it's – you can think of it as the number of towers.	
	274	Stephanie	Um hm.	
	275	R1	Right.	
	276	Stephanie	Um hm.	
	277	R1	'Cause that always is that way, isn't it? So if the towers are *n* high,	
	278	Stephanie	It would be *n*.	
	279	R1	With exactly one, wouldn't you have anything, you could have – 'cause there are *n* positions, right?	
	280	Stephanie	Um hm.	
	281	R1	Okay. So. So. Towers *n* high, there'd be *n*. That helps me. If they're *n* high, how many positions are there to place that second one?	
	282	Stephanie	*n* minus one?	BR-S
	283	R1	Interesting? So it would be *n* times *n* minus one. Why don't we just write that down while we have it in our heads?	
	284	Stephanie	You want me to write *n* times *n* minus one?	PAH
	285	R1	What do you think? Do you want to? If we talk about towers *n* high, this – since we – you're doing algebra let's try to connect algebra.	
	286	Stephanie	Do you want me to write – ' For towers *n* high…'	PAH
	287	R1	For towers – yeah – why not? Let's think about these simple cases and see what we can do with *n* high.	
	288	Stephanie	For towers *n* high, um, like, if you're putting	
	289	R1	If you're moving from – what – what's the question here? If you're moving from um *n* things taken one at a time to *n* things taken two	

Time	Line	Speaker	Transcript	Code
			at a time	
	290	Stephanie	Okay.	
	291	R1	Right? Isn't that right?	
	292	Stephanie	Um hm. [*Stephanie is writing down all the information as it is being said.*]	
	293	R1	If you're thinking of selecting those exactly one red	
	294	Stephanie	From	
	295	R1	to selecting those with exactly two red.	
	296	Stephanie	So if…	
	297	R1	Could you imagine this in your head?	
	298	Stephanie	Yeah.	
	299	R1	as we're talking	
	300	Stephanie	Yeah.	
	301	R1	about this?	
	302	Stephanie	I know what you're saying.	
	303	R1	That's – that's very impressive. I couldn't have done that at your age, Stephanie.	
	304	Stephanie	[*chuckles*] (inaudible)	
	305	R1	Could you have, Donna? - - So I'm I'm not – I really – if you don't – if you can't imagine it, tell me 'cause I have to think.	
	306	Stephanie	Towers one *n* at a time [*writes this as she speaks*]	
	307	R1	Well, the way – um – I don't know – the way I like to write it	
	308	Stephanie	Okay	
	309	R1	is this way. [*It isn't possible from this tape to see what R1 writes.*] But my son tells me you can write it this way. And he told me another way you could write it. - - This way: 'C' 'n" 'one'. Isn't that right? There are three ways you can write it. You have your choice.	
	310	Stephanie	Um. Yeah.	
	311	R1	This means *n* high. This means exactly one red.	
	312	Stephanie	Um hm.	
	313	R1	Right? How many – all – this tells you that there are *n* of them and what you are supposed to imagine in your mind – these *n* high towers with	
	314	Stephanie	Um hm,	
	315	R1	Right?	
	316	Stephanie	Yeah.	

Time	Line	Speaker	Transcript	Code
	317	R1	One position at *n* minus one. If you can imagine – if you can begin to think like that and imagine these things.	
	318	Stephanie	You want me to write that instead?	
	319	R1	Well. Any way you – it makes sense to you.	
	320	Stephanie	‘Cause I’ll just	
	321	R1	If you want to start trying	
	322	Stephanie	move from towers	
	323	R1	to use the notation, this is good practice, you know.	
	324	Stephanie	*n* – um - one at a time [*Stephanie writes* $\begin{pmatrix} n \\ 1 \end{pmatrix}$.] to *n* two at a time. Um. It would be *n* times *n* minus one.	BR-S
	325	R1	Okay. So *n* would be	
	326	Stephanie	*n* would be like the tower – like the, the number of towers.	BR-S/V
	327	R1	Okay. So write that. *n* is the number total number of towers.	
	328	Stephanie	Okay. Well, it would be	
	329	R1	So what’s n minus one?	
	330	Stephanie	be *n* times *n* minus one. Um <u>*n*</u> be – *n*’s the number of towers. Ooooh. Wait. *n*’s the number of towers in like	BR-S/V
	331	R1	Do you know what just occurred to me? You can think of *n* as the number of towers. That’s true. That will always work. But it is also the total number of positions.	
	332	Stephanie	Yeah.	BDI
	333	R1	Alright? So with towers four high, you’d have four possible positions. Right? It just so happened	
	334	Stephanie	So. Should I write *n* being the number of towers and the number of positions?	
	335	R1	Yeah. I’m just trying to think what’s useful here.	
	336	Stephanie	Because it – the number of towers is useful, but the number of positions is useful when you’re talking about *n* minus one.	BEJ; BR-S/V
	337	R1	Maybe so. Yeah.	
	338	Stephanie	Okay.	
	339	R1	Okay. That’s a good idea.	
25:00-29:59	340	Stephanie	[*Stephanie speaks as she writes.*] The position – um – *n* minus one being the number of	PAH

Time	Line	Speaker	Transcript	Code
			positions – minus one?	
	341	R1	Let's try to think what the *n* minus one means. The *n* minus one in this problem is what?	
	342	Stephanie	The *n* minus one in this problem is?	
	343	R1	Um hm.	
	344	Stephanie	Is – like -	
	345	R1	What's *n*?	
	346	Stephanie	Well, *n* is this [*Stephanie indicates the entire tower.*] or	BEJ
	353	R1	What number is it?	
	354	Stephanie	*n* is four?	
	355	R1	Four. And so what's *n* minus one?	
	356	Stephanie	The red, I guess?	OBS
	357	R1	Well, if *n* is four -	
	358	Stephanie	Yes.	
	359	R1	We know what *n* minus one is.	
	360	Stephanie	Oh! It's three. [*laughs*]	BDI
	361	R1	That's what happens after a while, by the way. It really is 'cause you're thinking of something else. Now. Okay. So *n* is four. *n* minus one is three.	
	362	Stephanie	is three.	
	363	R1	But what does it mean in terms of moving from here to here? [*Dr. Maher moves from the towers with one red to towers with two reds.*]	
	364	Stephanie	That means that you're taking away – like – well, we're talking about – like – aren't you talking about like *n* minus one being like yellow minus space like yellow being like – replaced by red?	BEJ; PAH
	365	R1	Yeah. Right. That's exactly what I'm thinking about.	
	366	Stephanie	So like *n* minus one would be like this [*Stephanie points to the tower of all yellow.*] being replaced by this [*Stephanie indicates* $\begin{bmatrix} R \\ Y \\ Y \\ Y \end{bmatrix}$.] Right? 'Cause like – it's not taking away - -	BEJ; PAH
	367	R1	Okay. So wait a minute. I thought we were going from here to here. [*R1 indicates the towers with one red and then the towers with*	

Time	Line	Speaker	Transcript	Code
			two reds.] Let's do that one.	
	368	Stephanie	Yeah. Right.	
	369	R1	Let's go from here to here.	
	370	Stephanie	Alright.	
	371	R1	But what we did is, remember – this one belonged here. [*R1 moves* $\begin{bmatrix} R \\ R \\ Y \\ Y \end{bmatrix}$ *to beside* $\begin{bmatrix} R \\ Y \\ Y \\ Y \end{bmatrix}$.]	
	372	Stephanie	Um hm.	
	373	R1	Right?	
	374	Stephanie	Yeah,	
	375	R1	And then – uh – this one belonged here [*moves the* $\begin{bmatrix} R \\ Y \\ Y \\ R \end{bmatrix}$]	
	376	Stephanie	And this one [*Stephanie moves* $\begin{bmatrix} R \\ Y \\ R \\ Y \end{bmatrix}$.]	
	377	R1	This one belonged here. Right?	
	378	Stephanie	Um hm.	
	379	R1	Okay. So – when we moved from one to two, right?	
	380	Stephanie	Um hm.	
	381	R1	we ended up with three	
	382	Stephanie	Yes.	
	383	R1	Why three? Because – because why?	
	384	Stephanie	Well. If that's *n* – well – because we're putting – there's three places where you can put it.	BEJ
	385	R1	Right.	
	386	Stephanie	Yeah.	
	387	R1	Isn't that right?	
	388	Stephanie	Yeah.	
	389	R1	So you could put it here. [*R1 points to the second position.*] You could put it here. [*She points to the third position.*]	
	390	Stephanie	You could put it there.	
	391	R1	You could've put it there. [*R1 indicates the*	

Time	Line	Speaker	Transcript	Code
			bottom position.]	
	392	Stephanie	Um hm.	
	393	R1	Here, here, or here. So if you have *n* positions. Right?	
	394	Stephanie	Um hm.	
	395	R1	And really what you have here *n* minus one of them are yellow if one is red -	
	396	Stephanie	Um hm.	
	397	R1	Right. Because *n* minus one plus one is *n*.	
	398	Stephanie	Um hm.	
	399	R1	Does that make sense? *n* minus one plus one	
	400	Stephanie	[*laughs*] is just *n*.	BMP
	401	R1	Isn't that right?	
	402	Stephanie	Yeah.	
	403	R1	So we have *n* positions and one red and *n* minus one yellow.	
	404	Stephanie	Okay.	
	405	R1	Okay. So *n* minus one in this case – four positions – three yellow and of those three positions we could've put a red.	
	406	Stephanie	Um hm.	
	407	R1	And that's where the three came in. The three is the *n* minus one.	
	408	Stephanie	Okay.	
	409	R1	Alright. But that could've happened not just one time. It could have happened [*R1 taps the table in front of each of the four towers with one red and three yellows.*]	
	410	Stephanie	Four times.	
	411	R1	four times or *n* times.	
	412	Stephanie	Okay. – So like	
	413	R1	So it was *n* times *n* minus one.	
	414	Stephanie	Um hm.	
	415	R1	That's how we got twelve.	
	416	Stephanie	Yes.	
	417	R1	But we got duplicates.	
	418	Stephanie	Yes.	
	419	R1	Alright. So we know we know we ended up with *n*, right?	
	420	Stephanie	Um hm.	
	421	R1	*n* positions times *n* minus one – how many choices we have for red?	
	422	Stephanie	How many choices do you have for red? You have – um – I guess – *n* minus one number of	

Time	Line	Speaker	Transcript	Code
			choices?	
	423	R1	*n* minus one. Okay. So we have *n* times *n* minus one.	
	424	Stephanie	Um hm.	
	425	R1	If you're writing your formula down. But that's too many. Alright.	
	426	Stephanie	So *n* times.	
	427	R1	Four times three is too many.	
	428	Stephanie	Yeah. Divided by um	
	429	R1	In that case, what did you have to divide by?	
	430	Stephanie	Well, these number of positions.	BEJ
	431	R1	You actually ended up dividing by two.	
	432	Stephanie	Oh.	
	433	R1	When you found your duplicates.	
	434	Stephanie	Well. Oh. That's right. But I don't know how many -	OBS
	435	R1	Hmmm. That's the part. That's the tricky part. That's the part we haven't really worked out yet.	
	436	R2	Yeah.	
	437	Stephanie	[*in the background*] (inaudible)	
	438	R1	Why do we divide by two. Maybe we don't know that yet. Maybe that's something to keep in the back of our minds as something we're trying to figure out, right?	
	439	Stephanie	Um hm.	
	440	R1	But, we know we had to divide by two just by sheer working it out.	
	441	Stephanie	Yes.	
	442	R1	Isn't that true?	
	443	Stephanie	Yes.	
	444	R1	'Cause you found duplicates. So, so that time it was four times three divided by two. Which is what you wrote here, by the way.	
	445	Stephanie	Yeah.	
	446	R1	Four, right?	
	447	Stephanie	Yes.	
	448	R1	Four positions times three	
	449	Stephanie	Um hm.	
	450	R1	available postions to choose red.	
	451	Stephanie	Um hm.	
	452	R1	Right? Divided by the number of duplicates.	
	453	Stephanie	Yeah.	
	454	R1	Let's write down what these mean again so we	

Time	Line	Speaker	Transcript	Code
			don't lose track of that.	
	455	Stephanie	Here?	
	456	R1	So this number – the four – is like your *n* – the number of positions -	
	457	Stephanie	Alright.	
	458	R1	or the height of the tower – either way. How tall it is. However you want to think	
	459	Stephanie	Okay. Four is [*writing*] number of positions	BR-S/V
	460	R1	or height of tower. Right?	
30:00-34:59	461	Stephanie	Um hm. - - or height of tower. Um. Three is um the number of spaces you can put a red.	BR-S/V
	462	R1	Very good. So – that's very good. So the spaces available for red. [*Stephanie continues to write.*]	
	463	Stephanie	And two is 'cause they came in pairs.	BR-S/V
	464	R1	Number of duplicates. Right. And that what we we know how we know this is n and we know this is *n* minus one. [*R1 points to the four and the three.*] Right?	
	465	Stephanie	Um hm.	
	466	R1	That's very helpful. But. What is that two? …Hmmm…Right?	
	467	R2	That's a tough one (inaudible)	
	468	R1	I mean I studied probability at college and they told me you had to divide by these numbers - -	
	469	Stephanie	Um hm.	
	470	R1	I didn't know why. Did you know why when they just told you to divide by these numbers? No. Don't tell me. I'm not going to ask you. But – But do you see? That's – that's part of the problem. It would be nice to think about, is there a nice explanation that we can see as we generate this – that division by two. So let's keep that in back of our minds and go to the next step. Then we'll come to this one.	
	471	Stephanie	Okay.	
	472	R1	Just sorta of a reflecting on what we've done. So why don't we number that page one? And this will be page two. But we need to keep this in mind. So let's try to think of what happened after that.	
	473	Stephanie	Alright.	
	474	R1	So when we divided by two – all this stuff –	

Time	Line	Speaker	Transcript	Code
			this is the row we ended up with.	
	475	Stephanie	Um hm.	
	476	R1	Isn't that right? We had a row of six. Alright. Now before doing it – see if you can use the same kind of reasoning that we just used with the four times three divided by in that case two to think about what's happening here?	
	477	Stephanie	Um.	
	478	R1	Okay.	
	479	Stephanie	Well. It's six, because there's six towers.	BEJ
	480	R1	Um hm.	
	481	Stephanie	And it's going to be six times um, three because – wait – no it was six times two because there's two places to put it. Because it was four times three. Yeah.	BEJ; BR-S/V
	482	R1	It's getting a little trickier. Huh?	
	483	Stephanie	It's it's six times two because there's two places to put it.	BEJ
	484	R1	Okay. So you have two places to put it. You have your six towers four high. Right?	
	485	Stephanie	Yes.	
	486	R1	And so you're saying there are two possible ways – places of putting it as you go from two to three.	
	487	Stephanie	Um hm.	
	488	R1	So that's – two available positions – but see now – the the height of the tower seems to have nothing to do with this anymore. Does it?	
	489	Stephanie	Um hm.	
	490	R1	It's weird. You know what I'm saying.	
	491	Stephanie	Yeah.	
	492	R1	It had something to do with it before and it doesn't have anything to do with it now. Um. And I sorta you know before you know you had gee that was very nice. You had the height of the tower. Right?	
	493	Stephanie	Um hm.	
	494	R1	Times the available positions. Right?	
	495	Stephanie	Um hm.	
	496	R1	Divided by the duplicates. Right? I sort of liked that. The height of the tower, right? That was kind of nice.	
	497	Stephanie	Um hm.	

Time	Line	Speaker	Transcript	Code
	498	R1	The tower is still high – right – but that's, now if we talked about available positions, right? It wouldn't it, would be two – isn't that right?	
	499	Stephanie	Yeah.	
	500	R1	It sorta doesn't quite – it would give us eight.	
	501	Stephanie	um	
	502	R1	Right. And you're saying that what we have here is – um – height of the tower doesn't seem to enter into it. The available positions does. It's two. Right? But there's six towers. So maybe you have to think about this as the number of towers? Right. You started with four towers with exactly one of a color. Maybe that's what this has to be thought of. What do you think?	
	503	Stephanie	Yeah, you could	
	504	R1	'Cause if you want to be consistent	
	505	Stephanie	You can still get from four to eight – though. You'd have to divide by two. But – I don't know where the two comes from.	BEJ; OBS
	506	R1	Well, remember you want to get twelve up here.	
	507	Stephanie	Um hm	
	508	R1	You said from each of the six	
	509	Stephanie	Um.	
	510	R1	that have exactly two of a color, right?	
	511	Stephanie	Um hm.	
	512	R1	You have two available positions.	
	513	Stephanie	Yes.	
	514	R1	You multiply by two. –or you can say in exactly four with exactly one of a color	
	515	Stephanie	Um hm.	
	516	R1	Right? You have three available positions – you multiply by three. That's consistent. If you thought about it that way.	
	517	Stephanie	Um.	
	518	R1	It's a little bit different though.	
	519	Stephanie	Yeah.	
	520	R1	See it's a way to think about it. I don't know. But the problem then is you still have duplicates. Right?	
	521	Stephanie	Um hm. You have three of each.	BEJ
	522	R1	So here you divided by	
	523	Stephanie	three	

Time	Line	Speaker	Transcript	Code
	524	R1	three. Right? And you ended up with four.	
	525	Stephanie	Um hm.	
	526	R1	Well. Let's use that again. You ended up with four. Right?	
	527	Stephanie	Yes.	
	528	R1	And you have how many available positions?	
	529	Stephanie	Two. – For which one?	PAH
35:00-39:59	530	R1	When you ended up with four. One – [*Dr. Maher points to the groups of four – each with three red and one yellow.*]	
	531	Stephanie	Oh! You have one available position there.	BDI
	531	R1	So you have one available position and then you produce	
	532	Stephanie	four	
	533	R1	And how many	
	534	Stephanie	[*speaking at the same time as R1*] There are duplicates.	
	535	R1	duplicates? How many?	
	536	Stephanie	So you divide – four -	
	537	R1	You div – you have four then.	
	538	Stephanie	Yes.	
	539	R1	And that works. Right?	
	540	Stephanie	Um hm.	
	541	R1	Four times three is twelve, divided by two – you got six.	
	542	Stephanie	Um hm.	
	543	R1	Is that right? Did you get six the first time?	
	545	Stephanie	Yeah.	
	546	R1	And six times two is twelve divided by three, you got four. And four times one is four divided by four, you got one.	
	547	Stephanie	Um hm.	
	548	R1	So that was an interesting way to think about it.	
	549	Stephanie	Yes.	
	550	R1	But this number meant how many of that kind. Now if we used that same thinking here – I wonder if that works. Right?	
	551	Stephanie	Um hm.	
	552	R1	With with no reds, you'd have how many?	
	553	Stephanie	You'd have all yellow.	BR-S/V
	554	R1	But how many? - We said here this is how many with exactly one red.	

Time	Line	Speaker	Transcript	Code
	555	Stephanie	Um hm.	
	556	R1	This was how many with exactly two reds. This was how many with exactly three reds. [*points to the rows of towers on the table*]	
	557	Stephanie	With no reds, you'd have four. With no reds, you'd have all yellow. You'd have all - - I gu – okay here zer-	BEJ
	558	R1	You'd have one. That's the only one.	
	559	Stephanie	Oh! Well -	BDI
	560	R1	Alright. Isn't that right?	
	561	Stephanie	Yeah.	
	562	R1	Does that work? You'd have one.	
	563	Stephanie	Um hm.	
	564	R1	And from this one – right?	
	565	Stephanie	Um hm.	
	566	R1	How many available positions do you have for exactly one red?	
	567	Stephanie	Four.	BR-S/V
	568	R1	So from that one you have four.	
	569	Stephanie	Yeah.	
	570	R1	But you don't divide by – right? – right?	
	571	R2	That's a tough one (inaudible)	
	572	R1	[*to R2*] What would you do if you had a situation like that?	
	573	Stephanie	Wouldn't you divide, though? 'Cause -	PAH
	574	R1	By what?	
	575	Stephanie	Well. –Wait. But if you're doing it with each one -	BEJ
	576	R1	We're supposed to end up with four, remember?	
	577	Stephanie	Oh. Yeah. That's right. I was thinking 'cause we had all four. Forget it.	BEJ
	578	R1	We're supposed to end up with one, two, three, four. And you're starting with no reds.	
	579	Stephanie	Um hm.	
	580	R1	[*to R2*] What do you do when you teach this? What do you tell your students? Do you hand wave it? Most textbooks hand wave it.	
	581	R2	Well. I guess you look at it as how many places you have to move things around.	
	582	R1	How many places you have to move things	
	583	Stephanie	You have four.	
	584	R1	around	

Time	Line	Speaker	Transcript	Code
	585	Stephanie	We have four places. But – we can't divide by that.	OBS
	586	R1	Well. Here you had three places to move things around. That was the three. There's four places to move things around.	
	587	Stephanie	Um hm.	
	588	R1	Right? So that works.	
	589	R2	Maybe if we think about how you grouped things when you were finished. If they're related.	
	590	R1	Here you divided by two. To make this work – what would you have to divide by here?	
	591	Stephanie	Oh! These are groups of one!	BDI
	592	R1	Okay. So here you divided by two. Here you divided by three. Here you divided by four.	
	593	Stephanie	Oh!	BDI
	594	R1	To make this work – what would you have to divide by	
	595	Stephanie	Yeah. But – oh – 'cause here we divided by the groups. 'Cause here there were groups of two. Here there were groups of three. Here there's groups of one.	BEJ; BDI
	596	R1	I don't understand. Help me.	
	597	Stephanie	All right. For this one. For like the second one, where there were four times three. There were groups of two. Like they came in pairs. There were two of these. Right?	BEJ
	598	R1	Um hm.	
	599	Stephanie	So they came in groups of two. So we divided by two.	BEJ
	600	R1	Um hm.	
	601	Stephanie	And here for the six, they came in groups of three. So you divided by three.	BEJ
	602	R1	Um hm.	
	603	Stephanie	But here	
	604	R1	How would you know what they come in groups of? – Unless you were all done?	
	605	Stephanie	Because there were three duplicates. Here's the two duplicates.	BEJ
	606	R1	How would you know before you start how many duplicates there would be?	
	607	Stephanie	You mean -	
	608	R1	I mean here you divided by one; here you divided by two; here you divided by	

Time	Line	Speaker	Transcript	Code
	609	Stephanie	The number of reds in it?	PAH
	610	R1	But isn't that nice? It goes one, two, three, four.	
	611	Stephanie	Um hm.	
	612	R1	Say I wonder if we were doing it the next way, would it be one, two, three, four, five? You know if we were going five high?	
	613	Stephanie	Um hm. Uh.	
	614	R1	Do you know what I'm saying?	
	615	Stephanie	Yeah.	
	616	R1	That's an interesting question, isn't it?	
	617	R2	It is.	
	618	R1	And if we were doing six high, would it be divided by one, two	
	619	Stephanie	And	
	620	R1	Or does it – is it a symmetry thing? Does it come back? I'm curious about that. Wouldn't that be something to explore?	
	621	Stephanie	Um hm.	
	622	R1	Are you getting sick of towers or is it interesting?	
	623	Stephanie	No. It's interesting. And 'cause they come in groups of one. Like this.	BEJ
	624	R1	Yeah. [*to R2*] Do your students think about groups like that?	
	625	R2	Some of them do.	
	626	R1	Some of them do.	
	627	R2	Yeah.	
	628	R1	Some kids will think about it. Okay.	
	629	R2	This is a very interesting way, though.	
	630	R1	To think about it?	
	631	R2	Yeah.	
	632	R1	Yeah. So I guess the next thing is to try this with five. I mean you know what the answers are. So you know what you're working for.	
	633	Stephanie	Okay.	
	634	R1	You may not even have to build them. 'Cause if you think of the number of positions. I mean you just have to do it for one.	
	635	Stephanie	Alright. For the first one – if you're	
40:00-44:59	636	R1	Oh! You want to do it now?	
	637	Stephanie	Well. Oh!	
	638	R1	Okay. We can if you want to. Wow! Um.	

Time	Line	Speaker	Transcript	Code
			Alright. Let's do it. For the first one. Okay. We need some more. We'll build the basic ones. This destroys all the things I worked on. I tried this before I came. But we can rebuild these. So if we're going five – this is what we're starting with, right?	
	639	Stephanie	Um hm.	
	640	R1	One of these. There. [*R1 builds* $\begin{bmatrix} Y \\ Y \\ Y \\ Y \\ Y \end{bmatrix}$.] Okay.	
	641	Stephanie	Okay.	
	642	R1	So I'll let you be recorder. This is my attempt at recording. So we're going to start with the case with	
	643	Stephanie	The first one's one	BR-S
	644	R1	There's one of those	
	645	Stephanie	times five	BR-S
	646	R1	Why five?	
	647	Stephanie	'Cause there's five positions.	BEJ
	648	R1	Okay.	
	649	Stephanie	Divided by one, 'cause they come in groups of one.	BEJ
	650	R1	Um hm.	
	651	Stephanie	Five.	BR-S
	652	R1	Okay. So that's five things taken one at a time.	
	653	Stephanie	Yes. The second one	
	654	R1	Why don't we write that down? Five things taken – equals five things taken one at a time. [*Stephanie writes.*]	
	655	Stephanie	Okay. For the second one – um – there's four spaces. But there's – out of five – so it's five times four and they'll come in groups of – I don't know – um – that's what we don't know though.	BEJ; BR-S; OBS
	656	R1	Alright. So. Let's – Can we make these five? - Just - here	
	657	Stephanie	Well – maybe they'll come in groups of two?	PAH
	658	R1	One – Let's think about at least one of these.	
	659	Stephanie	They might come in groups of two, I guess.	BR-V
	660	R1	Hm. Interesting. It's not easy to imagine what	

Time	Line	Speaker	Transcript	Code
			they come in. That - - one more do we need? [*They have built* $\begin{bmatrix} R \\ Y \\ Y \\ Y \\ Y \end{bmatrix} \begin{bmatrix} Y \\ R \\ Y \\ Y \\ Y \end{bmatrix} \begin{bmatrix} Y \\ Y \\ R \\ Y \\ Y \end{bmatrix} \begin{bmatrix} Y \\ Y \\ Y \\ R \\ Y \end{bmatrix}$.]	
	661	Stephanie	Yeah.	
	662	R1	(inaudible)	
	663	Stephanie	One red on the bottom. [*They add* $\begin{bmatrix} Y \\ Y \\ Y \\ Y \\ R \end{bmatrix}$ *to the row of towers.*]	BR-V
	664	R1	Alright. So. That's these five, right?	
	665	Stephanie	Yes.	
	666	R1	Okay. so what you're saying here - - move some of this aside - - um – okay. Let's think of that one. [*R1 indicates* $\begin{bmatrix} R \\ Y \\ Y \\ Y \\ Y \end{bmatrix}$.]	
	667	Stephanie	Okay.	
	668	R1	There are five.	
	669	Stephanie	Yes. [*Stephanie begins to build.*] You have one like that. [*builds* $\begin{bmatrix} R \\ R \\ Y \\ Y \\ Y \end{bmatrix}$], one like that [*builds* $\begin{bmatrix} R \\ Y \\ R \\ Y \\ Y \end{bmatrix}$]	BR-V
	670	R1	Well, can you predict before you do it?	

Time	Line	Speaker	Transcript	Code
	671	Stephanie	Yeah. There's going to be four from each. There's gonna be	BR-V
	672	R1	four from each.	
	673	Stephanie	Yeah.	
	674	R1	Okay. So – and what's the each? How many make up each?	
	675	Stephanie	How – wh – what do you mean?	PAH
	676	R1	You're saying – it's four from this.	
	677	Stephanie	Yeah. Four from	
	678	R1	What does	
	679	Stephanie	one	
	680	R1	each mean in this case?	
	681	Stephanie	Oh! Like there's going to be four from this one. Four from that one. Four from that one. Four from that one. Four from that one.	BEJ; BDI
	682	R1	Okay. So how many eaches?	
	683	Stephanie	There's five.	BR-V
	684	R1	Five eaches. Okay.	
	685	Stephanie	Yeah.	
	686	R1	Alright. So – so that – you say – five times four	
	687	Stephanie	Yes. I have that. I just don't know what the -	
	688	R1	right	
	689	Stephanie	bottom part. It	OBS
	690	R1	So - - and by groups you mean. The groupings you mean.	
	691	Stephanie	Groups like – one after we've put all of them out. Like how many groups there are going to – come in -	BEJ
	692	R1	I don't know. I'm	
	693	Stephanie	duplicates?	BEJ
	694	R1	I'm wondering – when you say you divide by	
	695	Stephanie	Oh! 'Cause that's the number of duplicates. That there are.	BEJ
	696	R1	But how do you know that before hand? Do you think there's a way?	
	697	Stephanie	Oops. [*She is building towers.*]	
	698	R1	So if this um, is going to be a pattern to this – the five times four – what do you think you would divide by?	
	699	Stephanie	Five times four – what do I think I'd – um – maybe two.	OBS
	700	R1	You would guess two? Let's see, what does Pascal's triangle say? Do it quick. Right? [*R1*	

Time	Line	Speaker	Transcript	Code
			draws Pascal's Triangle as she speaks.] Have to make that a four. Okay. – Is that right? Did I do that right?	
	701	R2	Um hm.	
	702	Stephanie	You've – well – there's the ten, so …if – I guess this is the	OBS
	703	R1	So this is	
	704	Stephanie	Five, 'cause that's zero. So that's one. That's like this one. [*It is not apparent to which she is referring from this vantage point.*] Right? Or that's	BEJ; PAH
	705	R1	This is	
	706	Stephanie	Like -	
	707	R1	Um hm.	
	708	Stephanie	That's this one. [*She points to the row of towers of four yellow and one red.*]	BR-S/V
	709	R1	Alright. So if we were writing a formula -	
	710	Stephanie	This is this. [*points to the one and then to the all yellow tower.*]	BR-S/V
	711	R1	Um hm.	
	712	Stephanie	This is these. [*points to the five and then to the row of the five towers, each with four yellows and one red*] So that – it – you would divide by two.	BR-S/V
	713	R1	So it works with the result. Okay. We'll explore that later.	
	714	Stephanie	Um.	
	715	R1	Okay.	
	716	Stephanie	And the next one, there's three spaces to put it. And there's	BR-S/V
	717	R1	But you have how many of them?	
45:00-49:59	718	Stephanie	Ten. So it would be ten times three and you divide by three. [*Stephanie writes as she speaks.*]	BR-S/V
	719	R1	And it worked?	
	720	Stephanie	Yeah. And the next one, there is two spaces to put it and you have ten. So that's ten times two and you divide by two? [*Stephanie writes on the paper in front of her.*] And the last one – there's one space to put it – it's five times one divided by five equals one.	BR-S/V
	721	R1	Okay. Why did you switch to dividing by two here? Why didn't you divide by four? Why didn't you go one, two, three, four, five? Here you went two and here you went five.	

Time	Line	Speaker	Transcript	Code
	722	Stephanie	Because – um – well, I was kind of thinking that if there there was one red and there were two reds and – I don't know – I guess I should've divided by four. Oh! – Duh – Yeah. [*Stephanie changes the two she wrote to a four.*] Yeah. That's right. It should be a four. I just wrote – I wasn't dividing right.	BEJ; BDI
	723	R1	So is that right then?	
	724	Stephanie	Yeah.	
	725	R1	And then you started with the one here, because there was one way of doing that?	
	726	Stephanie	Um hm.	
	727	R1	Should we try the next row? You think -	
	728	Stephanie	Okay.	
	729	R1	You think that makes sense? One – six – fifteen – twenty – fifteen – six – one. So we know the one and six. That's easy, right?	
	730	Stephanie	Times six divided by one – six – [*Stephanie writes.*] The next one is six times five divided by two. That's fifteen. The next one is fifteen times four divided by three. Gosh. Fifteen times four – sixty divided by three – twenty. The next one is twenty times three divided by four. Oops. Sixty. Fifteen. Next is fifteen times oh and there's two spaces. That's thirty um divided by five. That's um six – six – [*Stephanie is writing very quickly as she is speaking.*] is one. Yeah. That works.	BR-S/V
	731	R1	Okay. What do you think?	
	732	Stephanie	Um hm.	
	733	R1	Um. There's a nice pattern here. Um. Why are those – that many duplicates coming up?	
	734	Stephanie	Um.	
	735	R1	You know what I'm saying? We – we know – you know it after you do it and then you can count them.	
	736	Stephanie	Um hm.	
	737	R1	You understand what I'm saying?	
	738	Stephanie	Um hm.	
	739	R1	You know it after you do it and you count the next one. Right?	
	740	Stephanie	Um hm.	
	741	R1	But the question is – um – which is not a trivial question, is – how can you think about those duplicates if you'd – and I bet you can justify	

Time	Line	Speaker	Transcript	Code
			it by going through all the tediousness of finding it, but I don't think any of us want to do that, right? Um. But it would be nice to know in advance, how many duplicates come up. And that's – we have a good conjecture. It matches this and this, so it's a good tool, but we still haven't thought about why that works. Let's put that aside for a minute. Okay? Can I switch gears?	
	742	Stephanie	Um hm.	
	743	R1	'Cause I was interested in something like this piece of the triangle. Let's take – um – this is a four. [*R1 draws in the diagonal lines from one to four and three to four.*] Right?	
	744	Stephanie	Um hm.	
	745	R1	Or this piece. Right? Or this piece. – um – what did I do wrong here? Um. [*lines drawn in the opposite direction*]	
	746	Stephanie	(inaudible)	
	747	R1	What did I put there?	
	748	R3	It's two from the top, one from the bottom.	
	749	R1	I know it. Which is – I wanted to do this one – Thank you, Steve. I want to do the ten. The four to the ten. And um, do you understand what I'm saying?	
	750	Stephanie	Um hm.	
	751	R1	We went – we went this way. And what we still have to think about is – we have a pattern. We have a rule. We have a way of generating it and we know what to divide by because of a pattern, but I – and you said groupings.	
	752	Stephanie	Um hm.	
	753	R1	And I'm not so sure I follow the groupings stuff yet. Like I see the um – let's say here: six times two divided by three. Right?	
	754	Stephanie	Um hm.	
	755	R1	But I didn't see those groupings of three until you were all done and grouped them as three.	
	756	Stephanie	Yeah.	
50:00-54:59	757	R1	And I would've liked to have known why there'd be groupings of three before you did it and just counted. – I'd like to be able to have some way of thinking about that, to know that.	
	758	Stephanie	Um.	
	759	R1	Do you understand what I'm saying?	

Time	Line	Speaker	Transcript	Code
	760	Stephanie	Yeah. But it's like because there's three places that have the same – one in the same positions. Like -	
	761	R1	So you're telling me that when you move from here to here [*R1 indicates from the towers that are two reds and two yellows to the towers that are three reds and one yellow.*]	
	762	Stephanie	Like this $\begin{bmatrix} R \\ R \\ R \\ Y \end{bmatrix}$ one	
	763	R1	from two reds to three reds.	
	764	Stephanie	[*as she moves the appropriate towers*] This can be from this $\begin{bmatrix} R \\ Y \\ R \\ Y \end{bmatrix}$, it can be from this $\begin{bmatrix} R \\ Y \\ Y \\ R \end{bmatrix}$. Oh no, it can't be from that. It can be from this $\begin{bmatrix} Y \\ R \\ R \\ Y \end{bmatrix}$ or it can be from this $\begin{bmatrix} R \\ R \\ Y \\ Y \end{bmatrix}$.	BEJ; BR-V
	765	R1	So how do you know that in advance?	
	766	Stephanie	I guess you could look at 'em and say these – have I guess this space in common, but	BEJ
	767	R1	Okay. So it came from those three.	
	768	Stephanie	Yes.	
	769	R1	Um hm. I could see that. And so I have to imagine that this $\begin{bmatrix} R \\ R \\ Y \\ Y \end{bmatrix}$ could come, could come from the shorter one like that or the other one.	
	770	Stephanie	Um hm.	
	771	R1	And only two.	
	772	Stephanie	Yeah. (inaudible)	
	773	R1	Uh huh. – Okay. So – so – you would, you would know, for instance, something like this. – That there are five positions that could've	

Time	Line	Speaker	Transcript	Code
			generated this? [*R1 points out* $\frac{15 \cdot 2}{5}$.] How would you know that?	
	774	Stephanie	Um.	
	775	R1	It could've come from you know what I'm saying that [*Stephanie sneezes.*] God bless you.	
	776	Stephanie	Thank you.	
	777	R1	Hmmm.	
	778	Stephanie	I guess you'd – I mean – you'd really I guess have to be looking at it. I really – I probably	BEJ
	779	R1	A visual picture helps?	
	780	Stephanie	I probably couldn't like say well that's gonna come from that, that, that, that, and that.	
	781	R1	Yeah. Let's go back to this, so I can make a neater picture. [*redraws Pascal's triangle on a new sheet of paper*] I'm doing it again. I keep making this a six and I want it to be a four. Okay. Um. Let's explore um – which one should we explore? Let's do this one. [*R1 selects* 1 3] \ / 4 Okay?	
	782	Stephanie	Um hm.	
	783	R1	Do you know what this one means? If you had to build this one, what would that tower look like?	
	784	Stephanie	That one?	
	785	R1	What would that tower look like? What would these two look like?	
	786	Stephanie	[*There is a pause as Stephanie begins building towers.*] I think that one would be like this and – that one [*Stephanie indicates the one in R1's selection from Pascal's triangle. Stephanie has built this tower.* $\begin{bmatrix} Y \\ Y \\ Y \end{bmatrix}$] And that one	BR-S/V
	787	R1	(inaudible) three high, no red.	
	788	Stephanie	[*Stephanie continues to build and move towers.*] like this. [*She makes* $\begin{bmatrix} R \\ Y \\ Y \end{bmatrix}\begin{bmatrix} R \\ R \\ Y \end{bmatrix}\begin{bmatrix} R \\ Y \\ R \end{bmatrix}$.]	

Time	Line	Speaker	Transcript	Code
	789	R1	Okay. Three high. – Exactly one red.	
	790	Stephanie	Yes.	
	791	R1	Okay.	
	792	Stephanie	Oh! Wait! [*Stephanie corrects the two towers with two red cubes and builds the missing tower.* $\begin{bmatrix} R \\ R \\ Y \end{bmatrix}\begin{bmatrix} R \\ Y \\ R \end{bmatrix}$ *to* $\begin{bmatrix} Y \\ R \\ Y \end{bmatrix}\begin{bmatrix} Y \\ Y \\ R \end{bmatrix}$.]	BR-S/V
	793	R1	Okay, makes you dizzy after a while, doesn't it? 'Cause I think I see exactly one also. Even when you make it, I just believe you're gonna do it. Okay. Now. When we – doing this [1 3] \ / 4	
	794	Stephanie	Um hm.	
	795	R1	What's different about these and this tower here [*tapping the number four from Pascal's triangle*] that I call four? There	
	796	Stephanie	Well – it's four high.	BR-S/V
	797	R1	Okay. So there's one of these. [*indicates* $\begin{bmatrix} Y \\ Y \\ Y \end{bmatrix}$] That's one. Right?	
	798	Stephanie	Yes.	
	799	R1	There are three of these. [*indicates the towers with two yellows and one red*]	
	800	Stephanie	Yes.	
	801	R1	And that's exactly one red.	
	802	Stephanie	Um hm.	
	803	R1	And and that's four, but what else about it?	
	804	Stephanie	Like?	
	805	R1	They're four high.	
	806	Stephanie	What else now?	
	807	R1	What else can you tell me about this? They're four tall.	
	808	Stephanie	Um hm.	
	809	R1	What about the coloring of this?	
	810	Stephanie	Well, there's going to be three of one color and one of the other instead of two and one like for three.	BEJ

Time	Line	Speaker	Transcript	Code
	811	R1	Okay. So these are going to be four tall.	
	812	Stephanie	Um hm.	
	813	R1	And next, and there's going to be three of one color.	
	814	Stephanie	And one of another.	
	815	R1	And what's what's the – what's the one – what's the one color?	
	816	Stephanie	One can be red.	
	817	R1	Well	
	818	Stephanie	And three could be	
	819	R1	Well, we have to be consistent.	
	820	Stephanie	Alright. One is red. Three is yellow.	BEJ
	821	R1	One is red and three is yellow.	
	822	Stephanie	Yes.	
	823	R1	Okay. Now study that.	
	824	Stephanie	You want me to tell you why those give you four.	PAH
	825	R1	I want to know from here	
	826	Stephanie	Uh huh	
	827	R1	What would you do to these	
	828	Stephanie	Well	
	829	R1	to get me, to get me	
	830	Stephanie	Well, I'd build them higher.	BEJ
	831	R1	Well, don't don't do it yet. Just think about it for a minute. Remember what they're going to look like.	
	832	Stephanie	Yeah.	
	833	R1	There's going to be exactly one red.	
	834	Stephanie	This would go here. [*moves the* $\begin{bmatrix} Y \\ Y \\ Y \end{bmatrix}$ *over*] and there would be red	BEJ; BR-S/V
	835	R1	No. No. We start with these [*R1 indicates the four again and moves the* $\begin{bmatrix} Y \\ Y \\ Y \end{bmatrix}$ *back.*] I don't need to touch these. I want you to tell me what you're gonna do to these so that when you're all done	
	836	Stephanie	Um hm.	
	837	R1	you end up with exactly one red. But you got to make them all four tall.	

Time	Line	Speaker	Transcript	Code
	838	Stephanie	I'm going to put a yellow here [*points to* $\begin{bmatrix} R \\ Y \\ Y \end{bmatrix}$]	BEJ; BR-S/V
	839	R1	Okay.	
	840	Stephanie	I'm gonna put a yellow there. [*points to* $\begin{bmatrix} Y \\ R \\ Y \end{bmatrix}$]	BEJ; BR-S/V
	841	R1	Right.	
	842	Stephanie	I'm going to put a yellow there. [*points to* $\begin{bmatrix} Y \\ Y \\ R \end{bmatrix}$] and I'm gonna put a red there. [*points to* $\begin{bmatrix} Y \\ Y \\ Y \end{bmatrix}$]	BEJ; BR-S/V
	843	R1	Okay. So how many ways – how many do you end up with?	
	844	Stephanie	Four	
	845	R1	Four. – So from the one three tall with no reds	
	846	Stephanie	Um hm.	
55:00–59:59	847	R1	And the three three tall with one red, right?	
	848	Stephanie	Yes.	
	849	R1	You end up with four four tall with one red.	
	850	Stephanie	Um hm.	
	851	R1	Isn't that neat?	
	852	Stephanie	Yeah.	
	853	R1	Okay. Let's do another one. Which one should we do? Um. [6 4] \ / 10	
	854	Stephanie	Okay.	
	855	R1	That's a little hard.	
	856	Stephanie	Um. That's – well we had that. That would be -	
	857	R1	So, what's this one? Tell me what this one is. [*R1 points to the six.*]	
	858	Stephanie	Those are four high with two red.	BR-S/V

Time	Line	Speaker	Transcript	Code
	859	R1	Okay. And there are how many of those?	
	860	Stephanie	Six of them.	BR-S/V
	861	R1	Okay. I can help you a little bit. [*They pull already built towers from the pile on the table.*] Um. Here's another one.	
	862	Stephanie	We already have this one.	
	863	R1	A bunch of duplicates here.	
	864	Stephanie	Two, three, four, five. We need one more. What one do we need?	BR-S/V
	865	R1	Which one is missing?	
	866	Stephanie	Um. The one with two on the bottom. I'll just make it.	
	867	R1	Here.	
	868	Stephanie	Oh! Okay. Oh! Wait! [*Stephanie sees that the tower is upside down.*]	
	869	R1	Oh wait! [*She reverses the order of the cubes.*] There you go.	
	870	Stephanie	Alright.	
	871	R1	Okay. We better move these a little bit. Are you sure we have them all?	
	872	Stephanie	Yes. There's the six. And this – [*Stephanie points to the four.*] is um one with three red and one um one yellow.	BR-S/V
	873	R1	Okay. Three red and one yellow. These are the same?	
	874	Stephanie	Yes. [*R1 moves over towers as Stephanie builds.*] Um. (inaudible) one and – that one	
	875	R1	So there are four of them.	
	876	Stephanie	Um hm.	
	877	R1	And in order to help me [*R1 rearranges the order of the towers.*] Do you mind?	
	878	Stephanie	Yeah. Go ahead. [*The towers are arranged:* $\begin{bmatrix}Y\\R\\R\\R\end{bmatrix}\begin{bmatrix}R\\Y\\R\\R\end{bmatrix}\begin{bmatrix}R\\R\\Y\\R\end{bmatrix}\begin{bmatrix}R\\R\\R\\Y\end{bmatrix}\quad\begin{bmatrix}R\\Y\\Y\\R\end{bmatrix}\begin{bmatrix}Y\\R\\Y\\R\end{bmatrix}\begin{bmatrix}R\\Y\\R\\Y\end{bmatrix}\quad\begin{bmatrix}Y\\Y\\R\\R\end{bmatrix}$ $\begin{bmatrix}Y\\R\\R\\Y\end{bmatrix}\begin{bmatrix}R\\R\\Y\\Y\end{bmatrix}$]	

Time	Line	Speaker	Transcript	Code
	879	R1	Okay.	
	880	Stephanie	Okay.	
	881	R1	So I believe that. In combina – combinatorics how would you write this? This six?	
	882	Stephanie	How would I write that one?	PAH
	883	R1	Yeah.	
	884	Stephanie	Um.	
	885	R1	Just write it with an arrow and tell me what this is – what these numbers are.	
	886	Stephanie	The six the six is the two – so that would be be um – you want me to write it here?	BR-S/V
	887	R1	Sure.	
	888	Stephanie	[*writes* C_2^4] Four. Two.	BR-S
	889	R1	And this one is? [*Stephanie writes* C_3^4.] Four three and we're adding them together. Plus.	
	890	Stephanie	Oh.	
	891	R1	to get	
	892	Stephanie	to get [*Stephanie writes* C^5.] um. [*She then adds the three:* C_3^5.]	
	893	R1	Is that three?	
	894	Stephanie	Um. That's	
	895	R1	Another one, two, three.	
	896	Stephanie	Yeah.	
	897	R1	It is three. Okay. So. Um. What does that mean? What is -	
	898	Stephanie	That means you have four and you're selecting two. You're taking – well you're taking two red	BEJ
	899	R1	Okay. Exactly two red.	
	900	Stephanie	and	
	901	R1	And then you have exactly three red.	
	902	Stephanie	Yes.	
	903	R1	And now you're making them – how tall?	
	904	Stephanie	Five tall.	BR-S/V
	905	R1	Five tall. And how many reds are there going to be?	
	906	Stephanie	Three.	BR-S/V
	907	R1	So how can you make them five tall with three reds?	
	908	Stephanie	Red there. Red there. Red there.	BEJ
	909	R1	So here you get three ways, right?	

Time	Line	Speaker	Transcript	Code
	910	Stephanie	A red there. A red there. A red there. A yellow there. A yellow there, a yellow there and a yellow there.	BEJ
	911	R1	There's your ten.	
	912	Stephanie	Yes.	
	913	R1	Isn't that neat!	
	914	Stephanie	Um hm.	
	915	R1	That's what it is. I think that's so neat.	
	916	R2	It is.	
	917	R1	Do you like that?	
	918	R2	Yes.	
	919	R1	So the question is – think about these in general ways – you know. Are there general ways to be it? - And you see, we, we could do it in arithmetic with these combinatorics. We're saying four things two plus four things three is five things three.	
	920	Stephanie	Um hm.	
	921	R1	It's kind of arithmetic, isn't it? [*Stephanie laughs.*] I mean if you just start writing these as combina – or do we say here – we said – this is which row?	
	922	Stephanie	Um. That's the three. So that [*writes* C^3]	BR-S
	923	R1	Three things – each one is	
	924	Stephanie	That's none? [*writes* C^3_0]	BR-S
	925	R1	None. Right.	
	926	Stephanie	And that means one.	BR-S
	927	R1	plus	
	928	Stephanie	Three. One. [*writes* C^3_1]	BR-S
	929	R1	Right? And we said that's gonna give you	
	930	Stephanie	Four. One. [*writes* C^4_1]	BR-S
	931	R1	Isn't that an interesting kind of arithmetic?	
	932	Stephanie	Um hm.	
	933	R1	Now what I'm going to ask you to do – to think about	
	934	Stephanie	Okay.	
	935	R1	is to, is to write as many of these and convince yourself and see if you can come up with a general rule with your *n*'s and *n* minus one's or whatever.	
	936	Stephanie	Okay.	
	937	R1	You can call – what you can do is call this *n* and this *r*. [*writes* C^n_r] Right? Or you can	

Time	Line	Speaker	Transcript	Code
			call this *r* and then you can call this *r* minus one.	
	938	Stephanie	Okay.	
	939	R1	Right? If this is *r*, this is one mi – whatever – you understand? If this is *r*, this is *r* minus one. If this is *r*, this is *r* plus one.	
	940	Stephanie	Yes.	
	941	R1	You can do it any way you want to.	
	942	Stephanie	Yes.	
	943	R1	Do you see what I'm saying?	
	944	Stephanie	Yeah.	
	945	R1	You could go one either way or the other. And see if you can develop a prediction of a rule.	
	946	Stephanie	Okay.	
	947	R1	Won't that be – see if you can play around with the algebra.	
1:00:00–1:04:59	948	Stephanie	Okay.	
	949	R1	This is great. You did so well, Stephanie.	
	950	Stephanie	Thank you.	
	951	R1	Do you have any questions?	
	952	Stephanie	Not really.	
	953	R1	Steve, you had a question.	
	954	Steve	Hmmm.	
	955	R1	You had a division question about that other one? Did you have a question back there?	
	956	Steve	No. No. I'll talk to you later.	
	957	R1	Okay. I think probably we've done enough. We've we've been working really hard. It's very hard.	
	958	R2	Very hard.	
	959	R1	You've been doing really well. This – you might like this area of mathematics. It's called – combinatorics. It's the whole basis for probability. Which you can you can specialize and study as a whole field. Those people who work in insurance companies, actuaries, have to study all of this probability and combinatorics. –and to study statistics as a field you need to know all this counting and combinatorics and it's all so kind of a basic math kind of set of ideas which I find fascinating. There's a lot of work in number theory that has to do with this. I don't know – do you like to read histories of math or	

Time	Line	Speaker	Transcript	Code
			anything?	
	960	Stephanie	I have to. We have to do a report.	
	961	R1	Oh, you do?	
	962	Stephanie	Yeah. We just got it today – yesterday. We got a list of these mathematicians.	
	963	R1	Well, you know – the problem you worked on was conjectured by Fermat.	
	964	Stephanie	Yeah. That's – I think that's on the list. – I don't – she either did put that on the list or didn't and asked me about it. 'Cause she was telling me about it.	
	965	R1	Now I don't know the reference for this. But I'm going to talk with Dr. Speiser tonight.	
	966	Stephanie	Okay.	
	967	R1	And see if he does. Maybe you want to make that as your project.	
	968	Stephanie	Yeah. That'd be	
	969	R1	Maybe, he can, he can fax me some things. Wouldn't that be interesting?	
	970	R3	(inaudible) find something like that (inaudible)	
	971	R1	Yeah. That would be fantastic. So. I would – I haven't seen the Fermat materials myself. Fermat is, has become very much of interest to mathematicians because, he had this habit of writing in the margins of books that he has a proof, but there's not enough room. And so for centuries, people can't do the proof. And they wonder 'Did he really have a proof?' or not. And there was a problem that um mathematicians have been working on for centuries and they thought they solved it recently – in the last couple of years ago they had these special um, um, I guess colloquia in Princeton. And he did all but a piece and then, then they found out that it wasn't quite right. This is all after they thought he proved it – this famous mathematician. There are some newspaper articles about that um which would be interesting to track also. But it was dealing with what's called Fermat's Last Theorem.	
	972	Stephanie	Um hm.	
	973	R1	His last one that he conjectured. Apparently it wasn't a trivial proof.	
	974	Stephanie	Um.	
	975	R1	But he also made the conjecture – that – of, of	

Time	Line	Speaker	Transcript	Code
			what you've just done uh from here to here to here to here to here. The towers. But he provided some of the proofs. So we'll leave that for your doctoral dissertation. Okay?	
	976	Stephanie	Okay.	
	977	R1	So no one has any questions? You have no questions?	
	978	R2	No. I'm just	
	979	R1	Do you have anything to ask us?	
	980	Stephanie	No. I'm fine.	
	981	R1	Um. Yeah. I really would like you to kind of write this up and keep track. But this is also a nice – if you can pull some of these ideas together, Stephanie, these are great science/math fair projects.	
	982	Stephanie	Okay.	
	983	R1	to pull together and keep records. I just wonder if there's a way to do this on the computer. You know – to generate some of this -	
	984	R2	You mean in this kind of detail?	
	985	R1	Right. I mean to keep track. Um. I know that uh do you remember a few years ago um Dr. Davis was in and you were dealing with the Tower of Hanoi?	
	986	Stephanie	Yes.	
	987	R1	That was – you have these towers and you were looking at moves	
	988	Stephanie	Yeah.	
	989	R1	and trying to predict, right? - how many moves it would take to move the towers, right?	
	990	R2	(inaudible)	
	991	R1	Um. And then I remembered, I, you all were coming up with it trying to come up with a procedure for doing it. I think we have a video tape of that. Of um - and your procedure was like um sorta this idea – that you could predict the next row if you knew the row before.	
	992	Stephanie	Um hm.	
	993	R1	And what I'm asking you to think about is the way of – you know, if you know the one before you could do the next one. It was like um predicting how many towers you can build twenty tall.	
	994	Stephanie	Um hm.	

Time	Line	Speaker	Transcript	Code
	995	R1	Do you remember that?	
	996	Stephanie	(inaudible)	
1:05:00–1:09:59	997	R1	There was a point that you said you had to know nineteen before you could do twenty. Or eighteen before you could do nineteen. And then you came up with a kind of theory about it. Do you remember that?	
	998	Stephanie	Yeah. For the tower – just tower problems?	
	999	R1	Yeah.	
	1000	Stephanie	Yeah.	
	1001	R1	Do you remember what the theory was?	
	1002	Stephanie	Yeah. It was multiply the number of the last one by two.	PPK
	1003	R1	Right. But what if you didn't know the last one? What if all you knew was how tall the tower was? Suppose the tower was twenty tall. And you didn't know nineteen tall. We want to know how many towers you can make that are twenty tall that are different.	
	1004	Stephanie	Um hm.	
	1005	R1	Alright. So you know them from – how many can you make two tall?	
	1006	Stephanie	How many can I make two tall?	PAH
	1007	R1	Well, one tall.	
	1008	Stephanie	One tall? I can make one – two. Um. Am I using – oh! For two colors, I can make two.	BR-V
	1009	R1	Two tall?	
	1010	Stephanie	I can make four.	BR-V
	1011	R1	Three tall?	
	1012	Stephanie	I can make eight.	BR-V
	1013	R1	Four?	
	1014	Stephanie	Sixteen. Thirty-two. And it just keeps going like that.	BCA
	1015	R1	Right? But can you do this in a general way? So this would be – this is the height, right?	
	1016	Stephanie	Um hm. Yeah.	
	1017	R1	And this is the total. Is there another way to write this? With exponents?	
	1018	Stephanie	Oh!	BDI
	1019	R1	Remember that?	
	1020	Stephanie	Well. Yeah. Um. The first one, just two to the first. Two to the second. To the third. Fourth. Oh! Two to the twentieth.	PPK; BCA
	1021	R1	Okay. So	

Time	Line	Speaker	Transcript	Code
	1022	Stephanie	Oh! Well, or	
	1023	R1	Twenty would be two to the twentieth.	
	1024	Stephanie	[*at the same time*] Two to the twentieth.	BCA
	1025	R1	And *n* would be	
	1026	Stephanie	Two to the *n*.	BCA
	1027	R1	Two to the *n*..	
	1028	Stephanie	Yeah.	
	1029	R1	Right.	
	1030	Stephanie	Um hm.	
	1031	R1	We were sort of playing with that idea. And so zero tall would be?	
	1032	Stephanie	Zero tall? It would be zero.	BMP; OBS
	1033	R1	It would be two to the zero and that's one.	
	1034	Stephanie	Yeah.	
	1035	R1	But it almost doesn't make any sense.	
	1036	Stephanie	It doesn't. It makes	
	1037	R1	But if you wanted to make your nice pattern	
	1038	Stephanie	no sense.	
	1039	R1	to – but you want your pattern to work.	
	1040	Stephanie	Yeah. But that's not – that's not fair. 'Cause everything else has to make perfect sense except for that.	
	1041	R1	But this is what – why they make certain definitions.	
	1042	Stephanie	Hm.	
	1043	R1	You don't like that.	
	1044	Stephanie	No.	
	1045	R1	It's sort of really hand waving.	
	1046	R3	There was a big discussion in Dr. Davis' class about why *a* to the zero is one.	
	1047	R1	This is exactly Stephanie's point.	
	1048	R3	I think that I think I remember it – that it was uh – I mean you have to define the terms and if you think about um dividing -	
	1049	R1	[*R1 chuckles as the camera focuses on Steve causing him to stumble over his words.*] Now you know what Stephanie feels like.	
	1050	R3	If you think about dividing by different things like uh maybe you have uh if we if we define two to the negative two as one over two squared	
	1051	R1	Um hm.	
	1052	R3	Then if you take two squared, okay?	

Time	Line	Speaker	Transcript	Code
	1053	R1	Um hm.	
	1054	R3	Divided by two squared or two squared times two to the negative two, we know that's four divided by four which we know is one. So two squared times two to the negative two – what we do there is we add the exponents. Two minus two is zero and we get two to the zero and since we know four over four is one, we have two to the zero has to be one. So I mean I think that sometimes its just like	
	1055	R1	Consistency?	
	1056	R3	Right. And having a definition for it.	
	1057	Stephanie	Hmm.	
	1058	R1	So. Do you see what he's saying?	
	1059	Stephanie	Yeah.	
	1060	R1	That we know that this is one. Two squared divided by two squared is one.	
	1061	Stephanie	Yeah.	
	1062	R1	Right. We know that when you divide you're supposed to subtract exponents. Right?	
	1063	Stephanie	Um hm.	
	1064	R1	So you better make that zero or the whole system is going to flop.	
	1065	Stephanie	Hmm.	
	1066	R1	It's convenient. And it sorta works. But you can't always have a um a physical model	
	1067	Stephanie	Yeah.	
	1068	R1	to represent it. That we know of. Not with this. I mean there might be a more sophisticated abstract models that mathematicians and physicists have. Good. Thank you. That's very good.	
	1069	R3	You could just look at a graph.	
	1070	R1	What do you mean?	
	1071	R3	Hm.	
	1072	R1	What do you mean by looking at the graph?	
	1073	R3	Well, um.	
	1074	R1	y equals two to the x?	
	1075	R3	Well, y equals a to the x.	
	1076	R1	Okay. So if you plot these points on a graph, where would you want that to be? That's interesting. Do – do you do anything much with graphs?	
	1077	Stephanie	Sometimes. But not really. We haven't done a	

Time	Line	Speaker	Transcript	Code
			lot with them this year at all.	
	1078	R1	Do you think you'd like to be playing with a computer?	
	1079	Stephanie	Well	
	1080	R1	If you had a computer, would you play with it?	
	1081	Stephanie	Yeah.	
	1082	R1	Yeah. We're working on it, Stephanie. Right. We should be working on it.	
	1083	R3	Yeah. Actually um I, I did find some stuff out about that I forgot to tell you. But (inaudible) I wrote something down for you.	
	1084	R1	Good.	
	1085	R3	(inaudible) at home.	
	1086	R1	Great. Okay. Well, we need to number these pages and get them copied.	
	1087	Stephanie	Okay.	
	1088	R1	That's all going to take a little time. And this was really great fun.	
	1089	Stephanie	Alright.	
	1090	R1	And do you have anything you want to ask us about – anything you're doing in math that we could help you with?	
	1091	Stephanie	What am I doing in math?	
	1092	R1	What are you doing in math?	
1:10:00-1:11:09	1093	Stephanie	We just got back from vacation. I don't – what are we doing?	
	1094	R1	Well this is already this is already Wednesday.	
	1095	R2	(inaudible)	
	1096	Stephanie	We're doing we're doing rationals. And um	
	1097	R2	Rational what?	
	1098	Stephanie	Addition and subtraction. I think we're doing. We did multiplication and division. And now we're doing addition and subtraction of different denominators, or something. It's easy. It's not hard, though. (inaudible) but, I – it's – [*softly, almost to herself*] what are we doing? Is that what we're doing?	
	1099	R1	That's reasonable that you'd be doing that now.	
	1100	R2	Yeah.	
	1101	Stephanie	I think that's what we're doing. Because we really haven't been doing much like this whole week yet. 'Cause	
	1102	R1	Factoring is really important so you can get	

Time	Line	Speaker	Transcript	Code
			your common denominators	
	1103	R2	(inaudible)	
	1104	R1	Right?	
	1105	Stephanie	Yeah.	
	1106	R1	And you can put it in simplest – that's really an exercise in	
	1107	Stephanie	Yeah.	
	1108	R1	Knowing how to factor a lot. Don't you think? A lot of those problems have less to do with how to add, subtract, multiply and divide	
	1109	Stephanie	Yeah,	
	1110	R1	And more to do with simplifying. You're looking for common factors.	
	1111	Stephanie	'Cause that's what the whole thing is. You have to simplify it.	
	1112	R1	Right. Well, that's that's what they do. Your name's not on here, Stephanie, is it? And the date.	
	1113	Stephanie	Alright.	
	1114	R1	So I want to make sure this gets on here.	
	1115	??	Stephanie's mom is outside.	
	1116	R1	She can come in.	
	1117	Stephanie	Oh!	
	1118	R1	She can come in.	
			[*general conversation; end of tape*]	