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TRACING THE BUILDING OF ROBERT'S CONNECTIONS IN MATHEMATICAL
PROBLEM SOLVING: A SIXTEEN YEAR STUDY

by

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ABSTRACT OF THE DISSERTATION

TRACING THE BUILDING OF ROBERT'S CONNECTIONS IN MATHEMATICAL PROBLEM SOLVING: A SIXTEEN-YEAR STUDY

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This research analyzes how external representations created by a student, Robert, helped him in building mathematical understanding over a sixteen-year period. Robert (also known as Bobby), was an original participant of the Rutgers longitudinal study where students were encouraged to work on problem-solving tasks with minimum intervention (Maher, 2005). The research demonstrates how Robert built robust counting techniques by tracing the evolvement of his problem-solving heuristics, strategies, justifications and external representations. The study also examines how Robert made connections to his earlier problem solving. In addition, the origins of Robert's ideas related to Pascal's Triangle and Pascal's Pyramid are investigated.

Fifteen sessions were selected between Robert's fifth grade (February 26, 1993) and post-graduate interviews (March 27, 2009) yielding more than twenty hours of video data. Powell, Francisco, and Maher (2003) model was used for analysis where by each session was viewed, transcribed and coded for critical events to create a comprehensive narrative.

The study reveals that mature combinatorial techniques were a part of Robert's counting strategies as early as middle school. Robert used binary notation to count two-

colored candle arrangements and later to count the number of ways a team could win a World Series; modified exponential formulae to account for combinations for a garage door opener, arrangements for n -colored candles and n -toppings pizzas; discovered the combinations formula, $C(n, 2)$, in his eleventh grade; and connected these solutions to Pascal's identities. In general, Robert looked for patterns in his solutions; generalized the findings; and identified structural similarities in tasks presented to him as he connected three-position garage door opener to three-colored candles arrangements, pizza with four toppings to towers four high, and directions on Pascal's Triangle to routes for a taxi on a two-dimensional grid. External representations created by Robert served as communication tools for him and provided insight into his problem solving heuristics and mathematical understanding.

The research contributes to the growing body of case studies from Rutgers longitudinal study providing evidence that building of early mathematical ideas is the foundation of more advanced learning (Davis & Maher, 1997).

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My father had hoped that I would become a physician one day. I tried very hard to get into the medical schools in India but the competition was too fierce. Eventually, I realized that I loved mathematics and physics much more than biology and chemistry. I chose to study mathematics and it was clear, I was not going to be a physician after all. Even after all the efforts and hopes my father had put into helping me become a physician, he never gave me a chance to feel that I had disappointed him. For all the trips we made to New Delhi, all the tuitions he paid on top of schooling expenses, and all the hours he spent in scorching Indian sun waiting for me to come out of exams, I want to first and foremost thank my father, Mr. Gurusharan S. Kalsi. Thank you Papa for always believing in me and supporting everything I chose to do in my life. Your endless love for me has and always will be my inner strength.

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DEDICATION

To the most wonderful kids in the whole wide world!
I love you Nihal and I love you Ruhee!

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1. INTRODUCTION

1.1 Statement of the Problem

To help students become better problem solvers in mathematics, understanding how representations play a role in problem solving can be of great significance. A deeper understanding of how representations assist a student in solving problems can help practitioners design lessons that invite students to build their own representations to solve problems in mathematics. By representations, I mean how ideas are represented externally by what the student says, writes, draws, or builds to solve a mathematical problem.

The National Council for Teachers of Mathematics (NCTM) in its document, *Principles and Standards for School Mathematics* (2000a), identifies five process standards, with one of them focusing on representations. This document recommends that all students kindergarten through grade 12 should be able to: “1) create and use representations to organize, record, and communicate mathematical ideas; 2) select, apply and translate among mathematical representations to solve problems; and 3) use representation(s) to model and interpret physical, social and mathematical phenomena” (NCTM, 2000a, p. 67). The NCTM standard also states that the mathematics curriculum should provide students with opportunities to “represent their ideas in ways that make sense to them, even if their first representations are not conventional ones” (p. 67).

The NCTM standard suggests that creating representations “can play an important role in helping students understand and solve problems” (p. 68). The standard further states that students need to develop an understanding of the strengths and weaknesses of

various representations and use them as meaningful ways to “record a solution method and to describe the method to others” (p. 68). The standard also affirms that the representations that students develop can help teachers understand “students’ ways of interpreting and thinking about mathematics” (p. 68). That is, observing how students represent their ideas as they do mathematics or solve problems can help educators and researchers gain knowledge of how they “understand” or “do” mathematics.

Research has shown that use of particular modes of representation, visual or concrete, can improve a student’s mathematical success, reasoning skills and problem solving skills (Cifarelli, 1998; Niemi, 1996; Novick 1990). Students’ use of representations can play a central role in problem solving by serving as an illustration of symbolic results, helping a student resolve conflicts between correct and incorrect intuitions and helping a student understand the conceptual fundamentals of a mathematical task (Arcavi, 2003). Also, Pape and Tchoshanov (2001) contend that the brain works more effectively when it makes representational patterns for encoding (internalizing) and decoding (externalizing) information. They explain that when a student can “see” the relationship, he/she can easily memorize (internalize) and reproduce (externalize) the idea. For example, students often can’t remember a number like 1123581321345589 unless they see the Fibonacci pattern where each term is the sum of the two immediately preceding terms (Pape and Tchoshanov).

Goldin (1987) says that competent problem solvers have systems of representation that help them access “the world of imagined experience” to resolve problem solving situations. Cifarelli (1998) conducted an empirical study to show that the success of capable problem solvers is highly related to their ability to construct

appropriate representations and use them for the information and relationships described in the problem. Pape and Tchoshanov (2001) explain this relation between problem solving and construction of representations by asserting that representations help students keep track of their ideas, inferences and the intermediate steps and act as an important tool for absorbing some of the “cognitive load” during problem solving. Hence, they believe that students who are fluent in constructing representations can free up their mind by jotting down some of the important information in form of representations and then focus on the big picture of the problem. This can help students perform better on problems that have many different parts or various aspects that need attention at the same time.

The Rutgers Longitudinal Study has several examples where students used their own representations to communicate ideas and convince peers and researchers of their solutions and justifications for a mathematical problem (Maher, Martino & Alston, 1993; Davis & Maher, 1997; Maher & Martino, 1996). The longitudinal study research was conducted with a belief that students can construct their own mathematical understanding and build their own representations when they are given ample time to work on meaningful mathematical tasks with minimum intervention (Davis & Maher, 1990). Davis, Maher, and Martino (1992) contend that in order to build abstract ideas, the students need to build, rebuild, revisit and talk about their representations over a long period of time. For the Rutgers longitudinal-study students, using representations to solve a problem and to communicate their ideas was a natural part of their learning as they were given several opportunities to rebuild and revisit their representations.

Maier, Martino & Alston (1993) demonstrated that as students construct more elaborate and sophisticated representations for new problems, they expand their existing knowledge. Maier and Yankelewitz (2010) concluded that representations helped young students effectively communicate with others about their problem solving schemes much like an adult, active member of the mathematical community. Muter and Uptegrove (2010) pointed out that students from the Rutgers longitudinal study were able to connect many different problems and discover the mathematical connection by co-constructing and implementing a common representation.

Studying the initial representations offered by a student while working on a mathematical problem can give an insight into how his/her mathematical ideas originate, get connected and are extended through problem solving experiences.

1.2 Background of Longitudinal Study

This study is grounded in extensive research done in the public schools in Kenilworth, New Jersey, a working class community, as part of the Rutgers longitudinal study on the development of mathematical ideas in learners. The longitudinal study is now in its 24th year. The first purpose of the study was to explore the development of student's mathematical ideas that later lead to a study of their reasoning. The first eight years of the research were conducted in classrooms at the Harding Elementary School. At this elementary school, the students were being taught mathematics in half hour sessions and mostly rote methods were used. As such, the students were having difficulty with mathematics. The student difficulties led the school to a partnership with Rutgers University that began in 1984. The first step for this partnership was to conduct

professional development at the school and later, to study the effect of intervention on student learning.

For the first three years, this collaboration took the form of a teacher development project. Dr. Carolyn A. Maher, professor of mathematics education at Rutgers University and director of the longitudinal study and her team of graduate students worked with teachers to help them build an understanding of the mathematics they were teaching their students. Dr. Maher initiated a formal longitudinal study in 1987. In 1992, when students in the study were in fourth grade, the National Science Foundation funded the Rutgers research team with a grant for the longitudinal study. The various pieces of the study were partially funded by NSF grants MDR 9053597, directed by Robert B. Davis and Carolyn Maher, and REC9814846, directed by Carolyn Maher, as well as by grant 93-992022-8001 from the New Jersey Department of Higher Education.

As a part of the longitudinal study, the students were followed from elementary grades all the way up to high school years. Some were even interviewed and studied beyond high school. The topics that students investigated involved counting, combinatorics, algebra, probability pre-calculus, and calculus. The activities were open-ended in nature and provided the students with meaningful problem-solving experiences (Maher, Powell & Uptegrove, 2010). The researchers took up the role of a facilitator while students worked by themselves or in small groups to make sense of the new mathematical ideas. The students were not graded for their work but were encouraged to share their work with their classmates and the instructors by talking about their ideas or by presenting ideas on the projector to the entire class. The students were encouraged to explain their understanding and convince themselves, their peers, and the researchers of

their solutions or ideas about how they would solve the problems. The students were aware of the high expectations that involved defending their work to the peers and researchers and took it upon themselves to come up with satisfactory solutions and explanations (Maher, PME 2002). The sessions at all the sites were videotaped and the work of students, along with the field notes from researchers, was preserved in an archive.

The investigations were designed by the researchers and contained mathematics that had not yet been introduced into the school curriculum at the time of the study. The task design was crafted to provide a platform where students were invited to build mathematical ideas in ways that encouraged sense-making. Students worked in pairs or small groups to build not only individual but also a collaborative understanding of mathematics. Some tasks, like building towers of a specific height selecting from Unifix cubes of either two or three different colors, were revisited many times with new variations or extensions added to the original task. In addition, tasks that had apparently different features but shared a similar mathematical structure were given to students so that they could have an opportunity to connect mathematical ideas and discover isomorphism.

Throughout the study, researchers conducted several interviews with one or more students at a time to enable deeper exploration of students' ideas. They challenged students to explain their ideas in a clearer manner through verbal or written explanations and justifications. The problem-solving sessions and interviews were videotaped. In order to get a detailed view and to capture the student actions, multiple cameras were

used in the study. This enabled researchers to study an event through many different viewing angles.

1.3 Purpose of the Study

This study traced and analyzed the external representations used by Robert Sigley (Bobby) as he tried to discover and understand problems related to the properties of Pascal's Pyramid. Robert was part of the original cohort of students from Kenilworth and participated in the longitudinal study since the first grade. Robert is currently 27 years old and works as Applications Developer at the Graduate School of Education at Rutgers. His willingness to re-visit the tasks that were introduced to him during his school years, along with his continued interest in solving new mathematical problems, made him an appropriate participant for the study. Also, Robert went on to pursue a baccalaureate degree in mathematics and a graduate degree in statistics. His continued education in fields related to mathematics made it possible for me to introduce challenging combinatorial tasks that were aimed at eliciting creation of multiple representations. Furthermore, Robert was exposed to many challenging tasks at an early age through the longitudinal study and there is video data from his school years that provided an interesting insight into Robert's work as a young student.

The central activity that Robert worked on during his interviews with me was to understand properties of Pascal's Pyramid and how they can be used to solve a set of combinatorial tasks like the *Building Towers Activity* for three-colored Unifix blocks, *Ankur's Challenge* and the *TaxiCab Problem*. From his earlier work in the longitudinal study, there was video data available where Robert works with problems like the *World Series Problem*, *The Garage Door Problem*, *The Candles Arrangements Problem*, *Guess*

My Rule, *Pizza Toppings problem*, etc. Specifically, this study analyzed Robert's problem solving strategies through a lens that focused on the role of external representations. That is, to understand Robert's mathematical ideas, his external representations were used as a lens to analyze his problem-solving heuristics, strategies and justifications along with his hand-written work and verbal explanations. This research also analyzed some of Robert's earlier ideas from previous explorations and attempted to trace where his ideas related to Pascal's Triangle and Pyramid originally came from, how he built on these ideas and how he connected them across the range of various activities.

1.4 Research Questions

The following research questions guided the investigations:

- 1) What are the external representations that Robert uses to help him understand problems related to the properties of Pascal's Pyramid?
- 2) How, if at all, do these external representations help him in solving problems related to Pascal's Triangle or Pyramid?
- 3) How does he use, modify or reuse his external representations over time to provide justifications for his solutions?
- 4) What connections, if any, does he make to earlier problem solving?

2. Literature Review/ Theoretical Framework

2.1. Introduction

Davis and Maher (1990) indicate that when students are given the opportunity to work with problems that are well-designed and challenging and work in a supportive environment, they begin solving a problem by making personal representations. Maher (1998) further indicates that students develop their reasoning as they build connections between and among the representational systems. Novick (1990) suggests that it is not always easy for an observer to distinguish between the fine point “at which solvers stop representing information and start solving the problem” (p. 129). As such, understanding students’ representations and how they use them to solve a problem are complex ideas but important for providing important insight into how children reason about mathematical ideas. This literature review will first discuss theoretical views related to learning mathematics with understanding and how students use representations to build this understanding. Then, research that has been carried out in relation to these views will be outlined in the context of how representations can help a student build mathematical understanding and successfully solve problems.

2.2 Theoretical Framework

2.2.1 Mathematical Understanding

Davis proposes a theory to explain what happens in students’ mind when they “do” mathematics. In his theory, he explains some mechanisms that capture “understanding” of mathematics (Davis, 1992). Davis, in his book, *Learning*

Mathematics: A Cognitive Science Approach explains that understanding mathematics is a matter of fitting a new idea into larger framework of ideas that are assembled previously (Davis, 1984). This is Davis' idea of an "assimilation paradigm." He states that students create their own ways of understanding and what they learn is built upon the "previously-built-up understanding" which can possibly control or even limit future learning (Davis, 1992). He compares this to solving a jig-saw puzzle where a new piece is only helpful if it fits part of the puzzle that you have already figured out. That is, the new ideas are only useful when they act as an "answer" to a question that was already of interest to the learner (Davis, 1992).

Skemp (1976) provides a view on understanding that is particularly applicable to this study. He identifies *instrumental* and *relational* understanding as two types of understanding. He defines instrumental knowledge to be the basic knowledge about different rules and procedures in mathematics in contrast to conceptual understanding and understanding of relations between and among various concepts. Skemp (1976) holds that instrumental understanding is easier to teach and learn and can also provide immediate rewards. Skemp (1976) indicates that instrumental understanding is more prevalent in school mathematics teaching. He defines relational knowledge to be conceptual knowledge where students can use concepts to make relations, adapt and modify ideas and use them in new situations. Skemp (1976) suggests that although relational understanding is more difficult to learn, it is more enduring than the instrumental understanding. Furthermore, relational understanding can be effective as a goal in itself. It can, according to Skemp (1976) act as an "organic" agent in motivating students to explore new relations and build conceptual understanding on their own.

Hiebert and Leferve (1986) also make a distinction between procedural and conceptual knowledge. They describe conceptual knowledge as “rich in relationships” whereas procedural knowledge might not be linked to other knowledge (Hiebert & Leferve, 1986, p. 3).

Zoltan Dienes outlines a six-stage theory of learning mathematics in his book, *Building Up Mathematics* (Dienes, 1971). The six stages are: *free play*; *games*; *search for communalities*; *representation*; *symbolization*; and, *formalization*. He describes the “trial and error” activity as the first stage of *play*. When a learner realizes the rules to *play* and uses them, the activity becomes a *game*. The next stage is when the learner begins to see that many different games share the same structure. This is the stage where a learner is searching for isomorphisms. When a learner is able to make successful distinctions between the relevant and irrelevant features of the many games, he/she is ready for a *representation*. Dienes recommends that formal symbolic language should only be introduced after the learner has reached the representation stage. And finally, when a learner works with symbols and formalizes them, he/she is able to reach an abstraction for the idea at hand (Dienes, 1975, p. 83-84).

In order to achieve mathematical understanding, one needs to consider the role of the teacher and the environment for learning. Davis and Maher (1997) indicate that there is a need for change from a teacher-centered learning environment to a more student-centered environment. According to Davis, in this new type of environment, students would be able to build up mathematical ideas and conceptual understanding themselves. As students share their ideas and make their thinking public through group work, interviews, or class discussions, the teacher has an opportunity to observe and explore

student's ideas in development. As the teacher asks students to explain their ideas and probe their justifications, they help students build for themselves a deeper understanding of the mathematical concepts (Maher & Martino, 1992).

2.2.2 Representations

The NCTM (2000a) defines representations as following:

The term *representation* refers both to process and to product – in other words, to the act of capturing a mathematical concept or relationship in some form and to the form itself. (p. 67)

The NCTM further states that the term “representation” can apply to externally observable as well as internal processes and products that people use while doing mathematics. Goldin (1998) proposed a model for systems of representations that outlines the many attributes of external and internal representations.

Goldin (1998) describes natural languages, notation systems and computer environments (“microworlds”) to be external to the individual and recommends that the external representational systems should be analyzed independently of the cognitions of the learner. Goldin (1998) notes that external representations can be very structured and the study of these systems can provide significant insight into problem-solving behavior.

Goldin (1998) proposes five categories of “mature internal representational systems” (p. 148). He describes *verbal/syntactic* system to be an individual's capability to process “natural language, on the level of words, phrases and sentences (only)” (p. 149). In this system of internal representations, Goldin includes an individual's ability to understand and work with common definitions, verbal descriptions, synonyms, antonyms along with grammar and syntax information of a language. In the *formal notational* systems, Goldin suggests that formal notations of mathematics have external as well as

internal aspects of representations that can be studied. He contends that a skilled problem solver can make relationships between the formal “symbols, imagistic configurations, and words” (p. 153). That is, a competent problem solver can not only understand a problem and convert it to formal symbols, but can also “imagine” the situation of the problem by constructing imagistic configurations internally. Then, the solver is further able to translate the imagistic configurations back to formal notations. Although use of the imagistic competencies is mostly “tacit,” Goldin suggests that a careful analysis of words and symbols of the problem solver can help one describe his/her imagistic processing (p. 153).

In the *planning, monitoring and executive control* system of internal representations, Goldin includes the various strategies and heuristics used by an individual to solve a problem. For example, he includes the planning, monitoring and decision-making steps used by the problem solver in this system of internal representations. Also, a problem solver’s heuristic processes like “trial and error”, “think of a simpler problem”, “explore special cases”, etc are considered a part of this system. Goldin points out that these heuristics can be complex to study or isolate as “ heuristic processes can access each other, and act on each other in the course of their use” (p. 154).

Goldin and Shteingold (2001) write:

A mathematical representation cannot be understood in isolation. A specific formula or equation, a concrete arrangement of base-ten blocks, or a particular graph in Cartesian coordinates makes sense only as a part of a wider system within which meanings and conventions have been established (p. 1).

As such, they suggest that to understand representations offered by a student, one needs to pay attention to cognitive, social, and affective attributes that the student assigns to these representations. The representations should be studied not in isolation but as a

part of all the various problem-solving strategies that a student implements for a particular task.

Maher and Weber (in press) use Goldin's theory of representation systems to offer a definition of conceptual understanding; they used data from the Rutgers longitudinal study to show how students can construct this conceptual understanding. Maher and Weber assert using Goldin's model that students understand new representation systems by forming links with representation systems that are intuitively meaningful to them. They further state that participants in the longitudinal study respected each other's representation systems and used these systems to build understanding. Maher and Weber contend that asking students to justify and explain their work to others allows the students to reflect on their representation systems. In process of explaining the representation system to others, the students are able to build an explicit understanding of their representations.

Davis, in addressing "mental representations," indicates that the building of these representations is usually so quick that one may doubt if anything has actually happened (Davis 1984). However, this process is rather complicated and deserves a closer inspection. Davis (1984) suggests that students move through a cycle, once or many times, to think about a mathematical situation.

Davis suggests that learners build new knowledge from existing knowledge in the process of thoughtful problem solving, beginning by first building a representation for the data of the problem. A memory search for a representation of knowledge that are relevant to the problem at hand, brings forth certain representations, and if such representations do not already exist, the solver might construct them. Then, the solver

tries to map the representations from the data to the knowledge representations and judge if the representations are satisfactory. The process can lead to a successful solution to the problem if the constructions are appropriate. However, if the mapping between data and knowledge representations is not satisfactory, the solver might have to repeat this cycle a few times or create new representations (Davis, 1984).

Davis further suggests that visualization or representations can help a student foresee the “unseen” in problem solving situations. Davis uses the phrase visually-moderated-sequences (VMS) to explain this; he describes VMS as a procedure where one may use visualization as a tool to remove oneself from situations in which one may be uncertain of how to proceed. He gives, as an example, how one might not remember exactly how to get to a destination that one visits very occasionally. However when one sets out to drive, the visual reminders on the way like the ice cream shop, help one remember the next turn and give clues to the subsequent sequence of directions. In a similar manner, when a student sets out to solve a problem, the intermediary steps like making representations for the information in the problem can provide clues to the next step in problem solving.

Raymond Duval (1999) says that the role of representation and visualization lies at the core of understanding mathematics. Duval sees visualization as the intuition one might have for an object and representation as a way to denote that intuition. Duval explains representations through the idea of a “register” where he defines a semiotic system that leads to a specific way of representing and processing for mathematical thinking as a “register of representation” (p. 6). For example, he considers Cartesian graphs a “register.” Duval states that in a problem-solving situation, a student needs to

be able to change registers because another representation might be needed or because two registers might be needed simultaneously. Duval emphasizes that understanding of mathematics requires that one does not confuse the mathematical objects with their representations. He also cautions that the distinction between internal and external representations only refers to their mode of production and not to their form or nature.

Greer and Harel (1998) contend that “a central goal of mathematics education is to increase the power of students’ representations” (p. 22). They emphasize that isomorphism plays an important role in helping students to notice and develop the structural relationships in mathematics. They recommend that students should be encouraged to recognize isomorphism as a means to generally improving their representational skills. Greer and Harel view “isomorphisms as components of mental representations” that are constructed by individuals as they gradually “assimilate” and work with given situations (p. 11). They propose three models for construction of isomorphism and explain how an individual might make a *surface-level isomorphism*, *deep isomorphism* or a *mediated isomorphism*. For example, they explain that a student is making a *surface-level isomorphism* when he/she uses the “conservation formula” to solve a problem with “nasty” numbers by substituting “friendly” numbers in there (p. 13). As an example of *deep isomorphism*, they cite a case where a student makes a diagram to illustrate the problem: “how many ways can you distribute 8 candies among 3 children?” and realizes that this problem is the same as asking: “how many combinations can you position two sticks in the spaces between them (the candies)” (p. 14). In a *mediated isomorphism* there is a third situation that acts as a mediator to help construct isomorphism between two other situations.

Gerard Vergnaud (1998) proposed a representational theory for understanding mathematics in terms of schemes, “theorem-in-action” and “concept-in-action.” Vergnaud postulates that a student brings a full collection of schemes to problem-solving situations that were developed from previous learning. Vergnaud indicates that while learners are working with peers on new mathematical problems, new schemes are developed or former schemes are enhanced. This idea of building on schemes is similar to building an assimilation paradigm where the learner assimilates new knowledge into existing knowledge (Davis 1984; Davis & Maher, 1996).

In a classroom context, Vergnaud (1998) describes the teacher as a mediator whose most important responsibility is to help students develop their collection of schemes and representations by providing students with appropriate tasks to explore. Vergnaud notes that the teachers are largely responsible for clarifying the goal to be reached, providing a model for action, or helping students choose the relevant information and reason with that information. Davis and Maher (1997) outline a similar role of guidance for a mathematics educator and suggest using a “paradigm teaching strategy” where students are exposed to carefully designed experiences that reflect the structure of relevant mathematics. They contend that when students work on specifically chosen activities, they create conceptual frameworks that later enable student to build more abstract representations.

In a paper addressing goals for learning mathematics in schools, Davis and Maher (1996) emphasize the need for changing mathematical instruction in classrooms to encourage building representations and models that enhance students’ thinking rather than memorizing dry facts. In addressing children’s construction of mathematical ideas

in a conference in Brisbane, Australia, Maher, Martino and Alston (1993) report from their data that as students continue to work on new problems, their representations become increasingly elaborate and sophisticated and help them in expanding their existing knowledge. NCTM (2000b) also encourages educators to use representations as tools for learning and doing mathematics rather than teaching representations like graphical displays and symbolic expressions as if “they were ends in themselves” (p. 14).

2.3 Related Research

2.3.1 Mathematical Understanding

Maher and Alston (1989) and Davis and Maher (1990), in a case study about a student named Ling Chen, illustrate a distinction in the two types of understanding: instrumental and relational. Ling Chen was interviewed during the summer after she finished fifth grade when she was participating in a program for gifted students. This interview was videotaped as she worked on the following problem:

Karen had a whole candy bar. She gives $\frac{1}{2}$ to Kathy. She also gives $\frac{1}{3}$ to Paul. How much does she have left?

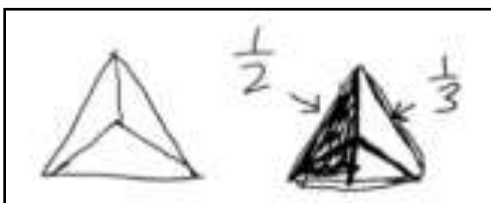


Figure 2-1 Ling Chen's Drawing (Maher & Alston, 1998, p. 244)

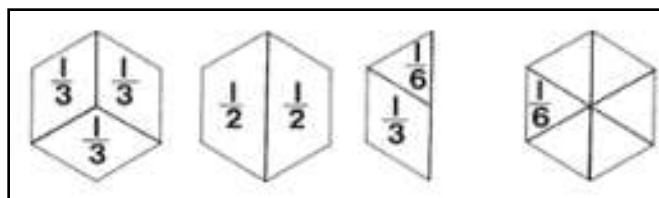


Figure 2-2 Ling Chen's Pattern Block construction (Maher & Alston, 1989, p. 244)

Ling Chen used pattern blocks to build a representation of the problem (Figure 2-2). With the blocks, Ling Chen concluded that one-sixth of the candy bar is left. Later, the interviewer asked her if she could do the problem with numbers. Originally, Ling Chen incorrectly used the algorithms she learned for fractions and she came up with an answer that did not match her representation. Ling Chen was using her instrumental understanding at this point. Later, she was able to match her numeric answer to her representation by writing one-third divided by one-half equals one-third multiplied by one-half, as she “reached” to go with her answer of one-sixth (Figure 2-3) (Maher & Alston, 1989, p. 247).

$$\begin{array}{l} \frac{1}{3} \div \frac{1}{2} = \frac{1}{3} \times \frac{2}{1} = \frac{2}{3} \\ \frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} = \frac{3}{2} \\ \frac{1}{3} \div \frac{1}{2} = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6} \end{array}$$

Figure 2-3 Ling Chen's three attempts to solve the problem (Maher & Alston, 1989, p. 247)

Although Ling Chen knew the division algorithm for fractions, she did not understand its relation to the candy bar task. Nevertheless, she was confident in her

reasoning and representations and was able to choose the correct solution for the problem. The researchers concluded that in this instance, as Ling Chen had chosen a “concrete imagery,” she had access to mental representations that enabled her to think “reliably about problems of this type” (Maher & Alston, 1989, p. 248).

2.3.2 Representations

Robert B. Davis emphasized the importance of the representations that students build, as he monitored growth in understanding when students “do” mathematics (Davis, 1984). Davis and Maher (1997) suggest that students’ representations can take many different forms and modes. Also, as students build new schemas for their mathematical understanding, these representations become increasingly sophisticated. As students construct more elaborate and sophisticated representations for new problems, they have further opportunity to expand their existing knowledge (Maher, Martino & Alston, 1993).

Davis and Maher used Brian’s work from fifth grade to explain how the cycle of building representations that is outlined in the theoretical framework might be carried out. Brian was a fifth grader when he solved the following pizza problem with Scott as his partner:

At pizza hut each large pizza is cut into 12 slices. Mrs. Elson ordered two large pizzas. Seven students from Mrs. Elson’s class are to eat one piece from each of the pizzas. What fraction of the two pizzas was eaten? (Davis & Maher, 1990)

Brian had the option to work with pattern blocks that were shaped like Triangles, hexagons, trapezoids, etc. The shapes of the blocks were such that six of the green Triangles fit on one yellow hexagon; two of the red trapezoids fit on the yellow hexagon, etc. There were many blocks of the same kind available. Brian first picked up a yellow hexagon to represent a pizza and ignored the matter of representing the students who will

eat the pizza. Once Brian had settled on representations for the pizza, he tried to work on building representations for the children who were to eat the pizza. He initially took the idea of “children in a class” and associated it with actual boys in his class at that moment (Davis and Maher, 1990). Davis and Maher (1990) conjecture that Brian perhaps built up a preliminary “primitive” representation of the problem in which the idea of “children in a class” was not well represented at first. When Scott reminded him that there are only seven students from Mrs. Elson’s class who will eat one slice from each of the two pizzas, Brian worked to model the students eating the slices very concretely. Once Brian had built representations for students who will eat the pizza, the knowledge representation he had about fractions matched his data representations. Using his fractional knowledge and representational model, Brian was able to finish the task correctly.

Davis and Maher (1990) observed that Brian exhibited a deeper understanding of the problem as he built representations of the data using concrete models, compared to his partner Scott who worked on paper and pencil. Also, this excerpt showed that Brian built his representations in parts, first for the pizzas, ignoring the children who will eat it, and then for the children, ignoring the pizzas. Davis and Maher (1990) conjecture that this behavior is common in adult experts as the problem solvers usually try to build representations in pieces.

Brandon in the fourth grade built powerful representations that Greer and Harel (1998) used as an illustration of an instance in research where a student not only created his own representation but also discovered and proved the similarity in the structure of two “unrelated” problems.

Brandon had previously solved the question:

How many different towers can you build, if each tower is 4 cubes high and you have as many red cubes, and as many yellow cubes, as you want?

Later, he worked on the following pizza problem:

How many different pizzas can be made if every pizza has cheese, but to this you can add whichever of the following toppings you wish and in any combination you wish: peppers, sausage, mushrooms and pepperoni? (Maher, Martino and Alston, 1993)

While working with the second problem, Brandon invented his very own notation where 1 meant that the topping was present and 0 meant that the topping was missing. He made a table with these “binary” symbols to come up with all the different possibilities for pizza toppings. Furthermore, Brandon was actually able to see and prove that the pizza problem was same as the problem of building towers.

P1	P	M	S	Pepperoni
1	0	0	0	0
2	1	0	0	0
3	0	1	0	0
4	1	1	0	0
5	0	0	1	0
6	1	0	1	0
7	0	1	1	0
8	1	1	1	0
9	0	0	0	1
10	1	0	0	1
11	0	1	0	1
12	1	1	0	1
13	0	0	1	1
14	1	0	1	1
15	0	1	1	1
16	1	1	1	1

P2	P	M	S	Pepperoni
1	0	0	0	0
2	1	0	0	0
3	0	1	0	0
4	1	1	0	0
5	0	0	1	0
6	1	0	1	0
7	0	1	1	0
8	1	1	1	0
9	0	0	0	1
10	1	0	0	1
11	0	1	0	1
12	1	1	0	1
13	0	0	1	1
14	1	0	1	1
15	0	1	1	1
16	1	1	1	1

Figure 2-4 Brandon's Work (Maher, Martino & Alston, 1993, p. 34)

The researchers summarize:

At this point Brandon's insight moved him to exhilaration. He pointed out that the group of four towers with exactly one red cube was like the four pizzas with one topping in his chart. He carefully moved each tower and placed it on top of the corresponding pizza code on the chart thereby validating the relationship he had organized. He then explained how the red cubes in each tower correspond to the "0"s in his pizza chart and how the yellow cubes in each tower correspond to the "1"s on his chart. (Maher, Martino & Alston, 1993, p. 36)

This discovery is very impressive considering that the two problems, the *towers* and the pizza problem are very different on the surface. Greer and Harel (1998) use Brandon as an example to assert that recognizing and exploiting structural relationships between situations that look different on the surface is an innate part of mathematical cognition. Greer and Harel claim that Brandon's insight into the isomorphism is not a sudden recognition, but actually the result of a long process of construction that he had mastered by working on his notational system while discussing and explaining his ideas to his peers and the interviewer.

In the Rutgers longitudinal study, students often worked in groups to discuss problems. As an example of how students working in groups can help each other build on their representation systems and communicate their ideas, Davis, Maher and Martino (1992) analyze the work of three students who worked on the following "Shirts and Pants" activity.

Stephen has a white shirt, a blue shirt, and a yellow shirt. He has a pair of blue jeans and a pair of white jeans. How many different outfits can he make? Convince us that you have them all. (p. 178)

In second grade, Dana, Stephanie and Michael all made pictures of shirts and jeans to represent the items of this problem. Michael suggested that only two outfits were possible as he held that blue shirt only goes with blue jeans and the white shirt only goes with white jeans. Dana and Stephanie told him that they were expected to find all combinations possible. Stephanie labeled her shirts with "w" for white and "y" for yellow and "b" for blue respectively and labeled the jeans similarly using a letter to represent the color. She then made a list writing letters that symbolized shirts together with letters that symbolized jeans and started counting her combinations. Dana drew

pictures of shirts and pants and labeled them with a letter to represent colors also, but in addition, she drew connecting lines to find outfits. Stephanie was able to find five combinations and was convinced that she had found them all. Dana also found five combinations as she did not include a line connecting a yellow shirt with white jeans. Dana said that yellow shirt and white pants can't go together. Davis, Maher, and Martino (1992) contend that Dana's list of outfits originated from her personal experience and sense of style. Maher and Yankelewitz (2010) note that in second grade, none of the three students counted all six outfits possible but they all showed that they were building schemes to solve this problem.

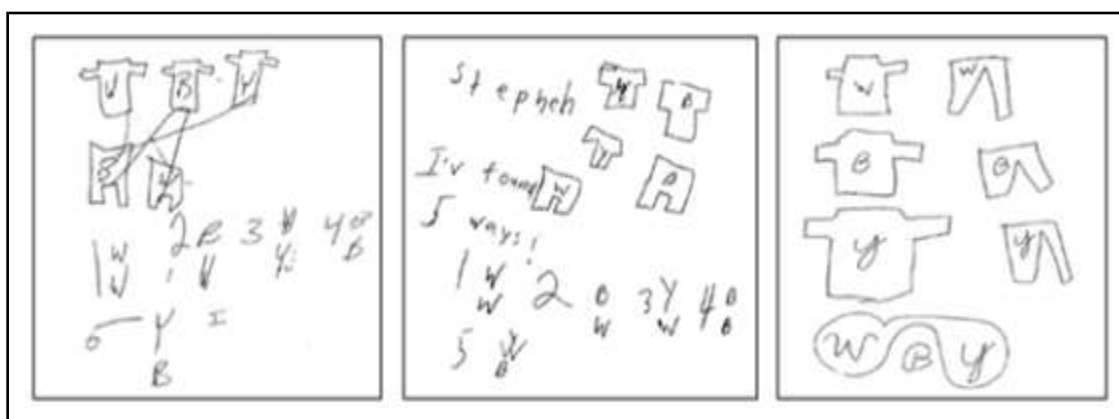
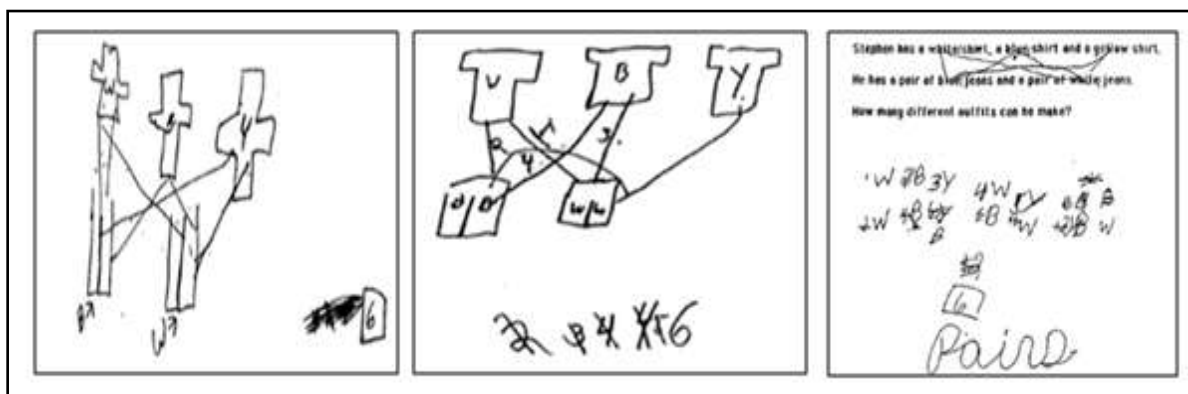


Figure 2-5 Dana's, Stephanie's and Michael's Second Grade work (Maher & Yankelewitz, 2010)

Davis, Maher, and Martino (1992) contend that in order to build abstract ideas, the students need to build, rebuild, revisit and talk about their representations over a long period of time. In grade three when Dana, Michael and Stephanie revisited this problem five months later, they found all six possible outfits. This time around, Stephanie and Michael also used the strategy of connecting lines that Dana had used in second grade and modified their representations. The students in this example had worked together on

a problem and were given ample time to explore and justify their solutions along with an opportunity to revisit the problem. Through co-construction of representations and understanding, the students were able to build their knowledge, find the correct solution, and convince researchers and peers of their solution.



become an “even” group. Similar reasoning was used to help students discover that “even” + “odd” = “odd.” Her approach was skillful and it gave students a chance to work creatively and build on the representations they already knew well.

All through the examples outlined above, it is clear that there were many occasions during the longitudinal study where students successfully built their own representations to solve a new problem. The social co-construction of representations was an important part of student discovery and sense making as in the example of Dana, Michael and Stephanie. Students often relied on their external representations to defend their solutions to peers or the researchers. They often worked with manipulatives or drew pictures to help them break the problem into tangible pieces and built visual evidence for their arguments as in the case of Brandon and Brian. Francisco and Maher (2005) emphasize the important role that the classroom culture played in the longitudinal study where students were invited to work on well-defined and open-ended mathematical tasks and encouraged to justify their solutions.

David Niemi (1996) says that, “Students who cannot use mathematical symbols appropriately and do not know the mathematical meanings of the symbols cannot be judged to know mathematics” and calls for a greater emphasis on understanding symbols and representations related to a mathematical concept (p. 361). Working with fifth-grade students from Washington State, Niemi discovered that “greater understanding of a written representation is closely related to number of different synonymous meanings one can associate with the representation” (p. 354). The statistical results from his study showed that there was significant superiority of the “high-fluency group” that did very well in relation to the concepts/principles and misconceptions dimensions. The “high-

representational-knowledge group” came up with more correct solutions and provided their answers with more graphical and verbal explanations. However, as Niemi had anticipated, the overall performance level was quite low, as the majority of the students did not have the desired synonymous meanings for the idea of a fraction. Niemi recommends that students should be exposed to multiple meanings of a concept so that they can build connections between various representations of a concept.

Thomas, Mulligan, and Goldin (2002) studied the representations of young children by examining their drawings and verbal explanations of the meaning of numbers 1 – 100. In one task, the students were asked to think of the number 100 and draw whatever representation comes to their mind. One of the students, Anthony, drew a truck with an explanation “...cause my Dad’s truck does a hundred.” (p. 122) The researchers coded this as an example of pictorial representation as Anthony connected the number 100 to speed of his Dad’s truck and saw the truck as a representation of it. Another student, Andrew, drew a picture of 100 shells and researchers inferred that as the “shells” were depicted with minimal detail, his representation was partially iconic. This study provides evidence for Goldin’s (1998) conjecture that external representations can be very structured and provide insight into the internal representations used by the student. Further, the researchers contend that the representations that students had previously developed provided a framework on which new cognitive structures could be built. They proposed that the active processing of images plays an important part in the numeration understanding of a child. From this study, Thomas, Mulligan and Goldin (2002) concluded that as the internal representational systems are developed, more “cohered, well-organized and stable” external representations emerge.

Gagatsis and Shiakalli (2004) conducted a study focusing on 195 students from the University of Cyprus to see if the “translation ability,” which students have for different representations of a function, is correlated with their success in problem solving. The results of Gagatsis and Shiakalli’s work reflected that the students see the verbal and graphical representations of the function as two different tasks. They concluded that this indicates that the students do not “understand” the idea of function, as they fail to recognize it when it is embedded in representational systems with varying qualities. The students in this study were not able to make connections between the external representation of a function, the graph, and the internal representation related to the verbal description of a function. Gagatsis and Shiakalli suggest that the instructor should encourage students to perform simple translations from one representation of a concept to the other. They advise that this will foster the students’ problem solving ability. Also, they recommend that all modes of representation should be emphasized as each representation has its own characteristic and poses different challenges for different students.

Goldin (1998) emphasizes a need to address qualitative aspects of mathematical performance (like abstract levels of solution activity) that are not usually considered an important part of the study of representations. Cifarelli (1998) agrees with Goldin and suggests that mental representation processes play a major role in problem solving. Cifarelli observed fourteen first-year students at University of California as they worked with a set of similar algebra word problems. Students were asked to talk aloud, and as such the researcher was able to observe their dilemmas through verbalized reasoning. Cifarelli concluded that there is a need to acknowledge a “constructive function of

representations in the development of conceptual knowledge and the resulting mental objects that solvers can then reflect on and transform as they interpret problem situations” (p. 261). In a particular instance, a student in this multiple case study was able to run through her potential solution activity to anticipate a problematic situation. Cifarelli points out that this is an example of Goldin’s imagistic processing as the student was able to reflect on her potential activity as an object. Cifarelli suggests that problem solving situations provide opportunities for solvers to “modify existing representations which may have outlived their usefulness” (p. 241). This is in sync with Davis’s (1984) view on how mathematical understanding is built.

A study by Rubel and Zolkower (2007) involved an interesting classroom activity with beginning teachers that highlighted the significance of spatial versus kinesthetic mathematical representations as outlined by Goldin (1998). The researchers gave the participants the following two non-routine tasks and asked them to work in groups on one problem of their choice.

Staircase Problem: Suppose that a staircase comprises ten steps and that you can climb the stairs one or two steps at a time. In how many different ways can you climb these ten steps?

Blocks Problem: You have a 2-by-10 rectangular frame as well as ten rectangular blocks, each having the dimensions 2 by 1. Your task is to fill the frame with the ten blocks so that no blocks overlap and the frame is entirely filled. In how many different ways can you arrange the ten blocks? (Rubel & Zolkower, 2007, p. 341)

The first interesting thing about this experiment was that many participants had immediate strong preference for one problem or the other. Each group spent forty minutes working on their selected problem and summarized their work on a poster to present to the class. The first group to present came up with the correct answer of 89

ways by adding the correct number of combinations for each simpler case. As soon as this solution was shared, the class felt curious and intrigued to find that the blocks problem had the same solution as the stair problem. Many students were surprised by the isomorphism between the two problems given that they had had strong preference for one of the problems initially. This experiment showed that when a problem solver is engaged in representations, the solutions depend on how each participant situates himself or herself within the task at hand (Rubel & Zolkower, 2007).

Interestingly, although both tasks in the Rubel and Zolkower study involved an iterative process and had an inductive solution in combinations, they had very important differences. The blocks problem was spatial, as it called for placing blocks within the fixed space, and the staircase problem called for climbing, which is a very dynamic process. The nature of this difference made “each problem more imaginable, accessible or challenging for different students” (p. 344). Rubel and Zolkower’s study further suggests that students should be invited to work on a variety of problems that bring forth a variety of representations.

Pape and Tchoshanov (2001) contend that the students usually come up with initial representations based on the “purpose” for creating the representational form. These initial representations are later refined by interacting with peers and the instructor. The task of building representations from the ground up requires intensive “social co-construction of meanings” and therefore, teachers and students need to work together to build understanding of the mathematical operations as they manipulate the concrete materials. Pape and Tchoshanov hypothesize that use of manipulatives facilitate the building of new understandings and representations through the use of “analogy,

transformation and simplification”. When working with manipulatives, the task of the learner is to construct the mapping between the manipulation of concrete material and the internal abstraction.

It has been supported through research that use of manipulatives can greatly assist students in building their own representations (DeGeorge & Santoro, 2004; Hall, 1998). DeGeorge and Santoro (2004) feel that using a multisensory approach to education targets the “strongest learning channels of individual students.” (p. 28) Also, it has been shown that students who use manipulatives in their math classroom usually outperform those who do not (DeGeorge, Santoro, 2004; Hall, 1998). Although this benefit might be slight, it holds across grade levels, ability levels, and topic, given that the student can make sense of the use of manipulative for that topic (Clements, 1999). Furthermore, manipulatives can increase scores on problem-solving tests along with improving student attitudes towards mathematics (Clements, 1999). These findings resonate with experiences of the Rutgers longitudinal study researchers as manipulatives were widely used in the study and were found to facilitate student sense-making, reasoning and creation of representations (Maher, Powell, & Uptegrove, 2010).

However, manipulatives are not always introduced or used effectively in a classroom. Hall (1998) states that in a typical classroom, the teachers expect that the “mathematical ideas embedded in these (concrete) materials, and in actions on them, will be absorbed by porous and inanimate students.” (p. 33) Educators often do not realize that the materials themselves are not enough to convey concepts, and that they need to impose mathematical structure on them to help students get mathematical understanding out of using them.

Although NCTM (2000a) emphasizes that the students should be able to use different forms of representations flexibly to work with real-world phenomenon, students have a rather difficult time in developing mathematical representations. Hiebert (1988) proposes a few things that can help students achieve representational competency. He contends that students should first be able to connect symbols with the objects that they represent. Then, the student must develop “manipulation procedures” for symbols that should become routine over time. Finally, a student should be able to elaborate on these symbols and rules to make more abstract systems using them as a building block. Pape and Tchoshonov (2001) recommend that for a fair chance at benefitting from construction of representations, students need ample time and opportunity to explore and understand the mathematical concept at hand and build multiple representations. Furthermore, Pape and Tchoshonov suggest that the students should have the freedom to negotiate the meanings of the symbols they create as well as the meanings of the standard representational forms with peers and the instructor.

Davis and Maher (1990) outline some very helpful hints for the teachers so that they can help students build their own representational blocks. They point out that how a teacher introduces a new idea is crucial to student success thereafter. Furthermore, if a teacher can understand the student’s representation and discuss concepts in terms of these representations, it can greatly benefit the student. On the other hand, Davis and Maher (1990) warn that if the teacher misunderstands the representation, the discussion can be very damaging to student progress. As such, they recommend that teachers should be trained to encourage students to create their own representations and pay attention to the meanings that students give to their representations.

3. METHODOLOGY

3.1. Setting and Background

This study is situated in the longitudinal study that began in 1987 in Kenilworth, New Jersey. Robert Sigley was a participant in the longitudinal study since first grade at the Harding Elementary School in Kenilworth. Prior to the study, this K-8 elementary school had half-hour sessions devoted to mathematics and the mathematical instruction involved rote memorization of procedures or drill and practice of computational skills. The principal of the Harding elementary school approached Rutgers University for help with instruction in mathematics as most of the students from the school did not excel when they moved on to high school mathematics classes (Maher, Powell, & Uptegrove, 2010).

The focus group initially picked by the Rutgers team consisted of 18 students from first grade that were randomly selected. These 18 students were together for grades 1 through 3. After grade three, the principal helped the Rutgers team keep a focus group of twelve students together for grades 4 through 8. During middle school, the group of students continued meeting with researchers, during school hours, about four to six times a year. Each time, they met for two 90-minute sessions and one 45-minute session and worked on mathematical tasks in small groups or pairs. In 1996 the high school in Kenilworth was closed and students became part of the regional system during their ninth grade. After a year, though, the community successfully protested the merger and students returned to Kenilworth for the remaining three years of high school. During high school years, fourteen students made time in their schedules to meet after school with researchers for informal sessions about four to six times a year and spent time on

problem-solving activities (Maher, Powell, & Uptegrove, 2010). Robert Sigley was one of the original ten students who had been with the study since first grade and continued working with the team during high school years on their own time.

The students in the Rutgers longitudinal study were encouraged to communicate their ideas and justify their solutions to their peers and researchers. They were not told if their answers were correct; instead it was up to them to convince themselves and others of the validity of their arguments and correctness of their solutions. The various experiences involving justification of solutions in the classroom helped students in developing sound mathematical reasoning skills and even facilitated building of proofs (Francisco & Maher, 2005; Maher 2002; 2005). As students worked in pairs or small groups, teachers and researchers questioned them to further explain their ideas. Martino and Maher (1999) assert that questioning students about their reasoning helps a teacher/researcher monitor the present thinking of a student.

3.2. Data Source

The data for the present study came from three main sources. The first source was the database of video recordings of every session and interview that was held with the students of the longitudinal study and maintained in an archive. For these sessions, one to three cameras were used to capture the data. Many times, one of the cameras was focused on capturing the students and their expressions while another camera focused on their work. The video recordings served as the source for Robert's early work in the longitudinal study. Video data allowed screen-shots that facilitated capturing participant's actions and representations as they happened during the actual session.

I recently interviewed Robert three times (7/2/2008; 11/14/08; 3/27/09) under the guidance of Professor Carolyn A. Maher, a principal investigator of the longitudinal study. These interviews were videotaped, and they provide the second major source of data of approximately six hours in length. These interviews followed and elaborated on an informal session where Robert began to explore the structure of Pascal's Pyramid with mathematician, Professor Todd Lee. This session provided additional two hours of video data that was also analyzed for this study. The third source of data came from field notes taken during interviews by me and Robert's work on paper preserved from his early years and recent interviews. The variety of sources provided a triangulation and helped enhance the validity of data collection.

To map out how Robert made any connections to his earlier problem solving, many sessions from the archived video data of the Rutgers longitudinal study were analyzed. All the sessions examined are listed at the beginning of Chapter 4. As a total, more than twenty hours of video data was analyzed for this study to construct a comprehensive account of how the participant created and used representations to solve mathematical problems.

3.3. Tasks

The tasks used in this research were combinatorial in nature and many of them shared an isomorphic structure. These tasks were chosen with the aim of providing meaningful instances during which the participant could be anticipated to create external representations and build understanding to solve a problem. Many of these tasks involved the Pascal's Pyramid and its relation to combinatorial problems. Pascal's Triangle was discussed by longitudinal study participants in relation to problems like the

Building Towers activity, the Pizza problem, the Taxicab problem, etc; and Robert was already familiar with the Pascal's Triangle. In this case study, he was given an opportunity to extend his understanding of the relationship between Pascal's ideas and combinatorics from the two-dimensional Triangle to the three-dimensional Pyramid.

3.4. Method of Analysis

I used the video analyzing model reported by Powell, Francisco, and Maher (2003) to analyze this data. This model consists of several steps that might be conducted in a non-linear fashion: viewing video, describing segments, identifying critical events, transcribing of video data, coding the data, constructing a storyline, and finally composing a narrative (p. 413). I briefly describe these steps.

3.4.1. Viewing

I viewed and listened to the video data many times so that I could become familiar with their contents. Powell, Francisco and Maher (2003) suggest that at this phase, the researcher should view the data without a particular lens for analysis in mind. This first step can help the researcher highlight the important parts of the video that can be analyzed further. These important pieces of the data are called critical events. Once the critical events surface, the researcher knows where to look in the video data for information-rich events and may then focus on analyzing them.

I watched all the videos at least two times and took notes on the important events. This helped me know what parts of the video were to be focused on. I also took time to write short summaries of the videos for myself as I viewed them. This was an important

step that assisted me in efficiently keeping track of the information in the sessions viewed and how it related across the sessions. In addition, I spent time working on the combinatorial problems in the sessions myself so that I could follow Robert's solutions and ideas.

3.4.2. Transcribing and Verifying

A detailed transcript provides a researcher an opportunity to pay attention to minor details of the video. Also a complete transcript is necessary to code critical events and keep track of the chronology of events as they occur. This step can play an important role in constructing a story line. All transcripts for this study were created by this researcher or other graduate students and they were verified by an independent researcher. The verification process assures that the researcher has an accurate transcript to analyze the events of a video session.

The transcripts helped me in accurately capturing the conversation that took place in the videos. It was also helpful to refer to the line numbers in the transcripts to verify the analysis provided in the results chapters.

3.4.3. Identifying Critical Events

The events that are coded as being critical depend on what the researcher is looking at. According to Maher (2002):

The analysis begins with the identification of critical events. The mathematical content of each critical event is identified and described, taking into account the context in which the event appears, the identifiable student strategies and/or heuristics employed earlier evidence for the origin of the idea, and subsequent mathematical developments that follow its emergence. (p. 35)

In this case study, the events that shed light on Robert's strategies, heuristics and justification along with his use of external representations to solve a problem or discover a pattern, were considered critical events.

Maher (2002) suggests that the critical events are part of a continuous story and that "Each critical event defines a timeline, consisting of a past, a present and a future" (p. 35). She uses the following illustration to represent the timeline of events (Figure 3-1).

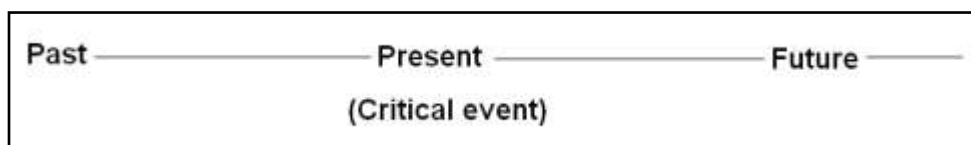


Figure 3-1 Maher's illustration for timeline of events

According to this model, as the critical event represents the present, it is important to study the events prior to the critical event and the events after the critical event to see how it might influence the future events.

3.4.4. Coding

A coding scheme was developed to help identify the themes that occurred in the data. In the Powell, Francisco and Maher (2003) model for analysis, the code is used to help the researcher make interpretations of the data by identifying recurring and important ideas. This stage of analysis is a major stepping stone to answering the research questions. As common themes emerged through the video data, the following codes were used.

Table 3-1 Codes used for the study

Codes for heuristics		
1.	Guess and check strategy	H1
2.	Try a simple problem	H2
3.	Think of a similar problem	H3
4.	Think of a special case	H4
5.	Create an external representation or notation	H5
Codes for explanations and justifications		
6.	Explain/justify answers using formal notation/formulae learned previously	E1
7.	Explain/justify using his external representations (diagrams, sketches, pictures)	E2
8.	Explain/justify using a manipulative	E3
9.	Explain/justify using solution of a similar problem done previously	E4
10.	Explain/justify how a previously solved problem (related to Pascal's) can be extended to three-dimensional case of Pyramid	E4
11.	Explain/justify how a problem can be solved using the external representation/diagrams/notations	E5
12.	Explain/justify using a formula devised by Robert during a session	E6
Codes for monitoring answers		
13.	Check to see if an external representation(s) are adequate or correct	M1
14.	Modify external representation or notation to take care of new ideas or discoveries	M2
15.	Count all possibilities to check answer	M3
16.	Consult with peers to verify conjectures or solutions	M4

3.3.5 Constructing a Storyline

Once the data is coded and verified for reliability, the researcher can begin to weave together pieces of information to construct a storyline. In this phase, the researcher tries to take results from the coding phase and begins to identify an “emerging narrative about the data” (Powell, Francisco & Maher, 2003, p. 430). At this time, other data sources, such as field notes and student's work should be used to come up with a comprehensive viewpoint of the video events. During this phase, the researcher attempts to identify a collection of events, called *traces*, that provide an insight into growth of student's mathematical understanding.

As I tried to construct a storyline for this study, a trace of Robert building an understanding of the structure of Pascal's Pyramid began to emerge.

3.3.6 Composing a Narrative

This is the final stage of data analysis where the researcher uses the outlined theoretical framework and attempts to answer the findings in terms of the research questions posed. This process, even though it is listed last, is usually intertwined and imbedded in the coding stage and the construction of story line stage (Powell, Francisco & Maher, 3003). The attempt of this stage is to provide overall general conclusions of the research study and synthesize the findings for the reader. My final narrative composition involved constructing an interpretation of the events described in the storyline, from the perspective of mathematical understanding and the role of external representations. As a final step, the findings from coded data and other non-video sources were narrated in an attempt to provide a comprehensive overview of the results of the study.

3.4 Verification of Validity

To ensure validity and trustworthiness, triangulation of data with researcher field notes, participant work, and video recordings along with transcripts was used to construct an accurate storyline of the events. Also, an independent researcher was used to assist in creation and verification of the coding scheme. Finally, results of the study were shared in a detail-oriented manner that allows the reader to conclude similar facts independently.

4 Chapter 4

4.1 Introduction

Robert (known as Bobby in his earlier years) was a participant in the longitudinal study since first grade. To analyze his problem solving strategies and heuristics related to combinatorial tasks leading up to his work with Pascal's Pyramid, this study analyzed his work over a sixteen-year period (from 1993 to 2009). One major focus of this study is to investigate how external representations used by the participant facilitate his building of mathematical understanding. The mathematical understanding relevant here is the development of Robert's ideas for an array of combinatorial activities leading up to the ideas of Pascal's Triangle and Pascal's Pyramid. The second major focus of this study is to trace any connections that Robert makes with his earlier problem solving and provide a longitudinal overview of interplay between his external representations, knowledge-building and problem solving.

To map out how Robert makes any connections to his earlier problem solving, many sessions from the archived video data of the Rutgers longitudinal study were analyzed in addition to his post graduate work with the researcher and Professor Lee. In specific, the following sessions were examined.

Table 4-1 Sessions Reviewed

Session No.	Grade	Date	Task/Interview Topic	Type of Session	Approximate Time Length
1.	5	2/26/1993	Guess My Towers	Whole class session	132 min
2.	5	3/1/1993	Pizza Problem	Whole class session	37 min
3.	7	12/13/1994	Garage Door Problem	Whole class session	91 min
4.	7	12/14/1994	Candles	Whole class	42 min

			Arrangement	session	
5.	7	12/15/1994	Candle Arrangements Problem	Whole class session	108 min
6.	11	11/13/1998	Revisiting Towers problem	Small group interview	120 min
7.	11	3/1/1999	Pizza Problem	Small group interview	97 min
8.	11	4/26/1999	Towers Extensions	Small group interview	118 min
9.	12	7/7/1999	Combinations for choosing committee of two out of five people	Two people presentation to class	27 min
10.	12	8/31/1999	World Series Problem and the Problem of Points	Two people interview	95 min
11.	16	9/12/2003	Towers: Each has two of one color	Two people problem solving session	46 min
12.	18	8/8/2005	Exploring Pascal's Pyramid	Informal session	82 min
13.	Post Graduate level	7/2/2008	Explaining layers of Pascal's Pyramid	First Post-Graduate interview	49 min
14.	Post Graduate level	11/14/2008	Explaining Pascal's Pyramid and connecting it to building towers and Ankur's Challenge	Second Post-Graduate interview	86 min
15.	Post Graduate level	3/27/2009	Explaining Pascal's Pyramid and connecting it to building towers and exploring the Taxicab problem	Third Post-Graduate interview	82 min

The research questions that guide the investigations of this study are: 1) What are the external representations that Robert uses to help him understand problems related to the properties of Pascal's Pyramid? 2) How, if at all, do these external representations help him in solving problems related to Pascal's Triangle or Pascal's Pyramid? 3) How

does he use, modify or reuse his external representations over time to provide justifications for his solutions? 4) What connections, if any, does he make to earlier problem solving?

The informal session (Session 12) with Todd Lee is the episode where I had started my study of Robert's work. In Session 12, Robert was informally discussing how to represent the layers of Pascal's Pyramid on white board with markers. To build on and understand Session 12, I conducted three interviews (Sessions 13, 14, 15) with Robert under the supervision of Carolyn A. Maher, a principal investigator of the Rutgers longitudinal study. An objective of these interviews was to understand the external representations and Pascal's pyramidal ideas that Robert had shared during Session 12. The second objective of these interviews was to provide Robert with an opportunity to revisit and extend various combinatorial problems related to Pascal's Triangle and Pascal's Pyramid that he had encountered as a part of the Rutgers longitudinal study. To begin investigating the first two research questions concerning the nature of external representations and how these external representations help Robert in building understanding, Sessions 12, 13, 14, and 15 (dated 8/8/2005, 7/2/2008, 11/14/2008, and 3/27/2009 respectively) are analyzed. In addition, to answer the remaining research questions about how Robert modifies his external representations over time, and to investigate if and how he makes connections to his previous problem solving, Robert's work related to the counting strand since fifth grade (sessions 1 – 11) are analyzed.

As a starting point, this chapter focuses on Session 12 where Robert informally explored the layers of Pascal's Pyramid with Professor Todd Lee. I want to use this session as a starting point as it is chronological with my journey of studying Robert's

work. Furthermore, this session includes a variety of external representations that Robert used for Pascal's Pyramid as he attempted to construct its layers. As such, Session 12 provided considerable insight into the first two research questions of this study to explore the nature and role of external representations that Robert used. In the next (fifth) chapter, I analyze the three post-graduate interviews (sessions 13, 14, and 15) that build on session 12 and provide opportunities to observe how Robert modified and reused his external representations over time. Once the recent work (Sessions 12 – 15) is reviewed to initially address the nature and role of external representations used by Robert, I explore the connections between his new and old ideas in the following chapters. Overall, this study begins with the critical event of Robert sharing ideas with Professor Lee in an informal conversation, and then tracing the origins and extensions of these ideas.

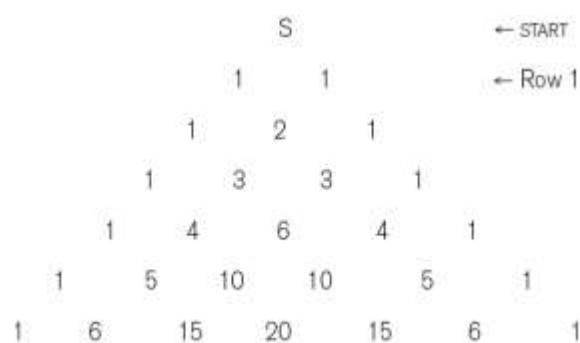
I present a brief overview of some of the properties of Pascal's Triangle and Pyramid that will be discussed as a part of the following analysis before sharing results of the Session 12.

4.2 Pascal's Triangle

Pascal's Triangle bears the name of the seventeenth-century French philosopher and mathematician Blaise Pascal who explored many of the Triangle's properties and related them to the area of probability (*Navigations II*, p. 41). The Triangle was known to mathematicians for centuries before Pascal. The Chinese mathematicians described this Triangle in eleventh, thirteenth and fourteenth centuries and Islamic mathematicians had worked with it in eleventh and fifteenth centuries (*Navigations I*, p. 157). The

earliest explicit depictions of a Triangle of binomial coefficients occur in the 10th century in an ancient Indian book on Sanskrit (Pascal's Triangle, 2009).

Pascal's Triangle is a triangular array of numbers. The top row, which is the top vertex of the Triangle, consists of the single number 1 (marked as S for start here). Each succeeding row begins and ends with a 1, and remaining each entry in the row refers to entries in the rows above it. Each number in a given row, except the 1s at the ends, represents the following sum: (Number just above and to the left) + (Number just above and to the right) (*Navigations II*, p. 41). Look at the figure below for further reference.



There is a deep connection between ideas of combinations, Pascal's Triangle and the binomial theorem. Pascal's Triangle determines the coefficients which arise in binomial expansions. In general, when a binomial like $x + y$ is raised to a positive integer power, n , the binomial theorem gives:

$$(x + y)^n = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots + a_{n-1}xy^{n-1} + a_ny^n,$$

where the coefficient a_i is the number of combinations $C(n, i)$. Interestingly, numbers on the rows of Pascal's Triangle are in fact the combinations a_i in the binomial expansion

where n corresponds to the row number. The index i corresponds to the position in a particular row as we move from left to right. Look at the figure below for reference.

Row 0						$C(0, 0)$					
Row 1					$C(1, 0)$		$C(1, 1)$				
Row 2			$C(2, 0)$		$C(2, 1)$		$C(2, 2)$				
Row 3		$C(3, 0)$		$C(3, 1)$		$C(3, 2)$		$C(3, 3)$			
Row 4	$C(4, 0)$		$C(4, 1)$		$C(4, 2)$		$C(4, 3)$		$C(4, 4)$		
Row 5	$C(5, 0)$		$C(5, 1)$		$C(5, 2)$		$C(5, 3)$		$C(5, 4)$		$C(5, 5)$

(Source for Figure: Pascal's Triangle, 2009)

As such, the coefficients of a binomial expansion can just be read on rows of the Triangle. For example, if we consider the binomial expansion of $(x+y)^3$, the binomial theorem gives the expansion to be: $C(3,0)x^3y^0 + C(3,1)x^2y^1 + C(3,2)xy^2 + C(3,3)x^0y^3$. When the combinations are calculated in this expansion, we get: $1x^3 + 3x^2y + 3xy^2 + 1y^3$. So, if we look at the coefficients in the expansion of $(x+y)^3$, they are 1, 3, 3, 1, which is the third row of the Pascal's Triangle.

Another interesting consequence of the binomial theorem is obtained by setting both variables x and y equal to one. In this case, we know that $(1+1)^n = 2^n$, and so

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$$

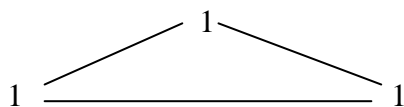
As such, sum of the entries in the n th row of Pascal's Triangle is the n th power of 2. So, sum of the entries in the third row is expected to be $2^3 = 8$ and it is easy to verify that:

$$1 + 3 + 3 + 1 = 8.$$

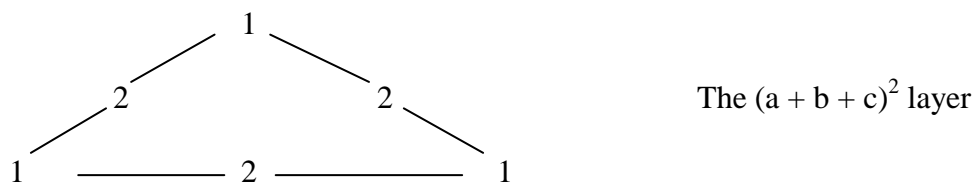
Another useful application of Pascal's Triangle is counting the number of combinations possible for a given n . Suppose there is a team of ten basketball players and one wants to know how many ways there are of selecting eight. Rather than working out the calculation: $C(10, 8)$, one can just refer to the row 10 (keeping in mind that the topmost row is numbered 0) of the Pascal's Triangle. Looking at the coefficient of the eight entry, (keeping in mind that the first entry is numbered 0), the value is 45. Therefore, the solution of "10 choose 8" is 45 (Problem adopted from Pascal's Triangle, 2009).

4.3 Pascal's Pyramid

The Pascal's Triangle has higher dimensional generalizations. The three-dimensional version is called Pascal's Pyramid or *Pascal's tetrahedron*. Just like the Pascal's Triangle gives coefficients of binomial expansion, the Pyramid gives coefficient of trinomial expansion. The faces of the Pyramid are Pascal's Triangle. As such every row on the outer surfaces of the Pyramid is a row corresponding to a row of the Triangle. Every entry in the middle of the Pyramid is the sum of three entries above it. The following figures show first two cross-sectional views of the Pyramid.



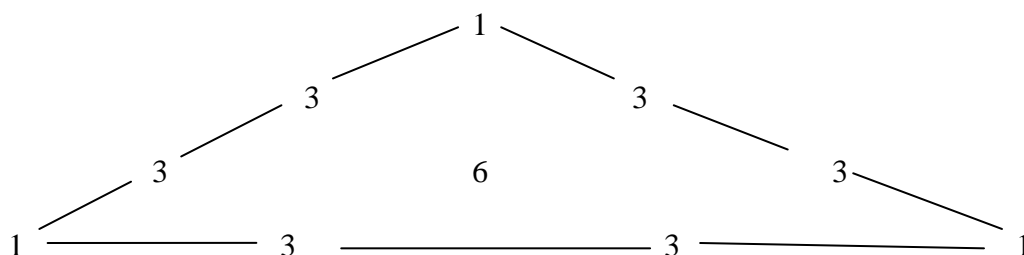
The $(a + b + c)^1$ layer



Again, if we consider the trinomial $(x + y + z)^n$ the coefficients are given by the trinomial coefficients. For example, for $n=3$, we have the coefficients:

$$\begin{aligned} & \binom{3}{0,3,0} \\ & \binom{3}{1,2,0} \binom{3}{0,2,1} \\ & \binom{3}{2,1,0} \binom{3}{1,1,1} \binom{3}{0,1,2} \\ & \binom{3}{3,0,0} \binom{3}{2,0,1} \binom{3}{1,0,2} \binom{3}{0,0,3} \end{aligned}$$

These trinomial coefficients simplify to the third cross-sectional layer shown in the figure below.



Also, the sum of entries in the n -th of cross-sections is 3^n . For example, all the terms in $n=3$ cross-section add up to $3^3 = 27$. Next, I present results from the Session 12.

4.4 Layers of Pascal's Pyramid: August 8, 2005 (Session 12)

4.4.1 Setting and Background

Note: For this session, Todd Lee is coded as R1, Elizabeth Uptegrove is coded as R2 and Carolyn Maher is coded as R3.

On August 8, 2005, Robert, working for the Robert B. Davis Institute for Learning as videographer, was planning to videotape some of the middle-school participants in an NSF funded Informal Mathematics Study conducted in Plainfield, New Jersey. The children were designing posters in preparation for sharing their work. Waiting for an earlier session to end, Robert was engaged in a conversation with Professor Todd Lee (R1), a mathematician from Elon University who was visiting Rutgers and who had observed videos of participants in the Kenilworth longitudinal study the previous evening. R1 learned that Robert was a participant in the long-term Rutgers study and they began chatting about his mathematical activity. What followed was an informal session where Robert and R1 talked at length about the Pascal's Pyramid (approximately 70 minutes) and how to represent it on the white board with markers. R2 and R3 observed parts of this session, arranging that it be captured on video. There was no specific task designed for this session other than trying, informally, to capture the conversation between a mathematician and participant in the longitudinal study that occurred rather spontaneously.

4.4.2 Initial Exploration

When the camera began to capture the conversation between Robert and R1, they were already engaged in a conversation about “duplicates” and adding a “b” or adding a “c” to the terms they had written out on the white board. Robert had already created a

couple of times (lines A25 – A30) but Robert continued to utter “addition” when he showed “multiplication” several times in the session. This language use was observed in other sessions as well, in particular, the three post-graduate interviews (Sessions 13, 14, 15).

The use of “addition” is not literal but reflects how Robert uses language to represent a product of factors using the notation of letters. It is conjectured that Robert may be drawing upon his earlier problem-solving experience, in particular the *towers building* activity where students used Unifix cubes to answer questions like how many different towers are possible that are three tall when using two colors, etc. This became evident at end of this session when Robert discussed a solution to *Ankur’s Challenge* and drew out towers four tall. While explaining *Ankur’s Challenge*, when Robert multiplied a term by a variable, he compared it to adding a block to the Unifix tower. It is reasonable to infer that Robert may mentally be *adding* a Unifix cube to make a tower taller as he moved between layers. This might explain his articulation of multiplication as addition.

Also, Robert was concerned that when he multiplied terms by a certain variable, there would be some “duplicate” terms that he thought would be counted twice. He considered dividing by two later as a strategy to correct the doubling (lines A47 – A51). Robert’s concern about “duplicates” also hints that Robert was drawing upon his earlier experience with building towers activities. In building towers with Unifix cubes, students would occasionally get two identical towers that they would refer to as “duplicates.” Robert was anticipating, perhaps drawing upon his previous experiences, that there will

be a repetition in terms that were generated by the multiplications with the three variables, a , b , and c .

4.4.4 Let's draw it 3-D! Concentric layers

Three and a half minutes into the session, R1 noted that Robert's picture (Figure 4-1) was not "beautiful" anymore and Robert said that they needed to draw in three dimensions. So, R1 encouraged Robert to try and create a three-dimensional representation for his two-dimensional drawing of Pascal's Pyramid. Robert hesitated and said that he can't draw in three dimensions and that he was a "terrible drawer" (line A57).

Note: For all sessions, the layer $(a + b + c)^0$ is referred to as the "layer zero", $(a + b + c)^1$ is referred to as the "first layer", $(a + b + c)^2$ is called the "second layer" and so on.

R1 suggested that Robert try to construct just the first layer of the Pyramid and Robert wrote a , b , c in a straight line. R1 questioned why the variables were in a straight line and Robert asked him to imagine that a is actually coming out of the board while b and c are in the plane of the white board to "visualize" his "three-dimensional" sketch. When asked to draw the next level out, Robert made the following representation of the second layer to represent the terms in expansion of $(a + b + c)^2$ (Figure 4-2).



Figure 4-2 Second Layer

Robert explained that the “ones” in the yellow circle represent $1a^2$, $1b^2$ and $1c^2$ in the expansion of $(a + b + c)^2$ and the “twos” outside the yellow circle (enclosed in green circle) were the terms $2ab$, $2bc$ and $2ac$. Interestingly, Robert suggested that the drawing in Figure 4-2 not only represented the second layer, $(a + b + c)^2$ but also encompassed the first layer $(a + b + c)^1$. That is, Robert suggested that the “ones” in the yellow circle could also mean $1a$, $1b$, $1c$ as he remarked that “It’s kinda like this is its own level but then green has these two levels” (line A87). This prompted R1 to ask Robert to carry his ideas further and draw out the third layer. Before Robert could draw the third layer out, R1 asked him to explain the coefficients of his second layer one more time. This time around Robert realized that he could not represent two layers simultaneously (A94 – 95). Robert had initially used a yellow circle to represent the first layer and alleged that the green circle was not only the second layer but it represented the first layer inside of the second layer. When Robert was convinced that the layers cannot coincide, he erased the inner yellow circle.

Shortly afterwards (seven minutes into the video), R1 encouraged Robert to think about layers of the Pascal’s Pyramid one at a time. When asked about the very topmost

layer, Robert replied that the level at very top was the $(a + b + c)^0$ layer which equaled one; this was followed by the $(a + b + c)^1$ and the $(a + b + c)^2$ layers and so on. R1 jokingly put 1 on the white board and said that he will help Robert with the very top layer and that Robert should try to create the next layers. Robert put three ones in a circle to represent the $(a + b + c)^1$ layer. He kept Figure 4-2 as his second layer and drew the third layer as shown in Figure 4-3.

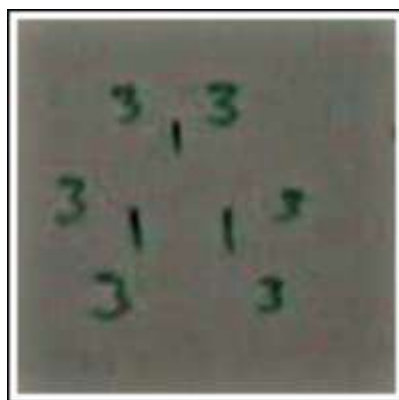


Figure 4-3 Third Layer

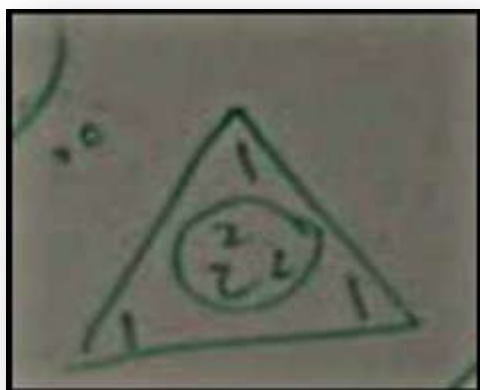
Once again, Robert tried to include the second layer within the third layer and wrote six 2s around the Figure 4-3. However, he changed his mind shortly afterwards and erased the “twos” to leave the third layer as in Figure 4-3.

At around ten minutes, Robert shared with R1 that he thought there would also be a six in the third layer and placed a six in the middle of Figure 4-3. Then, R1 encouraged Robert to explain how the layers he had created would fit in a real three-dimensional space. Robert explained that the layers, the single 1, three 1's in a circle, Figure 4-2 and Figure 4-3, would be stacked one on top of the other to create the three-dimensional

Pyramid. Robert was able to visualize how the two-dimensional representations of the various layers would be stacked in three-dimensional space to create the Pyramid.

4.4.5 “What’s the dominant geometric shape?” From circles to triangles

Next, R1 began to prompt Robert to make his representations in the most “dominant” geometric shape of the Pascal’s Pyramid. Robert remarked that a Triangle was the most “dominant” shape for the Pyramid and began to look for triangles in his representations. As one of the early attempts, Robert created the second and the third layers as shown below.



*Figure 4-5 "Triangular"
Second layer*



*Figure 4-4 "Triangular"
Third layer*

Robert represented his layers with circle(s) inside a Triangle with ones written in the vertices of the outer Triangle (Figure 4-5, Figure 4-4). R1 asked him to think if the numbers inside the circle actually form a triangular pattern as well. Robert suggested

they might and replaced the circles in his figures with smaller concentric triangles (Figure 4-6). Up to this point in the session, Robert maintained that a layer of Pascal's Pyramid would have "ones" on the edges and that the other numbers, "twos" and "threes," would be in middle of the surface of a layer.



Figure 4-6 Concentric Triangles

Then, Robert began to explain to R1 that he thought of an edge of the Triangle as a "direction" which guided what variable out of a , b , and c should be multiplied with terms of a previous layer. He said that as one followed a certain edge of the Pyramid, one went on to multiply with one of the three variables. He equated picking an edge of the Pyramid with choosing a particular "direction" out of the three possible "directions" (A 167). Thinking of each edge of the Pyramid as a "direction" was an important heuristic that Robert used to construct terms in the following layer of the Pyramid. Robert used his definition of "directions" to arrange the terms he created for the next layer. Thinking of a particular direction as multiplication with a particular variable seemed to provide a cue to Robert about what the next generated term would look like. It seems plausible that

this is not a novel idea for Robert as he explained it with ease to R1. Robert said that he had previously thought of Pascal's triangular edges as a "direction" that guided multiplication by a or b to yield terms in the expansions of binomials $(a + b)^n$ (A181-186).

When R1 asked Robert to explain what the threes in Figure 4-4 meant, Robert explained that the threes were the terms $3a^2b$, $3b^2c$, etc. in the expansion of $(a + b + c)^3$. This reveals that Robert has an understanding of the numbers he used in the representation of the third layer as coefficients of the terms in the expansion of $(a + b + c)^3$ (Figure 4-4).

4.4.6 Is there a pattern? Getting to the "prettiest picture"

At about fifteen minutes into the conversation, R1 pointed out the addition property of Pascal's Triangle where one could add successive terms from previous layer to generate the next layer and questioned Robert how they might do the same thing for the Pyramid. Robert wrote three "twos" inside a separate yellow Triangle and explained that each of the "twos" went to two places (using two arrows from each of the "twos") in the next layer and generated the "threes" (Figure 4-7).



Figure 4-7 Robert explains how third layer is built from second layer

At this point R1 asked the other observers in the room if there was a transparency they could use. He suggested that Robert should try to put together or “brick” a previous layer to show how the next layer might be generated and try to get the “prettiest picture” possible (line A187 - A197). R1 continued to encourage Robert to find a representation that was more convincing to them visually as they continued to discuss Robert’s existing diagrams.

R1 summarized that Robert had many correct ideas as he said “This may be what you want, ‘cause it certainly works out in our heads” (A229) but R1 indicated that he was not sure that they had found the “prettiest thing that makes you (Robert) happy” at that point (A217). And therefore, Robert continued to modify and edit his external representations to get to a mathematically “pretty” picture.

Simultaneously with placing numbers in a layer of the Pyramid, Robert tried to represent the addition pattern between terms of different layers. For one attempt, Robert moved the “ones” to the middle of the second layer and wrote the “twos” on the outside instead. Pretty quickly Robert realized that if “ones” were in the center, he would have

no space to put the six from the third layer in the middle. This was a guess and check strategy where Robert used his external representation as a sounding board and rejected the idea of placing “ones” in the middle of a layer. Robert projected that the representation with “ones” in the middle would fail for the third layer. At this point an observer handed R1 a transparency. R1 asked again if Robert’s representation of the second layer was a Triangle of some sort; Robert then created two triangles that together made a “star of David” (line A279, Figure 4-8).

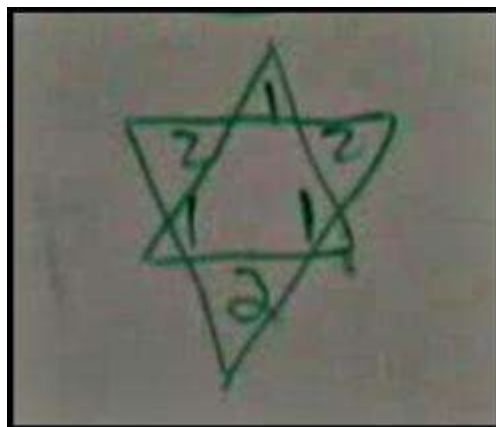


Figure 4-8 "Star of David"

R1 continued to ask if there was an overall single shape to the layers that Robert had created. Robert indicated that for the third layer the “threes” would sit outside of the “star” but the six would still go in the middle. Robert however indicated puzzlement and wondered why the “six” still has to be placed in the middle. R1 suggested that Robert should highlight all his terms related to a , b and c separately and see if that would yield a pattern or help him answer his question about the placement of “six.” R1 was suggesting to Robert to try a simpler case here as a problem-solving strategy. Robert created three

triangles with different colored markers to show all the terms related to the variables a (red triangle), b (yellow triangle) and c (black triangle) (Figure 4-9).

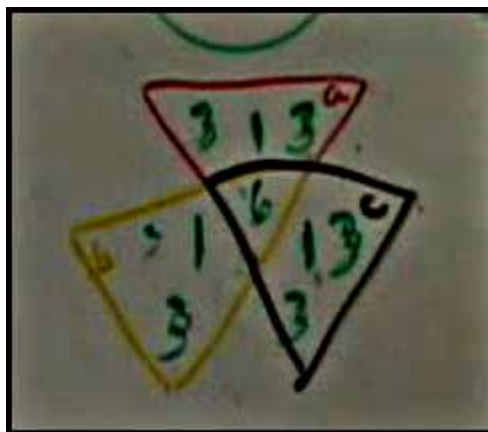


Figure 4-9 Separating the a , b , c terms

R1 questioned if Robert was convinced of his picture (Figure 4-9) and Robert changed his mind after trying to explain some of the terms in the diagram. He created another grouping of the terms in the third layer that R1 suggested were “asymmetric” (line A338). R1 asked Robert at this point in the conversation (about 30 minutes into the session), whether Robert would choose to work alone or with a friend if he was on the game show: “Who wants to be a Millionaire?” Robert replied that he would choose a friend because he claimed that if R1 was not there, he would not have gotten this far and that he would have stopped working on the representation in absence of R1’s questioning (lines A344 – A353).

R1 persisted in his quest to see a triangular representation that would yield the “prettiest picture” and encouraged Robert to come up with a mathematically appealing picture. R1 asked Robert to try again to write out the piece of a layer if one of the

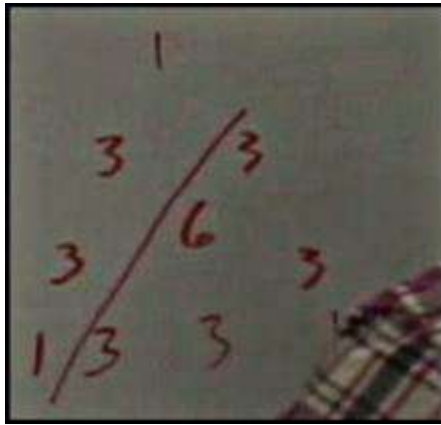


Figure 4-11 The "pretty" third layer

R1 then put another transparency on top of Figure 4-11 and asked Robert to explain what the layer previous to this one looked like. Robert placed the second layer on the transparency (Figure 4-12). R1 then repeated his questions about how the next layer might be generated from a previous layer in the Pyramid. At this time Robert clarified the pattern of addition between the layers by explaining that one of the black *1*s along with one black *2* from the second layer would add to give a red *3* of the third layer.

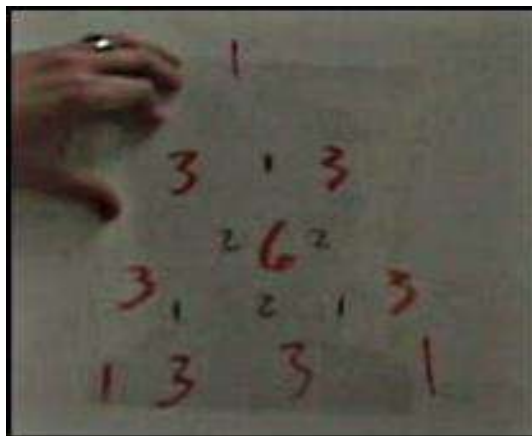


Figure 4-12 Placing second layer on top of the third

This is a remarkable moment in the session as Robert was able to very easily explain the pattern of addition in the Pyramid moments after he has created an appropriate and sufficient external representation for it (A 478 – A 481). All through the earlier parts of the session (first 35 minutes approximately), Robert had ideas about adding terms to produce the next row but he was not able to convey them to R1 in a clear manner. Occasionally, Robert's external representations hindered the communication as R1 tried to follow what terms Robert verbally proposed to add and create the new term. Once the external representation was concise, and R1 seemed to agree with it, Robert was able to not only see the pattern clearly but also communicate his ideas in an effective manner to R1.

4.4.7 **“We always ask the next layer”: Robert finds the fourth layer of Pascal's Pyramid**

Although R1 seemed satisfied with the third layer's representation that Robert created, he and Robert together decided to go ahead and discuss one more layer. Robert claimed that this was “fun” and he was happy to continue (line A489). Robert created a layer for $(a + b + c)^4$ by placing 4's and 6's around the Triangle enclosing the third layer (Figure 4-13). It is evident that Robert was adding terms from the previous layer to generate and place his 4's and 6's on the circumference. At this point, Robert had figured out that the outer rows of the next layer would be 1, 4, 6, 4, 1 on each side, which was a row of Pascal's Triangle.

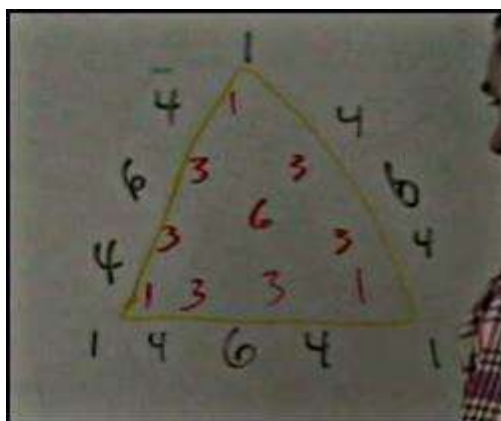


Figure 4-13 Attempting to construct fourth layer from the third layer

R1 asked Robert if he knew what would be the sum of all the numbers in the fourth layer. Robert quickly responded that it would be 3^4 or 81. This response was very quick on Robert's part, suggesting that he had previously worked with a similar idea. Perhaps Robert was retrieving earlier knowledge related to the sum of entries in a row of Pascal's Triangle.

It still remains to investigate how Robert might have generalized from 2^n for the sum of entries in a row of a Triangle to 3^n for sum of entries in a layer of the Pyramid. After settling with 81 as sum of the entries in the fourth layer of the Pyramid, Robert tried to calculate what was missing in the center part of this layer. He figured out that the outside entries all add up to 45, so the entries in the middle add up to 36 as $81 - 45 = 36$.

R1 pointed out that there were many kinds of additions going on here and Robert decided to start writing out "options" like "4-0-0". Robert explained that "4-0-0" meant there were four of the first variable (a) and none of other two variables (b , c). In other words, "4-0-0" was the term a^4 . The idea of using co-ordinates to list all the options was

used very fluently by Robert and hints that it was a notation that Robert had developed earlier.

4.4.8 Ankur's Challenge: Where does the solution lie on the Pascal's Pyramid?

At this point (about 42 minutes in to the session), R3 interrupted the conversation to divert Robert's attention to *Ankur's Challenge* upon a suggestion by R2. R3 asked Robert to recall what *Ankur's Challenge* was and Robert responded that the problem asked for "four tall, three colors, you have to use one of each color" referring to the building towers activity that he had seen in his earlier years (line A547)¹. R3 asked Robert to explain where the solution to Ankur's problem would lie in the Pyramid.

As a first response, Robert thought the solution would lie in the middle of the fourth layer; then he quickly offered that it would be on the outside. At this point, R3 rephrased *Ankur's Challenge* to be a problem of finding towers that were "four tall" with "exactly two of one color" and Robert agreed with this interpretation (line A552). Again, trying to locate solution to *Ankur's Challenge* on his graph, Robert responded:

A 555.	Robert	Umm...I the... this ring... the threes... cause those are like, that's like... no wait. This won't work 'cause this is three tall, so there can't be... this is the three tall case. The yellow Triangle, so that... you can't have two of one color in there.
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Robert realized quickly that the solution to Ankur's challenge could not be in the third layer as it only represented towers three tall. This instance suggests that Robert had previously mapped his ideas from building towers three tall to the third layer of Pascal's

¹ *Ankur's Challenge (January 1998, Grade 10)* – Find all possible towers that are four cubes tall, selecting from cubes available in three different colors, so that the resulting towers contain at least one of each color. Convince us that you have found them all.
(Maher, Powell, & Uptegrove, 2010)

the triplets of 3, 6, and 3 that he believed would produce one of the 12s in his next layer (Figure 4-15).



Figure 4-15 Where the 12s come from

At seven minutes into the second part of the session, R2 intervened and explained which group of 3, 3, and 6 from the third layer produced the twelve in the fourth layer. She was drawing upon her experience of working with the Pyramid and walked Robert through how one of the 12's was produced (A653 – 661). Robert quickly absorbed R2's explanation of the addition in the Pascal's Pyramid. However, he wondered if "her choice" of 3, 3 and 6 was the only choice that would work. He questioned why a particular 12 had to come from a particular set of 3's and 6. R1 was also unsure about an answer to Robert's question. However, to address the doubling issue, R1 guided Robert to match up his numbers in the Figure 4-15 with the terms in the expansion of $(a + b + c)^3$. This led R1 to discover that Robert was counting $6abc$ twice. R1 asked R2 if she had meant for Robert to count $6abc$ twice and she said that she had not. R2 said that she was not sure that she understood Robert's concern about the "doubling" at that point.

To clarify things, R2 began to talk about the variables, a , b , c as colors of the Unifix cubes. She talked about building towers and claimed that there were no duplicates. R2 contended that there were no duplicates as going from one layer to next, one multiplied by each of the variables (a , b , c) exactly once and they get “used up.” She suggested that Robert should remember to not multiply any term by a particular variable more than once. In other words, R2 was explaining that the $6abc$ term would produce $6a^2bc$ only once when it was by multiplied by a . Robert agreed that R2 was proposing an addition that will not yield any duplicates. Robert used his representation of towers to convince himself that there are no duplicates and he began to create towers corresponding to the terms in the fourth layer (Figure 4-16).

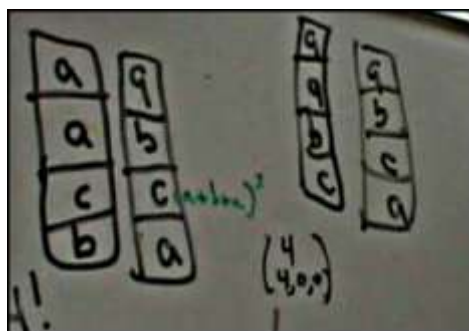


Figure 4-16 Robert uses towers for terms of fourth layer

From the towers listed, Robert saw that each tower was unique even though they all had two a s, one b , and one c and therefore there would be no over counting. However, Robert was still not sure why one set of 3, 3, and 6 from the third layer was better than any other set to produce a 12 in the fourth layer. R2 explained that although a sum of 12 can be obtained by picking any two 3s and one 6 mathematically, it will not work in terms of building towers (A705 – 709). Robert was not completely satisfied and

began to list each term from the third layer to explicitly create a 12 of the fourth layer. R1 encouraged him to write two alternate set of three's and six that he could have picked to create a certain 12. Robert picked a green Triangle and a red Triangle of 3, 3, 6 as two alternative sources of a 12 (Figure 4-17). He elaborated the green Triangle using a black list of terms and the red Triangle with a red list of terms (Figure 4-18).

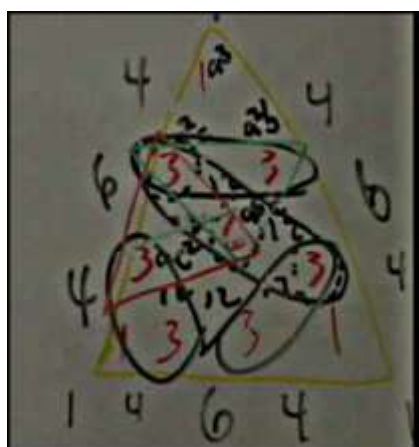


Figure 4-17 Two alternative triangles that may produce a 12

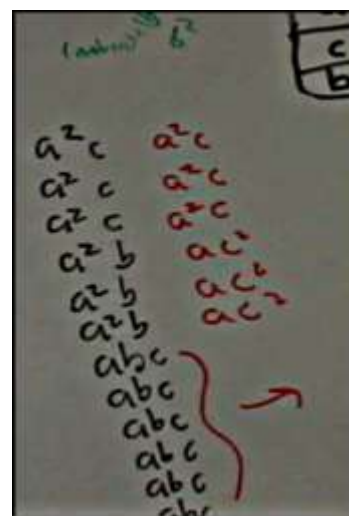


Figure 4-18 Listing terms for two triangles

R1 asked what was the property that one list had but the other did not. Robert continued to say that as long as there was no overlapping it did not matter and both the lists were equally sensible. However, through further discussion with R1, Robert came to agree that the red list would not work. Robert was eventually able to verify that the black list will yield a $12a^2bc$ when the first three a^2c terms are multiplied by b , the next three a^2b terms are multiplied by c , and the six abc terms are multiplied by a . However, carrying out a similar multiplication with the red list will produce a^2bc and abc^2 terms

that could not be added together. R1 concluded with Robert that “So, it’s (red list) not just ugly, it’s not working” (A824). This particular strategy of listing each term convinced Robert that only a particular set of 3, 3, 6 (in this case the green Triangle in Figure 4-17) could be counted to yield the 12 in the next layer. Although R2 had explicitly showed Robert how the addition was carried out in the Pascal’s Pyramid, Robert was only convinced after trying and testing out his own hypothesis through a term by term listing.

Finally when R3 inquired about *Ankur’s Challenge* again, Robert explained its solution is the three 12s on the inside of the fourth layer (A870 -883). Robert explained that he was counting the 12’s in the middle as they represented four-tall towers with exactly two of one color. Robert associated the variable a with the color blue, the variable b with the color red, and the variable c with the color white. He explained that a tower with two blues and one red (a term with a^2b) was missing a white (or c), so you add a white block to it (multiply by c to get a^2bc). Similarly, the $6abc$ term had one of each color and you were adding the blue (or a) to it to produce a $6a^2bc$ term. Finally when the two $3a^2bc$ terms and one $6a^2bc$ terms were combined, it resulted in one $12a^2bc$ term. Robert explained that this term represents 12 towers that were four tall and had exactly “two of one color” (here a^2) in them and hence a solution to *Ankur’s Challenge*. As a total there would be 36 towers from the three 12s added together. R3 claimed that it was a different solution than any one has shared with her before and requested Robert to write up his solution. The session concluded shortly afterwards with Robert discussing his math courses at the time and what his future plans for education were.

5. Chapter 5: Three Post-Graduate Interviews

5.1 Introduction

Three post-graduate interviews with the participant were conducted on 7/2/2008, 11/14/2008 and 3/27/2009 under supervision of a lead principal investigator of the Rutgers longitudinal study, Carolyn A. Maher. Robert revisited properties of Pascal's Pyramid as these interviews explored his ideas and external representations that were observed in Session 12 (8/8/2005) as discussed in Chapter 4. In addition, the interview sessions gave him an opportunity to revisit some of the combinatorial tasks that he had seen during his earlier participation in the Rutgers longitudinal study. The first interview (Session 13) was conducted with Robert sharing his ideas on the paper and white board; however, in the other two interviews, Robert was provided with concrete materials (*Zome-tools*) to build a three-dimensional model of the Pyramid. The three interviews will be analyzed at length in this chapter to further shed light on nature and role of the external representations in Robert's construction of understanding for properties of Pascal's Pyramid.

5.2 First Post-Graduate Interview: July 2, 2008 (Session 13)

5.2.1 Setting and Background

Note: For this session Anoop Ahluwalia is coded as R1 and Carolyn Maher is coded as R2.

The first post-graduate interview with the participant was conducted by R1 and observed by two graduate students, Erika Bilyk and Scott Rutherford. Scott also video

recorded the session while Erika observed the session and took notes. R2 had co-authored and overseen the interview protocol for this session.

R1 had transcribed Session 12 and watched its video several times prior to conducting this interview. Along with R2, R1 designed questions to probe further into the representations that Robert had created with Professor Lee. Several other questions were designed to encourage Robert to reflect on his initial introduction to the terms “Pascal’s Triangle” and “Pascal’s Pyramid,” the hurdles and triumphs from Session 12, his opinions about group-work and his experience studying mathematics. To facilitate discussion about Session 12, Robert was shown some clips to refresh his memory and then asked about those specific clips. Robert was provided with a white board and markers as well paper and pens to choose and write on. The interview lasted 48 minutes and a detailed discussion of the critical events from the session follows.

5.2.2 Robert’s first recollection of “Pascal’s Triangle” and “Pascal’s Pyramid”

R1 started the interview by asking Robert to recall where he had first learnt about properties of Pascal’s Triangle. Robert claimed that he “subconsciously knew it” since fourth or fifth grade but it was in high school that he connected several ideas, like the binomial expansion, to structure of Pascal’s Triangle (B3). R1 inquired what in specific about the Pascal’s Triangle was of interest to Robert. Robert replied that he and his friends saw the Triangle as a neat way of getting answers to many combinatorial problems like the *Towers-Building* activity, the *Pizza problem* and the *Taxicab problem* (B4 – 9). Robert recalled that the name “Pascal’s Triangle” was shared with the class by their teacher Ralph Pantozzi in eleventh grade. Robert expressed amazement that the

class came to know about a name for the triangular array of numbers long after they had worked at length relating it to the combinations formula $C(n, r)$ (B15).

When asked about the first time Robert had heard about Pascal's Pyramid, Robert recollected that it was only when he took a combinatorics course at undergraduate level that he came across the term "Pascal's Pyramid." Robert also recalled that in Session 11 he discussed some ideas related to Pascal's identities when he worked with Elizabeth Uptegrove and Brian, a fellow student. Robert explained that as an undergraduate he came to realize that it was possible to list terms in layers of the Pyramid in a methodological way and that the terms were related to expansion of a trinomial (B 25). Furthermore, Robert recalled that his experience with the Pyramid was related to mapping particular extensions of the *Pizza problem* and the *Building Towers activity* to terms of the Pyramid and not necessarily thinking of a trinomial expansion (B 25). Finally, Robert recollected that some student in high school had attempted to make a three-dimensional Pascal's *Triangle* which was hollow from inside and had Pascal's Triangle on each of its three surfaces. Robert explained that the students decided plainly that a three dimensional model was easier to discuss and share with a group rather than a two-dimensional sketch (B39 - 43).

5.2.3 Reconstructing layers of Pascal's Pyramid after three years

Around ten minutes into the session, R1 invited Robert to draw some layers of the Pascal's Pyramid on the white board. The last time Robert had done this particular activity was three years earlier in Session 12 (8/8/2005). Robert began by saying that the shape of each layer does not have to be a triangle and that it can be a circle or anything else; however, Robert claimed that a layer looked "nicer" as a triangle (B 47). Robert

suggested that properties of Pascal's Pyramid would be preserved even if they were not represented in a triangular arrangement. This is a significant observance as Robert is thinking beyond the conventional representation for Pascal's three-dimensional ideas and realized that the pyramidal shape might be convenient but not necessary.

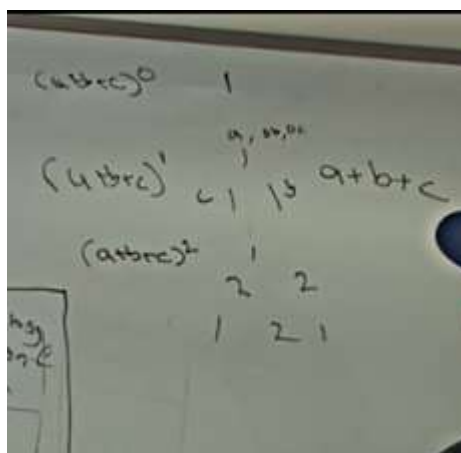


Figure 5-1 First three layers of Pascal's Pyramid

As Robert sketched some layers, he explained that the very top layer was 1 followed by a layer of $(a + b + c)^1$. He quickly sketched a representation for layer zero, first layer and second layer as in Figure 5-1. It is interesting to note that Robert placed 2's on the circumference of his triangular representation of the second layer immediately. As was observed in Session 12, Robert again started with a circular arrangement for numbers in a layer of the Pyramid and eventually "discovered" and adjusted to the triangular shape. Here in Session 13, Robert is picking his external representations up from the point of closure he had attained three years ago in Session 12.

When R1 asked Robert how he thought the first layer produced terms in the second layer, Robert drew arrows from the first layer as in Figure 5-2 to show the addition property. He drew two arrows from each of the 1's in the first layer and connected them to the 2's; he claimed they would generate in the next layer. He explained, for instance, 1 for a in the first layer goes to the $2ab$ term and the $2ac$ term in the second layer when multiplied by b and c (B 61).

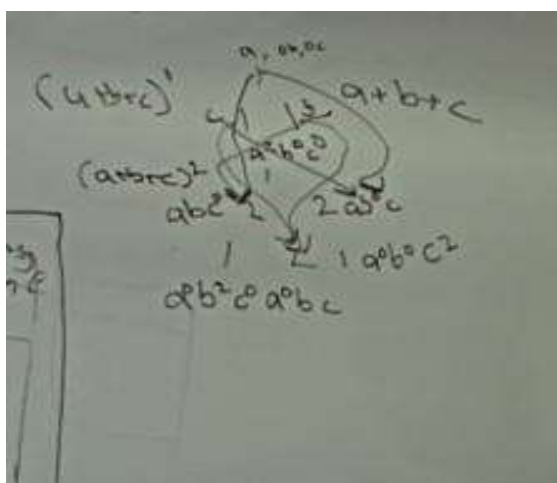


Figure 5-2 How addition property works to generate the next layer

Along with the numbers, Robert also used co-ordinates to list his terms. He used an exponent of zero to imply that a variable is missing. For instance, he wrote $a^0b^2c^0$ to imply b^2 . At this point Robert again reflected on the shape of the pyramid and concluded that “the pyramid makes more sense, cause you’re drawing it, all the arrows kind of go down” (B 63). Robert continued to justify to himself and R1 that Pyramid’s shape is arbitrary and went on to collect further evidence why pyramidal arrangement might be more convenient. He concluded that drawing all the “arrows” going down for the purposes of tracking addition between the terms might be most efficiently done on a

pyramid (B 63). R1 agreed that directions might be most conveniently represented as linear segments giving rise to a Pyramid.

5.2.4 What's in the middle of the third layer?

When asked to draw one more layer, Robert made Figure 5-3 as indicated. He arranged the 3's and 1s as a triangle and placed three 6's in the middle. He quickly remarked that he might have made an error.

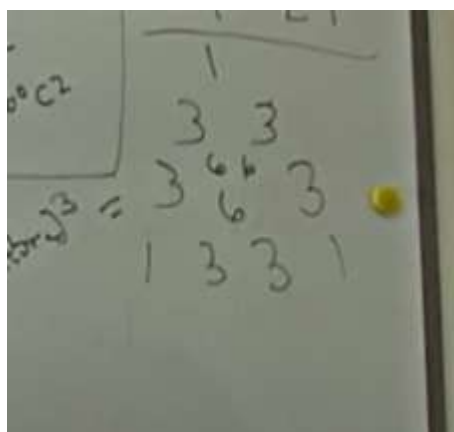


Figure 5-3 The third layer

R1 questioned Robert about whether there is a way to check how many 6's would go in this layer. Robert began to list the variables that would go with the coefficients of 6. He initially guessed that the three 6's represent $6a^2bc$, $6ab^2c$ and $6abc^2$. R1 pointed out that this layer corresponded to expanding third power of a trinomial and that $6a^2bc$ had four factors in it. Robert agreed but indicated that he was unsure what would go in center of the third layer. To figure out the middle, Robert began to list all the terms in the second layer. He wrote out the list as: a^2 , $2ac$, c^2 , $2bc$, b^2 , $2ab$, $2a^2b$, $2b^2a$. R1

questioned where $2a^2b$ might come from and Robert quickly realized that it does not exist in the second layer and reduced his list to $a^2, 2ac, c^2, 2bc, b^2, 2ab$. It is interesting to note that terms in the list followed a clockwise order of 1's and 2's that Robert created for the second layer (Figure 5-1). It is as if Robert was mentally walking on periphery of the second layer to generate his terms in the expansion of $(a + b + c)^2$. Robert also used his idea of "direction" discussed in Chapter 4. As he went from the term a^2 in the c "direction" terms begin to gain a c , that is, a^2 gained c and lost an a to become $2ac$ which in turn gained another c to become c^2 and so on.

Once he had settled on terms that lied on the circumference, Robert continued to explore terms for middle of the Pyramid's third layer. Robert indicated that the second layer had no abc term. He indicated that in absence of the abc term in the second layer, there is nothing that can multiply with a variable to produce the terms ($6a^2bc, 6ab^2c$ and $6abc^2$) that he had intended to be in middle of the third layer (B 96 – 99). At this point it was clear that Robert was using multiplication of terms from the second layer by a variable to check validity of terms he created for the third level. When R1 asked him about what the 3's in the third layer represented, Robert explained that the $1a^2$ term multiplied by b along with $2ab$ term multiplied by a gave rise to $3a^2b$ term when they were added together. Robert explained with similar reasoning what the six 3's in the third layer represent (B 94).

R1 questioned Robert again whether there was anything in the middle of the third layer, and Robert initially responded that there wasn't. He continued to say that each term in the layer had to have exponents "add up to three" (B 104). In the course of mentally counting options, Robert realized that the only option for the center would be

abc where each variable is represented once to give a total exponent of 3: 1 (for a^1), 1 (for b^1) and 1 (for c^1) (B 104). Then, he continued to use the multiplication and addition property of the Pyramid and multiplied $2ab$ by c , $2ac$ with b and $2bc$ with a to produce a single $6abc$ for the middle. Here, Robert monitored his work by counting all possibilities for exponents that add up to 3 and found the missing term for the third layer. At this point it is critical to note that although Robert initially guessed the middle terms incorrectly, he was able to reason and correct his work using his previously constructed understanding for exponents in a layer of Pascal's Pyramid.

R1 continued to ask Robert if he could explain one of the 12's that would be generated in the fourth layer and Robert easily explained that there would be a set of 3, 3, and 6 that would yield a 12. For one example, he lists that $3a^2c$ multiplied by b , $3a^2b$ multiplied by c and $6abc$ multiplied by a would yield $12a^2bc$ when added together. It is reasonable to conclude that Robert had a clear understanding of how terms from one layer generate terms in the next layer of the Pyramid in Session 13. In Session 12, however, Robert had placed six 12s in his fourth layer and only after a continued discussion discovered that there are three 12s that come from a specific choice of 3, 3 and 6. It is noteworthy that even after a span of three years Robert was fluently drawing upon his experience from Session 12 to explain the 12s in the fourth layer.

5.2.5 Reflecting on Session 12 and Robert's claim: "I am horrible at math"

The remaining part of the interview was designed to probe ideas that Robert had shared in Session 12 about the Pyramid and his attitude towards the subject of mathematics. R1 asked Robert if he had spent any time trying to construct a three-dimensional model of the Pyramid. Robert replied that Elizabeth Uptegrove had once

shown him a model but he had not made one himself. R1 then asked Robert to try and remember why he was trying to make concentric layers for the Pyramid and how he eventually thought that it would not work. To this Robert replied: “I think it can be done, but there’s no point to do it because it’s going to get messy” (B 188). Robert suggested that it is possible to create a representation for the third layer in a manner such that it contained the second and the first layer in it; however, he expressed uncertainty that it would be very helpful. One reason that he expressed for giving up on the attempt was that the representation became “messy.” He again claimed that “shape doesn’t matter” and that the layers could be represented as circles and the entire pyramid could be a sphere instead (B 192).

Then, R1 inquired about Robert’s representation in Session 12 where he had placed all the 1’s in the middle of the layers. Robert explained that he was “all over the place” and “just trying different things out” (B 208). That is, Robert was using a trial and error strategy to see if he could clearly represent the various cross-sections of the pyramid by placing 1’s in the middle. As noted in Session 12, Robert realized that 1’s in the middle would be problematic rather than useful and had abandoned that representation.

When R1 asked what the co-ordinates in his sketches from Session 12 meant, Robert explained that he was thinking of a three-dimensional space that has x , y and z axes. He was using his (x, y, z) co-ordinates to guide him in the placement of terms in a three-dimensional space (B 226). For example, he explained that one vertex of the pyramid in the fourth layer would be $(4, 0, 0)$ to represent a^4 and that it could be drawn as a “dot” on the white board (Figure 5-4).

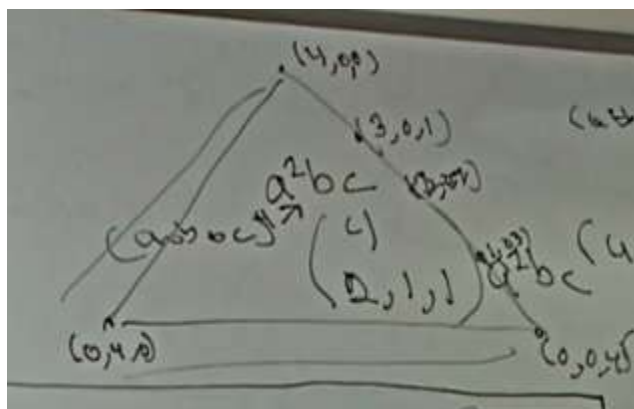


Figure 5-4 The co-ordinate system

Robert explained that $(4, 0, 0)$ would be a^4 and $(0, 4, 0)$ would be b^4 and $(0, 0, 4)$ would be c^4 . And the edge between a^4 and c^4 would have co-ordinates like $(3, 0, 1)$, $(2, 0, 2)$ and $(1, 0, 3)$ to represent terms with a^3c , a^2c^2 and ac^3 . Again, Robert was using his idea of “direction” to list the terms on the circumference, indicating that as one “walks” from a^4 to c^4 , one is “walking” in the c direction and as such, each term replaces an a with a c (becoming a^3c , a^2c^2 and ac^3) to finally yield a c^4 .

At this point in the interview (thirty minutes approximately), R1 encouraged Robert to share why he picked statistics over mathematics for his graduate level studies. Robert stated that in statistics he was able to use the data and “actually get something useful that makes sense” whereas in mathematics, he was not sure of the applicability of the proofs he had to do (B 266). Robert also remarked that as he was “not very good at English” he “had” to do math (B 359). Robert also shared with R1 a book he wrote that was recently published on the topic of gambling. R1 expressed amusement that Robert claimed to hate math even after an undergraduate degree in mathematics and a graduate

degree in statistics. Robert explained that he used the strategy of “I hate math” to motivate himself and said “You know that kind of influence, you know I am horrible and then you’re like no wait I can do this...so it’s a form of influence” (B 363). It is an interesting and modest statement by the participant as he also shared with R1 that he earned a 4.0 G.P.A in his math courses despite wondering “how the heck did I (Robert) get these grades because I (Robert) just don’t know simple things” (B 353 - 357). At this point, R1 interrupted the interview to show Robert some more clips from Session 12.

R1 asked Robert to explain why he claimed in Session 12 that there would be overlap between the terms. Robert explained that in Session 12, he was not using the “identities” about the expansion of a trinomial and therefore confusing himself (B 317). He reflected that in this session (Session 13), he was using the complete terms instead of the coefficients from the very beginning and therefore, he made fewer errors (B 317). Then, R1 asked Robert why he was doing many alternate additions picking different sets of 3s and 6s to produce a 12 in the fourth layer during Session 12. Robert said that he was trying a guess and check strategy to see where the 12 could come from, for example, four 3’s, two 6s or a group of 3, 3 and 6 could all make a 12 (B 322). However, as he was not writing the entire terms out, he was trying all groups of numbers that mathematically added to give a 12 (B 329 - 331). In Session 13 though, Robert was consistently reflecting on what term a 12 would represent and used it as a guide to pick a particular group of 3’s and 6.

R1 asked Robert what his biggest hurdle was in understanding properties of Pascal’s Pyramid during Session 12. Robert claimed that representing a three-dimensional object on a two-dimensional white board was the most difficult part. He

explained: “one, I can’t draw, two, you know it was in three dimensions, which if you don’t have a computer you need to actually have manipulatives to draw in 3-D” (B 335). Robert claimed that a person can draw the Pyramid on the board if he/she is a “good drawer” which he declared he is not (B 335). It is an interesting statement as the participant deemed it necessary to have good drawing skills to represent the Pyramid as a two-dimensional object. However, he was able to create clear and mathematically-correct two-dimensional representations for the layers of the Pyramid. Robert further stated that it was easier for him to work with the *Towers problem* as he had manipulatives (Unifix cubes) to play with and did not need to imagine what they looked like (B 335). Similarly in the *Pizza problem*, he stated that a two-dimensional representation was enough and that a third dimension was unnecessary (B335). Robert also recollected that while taking calculus courses, he had a difficult time visualizing the three-dimensional objects (B 335). R1 agreed that it is challenging to use a white board for representing the Pyramid. R1 also reflected that Robert’s attempt to create coincident layers played a role in delaying the creation of his final representation.

When R1 asked Robert what was the most helpful thing in understanding the Pyramid’s structure and properties; Robert asserted that his notation for the coordinates was most helpful (B 343). He went on to say that if he was taught to think of choosing

two things out of four (in his notation $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$) as $\begin{pmatrix} 4 \\ 2, 2 \end{pmatrix}$ instead, he would have an easier

time generalizing the notation to three variables. He explained that $\begin{pmatrix} 4 \\ 2, 2 \end{pmatrix}$ is same as

$\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ but it includes a representation for the variable that was not picked. In the three-

variable case Robert used notation like $\begin{pmatrix} 4 \\ 1, 2, 1 \end{pmatrix}$ to represent picking four things where one item is of the first kind, two items are of second kind and one item is of the third kind (B 343). In other words, $\begin{pmatrix} 4 \\ 1, 2, 1 \end{pmatrix}$ were his coordinates to represent ab^2c . Robert also reflected that he did not “put together” for himself that the “squared identities (implying binomial expansions)” could be extended to “more dimensions” (B 343). He claimed that if he had realized that the “triangle” can be extended to the three dimensions and if he were taught to think of $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ as $\begin{pmatrix} 4 \\ 2, 2 \end{pmatrix}$ “from the start,” he would have had an easier time creating a two-dimensional representation for the Pascal’s Pyramid (B 343).

Towards the end (around 45 minutes), R1 asked Robert if he would have chosen to work with a friend to draw out and represent the Pyramid like he claimed in Session 12 to Professor Lee. Robert confided that he was misrepresenting. He explained that he preferred to work by himself on a problem before having to participate in a group discussion (B 365). He reflected that groups work best when everyone first works on the problem individually and brings different ideas for the group to share (B 365). He explained that if students “just start working in a group right away there is always one or two people who immediately just shut down and become very passive” or end up passively copying work from other students (B 365).

Finally, R1 inquired about the ease with which Robert was able to point out a solution to *Ankur’s Challenge* in Session 12. Robert stated that he understood the “constraints” of *Ankur’s Challenge* and as long as you have the “abc’s written out there

you can just circle the ones that meet those requirements” (B 369). He explained that *Ankur’s Challenge* had solutions of the nature a^2bc , ab^2c and abc^2 and since he knew that, he could circle them right away (B 369). This statement provides evidence for Robert’s clear understanding of the isomorphism between towers that are four tall and have exactly two of one color to the terms that have exactly two of a variable that is the terms a^2bc , ab^2c and abc^2 . The session ended with Robert reflecting that he had worked on *Ankur’s Challenge* prior to the Session 12 when R2 had asked him to solve certain problems in a notebook.

5.3 Second Post-Graduate Interview: November 14, 2008 (Session 14)

5.3.1 Setting and Background

Note: For this session Anoop Ahluwalia is coded as R1.

A second interview with the participant was conducted to provide him with an opportunity to build a three-dimensional model using concrete materials, in this case the *Zome tools*. The *Zome tools* consisted of plastic rods and spheres that could interlock with each other to make a three-dimensional object. Carolyn Maher had recommended that Victor Liu should be an observing learner in this session as Robert built a model for the Pascal’s Pyramid and explained to him some properties of the Pyramid. Marjory Palius sat in as another observer for the session. A graduate student, Lou Pedrick, recorded the session on video camera. R1 along with Carolyn Maher designed the interview protocol for this session. The interview lasted an hour and 25 minutes. R1 had already transcribed and watched Session 13 several times before she conducted the Session 14 interview. R1 had also spent time building a two-color model of the Pyramid

herself using *Zome tools* prior to this session. The two-tier goal of this session was to allow Robert to experience building a three-dimensional model for the Pyramid and also map solution to *Ankur's Challenge* and the *Taxicab* problem to the Pyramid.

5.3.2 Robert makes a three-dimensional model for Pascal's Pyramid

R1 had already given Robert time to construct a three-dimensional model for the Pyramid before the interview recording began. Robert had constructed Figure 5-5 at the beginning of the interview while R1 had observed him construct the model and took notes. When camera began to roll, Robert was explaining to R1 that he thinks that the *Zome tools* will not allow a two-colored model for the Pyramid as the rods were specific lengths and that might hinder the layers from touching each other. R1 decided to address this comment later in the interview and proceeded on to request that Robert explain his model in Figure 5-5 to Victor. Victor commented that it was “very hard” for him to pretend that he did not know the problem (C11). R1 explained to Victor that he had to act as a student who had not seen the Pyramid before and encourage Robert to explain the structure of his model in great depth (C12).



Figure 5-5 Robert's first 3-D model for the Pyramid

Even though the *Zome tools*' kit had several other colors, Robert chose to create his model using blue rods of two different lengths along with the connecting white spheres. It is noteworthy that from each sphere in his model, three blue rods were hanging. Robert later explained that each of those rods represented a "direction" or one of the three variables in the trinomial $(a + b + c)^n$. Interestingly, Robert used longer blue rods only for the three outer edges and as such, his layers with shorter rods in the middle did not touch each other. Robert was able to move his hand inside the model to point out some of the spheres in middle of the layers. Finally, R1 had observed that Robert built his model from bottom up. That is, he had constructed the fourth layer (the bottom most layer in Figure 5-5) first and then constructed the third layer above it and so on. R1 addressed these observations in a later part of this session.

Robert had been briefed about the interview format and expected to share his ideas with Victor. As such, Robert had already prepared a sheet of paper with first four

layers of the pyramid drawn out as in Figure 5-6. It is remarkable that Robert had sketched all cross-sections of the four layers correctly in a short period of time. This reflects that Robert remembered what the layers of the Pyramid were and was confident about the coefficients that belonged in the middle and the periphery of each layer. Also noteworthy is the fact that he drew his layers as triangles from the very beginning and in the fourth layer he arranged his 12s in a smaller concentric triangle. This corresponds well with his three-dimensional model (Figure 5-5) where 12s in the fourth layer were connected with blue rods in a triangular arrangement in middle of the layer. It seems evident that making a concrete model of the Pyramid, along with Robert's prior experience exploring a two-dimensional representation for the Pyramid, allowed him to sketch Figure 5-6 confidently as a starting point.

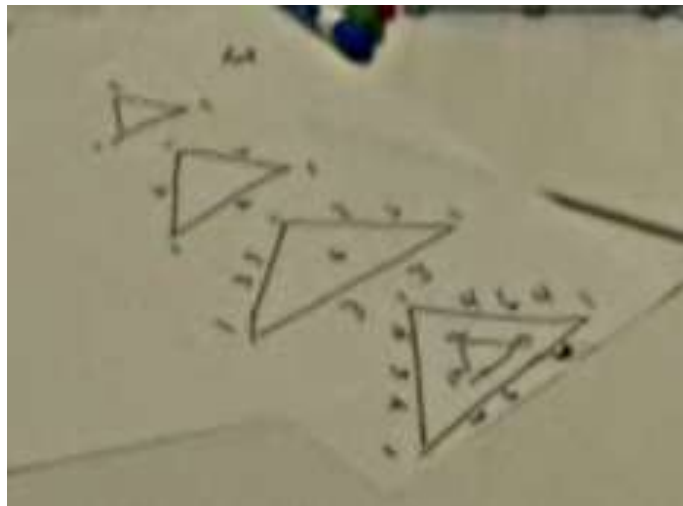


Figure 5-6 Robert's sketch for first four layers

5.3.3 Robert explains layers of his model Pyramid to Victor

Robert explained to Victor that the top most “circle” (referring to the white sphere on top of Figure 5-5) represented the $(a + b + c)^0$ layer which was just a 1 and that the layers underneath it were the layers corresponding to $(a + b + c)^1$, $(a + b + c)^2$, $(a + b + c)^3$ and $(a + b + c)^4$ respectively. Robert went on to explain to Victor how terms get added between the layers and he again remarked that “ ab plus b is ab -squared” (C 20). Robert was multiplying ab by b to produce ab^2 , however, he uttered “plus” a b . This is a significant statement that was discussed as a critical event in Session 12 and Session 13 where it was conjectured that Robert is using his experience with the *Towers-Building* activity when he refers to multiplication as addition. That is, it was conjectured that Robert is mentally “adding” a Unifix cube to a tower when he carries out multiplication by a variable.

R1 asked Robert what was the coefficient of the abc term is in the third layer and he responded that it was a six. When asked to explain to Victor why it was a six, Robert claimed that he would have to start with the second layer (the layer corresponding to $(a + b + c)^2$). Robert explained that the second layer was “ a , plus b , plus bc ... plus c ...plus ac ... a plus b plus c , so the corners would be a , b , c again” (C26). R1 asked if he meant there was “one” ab , “one” bc and “one” ac in the second layer and Robert corrected himself to say that it was “two” of each of the ab , bc and ac terms (C 29, 30). Along the same lines, while talking about spheres on the edges, Robert claimed that the “circles” (white spheres) on one edge all represented an a (C 38). R1 guided Robert to discuss if anything happens while going down an edge from one layer to the next. Robert claimed that “you’re not adding anything on top of it” and believed that the edge of the model

Pyramid being considered was a series of a , a , a , and a , for all four layers (C 36). That is, at that moment, Robert did not think that exponents of a grew from a to a^2 , a^3 , etc going down an edge of the Pyramid. When R1 re-uttered Robert's claim, "this is also an a , and this is also an a ," Robert responded "No. Oh, this is a -squared" and corrected himself (C 39, 40).

This discussion revealed that although Robert had carefully listed the coefficients in Figure 5-6, he had not mapped out what terms these coefficients represented for a moment. It took him some dialogue with R1 to modify his claims about what the spheres in the layers represented and conclude that the second layer contained a^2 , b^2 , c^2 along with $2ab$, $2bc$ and $2ac$ terms. This is a significant observation as in the first post-graduate interview (Session 13) Robert had contended that listing entire terms with coefficients at beginning of that session had aided him in making an accurate representation without too many errors. In this session (Session 14), Robert did not list the terms out initially and it took him some time to clarify what the spheres in his model represented (Figure 5-6).

At this point (ten minutes into the session), R1 conjectured that the long blue "bars" (rods) of the model Pyramid are "actually saying something" and asked Robert to explain what he thought they represented (C51). Robert claimed that the "bars" were saying "multiply by itself" and that going down an edge, following the longer blue rods, one would multiply a variable by itself to produce a , a^2 , a^3 , etc. Robert further explained that the three edges of the model pyramid corresponded to the three variables a , b and c (C 52 – 63). At this point Robert also conjectured that the model could be built with three different colors to represent the three variables. However, he suspected that the *Zome tools* might structurally obstruct the use of three colors to make a pyramid. R1

handed some more colored pieces of the *tools* to Robert as he tried to add a third color to a small model he had built with red and blue rods. He concluded that the third color would not have the right length to touch the next level and therefore, a three-colored model for the Pyramid might not be easy to build (C 72 - 81).

R1 asked Victor if he was following the conversation so far and Victor replied that he was comfortable up to the second layer in Robert's model. Robert had started to talk about each edge of the Pyramid as a "direction" and he explained to Victor that each of the blue rods in a particular direction represented multiplication by a certain variable. To explain the third layer, Robert showed Victor that when a^2 "travels" in the b direction it would become a^2b and when $2ab$ "travels" in the a direction it would become $2a^2b$ and when both (a^2b and $2a^2b$) are added, they would make a $3a^2b$ (C95 - 98). Victor was able to explain following a similar logic that the sphere next to the one representing $3a^2b$ would be $3ab^2$ (C 99 – 102).

Finally, R1 asked Robert why there were some spheres (terms) in middle of certain layers and not in middle of the first and the second layer. Robert went back to his "exponents idea" and explained that in the second layer exponents add up to two; as such, there is no term with a , b and c in it. He claimed that for three variables to be present, the minimum exponent size needed is "one, one, one" (or three); and therefore, the first term in the middle shows up in the "cube term" (third layer) (C 106). This is also a strategy that Robert had used in Session 13 to figure out the missing coefficient for middle of the third layer.

5.3.4 Would two colors work for the Pyramid's model?

When R1 asked Robert to explain one more time why the term in middle of the third layer is a $6abc$, Robert went on to say “two ab , two a ... two ab , two ac , two bc , and you’re adding the other term to it. Yeah, I think it would be much better if it was this block in three different colors” (C 112). Robert was reflecting at this point that if he used three colors of rods for the three variables a , b and c , then rather than having to think of “directions” he would be able to use colors to determine what variable a terms was getting multiplied by. R1 then addressed the gap between layers of Robert’s model and asked if the shorter blue rods represented a mathematical operation as well. Robert remarked that both the short and the long rods represented multiplication. As Robert played with a blue and red colored model (Figure 5-7) he held in his hand, Robert exclaimed that he could have built the entire model with red and blue rods instead (C 120-122). R1 pointed out that Robert did not initially believe that two colored rods could work. Robert replied that “that was off the record though” and on the record, “red works” (C 126-127). Robert had changed his mind that he could only use one color with *Zome tools* to build the Pyramid.



Figure 5-7 Robert's incomplete blue and red model

R1 inquired if Robert preferred one of the two models (all blue versus red and blue), and Robert stated that he preferred the two-colored model as it was more “compact” (C130). He further stated that if he had to rebuild a model for the Pyramid, he would use two colors (C130). R1 asked if the red and blue rods in his new model had any meaning and eventually Robert stated that “blue could represent addition and the red means multiplication...You know ‘cause this (pointing to blue rods in the layer) is really a -squared plus ab plus b -squared...” (C 138). Robert was clarifying that in a particular layer the terms are being added; and as the blue rods are holding spheres of one layer, they represent addition. Similarly, the terms get multiplied going down from one level to the next and as the red rod held spheres from two different layers together, the red would mean multiplication. Robert had reached this conclusion on his own although a few moments earlier he had decided that blue was meaningless and that it was just “holding it (the model) together” (C 132 - 134). R1 agreed that roles of red and blue rods were multiplication and addition respectively and shared with him the model she had created earlier (Figure 5-8). R1 noted that she knew that two colors would work because she had seen another student (Kevin Merges) build a similar model.

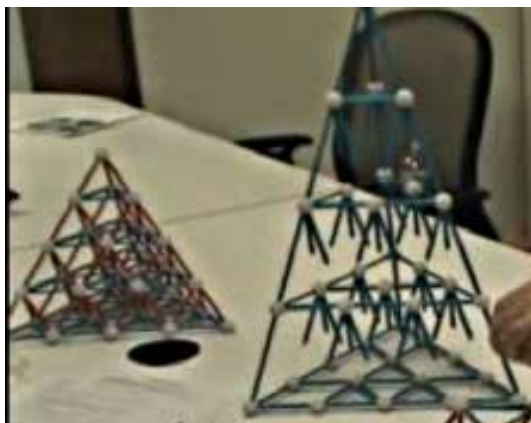


Figure 5-8 R1's and Robert's Model

At this point R1 asked Robert where the Pascal's Triangle was located on his model. Robert explained that each face of his model was a Pascal's Triangle (C 148). R1 also asked Robert to explain why he had built his model from bottom upwards. Robert clarified to R1 that he actually made all the layers separately and then he stacked them on top of each other (C 158). Robert also shared that he took about thirty minutes to make his model and that he was talking to people during the model-building about unrelated things (C 162). This comment reflects that Robert was making the model according to a plan he had developed prior to its construction. That is, he already had a good idea of what his layers would look like and was using tools to fit his sketch of the Pyramid rather than using tools to understand structure of the Pyramid.

5.3.5 “Any future Ankur’s” on Robert’s model for the Pyramid

R1 then encouraged Robert to explain to Victor how the *Building Towers* activity could be carried out using Pascal’s identities. Robert started off by explaining that not all *Towers* problems can be solved using the Triangle/Pyramid as “you can’t represent, you know, four colors case, there is no fourth dimension. Well, we didn’t make a fourth dimension” (C 172). Robert explained that only two and three colored *towers* can be mapped to Pascal’s Triangle and Pyramid respectively. He contended that as number of variables in polynomial expansions correspond with number of colors to choose for a tower as well as the dimensions one needs for the model, it would not be physically possible to create a Pascal’s model for four-tall towers. He further explained that he could work with a problem that had four colors as he understood “what those values (terms in expansion of $(a + b + c + d)^n$) would be” (C 178). In other words, Robert claimed that he was comfortable mapping four-colored *Towers* problems to expansions of four-variable polynomial and working with coefficients of the resulting terms.

Then R1 asked Robert to tell Victor about the *Ankur’s Challenge*. Robert explained to Victor that the problem was “towers four tall, with ah, three colors, there has to be one of each color in the tower” (C 193). R1 gave Robert Unifix cubes in three different colors in case he wanted to build towers with them. Robert did not use the Unifix cubes. Instead, Robert pointed out that three spheres in middle of the fourth layer would be a solution to *Ankur’s Challenge*. Robert knew immediately to use the fourth layer of the Pyramid for towers that are four tall. Robert tried to use long red rods and orange rods to enclose the three spheres in middle of the fourth layer of his model. Robert explained that the solution had to lie in middle of the layer as “you can’t have

anything from the two dimensional Pascal's Triangle involved. So that will eliminate all the sides and it would just be the inside" (C 217). Robert understood that *Ankur's Challenge* needed one each of the three colors and that the "faces" of the Pyramid were only two colored as they represented Pascal's Triangles. Therefore, he deemed it necessary to eliminate the spheres on the circumference of the fourth layer for *Ankur's Challenge*.

R1 encouraged Robert to explain what terms went with the 12s and how they made up a solution to *Ankur's* problem. Robert explained to Victor and R1 that he was counting 12s that represented the terms $12a^2bc$, $12ab^2c$ and $12abc^2$. He repeated that the "anything on that (any) face" is "missing a color" and "the only ones that have all three colors are the three in the center" (C 258). To this R1 responded that Robert was using the three variables to be equivalent to the three colors and Robert agreed with that implication (C 259 – 260). Robert then went on to generalize that for "any future *Ankur's*, you know whatever tall, it's just going to be what's in the center, it's not going to be what's on the outside" (C 264). For instance, Robert explained that if *Ankur's* problem was reduced to three tall, the answer would be a 6 corresponding to the $6abc$ term in middle of the third layer. This is a critical moment in the session as Robert had for the first time extended solution from a specific *Ankur's Challenge* with towers four tall to "any future *Ankur's*." He claimed that middle of the layers of the Pyramid would always be a solution to *Ankur's* problems as terms in the middle will always have one cube of each color. This is evidently a new observation by Robert about the Pyramidal properties. Although Robert knew the solution to *Ankur's Challenge* in Session 12 and Session 13, he did not generalize his solution to different height *towers* when he worked

with his two-dimensional sketches. Perhaps the three-dimensions of his model played a role in Robert's new conclusion about the Pyramid's "inside" and "outside" properties.

5.3.6 The Taxicab on Pascal's Triangle and Pyramid

At around thirty minutes into the session, R1 gave out a handout of the *taxicab problem* and noted that Victor had previously worked on the problem. The problem was stated as follows.

The Taxicab Problem (May 2002, Grade 12) – A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated on the map. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route. What is the shortest route from a taxi stand to each of three different destination points? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.

(Maher, Powell, & Uptegrove, 2010)

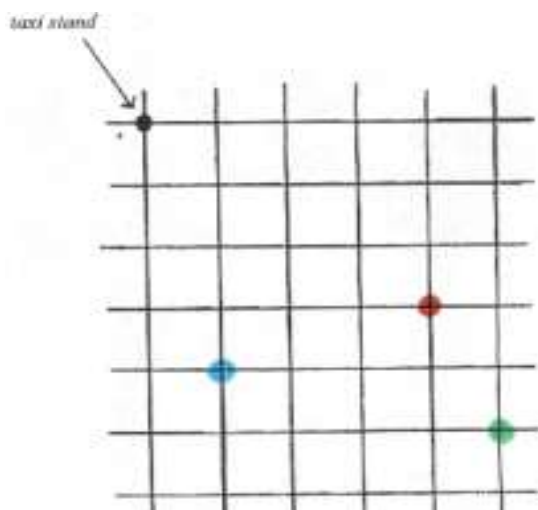


Figure 5-9 The Taxi-Cab Handout

R1 asked Robert if he would like to work on the problem prior to continuing with the interview. Robert said that he knew the solution and that he did not need extra time.

He recalled that he had previously worked on the problem in Carolyn Maher's class (C 278). Immediately, Robert started to write out terms of Pascal's triangle on top of the *Taxi-Cab* grid as in Figure 5-10.



Figure 5-10 Robert lists Pascal's triangle on taxi-cab grid

Robert quickly noted that the problem could not be solved for the blue dot in Figure 5-10 because the model he had constructed only went up to four layers. He noted that as the blue dot belonged to the fifth layer, he would need the fifth layer of the Pyramid. To adjust for this, R1 made a black dot on the fourth row, second intersection, right above the blue dot (in Figure 5-10) and called it the new destination of interest. Robert quickly responded that there were 4 shortest routes to this new destination.

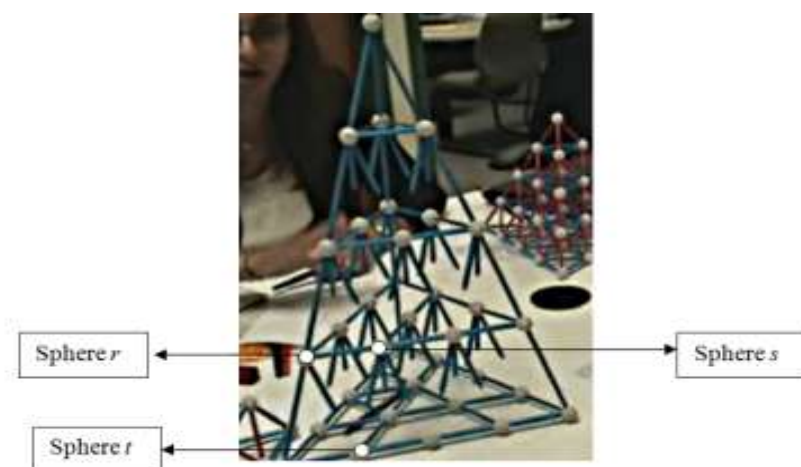


Figure 5-11 Robert explains taxi-cab on his model

R1 suggested that Robert should first explain where the black dot was on the model. Robert pointed out that the sphere t in Figure 5-11 was the black dot or the point of destination. Robert took the dot's placement from the sheet of paper where it lay on the second intersection of the fourth row and mapped it to the second sphere in the fourth layer of his model. It is conjectured that Robert had some previous experience connecting ideas of the *Taxi-Cab* to the Pascal's Triangle as he was immediately able to visualize the location of the destination on his model. Then Robert explained that to get to sphere t , there were only two lead-in routes, one coming from sphere r and one coming from sphere s . He went on to further explain that as there was only one way to get to sphere r from the taxi stand and three ways to get to the sphere s , there are total $1+3 = 4$ ways to get to the sphere t . Robert was using the addition property of the Pyramid where coefficients of sphere r and sphere s add up to produce the coefficient of sphere t .

R1 asked Victor if he agreed with Robert. Victor seemed unsure and Robert volunteered to sketch the paths on paper for him. Robert used different color markers to show Victor what the four paths would start from the taxi-stand and end at the black dot to give the shortest routes (Figure 5-12). Robert explained to Victor that going right on the taxi grid was same as going right on the model Pyramid and that going down on the taxi-grid was same as going left on the model. Robert was fluently pointing out paths on the paper and model simultaneously to Victor. It was apparent that he was comfortable with both the modes (paper and model) of representing solution to the *Taxicab* problem.

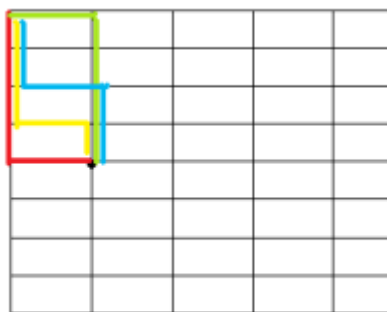


Figure 5-12 Four shortest routes to destination

Robert used a case by case listing when he did not immediately see his fourth path on the grid and said “so went down three, we went down two, and down one, so we want to go over right away” (C 344). He used an exhaustive listing of the routes and counted the cases where routes go “down three,” “down two” and “down one” and realized that he was missing the case “down zero” and therefore, he should go “over right away” to find his fourth route. Robert had used a similar strategy in Session 13 when he listed all possible coordinates to figure out that the missing term in middle of the third layer is *6abc*.

At this point (forty minutes into the session), R1 asked Robert if the variables a , b and c played a role in the *Taxi-Cab problem* as well. Robert responded that the *Taxi-Cab problem* is only two-dimensional unless the taxi can fly or dig underground to get to the destination (C 350). Robert proceeded to say that in the *Taxi-Cab problem*, variable a can be the direction going left on the model (or down on paper) and variable b can be the direction going right on the model (or right on paper). For instance, he explained that the red route in Figure 5-12 would be a^3b as it has three blocks going down and one going right.

Robert later observed that all paths he drew in Figure 5-12 involved one move to the right and three moves to the left (or down). Simultaneously, Robert started calling the down direction “south” and the right direction “east” (C 366). Robert wondered why three of the four shortest routes that he had found involved going “south” first and only one involved going “east” first. Then he realized that location of the black dot was exactly three “south” and one “east” from the origin of the taxi-stand. At this point Marjory asked Robert to again address his idea about the two different “starts,” one involving going “south” first and other involving going “east” first. Robert responded that it doesn’t matter as all the routes were supposed to take the same amount of time (C 393). R1 at this point offered an explanation to Marjory and said that Robert was counting all routes that had three “south” and one “east” in them. However, Robert remarked that he was instead thinking about location of the taxi stand with respect to the black dot. For instance, he elaborated that if the taxi stand was two blocks “south” and two blocks “east,” there would be six different shortest routes.

At this point, Robert was just reading coefficients of the spheres on his model to count the total number of shortest routes. R1 again asked Robert to show her on the model where his solution was. Robert placed a marker on the table to point out the sphere that he considered being the black dot (Figure 5-13). He said the solution would be a four as coefficient of that sphere was four. Robert also noted that other faces of the pyramid would produce a similar answer and placed other markers to represent the black dot on them. Here, Robert was just sharing his understanding that all faces of the Pyramid are the same, that is, they are all Pascal's Triangle. R1 asked Marjory if she was able to follow Robert's remarks and Marjory said that she was able to follow Robert better when he talked about the variable and coefficients together. That is when Robert said $4a^3b$ rather than just a 4, it became clear to Marjory where he was locating the black dot and how a^3 and $3a^2b$ terms led to it.

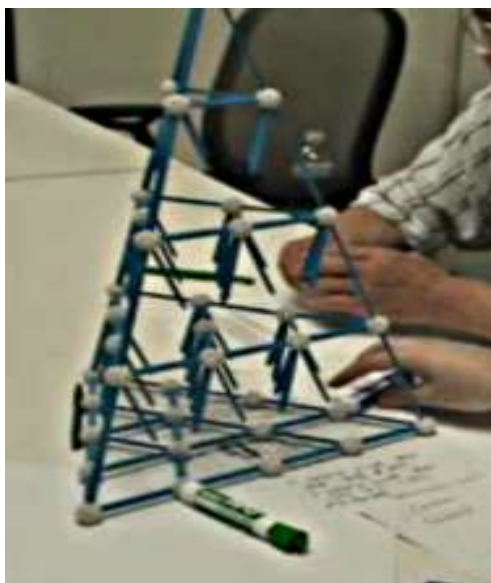


Figure 5-13 Robert points out the solution for taxi-cab on his model

Robert began to wonder how the problem could be extended to three dimensions. He proposed that if each of the grids was to include a diagonal between the squares, each vertex would have three routes originating from it and perhaps, that would be a three-dimensional extension of the *Taxi-Cab*. Marjory suggested that they would need a “cubical” grid rather than a “square” grid so that the taxi could go “up” (C 455). Robert continued to evaluate his idea of placing diagonals in the grid. He thought that if both the diagonals were placed in each square, that is, “each box had an x in it” it would become the fourth dimension (C 459). However, Marjory pointed out that the problem will not be the same if diagonals are introduced. She claimed that the diagonals will always be the shortest route between any two points sharing the diagonal. R1 decided to leave the three-dimensional *Taxi-Cab* discussion at that point and moved on to other questions about Robert’s experience with *Zome tools* to construct the model.

5.3.7 Was it helpful to build the Pyramid with Zome tools?

R1 asked Robert if he thought building a model for Pascal’s Pyramid told him anything new about the Pyramid. Robert responded “Nah, I don’t think so because I just drew it first and then I built it from my drawing I didn’t like ... but it was good I guess of in sense of the *Taxi-Cab* because you can kind of trace the routes, can’t really do that here (pointing to the drawing in Figure 5-6)” (C 494). Robert observed that he did not use the tools to discover structure of the Pyramid. He used his prior knowledge to draw out the layers and then used the sketch as a guide to make a three-dimensional object. He reflected that it was visually more convincing to show solution to the *Taxi-Cab* on the model rather than on the handout as colored routes overlapped. Robert also shared that if

he could get the tools to just fit right, he would use three colors to make the model where a , b , and c variables could be represented with different colored rods.

Then Robert claimed that he could have used his Figure 5-6 to solve the *Taxi-Cab* problem as long as he realized that “each face of the triangle (implying Pyramid) represents you know Pascal’s Triangle” (C 508). Robert explained that it was possible to just use his sketch of the four layers (Figure 5-6) to answer the question posed as long as one could visualize the Pascal’s Triangle across the cross-sections. He went on to claim that “once you know the formula you don’t really need to draw anyway” (C 512). Robert contended that if one knew the “multinomial formula” one could just figure out the answer without the aid of a sketch or model (C 514). For instance, he talked about the green dot that is placed on the fifth row of the grid at the fifth intersection on the grid and explains that the answer would be just “ten choose five, five” (C 516). Marjory questioned him how he came up with the choose notation so quickly. Robert replied that he had talked to R1 about this notation in Session 13. At this point, Robert was putting many different ideas together along with multiple ways of answering the *Taxi-Cab* problem. He had concluded that the model, the sketch and the trinomial expansion are all going to yield equivalent solutions. This reflects that Robert had a clear understanding of how the algebraic expansion of the trinomial, the two-dimensional sketch and the three-dimensional model could all be used in an inter-exchangeable manner to answer the *Taxi-Cab problem*.

5.4 Third Post-Graduate Interview: March 27, 2009 (Session 15)

5.4.1 Setting and Background

Note: For this session Anoop Ahluwalia is coded as R1 and Carolyn Maher is coded as R2.

The third and final interview with Robert was conducted on March 27, 2009, about four months after the second interview (Session 14). One of the objectives of the interview was to provide Robert with an opportunity to use *Zome tools* and construct a model for the Pyramid while a video camera captured the steps of construction. The other objective of the session was to invite Robert to explain to another graduate student, Marjory Palius, some properties Pascal's Pyramid. In specific, during this session, Robert helped Marjory map solutions of *Building-Towers* activity with three colors to his model for Pascal's Pyramid. I had watched the second interview and transcribed it prior to conducting this interview. Lou Pedrick, another graduate student, recorded this interview session. The interview protocol was designed by R1 and R2. The interview lasted one hour and thirty minutes.

5.4.2 Robert builds a Pyramid again with Zome tools

R1 asked Robert to use *Zome tools* and talk aloud as he constructed first few layers of Pascal's Pyramid. R1 pointed out to Robert that Marjory had never built a model for herself and that she will try to follow his construction. Robert started with observing that he was using the medium-sized rods as there were no smaller rods available to construct the model. Robert had started to build with red and blue rods and R1 asked him why he had picked two colors. Robert replied that he had built the model

before and that he was going to use two colors to represent two different things (D 13). This reflects that Robert was using his experience from Session 14 where he had concluded that two colors were helpful in explaining some properties of the Pyramid to rebuild the model in this session.

Robert also noted that he planned to start with building one of the middle layers first as “it’s easier to build down to up” (D 9). R1 encouraged Robert to further explain why he considered building from bottom up to be easier. Robert explained that “if you try to build it upside down (going from layer zero to first layer and so on) then you have more on the top and it gets, you know hard to put the blocks in because it’s heavier on the top than on the bottom” (D 28). Robert was making a point that pyramid is structurally more spread out at the bottom and once he had a big layer constructed, he would know where to place connectors for the smaller layer above it (Figure 5-14). He also contended that he made “a lot of mistakes” when he went the “right way (top to bottom)” (D 35).

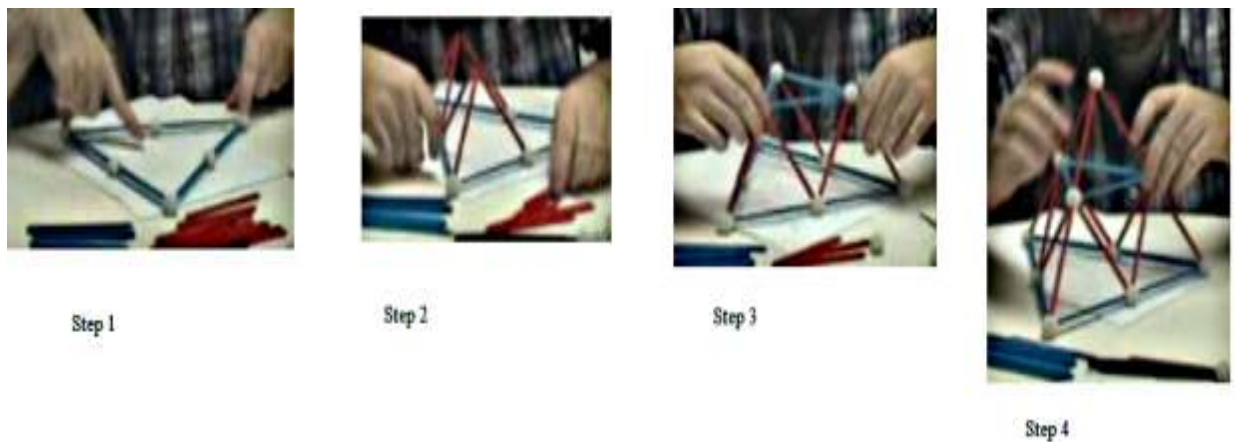


Figure 5-14 Step by step construction of the Pyramid Model

R1 shared that she always used induction to figure out what the next level would look like and, as such, she always built her model from top to bottom. To this Robert responded that there were “too many holes (in the spheres of *Zome tools*) and if you don’t put them in the right one then it gets...” (D 37) Robert was sharing that he had tried to construct the model from top to bottom at some point (perhaps in Session 14, off camera). However, reflecting on structure of the white connecting spheres in *Zome tools* he had noticed that there were too many spaces (“holes”) where the rods could be inserted. As such, if Robert had to begin from the top he would not know where to place three rods in a sphere such that a symmetric pyramidal model would follow.

Marjory asked Robert to explain to her what the top most sphere on the model represented. Robert did not answer Marjory’s immediate question and went on to explain that going around the model you are “adding a blue, and then you are adding a red” and he repeated this logic for all three faces of the Pyramid. R1 intervened to ask Robert if the Pyramid was only dealing with two colors and Robert reflected that it was actually three colors. At this point R2 suggested that Marjory should bring some three-colored Unifix cubes so that “she can map her understanding to what Robert is explaining” (D 65). Marjory brought three-colored Unifix cubes to the table. R1 asked Marjory if she understood why she needed three colors for the Pyramid. Marjory did not respond to the immediate question, however, she began to reflect on her experience with building towers. She decided to write out a few rows of Pascal’s Triangle on paper. Robert continued to work on the next layer of his model while Marjory worked on paper and with Unifix cubes.

5.4.3 Mapping two colored towers to Pascal's Triangle

Marjory wrote out the first three rows of Pascal's Triangle on paper (1; 1, 1; and 1, 2, 1). She commented that she was not sure of why there is a one at the very top and decided to explore it later. Marjory called the variable a "yellow" and the variable b "blue" and started to place Unifix cubes on the paper next to the coefficients she had listed (Figure 5-15). Marjory read her towers from left to right in Figure 5-15 as a^2 , ab , ba and b^2 . She went on to explain that when ab and ba terms are combined, they gave the $2ab$ in middle of the second row. This reveals that Marjory had not only a good understanding of rows in Pascal's Triangle but she also knew how to map the two-colored towers to terms in the Triangle.



Figure 5-15 Marjory maps Unifix cubes to the Triangle

In the meantime, Robert had finished his third layer and built Figure 5-16 as his model. Interestingly at this point, Robert's model was hollow inside. That is, it only had three surfaces of Pascal's Triangle and no middle terms. Robert continued to think about his model as Marjory went on to explain to R2 how terms from the first layer generated terms in the second layer.



Figure 5-16 Robert's hollow model up to the third layer

Marjory reversed the order of two towers in middle of the second layer (Figure 5-15) and explained that going from one level to the next, “you’re either adding a yellow cube on top” or “a blue cube on top” (D 78). Marjory had invented a strategy where she added a new block on top to make towers taller. As she went from left to right, Marjory had arranged her towers as: a yellow-yellow tower, a yellow-blue tower, a blue-yellow tower and a blue-blue tower. At this point R1 encouraged Marjory to connect her terms with line segments in a triangular manner that would reflect the pattern of Robert’s model. While Marjory worked on paper, Robert continued to reflect on his model. Although Robert had placed the hollow model (Figure 5-16) on the table for a while, he realized that he was missing a term in the inside of the third layer. He silently went on to finish his model and inserted one sphere in middle of the third layer as Figure 5-17.

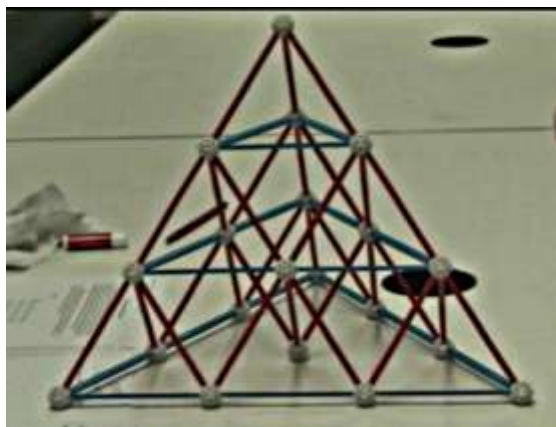


Figure 5-17 Robert's corrected third layer

Marjory again commented that she was not sure what the topmost 1 on the Triangle meant. At this point R1 asked Robert if he knew what he topmost 1 meant. Robert replied that it was “Nothing” (D 90). R2 claimed that all rows were powers of the binomial. Robert replied that the power of the binomial was a zero for the top-most layer. So, Marjory concluded that the top most layer was $(a + b)^0$ which she knew was equivalent to 1 from her previous understanding that “anything to the zero power was one” (D 111). Robert mentioned that “laws” about the role of zero in exponents and factorials were invented so that formulae like “four choose four” could exist (D 102). That is, he implied that in absence of a universal definition of zero factorial or raising terms to the zero exponent, many formulae would not work or that “math fails” (D 94).

R1 encouraged Marjory to connect the terms in her layers with segments so that the triangular pattern could become visible. Marjory chose to draw dotted lines for red rods in Robert’s model and solid lines for the blue rods. Her first two layers on paper connected with the dotted and solid lines looked like Figure 5-18.

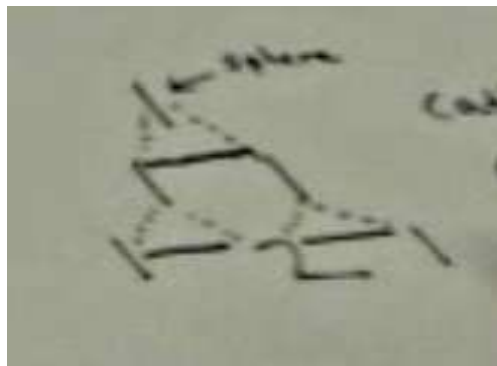


Figure 5-18 Marjory connects her Pascal Triangle's rows

Marjory also wrote out the third layer as 1, 3, 3, 1 when R2 asked her to go to the next level. R1 asked Robert if the two colored rods meant different things in his model. Robert replied that red rods represented multiplication and blue rods represented addition. When R1 asked Robert to explain what he meant, Robert chose to write on paper and show the first two layers of the triangles as Figure 5-19. Robert explained how terms from first layer get multiplied by a and b variables to produce terms in the second layer. In Robert's sketch (Figure 5-19), each variable had two line segments originating from it, one (towards left on the paper) to represent multiplication by a and other (towards right on the paper) to represent multiplication by b . Robert explained the $1ab$ and $1ab$ (that he later called $1ba$) terms in the middle add up to produce $2ab$. At this point R2 noted that Marjory spoke about "making the towers taller" and Robert talked about multiplication by a variable to communicate the same property of Pascal's Triangle (D 175). She contended that "the language can get in the way because you, the word *adding* (or *building up*) sounds like an addition but you don't really mean that, you really mean the multiplication" (D 175). This comment by R2 explains the observance from previous

sessions (Session 12, 13 and 14) where Robert repeatedly used the phrase “addition” (for tower *building up*) to imply “multiplication.”

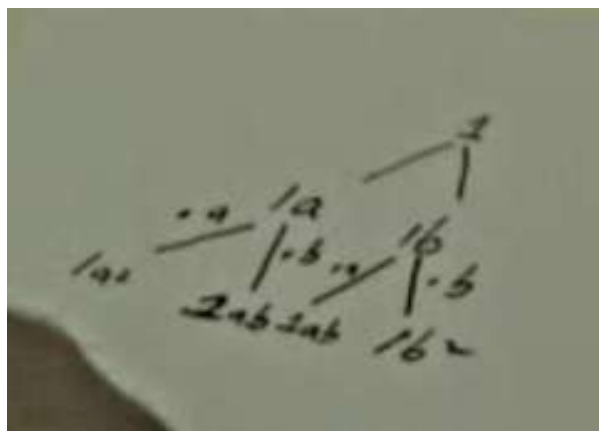


Figure 5-19 Robert explains multiplication in Pascal's Triangle

At this point Robert also explained that his blue bars represent addition between the terms of a binomial expansion. That is when $(a + b)^2$ was expanded to yield $a^2 + 2ab + b^2$, then the two $+$ symbols in the expression were represented by the blue rods placed horizontally in his model. Then, R2 asked Marjory to map her understanding of building towers with Unifix cubes to the third layer of Pascal's Triangle. Marjory explained that the entire second row will get multiplied by an a and a b . So, Marjory decided to make “two of each” term and made duplicates for her towers from the second layer (D 186). That is, she made two yellow-yellow towers, two yellow-blue towers, and so on. Then, she placed a yellow block on top on one tower in each pair and a blue block on the other. As such Marjory produced eight towers in four pairs for the third layer that looked like Figure 5-20.



Figure 5-20 Marjory builds third layer of the Triangle

Then R2 asked Marjory to place her towers on Robert's model. As Marjory started placing towers next to the *Zome-tools*, she placed the yellow-yellow-yellow tower at a corner to represent a^3 . For the second sphere on the model, Marjory found two towers that had one blue and two yellow blocks in them. Before Marjory could look for the missing tower, Robert handed the third tower with two yellows and one blue to her. This reveals that Robert understood what Marjory was missing in her set of towers to represent the $3a^2b$ term of the third layer. Marjory reflected that combining like terms can be "confusing" when the polynomial is expanded (D 231). At end of the conversation, Marjory had her towers lined up in front of Robert's model as in Figure 5-21.



Figure 5-21 Marjory maps towers to third row of Pascal's Triangle

5.4.4 Mapping three colored towers to the Pyramid

Robert explained that for the Pyramid there are three variables and therefore you need red, yellow and blue blocks for the towers. R2 encouraged Robert to use Unifix cubes and explain how towers grow in the Pyramid from the first layer to the second layer. Robert first clarified that the representations he would create would represent a cross section of the Pyramid. He then used three Unifix cubes, one of each color, to represent the variables a , b and c and placed them on the table. In addition to the cubes, Robert used blue and red rods from the *Zome tools* to represent addition and multiplication respectively to demonstrate how $(a + b + c)^1$ layer generated the $(a + b + c)^2$ layer. Robert had three red rods coming out of each cube to represent multiplication by a , b and c . He placed a cube next to the red rods to mark what variable was being multiplied by the term (Figure 5-22). For example, he took the red block and “multiplied” it by a yellow block to create the red-yellow tower in extreme right of Figure 5-22.



Figure 5-22 Robert builds nine towers for second layer of Pyramid

R2 asked Marjory to guess how many towers Robert was going to build (before he finished Figure 5-22) and she guessed 9 as “three things are going to grow by three” (D 304). That is, looking at Robert’s representation in Figure 5-22 Marjory had concluded that as each of the three blocks had three red rods coming out of it, Robert is going to produce 9 towers. Then R2 proceeded to ask Marjory if she saw 9 spheres for the second layer on the model. Marjory tried to go around the Pyramid’s model and observed that there were only 6 spheres in the second layer. R2 asked Marjory to consider if any of the towers that Robert had built could be grouped together to take care of addition between like terms. Marjory reasoned that a sphere in the second layer that has “two points leading to it” is “going to have two towers” and perhaps represent a point where like terms were collected (D 337). At this point Marjory held Robert’s model in her hand and counted how many red rods were “leading in” to a particular sphere from the layer above it. She concluded that number of red rods leading to a sphere yielded the number of towers that correspond to it. Hence, Marjory found evidence for her earlier conjecture. Robert agreed with Marjory’s interpretation. R2 encouraged Marjory to group her like towers. Marjory arranged the towers in a triangular fashion on the table as in Figure 5-22.



Figure 5-23 Marjory arranges towers for second layer of Pyramid

At this point R2 suggested that Robert should guide Marjory to build the third layer of the pyramid. R1 intervened at this point and requested that before constructing the third layer Robert should explain why he never put blue connectors in the “inside” of the pyramid. Robert explained that connectors on the periphery accounted for all the additions in a layer. He claimed that placing extra connectors in middle of the layers would mean that some terms are incorrectly repeated in the polynomial expansion. This reflects that Robert had carefully chosen to eliminate adding blue connectors in middle of the layers. This is noteworthy as in Session 14 R1 had shared with Robert a two-colored model of the Pyramid where blue connectors were placed between all adjoining spheres in a cross-section. For instance, Robert had added only one blue connector in the third layer to hold his $6abc$ sphere whereas R1’s model had several connectors to $6abc$. Robert explained that only one blue connector was needed to represent addition of $6abc$ in the polynomial expansion once. This reveals that Robert was not merely using two colors because he had seen them in Session 14. He was carefully building his model where all pieces were thoughtfully added or skipped.

At this point (49 minutes into the session) Marjory started to build towers three tall to represent the third layer. Marjory again predicted correctly that she would create 27 towers as she “multiplied” each of the nine two-tall towers by three colors. Marjory repeated her strategy of building copies of each tower. She made three towers of each kind and added a blue, a red and a yellow cube on top of them. While Marjory built her towers, R2 reflected that she had visited a class where another graduate student, had brought in a constructed model for the Pyramid. She noted that there were no Unifix cubes at that time. Lou Pedrik responded that the use of Unifix cubes in this session really helped him understand structure of the pyramid. Interestingly, Lou had recorded Session 14 as well and was sharing that in absence of mapping Unifix towers to the Pyramid, he had a difficult time making sense out of some properties of the Pyramid.

Once Marjory had built all 27 towers, she started to place them around the model that Robert had built. She placed the solid colored towers on vertices and the two-colored towers on the edges. Then she placed the three-colored towers in the middle of the Pyramid as in Figure 5-24. Marjory was further able to identify terms on periphery of the third layer using three colors as three variables. For example, she called her yellow-yellow tower to be a^3 and so on. Robert carefully listened to Marjory’s interpretation and agreed with her. At this point Professor Alice Alston came to observe the interview session briefly. Alice wondered if Marjory could systematically group the 6 three-colored towers that she placed in middle of the pyramid.



Figure 5-24 Marjory places 27 towers of the third layer on the model

Marjory did not understand Alice's request at that point. Instead, she went on to conjecture that the $6abc$ term (or six towers in the middle) would have six in-coming red bars. At this point R2 asked Robert to explain to Marjory where the $6abc$ came from. Robert explained that three 2s from the periphery of the second layer lead to the 6 in the third layer. Robert then guessed that if he placed all the blue rods in middle of the third layer as R1 had for her model in Session 14, that it would be 27 connecting rods. He conjectured that counting all possible blue connecting rods in a layer equaled the number of possible towers for that layer. R1 had questioned Robert about this idea earlier and now he was trying to verify his conjecture. Robert decided to make a skeleton of the third layer using *Zome tools*. This is an interesting point in the session as Robert voluntarily reached for the *Zome tools* rather than trying to sketch out a layer on paper. It suggests that Robert deemed it useful to build a concrete model of the third layer in order to count all the connecting rods.

As Robert worked on building a separated third layer, Marjory tried to explain to Alice briefly how she had used Pascal's Triangle to create towers for the third layer. As

she talked to Alice, Marjory stumbled upon the idea that the 2, 2, 2 from the second layer might add up to produce the six in the middle of the third layer. Although Robert had already explained this fact to R2 earlier, Marjory had not picked up on the idea at that point. This is a significant event in the session as Marjory was just beginning to discover the addition property of Pascal's Pyramid. She had already created the towers with three colors and mapped them to terms of the third layer successfully. However, she had not used the second layer to produce terms in the third layer. Robert asked her if she could explain why the 2, 2, 2 from the second layer added up to make $6abc$. Marjory chose to write some terms on the paper and reasoned that:

609. Marjory So, if this one on that side was $2bc$ and then over there that one was $2ab$ and then on that side was $2ac$ and then when you are combining them when you are actually what you have done is then you have multiplied each of those by a plus b plus c and that's how it grew in to six and I have got six $a b c$ towers there in the middle once I have combined the like terms because once you have put each umm, no

Marjory had finally put together for herself how multiplication of $2bc$, $2ac$ and $2ab$ terms from second layer with a , b and c respectively led to $6abc$ term in the third layer. However, she was a little unsure of her discovery. Robert guided her to understand that $6abc$ is only going to come from terms in the second layer that did not have a squared quantity in them. He explained that terms (or towers) in the middle of the third layer have one cube of each color and as such, they cannot have a squared variable in them. At this point, Robert was explaining the addition property of the Pyramid using three colored towers as well as his model. Robert had concluded in Session 14 that terms in the "inside" of the Pyramid were different from terms on the "outside" of the Pyramid, as the outside terms had two colors and the inside terms had three colors. Marjory agreed

with Robert's explanation of the addition property and seemed to follow his distinction between the "inside" and "outside" terms.

R1 asked Marjory if she could arrange the six three-colored towers in middle of the Pyramid and tell the group "which face (of the Pyramid) they belong to" (D 616). Marjory said that she finally understood what Alice had asked her to do and placed her six towers in middle to match the face they belonged to (Figure 5-25). For instance, Marjory placed the towers such that the blue-yellow-red and the yellow-blue-red towers sat on the same side as the blue and yellow towers (Figure 5-25).



Figure 5-25 Marjory arranges six towers to match the face of Pyramid

At this point R2 asked Robert to share with the group the third layer he had constructed on the side (Figure 5-26). Robert had conjectured earlier in the session that if all the blue rods are in place, there would be 27 rods, one for each of the 27 towers they had built for the third layer. However, there were 18 blue rods in the third layer. The group discussed briefly if there was a neat way of counting different combinations of blue rods, red rods and spheres to get the 27. However, they did not conclude with a

consistent strategy that could explain number towers for each layer in terms of the number of rods and spheres present in the layer.



Figure 5-26 Robert's third layer with all blue connectors

At end of the session, R2 decided to ask Robert about *Ankur's Challenge* one more time. Robert replied that it was “things in the center because *Ankur's Challenge* solution does not exist in the 2-D sense, because you have to have three variables, so, it would be everything in the center of this pyramid” (D 705). Robert was able to repeat the argument he first proposed in Session 14 and explained that inside of the pyramid had towers with one of each color. Therefore, he concluded that towers in the middle of a Pyramidal layer were a solution to *Ankur's Challenge*. The session concluded with a group discussion about *Ankur's Challenge* for the fourth layer. Robert explained that the 6 towers from the third layer will produce 18 new towers for the middle of the fourth layer as they will get one more cube of each color. Marjory pointed out that another 18 towers will come from spheres on the periphery of the third layer that had a squared variable in them. She explained that each of the 18 two-colored towers would get a cube

of the third color and then qualify as a solution to *Ankur's Challenge*. Everybody agreed that the fourth layer would have 36 towers in the middle that have one of each color.

6. Chapter 6: High school and undergraduate sessions

6.1 Introduction

Working backwards, in this chapter I analyze sessions 6 – 11 that include three sessions from Robert's eleventh grade (11/13/1998; 3/1/1999; 4/26/1999), two sessions from his twelfth grade (7/7/1999; 8/31/1999) and one session from his senior year in the undergraduate degree (9/12/2003). Tracing the origins of Robert's ideas, in the next chapter I analyze sessions 1 – 5 that include two sessions from his fifth grade and three sessions from his seventh grade. As I trace the origins of Robert's ideas related to Pascal's Triangle and Pyramid, I will analyze the recent most session (Session 11) first and then address the remaining sessions in a reverse chronology.

Some sessions provided more insight into Robert's ideas related to Pascal's identities than others and detailed analyses are given in Chapter 6 and Chapter 7. In some of the sessions (e. g. sessions 1, 2 and 7), Robert only participated briefly. Therefore, for sessions 1, 2 and 7 a summarized description of the events is included so as to report what mathematical ideas Robert was exposed to as a participant in the longitudinal study.

6.2 Session 11 (9/12/2003): Robert and Brian discuss a towers problem

6.2.1 Setting and Background

Note: Elizabeth Upetgrove is coded R1 for this session

R1 invited Robert to share his ideas about a *Towers* problem with another student, Brian. The problem explored in the session was: How many different two-colored towers

are possible when each tower must have two blocks of one specific color? Robert reported that this problem was first introduced to him in eleventh grade (1998 -1999) and that he had found a formula for a solution to this problem. Robert shared with Brian that he was not sure why the formula worked in his eleventh grade problem solving. In this session, Robert invited Brian to figure out a solution for the problem from scratch and discuss why the formula he had discovered in eleventh grade made sense. R1 posed the problem and observed the work during the session but did not intervene. Instead, she asked Robert to talk through the problem with Brian.

6.2.2 Brian tries to find a solution

Robert mentioned at the beginning that he had been working with a graduate student at that time, Lynn Tarlow, on this problem for a week prior to Session 11. He explained to Brian at length what the problem meant. For instance, he said “if you have hundred high, then it could be two of this color and ninety eight of the other color and how many different combinations are there for that?” (E 21) Robert requested Brian to try and solve the problem for himself before Robert could share his formula with him. Brian asked Robert if he needed to draw all the combinations out and Robert replied that “if you don’t know the numbers, then just say it or whatever, draw it all out” (E 33), perhaps suggesting to think about “numbers” without having to draw out all the combinations.

As a response, Brian created the list in Figure 6-1 using *B* to represent the color blue and *R* to represent the color red. Brian remarked that his *B* and *R* look the same and as such it is confusing to look at his list. Robert suggested to Brian that he might want to try towers only three high first. This is an interesting suggestion as Robert is

recommending that Brian try a simpler problem first. This indicates that Robert was aware of the heuristic of thinking of a simpler problem as a useful problem-solving strategy.

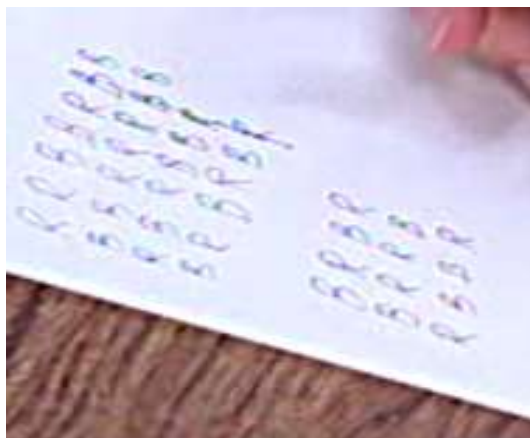


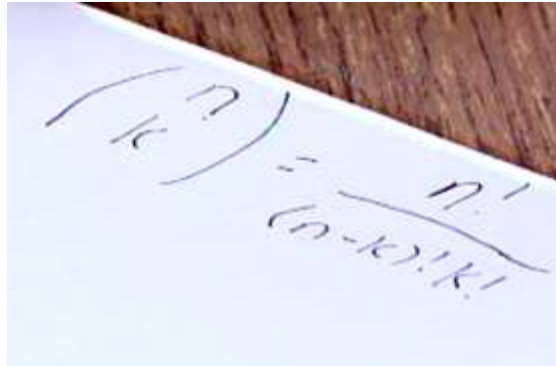
Figure 6-1 Brian lists the possible combinations

Brian modified his four-tall list to create a three-tall list as *RRB*, *RBR* and *BRR*. Robert agreed that there were only 3 towers possible such that they have exactly two red blocks in them. Robert shared with Brian that two-high towers would only have 1 possibility and four-tall towers would have 6 possibilities. He asked Brian if he saw a pattern in 1, 3, 6 that were solutions to towers two tall, three tall and four tall respectively. Brian guessed that solutions were doubling as towers grew taller. Robert indicated that solutions were not doubling and shared his formula with Brian. Robert explained his formula was $h\left(\frac{h}{2}-0.5\right)$ to find the number of combinations with exactly two of one color where h was height of the towers. Brian worked on the formula for towers four tall and verified that when $h = 4$ the formula gave 6 as an answer.

Robert then asked Brian if he could see why the formula works. Brian guessed that the height is getting divided by two as there are two colors. At this point Robert suggested to Brian to factor out the half in the formula. Robert explained that the resulting formula was $\frac{h(h-1)}{2}$ (E 76). Brian did not see an advantage to this new version of the formula. Robert then suggested him to think about “how we used to figure out towers, where we would draw like empty boxes and then” write things in (E 81). Brian drew four empty boxes on top of each other. Robert encouraged Brian to guess how many ways the first block can be filled in. Brian replied that the first block could be filled in with a 2 as there were two colors for the blocks. Robert then rephrased his question and asked Brian to consider the first red cube and think of “how many spaces are there for the first one (red cube)” and the second red cube (E 99-1119). Brian responded that there were 4 spaces for the first red block to go into and only 3 spaces for the second red to go into. As such, there were 4 times 3 or 12 total possibilities. Robert continued to help Brian generalize that for any height of a tower given, the first block would have h possible spots to go to and that the second block would always have $(h-1)$ possible spots available to go to.

Robert then questioned Brian why the formula had a division by 2. Brian guessed that it was because there were two colors for the blocks. Robert agreed with Brian’s conjecture (E 120 -121). Then Robert asked him “does that look familiar at all?” (E 123) Brian replied that the formula did not remind him of anything. Then, Robert went on to hint that the formula was related to the “choose stuff” (E127). Robert went to write the formula for combinations as in Figure 6-2. R1 suggested that Robert should show Brian

“n choose two” explicitly (E 141). Robert filled out 2 in his formula to show that n choose 2 would look like: $\frac{n!}{(n-2)!2!}$.



A photograph of a piece of white paper with a wooden surface in the background. The paper has the formula for combinations written in black ink: $C(n, k) = \frac{n!}{(n-k)!k!}$. The handwriting is somewhat cursive and the paper is slightly tilted.

Figure 6-2 Robert writes out combinations formula

Brian did not follow what Robert meant by the factorial notation. Robert went on to discuss that $n!$ represented multiplication of the following factors: $n(n-1)(n-2)(n-3)\dots(n-n+1)$. Robert then simplified his n choose 2 formula to show Brian that the resulting terms were just $\frac{n(n-1)}{2}$ similar to the formula he had shared initially (Figure 6-3).

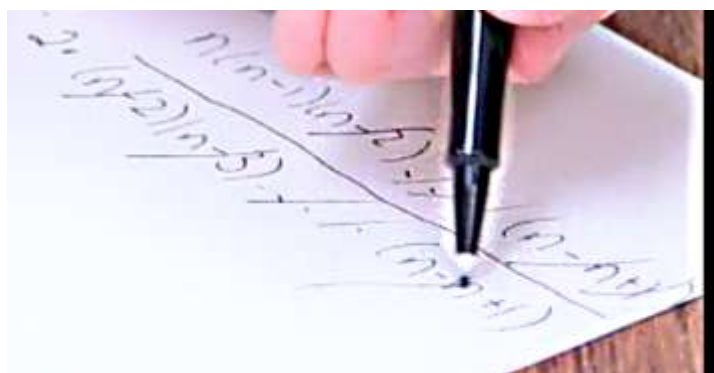


Figure 6-3 Robert simplifies n choose 2

Brian then went on to ask Robert if “blue has to be in before anything” or if the “blue can be mixed in with the group” (E 188). Robert explained that his strategy was to first place red blocks in the tower and then figure out where the blue went. He wrote down two R s on the paper and explained that the blues could either go on ends or in the middle (Figure 6-4). The final representation does not tally with the four blocks case as Robert drew three \wedge s for blue. Robert explained to Brian that the blue blocks (\wedge) could be on the ends of the red blocks, one in the middle or two in the middle of red blocks. This is an important strategy where Robert fixed a variable, in this case the red blocks, to systematically count the possibilities.

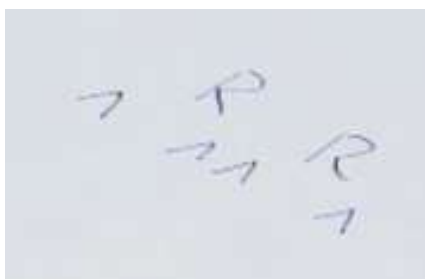


Figure 6-4 Robert shows where blue blocks can go

6.2.3 Draw out the towers

Brian and Robert then briefly discussed other conditions they could impose on the given problem. Shortly afterwards, R1 requested Brian to actually draw out the towers that he had mentioned earlier. Now, with guidance from R1, Brian made three sets of four towers and filled R in the first, second, third and fourth spot as in Figure 6-5.



Figure 6-5 Brian makes 12 towers

Then R1 asked Brian to fill in where he thought the second red could be placed in those towers. Brian filled the second R in the remaining possible spots so that each tower would have exactly two R s in it (Figure 6-6). Robert pointed out there were “duplicates” or identical towers in Figure 6-6. Brian went on to cancel the duplicates and had final six unique towers (Figure 6-6). Robert noted that the rows in Figure 6-6 yielded one unique tower, two unique towers and three unique towers. Robert and R1 wondered if there was a pattern to the number of unique towers in Figure 6-6. However, they did not further investigate a possible pattern.

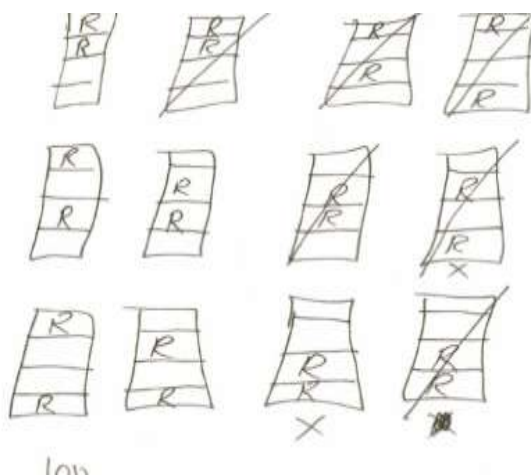


Figure 6-6 Brian crosses out the repeating towers

Robert then went on to ask Brian if he could figure out a similar solution for towers that were “a hundred tall” (E 278). Brian replied that he could use the same strategy they used for towers four tall. Robert then briefly discussed with R1 if there were three colors to choose from for the same problem, how the solution might change. However, Robert did not finalize an answer to the three-colored case as R1 noted that she was not too sure of the problem Robert was referring to.

6.2.4 Robert recalls Pascal’s Triangle

As Robert continued to chat with R1 and Brian, he mentioned “I guess, have you heard about Pascal’s Triangle?” (E 331) Brian said he had. Robert encouraged Brian to write a first few rows of the Pascal’s Triangle. Brian wrote the first four rows of the

triangle as in Figure 6-7. Robert inserted the 1 on the very top to remind Brian that there was a single 1 above the 1, 1 row.

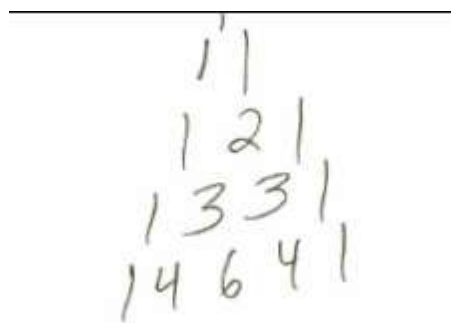
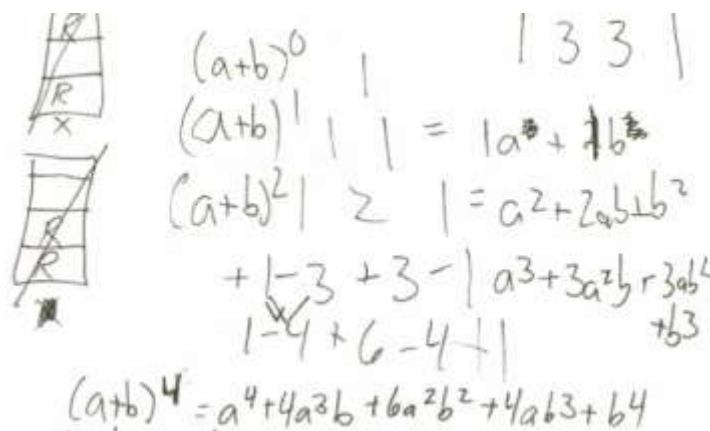


Figure 6-7 Brian writes rows of Pascal's Triangle

Robert pointed out to Brian that they only needed the rows listed in Figure 6-7 as they had built four-tall towers. Robert encouraged Brian to explain whether there is a connection between the combinations formula they just used and entries of Pascal's Triangle. Robert remarked that he knew there was a relationship but that he could honestly not remember it. Robert then wrote out how the binomial expansions were related to the Pascal's Triangle as in Figure 6-8. This reveals that Robert already understood the relationship between coefficients of a binomial expansion and entries in the rows of Pascal's Triangle.



Handwritten notes showing Pascal's Triangle and binomial expansion for $(a+b)^4$.

Pascal's Triangle (rows 0 to 4):

1				
1	1			
1	2	1		
1	3	3	1	
1	4	6	4	1

Binomial expansion for $(a+b)^4$:

$$(a+b)^0 = 1$$

$$(a+b)^1 = a + b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Figure 6-8 Robert explains binomial expansion on Triangle

Robert also noted that $(a + b)^4$ in expansion would be relevant to towers that are four tall. At this point Brian pointed out that “ a to the fourth right, so it’s all reds” and Robert exclaimed “yeah, that’s what, that’s what I was getting at” (E 383 – 384). They both discussed that the coefficients a and b represented two colors for the towers. For instance, Brian explained that $4a^3b$ is a tower that has three reds and one blue. Robert further explained that for towers that have two of each color, there would be 6 towers corresponding to the $6a^2b^2$ term in the fourth layer. This connected Robert’s formula to the binomial expansion and the fourth row of the Pascal’s Triangle. Robert did not explicitly address this connection at this point.

Brian then asked Robert why the topmost layer for Pascal’s Triangle was a 1. Robert replied with ideas that included “one way you can have zero things and that’s nothing”, “it’s mostly for like, so it doesn’t mess up formulas and stuff” and “cause zero is complex ... you have to invent a special rule just so that it (implying zero) doesn’t go messing things up” (E 404 -410). Robert responded that there was “one way” to have

“nothing” as zero-tall towers could not exist physically. However, he also concluded from his previous experience that “zero is complex” and if special rules are not invented for it, many formulae will not work. In Session 15, Robert had re-iterated this explanation for the topmost 1 in the Triangle and expressed a similar opinion about properties of zero.

Robert then asked Brian if he could see a connection between entries of the fourth row: 1, 4, 6, 4, 1 and the solution to *Towers* problem that they had discussed earlier in the session. Brian was not sure what Robert expected him to do. Robert then listed all terms from the fourth row as combinations (Figure 6-9). He explained to Brian “that makes sense too ‘cause if you want to choose three apricots, from four tall high, four choose three” (E 432). At this point Robert was able to recall the relationship between the combinations formula and entries of the Triangle. He shared with Brian that if the problem was to place “three apricots” (using “apricot” as a color) in four spaces, the answer would be “four choose three” corresponding to 4 which is the coefficient of the term $4a^3b$.

The image shows a handwritten list of five combinations: $\binom{4}{4}$, $\binom{4}{3}$, $\binom{4}{2}$, $\binom{4}{1}$, and $\binom{4}{0}$. Below the first combination, $\binom{4}{4}$, there is a small handwritten '1'.

Figure 6-9 Robert writes combinations for fourth row of the Triangle

This is a significant event in the session as it provides evidence that prior to post graduate sessions, Sessions 12 – 15, Robert had spent time thinking about entries of

Pascal's Triangle. It is also evident that he understood the multi-fold relationship between the two-colored *towers* activity, the binomial expansion, the combinations formula, and Pascal's Triangle.

Finally, another male observer in the room asked Brian if he could figure out an answer for two of one color in a tower that is hundred high when there are two colors to choose from. Brian solved the answer as in Figure 6-10. He did not use the combinations formula. Instead, he used the formula that Robert had shared with him in the beginning where number of possible combinations is given by $\frac{h(h-1)}{2}$ (h is height of the tower). This indicates that Brian was applying Robert's explanation for the equivalence of the two formulae.

$$\begin{array}{r} 100(100-1) \\ \hline 2 \\ 100(99) \\ \hline 2 \\ 9900 \\ \hline 2 = 4950 \end{array}$$

Figure 6-10 Brian solves for “two reds in a hundred tall” tower

Robert encouraged Brian to write out the solution using the combinations formula and Brian verified that he got the same answer of 4550 when he tried to solve $C(100, 2)$. The session concluded with Robert and Brian praising each other for a “good session” (E 532).

6.3 Session 10 (8-31-1999): Robert and Michael discuss World Series Problem

6.3.1 Setting and Background

Note: For this session Carolyn Maher is coded as R1.

This session is a two-person interview with R1 when Robert was in the summer prior to Robert's twelfth grade. Robert worked with another student, Michael, in this session and they discussed the *World-Series problem* and *the Problem of Points*. *World-Series problem* was a combinatorial task used in the Rutgers Longitudinal study when the students were eleventh graders. Michael had worked on the *World-Series problem* prior to this session but Robert had not. The question posed to them was: What is the probability of winning a world series in exactly four, exactly five, exactly six or exactly seven games? Michael had two different solutions for the problem from a previous session where he had worked with a group. Robert was expected to be a sounding board in this discussion to help Michael choose one of his solutions. R1 stepped out of the room occasionally to allow Robert and Michael to work on parts of the problem and then returned when they indicated that they were ready to discuss their ideas.

I report on the first hour of this session where Robert and Michael solved the *World-Series problem* and came up with a correct solution. They also decided which of the two conflicting solutions that had troubled Michael earlier should be picked to be the correct solution. It is noteworthy that Robert did not directly address Pascal's Identity in this session. Nevertheless, the session provides great insight into Robert's use of binary notation and the combinations formula to list possibilities for parts of the problem. It

becomes evident that Robert used his binary representation in conjunction with the combinations formula to solve the *World-Series problem* with Michael.

6.3.2 Using binary notation versus combinations formula to list possibilities

After Robert understood the statement of *World-Series problem*, R1 stepped out to allow Robert and Michael to work together. Michael started to use binary notation and represented 1 as one of the teams and 0 as the other team. For instance, he used 1111 to represent the event that “team 1” won first four games and therefore won the World Series. Michael also shared with Robert that they could list possibilities for team 1 winning and then multiply the number of possibilities by 2 to account for the event that team 0 would win. Robert began to list possibilities using the binary notation as well. He listed 10 possibilities for a six-game series where team 1 won (Figure 6-11). Then, he multiplied 10 by 2 and concluded that there are 20 ways for the series to end in six games as each team could win in 10 ways. Michael was trying out different multiplications and divisions on the side when Robert finished his list. Michael reflected that he remembered 20 as an answer from his previous experience with the problem.

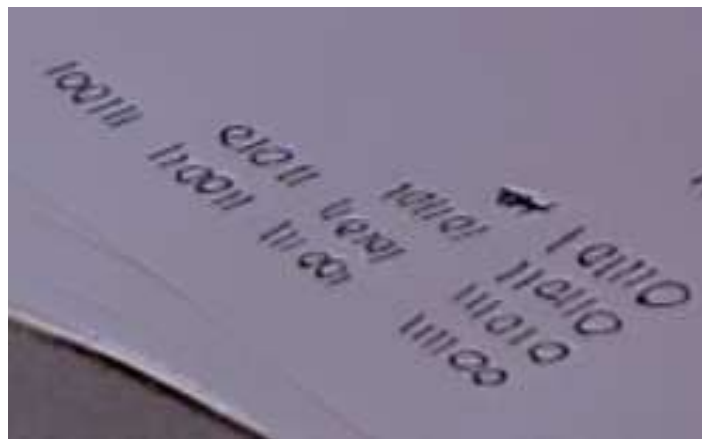


Figure 6-11 Robert's binary list for six games series

Robert used a system to generate his list in Figure 6-11. He started with placing 00 together in the first spot (001111) followed by one 1 in between the 00 (010111) followed by two 1s in between the 00 (011011) and so on. Then he moved the placement of 00 from first two spots to second and third spot (second column in his list) and repeated the pattern of inserting 1s in between the 00. Once Robert had placed 00 in fourth and fifth spot, he knew he was done. He knew that team 1 had to win the sixth game for the series to run six games. He carefully did not write the cases where the game could end sooner than six games (like 111100). Once his list was complete, Robert was confident that there were 20 ways for the series to end in six games. This reveals that Robert had previously worked with the binary notation to represent various cases in a problem.

Michael suggested that there had to be a “reason for the twenty” and went on to work by himself silently (F 121). Michael worked with some multiplications and divisions on the side as Robert tried to list possibilities for a seven-game series using the binary notation. Robert suggested that adding a 0 and then a 1 in front of the list of six games (Figure 6-11) he could list some possibilities for the seven-game series. This reflects that Robert wanted to continue using his counting technique and the use of binary notation. Michael worked with some factorials on the side and did not comment on his work at that point. However, Michael pointed out to Robert that he might miss cases for a seven games series if he used his strategy of adding 0s and 1s in front of the binary list he had created in Figure 6-11. Robert continued to discuss how he could create an exhaustive list for a seven games series, however, Michael suggested that Robert should

consider the combinations formula. Michael said “to figure out how many different possibilities you can have four and like six spaces or something, that’s like choose right?” (F 160) Robert agreed and they both started to work with the combinations formula instead. This reveals that Robert had decided to give up the binary notation for a seven-game series as he deemed it would be involved. Also, it shows that Robert and Michael were both comfortable thinking about the combinations formula to answer the question at hand.

Michael explained to Robert that a five series game is “five choose four” and a six series game is “six choose four” as a team needed to win four games to end the series (F 176). Michael went on to say that “five choose four” is a 5 which should be multiplied by 2 (creating a 10) to take care of the other team winning (F 180). And then, Michael suggested that a 2 should be subtracted from 10 to take care of the ways a series could end in four games (F 180). As such, Michael calculated that for a series to end in five games, there are $10 - 2 = 8$ possibilities. This discloses that Robert and Michael both agreed that there were only two ways to end the series in four games, that is either team 1 wins all four games or team 0 wins all four games. Robert had agreed that Michael needed to correct his answer to account for cases that were already accounted in the four-game series and subtract a 2 from 10.

Michael continued to work on six-game series and calculated that $C(6, 4) = 15$. Then, he multiplied 15 by 2 to get 30. Then, Michael suggested that 10 that they had gotten for five-game series ($2 \text{ times } C(5, 4)$) should be subtracted from 30 to get 20 possibilities for a six-game series. This answer matched the binary list that Robert had created to get 20 possibilities for a six-game series. For a moment, Michael thought that

they needed to subtract another 2 from the 20 to take care of four-game series. Robert explained to him that the 2 resulting from four-game series was already a part of the 10 they had calculated for the five-game series. As such Robert contended that they did not need to subtract the 2 again. Michael agreed to Robert's argument after some thought. It is conjectured that Robert had spent time convincing himself of 20 as the correct answer for a six-game series. As such, he was able to quickly dismiss Michael's suggestion to subtract another 2. In absence of the binary listing, it is possible that Robert would not have believed Michael's calculations quickly or provided feedback instantly. It is reasonable to suggest that Robert used his representation in Figure 6-11 to verify the ideas shared by Michael.

Within the first twenty minutes of the session, Robert and Michael had together come up with a strategy to count the number of ways a series could end. Robert wrote out a general formula to capture this strategy. They figured out that for seven games there would be $2C(7,4) - 2C(6,4) = 70 - 30$ or 40 possibilities. As such they figured out that the series could end with 4 games in 2 ways, 5 games in 8 ways, 6 games in 20 ways and 7 games in 40 ways. That is there were total $2 + 8 + 20 + 40$ or 70 possibilities for all the ways a world series could end. At about thirty minutes into the session, Robert and Michael invited R1 back to share their solution. As Robert and Michael explained some of their ideas to R1, R1 invited them to consider the following question: "Is it as likely to win the World Series in four games as it is in five games as it in six games?" (F 350) Robert quickly responded that it could not be equally likely to win the World Series in four, five, six or seven games. Michael agreed that it is "hard for a team to keep up

winning all four games” (F 358). Robert and Michael conjectured that it was less likely for the World Series to end in four games and more likely to end in seven games.

6.3.3 Which solution makes sense for the World Series Problem?

R1 wanted Robert and Michael to reflect more on their conjecture and left the room to give them time to discuss the question she had put forward. Michael began to reflect on his older solution from the last time he had done the problem with a group of students. In the group solution, Michael had considered that there are 16 ways for the series to end in four games. He and his group had conjectured that as each game had two possibilities, team 1 or team 0 could win, there were $2 \times 2 \times 2 \times 2 = 16$ ways for a series to end in four games. As a result, in the old solution, Michael would have guessed that probability of World Series ending in four games was $2/16$. In the new scenario where Robert and Michael had counted 70 possibilities, they calculated that the probability for the series to end in four games would be $2/70$ instead. Look at Table 6-1 for a comparison of how solutions would work out for probabilities in older solution versus the new solution proposed in this session. Robert and Michael did not create this table but they wrote some probabilities listed in the table on their paper and discussed others verbally.

Table 6-1 Old and New Probabilities

Number of games played to end the series	Old Probability	New Probability
4	$2/16 = 0.125$	$2/70 = 0.02857$
5	$8/32 = 0.25$	$8/70 = 0.1143$

6	$20/64 = 0.3125$	$20/70 = 0.2857$
7	$40/128 = 0.3125$	$40/70 = 0.5714$

Robert pointed out that 16 as total number of possibilities for a four-game series perhaps included non-realistic cases where the series would end in one, two or three games. Robert further noted that for any experiment all probabilities should add up to 1. Robert and Michael converted their fractions to decimals and realized that probability went up as number of games played increased. Also, Michael realized that the probability for series to end in seven games was twice the probability of series ending in six games in the new solution ($40/70$ versus $20/70$). Michael thought that was more convincing than the old solution case where series had the same probability of ending in six or seven games (highlighted red in the Table 6-1). Michael believed that it made sense intuitively that “it should be easier to win it (World-Series) in seven” (F 500). R1 encouraged Michael and Robert to reflect on how they listed possibilities for a six-game and seven-game series. She encouraged them to talk further and make a choice between the two solutions.

Michael conjectured that the new solution made more sense as by the end of six games, each team had won three games. While talking to Robert, Michael conjectured that “three and three and you need one more if you get the one or the zero when you add that extra game to the end, it doubles everything up” (F 544). Michael invited R1 back to present this logic about the doubling effect. Michael explained that there were “six choose three” or 20 ways to end up with a tie from six games (F 570). And after those

twenty ties, when series goes on to the seventh game, “it could either be a win for one team a win for other” to create a tie breaker (F 572). Michael explained that “they have twenty different ways that A would win twenty different ways that B would win” and as such, the probability of winning in seven games just doubles to give a 40 (F 574). R1 applauded the explanation that Michael had offered. To conclude the *World-Series* conversation in this session, Michael and Robert picked the new solution as the valid one.

It is significant to note that Robert had not worked on the *World-Series* problem prior to this session. It is remarkable that he was able to grasp the meaning of the problem and devise a solution with Michael in a short period of time. He was also able to convince Michael that the formula they devised in this session made sense. However, it is important to point out that the solution they picked is not the correct solution for the problem. The old solution is, in fact, the correct solution to the problem.

6.4 Session 9 (7- 7- 1999): Robert and Michael presentation

6.4.1 Setting and Background

Note: For this session, Regina Kiczek is coded as R1 Carolyn Maher is coded as R2 and Lynn Tarlow is coded as R3.

This session is a presentation by Robert and Michael to their classmates during the summer of 1999 while attending a two-week summer institute. R2 invited them to discuss their solution to the following problem: How many committees of two people can be made from a group of five people? The presentation lasted about 25 minutes. Robert discussed the combinations formula and how it was related to the terms in Pascal’s

Triangle in the session. A white board and markers were provided for the presentation. R1 and R3, graduate students at that time, also observed the presentation.

6.4.2 Robert shares three different methods to find the number of committees possible

Michael started the presentation by writing out the formula for combinations as

$\frac{x!}{n!(x-n)!}$ and commented that he was not sure if it was “x choose n or n choose x” (G 7).

Robert replied that it was “x choose n” (G 8). Robert asked Michael if he just wanted to “draw all the stuff on the board first?” (G 10). This suggests that Robert thought that making a representation for the solution at hand would be a good starting point. Michael did not appear to hear Robert and continued asking his classmates if they agreed with his formula. The students in the class nodded their heads in agreement. Michael then prompted Robert to explain the problem to the class. Robert replied that they were trying to “know how many groups of two we can make without writing all the combinations” when there are five people to choose from (G 20). Robert shared that he and Michael calculated 10 when they wrote out all the possible combinations for making a committee of two people. Robert then explained that he and Michael were trying to find a “faster way in case like we have like a hundred people choose ten or something like that” (G 20). This reveals that Robert and Michael were thinking about an extension of the problem in which listing all the possibilities would become cumbersome.

Robert then went on to write on the board as he explained that he was going to call the five people in the problem A, B, C, D, and E. Then he drew five circles beneath the alphabets and filled them in one by one (Figure 6-12). Robert explained that the first circle could be filled in by any of the five people since no one had been chosen yet and

wrote a 5 in the circle. Similarly, he explained that the second circle would have four people available to choose from as one person had already been selected for the first circle. Accordingly, he filled in a 4 in his second circle, a 3 in his next circle and so on. He calculated that this product was 120. He claimed that there were 120 possible ways to arrange all five people. Robert also pointed out to the class that he had actually written a product that could also be called “five factorial” (5!).

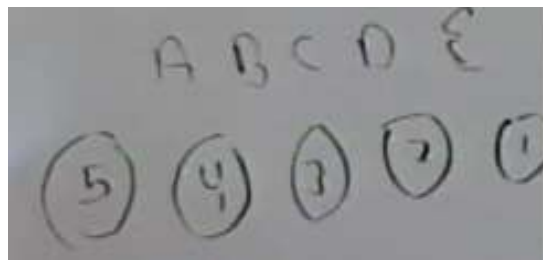


Figure 6-12 Total combinations to arrange five people

Robert then explained that they only had the first two “spots (referring to the circles)” and that last “three spots didn’t exist” as only two people were to be chosen for the committee (G 20). He claimed that to take care of the absence of the three spots, they needed to divide the product $5 \times 4 \times 3 \times 2 \times 1$ by the product $3 \times 2 \times 1$. As such, the answer simplified to a 5 times 4 or 20 combinations for selecting two out of five people. Robert then reminded the class that when he had listed all possibilities with Michael, they only found 10 combinations. As such, Robert recalled that he and Michael were puzzled by the 20 left in this calculation. Robert then explained that they realized that there were going to be “duplicates” as the “order doesn’t matter “if like I am in the first spot and he (referring to Michael) is in the second spot or if Mike is in the first spot and I am in the second spot, it’s like the same thing.” (G 20) Robert was explaining to his classmates

that in the given problem the order did not matter as they were just making committees of two people. In other words, a committee with Robert and Michael was going to be the same regardless of the order in which the committee members were selected.

To fix the over count due to duplicates, Robert explained that they needed to divide 20 by how many total ways two people can be arranged which is $2!$. He further stated that as $2!$ equals a 2 and consequently the final answer is a $20/2$ or 10. Robert explained that reasoning through this concrete problem, they had convinced themselves

that $C(x, n) = \frac{x!}{n!(x-n)!}$. This is a significant event in the session as Robert shared how

he and Michael had *discovered* the combinations formula for themselves. They had justified by working on one particular problem what the numerator and denominator of the combinations formula ought to be. It is noteworthy that in the combinations formula Robert and Michael used x to represent total things whereas most of the textbooks use n to represent total things. This further suggests that Robert and Michael had constructed a representation for the combinations formula using a convenient notation of their own rather than use the notation from a conventional textbook.

Robert then mentioned that he and Michael were also exploring the “Pascal’s Triangle thing” and drew out the first three rows of the Pascal’s Triangle (G 20). He shared with the class that in order to use Pascal’s Triangle to answer the problem “three choose two” one needed to go to the third row of the Triangle (1, 3, 3, 1) and take the second entry. As such he explained “three choose two” would be a 3, the second entry of the third row. He explained that the combinations formula they wrote for $C(x, n)$ also produced a 3 when x was 3 and n was a 2. At the end of this part of the presentation, Robert concluded that “there is like three ways to solve these problems, you can either

use this (the combinations formula), the triangle (Pascal's Triangle) or you can just write all the combinations" (G 20). This is an important critical event in the session, indicating the ease with which the various representations were used. This might be the earliest instance available in all the sessions (Sessions 1 – 15) where Robert connected the combinations formula with entries of the Pascal's Triangle. Michael applauded Robert's presentation with "That's very good" (G 21). Overall, it took Robert about four minutes to present the ideas described above. This reflects that Robert had a clear understanding of the connections between the ideas he presented.

R1 encouraged Michael to explain the ideas that Robert had summarized and Michael once again stated that order was not relevant to this problem. Robert pitched in to explain that if it was five choose three, then they would need to divide by $3!$ as there can be many "duplicates" when three people are chosen for a committee (G 33). This reflects that Robert had not only solved the problem at hand that involved choosing two people out of five but also worked on generalizing his solution to other cases.

6.4.3 Why is "5 choose 2" same as "5 choose 3"?

At about ten minutes into the session, R1 asked Michael if he could explain how the formula would work for five choose three as opposed to five choose two. Michael explained that when using the formula, the denominator for five choose two was $2!3!$ and in the case of five choose three, the denominator would change to $3!2!$. Michael pointed out to R1 that mathematically, both the combinations $C(5,2)$ and $C(5,3)$ would yield the same answer. R1 commented that she understood how $C(5,2)$ and $C(5,3)$ were the same mathematically; however, she wanted to really understand why she should divide by $3!$ versus a $2!$ when a committee of three people was being chosen. R2 took R1's question

further and asked “with the committees of two, committees of three, why would they come out to be the same answer?” (G 43) R2 encouraged Robert and Michael to explain why this would practically happen in the real world. Robert replied that he was not sure and replied that it was “kind of coincidence” that five choose two was same as five choose three (G 44). Michael agreed with Robert that this was a coincidence.

R2 opened the floor for input from all classmates and asked them to think why five choose two was same as five choose three. Another student, Magda, explained that when you choose a group of two people out of five, “then you are left with a group of three” and when “you have a group of three, but you also have a group of two” (G 64). Magda explained that choosing two people out of five people led to the creation of two groups, a group of two that was chosen and a group of three that was not chosen. This was true even if the order was reversed and three people were chosen instead and two were left behind. As such, Magda contended that whether the problem was stated as “five choose two” or “five choose three” the same two groups were created resulting in the same number of combinations. Robert and Michael both agreed with Magda’s reasoning.

6.4.4 Combinations on Pascal’s Triangle

At this point R2 invited other researchers to ask questions. R3 asked Michael and Robert to show her where five choose two or 10 was on Pascal’s Triangle. Michael drew out the first few rows of the Triangle as in Figure 6-13. He explained that the top most 1 is “zero choose zero” and in the second row the first 1 is “one choose zero” and the second 1 is “one choose one” (G 77). Continuing with his explanation that everything in

Pascal's Triangle was a “something choose something,” Michael stated that in “the fifth row, two over, is your five choose two” (G77). This reflects that Michael also had a good understanding of how Pascal's Triangle was related to the combinations formula.

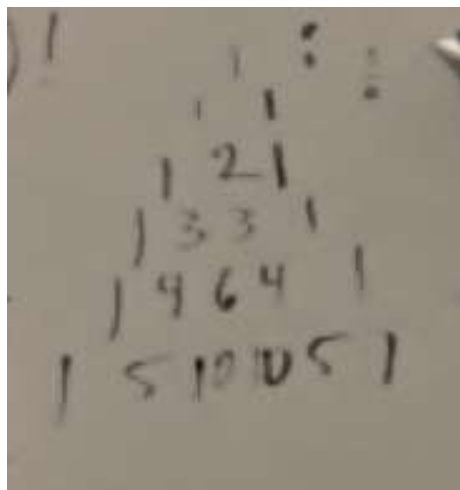


Figure 6-13 Michael writes out rows of Pascal's Triangle

R3 then asked Michael how he knew to write the entries so quickly for rows of the Triangle. Michael commented that he was adding terms from one row to create terms for the following row. Then, Michael guessed that R3 might ask him to explain why the addition of terms worked. In anticipation, he volunteered to rewrite the pyramid with choose notation (Figure 6-14).

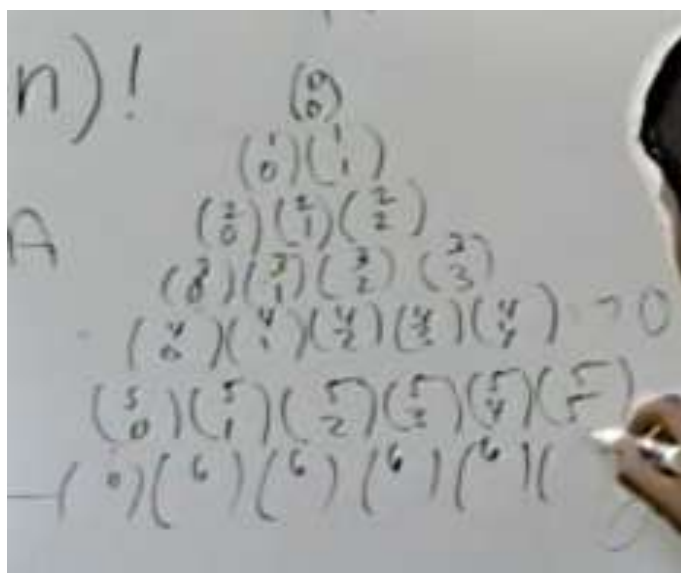


Figure 6-14 Michael writes combinations for Pascal's Triangle

Michael began to explain how the addition property of Pascal's Triangle worked. He seemed unsure of how to proceed and asked Robert if he remembered a reason why the addition property made sense. Robert said he did not know for sure why the terms are added. This reflects that Robert had not yet figured out why the addition property of Pascal's Triangle worked. Michael, then, introduced the metaphor of pizzas with and without toppings to explain the addition rule in Pascal's triangle. Michael used zero to represent absence of a topping and 1 to represent presence of a topping.

Michael introduced, as an example, a four-topping pizza with mushrooms, peppers, sausages and artichokes as the topping choices. He explained that in his notation 1000 would be a pizza with only mushrooms on it, 0100 would be a pizza with only peppers on it and so on. Then he went on to talk about a five topping pizza and chose onions as the fifth topping. Michael explained that the pizzas from the four

toppings case (or fourth row of the Triangle) could either get the fifth topping or not. For example he stated, “this pepper pizza can either be a pepper pizza or it could be a pepper and onion pizza” and as such “number of pizzas would double” (G 97). Michael explained that each pizza from the fourth row could go to two places in the fifth row. For instance, a pizza with no toppings from the fourth row represented by $C(4,0)$ would go to $C(5,0)$ if it did not get onions and would go to $C(5,1)$ if it got onions. R3 did not probe Michael’s ideas further. Instead, she asked why a $3!$ was needed in the denominator of $C(5,3)$. Robert explained that there are 6 “duplicates” when three people are arranged and as such they needed to divide by $3!$. The class applauded the presentation and the session concluded shortly afterwards.

6.5 Session 8 (4-26-1999): Towers extensions in a small group

6.5.1 Setting and Background

Note: For this session Carolyn Maher is coded as R1.

This session is from Robert’s eleventh grade where he worked in a small group setting with R1. The other students present were Angela, Amy-Lynn and Magda. The students were working on extensions of *the Building Towers* activity. The question that students worked on for the session was: If the towers had a height of n blocks and there were $n-1$ colors to choose from, how many towers can you build so that each color is used at least once. R1 pointed out that this question was posed by the students themselves in another session where they were “inventing” extensions to the *towers* problems. After 50 minutes Magda and Angela left the session while Amy-Lynn and Robert continued to work on the problem and talk to the researcher.

6.5.2 Listing all possible towers that are n tall with at least one each of the $n-1$ colors

R1 explained the problem to the students in the beginning and left the room so that students could work on the problem on their own. Magda decided to assign each student a part of the problem. She asked Robert to work on the case when $n = 3$, Amy to work on the case $n = 4$, Angela to work with $n = 5$ and herself took up the case $n = 6$. All students worked silently, writing on their sheets of paper for a little while. Robert finished quickly and reported that there were six towers that were three tall and had at least one of two colors. He listed his towers as in Figure 6-15.

3 tall
2 colors
have to have at least 1 of each

AAB BBA
ABA BAB
BAA ABB

= 6

Figure 6-15 Robert's six towers for three tall

Magda asked Robert if there was a generalization to what they were doing and Robert responded that he didn't think so. Robert began to look over Amy-Lynn's work and guessed that she should have 24 towers. However when Amy-Lynn and Robert checked the list together they realized that some towers were missing. Robert pointed to Amy-Lynn some of the towers that she was missing and Amy-Lynn finally generated 36 towers (Figure 6-16). In the meantime, Magda and Angela struggled with their listings for five and six tall towers. They started to realize that there were too many towers to list, especially for six tall towers.

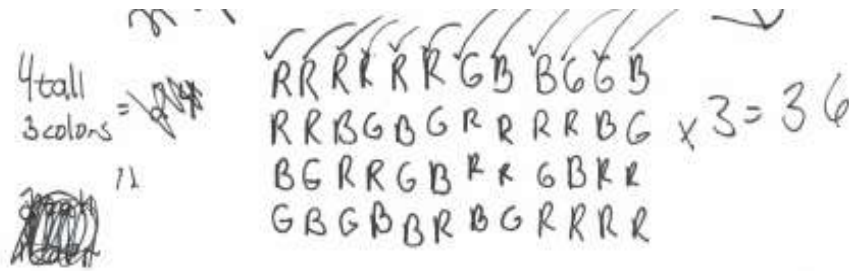


Figure 6-17 Robert's representation for four tall towers

The Figure 6-17 is very similar to Figure 6-11 from the World Series session (Session 10) in terms of the strategy used to list the possibilities. In Session 10, Robert had used binary numbers to list the ways a series could end. He listed the first few possibilities for a series by keeping 00 together in the first and the second place. Then he listed possibilities that resulted from separating 00 by a single 1, two 1s and so on. This is a significant observation as it reveals that Robert reused his representations to list all cases in the *World Series problem*. It further suggests that Robert had observed a similarity in the structure of *Towers problem* and the *World Series problem*.

In the meanwhile, Magda and Angela tried to count their possibilities without having to draw out all the towers. Angela encouraged Robert to think of a pattern and exclaimed "Let's just think for a second. Especially you, Bobby, Robert, I'm sorry, you always come up with these extravagant answers that make no sense to anyone and you're just like 'yeah I get it'." (H 207) The group agreed with Angela's comment. Robert modestly replied that his classmate Michael also got the "extravagant answers." This is an interesting event in the session as Angela called Michael and Robert "super genius people" who were often successful at finding patterns to a solution (H 209). This reflects that Robert had shared some interesting ideas with his classmates over the years to earn

the title of a “super genius.” Angela concluded that there were 360 towers for the five tall towers and Magda concluded that there were 2880 towers for the six tall towers. They were both not completely sure of their answers and they did not explicitly comment on why their answers were trustworthy.

Around 35 minutes into the session, Robert suggested that there was a pattern. He suggested that if n was the height of the tower, then $\frac{n!(n-1)}{2}$ might be a formula that produced the number of towers possible that had at least one each of the $n-1$ colors. Robert checked that $n = 3$ yielded a 6 and $n = 4$ yielded a 36. However the formula did not yield a 360 for $n = 5$ or a 2880 for $n = 6$ as Angela and Magda had proposed. According to Robert’s formula there would be 240 towers for $n = 5$ and 1800 towers for $n = 6$.

When R1 came back to the room to chat with the group, they all confessed that they were a little lost. R1 asked them if they were sure of anything. Robert said that they were all sure that for $n = 3$ there were 6 towers that met the requirements of the problem. Amy-Lynn said that as she had listed all the towers too, she was confident that $n = 4$ had 36 possible towers. R1 began to guide the group to some of the simpler towers problems that they had worked on. For instance she asked them how many four tall towers were possible when there are two colors. Robert replied that it was 16 or 2^4 . When R1 asked Robert how he knew it was a 16, Robert said that each time the height of towers was increased, the number of total towers possible was doubled. He used induction to explain this idea. Robert started with a one tall tower for two colors, blue and red, and wrote an R and B at the bottom of Figure 6-18. Then he explained that to make towers two tall, each one tall tower could get either a red block or a blue block. This he

explained would result in four towers (Figure 6-18). The group quickly followed Robert's argument. This suggests that all students in the group were familiar with the idea of building on an existing tower by using two colored blocks.

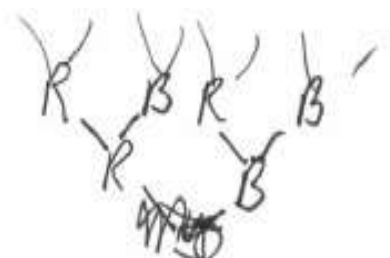


Figure 6-18 Robert explains how towers double in number with two colors

R1 asked Robert if there were three colors, how many two-tall towers were possible and Robert replied that it would be 9. He explained that his representation in Figure 6-18 would now have three towers to start with and each tower would get one of the three colors yielding 3^2 or 9 towers. This reflects that Robert was able to generalize from 2^n to 3^n and adjust for three colors to count all possibilities for the towers. It is evident that Robert used a representation for how towers are built inductively to justify the total possibilities. At this point Magda and Angela left, promising R1 that they would think more about the problem at home.

Once Robert and Amy-Lynn were alone with R1, Amy-Lynn shared Robert's formula with R1. Robert explained to R1 that the formula "just works" and did not know why it made sense for the problem (H 444). R1 then guided Robert to consider a problem where three different books A, B and C were to be arranged on a shelf. R1 asked him how many arrangements were possible for the three books. Robert replied that there were 6 arrangements. Then, R1 changed the problem to arranging three books when two

of the books were identical. R1 explained that the books were now A, A and B and asked Robert how many different arrangements were possible. Robert realized that there were only 3 different arrangements possible with A, A and B. Robert concluded that when two of the three things were identical, the number of possibilities went down by a factor of 2 (from 6 to 3). R1 encouraged Robert and Amy-Lynn to generalize a rule for this reduction effect. R1 hinted that thinking about a simple problem with identical items to be arranged might help Robert and Amy-Lynn find an answer to their original problem.

R1 stepped out for a little while to allow Robert and Amy-Lynn to work on the problem some more. However, the session was running long (almost an hour and forty five minutes). With another fifteen minutes left to discuss, Robert and Amy-Lynn did not come up with a conclusive argument. They discussed some ideas with R1 when she returned. They explained to R1 why the multiplication principle worked to list possibilities in a counting task. R1 encouraged them to think more about the problem at home and the session ended.

It is important to point out that this session sheds light on the great extent of time that Robert had spent working on *Towers problems* as a participant in the Rutgers Longitudinal study. He used many ideas from his understanding of how towers are counted, built or organized to solve other combinatorial tasks in several sessions (Session 10 -15). It is conjectured that Robert had a deep understanding about the towers activity as a result of spending extensive periods of time revisiting the *Towers problem* and its extensions.

6.6 Session 7 (3-1-1999): The Pizza problem

6.6.1 Setting and Background

Note: For this session Carolyn Maher is coded as R1, Regina Kiczek as R2 and Ralph Pantozzi as R3.

This session was a small group session conducted when Robert was in the eleventh grade. The other students present for this session were Stephanie, Amy-Lynn and Shelly. R1, the lead investigator, along with a graduate student, R2, and the classroom teacher, R3, observed the session. The session lasted over ninety minutes where students explored the following pizza problem:

A local pizza shop has asked us to help design a form to keep track of certain pizza choices. They offer a plain pizza that is cheese and tomato sauce. A customer can then select from the following toppings: pepper, sausage, mushrooms, and pepperoni. How many different choices for pizza does a customer have? List all the choices. Find a way to convince each other that you have accounted for all possible choices. Suppose a fifth topping, anchovies, were available. How many different choices for pizza does a customer now have? Why?

6.6.2 How many pizzas are possible with 4 toppings and 5 toppings?

R2 began the session with introducing herself to the students. She explained to the group that they would revisit the pizza problem that they had seen in their elementary grades. Robert said he remembered the problem from a while ago but he did not remember the solution. Every student was given a handout with the problem typed up on it. After reading the problem, Stephanie suggested that they create a tree diagram. Shelly mentioned that it is a combinations problem. Stephanie remarked before starting to work, “every time we do one of these, at the end we find an easier way and then every time we go back to do the one we have no idea how to do it. Except to start from scratch” (I 91).

This is an interesting statement by Stephanie that reveals that the longitudinal study participants often started from “scratch” to build a solution. That is, they actually started with listing possibilities in a combinatorial task rather than looking for a formula to apply as a starting point.

For the four toppings pizza, Robert quickly wrote an answer of 16 for total possibilities. It is conjectured that he used 2^4 to calculate this answer. Robert then listed systematically what the pizzas would look like. He counted the plain pizza as one option and all four toppings pizza as another option. He used P for Peppers, S for sausage, M for mushrooms and P_2 for pepperoni (Figure 6-19). He listed that one-topping pizzas would be 4 in number, P, S, M and P_2 . For the two-topping pizzas he made three little tree diagrams. First he paired P with S, M and P_2 . Then he paired S with M and P_2 and finally paired off M with P_2 . This final list had 6 pizzas with two toppings. For three-topping pizzas, he just listed a 4 without writing out all the combinations. Perhaps he used his total of 16. For all other cases, he had listed $1 + 4 + 6 + 1 = 12$ possibilities. It is noteworthy that Robert wrote 16 as an answer before he made his lists for second part of the problem.

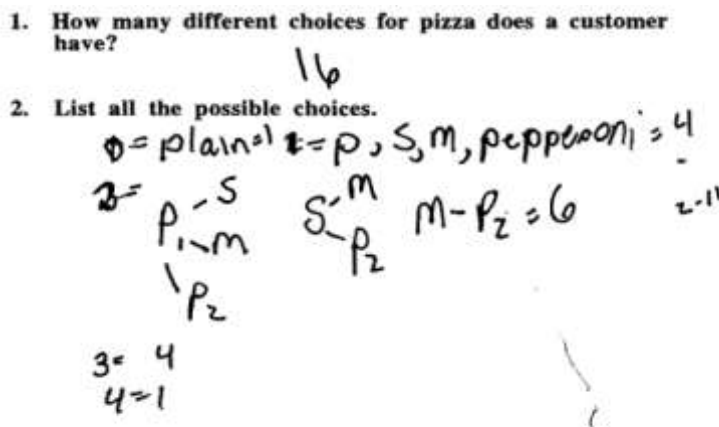


Figure 6-19 Robert's solution for 4-topping pizzas

At about 20 minutes into the session, all students finished counting their pizzas. They all agreed that for four toppings there are total 16 possibilities. Shelly asked Robert if he got 16 combinations as well and Robert replied that he did. At this point Stephanie invited Shelly to actually tally their lists. Together Shelly and Stephanie convinced each other that they had a correct answer as they discussed all of the sixteen combinations. Amy-Lynn worked on her paper silently and found 16 combinations as well. Then the group moved on to solve the problem for five-topping pizzas.

Interestingly, Shelly noted a pattern in number of possible pizzas. She realized that as number of toppings increased from 0 to 4, the number of pizzas possible followed the pattern: 1, 4, 6, 4, and 1. Shelly realized “this is thetriangle” (I 273). Stephanie agreed and conjectured that next row would be 1, 5, 10, 5, 1. Stephanie and Shelly then added their entries 1, 4, 6, 4, 1 to get a 16 and 1, 5, 10, 5, 1 to get a 32. They were not sure how to explain number of toppings on the Triangle at this point. However, they were convinced that there are 16 four topping pizzas and 32 five topping pizzas possible. At this point R2 came to the table. Shelly and Stephanie shared their answer with her. R2 then asked Robert if he had done the problem in the same way. Robert said he did. Addressing his tree-diagram, R2 noted that Robert used 0 toppings to represent a plain pizza. Robert then explained that he used a case by case listing where he first listed “one topping on each” and then “used pepper with everything” and then as pepper was used with everything already, he “just left it (pepper) out and went on to sausage” (I 353, Figure 6-19). Amy-Lynn also shared with R2 that she had used the “tree and the 1, 4, 6, 4, 1” (I 365). This is an important event in the session as it reveals that all students in the group were familiar with rows of Pascal’s Triangle. They all called it “the triangle”

rather than Pascal's Triangle. It is possible that they did not know a name for the triangular array of numbers at this point.

6.6.3 Why addition property of Pascal's Triangle works for pizzas and toppings

At about thirty minutes into the session, R1 came to the group. She encouraged them to think about how they get from one row of the Triangle to the next. Stephanie explained to R1 that they were using the addition property of Pascal's Triangle. R1 then encouraged them to explain in terms of pizza toppings why it was meaningful to add the entries in one row to generate entries in the next row. To begin with, Robert asked Stephanie whether 1 on top of her Pascal's Triangle was talking about zero toppings. Stephanie first replied that "No, that's one plain" and then pondered "guess that's zero toppings" (I 419). Shelly did not think that plain pizza was zero toppings as she considered "plain" to be a topping in itself. Stephanie clarified to Shelly that "plain" is not a topping as you cannot "add plain to the pepperoni and make a sausage all of a sudden" (I 458). Stephanie was eventually convinced that a plain pizza was zero toppings as Robert had suggested.

Stephanie and Shelly tried to understand why 1 and 3 from the third row produced a 4 in the fourth row of Pascal's Triangle. They were getting lost in the conversation when Robert suggested that they should try to "do it for one topping then two topping and then see if we see a pattern? You know we can do something based on the pattern" (I 477). Robert was suggesting that the group try a simpler problem and then build up to more involved cases, and look for a pattern. This is a strategy that Robert has used several times in the sessions observed. Robert often tried a simpler problem to understand a bigger problem or see a pattern. The group agreed to follow his suggestion.

Stephanie remarked that she did not understand how “you can make a pizza out of other pizzas” and if the problem “was applied to something else” she could look at it differently (I 483). Robert at this point suggested that they could think of the *Building Towers* activity. This is a critical moment in the session as Robert conjectured that *Building Towers* activity shared a similar underlying mathematical structure with the *Pizza problem*. At this point R3 gave the group some colored markers to work with so that they could draw out different colored towers on paper.

Stephanie invited R3 to join the group and talked at length to him explaining her ideas. She was not sure how the pizzas could be added. For instance, she reasoned, “This is one plain pizza, and this is four pizzas with one topping. If you add one plain pizza to three pizzas with one topping, you get, like, one pizza with no toppings and three pizzas with one topping” (I 524). This reflects that at the moment Stephanie did not see going down the row of Pascal’s Triangle as adding another topping. She argued that one plain pizza from the third row continued to be a plain pizza in the fourth row. As such, she did not feel comfortable adding unlike terms: no-topping pizzas with one-topping pizzas. However, later talking to R3 she stated that “we understand that when we’re doing the problem we’re adding toppings on, and that’s why this is working out here” but “this (1 and 3) doesn’t produce that (4)” (I 550). The later statement suggests that Stephanie was now adding a topping to the pizza as she went down the rows of the Triangle. However she was still unsure of why she could add the resulting pizzas together and make a group of four pizzas.

R3 shared with the group that his father owned a pizzeria. He explained that he was thinking of the problem as “you’ve seen what you can do with three toppings and

then my dad comes along and says well, I need you to put on a fourth topping” (I 547). R3 began to discuss with Stephanie the notation she was using for the Triangle and suggested, “put the pepperoni on top of the plain and maybe that’s one of the four or something” (I 554). Shelly and Stephanie liked this suggestion to think of adding a new topping to the pizzas. Shortly afterwards, Stephanie was able to explain addition rule of Pascal’s Triangle to R3. She stated that “this becomes three pizzas with three toppings, so then three pizzas with three toppings plus one pizza with three toppings is four pizzas with three toppings” (I 580). Stephanie had mixed up one topping with three toppings in her explanation. However, she was able to correct herself and re-explain that “three pizzas with one topping and the plain pizza becomes the pizza with the new topping” and therefore the 1 and 3 from the third row together make 4 one-topping pizzas (I 585).

6.6.4 Why is the number of total pizzas 2^n ?

At around fifty minutes into the session, R3 asked Robert about what he was writing on his paper. Robert explained that he was adding terms in rows of the triangle to see a pattern. He noted that two-topping pizzas were 4 or 2^2 in number and three-topping pizzas were 8 or 2^3 in number and so on. So, he concluded that “it’s two to however many toppings is the number of combinations there are for pizza toppings” (I 616). Robert had figured out that terms in the Pascal’s Triangle added up to 2^n which was also equal to the total number of pizzas possible. R3 asked Robert if he could explain why it was a 2 that needed to be raised to the power of n . Robert remarked “I remember something with towers that we did to find the total number of combinations, it was two to the something” (I 618). This is a significant moment in the session as Robert is starting to think of an isomorphism between the *Building Towers* activity and the *Pizza problem*.

Robert recalled that the *Building Towers* activity also led to answers with a pattern of 2^n . R3 pointed out that there were more than two toppings in the problem and encouraged Robert to explain why 2^n worked here. Robert was not quite sure at that point. Then R2 approached the group and Stephanie repeated her reasoning of why the addition rule of Pascal's Triangle made sense with pizzas and toppings. Robert did not justify the 2 in the 2^n for the pizza problem at this point.

However, later in this session at around eighty minutes, Robert explained to R1 that "four toppings would be a tower four high and then two colors would be with or without toppings" (I 982). Robert had connected the 2 in the 2^n to absence or presence of a topping and n to the number of toppings. At this point session was almost coming to an end. R1 had started to discuss with the group a concrete example to build on Robert's statement. However, Robert was not able to explain the connection in a clear manner as the group began to summarize their findings for the session.

6.6.5 How is building towers activity related to the pizza problem?

R2 then asked Stephanie if this activity with pizza toppings reminded her of something they had worked on earlier. Stephanie recalled that they had seen the Triangle while working on the *Towers* activity as Robert had mentioned. R2 then encouraged the group to think about how the pizza problem was related to the *Building Towers* activity. Stephanie began to explain that in the third row (1, 3, 3, 1 row), 1 would be a tower with "all blue" (I 756). Before Stephanie could explain the 3 next, Robert pitched in to say "two blue, one red" (I 757). Stephanie then asked Robert to finish his explanation. Robert explained that three tall towers were related to three-topping pizzas. Robert

pointed out that he was working with towers before but he was not sure how to use them in the pizza problem (Figure 6-20).

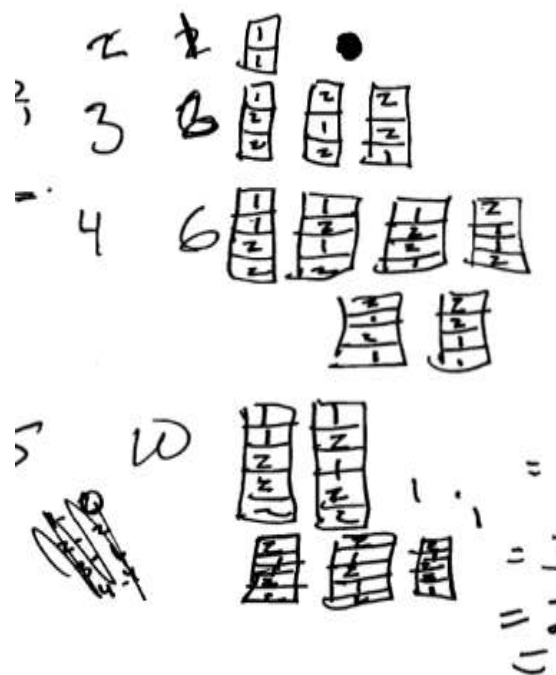


Figure 6-20 Robert's sketch of towers

It is interesting to note that Robert used 1 and 2 as colors to make his blocks. At a close observation of Figure 6-20 it becomes evident that it was an incomplete representation. Robert had only listed 1-1 as a tower that was two tall and he had incorrectly guessed that there were 6 three tall towers as he did not include 1-1-1 and 2-2-2 in his listing, and so on. This is an interesting occurrence as Robert commented that he did not see a connection between the *Towers* activity and the *Pizza problem* (I 767- 769). It is conjectured that an incorrect and incomplete representation for the two-colored towers hindered Robert from seeing an isomorphism between the two solutions.

As students discussed with R2 how the *Towers* activity might be related to the *Pizza problem*, Robert pointed out that Stephanie and Shelly had not correctly interpreted the 1, 1 in the first row as zero and one-topping pizzas respectively. He was offering an explanation to the hurdle Stephanie and Shelly faced in uncovering the relationship of Pascal's Triangle to *the pizza problem*. Stephanie and Shelly reacted with "That's why our thing wasn't working." and "Yeah Bobby (Robert), thanks for telling us before." (I 824 - 825). This instance reveals that Robert had uncovered the connection between Pascal's addition rule and the pizza toppings a while ago in the session. However, Robert did not stop Stephanie and Shelly from taking their time and working out the ideas for themselves. Robert had quietly worked on his paper while Stephanie had talked to the three researchers. He had deemed it appropriate to let his fellow students work on their own and shown faith in their capabilities.

Session 7 is an interesting session where Robert had just begun to make connections between the *Building Towers* activity, the *Pizza problem* and entries in the Pascal's Triangle. He shared ideas about how the sum of entries in a row of Pascal's triangle, yielded total number of pizzas possible. He also started to explain what 2^n meant in terms of the pizza toppings. It is evident that this session was critical in helping Robert build an understanding for the tri-fold isomorphism between the Unifix towers, the pizza toppings and the entries in Pascal's Triangle.

6.7 Session 6 (11-13-1998): Revisiting Towers Problems

6.7.1 Setting and Background

Note: Alice Alston is coded as R1 and Carolyn Maher is coded as R2 for this session.

This is a small group session that was conducted in the first half of Robert's eleventh grade. The session was led by R1, and R2 came to the visit the session towards the end. The other students who were present were Michelle, Magda, Angela, Sherly and Ashley. The session lasted approximately one hour and forty minutes. R1 asked students to recall some of the problems that they had worked on with the Rutgers team during their earlier grades. All students remember the *Towers problem*, Robert recalled the *Garage Door problem*, and Angela remembered the *Dice problem*.

6.7.2 How many four tall towers can you make with two colored blocks?

Ashley and Sherly shared with R1 that they had not worked with the Unifix cubes prior to this session. R1 then asked students to recall one of the *Towers problems* that they could remember. They all recalled problems where they had to find all combinations possible for towers of a certain height choosing from two colors. R1 decided to start with requesting the group to find all combinations for two colored towers that are four high. She paired students into three groups: Robert and Michelle, Angela and Magda, Sherly and Ashley. R1 emphasized that the students would be asked to justify to themselves and the group that they have found all possible combinations. Magda and Angela were the first to be done with their towers. They had found 14 towers as in Figure 6-21..



Figure 6-21 Angela and Magda find 14 towers

They explained to R1 that they had organized their towers in groups that had one blue, two blue and three blue. Then they created two solid towers. As R1 encouraged them to convince her that they have all the towers, Angela realized that they were missing two towers where blue and yellow alternate.

Then R1 moved to Michelle and Robert. Michelle said that they had “them all” as they “can’t think of any others” (J 176, 180). R1 asked them if they had a systematic way to build their towers. Michelle said that they did not have “any set plan.” However, Robert pointed out that they had a strategy. He explained that “we started with four yellow and zero blue, then we did three yellow and one blue” and then “two yellow and two blue” followed by “one yellow and three blue” and finally “four blue and no yellow” (J 186 – 188, Figure 6-22). Robert was systematically controlling for one color to produce his towers as he went from four yellow to zero yellow.



Figure 6-22 Robert and Michelle's towers

R1 then asked Robert and Michelle how they figured out the towers that had two blue in them. Robert explained that he had a strategy for that as well. He explained that he took two blues together on top of the tower and then began to separate them by one yellow and then two yellow blocks. So, Robert created towers in the following order: BBYY, BYBY, and BYYB (where B is blue and Y is yellow). Then he moved the two blue cubes down from the top most position to the second position. This led him to start with YBBY and insert a yellow in between the two blues to create YBYB. Finally he placed two blues in the third position and got YYBB. He pointed out that when BB was moved to the third position, no more yellows could be inserted, as towers were only four tall. As such, he expressed that he knew there were 6 four-tall towers that have two blues in them.

This is an interesting moment that lends insight into the binary notation that Robert used for the *World Series* session (Session 10) and the *Towers* problem (Session 8) that was described in earlier parts of this chapter. In both sessions, Robert had fixed one variable to stay together (two 1s or two blues) as he inserted the other variables in middle of them to generate other combinations. This is a consistent strategy that Robert

used to list combinations for different problem-solving tasks. It is evident that Robert trusted this method in several problems to account for all cases possible.

R1 then asked Robert and Michelle to think about how many towers would have exactly two blue blocks in them if the towers were only three tall. Robert first guessed 9 and then said he knew it was 8. R1 encouraged Robert and Michelle to check their conjectures and then went on to check Sherly and Ashley's work. Michelle used Unifix cubes and verified that there are 8 three-tall towers when choosing from two different colors. Angela and Magda drew towers on paper to verify that there were 8 towers for three tall. R1 then asked if the towers were five tall, how many combinations would be possible. Robert promptly replied that there would be 32. This suggests that Robert had a formula in his mind about how towers grow in number as the height is increased. However, he did not share the formula at this point.

6.7.3 How many five tall towers have exactly two of one color in them?

R1 then asked to the students to think about the number of the five tall towers that have exactly two blue blocks in them. At this point Michelle asked Robert if they should build something "Or you (Robert) can sit and figure it out 'cause you're a math like genius over here?" (J 296) This comment reflects that Michelle thought Robert might have a way to find the answer without building the towers with Unifix cubes. She also considered Robert to be an expert in solving the towers problems as she called him a math genius. As a response, Robert encouraged Michelle to herself guess how many towers would have exactly two blue blocks in them. He said that they could take her guess and verify it with their work. This is an interesting moment that reveals that Robert liked to encourage his peers to think of solutions to problems for themselves.

Robert initially guessed that for two tall towers, there are 2 towers with exactly two blue in them; for three tall there are 4 such towers; and for four tall there are 6 such towers. As number of towers went up by 2 each time, he conjectured that there would be 8 towers that are five tall and have exactly two blues in them. Robert then realized that for three-tall towers there are only 2 towers with exactly two blue and not 4 as he had thought. So, he changed his guess from 8 to a 10. However, he was not sure of his answer. At this point, Robert chose to build towers that are five tall with exactly two blues in them. This is a significant occurrence as Robert chose to build with Unifix cubes when he was not sure of the pattern he had conjectured on paper. Robert had decided it was helpful to use the concrete materials to actually see all the cases.

Eventually, Robert and Michelle decided that there were 10 such towers and they built them with the Unifix cubes. Once again Robert used the same strategy of keeping two blues together and inserting yellows in between the two blue blocks. Then R1 asked them to think about an answer for towers six tall that have exactly two blue. Robert guessed that there would be 15 such towers and then verified his answer by building the towers out (Figure 6-23). Robert again used the Unifix cubes to verify his conjecture. Robert did not disclose at this point how he guessed that six-tall towers would have 15 towers with exactly two blues in them.

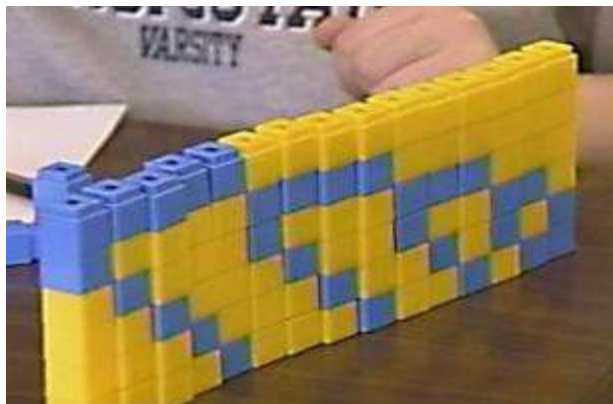


Figure 6-23 Robert and Michelle's towers six tall with exactly two blue

In the meantime, the other two groups of students were creating all towers possible for six tall in order to count the towers that had exactly two blues in them. As a result, it took them much longer to finish counting the cases. Then Robert generalized that there was a pattern as heights of the towers went up. He concluded that going down the second column which lists number of towers in Table 6-1, one had to progressively add the next positive integer. He explained that when first entry 0 was increased by 1, it produced result of 1 for the second row. Then, the second entry of 1 needed to be increased by 2 to get a 3 for the next entry and so on. As such, he conjectured that for towers that are seven tall, there would be $15 + 7$ or 22 towers that have two blues in them.

Table 6-2 Robert finds a pattern in towers with two blues

<i>Height of the tower</i>	<i>Number of towers with exactly two blues</i>
<i>1</i>	<i>0</i>
<i>2</i>	<i>1</i>
<i>3</i>	<i>3</i>

4	6
5	10
6	15

While the other groups worked on finding an answer to the five tall towers with exactly two blue in them, Robert talked to Michelle about towers seven high. Michelle and Robert chatted with each other about a horizontal pattern in Table 6-2 while waiting for the other two groups to catch up with them. After fifteen minutes, R1 invited all groups to share their answers with her. When Robert shared his answer with R1, he had found a horizontal pattern in Table 6-2. Robert explained to R1 that his vertical pattern where he found a rule between the successive outputs was not good enough. He stated that if they wanted to go “two hundred high” they could not keep on adding the next positive integer two hundred times (J 655). As such, Robert had decided that he needed a horizontal rule that would relate outputs to inputs.

Robert explained that the horizontal pattern he noticed was to multiply the heights of the tower by 0, 0.5, 1, 1.5, 2, 2.5, and so on (Figure 6-24). Robert then explained that height of the tower (h) needed to be multiplied by a factor of $\left(\frac{h}{2} - 0.5\right)$ to figure out how many towers have exactly two blues in them. This is a critical moment in the session. This is where Robert had first invented the formula $h\left(\frac{h}{2} - 0.5\right)$ that he had discussed with Brian in Session 11. The formula simplifies to $\frac{h(h-1)}{2}$ when the 2 in the denominator is distributed. In the first hour of this session, Robert had discovered for

himself the combinations formula where $C(n, 2) = \frac{n(n-1)}{2}$ when factorials are simplified.

This is a remarkable discovery for an eleventh grade student.



Figure 6-24 Robert's pattern for towers with exactly two blue

R1 then asked Robert to explain how he built his towers from one tall to the two tall case. Robert used the same strategy that Marjory had used in Session 15. He built duplicates of one-tall towers as Y, Y, B, and B. He then added one blue and one yellow on top of them to make YY, YB, BY, and BB. This generated four two-tall towers. Robert then explained this strategy to Angela and Magda. Later in the session R2 came in to the room with another observer. Michelle and Robert explained how Robert created the towers by creating duplicates and adding a yellow and a blue on the top. Then R1 shared with R2 that Robert had found a pattern for exactly two of a color in different height towers. Robert explained his vertical and horizontal patterns to R2 again. The session ended shortly afterwards.

7. Chapter 7: Elementary and Middle School Sessions

7.1 Introduction

In this chapter I analyze two sessions from Robert's fifth grade (Session 1 and 2) and three sessions from his seventh grade (Sessions 3, 4, and 5). All sessions are whole class discussion sessions. In Session 1 and Session 2, Robert's participation was captured on camera only briefly. As such, a full transcript of these two sessions is not included here. I use his paper work that was retrieved from the preserved archives at Rutgers to describe his solution and representations from Session 1 and Session 2. It is also important to point out that Sessions 3, 4 and 5 are whole class sessions that were carried out on three consecutive days: 12/13/1994, 12/14/1994 and 12/15/1994. It is anticipated that looking at Robert's work on consecutive days will shed light on how he modified his ideas and representations as he worked through particular problems. As three sessions in this chapter, Sessions 3, 4 and 5, build on each other, it is important to trace their events in a chronological manner. Consequently, for this chapter I will begin with the earliest session, Session 1, and then describe Sessions 2 through 5 in an ascending order.

7.2 Session 1 (2 – 26- 1993): Guess my towers

7.2.1 Setting and Background

Note: For this session Carolyn Maher is coded as R1 and Alice Alston is coded as R2.

This session was led by R1 and R2 as a whole class activity when Robert was in his fifth grade. Amy Martino, a graduate student at the time, was also present at this session. The researchers encouraged students to explain their ideas to each other as well

as to the researchers. In this session Robert worked with Michelle on the following problem:

You have been invited to participate in a TV Quiz Show and have the opportunity of win a vacation to Disney World. The game is played by choosing one of the four possibilities for winning and then picking a tower out of a covered box. If the tower you pick matches your choice, you win. You are told that the box contains all possible towers that are three tall that can be built when you select from cubes of two colors, red and yellow. You are given the following possibilities for a winning tower:

- 1. All cubes are exactly the same color; (Referred to as Option 1 in the chapter)*
 - 2. There is only one red cube; (Referred to as Option 2 in the chapter)*
 - 3. Exactly two cubes are red; (Referred to as Option 3 in the chapter)*
 - 4. At least two cubes are yellow. (Referred to as Option 4 in the chapter)*
- 1. Which choice would you make and why would this choice be better than any of the others?*

Assuming you won, you can play again for the grand prize which means you can take a friend to Disney World. But now your box has all possible towers that are four tall (built by selecting from the two colors yellow and red). You are to select from the same four possibilities for winning a tower.

- 2. Which choice would you make this time and why would this choice be better than any of the others?*

In this session the video camera is mainly focused on Stephanie and Matt. Briefly at the end of the session, camera captures Robert and Michelle's table.

7.2.2 Stephanie's "Family Tree"

The session began with R2 explaining to the class what the problem meant. The students were given sheets of papers with the problem typed on it. The students were also provided with red and yellow Unifix cubes. Stephanie and Matt worked together on the problem. They found all towers that were three tall and four tall to answer questions in the problem. Stephanie, working with Matt, came up with a strategy to account for all possible towers of a certain height when choosing from two colors. She took towers of a certain height and made duplicates for them. For example, in one-tall case there are only two towers: Y and R (where Y is yellow and R is red). Stephanie made duplicates of one-tall towers as Y, Y, R, and R. Then she added a yellow block to one of the two

duplicates and a red block to the other. As such, she produced four towers that were two tall as: YY, YR, RY, and RR. She carried out the same strategy again to make eight towers that are three tall using duplicates for the two-tall towers. She called the starting towers as “parents” and the resulting towers as “children.” Stephanie used this induction technique and produced 16 towers for four-tall towers by building on three-tall towers.



Figure 7-1 Stephanie's Family Tree for towers

Robert and Michelle were working together in this session. They were also trying to find all towers that are four tall. Together they only found 12 four-tall towers. R1 invited Stephanie and Matt to share their ideas with Robert and Michelle. Stephanie explained her induction idea to Robert and Michelle using Unifix cubes and constructed her “family tree” as in Figure 7-1. At the base of the “tree” she put one red and one yellow block. The next set of “branches” had RR, RY, YR and YY towers. She continued to list towers three and four high in the same manner. Later Stephanie was invited to share her strategy with the entire class. This is a significant event as Robert later used this induction strategy several times to produce towers of a certain height. In the sessions observed, this was the first instance where he observed the induction strategy being used to systematically list possibilities for two-colored towers. This episode

reveals that students in the longitudinal study openly shared their ideas and often absorbed each other's ideas. Robert had assimilated Stephanie's induction technique as an important and reliable counting tool for years to come (Sessions 12 – 15).

7.2.3 Robert's solution to the problem

Robert worked on paper and solved all parts of this problem. It is not clear if he finished all the work during the session or later at home. I use his written up solutions from the day to analyze his work for the problem. Robert figured out that for three-tall towers the Option 4 in the problem that called for at least two yellow blocks had most possibilities (Figure 7-2). He wrote on the paper "I would pick number 4 because it has the most possibilities" (Appendix: Paper work from 2-26-1993).

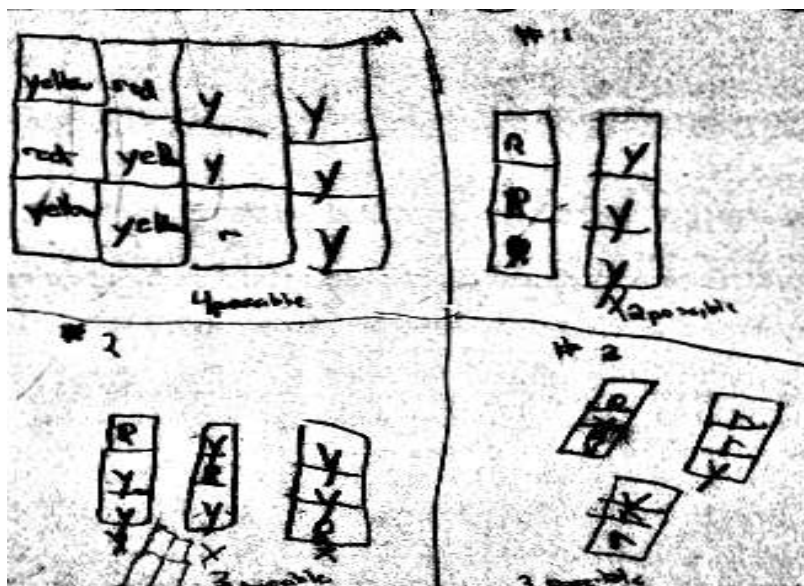


Figure 7-2 Robert's solution for three-tall towers

It is interesting to note in Figure 7-2 that Robert listed the towers as RYY, YRY and YYR for Option 2 (only one red) of the problem. He listed the three towers by systematically moving the R block from the first to the second to the third position in the tower. For Option 3 (exactly two red), he listed the towers as RYR, RRY and YRR. He did not systematically move the Y down in position this time. This suggests that Robert did not use equivalence of the case of two reds to the case of one yellow to create this list. It is conjectured that Robert used a trial and error method to list all towers with two reds in them. For Option 4 (at least two yellow), he found 4 towers as: YRY, RYY, YYR and YYY. Again, there is no visible strategy in this listing other than trial and error. Robert had correctly interpreted that at least two yellow in three-tall towers would mean two yellow blocks or three yellow blocks. As such, he had a correctly picked Option 4 as solution for the three-tall case.

For the four-tall case, Robert listed two possibilities for Option 1 (all cubes same color) by writing a big Y and a big R across four blocks he drew in Figure 7-3. For Option 2 (exactly one red), he listed his towers as RYYY, YRYY, YYRY, and YYR by moving R down one position each time. For Option 3 (exactly two red), he listed 6 towers as RYRY, RRY, RYYR, YRRY, YYRR, and YRYR (Figure 7-3). It seems that Robert used the “opposites” strategy to list these six towers. He took the opposite of third tower RYYR to write YRRY as the fourth tower next to it. Then, moving out to the second tower RRY, he created the opposite YYRR and wrote it as the fifth tower. Finally he moved to the leftmost tower and created opposite of the first tower RYRY as YRYR and listed it as the sixth tower. For Option 4 (at least two yellow) he found 10 towers (Figure 7-3). There does not seem to be a systematic counting pattern about how

he listed these 10 towers. There are some towers that were written together as opposites (YRYR and RYRY), but overall, there seems to be no particular order in the list. However, as the video did not capture Robert's listing, it is not possible to make conclusions about a strategy he might have implemented. This reflects that in his fifth grade, Robert was just beginning to discover some techniques for listing out the cases in a counting task.

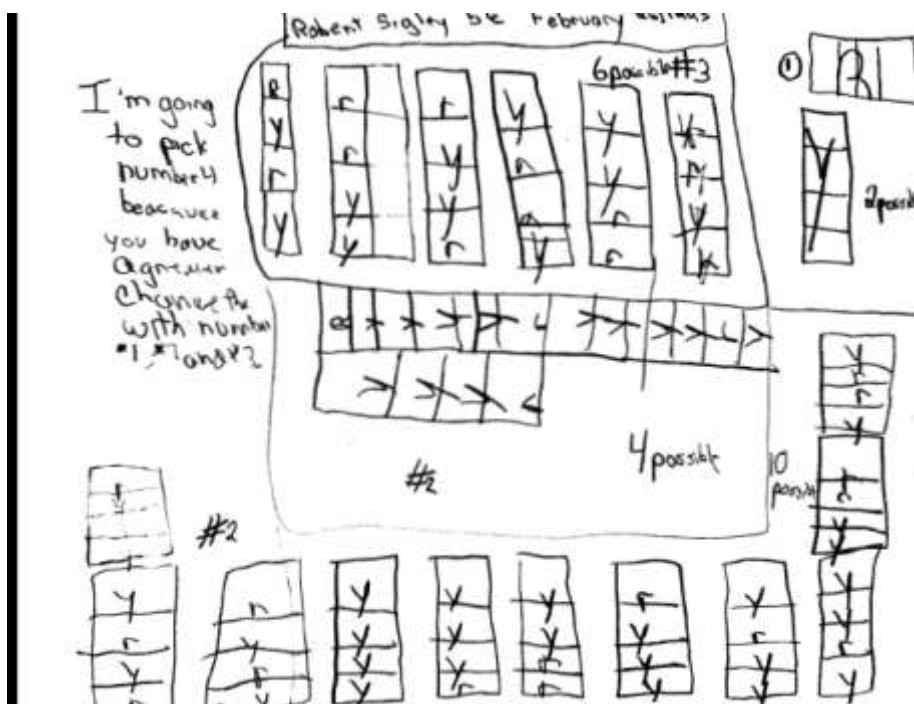


Figure 7-3 Robert's solution for four-tall towers

Interestingly in the paperwork from this day, there is a sheet of solution for towers that are five tall. Other students also have work for five-tall case on their sheets of work. It is not clear if students were asked to think about extending the problem to five-tall towers or if some of them worked on five-tall towers on their own. Again Robert listed

Option 1 (same color) as a big Y and a big R across five blocks he drew for the two towers. For Option 2 (one red), he listed five towers by moving R down in position each time as RYYYY, YRYYY, YYRYY, YYYRY and YYYR. This is one of the strategies that Robert used repetitively in this session when there was only one R in the towers.

For Option 3 (exactly two red), he created 11 towers as: RRYYY, RYRYY, RYYRY, RYYR, YRRYY, YRYRY, YRYR, YYRRY, YYRYR, YYYRR and YRYRY. In the first four towers listed here, Robert started with RR together on the top position and then gradually inserted one, two and three Ys in between them (RRYYY, RYRYY, RYYRY, RYYR). Then, he moved RR to the second position as in YRRYY and again inserted one and two Ys in between them (YRYRY and YRYRY). Next, he moved RR to the third position as in YYRRY. This time he inserted one Y in between them to get YYRYR. Finally he moved RR to the bottom most position to get YYYRR. For some reason he repeated the tower of YRYRY. This listing for five-tall towers is perhaps a foundation of the counting scheme that Robert used several times in his later years (Sessions 8 and 10). This episode shows evidence of his using a system to monitor keeping track of possibilities that had exactly two of one element. Later sessions (Sessions 8 and 10) revealed that Robert confidently used this method of listing when exactly two of one color was to be selected. For Option 4 (at least two yellow), he created 14 towers that did not have an observable pattern. It appears that Robert did not get to finish Option 4 for five-tall towers as he did not write a conclusion for the part and left a lot of space blank on the sheet.

7.3 Session 2 (3-1-1993): The Pizza Problem

7.3.1 Setting and Background

Note: Carolyn Maher is coded as R1 for this session and Alice Alston is coded R2.

This session was also conducted when Robert was in his fifth grade. R1 and R2 led the session. After the researchers introduced the problem, students worked in small groups. The problem being discussed was as follows.

Capri Pizza in Kenilworth has asked us to help them design a form to keep track of certain pizza sales. Their standard “plain” pizza contains cheese. On this cheese pizza, one or two toppings could be added to either half of the plain pie or the whole pie. How many choices do customers have if they could choose from two different toppings (sausage and pepperoni) that could be placed on either the whole cheese pizza or half of the cheese pizza? List all possibilities. Show your plan for determining these choices. Convince us that there could be no more.

The students were provided with sheets of paper with the problem typed out. In this session, Robert shared his table with Ankur, Brian, Michael, Michelle, Amy-Lynn and Romina. In this session Robert listed out some of the pizzas on his sheet of paper and contributed to the group solution and a group representation for the solution. The students did not finish the problem at the end of this 35 minute session; however, they came up with partial solutions that are discussed here.

7.3.2 Group Solution for the problem

At the beginning of the session R1 asked students to read the problem. She clarified that plain pizza includes cheese pizza for the problem. Soon afterwards the students started to work in their groups. At Robert’s table, Ankur started listing pizzas and counting them on his fingers. As Ankur was listing possibilities, other students at the table suggested other possible pizzas. As they did not reach a consensus talking about the pizzas verbally, the group decided to record the possibilities on paper. This is an

interesting occurrence as students' first attempt to solve the problem was a verbal discussion of possibilities. When they did not find it convenient to keep track of the pizzas orally, they all began to write on their separate sheets of paper. Ankur continued to list some pizzas verbally as he wrote down his list. The group asked R2 if they could mix pepperoni and sausage together, and she said yes. The group discussed that it would be good to make one sheet of results for the table to be turned in at end of the session. Romina suggested that everybody should have their own list even though they were going to turn in one common solution. Ankur took the lead and directed Michael to draw out the pizzas, and Amy-Lynn volunteered to write out the list of possibilities. The group considered drawing out with pencil and then decided to go on and use markers. The group gave instructions to Michael as to what pizzas he should draw out. Ankur suggested that they should make pies for the pizzas and as a result Michael started to sketch circles for pizzas.

Romina decided to finish her own list and got 8 pizzas. Her list included the following pizzas as in Figure 7-4.

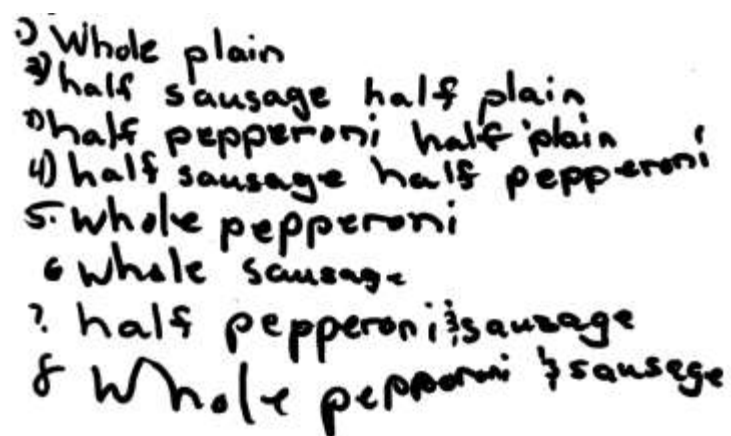
- 
- A handwritten list of 8 pizzas, numbered 1 through 8. The text is written in black ink on a white background. The list is as follows:
- 1 Whole plain
 - 2 half sausage half plain
 - 3 half pepperoni half plain
 - 4 half sausage half pepperoni
 - 5 whole pepperoni
 - 6 whole sausage
 - 7 half pepperoni 1/2 sausage
 - 8 whole pepperoni 1/2 sausage

Figure 7-4 Romina's list of 8 pizzas

Brian pointed out to Romina that there are more pizzas than the ones she had listed as she did not consider the pizza with pepperoni and sausage mixed on half and just pepperoni on the other half. Amy-Lynn continued to write down possibilities on her sheet of paper as well and Michael waited for further instructions from the group to draw more pies.

At this point, around thirteen minutes into the session, Ankur had come up with his own list. He had started to consider quarters of the pizzas (Figure 7-5). For instance, Ankur was considering a pizza where a quarter was cheese and a quarter was pepperoni and the other half was sausage.



Figure 7-5 Ankur's list of pizzas

Robert pointed out to Ankur that pepperoni covers the cheese and it is not reasonable to consider a pizza with cheese and pepperoni as being different from a pepperoni pizza. In the meantime Brian continued to list his pizzas and Romina watched

his list. She tried to check Brian's list with him. Romina also reminded Ankur that plain pizza was same as cheese pizza. Ankur started to list half "plain and sausage" and half pepperoni as a possibility and Brian said that he was thinking of the same thing. As a result, Brian came up with 12 possible pizzas. In addition to Romina's list of 8 pizzas in Figure 7-4, he also had the following four pizzas in his list: one half pepperoni and sausage with other half pepperoni; one half pepperoni and sausage with other half sausage; one half pepperoni and plain and other half sausage; and, one half sausage and plain with other half pepperoni. Brian considered "plain" as a topping and he listed pepperoni and plain as a different option from just pepperoni. Michelle recorded the same list of 12 pizzas on her paper and at the end Michael had pictures for these 12 pizzas on two sheets of paper as in Figure 7-6 and Figure 7-7 .

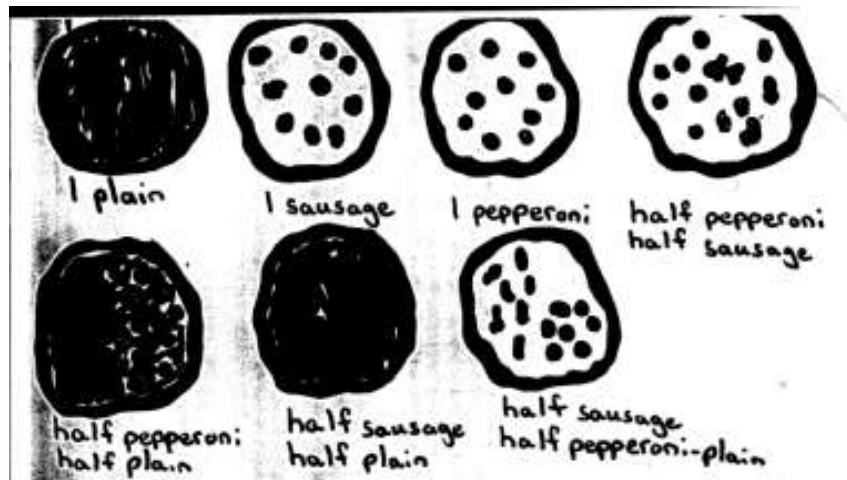


Figure 7-6 Group Solution - Page 1

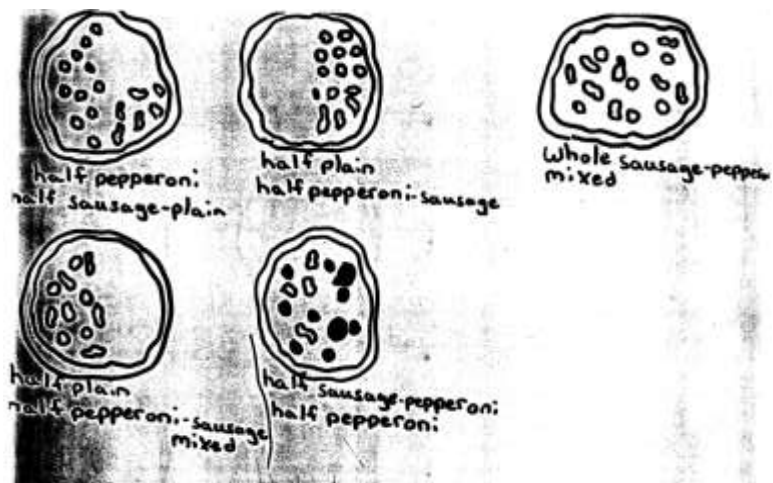
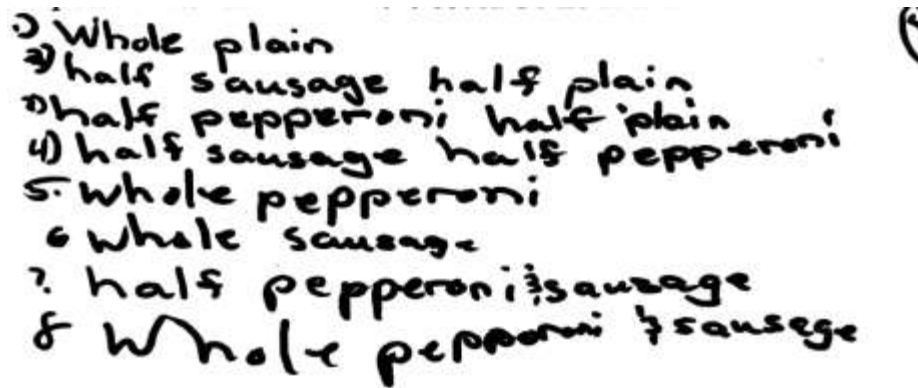


Figure 7-7 Group Solution - Page 2

In contrast, Robert's representation was the listing of eight pizzas as indicated in Figure 7-8. He was missing two pizzas where one half had two toppings, pepperoni and sausage together, and the other half had one topping, pepperoni or sausage. It is interesting that Robert did not create a list of 12 pizzas like Michelle and Brian as he believed that plain and sausage is same as sausage. However, he did not offer an objection to the group's picture with 12 pies in it.



Handwritten list of eight pizzas:

- 1) Whole plain
- 2) half sausage half plain
- 3) half pepperoni half plain
- 4) half sausage half pepperoni
- 5) whole pepperoni
- 6) whole sausage
- 7) half pepperoni & sausage
- 8) whole pepperoni & sausage

Figure 7-8 Robert's list for pizzas

R1 pointed to the class that this session was getting them ready for a more challenging pizza problem they would work on in next few sessions. She encouraged the class to think about how they would convince the researchers when they meet next time that they have found all possible pizzas.

7.4 Session 3 (12 -13 -1994): Garage Door problem

7.4.1 Setting and Background

Note: For this session Carolyn Maher is coded as R1 and Alice Alston is coded as R2.

The classroom teacher Mrs. Toy is referred to as T1 in the session.

This session was conducted when Robert was in the seventh grade. R1 and R2 observed the students working while the classroom teacher introduced the problems to the class. This session ran for about an hour and thirty minutes. The problem that students had worked on before this session was called the *Candles Arrangement problem* where they had to figure out how many five-candle arrangements are possible using two colors of candles: red and golden. In this session, Robert shared the table with Magda, Amy-Lynn and Brian. For this session, the class was given the following *Garage Door problem*.

Part I: Mrs. O'Brien lost her garage door remote control gadget. She was told that she could purchase a new gadget on which she could enter the code of her garage door. The remote control gadgets are designed with nine switches, each of which must be set to either low, medium or high. The positions of these switches determine the code. Mrs. O'Brien wanted to set the new gadget to match the code of her garage door opener. Unfortunately, Mrs. O'Brien did not keep a record of the original garage door opener combination. So she sat on the lawn for half an hour and entered different codes to see if she could find the one that matched the garage door. What do you think happened? Explain.

Part II: Would you advise Mrs. O'Brien how many combinations she might need to try and how long it might take? Prepare an argument to convince her.

Part III: Mrs. O'Brien remembered that the first of the nine switches had a high position and the last had a low position. Does this help? Explain.

Part IV: Make up another version of the Garage Door problem for the class to solve. Include on a separate sheet your solution.

7.4.2 Why use binary strings for Candles Arrangements problem?

The session began with T1 discussing solutions to the *Candles Arrangement problem* from the day before. Some students had concluded that for five candle arrangements using red and golden colors there were 30 combinations while others had reasoned that there were 32 combinations. Michelle shared with T1 that students in her group were using binary notation where 0 was used to represent the color red and 1 was

used to represent the color golden. Robert whispered at his table that he was the one who had initially observed that binary notation could be used for the candles arrangements to list combinations. T1 pointed out that those students who chose 30 as an answer had interpreted the problem to require both colors in each arrangement. Therefore they had not included all red or all golden candles as possible arrangements.

Then, T1 enquired how students had come to think of binary notation for the solution. Robert raised his hand and when called upon, he explained, “I saw all these five-digit numbers so I thought maybe you should just change it cause there’s only two different numbers in it and it’s base two so that means you wanna use like 0, 1 and there’s two different colors red and gold” (K 55). Robert was sharing that he was able to see a similarity between the binary numbers that use 0, 1 and the two colors red and golden. Robert further explained to T1 that as five candles were needed in an arrangement, each binary string needed five digits in it. Robert also shared that he had put down 32 as a final answer as he included the case of all reds and all golden.

It is conjectured that Robert had used the binary notation before these sessions were conducted. This reveals that Robert knew how to confidently use the binary notation to list cases as early as his seventh grade. Also, it is evident that he had a clear understanding behind how the strings of 0s and 1s mapped to red and golden candles for the *Candles Arrangement problem*.

7.4.3 Robert finds solution to the Garage Door problem

T1 went on to introduce the *Garage Door problem* to the class. Robert and Amy-Lynn shared with the class real life examples set in a context in which a teacher was unable to open her garage door. As groups started to work on the new problem, Robert

claimed almost right away, “I got the answer. There would be nine different combinations for it. And you put low, high, and medium. So, it’d be 1395” (K 141). When his group asked for explanation, Robert said “you can only use one and zero” and began to justify that binary notation would work in this problem as well (K 145). The group pointed out to him that there are more than two possibilities for each switch in the problem as it can be set to high, medium or low. Robert agreed with his peers that binary notation would not apply to the *Garage Door problem*.

After thinking about the problem a little longer, Robert explained to his group that “there’s low, medium or high and there’s nine different switches, so there could be nine possibilities, there could be a lot of possibilities where the switches could be”... “so the low medium high is three and the nine switches so it’s three to the ninth power” (K 172 - 174). Robert was trying to devise a formula to count the possibilities. He had not listed the cases as he was able to project that there were too many combinations possible. He understood that exponents would play a role in this counting and correctly guessed that there would be $3^9 = 19683$ combinations.

R2 encouraged Robert to explain his answer to the group and write down some of his ideas. Robert explained that “there’s three possibilities that you can switch each switch on. If there was one switch, it’d be low, medium or high, it’d be three to the one, there’d be only three possibilities. But there’s nine switches so you gotta put three to the nine” (K 194). Robert was using an induction argument to justify his solution. He started with convincing R2 of the simplest case where there was only one switch. R2 then asked him how many possibilities were there for two switches and Robert explained that there would be 3^2 or 9 possibilities for two switches. He listed the nine possibilities

when R2 encouraged him to do so. The students at the table agreed with Robert's solution.

Then, Robert drew a tree diagram to show that the first switch had three branches of possibilities, high, medium, and low. He illustrated that each branch will then get another set of three branches for two switches (Figure 7-9). He explained his tree diagram to R1 as well when she came around to talk to him. R1 asked him to write up his solution. The group took time to write out their solution on the overhead as Amy-Lynn drew a tree diagram on the overhead. It is significant that Robert was able to find a solution to the Garage door problem so quickly. He explained his ideas efficiently to his group and the researchers using the tree diagram. As Robert was able to quickly find a solution to the problem before sketching any representation on paper, it is conjectured that perhaps Robert had a clear mental representation for switches of the garage door and their positions.

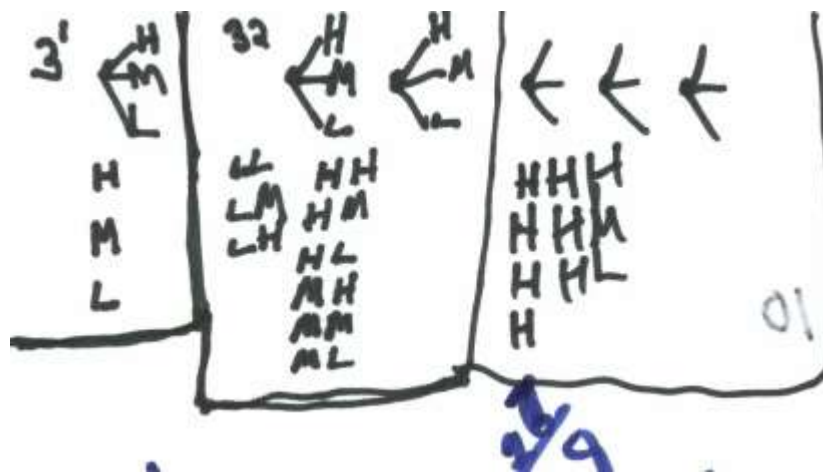


Figure 7-9 Robert's tree diagram

7.4.4 How long will it take Mrs. O'Brien to open the door?

R1 encouraged students at Robert's table to think about how long it would take Mrs. O'Brien in the problem to try out all the combinations. They all guessed that it would take her a very long time. Brian guessed that it might be a whole year before she can try every combination. Robert guessed that if Mrs. O'Brien worked 12 hours a day and changed the combination every minute, it would take her 27 days provided she makes no repetitions. He explained that he did $19683 \div 60$, sixty minutes in an hour and got 328 hours (K 577). Then he divided 328 hours by 12 as he reasoned that in real life a person could only work 12 hours in a day. This gave him an approximate answer of 27 days. The group agreed with Robert's reasoning. They rounded up the answer to a month to account for weekends or days that Mrs. O'Brien might take off.

Then R1 guided the group to work on the third part of the problem where Mrs. O'Brien remembered what the first and last switch's positions were. Robert quickly guessed that it would 2187. Brian asked Robert if the problem now just took "away two switches" and Robert said "No, she just knows the position of two switches" (K 678 – 679). Robert explained that this helps because in the third part, there were "only seven switches to figure out so makes it less possibilities" (K 681). He further explained that for this part he took 3^7 to find that there are 2187 combinations. Robert explained his reasoning to R2 when she requested him to do so.

For the fourth part of the problem, Robert devised a problem that stated his mom, Mrs. Sigley, had forgotten the combination to their garage door that had keypad like a telephone pad. As such, she has to punch in numbers 0 through 9 to figure out the code. He further added to the problem that Mrs. Sigley already knew what the first two numbers in the nine or ten digit combination were. Robert further said they should add

the pound and the asterisk symbol to the code so that there are 12 things to choose from for each spot in the combination. This challenging problem that Robert created suggests that he perhaps knew how to generalize his solution for the *Garage Door Problem* from three settings of high, medium and low to a more complicated case with twelve possibilities.

The session ended shortly afterwards with the researchers encouraging students to summarize their work on sheets of paper. It is important to note that Robert was extremely quick with the *Garage Door problem*. He was clearly drawing upon the skills he had previously assimilated for combinatorial tasks like using the multiplication principle, building up an inductive argument or using the binary notation to account for possibilities. He also was able to verify his solution by drawing out representations for simpler cases to convince his peers of his ideas.

7.5 Session 4 (12/14/1994): Candles Arrangement

7.5.1 Setting and Background

Note: Carolyn Maher is coded as R1 for this session, Alice Alston is coded as R2.

This is the second session from Robert's seventh grade. Robert shared his table with Amy-Lynn, Magda, and Brian for the day. In this session the group discussed the solution to *Candles Arrangements Problem* which was stated as follows:

Mr. Cliff is planning to use a theme of light to decorate the windows of Harding School for the Holiday Season. He usually creates designs for the windows on the Boulevard side of the building only. He thinks a design of 5 candles in 2 different colors (red candles or gold candles) will be most appropriate in each window.

*1) How many different 5 candle arrangements can he make?
Explain how you know that you have found all possible arrangements.*

2) Mr. Cliff plans to decorate 48 windows in the front of the school with each window having a different candle order. Will he be able to decorate them all differently?

3) Mr. Cliff may be looking for a different idea than candles this year and would like your help in designing a holiday theme for the Boulevard windows. Draw a large version of your design as well as plans of how it should be organized in the 48 windows.

R1 pointed out that the class had worked on this problem two days prior to the session (12/12/1994). She wanted students to discuss and convince her in this session how they knew that there were only 32 possible combinations.

7.5.2 Why there are 32 candle arrangements for five candles and two colors?

Angela responded first by saying that everybody used the binary system. She said that Michael and Robert helped her think of red as 1 and gold as 0 (L 6 - 10). R1 asked the class “how you know you had them all?” (L 11) Robert responded that “base two uses two different numbers zero and one and when you’re using the candles you’re using two different colors red and green, so it’s the same” (L12). Robert was explaining that base 2 corresponds well with two colors of the candles as there are two digits, 0 and 1, in the binary system and each can be assigned to a color. R1 again asked “How do you know you’ve accounted for all possibilities?” (L 13) and Robert replied, “Well, if the system is right and that’s all the numbers then all the candles would be right, the same amount.” (L 14) Robert was pointing out that he had listed all numbers that had five digit representations in base 2. Robert conjectured that as he had listed all numbers in base 2 that were five-digit strings, it would produce all possible five-candle arrangements.

R1 then discussed with the class a student’s solution where he/she had used g for gold and r for red to list the candle arrangements. Then, Michael pointed out that R1 was also holding one of his solutions. R1 invited Michael to explain his work. Michael

commented that “You would have to use the binary system” (L 39). This reflects that Michael also agreed that use of binary numbers to list possible candle arrangements was appropriate. R1 then asked the class if there were seven candles, how many arrangements were possible. Robert raised his hand and told Brian at his table that it would be 256 (L 41). R1 asked if there was a pattern in how many arrangements could be made with a certain number of candles. Robert and Brian both replied that it “doubles” (L 51).

7.5.3 What if there are three colored candles?

R2 asked the class if there were three colors and five candles, how many arrangements would be possible. Amy-Lynn replied that “you do base three” then (L 56). Robert further stated that the pattern would be “three, nine, twenty seven, eighty one” and Brian added that “it would be tripled” (L 58 – 59). Robert then conjectured that three colors for candles were same as the *Garage Door problem* and remarked, “cause there was three switches, high, medium and low in each, and three to the ninth is nineteen thousand six hundred eighty three (19683)” (L 60). This is a critical moment in the session as Robert had pointed out the isomorphism between the two problems, *Candles Arrangements* with three colors and the *Garage Door problem* with remarkable ease. R2 then asked the class if there were four colors what would happen and Robert replied that it would go “four, sixteen, sixty four” (L 61 – 64). Robert was changing the base to equal number of colors in order to obtain a solution to the *Candles Arrangements* problem.

7.5.4 How many arrangements are there for seven candles choosing from two colors?

R1 then invited the class to discuss their solutions to a problem where there are seven candles and two colors to choose from. R1 allowed students some time to discuss the problem in their groups. After about fifteen minutes, R1 called upon the students to discuss solutions with the whole class. First, Michelle explained that her group used the binary notation and they “just add zeros until there’s seven...then at the end of that you use the counting chart and we have the number right here you get 127” (L 90). Michelle was explaining that they added zeros to the binary strings so that they had strings of length seven and then counted the 127 possibilities. As camera was focused on Robert’s table, it is not clear from the video observed what Michelle’s list looked like. At this point, another student (not visible on camera) pointed out that he/she got a 128. R1 then again asked the class why they believed 128 or 127 was the correct answer. Stephanie explained that in binary system, things grow as 1, 2, 4, 8, 16, 32, 64, 128 (L 125). R1 asked all students “What makes the binary system go, one, two, four, eight, sixteen?” (L 130) Stephanie replied that “That’s how many there are” (L 131). At this point, Amy-Lynn raised her hand and when R1 invited her to share her ideas. Amy-Lynn said that each column in the list of binary strings had a “pattern to it” and “that’s how it’s done” (L 135). A few minutes later, Robert tried to explain at his table that they had already figured out that there were 32 combinations for five candles. He suggested that when 32 is doubled, it gave 64 combinations for six candles. He explained that when the 64 combinations for six candles were switched around, replacing 0s with 1s and vice versa, it made 128 combinations for seven candles (L 148). Amy-Lynn encouraged Robert to share his answer and he said that he would wait for the researchers to call upon him. He

told Amy-Lynn the answer is 2^7 or 128 as “There’s seven candles, two different colors” (L 169). Robert repeated this answer to the class when R1 called upon him (L 186).

At the end of the session, R1 encouraged students to write up their solutions and think about the problem’s extension to more than two colors.

7.6 Session 5 (12/15/1994): Candles Arrangement

7.6.1 Setting and Background

Note: Carolyn Maher is coded as R1 and Alice Alston is coded as R2 for this session

This is the third whole class session selected from Robert’s seventh grade. Robert shared the table with Brian, Magda, Michelle and Amy-Lynn for this day. R1 asked the students to recall the *Candles Arrangement problem* that they had seen the day before (Session 4). Amy-Lynn summarized that they had to find all combinations for six candles choosing from candles of two colors. Amy-Lynn also recalled that students had different answers of 128, 126 and 127 the day before. This session lasted two hours.

7.6.2 The dilemma: How many “places” to have in binary notation for candles?

At the beginning of the session Robert pointed out to R1 that there was a dilemma from the previous day that the class had not finished discussing. Robert explained that the dilemma involved how many digits or “places” an exponent of 2 should be assigned in binary notation (M 51). For instance, Robert explained that “two to the first it was really, it was supposed to be one but there’s really two, one and zero” (M 51). Robert was explaining that 2^1 is represented as 10 in the binary notation, taking up two spaces. However, relating 2^1 to *Candle Arrangements problem*, Robert explained that 2^1 represented the case where only one candle was chosen for the arrangement. As such,

Robert pointed out that the number of digits (two in 10) did not correspond with number of candles chosen (one in 2^1) (M 53). Robert further explained that “two to the first, it’s supposed to be a one digit number but it’s really...like say this is a one...one, zero. And two to the second that was a one, it would be one, zero, zero...if it’s two to the third then there’s four places here and two to the fourth there’s five places...” (M 64) Robert was referring to the fact that each binary string has a place saved for 2^0 at the end of it. For instance, he explained that 2^4 has five digits in binary notation as it has four zeros after a one and is written as 10000.

Robert suggested that to get digits in binary strings to match the number of candles, they should skip “two to the zero, cross that one (last zero) out, two to the first it would be just one number” (M 87). Robert was suggesting eliminating the last space for 2^0 so that 2^1 could be represented as 1 rather than 10 in the binary system. A few minutes later, Michael responded to this suggestion and remarked that you can’t take away the space for 2^0 as “you need the two to the zero to make odd numbers” (M 113). Michael was projecting that in absence of 2^0 , it would not be possible to represent odd digits as binary strings. This reveals that Michael had a good idea of how binary digits can be converted to base ten numbers and vice versa.

R1 encouraged Robert to summarize his dilemma and asked Robert, “if we have six positions for candles, it can be represented by two the sixth or it can’t?” (M 95) Robert replied that “it matters because if you like do it with the twos to the zero in it, it can’t...If you don’t do with two to the zero then you can. But then if you did it with two to the zero, then you just have to go up to five. It would be the same.” (M 96) Robert reflected that if the binary numbers were not to have a spot for 2^0 , then number of candles

in the arrangement would be same as number of digits in the binary notation. However, if a spot was used for 2^0 then there would be one extra digit than the candles in the arrangements. He said that “It would be the same” implying that it could work either way as long as one could keep track of whether the 2^0 was included or not. R1 then encouraged all students to work in their groups and try to address the dilemma that Robert had shared with them. She provided them with a hint and said, “if you can’t solve a hard problem, think of a simpler problem that’s like it.” (M 144)

Working in his group, Robert suggested that binary digits could have a decimal before the last space for zero to address the dilemma. Robert and Amy-Lynn started writing the following statements involving zero on paper: $2 + 0 = 2$; $2 - 0 = 2$; $2 \times 0 = 0$ and $2 \div 0 = 0$. Amy-Lynn pointed out to Robert that “zeros like counts as nothing here (addition and subtraction) but they count as something here (multiplication and division)” (M 182). Amy-Lynn was reflecting that adding or subtracting a zero had no effect on a number in base ten. Robert agreed with her and concluded that zeros “count as nothing in our system (base ten numbers) but they count as something in binary system” (M 183). Robert had concluded that place for zero in the binary string was meaningful and that zero was “something” in binary notation.

7.6.3 What is 2 divided by 0?

Robert then started to reflect on division property of zero. Amy-Lynn had written on paper that 2 divided by 0 is 0. At around 35 minutes in to the session R1 came to Robert’s table and discussed the four statements that Amy-Lynn had listed. R1 pointed out that 2 divided by 0 is not a zero. She encouraged Robert to think of simpler cases like 2 divided by one half, one third, one tenth, one hundredth and one millionth (M 234 –

264). Robert calculated the answers to these divisions accurately without a calculator. R1 asked him “what’s happening as you are getting closer and closer to zero” and Robert replied that “It (the final answer) goes up” (M 262 – 263). This is a significant event in the session as it reveals that as early as seventh grade, Robert was discussing beginning ideas of a limit and the concept of infinity.

Amy-Lynn interpreted that 2 divided by 0 implied as “you have nothing, and you want to divide it by two, you can’t ‘cause there is nothing to divide it by” (M 268). R1 responded that if nothing was to be divided amongst two people, everybody would get nothing. She explained that 2 divided by zero implied 2 divided by nothing (M 269). Robert agreed with R1 and said, “Yeah, because nobody has it, so you can’t divide it” (M 270). Robert explained to Amy-Lynn that in 2 divided by 0, 0 is a “nobody” and as such, there is no way to divide something between a “nobody.” At this point, Robert went back to his suggestion to take away the space for 2^0 from binary system. R1 encouraged him to discuss his ideas with his group. R1 then explained to Robert that any number to the power of zero is defined to be a 1 for convenience. She pointed out that this definition is “not really logical or anything” but it’s a convention that helps maintain consistency in listing numbers (M 273). R1 also pointed out to Robert and Amy-Lynn that 0^0 was a special case that did not equal a 1. R1 then encouraged Robert and Amy-Lynn to think more about the *Candles Arrangements problem* and see how the properties of zero they had listed related to the candles arrangements.

After R1 left, Robert and Amy-Lynn tried to divide some numbers by zero using a calculator. The calculator gave them an error message. Robert exclaimed “Nah, it’s impossible like see there’s no such number cause if you do like, if you multiply anything

with zero,... it equals zero, say if you put a million in this box, zero times million still equal a zero...darn! So, like, it's a question mark" (M 315). Robert realized that 2 divided by 0 was "impossible." He reasoned that there was no number that could be multiplied by zero to make a non-zero number as even a million multiplied by zero would produce a zero. Robert had concluded that division by zero produces a "question mark." Again this instance reflects that Robert was beginning to understand the complexities involved in dividing by zero in the seventh grade.

7.6.4 What can be done about the extra "place" in binary numbers?

Robert was approached by another male observer around fifty minutes into the session. Robert shared with him that if 2^0 was eliminated each binary string will be shorter by one digit. He further explained that if last three spaces were taken away (for 2^0 , 2^1 , and 2^2), then "2 to the third would equal one" (M 361, Figure 7-10). Robert was suggesting that binary notation can be adjusted if some of the spaces are "taken away." For instance, if 2^0 , 2^1 , and 2^2 were eliminated from the binary system, then 2^3 could be used as the unit (or 1) in base 2. This is an important event in the session. Robert was devising ways to challenge the conventional notation for the binary system and create a new unit for his version of binary notation.

The male observer encouraged Robert to write out candle arrangements to explain his reasoning. Robert created Figure 7-10 to explain his ideas. He wrote at bottom of the figure that "If you take out 2^4 , 2^5 , 2^6 , etc will be messed up. If you take out 2^0 you'll mess up the whole thing." (Figure 7-10) He explained that if 2^0 were taken away from the binary notation then all powers of 2 will have to be adjusted, but, if 2^4 were eliminated, only things beyond 2^4 would need to be adjusted.

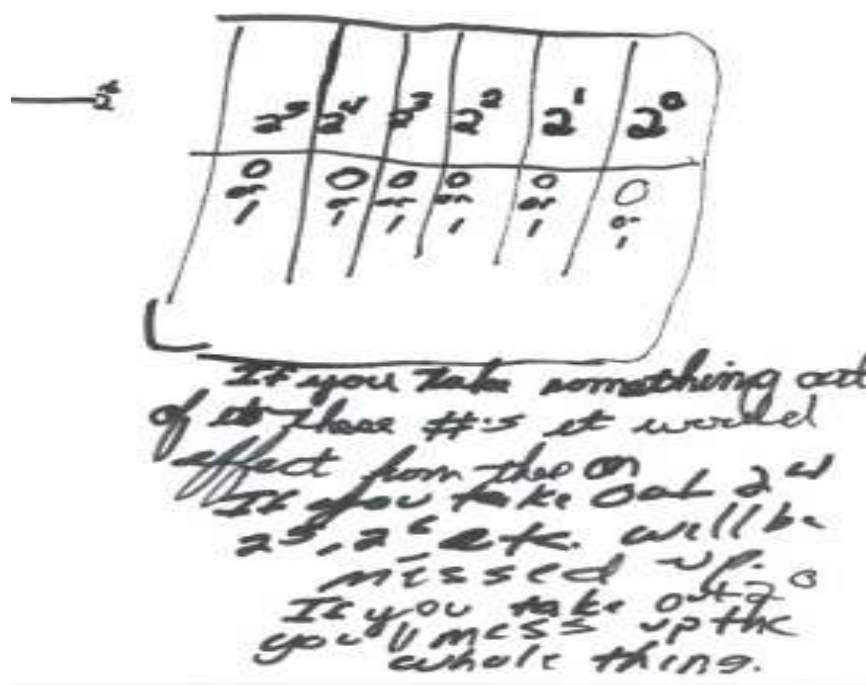


Figure 7-10 Robert's explanation for "places" in binary digits

Later R2 came to Robert and he re-explained the idea of eliminating a power of 2 from the binary representation and the idea that 2 divided by zero is undefined. R2 then encouraged Robert to slowly explain how the powers of 2 related to the candles arrangements. R2 asked Robert to show her what the candle arrangements would look like. Robert took out a sheet of paper from his bag that had his work from the previous day. He had listed 16 five digit binary strings on the sheet of paper (Figure 7-11). Robert explained that 1 meant gold here and 0 meant red. For instance, he used 01010 to represent the arrangement of red and gold candles alternating. Robert further explained that these 16 combinations would be doubled in number when 1 was switched to

represent red and 0 was switched to represent gold. As such, he claimed there were 32 total possibilities for five-candle arrangement using two colors.

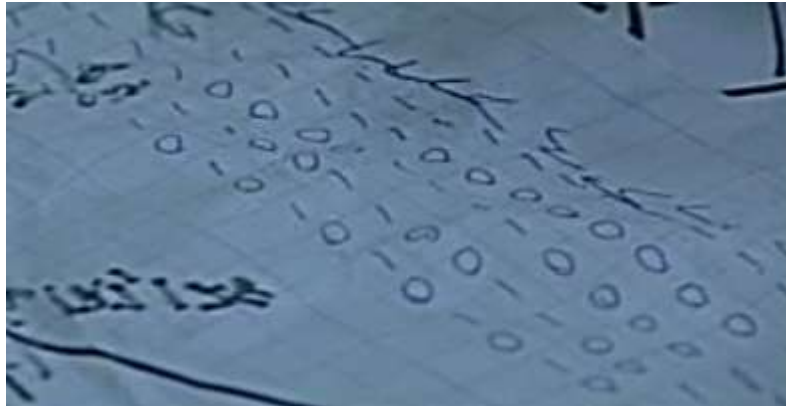


Figure 7-11 Robert's binary list for candles arrangements

7.6.5 Robert extends his solution to three colored candles

When R2 asked him to list the combinations for candles arrangements, Robert explained that if there is only one candle, there would be 2 combinations possible, with two candles there are 4 combinations and so on. He pointed out that as the number of candles increased by one, the number of combinations doubled (M 530). Robert pointed out that the combinations doubled as there were two colors to choose from and that with three colors, it would not be a doubling effect (M 564). R2 then asked Robert to think of three colored candles, black, green and blue. She asked him how many arrangements were possible with one candle to choose and Robert replied that three combinations were possible. She then asked him if two candles were chosen, how many arrangements would be there and Robert replied six. R2 asked Robert to show her what those six combinations would look like. Robert listed the arrangements as: blue-blue, blue-green,

blue-black, green-green, green-black, green-blue, black-black, black-green and black-blue. Robert realized that there were nine combinations and not six as he had guessed. He was quick to observe, “Oh, I know why it doesn’t work. That’s base two (referring to the original problem with two colors). But there’s three different colors so you have to go to base three...if you had four different colors, it would be base four” (M 594). Robert had quickly generalized the solution to three and four color cases of candles for arrangements.

Towards end of the session, R1 invited the class to watch presentations by several students. First, Michelle presented her solution for the *Garage Door problem*. She explained that the solution is 3^9 as there are three setting and nine switches (M 671). Then R1 invited Michael and Jeff to present their solution to the dilemma of the day. Michael and Jeff explained to the class that they had used the binary strings were 1 and 0 represent two colors and arranged them to get 32 combinations for five candles. They further explained that going from 16 to 31 in binary numbers gave them 16 different combinations. They also explained that numbers 16 through 31 only take five spots to represent in the binary notation. Finally they explained that when the 1s and 0s are interchanged, they get twice as many or 32 combinations (M 714 – 719). Jeff also pointed out to the class that Robert had initiated the use of binary notation (M 722). Next, Stephanie and Ankur explained to the class that number of combinations possible for candles arrangement is given $2^{(x-1)}$ times 2 where x is the number of positions for the arrangements (M 764 – 794). Finally Robert presented to the class and stated that 2^0 cannot be taken out of the binary numbers as it would affect all the numbers (M 802).

The session ended shortly afterwards with R2 encouraging students to carefully write up their solutions.

8. Conclusions

8.1 Introduction

Robert's work on various combinatorial tasks was examined over a sixteen-year period using more than twenty-hours of video data. In this chapter, I summarize the findings to address the research questions that guided this study. Interesting themes emerged in external representations and heuristics that Robert used to solve problems from the combinatorial strand of the Rutgers Longitudinal study. In this chapter, I address each of the four research questions in depth and summarize findings from all of the fifteen sessions. For reference, I repeat the list of sessions analyzed in this study.

Table 8-1 Sessions Analyzed

Session Number	Grade	Date	Task/Interview Topic	Type of Session	Approximate Time Length
1.	5	2/26/1993	Guess My Towers	Whole class session	132 min
2.	5	3/1/1993	Pizza Problem	Whole class session	37 min
3.	7	12/13/1994	Garage Door Problem	Whole class session	91 min
4.	7	12/14/1994	Candles Arrangement	Whole class session	42 min
5.	7	12/15/1994	Candle Arrangements Problem	Whole class session	108 min
6.	11	11/13/1998	Revisiting Towers problem	Small group interview	120 min
7.	11	3/1/1999	Pizza Problem	Small group interview	97 min
8.	11	4/26/1999	Towers Extensions	Small group interview	118 min
9.	12	7/7/1999	Combinations for choosing committee of two out of five people	Two people presentation to class	27 min
10.	12	8/31/1999	World Series Problem and the	Two person interview	95 min

			Problem of Points		
11.	16	9/12/2003	Towers: Each has two of one color	Two person problem solving session	46 min
12.	18	8/8/2005	Exploring Pascal's Pyramid	Informal session	82 min
13.	Post Graduate level	7/2/2008	Explaining layers of Pascal's Pyramid	First Post-Graduate interview	49 min
14.	Post Graduate level	11/14/2008	Explaining Pascal's Pyramid and connecting it to building towers and Ankur's Challenge	Second Post-Graduate interview	86 min
15.	Post Graduate level	3/27/2009	Explaining Pascal's Pyramid and connecting it to building towers and exploring the Taxicab problem	Third Post-Graduate interview	82 min

8.2 What are the external representations that Robert uses to help him understand problems related to the properties of Pascal's Pyramid?

8.2.1 Introduction

Robert created several representations to understand structure and properties of Pascal's Pyramid in Sessions 12, 13, 14 and 15. He created two dimensional sketches and three dimensional models to understand how the various cross sections of the Pyramid are related to each other. In other videos selected for this study, Robert used Pascal's Triangle but he did not create a representation for the Pascal's Pyramid prior to Session 12. To answer the first research question, I summarize the representations he created to understand problems related to properties of Pascal's Pyramid in sessions 12 through 15.

8.2.2 Session 12

The first session analyzed for this study was Session 12 where Robert informally discussed the representations for Pascal's Pyramid with mathematician, Professor Todd Lee. In Session 12, Robert created and edited several representations before he created a mathematically "beautiful" representation. Although Robert indicated that the dominant shape of the Pyramid was a triangle, he started with creating circular cross-sections (Figure 8-1). He also sketched many triangular shapes that had circles inside of them (Figure 8-1). Robert also conjectured that he could represent cross sections of the Pyramid in a coinciding manner where third layer would contain the second and the first layer in it, and so on. However, he indicated later in the session that it would not be possible to impose layers on one another in a two-dimensional sketch. For example when he sketched his second layer, he concluded that there would be no space to place the middle term of third layer when it is imposed on top of the second layer.

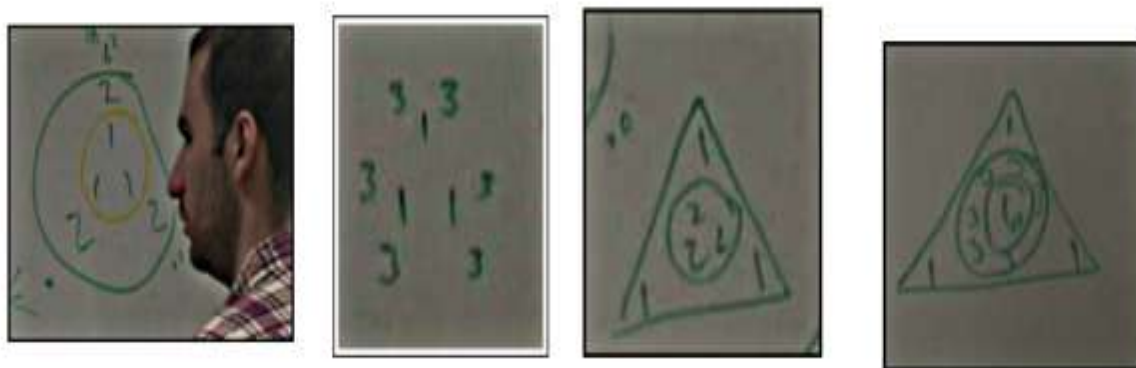


Figure 8-1 Robert's circular and triangular cross sections

Robert was creating a two-dimensional representation of the Pyramid for the first time in Session 12. He was using properties of trinomial expansions and Pascal's identities to place coefficients in cross sections of the various layers. He verified his cross sections with terms in the trinomial expansion to take care of terms that were either missing or repeating. It took Robert approximately 35 minutes to create a precise triangular representation for the third layer after he considered representations like the "Star of David" and other interesting arrangements (Figure 8-2).



Figure 8-2 Star of David, Three Triangles and "Pretty" Third layer

Finally, Robert arranged his terms in a triangular manner that was close to the conventional notation for the Pascal's Pyramid (see third figure in Figure 8-2). Robert commented that shape of the Pyramid was arbitrary and that a sphere could be used to represent Pascal's properties for the trinomials as well (B 47). This might explain why Robert started with circular cross sections at the beginning of Session 12. Robert successfully created triangular representations for the Pyramid up to its fourth layer. In this session Robert talked about three edges of the Pyramid as three "directions" and used the "directions" to guide multiplication with one of the variables. In Session 12 Robert

often used his understanding of Pascal's Triangle to figure out properties of Pascal's Pyramid. For example, he had previously used "directions" for variables in Pascal's Triangle and used it again to understand terms in the Pascal's Pyramid.

8.2.3 Three Post-Graduate interviews (Sessions 13, 14, 15)

8.2.3.1 Session 13

In Session 13, the first post-graduate interview, Robert was asked to again create representations for the Pyramid on the white board using markers. After a span of three years since Session 12, Robert was able to sketch the triangular arrangement of coefficients in layers of the Pyramid accurately. Robert again mentioned that a circle could be used instead of a triangle to represent cross sections of Pascal's Pyramid but the triangular arrangement was just "nicer" (B 47). Robert was able to assess that the choice of a pyramid was convenient but not necessary for Pascal's Pyramid.

In Session 13, Robert explained how terms in a layer add up to produce terms in the next layer. Robert was able to use the fact that the third layer had terms where exponents add up to 3 to figure out the missing $6abc$ term. Robert also used a coordinates system to list his terms in the fourth layer (Figure 5-4). For example, he used $(4, 0, 0)$ to represent the term a^4 . Robert commented that the coordinates helped him in listing all terms in the layer efficiently (B 343). Robert also said in Session 13 that it was difficult to create a two-dimensional representation for a three-dimensional object (B 335). Furthermore, he commented that when he tried to list only the coefficients and not the entire terms, he had a harder time creating a correct representation for a cross section (B 317). This session revealed that Robert relied on explicit trinomial expansion, his

coordinates' system along with Pascal's Triangular properties to discover and represent the terms in a layer of Pascal's Pyramid.

8.2.3.2 Session 14

In the second post-graduate interview, Session 14, Robert worked with *Zome* tools to create a three-dimensional model for the Pyramid. He chose to use all blue rods of varying lengths for the model. He first sketched four layers of the Pyramid on paper and used this sketch as a guide to create the three-dimensional model. Robert listed cross sections of the Pyramid without any difficulty in this session. Interestingly, Robert only listed coefficients in his two dimensional sketch and not the entire terms (Figure 8-3). As Robert talked about addition property of the Pyramid on his model, he mixed up some terms of the second layer. This instance supported Robert's earlier comment that listing entire terms for a layer was more efficient than just listing the coefficients.

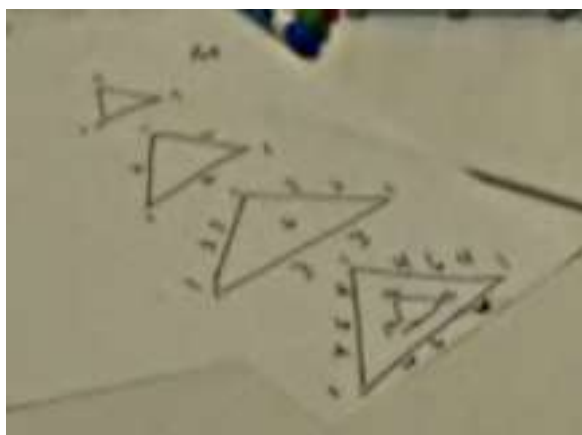


Figure 8-3 Robert's cross sections for first four layers of the Pyramid

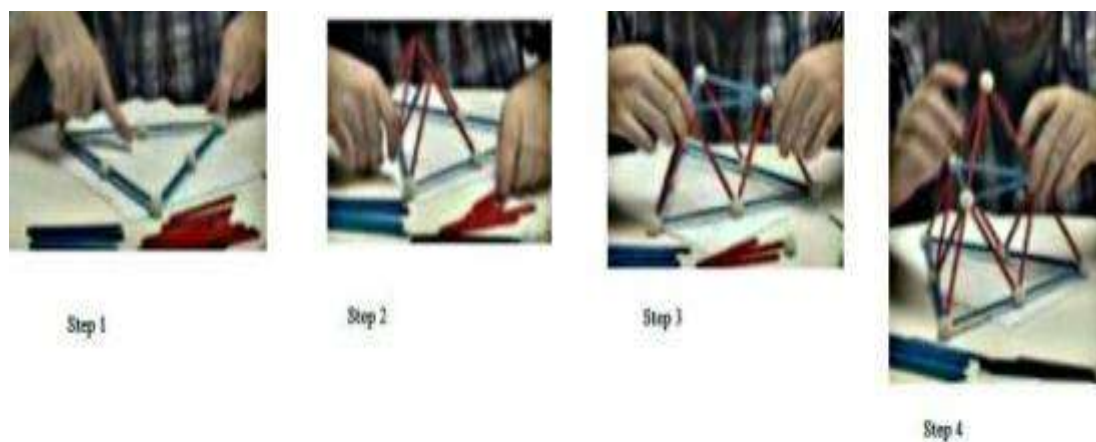
Robert also used an analogy of “directions” to explain multiplication on his model where going left meant multiplication by one variable and going right meant multiplication by another variable. Robert also indicated that Pascal's Triangle was on

all three faces of the Pyramid. In Session 14, Robert also built a smaller two-colored model with blue and red rods. He said that the horizontal blue rods represented addition where as the vertical red rods represented multiplication.

Robert used the sum of exponents in a layer to explain that the Pyramid did not have terms in the middle till it reached the third layer. That is, Robert said that terms in middle of the layer need a minimum exponent sum of three and as such the first two layers had no terms in the middle. Robert reflected in Session 14 that the *Zome* tools were helpful in understanding and communicating ideas about properties of the Pyramid. However, Robert conjectured that in a first attempt to create a representation of the Pyramid, the tools would not have necessarily saved him time or effort (C 494, C 588). Robert further indicated in the session that the two-dimensional sketch, the three-dimensional model and an explicit expansion of the trinomial were all interchangeable once one understands the properties of Pascal Pyramid (C 514).

8.2.3.3 Session 15

In the third post-graduate interview, Session 15, Robert was again given an opportunity to use *Zome* tools to build a model for the Pyramid. In this session, he used blue and red rods to build a two-colored model. Robert built his model from the bottom up as he considered construction of the model to be more practical in this manner (Figure 8-4). He commented that the *Zome* tools had too many positions in connectors and he was not sure how to construct a symmetric-pyramidal model if he started from the top.



►

Figure 8-4 Robert's bottom up construction of the Pyramid's model

Robert explained again in this session how terms of different layers are related in Pascal's Pyramid using his analogy of "directions" for multiplication with the three variables. He also explained the additive and multiplicative roles of blue and red rods respectively in his model. Robert used the rods in his model very explicitly. The number of blue rods in his model corresponded with the number of terms in the layer and the number of red rods corresponded with the number of products between two layers. Interestingly in Session 15, Robert created a model with Unifix cubes and *Zome* tools to show how the second layer of the Pyramid is generated from the first layer (Figure 8-5). The Figure 8-5 put Robert's paper sketch of the Pyramid, his understanding of the *Towers activity* and his construction of the model all together in one representation. Robert explained that each one-tall tower got multiplied by three variables, represented by three Unifix cubes, to produce nine towers of the second layer and so on. He carefully used red and blue rods to represent multiplication and addition between the terms.



Figure 8-5 Robert puts Unifix cubes and Zome tools together

8.3 How, if at all, do these external representations help him in solving problems related to Pascal's Triangle or Pyramid?

8.3.1 Introduction

In the sessions selected for this study, Robert solved *Ankur's Challenge*, *Taxi-Cab*, *Building-Towers* activity and the *Pizza problem* using Pascal's Triangle and/or Pyramid. During Sessions 12, 13, 14 and 15, he mapped his solution for *Ankur's Challenge* onto Pascal's Pyramid. In Session 14, he solved the *Taxi-Cab problem* using Pascal's Triangle. In Session 15, Robert helped Marjory map two colored towers to the Pascal's Triangle and three colored towers to the Pascal's Pyramid. In Session 11, Robert commented on connections between Pascal's entries and the binomial formula while working on a *Towers* activity with Brian. In Session 9, Robert and Michael presented a solution to their classmates about choosing a committee of two people out of five and connected the combination's formula to Pascal's Triangle.

To address how Robert's external representations helped him in solving problems, I summarize his work for the following problems: *Ankur's Challenge*, *Taxi-Cab*, *Committee of two out of five*, *Towers with exactly two of one color* and the *Pizza problem*.

8.3.2 Solution to *Ankur's Challenge*

Recall that *Ankur's challenge* called for counting total number of towers that are four tall choosing from three colors such that each color is represented at least once. In Session 12 Robert created a sketch of the fourth layer of the Pyramid. He also sketched four-tall towers to relate his terms in the fourth layer to the three colored towers by using the three variables a , b and c to represent three colors. Robert indicated that three 12s in middle of the fourth layer were a solution to *Ankur's* problem. He explained that terms on the periphery of the Pyramid were missing a variable (or a color) and as such, they could not be a solution to *Ankur's Challenge*. Robert pointed out that four tall towers corresponded to the fourth layer of the Pyramid and that terms in middle of the fourth layer had at least one each of the variables.

In Session 13 (first post graduate interview), Robert found the solution by pointing out that *Ankur's Challenge* had solution of the nature a^2bc , ab^2c and abc^2 . He commented that as he knew the relationship between the terms with three variables and the towers with three colors, he could just circle them right away and get an answer for the problem (B369).

In Session 14 (second post graduate interview), Robert extended his solution to “any future *Ankur's Challenge*” as he constructed a three-dimensional model for the Pyramid. Robert commented that each surface of the Pyramid only had two variables or two colors. As such, he contended that periphery of the Pyramid would not be a part of the solution to *Ankur's Challenge* even when the problem was extended to other heights for the towers (C 217). In Session 15, Robert repeated this argument again as he generalized his findings to any future version of *Ankur's Challenge*.

Ankur's Challenge is a problem that involves building out towers and counting how many towers meet the very specific requirements for the solution. Once Robert had created an accurate representation (sketch or the model) for the Pyramid, he was able to use the isomorphism between terms with three variables and towers with three colors to quickly isolate his solution for *Ankur's Challenge*.

8.3.3 Solution to the *Taxi-Cab* problem

In Session 14 (second post-graduate interview) where Robert first built a model for the Pyramid, he was invited to solve the *Taxi-Cab* problem. Robert immediately started writing terms of Pascal's Triangle on the grid of the handout. Robert indicated that there was an isomorphism between Pascal's Triangle and solution to the *Taxi-Cab* problem. Robert pointed out that solution of the given problem did not exist on his model as the destination was too far (fifth row) from the taxi stand. He explained that a dot on fifth row of the grid would be on the fifth row of Pascal's Triangle, a row that he had not constructed in his model. Robert commented that the solution to *Taxi-Cab* will only be on surface of the Pyramid as the taxi cannot fly and has only two dimensions in which to travel. Robert then mapped the new destination from the fourth row of the grid to a sphere on the fourth row of the Pyramid's model.

Robert used the addition property of Pascal's Triangle to figure out that there were 4 shortest routes to the destination of choice. He was also able to show the observer, Victor, what these 4 shortest routes were by sketching them out on the paper and simultaneously tracing them with his fingers on the model. Robert also used the analogy of directions "south" and "east" for the taxi and connected them to the directions "left" and "right" for variables a and b on his model. Robert worked with several other

destinations on the grid and quickly used coefficients of Pascal's Pyramid to figure out the number of shortest routes possible.

Robert commented that the three-dimensional model was more convenient to visualize the taxi routes than his disconnected sketches for cross sections of the Pyramid. Robert also talked about an extension to the problem where the taxi could fly so that the solution could be mapped to the three-dimensional Pyramid.

Solving the *Taxi-Cab* problem by listing or drawing out each route is tedious when destinations are far from the taxi stand. Robert was able to use isomorphism between the taxi routes and coefficients in Pascal's Triangle to efficiently solve the problem and illustrate the various routes.

8.3.4 Solution to the *Committee of two* problem

In Session 9 (grade 12), Robert and Michael presented a solution to the problem where a committee of two people had to be chosen from a group of five people. Robert explained to his classmates that combinations formula $C(5,2)$ was a solution for the number of two-people committees possible. He explained why various pieces of the formula made sense and why certain factorials were used in the denominator of the formula. Then, Robert also shared with his classmates how Pascal's Triangle could be used to find $C(5,2)$. He pointed out that the second entry in the fifth row of the Triangle was the same as $C(5,2)$. This was the first time in all the sessions observed that Robert had connected Pascal's ideas to the combinations formula. Robert explained to his classmates that there were three ways to solve the committee problem as they could use the combinations formula, or write out all possibilities or use entries of Pascal's Triangle.

8.3.5 Solution to *Towers with two of one color* problem

In Session 11 (grade 16) Robert worked with Brian on the following problem: find the total number of two-colored towers that are possible when each tower must have two blocks of one particular color. Robert shared a formula with Brian that he had invented in the eleventh grade to solve this problem. The formula was in fact the combinations formula, $C(h, 2)$ where h is height of the tower. Robert helped Brian solve the problem and see how the solution was related to the formula of $C(h, 2)$. Robert also shared with Brian that Pascal's Triangle was related to the combinations formula. Robert first started with writing out some binomial expansions and connecting the coefficients of the binomial to entries of the Triangle. He explained to Brian that the variables a and b represent two colors of Unifix cubes. Robert explained that for four tall towers, the number of towers with exactly two of one color would be given by $C(4, 2)$ which is 6. He pointed out to Brian that 6 was also the coefficient of the second entry in the fourth row of Pascal's Triangle, $6a^2b^2$. Robert explained to Brian the four-way relationship between the *towers activity*, the combinations formula, the binomial expansion and Pascal's Triangle.

In this session Robert used the combination's formula to count all possibilities without having to create individual towers. He also used his understanding of Pascal's Triangle to find values for the combination's formula.

8.3.6 Solution to the *Pizza* problem

In Session 7 (grade 11), Robert worked on the *Pizza problem* in a small group setting where students had to figure out total number of pizzas possible choosing from four or five toppings. Shelly, another student in Robert's group, pointed out that as the

number of toppings increased from 0 to 4, the number of possible pizzas were 1, 4, 6, 4 and 1. Shelly commented that Pascal's Triangle was related to the number of pizzas generated by a certain number of toppings.

When the group tried to figure out why the addition property of Pascal's Triangle made sense in terms of pizza toppings, Robert suggested that they could think of the *Towers activity* instead. Robert indicated to the group that there was an isomorphism between towers and pizza toppings. Robert also reported that total number of pizzas was given by 2^n where n is the number of toppings, and explained that this was same as the sum of entries in a row of Pascal's Triangle. Robert said that a 2 was raised to the power of n as "four toppings would be a tower four high and then two colors would be with or without toppings" (I 982).

Robert used the isomorphism between toppings for the pizzas and colors for the towers to solve the *Pizza problem* and provide an explanation for his solution. Also, he extended this isomorphism to Pascal's Triangle, indicating that the total number of pizzas possible was same as the sum of the entries in a row of Pascal's Triangle.

8.4 How does Robert use, modify or reuse his external representations over time to provide justifications for his solutions?

8.4.1 Introduction

Robert used external representations such as sketches, formulae and physical models, to explain his reasoning to peers and researchers. There are certain representations and heuristics that Robert used in more than one session to solve the

combinatorial tasks. I summarize the representations and heuristics that the mathematical behavior of Robert illustrated.

8.4.2 The Binary Notation

In Session 3 (grade 7), Robert commented that he used binary notation to solve the *Candles Arrangements* problem. He used the numbers 0 and 1 to represent the two colors, red and gold for the candles. Robert pointed out that five candles in an arrangement would correspond to a string of five digits in the binary notation. He figured out that 32 arrangements were possible for five candles using two colors. In Session 10 (grade 12) Robert used binary notation to list how many ways a World Series could end. He used 0 to represent one team winning and 1 to represent the other team winning. Robert first listed all the ways one team could win and then multiplied it by two to account for the other team winning.

8.4.3 The “family tree” for towers

In Session 1 (grade 5), Robert observed Stephanie use the strategy of creating a “family tree” to account for all towers possible using two colors of Unifix cubes. Stephanie built duplicates of each tower and called them “parent” towers who created the “children” towers when one block of each color was added to them. In Session 6 (grade 11), Robert used this strategy to show the researcher how he built towers two tall from one tall and then how he built three tall towers from two tall towers. Robert had assimilated the strategy that Stephanie shared in grade 5 with him and used it to count possible towers later². In Session 8 (grade 11), Robert again explained using a tree

² This strategy was first discussed by Milin (Maher & Sran, 2010)

diagram (Figure 8-6) how he knew that there are exactly sixteen towers that are four tall when choosing from two colors.



Figure 8-6 Robert's sketch of "family tree"

8.4.4 “Exactly one” or “exactly or two” of something

As Robert solved counting problems that involved listing out possibilities that had restrictions like exactly one of a color, or exactly two of a color, he came up with four observable strategies. The strategies were to move a variable down the list, use opposites, use a special listing technique and control one of the variables to list all cases. I summarize how Robert used these strategies to list possibilities.

8.4.4.1 Moving the variable down

In Session 1 (grade 5), while working on the *Guess My Tower* problem, Robert created the list RYYY, YRYY, YYRY and YYR (where Y is yellow and R is red) to list possibilities for four-tall towers with exactly one red. Robert moved the single R in a four-letter arrangement until it reached the very bottom. He repeated this strategy for towers five tall as well.

8.4.4.2 Using opposites

In Session 1 (grade 5), Robert used “opposites” to list towers with exactly two red cubes while working on the *Guess My Tower* problem. He listed the towers RYRY, RRYY and RYYR followed by YRRY, YYRR and YRYR for the four-tall towers. The later three towers are opposites of the first three towers as Ys and Rs are interchanged (compare third tower to fourth tower, second tower to fifth tower and first tower to sixth tower). He again used opposites in Session 8 (grade 11) as part of his listing technique, which is discussed in the following section.

8.4.4.3 Robert’s special listing technique

In Session 1 (grade 5), Robert created 11 five-tall towers with exactly two of one color (red) in them as RRYYY, RYRY, RYYRY, RYYR, YRRY, YRYRY, YRYR, YYRRY, YYRY, YYRR and YRYRY (where R is red and Y is yellow). The eleventh tower was repetitive. He started his list by keeping two Rs together on the top and increasingly inserted Ys in between them. When all three Ys were inserted, he moved the two Rs to the second position and repeated the same strategy. This strategy is called Robert’s special listing technique.

In Session 6 (grade 11) Robert had to create four-tall towers with exactly two blues in them. Robert created towers as BBYY, BYBY, BYYB, YBBY, YBYB and YYBB (where B is blue and Y is yellow). This was Robert’s special listing technique again. When in the same session Robert had to find towers that are five tall and have exactly two blues in them, he used the technique one more time.

In Session 8 (grade 11) Robert worked on the problem where they had to count how many four-tall towers were possible choosing from three colors such that each color

is represented. Robert used “opposites” and his special technique together this time. He listed towers as RRBG, RRGB, RBRG, RGRB, etc (where R is Red, B is blue and G is green). He started with placing two Rs on the top together, then switching the bottom from BG to GB. He was creating the “opposite” of RRBG as RRGB. He then moved the two Rs to a lower position and repeated the strategy of inserting G and B in between them and taking care of the opposites of the towers he had listed. Robert found 12 towers with exactly two Rs in them. He contended that there are three times as many or 36 such towers to take care of the case where two Bs or two Gs are used instead of two Rs.

In Session 10 (grade 12) Robert worked on the *World Series problem* using the same special technique of listing. For a six games series, Robert listed his possibilities with two zeros together on the top and then he started inserting ones in between them. He used 0 to represent one team winning a game and 1 to represent the other team winning. His list looked like 001111, 010111, 011011, 011101, etc. Then he moved the two zeros to the next spot and repeated the insertion of ones in between them. By this time (grade 12), Robert was very quick with this listing technique.

8.4.4.4 Controlling for a variable

In Session 6 (grade 11), Robert revisited the problem of how many four-tall towers can be made using two colors. Robert explained to the researcher that he found his 16 towers by first using “four yellow and zero blue” and then increased the number of yellow to get “three yellow and one blue” and so on (J 186 -188). Robert controlled for the variable by focusing on yellow colored blocks as they went from four to zero in his towers. He used this as a strategy to convince others that no other towers were possible.

In Session 7 (grade 11) Robert worked on the *Pizza problem* where he had to count all pizzas that can be made with four toppings: peppers, sausage, mushrooms and pepperoni. Robert first counted the plain and one topping pizzas. Then, he fixed peppers as a topping and found all other two-topping pizzas he could create with peppers as the first topping. Next, he paired sausage with other toppings followed by pairing mushrooms with the only topping left, pepperoni. He found two-topping pizzas by controlling the first topping and exhausting all possibilities with it and then moving on to the next “first” topping.

In Session 11 (undergraduate level) Robert worked with Brian to find towers of different heights that have exactly two blues in them. Robert shared with Brian that he first fixed spaces for red blocks in the towers and then tried to figure out where the blue blocks could go. That is, Robert fixed position for one color and then counted how many ways the other color can be inserted in the towers.

8.5 What connections, if any, does Robert make to earlier problem solving?

8.5.1 Introduction

Robert revisited his earlier ideas and problem solving with the counting tasks and used them in later years. As is illustrated by the examples that follow, some of the heuristics and techniques that Robert developed in elementary and middle school were used by him as a college student and beyond. I describe some of the connections he made with his earlier strategies and heuristics that were found in the work studied.

8.5.2 Looking for a pattern or a formula

In Session 3 (grade 7), Robert solved *the Garage Door problem* indicating 3^9 for the solution, explaining that there are three positions, high, medium and low for nine different switches. Robert reasoned to his peers that if it was one switch, it would be three or 3^1 possibilities and argued inductively that for nine switches it would be 3^9 . In another part of the problem, where a condition was that Mrs. O'Brien remembered two of the switches, Robert indicated that it would now take 3^7 combinations to open the garage. He modified his rule of 3^n quickly to accommodate for the new condition of the problem. Also, Robert calculated that if Mrs. O'Brien were to work twelve hours a day trying to find a combination for the lock, it would take her approximately 27 days. He converted 3^9 or 19,683 minutes to 328 hours and divided by 12 to compute the number of days to be 27. This work suggests that Robert had a good understanding of how units are converted.

In Session 4 (grade 7), Robert used 2^n to find the number of candle arrangements possible with two color candles (n is number of candles in the arrangement). When the researcher asked students to think about three colored candles for the arrangements, Robert conjectured that the pattern would be "three, nine, twenty seven, eighty one" and offered that the solution was 3^n (L 58 - 59). Robert was quick to observe that the three-colored candles arrangement was similar to the *Garage Door problem* where switches had three different positions. Robert was further able to generalize that for four colors of candles, the formula would change to 4^n , and for seven candles with two colors, 2^7 . Robert showed understanding of the structure of the problem. With ease, he adjusted the base and the exponent in the exponential formula for variations to the *Candles Arrangements* problem.

In Session 7 (grade 11), Robert worked on the *Pizza problem* where he quickly offered that there are 2^4 or 16 pizzas possible with four toppings. He commented that the 2 represented two choices, the presence or absence of a topping.

In Session 6 (grade 11), when Robert was trying to figure out how many two-colored towers were possible with exactly two blues in them, Robert conjectured that there were 15 towers for the six-tall case. Robert listed his findings for towers that were one through six high and looked for a pattern. The formula he built was $h \left(\frac{h}{2} - 0.5 \right)$ where h is height of the tower. He used this formula to compute the solution for seven-tall towers with exactly two blue cubes. Robert's formula, when simplified, is equivalent to the combinations formula, $C(h, 2)$. It is rather impressive that Robert had constructed the combinations formula in eleventh grade in searching for a rule to describe the general solution to the problem.

In Session 8 (grade 11), Robert worked on extensions to the towers problems in a group setting. One of the problems that day involved finding total number of towers of height n choosing from $(n - 1)$ colors such that each color was represented. Robert first created a list of four-tall towers to conclude that when $n = 4$, there are 36 towers such that all three colors are present in each of the towers. As the group considered five-tall and six-tall towers, Robert proposed the formula $\frac{n!(n-1)}{2}$ as a solution for counting the towers possible. He verified that this formula yielded a 36 when n was 4. He did not get an opportunity to discuss why the formula made sense in this session but commented that the formula worked.

In Session 9 (grade 12), Robert was working with Michael. They proposed the formula $C(n, x) = \frac{x!}{n!(x-n)!}$ for how many n -people committees can be created when choosing from x people. Robert commented that he and his partner devised the formula to find a “faster way in case like we have like a hundred people choose ten or something like that” (G 20). Their interest in searching for a general rule suggests that Robert and his partner were considering an extension to the problem where listing cases would be too tedious when x gets large.

In Session 11, Robert shared his formula $h\left(\frac{h}{2} - 0.5\right)$ with Brian and explained why it made sense for counting towers that have height h and have exactly two of one color in them. Robert simplified the formula to get $\frac{h(h-1)}{2}$ and explained to Brian that this was the same as $C(h, 2)$. Robert commented that the problem involved placing two blocks of the same color in a tower of height h . He told Brian that as one needed to count how many ways the two positions out of h positions could be chosen, they could use $C(h, 2)$. Robert encouraged Brian to solve the problem for towers that were a hundred tall and had exactly two of one color. Brian used Robert’s formula and found the correct solution.

It is evident that Robert liked to discover patterns in solutions to the problems and look for formulas to generalize the solutions. He was careful to monitor his conjectures by using counting techniques such as creating exhaustive lists, tree diagrams or using the manipulatives to build models of the solution for simpler cases.

8.5.3 Thinking of a simpler problem

In Session 3 (grade 7), Robert worked on the *Garage Door* problem, and quickly offered the solution of 3^9 . He explained to his peers that if it were one switch, the solution would be three or 3^1 possibilities because the switch could be at the high, medium or low position. With the researcher, he worked on a solution for two switches. He drew out a tree diagram with branches to show that there would be 3^2 or 9 possibilities for two switches with three settings. Once he had presented solutions for one and two switches, he contended that that the solution was 3^9 for nine switches.

In Session 4 (grade 7), the researcher asked students to find out how many seven-candle arrangements were possible choosing from two colors of candles. Robert pointed out to his peers that they had already listed all binary combinations for five candles. They had concluded as a group that there were 32 five-candle arrangements. He said that when 32 combinations were doubled, it gave 64 combinations for six candles. He explained to his group that when these 64 six-candle arrangements were switched around, that is red and golden candles were interchanged, there were 128 resulting arrangements for seven candles. Robert used the solution for the five-candle arrangements to build a solution for seven candles arrangements.

In Session 7 (grade 11) Robert worked in a group to explain why Pascal's addition property made sense in terms of adding toppings to the pizza. Stephanie was trying to explain how toppings could be added from one layer of the Triangle to the next. Robert suggested to her that she should work with "one topping, then two toppings and then see if we see a pattern? You know we can do something based on the pattern" (I

477). Robert was recommending to Stephanie that she should try to work with a simpler problem and then build on it to see if a useful pattern would emerge.

In Session 11 (undergraduate level) Robert worked with Brian to find all towers of a certain height that have two of one color in them. Robert suggested to Brian that he should try towers three high before trying towers four high.

These examples reflect that Robert went back to a simpler case to either convince his peers or find a pattern in the solutions.

8.5.4 Using isomorphism between structure of problems

In Session 5 (grade 7), Robert saw a connection between the three-colored *Candles Arrangement* and the *Garage Door problem* where the switches had three settings. He commented that three colors of candles were same as three positions for garage door switches. In Session 7 (grade 11), working on the *Pizza problem*, Robert recommended to Stephanie that she should think of adding blocks to a Unifix tower to understand why Pascal's addition property made sense in terms of adding toppings. He pointed out to her that all three-tall towers were related to pizzas with three toppings.

In Sessions 12 – 15, Robert used the three-way relationship between the binomial expansion, combinations formula and Pascal's Triangles' entries. He mapped the solutions of *Taxi-Cab*, *Ankur's Challenge*, and the *Building Towers* problems to Pascal's Triangle and Pascal's Pyramid. For the *Taxi-Cab* problem, he used the variables a and b as the directions on grid of the *Taxi-Cab problem*. He showed the relationship of the the "south" and "east" directions on the grid to the "left" and "right" directions of the Pascal's Triangle. For *Ankur's Challenge*, he represented the variables as colors of the Unifix cubes. He commented that the terms in which each of the three variables were

represented were equivalent to the towers in which all three colors of cubes were represented. He also helped Marjory in Session 15 to map the three-colored Unifix towers to Pascal's Pyramid by relating the three variables to three colors in the towers.

Robert was quick to observe similarities in structure of the problems and their solutions. He uncovered interesting isomorphic properties of the combinatorial problems presented to him.

8.5.5 Proposing changes to conventional notations

In Session 5 (grade 7), Robert worked on the dilemma where students were not sure why binary representation of 32 took up six digits when it represented a solution to the five-candle arrangements. Robert proposed that binary notation could give up the space for 2^0 so that length of binary strings was same as the number of candles used in the arrangement. He also suggested that other powers of 2 might be used as a unit other than 2^0 in the binary system. Robert also explored division by zero in this session and expressed interest in the ideas of limits and infinity in conversations with the researchers.

In Sessions 12 – 15, Robert commented that a pyramid was not absolutely necessary to represent properties of Pascal's Pyramid. He also explored the possibility of representing all layers of the Pyramid in a concentric manner. He attempted to map *the Taxi-Cab problem* to the three-dimensional Pascal's Pyramid. In Session 13, Robert suggested that conventional notation for $C(n, x)$ should be represented as $C(n; x, n-x)$ to help students generalize the formula to the multinomial case. He created his own notation for coordinates to list terms in the Pascal's Pyramid in Session 13 where $(4, 0, 0)$ referred to the location of the coefficient of a^4 .

8.6 Conclusions

NCTM (2000a) defines representation as both the process and the product of “capturing a mathematical concept or relationship” (p. 67). Robert traveled an interesting journey to capture the structural properties of Pascal’s Pyramid. He used different notations and sketches in his work to capture the Pyramid in a two-dimensional representation. Robert refined his representations for Pascal’s Pyramid when he was asked to justify and explain his comments, sketches and models (see Sessions 12 through 15). Robert showed how different layers of the Pyramid were related and successfully translated his ideas into words, notations, physical sketches and models.

Representations that Robert created to solve combinatorial tasks played an important role in his problem solving. When Robert created a sufficient and correct representation for the Pyramid he communicated his ideas to the researchers and observers effectively, as detailed in Sessions 12 through 15. Also, Robert tended to modify his representations when new information emerged, as in Session 12, when he changed cross sections of Pascal’s Pyramid several times to take into account the new properties of the Pyramid that he discovered along the way.

Robert did not accept a solution until it made sense to him. For example, in Session 12, one of the researchers shared with him how a $12a^2bc$ term for the fourth cross section of Pascal’s Pyramid was created. She explained to Robert that only a specific set of two 3s and one 6 from the third layer when multiplied with appropriate variables would yield the $12a^2bc$ for the fourth layer. Robert questioned why her choice of 3s and 6 worked and explored if other sets of 3s and 6 could be chosen instead. Robert continued to test and verify all other possible choices. As he worked with other possible

sets of 3s and 6, he found out why the researcher's choice worked. He figured out that only the suggested choice of 3s and 6 produced like terms that could be added to make a 12 in the fourth layer.

Robert was usually very quick at verifying and evaluating other people's ideas and exhibited careful counting and reasoning skills. For example, in Session 10 (grade 12) Michael shared with Robert his solution to the *World Series* problem. Michael suggested that the combinations formula was reasonable to use in solving the *World Series* problem. Robert verified Michael's suggestion by comparing the outcomes of combinations formula to his binary list. Once Robert had verified that combinations formula yielded the same numerical value as his binary list, Robert used the combinations formula to solve other parts of the problem. Also as summarized above for Session 12, Robert verified the researcher's claim about producing a 12 for the fourth layer by writing out all possible cases.

The problem-solving heuristics that were exhibited by Robert showed evidence of his growth in understanding of concepts related to Pascal's Pyramid over the years. Tracing back to Robert's earlier problem solving with counting tasks, we learn that he was engaged in building five-tall towers selecting from two colors, with his classmates in grade 4 (Maher, 2005). We know this because Robert was a participant in the longitudinal study at Kenilworth since his first grade. While there is no video record of Robert and his partner's working together, we do know that he was in the class that worked on the problem and shared their solutions and strategies. The video data that was available to study in detail Robert's selection and application of external representations

revealed that he made use of strategies and patterns that were used by the younger children in his later problem solving.

Robert's external representations also served as an effective tool for communicating ideas to his peers and the researchers. For example, he referred to his tree diagrams, sketches, lists and models to make a point. In Session 14, Robert traced the Taxi-Cab routes on his model for the Pyramid along with sketching out colored routes on the paper grid to share his solution with the observer. In Session 15, he drew out an explicit "family tree" for three colored blocks using the Unifix cubes and Zome tools (Figure 8-5) to illustrate how the Pyramid's layers were related to the *Building Towers* activity. In Session 10, he used his binary list to convince Michael that there were 20 and not 18 ways for a World-Series to end in five games. He used his tree diagram in Session 8 (Figure 8-6) to convince the researchers that there are only 16 four tall towers choosing from two colors of the Unifix cubes.

8.7 Implications

The case study from this research provides some insight into the importance of building mathematical understanding consistently over the years. Robert built robust counting techniques as he spent time working on his combinatorial skills and representations as a young student. Once he mastered these skills and representations, he used them fluently as an adult. Furthermore, Robert was able to recognize equivalent structures in numerous tasks and, consequently, successfully generalized several solutions. The external representations created by Robert assisted this researcher in making reliable inferences about his problem-solving strategies and understanding of the task. Robert is an example of the success of a student who was encouraged to create his

own representations in building meaningful solutions to problems as he was given opportunities to revisit ideas, tasks and representations over time.

NCTM (2000) standards and the literature reviewed for this study document a need to provide students with opportunities to create and own their mathematical representations as an important constituent to building strong problem-solving skills and mathematical understanding. As we routinely revise policies in mathematics education and try to capture and inspire young mathematical minds, the success stories echoed by individual case studies become crucial to consider. Robert was fortunately provided environments and opportunities that the recommendations outlined by the standards and research entail. As such, Robert's success, although very unique and individualistic, provides important evidence for how and why these recommendations should be broadly implemented.

8.8 Limitations

Any single study captures a part of the story as many other lenses are possible and many questions are left unexplored. Robert's work has many untold stories that could be unraveled and shared. For some of the earlier sessions, Robert's participation was not captured on video camera. This is a limiting factor for the detailed analysis of Robert's early work. Because of the vast quantity of video data (approximately 4000 hours) captured from the longitudinal study that is now in its 24th year, other video data can be examined that characterizes Robert's mathematical behavior, including data from the counting strand.

This researcher selected data from sessions that spanned 16 years, with earliest data from Robert's fifth grade. It should also be noted that a variety of sessions were

chosen to study Robert's participation in whole class, small group, partners and one on one settings. Any effect that the nature of a session might have had on Robert's participation was not addressed by this study and might be a topic for further study.

In addition, the results of the study are specific to the participant, the nature of the tasks and the learning environment he was provided. The combinatorial tasks that were chosen in these sessions elicited Robert's building of rules,, notations, sketches and models. While this study focused on the mathematical activity of Robert in the counting/combinatorics strand of the longitudinal study, further analysis of his mathematical activity in the other strands would be interesting. It is important to note that the sessions observed were separate from Robert's regular mathematics classroom. The longitudinal study sessions were specifically designed to encourage students to provide justifications for their solutions to non-routine tasks, often working in a group setting. The impact of the setting of these sessions was not analyzed and might be an important topic for a future study. Finally, Robert is unique as any individual is and his strategies, heuristics, external representations along with the challenges and successes that he encountered are not replicable. Nevertheless, a detailed study of the problem solving and mathematical growth of his and of other participants in the longitudinal study can shed light on how mathematical ideas and ways of reasoning develop over time, under conditions indicated for this study. However, together, a collection of case studies provides insights into the benefits of early learning of meaningful mathematics (Sran, 2010; Steffero, 2010; Aboelnaga, 2010; Maher, 2005)

8.9 Future Studies

Robert's mathematical behavior and problem solving could be examined using other lenses. For example, one might look for the influence of other characteristics such as affect, motivation, and collaboration in his problem solving across the years. It would be interesting to investigate the effect of particular interventions in his learning about counting, such as the early algebra strand in grade 6. Robert and his classmates were introduced to ideas such variable and function through engaging in *Guess My Rule* activities introduced first by Robert B. Davis (Giodarno (2008), Spang (2009)), and then continued by Emily Dann with inverse functions (grade 7) (Dandola-Depaolo, in progress), and Carolyn Maher with applications to geometry (grade 8) (Marchese, 2009). Robert had available some powerful mathematical knowledge at a young age. He knew how to represent relations and functions in algebraic form; he was accustomed to examining patterns to represent functions and their inverses. In grade 6, he worked with linear, quadratic and exponential functions, and explored, as a middle-school student many challenging problems, such as the *Tower of Hanoi* (Mayansky, 2007). How Robert built these ideas and shared them with his peers may have important bearing on his success in working on counting and probability problems later. These are questions for further study.

Other areas to explore are the nature and consequence of researcher intervention and peer collaboration. For example, how might have Robert's interactions with the various researchers and with his peers shaped his learning and facilitated his understanding of mathematical ideas? Another question for study involves Robert's mathematical learning across strands. Robert participated in the algebra, probability and

early calculus strands as well as the counting/combinatorics strand. It would be of interest to investigate whether Robert's mathematical behavior was consistent across other strands and where knowledge of fundamental ideas and concepts intersected across domains and was helpful to him. Furthermore, it might be of interest to investigate the nature of Robert's participation across the strands. Did a particular interest in counting and probability surface early?

Robert continues to pursue his education and is currently a doctoral student in mathematics education at Rutgers University studies. He has chosen mathematics education and probability and statistics as specializations. His undergraduate major is mathematics and his master's degree is in statistics, where he is pursuing further graduate work. He works as Applications Developer at the Robert B. Davis Institute for Learning and is responsible for managing the video database and supports data collection for research projects. It would be an interesting study to follow Robert as he transitions into the role of an educator and researcher. The possibilities to continue to follow Robert's story are enticing.

The case study from this research provides some insight into the importance of building mathematics understanding consistently over the years. Other case studies of longitudinal participants could be conducted to see if similar patterns in building mathematical understanding emerge. The recent, seventeen-year case study of Romina highlighted the importance to her of building personal understanding (Steffero, 2010). Another study traced Milin's building of an inductive argument and how that argument traveled through the community (Sran, 2010). Another study, in progress, traces Stephanie's building of the binomial theorem and how she relates it to Pascal's Triangle

through tower building. This study is an important contribution to the body of research that is accumulating increasingly more evidence that building of early mathematical ideas is the foundation for more advanced learning. This understanding is, in my view, prerequisite for later, deeper understandings.

I am a mathematics faculty member at Brookdale Community College. I often reflect on how I might establish effective learning environments similar to those that were created for the Rutgers longitudinal study. Barbara Glass (2001), in her research, went a step further and actually integrated a strand of counting tasks from the longitudinal study into her Math for Liberal Arts class at Sussex Community College. Glass' work demonstrates that students at college level were successful on the same combinatorial tasks and learned to reason and support their solutions with convincing justifications. The students identified an isomorphism between the *Pizza problem* and Pascal's Triangle and between the *Towers Problem* and the *Pizza Problem*. Glass' (2001) study demonstrates that her students were successful and moved efficiently through the problem strand.

Glass' research suggests that it is never be too late to help students learn to build their own mathematical ideas and representations (Glass 2001; 2002; Glass & Maher 2002; Glass & Maher 2004). While early understanding of mathematics is ideal, it is important to strive to achieve understanding at any age. The case study of Robert provides a powerful example of the benefits of starting early.

Appendix A

Date of Session: 08-08-2005

Author: Anoop Ahluwalia, Ericka Blyck, Scott Rutherford

Verified by: Anoop Ahluwalia, Ericka Blyck, Scott Rutherford

Date of transcript: June, 2008

Researchers:

R1: Todd Lee

R2: Elizabeth Uptegrove

R3: Carolyn Maher

Line	Time	Speaker	Transcript
1.	0:27	Robert	And this would be like... a $3ab^2$...
2.		R1	Yeah, yeah, yeah... ok
3.		Robert	And then something like this, which would be $3b^2c$, ...and this which would be $3abc^2$... then this which is $...2a^2c$
4.		R1	Oh, oh, oh... yeah... yeah yeah yeah
5.	1:00	Robert	This would be b^3 , so... we're missing something...oh its 3 sorry, and then I guess I can combine these two, just gonna be up here. Right? No...oh yes this will be $2abc$... this way? Yeah then...
6.		R1	2? 2?
7.		Robert	4? I think it's 2
8.	1:28	R1	Well what, what, what's occurring to get them together.
9.		Robert	Because you get duplicates if you umm...
10.		R1	What are you gonna be multiplying by?
11.		Robert	I'm only taking one of these and adding a b.
12.		R1	You're adding a b.
13.		Robert	And this one I'm adding a c.
14.		R1	So you get $2ac$ plus b...
15.		Robert	Yeah, but if you do it twice
16.		R1	and $2ab$ plus c those two can't combine
17.		Robert	Really?
18.		R1	Well, can we, can I write down that? So you said $2ac + b$
19.		Robert	Or times b, yeah whatever
20.		R1	Oh...Ok, now I am OK. Oh, so write that...let's write that down. What is that... $2abc$ ended up there?
21.		Robert	$2abc$? So... but I think...
22.	2:09	R1	Then, you're adding them?
23.		Robert	Yeah but the thing is like, if there's already 2, I guess yeah it'd make sense so there'd be 4.
24.		R1	Well where did the 3 come from? How about... that's a better question.
25.		Robert	3. This plus...you added...you added a to this...
26.		R1	I, I did not.

27. Robert Yeah?
28. R1 You didn't either
29. Robert Oh yeah, I, I multiplied a by it. Yeah. You know what I mean. And then umm...
30. R1 (Laughing) Now the teacher mode is kicking in, I'm sorry. That's right.
31. 2:33 Robert Yeah...and times b by these two, so I guess you're right, yeah so this will be 4 right?
32. R1 The coefficients, right. 'Cause we're counting how many of those things we have right. So, I see you, you...you multiply that by a and you've got how many of those ab^2 terms? Just one there...
33. Robert One there
34. R1 and I got two there, so how many total? 3.
35. Robert Okay... and then...
36. R1 So you've got 4 over here? And you're creating 4...
37. Robert And so now this is...I think there's gonna be...
38. R1 Ooh...how come we're not getting up to 12 total?
39. Robert I think there's gonna be duplicates. That's what I thought, 'cause then this is gonna be a 12.
40. R1 Well, you you've added that already once, now you are adding it again, so I'm thinking...we should add, we should have like a tri-curve or something...
41. Robert Yeah cause like it's
42. R1 where they're all adding together, 2, 2, 2 is...
43. 3:13 Robert What I was gonna do... what I was doing was sending one this direction and one this direction (making connections on the diagram on the board)...
44. R1 I gotcha... aah
45. Robert that's why I had 2 there, 'cause I was sending one here and one over here.
46. R1 Very clever.
47. Robert But, I guess we could just send it twice and then we divi... and then there'd be ...this is gettin' a mess...it'd be $4abc$ here. But then like, ya know, they'll be each counted twice cause ...
48. R1 So, you divide by 2?
49. Robert Yeah, so that's why... so it's really like...
50. R1 12 divided by 2 is 6, is that what we got?
51. Robert Yeah. Or yeah 12, yeah we got 6.
52. R1 What happened to your beautiful uhh picture
53. Robert Yes, and then I don't know...I guess like, you kinda need 3 dimensions cause then...
54. 3:49 R1 Ok, let's do it.
55. Robert I can't draw in 3D.

56. R1 Sure you could.
57. Robert I don't know. I'm a terrible drawer. You couldn't tell? Like but see this...
58. R1 Well which...I tell you what, I'll draw the shape and then you fill it in?
59. Robert Alright, 'cause it would be like coming outward at us as you go up.
60. R1 What would be ... the shape?
61. Robert Umm, the levels... like the first level, uhh try...I wish like...
62. R1 Do...do the first level.
63. Robert The first level? Like you'd start...
64. R1 The very first level is what?
65. Robert Uhh one, one, one...or a, b, c. A...b...c...
66. R1 And it'd be in line like that?
67. 4:20 Robert Yes, then imagine like this is really coming out at us, so like, it's really here now and then here...
68. R1 Yeah, yeah... So then tell me what's here now?
69. Robert Umm, where? 2...you know a-squared, b-squared, c-squared, 2ab, 2ac, 2bc.
70. R1 And how you gonna... how you gonna arrange that? You can still gonna arrange it in a row?
71. Robert I guess umm, the way I would do was put the a^2 , b^2 , c^2 in the center.
72. R1 Ah-ha
73. Robert And then have the out...so it would be like... something like this...1, 1, 1, and then I got like 2, 2, 2...[Draws a circle around 1,1,1 in the center and writes 2s outside and encloses them with a bigger circle]
74. 4:59 R1 Ok, so let's circle that so that you don't erase it. Now you can erase anything else, if you got this in your head.
75. Robert Yeah, I can reconstruct these
76. R1 Then, how you're going to arrange these? You should tell me, tell me just coefficient wise how you gonna do those.
77. Robert Well, I guess...
78. R1 So, that's going to be level what? Level 2?
79. Robert 2
80. R1 Oh, this isthis is this fooled me didn't it, well this is...this is, we better us a...l
81. Robert Like here. This will be level 1, that'd be level 2
82. R1 Those are a^2 , b^2 , c^2 's right?
83. Robert Yeah, this is what a^2 ... b^2 ... c^2 [Writes a^2 , b^2 , c^2 in circle]
84. R1 But those are part of this level here
85. Robert Yeah so like this green encompasses...
86. R1 This, this,
87. 5:40 Robert It's kindda like this is it's own level but then green has these two levels

88. R1 Alright, so make the next level.
89. Robert The next level, so then...
90. R1 Oh no I wanted to know what that level was, what is that level going to look like?
91. Robert This ...
92. R1 That's a^2 . That's a.
93. Robert Oh, my bad... and then I guess this becomes a^2 ...
94. R1 That's 2 what? Tell me what those ...oh,oh...so that, is that 1 times what?
95. Robert I guess like 1 times a, 1 times b, 1 times b. So, I guess I can't represent like...I was trying to make it like so you could encompass them all but it I guess I'll take this out.
96. R1 Oh I see, yeah, yeah. That's clever, that's very clever but that really is a^2
97. Robert So this is really a^2 , c^2 , this is ab, wait...yeah bc, ac
98. 6:24 R1 Now I got it, so you're puttin' the 2 here, why did you decide to put this term outside that corner?
99. Robert Well I guess it would be umm, ca, well, I'm just choosing a direction 'cause I'm starting going like ca, ab...
100. R1 Oh, so your going ca...
101. Robert Yeah then ab
102. R1 but you're putting it there...that's a good place for it...
103. Robert Yeah, where it ends, so it like it ends...
104. R1 'cause you're going clockwise
105. Robert Yeah, then ab, and... so then there's...how many? 1, 2, 3,4, 5, 6... right?
- 106 6:52 R1 6, 6, yes you got 6 and show me again... no, no, show me the first time, how you going to arrange those? Right here, those...
107. Robert Oh you want to do this all over again? Alright so, 1, 1, 1...
108. R1 Okay, no, no. I, I just want this: ab, a plus b plus c. There's only 3
109. Robert Oh, okay.
110. R1 The...I mean, what's the very very top level here... I mean, this is a plus b plus c squared, what's the level before it?
111. Robert a plus b plus c
112. R1 Okay. And what's the level before that? Is there one?
113. Robert Ummm
114. R1 Is there a coefficient there?
115. Robert 1
116. R1 Ok, so what's this,
117. Robert $a+b+c$ to zero, which is just 1
- 118 7:30 R1 Oh, so I'll do that one.
119. Robert So, this is like a...
120. R1 There.

- 121 Robert plus b plus c and then a^2 . Here's 1, 1, 1, a plus b plus c
 122 R1 And look how kind I've been. I've done that. And so next one.
 123 Robert And then a...b...c...so you have one...I guess I can put it in...I guess
 it'll be easier if I put it here...like in the gaps in between the
 letters...so then like, it looks neater I guess...this is ac, ab, bc
 [moves 2s in middle of 1s]
 124 R1 I think better is a great word.
 125 Robert Yeah better, you know what I mean. So then like, this will go ...
 126 R1 Yeah, yeah, better is a judgment call, that's great, you like it
 127 Robert 2...2...And then I guess you can see it better and do like...and then
 the 6 is like these 3, so I guess...
 128 8:36 R1 No, I'm confused by the 2. Help me there.
 129 Robert Oh wait, there is no 2s...
 130 R1 that's why I put...alright
 131 Robert [looks at his work for a few seconds] Well see, what I was doing, I
 was showing ...you know how the triangle when you show the third
 row, you just don't show you know, 1 3 3 1. You know what I mean.
 I was gonna show the 1, 2, 1...1, 1. I was going to show it all, and
 tryin' to show it all in one.
 132 R1 I know what you're tryin'. Yeah I mean I hear you
 133 Robert Yeah, like this is 2^3
 134 R1 I don't know what you are tryin' to do, but now I hear you, but
 135 Robert So I was trying to show 3 to 3^{rd} with the 2s in there. I guess I'd have
 to cross them out.
 136 R1 But, but, but...Ummm you're not drawing a 3 dimensional object, you
 were just giving me a slice, so it's okay. So, are you picturing how
 this slice...I mean, that's a slice right?
 137 Robert Uh huh. Yeah.
 138 R1 And then what comes...so I guess if I put that on the board what
 would come on this next slice?
 139 Robert I guess, this would be the top, and this would go underneath it, this
 would go underneath it, and then what ever this is, is gonna
 underneath it...so I guess, what I am trying...it's something like
 this... these two go here you know what I mean?
 140 R1 Yeah, yeah, yeah, yeah.
 141 Robert And I guess there'd be a 6...and... where would it go...I wanna put it
 directly in the center.
 142 10:04 R1 I can see why because it's all everything...
 143 Robert Yeah, then everything fits neatly if its in the center [1s in the center
 and 3s around them]
 144 R1 What the dominant...uhh... geometric shape then,
 uhh...here...what's the dominant geometric shape
 145 Robert Triangle
 146 R1 Right, but the triangle is the dominant shape in Pascal's triangle too,

- ... I guess
- 147 Robert I guess, Some people draw it this way right
- 148 R1 I don't know
- 149 Robert I have seen people draw it this way where it's like 1, 1,1; etc [draws a graph with all rows of Pascal's triangle aligned to the right]
- 150 R1 Oh, oh,... oh probably software is forcing that...there is a line of justification there...can you put this in any shape other than...cause I see you got a triangle here, I got a triangle here, I got a triangle here, I see that, can you put it in any shape...one shape that accommodates all on this one... but I think you already get in trouble here but maybe you see it there
- 151 11:00 Robert Actually you can do a triangle here sort of like...no, I am sure you're fine but I don't ...
- 152 R1 How about one shape that accommodates them all
- 153 Robert Yeah, one shape that accommodates them all...a pyramid, that's what I would think you put,... like a Pascal's pyramid
- 154 R1 Oh oh oh...it would be like a hyper pyramid cause as each stage would be a pyramid
- 155 Robert Yeah, I think like, like the way I would do it is something like this with the circle inside of it...like like these would be the outside numbers on it 1,1, 1, you know and then this would be the 3s, six of them and then you'd have a 6 and then I guess [makes a triangle with 1s on the vertices with two circles inside it, the inner circle has a 6 and the outer circle has six 3s]
- 156 R1 I see your levels. Does that work out, the levels? Is that right?f
- 157 Robert Yes, cause when you have the triangle here, you have a 1, 1, 1 and have three 2s inside the circle
- 158 R1 And you'll always have the 1, 1, 1s...ah ah ah...can you carry it one more level? Is there going to be a middle level for the next round up?
- 159 Robert I think...yeah, it's too early to see like a pattern type of thing but I think that will add another circle, ... 1, 1,1 so if you stack it, this be the top, this be underneath it, so then it will go 1, 1, 1...
- 160 R1 That part I don't agree with...is this a triangle here? Is that one of your triangle?
- 161 Robert I guess you could make it your triangle like,...yeah make a triangle like put a triangle inside the triangle...or is that a satanic symbol and we shouldn't do that?
- 162 13:00 R1 I am pretty sure that in math it doesn't invoke anything...but does it preserve the other properties...what kind of properties do we want these 3s...there's a reason why you put those 3s like that
- 163 Robert Umm...
- 164 R1 I am in the water here, I don't, as it seems, [pause] ... do you want the 3 right under there, do you want that?
- 165 Robert I say, so...like, I think you look these cause they are all the, these are coming from the... let's say this is the a direction, you know what I mean, this is the b direction

- 166 R1 Now you are back to the other way you had it back here, remember when you put a 2 there, a 2 there and a 2 there, that's kind of corresponding with that but you deliberately put a 2 here and then I didn't have to remember directions, I think that kind of locked in, I think we were happy with that, you're taking it away, from me. But maybe this is will be better
- 167 14:06 Robert Yeah, I think...cause, see like, alright, this part of the triangle is like the b direction, adding bs as you go this way and I guess as you're going this way, you're adding as and you go this way, you're adding cs, so like lots of 3s, 3, 3, 3 ... and they kindda like all
- 168 R1 So, tell me how the threes occur there so I can follow you. We did that 3 right there, how do you get that 3? From up...from up above? How do I get that 3, tell me how I would know that 3 goes there?
- 169 Robert I guess this is 3 a-squared...see, then I don't...I thought maybe ...doesn't seem to sense now...this was a^2 , a^2 , a and a^2 again, you know what I mean, I guess, let me circle all the a^2 right there, this was all the a^2 , so there's only 2 a^2 so I guess, alright, this is a^2 , a^2 , b^2 , b^2 , c^2 , c^2 , so like
- 170 R1 Ok. Use yellow to connect what's giving you what or something
- 171 Robert Should have drawn it bigger...so like this is a^2 , a^2 , b^2 , b^2 , c^2 , c^2 and then bc, ac, ba, I don't know like any
- 172 15:48 R1 But, there is a property of the Pascal's triangle where you can use the original, the previous layer to generate... so, how would I use...what would be previous layer be here for that layer and how would I use that?
- 173 Robert This would be 2ab and you're adding "b", right? So that would be 2...
- 174 R1 Which 2 are you add..., what are you adding to this layer to give you that layer?
- 175 Robert Ok, wait, can I redraw this
- 176 R1 You can do...absolutely yeah
- 177 16:19 Robert So, let's just look at the inner circle. Like, 2ab, 2bc, 2ac, so like I guess you add the b, you add the a, ...I guess, this, maybe I was writing it wrong but this b, this is c, so this like kindda goes here and here, you know what I mean and this goes here and here and this goes here and here but I don't know how to like..
- 178 R1 You think it gets used twice
- 179 Robert Yeah
- 180 R1 Yeah, I guess then Pascal's triangle kind of works
- 181 Robert Yeah, but then I don't know, well, in the Pascal's triangle they go two directions too, they go left and right
- 182 R1 That's right, that's right, I am not too uncomfortable with that
- 183 R1 And then the 2s add together to give you a 6...
- 184 Robert But I don't know a way to represent it, like you know what I mean...
- 185 R1 I know, is the direction thing, is the direction thing the issue here

- trying to keep track of what those two are?
- 186 Robert Yeah, cause like if I put, I always put guess stays the same goes to right, no I can't say the same, you know I mean, I guess add c goes to right, add b goes to left and then a goes which ever direction, you know what I mean, cause its like you know its like in Pascal's it's easier if you don't add, you go this way, if you add, you know what I mean, no add... a, it's like ... actually this is like
- 187 R1 You're now showing the other end, ...they're kind of lined up between the two, it's like a bricks
- 188 Robert Yeah
- 189 18:03 R1 Why can't you brick this [1, 1, 1, from layer 1]? Brick that, go up another layer? and brick that
- 190 Robert That was how represented it before where you just got
- 191 R1 And we like that,
- 192 Robert But then I couldn't
- 193 R1 but imagine it coming out here, where would it lie exactly in standing out, how would those two, ...so, if you're headed like this, if this is the board right here, where exactly would you put the 1s, the 2s, the 1s you've got 2s and 1s out here, where would you put them?
- 194 Robert Like, Directly above?
- 195 R1 Do we, do we have a transparency?
- 196 18:40 Robert Like you know how you have a triangle and it goes like this, or a pyramid was it, like this is that a top of the pyramid?
- 197 R1 So, that would be the top there and then we slice it again and we got that, so, I want this layer here, this is the thing you're constructing, you have puts 2s and 1s out but, I mean, when it's sitting here where do you really want to put those 1s and 2s to get the prettiest picture? That's what I am asking. Can you imagine that in your head?
- 198 Robert Yeah, Is this a pyramid top or something
- 199 R1 It is
- 200 Robert Sortta
- 201 R1 I guess it's kind of full flat? out...and then it will help to draw a line back there, that's what you are thinking
- 202 Robert Maybe this isn't right, I think, that's why I kind of wished I had a pyramid so that we could really
- 203 19:20 R1 But that layer there, just the layer, vision that layer right here, where would you put the 1 on my hand, point at it
- 204 Robert In the corner, here like lets say
- 205 R1 Would you put it right above that one?
- 206 Robert I say, ...Yes
- 207 R1 But if you put it right ...oh, you would?
- 208 Robert Yeah, cause you know like when you draw
- 209 R1 When you do anything on the pyramid, it always comes out though
- 210 Robert But there's always one corner straight down, isn't there, I guess it

depends on how you look at it

211 R1 What about the corners every time... all these

212 Robert So yeah if, like the three corners will always be the one, so I guess, if you go out

213 R1 So where would you put it, put it on my hand

214 Robert Let's say it's top would be here

215 R1 So, now let's draw it here and do that

216 Robert So, you want me to, I see what you're saying

217 20:05 R1 I don't...don't want anything. I want you to get the prettiest thing that makes you happy.

218 Robert Well actually let's just start with...

219 R1 Oh, so you're happy here?

220 Robert With the ones on the outside...

221 R1 Yes

222 Robert And then

223 R1 yea yea yea

224 Robert the three...

225 R1 But... but now, so ... so we'll do that. Let's do it one more time here.

226 Robert Ok, I don't like that one

227 R1 Put those ones.

228 Robert So 1...

229 R1 One. Oh, ok so we've come out...now where do the twos...where would you put the twos? Think about your pyramid. This is a new slice in the pyramid. Where would you put the twos? And think about how Pascal's works. And then, when you write it down, you want to have that math gut of this is ...what I want! This may be what you want 'cause it certainly works out in our heads right?

230 Robert Yeah. I want them on the outside too? No.

231 R1 Oh, now your thinking I'm trying to give you an answer. I don't know. I've never done trinomials. I don't know.

232 Robert Alright. Umm...

233 R1 Maybe we're done over there.

234 Robert Like...the thing you're looking at. You said it went outward, since this is out more I think it would go out more too, so actually the one's wouldn't be on the outside they would be in the center.

235 21:06 R1 Oh.

236 Robert 'Cause the way I'm seeing it, it's like you said that umm...

237 R1 Where are the ones on the Pascal's triangle?

238 Robert On the ends. But umm, you said these are out further, right? Or, they are, obviously.

239 R1 I was...I was asking you.

240 21:18 Robert Obviously they're out further, so if you're kinda like... putting this

on like a line, you know like an angle, everything would go on the same...

241 R1 Ok, so show me.

242 Robert Otherwise it wouldn't be a pyramid right? It would be something weird. So if it's like 1,1,1, than this would actually get put outward to here and get added. So it would be like...

243 R1 So you've pulled the ones in.

244 Robert Yea.

245 R1 Ok.

246 Robert So it'd be...

247 R1 So the ones are always...aren't always on the corners.

248 Robert Nah, I guess not, yea.

249 R1 Well no no, I think this is...

250 Robert I think they're in the center now, all the time.

251 21:45 R1 Ok do it, do it. Show me the twos

252 Robert But then I don't know where to put the six, that's the only problem.

253 R1 You see it coming don't you?

254 Robert Yea. Let's see, this is what... six of them?

255 R1 Alright, so you're doing that next layer....

256 Robert Oh wait, there's five of them, sorry

257 R1 Oh excellent thank you (receives transparencies). So is that the, that's the first layer. Oh you're tryin' to, to compact them all down, I gotch you.

258 Robert I'm just trying to see like where it would go that would make sense. How many is there? One two three four five six. Alright, so...

259 R1 And how's it adding from this layer?

260 Robert Umm, the two...oh I see. Ok so the two goes with this one and it ends up here and the two goes with this one and ends up in the middle of them, I guess. Cause this like...oh, actually alright this one is in between this one and this one. Let me just...can I just draw little twos?

261 R1 Do whatever you want. Yea.

262 Robert You know, that would be erased.

263 R1 Oh, where are those two exactly?

264 22:46 Robert So, they're actually in between these, right?

265 R1 So do the one two....do this one like you really... I'm going to erase this. Okay? And I want you to write it like you want the second level. Actually, I still am a little confused what you want to do here. Put the 1 1 1 in the triangle, like you had. Now, here's the next level. This level right here. No no no.

266 Robert No?

267 R1 I want, there's...put a 1 1 1, like a 1 1 1. 1 1 1. Here's, here's layer two.

268 Robert Uh huh.
 269 R1 Show me where the, all 6 coordinates, all these coefficients go... on layer two. So the pyramid is peaking into the board right, and coming out. Are we agreeing? So, where would layer 2 go? 'Cause I'm just feeling confused.
 270 Robert I think. Can I draw on this? Alright. Put it up against the board?
 271 R1 So there's going to be a one there again. (referring to transparency)
 272 Robert One.
 273 R1 Alright.
 274 Robert And then there'd be 2, 2, 2...
 275 23:42 R1 Ok, now let's transcribe that. Let's erase this, put it back on the board. I want it to match up with this nicely. So there it is, can you do that? I bet I could turn this over and it'd probably leak on the board. Ah hah. Ah hah, ah hah. Is that close enough?
 276 Robert Almost. You want me to move this in closer?
 277 R1 Alright. Alright, is this a triangle of some sort? I mean what shape is that? Show me the shape that you're drawing.
 278 Robert This is a triangle.
 279 R1 So, oh oh oh oh. It's like a Star of David?
 280 Robert Yeah. Here's a triangle too.
 281 R1 Oh, that's interesting. I see it. I see it.
 282 Robert Yeah I guess...
 283 R1 So there's like vertices on some shape, I don't know what shape that is.
 284 Robert Mmm hmmm
 285 R1 What shape is that? Do we have a shape for that? That one's really out here? Is that one really out there?
 286 Robert Yea actually. That one should be out in the corners.
 287 R1 So what shape is that? Is that a shape?
 288 Robert I don't know. Circle?
 289 24:46 R1 It's certainly a circle filled in...
 290 Robert No, 6 sided circle.
 291 Robert But then, before I came up with a triangle then can you do...does it form like...like see how these ...so the threes will go 1, 2, 3, 4, 5, 6.
 292 R1 Ahhhhh. And then the 6 would go?
 293 Robert Is that it?
 294 R1 And, so erase out everything that didn't...
 295 Robert Didn't come with us?
 296 R1 That doesn't come. Yea. So the ones are always staying in the same spot huh?
 297 Robert Yeah I guess so. Oh so I guess like, that's what I thought they went straight down and then everything else kinda goes out. But I don't know how the 6 like... why does it end up in the center?

298 R1 And then what's the next level?

299 Robert Then I guess. Well I guess let's erase this triangle. Does this form like...so this is a one, one. Now this, if I drew this right would this form a triangle? And then would this form a triangle? And then this form ...you know what I mean, would this form a lot of triangles? Or would you know, you kinda got a bigger triangle that encompasses all 3 or all 4 numbers, you know what I mean? Something like this. This is all getting messy.

300 26:10 R1 Can you take the red and show me all those things that involve "a"?

301 Robert Yea, let me just clean this up a little.

302 R1 Like, just go through and circle, maybe on all of the layers the things that involve "a".

303 Robert Alright, let me just redraw this. Oh I guess...

304 R1 Lets do red.

305 Robert Red, my bad.

306 R1 Let's still do red.

307 Robert That's, that just a different green.

308 R1 I think this is a... there, that's a safe color.

309 Robert Oh no, 'cause I think I just drew over it. Oh wait, that's green. Alright. But it's got a red cap.

310 R1 Oh yeah, that's a, that's a fool me cap.

311 Robert So these are "a's".

312 R1 Ahhh..

313 27:00 Robert And then, oh I see what you... oh not, not what you're trying to say, but I know what...

314 R1 I didn't

315 Robert Yea. No, it was kinda like... this is "b's".

316 R1 Ah hah.

317 Robert And then I guess that...oh that's black. That's green. And then this is "c's"?

318 R1 Hmm...are you convinced of that?

319 Robert Well I guess 'cause you can break this into 3 triangles. You can break this into 3...

320 R1 Talk, talk me through your "a", just talk me through you're "a".

321 Robert Ummm, this is a^3 ...

322 R1 Okay.

323 Robert And then. Oh wait, no I'm not convinced of that...because the b would have to overlap the a in more than one spot, you know what I'm saying. Cause like...

324 R1 I think.

325 Robert This is why I do horrible in math, 'cause I'm just...

326 R1 You do horrible in math?

327 Robert Yes

328 R1 Ahhh...

329 28:01 Robert Ummm, alright so...this would be...alright I know this one will have it, this one will have it.... I'm guessing both these will have it...but then something over here's gonna have it too. Probably...this and this...so its kinda like just...Well actually, the triangle would be the ones that have a 4 is the a^2 or above. No actually, 'cause then that would include the center piece. So...

330 28:56 R1 So that, that, what's special about that three and that three?

331 Robert These are a^2 's

332 R1 Got you. And what is that one?

333 Robert These are just single a 's...

334 R1 And what's...?

335 Robert and this is a single a too.

336 R1 Oh. Oh oh oh oh.

337 Robert This is triple a , a^3 . And then I guess with the ...

338 R1 What happened to the, the last one there. Why does that look asymmetric?

339 Robert Oh, 'cause I deleted it to... 'cause I didn't want to include it in there. So...I'll just pull it out...

340 R1 Maybe it could be like....up, down like a batman something or other.

341 Robert Yea. Ummm. Well it's not drawn perfectly, right?

342 R1 So if you umm...

343 Robert So these are a 's right?

344 R1 Did you ever watch "Do You Want to be a Millionaire"?

345 Robert Yea.

346 R1 You've seen that once? If you had, if you had right this moment, right this moment...

347 Robert Final answer ...

348 R1 No, no I know you, I know you, I know you wouldn't take that, you would be like...huh what's the point on that 'cause you just... If you could either one have a couple of hours by yourself thinking about or two, have some...one of your friend's with you, which would you pick right now?

349 29:55 Robert I guess friend because if you, well if you weren't here I would have said that was the ... I would've never gotten this far, I would have just concluded that that was the answer. You know I kinda needed like...

350 R1 Whatever comes after that...

351 Robert like your questioning to, you know, kinda make me question myself. You know what I mean, 'casue I don't...

352 R1 I would help you, yea.

353 Robert I don't have the... I mean maybe I do...you know I wasn't seeing it perfectly, but then you know, even though you didn't, you didn't

- directly contribute anything you're just questions like kinda made me think about it differently.
- 354 R1 But a friend can contribute?
- 355 Robert Yeah a friend, yeah more...
- 356 R1 A colleague.
- 357 Robert A colleague, yeah.
- 358 R1 That's the difference I'm not being your colleague. I'm not being (inaudible) your colleague.
- 359 Robert So you know, I like guess this would be all of the a's, and I guess there'd have to be, you know the b thing would kinda encompass this
- ...
- 360 R1 Is it hopeless for the next level?
- 361 Robert I think... I could construct a...
- 362 R1 What would the next level be? Can you see it?
- 363 Robert Yea... I can construct like a pyramid out of paper mache that would be...
- 364 R1 That pyramid's gotta have an inside...
- 365 Robert Yea...
- 366 30:54 R1 And where, how is...how...if I slice a pyramid, like if I set a pyramid on the floor and sliced it parallel to the floor, what shape is the face that I get when I slice it?
- 367 Robert Triangle?
- 368 R1 And then where's the triangle in this?
- 369 Robert I guess...the floor...the whole thing? Is it kinda...
- 370 R1 Show me... take another color and show me the triangle that's sliced. I mean I see the triangle here.
- 371 Robert Yeah.
- 372 R1 I think, unless your saying this is the triangle. I guess there could be other triangles. So here's...here's top, here's the next layer, the next layer, and the next layer and I wanna see those faces. How are they going to sit on the face each time?
- 373 Robert I just don't think that it's always gonna to be in the direct corner of the triangle. That's my... I don't it's always gonna
- 374 R1 There's not going to be numbers in the corners?
- 375 31:42 Robert Like alternating steps might have numbers in the corners.
- 376 R1 Does that have numbers in the corners?
- 377 Robert Ummm... Well I guess it has number in a corner, right? Cause it got this one.
- 378 R1 Ohhh... That's not very symmetric, huh? Mathematicians are never happy with asymmetry
- 379 Robert Yeah... I wish Liz was here 'cause she made one.
- 380 R1 Did she?
- 381 32:02 Robert Out of papier-mâché. She made a pyramid... and it was very complicated but...

382 R1 Only I mean this... we can write slices on the board. It's not that bad.

383 Robert Yeah... (drops pencil) oops. And ummm... yeah so I guess that's uhh, that's the hard part is tryin' to see it.

384 R1 Sometimes the math we go through when it gets difficult and we say ... well let one of the variables be zero, right? So if I went through and let c be zero in all these what would I, what would happen?

385 Robert You'd have Pascal's triangle in 2-D, but just the old one if c was zero. That'd just be $(a+b)^2$.

386 R1 Is that right? Is that right? I would have one right and then here I get $a+b$, ok I give you that. What would this give me?

387 Robert Well I guess could I just kinda like ...

388 R1 Circle what?

389 Robert cross out what would, not cross out just put like a little

390 R1 Oh yeah yeah sure sure

391 Robert thing above what would disappear

392 32:45 R1 Accent, that would disappear, that would disappear, that would disappear

393 Robert These would all disappear except for that

394 R1 Oh that does look...uh...ohhh

395 Robert Why did you mark that?

396 R1 Oh that was an accident.

397 Robert Oh ok.

398 R1 Oh it'd wait. These, these are equivalent statements, never mind.

399 33:00 Robert There, there are equals here

400 R1 Yes, so I...

401 Robert So what, whatwhat would disappear on this line? This is the line.

402 R1 It's easier to look at this one. So this would go away, this would go away.

403 Robert Oh yeah look at that.

404 R1 So you would have this which is just like.

405 Robert Well how about all of this then?

406 R1 That's just... that's one...that's two cases that can't be right.

407 Robert I already crossed everything out

408 R1 We'll just come down here

409 Robert Oh yeah, ok.

410 R1 This is the line. This is an equivalent line

411 Robert So everything..

412 R1 And then what would you...Ohh so when you circle the things that have just a and b, at each level, so this is just ...the things that just have either a or b but no c

413 Robert Oh see the way.. oh

414 R1 The compliments of the things that have ... so here when you circled

what was in a
 415 Robert Mmm hmmm I circled things that had a..
 416 R1 The compliment should be Pascal's triangle shouldn't...Pascal's line
 shouldn't it? Because here you got rid of all the c. If you circle...
 417 33:43 Robert And it actually is $1 - 3 - 3 - 1$
 418 R1 Right, but it is arranged funky.
 419 Robert Mmm hmm
 420 R1 Can you arrange it... so I'm... I'm pushing to the very limits of .. of
 human endurance here. This should be a Television show. Can you
 arrange this so that every time you blank out one of the variables,
 what's left over looks like a line ... a *line* out of Pascal's triangle?
 421 Robert Yeah, I could have uhhh... one, three, three, one, three, three, one,
 three, three, six
 422 R1 Now that one there, that doesn't... ok I'm gonna blank out one
 423 Robert One...so then
 424 R1 I... I think you've almost... no I think you're tryin to cheat me,
 show... if ... what, what...that a is what? I mean that one is what?
 425 Robert a... b...c
 426 R1 Oh so this is a cubed right there
 427 Robert Yeah a cubed
 428 R1 So suppose I wiped out
 429 Robert $a^2b \dots ab^2 \dots b^3$
 430 R1 Ohhh, ohhh... I like that
 431 Robert Umm 3 this is $b^2c \dots 3c^2b \dots$
 432 R1 Uh huh
 433 Robert and this would be c^3 and this would be $3c^2b \dots$ do we already have
 that? Yeah we do.... c^2a
 434 R1 Uh huh
 435 Robert Then ca^2
 436 R1 So if I wiped out c, what would be left over?
 437 35:02 Robert So if I wiped out... so you want me to write.. I guess I'll just
 438 R1 No, no use a yellow. Be careful.
 439 Robert Oh use yellow... alright.
 440 R1 Something
 441 Robert I'll use the real red
 442 R1 Yeah
 443 Robert So the c...
 444 R1 If you wiped out c, what would be left? Wipe out the numbers that c
 are involved with.... Yeah, the numbers that c is involved with
 445 Robert Wait... is that right?
 446 R1 Is that one got a c involved with it?
 447 Robert Yeah

448 35:24 R1 Strike it
 449 Robert 1.. so these are like
 450 R1 Does that number have a c in it?
 451 Robert Yeah
 452 R1 Strike it
 453 Robert Does that number have a c in it?
 454 R1 Ummm
 455 Robert This three right here
 456 R1 Wait so what I did... I'm gonna redo this because I rewrote three's...
 Can I erase this now?
 457 Robert I think
 458 R1 Cause now I've asked ... I've really just been nasty and asked you
 beyond reason.
 459 Robert Nah nah
 460 R1 Beyond... So draw the whole thing again, but remember my, my...
 my, I want all the Pascal Triangle things to be a line segment
 461 Robert Ok, so
 462 R1 And I... show me one of the Pascal's things.
 463 Robert Wait.. let me just write it big.. that's probably my
 464 R1 I think... I think
 465 Robert So one three... three ...
 466 R1 Excellent, show me one of the Pascal lines.
 467 Robert Oh wait, let me make it more triangular.
 468 R1 Ahh... ahhh. I assume if someone complains enough you...
 469 Robert No, no 'cause now it would be easier for me to see
 470 R1 Look at that... my math gut kicks in and says ah ha ha that's pretty.
 Is that... that... so now that's a layer... what's the uhh, what's the
 uhh, use another color and put in the old layer, show me what the
 old... like it was a trans... oh no no no actually... ummm... thank
 you. I've got... suppose this was the previous layer.
 471 Robert Oh there's Liz. You made a Pascal's Triangle in 3 -D right?
 Remember you brought that?
 472 R1 Yes, Pascal's Pyramid.
 473 Robert Pyramid.. yeah
 474 R1 So where, where does, where does the... the previous layer lie on that.
 And hold on a second.. before you right on that. Before you write on
 that let's use this other pen because I think it erases easier
 475 Robert ??Yeah, these other ones are for the dry erase board
 476 R1 Try this one... let's stick... it absorbs. Write in what the previous
 layer looks like. Sticking to my... my need for everything to be a
 line. Oh...
 477 37:19 Robert Like that
 478 R1 And then how does that add work? How does the adding work?

479 Robert These two go here, these two go here, these two go here, these two go
here, here... here ...

480 R1 Ohhh...and then where did the six come from?

481 Robert These three make this one (points to 6)

482 R1 Now what is the last step before were gonna wipe our hands of this?
What are we gonna ask for now?

483 Robert Uhh

484 R1 What would you ask?

485 Robert Oh you want me to draw the thing that you said where you take out
all of the c's?

486 R1 No, I am ... I am satisfied. It isn't that pretty, but I uhh

487 Robert The next layer?

488 R1 Aah, we always ask the next layer. She me the next layer and then
we'll all... we'll all applaud and ...

489 Robert Alright, this is fun

490 R1 I have never seen this, so I don't

491 Robert One

492 R1 I have no idea what the...

493 38:02 Robert Actually, I'm just gonna draw like a barrier

494 R1 And then do another color so that we don't, so we don't... You can
stick with yellow so that we don't get confused or something.

495 Robert Oh, I'll just use black.

496 R1 Oh yeah, yeah, yeah. Black says we're sure of what we're doing.

497 Robert So it'll be...one

498 R1 So this is the layer out, up, down, something, right?

499 Robert Yeah, like the next layer...oops six, oh what'd I do, four

500 R1 That's

501 Robert Oh I see, so that's gonna be four, one, four, six, and then in the center

502 R1 Now this is conjecture because we haven't computed it out right?
Your conjecturing. Your using your pattern to conjecture what its
gonna be if we actually computed it out.

503 Robert Yeah but I think it... it's gonna hold... it'll hold

504 38:45 R1 So you're a believer now, ohhh... ohhh

505 Robert But then see ... I don't know ... Is there going to be another side of
it? Where it's like nine, nine, nine, nine? I'd say yes.

506 R1 I'm gonna ..

507 Robert Well let's see...

508 R1 Can you at least compute how many numbers total there will be in
the next layer?

509 Robert I'm sure there's ...

510 R1 Is that an easy thing to do?

511 Robert yeah 3 to the 4th... oh there's going to be a lot

512 R1 81

513 Robert Yeah 81
 514 R1 Ya think. Or do they add up to 81?
 515 Robert Oh they add up to 81. So what do we have here? 10...
 So what is this? 10... 14...15... Oh wait, I don't want to do that
 because that would be confusing. Well these are 14 times 3.... which
 is 2...42...plus 3 so we have 45 already
 516 R1 14 times 3... help me on that....oh plus the three
 517 Robert These 4... 6 four's the 14 and there's three of them plus three for the
 three so ...that's 45 ... so how much more do we need 36?
 45...6...36...36...
 518 R1 36...
 519 Robert I'm gonna say... we can't have it for everyone cause then..
 520 R1 'cause we see all sorts of addings go on don't we..huh huh huh
 521 40:01 Robert Yeah I don't....'cause this would be 9, 18, 27, 36, 45... you know
 what I mean. ... that would be too much.
 522 R1 It's like thousands.
 523 Robert Yeah... I guess the easy way what...like what's...
 524 R1 Uhhh
 525 Robert Do what? (laughs)
 526 R1 Oh, whatever. I thought I read your mind for a second.
 527 Robert I know, I know. I'm gonna write out the umm ...the options. Like
 what 4-0-0. But that's alright we're gonna know that one. So I
 guess the...the case we're tryin' to look at is $2 - 2 - 0$. Right?
 528 R1 Do we know that one?
 529 Robert Well let's see $4 - 0 - 0$ is umm....
 530 R1 What's gonna be inside this on the layer? On the outside layer...
 what's the difference between the things that appear on the outside
 layer than the things that appear on the inside?
 531 Robert I guess this is $4 - 4 - 0 - 0$ right? That's what the corners are right?
 532 R1 Yes, I see it ... I see it. You've got some kind of coordinate system
 going on. I gotcha ya.
 533 Robert And this is uhhh...4... Which one has four options? $1 - 2 - 1$, $2 - 1$
 $- 1$, $1 - 1 - 2$, $2 - 1$...
 534 R1 Well, all of these have what property?
 535 41:05 Robert That they have noooo, oh whatever letter. It's missing one letter.
 536 R1 Right, right. So the outside is always missing at least what? One
 letter...
 537 Robert Yeah..
 538 R1 and this is missing two letters so the insides gonna have what?
 539 Robert It's gonna have all three?
 540 R1 There was a question mark at the end of that.
 541 Robert Uh exclamation point. (laughs)
 542 R2 Robert, umm.. Liz said that uhh... something about Ankur's problem

			is in there ... the solution.
543	Robert		Yeah, cause ummm
544	R2		Can you explain that to me... I don't know if it's...
545	Robert		Use three colors.
546	R2		What was that again?
547	Robert		Umm... the Ankur's... four tall, three colors, you have to use one of each color.
548	R2		So how is it in there?
549	Robert		Ummm...it's the inside of it, not the...the outside.
550	R2		Why?
551	Robert		Because these only have two of each color, you know what I mean? Whereas the inside has, you know, all the colors.
552	R2		Right but what was Ankur's condition? Didn't it have to have exactly two of one color?
553	Robert		Well, it's four tall and you have to use every color so yeah exactly 2 has to be...
554	R2		So where are those numbers in there?
555	Robert		Umm...I the... this ring... the threes... cause those are like, that's like... no wait. This won't work 'cause this is three tall, so there can't be... this is the three tall case. The yellow triangle, so that... you can't have two of one color in there.
556	R2		So Liz
557	R3		Well, they haven't done the inside yet, it's the inside of the black one
558	Robert		Yeah I haven't done the inside of this.
559	R1		Yeah we're trying to ...yeah. Keep... keep
560	R2		So can you do that?
561	R1		He can.
562	Robert		Ok
563	R2		I'm curious to know how Ankur's problem fits in there.
564	Robert		Umm...well I can write it... it would be like...what 4 factorial over 2 – 1 factorial. 3... it's gonna be this...
565	R1		Ok, there's one
566	Robert		Umm three times, right? So what is that...
567	R1		I don't know
568 42:54	Robert		9.. 6 times 24... 12. So where's the twelve gonna come from?.... oh... these... I get it... this is gonna be 12...12... 12...
DISK 2			
Line	Time	Speaker	Transcript
569	0:40	Robert	There's like...you know there's 12s everywhere
570	0:41	R1	I thought there's lots of places where 12s made sense
571	0:43	Robert	But then, this 3 is included in this one and this one so it will be counted twice, like there's going to be an overlap
572	0:54	Robert	Let's say this is a-squared b and this is b-squared a, if you add an a, add a b, there's gonna be...

573 1:00 R1 You're adding a's and b's
 574 1:01 Robert Multiplying, sorry...It's gonna be duplicates. Like take this one case...let's just say this is a^2b right, and this is ab^2 ...and this is ac ...well let's multiply this by b ...
 575 1:26 R1 What letter is this...what letter does this corner represent of the pyramid?
 576 1:31 Robert a's
 577 1:36 R1 Ok, You really think...are we agreeing on that
 578 1:40 Robert NO
 579 1:45 R1 No, we are not agreeing on that?
 580 1:50 Robert This is a^2c , right?
 581 1:51 R1 aha, I see symmetry
 582 [changes the second row to show a^2c and a^2b]
 583 1:52 Robert So, 12 is going to be the one that includes all three...then you multiply this by b and this multiplies by c and then this is abc [to the 6 in middle]
 584 2:09 R1 What would you multiply that by ?[pointing to the 6]
 585 2:10 Robert a, b and then c. So 'a' goes times that, yeah
 586 2:14 R1 What goes times that? [pointing to a^2c]
 587 2:16 Robert B
 588 2:17 R1 What goes times that? [pointing to a^2b]
 589 2:17 Robert C
 590 2:21 R1 So, you're saying the 12 does belong there
 591 2:22 Robert Yeah, but then when you go around, I think you will count everything twice
 592 2:25 R1 I have choices about... you mean..ahh..I get a choice about which 12 I wanna use
 593 2:31 Robert Yeah like when you go down later here ...this one and this one and this one [pointing to a^2c , $6abc$ and ac^2] will produce two of the same...cause this is c^2a right..so like you multiply this by b [ac^2] and you multiply this by [$6abc$] c , this is already going to be covered
 594 2:55 R1 You multiply this by what [$3ac^2$]
 595 2:56 Robert B
 596 2:57 R1 and, you multiply this by what [$3a^2c$]
 597 2:59 Robert B
 598 3:00 R1 This doesn't seem to be the same game then

 599 3:02 Robert But its gonna with this one right?
 600 3:04 R1 And that would be
 601 3:21 Robert You multiply this [ac^2] by b so you get abc^2 and you multiply this [$6abc$] with c and you get abc^2 ...and...
 602 3:27 R1 What was that?
 603 3:30 Robert You get a^2bc [multiplies a^2c by b]

- 604 3:37 R1 Oh That's not important, that one is [pointing to a 3 in the last row]...what's the other one important for these two..
- 605 3:39 Robert So when you multiply this $[ac^2]$ by b, and this $[6abc]$ by c, you get the same thing
- 606 3:47 R1 Which 12, I am sorry, what 12 are we looking at?
- 607 3:51 Robert Ok, I will write it like this...[writes ac^2 , bc^2 , abc at the board]
- 608 4:00 R1 Yes, please...Show me which three those are
- 609 4:11 Robert [points to the triangle], bc^2 is right here, and then...[circles it, circles ac^2 and $6abc$ on the triangle]
- 610 4:21 R1 ah-huh, ah..
- 611 4:27 Robert and then you multiply this $[bc^2]$ with a, this $[ac^2]$ with b and you get abc^2 , you do these two right, these are two separate cases right, you multiply this $[abc]$ with c, you get abc^2 but then here,...these two right $[bc^2$ and $abc]$, these are two separate cases, but here you are going to get abc^2 again when you do these two, there will be two cases from which you get abc^2like when you combine these two $[abc$ and $ac^2]$ and these two $[abc$ and $bc^2]$, they are going to make the same thing twice
- 612 5:11 R1 You have argued that...I am echoing I think...you tell me if I am wrong...you can't have all those 12s up there
- 613 5:23 Robert You can have them all up there it's just that things are going to be counted twice...cause...like
- 614 5:26 R1 and, we don't want that
- 615 5:29 R1 You already argued to me that I only get 3, I feel you, I feel you that of course we can only get 3, you give good arguments about that...What does your math gut or your formal reasoning say what 3, between that one and that one, which one? [points to 3 in the beginning of rows 2 and 3] Maybe both of these and one this one down there and leave the other out...
- 616 5:55 Robert These are the ones that are causing problem
- 617 5:57 R1 They are the trouble maker 12s
- 618 6:00 Robert Yeah
- 619 6:02 R3 How are they causing troubles?
- 620 6:03 R1 How are they causing troubles? You don't like 'em?
- 621 6:06 Robert Cause that's where this problem came up
- 622 6:09 R1 Well, how did this 12 appear, it is the sum of those three? [pointing to a 3, 6, 3 on the diagonal]
- 623 6:15 Robert Yes [draws a solid oval around 3, 6, 3 on the diagonal]. Maybe something like this 12.
- 624 6:18 R1 Is that how, you...
- 625 6:24 R1 Where did that 12 come from
- 626 6:25 Robert This 12. The way I did it, it came from these three [points to a triangle of 6 and 3s]
- 627 6:31 R1 Can it come from there? Let me ask you this
- 628 6:32 Robert Yes, it can...
- 629 6:33 R1 It can and you intended it to

- 630 6:35 Robert The ones alternate with each other. These two are going to have a problem too...these [circles 3s on second row and 3s on the diagonals of row 3 and 4]...these will have the same problem
- 631 7:15 R1 I have a problem, now... [laughing] I am lost, but, it's Ok, I get lost easily
- 632 7:24 R2
intervenes Want me to try to figure something out, I can tell you what I remember from making it, from making it, you try to find visually two numbers that add together to give you the 12, and visually looking at it, looks like these three give you a twelve here and these three will give you a 12 here and these three will give you a 12 here [points to triangles of 3s and 6s that add to 12]
- 633 7:46 R1 What do you say to that?
- 634 7:50 Robert ...The problem was...I was thinking why is it these three and not these three or these three...
- 635 7:53 R1 Yeah, yeah..
- 636 7:57 Robert That was the problem I just came up with..
- 637 7:58 R2 I don't know, do you want me say anything
- 638 8:00 R1 Sure!
- 639 8:04 R2 Well, I am not so good at looking at these things [pointing to bc^2 , etc on the side] but when you pointed to this, I said if you but the b here, multiply this [a^2c] with a b, and multiply this [a^2b] with a c here and multiply this [abc] with an a, that will give you $12 a^2bc$
- 640 8:16 Robert ...but those will all have duplicates...I don't think they give you 6 unique ones...so I think you need these[points to three 3's and the 6]
- 641 8:21 R2 What are your duplicates?... Well, let's stay to start with that will give us 12 and then, we don't have any duplicates yet
- 642 8:29 Robert Oh yeah, yeah
- 643 8:30 R2 and leave out something else that will give us duplicates now
- 644 8:34 R1 She's giving an argument of how these three [points to the top row 3s and the middle 6 triangle] can produce the 12, and, I am still sitting where...well, does her argument fail with these three? [pointing to 3s on different rows and the 6]
- 645 8:45 Robert I think it would be these three [the top triangle] because it's the same, same thing, maybe, except you have a^2c and a^2b ?
- 646 8:54 R1 [Repeating and pointing to the triangle] that's a^2c and that's a^2b ...no, no,... I don't mean to...
- 647 9:00 Robert and that's abc and then times [a^2c] b, [a^2b] times c, and these two[abc written twice] times a and then times a
- 648 9:14 R1 ...and that times a, just what she said, when b hits that you will get a^2bc ... I am buying that..
- 649 9:17 Robert and you will get a^2bc
- 650 9:22 R1 and these are all different coefficients, so there is no overlap
- 651 9:23 Robert You will get a^2bc twice, for both of them
- 652 9:24 R1 What's both of them
- 653 9:26 Robert ahh..Maybe if I arrange it, like you have a^2c , abc and over here you have a^2b and abc and then you add an a, no, you multiply an a, I am

sorry, no multiply b, then you get a^2bc , multiply with a here, get a^2bc , same logic, again a^2bc , a^2bc ,

654 9:57 R1 So, when you add, tell me exactly, use your fingers to show me, how she... you're hearing her say you are gonna add these things ...how many additions are you seeing as occurring here

655 10:10 Robert This is ..

656 10:11 R1 This is multiplication...ok, so..

657 10:13 Robert This is 2 of 12 ...and I guess you do this 6 more times, ...ten more times, ...you are going to do these 5 more times

658 10:28 R1 This will be three times, there's a 3 here [a^2c], agreed?

659 10:29 Robert Yeah yeah..

660 10:30 R1 So, let's put it there, if you agree..Is there a 6 there [abc]? Is there a 3 there [a^2b]? Is there a 6 there [abc]? How many times does this appear in the equation, in the previous equation, this term right here [a^2c],?

661 10:43 Robert Um...3

662 10:44 R1 Really, here's one show me the other two

663 10:49 R1 Oh, I am sorry... a^2c is right here, where's the other place it occurs, this whole term?

664 10:58 Robert This whole term?

665 11:01 Robert Once

666 11:03 R1 How many times does the 6abc appear?

667 11:06 Robert Once

668 11:07 R1 How many times does that appear?

669 11:07 Robert Once

670 11:09 R1 So, show me how its, 6abc term is counted twice

671 11:11 Robert Cause, going to here and where's the other term a^2b [points to 6abc, $3a^2b$ and $3a^2c$ terms]?..So that's why....

672 11:19 R1 So, I see what you're doing...You are adding these two [going across], you are adding these two and then you are adding the two results [going down] Is that what you were claiming that you wanted him to do? [talking to R2]

673 11:27 Robert Yes,

674 11:29 R1 I am not quite sure what he's doing actually now that he explained it

675 11:33 Robert You see a a^2c and an a^2b here right?

676 11:34 R2 Yeah

677 11:35 Robert So, when you add a^2c and abc you get a^2bc , right

678 11:43 R2 I'm sorry

679 11:45 R3 He's multiplying by b and then he's multiplying by a

680 11:50 Robert and you're getting the same answer twice, you're getting a duplicate..it's like getting a tower with two 'a's on top

681 11:58 R2 Let me go back to it, $3a^2c$, b is like blue, so we have a^2c is.. that's like ...two whites and a...

682 12:10 Robert Let's call it apple, blue and what's a c? ...a cherry, yeah

683 12:17 R2 Two apples and a cherry and then you put a blue on top

684 12:19 Robert Yeah

- 685 12:20 R2 Ok, then we have one with each color and you put an apple on top of that...I still don't see why you get dupli..., so, that's fine, you are not duplicating anything, you got blue on top of one and you got apple on top of one..
- 686 12:36 Robert Yeah...but then how is...apple's a color ... [laughing]
- 687 12:51 R2 That color goes with "a"
- 688 12:59 R1 You almost got me convinced...I think I am understanding you better now and not really her...
- 689 13:08 R3 That helps me Robert, thank you...
- 690 13:09 Robert She's saying like this is a, a, c, this is the way it is arranged and then you are adding a b at the end and ...and then you have a...then I guess you could too duplicate this way, cause then you have an abc and then you are adding an a at the end, and then you have ...that's the first case and then the second one you have 'a', 'a', 'b' right and then you are writing a c at the end and then you have abc and writing an a at the end
- 691 13:42 R2 I still don't understand why you are doing it again....can I go back to what we are doing with the trinomials to get from one row to the next is that we are multiplying every term in here by a and then we are multiplying every term here by a b and then we are multiplying every term here by c and we are only multiplying this [6abc] by an a once, and the one time we do it, we get a a^2bc and that's the thing I want you to add to the a^2bc that comes from here and the a^2bc that comes from there
- 692 14:10 Robert OK, but then...
- 693 14:15 R2 And to get there we have used the b on this one and a c on this one, so we have used up the a..the b here and c here, so have to remember that, makes it a lot easier to get...so, we are gonna multiply this by a b and this is going to add somethings here, and then we multiply this by a c and its going to add somethings down there, and that's our three times, does that make sense...
- 694 14:37 Robert Yes,..I guess but....
- 695 14:45 Robert But she's saying you only want to do these three, these three and these three, but could you...look at all 3s and then you add this [6] to itself...would that also work? but that will give me 18...add it to itself three times
- 696 16:00 R2 But, I don't want 18, I want 12
- 697 15:08 Robert Oh yeah, well, 18 plus 18 is a 36 too, I guess you could add one to [circling his finger over all the 3s]...then you are adding a to... you are multiplying a to it, multiplying b to it, multiplying c to it, so then, that's 6 times 3 and then you add..multiply 1 to each of these [circling the 3s], so then this is like 18, you add this together, and you this [6] to itself 3 times? Or multiplying...saying add..
- 698 15:34 R2 Well adding it 3 times is multiplying by 3
- 699 15:35 Robert Yeah
- 700 15:36 R2 Yeah that would work mathematically, but in terms of building

- towers and in terms of multiplying by these, I see it better the other way
- 701 15:41 Robert The only thing was.. how come you choose these three [the top triangle] and not these three [the side triangle]? Cause like...the way you said it, you just look for something to add to 12, right? ...I don't know why this one [the top triangle] is better than this one [side triangle], you know...
- 702 15:58 R2 I am not sure I can tell you that
- 703 16:00 Robert The only reason why I decided not to choose this one is because this has two a ... the two a-squares and I thought that would cause the problem, but then, I guess you could look at it this way, like this would cause a problem because you know, I have a c-squared and one c, so I think... I don't know if its like arbitrary the one you chose or what..
- 704 16:19 R1 Well write down. ...pick two of the twelves that you are trying to choose from, and let's...let's make some space so we can all see the...let's take this space here and write down the..the three terms for one choice and the three terms for another choice, and let's... so we can ... we can make our decision deliberately
- 705 16:40 Robert So you got 3 a^2c 's right? (writes 3 a^2c 's) .. you have three a^2 ... (writes 3 a^2b 's)
- 706 16:47 R1 and let's put those three's there
- 707 16:49 Robert Oh, I'm writing three of them. Right?
- 708 16:51 R1 Oh yeah. You got me.
- 709 16:53 Robert So we can just do it that way and then you've got abc ..abc.. I'll just write six of them. [writes abc 6 times]
- 710 17:04 R1 And.. and, could you show me... umm...here take the green and show me the three, ...the ones that you have selected for that batch
- 711 17:11 Robert So, we want this triangle...this triangle that we said [top triangle]
- 712 17:14 R1 Okay so put that triangle there... so that big green triangle. Now show me the other that your... that you could choose. Something else you could choose that's not something that she'd want.
- 713 17:24 Robert Oh, the one I wanted?
- 714 17:26 R1 Yeah, oh I didn't know you wanted it but..
- 715 17:27 Robert I am not sure which one I wanted.
- 716 17:27 R2 Maybe with a different color.
- 717 17:29 R1 Well something different than, that you argued could also have been used
- 718 17:32 Robert Oh ok umm. I'm not sure which one I want right now but..
- 719 17:34 R1 Right right right, I'm not, I meant I'm not forcing you to buy something.
- 720 17:38 Robert Like you know we can pick this one [side triangle in red]
- 721 17:39 R1 So let's write those down
- 722 17:43 Robert Umm...I guess, can I just write next to it in red?
- 723 17:44 R1 Why not?
- 724 17:46 Robert [writes a^2c in red] $a^2c, a^2c, a^2c \dots b^2a$ or ... c^2a , I'm sorry. [writes ac^2

- 3 times in red] and then these just kind of carry over [writes abc 6 times in red]
- 725 18:02 R1 Which is better?
- 726 18:06 Robert Umm...I personally think, this one [pointing to the red column]
- 727 18:10 R1 The orange?
- 728 18:12 Robert Because like if we continued this way..Is this orange? Red? Whatever. You will have one set...I don't like how there's all a-squares here. That's...that's I guess my...
- 729 18:22 R Uh-huh, why don't you like that?
- 730 18:23 Robert I don't think it really makes a difference, but...
- 731 18:25 R1 But, why don't you like it?
- 732 18:27 Robert Umm...it feels like, cause the next one will be all b-squares, next one will be all c-squares, so maybe I do like it... Yeah, I guess it doesn't matter.
- 733 18:35 R1 Yeah but it, but no it it...I think it could matter. Liking something matters a lot you'll find. So what is guiding your judgment about "like"?
- 734 18:48 Robert Umm. Oh, cause I wanna, you know for this one [black column], I don't wanna...I can't multiply any of these by a cause then I will get a-cubed which is not possible so, I just kinda like this one [red column] cause I can multiply by b or c, you know what I mean and this one I can multiply..
- 735 19:04 R1 and, the last one ..
- 736 19:06 Robert by a or b
- 737 19:07 R1 So, what does this [red list] one have, can you put it...,I mean I hear you explain it, but is there... is this a kind of property that this [red list] has that this [black list] you're claiming doesn't quite have
- 738 19:20 R1 Like here you'd multiply that [a^2c in black] by what?
- 739 19:24 Robert Umm...B
- 740 19:25 R1 and you multiply that [a^2b in black] one by what?
- 741 19:26 Robert C
- 742 19:27 R1 and you multiply that [abc in black] by what?
- 743 19:29 Robert A
- 744 19:30 R1 and you multiply that one [a^2c in red] by what?
- 745 19:30 Robert Oh yeah so I guess when,... this one [red abc's]you kinda split it up, like this will be multiplied by a, this will be multiplied by c [splits the 6 abcs into two halves]
- 746 19:38 R1 and then what does that [a^2c in red] multiply by?
- 747 19:39 Robert B and then b
- 748 19:40 R1 And that one by b
- 749 19:42 Robert Yeah, and then this one by a
- 750 19:42 R1 So, which one is prettier?
- 751 19:44 Robert So, I guess this one is actually prettier [the black one] cause then you don't have to
- 752 19:45 R1 Do you like pretty?
- 753 19:48 Robert It's simpler, I guess it doesn't...

754 19:49 Robert I guess what I'm trying to say is, it doesn't matter which one you choose as long as there is no overlapping

755 19:56 R1 Which one... which one are going we going to choose though, this is your triangle, which one should we choose

756 19:59 Robert I'll say the one she said, this one [black list] cause it seems easier now that I think about it.

757 20:03 R1 Oh...so, easier, simpler or prettier?

758 20:06 Robert all three

759 20:07 R1 and you like all those, huh?

760 20:08 Robert Yeah [chuckling]...So, I guess the thing is...it doesn't matter which one you choose

761 20:11 R1 I don't see it's easy, I think you're... I think you're trying to snow me...the reason why is ... is uh here, let's say I multiply that [red a^2c] by what?

762 20:18 Robert B

763 20:18 R1 and I multiply that one by what [red ac^2]

764 20:19 Robert A

765 20:20 R1 ah..that one by

766 20:22 Robert Uhh, I mean B, B..

767 20:23 R1 and that [abcs] one by ..

768 20:24 Robert a or c

769 20:26 R1 Uh yeah... So, I... I think that's easier

770 20:28 Robert Really?

771 20:28 R1 Yeah well...cause I'll only have to remember two things to multiply by

772 20:33 Robert Well, I guess this is 50% , is multiplied by the same, right and then this is all mult...so it's 25 and 25

773 20:42 R1 Yeah, I have to remember three different letters

774 20:43 Robert And here...It's the same thing as

775 20:44 R1 You're trying to hurt me here, it's 50 and 50, only have to remember two letters and matter of fact, what letter is it? Show me the triangle... point to the triangle this represents.

776 20:52 Robert Umm, the red...orange one

777 20:54 R1 and what am I multiplying by here?

778 20:55 Robert Uhh...B

779 20:56 R1 Show me where b is.

780 20:57 Robert Which is the one over here, yeah.

781 20:58 R1 So, all I have to remember is go opposite, you are trying to hurt me, I know what's going on, so, it's 'b', and then what's... I multiply by, oh, oh 'a'...how did I remember to do 'a' there? Oh.

782 21:09 R2 Why 'a'?

783 21:10 R1 I don't know, I thought he said 'a'

784 21:13 Robert Is it a?

785 21:14 R2 Why not c?

786 21:15 Robert oh, you have to split it in half like, 3 of them have to be multiplied by a and three of them have to be multiplied by c, so it's the same

distribution

787 21:24 R1 What's the final thing. What's that 12 going to represent? There's my question then.

788 21:28 Robert 12?

789 21:29 R1 What would 12 represent in that orange triangle? 12 how many...of what

790 21:33 Robert a-squareds

791 21:34 R1 a-squared?

792 21:35 Robert bc's?

793 21:36 R1 a^2bc 's, huh.

794 21:38 R3 He said it like a question

795 21:40 R1 Yeah

796 21:41 Robert Oh, exclamation point

797 21:42 R1 Oh really, you sure about that?

798 21:43 Robert 12 a^2bc 's

799 21:43 R1 You're absolutely sure about that?

800 21:45 Robert Yes

801 21:45 R1 Exclamation point means something to me

802 21:46 Robert Yes, exclamation point.

803 21:47 R1 So if I multiply that by [red a^2c], just one more time help me out

804 21:50 Robert This would actually...this would actually

805 21:51 R1 I multiply that by what?

806 21:53 Robert B, these are all b

807 21:54 R1 So, I am gonna get what when you multiply that by a b, help me

808 21:56 Robert a^2bc

809 21:57 R1 And when I...and we'll multiply this [ac^2] by what?

810 21:59 Robert a^2bc , I mean abc^2

811 22:03 R1 and that and that add to same thing?

812 22:06 Robert No cause you have,

813 22:07 Robert So, this is actually going to be like a 6, 6 a^2bc and $6abc^2$

814 22:15 R1 6, 6,6, help me, I've got one, two, three [counting a^2c] of a^2b ,

815 22:19 Robert One, two, three

816 22:19 R1 You're gonna multiply these by what? Those are just 6, so you're gonna multiply by one thing, you're cheating me, I am not letting you split 'em up [Robert laughing], you're gonna multiply by one thing, what are you going to multiply it by?

817 22:30 Robert That's not gonna be...that's not gonna to be as pretty as you said or simple

818 22:33 R1 How 'bout...Let's forget about this, how 'bout these up here, what are you gonna multiply this by [a^2c] and multiply that by so that they add up to be the same thing.

819 22:36 Robert B

820 22:39 R1 B?

821 22:39 Robert You can't add them up to be the same thing... cause...

822 22:41 R1 But they're all adding up to be 12 of something, 12 of what? That's what I am asking... 12 of what?

823 22:46 Robert Yeah so, I'm gonna say I'm convinced this is simpler [the black list], this triangle

824 22:51 R1 Ahh. So, it's not just ugly, it's not working

825 22:55 Robert So, you can't pick any triangle you want, you have to pick this, this and this.

826 23:00 R1 So that they'll add up huh, so she might be a pretty smart cookie,

827 23:02 Robert She tried to trick me

828 22:04 R1 I tried to trick you ...your the one giving this to me

829 23:06 Robert But, you said this one was so much prettier and then

830 23:09 R1 No, no, no, I thought...I was confused

831 23:10 Robert I switched my answer, I was like this one looks prettier

832 23:10 R1 No, no, no...I was confused...I get confused...I just wanted you to reflect upon it

833 23:16 Robert Yeah. So, I think it does matter which one you choose

834 23:18 R3 Say that again Robert

835 23:20 Robert It does matter which one's you choose because, your trying to get 12 of something, so, I guess if you did this way [the red way] , you'd have to write sixes everywhere, you know what I mean...but then...no it wouldn't work...

836 23:31 R1 But just remember why are these multiplications,... it's coming back down to here, this everything times, this distributive law

837 23:37 Robert It wouldn't make sense if you did it like that.

838 23:38 R1 It wouldn't make sense.

839 23:39 Robert Yeah you could write it like that but it wouldn't work

840 23:41 R3 But you didn't tell...You 're talking to him, explain it to me

841 23:44 Robert Oh, I am sorry...ahh..this one makes more sense because you're trying to get 12, and since they're all be a-squareds, these are all the a-squareds so there'll be 12 of them altogether, whereas this is gonna be 6 a-squareds and 6 c-squareds

842 23:54 R1 I see what you're saying, so, you can split it up, but there...they don't come to...

843 23:38 Robert That wouldn't look pretty

844 23:39 R1 but it's not giving me a 12, I mean I don't...I know there is a 12 something there

845 24:00 Robert You have like 6 comma six...you know something like that...she is...I think...did you just randomly pick this one or did you know that for...cause that's the one I initially picked only cause it seemed easier but then I didn't...

846 24:18 R2 It looked prettier to start with and then as you were going through it, I sort of remembered that it made sense

847 24:25 Robert So yeah, you have to pick this one [the black one] cause then otherwise the 12 wouldn't make sense.

848 24:32 R3 Well, that was very interesting

849 24:34 R1 Thanks Robert

850 24:35 Robert No, thank you

851 24:36 R1 Now, I want to know what the next layer is

852 24:39 Robert Now I understand the triangle
 853 24:40 R1 I'll figure that out on the plane
 854 24:41 R3 Now, you understand the triangle?
 855 24:42 Robert Yeah
 856 24:43 R3 You didn't tell me where ankur's... where's ankur's solution?
 857 24:46 Robert Oh, ankur's solution is the 12s, the triangle of 12s and there's...
 858 24:52 R3 any triangle of 12s?
 859 24:54 Robert No the triangle of 12s that we picked
 860 24:46 R1 The three on the top
 861 24:59 Robert This one...
 862 24:59 R3 And why is that again? Can you show me a picture of why?
 863 25:03 Robert This... all the yellows... Umm because these are all the ones
 containing two
 864 25:09 R3 So are there 6 of them?
 865 25:11 Robert It's three triangles. Right?
 866 25:14 R2 Yeah. Those are twelves
 867 25:16 R3 So you're counting six three times?
 868 25:17 Like the 6 is the, like I don't know just give me three colors red,
 white and blue, right? So the 6 is uhh, all of the red, white and blues
 that has one of each and then you are adding, in this case you are
 adding, you know, a blue to it? Is that how it goes? Like, you're
 adding a blue to ... alright let's say like a is blue, right? so you are
 adding a blue to all the a b cs, you know in all the 6 spots, and then
 you're taking the one that already has two blues and adding a red, and
 then taking the one that has two blues and adding a... whatever the
 other colors is... white... you know what I mean..so, the one that has
 two blues and a red, you're adding three whites to those and then the
 one that has two blues and a white, you are adding a red, three times
 869 26:07 R3 That's a different solution than any one gave me so far, could you
 write that one up
 870 26:11 Robert Uh yeah, the way I did it was is actually different too... I did it with
 like gaps, like you have ...you already have the initial condition
 satisfied , red, white, blue...and then there's you know an asterisk
 here and an asterisk here and an asterisk here an asterisk here... and
 then you know, there's like this many spots, right? And then I wrote
 this the 6 different ways, you know what I mean.., so kinda like...
 where you could put that extra block in there, you know the...I called
 it like random block, you know what I mean and I just multiply three
 for the 3 different colors, so, you know the random block can go
 anywhere here and then...
 871 26:52 R3 How many asterisks do you have there?
 872 26:54 R1 Four? Four.
 873 26:55 Robert Four asterisks, 6 of these and so it is 24 times 3 which is 72 and then,
 its gonna be duplicates, which is half so it's 36
 874 27:03 R3 Is that how you did it?
 875 27:04 Robert Yeah

- 876 27:05 R3 Write up the other way too, when you get into that. I'd like to see that. That's interesting.
- 877 27:07 Robert Okay. Okay. Yeah so I thought this was... this only 'cause like it's a special case where like you know, it's not that hard. I think this is more like a more generic ...you know what is it called when it's everything?
- 878 27:22 R1 General
- 879 27:22 Robert General, yeah.
- 880 27:23 R3 It's nice to see the simpler ... what wasn't a simple problem in grade 10 to fall into the more complex problem. Don't you think?
- 881 27:29 Robert Yeah I know. This is like a simple way to look at it. You know you have to..
- 882 27:32 R3 Well would that have been a simple way for the kids that did it in grade 10? Ankur's problem?
- 883 27:36 Robert Umm, I think.
- 884 27:38 R3 You think so?
- 885 27:39 Robert I think well once you realize you have to ...there's like an initial condition that you have to satisfy that they all have one of each color and then since there is only one more color and it's like the special case... there are four spots to put that one more color, you know, and then there's 6 ways to arrange these and then there's going to be duplicates
- 886 27:59 R1 How many colors where there?
- 887 28:01 R3 I thought there were...
- 888 28:01 Robert Four high, three colors, one of each color
- 889 28:03 R3 Right.
- 890 28:04 R1 So, I've got a question on that, so I want to go to 5 high, because I want a little room, but I only want three colors ok and I'm gonna let you...I don't want the colors to be split up ...
- 891 28:19 Robert So you want like ...two two
- 892 28:21 R1 and you don't even have to have three colors, you can have all of one color, but I don't, I don't want them to split up and I want them in the same order so umm...
- 893 28:28 Robert Oh I know, I would do that like a donut problem
- 894 28:31 R1 What's a donut problem?
- 895 28:32 Robert You know you go get 12 donuts...and then you know you want the certain...like umm, you do like, it's like solving multiple linear equations, so let's say you have, forget how to do this...let's say you want two blue...or you just want any amount of colors?
- 896 28:48 R1 Yeah, I want a maximum of three colors
- 897 28:50 Robert Or do you want ... max three colors?
- 898 28:52 R1 Do you like donuts better?
- 899 28:53 Robert Yeah, no, no... I'll do it with towers but does it have to be one of each color or something?
- 900 28:57 R1 Nahh...
- 901 28:58 Robert Oh, so then you just do like x_1 plus

902 29:00 R1 that would be another one, you can throw that in then, you could help me...Ok so that has

903 29:03 Robert Equals five, and then solve the, how many solutions are there to this.

904 29:09 R1 But I don't want them split up ,.. I only want to count them...

905 29:12 Robert Yeah doesn't, doesn't, Like when it arranges it, it'll do it like this ...and it's just looking where to put dividers, like you put a divider here and a divider here...so it kinda groups them together,...

906 29:22 R3 That's partitioning

907 29:23 Robert yeah...that's how we umm did it, that's what me and chuck and were trying to do... I actually asked chuck for help because I forgot how to do the donut problem, so I was trying to apply it to this...but then we realized that the problem was that it groups together, you know what I mean, it doesn't count for the case where this is like red, red over here, you know what I mean. It only counts like... these are all reds, these are all blues, and these are all whites.

908 29:48 R3 I see.

909 29:49 Robert So...and then if you wanna solve it like where umm..

910 29:51 R1 What...what about the case when it is red red though... I mean..

911 29:54 Robert Oh then...

912 29:55 R1 So you...that's it?

913 29:56 Robert Then the partition will go here and this would be like red, red. You know there's no way to have them separate from each other. And then...

914 30:04 R1 No, I don't know that, say that again. I'm not...

915 30:06 Robert Like order doesn't matter... or ...

916 30:09 R1 What...what if you let... if you made order matter? The red would come first, blue would come next and white would come after that.

917 30:15 Robert Oh, that doesn't...I mean like order doesn't matter like ...uhh if you have red red, it can't have, you can't have red blue red. I don't know what that's called.

918 30:24 R1 That's right. I'm asking with my donuts, never mix 'em up never split 'em up, so I want red donuts, I want blue donuts, and I want white donuts always in that order, but you don't have to have, I mean, I may only want blue or white, I may only want all red, I want to know how many ways can I do that...

919 30:41 Robert Yeah, then the partition would just go at the end

920 30:45 R1 OK

921 30:46 Robert Like I said if here... so then you have red, white and then zero blues

922 30:50 R1 So, so the answer is 5?

923 30:53 Robert Umm, whatever...

924 30:55 R1 17?

925 30:56 Robert What's the word? Oh the solution. How many real number solutions or integer solutions there are to this.

926 31:00 R1 and what's the answer to that?

927 31:01 Robert Umm...what is it like ... 5 what is it minus 3 choose 3 or 5 plus 3 choose 3 ...something like that, there...theres a formula that I can't

remember. Its something like ...

928 31:15 R1 It's just a formula
 929 31:16 Robert Yeah, this plus this and I know I, I...
 930 31:18 R1 How come I can't look at that and see the formula?
 931 31:22 Robert Umm
 932 31:22 R1 I've got seven years of graduate experience and I can't just see a formula there, I think it's..
 933 31:26 Robert Actually, I think it's...I think the formula comes into play when you like ...do change of variables and make this like... you know like if you set a restriction that there has to be one of each. Right?
 934 31:40 R1 You keep coming back with that. I would be more than happy to give you that, but eventually ...
 935 31:41 Robert No, no, no...
 936 31:44 R1 Ok, suppose you did that, could you get back to the one I wanted? Suppose I told you how to do then, how would I get back to one where it's, it's...I didn't have to have all the colors?
 937 31:52 Robert Well, or you didn't have to have all the colors?
 938 31:55 R1 Well, suppose, you... you keep wanting me to have at least one color of each
 939 31:59 Robert No, no let's forget about that
 940 32:00 R1 Oh, ok
 941 32:00 Robert Yeah
 942 32:01 R3 Maybe we have to stop
 943
 944
 945 Talks about liking stat more, etc

Appendix B

Date of Session: 7-2-2008

Author: Anoop Ahluwalia, Ericka Bilyk, Scott Rutherford

Verified by: Anoop Ahluwalia, Ericka Bilyk, Scott Rutherford

Date of transcript: August 2008

Line	Time	Speaker	Transcript
1.			WE showed him clips from the Disc 1 then interviewed him. We stopped after Disc 1, showed him clips from Disc 2 and then interviewed him
2.	1:35	Anoop	So, here we go, where did you first learn about Pascal's triangle
3.		Robert	Uhh I think we subconsciously knew it in like 4 th , 5 th grade but we never really... knew it was a triangle until high school, like guess 11 th grade maybe, we never heard of it but we had done the patterns before you know and we had also done the binomial expansion before but we just never related the two to the triangle form, so I guess high school
4.		Anoop	Ok, what about it interested you...about the Pascal's triangle and how are other math ideas connected to the Pascal's triangle?
5.		Robert	The only thing I think that interested me to the Pascal's triangle but I guess the one thing was we could get the answers easier than you know having to redo it every time. It kind of gave us since...we did a lot of same problems you know probability and they all dealt with powers of 2 so we're able to and the answer was just like oh, just make the triangle and then here you go...so, I guess it gave a more like a picture presentation of answers than just numerical which you know helps some other people understand it, and I guess it also helped uhh...we had all these theories like you know you add when you get, when you move from one tower to another what you are actually doing, you're adding one color, you're adding a blue and a red, if you have two colors, and then you can kind of see that in the triangle if you draw the arrows, you know like if you have one row, you draw arrows from and then plus one you get the one below it
6.		Anoop	Right, so towers problem was related can you think of any another problem?
7.		Robert	Yeah, The pizza...
8.		Anoop	The pizza problem
9.	3:20	Robert	I guess the taxi cab geometry it's the same thing it's like the pyramids, it's triangle just on the grid... in a square
10.		Anoop	And, you started doing these problems in second grade about? And you were doing things like towers in second grade?
11.		Robert	Yeah, 2 nd 3 rd don't remember when I started
12.		Anoop	Oh, Ok, oh Ok
13.		Robert	But, I think the big time was 6 th 7 th 8 th grade, so yeah
14.		Anoop	Oh Ok, So, you're saying in 11 th grade you actually named it and you kind of knew the name of Pascal's pyramid, triangle

15. 3:50 Robert Yeah, what is weird is we figured out all those formulas like n choose r , and all the identities before we actually knew what the triangle was and I think the teacher is Mr. Pantose and he is like a Rutgers and he's friends with Carolyn and stuff so one day he just tossed it out there one day like that's what it's called
16. Anoop Oh, Ok, ok and when did you first learn about the Pyramid, Pascal's pyramid?
17. Robert Uhh Actually when I came here, I took combinatorics and it was just kind of like a little blurb. It wasn't like the pyramid wasn't in there but the formula was and then Liz was, Liz Uptegrove was going over, I think she was doing it partly for her dissertation and she introduced us she brought in the pyramid and a book
18. Anoop Ok, was she teaching that course, the combinatorics course?
19. Robert Ah, no, no she was still you know a PhD student I think she wanted to do more research so and she started asking, talking to me and Brian about that and we filmed a couple of interviews, 3 or 4
20. Anoop Yes, and I have to get a hold of that (laughing)
21. 5:00 Robert And, yes and that may be the first time I was really like you know
22. Anoop And then did you explore it further when you saw the blurb or you waited for Liz to actually introduce it to you guys, like I mean...
23. Robert I think we kind of because of we do towers stuff with 3 colors which is the pyra...you know it's the same thing so, uhh but I don't think we really ever thought you could go to the Pascal's pyramid or the 4th, that you can actually the infinite, you know the Pascal's 20th whatever, you know you can draw..
24. Anoop Ok, so you kind of answered this question but I mean did you ever discuss how the triangle and the pyramid are related?
25. 5:42 Robert Uhh...Yeah, we ...I think a lot, but we kinda talked about the same stuff like you know this is Pizza with four topping so you know this row, but the one thing I don't think we ever did was, we never did the formal way where its like is a plus b to the 4th you know this row is a plus b to the 4th We never really did that. But, that's what we learned when I got to college so, but actually when we did it, we looked at like this is you know pizza with one topping, you know these two toppings so, or towers with like...a red in position one, or two reds and two blues or three reds, so we were actually doing it but we never actually formally said that this is a plus b squared or a plus b cubed, you know we just...
26. Anoop The notation was not there yet
27. Robert Yeah, the notation wasn't there which is kind of weird because...
28. Anoop But, you were using the pyramid, and like because you went to 3 colors and that was something that didn't work on the triangle anymore right? So you were jumping on to the pyramid but not...
29. 6:44 Robert Yeah, the pyramid I think we knew that it was 3 you know cubed or whatever, 3 to the 4th but I don't think anybody knew like if you are like what's 2 on 1, no could just be like oh, this will sit on pyramid?

30. Anoop You know, so that was the problem
And my next question also you partly answered, I mean you remember the sessions when you sat down with Brian and Liz. How did that come up? How did those sessions happen?
31. 7:11 Robert I just started working here part time maybe it's 03 or 02 and Liz was like oh rob do you wanna.... Think she was just going to interview Brian and she's like... and I think we had a new guy here Toby from Howard Hues like and we were starting new grant and they wanted to use the equipment, so we just made up these interviews, we're like so we can test the equipment and Liz could ... and Liz was interesting in this so it was like one of those you know we want to practice with the new equipment and me and Brian were just around so figured why not go for it
32. Anoop This is interesting
33. Robert I think Liz write a book or something like a children's story which kind of loosely taught the idea in like first grade with peg or something and she thought that was interesting and she brought that in, anything was... she didn't know that much about it either...cause she was still learning it, so we were all kind of learning together
34. Anoop Wow, wow, umm....do you remember the first time a model for this Pascal Pyramid was built
35. 8:15 Robert Umm...I'm gonna say high school
36. Anoop Oh really? You made, you were involved in making one?
37. Robert I didn't make one, no I never... I am not really a...
38. Anoop Hands on (laughing)
39. Robert Yeah, I don't like making stuff but I think there was one session where they are filming Pup Math, and they came to the classroom and they are like three days just do anything and I think one group made Pascal's triangle but when they made it they made it as a Pyramid not as a...but they didn't make it as a Pyramid to talk about that, they just made it as a pyramid because they thought you know you can show something better as cause in 3-d not 2-d so they made the pyramid, but they just made triangles on the faces and they didn't really you know
40. Anoop I wonder if they had those sub layers
41. Robert Yeah, they didn't have that...they just had like a ...they took a Pascal's triangle and just made it 3-d but they didn't you know take it to 3rd power
42. Anoop Like with a hollow center, pretty much
43. Robert They thought it would be more interesting to show it like that
44. Anoop Interesting. Can you show me how the Pascal's pyramid is built, you know, I mean
45. Robert This is where I use the board?
46. Anoop I think so (chuckling)
47. Robert I guess this goes back to your concentric circle questions, I didn't, you know can really draw, I think you can really draw it any way doesn't really matter, same thing with the triangle doesn't have to be a

- triangle, it could be a circle, you know, it just looks nicer, plus I have seen it in books as a right triangle or an equilateral triangle doesn't... you know what I mean it's just all
48. Anoop Right, right
49. 9:56 Robert Isn't it like, this is a, b, c or something. This is $a+b+c$ to the 1. So, There is one a, one plus b plus c, So this is like a one, zero b, zero c, you know etc
50. Anoop I see, I see, ... and is this the very first level?
51. Robert The second level
52. Anoop It's the second level, OK
53. Robert Cause it's just one on the top for it's a plus b plus c to the zero
54. Anoop Oh, Ok, what would be the next level I guess
55. Robert Ahh...(writes $(a+b+c)^2$ and make a triangle of 1s and 2s) yeah, that's it
56. Anoop That it, and what are these twos?
57. Robert Ahh, this is a-squared, b zero, c zero, a, b zero c, a zero b zero c squared a zero bc, a zero b squared c, ab c zero, so it's a-squared wait a second this is c zero
58. Anoop Wow!
59. Robert a-squared plus b-squared plus c-squared plus 2ab plus 2bc plus 2ac
60. Anoop And how would you say this level, the squared level how did it come from the level before that?
61. Robert Aah, so this would be, where's there b...this one plus (b) this one (c) goes here (2bc term) right and then a , b goes here and then c a goes here (makes arrows from level one and connects them to level two)
62. Anoop And do you mind drawing just one more level
63. Robert Yup, see that's why I guess the pyramid makes more sense, cause you're drawing it, all the arrows kind of go down, but it's 2-d here so it's kindda
64. 12:24 Anoop It's harder to see...
65. Robert Yeah, so actually I am just gonna redraw this one so when I show you how it goes to the 3rd, makes more sense
66. Anoop Sounds good, sounds good
67. Robert (mumbles to himself and draws a triangles with 1, 4, ,6, 4, 1 on each side of the triangle with a single 12 in the middle)
68. 13:00 Anoop Ok, And this would be what level, I mean you're writing here notations...
69. Robert This would be $(a+b+c)$ to the 4th
70. Anoop Oh, oh, wow, so you went to the 4th power
71. Robert Oops 3rd, and then this is actually the 4th yeah I messed up, let's go to the 3rd first (erases his 4th level)
72. Anoop Oh, Ok (laughing)
73. (makes a triangle with 1 3 3 1 on the sides and writes three 6s in the middle)
74. Robert I don't know if that's right but
75. Anoop How many sixes do you think we get in the middle? Is there a way to

- check
76. Robert Yeah, I guess (starts to write a^3 , a^2c terms with the coefficients 1, 3, etc and writes terms like a^2bc with 6 in the middle)
77. 14:39 Robert So there's only three combinations with that with a-squared in all 3 uhh letters
78. Anoop But if you look at a^2bc , ...how many factors...is that
79. Robert Yup,... What you mean
80. Anoop Like a^3 is three factors a, a, a. ba^2 is three factors, so when you're cubing the things, to me a^2bc is four factors
81. Robert Ahh...Oh wait here I think you're right
82. Anoop Like can you get a^2bc ...
83. Robert I got it, like which ones are abc in this one, right there's two, ...can I just rewrite this out
84. Anoop Please, I mean you can erase some of it, that's fine
85. 16:30 Robert (writes out $a^2+2ac+c^2+2bc+b^2+2ab+2a^2b+2b^2a$...)
86. Robert Forgot what you call the center now
87. Anoop That's OK, it'll come back to you
88. (Researcher walks to the board and points out)
89. Anoop Uhh...I understand where all these ($a^2+2ac+c^2+2bc+b^2+2ab$) come from and they are right here, I am not sure where these ($2a^2b+2b^2a$) come from
90. 17:03 Robert Yeah, I don't think those are in there. Aah, I see what I did, you're right cause these is 1, 1, I was looking for the inside 2s that weren't there
91. Anoop Exactly (Researcher sits down)
92. Robert So, this a^2bc comes from abc0, a^2c0 , is there a zero? This is a^2b0 , I don't know, I don't know
93. 18:00 Anoop [After a pause] Ok, I think you kind of showed me how you know how this bc comes to sit...
94. Robert I think this is wrong, that's the problem...yeah... and there has to be another number, has to be a c-squared b, and there is 0abc a-squared, those ones I had before are zero then right, so, there's we need a^2c , oops (erases and rewrites the following list)...so, we need a^2b , a^2c , b^2a , b^2c , c^2a , c^2b and there was 2ab, and so you add, so this will be 2, and there was, how many a-squareds, one a-squared so there will be 3, there's same, same, same, same so there'd be six 3s
95. 19:21 Anoop Exactly
96. Robert And there'd be zero a^2bc , or a-squared c, zero, zeros
97. Anoop Zeros, because....
98. Robert There is no abc to add an a onto to get a^2bc there's no abc to add a b etc
99. Anoop Ok, so no abc term in this level that could lead to any of these that's what you're saying
100. Robert Yeah
101. Anoop And I see all the threes here right, I mean that's what you wrote and you got them all covered. Now my question is is there anything in

there

102 Robert I am gonna say no, only because what other term can it be?

103 Anoop OK

104 19:59 Robert We've got a^2b , a^2 and we've got all the a-cubes ... you know with three, but the... the exponents have to add up to three. Like this it'd be one, one, one, one... so the only option for the center would be abc,

105 Anoop Okay.

106 Robert Which there is, right?

107 Anoop Okay.

108 Robert So we have to find out how many ac's there are.

109 Anoop Okay.

110 Robert 2.

111 Anoop Right.

112 Robert $2ac$'s... $2ab$'s and $2bc$'s so there's six in the center.

113 Anoop Okay.

114 Robert And... so it's $6abc$.

115 Anoop $6abc$'s. Okay. Because...

116 Robert And I think that's all.

117 Anoop Because this will get multiplied with a ... $2ac$ would get multiplied with a b.

118 Robert b , and that would get multiplied by a c and a.

119 Anoop A

120 Robert So 2, 2, 2. 6.

121 Anoop Yeah, I think that's, that's the level.

122 Robert Yep.

123 Anoop Ok, I'm not gonna go to the other one. I think this is.... What do you think, yeah? (asking Scott and Erica if we should continue to probe the levels?)

124 Robert That one was a twelve in the center right?

125 Anoop That has twelve. More than one twelve.

126 Robert Yeah. Three twelve's.

127 Anoop Three twelve's. You remember that.

128 Robert $12a^2bc$. Well, I just remember from this

129 Anoop Yes, what you wrote first, yeah.

130 Robert Yeah.

131 Anoop Yeah.

132 Robert And there's $12ab^2c$'s and $12abc^2$'s.

133 Anoop Exactly.

134 Robert Just from knowing that they'll have to be an ab.... cause there's ... you'll have to find the 4 somehow, so there can be a 2, 1, 1. Where it's a, b, c or 2, 1, 1 b, a, c ... c, a, b. And that will come from...

135 Anoop Right, because that's going to have four factors and that's what you're saying 2, 1, 1. Okay.

136 Robert Yep.

137 21:41 Anoop Umm. Can you show me just umm, what you drew here? One of the twelve's that would come up? Like, I mean, how would these

- coefficients from this level, give me any one of the twelve's in the next level? Like, whichever twelve you want to talk about.
- 138 Robert Umm. Sure. Uhh you have a^2bc right?
- 139 Anoop Okay.
- 140 Robert That would be this three.
- 141 Anoop Uh huh.
- 142 Robert So it would be $3a^2c$...
- 143 Anoop Okay.
- 144 Robert times b.
- 145 Anoop Yes.
- 146 Robert And you have $3a^2b$ times c, which is umm a^2 ... is this it?
- 147 Anoop Yeah.
- 148 Robert Yeah a^2b
- 149 Anoop It's very hard to see things on the board.
- 150 Robert And then two more three's.
- 151 Anoop Okay.
- 152 Robert There'd be... oh no...abc
- 153 Anoop Okay.
- 154 Robert Which is 6.
- 155 Anoop Oh, okay.
- 156 Robert And... that's 12.
- 157 Anoop Oh.
- 158 Robert This would be times a.
- 159 Anoop Times a.
- 160 Robert Yep.
- 161 Anoop Ok. Thank you.
- 162 Robert So that's where this all comes from.
- 163 Anoop Yes.
- 164 Robert And then, for the ab^2c it'd be almost the same thing, it'd be 3...
- 165 Anoop ab...
- 166 Robert Or 6 abc
- 167 Anoop Right.
- 168 Robert Uhhh. 3, 3, b^2c , b^2a , times a, times c.
- 169 Anoop Right.
- 170 Robert So yeah.
- 171 Anoop Okay. Exactly. Great. Thank you. Ok, which reminds me... kind of skipped a couple of questions.
- 172 Robert Ok.
- 173 Anoop Uhh, do you remember the discussion where you discussed the Pascal's Pyramid with Todd, when was it?
- 174 23:22 Robert Umm yeah, it was the summer institute, we were teaching kids probability? And they were working on their final posters so we didn't need as many cameras so we were just kinda sitting around and I forget how it started by we just started talking about it... and then one of the other camera people were sitting around just started filming it.

- 175 Anoop Oh, interesting. So have you ever spent time playing with the 3-d model of the Pyramid?
- 176 Robert Nah, I haven't looked at it since the Liz interview. So, that was...
- 177 Anoop Even then she didn't have it. Right? Oh maybe...
- 178 Robert Yeah, she brought one in eventually... she made.
- 179 Anoop When, when you were talking to Todd or when you were talking with Brian?
- 180 Robert Oh, when I was talking with Todd? I was talking to Brian. I guess the Todd interview happened after Liz so that was last time so...
- 181 Anoop I see.
- 182 Robert That was about five years ago.
- 183 Anoop Oh...
- 184 Robert I haven't done that since. Yeah.
- 185 Anoop Okay. So I'm going to take you to the clips that I showed you. Umm you were trying to represent all the layers as concentric circles, so when you wanted to show $a+b+c$ cubed it was supposed to contain $a + b + c$ squared, that's what you were doing.
- 186 Robert Mmm hmmm.
- 187 Anoop Can you remember why you thought it cannot be done?
- 188 24:26 Robert Ummmm, I don't know, but I think it can be done, but there's no point to do it because it's going to get messy, whereas this... I guess like if you're building a pyramid it makes more sense because they go down in a straight line, but if you're building ... actually if you're building a circle it shouldn't really matter either because... except the only thing is you're not gonna have like coordinates, you know like to the pyramid or whatever. Like you know, you can draw the pyramid on graph paper in 2-D if you just give it an x, y, z coordinate.
- 189 Anoop I see.
- 190 Robert But, you're doing circles.
- 191 Anoop But, you're saying this would work on a sphere, rather than a pyramid? That's what you're saying, like I could have a 3-D sphere?
- 192 Robert I think it can work on any, any uhh... hmmm. [Pause] I think the shape doesn't matter but, it's not going to grow at the same rate. You're not going to have like consistency like, you know, it's not going to build outward the same sphere. Like you could just take everything and throw it in the... constrain it by a sphere.
- 193 Anoop I see.
- 194 25:31 Robert You know what I mean, like 'cause well doesn't... can't you fit a pyramid inside sphere or can you, I don't... you know, whatever you can fit inside a... like I think the shape is arbitrary but you know, it's for a good reason...
- 195 Anoop But you think it's like, I think what you were doing was making a 2-D representation of the next layer containing the previous layer. You know, so when I look at $a+b+c$ Oh, cubed, I can see the $a+b+c$ squared in it and the $a+b+c$ to the first power in it. Do you believe that can be done?

196 Robert Mmmm, I don't know. I know you can see the Pascal's Triangle in the pyramid, but that's ...

197 Anoop Yes. That's by...

198 Robert Yeah, that's how I constructed it on the ... the outer edges.

199 Anoop Right.

200 Robert Or just all Pas...,but yeah I don't know.

201 Anoop Okay. Okay. Umm, if you remember the next clip I showed you , you were making triangles with one's on the vertices and then there was circles in the middle, so you're one's were at on the corners, but later you convinced yourself that the one's are always in the center.

202 Robert Yeah, that's wrong.

203 Anoop Okay.

204 Robert Yeah.

205 Anoop But okay, I mean yes it's wrong but do you know what convinced you that they were in the center?

206 Robert Umm.

207 Anoop Do you remember a little bit about the ...

208 Robert I don't know from watching the video, like I don't know ... it seemed like I was all over the place, so I guess I was just more ummm ... just trying different things out.

209 Anoop Okay.

210 Robert And, then eventually I think I saw that that's not the best way to represent it, 'cause...

211 Anoop Okay.

212 27:06 Robert You know, I, I think when we did the Pascal's Pyramid, you know we only represented it in like one day and I didn't really remember what it was like so I was like trying to create it from you know, scratch which was kinda stupid because I knew it was Pascal's Pyramid but I was still drawing it in circles. You know like...

213 Anoop No, but it made sense cause it worked for a few ...

214 Robert Yeah.

215 Anoop A few layers.

216 Robert Yeah.

217 Anoop Okay.

218 Robert I'm sure you can get it to work too, but it just looked like a mess. You know, it wouldn't be like... uhh recur... like algorithm, you know what I mean with the pyramid it kinda becomes like a algorithm when you're building the next...

219 Anoop Right.

220 Robert rows but, if you have it in a different shape, you know it doesn't really become that, it's just a bunch of numbers in a circle and you're like what the heck is this. So that's why I mean you can do it in anything but I guess the pyramid makes the most sense or tetrahedron or whatever I think they call it, yeah.

221 Anoop Yeah

222 Robert So, I guess those two shapes are probably the, the best but yeah that's

uhh... so you can put it in any shape but that's the best choice.

223 28:05 Anoop Okay. OK

224 Robert Yeah, that's what I feel.

225 Anoop Oh, I agree. 'Cause I mean to have three directions it's good to have linear directions. Umm, you used the coordinate system and I saw it a little bit even now, so you had things like 4, 0, 0. What did that represent?

226 Robert Ummm, drawing a pyramid like x, y, z so 4, 0, 0, would be you know x four, y zero, z zero, and I could just draw a dot there.

227 Anoop Okay.

228 Robert And you know, you draw all the dots and I guess you connect them it must make a pyramid. I hope.

229 Anoop Okay. Okay, sounds good.

230 Robert I have never done it but, so yeah.

231 Anoop But do you remember where you saw this coordinate idea and the pyramid put together in the first place? Or maybe...

232 Robert No, I don't think I've ever seen that. I think in the... when watching the video I was more drawing it like ummm just four a, zero b, zero c and it just looked like a coordinate.

233 Anoop I see.

234 Robert I don't think I actually, specifically went out and you know did it like a coordinate system, I think it just looked that way 'cause I used, 'cause the way I did is when I wrote it I... Instead of ...oh I'll show you ... instead of writing uhh... instead of writing a^2bc for...

235 Anoop Right.

236 Robert you know the four. I would write it 4 choose 2 comma 1 comma 1. You know so that's 4 choose 2 as, 1b and 1c. So I think I was writing them like that.

237 Anoop Uh huh.

238 Robert And it just kinda looked like coordinates.

239 29:38 Anoop Okay and by 4 you meant the fourth power...

240 Robert Yeah...this is like ...

241 Anoop It kinda represented

242 Robert $(a + b + c)$ to the fourth

243 Anoop Right. So that's the ...

244 Robert This is one entry of the, you know...

245 Anoop Okay.

246 Robert Is the coefficient for a^2bc .

247 Anoop I see.

248 Robert But if you're drawing it, well I guess I'll just start..., like you can have this be a dot and it'd be four comma zero zero ... and you can have this and this would be zero four zero and...

249 Anoop I see.

250 Robert Zero zero four and then this would be like three, you know there'd be a dot here, dot here, three... oops two, one zero three, you know etcetera.

251 Anoop I see. To make sure you didn't miss a term basically. You were...

252 Robert Well yeah, 'cause I guess then this is like you're graphing the pyramid onto the 2-d...

253 Anoop Right.

254 Robert Piece of paper cause...you know so if you have like a computer program you type in those coordinates I guess it'll make a pyramid... a layer of the pyramid which would be the $(a+b+c)$ to the fourth?

255 30:50 Anoop That's interesting.

256 Robert I think so, I don't know.

257 Anoop Yeah, it will tell them what the exponents should be right?

258 Robert Yeah...

259 Anoop Because your coordinate system is talking about the exponents. It's another story how to get the constants.

260 Robert Yeah.

261 Anoop Okay.

262 Robert Yeah, but I don't have a computer that can...like I don't know how to graph in 3-D so...

263 Anoop Right. Okay, I'm gonna stop here and show you more clips.

264 Robert Okay.

265 Anoop ...picture of where you are and what you did with math. So you have a Masters now?

266 Robert Yeah I have a masters in statistics and umm I picked statistics I guess for two reasons...uhh, one is because I felt like what they teach you ... there's so much data out there on the internet now that you can actually use the data and use the theories you learned and actually get something useful that makes sense, whereas in math I was learning how to do proofs..

267 Anoop Uh huh.

268 Robert for things that I didn't see any use for ...

269 Anoop I see.

270 Robert Like, oh I can prove you know, all these crazy things, but I don't know what they mean.

271 Anoop Right.

272 Robert What's the purpose of them? I'm sure there is some purpose but, I mean I didn't take enough Math to do it but it seemed like all I was doing is proofs and not really... I like working with data and actually getting meaning out of something, you know. Like it seems like I can just take what I learned in stat and apply it to anything.

273 Anoop Uh huh.

274 Robert Whereas in math, it didn't seem that way to me.

275 Anoop So is your job now stat related?

276 Robert Oh kinda, I guess data base, you know but ummm, I'm really big on uhh...sports betting

277 Anoop Uh huh.

278 Robert And it comes in real useful there and it's a good source of income so, yeah.

279 Anoop Oh really? So you do that seriously?
 280 Robert Yes, plus I have a book that's out if you ever want to buy it, it's uhh
 281 Anoop Oh wow.
 282 Robert Oh yeah. It's uhh...
 283 Anoop Oh wow.
 284 Robert Umm, yeah it's about like I guess theory of sports betting...specially tied to hockey.
 285 32:43 Anoop Oh, I'd like a copy of that.
 286 Robert Okay, it's... you have to get it from the gambler's place in Las Vegas...hopefully from Amazon soon.
 287 Anoop Oh really?
 288 Robert But, yes it has a very uhh small run, only a thousand copies.
 289 Anoop When was it published?
 290 Robert Like eight months ago, I think we sold like two.
 291 Anoop Ohhh....
 292 Robert Yeah so, big money losing venture. Yes.
 293 Anoop That's ok.
 294 Robert But umm, no, well I guess it was more like to waste uhh time I guess. I don't know, I thought it would be a good idea, but... actually looking back it's probably not because umm information is kinda vital, you know, and you don't want to be giving people ideas 'cause you're competing against other people so it's ...
 295 Anoop So you hold back your genius in the book?
 296 Robert Ahh, no.
 297 Anoop Okay.
 298 Robert Well, you use a lot of statistics to find like... there's so much data out there...
 299 Anoop Right.
 300 Robert Like in football there's spreads or whatever...
 301 Anoop Right.
 302 Robert But then there's always alternative markets where they're not that you know, precise so you can use history to you know...
 303 Anoop To make predictions
 304 Robert like if the spread's seven or whatever and they have like a first half spread you could see like all the games where the spread was seven, what was the first, you know, first half and then you can use that to exploit like small edges and ...
 305 Anoop I see, wow.
 306 Robert Yeah, it's very interesting.
 307 Anoop That is interesting.
 308 Robert It just involves sitting in front of a computer all day and waiting for something to happen.
 309 Anoop Ohh...
 310 Robert Yeah.
 311 Anoop Alright, we can stop here Scott and I will show him the clips.
 312 34:07 Anoop Okay, so in disc two when you start off the triangle, you have six of

- the twelve's there.
- 313 Robert Mmm hmmm.
- 314 Anoop Right? And you say that they can all stay because there will be some overlaps and you go on to say that things will be counted twice. What did you mean by this?
- 315 Robert Ummm, no I watched the first video over and I think kinda what happened was when you don't know what you're doing and you just start drawing stuff, it gets confusing and then you confuse... by drawing like stuff on top of each other I think I confused myself even more ..
- 316 Anoop I see.
- 317 Robert So, I knew all the identities and the, you know the theoretical but I wasn't going back on that for some reason instead I was just so lost and I... you know, just throwing stuff out there and hoping it works, whereas, when here when I got stuck for a second I immediately went back to you know the identities.
- 318 Anoop Right.
- 319 Robert and was like, you know, I'm fine now, you know 'cause I also drew it less confusing so I think that was one of the problems where I just, you know 'cause I watched that part to and I was like umm, I don't now what I'm saying 'cause I think it was just more confusion than anything.
- 320 35:19 Anoop Okay.
- 321 Anoop And um also too later Todd shows some work on the board where you are adding some things across and down. Are you really doing that kind of addition he was saying?
- 322 Robert I probably was at the....I probably was doing that sort of addition just because I saw 12 and I was like how did I get the 12. well its 4 3's, 2 6's a 6, 3, 3 you know, so I was kinda like what numbers can I manipulate to somehow get to 12 to maybe trick him to think I know what I'm talking about type deal. You know like. It happens a lot in Combinatorics, where especially if you take a class, and they have the answers and you see 81, and then you go back to the problem, you're like well, I got these numbers how do I get back to 81 from here. And you just start multiplying and adding stuff and dividing and then some how you get 81, and you're like Oh, Ok that must work every time and you get the test and its like no that's not how it works. You know.... So I think that's one of the flaws with combinatorics in general, it's just
- 323 36:27 Anoop So basically you were just saying whatever way you could get that 12? And you were just trying
- 324 Robert Yeah the problem is....Yeah the answers are always somehow related to every number in the thing. There are multiple ways to get there, so you just...you know... take that answer, and then like its $\frac{1}{2}$ this so you must multiply by 2 for some reason. You know, you just
- 325 Anoop I see....I see. Ok..so you were focusing on the number 12. and how

- you could get it
- 326 Robert how do I get 12? If that makes sense.
- 327 Robert I think part of it too was that I just had numbers up there I didn't like have that many...I think when I started writing it out what the numbers mean
- 328 Anoop Yes
- 329 Robert how I put the meanings right next to it. It actually makes more sense. Its not 3 its $3a^2b$, ...
- 330 Anoop Yes
- 331 Robert You know 3 means nothing, but when you have the you know the as bs c's attached to it actually has meaning and you're able to see it better
- 332 37:21 Anoop Yes and I agree here you started with writing them out in the very first triangle....ok...you were counting the 6abc term twice and then you decided you could split it multiply half with an A, half with a b, Now what where you thinking about the 6abc term, I mean, you were doing lots of different things with it.
- 333 Robert I think that was something like so for, where. Um...we got the 12 but it didn't make sense to include like the whole term so like we thought half goes here, half goes here, you know...so we just trying to...actually we were, I think when I did it I was looking at how to get the 12 from numbers that were in the same level but not the previous level, you know...so...it was just one of those confusing myself and I was just, you know I thought $\frac{1}{2}$ would go here and $\frac{1}{2}$ would go here, but after you say it you like wait, that's not right.
- 334 Anoop Ok, Ok. Now, At least for this session do you recall the biggest hurdle you had in understanding Pascal's Pyramid?
- 335 38:27 Robert Uh...I think the biggest problem was just representing it because One, I can't draw, two, you know it was in 3-d which if you don't have a computer you need to actually have manipulatives to draw in 3-d, I mean you can draw it on the board, but you have to be a good drawer and I'm not. So, Plus I've always had problems with 3-d. Like I never did good in Calc III which was all like you know third dimensional. I could do all the formulas but visualizing stuff, that was the problem. You know, they always talk about how these problems are good because you can you know work with these and easily get the answers. But when you work on towers you have get the manipulatives, pizzas you know, you can draw your own manipulative because it's 2-d. You know... so I think the combinatorics problems are easy if they are simple, but when you actually get harder you need those theories to back you up. Otherwise your just gonna get a problem. But if you had paper and tape, I'm sure you can make a real pyramid and go from there, but yeah working on a board isn't a good idea.
- 336 Anoop (Laughs) I agree, I think paper works even better sometimes than the board.

- 337 39:44 Robert Well if I was gonna work on the board too I should have drawn it like I drew it this time instead of on top of each other, like each level has its own square, you know its own place, instead of...you know, I think when I drawing it there I was kinda just drawing triangle inside of triangle and trying to create a 3-D representation and it didn't really work, it just gets confusing.
- 338 Anoop. Well I just want to say this, it might not be useful at all but it seemed to me that you wanted to squish the pyramid and looking at the top see all the layers
- 339 Robert Yeah. Here's one row and there's another row around it.
- 340 Anoop So like here's 1, here's 1-1-1
- 341 Robert And you can't look at the center stuff because technically it would be covered up by the.....
- 342 Anoop Exactly I think that's what was happening. Ok what was the most helpful thing about, I mean that helped you understand the Pascal's pyramid?
- 343 Robert Um....well...I Knew all the, the squared identities, but for some reason I never put together that you could extend it to more dimensions. And once like I understood that, then it made it easier. Like how that choose right there is the 4 choose 2, choose.....like in precalculus whatever we just you know 4 choose 2, that's all we were taught, that's really 4 choose 2 choose 2. You know, I think that if we were taught that from the start, it think it would have been better or like the Pascal's identity the one you know the one where the triangle shows the 2, that could be extended to 3. Looking back that seems obvious but when you don't know, you are like it doesn't really make a lot of sense.
- 344 41:20 Anoop But do you know from the session that you had with Todd what was your moment when you said Oh, I get it
- 345 Robert Uh did I get it at the end I don't even know
- 346 Anoop You did. Yes
- 347 Robert Really? I'm gonna guess its just because we did it so many times and we exhausted so many options and there were other people there helping too. And, I don't know if Liz, she looked generally confused, she didn't really know that well but I think Todd was more like pretending to be you know so to lead questions. Like when you interview a kid and you're like what's this...Todd was...But I think Liz more generally was asking inquisitive questions because she was just as lost as me, which is Ok. So, I think that kind of helped too. Cause she was asking questions that you know and she was answering stuff that I didn't ... so we were working together like a group thing
- 348 Anoop Ok...ok...good. What did you think of the questioning style of Todd? I mean did you think he gave you enough time to respond? How did it feel?
- 349 42:27 Robert I watched about 40 mins. I thought it was alright, but I think his method was probably the best method when you act like you don't

- know and get some one to explain it to you. But same time, when you see someone going to far in the wrong direction he would say something that would pull you back. Nah I think it was great he's a math educator so he has got a lot of experience with that.
- 350 Anoop That's great. You make a comment on the video, you say you are horrible with math (laughing). What did you mean?
- 351 Robert Obviously, I'm just bad at it
- 352 Anoop You hate math and you love stat
- 353 Robert No I mean it's the same thing, but sometimes like I wonder how the heck did I get these grades because I just don't know simple things. Like before I was just trying to count the number of years between you know two years I had to use my hands cause I keep getting the wrong answer . I am saying like what, you know just like one of those moments when you are like why, you know you don' t know the simple things when you are not thinking correctly.
- 354 Anoop You have a Bachelor's in math Masters in stat, and even today you say you I hate math
- 355 Robert Yeah I can't find the difference between 2000 and 1990 how many years there are its like, you know what I mean 10 years, no wait 11.....
- 356 Anoop So how did you do in your math classes
- 357 Robert I had good grades, I had a 4.0 so I'm happy with that
- 358 Anoop Wow....ok
- 359 Robert English classes weren't the good ones so they kinda brought the averages down. Yeah I'm not very good at English. Can't write that's why I have to do math...because yeah
- 360 Anoop So no papers, right. Ok
- 361 Robert Well the one thing I always found was its funny how people say I'm horrible at math and it just gets accepted, but when they say I'm horrible at English they think you're stupid or whatever. You know what I mean. Yes, Its ok to be bad at math but you're bad at English is the worst thing you do
- 362 Anoop It's a taboo....yes...yes...I don't know how ok it is to have a math degree and still say that. Ok
- 363 Robert Well its mostly umm... negative influence. If you kindda bring yourself down you motivate yourself to do better, you know. You know that kind of influence, you know I'm horrible and then you're like no wait I can do this....so it's a form of influence, or it could be you just loose all yourself confidence and give up. Just cause you thought something was simple and your like wait a second I should know this and you don't. So
- 364 Anoop I understand what your saying ok....when Todd gave you the opportunity, I mean he just asked you hypothetically....Would you work with a friend or work on the problem a few hours by yourself you said with a friend. why?
- 365 Robert Uh...I was probably lying because I'd rather work by myself. I don't know. Whenever we did all the sessions I always liked to work on the

- problem myself and then work with the group. Just because you know there is certain people like they won't work, because they would just be like Oh I will just wait until...they'd just sit there and listen to other people that actually work. But I think when everybody kind of work by themselves and they bring different Ideas to the table, it's better. Whereas if we just start working in a group right away there is always one or 2 people who immediately just shut down and just become very passive. Where as when you work by yourself and then the group everyone is bringing ideas or questions they have to the group. That's the best way for people to understand Because you either didn't get it or got it, and can say this is how I got it or what was tripping you up. I've been in groups before in classes where I didn't know the materiel that well because I didn't do the homework for the week, and in the group I would just be passive and copying. Then when test time you just kinda like what going on. Right? Its one of those, so I think if I had a choice I guess I'd work in a group, but if you could meet those conditions that'd be best.
- 366 Anoop First have time to yourself
- 367 Robert Yeah...because if you don't attack the problem by yourself you won't know if you do or don't understand about. That's how I feel about it.
- 368 46:50 Anoop Ok...Dr. Maher mentions that Liz said the solution to Ahnkur's problem is in the pyramid. And you saw it right away. Had you seen that before? I mean
- 369 Robert Uh...No, I mean I just know Ankur's problem what the constraints are. So as long as you know what the constraints are you can easily...as long as you have if you have abc's written out there you can just circle the ones that meet those requirements. Ankur's were either a^2bc , ab^2c , abc^2 . so since I know that I can just circle the ones right away and be like there you go.
- 370 47:28 Anoop Ok...was that the first time you made the connection between Ankur's problem and the Pascal's pyramid?
- 371 Robert Yeah I think, I think I never, I don't think I ever worked on Ankur's problem. The only reason I know what it is, is from working here cause it was a very popular clip for a while. You know...it was in the PUP math Carolyn would always comment on it... so I watched the clip you know a couple of times. So...But I don't think I ever worked on Ankurs problem you know in group or by my self in an interview. So, That'd probably be the first time
- 372 Anoop Yeah, ok...ok
- 373 Robert Well actually, I lied, I did work on it once. Cause Carolyn gave me all the problems we did in Kenilworth and she asked me to do them all and give her a notebook and then I don't know what became of that. So I did it that time
- 374 48:16 Ok because in the disk you remember doing it another way. You do mention another way like asterisks and spaces, how many ways can you fill the asterisks and spaces or you said the donut problem,

remember?

375

Oh yeah I was thinking what I was talking about was the you know you have the constraint where there has to be one of these or something. Since there is two the....its more like a 3 tall tower and then you have to use the other....so one space always has to be a, and for the other 3 you have to just arrange abc. Something like that. That's how I do the world series problem. I think I did it in the same way

376

Anoop

Ok, we're gonna stop here again

Appendix C

Date of Session: 11-14-2008

Author: Anoop Ahluwalia and Mathew J. Cann

Verified by : Anoop Ahluwalia and Mathew J. Cann

Date of transcription: July 2009

Line	Time	Speaker	Transcript
1.	0:00:00	Robert	We needed to find another color that worked.
2.		Anoop	I see what you're saying.
3.		Lou	I'm recording. Is that ok? Or do you want me to not?
4.		Anoop	We're starting now but if you recorded it, what he just said is fine. We'll need it.
5.		Robert	I think the way it's set up is like every other one is the same hole for the most part. So, you know.
6.		Anoop	You're talking about the little ball.
7.	0:00:30	Robert	The distance between the...you know rectangle and the next rectangle. Distance between the pentagon and next pentagon so I don't think you can use a different color
8.		Anoop	Ok. You know I'm going to come back and ask you more of that.
9.		Robert	Ok.
10.		Anoop	Ok. I'm just gonna to go with my list now and I'm really going to come back and ask you more of that.
11.		Victor	So just pretend I don't know the problem? Ok, it's very hard (laughing)
12.	0:01:00	Anoop	Well you just want to make sure if he was talking to one of your students, who had never seen this problem, and they were very frustrated now, and Robert had to explain it to them now. You know, did he go to that depth? So you really really want him to explain it well.
13.		Victor	Oh.
14.		Lou	Maybe me and Victor should switch.
15.		Anoop	You want to switch?
16.		Lou	No, no. I'm just saying. I definitely don't get it.
17.		Anoop	Really? Ok. That's what Marjory was saying too. That she could be a better student than Victor. Oh, boy. Well, um, I'm ready to start. Ok we're going to start with this model you just built Robert and I want you to explain this model to Victor. What have you built? What does it mean?
18.	0:01:30	Robert	Uh, ok. Just show this piece of paper. This is... each row of the Pascal's pyramid and this is trinomial expansion a plus b plus c squared. And this is... well this is actually the row that's labeled top there is one. Each of these little, ah, circles represents a spot on the triangle where there would be a number and this is you know a plus b plus c to the zero, which is one, to the first, to the second, to the third, to the fourth and... I put these in here to show how you get this one it comes from this one and this one. The ones that are

pointing out and like this one, this two, come from this one and this one. And, you know, etcetera. And these are just holding the blocks together... These are... these are just like, these are all ones down the side. And, you know, these are just connecting together, so this... You know, like this blue has points down here... and...

19. 0:03:00 Victor So where are you mention, this connect with this?
 20. Robert Yeah, like, ah, well I guess this is, you know, this would be a, a to the one, b to the zero, c to the zero, so it would carry the a over here and this would be a to the zero, or just be b, so then they combine to make ab. And you know a and this is c so this is ac, bc. And then, down here, so this is ab and this would just be, um, doesn't really matter I guess, this would just be a again. So this would be a squared b. And then this would b, so ab plus b is ab squared. And in the center, see how this... um, ab, ac, and bc... and you get the abc right here. (Victor listens and agrees to everything)
21. 0:04:20 Anoop Do you want to tell victor what is the coefficient of this A B C term?
 22. Robert Six.
 23. Anoop Why is it a six?
 24. Robert Um, because, well, I start up here, here you have one a like if you had a...but if you had this. Hand me that. I didn't set that up right. Ok.
 25. Anoop That's ok. It was my fault.
 26. Robert Ok I got the top row you we have a, plus b, plus c, to the one and you expand that it becomes a, plus b, plus c. And that's the, so, this is a, this is b, and this is c. And if you have a, plus b, plus c squared you get a plus ab plus b plus bc... plus c... plus ac...a plus b plus c, so the corners would be a, b, c again, and then this would be ab, bc, ac.
27. 0:05:40 Anoop So this is one a?
 28. Robert Yup. That is this term. So this, if we drew it like this. We'd... drew a circle here, here, here. So this is a, b, c. Well I think I said differently but yeah. I said this was c or whatever, as long as you can see... this would be ab, bc, ac.
29. 0:06:10 Anoop Oh, so this is one a, one ab, one b, one bc, one c, one ac.
 30. Robert Oh whoops. There's two.
 31. Anoop There's two. Ok. Why is it two?
 32. Robert Um, oh, cause you have... this is already...(Robert thinking and writing something for himself)
33. 0:07:15 Anoop Ok, but this was to the power of zero, right? That just gave me a one. This was to the power one, so it gave me linear a, b, c. So what do you think this level is talking about?
 34. Robert Squared.
 35. Anoop The squared, so this would be, a plain a, you think? I mean this

- was an a, and it went down again. What are these bars talking about? When you go down these long bars, what's happening?
36. Robert Ah, you're not adding anything on top of it.
37. Anoop Ok.
38. 0:07:45 Robert So, it's just bringing a down all the way.
39. Anoop So, this is also an a. And this is also an a.
40. Robert No. Ooohhhh. (Writing some more) This is a-squared.
41. 0:08:15 Anoop Which one is a squared, Robert? Could you point it for the camera?
42. Robert Ah, it's right here. This is c-squared and this is b-squared.
43. Anoop Ok.
44. Robert So, you're adding a this way and this way... Ok nevermind, when you go this way, you're adding what this is.
45. Anoop Yes
46. Robert So you're adding a. If you go this way you're adding the one next to it, so you're adding a b and, so this is an ab. And here you're taking a B and you're adding an A, so A B and you're adding a B here for B squared. So, ...(pause)... yeah I guess if it's pointing in the direction of the one next to it, it goes that way, so this is A, then A because it's pointing in the direction of A. Here it's pointing in the direction of B, so it's adding a B. And here this B is pointing in the direction of A, so it's adding A.
47. 0:09:05 Anoop So, but you're saying adding an A... B and adding an A should give be B plus A. But I think you said this was two ab
48. Robert Yeah I'm multiplying it by B.
49. Anoop Ok multiplying.
50. Robert So, I guess it's like this, right? Then you get A, b , A B, B squared... So that's where the two, so if you're going to the left, in this case, you're multiplying by A and if you're going to the right, you're multiplying by B. And same thing here. And then, on this side, this way is multiplying by B and this way is multiplying by...
51. 0:09:50 Anoop So, I am understanding that these bars are actually saying something now, what are they saying?
52. Robert Yeah they are saying, uh, multiply by itself.
53. Anoop Multiply by itself, so these on the line...
54. Robert Multiply by one instance of itself.
55. Anoop Ah.
56. Robert Cause A squared multiplied by itself is A fourth, it's not right.
57. Anoop So this whole thing is one.
58. Robert Well this is A.
59. Anoop It's all A. Where's the B?
60. Robert Yeah this one too. So...
61. Anoop So that's B?
62. Robert Uh Huh.
63. 0:10:20 Anoop And where is C?

64. Robert On the other side. I guess if I... I don't know if it would work cause there is only the colors... I guess you could have different colors for each. You know like have blue be A and red be B and yellow be C.
65. Anoop But you think technically, these tools might not shape up for that?
66. Robert Um, yeah I don't think.
67. Anoop Because our kit only has certain colors, in certain sizes, right?
68. 0:10:50 Robert Yeah we have red but...
69. Anoop Ok.
70. Robert Uh. Maybe.
71. Anoop I really want to jump to what you have in your hand before you break it (laughing) If you don't mind.
72. Robert Ah ok. Well I don't think it will work because you normally can only... can't use two... you can't connect these two. There is no way to connect these two.
73. Anoop But you have them connected. You have them connected with a red. What do you mean by you cannot connect them?
74. Robert With a different color.
75. Anoop Ah. So you're saying... Show me what you are saying cause I am kind of misunderstanding.
76. 0:11:25 Robert Uh take the red out.
77. Anoop Ok.
78. Robert That doesn't work.
79. Anoop Oh, is it... Are you sure?
80. Robert Umm, this might work. No. Yeah so...
81. Anoop Ah, I see what you are saying, like technically these tools are probably not built for what you are trying to do in three colors, right? But we haven't explored it completely. Ok. So, Victor do you have a question for Robert at this point?
82. 0:12:05 Victor I understand this, so far up to this level.
83. Anoop Ok. So then... Do you...
84. Victor So this level would, two connection, so, this would be that one, right? That would be one.
85. Anoop One
86. Victor So that's two. Same thing, like, B. And this two. And this, this, this, that.
87. Anoop That's, that's exactly it. So, I think
88. Victor So I know a plus b plus c, squared.
89. Robert I guess but in this case there'd be three of them because you know there are two here and there is one there.
90. 0:12:55 Anoop I want to see if Victor understands this. So if he just had one here and two here, can you guess Victor what would this be, on this model? What is this ball talking about?
91. Victor So basically. This will be, explain two ball, two connection, right?
92. Anoop Right.
93. Robert Well it would be two ab

94. Victor Two ab, so this one a, this would be, this one and... So that's two ab. So that's uh... a... and two... a squared... and uh... b there and here is also two already.
95. 0:13:45 Robert Ok, so, two a squared b? Cause you have two AB and you're adding an a to both of them, so that gives you two a squared b. And you the other is a squared and you're adding b
96. Victor Ah ok. So, one plus three equals three. So that's got three.
97. Robert And for the same. With this one.
98. Victor Ah ok. The same.
99. Anoop Can you guess that one now, Victor? Like I think you kind of understood that this would be 3 A squared B but can you guess what this one will be? From...
100. Victor Ah, this will be, from the b point this is symmetric, so that is the same answer.
101. Anoop So this would also be three A squared...
102. Victor Three A B squared
103. 0:14:32 Anoop Ah it would be B squared. Is that correct Robert?
104. Robert Yup.
105. Anoop Ok. So this side is A, that side is B but what is this middle? Why are there things in the middle here and not here and not here?
106. Robert Ah because these are squared so you can't have three terms cause you can only have, ah, you know... if you add the exponents up it would have to equal two. So you have either A squared and zero for the other two or you have one one zero. You can't have one, one, one, that wouldn't work. You know? So, that's why the middle row starts showing up on the cube term because you can have one, one, one with exponents then ...and you know for the fourth term you can have it too. You can have two, one, one; one, two, one; or one, one, two.
107. 0:15:20 Anoop So what is this in the middle?
108. Robert Ah. ABC
109. Anoop ABC term. So what is the coefficient on that ABC term?
110. Robert Six.
111. Anoop Why?
112. Robert Um, two, two, two. So, you have two AB, two A... two AB, two AC, two BC, and you're adding the other term to it. Yeah I think it would be much better if it was this block in three different colors.
113. Anoop I see what you are saying.
114. Robert Because then you can have the, you know... this is, this is AB and you want to add a C. I guess it's kind of pointing in the direction of C.
115. 0:16:05 Anoop Yeah
116. Robert And this is AC and it's kind of pointing in the direction of B you know, it's kind of pointing to the right. And this BC is kind of pointing, you know. So...
117. Anoop But they are not touching. Right? I mean there's gap. And my

- guess is the gap is because we used two sizes of blue. Now do you think these two different blues that you have, one of them has meaning? Or none of them has meaning? Like, I think we just said this means multiplication in the same direction as the variable. Correct? What about these shorter blues, the ones that are...
118. Robert They mean the same thing.
119. Anoop Ah ok. So these are still representing multiplication, what about
120. Robert So we could have put it like this.
121. 0:16:45 Anoop We could have!
122. Robert And it worked to the same thing.
123. Anoop But earlier you said...
124. Robert Well yeah I...
125. Anoop Ok I just want to say it, so that... I can type it later but you had said that red would not work. That is what you had said before. Two colors would not work.
126. Robert Yeah that was off the record though.
127. Anoop But now on the record red works.
128. Robert Yeah it works.
129. Anoop Do you prefer your, one of the two models that you built?
130. Robert This one's more compact. So if I was going to rebuild that one, I would use this one. But...
131. 0:17:20 Anoop I see. So what's... Now that we have two colors. Maybe I can ask you this, do you think one is more meaningful than the other? The color... The two color bars that you put there. Is red more meaningful than the blue?
132. Robert Blue doesn't mean anything.
133. Anoop Blue doesn't mean anything at all?
134. Robert It's just holding it together.
135. Anoop It's just holding it together. And red means?
136. Robert Multiplication.
137. Anoop Multiplication. Ok.
138. Robert Well I guess the a could, blue, could represent addition. You know cause this is really A^2 plus AB plus B^2 ... so the blue could represent addition and the red means multiplication.
139. Anoop I agree.
140. Robert That makes more sense.
141. 0:18:00 Anoop Ok. So Robert what I want to ask you next is, you know, you first saw a Pascal's triangle and the pyramid came much later. So where is the triangle on this model?
142. Robert Ah, the blues. Like the whole row, I guess I'll use this one. It's bigger. This row. So,
143. Anoop Can I go all around this row and get...?
144. Robert No just on the side. Each side of the triangle, uh, pyramid represents.
145. 0:18:35 Anoop Do you understand Victor, what he is saying?
146. Robert Because the Pascal's triangle is only two dimension, so you take

- each one of these rows and you have only two variables. And like A and B and then C
147. Anoop Oh ok.
148. Robert So, And this is A and C. So each face. Well it's not really a face. Yeah I guess it is. So each face of the pyramid, so this is one. This is the first row or the zeroth row, first row, second row, third row, fourth row. And, so, it's a pyramid with three triangles, Pascal's Triangles on each face. I guess the bottom is nonsense.
149. 0:19:15 Anoop The bottom is nonsense? In two dimensions.
150. Robert Yeah in two dimensions. In three D it's that row of Pascal's Pyramid. So it's just the fourth.
151. Anoop You were in the zero, one, two, three, fourth, Yeah, fourth. Ok
152. Robert If I was to rebuild it, I would do it this way.
153. Anoop You would? How is this?
154. Robert Yeah like that. I built this off camera. I think one thing though if they made different colors, you could, you know, just make it look better.
155. 0:20:00 Anoop I see. Like if A, B, and C could be their own color.
156. Robert Yeah but I guess these tools aren't really built for it.
157. Anoop You know what I am going to ask you next is, you just said the very base of this is meaningless when we are talking about two dimensions. But when I came in, and you were still building this, you had put some balls on the bottom, already. Where did that come from? Like, I mean, this was not completely but you had put some spheres down there. Why did you do that first is my question, I was very curious.
158. 0:20:35 Robert Oh, I built them, ah, I built them all separately and then I stacked them afterwards. So I built each row, independently, and then I had to stack 'em. And that didn't really work too well cause they don't line up correctly. If you ah...
159. Anoop If you don't build them...
160. Robert Yeah apparently there is a difference. Even though it looks like the same but the circles not tilted the right way it doesn't build correctly.
161. Anoop How long did it take you to make this?
162. Robert Ah maybe like thirty minutes but I was talking to people while I was doing this.
163. Anoop Ok. Could have gone faster, is that what you are saying?
164. 0:21:10 Lou Hey. Hey. Hey.
165. Anoop What do you know about the origin of Pascal's Pyramid? Like do you have any idea?
166. Robert Nah. I don't know anything about it. I guess Pascal found it but I bet he probably didn't and someone else found it earlier but he got the credit for some reason. Cause that's how it always works, right?
167. Lou Pretty much.

168. Robert There is always some random priest from like the nine hundreds who figured it out and then...
169. Anoop Yeah like I am taking credit for this but I didn't make it, Kevin did. I did break it and make it again. But I was cheating cause I knew exactly what I needed and you didn't.
170. 0:21:45 Robert So I'm guessing he must have used it for something and that is why he got his name attached to it but yeah I don't really know.
171. Anoop You had said problems like building towers could be solved by using Pascal's Pyramid. How is that? Explain it to Victor. How can I take the towers problem and how can we solve using Pascal's Triangle or pyramid?
172. Robert Ah, well I guess you can only solve certain towers problem but you can't solve them all cause this would be, I guess solutions for towers, like this bottom row would be towers four tall with three colors available to choose from and this face here would be towers four tall with two colors to chose from. But you can't represent, you know, four colors cause there is no fourth dimension. Well we didn't make a fourth dimension.
173. Lou You can't make one.
174. Robert You can make a fourth dimension.
175. Lou You can?
176. Robert I can't
177. 0:22:50 Lou I was going to say, if you can let me know.
178. Robert I mean I know what the values would be but I don't know...
179. Anoop You do know what those values would be?
180. Lou For the fourth dimension.
181. 0:23:03 Robert For X, N dimensions yes.
182. Anoop So you're saying, if I said A plus B plus C plus D to the fourth, you would know what that would expand to.
183. Robert Yup.
184. Anoop But how would you actually say that you know how it would work in four dimensions? What do you mean by that?
185. Robert Well I don't know how to represent it with the, you know, this cause ah... can you draw it in the fourth dimension? I don't know.
186. 0:23:30 Anoop I read somewhere that for four dimensions you take shadows. Shadows are three D and we can handle three D shadows. So, they do graph some four D stuff but you know I'm not an expert on that of course.
187. Robert But I mean, if you say, give me the tower that is four tall with two blue, one yellow, and one red, I can do that. You know, if you say...
188. Anoop And that is exactly what we are going to do next.
189. Robert Ah ok.
190. Anoop Do you remember Ankur's problem?
191. Robert Yup.
192. Anoop And could you tell Victor what that problem was?

193. 0:24:00 Robert Ankur's challenge is, or problem is, towers four tall, with, ah, three colors, there has to be one of each color in the tower. Is that right?
194. Anoop Yes. Exactly. So here are the blocks, if you need them, feel free to use them but where is the solution? Victor you got the problem? You've heard of it before?
195. Victor Yeah.
196. Anoop Ok, so where is the solution on your model for that Ankur's problem?
197. Robert Ah, this eliminating the edges...
198. Anoop Ah, what do you mean by that?
199. Robert And certain ones of these, I'm not sure which ones. I'm gonna guess, without thinking about it, I'm just gonna guess...
200. 0:24:45 Anoop You're free to think about it.
201. Robert Well start with my early guess without thinking about it. Um, can I get two of these? Two red.
202. Anoop Going to make sure you get lots of toys.
203. Robert I'm going to say, the area in closed here. What about you Victor? No, wait. They're too big, cause I don't want to include these, so I just want to include the inner triangle. This thing.
204. 0:25:22 Anoop Can you put the red again for Lou, so it is captured? Sorry.
205. Robert I was trying to...
206. Anoop What you were initially saying.
207. Robert I was trying to just say these in here but I don't want to include these, the ones on the outside. I just want the ones on the inside.
208. Anoop You said it was too big.
209. Robert Yeah, I'm trying to just get these three.
210. Anoop Ok, so you're...
211. Robert I don't know how to.
212. Anoop Ok, the little one, do you think orange ones would enclose it?
213. Robert Yeah see I put it in front, so it doesn't work too well. I don't know.
214. Anoop Ok.
215. Robert So I want to include... Yeah this doesn't work either. Well I just think it's going to be these three. I don't know.
216. 0:26:03 Anoop Victor, are you convinced?
217. Robert Because... because, well here it is. Cause these are the two dimensional Pascal's Triangles, so you have to have at least one, so you can't have anything from the two dimensional Pascal's Triangle involved. So that will eliminate all the sides and it would just be the inside. This is what I am trying to say.
218. Anoop So can you explain to Victor, how many total towers are possible to Ankur's solution?
219. Robert Thirty six.
220. Anoop Thirty six. Where did that thirty six come from?
221. Robert Ah, six plus three plus three. Six plus three plus three. Six plus

- three plus three.
222. Anoop Oh the six, three, three adds up to twelve...
223. 0:26:48 Robert Twelve.
224. Anoop And where is that twelve? Where is this twelve?
225. Robert Ah, right here.
226. Anoop Ah, ok.
227. Robert These three pointing things. This... This, this, and this. Oh wait, these three. I guess it will work...
228. Anoop These three is this, right?
229. Robert Yeah the ones my fingers are touching. These three. And these three would go here. And then the same for the other one.
230. 0:27:15 Anoop Does this model do a better job at what you are saying cause things are connected? Or...
231. Robert Well I guess the only difference is I can't get my hand in there. I could get my hand in here.
232. Anoop Right.
233. Robert Yeah, so...
234. Anoop So, that's better cause you can get your hand in there.
235. Robert Yeah I can't get my hand inside there.
236. Anoop But the answer is...
237. Robert Those three, yeah.
238. Anoop But, Victor, do you understand? He is saying these three. Why these three (in the center)? And not these (on the edge)?
239. Victor That's the top, right?
240. Anoop That is the top
241. Victor Ah, ok. Why it's this one?
242. 0:28:00 Robert Think about the conditions of Ankur's challenge and maybe it will...
243. Victor That's three and that's six and that's three, right?
244. Robert Well this is 3, ah, $A^2 B$
245. Victor This will be six.
246. Robert Yeah this is three $A^2 B$ squared and this is three $B^2 C$ and that's six $A B C$. So you're adding, ah, you know, a B to it. No, you're adding an A to it. Oh no, a C . I'm gonna to say C . No, huh.
247. 0:28:48 Victor Ok, three plus three plus six equal twelve.
248. Anoop Twelve of what?
249. Victor Ah on the coefficient.
250. Robert Ah yeah but this is A , $A^2 B$, so this must be $A^2 B$ squared. So you have to add a C to it. Right?
251. Victor Um hum.
252. Robert And that's... But that one over there is $B^2 C$, is it?
253. Victor Yeah.
254. Robert And the middle is $A B C$, so that doesn't really work. Oh cause you have to add A ... Ah ok, I got it, you're adding an A to this one, times A . Adding a C to this one, you're adding a... B . Yup. So, ok, that makes sense. And then this one over here is $A^2 B$ squared

255. 0:29:46 Anoop C and this is, (pause) well actually I can't see, is that going...
I think Victor just agreed with you but I didn't really see what it added up to. Could you just finish the thought? Like what do they add up to? Twelve of what?
256. Robert Oh six, three, three. So this is six A squared B. A B squared C. Three A B squared C. So it becomes twelve A B squared C.
257. Anoop Ah, ok. So one of them was twelve A B squared C. Ok.
258. Robert And the other ones are twelve A squared B C and then A B C squared, twelve. Well I think if you... the condition of Ankur's challenge is that there has to be one of each color. So all the faces are only two variables, not three, so they can't be included at all onto Ankur's solution because see this is like, this is A fourth, you know, this is. Ah wait. So this is A to the fourth this is A cubed B, A squared B squared, see there is no C in anything here. So it doesn't count. There is no B in anything here and there is no A in anything on that face. So all of those can't be solutions cause you're lacking, they are missing a color. So the only ones that have all three colors are the three in the center, which are...
259. 0:31:16 Anoop So my variables are just like my three colors? The A B C are colors?
260. Robert Yup.
261. Anoop Ok. And none of the faces have all three colors.
262. Robert Yup, so...
263. Anoop Ok.
264. Robert So, for any future Ankur's, you know whatever tall, it's just going to be what's in the center, it's not going to be what's on the outside.
265. Anoop Ah, so if we took it to, not four tall but five tall.
266. Robert And I guess if you reduce Ankur's challenge to three tall, there is only one solution that has, or ah six, you know that have... and if you, you can't do Ankur's challenge if you have two tall because you have three colors and you can't have one of each color, you know. And then, so...
267. 0:31:57 Anoop That's good.
268. Robert For the future, it's whatever is in the center.
269. Anoop So wanna to try another problem on this triangle?
270. Robert Oh yeah.
271. Anoop How about the taxi cab problem? If you need a handout I'll give it out to so you can take a look at. Here, Victor.
272. Victor Thank you.
273. Anoop No problem. Um, Victor has played with this before but...
274. Robert I don't think I have ever did taxi cab, I think I did it once.
275. Anoop Never did the taxi cab problem. Hmm...
276. Robert I did in Carolyn's class though.
277. 0:32:43 Anoop Ok. Would you think we give you time to play with it and come back to it?

278. Robert Ah nah, I think I got it.
 279. Anoop Ok. (Robert working on paper writing out pascal's coefficients on the taxi-cab grid) Do you need a pen, Victor?
 280. Victor No thank you.
 281. 0:34:12 Anoop Ok. When I interviewed you last, you said taxi cab problem can be solved by Pascal's. And that's why I brought it up.
 282. Robert Yup
 283. Anoop And you still believe that it can be solved using the Pascal's?
 284. 0:35:00 Robert Yup!
 285. Anoop Oh, OK
 286. Robert So do you want the answers for all three?
 287. Anoop You know we can just play with one. We can play with one destination, right? And coming back...
 288. Robert The problem is this is only four tall, so I guess the only one we could show is... none of them.
 289. Anoop Umhhh..
 290. Robert We'd have to move... well if we move the blue dot up one we can do it because...
 291. Anoop Ok so the given problem cannot be solved. Why is that?
 292. Robert It can be solved. It can't be solved...
 293. Anoop On the model
 294. Robert With this particular model.
 295. Anoop Why? Ok, could you explain that to me, why it cannot be solved?
 296. Robert Ummm, cause it's too far away from the starting point that... It would have to be the fifth row. Wait, one, two, three... ok, wait...did I do it right... well maybe I did it wrong and then...
 297. 0:35:54 Anoop One, two, three, I mean the blue dot compared to the taxi stand is in the fourth row. Wha... Isn't it?
 298. Lou Unnnhh...
 299. Anoop No, fifth row? Lou?
 300. Robert The fifth row. One, two, three, four, five, I think it's the fifth row.
 301. Anoop Fifth row. We get, oh because I am not counting the very first one. Ok so let's move the blue dot up. How's that?
 302. Robert Yeah to four
 303. Anoop Ok so let's move the blue dot to this position.
 304. Robert Or we say the black dot.
 305. Anoop I made it a big black dot. Yes. So, do you need another sheet of paper to work it out on?
 306. Robert No, I'm ok.
 307. Anoop You're ok. Ok, so explain to Lou what this, or no not Lou, Victor, what the solution is for some?
 308. Robert For the black dot?
 309. Anoop Yeah.
 310. Robert Four.
 311. Anoop Four. Why is it?
 312. Robert Well I guess if this is the taxi stand up here, at the top. The four

- ways you can go, there is, you can go to here. Can you? Can go... Well I guess you go one, two, three and over. So, one, two, three and then here.
313. 0:36:58 Anoop So, that's where the blue dot is?
314. Robert No it won't work.
315. Anoop I mean my big...
316. Robert Ok, I guess here, ummmm, There is one way to get here, right? And there's two ways to get here. Three ways to get here. Oh, I guess, hmm.
317. Anoop I think we have to first convince ourselves where this black dot is on this model, right?
318. Robert I think I just did.
319. Anoop Ok.
320. Robert I didn't put... I think what happened was... There we go, whoops. Wrong spot. Ummmmm. (Adding a white sphere to the model) Now it makes sense. Alright.
321. Anoop Alright.
322. 0:37:47 Robert There is... Remember this is one and this is three, so to get to four, there is one way to get to here that is either all the way down here. There is three ways to here, you can go This, that's one, or you can go, that's two, or you can go this way, that's three. Right? And there is only one way to get here and then, so there is one way to from here from up there and there are three ways to here from up there, so that's four.
323. Anoop Victor, do you agree with him?
324. Victor Yes.
325. Anoop So what Robert, says is move this blue dot one up, right here.
326. Victor Ok.
327. 0:38:37 Anoop And he is saying there is only four ways to get there. Do you agree with that? There is only four ways to get there.
328. Victor Four way.
329. Anoop Four ways, you agree?
330. Victor Ahhhh, yes.
331. Anoop Four, so I think...
332. Robert I could draw them, if ah...
333. Anoop Robert almost convinced me but I don't think you watched him on the model.
334. Robert There is one. So, on the model it be one... You're going down three and over one. So that's the first one. That's the blue one. And then, you want to go down one, so you go down one, and then you wanna go over, so you are going over, then down, down, so that's that one. So...
335. Victor Ok.
336. 0:39:28 Robert And then you can go down two, over...
337. Anoop Take another color, that would be great.
338. Robert Oh yeah, shoot. You can go down two over one, then down. See

- you go down two, to the right
339. Victor So we have three now.
340. Robert Well I guess in this down would be left, so you're going left and going the other, perpendicular to it would be going right. On this, down the face. And I don't know what happened to the top of this.
341. 0:40:00 Anoop What do you need?
342. Robert Oh can you get that top?
343. Anoop Oh right here. Yes.
344. Robert And then, what's the final solution is, you go... so we went down three, we went down two, and down one, so we want to go over right away.
345. Anoop Ah.
346. Robert And then down, so that would be like going to the right here and left, left, left. So there is four solutions.
347. Victor Yes
348. Robert To get there.
349. Anoop You know. I am very convinced of that but we had all this business of the variable, As Bs and Cs What is ... Does that just go away now?
350. 0:40:38 Robert Ah well this is just two dimensions, you can't. You can fly like... You know if you could somehow get here by I guess digging underground but yeah the streets two dimensions, so the third dimensions wouldn't come into...
351. Anoop Driving on the street is two dimensional. So I really just need one face of this pyramid. I only need my triangle. So I understand that. But... I mean... Ok so this A and this is B and the location I wanted to be is a crazy location. Would it make sense to even think of those variables when you are doing the taxi cab? Like you very nicely took these colors and compared them to the A B C, now how would you take these coordinates, these positions on that grid, and compare it to that A B business? You know? The variables?
352. Robert Well, I guess you could say A would be, uhhhh, be a route that starts going down or I guess let's just say down is left.
353. Anoop Down is left.
354. 0:41:53 Robert And, uh, right is right.
355. Anoop And right is right.
356. Robert So starts going left and B would be a route that starts going right, so this would be A cubed B because there's three routes that start by going down.
357. Anoop Wow, I like that.
358. Robert Like this one and then... you know I guess there is this one, yeah so there are three routes that start by going left first and there is one route that starts by going right first.
359. 0:42:34 Anoop So A cubed B, if we could write the meaning down, as well. A cubed B is talking about three...

360. Robert Three routes with a first turn...

361. Anoop Right.

362. Robert Or is going left.

363. Anoop We can just call it steps if you want. Three steps down. Is that good language or no?

364. Robert Uhhh.

365. Anoop But not steps.

366. Robert It not... Three of the routes start out by going left and one route starts out by going right. Because I mean they switch afterward, you know. Or it could also mean the spot is three blocks to the south and the other ones, you know it's one block to the east. I guess is the...

367. 0:43:17 Anoop So how many ways can I start from a taxi stand and get to this passenger who is standing three down, one right?

368. Robert Yup. So, er, yeah, three down, one right. I don't know if that always holds true with the, there's three routes that's... I guess it is, there is three routes that start with going down and there is one that starts with going... I don't know.

369. Anoop Wouldn't I always have to...

370. 0:43:45 Robert I think it will always work in this row but I don't know if it would always work in this row. I don't see why it wouldn't.

371. Anoop Well all the pictures you made for me where going south, south, south and there was only one west. Or east.

372. Robert Right is west.

373. Anoop You tell me. I don't know.

374. Robert I think it's east.

375. Lou If down is south, right is east.

376. Robert Yeah, ok. But yeah, I guess, but it also means that the location is three blocks to the south and one block to the east.

377. 0:44:22 Anoop And that's what the A and the B can mean now. They can mean south and east

378. Robert Or they can mean...

379. Anoop Lou are you, I keep on calling him Lou. Victor, are you convinced?

380. Victor Yes, Yes.

381. Anoop You see the taxi cab.

382. Marjory I'm confused.

383. Anoop You're confused? Explain it to Marjory, she probably did...

384. Marjory Well I don't understand what significance it has which direction you start in. Like you were saying there are three routes when you start going this way... I mean I guess I'm not... Does that matter? Does it matter where you start? I mean, does it matter what move... Do you know what I mean? You said there were three ways to get to the black point from the taxi stand.

385. 0:45:03 Anoop Can you see his picture?

386. Marjory Yeah

387. Anoop Oh, ok.
388. Marjory The taxi stand when you move down first and only one route when you move right first. Right?
389. Robert Yup.
390. Marjory So, um...
391. Robert I guess it is...
392. Marjory I guess what I'm asking; I don't understand why it's important that, which way you move, like your starting route. Like why does that matter?
393. Robert I guess it doesn't. In this example, if you assume that all the... there's not... you know all the routes are traffic, it takes the same amount of...
394. Marjory Cause it's sort of... Cause I guess it's not... I'm still not sure about the As and Bs mean on the taxi cab.
395. 0:45:48 Anoop Oh, ok. All the routes that Robert made and I think Robert should try to say this.
396. Robert Uh, I made four routes, right? Three started going south and one started going...
397. Anoop They involved three blocks that you are driving south, now it doesn't mean you drove three south in, one after the other, it wasn't consecutive, but if you look at the routes, only one time you turned. Right? Whether the turn happened right away or afterward but the other directions where south, south, south and one east. So east, south, south, south. South, south, south, east.
398. Marjory Is that what you were about saying? That's what...
399. Robert Yeah. I guess it could...
400. Anoop To me, that's what I understood from what he was saying that you need to go south, south, south...
401. 0:46:40 Robert And I guess it also means that the taxi stand is three blocks to the south and one block to the east. And if it was A squared B squared the taxi stand would be two blocks to the south and two blocks to the east. So it would be the six here. So this is...
402. Marjory So the A, so then A stands for moving in one direction and B stands for moving in the other direction.
403. Robert Yup and you can only go in two directions.
404. Marjory Right.
405. Robert Cause why would you go backwards?
406. Marjory Right, no cause I know it's part of the problem, you have to have the shortest route, so you wouldn't have any back and forth movement and you wouldn't go... You wouldn't travel further than you needed to go.
407. 0:47:20 Robert Yup, so there is only two choices to go... So I guess you could make it into this if you include that you could go backwards but then that wouldn't... You would get caught in an infinite loop. Right? I mean if you really looked at it that way. I don't know. You could extend this too. I mean if it's like those crazy towns

- where, you know you have a side street.
408. Anoop Oh, no. We're not going there (laughing).
409. Robert Would that work? I don't know. Would that work if you had instead of this just being a grid, you had side streets that go there? Then maybe you could make it so this model would represent it. I don't know though.
410. 0:48:00 Anoop Marjory, are you convinced now? Like with his two variables are two directions he's talking about, south and east. He convinced you of that but what he didn't convince me of yet is the four. Where is the four on this triangle? I don't see the four. Where is the four?
411. Robert Right here.
412. Anoop Oh it's a four.
413. Robert Or here. Or there. Or there. Or there. Or there.
414. Anoop Because this was A cubed.
415. Robert Well cause in two dimensions, you know, these would be equivalent. Actually no, this four, I'm sorry, this four would be over here and this four is over here. So, it would be these three. Um, Actually here.
416. 0:48:47 Anoop So you're saying the coefficient of the sphere, right? Was the answer.
417. Robert Ah which one? I'm saying this one, this one, and this one, would all be equivalent answers, just depending on what side you are looking on cause... and then this four, this four, and this four, wrong answers? Or the other four.
418. Lou Just mark them with a marker so we can see. Not with a the marker, just put it next to it, so we can see what ones.
419. Robert This one, this one...
420. Lou Cause it's hard to see the three dimensions in camera world.
421. Anoop Oh yeah I would imagine.
422. Robert And this one are the same four.
423. Anoop Oh yeah.
424. Robert That we chose for the taxi stands. Or it could be this one, this one, and this one.
425. Anoop Cause those would be the same.
426. 0:49:30 Robert But if you did that then A would have to represent going east and B would represent going south.
427. Anoop Do I understand that?
428. Robert Cause this is A B cubed, A b cubed, well you end up with... So if you chose this one...
429. Anoop Ok the reason I understand that is because right away I know these are just Pascal's Triangles and you are saying if I play with A B versus B C, it's the same game.
430. Robert Yeah.
431. Anoop So that I understand. Um, but what about this guy? Where is that one the grid? This guy.

432. Robert Ummmm.
433. Anoop Cause I don't...
434. Robert He's right there. It's six. He's, uh, two down, two to the... two south, two east. This is one south, three east. And this is no south, just going east. And this is all south.
435. 0:50:21 Anoop Is he making sense to you Marjory?
436. Robert And I guess...
437. Anoop Victor, are you convinced?
438. Victor Yeah
439. Marjory It makes more sense when you talk about numbers, when you talk about the coefficient with the variables at the same time because it's confusing to me when you talk about there being fours and then later you throw in the As and the Bs...
440. Anoop True.
441. Marjory Well it's because it's not always sure, it's not always clear how the coefficient and the two variables are going together, if you don't talk about them all as you know one thing. Cause if you say four A B cubed then that makes more sense to me than saying that there is four and then later to talk about the A and the B cubed. And that's why I was trying to ask questions before because I was hoping it would come together.
442. 0:51:11 Anoop Ok. Did that help?
443. Marjory I think so.
444. Anoop Ok. Ok, so that is great. Do you want to say more about this? Yes
445. Robert That would work. I was just saying, trying to think how this would represent this. So I guess since there's two ways of get here, you would have to have three ways to get here. So if you had those, I guess if you drew a, cut each square in half it would work. But then how would you have three ways of getting here? I'm just trying to think of how we could make this into this.
446. Anoop I don't understand. By this you mean the three D model?
447. Robert Yup.
448. Anoop How the three D model would fit on the taxi cab?
449. Robert Yeah you would have to have Three routes to each point.
450. Anoop Ahhh,
451. Marjory I have a thought.
452. Anoop Yes, what's your thought?
453. 0:51:53 Marjory If instead of there being a grid, like a square, if you had a cube and you could go up.
454. Robert Up buildings.
455. Marjory Do you know what I mean? So if you could go down, you know if you to the south, you could go east, and you could go up.
456. Anoop If you could fly.
457. Marjory Yeah. Yeah. Or dig underground.
458. Anoop And what... and you said that too, before. But I think splitting, like you were saying, splitting the square would give every vertex

- three options. So that might be the same as flying. Right? So, that is the same thing, right?
459. Robert Uhh, yeah, we could go, I guess, I think it's the same cause, let's see, this is..., you go, here, here, here and then when you are here you can go here, here, or here. But it would have to be going from the top left to the bottom right cause you can't have, if you start doing this, then you're in the fourth. I guess if it looked like this then each box had an x in it, would it be the fourth dimension thing we were trying to think of, the shadows, you know?
460. Anoop Oh boy.
461. 0:52:58 Robert Well, uh, yeah, cause each one has four ways to get... does it... let's do this...use this box for an example.
462. Anoop I don't...
463. Robert So this point here.
464. Anoop Yeah.
465. Robert You can get from here, here, here.
466. Anoop What do you think, Victor? Is that four dimensions? What he is trying to do?
467. Robert No I think it's more.
468. Victor I don't know.
469. Robert Cause you have one, two, three, four, five, six, seven, eight. So yeah.
470. Marjory But the thing is, if you move on a di... If you could move on a diagonal, like if you made that as another directional choice, then that would change your solution to your problem because instead of... because then you would have different shortest, different ... There would only be one shortest route and it would be the one that made the most efficient use of the diagonals as well as the up and down. You know as long as the downs and to the rights.
471. Anoop Yeah.
472. 0:53:55 Marjory Do you know? It could change your solution... Well it would... Like here think about one of these that no one has drawn on yet and you're trying to get to this point and if you could use a diagonal to get there, you would shorten your route instead of there being four moves, like here to here, here, here and over...
473. Anoop But Marjory...
474. Marjory If you could across a diagonal there would be one shortest route... I think there would be two shortest routes, you know what I mean? You could go diagonal. It would change the problem. And it wouldn't fit with Pascal's Triangle.
475. Anoop But if this was my point and your option was available then this is the shortest and those two are not, obviously.
476. Marjory Right.
477. 0:54:35 Anoop Right?
478. Marjory That's what I was trying to say.
479. Anoop Ok.

480. Marjory That is interesting cause there is some way you could do it but I don't know though.
481. Anoop I'm going to put taxi cab to rest.
482. Robert Alright.
483. Anoop Because, um, I think it can be very interesting but I probably won't be able to capture it all for my analysis. Ok, um, how did you decide how many colors you need for making this model that you made?
484. 0:55:00 Robert Well, going back I'd probably need two but I couldn't really get the other colors to...if I could get the other colors to fit and it would take a pyramid shape, then I would use four colors? Four colors.
485. Anoop And what would those four colors do?
486. Robert Ah, one would represent addition, one would represent multiply by a, one represent multiply by b, one be multiply by c, so those would be the 4 colors... Uh, I guess 5 if you include the white ball because that would represents a color
487. Anoop Cause that is a color (laughing)...Yes, oh boy, but you wouldn't want different colors of balls would you?
488. Robert No, I don't think so... well, actually if you had like you know infinite colors, or even in this cases if you had well how many different are there, ten maybe, one could represent the number 1 and one could represent the number 2, etc but..
489. Anoop So, the balls could represent the coefficients...the different colors could represent different coefficients. That'd be very colorful...So was it helpful to play with these tools at all?
490. Robert Um, yeah, it was alright. I guess I shouldn't have given up so quickly once I had a semi-working model cause I probably would have gotten that but uh... (pointing to Anoop's two-color model)
491. Anoop You did, here, you know, you did (pointing to Robert's small 2-color pyramid)
492. Robert That was after I built this then I started working on this and I was like, Oh! this makes much more sense and then I saw your model and I said oh it look exactly like it.
493. Anoop You did, you did. But does this tell you anything *new* about the pyramid or did it take you to a different level?
494. Robert Nah, I don't think so because I just drew it first and then I built it from my drawing I didn't like...but it was good I guess for in sense of the taxi-cab because you can kind of trace the routes can't really do that here (pointing to the drawing)
495. Anoop Very confusing here (the drawing)
496. Robert Yeah, I guess, you know...one of the benefits is that you can trace how you get to this number.
497. Anoop Uh, but, what have you drawn here?
498. Robert Uh...The individual rows, these...
499. Anoop Oh, the individual rows? What are these rows?

500. Robert Uh I guess this is a plus b plus c to the one, a plus b plus c squared
 501. Anoop So they are really the cross-sections of your model which is very different, which is very different than what you just used for the taxi-cab...you didn't use the cross sections
502. Robert No, I used the faces
 503. Anoop You used the faces, which is, uh...
 504. Robert Which is part of the cross section...so
 505. Anoop So, this is not the triangle, this is Pascal's Triangle here...
 506. **CHANGING TAPES**
 507. Anoop Ok, because we changed tapes, I am going to repeat what we were talking about...you were just saying to Victor or to me that, how you could use this sheet for the taxi cab problem and you just said that these are actually cross-sections of the pyramid but that's not what you used to explain the taxi cab problem, you used something else...I mean...could you have used these to talk about the taxi cab problem?
508. Robert Yeah, as long as you know that each face side of the triangle represents you know Pascal's triangle, then you could...but you would have to know the full number of spaces away it is ... I guess you know, if you know, that like this one was three down one to the right so you'd have to know where uh...something to the third and then something to the one is on this triangle so that would either be this one or this one depending ...depending on what your going to call down and which one you call right so, if you knew but I think uh..since this is two dimensional, it would be easier to work with.. the right way, yeah... I guess you still need to know that though, so, I don't know, you know there's total three plus one is four so you immediately know to go to this one and this is the same thing and you focus on that, the correct row so, I don't think there's a difference
509. Anoop It can be used is what you are saying, right?
 510. Robert Yeah, if you know about it, yeah.
 511. **VICTOR LEAVES**
512. 2:13 Robert I guess once you know the formula, you don't really need to draw anyway
513. Anoop Once you know the formula? For what?
 514. Robert For the pascal's... uh, what you call it, the multinomial formula, theorem, something like that
515. Anoop OK, what you are saying is if you knew the expansion of the polynomial like a plus b to the fourth
 516. Robert Or like four choose three that's what this is so once you know that you're like (inaudible)...you say you pick up the green dot, ten choose five, five.
517. Anoop What is the ten choose five, what do you mean by ten choose five?
 518. Robert Cause you are five to the south and five to the east, I think, one, two, three, four, five, one, two, three, four, five so the total number

- of moves is ten, so ten choose 5 south, or 5 east whatever the solution is, it is 10 times 9 times 8 times 7 times 6 over 5 factorial Is that right? ...I don't know what that number is ...
519. Anoop OK
520. 3:18 Marjory It is interesting that you said that because I remember from Arthur Powell's dissertation which was an examination of students that from Kenilworth that worked on the taxi cab problem that in their solution they had...they came up with this combinatorial notation you know where Robert was talking about was choose this choose that kind of thing and I...I don't know that Robert...so he like brought that up but that was like the first time we talked about any of that today you know what I mean so I wonder how did you so quickly come with you come up with this...you know what I mean
521. Robert I talked about this at last interview I think and... I took math here so I just...uh...I took math here...yeah I took Math here so I took a course where it came up
522. Anoop He has talked about that before but what is still throwing me off is 10 choose 5 for the green dot, 10 choose
523. Robert 10 choose 5, 5
524. Anoop What do you mean by 10 choose 5?
525. Robert You are choosing 5 south and 5 east
526. Anoop Why does it have to be 5 and 5 and why isn't it 10 choose something else?
527. Robert Because then you will be somewhere else on the grid
528. Anoop So, the 5 is coming from the position of the green
529. Robert Yes
530. Anoop The green is 5 south and 5 east
531. Robert From the taxi stand
532. Anoop From the taxi stand...so if I am gonna guess for the red and I am not so good at the problem
533. Robert 4, 3 or seven, four, three
534. Anoop (Pause) Oh would it be 7 choose 4?
535. Robert Choose 4 comma, Yeah, same thing cause you are only choosing...since it is two-dimensions, 7 choose 4 or 10 choose 5 or 4 choose 3 or it could be 7 choose 3, 10 you know 7, 4 and 7, 3 are equivalent
536. 5:17 Anoop They are equivalent? They are they give you the same number
537. Robert Would it be 21
538. Anoop That's very interesting. Yes! I see it now. Well, like for myself I see it now. I wasn't sure why you are pulling in those combinations. Um...Here's a weird question for you. How do you think this Pascal's Pyramid can be used for the teachers who are working on the strand of counting problems?
539. Robert Ankur's Challenge
540. Anoop Ankur's Challenge
541. Robert But may be we remove the condition that there must be one of

- each color. You wanna show the whole row. So just 4 tall, choose from 3 colors and that's this entire bottom
542. Anoop Aah..
543. Anoop Would you recommend those teachers to actually make this model?
544. 6:12 Robert Um, it will be worth, I think...Maybe...one thing I find useful with messing up is that you can actually go in here and touch this...touch the center ones and you can't...I guess you could do it with a stick or something there...so it's hard to see...but...if there are some tools out there where you know you can make this whole thing fit in there and if they're like what is this, you can point to them you know and here you are like uhh and stuff...I guess you could turn it upside down but then you have to you couldn't focus on the third row...you'd have to remove the whole bottom row to talk about this six
545. Anoop I see what you are saying, I can't touch this ball in there
546. Robert Yeah and you are holding up in front of the whole class, you are like what?... and I just see a bunch of sticks and I don't see a center thing, you know, so
547. Lou 3-dimensional LCD
548. Marjory Is there anything that's an advantage of this model compared to yours?
549. Robert Yeah, each stick has different meaning like the blue stick represents addition, red represents multiplication whereas in this one blue stick has dual meanings depending on which direction it's facing when it's facing down it is multiplication and when it's facing left or right, it's addition, so that can be just confusing plus I didn't put a...I didn't connect this to this to this so I have to do that too, and connect these...(inaudible)
550. Anoop And suppose you have to go to, uh, Intro to Math education class and you were going to teach them everything about Pascal's triangle and Pyramid
551. Robert That almost happened the one day...
552. Anoop Really? You did? Really?
553. Robert But Carolyn is coming back and she is like teach them the Pascal's triangle and the taxi-cab
554. Anoop So, you have all the tools and you have internet and you have everything, the towers and everything, where would you start?
555. Robert Uhh...I guess we'd have to...if you wanted to do it slowly...cause so many of the problems Carolyn gives in intro math ed revolve around Pascal's triangle, that you probably have to do all the problems first and then ask them if they see any connection and then build it in that way because if you start doing the towers problem, they immediately will be like hey here's this thing Pascal's triangle, either you're going to get a lot of people trying to use it on problems that it doesn't work for or they are going to

- do right away on the pizza problem and get the solution or they are going to do it right away on the taxi cab problem get the solution so... if they recognize that is it's that but..
556. Anoop That's what I am thinking even if you tell them, I am not sure they always recognize it for a new problem
557. Robert So, I guess if you are taking like an intro to math Ed course and you want to teach it like Carolyn teaches it, you probably have to teach it at the end otherwise some of the problems are going to lose their meaning
558. Anoop Keep it a secret
559. Robert Yeah, I think with this class a lot of students already know it so it's kinda and they tend to fly to the problems
560. 9:33 Anoop They do? We didn't, I remember we didn't
561. Robert Yeah, this class, they kinda
562. Lou But I still think, not that I... I still think that they can use the choose notation and they can do the formula but if you ask them to actually tell you what each of those things represents, it is not a trivial task
563. Anoop No (agreeing)
564. Lou When you ask to explain what each piece of the model means compared back to the problem it's not as simple as... maybe some people have an easy time with it, but I think that if that's the expectation, then they really have to think outside of their procedural knowledge
565. Anoop Yes. What would you want to accomplish if you were going to teach this to the class of intro to Math Ed? What would be your goal behind it?
566. Robert Hm...
567. Anoop Why? Why teach this?
568. 10:40 Robert Because it's like the Binomial distribution, like it comes up a lot. A lot of things where you only have two choices, or three choices, you know, so... I don't know how it is useful... if you want to teach students easy ways to, you know... you know write up you know a plus b cubed but other than now a days this kind of stuff isn't useful...
569. Anoop Isn't?
570. Robert No, but I think maybe not now, but maybe in couple of years
571. Anoop Technology might be useful
572. Robert Technology now be useful in identifying what to use and then you just put it in the computer... I don't think you will... I think problem recognition is gonna be a bigger thing that actually doing the problem... cause I can just feed in this is 7 choose 4, maybe I don't need to know what that value means
573. 11:37 Anoop You'll be able to get the answer
574. Robert If that is the goal, you got to get the answer
575. Anoop Yeah, but would that be my goal when I am teaching an intro to

- Math Ed class? What would be my goal for teaching these pre-service and in-service teachers all of this stuff about Pascals' and giving them...I am kind of putting you in Carolyn's shoes, am I not? Why does she do this? Has it helped you...sorry
576. Robert Must come up a lot...you know so I don't know like what...especially when I went to school we didn't have a traditional...you know route...like teachers were different...like in high school I had a different type of class like I had Mr. Pantosie so I don't know how...
577. Anoop How this works out in a traditional classroom?
578. Robert Yeah, but then again she has a lot of middle school teachers, and I don't remember it coming up then but I am sure it did...but, I am sure there is some use to (inaudible)
579. Anoop You just said to me that you actually did this first (the hand sketch) and then build the model. Could you have built the model without this?
580. Robert Oh yeah
581. Anoop You could have? And if you had never played with this, and somebody said go build a 3-D Pascal's, would that even be a possible task
582. Robert Yeah, I think that's what Liz did
583. Anoop Liz did that.
584. Robert I don't remember but I think it is... she just came here and noticed things and we just went from there...and, I think that if you have understanding of Pascal's triangle, you could do it, but I don't think you can go from scratch to it
585. Anoop But tell me this, you had this long interview with Todd and Liz and it was about working on the board and imagine that same interview with these toys if they gave you these toys and you were talking about the pyramid with them at that point with these tools, would it have been a different interview? I mean, what would have been different?
586. Robert I remember that interview with Todd and one of the problems was you getting confused cause you are trying to draw and keep track of everything and you know, it doesn't really, you are trying to draw 3-D on the board and plus I have very sloppy handwriting so it's a little confusing whereas with this you know (the tools) you can build it in 3-D
587. Anoop And this stands
588. 14:08 Robert I guess the only difference would be, you know the first time I build it, I wouldn't know...maybe I would/wouldn't know to stack it like this where each cross section is a pascal's, you know as a different triangle, I might you know start building it huge like outwards like disconnected, you know like coming out, you don't know... but hopefully eventually I would end up with something like this

589. Anoop You were doing very interesting things there, remember you were trying to build these cross-sections and at one point you believed that these cross sections can be co-centric? Or whatever the word is. That if I looked at the fourth level, it could contain within it the third and the second, and the first all these pictures could be squished and you could make one picture that had the zeroth power and then the first power around it and then the square around it
590. Robert You could probably do it but I don't know why you would want it.
591. Anoop (Laughing)...Ok, I just wanted to bring that up cause it was so interesting what you were talking about
592. 15:14 Robert I think you could... Yeah because this triangle fits inside this triangle, this triangle fits inside this triangle, so, maybe you can like just push it down you know, but it would have to be bigger you'd have to draw it so huge that
593. Anoop And if it's pushed it would be spaced...
594. Robert I guess you start with this one and you work expanding you know...but then the problem will come here with the center thing because this is gonna squish on to the center so you have to have like different balls I guess but, yeah
595. Anoop Nah, I just wanted to stretch the idea, that's all. Wow, interesting! So, I mean in general right, when you are solving problems, where do you start, how do you start a problem that you have not seen before or even a problem that you have seen before, how do you start?
596. Robert What is it asking and what do I know that I can apply to this?
597. Anoop Ok, and, how do you become convinced of your solution that this is the answer?
598. Robert Happens, some times still not convinced but I just...
599. Anoop See, you were very confident about this taxi cab that it could be solved by Pascal's and you proved it and you when talking to me just a little while ago, at what point did you say, Oh yeah, I am sure that this is the answer. Like, what tells you to be sure?
600. Robert I have seen this problem so many times that, yeah, Carolyn's intro class, you know, I have been there for like 3 years, every class, and so they do this problem every time and you don't get many unique solutions plus I was getting video for someone of this yesterday
601. Anoop So, you had some prior knowledge of this to kind of compare it to
602. 17:17 Robert Yup, but I don't think I have really sat down and done the problem I have just seen it done so many times that I just know what the answer is
603. Anoop So, you just did it now for the first time, the taxi cab
604. Robert I think there might have been one time I did with Brian maybe, but yeah but I think this was probably the first....I wasn't in the group that did it in 2000

605. Anoop But Robert, as a math student, and you have been doing a lot of Math and a lot of Stat, playing with all of these games years ago, how has that shaped you as a Math student?
606. Uhh...well, when I took combinatorics here, it was real easy cause I mean like a lot of problems...you know, not... they didn't all breakdown to this but the whole idea of breaking down the problem and finding the useful information, and you know stuff like that and tricks like you know the problem we did the world series problem, you know the trick that we always did or I did, (inaudible) was that we fixed the last game, let's say how many times does team A win in five games, you fix team A to win game 5 and then you use, you know, combinatorics for the rest
607. Anoop Right
608. Robert That came up a lot, in combinatorics class I took, so, I mean a lot of people didn't get that idea that you can just fix one thing and then randomize the rest, look at all the possible solutions for the rest, so, that helped. But, I think mostly, you know, problem solving.
609. Anoop It helped you be a better problem solver
610. Robert Yup!
611. Anoop So you would recommend this activity, the whole bunch of activities for kids in middle school and elementary schools, it's a good thing to do versus...
612. 19:05 Robert That's fine cause I always...I don't know if ...but I don't think ALL math can be taught like this...I think...I think... there should be like a class specifically with problem solving that this would be a part of and then you can do problem solving in math and you do problem solving in science and hopefully it would run ...concurrently with, is that the word I don't know ...like if you are doing...if you're in fifth grade, you're doing some thing in algebra and so, I don't know what grade algebra is, you're in seventh you are doing algebra and problem solving and this entirely different class, you would work on problems like this that would reinforce the drills and other things that you learned in a regular math class so kindda like...anyway...but the whole class wouldn't be just math, it would be science and you know, there'd be problem solving and reading and you know critical... yeah, stuff like that so, I think, ... kindda hard in many ways because this can take days and you don't have days... and there is a problem we run out of time all the time, and plus if a kid misses the class, it would you know, be harder for him to make it up then...
613. Anoop Cause he wasn't there for the activity, the toys are not there...
614. Robert Yeah, then you come after school, work on the activity whereas when you are teaching straight from the book, you know you can learn from the book, so

615. Anoop Right
616. Robert I think this should in compliment to you know regular
617. Anoop The drill and practice
618. Robert Yeah
619. Anoop It needs to be there but this could be added on to as a problem solving
620. Robert Yup
621. 20:52 Anoop Yeah...I just want to say that listening to you and watching you do math, is very ahh ... impressive to me because I come from all drill and practice background and I can always find my answers and I know they are my answers because I am doing math, but you go beyond that and you say why, and you are able to put it in words and context of the problem, you know and I was never trained to do that so, in that sense, I really think somewhere all of these experiments and all the time you spent doing these fun things, got you to a point where you communicate your understanding of math better than a person who would just come from drill and practice environment, that's just...
622. Robert I guess that's true in intro to math Ed, when they do a problem, a lot of students just write this choose notation and
623. Anoop Then it's done, right?
624. Robert Yeah, I have seen a lot of that this semester actually, I was surprised
625. 21:43 Anoop That's what Lou was saying too. That they have the number but what does that really mean? How you convince some body who doesn't understand your notation or number?
626. Robert And if you change the problem slightly, they get stuck, because they didn't really do the work so then they just have a page like of you know just random guess and check basically instead of actually trying to do the problem, they are actually just guessing and checking
627. Anoop Thank you Robert
628. Anoop leaves to feed the meter
629. Marjory Well, because these connecting cubes have the different shapes, they are triangles, rectangles, and pentagons and the colored sticks have different shapes on their end so you're limited on what you can connect, and, so is there a way to build a model with these tools that still lets you build something that has you know three colors in choosing, and can you , I don't know if it would ...but see it wouldn't end up being a pyramid but then there's something about the triangularity of Pascal's triangle that's important when you are talking about choosing from two variables and similarly when there is a pyramid and you are choosing from three variables, you know, say a, b and c or say this that and the other you know that you know gives you this (2 color model)
630. 23:23 Robert I'm gonna go with Kevin Merges is very thorough and spends a lot

- of time on this, and if there was a way, he would have figured it out
631. Marjory Right
632. Robert Yes, I'll go with that proof
633. Lou Or, if you wanted to use these same rods to try and do it another way, I think you just have to get like some sort of rubber balls that you could poke in from whatever angle
634. Marjory Styrofoam balls, but you are not allowed to them anymore right, they are environmentally unfriendly in ...
635. Lou Or what you call it, silly putty
636. Robert Or that or dump a bunch of the blues in paint and you know, take them out and change the color
637. Marjory Oh, you know what, I realized as I am pulling on this, not only that, but the yellow rods are not the same shape as the red and the blues
638. Robert Yeah, I think it's a little shorter too
639. Marjory So, these are meant for something else, but I am not sure what
640. Lou And, the interesting part is that I thought you should be able to make a cube. Cause, from what I have seen of the hypercube that it is a cube with basically a flat-top pyramid on each side, so...not a pyramid, but like a four-sided pyramid, what's that called?
641. Marjory Tetrahedron
642. Lou Tetrahedron. They have tetrahedrons on each edge, on each face
643. Marjory Technically, tetrahedron is a pyramid, umm...because face one, two, three and the bottom is the fourth...but what you are talking about a solid that has four faces, not three
644. Lou It's basically a 3-dimensional trapezoid. What I am talking about. Something like this would be interesting to try and make a hypercube
645. Anoop enters the room
646. Anoop (Anoop's start to talk about Hannah's ball that was built by Kevin Merges' daughter) They are very flexible tools from what I understand
647. Lou But I don't think you can make a cube.
648. Anoop You can not make a cube with these
649. Anoop We would have to interview Lou about that or did we already answer that?
650. Marjory I think, we might have already answered that, but some of that...
651. Robert We also concluded that we can not use three different colors for the different shapes
652. Marjory Not only that, the yellow connecting pieces, there is something structurally different about yellow pieces, because they
653. Marjory concludes that yellow pieces have a twist unique to the color. Anoop concludes that you can't get pieces of the same color to make right angles using the connecting spheres. Interview Concludes.

Appendix D

Date of Session: 03-27-2009

Author: Anoop Ahluwalia and Mathew J. Cann

Verified by : Anoop Ahluwalia and Mathew J. Cann

Date of transcription: August 2009

Line	Time	Speaker	Transcript
1.	0:00:00	Marjory	Yeah and at some point Bob Search is going to emerge from that room but there is not much we can do about that.
2.		Carolyn	We might want to stop if this doesn't...
3.		Marjory	Well, yeah, I'm saying when the door opens. I'm just... a heads up that at some point it will open and people will come out. Ok?
4.		Anoop	Ok.
5.		Marjory	We're ready.
6.		Anoop	Here we go. So Robert, Marjory has never built this Pascal's Pyramid and she is going to try and understand how you build it, so what we want you to do now is make it for us, while the camera is rolling, talk loud as to what you're doing, and see if Marjory can follow, how you build it, what it's purpose is.
7.		Robert	Ok, so let's (inaudible)...
8.		Anoop	We want you to start from scratch. We can go to a different unit later.
9.	0:00:50	Robert	Um, I'll pick the medium blocks because we don't have any small blocks, so... I start by building by building one of the middle rows first. I don't know why. Cause it's easier to build down to up, so, ummm... I have each one of these represent a number on the pyramid so, all the balls are numbers and these are just connecting. You know the numbers to each other.
10.		Marjory	Ok, so, maybe as they start to come together that will make more sense. Otherwise I will ask you.
11.		Robert	Yup.
12.		Anoop	Well I'll ask you, how you just went and grabbed two colors?
13.	0:01:37	Robert	Oh yeah cause I already built it before and I know. Well, I want one color to represent something else, these are, they're just connecting, umm the numbers you show but these are going to represent something later on, so I guess when we get there I will explain what...
14.		Anoop	Ok.
15.		Robert	So this is actually the one, two, one row, so it would just be one, two, one and one, two, one. So, is that... squared? Yeah.
16.		Marjory	Ok, so, cause, um, cause I know Pascal's triangle starts with a one at the top of the triangle, right? And then it, cause I have seen the triangle. I am familiar with the triangle, as numerical,

- not as a model though.
17. 0:02:33 Carolyn So what squared? You said something was squared.
 18. Robert Uh, it's the A plus B squared, I guess.
 19. Anoop A plus B squared?
 20. Robert Yup.
 21. Carolyn Talk more.
 22. Robert So this is A squared, one A squared, two A B, one B squared, one B squared, two A B, one A squared.
 23. Anoop A squared.
 24. Robert Yeah this would be B squared. This would be B squared and then this doesn't make any sense. Ummm...
 25. Marjory Do you have to make more of it before you can talk about it? As, cause, I think you are talking about stuff that is not yet built. Cause I don't see it. Ok.
 26. 0:03:29 Robert So I want to connect the previous row to this row.
 27. Anoop Ok. You're building it bottom, up because, it's just more convenient or is there...
 28. Robert I think it's just easier to use the blocks that way cause if you try to build down... Well I guess you got to build it upside down anyway. I don't want to be like that and if you try to build it upside down then you have more on the top and it gets, you know hard to put the blocks in because it's heavier on the top than the bottom.
 29. Marjory I wonder if you start at the top if you'd... I don't know you'd have to maybe know what you were doing already. I don't know, it's like a visual/spatial thing.
 30. Anoop I always start at the top cause to me it is induction starting with the smallest and then building... To me... You always do this you start bottom up. You go backwards. You are going from a big layer to a smaller layer.
 31. 0:04:36 Robert Yup.
 32. Anoop Something...
 33. Robert Well if I have just this, I mean I don't know where to stick everything. Some there and then there and then and some... I did before
 34. Anoop Symmetry doesn't...
 35. Robert Yeah doesn't get, doesn't... I make a lot of mistakes when I go backwards or when I got the right way, for some reason.
 36. Anoop Yeah ok.
 37. Robert I think cause there is too many holes and if you don't put them in the right one then it gets...
 38. Anoop Too many holes on these spheres.
 39. Robert Yes.
 40. Anoop And if you start from the top, you don't know which hole would be symmetric to which? Is that what you're saying?
 41. Robert I guess cause they're all the same pattern, right? It's pentagon,

- rectangle, and triangle, so I don't know, why it's so hard going the other way. But anyway, so this is done. So the red, um, so this is just the first row, second row, third row. So, the red is going to represent this, um, going down to here and then going down to here, like putting an A on top and putting a B on top. Putting an A on top, putting a B on top, etcetera.
42. 0:05:53 Marjory Is that, is this like the sphere at the top, is that like the one at the very top of Pascal's triangle or is it something else?
43. Robert Well if you look at it like towers, this would be a tower with blue on top and then you are adding a blue, and then you are adding a red. And then if you look at this way you are adding a blue, you're adding a red. You're adding blue, red. So...
44. Anoop So, you just said blue and red, blue and red...
45. 0:06:24 Robert Maybe I should pick different colors.
46. Anoop No, it's fine. Blue and red is fine but you think this would represent two color towers.
47. Robert No that's wrong. Yeah. Ah...
48. Anoop How is this different from Pascal's Triangle?
49. Robert Is it?
50. Anoop Is it different?
51. Robert No, it's just three of them.
52. Anoop It's just three of them.
53. Robert On the faces.
54. Anoop The faces...
55. Robert Are all Pascal's Triangles.
56. Anoop Are all Pascal's Triangles. Ok. So why even bother having three faces, if they are just Pascal's Triangle? Why even concentrate on a three d model?
57. Robert Cause you want to know what the third power would be. If it is A plus B plus C. Yeah A plus...
58. Anoop Do you want to write some of this that you are saying down? Cause all of the sudden I heard you say...
59. Robert Whoops, yeah. It's three variables not two. My bad, that's why it didn't make sense.
60. 0:07:34 Anoop Ok. Three variables not two.
61. Robert Yup
62. Anoop So...
63. Carolyn Excuse me Anoop, can I make a suggestion?
64. Anoop Yes.
65. Carolyn It might be helpful if Marjory had some unifix cubes available in three colors, so she can map her understanding to what Robert is explaining.
66. Anoop Good idea.
67. Marjory Ok can you just hit pause for a minute and I'll...
68. 0:07:52 **Tape is paused while Marjory gets unifix cubes.**
69. Anoop Ok.

70. Marjory Ok, so...
71. Anoop So, why do you have three colors blocks Marjory?
72. Marjory Ok, cause, ah, a minute ago Robert was saying something about variables and, um, I've spent a lot of time working with unfix cubes and building towers, and Robert has too, and so, um, when we talked about Pascal's Triangle and like on the qualify exam had to think about, um, towers, you know when there were two different colors available. So, let's say we start by having red and, excuse me, blue and yellow available. Then, um, you could think about the notation... I'm going to write down a couple of rows of Pascal's Triangle. So, there is a one at the top and then the next row is a one and a one and I don't really have a good explanation for the one at the top. I'm going to differ. I am going to come back to that. But this one and this one could be like one yellow cube and one blue cube, right? And then when you get to the next row of Pascal's Triangle and you get to that by adding, you got the one of the outside and then you are adding this one and that one together to get a two in the middle and then another one on the other side. And so, if that's this row here and then what you get in... for the second row, I'm going to call the top the zero row, right? But in the second row you are going to have, um, a tower that's two yellow cubes tall and then you would have in the middle, there are two different kinds of towers, I guess, you could make that have one yellow and one blue cube and they look like that, right? And then there is tower that gets two blues cubes. And, um, so even just that far, right if you were to just talk about yellow and blue cubes, if we called yellow A and, um, the letter, variable A and blue was a B. Then this second row would look like A squared plus, well it is really plus A B plus B A, right? Then plus B squared but we like to combine the terms that are the same, so you know, you would do A squared plus two A B plus B squared. And I don't know when I looked at video of students and stuff, and then they deal with, well, you know, if you have like terms your towers... you've basically done that to your towers but that's not really the towers solution. You have to keep them like that.
73. 0:10:56 Carolyn But Marjory can I go back and ask you a question?
74. Marjory Ok.
75. Carolyn So you have that, what you call the first row with one yellow and one blue?
76. Marjory Right.
77. Carolyn Can you show how the second row gets built with the cubes to get the...
78. Marjory Oh, yes. To do that I am going to reverse this order and placing them down. Because on that yellow cube, I guess you

- could talk about adding in the triangle because you're adding the next level. You're either adding a yellow cube on top and you get this one or if you add a blue cube on top of the one before, you get to that one. Um, but you're adding cubes but you're multiplying terms.
79. 0:11:47 Anoop Adding cubes and multiplying terms.
 80. Carolyn Well that will show... Continue.
 81. Marjory And then on this side, um, you are doing the same thing but in reverse, so on top of the blue... I guess you're adding a cube that is growing in height, you're really multiplying, like one blue cube times one yellow cube and they just get put together going vertical cause they are unifix cubes. That's the only way that you could put them together. (laughing) Um, and then otherwise, and that's this term and otherwise on a blue cube you can add one more blue cube and so, it adds and then it grows taller but you're really, like multiplying this times that and coming down to get the next row. Like you, to go back to letters, like if that row with one and one, there is the quantity $A + B$, right? One yellow cube and one blue cube and you're multiplying it times itself. And then that is how you get like from this is where you first go to that, you know, this term $A^2 + AB + BA + B^2$ cause you mult...
 82. 0:13:08 Anoop Marj, I can... yeah I understand that you see going from this layer to the next as an addition of the block or the multiplication by the variable, A or B . Now I wonder if you can actually put this in a real triangle and connect them with lines. Would you be... like how you wrote them and they're just numbers, can you draw lines who's coming from where? Like, how is it connected? Like talk to me that this gets added down there but Robert is making something that has lots of lines and connections.
 83. Carolyn Well that's a triangle.
 84. Anoop So, can you make those connections on these numbers you have put on the paper, for us?
 85. Carolyn Can you do that too, Robert? To help her, on her paper.
 86. 0:13:53 Marjory Um, I gotta move these away, so I can see what I am talking about here. So, like are you asking me how does this look like the top of that?
 87. Anoop Yeah like how would I connect these numbers with straight line segments?
 88. Marjory Well I am going to use a dotted line. So, it comes down like, this is the sphere at the very top, one that we don't really know what the call.
 89. A Ok, we will ask Robert if he knows what to call it. And what it means and why it's a one. So why don't you try to make those lines and Robert, Marjory didn't know why this top thing is a

- one and she didn't know how to use the cubes to represent the very first one.
90. Robert Well, here. Nothing, there you go, that is how you represent it.
91. Carolyn Well aren't they all A plus B to some power.
92. Marjory Exactly.
93. Robert It's a zero.
94. Marjory Yeah.
95. Robert Yeah otherwise math fails if two to the zero doesn't equal one.
96. 0:14:52 Carolyn Stephanie thought that was cheating when Bob Spizer interviewed her. She thought that was really cheating, she was in the eighth grade, that you're just doing it that way to make it all work and that's really cheating.
97. Marjory Well...
98. Robert Well if you do that...
99. Marjory Somewhere...
100. Lou Another way...
101. Carolyn Finish your sentence Robert.
102. Robert I was saying if you don't have those laws then four choose four wouldn't exist under zero factorial which would just be zero and then you divide by zero, so you would have to have the laws otherwise...
103. Lou One of the students in Kathryn's class had an interesting enough, that the one means that there is one tower that is of nothing.
104. Anoop One tower of nothing.
105. Lou Which I thought was an interesting argument.
106. Carolyn What did you say, Robert?
107. Robert Wouldn't there be infinite towers of nothing cause, I don't know...
108. Marjory Ok so...
109. Carolyn Indeterminate form.
110. Robert Yeah, I can't, I don't think...
111. Marjory Ok, so the top, if we're going to call that A plus B to the zero power and that's equal to one and then I don't know I was probably taught around somewhere in the eighth grade that anything to the zero power was one and it's hard to believe it.
112. Carolyn Except zero.
113. Anoop Except zero, that is not defined.
114. Robert What about e to the zero? Is that zero?
115. 0:16:00 Anoop It's also one.
116. Marjory Well zero to any power would be zero, so I guess no one really dealt with that.
117. Carolyn You can't have zero to the zero, that's indeterminate.
118. Marjory Oh, ok. Um, and this second row is A plus B to the first power, which is just equal to A plus B. And then, this is the quantity A plus B squared and that's all this stuff I drew already down

- there.
119. Carolyn And showed us how to make.
120. Marjory And then, if this is like a sphere...
121. Anoop Right.
122. Marjory Right and this. Here maybe I will do a solid line for the blue ones, so that's a solid line in that tower and the dashed is a red in that tow, in that model.
123. Anoop Ok.
124. Marjory So that's what you're... but I can only, this triangle I think is only this face of that.
125. Anoop Is that right, Robert?
126. Marjory Is that right?
127. R Yup.
128. 0:17:03 Anoop Ok.
129. Marjory Alright but now you have to...
130. Anoop We want you to finish your triangles, so that we...
131. Marjory And then, well you're coming down the face, and then, so this is a red bar, or a red rod there, and this is also... I guess anything that is a number is a sphere.
132. Carolyn Sphere?
133. Marjory The connecting thing.
134. Robert Dot. Yeah. Circle.
135. Carolyn Oh, ok. Sorry didn't mean to, didn't know what you were talking about.
136. Anoop Hmm, ok, well. She got this side of what you built.
137. Marjory At least the beginning of it.
138. Carolyn And how far can you go from this picture?
139. Marjory Oh wait and from that picture, then there is going to be another...
140. Carolyn So you are going to have this, you have this, and you just did this.
141. Robert Now add a row to what you're tracing.
142. 0:17:48 Marjory And that was that and this is going to be a three...
143. Anoop Ok while you do that I am going to ask Robert a question. Robert this is, like, peculiar to me. She is making dotted lines and solid lines, you used two colors. Are these bars really different? Are they saying different things?
144. Robert Yes.
145. Anoop They are saying different. What are those different things? What is the red saying versus the blue?
146. Robert Um, red is saying go down to the next, like add one to it. Like, ah, they are both saying addition but different types of addition.
147. Anoop Oh, I want you to explain that to Marjory, once her picture is complete.
148. Carolyn And to me too.

149. 0:18:48 Marjory Ok, I think completed the picture
 150. Anoop You got the one... one, three, three, one.
 151. Marjory Right.
 152. Anoop Marjory, he just said the dotted and the solid are both additions but maybe different types of additions. What do you mean, Robert?
 153. Robert Um, like, for this one you're adding you know another of the same type of block in there but the addition for the blue is adding the entire term together. I don't know how to explain that. Like that's adding another A plus B, all the reds together are adding another A plus B or multiplying to an A plus B. Whereas... I guess no red is multiplication and blue is addition.
 154. Anoop Oh you just changed your answer, ok. Red is multiplication and blue is addition.
 155. Robert Yes.
 156. Anoop What does that mean? I don't think Marjory got that.
 157. 0:19:38 Carolyn I don't have it either so...
 158. Marjory No.
 159. Anoop Can you show them like on paper? Like convince them?
 160. Carolyn Or make cubes.
 161. Anoop Or make cubes.
 162. Carolyn I like cubes.
 163. Robert I guess starting from the... if you want to look at it like this... this is A B and this is just you know, one... Um, multiplying by A here to get one A squared. And multiplying by B here to get, there's a one, two A B. Well you're really getting one A B but you are multiplying by here and getting another one A B, times A. And this is times B to get one B squared. Yeah, so...
 164. Anoop Ok, Marj can see it now.
 165. Carolyn Ok, can you explain it?
 166. 0:20:37 Robert I mean she just has the numbers one but...
 167. Carolyn Ok.
 168. Robert The one is in reference to the variable, A in this case.
 169. Carolyn Right
 170. Robert So there is one A, one B...
 171. Carolyn Ok.
 172. Robert You multiplying the A, one A by A and you're getting one A squared which is right here. Multiplying A by B to give you one A B, multiplying the B by A to give you one A B which, you know, goes down to two A B. Well actually this would be one B A, you know, same thing. And then, multiplying by B here to get one B squared which is the one over there, so you get one, two, one.
 173. Carolyn So, so the multiplying, um, so the A plus B quantity squared distributed is like multiplying, distributing the A or

- distributing the B, that's what you're saying.
174. Robert Yup.
175. 0:21:23 Carolyn And what Marjory showed us is that is equivalent to making the towers taller. Um, and so she used the term, I'm adding a block on the tower to make it taller. So that's why the language can get in the way because you, the word adding or building up sounds like an addition but you don't really mean that, you really means the multiplication.
176. Marjory Yeah it does.
177. Carolyn Do you see that Marjory? Does that make sense?
178. Marjory Yeah I think it does but you know, the word adding also, like, maps into when people talk about the addition rule of Pascal's Triangle, so that, to me, makes it really hard to pick this apart and explain what you are saying because...
179. Carolyn Well that's hidden isn't it?
180. Marjory It is. It's hidden. It's hidden.
181. Carolyn Is that what you think Robert?
182. Marjory So you know as you are starting there with the, not the one at the very top of the triangle but you're starting here with, you know, one yellow cube and one blue cube. And this is A and this is B and you're, the coefficient, this is one A and one B, to make it very explicit there. And that when you multiply that quantity times itself you're getting one A squared plus...
183. 0:22:47 Carolyn That's A times A
184. Marjory Plus one, one A, A B plus one B A plus one B squared. And when you combine this, you know, with the um, associative, commutative...
- Carolyn Just adding.
- Marjory The property. Exactly. You're combining it, the like terms and you're going to get two A B and that gives you these coefficients, one, two, one and the next row of the triangle.
- Carolyn Could you have gotten two B A?
- Marjory Yeah sure, it would be the same thing. So is that what you mean by you add this way?
- Robert Um, blue is add so...
- Carolyn So she just added to get two and the red is what you are showing by the multiplication, which is kind of...
- Robert Um, no I think what she was saying, you were saying adding the coefficients to get... No, I think the blue represents adding the equation to get the solution to A plus B squared. Like, uh, factor out A plus B squared equals A squared blue block...
- 0:23:47 Carolyn That's what I meant.
- Robert Oh sorry. Plus you know two, or no it's not the blue block, A B.
- Anoop The terms.
- Robert Blue block

- Carolyn You're combining terms.
 Marjory So, what you're saying is you're adding terms...
 Carolyn So, that's what you're saying. I didn't get that.
 Marjory So, if I'm hearing this right then you're saying Robert the blue bars like these are used to add the terms of the, um, ah, expanded binomial. Right?
- Anoop Yes.
 Carolyn Very nice, Marjory. Isn't that interesting?
 Robert Actually that's interesting. I never thought of that and the coefficients, people just say add but if you really just add you get two A over here and not one A squared.
- 0:24:34 Carolyn That's right. I think that distinction is really nice. That's very... the first time I've ever heard that distinction so well expressed is by the team here.
- Robert It's great that the blocks are different colors and they add to like...
- Carolyn I think so.
 Anoop So would we agree that we understand Pascal's Triangle? We understand this dotted and solid line business but this is obviously more than just Pascal's Triangle.
- Carolyn We're just trying to understand one piece now. So, we just understand this piece pretty well, right?
185. Marjory Yeah.
 Carolyn So, Marjory can you show me how adding and multiplying, and Robert is going to monitor, how adding and multiplying get extended to the next row?
186. 0:25:30 Marjory Ok, so, in the next row you're going to multiply the term A plus B squared times A plus B. Right? Cause it's that, that term A plus B that it's getting, increased you know, those powers as it goes down. Right? So then... All right, so then, for every, but that means is then, for every term in the um the previous row it's... maybe the easiest thing to do is to make two of each of those terms. Right. So, there are going to be two of that term and...
187. Anoop Why two of each term?
 188. Marjory Because in each one I got to multiply by A and then times B, like, kind of like taking, ah I'm going to write it out this way. So, to get to the next row I've got to take the quantity A plus B and multiply it times the term before it which is A squared plus two A B plus B squared, right?
189. Anoop Ok.
 190. 0:26:31 Marjory Which it makes more sense if you think about it before, if you uncombined it cause it's really adding, I've got to multiply the quantity A plus B, so this plus this...
191. Anoop Uh Huh.
 192. Marjory And I've got to multiply it by each of these towers.

193. Anoop Ok.
194. Marjory And what I am going to get as a result is the tower is going to grow one cube taller.
195. Anoop Ok.
196. Marjory And I've got to have room to put a yellow block and a blue block on top of each of the terms, so I'm going to need two of them because this one is going to get yellow on top and this one is going to get blue on top. And I need to do that for the other three terms. So if this was the second term then it's going to get, first a yellow block and then a blue one.
197. Anoop So there was four terms and you are going to get two towers for each term
198. 0:27:28 Marjory For two. Yeah for each term and that's why you get this doubling when the towers grow taller.
199. Lou Where you adding to the top of the bottom before?
200. Marjory What do you mean? I'm adding to the top. So, except I put the term, I put the wrong term.
201. Lou I just, I didn't know that can stick you if you don't add to one or the other.
202. Marjory No I had it. Cause this, this was hiding. This was the term I was adding to. I had 'em right. Don't scare me that way.
203. Lou It just looked like these two were...
204. Anoop These two need to be switched around is what Lou is saying.
205. Carolyn To be consistent from the before.
206. Anoop Cause this is adding yellow and then blue and then yellow and then blue. Right?
207. Lou So the bottoms need to be blue.
208. Carolyn It's just the order of those two.
209. Marjory Sure, ok. Once you move them around you forget where they were.
210. Anoop Exactly so but you were doing yellow, blue. Ok, so. Alright.
211. 0:28:24 Marjory Maybe, actually remember that awful acronym was that foil first, outside, inside, last. Maybe I actually did my multiplication that way.
212. Carolyn So are you adding or multiplying that by putting those plastic cubes on top?
213. Robert Multiplying
214. Carolyn So, Robert says multiplying, do you think it's multiplying, Marjory?
215. Marjory Yeah because I am multiplying the quantity A plus B by A squared plus A B plus B squared. So, I'm taking two terms, you know, an A and a B and multiplying by each of these four terms.
216. Anoop Robert, does that sound right?
217. Robert I think it sounds great.

218. Carolyn She's doing great, isn't she?
219. Robert And then you're adding the doubles, like, whoops, the number then double...
220. Marjory Yes.
221. Robert Cause you're adding the one, one. It's just coincidence that it doubles cause there we are two of them but...
222. Marjory Well it's doubling right cause we have two colors or two variables.
223. Robert Yup.
224. Anoop Um, but I want to ask Marjory and Robert both I guess...
225. 0:29:30 Carolyn Yeah, where are we? On here, I need to...
226. Anoop Where are we on this triangle?
227. Marjory Ah and so I just built this row, I think cause that was the one at the top, this was the one yellow and one blue, this is the term A^2 or one A^2 , this is the term... There is actually two towers I guess that sit right there, right? Then that one is that over there.
228. Lou Maybe that's your fourth dimension.
229. Marjory Now let's see, now what am I getting, so this is one A^3 and it's represented on that sphere, and then this one is going to be, um, this term, the two, so it's going to be...
230. Carolyn Two?
231. 0:30:29 Marjory Well, um, the two in the no wait I have to have a three, hang on a second. Oh I needed one from the other side. Ok, so... Following down the triangle sort of makes you jump through the mass steps in the towers and you gotta keep expanding and simplifying your polynomial and so, umm... That's confusing. So, yeah...
232. Lou Combing like terms.
233. Marjory Yeah. So you got to combine the like terms and then, thank you Robert for helping that because then these are those and there is that one there. Ok, so, um, this is...
234. Anoop This is?
235. Marjory One A^3 plus three $A^2 B$ plus three $A B^2$ plus one B^3 .
236. Anoop Excellent.
237. Carolyn Beautiful. Yeah
238. Anoop Good job.
239. Robert (inaudible)
240. 0:31:26 Marjory No, well see the I don't... you have to talk about what's in the rest of this.
241. Carolyn I know, so far we have established the baseline for mapping what you understood from mapping it to a new, really from a new model and that's great. So, now what Robert? What do you have in mind for us now to help us understand? We only needed to have this in two dimensions and not three.

242. Robert Ah, yeah.
 243. Marjory Yeah.
 244. Carolyn So what does all this three dimension stuff?
 245. Marjory Or maybe it's what happens when there is a third color of cubes.
 246. Robert Yup. You get three colors, so instead of multiplying, you see here you're multiplying by two, you'd actually multiplying by three.
 247. Carolyn By two?
 248. Robert Well two different variables, I'm sorry.
 249. Carolyn Ok.
 250. Robert Ah, and in this instance you are going to multiply by three variables because you are going to add red, yellow or blue to the... So, you start with this.
 251. Carolyn So is that A, B, C?
 252. 0:32:28 Robert Yup. And then you would multiply them all by red.
 253. Carolyn So is that A plus B plus C?
 254. Robert This is A, one A plus one B plus one C. Maybe...
 255. Carolyn Don't skip steps with us. We're not very...
 256. Robert There you go.
 257. Carolyn We're slow. You have to go very...
 258. Marjory Oh, that's cool. So, now you're, now you're, hmmm...
 259. Anoop So, can you show quickly before you build more of this, how this A plus B plus C sits on this three d thing? Where's the A plus B plus C?
 260. Robert Ah, this row, A, B, and then the one furthest from me and on that row.
 261. Carolyn Show me.
 262. Robert Well, you know, it doesn't matter this could be A B or C, this could be B, A, C.
 263. Carolyn So, we're in a different plane now. We're not in this plane, we're now in the plan of this Triangle, is that right?
 264. Robert Yes. We're on the inside.
 265. Carolyn So in other words, if we put cardboard, you know, we're in here. There is one A, one B, one C.
 266. Robert Yeah we're inside the Pyramid. I guess.
 267. Carolyn It's this plane.
 268. Marjory So, like that might sit at that spot and, I have, this, I need to be an octopus, I need another hand.
 269. Robert And this guy would be over here.
 270. Marjory Yeah, and so, what had been here and here and now there is one there.
 271. 0:33:33 Robert Yeah there you go.
 272. Marjory Ok.
 273. Robert Whereas before this whole... We were just looking at this cross section and, you know, all this stuff behind it we didn't

- see. It was like covered up. Now we are looking at the whole thing.
274. Anoop Ok, we can take her to the next level maybe. How do you want to do that? Do you to go back to your unifix cubes like you had started?
275. Robert Yeah if you want.
276. Carolyn That helps me. I'm slow.
277. Robert We had, uh, three different unifix cubes, which is called A, B, C or R, Y, B whatever. Actually it should be B cause it's blue. And then we are going to multiply this thing by red, blue and yellow. Ah, see I don't want to say that. See if I maybe put it like this.
278. 0:34:34 Marjory Ok.
279. Robert So and then this gives you, red yellow, um, red blue, and red red. So, I don't have to, so on this, say this was the red one, we are multiplying it by yellow, so this would be red yellow. Multiplying it by blue, so this would be red blue. And back here we are multiplying it by red, to give you red red. Actually, it would probably be easier if this is red, so you're multiplying by red and this gives you red red.
280. Marjory Are you trying to say, you are going to put the red red here?
281. Robert Yup.
282. Marjory Ok.
283. Robert And then the red blue would go here.
284. Marjory I guess it depends on what you are calling...
285. Robert Yeah this is blue, red, yellow, blue, so red blue, blue blue, yellow blue, yellow yellow, yellow red, red red.
286. 0:35:45 Anoop Ok.
287. Carolyn Wait, wait.
288. Marjory How many terms are there going to be now.
289. Carolyn I'm not so sure Marjory...
290. Robert Six.
291. Carolyn Slow down, Marjory, tell us what you follow.
292. Marjory Cause you built these but then you talked about things that you hadn't yet built in the towers.
293. Carolyn Right. So can we...
294. Marjory So maybe we to, ah...
295. Robert So I have to build all of these.
296. Marjory Deal with these too.
297. Robert Maybe we should space them out more.
298. Carolyn We got, yeah.
299. Robert And, B...
300. Carolyn So, how many would you guess there would be? How many of these would you guess there would be? Before you build them, would you conjecture?
301. Robert Me?

302. Carolyn In the second dimesion. No Marjory.
 303. Robert Oh.
 304. 0:36:37 Marjory Well, there's a quantity of three things being multiplied by itself, I think. So, three things are going to grow by three, so nine.
 305. Carolyn So, if you look up here where you started your red, blue, yellow, do you think you could track nine other end points? Is that a reasonable question? To show where they would be on the next level. Hold it and look at it and trace it.
 306. Marjory See this is where it can get confusing because you have to keep track of things in your head.
 307. Carolyn Well let's help you.
 308. Marjory Because these things... Cause kind of what can happen when I went down one face of it, when it was only red and blue and there was some spots on here that there was going to be... so of these spots, I've been calling them spheres, so some of these white place here...
 309. 0:37:40 Carolyn They're called points.
 310. Marjory Points have, um, one represented by one tower and others of them are represented by more than one tower.
 311. Carolyn Ok but just pick one. Just choose one.
 312. Marjory So, ok. I going to pick the one closest to me as red and say this blue and this is yellow.
 313. Carolyn Ok just hold on to the red one for a moment.
 314. Marjory Ok.
 315. Carolyn Just keep your finger on the red.
 316. Marjory So from the red.
 317. Carolyn Can you see three points going out of there?
 318. Marjory Yes. So, there is one where it gets a second red and it goes over here and I am going to say this one is going to get a red with a yellow and it's going to go over there and a red with a blue is going to go there.
 319. Carolyn Ok. There's three.
 320. Marjory So there is three things that come out from there.
 321. Carolyn Now go around.
 322. Marjory And then, um, from the next one, let's say we are calling it yellow, and there are three places that extend down from there. So, yellow with a red on top is going to go there and yellow with a yellow on top is going to go there and yellow with a blue on top is going to land there. And then the last point is blue and that one when it gets a yellow on top it's going to go this way, when it gets a blue on top it's going to go that way, and when it gets a red on top it's going to go that way. Right? Is that...
 323. 0:38:57 Carolyn You see the nine?
 324. Marjory Well this has one coming down to it. This has two coming

- down to it and that makes three.
325. Anoop Three?
326. Marjory I'm adding. Right? I'm trying to get to nine, so one...
327. Carolyn Well lets go from that point, how many went from that point?
328. Marjory Well there were three from this point...
329. Carolyn That's three.
330. Marjory Three from that point is three more and three from that point is three more.
331. Anoop So nine things, like these nine towers that he made.
332. Carolyn Are any of them alike? That you can combine.
333. Marjory Yes.
334. 0:39:29 Carolyn When he does his adding. Cause I look at some of the points there and you counted them more than... right? Did you see that, Robert?
335. Marjory Yes.
336. Carolyn So, how does that work? Just look at what you built here, the nine and could look there, the ones, you can see some common ones. Right? It's like the A B, B A idea.
337. Marjory This one... Well this one has two points leading to it, so it's going to have two towers there and this one only had one, this one only have one way to get to this points, right? Down the outside. There are two ways to get to that point...
338. Carolyn So what would be the one way? What would they look like, those towers?
339. Marjory I think I made the one closest to me red, so that's going to be this tower.
340. Carolyn Ok.
341. Marjory Cause I said it was going to be red, right? And if this was one yellow and if you combine these you are going to get, there are two leading to it, you are going to get one that looks like this and one that looks like that over there on that point.
342. Carolyn Is that ok, Robert?
343. Robert Yes. I agree.
344. Marjory Then what's... Then this one is going to be yellow, right? There is only one and wouldn't it be the same on the other side, Robert?
345. Anoop Funny thing...
346. 0:40:43 Robert Well, it would be yellow and red.
347. Marjory It would be similar on the other side, right? Cause that's where the blue comes
348. Robert Well it would be blue and yellow instead of red and yellow and then this side would be blue and red instead of yellow red or yellow blue.
349. Carolyn And it's easier for me if you look at the towers cause I can combine like terms much more easily. I can see the different ones and I can see the one I can combine.

350. Robert I think it gets more confus... sorry, if you have this one similar to that one, back there, when you go down. So actually this one and one of the ones over there are actually going to be the same term and you are going to have to combine them across the pyramid instead of, ah...
351. Marjory Ok now you lost me cause I was, I think I was...
352. L So there would be some connecting lines in the middle?
353. Marjory Now that we built this row.
354. Carolyn Why don't you combine those terms? Cause you have A plus B plus C quantity square. Right?
355. Marjory Yes.
356. Carolyn So, Right? So, you have... isn't that right? A plus B plus C. So you have shown me A squared, B squared, and C squared are the solid color towers can you show me what the coefficients of the others would be? In that you have them represented here. In other words, start to combine...
357. 0:41:58 Marjory Are you asking me for something... Are you asking me to see like, ah, and expression with coefficients and variables or are you asking to see something with the towers? I don't understand.
358. Carolyn I'm saying to use the towers.
359. Anoop Like how you prepared these and came up with the coefficients of three of what and three of what.
360. Marjory Oh.
361. Anoop She is asking you to do the same thing.
362. Marjory Me to do it. Ok, so then...
363. Carolyn Robert is going to monitor you.
364. Marjory Alright, I don't see how much of it I can do outloud.
365. Carolyn Well just use your towers.
366. Marjory Right, so...
367. Carolyn Do it with what you've built.
368. Marjory I know. So, I'm going to have, is that one A now? Is the red A?
369. 0:42:37 Robert Yeah sure.
370. Marjory So I am going to have one A squared plus... oh no wait, blue is B, so then I'm going to have (inaudible) then this is going to be two A B plus B squared plus... I'm missing something, oh no wait, two... I don't know how much it matters and what goes in, I want this to be a circle, I want this to be a pyramid. This isn't to... Now it will, wait hang on. These are on the...
371. Robert Well if...
372. Lou Ah I see.
373. 0:43:26 Marjory I have to do it this way cause otherwise it doesn't work. Cause the things don't match up. So this is one A squared plus two A B plus one B squared plus two B C plus C squared plus two A C.

374. Anoop Is it good, Robert?
 375. Robert (inaudible)
 376. Carolyn So the next challenge will be to take it to the next row and that is a challenge that Robert has sort of, I think, predicted will be, what piece of the challenge is but we are not going to repeat what he predicted because we want to see if you see, maybe it's not a challenge for you. You may just be able to do it.
377. Anoop You want Marjory to try the next level? Is that what you are saying?
378. Carolyn I do and I want Robert to watch her.
 379. Anoop Ok.
 380. Carolyn Be sure she doesn't, you know, miss anything.
 381. Marjory Ok, so, um...
 382. Carolyn You can do them with the tower too.
 383. 0:44:22 Marjory Right. So now I got to take this row, right? We're ok that we established it's the, this plane, if you will, of the pyramid. You know, that encompasses you know these points and what, the blue pieces connecting the points, right? Cause the blue pieces, you might think of that as the addition sign between the terms.
384. Anoop Yes. I'm going to stop you, just for a second cause then we got to go to the next level. You never put any blue pieces in these.
385. Carolyn Yeah.
 386. Lou I was just going to ask that.
 387. Robert It only matters in the bottom case. It doesn't matter up here.
 388. Carolyn Why?
 389. 45:00 Robert I mean...you're just gonna put additional, it's you know a plus b plus c. It's not a plus b plus ab and you know when you get all the way back here you don't add another ab again. You know what I am trying to say. Like you only add ab once, you're not adding it twice. So, I don't know how that will work for this one because
390. Marjory Hey, but if you put in a piece here, would that be like what you do when you combine a like term or no, those aren't like terms on the other side
391. Anoop But, didn't we put these here? These were not like terms, this was a, b and c and we put the blue connectors between them
392. Robert Because the equation is a plus b plus c
 393. Anoop OK, so, these are not like terms, that's what Marjory is saying but why can't they be connected if these can be connected on the triangle
394. Robert Because if you write out a-squared, a, b, b-squared or 2 ab, 2 bc, c-squared, 2ac plus, plus, plus, plus, plus, plus. Now, if you connect them, you're gonna have so, if this is what we have here, a-squared plus b-squared (talking to himself) I think that if you connect this, then you have to put, when you get to here you have to plus another 2bc and when you get

- there you have to put another plus 2ab and when you get here another 2 ac
395. Anoop Huh?
396. Carolyn Marjory you following that? So, he...so, the connectors have a special meaning to you of why they are there and they are telling you about adding. Right? Is that what I am hearing? And, so once you have the terms you added, then you don't want to add them twice
397. Marjory And only some things can be added to one another maybe that's another, right, because there's some things that it's we could say it would be valid to add right maybe, and maybe some things wouldn't be valid to add cause it's gonna make...because it is going to interfere with the expression
398. 47:27 Anoop Ok, You think, you think this addition is adding a new term? It's bringing in a new term
399. Robert No, these are all the terms, and then, I mean when you yeah, what is a plus b plus c to the one, it's not, that does not equal a b c you know a comma b it equals a plus b plus c so if you don't have the, this doesn't represent addition then this row is just a-squared comma b-squared, comma, you know there'd just be no relationship between all the variables and then coefficients and it would just be a variable that is not related in any way to this variable and I don't...you connect them because this is related to this one through this and if you connect them, it's saying the same thing twice and cause with addition, it doesn't matter, order doesn't matter
400. Anoop So, by the time I walk around this face, I would have all the additions I need and all of the terms I need
401. 48:35 Carolyn That bottom one, you got something blue
402. Anoop You got something blue in here
403. Marjory He just added that
404. Carolyn Yeah, he just did. I wanna know why
405. Robert Only one though, because you have to get to this term eventually
406. Carolyn Somehow
407. Robert Somehow, So, you take a route that starts from here and goes out but you only wanna go there once, I think, that's why I wanna talk to Kevin cause I remember last time, I think he connected them, I don't know if he did, I can't remember, I think he connected them all
408. Anoop He did. He had a...
409. Marjory He had connectors in every place
410. 49:04 Carolyn Well, this is interesting, this is interesting. So, let's follow it through. Can we just kind of follow this last cause this is very interesting and I follow the way you're representing. Does everyone else follow the way Robert is representing?

411. Lou I do
412. Carolyn Isn't this fun? Doing great job guys!
413. Lou I am glad I did towers this week too with three colors
414. Marjory There's cubes and stuff every where, I am going to move these
Ok Robert? Cause....NO, no the triangle, the model is good.
The model of the pyramid is good but we need all the terms
from the previous equations.
415. Carolyn She needs to rebuild them again.
416. Marjory Because, there's hang on there was one of these, one of these
and one of these and from that the next row right, was this, and
these and these and these, well, hang on a second
417. Anoop It was beautiful how you had them as a Pyramid if you want to
go back to it?
418. Carolyn She needs to do it the way she's ... She'll re do it
419. 50: 04 Marjory No, I'm gonna, I'm gonna try to do that because that worked
like this right or these were the, umm...and, well we don't
have a way of holding 'em up here, but this was above this
right?
420. Carolyn We can imagine that
421. Marjory Now, you can imagine that now I am going to the next row
that gonna go underneath those
422. Carolyn Can you imagine that Lou?
423. Marjory Ok, so then, when I take the tower that was two reds tall and it
gets multiplied by the term a again cause now I am multiplying
this expression by the expression a plus b plus c, right? So,
that's gonna...this term is gonna grow into that one and it's
gonna be out there, right? And then...
424. Lou Wouldn't it be on this side?
425. Carolyn Red
426. Robert Yeah, I think we had red over here and then but then we tipped
in and twirled it yeah, right? It doesn't matter
427. 51:00 Lou I think that actually makes more sense though because you're
putting red outside of red and before you were putting red
outside of blue
428. Anoop You're right, she had the red here but she just placed it there. I
think eventually she would have moved it.
429. Lou I am not doubting that but
430. Anoop Doesn't matter Marjory
431. Marjory You're making me crazy
432. Carolyn you're doing great Marjory, you're doing great, you're doing
great
433. Marjory This is growing one, right? And then....Well, hang on, I gotta
think about all the ways that...then this term...first it got the
red I have to think about the towers, if I am not systematic, I
am won't make all the towers and then it's also gonna grow
into this and I'll figure out where they go after I build the other

- ones
434. Anoop Ok
435. Marjory Ok, and then the one that looks like this, this also has to get all three colors,
436. Anoop Can we help you build the...
437. Carolyn No, let her do it herself, but I would like her, before she starts building to estimate how many there would be three tall before you start building, you know what I am saying
438. Marjory So, OK. I had nine towers right, one, two ... (counting them loud one through nine), and each of those nine is gonna have three possibilities of what color could go on top when it is one taller, so it's nine by three is gonna be twenty seven
439. 52:29 Anoop Twenty seven towers altogether
440. Marjory Twenty seven towers altogether
441. Carolyn Oh, I know there are not twenty seven dots there but we would expect some other activity
442. Lou Um-huh that's what I was looking
443. Marjory NO, but there wasn't nine at this level either, so
444. Robert Cause there isn't... Why is there nine? There isn't... why there's only six?
445. Marjory ME? You think about that
446. Robert Oh, No, I know the answer
447. Marjory Oh, You know the answer
448. Carolyn What happened to the terms
449. Marjory Um, well the terms... Because three of the terms could be combined with three of the other terms. So, there is like, one... there are three single terms and three double terms
450. Robert So, six, so three towers
451. Marjory Three singles and three doubles are three and three is six, so that's six points for but some have two things, two towers that sit on a point
452. 53:25 Robert So, its unique towers I guess would be the term
453. Marjory Yes!
454. Carolyn Very good! Good question, good answer
455. Lou So, it's one two, one two, one two
456. Robert Yup or two two two and then one one one or one two one
457. Lou The corners aren't ones?
458. Robert Yeah the corners... well, they don't have to be like with Pascal's right triangle you can really represent it any way, it doesn't really represent a pyramid... but if you view it as a pyramid, then the ones would be the corners
459. Marjory Yeah, like the one exactly, when there is only one term, they are going to be on this vertex? Or this Right, no vertex doesn't
460. Lou Well, it doesn't get until the next row
461. Carolyn Face?

462. Marjory What would you call, the
463. 54:10 Carolyn That would be an edge, that would be an edge
464. Marjory The edge, OK
465. Lou But each face is still the terms in the triangle but then when you start getting to the fourth row and getting in the middle, then you get
466. Carolyn Isn't that clever
467. Robert I think so, I think last time we built one more row, right?
468. Anoop Yes, you had
469. Carolyn We're not going to do that today now
470. Robert Cause, if we had the smaller blocks, it'd be easier cause there's not enough of the smaller reds you could only build first two
471. Anoop Last time you remember you just used one color, it was all blue
472. Carolyn I like the way you did the two colors
473. Robert I stole it... I think half way through the interview, I started seeing it and then we saw Kevin's at the end and I ohhh yeah
474. Carolyn Did he use two colors?
475. Anoop Kevin used two
476. Carolyn Did he use them that way?
477. Robert I don't know, he never talked about it if he did. He wasn't here for it
478. Anoop He wasn't here for the interview
479. Lou He built it and Anoop had it hidden or the best he could
480. Anoop Yeah
481. Carolyn We'll interview him next time and we'll ask him
482. Lou Is he supposed to today?
483. Anoop He is coming in at 3:30
484. 55:10 Carolyn He'll be here soon, so let's ask him
485. Robert Maybe ask him about why if he connected the bottom and why
486. And we will ask him about the use of colors
487. (Talking to Carolyn about her class)
488. Robert I guess if we built the next one, I know that it has that little island in the center, I guess it would only connect once and then you go around the island.
489. Anoop Well, we can ask Kevin because he always just made those little triangles everywhere maybe he just made them so that it would be sturdy structurally yeah, so, we can ask him that and I think he also said using three different colors so a being one color, b being one color and c being one color and last time when we were finishing up our interview, you and Marjory were playing with it and kind of concluded that
490. Robert You can't just, the yellow block doesn't go in the right direction
491. Lou We said that we would use like something like play dough for the balls...

492. Marjory One of the issues you were limited on how you could connect the rods into the sphere
493. Lou Silly putty in the corners
494. Marjory Yeah exactly...we needed something alright, we are almost there
495. 56:35 Carolyn When you guys did this before in Elena's class, you did, I remember I heard your presentation but I didn't see the actual building
496. Anoop There was no building at that point
497. Carolyn So, there were no unifix cubes around to make this
498. Anoop No, and, ah
499. Marjory And Kevin came in with a built thing
500. Anoop Yeah and Robert had now played with
501. Carolyn So, you think that matters, I mean in learning it? Do you think?
502. Lou I didn't get it. You guys went way way over my head
503. Anoop It was very difficult to do just see it and get it
504. Carolyn Ok, So, this helped with the unifix cubes, now you gotta do it with the zero, one pizza problem, the binary stuff
505. Robert It's base two
506. Lou You was able to to ... You were able to explain what each rod meant but it was hard to follow because you were like spinning it around saying, putting your finger on what term it was
507. Anoop Exactly, exactly when you hold this up and say that's my C, and I went from here to here, how? It's very difficult unless you build it
508. 57:30 Carolyn Shouldn't it be, shouldn't it be isomorphic?
509. Robert Yeah you could do it, I think cause when view it as a zero one pizza problem I view it as zero being off, one being on
510. Carolyn Like switching
511. Robert Yeah, here you get to view it as zero, one, two where zero is a in that place, one is b in that place, two is c in that place, so I think you can actually write it not the binary cause binary is base two but? Or is it base, I know you only use two numbers for base two
512. Carolyn Yeah, the binary is base two
513. Marjory Ok, so we're getting closer I have the towers and now they have to get mapped to the triangle
514. Carolyn Let's see if that works
515. Marjory So, this is one a-cubed, and then um let's see and if the blue the one that's three blue in that corner, so this is gonna
516. Carolyn Do you have papers to mark it now
517. Marjory we're on the bottom row now so
518. Carolyn Can we mark those with the paper now the a-cubed b-cubed and c-cubed Robert to help her get a little

519. 58:31 Robert They don't stick for some reason but we can like..., since it's the bottom row, we can leave it up right,
520. Carolyn We'll put it underneath, right onto the spot
521. Lou Since it's the bottom, maybe you can even place the towers outside of your corners.
522. Carolyn Maybe you can fit them in the right spot, how about that Marjory? Robert will help you. Tell him where to put it and he will put it there for you
(Marjory arranging towers)
523. (Marjory arranging towers)
524. Lou We have five minutes on this tape
525. (SWITCHING TAPES)
526. Marjory I have got the towers built and these have to sit in the middle
527. Carolyn Do You want to put them by the pyramid like in this
528. Marjory Yeah, Well, I can sort of take them from here so
529. Carolyn Give them to Robert, he will put them for you
530. Marjory So, that one is going over there, it doesn't really matter you can a b c you can give the colors whatever, those would go there , those would go there, where there's one cause there's going to be one, three, three, one along the three outsides
531. Carolyn Oh, there goes Pascal's again....triangle
532. 0:47 Robert Actually last time I started doing some reading on it, and I was telling that Pascal's wasn't even the second person to invent this, he was like you know there were the Chinese before him and you know the Greeks before him so
533. Carolyn Also the Hungarians did a lot of stuff independently which I was... how are you quoting all these people in Hungary, these very famous people, and then they showed me one of these books I couldn't read any of the language but I saw all the diagrams and I said "oh my gosh!"
534. Marjory See, now you hit on it, what language does somebody write in and who else reads or speaks that language
535. Carolyn I found that book from some one in Brazil
536. Robert Oh really
537. Marjory And then, these have to go ...I'll never get them in there
538. 1:29 Carolyn Give them to Robert
539. Robert Ohh...
540. Marjory These go inside
541. Robert I was going to lift it up yeah...and hope we don't put it on top of it
542. Marjory Ok, very clever
543. Anoop Well, you want to try to do it
544. Lou Actually, where she's putting them right now is good
545. Marjory Voila! (Anoop places the model on top of towers) Ok , um So, I don't know. Whatever, whatever term is if we were b, I remember blue was b, the other ones kept switching but let's say this is one a cubed and this is three a squared b and this is

- three a b squared and this is one b cubed and again these are, everything is being added its this plus this plus this and then you are plus um um three b cubed c
546. Carolyn B cubed?
547. Marjory Cause b is I'm sorry b squared c I'm writing and talking. this one is gonna be plus three b c squared plus one c cubed plus three b no c squared a plus three um a squared c
548. 3:06 Carolyn What's this stuff in the middle?
549. Marjory And then, those are the terms that have a bs and cs in them it it's gonna be oh, six abc
550. Carolyn Isn't that amazing? What's that little point there which is connected to adding, which is very clever
551. Marjory And if you think about that point
552. Carolyn You like that lou?
553. Marjory See, really there should be six ways some how of getting to that point
554. Lou There should never be a use of teaching foil again, should be using this
555. Anoop Six ways of getting there? Robert what do you think?
556. Robert But then are there, I guess then there would be three ways of getting here I guess it all depends on what you're using the blue blocks to represent
557. Lou What happens when you get to the fourth power
558. Marjory Well, unless what you are doing is is you're carrying something with you from the layer above to get to the one below it I don't know
559. Carolyn So, How are you getting to, I wanna know how are you getting and Robert maybe you can help me, how are you getting and Alice wants to know too cause she is very curious, um, you have those a b cs and six but you, she has them blocked as twos, twos, twos somewhere, are they coming from some place, do they all come from same...where they coming from
560. Robert Two from here, two from here, two from here, I just thought if you connect all of them including the center, the number of blue blocks would be the total number of blocks in two to the third you know they started at three squared, three to the third cause if you connect three are six there right? if you connect three in the center that would be nine which we showed there was nine, you know one, one, one, two, two, two and I am guessing if you connect to this to this, those six in the center, just based on what you said that there should be six blocks going there, which there can be
561. 4:50 Marjory Oh that would be so cool if that worked
562. Robert Then there'd be twenty seven
563. Carolyn Well, can you make it work, I mean can it work?
564. Robert Yes, we just need more

565. Carolyn Let's look at the other representation
 566. Lou Four, five six. Yeah, so, there's already one, so there's five left
 567. Robert I will just build the bottom not separate from this thing so we
 don't have to lift it up and mess with it
 568. Anoop Are you trying to build this level now?
 569. Robert The bottom level
 570. Carolyn Just the bottom level, he's just building the bottom level
 571. Marjory Didn't we have seventy seven towers in that bottom level?
 572. Lou Um huh
 573. Alice In there you've got a six ...
 574. Robert Yup
 575. Carolyn So the question is where do those, where do they each come
 from is the question
 576. Marjory Yeah, these are six a b c Right, because I think we said we said
 a was red, b was blue and yellow was c
 577. Lou Three in the middle above
 578. Alice Do they all need to sit over there by themselves in the middle
 579. Marjory Oh this was, sorry, we're having a
 580. Alice It's these guys in the inside that I am interested in
 581. Carolyn Yeah, we are at the last level now for this pyramid we did all
 the others already, you missed the fun
 582. Alice No, wel..
 583. Marjory And why are those all in there? beause they are not on
 584. Alice Are they are standing where they ought to be standing? Is there
 a way to arrange them that I could
 585. Carolyn That's the question, that's exactly the question
 586. Marjory Yes, and
 587. Alice Or all they need all to be there in one corner, that's what I am
 asking
 588. Carolyn That's a very good question
 589. Marjory Ok, hang on. Let's see if I can make an adjustment without
 ruining what we have made here No, wait a minute
 590. Alice Don't
 591. 6:18 Marjory No, no I had to thin... I have to take the thing away to think
 about where they would go
 592. Carolyn That's fair
 593. Marjory There's oh, but there's only one point here you can't have, you
 can't get a point in between these, no, they belong all where
 they are Alice
 594. Alice Oh really
 595. Marjory Yeah
 596. Carolyn Let's see what Robert comes up with
 597. Marjory Because, um, because what they are doing is that they have
 combined, it's... that represents a combination of like terms
 when you were taking terms that were, cause there, were two
 terms at this point and two terms at that point and two terms at

- that point. And so, when you combine them, oh, hey, Robert I just thought of something but you have to tell me if this makes sense, because is this like an addition rule of Pascal's pyramid that you can take this two plus that two plus that two and have that six. Like this two plus that two plus that two
598. Alice What two?
599. Marjory This, this There's a two
600. Carolyn Ab or ...that red
601. Marjory This is a two, this is a, this side happens to be two bc right?
602. Alice On this side it would be the two would be all in blue on the bottom, is that what you are doing here?
603. Lou That red line would represent that
604. Marjory Hang on, this point here is represented by this tower with two blue cubes, and this point in the middle of this is two towers each with one yellow and one blue cube and this point here was the one tower right, and we are talking from Pascal's and this above it, this was one blue and this was one yellow and so like thinking like Pascal's triangle you know, this is the one one and this row, one two one and so what I was wondering from Robert and we also established before you where here that what's represented on one face of the pyramid is similar to what's on the other two faces but with different terms right, cause these are, because this combines terms b and c, and this combines, that face combines terms a and c and this face combines a and b so, I am wondering if you can add, the way you can add terms on Pascal's triangle going you know you are adding the coefficients cause you are combining I guess like terms when you do that so, if those six towers that are inside middle of the pyramid, I am wondering, Robert, if we can as we combine those like terms, does it mean that you get to add the two here and the two here and the two here to get the six down there?
605. Robert Yes, do you know why, I mean can you think why?
606. Carolyn So, think of all those twos are and what the new ones look like and why that works
607. Marjory Because, if this is two, I have to write that down, start with a new paper
608. Carolyn I am having so much fun (whispering to Anoop)
609. Marjory So, if this one on that side was two bc and then over there that one was two ab and then on that side was two ac and then when you are combining them when you are actually what you have done is then you have multiplied each of those by a plus b plus c and that's how it grew in to six and I have got six a b c towers there in the middle once I have combined the like terms because once you have put each umm, no
610. 10:26 Robert I don't know if it's red or yellow, these three in the center are

- missing these three uh, the middle ones are missing something the end ones are
611. Marjory Oh, right right right because, because basically what's going to happen is, is this term that didn't have when it was over here like the two these two didn't have any of the yellow color in them and so they're gonna get one
612. Robert I was actually thinking cause these are all squared so there has to be at least two of one color and all the ones in the center have one of each color, so yeah
613. Marjory Right, I was only talking about the ones inI am only talking about here, here and here
614. Robert So, that's why they all come from there because those don't have any squared terms in them they are all singular like a to the one b to the one yeah
615. 11:19 Marjory Right, right and the thing is is the ones that are gonna come in the middle because when it does have a square term like, instead of being in the middle it is going to be out here or here or you know in one of these spots
616. Anoop So, maybe you can work with these six towers in here and tell us which face they belong to
617. Alice That was my question long ago, I wanted to know whether they were...
618. Anoop Well these six how they are sitting just all mixed up, can we
619. Marjory Oh, I think I understand the question now, hang on
620. Carolyn And I think she can answer that
621. Lou I don't think it matters
622. Carolyn But, I think she could answer
623. Marjory Well, it down in a way, right , be cause if they are growing from the bottom and then getting taller, then the ones, these two come from that side because they were these terms and so maybe you can put them over there and the ones on that side were these which are these over here and that makes these over there
624. Carolyn Let's applaud guys (Applaud)
625. Lou I mean, it makes them even, but they are still together at that point
626. Carolyn What do you think Robert?
627. 12:28 Robert That's amazing
628. Carolyn That's lovely. You guys did a gorgeous job
629. Alice What if it fitted here this way and put it so either that or
630. Marjory I am only trying to get it so that the thing can stand and not knock everything over but that is tough, that was the trouble in arranging them
631. Carolyn Robert you do that, you can do that. I wanna know Robert what you are gonna show us with the base if we did the base differently, can we explain

632. Marjory Oh, I see what you want
 633. Alice Oh, no, that's fine. That's fine
 634. Marjory NO, I see what Alice wants
 635. Carolyn Can you put it this base down instead of her whole pyramid or not yes, oh Alice wants that, oh OK
 636. Alice Or the other way, it doesn't matter but I what want to remember is what you just said that these two grew from ...from here
 637. Marjory From that point and yes, they grew from that face
 638. Alice Yeah. And these two, I can't remember where they came
 639. Marjory Well, they are blue and yellow on the they look like they grew from those towers and see if my, so they go over there and the last two have to come in on that side
 640. Anoop You got it, I can help you. Alright
 641. Alice I agree with Lou saying six is six
 642. Lou I agree that it makes it clearer, but I still think they all end up that point anyway
 643. Marjory They do
 644. Anoop Ok, I think we should explain to Lou because he is saying they are all the same way, but I think we were trying to figure out what direction things are coming from and so before we didn't see I think where those six towers grew from and the fact that she has paired them and put them aside, to me it tells where it grew from, isn't that right?
 645. 14:10 Carolyn What was interesting to me in this interview Alice, is Robert had made a distinction with how he built this pyramid with colors red and blue where he represented let's say one of the faces of the pyramid to be Pascal's triangle to show how you making your towers grow and how the multiplication is involved and then he has shown with the blue how he now collects his terms how he is adding in a sense, and it's so, he ended up with the question that we raised that was different than what Kevin had done earlier is that what happened, how come you don't have blues connecting on that base and you have certain things not being connected even on the face here and Robert gave a really lovely explanation about that why don't you share it with Jennifer and Alice why Robert
 646. Robert I was saying that to me the blue represents like an addition symbol and red represents multiplication so I thought if you connect this block to the middle term, like if you connect all the blocks to the middle term like in this case how they are all connected, that you will be adding the term six times instead of just once so you'd have
 647. Alice So, that means there'd be two...if you had another blue hanging in to the front, that would be 2 that came to here
 648. Carolyn Isn't that clever?

649. Robert Yeah, instead of having like, you know you would have plus six abc plus six abc,
650. Carolyn I love it
651. Alice That really helps me to think about it
652. Carolyn That's the question you asked
653. Anoop Yes
654. Robert I thought may be if you connect them all, it would give you the total number of things and it does but the problem is you can't connect this to this because you can't go through this block, so this is actually eighteen, but if you connect this to this, that's nineteen, twenty, if you connect every piece, every circle to every other circle, and it would be twenty-seven
655. Marjory Your every point right, kindda I like the word point
656. Alice Are you talking about that for the
657. Robert The total number of towers
658. 16:13 Marjory We are trying to see, like so for instance here Alice, um on this level right, there are three points and three connectors and there are three terms, right that look like these and here there are six points, but there were nine towers um and it was because the points on the outside only had one tower but the points that were, you know the midpoint, those had two so you are adding one plus two more is three plus one, two is another three and a one and a two is another three, that's how you are getting to nine towers
659. Al OH, OK
660. 17:01 Marjory And then here, if you were to count all the towers that I built, that there are twenty seven of those towers that are three tall but like there's one here, three here, three here, one here, three, three and then the six in the middle
661. Lou Is there 27 spheres and red connectors from level two to three?
662. Robert No, there's six times three, 18
663. Lou Plus plus the stairs though
664. Robert Oh
665. Marjory I think lou is asking what if you count these also, so if you count all of, if you count all of these, did you say those were 18
666. Robert Yeah, these are 18 but the thing is you can't connect this one to this one cause there block
667. Marjory Is Lou asking what if you also counted these
668. Robert Wouldn't that be 18 plus
669. Lou I get 26 just at a quick glance, I think it is 27
670. Alice counting
671. Marjory 6, 3, 9 And this is 6, 3, 9, it is! hey, that was, did you see this? If you count... This was 18...
672. Robert Oh, red plus the blue
673. Marjory Yeah, and that's how you are getting to 27 because you are

- counting all of these on this level like the plus you are counting, it's these you're counting so this is 3, 6, 9 um , 12, 15, 18,
674. Robert So does that mean there's six for this one? One, two, three, four, five, six
675. Anoop So, how many for this one?
676. Robert Three times, nine plus 10, 11, 12, 13, 14, 15, I think it's just a coincidence
677. Marjory Then, you wouldn't have these in the middle if you are counting these because I counted 18 of that and then you would want nine around the outside, so, I don't know
678. Lou No, I was saying count the balls and the red (Alice leaves)
679. 19:00 Robert Balls and the red, 1, 2, 3, 4, 5,6 instead of 3
680. Lou And, there's supposed to be just three, well no, you just start with balls at that level cause you don't come from anything, you come from zero and then you get three, you count the three red rods
681. Robert Then this one Six plus nine is fifteen. Right, cause number of balls equals number of blue rods
682. Lou So, 3, 6, 9, 11, 12, 13,14, 15, 16,
683. Robert That's going to be 12 plus nine times three which is 39, right, as each of these ones has three coming out of it
684. Marjory No, you know what it comes to is, is cause you have the three that come here and then what you have when you get to this level is, is you have to count may be if you are counting connectors, you have gotta take what is above it and that was 3, 6, 18, then you have to multiply it times, no you have to add the nine from above to get the 27, then you get the connectors
685. 20:07 Lou True
686. Robert I think you are adding three times the previous number of connectors plus not the previous number of connectors, plus three cause you are just adding three connectors each time cause you are adding the ones on the outside the fourth term, because the fourth term didn't exist previously whereas the squared cc existed, like, two, one kind of existed before except for the center, cause the center existed before
687. Anoop I'm kind of getting lost
688. Marjory Yeah, no, I'm getting lost too
689. Anoop I think the idea was can we count the red and the blue
690. Robert I would say no
691. Anoop To give us how many towers there are before we collect like terms that's what we are trying to answer. If the reds and the blue segments, if we can count them in some funny way if they can tell us how many total towers are there before we combine like terms, that's what we are trying to answer
692. Robert Ok. I guess we could do

693. Anoop So, if we just started with this level there's three towers, one, one, one, and they came from three of these, so the blues were not really counted, I mean, and then, we went down here, how many towers did we have for this one
694. Carolyn The blues show that you are adding them together, so if anything, you are looking at reds
695. Anoop So, just counting the reds right? I mean, right now we were talking about counting the blues and the reds. No? Just counting the reds?
696. Lou Just the reds and then the connectors
697. 21:40 Anoop reds and the connectors in the cross sections, Ok, see that's what I missed, I thought you were talking about all the connectors
698. Marjory Inner connectors, that would give it to you, because then you have to get to this bottom to get to this row here, you have the 18 red connectors and the 9 in the middle
699. Anoop Right, and that would nicely explain why Robert had argued that there are no connectors here cause, I had 9 towers, and I have nine reds already. So, that's like going back to kind of defend again why we shouldn't have connectors there Right Robert or am I totally
700. Marjory Well, it doesn't make any sense to say they belong in one level or row of the pyramid/triangle and then not to have them elsewhere, so
701. Carolyn Consistency is important
702. Marjory Yeah, yeah. If it makes sense in one place it should make sense in other places, it might just be a coincidence that the numbers work out if only works in one place
703. Carolyn That's what I have heard Robert say. Robert Thank you and Marjory, thank you!
704. Everyone chatting (22:50 – 26:37) Anoop explains to Alice the background of this interview. Alice explains that many students have been talking about the pyramid in their qualifying exam. It is discussed that next interview would involve taxi cab problem. Dr Maher gets interested in Ankur's challenge. Robert begins to answer
705. 26:40 Robert I would say that it would only be the things in the center because Ankur's challenge solution doesn't exist in the two D sense, because you have to have three variables, so, it would everything in the center of this pyramid
706. Anoop It's everything in the center and but I just am very intrigued as to how I am going to get there
707. Carolyn So, how many things are there in the center?
708. Lou The thing that came up in the Monroe groups is what happens if I keep going, is there a generalization for the Ankur's challenge?

709. Carolyn Let's just worry about that part
 710. Robert There would be six and then the next one would be three?
 There would be nine unique solutions, like there'd be three
 unique solutions comprised of 18, 18 towers? And there would
 be
 711. Carolyn So, where did the 18 come from? You had the six here, so you
 are saying three, three, three
 712. Robert Three, three, three, add three to each of them
 713. 27:26 Marjory That doesn't sound right
 714. Carolyn And one of them has to be at least, they have to have at least
 one of each color remember and these have at least one of each
 color
 715. Robert I think there might be some duplicates
 716. Lou It's close but you do have to go to the next level
 717. Alice And that's the questions, I would really love to, but I have to
 catch a train...so I really want to
 718. Robert I would say 6,6,6 and that would be 18
 719.
 720. Lou The next level is is um after 36
 721. Alice I think we ought to play
 722. Carolyn But, let me see your conjecture, so you are saying 6, 6, 6 and
 you would guess 18 but why? For the record
 723. Robert Um, because it's currently six now and these all have one of
 each and Ankur's challenge was two, one, one or something ,
 they have to have two of one at least two of one or was it
 724. Carolyn What was Ankur's problem Alice?
 725. Alice Ankur's challenge was one of each
 726. Carolyn At least, one of each color, so notice we have here, in all of
 these, at least one of each color so what happens when you
 build down
 727. Robert You add one of each color
 728. Carolyn So how many do you get
 729. Robert 18
 730. Carolyn Becomes trivial almost
 731. Marjory Well no, not exactly because what...that's part of it because
 when teachers in Monroe worked on this task earlier this week
 and some of them started, they had...they built what we just
 what I just did, you know these 27 towers, and then they took
 these six that were three tall that already had one of each color
 in it and they thought about well, they would make you know,
 identical ones for this because then they could put on this a
 red, a yellow or a blue, and on this they could put a red, a
 yellow, or a blue and so with three times each of those six,
 they would get to 18. But, the thing is that that didn't include
 is is that some of these like this term that was on the outside,
 once you add a red to it, it has to get in there somewhere just

- like so really
732. Carolyn You don't want duplicates either
733. Lou They were really pushing for the generalization
734. 29:36 Marjory But you won't have duplicates um because you are going to be selecting from the outside Robert, you're gonna be, they are not just these six in the middle but there's gonna be the three here, the three here, the three here, these three hang on exactly so, that's two this side, two this sides so that's six more that are gonna get and there's six in the middle and there are six on certain faces I guess here, here and here, and when you add those six and
735. Robert That'd be 18
736. Marjory And they become 18 and then you are going to add them to the 18 in the middle, then you are gonna get
737. Carolyn 36
738. Marjory Right
739. 30:25 Robert And then you add those for the next one and there would be more coming in from the edge, the most center part of the
740. Marjory You already took some of them apart
741. Robert Sorry
742. Lou The center would already qualify, it's just which ones form the outside do
743. Robert Like the centers of the 2-D pyramid, I mean like the inner most
744. Lou Then you would, oh, go ahead
745. Robert Likes one in the center of the 2-d pyramid, in this case it would be two and next one would be one but it would send more than, I don't know
746. Lou And then for the next level you actually have to go back to this again and include ones that you didn't cause you are adding two on the top
747. Anoop That's a very good long conjecture and we have to play with it and put it to rest next time, right
748. Lou I don't now where I would represent this on the triangle though or on the pyramid though
749. Anoop On the pyramid, for that I think we really need to construct the next level, really big layer. Oh he wants to do a computer program.
750. Carolyn This was really nice though with being able to map the actual towers to you know, each of those points I mean, and to show how they evolved and I think that was really nice and it makes the activity very interesting at this level even. I think you get the idea but it is just a lot stuff to handle, people get you know, can lose one or two along the way. But you did really well, Marjory in keeping track and it keeping it ...
751. Lou The towers really helped, because without the towers, when this was explained to me, it went

752. (chatting, tape ends)
753.

Appendix E

Date of Session: 09-12-2003

Author: Anoop Ahluwalia

Verified by : Kiranjeet K. Sran

Date of transcription: 6-8-2010

Researchers: R1: Elizabeth Uptegrove

Line	Time	Speaker	Transcript
1.			R1 shows Book to Robert and Brian
2.		Liz (R1)	Let me show you this
3.		Brian	Oh Yeah, I was looking at that while I was waiting, that's pretty cool
4.	1:00	R1	Ok, yeah, it's called... let me see if you turn it this way so that the camera can see it too. This book was recommended to me by Richard Asky. Who I guess is a fairly famous mathematician and expert in combinatorics and so, Carolyn asked me to get in touch with him about the work I am doing on Pascal's triangle and he said this is the best book that's the introduction to combinatorics. And, let me show you some of the stuff he has. It's a ... the wolf wants to eat the three little pigs, and there's five houses that the three little pigs can be hiding in, so you get the idea of combinations, and they have all this stuff about, you know, all the different ways the three pigs can be arrayed in the three houses and then they talk about well, there's all the possibilities but then Mrs wolf doesn't care which pig is where because they are all the same to her so then they talk about counting them and not caring which is which, so
5.		Brian	You can't put all three pigs in one house?
6.		R1	Well, then they talk about yeah some, some of them say well what if we did that, what if we can't put them in the same house, what would it look like? And then talk about how to do those when you do and do not care about which pig is where. I don't know if any kid at the picture book age who would really get all this so, I think it's kind of a picture book for grown ups
7.		Brian	NO
8.		R1	But it's very hard to find. Yeah, right, I wanted to buy it. He told me I wouldn't be able to. And he was right. I got this from library but it is totally out of print, can't even buy it from tused book stores.
9.		Robert	Maybe check half.com they have some weird... like you can find anything there
10.		R1	Maybe we will. So I will have it for as long as I can have it from the library, couple of weeks, I might have it around if you guys want to look at it, it is cute. OK
11.		Robert	Alright, OK you want to write down the problem, I guess? Sure... Or you want me to write it down?
12.		Brian	Let me know what it is
13.		Robert	Ok, uh..I guess we'll start with this... uh tower...
14.		Third	What's the ...When is the problem from?

15. Robert Oh...The problem's from um eleventh grade, and uh I don't know, I don't really remember anything about ever thinking of this or coming up with it, but, Lynn Tarlow I think came across this and so we have just been working on this for a week now ..uh.. I found some formula and then we didn't know what it was but now looking further into it found a lot about it so, yeah,... the problem was a tower of four high and uh...
16. 3:54 Brian Two colors...
17. Robert And uh... And two of the blocks had to be the same color so... and the rest were the other color, so if you had five high, there'd be two red and three blue... you know what I mean?
18. Brian Yeah, you need at least two of one color, of both colors
19. Robert Yeah, you need to have both colors in there... well, no you don't have to, but, well, for this four high you do, but um you could have like two of the blocks be a certain color and that's it, you know what I mean? Like...
20. Brian And everything else could be...
21. Robert Yeah, It has be other color, so like if you have a hundred high, then it could be two of this color and ninety eight of the other color and how many different combinations are there for that?
22. [Brian writes on paper]
23. 4:43 Brian Just for four high and two colors?
24. Robert Yeah. And you want two of the same color, so like how many
25. Brian So, it's gotta be two and two
26. Robert Yeah, so let's say the colors are red and blue, how many have two red, you know what I mean and so...guess you understand the problem, I don't know, alright cool
27. Brian Yeah, I do
28. [Brian writes more on the paper]
29. R1 Did you want Brian to figure that out before you show him your equation?
30. 5:14 Robert Yeah, that's what
31. R1 OK
32. Brian Do I need to just, do I need to draw em all out
33. Robert Yeah, if you don't know the numbers, then just say it or whatever, draw it all out,
34. Third just draw them out
35. [Brian writes some more]
36. Brian See, I am putting them all randomly so I can't really see them
37. Robert Yeah, that's what I did too
38. Brian It's like oh, got that one already. Shouldn't use b's and r's
39. Robert Why
40. Brian Because b's look like r's, so now it's looking confusing, I don't know, is it six
41. Robert Uh, well, Yeah, I guess... you want to try like three high then may be two and maybe see a pattern you know

42. Brian Just three high
43. Robert Yeah
44. Brian Two colors and then
45. Robert Yeah
46. Brian Ok, alright
47. [Brian writes more on paper]
48. 6:44 Robert Oh we're only looking for like with one color, so like, you know what I mean
49. Brian What do you mean?
50. Robert Yeah, this is like two blues two red,
51. Brian Yeah
52. Robert there's only one of them, you know what I mean
53. Brian So, none of that the opposite stuff is coming into effect
54. Robert Yeah, so just like
55. Brian Alright well, then I can cross out some of the stuff
56. Robert Oh no no no, that's unique... that one's unique.
57. Brian But, alright so you have to do two reds and then a blue and then two blues and a red, is that what you're saying I can't do
58. Robert Yeah, you just have to have like two reds, or like how many have just two reds and like one blue. For three high, the question would be how many have two reds and one blue
59. Brian Oh, I got you
60. Robert As you said, four is like the unique case cause you know what I mean, cause like you would have two, two (Brian works on paper) I guess four is a bad one to start with. I guess two high, one, three, six... You see a pattern or anything...
61. Brian What... are they just doubling
62. Robert Nah...(chuckles) OK I will tell you the formula I came up in eleventh grade and then maybe you can see something with it
63. Brian Right on it
64. Robert I did h parenthesis, h over two, minus point five So, I guess we can try with four being h for this one and see if you get... you know...
65. Brian Alright (Brian works on paper)
66. Robert Good job. Uh... do you think it would work for three, two, ten
67. Brian Do I think it would work, I don't see why not; I also don't see why though
68. 8:45 Robert Oh yeah, Well, why do you think, do you have any ideas why it might work?
69. Brian Uh...most probably the h ...alright, overall height, maybe it is divided by two because of the colors, or is it divided by two because you can't have like the opposites, you know what I mean, I feel you are only looking for that one color
70. Robert It might be easier –
71. Brian It's like each pattern can two, two separate things except it is flip flop
72. Robert Umm...uh, you can try factoring out the half, cause that's what, first I didn't see that until I did that like that gets rid of the half in the middle

- so you get h times h minus one over two
73. Brian Wait, factor out a half?
74. Robert yeah
75. Brian So, I can get rid of that?
76. Robert Oh yeah, then it becomes, h , h minus one. All over two... the whole thing.
77. Brian The whole thing?
78. Robert No, no, Yeah
79. Robert See anything different?
80. 10:01 Brian Well, the only thing I see that's effected is now h in front of the parenthesis, right cause that is still the same, before you weren't doing anything with that except multiplying what was the inside, now you gotta divide it
81. Robert Um...well, let's see, do you remember, how we used to do figure out towers, where we would draw like empty boxes and then
82. Brian we write things in?
83. Robert And then like see how many possibilities there were, for a certain box? You know, And may be look at that and, see if it eyes it that way, I don't know
84. [Both laughing]
85. Brian Just draw the boxes out?
86. Robert Try drawing four empty boxes, you know, you know what I mean
87. Brian Like stack them up?
88. Robert Yeah. You can put space in between them or whatever
89. Brian OK
90. Robert So, like the first option, I mean how many go in there
91. Brian Like this?
92. Robert Yeah.
93. Brian Just put that in this box?
94. Robert Uh, like how many choices do you have for like the first box
95. Brian Two. Right? You can either fill it with either red or blue, right?
96. Robert Yeah, but....
97. Brian Depends how it goes further on down like
98. 11:10 R1 Maybe you want to think about how many, you got the first blue cube and where you can put it
99. Robert Yeah, yeah, like it's like you have a..., alright, if you wanna do two reds, you get the two red cubes right? and you can put them anywhere in this four towers to make a new tower
100. Brian Yeah
101. Robert Like, how many possibilities, like can you put in here, like you know what I mean, like for the first block, you can put it here, here, here, here, yeah
102. Brian So, for the second block I can only put it in three spots,
103. Robert Yeah, something like that like so how many ...
104. Brian So, each time, the places keep getting reduced
105. Robert Yeah so like, if you had the two blocks, where could you put them,

you know what I mean, so if I give you two red blocks and I give you four spaces, uh, how many spaces are there for the first one

106 Brian Four

107 Robert Yeah, so, then what about the second one?

108 Brian Three

109 Robert So, that four times three

110 Brian What did you just point to right there?

111 Robert Oh, the height, like that...

112 Brian OK

113 Robert Let's say you have a tower with h

114 Brian Is that why we subtracting one each time?

115 Robert Oh, yeah, because what I was gonna say if you had a tower say h high, for the first block, how many spaces would you have to put it

116 Brian Whatever h is

117 Robert Yeah...and then what about second one?

118 Brian H minus one

119 Robert And then you multiply that, to get the top right? So, so that's I figure out the top right? And then, why you think we divide by two

120 Brian Two colors?

121 Robert Yeah, so basically yes, that's what I thought that too, because you have two, um, I can either give you a red block or a blue block

122 Brian Ahh

123 Robert And well, that's how we came up, that's how we thought it would work for all... things. But, does that look familiar at all? No.

124 Brian Something, the...I have no idea

125 Robert Ha ha...Do you remember,

126 Brian Fundamental theorem of calculus?

127 13:14 Robert yeah, fundamental theorem of calculus, uh, anything about choose stuff

128 Brian No

129 Robert Alright, I guess I will show you, I'll try to show you all I know, I don't know that much

130 Brian Ok, Isn't choose the exclamation point?

131 Robert Yeah, oh sorta, this thing well there's a lot of way to write it, um

132 Brian OK

133 Robert Like decimals I guess, the accepted way

134 Brian Ah, I remember trying to do this with pizzas or something

135 Robert Yeah, and then

136 Brian Four possible, you just put a one or two or a three or a four underneath it right?

137 Robert Yeah, yeah and then like this is if you have like n objects and you want to put them into k subgroups I think

138 Brian OK

139 Robert And, what this comes out like the formula for this is um (writes formula on paper) so, like. Do you see any connection between this and this?

140 Brian (Sighs)
 141 R1 Maybe you can show him n choose two
 142 Robert Ok, so like n choose two, that would be (writes formula for n choose two) No?
 143 Brian No
 144 Robert Um, alright, do you know what n factorial like can you, any idea what maybe it stands for or, you know, if I had three factorial
 145 Brian light on me
 146 Robert If I had three factorial do you know what that is
 147 Brian Like, what three to the first, three to the second
 148 Robert Nah, it's like three times
 149 Liz You can write it down
 150 Brian Three, two
 151 Robert If you got three factorial, you got three times two times one
 152 Brian Ah Ok
 153 Robert So, if you got n factorial, what do you think that is?
 154 Brian N times n minus one times n minus two
 155 Robert Etc, right? Well it goes to n minus n plus one because you can't multiply by zero then you because you will destroy the whole thing so at the bottom like you got n minus two factorial... what do you think that is?
 156 Brian Wait, repeat that?
 157 Robert N minus two factorial
 158 Brian What do you mean by what I think that is... n minus two times n minus three
 159 Robert Yeah and then
 160 Brian So on
 161 Robert So on, I guess
 162 15:35 Brian So where does this n plus one like the whole time you are going down but the last one, the last possible one
 163 Robert Yeah, It's like the last, that's like where you stop that's like you do three times two times one
 164 Brian Does it not even come into play it's just a mathematical figure to keep it logical?
 165 Robert Yeah, I guess it's like the ending point
 166 R1 You always stop at one in other words
 167 Brian Ok
 168 Robert Yeah, yeah, yeah make sure you don't go over cause like you have three, three times and two and one ...so plug three and you get three minus three plus one
 169 Brian Gotcha
 170 Robert If you don't have the plus one, it becomes three minus three and this whole thing becomes zero and it's like uh
 171 Brian So what's the point of that
 172 Robert So, then two factorial is two times one so you get two times this but then well you can cross out like terms, if you just cross out

173 Brian Yeah, oh
 174 Robert And then you get n minus one...
 175 Brian Pretty much this then
 176 Robert Yeah, so
 177 Male So, pretty much what?
 Researc
 h
 178 Brian This equation except where the n is
 179 Robert The n and h , well n would just be h
 180 Brian Well, he's just showing how that's related to that who cares what
 ...variable we use
 181 Robert Yeah
 182 Brian So, how are the two related though?
 183 Robert Well
 184 Brian Well, Just cause of this? cause each time you got to take one away, is
 that how that becomes that like how you say when you start with two,
 you're saying alright first one has four, next one is three
 185 Robert Well, yes I was saying earlier like what is the verbal like when you
 have h choose two I mean you are taking h high and choosing two, I
 guess, colors, you know
 186 Brian Yeah
 187 Robert or two blocks, and you want to see how many ways you can you
 arrange those two blocks being the same color red or blue in the tower
 188 17:16 Brian So, What would happen if say like you start with two, two reds and
 you have four spots to put them like, say that the possibility of a blue
 comes in, you know what I mean, how would you work that like... if a
 blue has to be in before anything or the blue can be mixed in with the
 group, you know what I mean?
 189 Robert Yeah yeah yeah well...
 190 Brian Do your choices then automatically go down?
 191 Robert Uh, the way I thought was like you um lay the two reds first, like
 192 Brian Just to get your
 193 Robert Like here, here and then kind of like you have four spots to put the
 blues that's the way I always thought about, like you know what I
 mean, like you lay the two red first and you can get to put blue in the
 front of it and you can put one in the middle, or two in the middle and
 you put one at the end
 194 Brian OK
 195 Robert And kind of like what I always, thought like you kind of put these
 blocks first and then you add blues to it and wherever, but that's kind
 of hard with towers taking points and stuff, but... That's what I
 thought. What about, how do you think about it?
 196 18:22 Brian Uh, I like that idea. I think that looks good but otherwise I'd just be
 jumping at things

- 197 Robert Yeah, I see the way you do, that's the way I do that too, you put a red and you kind of go down and put the red throughout and that's how I list them too
- 198 Brian Could there ever be a way like with that blue thing, with one blue block, say I put the question out like alright, there's going to be one blue block and you don't know where it is going to be placed in here, know what I mean, so do your odds of... like... does your... how am I going to state this? If you don't have a specific spot where the blue would be, you know what I mean, or the...let's forget that, forget it cause I don't know how to phrase it
- 199 Robert Do you mean like if you fix a block to be in a certain place
- 200 19:15 Brian Instead of saying alright, all these blocks are free to use to put reds in and do your thing, let's say instead of having nothing here, you have the possibility that a blue can be anywhere in this box, in any of the boxes, so don't you only have like a like a fifty percent chance of being able to put it in that box to begin with
- 201 Robert I only think, I only think that it would reduce the odds of like you say oh, you have a tower like four high and you say oh, there must be a blue here, know what I mean, I think that's the only way it would affect its like
- 202 Brian Put the question marks in there alright, behind these question marks are letters, I am going to say r or b, but you have to like match them up, know what I mean, you say alright are you positive that you can put it there or you can put there or you can put it there... cause who knows if there's a blue behind it
- 203 Robert Yeah, I guess, I don't know
- 204 Brian I don't know
- 205 Robert Yeah But I don't see the probability being affected I only see it affected if you fix, if you say there must be blue in between two reds know something like that then that would be like
- 206 Brian This is too random, just being like hey
- 207 Robert Yeah, but I think with this is kind of like independent but that is like conditional, you oh well, let's take the alphabet for example, like if you got the alphabet you're kind of saying, well, like there's five vowels but in between every vowel there must be a consonant or something like that so how many ways can you arrange that, it's kind of like well it's a whole different problem than this one, you know what I mean, like then how many ways can you just arrange them all together so I think that's what, I don't think there's any difference in your one unless you say they like they must be a blue block between red
- 208 Brian Standard
- 209 Robert Yeah like conditions or
- 210 Brian Yeah
- 211 21:01 Robert It just makes it confusing so I don't know what else, so
- 212 Liz Can I ask you a question?

213 Robert Ok

214 Liz Uh I was going back to the first thing you talked about here with the you draw your empty boxes and then you can put the first red in it in one of those places and then you can put the second red in any one of the three places and that's how you got twelve

215 Robert Hmm hmm

216 Liz Would it be too much trouble for you to draw that out for me? You know like you draw four boxes four sets of boxes and put your first red in each one of those and then show me the way you're gonna put the where you're gonna put the next reds you know what I mean

217 Brian I see what you mean

218 Robert Yeah, (Brian writes on paper)

219 Robert I wish we had a pre-like... nice drawn tower paper and we just make photo copies and we just put them in there

220 Liz Yeah, that would help

221 Brian These are weak towers but maybe I should re-draw them

222 Liz No, that's good, I think that's good

223 Robert I don't even draw towers... I just write down letters.

224 Liz Ok, now for each one of those, you could do, three yeah you're doing a lot of work there

225 Robert And maybe we can make some on the computer like... you know what I mean?

226 Liz Yeah, you know it's even possible to do a computer program maybe to

227 Robert Oh yeah, really?

228 Liz Make copies of the things Ok

229 Brian You can just go anywhere right

230 Liz Now, do one more please

231 Brian Do one more row?

232 Liz One more just like you did cause you said for each of these you can do three so I want you to show me all the

233 Brian So, now I would have to do two more than right?

234 Liz No, I think you stop there cause you said twelve, there was gonna be twelve possibilities, so you got twelve there now, so you are going draw these with the first red in the same place and then for each one of those

235 Robert Put red in different places, like put a red here here, here

236 Liz But he's gotta finish doing the first ones first so and then you are gonna do this with the red, red, red, again

237 (Brian writes on paper)

238 23:15 Liz Now, if you work down here the first red and you can put the second red in any one of those three places right... and then there's the first red and you can put the second red in one of those three places

239 Brian If there can be

240 Liz Yeah, and so on.

241 Brian Gotcha

242 (Brian writes on paper)
 243 Liz Ok, that's very nice and organized
 244 Robert But yeah but it's... duplicates.
 245 Brian Yeah
 246 Liz But there's duplicates those are duplicates
 247 Robert You don't want to cross both out, you want to cross one out, right?
 248 Liz Oh, well, ok
 249 Brian That one out, that one out
 250 [Robert points to paper]
 251 Brian This one?
 252 Robert No, yeah, they are all gonna have duplicates any way
 253 Liz They're all gonna have duplicates
 254 Robert And yeah, so
 255 [Brian works on paper]
 256 24:20 Robert There
 257 Brian Yeah, good call
 258 Liz So, every single one had a duplicate
 259 Brian Yup
 260 Robert It's weird how it's three two one yeah, it just sounded weird, I don't know
 261 Liz One, two, yeah, OK, so you saw another pattern
 262 Robert yeah
 263 Brian That could be looks like something right
 264 Robert Yeah
 265 Brian Wonder why that is
 266 Robert Hmm....I had something that I thought about that was really good and I was gonna show you it but I can't remember a single thing about it
 267 Brian Ha ha ha
 268 Robert And like, I don't know, I have been trying to think of it the whole time, but I can't think of it so, sorry
 269 Brian It'll come
 270 Robert Nah, It won't cause I have been trying since the whole way here and sitting in the lounge and
 271 Brian Awww...sometimes if you try too hard you just can't get
 272 Robert Yeah, like when we are done like when I am home then all of a sudden it come, damn it
 273 Brian Man!
 274 Robert But it was good uh
 275 Brian So
 276 Robert So, what do you think? This like if I give you
 277 Brian I'm feeling light
 278 25:30 Robert Yeah, If I give you like uh a hundred tall and I want you to do you think you could figure it out?
 279 Brian Pretty sure, you could just do that right all the way down to one
 280 Robert Yup but you can't do one high, cause you can't have two colors
 281 Brian Two?

282 Robert Two and uh I don't know
 283 Brian It's cool, it's cool that I know that now but I guarantee you that I will
 not remember it in like a day
 284 Robert (Laughs), uh, what about like all I was thinking about what about if
 you have three colors or if you want no three of a certain color you
 know what I mean
 285 Brian Yeah
 286 Robert How do you think that would come into play?
 287 Brian What tell me, towers four high? Doesn't matter
 288 Robert Yeah, well the minimum has to be three I guess but you know what I
 mean like you think, might be any relation, or any of that stuff cause
 that's what I was thinking about not really
 289 Brian Is it possible to do the same thing except end at three?
 290 Robert Yeah so cause I was thinking like there... what I thought was
 291 Brian Same thing
 292 Robert Yeah, you add up all of the, the ones before it cause
 293 Brian What do you mean?
 294 Robert Like, I don't know what I was thinking but I was thinking like what
 you got h three you know what I mean? That's gonna be like all the h
 two h, ks, you know what I mean?
 295 Brian Yeah
 296 Robert Exclamation then k to infinity you know what I mean? I don't know I
 was messing around with it last... on Monday... but I
 297 Brian Did not really develop?
 298 Robert It was hard cause I was trying to do it in my head and wasn't really
 thinking
 299 Brian Yeah
 300 Robert So, do you
 301 Brian Nah, stuff like that's over my head right now
 302 Liz Can you write out, you started to write h choose three
 303 27:20 Robert Uh, I was just asking if you like let's say you have to have three of a
 certain color instead of two
 304 Liz So, you're thinking of formula h choose three or n choose three or
 something
 305 Robert Yeah, well, I don't know
 306 Liz You're not sure
 307 Robert I remember something like very vaguely like that was like you have,
 oh, cause if you do like you know one plus two plus all the way to you
 know n, that's a n plus one choose two it's like that you know what I
 mean?
 308 Liz Yeah
 309 Robert But then, I remember I was remember doing something like this was
 like one one, one plus two one or something like that so then you
 know all the way to n one and the answer is n plus one choose two, so,
 I thought if you add up all the towers like there are two colors like you
 know uh, one, oh the other tow, so it would be h two, n equals two to

infinity gonna be I thought this would be well h plus one choose three
and I thought like if you want to figure out how many you got three
colors you just would add all the ones previous, like still worked here,
I don't know if works up there cause it's like a rule or something you
know what I mean?

310 Liz I think I know what you mean but I am not sure,
311 Robert That's why I asked if he knew like thought anything about how you
get something like if I give you three red blocks instead of two red
blocks and I'm like put these anywhere in the tower

312 Brian Yeah
313 Robert Like
314 Liz I'm not sure your sigma will be from two to infinity just going from
that, you know what I mean, cause the other one went from one to

315 Robert Right... I don't know
316 Liz I'm not sure either
317 Robert Me neither so that's why I was asking if he had any ideas cause it
seemed interesting nah?

318 Brian Maybe I have to think about it for a little while
319 Robert Yeah
320 Brian Get some more information
321 Robert Yeah
322 Brian Cause
323 Robert (inaudible)
324 Brian Yeah
325 Robert Yeah, if there was any like correlation between like me giving you
one color, two colors, three colors, four colors, etc, you know what I
mean? If there's like a pattern between that

326 Brian Yeah
327 Robert That's what I was thinking but uh I don't really think about it much
when I leave stop working with that like I am working on you know
what I mean?

328 Brian Yeah
329 Robert Kind of like starts slipping and then I forget
330 Brian Yeah. It's been so long since anything like this has ever been done
with me

331 Robert Yeah, and uh Oh, yes, I guess have you heard about Pascal's triangle
332 Brian Yeah
333 29:59 Robert Uh, can you draw it maybe? Maybe something like a little first couple
of rows cause maybe see a correlation or

334 Brian Well
335 Robert With this
336 (Brain works on paper)
337 Robert That's good enough, like, uh, actually I guess
338 Brian Keep going
339 Robert Nah, it's good, it's good cause we only did four high so do you know
what it is, like you take away the sides, you know just look at the

numbers in the center
 340 Brian Here
 341 Robert No, include the top you'll need that
 342 Brian What more like
 343 Robert Let's say this is what I was like add these to six, two, was is even two
 is an answer? Yeah, it was two right? No, it's three, never mind. I
 forget how to do it
 344 Liz Just like that
 345 Robert Oh, yeah, yeah
 346 Liz It looked a little asymmetrical there
 347 Brian Yeah, it's going nice
 348 Robert Yeah, I totally forgot what I was gonna say so
 349 Liz Why don't you draw another row?
 350 Robert Yeah, sure
 351 (Brian writes another row)
 352 31:15 Robert Do you know like, uh, what Pascal's triangle has do with like
 coefficients of
 353 Brian I'll write it
 354 Robert I'll show you um, alright, like Pascal's triangle is like see when I write
 it, I write it like this cause this like cause at first I used to right it like
 that, now, every class I got to they write it like this like you know
 straight down I'll write your way , so
 355 Brian Either way, I could ,
 356 Robert Alright, doesn't matter what you can do like Pascal's triangle to find
 the coefficients to solve like if you have a plus b squared, a plus b
 cubed, so this is like
 357 Brian Uh , yeah
 358 Robert So, this is like a plus b zero
 359 Brian And you multiply it out
 360 Robert Yeah and then, this is like one a-squared what the heck, one b squared
 so let's say like
 361 Brian One a squared two a, b
 362 32:18 Robert Yeah, So, let's try to do , like
 363 Liz That top row is like one a and one b, maybe
 364 Robert Oh, yeah, yeah, thank you and then so this is a squared plus two ab
 plus b squared and blah blah blah This is a cubed right, plus three a
 squared b plus three a b squared plus b cubed so like do you seen any
 correlate...like let's say like
 365 Brian I see how this matches up into there
 366 Robert Oh yeah?
 367 Brian Yeah I remember that from a while ago
 368 Robert So yeah the only thing I was saying was like um, I forget, Ok, yeah,
 so, I am sure there's a relation, I just can't remember it and then
 between this and this and this and all this you see anything, help me
 remember, cause I forget
 369 Brian Don't recall that

370 Robert Ha ha I really don't remember
 371 Brian I don't see anything man
 372 Robert Oh really? Um
 373 Liz Can I ask you a question
 374 Robert Ok.
 375 Liz Now, here's where you figured out now how many ways can you put
 two red cubes in a four tall tower
 376 Brian Yeah
 377 Liz Ok, so now ask now show me or tell me how many ways can you put
 one red cube in four tall tower, exactly one
 378 Brian Four
 379 Liz Ok So why don't you write the stuff down so that you don't forget it
 380 Robert Well, actually, since we are doing like, uh, four high, you should
 actually do the $a + b$ to the fourth so like cause that this is like let
 four be the height you know what I mean?
 381 Brian Yeah
 382 Robert So, kinda like that would be eighty two? It's all screwed So, a
 squared, b squared...(mumbles)
 383 34:08 Brian Can you look at things like, uh, when you are building towers, alright,
 a to the fourth right, so it's all reds
 384 Robert Yeah, that's what, that what I was getting at
 385 Brian 4 a to the third b is like four where it's got three reds and one blue
 386 Robert Yeah, that's what I was trying to get you to say, actually I didn't know
 387 Brian You made it happen though
 388 Robert And plus then that's why I was saying let's call to be apricot so like,
 let's say you want a tower that's has two apricots and two blue well
 right there, there's six of them
 389 Brian There's six of them
 390 Robert And if you want three apricot and one blue yeah, that's what I was
 good job!
 391 34:53 Brian Good job to you!
 392 Robert Nah, nah, it was all you
 393 Brian Laughs
 394 Robert It's easier when Dr. Maher says she doesn't know though, yeah, like I
 don't know.... when she does it, it's weird, it's really weird. You see
 like this is three tall, four tall, two tall and so on, zero tall which is
 like an exception
 395 Brian Yeah
 396 Robert You can't have zero tall
 397 Brian I like that. So why it one then?
 398 Robert Why is what one?
 399 Brian Like you can't have something zero tall how can it have a one?
 400 Robert I think its most, I think it's four like
 401 Brian So, it's not a zero
 402 Robert The math doesn't screw up because the way I heard it was like
 because you can break everything down to like factorization of

numbers or something and then like you have to exclude zero equals
 one like will stop it creates all of these problems if you don't have it
 so I think it is mostly for
 403 Brian Theoretically, it is
 404 Robert Yeah, doesn't make any sense, I don't think or like there's zero way,
 one way you can have zero things and that's nothing you know, or
 405 Brian Ah
 406 Robert I think it's mostly for like, so it doesn't mess up formulas and stuff
 like that
 407 Brian yeah
 408 Robert Yeah, cause zero is complex
 409 Brian Zero just shuts everything down
 410 Robert I know it screws everything up so you have to invent a special rule
 just so that it doesn't go messing things up so, uh
 411 36:10 Brian Damn zero,
 412 Robert Ok, now, I know so like, now so I say you think if I give you like a
 Pascal's triangle, with like a hundred entries I say like how many
 thirty seven apricots and hundred minus thirty seven, sixty three blues
 are there, you could tell me? So, like
 413 Brian Yeah, if I wanted to, I'd write all of it out
 414 Robert Yeah, cause go across or whatever uh, yeah, so, one more thing, what
 where do you think this number four comes from?
 415 Brian What number four?
 416 Robert Uh, like the four that's like this is like one four six four what do you
 think these numbers come from? Hey, I will give you a hint, it has to
 do somewhat with the formula
 417 Brian You mean, where they come from
 418 Robert No, you know like they come from adding these two , like technically,
 but I mean like you think you could write them another way
 419 Brian I'm not following
 420 Robert Like I don't know like, this is like, do you see any correlation between
 this number four, like six remember you said there's six here
 421 Brian Yeah
 422 Robert And you got the six somehow from the formula or so, like, what was
 the way we wrote that six like four choose two, you know so like, let's
 say I'll just put four choose two there.. do you think maybe it could be
 written like this ... like that row of Pascal's triangle could be written
 like this
 423 Brian Yup
 424 Robert Really?
 425 Brian Yeah
 426 Robert So, like maybe a plus b to the fourth is really (writes on paper) etc
 427 Brian That's crazy
 428 Robert Uh, so
 429 Brian I have seen it, I've seen it done before, I think
 430 Robert Yeah, see so

431 Brian It looks familiar, that's the only reason I'm saying yeah, I don't really
 432 Robert So like, that makes sense too cause like if you want to choose three
 apricot, from four tall high, four choose three
 433 Brian Hmmm
 434 Robert Actually, so yeah that's pretty cool
 435 Brian I like that
 436 Robert Yeah, there's all this weird stuff too that I learned, like alternating
 sum equal zero, and stuff like that... like if you do minus plus, minus,
 plus, every row would equal zero
 437 Brian Phew!
 438 Robert Yeah
 439 Brian That's pretty interesting
 440 Robert It is, yeah... I can actually prove it too which sucks. It's a waste of
 time and then yeah so, there's a lot of weird stuff that doesn't make
 any sense
 441 Brian Is it just cause it's pretty much symmetrical though?
 442 Robert That's what I thought at first but then like, here, you know this one it
 works, because you have plus minus, plus minus, so these all cancel
 out but here these don't cancel out, these don't cancel out and there's
 stuff in the center I could show you why but I don't know it wouldn't
 mean anything alright, might as well show you
 443 Brian You don't have to, I get it
 444 Robert Oh, really, alright good
 445 Brian Can you think of a way, might as well show it
 446 Robert Ah! Do you have a way to show what? What we were just talking
 about?
 447 Brian Yeah. This has nothing to do with, I just was rambling around to kill
 time
 448 Robert Oh really? Well
 449 Brian I already forgot what we were talking about
 450 Robert Just let a equal one and b equal negative one
 451 Brian Ok
 452 Robert So, that creates an alternating sum cause this positive, negative,
 positive, cause you know what I mean?
 453 Brian Yeah
 454 Robert And then, in the center, the zero, the fourth, I don't know anything
 else about Pascal's triangle to show you?
 455 Brian I think you showed me enough.
 456 Robert Yeah, so, do you have any questions cause I have been like I don't
 have anything to say for a while now
 457 Brian I think you did good though, man
 458 Robert I think you did good
 459 Liz Yeah, I think so too
 460 Robert I think he did better
 461 Liz I think you guys did a lot of stuff, was very interesting
 462 Brian Surprised, picked up right away

463 Male I have one question though, for you Brian So, you did two colors, four high right, figured that out

464 Brian yeah

465 40:15 Male What about, what if, Robert asked before if you could do, uh, two colors a hundred high, Ok, can you figure that out?

466 Brian OK, right now?

467 Male Yeah

468 Brian Couldn't we just

469 Robert Just write, write what we figured out, how we figured out to use either formulas

470 Brian Phew! Where's that one? I can just

471 Male One second, let me just sorry, I'll ask the question again two, you did two colors four high can you do, can you give me an answer for two colors one hundred high? (Brian writes on paper) Two of one color, right, in every tower

472 Robert Yeah
What is that, forty- five fifty or something, forty four fifty?
Yeah, forty four fifty. Can you write that simpler way maybe?

473 Brian Hundred choose two

474 Robert Yeah, I guess you got like billion tall, the calculations could be a pain

475 Brian Yeah, you don't have to put any line underneath it?

476 Robert What for?

477 Liz No, he wants to know if to put a line in between, No it's just hundred choose two that way

478 Robert And there's other ways What was the other way? Something like hundred C two, that was uh Pascal's way and then some class they write it like this for me or like this

479 Liz And sometimes you see it just as a function C parenthesis a hundred comma two

480 Robert Oh, really, like this? That would confuse me

481 Liz Yeah

482 Robert Cause that looks like points on a graph or something

483 Liz So, can you continue with hundred choose two and write that, you know that formula with the factorials remember that

484 Robert Yeah, yeah

485 Liz Maybe you need Robert to show you again how that works

486 Robert This formula,

487 Brian I, I got to go through all that?

488 Robert I guess just write it like hundred factorial, whatever
(Brian works on paper)

489 Brian Now, what's n, just given or?

491 Robert Oh, it was h, I can refer to them all, yeah, so does a hundred

492 Brian And it can't k is what? Two

493 Robert Yeah so, it's like h, k

494 Brian Ok
 495 Robert So, then yeah
 496 Brian You want me to break it down?
 497 Robert If you want to I don't think that you need to write it all down
 498 Liz You guys can write a little bit, cause it makes it, I think it's interesting
 if you write hundred factorial, write the first couple of terms of that,
 499 Robert And then maybe like hundred times
 500 Liz You know what I mean?
 501 Robert Whatever times whatever times whatever factorial and see something
 that's crossed out
 502 Brian What?
 503 Robert Like reduce the bottom, you get ninety-eight factorial times two
 factorial?
 504 Brian Yeah
 505 43:13 Robert Like start writing out a hundred factorial out like a hundred times
 whatever times whatever factorial you know
 506 Brian Yeah
 507 Robert And then see if anything crosses out, you know?
 508 Brian Alright, so just mean a hundred times ninety nine times ninety eight
 509 Robert Yeah, maybe etc
 510 Liz Just a few more
 511 Robert Yeah, ...you put a dot, dot, you just leave the last one as factorial,
 doesn't matter, you know what I mean?
 512 Brian Yeah
 513 Liz Write a few of them out too
 514 Brian Now, I'd have to go ninety eight
 515 Robert Yeah, or put the factorial again, you know,
 516 Brian And yeah, then so, right, gone, gone, gone, gone, gone, ah it's just
 gonna break down to a hundred times ninety nine and then you would
 have to divide by two any way
 517 Robert Yeah, and so cause that's the one from this one
 518 Brian yeah
 519 Robert Yeah, so it's weird how this all stuff is like
 520 44:12 Brian Connected?
 521 Robert Yeah, seemingly unrelated stuff how is the binomial coefficients for
 two things, related to towers, related to pizza, etc,
 522 Brian Yeah
 523 Robert It's very interesting
 524 Brian Boggles my mind sometimes
 525 Robert So, maybe when you invented this, did you know this, I doubt it, but
 like it's weird how you invent something and then there are a million
 things you didn't think would spawn from it, spawn from it.
 526 Brian Yeah
 527 Robert Or maybe these two are so related that, uh, it's same as that problem,
 just worded differently
 528 Brian That is crazy, Pascal's triangle is

529 Robert Mad props
530 Brian Yeah... yeah Pascal!
531 Robert I know
532 Brian Good session
533 Robert Yeah, well
534 Brian Do you have any more questions,
535 Liz No, thank you that's very interesting
536
537 (Discuss writing names on back of the paper to be copied later.)
538
539

Appendix F

Date of Session: 08-31-1999

Author: Anoop Ahluwalia

Verified by : Kiranjeet K. Sran

Date of transcription: 5-26-2010

Researchers: R1: Carolyn Maher

Line	Time	Speaker	Transcript
1.	00:44	R1 (Carolyn)	Before they start really filming us let me tell you why we wanted you to come in Ok, to do this I mean uh...um Michael has been working on some problems last year Robert that you weren't working on OK, we know you two talk to each other about Math sometimes, don't you?
2.		Robert	Yeah
3.		R1	Sometimes, right? And um there was one problem that Michael worked on that he wasn't sure of. Maybe he'll remember. He wasn't really sure both the answers made sense
4.		Michael	Yeah, I know what it is
5.		R1	He knows what it is, right. And so, we're going to be giving these problems also to some of the students who didn't have them, you know some of the folks you worked with last year and so um, we kind of thought that it would be good to, to revisit the problem, again, with Michael and maybe Michael has to explain to you a little bit and you can be the sounding board, so that he has two answers and he says both of them make sense. Do you think there could be two correct answers to a problem?
6.		Robert	yeah
7.		R1	You do? Now I am worried. How could there be two correct answers Robert?
8.		Robert	I don't know
9.		R1	For the same problem
10.		Robert	Could it be
11.		R1	Like, could five plus three be eight and nine?
12.		Robert	Yeah but x squared can be one and negative one right? Ain't that two answers
13.		R1	Oh, Ok. You're right on that but what I am saying is two different answers
14.		Robert	Oh then
15.		R1	And both being correct, I didn't say it correctly
16.		Robert	Yeah you could I think
17.		R1	How do I want to say this? Help me Michael
18.		Michael	I think you just said
19.		R1	Help me guys, how do I want to say this?
20.		Michael	Um, two probabilities of something happening
21.		Robert	Oh
22.		Michael	You know how two chances it's either, you know

23. Robert Uh
24. R1 You could have a lot of sometimes there are multiple solutions to a problem right?
25. Robert Right then, no
26. R1 Sometimes there are you gave a good example plus and minus one, that's true or sometimes you have maybe a fourth degree polynomial right, and you have four solutions, some are the same are different, um, but for a particular problem,
27. Robert Like one plus one oh then one I think
28. R1 Right. Now, this is a probability problem right Michael, you remember the problem Michael?
29. 3:047 Michael Yeah, world series
30. R1 Do you, you know the world series problem too, don't you?
31. Robert No
32. R1 Tell us what you remember
33. Michael Uh
34. R1 We have the problem so, just try to, it'd be better if you can try to do it yourself
35. Michael There's two teams both have you know equal chance of winning each game um...did the teams like already play a couple of games, I don't remember how it went exactly. Or like one year ...It was like, let's say team one won series one year, what's the probability it won it in four games, five, six or seven. You know what's the probability it took that many games to win?
36. R1 Do you know how, what it takes to win a world series? What the rules are to winning a world series
37. Robert Yeah
38. R1 Tell me
39. Robert Best out of seven or something
40. R1 Ok, so what's the fewest number of games that
41. Robert Four
42. R1 And what's the max?
43. Robert Seven
44. R1 Seven right, so there's certain probabilities associated to winning, there's the probability of winning in exactly four,
45. Robert Five
46. R1 There's the probability of winning in exactly five, is there more than one probability of winning in exactly four?
47. Robert Uh, no
48. R1 No, we're assuming, we're making certain assumptions about the teams, aren't we Michael?
49. Michael (nods)
50. R1 What are we assuming?
51. Michael Assuming they're both equal
52. 4:21 R1 Yeah. And we know that's not practically probably anything we can ever really know for sure but for the purpose of doing the

- mathematical modeling, we'll make the assumption for this time, we can play with the problem and check that, sometimes we do have different, we won't change it for now we make the assumption that the teams are equally matched, right? And so what is the probability, maybe that's the one you want to revisit, um, first, you never worked on this one, did you
53. Robert Nah
54. R1 So, you might want to revisit that with Michael and then there is a second problem, but maybe we will go to the first
55. Michael How about that, how about world series
56. R1 No, the other problem
57. Michael That I wasn't sure about
58. R1 Yeah, that you and your, you're the people you worked with didn't reach consensus on, is that the way to say it?
59. R2 Right, right
(Gina?)
60. R1 Yeah that was Pascal's and
61. Michael The game?
62. R1 The problem of points
63. Michael I think we reached to a conclusion, I don't know what it was
64. R1 Ok, well, that's interesting so maybe we should revisit both of them. So why don't you, both of you, talk about the world series problem and then you can call me, that's another way you can do it, we'll leave
65. R2 Ok
66. R1 And then call me when you're ready to talk about it you so have Robert as your sounding board
67. Michael You have the paper
68. R1 Yeah
69. 5:36 R2 I didn't bring the world series one I didn't think we were doing it
70. R1 Well, write it down, I'll ...here's all your papers, what is the probability of winning a world series in exactly four, exactly five, exactly six, exactly seven right? Fair enough?
71. Robert Yeah
72. R1 Ok, we'll let you go. Call me when you need me.
73. (R1 leaves the room, Michael and Robert talk to each other about something, not clear)
74. Michael Alright, got to remember how we did it cause I guess
75. Robert Probably figure out
76. Michael I guess
77. Robert All the chances that somebody could win
78. Michael Not all the chances, there's a lot of chances, how many games could go, you know, I guess if you win in four games, you got to win all four of them, right
79. Robert yeah
80. Michael So, imagine there's team one and team two um...what is it, and if you win in five games,

81. Robert Zero, one, one, one, one, one zero isn't it when we move the zero, I think that when if it was blue, we just move blue down
82. Michael What? Can't remember anything
83. Robert Remember like how, how like
84. Michael No, they're different combinations though, you can't have the four four, four, four, one cause you have the, so you got that and same for two times whatever Now the thing is, well we'll get to it later, why controversy or whatever, (inaudible) see some kind of pattern going on?
85. Robert Not right now
86. Robert Zero, I see it's hard to write
87. (both work on the problem)
88. Michael Oh, you're writing them out actually?
89. Robert Yeah
90. Michael Ok go ahead, I am gonna try doing something else.
91. (both work on the problem separately, Robert mumbles to himself)
92. Michael Is there a choose thing going on here?
93. Robert Oh, I don't know, do you?
94. Michael Not really, I am trying to figure it out
95. (work on their own)
96. Robert Four in the first, I don't know
97. 8:44 Michael Is that, is that it?
98. Robert Nah, I didn't finish that part
99. (work on their own)
100. Michael That's one twenty right?
101. (Robert writes combinations using binary notation)
102. Robert I don't know if this means anything
103. Michael Is that everything?
104. Robert No, this is for the six games, six and then you times it by two, one, two, ...
105. Michael Is that everything, for the for one team, I guess
106. Robert Yeah, it's ten, it's twenty, think it's all of them
107. Michael I do, I do remember twenty I think, so do we see a pattern?
108. Robert Two, eight, twenty,
109. Michael No, the first one was eight
110. 10:04 Robert Yeah
111. Robert Let' try to guess what this one would mean
112. Michael What? The next one?
113. Robert Yeah
114. Michael And then we check it
115. Robert Umm...
116. Michael I think maybe like a forty four, I don't know
117. Robert Forty four? Where did you get that from?
118. Michael Right there
119. Robert Um-um try to find try to find you know like a
120. Michael Yeah

121 Robert I mean, there's a reason for that twenty or something Just cause there is

122 Michael I'll write it down and then see

123 (both work on their own, Robert continues on his binary lists)

124 11:45 Robert I still don't

125 Michael I told you, it was a pain

126 (continue to work)

127 Robert I know, why don't we take all this and just put a zero in front of everything

128 Michael Hmm?

129 Robert We could just take all this and put a zero in front of it like with that one

130 Michael Put a zero in front of this one?

131 Robert This and this would look like that like with a zero in front of it and then you put a one in front and then you put two ones and then three ones

132 Michael Go ahead, ok, doesn't this look like

133 Robert Uh, let me copy that down

134 Michael Uh,

135 Robert Zero, one, one, one, now let, if you just put like zero in front and like a one at the end, and then doesn't this

136 Michael But you can't do this for the next one

137 Robert But then we skip from four to six, that's why, I didn't do five

138 Michael What do you mean four or six?

139 Robert This is for six games and that's for four games oh, that's five games, never mind

140 Michael This is seven, right?

141 Robert Hold on, I just copied it down wrong so it's zero in front of it

142 Michael This is six, for five, four, five, six and seven

143 Robert And so, I just put a zero in front of it and you get the first set here, or here, or here, right? So, like this and this is the same thing

144 Michael So, yeah, then if you put a one in front of everything, one and a one and a one like

145 Robert Nah, if you put I think all the zeros in the middle and then all the zeros here and then all the zeros here

146 14:07 Michael But, the one you started with was, everything, without these zeros

147 Robert Yeah, and that's the same, I got that from here

148 Michael Alright, that would give you some more but then you could have more like this switch with that so, there's a lot more combinations to that

149 Robert Nah, it's got to be, there's only a few

150 Michael And we had it last time

151 Robert Oh, what happened to it?

152 Michael I forgot it, six times one twenty

153 Robert Oh, then what, so, you just like move this here, like you just take this one off and then you move this in here and this in here and this in here and you just take like this one off, this in here, and you take this one

- off
- 154 Michael cause I am moving the zero up the line?
- 155 Robert Yeah, but you always take the top one off. Cause then you move the zero over here, you got the this one, you move the zero over one, you got one zero, zero, one, one and one, that's what's here and if you move the zero over here, you got one zero, one zero, one, one and if you move it over here you got,
- 156 Michael Uh, yeah, but that way you still have to write everything out, you know,
- 157 15:16 Robert Yeah
- 158 Michael I think there's a way where choose something, choose something because um to figure out how many different possibilities you can have four and like six spaces or something, that's like choose right?
- 159 Robert I guess
- 160 Michael I remember something like that, just take off the zeros at the end, that's same as the ones before, um, like something (Michael writes on the paper and Robert watches)
- 161 Robert Well, this is the third thing
- 162 Michael huh
- 163 Robert Uh six...
- 164 (Michael writes on paper)
- 165 Michael Is that like seven twenty divided by forty eight?
- 166 Robert Like eleven something
- 167 Michael Yeah, that's what it looks like
- 168 (Michael writes on paper, Robert watches)
- 169 16:40 Robert What about if you just (Michael continues to write on paper)
- 170 Michael Fifteen right?
- 171 Robert What's fifteen?
- 172 Michael That's what I got, fifteen, what can I do with that number now? You got twenty for the one that's six?
- 173 Robert Uh, yeah. I don't know if it's right, looks right
- 174 Michael Let's see. Alright let's see, now I was thinking of it in terms of choosing
- 175 Robert Yeah, that's what I am trying
- 176 Michael So, this one is like um five choose four and next one is a six choose four, all right? So, answer to five choose four is um, wait it's five choose,
- 177 Robert Then what
- 178 Michael Yeah, five choose four, twwhat's that number, I think it's that number, five, right? And that will give you that and this one too you know?
- 179 Robert And you get the minus two because
- 180 Michael And then I guess you multiply by two to give you the other team and then you take away the two combinations from before, cause you minus two, so that's five times two
- 181 Robert What you had, two times won't be like constants...
- 182 Michael No, it's not two, it's just the number you had before, five times two

- you know, four, what are you, five is um the five choose four, four times two cause the other team can get the same combinations and then you take away the two combinations from before and then next one, I did six choose four, and that's fifteen. Fifteen times two
- 183 Robert (Watches Michael's work)
- 184 Michael The other team would have the same combinations then you... that's thirty... then you subtract the combinations from before which was
- 185 Robert And the combination before that too
- 186 Michael And the combination before that that's ten and that would give you twenty and I guess the next one would be uh seven choose four times two um minus forty...
- 187 Robert It's sixty four right?
- 188 Michael What?
- 189 Robert Wouldn't it be just minus six choose four
- 190 Michael Minus six choose four times this one minus this one you have to minus all of them
- 191 19:37 Robert No, just like if you do six choose four, and you don't minus anything from it you'll get all the other combinations too, wouldn't you?
- 192 Michael What, what are you saying?
- 193 Robert Like you know how you had a minus something to get the answer here what if you don't minus something like just take the five time two, you get your ten and you just subtract the ten from here
- 194 Michael Oh, yeah, cause then five minus two, then you have to add that on later
- 195 Robert Yeah, so instead of just doing it, you could just
- 196 Michael Yeah, you could just do like six choose four or minus five, five choose four, and right?
- 197 Robert Yeah, you wanna write this
- 198 20:18 Michael Not really, I guess I have to. Alright, so we got that
- 199 Robert Alright, so, (inaudible)
- 200 Michael So, we know, in four games, you have two ways to do it. Five games you have eight, six
- 201 Robert Seven
- 202 Michael Six games you have, what you have to
- 203 Robert Four
- 204 Michael Twenty, right?
- 205 Robert Yeah,
- 206 Michael And for seven, got to figure that one out
- 207 Robert Minimal
- 208 Michael That'd be seven choose four
- 209 (Both write on paper)
- 210 21:24 Robert This look kind of right?
- 211 Michael seven times seven
- 212 Robert Number of games times the, the minimum number of games you need to win, four times two minus one
- 213 Michael Number of games times the

214 Robert Minimum number of games to win like four
 215 Michael Yeah, four
 216 Robert Times two minus the number of games minus one
 217 Michael Gotta put this there too minus number of games minus one (looks at paper) just to multiply by two again though? No?
 218 Robert I don't know, I'd try it out
 219 Michael Anything
 220 Robert Subtract I try
 221 Michael Cause you have to multiply by two
 222 Robert Um, six
 223 Michael I don't think you have to do it cause if you figure, watch you got what like seven choose um seven choose four which is this um times two I guess um minus two times six choose four right? And I guess you could put the two now, I guess you do need the two in there. Never mind
 224 Robert So, then what did you get for six, the six factorial
 225 Michael Six factorial, see it's times two for, you know remember how we did that choose
 226 Robert Oh yeah, we got to do that too?
 227 Michael Yeah, we got to do that
 228 Robert So, we got to add something else down here
 229 Michael What do you mean? No, uh, just make a difference to that or you could just do number of games
 230 Robert Minus
 231 Michael Minimum games to win, six choose four, you know how they write it
 232 Robert Yeah
 233 Michael Times two (writes on paper) subtract I guess, I guess that's how you write it and it will give you how many combinations you got right?
 234 Robert Yeah
 235 Michael I am trying to figure out, the one for seven would be I guess I would write it down
 236 Robert That's when we use that c thing
 237 Michael Does that look right to you? No one has a calculator here, right?
 238 24:02 R2 Should I get a calculator
 239 Michael You don't have to get it now
 240 Robert I will check seventeen times six oh five
 241 Michael What?
 242 Robert Did you get fifty four?
 243 Michael Yeah
 244 Robert Over
 245 Michael You don't have to get it
 246 Robert Nah, it's OK. Over twenty four, divide by
 247 Michael Over twenty four times six
 248 Robert Six
 249 Michael It's one forty four ...formula again
 250 (Both write on paper)

251 25:05 Robert Here it is, it's thirty five. Is it, let me check
 252 Michael I don't know
 253 Robert One forty four, thirty five (mumbles to himself)
 254 Robert Yeah, I got fifty four, zero, four
 255 Michael So, it's thirty five
 256 Robert Yeah
 257 Michael So, that's seven, seven choose uh uh, seven choose four right?
 258 Robert Uh, yeah
 259 Michael So, it's thirty five? And I guess you multiply by two
 260 Robert Seven, yeah and then minus
 261 Michael And take away what we had before
 262 Robert Answer we had, twenty? You get fifty.
 263 Michael No, that's not twenty its ahh
 264 Robert No, thirty, thirty,
 265 Michael Thirty, right?
 266 Robert Yes, it's forty.
 267 Michael So, I guess that's forty, right?
 268 Robert It makes sense
 269 Michael Um alright, at least this looks
 270 Robert That's the total combinations for each amount of games
 271 Michael It can't be.
 272 Robert You can use that calculator
 273 Michael Cause I remember it was like seventy two games or something, not exactly seventy. I am thinking
 274 Robert What's forty four times thirty five?
 275 Michael That'd be forty and um this would be like seven (works on calculator) I like it, so you're fine (works on calculator)
 276 Robert (Watches the calculator as Michael works on it). Just, two, six, four...
 277 Michael Oh, I messed up the, supposed to delete it
 278 Robert Yeah, you have to take the answer from the last one
 279 Michael Hmmm (agreeing)
 280 Robert So, It's like taking answer from last thing, it's two point three, three, three
 281 Michael Very goodRight? What's that mean?
 282 Robert I don't know. Did you multiply the answer by two?
 283 28:00 Michael Alright ...Forty, alright, good, I guess you were right So,
 284 Robert We have to write it
 285 Michael So, the total, possible of winning a game, possible will chances like winning games that could possibly happen is seventy right? And it seems like right, right, seems like the right answer... Seventy... The way were doing it, we were having trouble was uh, we were thinking that uh, you have a one half possibility times one half then one half
 286 Robert Which is seven choose four.
 287 Michael Then, you have a
 288 Robert cause seven choose four is all these, all these and all these
 289 Michael Well, you have one out of a... like for the first winning in four games,

a one ... a team has one out of sixteenth yeah, alright, two times two right? So, one out of sixteenth and so then getting it in a four, whatever... so they took into every single possibility and um, seemed right when we were doing it like that but some ways I feel strongly about that one

290 Robert Yeah

291 Michael Because the question is not what's the chance that the team will do it in

292 Robert Yeah

293 Michael It's like, it has been won already so you know it's gotta be one out of these eighty possibilities

294 Robert Seventy

295 Michael Seventy

296 Robert So, wait there's like one over seventy chance that one team will go four and oh.

297 Michael Well, there's a

298 Robert Yeah, there is

299 Michael Yeah, it's not for the question was what's the chance that it was won in four games... So, we have two possible outcomes in two games, so it's two out of a seventy and next one will be eight out of a seventy. I am happy with that one

300 Robert Yeah makes sense to me

301 Michael You want to get her?

302 Robert I just want to write it all down

303 Michael Yeah, we'll talk to her

304 Robert Alright

305 30:26 Michael I don't know you fine with that?

306 Robert Yeah and total number of combinations, we can just do seven choose four, cause that's seventy. Because that includes like everything that's in there this one this one this one

307 Michael No, seven choose four is thirty five.

308 Robert Times two

309 Michael Yeah

310 Robert That's seventy

311 Michael I don't think I will explain it that way

312 Robert alright

313 Robert Well it makes sense cause it's taking in all the other ones too, then you just end up subtracting the seventy to get this number

314 Robert That was good, you can go ahead and explain it

315 Michael No, I didn't think of it, I knew what to think

316 Robert You explained it all (R1 enters)

317 R1 Pull up your chair Gina and join us

318 31:32 Michael I guess we sided with the... remember, when we saw that paper back then and had a different answer uh, I still, I still think that's right you wanna know why

319 R1 No, I wanna know what you have done and I wanna know what

Robert thinks

320 Michael Bobby explains it

321 Robert Uh,

322 R1 Go ahead Robert

323 Robert Uh, well we started out by writing out all the combinations which is

324 R1 Can I see? What were you writing combinations for go ahead, very slowly with me cause I am in a summer mode and

325 Robert This is for four games and there's two chances like team A could win four times

326 R1 What are these, the ones

327 Robert Team one wins four times and team zero going four times

328 R1 Hmm-hmm

329 Robert And for five games, it would be like this, like team zero going the first game and team one going the rest, the team zero won second game, third game, fourth game, and then we just and then we multiply it by two cause there'd be like team one just wins first game, team one just win the second game and then we got six two and then, tried doing it for seven but seemed like too much work

330 R1 I guess my question is, you said there are two

331 Robert Yeah

332 Michael Yeah, cause it is two team, I mean whatever one team does, so the other one can

333 R1 But I don't understand what's your reference to these two... two from what

334 Michael Like we could have

335 R1 Like, this is like winning in four games and winning

336 Michael Take it like one team has to win all four games,

337 R1 Out of how many

338 Michael Just out of four. Four out of four.

339 Robert That's the number of possibilities, if there was if world series only went four games that's how many ways it could win like it can only win two ways, like if the world series only lasted four games, it could either be this choice or this choice and it's only two combinations to be and if the world series went five games, then there'd be eight

340 R1 Yeah, but I am still coming back to these two, right, I don't understand you're saying they could win a team could win all of them the other team could win all of them right?

341 Robert Yeah

342 Michael Mostly all, in order to be four only four games

343 R1 I am just saying, I am a spectator and I am sitting in the stand I don't know what's going to happen right? Um and these are the only two things that can happen

344 Both No

345 Robert See when you

346 Michael You got the other seventy possibilities that can happen

347 R1 Now I am really confused

- 348 Michael But, if the aim, let's alright if you pick up the paper next day and it says world series won in four games, there's two possible outcomes how many games have been on it, either team A won all four of them or team B win all four of them Cause if you don't win all four of them, you'd be picking another paper and the seeing the headline saying the game another game today or whatever you know so, if you know it's won in four games that's all you can have just those two possibilities
- 349 Robert So, we did like, we know that like the world series is done already and if, if it does not end in four games, then five and six and seven just add them altogether
- 350 35:00 R1 Let me ask you a question. Is it as likely to win the world series in four games as it is in five games as it is in six games
- 351 Robert no
- 352 R1 As it is in seven games, as it is in ...they all as likely?
- 353 Robert When we figured out, we got
- 354 R1 Just answer my question and figure out the numbers, let's just think about the event of winning, here you are and they are having a chat about
- 355 Michael I think
- 356 R1 No, no, just listen to me for a minute just deal intuitively what does this mean
- 357 Robert It's very rare cause you like they always say oh they're gonna win it in five or six, you never hear like a
- 358 Michael Yeah, it's rare cause it's kinda hard for a team to keep up winning all four games
- 359 R1 So, you're saying that the probabilities of winning, in four games
- 360 Michael Are low
- 361 R1 Right? are not the same as winning in five or six games is that what you're telling me
- 362 36:00 Michael But the question is, is not is not what's the probability that a team would win it you know, it's a probability that how many games out of like the question was not like how man what's the probability that a team like a certain team like team one could win it in four, five or six games. The question that you're asking, I think you are asking is like what's the probability that they would win in four games it's, it's not likely
- 363 R1 You see difference in those questions
- 364 Michael I do
- 365 R1 Ok. I don't. I am a little
- 366 Michael You, do you understand that we can have
- 367 R1 Go ahead
- 368 Michael Seventy possible outcomes
- 369 R1 My question is are they equally likely?
- 370 Michael No
- 371 Robert No

372 Michael Because like
 373 Robert It'll be twice as much like you have like two teams like you know like
 chance of one team winning would be like rare then you add, you
 have to multiply add it again because like you have to other team
 winning it too
 374 R1 Hmm
 375 Robert So, I think it would just be like that
 376 R1 Ok, that's that one part, so you're saying that what I am hearing you
 say I think Robert is that you have to take into account that it could
 happen to either team
 377 Robert Yeah. That's why we multiply by two
 378 R1 I understand that part. The part that I am not clear about is the fact that
 you can treat all of these winning in four, five or six or seven as all
 having same likely, likelihood. We're treating all the wins as being
 equally likely, is that Ok to do? Does that bother you
 379 Michael I don't know, it depends,
 380 R1 Cause numerically you're doing that, intuitively, is that Ok to do?
 381 Michael I don't know the probability we had last time
 382 R1 That's another way to think, why don't we think about really
 important question is, do you understand what I am saying here? I
 want to make sure you're understanding, when I leave and come back,
 leave my stuff here, excuse me
 383 (R1 leaves the room)
 384 Robert So, what was the question?
 385 Michael Hmm?
 386 Robert What
 387 Michael Like, alright, let's say any given year, right, or like what do you think
 would be harder to do? Whether a team to keep up every single game
 and win it in four
 388 Robert Yeah
 389 Michael Or sometimes slip up and win it in seven
 390 Robert I think four
 391 Michael You think it's more likely to happen like that?
 392 Robert Nah, I think it's more likely to happen like high
 393 Michael Seven games is you know
 394 38:25 Robert Yeah. Teams are equal she said
 395 Michael But, I guess it has to do with just the question cause she's asking us
 ... What do you think?
 396 Robert I still think what we do works
 397 Michael Um, but maybe if there's a difference, in like winning it in two and
 three, not two, four, five and six, maybe that will change all the stuff.
 Maybe the thing they were doing with that, that might be right, you
 know? Who knows? Cause they had something going like one out of
 sixteen (writes on paper), but that one out of sixteen is one out of uh, a
 lot of possibility I mean, sixteen Wouldn't be sixteen, sixteen for four
 games, doesn't make sense cause it's only possibility that it can come

out that's like
 398 Robert They were including like the first
 399 Michael Yeah, this, so something like this you know for four games, they are
 including stuff like that. Didn't really make sense I thought it didn't
 make sense
 400 Robert I don't think it does
 401 Michael I don't know
 402 Robert Cause like uh they didn't count something, I don't think
 403 Michael Uh...try to think, make something up,
 404 Robert I have no idea I think that works fine cause like we can't disregard
 them but they're just like actively
 405 Michael I think something's up with that, cause she would not ask that. She
 would go and mess us up like that.
 406 40:21 Robert (looks around)
 407 Michael So, you have like, sixteen different possibilities, in four games, right?
 Correct
 408 Robert Yeah
 409 Michael And two of them could be winners right? One eight, no, so I guess
 you could say that's a one-eight chance of that happening, I am not
 sure, if we can even do that
 410 Robert yeah
 411 Michael I'm just guessing and then the next one will be eight out of thirty two I
 guess
 412 Robert It's one fourth, it's like
 413 Michael Even lesser you know what I mean
 414 Robert Yeah ...But that makes sense because the chances of them winning in
 more games is getting bigger
 415 Michael No, it's getting smaller
 416 Robert Yeah, it's getting smaller, no, it's getting bigger cause one eight is
 smaller than one fourth
 417 Michael Yeah, it is, I was thinking one half
 418 41:37 Robert And then this would probably this will make chance that this is one
 because someone's gotta win the seventh game
 419 Michael The next one, it would be uh twenty...thirty two sixty four right (types
 on calculator) it's a little bigger and that last one is
 420 Robert The last one should be one Because there is a one chance that
 someone is gonna win
 421 Michael Yeah, that's true
 422 Robert Don't you have to multiply like two also for the other team
 423 Michael No, cause, I'm figuring like you do that
 424 Robert Ok
 425 Michael You got, for four games let's say you play for four games, let's say
 you hear the team, it's four games in the fourth day in world series,
 you didn't know what the outcome is, there's sixteen possibilities,
 right
 426 Robert Yeah

- 427 Michael And two out of those possibilities are winners and that's uh about twelve percent, hopefully five percent, let's say you play five instead and eight out of those, eight out of those six, thirty two?
- 428 Robert Yeah
- 429 Michael Are winners. Or eight plus two, cause the last one
- 430 Robert I don't think...when I said the last one should be a one I think, I meant they should all add up to one, that makes more sense because
- 431 Michael All these percents?
- 432 Robert No, all these, when you add them, they have to equal one
- 433 Michael Why? That's the number of games though. Maybe your one is eighty, like your whole is like seventy
- 434 Robert Yeah. Seven should be one that's not right, cause it can end up like earlier
- 435 Michael I want to
- 436 Robert Check
- 437 (Michael works on calculator, Robert looks into the calculator)
- 438 Michael What's sixty four times two
- 439 Robert One twenty eight
- 440 (Michael works on calculator)
- 441 44:43 Michael That equals one if you do
- 442 Robert That's right
- 443 Michael If you do two over eight plus sixteen over thirty two plus twenty over sixty four plus forty over one twenty they all equal one. But what does that tell us?
- 444 Robert Uh...
- 445 Michael Out of this one alright you have twelve percent of the going happening in the there
- 446 Robert (Nods) like you said, four, five, six or seven, there's a hundred percent chance that one of two teams won or something like that
- 447 Michael I guess you can say
- 448 Robert I guess there's a hundred percent chance of ending either in of those four combinations
- 449 Michael Hold on (works on calculator and writes on paper)
- 450 Michael Maybe we had this too before, like the possibilities of six and seven are both the same, I don't know why. Just were, like before when we did that, and I guess it shows up again So, alright, I am looking at this and I am saying, alright maybe the answer to the question is, you win in four games you have a twelve point five percent chance I don't know if that's right but that's what came up. See I want to see what we did last time... I mean, what me and Romina did last time. I wanna see that
- 451 Robert Hmm
- 452 Michael You, Seeing it?
- 453 Robert Also, trying to wonder why those two are the same
- 454 46:05 Michael They just are, right, I don't know why. Actually cause the numbers double cause you figure this is six over I mean twenty over sixty four

- and this is forty over twice is twenty as, twenty over sixty four, you know like, you have to do is take the two in front of the first one so that's why they are the same but why? Why wouldn't like, I don't know
- 455 Robert You have point two five it's point two five point two five
 456 Michael I know, I was looking at that before but I don't think that's it
 457 Both look at paper
 458 Michael What do you think this one is, say something about that one
 459 Robert Um, I don't even know what to say about it
 460 Michael Let's see (types on calculator)
 461 Robert So, what do we do to make sure, are we backing up in six, five games or something
 462 Michael Oh wait, we had before we had, you have a two out of seventy two chance of happening in two games, that like two percent or three percent, next one is like uh, it goes up like the percentage no, what I am saying, it grows until this one is like fifty seven percent... That makes more sense
 463 Robert Yeah. What happens if you add all those percents when you add them together?
 464 Michael Should equal one cause eight plus twenty these would all equal one
 465 Robert I thought the first thing we did was fine
 466 Michael Well, let's see what she says about that
 467 (Both wait for Dr. Maher)
 468 Robert Maybe we're just explaining it wrong
 469 Michael Maybe we are, who knows
 470 Michael Here explain it, (throws the pen on the table towards Robert)
 471 Robert I don't even know what we just did so... It's a lot of books
 472 Michael Let's see what she says
 473 Robert What don't you talk, tell her how you got those numbers
 474 Michael No, I'll tell her (R1 comes back)
 475 Michael We didn't do actually much
 476 R1 Didn't do much, gee... you thought
 477 Michael Alright, our numbers were before, two say for first one but there's sixteen possible outcomes for four games with two teams right?
 478 R1 hmm
 479 Michael You give me that, so I guess those two out of sixteen I mean um that might be like that those two actually happen in those two sixteen, it's like it's the one eight point one two five
 480 R1 Hmm-hmm
 481 Michael And I did it the same with the eight you have eight out of like you got thirty two possibilities that can come out, that could happen in a, with a I guess with with five games and those eight that we picked were, you know, winners
 482 R1 Hmm-hmm
 483 Michael Possibilities that eight coming out in those five times is uh point two five or one forth and with the six and the seven, we both had the same

- probability which I guess happened last time too
- 484 R1 Hmm-hmm
- 485 Michael Then we took twenty over uh what was it sixty four
- 486 Robert One out of
- 487 Michael Uh, cause it was a hundred, the sixty four different possibilities of game that can happen in uh with six, you know six games
- 488 R1 Hmm-hmm
- 489 Michael And there's twenty of them we picked you know winners so possibility of those coming out is like thirty one thirty two uh, same for seven and I don't know if that's the kind of answer you're looking for, like
- 490 R1 I'm looking for correct answer. I mean if that's the correct answer I am looking for it, but if it's not
- 491 Michael Then we look at the possibilities that not regarding the other ones it would be like two out of seven, that's like three percent and eight out of seven, it's eleven, twenty out of seventy, it's twenty nine and seven out of this I mean forty out of seventy is fifty seven so, I don't know if that makes more sense
- 492 R1 Does that make more sense to you? Does that satisfy your intuition?
- 493 Michael I don't know which one makes more sense
- 494 R1 Robert, that's why you're here
- 495 Robert I think the percent ones makes more sense
- 496 R1 Why? Cause this is the dilemma, that's why we wanted to bring you in to be the sounding board to the dilemma
- 497 Robert Uh, because I don't think you should count the first three games because I mean they should matter but I don't think they should be included in the probability
- 498 51:48 Michael No, I know we're counting those, just like well, I don't have any trouble like cause you figure like the two out of seventy, the eight out of seventy those are like they come from three percent and goes up to fifty seven so, it's kind of makes more, kinda makes more sense cause, you'd have figured it'd be easier for them have a better chance of winning at seven rather than four
- 499 R1 So, intuitively that's reasonable?
- 500 Michael Or, yeah, I think it's reasonable, like in this one you have six and seven have the same exact probability uh, it should be easier to win it in seven you know
- 501 R1 How would you justify that
- 502 Michael I don't know, you know
- 503 R1 Why don't you think about that one
- 504 Robert That's why I don't like this one
- 505 Michael Cause it's like a feeling if you'd feel it, it'd be more easier cause probability
- 506 R1 Well, where do these numbers come from the twenty
- 507 Michael That's how many uh
- 508 R1 But can you imagine where those numbers come from? The origin of

those numbers? Can you, can you give some kind of image of

509 Michael Yeah

510 R1 Like what were you counting when you got

511 Robert I guess (Robert passes the paper to R1)

512 53:00 R1 Ok, what were you counting does that help you count or help you understand or question it understand what I am saying?

513 Michael Yeah when you figure counting, I guess intuitively two out of seventy makes more sense cause you know um don't really take, you don't write in fact the difficulty of winning it in all four, or just winning in five games

514 R1 Should you?

515 Michael I think so, like, I think that, you take into account the difficulty in winning it, and getting four games, and five, like it says what twelve percent, twenty five percent, I don't know if that's right, I'm not saying it is, I'm not saying it's not.

516 R1 Do you understand what he's talking about Robert?

517 Michael But, do you

518 Robert No

519 R1 I don't either

520 Michael Can I ask you a question

521 R1 Talk to Robert

522 Michael Can I ask you a question, do you think it is a hard twenty and six like it's easier to have the game when it's seven or six games or you

523 R1 Well, I think what you have to think about

524 Michael Well, naturally what do you think will happen

525 R1 I think when you're using that looked at exactly four and exactly five and exactly six, if you think about it that way and you begin to see that uh it gets more likely as you go up and then that seven hits you hard right so you have to think about why are they the same? Is there any is there any rationalization for them to come up to same, what are you counting when you get those numbers twenty and forty, is there any way to make sense that tends to happen or not. That's what, what I would need to think about next, do you understand what I am saying Robert?

526 Robert Yeah,

527 R1 If this is the part that troubles you if this is intuitively correct this sort of progression and this part troubles you, you need to really got back to always imagine why are you counting and why. You have to understand what you're counting this way, you have to understand what you're counting this way, right, what these numbers are

528 Michael After we start again

529 R1 And you think about does that make sense or not when you think about what you are counting and I would suggest you think about that a little bit more so you have a stronger feeling You're trying to just look at numbers now you're doing it looking at the basic feeling for the problem but the thing I learned about probability that you can't

- ever get away from that basic data, the basic what you're counting so I want you to spend a little more time thinking deeply about the what you're counting about these two and then see if that makes any sense to you, does that make sense? That's my suggestion to you unless someone has another one
- 530 Robert Yeah, that's good.
- 531 R1 Other suggestions?
- 532 John I think it's fine
- 533 R1 John spent a lot of time thinking about this problem too, he wants to talk about it with you a little bit but we'll leave for you a little longer to think about you understand what I am asking you to think about?
- 534 Michael Yeah
- 535 R1 OK
- 536 Robert Maybe because you're multiplying by
- 537 Michael Yeah, that's the same like why twenty four why exactly double, you know?
- 538 Robert Yeah
- 539 Michael And the other ones weren't I mean there's gotta be a reason because let's go back to where's all my stuff?
- 540 Robert Right here
- 541 Michael Wish we had more stuff down (Both think for a little while silently)
- 542 56:12 Michael Let's say you figure in six games, you have all your possibilities in six games twenty of them, in last game it could either be a loss or win for both you know, so figure all you want, so when you get to the sixth game, there it has to be either a win or a loss as you got three on one side and three on the other and that last game is one for both you now, could be a one or a zero so that when you add one to that you double up know what I am saying you double the amount
- 543 Robert Yeah
- 544 Michael And if let's say is you had the five games in a series, then all the ones that you just put a zero on, it wouldn't count that's why it wants to be double it'll be less than double what did it come out to more, I guess that's what I see in my head, I see like three and three and you need one more if you get the one or the zero when you add that extra game on to the end, it doubles everything up if you stopped at you know four, then you'd have to you know, no, if you went on,
- 545 Robert Yeah
- 546 Michael To five games all the ones that don't already have five games already combinations from before, like if you have four and two you would eliminate those cause you already won
- 547 Robert (Agrees)
- 548 Michael So, maybe I see why it doubles, I don't know why the other ones don't you know
- 549 Robert Yeah
- 550 Michael Yeah
- 551 (both think silently)

552 58:18 Michael Do you see anything why where's the other one where's the one with
two eight and four, oh, I got it, why does it go from two to four to
eight you know
553 (both think silently)
554 59:02 Robert I have no idea
555 Michael I am trying to think too
556 (Both look at the paper silently)
557 Michael I am not seeing anything
558 (look at the paper silently)
559 Michael Hey get her in, I want to tell her about you tell her about how it
doubles, (to an observer) can you just ask her We'll see how that
works with her
560 (R1 enters)
561 1:00:3 Robert With this has something to do with like this, like you put the zero in
9 front, the outcome, I don't know it's pretty simple it's like number of
spaces that this moves over. Like don't pay attention to the top ones,
it's three multiply by two
562 Michael I am trying to think about, explain how it doubles, why it doubles last
number
563 R1 Ok.
564 Michael Alright, I think I just forgot it, no, with the last two numbers um, yeah
it doubles from twenty to forty I was thinking like when you have six
and if you didn't win at six what you're gonna have is three and three
you have three wins and three losses
565 R1 Gotcha
566 Michael Whichever way they are, that's how it's gonna be oh, what's three
choose six
567 R1 Six choose three
568 Michael Uh, six choose three you would have, I am just thinking about it now
569 Robert So it's one twenty right
570 Michael That's twenty right, there's your like you different possibilities that
you can have those three aight, so twenty losing not yet winning
possibilities
571 R1 Ties
572 1:02:1 Michael Ties, yeah, twenty ties and when you go on with the game, it could
0 either be a win for one team a win for other if you know, that's why it
be like
573 R1 The tie breaker
574 Michael A tie breaker usually they have twenty different ways that A would
win twenty different ways that B would win, and I get from that six
choose three and six choose four are the same number... Ah, not the
same numbers, six choose three is the same number that we come up
with here, that's probably why, it's why the number are like the
probability or whatever of you winning in six is same of being in a tie
in six and when you know the tie you go in another game just doubles
575 R1 What do you think Robert?

576 Robert Yeah, that makes sense
 577 Michael For this one why it doubles
 578 R1 Yeah, that's the best explanation I have heard so far. Gina?
 579 R2 Yeah
 580 R1 Don't you think it was a good explanation John? Haven't hear
 anything better than that from anywhere else
 581 Michael I don't even know what I said
 582 R1 You should watch the tape, however, however, that was really nice
 but what do you think? What do you believe?
 583 Michael That I guess, I kinda believe there's an actual explanation why the
 numbers are the same
 584 Robert Yeah, that's I didn't think that one though
 585 Michael I was trying to think that one
 586 R1 Ok, now, here I don't mean to give you hard time but I am going to
 invite you to come to some classes next year here, really you gonna
 have some perspective teachers who have thought longer and deeper
 about the problem as you have are gonna give you the alternative
 answer What would you do to say to them that either one is OK
 587 Michael The one, the one that we were talking about before, that answer?
 588 R1 No, say you believe this answer
 589 Michael No, I don't believe this one
 590 R1 Oh, you don't believe this one
 591 Michael This one, I changed this one around to make it into the ultimate
 answer that we
 592 R1 So, explain to me what you believe. Can you recapitulate? Bobby you
 can speak for yourself if you don't believe me, you can say that too
 593 Robert This one
 594 R1 Tell me what you think, let you go first
 595 Robert I think this one, cause with the explanation with the tie breaker and
 with the you have twenty one way and twenty the other,
 596 R1 Hmm-hmm
 597 Robert It's the same thing
 598 1:04:3 R1 So, if I asked you, what is the probability, that the world series can be
 5 won in exactly four or exactly five or exactly six or exactly seven
 games, right?
 599 Michael I would say twelve point five, twenty five percent, thirty one point
 two
 600 R1 Ok, and you would justify that this intuitively makes sense because
 why? These numbers being different
 601 Michael Uh, why these numbers are all different?
 602 R1 Hmm-hmm
 603 Michael Who do you want to answer it?
 604 R1 Either one Robert, let's hear from Robert since you can always add on
 since he's looked at this for the first time,
 605 Robert I don't know, it just looks like it'd make sense
 606 R1 I maybe I should ask my question so that you can give an answer

- 607 Robert more specifically Could you explain where the numbers came from?
Uh we took the, didn't we take the total number uh alright, we took the one half to this number and put this as then we had the denominator right? Like one was like this one, sixteenth, and then we do two over sixteen, so we got this and here we do one thirty two and put eight over it, we get one fourth and then, there is next sixty four, and then we got that, and then one twenty and fourteen got that
- 608 R1 Ok, and you could explain to me where the two, eight, twenty and forty came from
- 609 Michael the two, eight, twenty, forty came from just the winning in four like but that two is a part of another group how many can have four combinations you know
- 610 R1 Hmm-hmm
- 611 Michael That's sixteen so, two in those sixteen and the probability of those two sixteen coming out of sixteen is that twelve point five like that's the explanation I would give
- 612 1:06:2 R1 Ok, Robert, you with that?
9
- 613 Robert Yeah
- 614 R1 You gave me an earlier explanation is that... when we talked the last time before you came here, one of the interviews you said that at the time you believed that either one could be right Is your feeling that either answer could be right, the same, Robert wasn't (inaudible) or would you feel that one is more right or one you feel better with if you had to pick now I we have something at stake here like a million dollars what would you do, million dollars is at stake here, cause I think money was the motivation for you, you were telling me this summer, what would you pick
- 615 Michael I had to pick I would go with this one like that
- 616 R1 Ok, now you're picking are you feeling pretty confident
- 617 Michael I do feel pretty good cause earlier I didn't understand it too well back then I guess now
- 618 R1 Ok
- 619 Michael I guess he doesn't cause he just started but I had the time before to think about it now
- 620 R1 Hmm-hmm so, let me ask you then another question so now suppose uh you had some folks coming with the other solution and you two came with this and now you had to not only present yours you had it for the judge here talk about the fallacy in the other one what was wrong with the other one, I mean both can't be right, they're just different answers, one of them is correct
- 621 1:07:4 Michael But I think with, how I came up with this it's not the kind of way we
9 did before
- 622 R1 What then what was wrong with it the other day?
- 623 Michael I mean, no, I am not talking about before like a couple of days before, I am talking about like a couple of months ago

624 R1 I understand that
 625 Michael Like before when we did this, we came up with the answer maybe the
 same numbers, I am not sure
 626 R1 Oh, Yeah, you did, then you changed your mind
 627 Michael Yeah, but
 628 R1 Cause I showed you what one of my students did and you thought my
 students were smarter than you
 629 Michael But like the numbers
 630 R1 Robert did you do that once
 631 Robert Yeah, with that door thing, the guy told me something
 632 R1 And you believed him?
 633 Robert Yeah, also
 634 R1 Is there a lesson here?
 635 Robert Yeah, and then the next week he came he said he thought he was
 wrong and I was still believing him
 636 R1 Is there a lesson here
 637 Robert Yeah
 638 1:08:3 R1 What's the lesson I want to get, what's the lesson here?
 3
 639 Robert Stick with your answer, don't allow
 640 Michael I figure that whatever the person told you like he's not sure either
 641 R1 Even the teachers have been known to do some things wrong you
 have to have a real good argument your numbers have to have
 meaning, not saying not to believe someone else
 642 Robert He had a pretty good argument
 643 R1 He had a pretty good argument that was Richard, wasn't it
 644 Robert I and yeah the next week he came in, he said, uh-no, I was wrong you
 need he told us what we were thinking before cause I felt kind of
 stupid cause
 645 R1 This stuff is pretty elusive isn't it probability is tough isn't it mean,
 once you start going into these formulas and don't really reason from
 the data, you can get really but I still am wondering, if you were able
 now able to talk to these students these graduate students and tell them
 why you think the other solution is wrong
 646 Michael Alright, cause you figure I came up with these numbers are the same
 and they came up with it, the wrong, I guess I don't know if it's wrong
 or right, but you call it the wrong solution
 647 R1 The seventy
 648 Michael Yeah, the seventy I took those numbers and You know I kinda
 converted to the other ones, so I guess it would be easier for me to
 show, instead of me coming up with how we did before, like starting
 from a different way to do it cause I started the same way they did and
 I guess I could take them further and show them how that connects
 with that, the other one in a way just by doing, I just retraced my
 steps and hopefully they would probably pick it up cause to me it
 sounds pretty simple it's not something that complicated

649 R1 Once you understand it
 650 Michael Yeah, once you understand it
 651 R1 Cause we're gonna hopefully Robert will visit one of our courses next semester and he's gonna see these students working on these problems right, Robert would be quiet, and tell them how to do it, but at some point when they think they are right, he doesn't he's gonna have to what to say to them, so we have to think about it
 652 Michael Can't we like just sit down and watch we have to actually interact?
 653 R1 Well some point, when the debate starts
 654 Michael I just want to sit there and watch
 655 R1 In the beginning you can watch, in the beginning you can do that
 656 Robert We can be like the students and just take notes
 657 R1 Well, you could do that too but I guess
 658 (Video calibration shows up)
 659 R1 Let's spend a couple of minutes with the points problem, that's the one we really came to give you and I don't think this one was done by, put your names on here and number them both
 660 Michael Don't remember. This is yours
 661 R1 Best as you can
 662 R1 Ok, so we'll leave you alone for bit and put this over here.
 663 (R1 leaves the room, Bobby and Michael read the points problem)
 664 Michael I remember this one and I remember I did it, I looked it up and I think it was right, so think we can do this
 665 Robert Cheated
 666 Michael But I left the work at home
 667 1:12:1 Robert Who's Fermat?
 3
 668 Michael Two mathematicians
 669 Robert Oh Were they alive at the same time?
 670 Michael I don't know... I don't think so... like could be
 671 Robert You need paper?
 672 Michael No
 673 Robert Do they get the money after each turn or just after the first one
 674 1:12:3 Michael No the thing is that, alright, they're playing this game one has eight points, one has seven and, one has eight, one has seven and they for some reason they had to stop at eight and seven and they need to know how to, no one won, so they have to split the money but they want to split it in an equal way you know,
 7
 675 Robert I think it'd be easier if they just flipped the coin one more time than doing all the math
 676 Michael That's not the point that would be any one else... that's what I would do but we have to figure how to equally divide the money you know, not cheating him like
 677 Robert Yeah
 678 Michael You're not gonna get from math everything cause he hasn't won yet I mean

679 Robert It's like sixty forty, something like
 680 Michael Something like, I don't know, it could be and Pascal, just cause he's
 losing doesn't mean he can't win so you gotta find out that if both
 have the probability of winning and equals to one, so you can divide
 the money using that probability or whatever
 681 Robert Alright, ...So, we just have to figure out, what chance of him getting
 two in a row and then like there's five flips left, right
 682 Michael No, there's
 683 Robert The maximum
 684 Michael The maximum could be five right
 685 Robert Yeah so you have to include that right
 686 Michael No, actually I think the maximum could be four, no
 687 Robert No, five cause yeah, it is four
 688 Michael Has to be four, cause you can't like win
 689 Robert Yeah
 690 Michael And the minimum is two and it's kinda like the world series problem
 691 Robert Yeah
 692 Michael I think it's just exactly like it
 693 (Agrees)
 694 Michael So, probably have to do the same thing
 695 Robert Alright, so Fermat would have to win two or more
 696 Michael So, I guess its two to four, it could be two three or four flips
 697 Robert Yeah and Pascal can't win in two and four, he can only win three and
 four and Fermat could win two, three or four
 698 Michael So, possibility for two flips can only be the next two are like that
 699 Robert Yeah, it could be
 700 Michael Can it
 701 Robert One for each and then
 702 Michael Hmm hmm And for three flips it could be
 703 Robert No for two flips
 704 Michael Pascal's
 705 Robert Pascal could win
 706 Michael Yeah, I am saying this is Fermat, Fermat wins this one, wins that one
 too and then the other guy wins that, and there's your three
 707 1:15:1 Robert Yeah, zero one zero one
 7
 708 Michael And then
 709 Robert And then it could be
 710 Michael Four flips could be um, no three not four
 711 Robert Is that right? No he's gotta... no, it's right
 712 Michael Yeah but Fermat can, you got that already
 713 Robert Yeah... It's gonna be sixty forty right? Cause it's six chances for
 Fermat to win and four for Pascal
 714 (Michael writes on paper)
 715 Robert I think that's all
 716 Michael I want to write it down

717 Robert No
 718 Michael It's kind of coming back to the world series one what's the different
 chances of coming out? Um, I don't like that
 719 Robert Oh yes, you can't take the whole thing like say, what' the chance of
 720 Michael Nah, you can't do that that's what we did in the world series one
 721 Robert Yeah. That seemed like an easy way to do it
 722 Michael I know what's that?
 723 Robert That's one zero,
 724 Michael Zero one
 725 Robert There's one zero but I actually drew it between the line
 726 Michael Hmm-hmm
 727 1:16:4 Robert Let's see for the two, it would be four flips so there is a one fourth
 5 chance of him getting that, alright,
 728 Michael Hmm hmm
 729 Robert And then it would be eight flips, so one fourth chance of him getting
 that too, so one eight for him winning
 730 Michael Alright
 731 Robert Is that right? And then it's three sixteenth for him winning the last
 one, three sixteenth for Pascal winning, now let's get count down,
 two, sixteen
 732 Michael Just add them together
 733 Robert Sixteen, and then six eight, eight sixteen
 734 Michael It all equals eight?
 735 Robert No, it all equals sixteen
 736 Michael Does it?
 737 Robert Yeah. Six, eight, twelve, sixteen
 738 (both look at the paper)
 739 Michael So, Fermat has a
 740 Robert So add up all Fermat's chances
 741 Michael Same chance of winning he's got a
 742 Robert Eleven sixteen
 743 Michael Another four sixteen chance and
 744 Robert A three sixteen
 745 Michael And a three sixteen
 746 Robert It's eleven sixteenth and another one is five sixteenth
 747 Michael The other one is five?
 748 Robert So,
 749 Michael That's what we came up with before, another way, a different way of
 doing that remember eleven five
 750 Robert So, he gets oh wait so, Ferm, Fermat guy gets I guess sixty nine and
 other one gets thirty one, it's like sixty eight point seven five to be
 exact
 751 Michael That one is pretty good
 752 1:18:2 Robert Is that what you got before?
 4
 753 Michael I think so eleven and five I remember that and we had, ours was a lot

longer

754 Robert So, wait

755 Michael So, I guess it's like a short cut

756 Robert Fermat, sixty nine

757 Michael Want to get her, this time you explain cause I don't know just what you said to me

758 Robert Alright

759 Michael Yup

760 (Both wait for R1 to come back)

761 Robert We always think we are right but we always end up being wrong

762 Michael That's why we're getting here

763 Robert Where's toll house

764 Michael Hmm?

765 Robert Where's toll house? I guess it's in France but I don't know where

766 Michael No, Toulouse

767 (R1 comes back)

768 R1 OK

769 Robert You want me to just explain

770 R1 Yeah, yeah

771 Robert Alright, we figured that there is going to be a maximum of four flips left you know

772 R1 Hmm-hmm

773 Robert And, um

774 1:20:0 Michael Minimum of two

1

775 Robert Yeah, minimum of two till someone wins

776 R1 How did you figure that out?

777 Robert Uh, alright, we had two to this one, we had both to nine one plus two is three and if it nine, nine there's gonna be one more so, that's how you get the four because this... like first if this thought was five was just like add these two up and wait till I get to twenty but then I realize what if one person wins then you won't be like you can't flip again when someone wins Michael told me that and then just came out four

778 Michael I mean the least could be, uh, Fermat getting two and winning and the most could be Fermat getting like one and Pascal getting the other three. Pascal can't get any more than three

779 R1 And how did

780 Michael And can't get more than one

781 1:20:5 R1 Is that how you represented it?

8

782 Robert Uh, yeah, so we figured like since there's a four there'd be one sixteenth chance of everything happening cause one half times one half times one half times one half so then, we did two, and the only way that someone could win in two this for Fermat to lose one to get two in a row

783 Michael It's one fourth

- 784 Robert Yeah, and we put all on one sixteenth cause common denominator, cause we were trying to add them up, and then for the game to end in three flips Fermat has to get either loose the first one and then win two or you could win one lose one then win one and for the game to end in four, you'd have to take all these Fermat could win then lose twice and then win or lose, win, lose win, lose twice and win twice and for Pascal to win, he has to win once or Fermat wins once and loses three times and he wins once and Fermat wins he wins twice, he wins twice then Fermat wins and then he wins, and up there could also win in three if he gets three in a row, so then we do all of this, it turns out one-fourth of this happening cause one half times one half is one fourth and this is over one eight and turn out to be one fourth to him and one eight for Pascal winning and for this one it would be three over sixteenth for Fermat winning and three over sixteen for Pascal winning and then we all put them all in the sixteen denominator and add them up and we get the chances for Fermat was eleven sixteen it um four sixteen, four sixteen, three sixteen, eleven sixteen and Pascal was five sixteen, plus two sixteen plus two sixteen is five sixteen and I put it in a calculator we got like that equals point six eight seven five and this equals thirty one point two five or thirty one two five over thousandths or whatever and we rounded up this and rounded down that so, figured out that Fermat he gets sixty nine francs and Pascal gets thirty one
- 785 R2 Hmm
- 786 R1 Gina?
- 787 R2 It's really interesting, I was looking at your representation and you explained it really well for me, he explained it really well So,
- 788 Robert And we rounded up because, can't split up a franc so, I guess he should get more if he's higher up
- 789 R2 Unless he wants to be generous to the poor guy that lost, give him a break
- 790 Robert He probably got a lot of money because he was good at math
- 791 R1 You have worked on this problem before Michael?
- 792 Michael (Nods)
- 793 RR1 So do you remember what happened last time?
- 794 1:24:0 Michael The last time was uh, longer but I think we came down to the same
4 eleven and five. Eleven possibilities that you know Fermat would win it and five that Pascal would win, I remember those numbers very distinctively and so I am so, I think that we spent like an hour on this problem or something and we came like with that and we spent ten minutes and you now, I think that it's correct
- 795 R2 Why do you think it only took you ten minutes this time?
- 796 Michael Um, because I guess I saw it like the world series problem kind of the same so I went that way, I was do it like this, the same way
- 797 R1 The last time you did the world series problem a different way
- 798 Michael I know, I am just saying, what we did right now

799 R1 Ok, so you modeled it on the world series, you think the problems are similar in that way?

800 Michael Yeah

801 R1 Can you say something about it

802 Robert In the world series you need to get the four games to win and this you need to get the two games to win and

803 Michael But, it's different cause one needs two and one needs three

804 Robert Yeah, and the max was seven in the world series and max was here a four and we knew by the fourth one that someone had to win and that one by the seventh one you knew someone had to win

805 R2 Interesting

806 R1 Do you have any questions, Jim?

807 Jim No

808 R1 Lynn. Do you think you could write up these solutions for us?

809 Michael I think Bob can write them up

810 R1 Bob and Michael, I mean

811 Robert I'll write this one

812 R1 Ok

813 Michael No, I will take this one

814 R2 There's one other sheet in there, you might just want to look at it for a second, there was an extension that we thought would be real easy for you

815 R1 Yeah, we would really like you to write it up cause make you famous, make another copy of this Ok, so we will take couple of another minutes and Ron scheduled so, we will wait for a few other minutes and... why don't you write your names on here and so we don't lose any numbers you have a piece of paper

816 Robert Should I put this on this paper three or one? Cause it's the first one

817 R1 Three

818 R2 Ok, thanks

819 (hand paper to R1, R1 leaves)

820 Robert This should be like the same thing right?

821 Michael Yeah

822 Robert Alright, there's how many three there'd still be four flips right?

823 Michael Hmm?

824 Robert The maximum number of flips would still be four, right?

825 Michael How many did... no can't... most now you can have is just one right it has to be four again? Cause you could win it in one or four, you can't win

826 Robert This is no, no, no

827 Michael No. You only got one flip, it's gonna be like this I got that one, when you got two flips, that's right, wait, what's his name could win in four too

828 Robert Yeah

829 Michael That's it, right?

830 Robert Yeah, I guess

831 Michael That's all
832 Robert One fourth, one eighth, one sixteenth and this will one sixteenth so
fifteen, sixteen, Fermat's sixteen, so you want to figure out
833 Michael (Work on calculator)
834 Michael How much is the money again?
835 Robert Hundred, ninety four francs
836 Robert writes on paper
837 Michael I don't know that one is that different
838 Robert What's the probability that number would be ninety six, that's five
flips later, that's five
839 Michael Um huh
840 Robert That's ninety six, so just do it
841 Michael It could be
842 Robert What is two to the fifth? Sixteen? No, it's thirty two right
843 Michael Hmm?
844 Robert Two to the fifth is sixteen, no thirty two
845 Michael I think this is pretty easy
846 Robert You just do it over thirty two right
847 Michael There are five basic ways to do it
848 Robert Five over thirty two
849 Michael Sixteen, about sixteen (Works on calculator)
850 Robert You think that works, just put it over thirty two
851 Michael (Agrees), cause you are looking for those five combinations, out of
thirty two, and um only five of them came up that would equal five
that would equal six that's your probability, it's time to knock on the
door
852 (write name on paper)
853 Robert What if you had to have an MA person, someone else was named MA
854 Michael Good for him. They'll figure it out. You wanna go?
855 Robert Uh, I guess, if you want, I'll explain the first one and you explain the
second one cause I did the first one and you did the second one
856 Robert Actually we could do this one easier
857 Michael Hmm?
858 Robert We could do this one easier there's only one way if he spends four
gets one and you just minus one from this
859 (R1 comes back)
860 R2 What was that Robert?
861 Robert I am going to explain the first one and he's going to explain the
second one Alright, I figured there's only one way that Pascal could
win that's if he gets four zeros, four in a row, so we just did that,
there's one sixteenth chance of that happening and to figure out
Fermat's, we did one minus one sixteenth, fifteen sixteenth and that
turns out be ninety four francs and it turns out to be six for him and
wrote them all out, to double check this is going to be one half, one
fourth, one sixteenth which is fifteen sixteenth
862 R1 Getting pretty fancy here

863 Robert Yeah
 864 R2 And fast too
 865 R1 Yes, so, why could you just subtract it from one? To get the fifteen sixteenth?
 866 Robert Uh, we figured, we actually we didn't do it that way, I just thought of it when you were coming in that like one will be the total number of choices and that if you flip four times, it would be sixteen possibilities, add that one and then like add those like if you just flip four times, Fermat could have won fifteen times even if you kept just flipping, even if he won fifteen
 867 Michael Wait, what's his name has only one out of sixteen chance of winning
 868 R1 Pascal
 869 1:32:0 Michael Pascal, so somebody has a fifteen out of sixteen chance yeah,
 6 someone's gotta have it
 870 Robert Yeah and you could actually like, you are flipping four heads in a row or whatever, that's what it is cause he's gotta do
 871 R2 Pascal, it says here Pascal is winning nine to six in the first scenario who's winning
 872 Robert Oh, I had to switch this, my fault
 873 R2 Ok
 874 Michael Switch the names
 875 Robert Yeah, I thought he was still winning
 876 R2 So, the guy that was losing
 877 Robert Is Fermat
 878 R2 Is Fermat and he's only got one way?
 879 Robert To win to get the
 880 R2 Whatever it was
 881 Robert Head, in a row and the only chance to get that was one sixteenth so then just all the other chances go to Pascal
 882 R2 I see,
 883 R1 Yeah, I guess so John, Elena, Lynda?
 884 R2 Look how fast you are getting at this one
 885 R1 What's the last one, you want to tell us Michael?
 886 Michael Yeah, five and five, the only way to get nine six, Pascal has to get four and Fermat has to get one, and there's only five different ways of doing that
 887 R1 How did you get that five
 888 Michael Just by writing it and it's like five chose four is also that uh so, I mean you got five different chances and out of five how many total possibilities with five is thirty two, five out of thirty two is what we have
 889 Robert It's about sixteen percent
 890 Michael Yeah sixteen percent for it to actually happen.
 891 R1 Ok. Do you ever think about Pascal or Fermat
 892 Michael Hmm?
 893 R1 Know anything about these people? Pascal,

894 Michael I think they are mathematicians
 895 Robert The lived in France
 896 R1 (chuckles) you know you might want to, you both are computer
 people, right?
 897 (Agree)
 898 R1 You might wanna look up some history about them they're pretty
 famous mathematicians
 899 Robert Were they alive during the same time
 900 R1 I don't know
 901 R2 Yes
 902 R1 Close. Pascal and Fermat have done a lot of work in Number theory, a
 lot of the earlier ideas, you know the Pascal's triangle
 903 Robert Yeah
 904 R1 You have been playing with it and using relationships way back
 historically and there are there are theorems named after these people
 and all there's Fermat's last theorem, this famous stuff about them and
 there's stuff you are working on were really major problems at the
 time You know these were adult mathematicians working on the same
 kinds of problems you are working on now. They put a lot more into
 younger years at school because there was so much more to work, you
 know the little math that has been developed since then
 905 1:35:1 I am watching TV and they say you need calculus now for like most
 8 like jobs now require like a math degree, it was on a commercial for
 some college
 906 (chat more about Pascal and Fermat)

Appendix G

Date of Session: 07-07-1999

Tape Name: KW-RobMik

Author: Anoop Ahluwalia

Verified by : Kiranjeet K. Sran

Date of transcription: 5-26-2010

Researchers:

R1: Regina

R2: Carolyn Maher

R3: Lynn Tarlow

R4: Unknown

Time	Speaker	Transcript
540	Michael	I am gonna write it on the board.
541	Robert	Bring the paper or no? X exclamation n... you should put each thing equals.
542	Michael	Ummm...do you know what that's for? Do you remember what that's for?
543	Robert	Um yeah.
544	Michael	What was it for? Remember the choose thing? You remember that?
545	Robert	Yeah
546	Michael	How does it go... X choose n or n choose X?
547	Robert	Uh...X choose n
548	Michael	Alright it was like that um
549	Robert	All right X's...Do you just want to draw all the stuff on the board first?
550	Michael	Do you remember that? The X goes there... That's how you right the choose thing, right?
551	Robert	Yeah... You want me to finish?
552	Michael	What... Alright we'll let her talk. I'm gonna erase it.
553		(indistinct group conversation in the back)
554 1:35	Michael	Alright, You remember that one, right?
555	Student	Yes
556	Michael	What does that mean? I just wanna make sure you know... Yeah, you choose... Like this is the amount of yeah...and that's how many groups of, n groups, you can make five choose two, out of five things how many groups of two can you choose, I don't remember what that is
557	Robert	10
558	Michael	Like 10 or something like that, uhh... you explain it cause you
559	Robert	Alright, we were getting the problem like that we have five people and we gotta make groups of two and we want to know how many groups of two we can make without writing all the combinations. I first started off by writing just all the combinations to see and we got like ten and it took like quite a while but we just wanted a faster way in case like we have like a hundred people choose ten, or something like that, so then we figured out that we're gonna take all the groups, you want me to draw? you want to draw? Alright we had like... for the first spot we have five people named A, B, C, D, E and we had like the first spot we had like all five people to choose from cause like no one was put in a

spot yet and then let's say we put A in the spot then so there'd will be only four people left for the second spot so it'd be four here and then if we chose B then there'd would be only three left for this one and two for this one and then one for this one and then if we multiply all these, you get hundred and twenty. That's how many like total combinations to arrange five people in like any spots. That's OK but that can be written like five factorial two. And then, That's it but if you only had like these two spots and like these three spots didn't exist, there is no way to write just these two spots multiplied So, instead we divide, decide to divide by two, one, the three we didn't need or three factorial and that gives us six and then like...cause this what this does is crosses these out, this out this out, and then you get the two that you wanted, the five times four and so then you divide this two you get twenty. But the answer we got when we did this was ten and so we wanted to figure out how come this was different this and we figured there's going to be like duplicates in this the order doesn't matter if like if I am in the first spot and he is in, mike is in the second spot of if mike is in the first spot and I am in the second spot it's still like the same thing. So we decide to divide by the number of these factorial, which is 2 and then we get ten. And then like because this is like, that takes into consideration like both of them switching if it were three spots, it would be three factorial in the bottom and then we just came up with this (The combinations formula) and then like at the same time we were doing also with Pascal's Triangle thing. And then, so like this (draws first three rows of Pascal's triangle) and there is cause there is zero row, one two and three then zero one two three and we were just saying like that if had three choose two like and were looking at this and we went down to three and over two and it was three and when we did this thing, we got three also, so like found out that if you go down to third row and over two, and then you get the answer so there is like three ways to solve these problems, you can either use this, the triangle or you can just write all the combinations that's really all we found out... That's it

560	5:41	Michael	That's very good
561		Robert	Yeah Sure
562		R2	Michael, how can you find out if they are getting it?
563		Michael	Ask them? I mean...
564		R2	you don't believe it
565		Michael	I don't think they get it, but...
566		Student	(Inaudible) we over it in class
567		Michael	You understand why we cancel out these three?
568		Student	Yeah
569	6:09	Michael	Because we are not worried about where these three are so you want to get rid of all the combinations where the three...those three are all mixed up and you just basically left with the combinations that he's doing. You just want them as a group so you want like five and four,

- you don't want you know four then five and you just want, you don't care which one is before it. So these two, they will switch spots as he said this one could be there, and he can be there or it could be switched, that's why so you want to get rid of how many combinations that they are that they you know do themselves and that is two factorial, you want to divide that to get rid of duplicates
- 570 Robert Yeah, duplicates
- 571 Michael We went over it in math class and that's why they understand it
- 572 Robert And then like if this was just like five choose three, then you just divide by three factorial because you may have like a lot of duplicates between those three and it's the same thing and instead of dividing by that you just divide by two factorial... Only this like x is the number of people in this condition then n would be number of groups... and then that's it
- 573 Michael Any questions?
- 574 R1 Could you explain to me, I am not quite sure I understand why... what five choose three, how would work out the same way. Could you explain that to me please?
- 575 7:41 Michael Well, basically, five choose two we write out five factorial over two, n , two factorial and five minus two is three so three factorial. And five choose three, three would be in that spot and five minus three is two so they're basically the same thing. I mean did you ever get two numbers that both equal up I mean, what am I trying to say, like if you get five minus two, and then there would always be duplicates, that's why there are duplicates in the triangle, you got a three and three and you will have the one, four, six, four, you'll have the four and four because where the difference in one will be taking off in the next one like, you are taking two off here you minusing, you take off two here but in the other one, umm, the other one, you'll have the three will be at here and two will be ... you know I am saying it's just like switching them I guess
- 576 R1 I understand the mathematics, now, I understand that you are saying that and I believe I remember you saying that you can see that the math works in terms of filling in the numbers in this case you know, substituting the numbers and I see how that works there but I am wondering in the first example that Robert came up with, with the five, four, three, two, one in circles, and then below that you divided by three, two, one, you didn't need them so how does that example work? I found that to be very helpful to me, to see how you started to get at this formula. How does that example work with five choose three as oppose to five choose two?
- 577 9:27 Robert Yeah, like say like you just instead of putting this in the bottom, this wouldn't be here and this one would be crossed out and then these two will just be crossed out and then on the top you're just left with five times four times three and so like, cause you just want three spots, not five spots, so want to get rid of two, and that's how you get rid of two

of them, like just, so this isn't here and then choose these three and then just divide by this to get by this and then like instead of dividing by two factorial this time you are gonna divide by three factorial because like you always divide finally by how many like spots you want, that's the n here

578 R1 I guess this is what I don't understand. How do you know why you always divide by that? Why do you divide by that? I understand you have three spots, but

579 Robert I don't know, we just did it, we just got rid of the duplicates, so, it seemed to work, so that's what we did

580 R2 Can I ask another one Robert?

581 Robert OK

582 R2 You started off with the example, you said like by forming committees, you were sitting at the table before remember that? You were explaining this to Suzanne, you were explaining this to me? I am curious about with the committee example, with the committees of two, committees of three, why would they come out to be the same answer? If you go to the next row, why, think of those committees, why does one's committee always turn out to be the same, you know practically, how does that happen? You understand my question? I am asking this of everybody, not just of you and Michael. Have you thought about that?

583 11:00 Robert No, I don't know. No, I think that's kind of co-incidence, you know what I'm saying?

584 R2 You think it's a coincidence Michael?

585 Michael Yeah

586 R2 You think it's a coincidence. That's not a reason you know, you think I should just say it's a coincidence and that's the way it is? Did you hear my question? Jeff, Romina do you understand my question?

587 Michael Anyone have an idea? Give us a hint

588 R2 They didn't hear my question, can someone repeat my question so that Jeff and Romina can hear it

589 Michael Why is that five choose two is the same as five choose three, why would it...

590 Robert It's the same amount of total possibilities and you are like dividing by the same amount

591 Michael Not mathematically, you know

592 Robert Yeah

593 Michael I don't know...

594 R2 Why does that turn out that choosing out of five people a committee of two turns out to be the same number of committees as choosing out of five people a committee of three? Let's go back to the six people, right, where does, where does the same answer come up?

595 Michael It kind of won't because, (students talking in the back ground) it doesn't, because

596 12:15 R2 I'm very confused.

- 597 R2 Does six choose four give you the same answer as six choose two? Mathematically, you could show that. By Robert's and Michael's explanation? Why does that work?
- 598 Romina You remember in towers, you have like three of one color and two of an another one, like you could split them up It's almost like it's the opposite, it's like building up the other color is the same, you know what I mean
- 599 R2 Interesting, I think Magda?
- 600 Magda If you have a group of two, you are choosing, like another group of three, and you have a group of three... (inaudible) but when you choose a group of three, you are left with a group of two, you are left with the same thing
- 601 13:09 R2 Oh, you know what Magda is saying? Magda is saying something different than than Romina. They both said something interesting. They are the same thing? One more time then. Someone say it one more time. Help me, that are they the same thing.
- 602 Robert No one was paying attention
- 603 Magda Ok, if we have five choose two, we have a group of two, then you are left with a group of three right and then you have five choose three, you have a group of three, but you also have a group of two. So, that's the same thing, a group of three a group of two or group of two and group of three
- 604 R2 But you didn't say that (to Romina), you talked about towers
- 605 Romina I am just trying to show you how I did (inaudible) like I was just trying something...
- 606 R2 You are trying to get to my level
- 607 Romina Yeah (everybody laughs)
- 608 R2 Thank you Romina, that really helped me. Anyone else have any questions? Does that make sense Robert? Michael?
- 609 Robert Yeah
- 610 Michael Yeah
- 611 Group claps
- 612 R2 Now you are going to get the questioning group. Go ahead Lyn, go for it. But you are asking everybody as everybody is supposed to understand this. So, but Robert and Michael will help you, but don't feel you are on the spot guys Go ahead
- 613 Robert Oh, man
- 614 R3 Well, I actually have two questions. First question, thinking about pascal's triangle, you said (inaudible) the combinations on the board was ten, and umm..you didn't show me where it is on the triangle so (inaudible) Try to ask my question, slightly to see...
- 615 Michael Mind if I erase some of this stuff (Michael erases a part of work on the board and writes first five rows of the Pascal's triangle, Robert watches his work, points out there's two tens)
- 616 15: 36 Michael Well, um, alright, well the ten, the five choose two is right here, cause um each of these things are something choose something, these little

spots right here, the top would be guess zero choose zero, doesn't make sense, cause you know...but that's what it is right?...umm.. the first one will be one choose zero and one choose one, alright, cause...that doesn't make sense either but there's only combination you can have here, one thing and you are choosing one thing, and that's right there... all of these don't make sense cause you can't have zero groups, let's just stick a one there, alright? you got two choose one and there is two different, you have one thing to choose from, from two different things, Ok, you can choose this one or that one, you got, there's your two...and if you have let's say two people to choose from and I have to choose two, I could only take one group so that's only one, the next one will be you have three different things, If you choose nothing, there is only one thing you can get which is nothing, second one if you choose one, you only take one of each, that's three, this one you could take, if you wanna have two, two different, two things to choose from out of three, you can do three different combinations, and if you want to take three, if you want to take all three, and that's how you get... and that's how it works for each one. when we get down to here the fifth row, two over, is your five choose two, you got your five things, your five bunches, markers, five different color markers, two groups, want to take two markers I could take ten different combinations of colors or markers what you want to call them, I tell them the choose things...

- 617 17:30 R3 Well, what I was wondering about here is when you do that triangle, you did it pretty quickly so it looks like you did it some way other than moving over all those the combinations
- 618 Michael Yeah, I add the two numbers, you are going to ask why we add the two numbers, that's what you are going to ask, right (students laugh)
- 619 Michael I am basically questioning me
- 620 R3 Well, actually what I was going to ask you about was, Michael, wasn't to make you crazy here, but I was really kind of curious after you drew the triangle, you showed me how you mapped these combinations. I wondered if that is also coincidence? Coincidence – coincidence now is this like a cool trick you did?
- 621 Michael Yeah that is really...NO, if you rewrite instead of numbers, you chooses, you can see how the next row comes ups, what? I can do that, what...alright, so, rather than writing one on the top, we write what it really means, zero choose zero, you get that? Next one will be one choose zero and one choose one, both being one, ...you with me?
- 622 Robert Keep on going
- 623 Michael writes first five rows with combinations filled in
- 624 Michael Alright, that's what the kind of triangle looks like, that's how you can look at it, to make another row, alright, you have a, you gonna have like another thing to choose from, so, all the top ones are going to be six, right? If you have another thing to choose from, it is going to be six choose something, right...umm...first one is going to be zero, they

- are all going to be one, two, three, four, five, six, whatever, I don't know how you want me to...like why you add the numbers, I lost my self...Dude why would add the numbers (to Robert)?
- 625 20:10 Robert I don't know
- 626 Michael I think, I kinda, I did explain it in that thing, in that paper though, I don't think I explained the whole thing, want me to do that?
- 627 R2 Uh-huh (students laugh)
- 628 Michael talking to himself
- 629 Michael Umm..how should I start? Umm... think of choosing things as pizza toppings, alright
- 630 R2 She's done that problem
- 631 Michael You're familiar with it?
- 632 R3 Vaguely
- 633 R2 But, you might state it
- 634 Michael If you have numbers one and zero, 1 means you have the certain topping, zero means you didn't pick it, you didn't choose it, you have five, four, I will do four, four different toppings, let's say this mushroom, peppers, sausages and anchovies, what a nasty pizza (students laugh)...umm....you want a mushroom pizza, here's your mushroom pizza (writes 1000), you want a, what did I say peppers? (writes 0100) here's your pepp...you can express each pizza by these four, these four numbers... now when you add a topping, what do you want today? Extra cheese? I am joking about that...
- 635 Robert suggests Onions as new topping
- 636 Michael Onions, alright, you have an extra spot, so this pepper cheese, not pepper, this pepper pizza can either be a pepper pizza or it could be a pepper and onion pizza, so that, so each combination I guess would double, no, not double, the number of pizzas would double because each combination could turn into two different things, so when you write down... when you write this triangle, you have one choose zero, pizza with nothing on it, three choose one, that'd be you have three different pizzas that only have one topping. Three choose two, three different pizzas that have two toppings. So when you write the next row... if you add these numbers and it doubles the amount of this, you are doubling the amount of pizzas, the possible pizzas that you can make, so these three pizzas right here, three of them will get another topping which will bounce it up to the next category of having two toppings, and three of the them will not receive anything, which will, which will leave it in the one topping category. Right? These three right here won't get anything, well one of the one of the...the three of the six will not get a topping, so they will stay behind and still be in the two topping category, and three would come up to the three topping pizzas and same with this, one of these guys will stay a cheese pizza and one will have a, one will have a topping, I guess...that's why you would add them all like that. You understand why we add them?
- 637 23:40 R3 I just want to ask you one more time, what really intrigued me with

- this. Originally, when you showed the example of five choose two and came up with twenty, and you said, you had done, the answer will be ten and you divided by two and Robert explained that because Robert and Michael was same as Michael and Robert and that was the duplication. (Inaudible), in the next example, five choose three, you divided by three factorial, you showed that it worked out mathematically but I wondered if you could explain why that would work, you gave a nice example I understand why you divided by two in five choose two...
- 638 24:31 Michael The reason why we divide by two because we double the amount there's there's...there's two different combinations you can put me and Robert in it, in two spots, therefore, it would double every combination we had. Like if I am over here, it would be switched, now if there was three people, there's three different combinations, you can put them in...three and then that would
- 639 Robert Nah, six
- 640 Michael Six, my bad, six different combinations you can put them in and three spots, so it would be six times as much your, your end result, and that's why we would have to divide it, like these people in three different spots, each and every combination would have six duplicates, whatever you want to call it, that's why you would divide by six
- 641 R2 Robert?
- 642 Robert It's cool. Yeah
- 643 R4 I have a quick question also. If you had towers of four in two colors, how many different towers can you make?
- 644 Robert One, two, three, four, six, Six, right, isn't that same thing?
- 645 Michael Yeah, I guess so
- 646 Robert Yeah, It is six
- 647 Michael Two colors?
- 648 R4 Two different colors, towers are four high
- 649 Michael And maybe you could have three of one color and one of another right...
- 650 Robert That's even more
- 651 Michael That's a different problem um... It would be like...
- 652 26:14 Robert What's two to the fourth, Thirty two, thirty two,
- 653 Michael Yeah, something like that
- 654 R4 What did you say
- 655 Robert Two to the fourth,
- 656 R4 Yeah, two to the fourth is sixteen. Think that's the fourth row of Pascal's triangle. Yeah. Why is that? Is that a coincidence or...
- 657 Michael Because this one here is every single combination you could have and this is broken down to categories, ones with, that only have one red, or cause they're red and blue, only have one red, zero red, ...zero this one only has one red, there's four, there's four towers which only have one red in 'em... it's broken up into categories, that's how it works
- 658 R4 And six would be...

659 Robert Two red in them
660 Michael Two reds, four with three and it has to equal sixteen because this
 includes every single one, ... (to Robert) I messed up some where
661 Everybody claps

Appendix H

Date of Session: 04-26-1999

Author: Kathleen Dougherty

Verified by: Kristen Lew

Date of transcript: 7-1-2010

Line	Time	Speaker	Transcript
1.	0:00:30	Dr. Carolyn Maher	Tell me what you remember.
2.		Robert	It was 4 high and 3 colors and you had to have at least 1 of each color in the tower. So you can have if there's 2 colors, you have 3. One time.
3.		Dr. Maher	Are you remembering anything? (mumbling) Ok, so now you have 3 piles of cubes, right? Three different colors you can select from. Building towers 4 tall. And every tower must have 1 of the cubes in each pile, at least. That was the first one and you worked on that. And maybe you want to think about it again for a couple minutes together, right? And then you're sort of inventing new problems.
4.		Magda	Yeah, there was a lot.
5.		Dr. Maher	Well, you were doing some interesting inventions but the one that Robert and I talked about when Robert came was suppose you were now building them 3 tall. Selecting from 2 colors and every tower had to have at least 1 of the colors. Suppose you were doing that 5 tall. Now you'd be selecting from...
6.		All	4 colors.
7.		Dr. Maher	That was 1 variation we talked about. And everyone had to have at least 1 of those. And so some people were talking about the generalization of that and I'd like you to state the generalization. You know, write out what you think the generalization for that one is. Ok? For when we're building them. You decide how tall and how many colors, which would be an extension. Wanna think about that again a little bit?
8.		Amy	Yeah sure.
9.		Dr. Maher	And I do have work from last time (pulls out work) but after you've thought about it you can see. Fair enough? (Group: Mhmm.) I'll leave you alone for a bit. (She gets up from table.)
10.	0:02:34	All	(Grabs paper.) Mumbles.
11.		Magda	What are you gonna do? How 'bout (points at Robert) you do all the combinations for like, like, you do like 2 tall for like. No.
12.		Bobby	I'll do 3.
13.		Magda	3 tall will 2 colors. (Points at Amy.) You do 4 tall with 3 colors. (Points at Angela.) You do 5. And how bout we switch it around so I don't have the hardest one.
14.		Angela	(Giggles) So wait, what am I doing?
15.		Magda	She's doing 4 tall 3 colors. You're doing 5 with 4. Great, I'm doing 6

- tall.
16. 0:03:22 Amy There's a restriction where all of them have to have one color?
17. Angela Yeah, that's what the problem is, right?
18. Amy Yeah. Ok.
19. Angela Sorry, I didn't mean to sound snippy there.
20. ALL (Work on their lists.) (Mumbling amongst themselves.) (Sounds of Dr. Maher falling) (All students turn and look.)
21. Dr. I'm alright. A little stupid, but alright.
22. Maher
23. ALL (Go back to working
24. Angela (Bobby is finished.) Bobby, just do one on 7 tall, 6 colors. (Laughs.) (Others continue working.)
25. Bobby No I'm alright.
26. 0:05:22 Magda There must be an easier way to do this. (Laughs)
27. Angela There has to be, we have to figure it out.
28. Robert Mine was pretty easy.
29. Magda (inaudible) (Mumbles to herself trying to figure out her part of the problem.) Bobby, is there an easier way?
30. Robert No. Don't think so.
31. Angela There has to be. There must be.... (to Amy and Robert.) Work on a general statement.
32. Amy Generalized statement dot dot dot. Did I get them all? I don't think I did.
33. Robert No, I think you missed some.
34. 0:07:38 Amy I think I did too. My brain is fried.
35. Robert I think you missed some cause there's 6 with 2 reds. So there should be 6 with 2 blues and 6 with... So unless you find more with 2 reds.
36. Amy Cause I don't have any that are red, that are RR. Like the 2 (mumbles).
37. Robert What are RR (mumbles)?
38. Amy MmMm. I don't have any with 2 R's.
39. Robert Up top?
40. Amy MmMm.
41. Robert You should have 24.
42. Amy (Mumbles while trying to work on the problem.)
43. 0:10:13 Robert Oh, there's more?
44. Amy Yeah, cause you don't have the two in the middle.
45. Magda I think I have fifty? I took an easy way out so I'm hoping that's right.
46. Amy Mine's like 30.
47. Angela I can find out. 1, 2, 3....
48. Amy You see any I'm missing?
49. Robert No, not so far.
50. Angela No there's more. Did you- did you do like, you have to do like red blue red yellow green. Ugh...
51. Magda You forgot those?
- Angela Yeah. Cause you have to do the other colors like the ones that aren't double in the other orders.

52. Magda Hold on. This is what I did. I did, took 2 colors whites together then all the other colors. And then put the one and the bottom and went through the whole thing.
53. Angela Yeah.
54. 0:12:22 Magda And then I stayed with the whites but separated them by blue. And then like switched them around.
55. Angela Yeah but what about white white blue yellow red green?
56. Magda White white... (looks and tries to find that combination)
57. Angela You have to switch these down here too.
58. Magda You know what Angela...
59. Angela There's like 2,000 of them. Oh my gosh.
60. Magda I'm not liking this.
61. Angela Ok, now I'll take care of those.
62. Magda I'm trying to figure out everything (??).
So, hold on, what you're saying is that I have to switch these and then I have to switch these in here.
63. Angela Mmhm. Like see, ignore these. These are the other colors.
64. Robert (To Amy) Do you have all these for 2 reds? That's all of them.
65. Amy (Checks her list against Robert's) I forgot RRG. (??)
66. Robert So that adds 3 more.
67. Amy Yeah.
68. 0:15:28 Angela You have to do separated by 2 and then you have to do separated by 3.
69. Magda No because separated by 3 is just in here.
70. Angela No.
71. Magda Oh that's separated by 1. I'm giving up on this.
72. Angela Don't give up now.
73. Magda There's too many of them. Just think. We're gonna need to write out 300 different combinations. There's more than 300 because with this. Like if you just add 1 like that there's like 30 extra ones.
74. Amy & Robert (Start checking off his list against her list to see what's missing.)
75. Robert You've got it right there.
76. Amy Thanks.
77. Magda Maybe we can figure out a pattern for all 3.
78. Angela 3 times 8 is... 24. I think there's 255.
79. Magda Yeah well I think there's 500 and something of mine.
Did I switch all of them? No I didn't.
80. Robert (To Amy) B-R-B-G. And B-G-B-R. Then G-B-G-R. G-R-G-B. So there's...
81. Amy 64. (??)
82. Robert So if it holds up, that's gonna be 216 (referring to the one Angela is working on).
83. Amy Yeah. But she has 258.
84. Angela 255.
85. Amy 255?
86. Angela (Nods.)

87. Magda I don't know how many I have.
 88. Amy Just remember Maggie, you're the one that gave out this assignment.
 89. Angela Is he taking mine?
 90. Amy He took mine too.
 91. Angela Want to find 255 of them? Is that what you want to do?
 92. 0:19:11 Robert Ok. I'll just sit here and do nothing.
 93. Magda You can help me if you want.
 94. Angela Help Magda, she needs it.
 95. Amy Help Magda, she's confused.
 96. Magda There's too many of them
 97. Amy What colors are you using?
 98. Magda Me?
 99. Amy Yeah.
 100. Magda White blue red yellow green.
 101. Amy Which one you doing? 512. (??)
 102. Robert No 15 36. (??)
 103. Amy I know. (laughs)
 104. Angela Wait, there's more. I did 3 colors instead of 4. No I mean I multiplied
 by 3 instead of 4.
 There's 304.
 105. Magda Well I've got like 6,000 of mine.
 106. Angela Is there a pattern in the numbers?
 107. Amy Mine is the square of his. We were figuring yours should be 216 but...
 It's 6 and 36 but...
 108. Magda Alright, I'm giving up. How many different ways is there to arrange 4
 different colors in (??).
 109. Angela All of the letters are starting to look the same.
 110. 0:22:53 Amy I don't like this.
 111. Magda If you don't like it?
 112. Amy Yeah, I'm doing your numbers for you.
 113. Magda Alright, Bobby, how many different ways can you rearrange 4 colors 4
 high.
 114. Robert Do they have to have like at least 1?
 115. Magda Yeah like all 5 colors.
 116. Robert Then 12.
 117. Magda I think there's more than 12. Is it? 16 maybe? 16 sounds good, no?
 118. Robert Alright. I think it's 12.
 119. Magda Blue, blue...
 120. Robert It's like the first spot -
 121. Angela Wait. Magda, how many are in your little moving them down down
 sets?
 122. Magda 6.
 123. Angela 6? Ok.
 124. Magda Yeah, but there's gonna be, yeah but I have to find -
 125. Robert You have 4 choices for the top one. 3 for this one. Now these 2 you
 need more than (??). And you have 2 for this one. And then this (??).

- Then you have 1. So it's 4 times 3 times 2 times 1, 24 or something.
Like you have 4 choices for what it could be: A, B, C, D.
- 126 Magda Yeah, yeah I know.
- 127 Robert So it'd be A, B, C, or D. So I guess that's... Cause you always have at least 1
- 128 Magda Yeah, mmmm. So that's 24 for this one. Let's try to figure this out. So 24 different ways to arrange -
- 129 Angela Wait. Do that again.
- 130 Robert Alright, for like this -
- 131 Magda You know for when we were doing the thing in line where you stand?
- 132 Robert Yeah.
- 133 Angela Yeah, I get that, but what does that have to do with this?
- 134 Robert Well, you have 4 high. 4 colors. So you have 4 colors choosing A, B, C, D. And here you have 4 to choose from.
- 135 Angela Yeah, but we only have 3 colors.
- 136 Magda No, we have 4. I'm doing the 4 cause if 2 whites on top. Only the bottom would be changing. I'm rearranging that. And there's 6 possibilities after you have that. So I need to find out how many times, how many different ways can you put this. And then keep rearranging. So it's 24 different.
- 137 Alright, yeah, yeah. That's why I was confused.
- 138 0:25:08 Robert D you only have 3 choices. And C you only have 2.
- 139 Magda It's 24 for that so 24 different of these. It can be 24 times 6? Anybody? 1, 2, so that's 144. So that's for with the whites together. Times all different colors.
- 140 Magda Times 5.
- &
- Angela
- 141 Magda I'm starting to lose it. Times 5. 2, 2, 2-7... 720 so far. And that's for the colors being together. If they're separated by 1, how many different times can they be arranged? 3, 3 times 2, 6 times?
- 142 Angela Mmmm.
- 143 Magda So it's...
- 144 Angela But then you can do it with each different color.
- 145 Magda Yeah but that, yeah. So that's 720 with the colors together, (writing) 2 colors together.
- 146 Angela Wow.
- 147 Magda Yeah, you wanted me to write all of them out? (Laughs) And then there's separated by 1, so that'd be... How many different ones? 6 different ones?
- 148 Angela And you have 4 colors to put up there. Times 4.
- 149 Magda 6 different times like that and 6 like that so that's 36. How many different colors can it be?
- 150 Angela 4.
- 151 Magda I thought 5? 4?
- 152 Angela 4.

153 Magda 4 between, 4, 2, 144. And then...
 154 Angela You can do it will 2 in between.
 155 Magda Hold on, but then these 2 colors can be different. Different colors.
 Instead of the whites up there.
 156 Angela Yeah but you take care of that when you multiply by the total number
 you're gonna get by 5.
 157 Magda Oh but here, I did it like that so.
 158 Angela Oh.
 159 Magda Then you do that by 5. So it's the same thing, It's 720.
 160 0:27:25 Angela Well it's gonna be the same cause you got like the same amount in
 each set thing-a-ma-jigger.
 161 Robert Set thing-a-ma-jigger?
 162 Angela Yeah ha. It's because I have a limited vocabulary. Ok? Don't make
 fun of me.
 163 Magda 1 color separated by, separated by 1.
 164 Angela Separated by 2.
 165 Magda Then, 1 color separated by 2. That's another 720.
 166 Angela Then you can do separated by 3.
 167 Magda Separated by 3.
 168 Amy (To Robert) What is this counting principle?
 169 Robert I don't know.
 170 Amy That's what I'm trying to use but it's not working.
 I had like 4 times 3 squared is 36, but to do yours, 3 times 2 squared is
 12, not 6. I don't know how to relate them except 6 squared is 36.
 171 Angela 720 times 4.
 172 Magda 1, 2, 3, 4 now.
 173 Angela You can't separate by 4. Then you'll just have the same as these two
 put together. See, cause it's like it's separated by 4 there.
 174 Magda Yup, yup, yeah I know. 720 4 times?
 175 Angela 2,880.
 176 Amy Draw them out, Magda.
 177 Magda Told you I'm not drawing them out. I was like Uhh... That's what it
 seems like. Let's see a pattern.
 178 Angela Ok, we've got how many for wa-wa-wait for. 2 tall 1 color we have
 179 Magda That's not what Bobby did.
 180 Amy Bobby did 3 tall.
 181 Angela Well that's ok. We can still include that. There's 2 right? No. 1. I'm
 a little slow today. 3 tall 2, you have?
 182 Magda 6.
 183 Angela 6. 4 tall, 3 you have
 184 Amy 36.
 185 Angela 5 tall, that was me?
 186 Amy Yup.
 187 Angela 340.
 188 Magda You sure you got all of them?
 189 Angela Yeah, positive. You have 2-8-8-0.

190 0:30:05 Magda Do they seem similar? That's a square root. That might be, uhh. What's 36 times 36?

191 Angela 216, is it, right? No. It's more than that isn't it?

192 Magda Yeah, it's too much.

193 Robert 1-2-96.

194 Magda What?

195 Robert 1-2-96. That's if those numbers are right too.

196 Magda Yeah.

197 Robert I know mine's right.

198 Angela Mine are right. They're right. They're all right. Magda's are right.

199 Magda How do you know?

200 Angela Would you like us to go about writing them all out? See I did it all for like 1 color. It's right.

201 Magda I didn't do that.

202 Angela Oh wait. Did I do separated by 2? Yes I did! Yes I did! Separated by 2. Phew!

203 Magda Separated by 3?

204 Angela Separated by 3 is the same thing as you separating by 4.

205 Amy If you multiply 5 times 4 times 17 you get 340.

206 All (Laughs)

207 Amy I'm playing with numbers here.

208 Magda I don't know.

209 Robert Maybe there is no pattern.

210 Angela I think we should write them all out every single time.

211 Magda There must be an easier way than all this.

212 Angela Hm, think.

213 Magda Yeah, there's 150, I wish. (Crosses something out.) Look at that. I don't know. Anybody?

214 Angela Let's just think for a second. Especially you, Bobby, you always-- Robert, I'm sorry, you always come up with these extravagant answers that make no sense to anyone and you're just like "yeah I get it".

215 Robert Mike (??) gets it too.

216 Angela Yeah, well, you two are super genius people. I can draw towers.

217 Magda I don't know.

218 Angela Maybe we can think of something.

219 0:31:19 Robert What's a set thing-a-ma-jigger? Like which one is it?

220 Angela (Laughs) This is a set thing-a-ma-jigger (points). Ok? This is one, this is one, this is one, this is one....and so on.

221 Robert See if that has something to do with it.

222 Angela Alright, great. I'll label it for you set thing-a-ma-jigger.

223 Magda Is this a set?

224 Angela No it's part of this one.

225 Magda (Starts counting Angela's sets)

226 Angela No, don't count these, these are different colors. That's when I was stupid. Well, yeah, stupider.

227 Magda So it's 1, 2, 3, 4, 5, 6?

228 Angela 7, 8 18. 18 times 5. Let's check my math. UGH! I was wrong!
 Ooops. So it was just 20 more added on.

229 Robert The factorial of the height divided by half the color. Cause it works for
 these 2. 3 times 2 times 1 is 6 and half the color is 1, so that's 6.

230 Magda Uh huh.

231 Robert And 4 times 3 times 2 times 1 is 24 and half of 3 is 1.5 and 24 times
 1.5 is 36.

232 0:35:19 Amy Well what's 3 gonna be? It'd be 240.

233 Magda 6 times 6. Wait, what'd you say? 6 times 5 times...

234 Robert Alright. 3 factorial is 6 and half -

235 Magda Yeah, I get that but does it work for the other ones? 240?

236 Robert Yeah it does. 4 times 3 times 2 times 1 is 24, and half of 3 is 1.5 .

237 Magda Yeah, I get it but does it work for Angela's?

238 Amy I got 240.

239 Angela Does it work for Magda's?

240 Magda Cause Angela might have messed up.

241 Amy Check 5. 5 times 4 times 3 times 2 times 1 is 120. Times 2 is 240.

242 Robert Yeah, so?

243 Amy Yeah but she's getting 360.

244 Magda 6 times, 30 times 4 120 times 3, 360, times 2 720. Times 2.5.

245 Amy You can check my math.

246 Robert No its right, maybe it just works for these two.

247 Amy 720 times 2.5?

248 Robert 2000 something?

249 Amy 18,000.

250 Magda 18,000? That's too much.

251 Magda (Talking to themselves.) 1800.
 &
 Angela

252 Amy I had an extra 0 in there.

253 Angela You just didn't put the decimal point in.

254 Amy Oh yeah, there you go.

255 Angela See. I don't think it'll work.

256 Magda Unless we did something wrong.

257 Angela I don't know.

258 Magda Which is quite the possibility.

259 Amy 36 times 9 is 360.

260 Angela Does it work for 2 tall with 1 color.

261 Magda No.

262 Magda Yeah it does.
 & Amy
 &
 Robert

263 Robert 2 times $\frac{1}{2}$ is 1.

264 Angela Maybe there's doubles or something.

265 Robert Maybe it's all wrong and just happens to work for those three.

266 Angela Yeah cause that also works if you times 6 times 6. But after that it all goes downhill.

267 Amy Does 360 go into 2880?

268 0:38:00 Magda Maybe you have doubles. Because if you divide that its 120 and 180, 180 yeah. And that times 6? No, Magda. You multiply. Never mind.

269 Robert Maybe you messed up on those set thing-a-ma-jiggers.

270 Angela Would you like to check it? (Angela passes her work to Robert.)

271 Robert Are these all different?

272 Angela Yeah. Ignore the ones that are crossed out. That's just for double reds. Then you multiply it by 4 cause there's 4 more colors. Double green double yellow double blue.

273 Amy I think we need the same (??) sheet as last time.

274 Angela The sheets from last time were wrong. There were like 2 hours of going on a wrong thing.

275 Robert Good work. (Passes paper back to Angela.)

276 Angela Thank you. See, I know what I'm talking about.

277 Magda Why'd I multiply this by 6?

278 Amy Uh oh.

279 Angela What'd you multiply by 6?

280 Magda 24 different things, oh yeah, 6 different things there. Okay, 5 different colors. That works. 720. Separated by that is...

281 Angela Did you do the ones where they're not separated? Correct?

282 Magda What?

283 Angela When you have 2 colors.

284 Magda 2 together?

285 0:40:05 Magda Makes sense. Might have done it wrong.

286 Angela I'm lost.

287 Magda Any ideas?

288 All (Start talking about plans after school.)

289 Dr. Are you done?

290 Robert Kinda confused.

291 Dr. Is there anything you're sure of?

292 Robert That 3 tall, 2 colors is 6.

293 Dr. Can you convince us?

294 Robert Yeah, you can check out what I got (??).

295 Dr. What do you have there, Robert? 3 tall...

296 Robert 2 colors.

297 Dr. So you're all sure of that?

298 Magda Yeah, that's about it.

299 Angela We're also sure that 4 talk 3 colors is 36.

300 Dr. How do you know that?

- 301 Maher
Angela Wrote them all out. 5 tall 4 colors, we had a little problem with 2 problems cause I can't multiply. But it's 360. I didn't write them all out, I just wrote -
- 302 Dr. You're sure of that?
- 303 Maher
- 303 Angela 144. Yes.
- 304 0:42: Dr. How do you know? Let's say the 4 tall 3 colors. You wrote them all out? So tell me, how many did you write out.
- 305 Maher 36.
- 306 Amy 36.
- 306 Angela Yeah, the wrote out 36.
- 307 Dr. And how many towers are there? 4 tall, selecting from 3 colors?
- 308 Maher
- 308 Magda All in all?
- 309 Dr. Mhmm.
- 309 Maher
- 310 Magda 36.
- 311 Amy I wrote them out.
- 312 Angela Let me see. You separated them, right?
- 313 Magda Using the colors 3 times?
- 314 Dr. No, how many towers are there 4 tall selecting from 3 colors?
- 314 Maher
- 315 All There's 36.
- 316 Magda No, cause you don't have to use the 3 colors.
- 317 Angela Ahh!
- 318 Magda Is that what you're saying?
- 319 Dr. Yeah, how many towers can you build 4 tall when you're selecting from 3 colors?
- 320 Maher
- 320 Angela Do you have to use all 3 colors?
- 321 Robert Is that the 4 choose 3 thing?
- 322 Dr. Just how many towers can you make that are 4 tall? When you can select from 3 colors?
- 322 Maher
- 323 Angela A lot more.
- 324 Dr. You said that already. You think there's a lot more?
- 324 Maher
- 325 Angela Well, there's more. I mean, you can do...
- 326 Dr. Do you know how many?
- 326 Maher
- 327 Angela All blue, all red, all green.
- 328 Amy (mumbles something)
- 329 Angela Write them out, Amy.
- 330 Dr. Do you want to have a conjecture before you start, Amy? That's something to think about.
- 330 Maher
- 331 Magda I guess something 20.
- 332 Angela Wait. You could do all one color. You could do the tower 4 of one color. 3 of one color. 2 of one color. And one of one color. No, you

can't do one of each color. I'm like a little slow. Well, you could do like one of that color and multiply it by 3.

333 0:44:08 Dr. How many towers can you make when you're selecting from 2 colors 4 tall?
 Maher

334 Magda I'm like really out of it today.

335 Angela Yeah, I'm like so tired.

336 Magda 4 tall 2 colors...

337 Robert 16.

338 Angela I think you're right.

339 Robert No. Maybe like 4.

340 Angela (Mumbles something) Bobby, Robert, sorry, everyone's picking on him today.

341 Dr. Where did you get that number from? Just remembered it? Or did you
 Maher figure it out?

342 Robert Wasn't it like x to the n ? Something we thought of.

343 Angela Oh yeah!

344 Magda Oh yeah.

345 Angela Derr. I forgot about that. That was like from forever ago, though.
 Right?

346 Robert Yeah.

347 Dr. So what does that have to do with 16?
 Maher

348 Robert 2 to the 4th.

349 Magda 2 to the 3rd.

350 Angela No, you said 4 high.

351 Robert Yeah.

352 Dr. How do you know that works, that rule?
 Maher

353 Magda That's what we did before.

354 Angela Yeah.

355 Dr. How do you know? I mean, you just gave a rule and sometimes you
 Maher give the remembered. (??) Do you remember why it worked before?

356 Angela Oh gosh. I don't know, we tested it. Everything...

357 Magda (To Robert) No, didn't you have that little rule where you go

358 Angela Yeah the 1.5 and all the other junk you were doing.

359 Robert Probably.

360 Magda No, weren't you doing where you started with one and then you added
 it on then you branched out and stuff.

361 Robert Yeah.

362 Angela Yes. That branching.

363 Dr. Amy, hunny, do you know what they're talking about?
 Maher

364 Amy Not really.

365 Dr. She doesn't know what you're talking about.
 Maher

366 Robert Like, I think it was 1 block or something. Then you put, you could put

a red one on top of it or you could put a blue one on top of it. And then you could put a red on top of this and then a blue on top of this, red on top of this, blue on top of this.

367 Angela And so on and so forth.

368 Robert And keep going. And I don't think this one really counted but this was like combination 2 and 4. This was like 1 high, 2 colors. This was like 2 high, 2 colors. And it kept branching out on top. That was it. And then you kept going.

369 0:47:07 Dr. What color was that bottom?

Maher

370 Robert Umm...

371 Angela Didn't you just start with a red and a blue.

372 Robert Yeah. That's kinda what these are. Just kinda started from 2 separate points.

373 Dr. Oh. I see.

Maher

374 Robert And then you just kept going. Forever.

375 Angela And ever.

376 Dr. So, how many can you make when you're selecting from 3 colors that are 2 high?

Maher

377 Robert 9.

378 Dr. How does that work?

Maher

379 Angela You mean like with the branching thing?

380 Robert Yeah, you have 3 on the bottom and you keep going.

381 Dr. (To Amy) Does that make any sense?

Maher

382 Amy Mhmm.

383 Dr. What does that have to do with this?

Maher

384 Magda You see a pattern? How many total is there here? Um. 2 colors 3 tall, 2 times 2 times 2 is 8?

385 Angela So we didn't have to write all these out.

386 Magda Yeah we did.

387 Angela No we didn't. We could have used that.

388 Amy No it doesn't work for these.

389 Angela Oh yeah.

390 Magda No cause you have to use all the colors.

391 Angela Never mind.

392 Magda 3 times 3 is 9 times 3 times 3 is? 81? 4 times 4?

393 Amy 16.

394 Magda Times 4.

395 Amy 64.

396 Magda I think we need a calculator. 64 times 4.

397 Angela Why can't we multiply?

398 Robert 256.

399 Magda 256?
 400 0:49:00 Magda And then 5 times
 401 Angela Wait – how is that 256?
 402 Magda That’s what Bobby said.
 403 Robert I don’t know.
 404 Magda 4 times 4 times 4 times 4. (writing it down.) 4 times 4 times 4 times 4 times 4.
 405 Angela No it’s not, it’s 5... No no no. You’re right, I’m sorry. I was thinking backwards.
 406 Magda Ok. I thought I was a little slow. 16 times 16? (Does the math) 356. Times 4. (Does the math) Umm...
 407 Robert You said 356, it’s 256.
 & Amy
 408 Magda I can’t multiply. (Crosses out work)
 409 Amy And then wouldn’t you have to multiply by another 4 again?
 410 Amy & Yeah.
 Angela
 411 Angela Hey, these numbers look familiar.
 412 Magda Yeah, that looks familiar. And then that one.
 413 Amy Isn’t it down the row or something?
 414 Dr. Well what time did you have to leave?
 Maher
 415 Magda 4:30.
 416 Dr. It’s past. See what happens when you’re doing math? You just lose
 Maher track of time.
 417 Angela Time does go when you’re having fun.
 418 Dr. Do you want to think about this some more?
 Maher
 419 Magda Yeah. I’m gonna write this down so I don’t have to do it all over again.
 I can just use this, right?
 420 Robert (To Dr. Maher) I think this is your paper.
 421 Magda Can I use this? Can I write on the back of it?
 422 All (mumbling)
 423 Magda 2 tall. Just need a minute to write this down. (Write all the numbers they found down.) Ok.
 424 Dr. So what you’ve done is that for the ones you have you’ve actually tried
 Maher to produce them. And you have some system for keeping, for keeping a record. And you all agreed on those? Did you have agreement? Did some of you work on it? Or did all of you work on it?
 425 0:52:07 Magda Like basically, each person worked on like
 426 Angela Like, we assigned, well Magda assigned.
 427 Magda Yeah, I assigned myself the hardest one.
 428 Dr. But did you check each other? While you.
 Maher
 429 Magda Yeah. He did. (pointing to Robert)
 &

- Angela
 430 Angela Yeah, and he checked mine. And we sorta all did Magda's together.
 She couldn't handle the burden.
- 431 Dr. What's the general problem? Can you state it? Anybody?
- Mahe
 432 Angela How many combinations are there if you like have to use one of each
 color and you're choosing from one less than the number of ...
- 433 Amy The height.
- 434 Angela The height. Yeah. One less than the number of the height.
- 435 Dr. Fair enough. So um, ok, now you remember what the problem is?
- Mahe
 436 Angela Yeah.
- 437 Dr. So if you have anything, you could e-mail me before next week. If not,
 Mahe would you make sure that you can number them so you can remember
 the order? Number them or something. Are yours numbered? Put
 your name on all of them. Have a wonderful dinner tonight.
- 438 0:53:15 Magda & Angela (Break up and leave. Only Amy and Robert and Dr. Mahe are left.)
- 439 Dr. Ok. So do you want to talk about this problem anymore? What do you
 Mahe guys think about it?
- 440 Robert I thought I found something that worked, but it only worked for the
 first 3. We don't know if like that's right. Like their numbers.
- 441 Dr. You're not sure?
- Mahe
 442 Amy & Robert Yeah.
- 443 Dr. What did you check?
- Mahe
 444 Robert We were writing them all out, so...
- 445 Amy Bobby was saying that if you take like 2 factorial and then like 2 times
 1. Then you take half of the color, like .5 is 1. Like 3 factorial is -
- 446 Dr. What is $\frac{1}{2}$ of the color?
- Mahe
 447 Amy If you take $\frac{1}{2}$ of the number.
- 448 Dr. Oh, $\frac{1}{2}$ of the number. There's 1 color. I don't understand the thoughts.
 Mahe
- 449 Robert Just seemed to work, so, I don't know.
- 450 Dr. But what does it mean? So, if you have 2 tall, right, and 1 color, how
 Mahe many are there?
- 451 Robert One. Cause you'd just have blue blue.
- 452 Dr. So what's the $\frac{1}{2}$ of a color? 2 tall, if you're making towers 2 tall, and
 Mahe you have 1 color, right?
- 453 Amy Mhmm.
- 454 Dr. You're not doing anything with $\frac{1}{2}$, it's still 1 isn't it?
 Mahe

455 Robert It just works.
 456 Dr. Right?
 Maher
 457 Amy & Yeah.
 Robert
 458 Dr. Okay, so, 2 tall if you have 1 color and it has the 1 color. Now how
 Maher does, talk to me about this one.
 459 Amy The 3 tall?
 460 0:55:12 Dr. Mhmm.
 Maher
 461 Amy Ok. It'd be 3 factorial, which is 6 and then $\frac{1}{2}$ of 2 is 1. So it's 6 times
 1 is 6.
 462 Robert It seems to work for the next one too.
 463 Amy Yeah
 464 Dr. Let's see now. Um, well, I'm not sure. What could the 3 factorial
 Maher mean?
 465 Robert The total of like using 3 in it? Like you have 3 tall and 3 colors
 466 Dr. You have 2 colors.
 Maher
 467 Robert to choose from.
 468 Dr. You have 2 colors.
 Maher
 469 Robert 2 colors. Well, it'd really just be 3 times 2. The times-ing by 1 really
 doesn't matter. Like you have 3 and 2 to choose from so it'd be like 3
 times 2.
 470 Dr. Ok, if you were, if you had 3 tall, right? Ok. If you had 3 things that
 Maher you were arranging, just 3 things. And they were all different. You
 could make them anything you want. 3 letters, A, B, and C. How
 many different ways can you arrange those letters, A, B, and C?
 471 Robert 6.
 472 Dr. Write them out. Is it that we don't have paper? Oh here's some paper.
 Maher (Hands them paper.)
 473 Robert 3 places and you can have A, B, or C?
 474 Dr. Yeah, show me that.
 Maher
 475 Robert Like do you have to use at least 1? Cause that's what we're doing in
 this problem.
 476 Dr. Yeah.
 Maher
 477 Robert You have to use at least 1? Ok. (Robert writes them all out. Amy
 agrees.) 6.
 478 Dr. Ok. Now, suppose 2 of them were the same.
 Maher
 479 Robert 2 the same.
 480 Dr. Right.
 Maher

481 Robert So it'd be
 482 Dr. So, if you did them with 2 of them are the same. Right? What would
 Maher that look like?
 483 Robert It'd be (??). Like you could use 2 of them but 2 (??).
 484 Dr. Yes, but...
 Maher
 485 Robert Out of 3 choices.
 486 Dr. Well, let's change the problem a little bit. We're stacking, um, we
 Maher could think of A's, B's, and C's as books on a shelf. Ok? And they all
 look different. So you're gonna think of the A book as a book by
 Albert, the B book as a book by Bobby, the C book as a book by Carl.
 And if it didn't matter, you'd put them on the shelf and you could make
 6 different arrangements, right? Suppose those 3 books, you still have
 3 books but 2 of them look alike.

487 Amy Mhmm.
 488 0:58:27 Dr. I don't care which 2 look alike. Right? So you don't have A, B, and C
 Maher now. You can think of it as having 2 A's and a B. Make it anything
 you want. Will that change how many ways you can arrange them?

489 Robert Well if you have, oh, go ahead.
 490 Amy Well it'd be like a 3 tall 2 colors. Right?
 491 Robert Well you still have 3 but they look the same. So you're saying that,
 492 Amy You drop the C and have A, A, B. Which is what you had there. The 3
 blocks with 2 would look the same.

493 Robert So C turns into an A or a B.
 494 Amy & Mhmm.
 Dr.
 Maher

495 Robert So where this would be. Instead of this you'd see "A". So A here, A
 here, no here would be B.
 496 Dr. Well, you're gonna make all the C's, A's.
 Maher

497 Robert This could be A or B.
 498 Dr. Don't switch them. Be consistent.
 Maher

499 Robert Alright. Ok, so A, A...(switches C's to A's). But that would--But
 there'd be duplicates then.
 500 Dr. There'd be duplicates then?
 Maher

501 Robert Yeah.
 502 Dr. So where'd the duplicates come from?
 Maher

503 Robert Um, this one and this one and this one.
 504 Dr. Ok. So how many are duplicates?
 Maher

505 Robert Half of them.
 506 Dr. Half? So you divided that by the duplicates. Right? So there aren't 6

507 Maher anymore. There are?
 508 Robert 3.
 509 1:00:22 Amy Ok.
 Dr. Ok?
 Maher
 510 Amy Mhmm.
 511 Dr. What about you should think of that same problem but now you have 4
 Maher books. All different, A, B, C, and D.
 512 Robert That would just be this, right?
 513 Dr. But now we're stacking 4 books A, B, C, and D and they're all
 Maher different. How many ways can you arrange them on the shelf? A, B,
 C, and D?
 514 Robert 24.
 515 Dr. (To Amy) Do you believe that?
 Maher
 516 Amy Yeah.
 517 Dr. Ok. Now, suppose we have a duplicate. Now suppose we have two
 Maher duplicates. Do you understand my question?
 518 Amy Mhmm.
 519 Dr. Can you come up with what happens when you have duplicates?
 Maher Suppose we have 3 duplicates? What would that do to the
 arrangements? Would it be more, or fewer, or the same? Do you
 think?
 520 Robert I think it's gonna be less arrangements this time.
 521 Amy Yeah.
 522 Dr. See if you can come to predict the way it would be fewer. Cause you
 Maher think it's fewer, right? (Amy nods.) Depending upon the difference.
 Why don't you think about that for a little bit. I know this is a
 divergence but I think if you think about that, it might help you go back
 to other things. Does that make sense?
 523 1.01.15 Robert Yeah.
 & Amy
 524 Dr. Cause you're the one that introduced factorials, and when I think of
 Maher factorials, I think of arrangements, so... All different objects. Where
 they all look different. And the order matters. Right?
 525 Robert Yeah.
 526 Dr. But now when you have some that look alike, you're suggesting to me
 Maher the theory is that the number of arrangements will be the same,
 increased, reduced?
 527 Amy Reduced.
 528 Dr. Reduced you said. And that's what Robert said before. It's something
 Maher to explore, right? But if you can come up with, predict the reduction.
 Here it was a half, right? When you had this. Can you explain why if
 you study what happens, where it came from and then try to test it.
 You know, the other problem. You might want to play with that a little
 bit. Have you thought about these ideas before? Are these new?

- 529 Robert Sorta.
 530 Dr. Sort of, right?
 Maher
- 531 Robert They sort of tie in to what we do in class.
 532 Dr. What are you doing in class?
 Maher
- 533 Robert Like, probability and stuff. But we had to, like, we had to take out the duplicates. And how if you want 5 factorial but you want to stop at 3, how you would do this. And that was hard to figure out. Seems, like, simple.
- 534 Dr. What do you mean 5 factorial and had to stop at 3?
 Maher
- 535 Robert Like if you just wanted to, like if you had umm 3 boxes. And like let's say you had like 5 layers to choose from. And you only had 3 spaces on the shelf. Like how would you arrange those 3? And like it'd be 5, cause you could be 5 here A, B, C, D, or E. Times 4, times 3. And then, if you wanted to express that how would you write it out? Cause you can't do 5 factorial because then that'd be like you'd have to add times 2 times 1. But that wouldn't be the real answer. So you gotta like divide by the 2 factorial to get rid of these 2 over here. And then that'd give you just 5 times 4 times 3. 'Cause um. What's that, 60? And then that 60 is what you're supposed to get. And then you take the 5 factorial. That's 120. Over 2 and that's the 60 that you wanted.
- 536 1:03:46 Dr. Can you explain to me why that multiplication works to give you all the different arrangements?
 Maher
- 537 Robert Ok cause like even if it's just – it's easier to explain if I just show like 2. So pretend it's not here. You just have 5 books, 2 places to choose from, so 5 times 4. And that's gonna be 20. You just put that -
- 538 Dr. But why 20?
 Maher
- 539 Robert Cause 5 times 4.
 540 Dr. But why 5 times 4? Do you understand my question?
 Maher
- 541 Robert Oh yeah, yeah, yeah. Cause you have 5 choices. Like if I had 5 books and let's say I put my math book here, and my science book here then I have no more spaces to choose from. But like when I had the first choice, I could choose from math, science, English, Italian, and Spanish. Or something. And then since I chose English, that's no longer my choice here. I'll just write A, B, C, D, E. Let's say for the first one, I choose A. See at first I have 5 choices. I choose A. And then I know I no longer get to choose from that. So I only have 4 choices – B, C, D, or E. And then for the next one I only have 3 choices.
- 542 Dr. But why aren't you adding them? Why are you multiplying them?
 Maher
- 543 Robert Cause we always seem to multiply.

- 544 Dr. But I don't understand why multiplication works. Do you, Amy?
 Maher
- 545 1:05:06 Amy Cause we just do it. Um...
- 546 Robert Cause we never seem to add when we're doing this.
- 547 Amy Yeah.
- 548 Robert Cause like
- 549 Dr. Can you, can you imagine all those possible 2 book arrangements?
 Maher
- 550 Robert You just add them. It's not gonna work because like you have 5 here to choose from and if you add them you get 9 and I can arrange them more than 9 ways. Just by switching them.
- 551 Dr. I agree with that but I still can't imagine the 20 unless you can pursue
 Maher me that there really are. Why is it 20? Why are you multiplying?
- 552 Robert Maybe cause you could have, like here you have A and B, and AC, and AD, and AE, and then you're multiplying by the 5 because you got the BA, the BB, the BC, the BD, BE and you have the C. So you multiply by 5 for each.
- 553 Dr. Why don't you write it out so that Amy can be sure what you're
 Maher saying. Ok? Right Amy?
- 554 Amy Mhmm.
- 555 Robert Ok.
- 556 Dr. I think that would make her happy and me too.
 Maher
- 557 Robert Um, cause the first one, you'd have A. A can go with the B, C, D, or E. And then you can have the B where that can go with the A, C, D, or E. See, you have the 5, and then there's 4 possibilities with this one. Like there's 5 here and there's 4 possibilities that can go with each of them. So that'd make it 5 times 4 because you can either have this for the first one and you'd have this many for the second one
- 558 1:07:34 Dr. What do you think, Amy?
 Maher
- 559 Amy I understand what he's saying.
- 560 Dr. Can you explain it to me?
 Maher
- 561 Amy Um, I guess. Like you have the 5 different books...
- 562 Dr. Is that the way you write it? Or would you write it differently? If these
 Maher were your books. And you had to explain it to someone in your class.
- 563 Amy I would probably set it up the same way. You have 5 different books like drawing A, B, C, D, and E. And then um you can put if 2, we can call the shelves you can stick them with. It'd be AB, AC, AD, or AE. And then it could be if you put B on one shelf you can stick with book A, book C, book D, and book E. The same thing with C. With A, B, D, and E. Then D is A, B, C, and E. And E is A, B, C, and D.
- 564 Dr. So you're convinced that explains the multiplication. 5 times 4.
 Maher
- 565 Amy Mhmm.

- 566 Robert Wanna know why you divide by?
 567 Dr. Yeah.
 Maher
- 568 Robert It's like here the (??) is just gonna be 3 2 1. That's just like multiplying this by 6. So if you divide by 6, you're just going to end up with what you wanted. Because you're just going to end up dividing by 6.
- 569 Dr. Which is 3 factorial.
 Maher
- 570 Robert Yeah.
- 571 Dr. So you're dividing essentially by a factorial also. Mhmm. That's very
 Maher interesting.
- 572 Robert Cause there's no symbol that they have for not all. If you want to go 5 to 3. You just go 5 something 3. You know what I'm saying? Is there anything like that?
- 573 1:09:32 Dr. So what does this mean, the symbol you're looking for?
 Maher
- 574 Robert Like if you just want to multiply numbers between 5 and 3. Like just whole numbers. There's nothing?
- 575 Dr. 5 factorial divided by 3 factorial. Right? Ok. So you have now some
 Maher interesting ideas. Wonder if any of these ideas could be useful in the problem you're solving right now. K? Because here you weren't dividing by 2, you divided by 6 or 3 factorial. Now, you're claiming you divided by those cause you didn't have any more spaces.
- 576 Robert Yeah. Out of room.
- 577 Dr. Mhmm. Now, the question I asked you earlier to think about to
 Maher arrange them and in this case you said you had half the amount. Right? When you had these duplicates. So suppose you took that up one step, you know, to 4. Think about 1 duplicate or 2 duplicates or 3 duplicates. How do you account for those duplicates in a way that's systematic? You know they are reduced, you're saying, when there's duplicates. You want to test that. But reduced how? You came up with a very neat way of showing the reduction. Right? Is there a way you can do that here? Does that work there? Is that a useful tool? Why don't you both think about that for a minute? You have a little more time to stay? You can stay another 15 minutes?
- 578 Amy Mhmm.
- 579 Dr. Ok. Half hour? (laughs) Why don't you play with that for a little bit,
 Maher that idea. Do you understand the problem I'm giving you?
- 580 Robert Yeah.
 & Amy
- 581 Dr. Ok. Good. (Leaves table.)
 Maher
- 582 Robert Instead of 5 here, it'd just be 4. And this would just be 4 times... And then the rest would just stay the same. It's just that this one, the first one, would change to 1 minus. Cause you have duplicates. You can

- have 4 here but then you don't need to cross it out because you choose it again.
- 583 Amy Mhmm.
- 584 Robert But then you don't use it twice. So you move on to 3 choose 1. I mean, I don't know. Duplicates are...
- 585 Amy A systematic way.
- 586 Robert 4 colors. There's A, B, C, and D.
- 587 Amy Mhmm.
- 588 Robert For the first one you choose 4. And then you choose 4 again. Choose 3. But then, it doesn't come out to what we got.
- 589 Amy Wait. Why is it two 4's?
- 590 Robert Because let's say you choose A for the first one. But since 2 look the same, you could have an A again. So you don't cross it off cause you can have any one of the 4 of them again.
- 591 Amy Mhmm.
- 592 Robert And for the second one you have B, C, and D. But then you can also have like instead of A, it could be B and then you could choose C (??).
- 593 Amy Mhmm.
- 594 Robert So, I don't know if it's right. And then, I don't know how to get out the duplicates. Hmm. (Robert does some calculations.)
- 595 Amy That number came up. I have that number somewhere. (Amy searches her papers to see if she can find that number in her work somewhere.) It must have come up from here. When I was playing with numbers. We had the 2080. And I had like 28 the 6 times the 5. And if you divide 2080 by that, the 6 and the 5, 30, you get 96. It probably doesn't have anything to do with it, but...
- 596 Robert 6, 36. We have to get 36 to 96. It's 2 and $\frac{2}{3}$. Like this time 2 $\frac{2}{3}$ gives us that.
- 597 1:14:55 Amy Then what could you explain the 2 $\frac{2}{3}$ as.
- 598 Robert Yeah, I don't know.
- 599 Amy Umm.
- 600 Robert Let's try with 3 just to see.
- 601 Amy Mhmm.
- 602 Robert With numbers cause I think it's easier. You have 2 there and 2 there and 1 there. (Comes up with the combinations.) But there's just 6. (Both sit thinking.) Hmm...if this is 1 $\frac{2}{3}$ times 3 equals the 6. That $\frac{2}{3}$ might...
- 603 Amy Mean something.
- 604 Robert No, but the thing is, with this you're multiplying 3 and it's by 2 $\frac{2}{3}$ to get 96 and here you're multiplying... (Sits and thinks. Does some more calculations.)
- 605 1:18:40 Amy Why does it go from $\frac{2}{3}$ to 2 $\frac{2}{3}$? How'd you get that?
- 606 Robert I don't know. How'd you get the $\frac{2}{3}$? Let's do it the other way.
- 607 Amy Here it's 6 squared. Relate it to the exponent? Somehow. I'm guessing. (Both sit and think again.)

608 Robert Looking at the $\frac{2}{3}$ seeing if I can see something.
(Contemplates the problem, thinks out-loud a little but it is hard to hear all of it.)

609 1:23:29 Robert (??) we did 5 and we got 96. If you do 4 it's (??). 3, 2 it's 18. It doesn't work like the other ones cause it's supposed to be 36.

610 Amy Mhmm. It's half of that.

611 Robert This needs to be like times... This is like times 1.5. And this is like times 2. You know what I'm saying? This is 4 times 1.5. And that's the half thing.

612 Amy Oh yeah!

613 Robert And for this it would just be 1 times 1 and a half is 1.

614 Amy Mhmm.

615 Robert If this works out then it'd be 96... (Does the math on a sheet of paper.) 240 and that's what we had before too.

616 1:26:05 Amy Where did that 240 come in before?

617 Robert I don't know. Maybe it's cause, the places you can have the 3's. Cause here you have the 3's here, the 3's on top or the bottom and you're counting by 2's here and you're counting by a half here. You know what I'm saying?

618 Amy Mhmm.

619 Robert And you just put a half there. (??)

620 1:30:45 Amy I'm not seeing a pattern.

621 Robert Any ideas?

622 Amy No. The 1800 like with 6 tall is 720 times 2.5 but (??).

623 1:31:41 Dr. (Comes back over to the table.) Wanna share what you're thinking?

624 Robert I went back to what we were doing this other way. I kinda came up with the same thing we're doing here. We like, since you only have 3 colors, and you have 4 tall, you can choose 3 here, 3 here, 2 here, and 1 here. Because you have at least 1 duplicate. So you can choose three twice. And we found that and multiply by this, you can still get that. Like what you're supposed to get.

625 Amy (Reaches down to grab something.) Excuse me.

626 Robert You wanna see why we did two 4's here?

627 Dr. Why?

628 Robert Well because, um, the first one you choose A and since one looks exactly like it, you can choose A again later on. So you're gonna end up with 4 times 4. And then you can only choose it twice so after that it's only 3, then it's only 2, then there's only 1 left.

629 Dr. So, so what are you saying? It's the product of those numbers?

630 Robert Yeah cause then you times them by something.

631 Amy Yeah.

632 Dr. Times it by something?

632 Maher

633 Robert Yeah.
634 Dr. Why?
Maher
635 Robert Well that's because this is just the combinations for this. And you still have the combinations it's down here maybe or like you have a different number up here. You know what I'm saying? Like if this was B instead of A, it might change. But we just found an easier way than doing all those combinations. You just multiply it by half the height. And you get, we think, the big answer.

636 Dr. Why half?
Maher
637 Robert We didn't figure that out yet.
638 Amy Yeah.
639 Robert That's what we were trying to figure out (do).
& Amy
640 Robert Cause we were just looking at the numbers this gave us. And the actual answers we had. And we just figured out some pattern between the two.

641 Dr. Mhmm. Have you two thought at all about why you've been getting
Maher more than you need? Why that many?
642 Robert Which one? This one?
643 Dr. Mhmm.
Maher
644 Robert There's duplicates. Like things used twice but instead of multiplying it out like all the way, we just took one short stack. Then we ended up with less but then we multiplied the less by something, we get what we want. The answers.

645 1:34:40 Dr. So suppose you had to have 3 of one color in there. Would you still
Maher divide by 2?
646 Robert Maybe.
647 Dr. Do you understand what I'm saying, Amy? You have 3 A's.
Maher
648 Amy Mhmm.
649 Dr. I'm sort of curious why you're dividing by the number you're dividing
Maher by. And will that always be the same?
650 Robert Oh no, we're not. We're multiplying.
651 Dr. You said when you're done, you divide by 2.
Maher
652 Amy & No, multiply.
Robert
653 Dr. You multiply?
Maher
654 Robert You multiply the answer you get from this.
655 Dr. Oh.
Maher
656 Amy & By half the height.

- 657 Robert
Dr. Wait. Why half the height?
Maher
- 658 Robert That's what we -
659 Dr. That's dividing by 2. Don't you think it's dividing by 2?
Maher
- 660 Robert Well, if you use 3, then you'd divide by...
661 Dr. Now I'm lost. See I want to know what you're dividing or multiplying
Maher by the numbers you do. Because you said to me earlier, if you have
duplicates you have to divide out those duplicates, right? And in the
example you gave me, you divided by 2 because you showed me that
each one produced another one just like it. Is that what you did? So
maybe what you need to do is figure out why you're dividing out by. If
you're getting duplicates, think about those duplicates. Move away
from those number patterns and think more about what is happening.
Try to create a situation so you can follow what's happening. You're
doing this with numbers you know, you don't lose it. Cause you can
always do that with a calculator, right? And test it later. See if your
theory works. But my suggestion is to think about, you know why
you're multiplying some of this. You went into great detail, the both of
you, explaining that to me before, right? And you maybe need to test
some of that. Um, see, when you say the A, I'm thinking of towers
now, right? You're supposed to be thinking of towers and how tall are
these towers.
- 662 Robert This one's 5.
663 Dr. Ok so now you're asking me to think about 5 tall towers. Right? And
Maher now you're telling me these towers have 4 colors.
- 664 Robert Mhmm.
& Amy
- 665 Dr. You're telling me the colors are A, B, C, and D. Right?
Maher
- 666 Robert Mhmm.
& Amy
- 667 Dr. Ok. And you're telling me that you have to have every one of these
Maher colors. Right? That's what you're telling me. Ok. So you're telling
me if you're building this tower, right? We're gonna build up. So this
is your table, right? So you're saying if you're going to build it, for the
bottom one you can put an A, B, C, or D. Is that what you're telling
me? Or are you telling me
- 668 Robert Yeah A, A.
669 Dr. Or are you telling me for the bottom one, you're going to put A, A, A,
Maher A in all these towers?
- 670 Robert No.
671 Dr. I'm trying to understand what you're telling me?
Maher
- 672 Robert The first one, this one.

673 Dr. So this is the bottom tower?
 Maher

674 Robert Yeah.

675 Dr. What I'm trying to understand, right? You could either be doing this
 Maher (writes a combination down) or are you telling me this?

676 Robert Mine was like 4 tall, 5 tall. You have an extra.

677 1:39:03 Dr. Well, I'm just saying here. So what do you want me to do? I'm not
 Maher making them tall now; I'm just starting at the bottom of the tower.

678 Robert Yeah, the bottom can be either one of those colors. This one.

679 Dr. So you're saying the bottom can be either one of these colors?
 Maher

680 Robert Yeah.

681 Amy Mhmm.

682 Robert And the next row can be either one of those colors again.

683 Dr. So I can put here an A
 Maher

684 Robert a B, a C

685 Dr. a B, a C, or a D?
 Maher

686 Robert Yeah.

687 Dr. And here I can put?
 Maher

688 Robert Same thing.

689 Dr. A, B, C, or D? Is that what you're telling me?
 Maher

690 Robert Yeah. And for this one, you can only put...if for this one here you put
 A, you can only put B, C, D. Or B, C...yeah B, C, D.

691 Amy B, C, D.

692 Robert And if you keep the B, you put A -

693 Dr. That would help me if you could explain to me. You see what I'm
 Maher saying? If you can show me what happens when you're doing this and
 this and convince me sort of that that yields it. That would be helpful.
 You understand what I'm saying?

694 Amy Mhmm.

695 Dr. Cause I have to think and understand it in very concrete ways. I'm not
 Maher saying you didn't have to start with these on the bottom for all of them.
 But I was just wondering what you were thinking. Maybe this is a
 place to stop. Cause you have a lot of things to think about. You can
 even talk on e-mail to each other, you know. If you think about it.
 And to me. So you have a very powerful idea with a simpler case.
 That becomes a very powerful metaphor. If you really believe it and
 are totally convinced of it. And then you can play with that. But make
 sure that every little piece of it you understand. You're not just doing
 operations. That you're not totally convinced that what they yield.
 That's what I want you to really really think about. Does that make
 sense?

696 Amy Mhmm.
697 Dr. Cause you're doing wonderful thinking here. There's a lot of very
Maher important ideas you see here. You're on a nice, interesting path. So
you have a lot to think about. Ok? So you might want to try to come
up with very careful solutions for the simple problems. Ok?

698 Amy (Nods.)
699 Dr. You wanna describe them in detail. The simple cases. Does that make
Maher sense?

700 Robert Yeah.
701 Dr. So which ones might you decide to do?
Maher

702 Robert 3, 2.
703 Dr. Yeah, exactly.
Maher

704 Robert That's what we do in class.
705 Amy Yeah.
706 Robert We always start with the easy ones and then work our way up to the
harder ones.

707 Dr. That's the way. But make sure you really assume that the person
Maher you're explaining this to has no understanding. So you can write it out
and show it one way or the other and what you're doing with the
numbers and why. To try to explain it with the models you've done.
You tired? Want some pizza now? So thank you very much and we'll
see you next week. You two were wonderful sports. Do you have any
questions for me? Or anything for anybody here? This is great.

708 1:42:35 Dr. It's hard to have a really big amount of time to think about something
Maher or have someone to talk about it, just a couple of people. Even class
periods aren't that long. You have to talk to somebody about
something for a long time. It's really a luxury. And you need to, I
think. I really do, I think you need to think about it and test your ideas
and talk to people. So if you can, learn to do that. I think its one of
those important keys to success.

709 1:43:36 Robert (Begins talking about the birthday problem that they were doing in
class.)

710 Robert (Talking about the birthday problem.)
& Dr.
Maher

711 Robert (to Amy) It's like if I know your birthday, then the chances would be
greater than if I didn't know your birthday. That's like the problem
we're talking about. Cause like if I picked 2 random people, I picked
you two, and didn't know your birthdays, it'd be different than if I
knew her birthday. And then I picked yours. And then I picked you,
see.

712 Dr. There's a lot of things to be thinking about, right. (They number their
Maher work and put their names and dates on each of their papers to get ready
to leave. Also, talk about having an open house at Rutgers for the

students' parents.)

Appendix I

Date of Session: 03-01-1999

Tape Name: - Reviewing Pizza Problem & Making Connections Authors: Elijah Brookes, Marcelle Farhat, Anat Even-Zahav, Gary Wenger

Verified by: Elijah Brookes, Marcelle Farhat, Anat Even-Zahav, Gary Wenger

Date of transcript: October 2009

Researchers: Regina and Carolyn Maher

Teacher: Ralph Pantozzi

Camera View: Stephanie, Robert, Amy-Lynn, and Shelly's Table

Line	Time	Speaker	Transcript
1	0:00	Gina	Yeah? No, OK, then we didn't meet. And Amy, I've never met you. Hi, I'm Gina Keycheck, how are you? I'm a, uh, I'm a teacher in Jersey City. I teach at, uh, Ferris High School in Jersey City. And, I'm a student of Dr. Maher's. I've been doing a lot of work with Rutgers, and, um, I've seen a lot of you over the years on some of the video tapes, you know you're famous, but, um, actually, at, uh, some of what I've been doing, some of my coursework through the years at Rutgers has had to do with things that you've been doing. So, I've seen a number of you. So you look familiar to me. Stephanie I've met actually in the flesh, and I've met Robert in the flesh, and so, I'm glad to meet you. Um, Amy, where are you in school now?
2		Amy	Um, I'm in Mother Seton.
3		Gina	Oh, OK, OK And Michel-Shelly, how about you?
4	0:52	Shelly	UC. Union Catholic.
5		Gina	OK. Alright. Well, we are, um, going to be working on, these, these next couple of sessions, we are planning on working on some combinatorics tasks, um, for starters. We don't know quite where we're going to go from there. Depends on where you all take us, actually, is what we are going to be doing, and we were hoping that we could revisit some of the things that you had done earlier on, like in fourth grade, and fifth grade, and sixth grade, and so forth. And, one of those tasks you might remember having done is the pizza problem? Does that sound familiar to you?
6		Shelly	No, I remember shirts and pants, but I don't remember anything else.
7		Gina	Shirts and pants, wow, that's, that was second grade, so you're really going back. Well, this is the statement of the problem, so I'll just pass that out to you all. Give you a second to read it, just the, at least the top of it, see if it sounds familiar to you.
8			(Students are reading from 1:46 until 2:07)
9	2:06	Gina	Is this familiar. Yeah? Shelly? You remember this at all?
10		Shelly	A little. A little.
11		Gina	A little bit, a little bit? OK, um, I think, let me think back. I

believe you first looked at this, or first explored this in the fifth grade, um, but it's our interest in finding out how you would go about doing this right now, how um, ah, how you would like to pursue this, and how you could answer at least the first, uh, two questions, if not the other ones. OK, you want some time to talk about that, think about that?

12 Shelly Sure.

13 Gina Do you understand the problem? Do you understand the problem?

14 Shelly Yeah.

15 Robert Yeah, we did this one in algebra. A couple days ago.

16 Gina OK, OK, well that's great. Then I'm going to leave you.

17 Shelly So you know the answer?

18 Stephanie Yes!

19 Shelly Yes! We're done.

20 (group laughs)

21 2:57 Gina Oh, but now is that all there is to it, after all this time working with Rutgers, is that all there is to it?

22 Shelly (Stephanie groans) No.

23 Gina OK, the answer is seventeen, I'm done. (laughs)

24 Shelly. I guess I'll write my name.

25 Gina OK, I'm going to leave you alone. (leaves the table)

26 Shelly We just did this in school. And stuff.

27 (unknown, We did.
not from
table)

28 Stephanie Did we really?

29 Shelly Combinatorics, stuff.

30 Stephanie Who do you have, Whitehead?

31 Shelly McKenzie.

32 Stephanie Oh, forget I said anything. (laughs)

33 Shelly The grades in there are like: A, D.

34 Stephanie I heard, I...

35 3:27 Shelly A, F.

36 3:40 Shelly This is combination, isn't it? Just, well...

37 Stephanie Mhmm.
?

38 Shelly So are we supposed to find an answer?

39 Stephanie I guess (then unintelligible, 'that's what it wants?').

40 Shelly Or are we just going to sit here?

41 3:50 Stephanie Mm. Um(drops pen, grabs soda and drinks). You did this already?

42 Robert Yeah, a couple days ago, er a couple weeks ago.

43 Stephanie So what was the quickest way to do it? Just...

44 (everyone laughs)

45 Stephanie I mean...

46 Shelly Want to help us out here?

47 Stephanie Do you want us to just sit around? Or, no, I mean, did you guys
just kind of like...

48 Robert I don't remember.

49 Shelly Geez, we're supposed to remember from fourth grade?

50 Stephanie And he can't even remember two weeks ago? Um...

51 4:20 Robert No, it's more, it's like two months.

52 Stephanie Whatever.

53 Shelly Oh, well, you know.

54 Stephanie Doesn't matter, you're useless. Um...

55 Shelly That's a nice way to put it.

57 Shelly Yeah, like...

58 Stephanie Where we just...

59 Shelly The tree diagram type thing.

60 Stephanie Yeah, kind of like that. Or is there an easier way to do it. That I'm
just not ... thinking of.

61 Shelly Can't you, wait, plain counts as one, right? So there's...

62 Stephanie Yeah, plain, you can order just a plain pizza.

63 Shelly So there's six choices...

64 Stephanie. Mhmm.

65 Shelly ... so it's combination six, isn't it, something like that.

66 4:52 Shelly (whispers)I need my calculator. Too bad I don't remember how to
do combination. We just did it. (two unintelligible words)

67 Stephanie Didn't we do a little bit of it in algebra too(two?).

68 Shelly Can I whip out my calculator, is that OK?

69 Stephanie Yes, hurry up and(to?) do that.

70 5:04 Shelly Oohh (unintelligible) combination with the big guns. (begins
typing into calculator) (whispers) I think we're cheating this way.

71 Stephanie (laughs) One second, they'll probably make us plot it out anyway.

72 Shelly No, but see, (sighs) I'm not doing this right.

73 Stephanie Well.

74 Shelly OK, we'll do it their way, cause I'm confused.

75 Stephanie Yeah, well, why don't we do it the other way, and then once we
get an answer we can see if the combination thing is the same
answer.

76 Shelly Is right, or if I'm just ... confused.

77 5:31 Stephanie Alright, I need a piece of paper.

78 Shelly (sighs) Sounds good to me.

79 Stephanie They offer a plain pizza...

80 Shelly (mouths 'paper' to Amy) No?

81 Amy Yeah, what the heck.

82 Shelly OK.

83 Amy My brain is fried.

84 Shelly I had a physics test today. I've done enough today.

85 Amy? Geez Louise, I only have chemistry. Don't have a physics class.

86 (difficult to tell who is speaking, the camera is focusing on
Robert's paper)

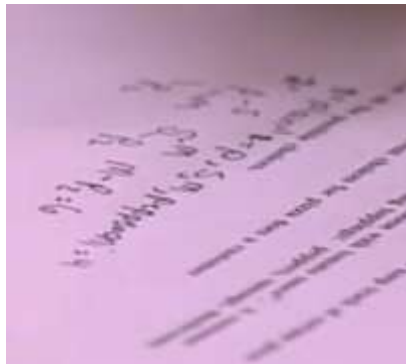
87 Stephanie OK, so your combinations are, plain pizza. Now everybody write
? plain.

88 Shelly? I know the next one, peppers.
Amy

89 Stephanie OK.
90 Shelly There's got to be an easier way.
91 Uh, see that's, you know what the thing is, every time we do one
of these, at the end we find an easier way, and then every time we
go back to do the next one we have no (laughing begins) idea
how to do it. Except to start from scratch.

92 6:17 Shelly See, that's good stuff.
93 Stephanie Huh?
94 Shelly P, and a peppers. Good stuff Bobby.
95 Stephanie What is Bobby doing, I can't see.
96 Shelly Sausage, mushrooms.
97 Robert No doing, just writing them out. All the (unintelligible).
98 Shelly Oh, pepperoni, that's going to be confusing. OK. Oh, you're
writing it like that. (whispers) I should try that.

99 6:45 Stephanie Tell them what?
100 Amy (unintelligible about "too many P's")
101 Stephanie Alright. So then you can have...
102 (camera shows Robert counting)
103 Shelly? (Amy says something unintelligible) Couldn't, oh, OK. What?
104 Stephanie I don't know how to organize it.
105 7:08 (Robert's paper, showing a branch diagram of options)

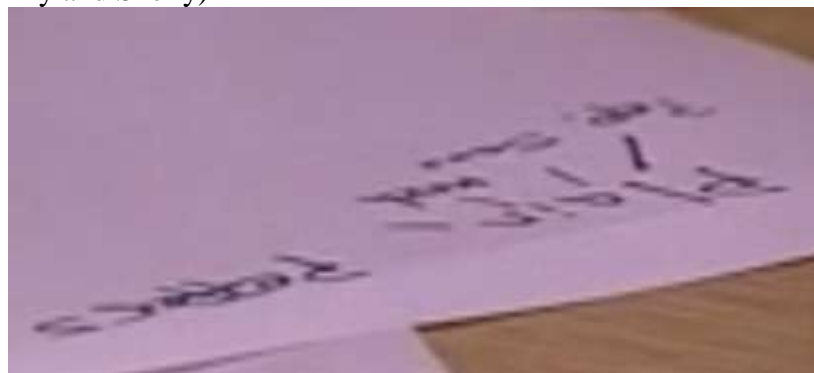


106 Shelly Wouldn't it be easier just to go like plain, and then just go like,
peppers, sausage, mushrooms, pepperoni? Or... I know there's an
easier way to do this.

107 Stephanie Yeah, but see then when you do peppers, yeah, but then we're just
going to go have to go back and eliminate it cause if you go plain,
peppers, and then you, it's the same as just having a peppers
pizza.
(Robert grabs a sheet of paper)

108 Stephanie Like plain with peppers is the same as a peppers pizza.
109 Shelly OK.
110 7:31

111 Stephanie Alright?
 112 Shelly Mhmm.
 113 Stephanie So, yeah, you can do it like that, but we're just going to have to go back and like cross things out when we're done. You know?
 114 Shelly My teacher's gonna kill me cause she knows I can do this.
 115 Stephanie Like she's ever going to see this.
 116 7:46 Shelly (exasperated sigh) Um.
 117 Stephanie So, yeah, let's just... let's just write it out and then double check with everybody else's answer.
 118 Shelly How are you writing it though, are you just putting, like...
 119 Stephanie I'm just going to like, plain, and then under it...
 120 Shelly OK, so, it's just like this...
 121 Stephanie ...plain with sausage.
 122 Shelly ...peppers...
 123 Stephanie This is going to take like eight years.
 124 (Stephanie begins her diagram, similar to those developed by Amy and Shelly)



125 8:06 Shelly (sighs an 'aw') You're doing it that way.
 126 8:28 Stephanie There's got to be an easier way to do this.
 ?
 127 Shelly I don't know if it's factorial or combination. I don't know if you just do like, five factorial plus four factorial plus three factorial plus two factorial plus one factorial. Cause OK, so, for, you have five, oh...
 128 Stephanie Right.
 129 Shelly So confusing. I don't know how to explain it. I don't know what I'm doing. Let's just go our old and easy way. Peppers...
 130 8:48 (Stephanie, who was listening, sighs and closes her eyes)
 131 Shelly I can't remember. It was the last section we did, so pathetic. Sausage... hmm.
 132 9:05 (camera focuses on Amy's paper)
 133 9:21 Shelly? OK, we have, we have plain, peppers, (whispers) sausage.
 134 10:03 Shelly? Doing this right.
 135 Shelly I think it's a factorial.
 136 Stephanie I think you're right, Shelly, cause.

137 10:15 Shelly So what did you get for the first one? Total answers. Oh, never mind, you're not there yet.

138 Amy? So I'll write it down.

139 Shelly Let's see how big that answer is. (enters into calculator) (sighs) (enters more numbers into calculator)

140 10:46 Shelly OK, so this way you get one-fifty-three. We'll have to see if that's right.

141 (Shelly is sighing again, says something unintelligible to herself, and Stephanie looks over her paper)

142 Shelly No wait, is that the same thing? (sighs an 'oh')

143 Amy? How many toppings on your one pizza? Can you just have two?

144 Shelly Two, three, four, fiv--it doesn't matter, you can have one, or two, or three, or four, or five.

145 11:41 Shelly (counting to self) One, one, two, three, one, two, three, four, and if you add...

146 Stephanie This is it, it's a factorial, right?

147 Shelly It *is* a factorial?

148 Stephanie Yeah, OK, (interrupted by someone moving drinks) cause if you have, if you have, OK, you have five choices for the first one, oh no, that, that might not be factorial.

149 Stephanie (recovering from a burst of laughter) Well wait, explain it.

150 Shelly I'm trying, I'm trying.

151 Stephanie Just wait, explain it to me.

152 Shelly I thought I was right.

153 Stephanie Well, cause I don't...

154 Shelly Hold on, OK, there's plain, and then so you just have two toppings. Then you have four choices there, then you have peppers, and then you have three choices there, right? (sighs) Sausage, and then you have two, right, or am I wrong? Plain, one, two, three, four...

155 Stephanie And then the mushrooms, you'd have one choice. And then...

156 Shelly Pepperoni none.

157 Stephanie Yeah.

158 Shelly Is that right? Mushrooms, one.

159 Stephanie Yeah, that's right cause that's the...

160 Shelly Pepperoni, none. And then from here you branch off and have three, and then two, and then one. (whispers) There's some way to do this, I hate this. OK. I'm trying, I'm trying. Haha.

161 Stephanie Calm down. It's going to be OK.

162 12:54 Shelly Five, one, two, three, four... (whispers) Six, seven, eight, nine, ten.

163 13:15 (camera focuses on Stephanie, who is now writing, camera switches to all but Robert around 14:00)

164 Amy? Um...

165 14:32 Shelly (camera focuses on Amy writing)

166 15:15 Shelly Wait a minute. I'm gettin' five, and then ten, and then ten, and

then five.

166 Stephanie (moans?) So then...
?

167 Shelly And then one.

168 Stephanie Then, well the next (unintelligible word) is just going to go like that?

169 Shelly Yeah. So then five and then ten(Stephanie says these two simultaneously), and then ten, and the one.

170 Stephanie Ahuh.

171 Shelly And that's it. Cause, well there's no one in the, in, in the front right? Cause... (sighs)

172 Stephanie (simultaneously)Wait, what are you talking about?

173 Shelly (simultaneously)Wait, there's got to be something because you, you...

174 Stephanie Wait, say it out loud cause I...

175 Shelly You can't have... you can't have just a pepperoni, oh...

176 Stephanie Just a pep...

177 Shelly Cause a pepperoni is a plain with a pepperoni.

178 Stephanie Yeah. So...

179 Shelly So that's why there's no one there, right, right? Is that what you're getting? For numbers?

180 16:06 Stephanie I'm doing, I don't know. I'm getting that there's one with four, and then there's, ah, three with three, and then the next one there's none with four, and there's two with three, and the next one there's number four, and one with three, and the next one'll have...

181 Shelly So I'm lost with that. But... (laughs)

182 Stephanie Yeah, it's just like, it's just like what you were doing before. Like here.

183 Shelly Yeah.

184 Stephanie Um... there's, oh, see you're working like, it'd be like doing it backwards. Like there's, we can have one pizza with all four toppings.

185 Shelly Mhmm.

186 Stephanie And that's it.

187 Shelly Mhmm.

188 Stephanie And then...

189 Shelly Oh, see that's where I'm doing it wrong. See, I have it plain, then I have them going there. (simultaneously) And then I have that. No because... OK, one, four...

190 Stephanie (simultaneously) No, you're not, you're just going the opposite way that I'm do it. Like, I'm starting from the...

191 16:50 Shelly See, these, all of these don't count. Cause that's with plain. That's like plain up here.

192 16:52 Stephanie Yeah, ahuh.

193 Shelly So all these don't count. So it's one, and then one, two, three, four, and one, two, three, four, five, six, and then four and then one.

194 Stephanie Yeah, you see, that's like, look, that's what it is...
 195 Shelly So it's one, four, six, four, one.
 196 Stephanie Like, see, there's four, and then there's three of those...
 197 Shelly Yeah.
 198 Stephanie Well actually yours is really (unintelligible word). I don't know.
 There's one with that, and then if you take it, like, from the
 peppers there's three with peppers and two other toppings.
 199 Shelly Yeah.
 200 Stephanie And then there's...
 201 Shelly (mumbles something unintelligible) Yeah, I think that's it. Five,
 five, ten, sixteen. It seems kind of small though. Well obviously
 that wasn't it.(crosses out the answer leading to 153) ... Does that
 number look familiar to you, sixteen?
 202 17:53 Robert It's what I got.
 203 Shelly 'Kay. (whispers) I'm done, that's it.
 204 Amy? (unintelligible, perhaps 'I don't have the same thing?')
 205 Stephanie Alright, hold on.
 206 18:20 Stephanie Oh. That makes more sense.
 ?
 207 18:21 (camera has been focused on Amy from 17:45, and she creates a
 branch diagram and circles sixteen)
 208 Stephanie Alright, wait, can I... try this. (hear paper moving)
 209 Shelly Feel free.
 210 Stephanie Oh, thank you. (Shelly laughs) Alright, you're going to have to
 explain it to me, cause I don't, under...
 211 Shelly OK, so that's plain.
 212 Stephanie So, I've got plain.
 213 Shelly Mhmm.
 214 Stephanie Alright. And then I've got one with pe...
 215 Shelly And then you have plain. Oh.
 216 Stephanie I've got one with peppers.
 217 Shelly Yeah.
 218 Shelly and Then one with mushrooms.
 Stephanie
 219 Stephanie Oh, wait. OK, I've got one with mushrooms.
 220 Shelly And then one with sausage, and one with pepperonis, peppers I
 mean. Pepperoni, yeah both.
 221 Stephanie Yeah, whatever. Same think. OK, then I have...
 222 19:00 Shelly Plain with peppers with sausage, and then plain with peppers with
 mushrooms.
 223 Stephanie You have plain with peppers and sausage. OK, so I have that.
 (Stephanie crosses the item off her diagram) OK.
 224 Shelly And then plain with peppers with mushrooms.
 225 Stephanie OK.
 226 Shelly Plain with peppers and (Stephanie echoes 'and') pepperoni.
 227 19:19 Stephanie OK.

228 Shelly Alright, and then you have, um, plain with sausage with, yeah, plain with sausage with, with mushrooms.

229 Stephanie OK.

230 Shelly Plain, sausage, and pepperonis.

231 Stephanie Mhmm.

232 Shelly Uhm, plain, mushrooms, pepperonis.

233 Stephanie OK.

234 Shelly OK, and then you have plain, peppers, sausage, mushroom.

235 Stephanie Mhmm.

236 Shelly Plain, peppers, sausage, pepperoni. Plain, peppers, mushrooms, pepperoni.

237 Stephanie OK.

238 Shelly Plain, sausage, mushroom, pepperoni. And then you have plain, peppers, sausage, mushrooms, pepperoni. With all five.

239 19:56 Stephanie This is... the same thing as there. OK. That's sixteen.

240 20:05 Shelly So we got that.(writing on hand-out) Sixteen. List all possible choices, see tree diagram. (laughs)

241 20:14 Stephanie Alright, if you're going to fill yours out, then I'm just going to leave mine.

242 Shelly Unless they wanted me to copy it.

243 Stephanie They probably will.

244 Shelly Pl equals plain. P equals peppers. (sighs) P equals... mushrooms. P equals..... pepperoni. Find a way to convince each other you have accounted for all possible choices. Does that mean?

245 Stephanie Are you convinced?

246 Shelly Each other?(points around) (laughs at Stephanie)

247 Stephanie Yes, I think that means each other. I'm not sure though.

248 20:59 Shelly Alright, sixteen, we're done. (reading from hand-out) Suppose a fifth topping, anchovies, were available...

249 Stephanie You know, there's got to be an easier way to figure out, you know, than just adding, anchovies.

250 Shelly Did we, we didn't repeat ourselves anywhere, right?

251 Stephanie What, with that?

252 Shelly Yeah.

253 21:14 Stephanie No. We're good.

254 21:24 Shelly No, wait. (looks over paper)

255 21:48 Stephanie Um.

256 22:01 Shelly There's got to be some easier way to do this.

257 Stephanie That's something. (unintelligible word)


258 Shelly I just can't go about it. Cause if you do f--, OK, so if you do five plus four plus three plus two plus one, just fi--you get fifteen and that's one off.

259 Stephanie That's wrong. So...


260 Shelly So... oh...

261 Stephanie Ah, um... Alright, you know what, let's just add the answer of these, and maybe that we'll see like a pattern.

262 Shelly OK.
 263 Stephanie Yeah. Bobby, do you know anything, that you're not like...(Shelly laughs)
 264 22:29 Robert No.
 265 Stephanie Willing to share, cause you're like off in your own little world.
 266 22:37 (The camera stops focusing on Robert's work as he puts down his pen, it had been on him since 21:55)
 267 Shelly (mumbling to self) Pepperonis, anchovies, (unintelligible word), anchovies, (unintelligible word), anchovies.
 268 Stephanie Are you done?
 269 Robert No.
 270 Stephanie Adding anchovies?
 271 Shelly Oh, I was going to say, wow that was quick.
 272 Stephanie I was like, alright, someone's super genius. Or I'm just retarded.
 273 22:55 Shelly One, four, wait a minute, one, four, six, four, one. So the next one would be one, this is the... triangle.
 274 Stephanie The triangle. Yeah, so the next one is one, five, ten, ten, five, one.
 275 23:05 Shelly We're done. Ahah.
 276 Stephanie But what does that mean to (mouths me, everyone starts laughing). What does that mean to me.
 277 Shelly I don't know. I don't know, but that's the answer. Um...
 278 Stephanie But i--like, what is one, four, six, four, what... that means nothing to me.
 279 Shelly It means nothing to me either. But, it's the pattern we saw.
 280 Stephanie Oh, dear lord. Uh, so we have a pattern, but how to we apply it to getting sixteen pizzas.
 281 23:32 Stephanie Are we even sure... (interrupted) that that's...
 282 Shelly That, that, that would be the problem.
 283 Stephanie ... the pattern. Like, are we positive that this is the pattern that's happening.
 284 Shelly Well, if our answers are right, then that's a pattern.
 285 Stephanie OK.
 286 Shelly If our answer is wrong, then it's not.
 287 23:42 Stephanie So, well, OK. Let's figure it. This is saying that we have one plain pizza, and then we have 4 pizzas with two toppings?
 288 Shelly With one. Because it's the plain and then we go one toppings.
 289 Stephanie O.K. So, we have 4 pizzas with one topping and we have 4 pizzas with two toppings, oh no-we have six pizzas with two toppings, 3 pizzas with 4 toppings...
 290 24:00 Shelly And one pizza with 4 toppings.
 291 Stephanie O.K. So, three of one pizza. How many piz...? See, but it doesn't add up...to sixteen.
 292 Shelly This? Doesn't? Am I just..
 293 Stephanie Oh, yes it does (Stephanie and Shelly laughing). So the answer..give me a second I'll add it...32?
 294 Shelly This or this? (pointing on the paper)

- 295 24:40 Stephanie Oh this (pause) well, I'm hoping I got this right the second time, but I'm saying this is 16 and this ...32. (rising hands, happily).
- 296 Shelly If you add this its 30(Staphanie joins) ..two . So you messed up on the first and I messed up on the secnd. Tying to figure out these problems and cant even add... So, it will be 32,
- 297 25:00 Stephanie O.K...(pause, they think)
- 298 Shelly There's got to be an easier way to do it.
- 299 Stephanie No, that was pretty easy, because look, look, now we know what all the pizzas are. What's this the three ones? Is that what it is/
- 300 Shelly Two ones, right? No that could be wrong.
- 301 25:28 Stephanie Is it 1, 2, 1 and then 1, 3, 3, 1 (filling on a paper the Pascal triangle) 4, 6, 4, 1 and then 1, 5, 10, 10, 5, 1.
- 
- 302 Shelly Yeah, that's it.
- 303 Stephanie That works right? (both watching the paper where Stephanie drew Pascal Triangle) If you have one pizza, you one pizza.
- 304 Shelly Yeah
- 305 Stephanie If you have one pizza with two toppings. You have one pizza with two toppings, you have plane pizza and pizza with both toppings. Yeah, all right...
- 306 Shelly That's good to me (Shelly is reading the written problem on the page) Why?
- 307 Stephanie Why?
- 308 Shelly (writing) See ditto.
- 309 Stephanie See diagram
- 310 Shelly See the triangle (both laughing)
- 311 Stephanie Whose class is it?
- 312 Robert Mr. Pentozzi
- 313 25:55 Stephanie It's kind of a big class.
- 314 Robert It used to be the sewing room.
- 315 Stephanie Used to be awhat room?
- 316 All except Sewing room
- Stephanie
- 317 Stephanie Oh sorry
- 318 Shelly So are we done. I hope so. Just sitting here I feel bad not doing anything. Oh, oh!! I draw attention

319		Stephanie	You should've acted like you were busy... (Gina is coming to their table)
320	26:13	Gina	(Looking at their work sheet) A lot of stuff going on here. So, what did you come up with?
321		Stephanie and Shelly	16 and 32.
322	26:34	Gina	16 and 32?
323		Stephanie	Yes.
324		Gina	Let's see, the question is "How many choices for Pizza a customer has, list all possible choices, and your answer is 16 and 32?"
325		Stephanie	Yes!
326		Shelly	All possible choices are on that diagram
327		Gina	Ahhaa, and how does that little diagram thing work?
328		Stephanie	Go ahead Shelly (referring to shelly)
329		Shelly	(she points on the diagram and explain). Well it's a tree diagram so, it's like, you have plain and then you have plain with pepperoni...speak (says to herself in low voice) then you have plain with sausage...
330		Gina	Now it's coming back?
331		Shelly	Yehe, now it's coming back. Now.. then you have plain with mushroom, then you have plain with ...peppers.
332		Stephanie	Yes that was pepper, What ever..
333		shelly	Too many p's...(continue to explain from her diagram) plain with pepper and sausage, that's how it works, then it branches down.
334		Gina	I see. O.K. So all your possibilities then ummm...if you had to list them all you would do what? How would that work out? If you actually had to list all of the different choices, write them all out?
335		Shelly and Stephanie	You mean like write them? (Stephanie shows the list they wrote down on the paper).
336	27:15	Gina	O.K. tell me for a minute, how this diagram will help you do that? Like if you were just working from this diagram.
337		Shelly	You put plain and then you put plain with pepperonis...with peppers, that's right. And then if you just keep...
338		Stephanie	(Both pointing to the paper) And if you just kept reading down, it just kind of builds up.
339		Gina	O.K. So you listed them this way and Shelly did the tree diagram.
340		Stephanie	I kind of went the long retarded route.
341		Shelly	Yeah, She did them to make sure...
342		Stephanie	(hesitating)She kind of looked at it in side out...
343	28:00	Gina	O.K. So you checked them?
344		Shelly	That way we could check.
345		Stephanie	Yes, we cross check them.
346		Gina	And you came out with how many?
347		Stephanie	16, I think Bobby and Amy...actually he said 16,so we were like

348		Gina	ah.. (Turn to Robert) How does yours compare with what they do to
			
349		Robert	solve it? How did you solve it?
350	28:25	Gina	The same way. The same way? I see a tree there. Did you make a tree also? O.K. (Gina reads from Robert's solution) Zero equal plain?
351		Robert	Yeah when there is zero toppings it's plain so...
352		Gina	I see.
353		Robert	One, there is one topping on each and then just use the pepper with everything. now since I used the pepper with everything, I just left it out and went on to sausage and instead of putting three, like I did here, I only put two cause I already have sausage and pepper.
354		Gina	I see.
355		Robert	And then I just took the remaining two of that I didn't use and put them.
356		Gina	O.K. So this P2 stands for what?
357		Robert	Oh, Pepperoni, cause there were two P's and I didn't feel like writing out pepperoni.
358	29:05	Gina	That is clever, O.K. That's interesting. O.K. So then you ohmm I guess, you kind of categorized them. Zero Topping, and then one topping. Is that what this is?
359		Robert	Yes
360		Gina	And two toppings, two toppings, four toppings. O.K, there is one with zero topping and there is..
361		Robert	Six oh... Zero there is one. One there is four, two there is six and three, four and...
362		Gina	I see, O.K. and Amy Lynn how did you solve?
363		Amy	Exactly the same way.
364	29:43	Gina	Exactly the same way?
365		Amy	With the tree and the 1,4,6,4,1
366		Gina	1,4,6,4,1, What is that?
367		Amy	The way it branches of where you start with one... Exactly that (she points to Shelly's diagram).
368		Gina	And what is exactly this? Where does that come from?
369		Shelly	So, you have one choice there and you have four choices and six and then one.
370		Gina	One choice or...
371	30:00	Stephanie	Yehe, one pizza.
372		Gina	Yehe
373		Stephanie	One pizza with no toppings, and then you have four pizzas with two toppings or with...one, no. (looking at Shelly to get

confirmation) You have four pizzas with one topping and one pizza with no topping which is just plain. Yes, and then you have four pizzas with one topping, like with pepper.

374 Gina Yes

375 30:10 Stephanie You have six pizzas with two toppings like pepper and sausage or whatever, and you have four with three toppings and one pizza with all four toppings.

376 Gina O.K., alright, and that's what you were talking about with the 1,4,6,4,1?

377 30:43 Stephanie Yeah, uh-huh.

378 Gina O.K. So your diagram....(turning to Amy). Can I just see your diagram for a second?

379 Amy I haven't drawn that yet.

380 Gina O.K, but you do..(Inaudible).

381 Amy Yes, I have like...

382 Gina So, you're drawing lines, from topping to topping.

383 Amy Yeah, I write that way...

384 Gina And that's fine. You don't have to do that....(turning to talk to all). So, you are all certain you have the exactly same pizzas and you know 16, you're 16 and you're 16...so you all have same,16. So, I guess what I'm interested in is what's the rest of that here, what are those numbers?

385 Stephanie We were just playing.

386 31:00 Gina Really? (all are laughing)

387 Stephanie We were all....just making sure...we were right.

388 Gina (Inaudible) there were seven or something? Haven't you? Well, what do you mean making sure?

389 Stephanie Because, well, we got, you know, we didn't want to do the same thing to the anchovies (?), and Shelly recalled that we remembered that we have seen it as like the 1,4,6,4,1 and so we just do the rest of it and then...we to make sure that the 1,4,6,4,1 was it...

390 Gina Where did you see the 1,4,6,4,1?

391 Stephanie Right after the 1,3,.....(all laughing)....I know

392 Gina I see, I know, O.K. You remember having seeing it before.

393 Stephanie Yes, we did stuff with it, before.

394 Shelly Yeah, this problem...(All turning to Dr. Maher, that is getting closer to their table).

395 Dr. Maher Can I ask you a question? Do you mind? (looking into their work sheet) 1,4,6,4,1 you talk. I was listening to your explanation. Do these (points to their paper) have some relationship to how many pizzas? When you can select from four topping? Right?

396 Stephanie Uh-hmm

397 32:00 Dr. Maher And this row?

398 Stephanie Oh, that's the answer...the next

399 Shelly That's the answer to the anchovies.

- 400 Stephanie That was the anchovies pizza.
and Shelly
- 401 Dr. Maher So tell me what these numbers mean?
- 402 Stephanie One plain pizza, oh...five pizzas with one topping, 10 pizzas with two toppings, 10 pizzas with three toppings, five pizzas with four toppings and one pizza with all five.
- 403 Dr. Maher O.K. Now my question. This is my question. O.K? How did you get this triangle so fast?
- 404 Stephanie Cause, we remembered, Oh we didn't like all of a sudden...
- 405 Dr. Maher How did you get from one row to the next? From the 3rd row to the forth? From the forth to the...
- 406 Stephanie Cause, you leave the one, and the one plus three is four and the three plus three is six and the one plus three is four and then the 1 and 1.
- 407 32:47 Dr. Maher
-
- Now, this is my question. You told me what that meant by pizzas and toppings, right? When you have four topping to choose from. And you told me what this meant when you have five toppings to choose from. Can you show me thinking about pizzas/ toppings, Why, for instance, the four plus the six is the ten? You told me what that meant in pizzas, right? Could you tell me what's that mean in pizzas? That four? (Pointing to their Pascal triangle) you know what kind of pizzas they are? And you know what kinds of pizza these are? And you know the kind of pizza these are?
- 408 Stephanie Uh-hmm
and Shelly
- 409 33:21 Dr. Maher I'd like you to explore why that works with the pizzas? and We're gonna leave you out. Do you understand my question?
- 410 Stephanie Uh-hmm
and Shelly
- 411 Dr. Maher O.K (she is leaving the table).
- 412 Shelly I think you explain it right after we do another tree diagram .
Another ...
- 413 33:34 Stephanie So go ahead Shell...
- 414 Shelly O.K.(they start to draw the diagram, Stephanie draw the triangle and signing arrows from the 4 and the six to the ten, in the next row).
- 415 Stephanie O.K , do you know what?
- 416 Shelly No. But I have the tree diagram done. (They both laughing).
- 417 Stephanie O.K. I got to explain...What if we start, what if we do it like up here. Because it is gonna go all the way down. And it's gonna be

lot easier to do it with one topping and one...

418 Robert What's the top number? Is that zero topping? Or what?

419 34:46 Stephanie No, that's one plain, that zero. I guess that's zero toppings? Well that's zero toppings, that's a plain pizza. The next row, we have a plain pizza, and then we have two pizzas with one topping, right?

420 Shelly Yeah.

421 Stephanie And then we have one pizza with both toppings.

422 35:07 Shelly Yeah, O.K.

423 Stephanie Right?

424 Robert So, this is no topping, one topping..that's how it goes?

425 35:14 Shelly Yeah

426 Stephanie Right, right, I ...yeah.

427 Robert Then I don't think it works.

428 Stephanie No, it works! We just don't know why but it works! (both laughing). Ohm-yes, cause this is a plain pizza, if we had plain...

429 Amy Plain is zero topping....(inaudible)

430 Shelly So, you see you count ach...you count plain as a topping, and if you don't count plain as a topping.

431 Stephanie So does that make....

432 35:41 Shelly We have to count it as a topping...

433 Stephanie But it doesn't really matter. Because even if we count a plain as a topping, here it will still be repeated...if we only had...if this row stands for one topping...this is wrong, right? (asking Robert)

434 Robert Yeah

435 Stephanie That's what you were thinking? (to Robert)

436 Robert Yes

437 36:04 Stephanie But, like past that it works. Right? (laughing). I think..the pattern works.

438 Shelly I know at the top...

439 Stephanie No. But if it stands for the two toppings then it works. Then this stands for three toppings, this works for four. And this works for five.

440 Shelly Yeah, so that's, this is one topping.

441 Stephanie Two, and this is...

442 Shelly No, because if that's one topping, then you have one with topping.

443 36:31 Stephanie Let's just ignore the top...Let's work from three to four.

444 Shelly O.K

445 Stephanie I guess we just have to...


446 Shelly No. The diagram oh... (all writing, and thinking in silent). ...1,3,3,1 and that's with three toppings.

447 Stephanie But do we understand how the one and the three get to be the four? O.K where is the one?

448 Shelly Plain.

449 37:01 Stephanie And these are the three

450 Shelly Yeah

- 451 37:06 Stephanie O.K, we have to do one for the four, the one and the four too, so....you know...
- 452 Shelly Uh-hmm
- 453 Stephanie Cause otherwise we just have one half of the....(thinking, looking into the diagrams...inaudible)
- 454 Stephanie How can you prove how this makes this when you're adding, when you don't have a fourth topping? (throwing her pen on the table) Do you know what I'm saying?
- 455 37:26 Shelly Yeah
- 456 37:58 Stephanie Like they are saying...It's like you are materializing a topping.
- 
- 457 38:09 Shelly Yeah
- 458 Stephanie Like I can't add plain to the pepperoni and make a sausage all of a sudden...and that's how one and three make four. Ohh yeah...I don't understand.
- 459 Shelly It's the...
- 460 Stephanie I need help, I don't understand.
- 461 Shelly There has got to be some way to do the top of the triangle(stop for a moment and continue) So, this was for four toppings, right? Five choices, four toppings this works for five toppings. This works for three...
- 462 38:36 Stephanie But this, this only five toppings if we count plain as a topping.... (continue to think silently)
- 463 Shelly No, that's if you don't count...
- 464 Stephanie No, because look..
- 465 Shelly Look...(turning to another page she wrote before) one, two, three, four, five and it's kind of stopping.
- 466 Stephanie O.K
- 467 39:04 Shelly Five and that's three so, let's try two plain. Pepperoni, sausage, sausage. So, that's for two toppings. So why doesn't it work with one topping?
- 468 Stephanie It works if...we count plain as zero.
- 469 Shelly Yeah (Stephanie writes zero in their diagram and both are laughing)
- 470 Stephanie Oh, so we understand why that works. But I still don't understand how we can answer the question, when you can bring a new topping into it.
- 471 39:29 Shelly I know...
- 472 Stephanie Cause, bringing up... I don't understand... Yeah, like I understand that one plus three equals four.(Shelly is laughing) and I understand that this row comes after this row, but I don't

understand, and no matter how many times I add pepperoni and sausage, I'm not going to get a mushroom pizza. And that's what we are doing, here...you know what I mean? We are getting another pizza.

473 Shelly Yeah

474 39:42 Stephanie Like a completely different pizzas, that cannot be...

475 40:03 Shelly Bobby can you...do you know?

476 40:14 (Inaudible..talking in a lower voice or together)

477 Robert Why don't we do it for one topping then two topping, then see if we see a pattern? You know we can do something based on the pattern.

478 Stephanie Yeah, that's fine.

479 40:33 Robert You can write it though.

480 Shelly You have like one topping..but

481 Stephanie But, see I don't, see I still don't understand like the point of the question. You know what I'm saying? Because..

482 40:47 Shelly I think I understand the question, I just don't understand how to explain it.

483 Stephanie Because I don't know how it can...You know. I think maybe if it was applied to something else it could be explained. I think maybe I can't get past the fact that you can make a pizza out of other pizzas. I think maybe if it was applied to something else I could look at it differently. I'd be like oh...(all continue thinking silently)

484 41:06 Robert Like towers.

485 Stephanie I knew, someone was gonna say that, it would be them. Alright, let's apply it to towers then. Go ahead Bobby. (all laughing)

486 41:29 Stephanie No, wait wait. So this one is just one tower?

487 41:37 Shelly uhmm (both laughing) We need colored pencils.

488 41:45 Stephanie Don't start...Thank you Mr. Pentozzi (He brought some markers to the tables)

489 Mr. Pentozzi Becareful what you asked for

490 Shelly This is going to lead us nowhere, I know where it's going to be exactly...

491 42:05 Shelly and Stephanie (talking together, half sentences, not understood)

492 Stephanie I have no idea how to do it, I just don't...(Shelly is mumbling something not understood)

493 42:14 Shelly Well atleast we get to have fun with colors

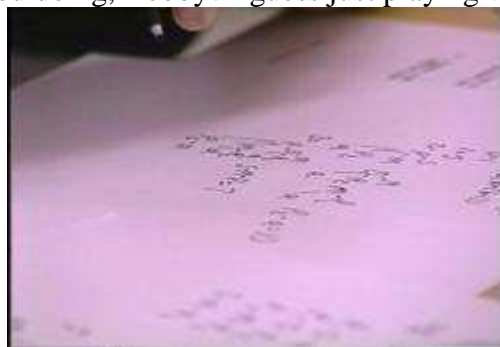
494 Stephanie But there is no point, do we think...it's going to help work with the towers?

495 Robert It would be fun though.

496 Shelly I think it's just going to get us to the same place we are now. (Robert say something not understood)

497 Stephanie So, go a head Bobby, here is a red pencil (all laughing). But I'm,

- I just don't understand. You know I can.. I can't see it..how do you get from the three to the four? We know four means that we have four pizzas with one topping, right?
- 498 42:25 Shelly Yeah, well, see, O.K, so this is the one...
- 499 42:37 Stephanie (Interrupting) .One..I'm sorry go a head
- 500 Shelly This is the one with two topping here, right? So, you have one with one topping, no that's with no toppings. And this..
- 501 Stephanie Wait, I forget we're counting, we are not counting, I'm getting confused. O/K we are not counting plain as a topping.
- 502 Shelly This is with no toppings.
- 503 43:23 Stephanie Yes.
- 504 Shelly So this is with one topping.
- 505 Stephanie Got it.
- 506 Shelly And this is with no topping.
- 507 Stephanie One top...and this and this are supposed to make this...but all we are doing is adding an extra topping.
- 508 43:28 Shelly So that's it, you are adding an extra topping and that's how you get the four
- 509 Stephanie But this doesn't explain that. We got understand that. But, see the problem is there is no like extra topping....you know.
- 510 43:44 Shelly You get an extra topping from one.
- 511 Stephanie But one doesn't have an extra topping (they put the pencils and think, kind of desperate then Stephanie continue). Like no, I know what you are saying, but it doesn't make sense like when you say it. You know because like one is no topping, so adding one to three doesn't materialized another topping, it just gives you a plain pizza. You know.
- 512 Shelly I don't really...
- 513 44:13 Stephanie No it's not we are right! That's not the problem. What do you think Amy?
- 514 Amy Im just as confused as you are. What do you think? (asking Robert)
- 515 Shelly What are you doing, Bobby? I guess just playing with the letters.



- (laughing).
- 516 Stephanie I think if Mr. Pentozzi can bring us markers, he can come over and help us, That's what I think. (Shelly is laughing and all are looking to the direction where Mr. Pentozzi sits)
- 517 Mr. (coming to their table and sit next to Stephanie) What are you

Pentozzi



doing?

- 518 Stephanie Yes. O.K we had to do the pizza problem, right? and we know the pizza and we know the pattern. But, now they want to know how like one and three pizzas make four pizzas? And I understand the question, and we understand how it works. Do you understand what I'm saying?
- 519 45:00 Mr. Pentozzi I think
- 520 Stephanie O.K. and we understand how like, this works, you know. And we understand how we got the answers to the pizzas problem. But we have no idea how one pizza and three pizzas make a whole new category of four pizzas. Because, this is one plain pizza, right? Like this one right here is plain and these are three pizzas with one...
- 521 45:17 Mr. Pentozzi One topping. Out of four toppings/What are the toppings? There's four possible toppings?
- 522 Stephanie Oh, this, there is three topping here, right?
- 523 45:38 Shelly In this row, yeah.
- 524 Stephanie And there is four topping here. There is three topping here. Um, let me just...cause, I'm like...O.K there is two topping here, there is three here, and there is four here. Ok, we know that this makes that. This is just one plain pizza. This is three pizzas with one topping. This is one plain pizza, and this is four pizzas with one topping. If you add one plain pizza to three pizzas with one topping, you get, like, one pizza with no toppings and three pizzas with one topping. You don't, you know, get four if you're using the pattern here, but like, in reality you don't get four. You know, so I don't know how to answer the question (drops her pencil to the table).
- 525 Mr. Pentozzi You're trying to explain what this four means in terms of pizzas, is that what you're trying to say?
- 526 46:08 Stephanie I don't know, she, they wants to know how we got four pizzas out of one pizza and three pizzas.
- 527 46:58 Mr. Pentozzi I overheard you saying that if this wasn't pizza then it might be possible to do.
- 528 47:02 Stephanie Yeah, if this wasn't pizza I could do it. But it's not possible with pizzas. So...
- 529 47:06 Mr. Well, using the notation you have here, you're saying this is one

530	46:09	Pentozzi Stephanie	Plain pizza.
531	47:15	Mr. Pentozzi	There's only one way to get a plain pizza, and there's three ways to get a pizza with
532	47:18	Stephanie	One topping.
533	47:20	Mr. Pentozzi	Pizza with one topping, choosing from
534	47:22	Stephanie	Three toppings.
535	47:23	Mr. Pentozzi	Three different toppings. And then this four, using the notation that you're using, this means that four ways to get pizza with one topping
536	47:25	Stephanie	One topping (together with Mr. Pentozzi). Out of four toppings.
537	47:26	Mr. Pentozzi	And when you go from one column, one row to the next, you're going two toppings, three toppings, four toppings.
538	47:35	Stephanie	Yeah.
539	47:42	Shelly	You could say it works because ok, you're discarding the plain pizza. So when you add the plain to the one toppings here, that's like another topping. It's just like plain was a topping. You know what I'm saying?
540	47:46	Stephanie	Yeah, I know what you're saying. But, like here, that doesn't stand for a plain pizza. Like these four pizzas don't stand for plain pizzas, at all.
541	47:48	Shelly	I know.
542	47:59	Stephanie	None of them are plain.
543	48:07	Mr. Pentozzi	The only thing I think I can add is that at this point is that my father used to own a pizzeria. Really.
544	48:10	Stephanie and Shelly	That's helpful (smiling sarcastically).
545	48:15	Mr. Pentozzi	I never figured out how to toss the things up (motioning like tossing pizza dough). My little brother learned how to do that but I couldn't learn how to do that. But, I mean, when I see this...
546	48:20	Stephanie	We've all got our handicaps (laughing).
547	48:22	Mr. Pentozzi	Yes, we all - that's true, that's true. Um, I would have to put the toppings on the pizza though. The only thing I'm seeing, the only thing that makes, the only thing your saying which makes me think is that - it's like you've done, you've seen what you can do with three toppings and then my dad comes along and says well, I need you to put on a fourth topping.
548	48:30	Stephanie	Uh-huh.
549	48:32	Mr. Pentozzi	That's just, I don't know, it might not be helpful at all. I'm just thinking about going from here to here seems like adding on another topping in some way.
550	48:49	Stephanie	Uh-huh. I think we understand that. Like, we understand that when we're doing the problem we're adding toppings on, and that's why this is working out here. And we understand that.

But it just doesn't make sense. This doesn't produce that. Even if your dad tells me to put another topping on it. Throwing that plain pizza in there isn't going to make a difference. You know? Well, what's the new topping on the fourth row?

551 48:50 Mr. Pentozzi

552 49:00 Stephanie Mushrooms. We had pepperonis, sausage, peppers and mushrooms.

553 49:26 Mr. Pentozzi Um, since I haven't really thought about this, um, when I look at that I'm thinking... well, now my dad says put pepperoni on it, but I had a plain pizza, so now I'm putting pepperoni on a plain pizza. And I'm seeing one there. Maybe this is the pepperoni being added on, or something.

554 49:30 Mr. Pentozzi You know I exhausted what I could do with three, and now, as you said, put the pepperoni on top of the plain and maybe that's one of the four, or something. But I don't know if that explains what

555 49:32 Stephanie So what you're saying is that this pizza is just the pizza with the topping thrown on it.

556 49:55 Shelly Yeah, like treat it like a pizza with a topping. Because you're discarding the plain, because the plain is here already. So treat it like a pizza with a topping, and just add it the other toppings.

557 Stephanie Alright.

558 50:10 Mr. Pentozzi But I'd ask then, does that explain like- three and three make six for instance.

559 Stephanie Well, so this is...

560 Mr. Pentozzi If you come up with something that could explain one of them, then maybe, as I keep coming up to against in class, it explains one thing but it doesn't explain something else.

561 Stephanie Ok, so this becomes, if this becomes a pizza with one topping, for this one right, then this becomes a pizza with two toppings, and three two-toppings plus three two-toppings equals six two-toppings.

562 50:29 Shelly Yeah.

563 Stephanie That's how, if we were explaining it how, so this is one pizza with everything. Right?

564 50:50 Shelly But then that would become one pizza with two toppings, right?

565 Stephanie No, one pizza with three toppings.

566 Shelly No, yeah.

567 51:00 Stephanie And this would become three pizzas with three toppings. For this row. Right?

568 Shelly I think so.

569 Stephanie To get four pizzas with three toppings.

570 Shelly Yeah, so that would become three toppings.

571 Stephanie Yeah, that's what I said.

572 51:20 Shelly Ok.

573 Mr. So what did you just say? Does that thing I suggested sort of

		Pentozzi	work for this too?
574	51:25	Stephanie	Uh-huh.
575		Mr. Pentozzi	So tell me what it means here. This had three toppings then it becomes four toppings. What happened there?
576		Stephanie	Ok. Well this is one pizza with three toppings. So this becomes three pizzas with three toppings. Is that, does he have it better, because let him do it (motioning towards Robert).
577		Mr. Pentozzi	No, this is something we did in class - that he's working on. Ok, I'm sorry.
578		Stephanie	So, um.
579	51:50	Mr. Pentozzi	I interrupted you.
580		Stephanie	So this becomes three pizzas with three toppings, so then three pizzas with three toppings plus one pizza with three toppings is four pizzas with three toppings
581		Mr. Pentozzi	This is three pizzas with...
582	52:00	Stephanie	Three toppings – for this one. For this one... right? (Looks at Shelly for help).
583		Shelly	(Mumbles inaudibly, looks frustrated).
584		Mr. Pentozzi	I'm just trying to get, I'm just trying to get the language right. This is...
585	52:15	Stephanie	I have to start all over, I have to go from here, because I'm forgetting what I'm doing. Ok this – to get four pizzas with one topping you already have three pizzas with one topping, and the plain pizza becomes the pizza with the new topping.
586		Mr. Pentozzi	Ok.
587		Stephanie	Ok, so this becomes, instead of one plain pizza this is one pizza with one topping – because this one's getting like the pepperoni thrown onto it.
588	52:35	Mr. Pentozzi	Ok, ok.
589		Stephanie	And that produces the one, the four pizzas with one topping.
590		Mr. Pentozzi	This is four pizzas with one topping. You didn't need to add anything to these? These just sort of became these?
591		Stephanie	Those were, those just got brought down.
592		Mr. Pentozzi	Those got brought down. Ok.
593		Stephanie	Those are the same three pizzas. So then here, you have six pizzas with two toppings. Now you already have three pizzas with two toppings, so these three pizzas with one topping get an extra topping added on.
594	52:55	Mr. Pentozzi	Ok.
595		Stephanie	So these become three pizzas with two toppings. And then three

pizzas with two toppings plus three pizzas with two toppings become six pizzas.

596 Mr. (Together with Stephanie) With two toppings. But now you're
Pentozzi choosing from four now, right?

597 Stephanie Yes. So now this is right, I'm not... right? Right?! (Looks
around).

598 53:21 Shelly Yes. (Students laughing softly).
599 Mr. How about that last one? Now this is three, this is, there's only
Pentozzi: one pizza that has all three of these toppings and-
600 Stephanie Yes.
601 Mr. -and how does that move into here?
Pentozzi:
602 Stephanie That just drops down.
603 53:40 Mr. Oh, it just drops down. Because it still has...
Pentozzi:
604 Stephanie: Yeah, so that's the one pizza with three toppings. And then you
need, then these become... these all get an extra topping added
onto them. Like these are three pizzas with two toppings, so they
all get the extra topping that you would have here. Like the
pepperoni that is here, or whatever, gets thrown onto these three
pizzas that don't have pepperoni, but have two other toppings. So
now there are three pizzas with three toppings.

605 Mr. Ok.
Pentozzi:

606 53:45 Stephanie: You add them to the one pizza with three toppings and you get
your four pizzas with three toppings.

607 Mr. Now I understand all of that, but I don't know if that's the answer
Pentozzi: to the question.

608 Stephanie: I hope so. Thank you. And here... (hands colored markers back
to Mr. Pentozzi).

609 Mr. Let me see that again (looks at Robert's notebook). That just
Pentozzi: made me think of a thing you were doing in class, with the adding
of the...

610 Robert: Oh yeah, it's kind of, uh... (inaudible).
611 54:30 Mr. Because we were adding, the thing you came up with, you were
Pentozzi: adding on, what did I ask you to do? Wasn't it something like
this? Doesn't adding up numbers like this...?

612 Robert: Yeah, but it was like adding, so... (inaudible).
613 54:40 Mr. You didn't use what you used in class on this?
Pentozzi:

614 Robert: Because... well, can I show you.
615 54:55 Mr. Yeah.
Pentozzi:

616 Robert: Alright, see this is four toppings right here. And one plus four
plus six plus four plus one equals sixteen, and two the fourth is
sixteen. And three toppings- one, three, three, and one, is eight.

And two to the third is eight. And then one, two, one is four, and two to the second is four. There's supposed to be two up here – and one plus one is two, and two to the first is two. And then we got thirty-two for the next one and we add that up there and we get thirty-two, and two to the fifth is thirty-two. So I guess it's two to however many toppings is the number of combinations there are for pizza toppings.

617 Mr. I wonder whether like, whether the two plays any role in it – like
Pentozzi: why all these have two. Is that something you thought about?

618 Robert: Yes, I remember something with towers that we did to find the
total number of combinations – it was two to the something....

619 Mr. Does that apply here?

Pentozzi:

620 Robert: Yeah, it's the same thing.

621 55:55 Mr. With pizza toppings?

Pentozzi:

622 Robert: Because...

623 Mr. But there are so many different pizza toppings, it's not like
Pentozzi: there's green and purple, or whatever colors you used.

624 56:00 Stephanie: Uh-huh.

625 Robert: So, I guess like, if you want to find out if there's ten toppings you
just do ten- two to the tenth, then you got how many
combinations there are.

626 Mr. Well, why does that work, I mean if there's ten different toppings,
Pentozzi: I'd figure you'd have to go through... I don't even know if there
are ten toppings you'd want to put on a pizza, well I guess there
are.

627 Robert: I think it's more like, it's something like if you have... (writing
something) and then you go to **b**, and you have **a**, you keep going
by two, like this is one, this two, and you keep adding... I forget
what it was, but we did it before.

628 56:20 Mr. We did? (Gestures around the classroom).

Pentozzi:

629 Robert: No – Rutgers.

630 Mr. Oh, Rutgers.

Pentozzi:

631 Robert: And then we made a branch or something, and it just kept going
by two. And then something... uh, I forget.

632 Amy- ...by two.

Lynn

633 Mr. What was that?

Pentozzi:

634 57:01 Amy- That was with a lot of problems- that went by two. (Trails off,
Lynn inaudible) ... with towers.

635 Robert: I forget how we figured out something.

636 Mr. Ok, now there is something that- that is interesting by itself. This

		Pentozzi:	I thought, I remember I think I asked you in class to add these numbers up and how you figured this out.
637		Robert:	Yeah, it was like, if it was like one four sixteen, or something, it was like all multiplied by four, and then, well how would you know, like, what the answer is?
638	57:08	Mr. Pentozzi:	What the total sum of it would be? Yeah, that came up (trails off, inaudible).
639		Robert:	I don't think that works here though.
640		Mr. Pentozzi:	Ok, that doesn't apply to what you're doing. Well I just wonder whether how the two, you know, if the two really does apply here as it seems to from what he came up with – what that might mean.... See, you bring me in then you get another question. I just can't...
641		Stephanie:	All I needed was help, then you should have just left.
642	57:31	Mr. Pentozzi:	Just had to ask another question, I understand.
643		Stephanie:	Um. Great, thank you.
644		Mr. Pentozzi:	Well, you can think about that. I just came in impromptu, this was not, um... some students in my class like to think I stage not understanding something, and sometimes I do, but not that time.
645	57:50	Stephanie:	Great.
646		Gina:	So you're able to answer Dr. Maher's question yet?
647	58:05	Stephanie:	Yes, sort of.
648		Robert:	Not why.
649		Gina:	Not why?
650		Robert:	I guess, got four but not three.
651	58:10	Gina:	What do you mean you got four but not three?
652		Robert:	Because three is convinced, but don't know why.
653		Stephanie:	What are you talking about?
654		Gina:	Dr. Maher's question was how you go from these numbers to these numbers, right?
655	58:20	Stephanie:	Yes.
656		Gina:	And were you able to answer that?
657	58:25	Stephanie:	Yes.
658		Gina:	Ok, could you explain it to me please?
659		Stephanie:	Yes. Ok, this row is a row with pizzas where you can have three possible toppings. Ok?
660		Gina:	Ok.
661		Stephanie:	The next row is a row where you can have four toppings on your pizza. The question was how to get from like one pizza with one topping and three pizzas with two toppings, or I'm sorry, one pizza with no topping, no I'm sorry, like just a plain pizza
662		Gina:	Uh-huh.
663	58:45	Stephanie:	And three pizzas with one topping to four pizzas with one topping. And we had trouble, well I had trouble, because I kept thinking that you guys were like, we were like, materializing this

pizza out of nowhere. But then Mr. Pentozzi came over and explained that what happens is we already have three pizzas with one topping and the three pizzas just kind of get dragged down because they're still going to be here. Like, we'd still have a pepperoni pizza, a mushroom pizza, and a sausage pizza here, but we're adding peppers. So the plain pizza becomes a pizza with peppers on it.

664 Gina: Ok.

665 59:10 Stephanie: You throw peppers on the plain pizza, so now you have – this becomes a pizza with one topping, and you add that to the three pizzas with one topping and you get the four pizzas with one topping.

666 Gina: Ok.

667 59:48 Stephanie: Then, to get from there to there, this is six pizzas with two toppings

668 Gina: And where do they come from?

669 Stephanie: You already have three pizzas with two toppings, so these pizzas, like if you're adding on peppers- these pizzas are all pizzas without peppers-

670 60:05 Gina: Ok.

671 Stephanie: So you add peppers onto them, and they become three brand new pizzas with two toppings. So if you add the three new pizzas with two toppings to the three old pizzas with two toppings, you get the six new pizzas, or the six whatever – the six pizzas with the two toppings.

672 60:15 Gina: Ok.

673 Stephanie: Um, (notes to herself) these are two toppings. The four pizzas have three toppings. We already have one pizza with three toppings – and that's the mushrooms, the peppers, and the sausage, or whatever. Here you're adding pepperoni, so you add the pepperoni to these three pizzas that don't have pepperoni, and you get three new pizzas with pepperoni plus the one old pizza, and you have the four new pizzas with three toppings.

674 Gina: Ok. Now, how is it that you can take these pizzas and move them here and move them there? I don't understand that.

675 60:33 Stephanie: Because, you're.... Ok, these in this column aren't changed.

676 61:06 Gina: Those are unchanged, ok.

677 Stephanie: If you're bringing them down to the sixth – these three pizzas remain the same. But, when you're moving them here, these three pizzas are getting an extra topping.

678 Gina: Ok, so these three pizzas... then, you have different things that could happen to them?

679 Stephanie: Yup.

680 61:35 Gina: They can either remain-

681 Stephanie: -the same

682 Gina: The same, or they can get another topping. So each of those

pizzas – what happens to each of those pizzas?

683 Stephanie: What do you...?

684 61:45 Gina: When you're moving down a row. For example, what's the...
could you give me an example of, let's do this one because it's
easier.

685 Stephanie: Alright.

686 Gina: Can you give one example of a pizza in that category?

687 Stephanie: In the three pizzas with one topping?

688 Shelly: She's saying you literally move it down. So if you move these
down, how can you move it down there, if they're already down
here? Because... isn't that what you're saying?

689 62:01 Stephanie: Oh.

690 Shelly: Is that what you mean?

691 Gina: Well, I'm not quite sure if that's what I mean. What I mean is-
how come you get to do two things with that? How can it be
there and there?

692 62:10 Stephanie: Because they represent, they're representing different pizzas, like
different kinds of pizzas, in each thing.

693 Shelly: Combinations.

694 62:20 Gina: So can you give me an idea of one pizza in this category?

695 Stephanie: Um, a pizza with peppers. Say you've got-

696 Gina: So here's the pizza with peppers, right?

697 Stephanie: Yes.

698 Gina: Ok, to move it down into this, you're doing...

699 62:33 Stephanie: -nothing to it. It becomes just a pizza with peppers.

700 Gina: To move it here, you are-

701 Stephanie: Adding the new topping so it becomes a pizza with peppers and
mushrooms.

702 62:40 Gina: Ok. So that's the pizza with peppers. It can go here or here?

703 Stephanie: Yes.

704 Gina: And the pizza with-

705 62:50 Stephanie: -peppers and mushrooms-

706 Gina: -can go here or here, and so forth.

707 Stephanie: Yes.

708 Gina: So each of these pizzas-

709 Stephanie: -has two, like, spots.

710 Gina: Oh, that's interesting.

711 63:00 Stephanie: One where it has, where it stays the same, and one where it gets
added topping.

712 Gina: So each one has two new things that happen.

713 Stephanie: Yes.

714 Gina: Amy?

715 Amy- Lynn Maybe that's where he got the two to the n. Isn't that what
Bobbie said before? Maybe that's where the two comes from?

716 Gina: I don't know, what do you think? You had how many pizzas up
in this row all together?

717 Stephanie: All together? Um, eight.
 718 63:21 Gina: And how many pizzas in this one?
 719 Stephanie: Sixteen.
 720 63:30 Gina: And the next one you said-
 721 Stephanie: Thirty-two.
 722 Gina: Hmm, I don't know...
 723 Stephanie: It's you know, more of an idea than I had.
 724 Shelly: It makes sense. That's where each of the twos come from. You described the one, because that's used once. Then you have two, two, two – two raised to the third, that's eight, so....
 725 Stephanie: Yeah. That was good, that was really good (looking at Amy-Lynn)
 726 Amy-Lynn You remember that, because I'll be the only one that... (trails off, inaudible).
 727 Gina: This one only goes here? Does it go here too?
 728 Stephanie: Yes. It drops down as a plain pizza.
 729 Gina: I see. Ok, so your drop down idea is that it stays the same?
 730 64:01 Stephanie: It stays the same once, and it changes once.
 731 Gina: Ok.
 732 Stephanie: That's where, I guess, Amy got the two.
 733 Gina: Very interesting, ok. Do you agree with this? (Looks at Robert).
 734 Robert: Uh-huh.
 735 64:15 Gina: Um, that's really really interesting. So where, um, did, I was going to ask you before- where did you first see them? You said you remembered something about these numbers.
 736 Stephanie: Yeah, we worked on it in eighth grade. But, like for a different problem I think.
 737 Gina: Do you remember what problem it was? What else you..
 738 64:30 Stephanie: We worked with towers. I remember Dr. Seiser (?) came, and we did something with towers and stuff. We plotted the towers out in the thing- in the triangle, you know what I mean? There was one tower, then there was one two, and there was one tower...
 739 Gina: Oh really?
 740 64:42 Stephanie: Yeah, I don't know like what the problem was or anything, but I remember the triangle is like imprinted on my brain, forever.
 741 Gina: What did the, can you remember what the towers looked like for this row?
 742 65:00 Stephanie: Um.
 743 Gina: Do you all remember the towers?
 744 Shelly: Uh-huh. I remember being in like, the home-ec room and making all the towers.
 745 Stephanie: Oh yeah, but I don't think we had known what it was then.
 746 Shelly: No, we didn't. That's how we learned that (pointing to Pascal's Triangle).
 747 Gina: Do you remember what you did with the towers, what the problem was?

748 65:25 Shelly: It was probably how many combinations, (speaking together with Stephanie and Amy-Lyn) yeah how many combinations, because there are different colors for the towers.

749 Gina: And how many colors did you work with?

750 Shelly: It depended, two, we probably had two, or three...

751 Stephanie: And then it kept growing and growing

752 65:40 Gina: The number of colors or the-

753 Shelly: Yeah, the number of colors, like how many colors you could... right?

754 Stephanie: Well I think it grew both ways. Because if it grows with colors it's got to grow with (gesturing the height) or you'd get stuck, right?

755 Gina: So, can you remember anything about what these, or can you imagine what these might mean, what these numbers might mean with respect to towers then?

756 65:52 Stephanie: I have to think. Um, this is probably, probably, I don't know, this is probably one with all blue, I don't know, right? Like say we have blue and red. This is one tower with all blue. I don't know how high it would have to be. And then this is three with-

757 Robert: Two blue, one red.

758 66:10 Stephanie: Thank you. Two blue, one red. And then this is three with two red, one blue. And this is one all red. So I guess they are three high, I don't know, go ahead Bobbie, you seem to know what you're talking about.

759 Gina: They would be three high?

760 66:30 Robert: And that's how many toppings there are, right? Like for this one there are-

761 Stephanie: There are three toppings. So next row there are four toppings, or four high, I guess.

762 67:00 Robert: Yeah.

763 Stephanie: With two towers.

764 Shelly: Yeah, they were three high.

765 Gina: They were three high? Would that work? Oh, what were you doing there? (Looks at Robert).

766 Stephanie: Oh look, we came prepared.

767 67:14 Robert: Yeah, I was doing it before.

768 Gina: You were doing it before.

769 Robert: But I didn't know what it had to do with it.

770 Stephanie: He was just playing (laughs).

771 67:25 Gina: So it's... when you were doing the pizza problem you started to think about it in terms of towers?

772 Robert: Yeah.

773 Gina: And what are these ones and twos here?

774 Robert: Oh, two different colors.

775 Gina: Ok.

776 Robert: Two different colored crayons, or something.

777 67:41 Gina: Ok, and how were you thinking about the pizza problem in terms
of what you were writing here?

778 Robert: I don't know, because like I just did one and saw that three and
three matched up, and then did the next one.

779 Gina: (Distracted, looks away). I'm sorry, say that again.

780 67:53 Robert: I don't really know.

781 68:04 Stephanie: Just talking?

782 Shelly: And it makes sense.

783 Gina: You don't really know, but you spent a lot of time on this, huh?
There must be something going on. Where do you, maybe you
can help us out here. What do you see here that makes sense, to
you, that relates to you?

784 Stephanie: What do you mean?

785 Gina: Well, Robert said that he was working on this

786 68:25 Shelly: Those are the toppings right, for the pizza, aren't they?

787 Robert: That's just one of them.

788 Stephanie: Were you doing it as towers or does it...

789 Robert: I guess yeah.

790 Shelly: It's the same thing, so it doesn't really matter.

791 Robert: But that only just gets to this number in the middle, I think.

792 68:35 Gina: Really?

793 Robert: Yeah, I just...

794 Stephanie: You never...

795 Shelly: O.K

796 Robert: I was thinking of something we did. It didn't work, so I just...

797 Shelly: O.K,yeah, I see why, because you're not including like the ones,
you're just saying with four here, you're not saying the ones that
had three with it. You know what I'm saying?

798 Gina: I don't know what you are saying.

799 Shelly: OK. Ok so these like they are four high but with those four high
you also still have the ones that were three high(Stephanie agrees)
You still have those possibilities to make them if you want to.
You understand what I'm saying? (speaking to Gina)...No
(Shelly laughs)

800 Gina: I'm sorry.

801 Stephanie: It's the same thing as with the pizzas, right? That what you are
saying. (speaking to Shelly)

802 Shelly: Yeah

803 Stephanie: Like you're bringing them down but your adding one on to it.

804 69:25 Gina: (pointing at paper) so your bringing these guys down

805 Stephanie: But your adding another one on to it

806 Shelly: Yeah but like his number is only the middle one

807 Stephanie: Yeah he only has the big one

808 Shelly: He only has that one one because he didn't like carry those down
too, like just those three

809 Gina: Ok so this is , if this is in fact the middle one what does that.. I

don't know I'm curious to know why you did this. How you saw this as being helpful to you doing the pizza problem?(Robert says I don't know while she asks) Was this helpful to you or no?(speaking to Robert)

810 Robert Umm well not this part but the towers were. Because like I remembered how we figured out the total combinations. (Interviewer says mhhmm)That the same thing to figure this out. [pause] so I guess its helpful in that sense but this wasn't it just

811 69:53 Gina Is there a.. yeah I was interested when you grabbed that

812 Shelly or It was useless

813 Stephanie We were hoping there was an easier way

814 Gina Laughing: well you never know

815 girls We were hoping there was a really easy way

816 Gina Well I think that the way you did it I mean you started to see some things that made it easier right?

817 Stephanie yeah

818 70:26 Gina I heard you say there's an easier way and look we could get this right away and then you started fiddling with these numbers here. So maybe you did find an easier way. (pointing at Robert's work) I'm not quite sure, I think this is very interesting and I think it is interesting that these numbers that you see them reflected over there(pointing to Stephanie's paper and Pascal's triangle) and I'm just wondering if there is anything any other relationship between these two? Hmm (while looking at papers and turns to Robert) You said something before uh also that interested me uh you said there is supposed to be a two over there or something when you were talking about what you di din Mr. Pentozzi's class?

819 Robert Oh no I think she left like one thing ok because like I have it 1 and then like two 1s, because this is like zero toppings

820 Shelly Yeah I was thinking that

821 71:00 Robert And then with one topping

822 Shelly and Yeah it's supposed to be like one one

823 Stephanie

824 Gina mhhh

824 Stephanie That's why our thing wasn't working

825 and Shelly

825 Stephanie Because we were having.. yeah bobby thanks for telling us before (girls laugh)

826 Shelly So we're all here, so you got anything bobby "nooooo I'm just not doing anything

827 Stephanie Bobby's just drawing and we're like why isn't it working

815 Gina Well he's drawing all kinds of stuff over here so he was very busy. So uh what wasn't working out before when you were missing that other one

816 Stephanie Oh it was just uhh we would jump from like plain to two toppings
and we had know idea why we were just uhh

817 71:46 Shelly Lets say this was plain..Yeah and wasn't even working out for
plain so we were like why isn't working for plain

818 Stephanie Yeah so were like that's not working. Of course bobby had the
answer all along and didn't care to share with us (Shelly laughs)

819 Shelly Got anything over there bobby, no I don't have anything

820 Stephanie But hey we wont hold that against him

821 Gina Well he wanted to give you an opportunity to think about it

822 Stephanie To look really stupid. Great!

823 Gina No no. So I guess I heard you say this but I'd like to go back to
this. If you had um in this problem ..uh oh I lost my problems. Ok
uh there's another one though that should be here in this problem
we have lets see we start out with peppers sausage mushrooms
and pepperoni. So those were 4 toppings. And you said there
were 16 pizzas

824 Robert Yeah

825 Gina Ok. And you listed them in various ways ok. Do you think that
you've convinced each other

826 All Mhmm yeah

827 72:35 Mrs. Yeah, ok and if you added a fifth topping

828 Stephanie yeah

829 Gina You came up with 32

830 Stephanie mhmm

831 Gina Ok what if we did that ten toppings. What if Mr. Pen tells his dad
is choosing from 10 toppings how many pizzas would there be?

832 Shelly Two to the tenth, (All begin laughing)

Amy

833 Amy Inaudible... calculator

834 Stephanie Where's the calculator when you need one

835 Gina What is 2 to the tenth? (Shelly turns pretending to get calculator.)
No (and begin laughing). Surely you could do 2 to the tenth in
your head no? well let's see If you had ten toppings you could
you write them out fairly quickly.

836 all No nervous laughter

837 Gina I mean I know it would take a long time I shouldn't say that I
mean it would take a long time but could you kinda picture what
they would look like

838 Shelly yeah

839 Gina Would this also fit in to this scheme that you have

840 Stephanie Yeah you'd just have to go down a lot (referring to Pascal's
triangle)

841 73:06 Shelly yeah

842 Gina Do you have any idea what the beginning of that row would look
like? Just the beginning

843 Stephanie 1 yeah , (both begin laughing)

& Shelly
 844 Gina 1 and then?
 845 Shelly 10
 846 Gina Ok why would it be 1 what would the 1 represent
 847 Shelly The plain
 848 Gina Ok and what would the 10 represent?
 849 Shelly Well with the 1 toppings because there's 10 toppings
 850 Gina Ok and then way at the other end
 851 Stephanie 10
 852 Shelly 1
 853 Stephanie 1
 854 Gina 10,1 ok the one at the very far right end what would that be
 855 Shelly/Stephanie All toppings
 856 Gina And how about the 10 before that
 857 Shelly/Stephanie 9 toppings
 858 Gina Ok so it would take a longtime but you probably could go all through them
 859 Shelly/Stephanie Mhmm yeah
 860 Gina Ok so then the ultimate question always what if you have n toppings
 861 Stephanie 2 to the n
 862 Robert 2 to the n
 863 73:47 Gina 2 to the n?
 864 Stephanie Yes. 1 n and then whatever
 865 MRs 1 n and whatever oh that's interesting
 866 Shelly And then n 1
 867 Stephanie There you go (laughing starts)
 868 Gina So you get the beginning and then no problem right
 869 Stephanie yes
 870 Mrs. Ok umm oh I had it slipped right out of my head..umm why do you think that its always this 2? Uh why do think this 2 is floating around
 871 Stephanie Oh, Amy
 872 Amy Because you're always bringing down um the toppings that you had before that are like unchanged and then you bring down another set and add another topping to it
 873 Gina Ok ok and that's why you keep getting this doubling affect. Very nice ok lets see if Dr. Maher has..I understand, I understand now you have to see if she understands. I I'm going to (inaudible) so you can have a seat.
 874 74:25 Dr. Maher Ok so you can tell me why you add in Pascal's triangle
 875 Stephanie yes
 876 74:40 Dr. Maher ok

877 Shelly I guess you want us too (begin laughing
 878 Dr. Maher Go Shelly tell me
 879 Shelly Me? Oh but Stephanie did it so well before
 880 Dr. Maher Ok well give you a chance
 881 Shelly Ok so umm ok wait I guess ill just start from this row
 882 Stephanie Yeah do it from that one
 883 Shelly Ok so this is a plain one and this is the three with one toppings.
 (moving down to the next row) So then this is the plain and you
 get this because you make the plain one topping and then you just
 add them to become the four one toppings, does that make sense?
 I can't explain it well
 884 Stephanie No you did it
 885 Dr. Maher Asking Robert: does that make sense?
 886 Robert yeah
 887 Dr. Maher It makes sense to me. ok
 888 Shelly Ok ok and then
 889 Dr. Maher Ok go ahead
 890 Shelly Ok umm and this would be the two toppings and this was the two
 toppings so you make that two toppings and that two toppings
 because like you're adding pepperonis or whatever on to it so that
 becomes a two toppings too. So you add them and become a 6
 891 Dr. Maher And they're all two toppings
 892 Shelly Because yeah because you had those three original two toppings
 up there and those three with one toppings but your adding the
 extra thing on to it so that has to become a 6 and then you just
 lepp explaining
 893 Dr. Maher Ok I understand what you said. And then I heard you say
 something about this looks familiar to you Stephanie when you
 had Professor Spizzer come in
 894 Stephanie Oh yeah we did we did it in 8th grade but we did it with the
 towers
 895 75:51 Dr. Maher Did you all know what she was talking about with the towers?
 896 Amy inaudible
 897 Dr. Maher Could you imagine that in your head what she was talking about?
 Because it's really hard when you don't have it in front of you
 without any pictures or anything you all can follow what
 Stephanie was saying with the towers?
 898 Stephanie Well rob-he was drawing it so
 899 Dr. Maher All of you were Able to follow it. Amy tell me what she was
 saying.
 900 Amy What?
 901 Dr. Maher With the towers how this works for towers
 902 76:19 Amy Like we had done it in class that we had in the sessions and so we
 could just kind of picture what it was like I mean bobby was
 drawing
 903 Dr. Maher So so tell what I'm supposed to picture if now I'm not thinking of

pizzas anymore here. I'm thinking of the same rows, can you tell what I am supposed to imagine in my head with towers?

904 Amy Isn't like so many high and then so

905 Dr. Maher Ok so tell me so this (pointing to a row in Pascal's triangle) would be how high?

906 Amy That would be three high right?

907 Dr. Maher Three high

908 Amy Three high and then its how many colors you have

909 Stephanie (speaking to Amy) remember we did it on the bottom

910 Amy yeah

911 Dr. Maher So

912 Stephanie If you had blue and red

913 Amy If you have blue and red if you had two colors

914 Dr. Maher So what does the one represent?

915 Amy One of all like say blue like three high of all blue cubes

916 Dr. Maher Mhmm ok I could imagine that and what does the three represent?

917 Amy Towers where you have like 2 blue and one red.. umm two red and one blue

918 Dr. Maher So now you have..which is it? Does it matter? If your making this(pointing to the one in Pascal's triangle) all blue and your making this

919 77:18 Stephanie Oh that would probably be. I think that would be one the way that it would work out it would be one blue (Amy and Shelly agree)and two reds and the next would be two blues and one red.

920 Dr. Maher How many blues are here?

921 Stephanie All

922 Amy three

923 Dr. Maher So this your going from three blue to one blue

924 Stephanie Oh I don't know

925 Shelly Three blues to two blues

926 Stephanie Too one blue to none

927 &Shelly

927 Shelly Or you could do it the other way around

928 Dr. Maher Ok. No reds to one red to two reds to three reds(all join here in saying three reds.) ok I could imagine that I n my head. Ok so let's decide on one. You said the three blues so no reds one reds two reds three reds. So tell me where the 4 comes in.

929 Stephanie ok

930 Dr. Maher Well let Amy do it I'm curious. Because she hasn't played with it for a long time right?

931 78:10 Amy Yeah I guess you could say that

932 Dr. Maher You too Shelly you haven't played with this in a while so you can help her.

933 Shelly That would be four like four blue three blue two blue one blue no blue. or

934 Dr. Maher Ok or no red
935 Shelly& No red one red two red three red 4 red
Dr. Maher
936 Dr. Maher Ok I could imagine that. Ok so why, why does one plus three give you 4. You have towers three tall now you have towers 4 tall how does the one plus three give you the four?
937 Shelly Because your just adding the extra block on
938 Dr. Maher What are you adding on from here to here and here to here(pointing the Pascal's triangle) when this is no reds right? And this is three with one red right? Do you see in your heads the three with one red. Could you imagine those? What do they look like the three with one red? Can you see them?
939 78:44 Stephanie mmmm
940 Dr. Maher You know there are exactly three? What do you see I am curious? What do you see in your heads?
941 Amy Blocks (all giggle)
942 Dr. Maher How do you see the three of them with exactly one red?
943 Stephanie Umm one with a red at the top. One with the red in the middle(other join in whispering) and one with a red at the bottom.
944 Dr. Maher You all could imagine that?
945 Robert Mmmm
946 Dr. Maher Very impressive. Ok so how do we now get these four with one red?
947 Stephanie Umm wait I have to think(pulls paper closer)
948 Dr. Maher right
949 Stephanie These are all blue and these are one blue is that what we are saying?
950 Dr. Maher These are
951 Shelly These are no red
952 Stephanie Laughs so these are all blue and these are oen blue
953 Shelly Yeah hehe same thing
954 Dr. Maher I have to switch I'm not as fast as you are Stephanie your much more expert on these tower problems than I am not having them in front of me
955 Stephanie Oh these are all blue these are two blue I'm sorry I was ok right
956 Dr. Maher Right or no red and one red
957 79:54 Stephanie So here all you're doing is adding one red
958 Dr. Maher Ok so this has to be a one red
959 Stephanie Yes so you already have um three with one red, so here this becomes the fourth one with one red because here there is no reds. And each of them get a block added to them how do I..ok
960 Dr. Maher I can see my little stack here and this little stack here there is only one of them and its all blue no red right?
961 Stephanie Yeah, where is your picture? Do you have your picture to that (speaking to Robert)?no?
962 Dr. Maher You'll have to help me why don't you make a picture

- 963 Stephanie (grabs paper and marker) ok (and begins drawing) so here is. You have the one three high with all blue. Then you have the three with one red so you have red blue blue, blue red blue, blue, blue red. And then these two make..this one is four rows gosh(others giggle) ok and these two together make umm the one with four the four with one red so this one gets a red added on because its already got three blues so it cant have any more blues. And then these three all get a blue added on to it.
- 964 Dr. Maher Because they already have one red.
- 965 Stephanie Because they already have one red
- 966 Dr. Maher Does that make sense? (looking at Robert)
- 967 Robert Mhmm
- 968 Dr. Maher Ok. That that helped me a lot. Ok now my next question is how many pizzas how many towers are these problems
- 969 Robert The same
- 970 81:48 Dr. Maher The same or
- 971 Shelly & Mhmm yeah
- Stephanie
- 972 Dr. Maher Coincidentally the same answer. How can you convince me that these are the same problem?
- 973 Shelly Its easier to explain the two thing with this because there are only two colors
- 974 Dr. Maher You like that huh?
- 975 Students yeah
- 976 Dr. Maher Ok I agree I think that's easier to see with the two colorsi think most people
- 977 Stephanie Its all the toppings it throws you off, you expect like 800 pizzas.
- 978 Others Yeah mhmm
- students
- 979 Dr. Maher Is there a way of thinking about the pizzas another way so that the toppings don't...
- 980 Robert Yeah
- 981 Dr. Maher Robert
- 982 Robert Toppings is the height. Like 4 toppings would be a tower 4 high and then two colors would be with or without toppings.(inaudible mumble)
- 983 Dr. Maher Say say more about that now so if I want to think of this as a pizza right lets think if these each of these as a pizza, right? So you're saying four toppings is a tower 4 tall. That's what you just told me right?
- 984 82:39 Robert Yeah
- 985 Dr. Maher And what would the b mean for instance on the pizza here if you were thinking of pizzas
- 986 Robert Peppers or ..oh no no it would be
- 987 Stephanie It would either mean you had a topping there or you didn't have a

topping there

988 Robert Yeah

989 Dr. Maher Oh So b means you either put whatever topping is

990 Stephanie Right? Is that what you are trying to say?

991 Robert Yeah but like it wouldn't work because you'd have more than one
a topping

992 Stephanie I think he's on the right track but

993 Dr. Maher Well you're saying that the color means you have a topping or
you don't. Lets follow that thinking right?

994 Stephanie So

995 Dr. Maher So b will, would mean you choose the topping right? And red will
be you don't choose it right fair enough?

996 All mmhmm

students

997 Dr. Maher Ok so you have four toppings so let's go through our toppings
right?.

998 Stephanie mhmm

999 Dr. Maher We have well what are they?

1000 Stephanie Well for which

1001 Robert Peppers

1002 84:00 Stephanie For four high?

1003 Dr. Maher For four high

1004 Stephanie We have

1005 Dr. Maher So this would me we choose peppers or we don't

1006 Stephanie mhmm

1007 Dr. Maher moves her pen pointing at a different topping

1008 Robert You choose peppers or you don't

1009 Stephanie No, you choose mushrooms or you don't, you choose sausage or
you don't . and you choose umm pepperoni or you don't because
your going to have four of them and each one is gonna have like
if it was written like that (she grabs a marker and begins writing)
like here would be mushrooms like going all the way across and
each tower you had with an "r" there meant that you didn't have
mushrooms

1010 Dr. Maher Ok

1011 Stephanie Like you know what I mean so your gonna have all the towers
four high like that and for each one with an "r" there you have
mushrooms on it or you don't have mushrooms on it you know
and then here's the peppers and if their you know and just like
that

1012 Dr. Maher Do the rest of you follow what Stephanie is saying? Its pretty
heavy stuff here were talking about. Were really into
sophisticated thinking. Ok I wonder I know that you have to
leave.

1013 Stephanie mmhmm

1014 Dr. Maher Absolutely at 5:15 right?

1015		Stephanie	Well I have to be home between 5:30 and a quarter to six so
1016	84:58	Dr. Maher	So we can take a couple of more minutes then
1017		Stephanie	Yeah
1018		Dr. Maher	Ok but what I would really like you to do even though Stephanie has to leave a little bit before you do. I would like you to kind of write up you solutions. Ok because then we're going to share so you're going to have to work together to present it ok. You want to sort of pull your ideas together on a piece of paper and order them. Ok but before that write your names on your paper the best you can or your initials any way you want to do it. The best you can order them the try and come up with a gen- your big idea you're going to share with the group ok? Is that fair? You've got some really remarkable stuff all of you. I'm so glad you're all here. We're going to see you all next week aren't we? Fantastic. If we don't have time to do this maybe you can stay a little later next week if you want to head out. See how much time we have to share. So why don't you think about sorta your big ideas for a presentation and get sort of your notes down. Ok
1019		Female student	ok
1020		Dr. Maher	Thank you
1021	86:02	Shelly	Do you want to just write it on one paper and have all of our initials on it?
1022		Robert	I think someone else should write it. You...
1023		Stephanie	Yea Shelly
1024		Robert	That's a good idea
1025		Stephanie	I like that idea
1026		Shelly	Yeah ok. One of the people with the worst hand writing. Lets have you write..Well besides you (Stephanie)
1027		Stephanie	Besides me hahahaha, what are you tryin to say?
1028		Shelly	So what do we want to say? Does any one want to draw the diagram and the towers?
1029		Stephanie	Well what are we trying to explain, I don't remember what we are trying to explain.
1030		Shelly	Yeah haha we are trying to explain our answer like everything
1031		Stephanie	Just our answer?
1032		Amy	Here generalize 2 to the n.
1033		Dr. Maher	If you don't get to do it this week you could do it next week. Just organize your ideas and you have more chance to think about it before you present it
1034		Shelly	I don't want to write, look at my hand writing.
1035	86:48	Stephanie	Amy you have nice handwriting, why don't you do it?
1036		Shelly	haha
1037		Stephanie	You do!
1038		Shelly	You are cursed because you have nice hand writing.
1039		Robert	You see I would do but I'm left handed so I'd smear

1040 Stephanie Oh yeah well I'm just messy
1041 Shelly Yeah So am I, I just I didn't use that excuse I just said its sloppy.
You see (pointing to Amy's work) look at the look at the
difference.
1042 Stephanie No, you said that mine was sloppy.(and both begin laughing)
1043 Mr. See some of my students still need to practice their handwriting(
Pentozzi Shelly and Stephanie begin laughing)
1044 Stephanie Was that like their punishment?
1045 Mr. No its just what one of my students drew on the board today.
Pentozzi
1046 Stephanie Oh wow
1047 Mr. So he decided he needed to practice his penmanship. I just told
Pentozzi him I'm saving that saving it as a memento. I hope you enjoyed
my class room
1048 Stephanie Its very big you have a big classroom.. You have a sink
1049 Mr. I know I can do all the experiments I want
Pentozzi
1050 Shelly& hahaha
Steph
1051 Mr. So you guys will all be back next week?
Pentozzi
1052 Stephanie Yes
1053 87:44 Mr. I was hoping Id be able to stay long enough and chat but
Pentozzi
1054 Stephanie Ok..I know
1055 Mr Math out lasts the teacher in this case
Pentozzi
1056 Shelly Oh.. I was trying to read the posters over there
1057 Amy I've been reading these three over here
1058 88:03 Shelly I like those posters that have pride and stuff. Ok anyway...So
what do we want to do?
1059 Gina Do you know what Dr. Maher wants you to do?
1060 Shelly Yeah to explain our answers I guess
1061 Gina To explain your answer but then talk ab- if you could you know
you could talk a little bit about the other things you found like..
1062 Shelly About her(Dr. Maher) questions?
1063 Gina Yeah
1064 Shelly What her questions were to us?
1065 Gina Yeah about the numbers the moving down the two and the
1066 Shelly ok
1067 Gina Amy-Lynn said this and hehe
1068 Amy (pointing at triangle) So this is the generalization right?
1069 Shelly Yeah
1070 Several seconds pass
1071 Shelly Oh hehehe so uh I don't know just draw the triangle so we have
like

1072 Amy So
1073 88:56 Shelly Like The one and the one one and (inaudible mumble).. so we have some general idea..We'' have to converse with them after words. We could be like (whispering) Do you guys have an easier way?(all giggle)
1074 Stephanie They were like done 4 hours ago you know hahah "oh look its 16"
1075 Shelly Hahaha "so what are you guys doin this weekend"
1076 Amy You wanna extend it to the 15 10 10 5 1
1077 Stephanie I'm sorry what?
1078 Amy Do you want to go down to the next row for the five
1079 Shelly Yeah because we have to do the Anchovies so
1080 Stephanie Yeah probably
1081 Shelly And if you want(pointing to first row in Pascal's Triangle) you could put equals 2 to the zero...
(Amy writes down, next to the rows- $2^0, 2^1, 2^3, \dots$)
1082 Shelly And then what else do we write?
1083 Stephanie Uh
1084 89:54 Amy Do we have to explain the 2?
1085 Stephanie I think a lot of it we just know.
1086 Shelly Yeah
1087 Stephanie Like it's hard to write down like in words but you know what you are doing.
1088 Shelly Unless you want to write real quick on top like one of them one topping two topping you know what I'm saying?
1089 Amy Yeah
1090 Shelly Or no that's not one topping
1091 Stephanie Or just write exponent equals topping
1092 Amy uh
1093 Stephanie Exponent equals topping that you don't have to go down to each—oh well then you know that they are toppings you don't need to go back to the other side and write that they are one topping. Do you?
1094 Shelly Unless you want to.
1095 Stephanie Unless you're like really
1096 Shelly Want to put for one of them the little arrow thing that shows that 3 and 1 go to the four, like you know what I'm saying like put yeah you know what I'm saying
(Speaking to Amy)
1097 Amy (as writing) so this arrow goes to the four
1098 Amy Do we have to explain here the 2 came from?
1099 Stephanie Uh
1100 90:44 Shelly Hi (waving to someone else) how you doin? hehehe
1101 Stephanie umm
1102 Robert Say because it can go two ways and whatever else we said
1103 Amy Oh yeah

1104 Shelly Just put a little foot note, a little star two ways
 1105 Amy Oh I love my little stars. This what I do everyday in my note
 book. umm
 1106 Robert You used to call them Suzie snow flakes right? Mrs. Inslin call
 them Suzie snow flakes
 1107 Shelly Ok
 1108 Amy That's what you used to call them?
 1109 Robert Yeah
 1110 Shelly Those were the part in the play? Isn't? weren't people Suzie snow
 flakes
 1111 Robert Yeah
 1112 Amy Yeah that was like in first grade
 1113 Shelly I wasn't one though I was like
 1114 Stephanie I think I was an elf
 1115 Shelly I don't remember
 1116 Stephanie I don't remember that play
 1117 Shelly I just remember doing Mr. Binder's play
 1118 Stephanie Oh Mr. Binder's play(pause) and Matt had chicken pocks.
 1119 Amy He's coming back.
 1120 Stephanie Oh I guess to do summer work.
 1121 Shelly He is coming for the money heheh
 1122 Amy Soo it would be 2.
 1123 Stephanie Are you coming?
 1124 Shelly Laughing-I cant say anything..well that'd be if he lives in like
 Wall it would be impossible for him to do it now. Unless he came
 like at 3:45. Depending on when his school gets out.
 1125 Amy Yeah it's a drive to get up here form there
 1126 Stephanie Does Matt drive yet? How old is he?
 1127 Amy Like 16 or 17
 1128 Shelly Do you drive yet?
 1129 Stephanie No are you kidding I haven't even
 1130 Shelly When's your Birthday?
 1131 Stephanie June. I haven't even done my six hours. I have no you don't even
 I have no desire to drive. Like I seriously like to be driven
 around.
 1132 Amy What? Oh man
 1133 Robert My birthday is next week I don't have a permit yet.
 1134 Stephanie You're kidding?
 1135 Robert No
 1136 Stephanie If you don't get it though don't you have to uh
 1137 Shelly You wait like 30 days yeah, you get your permit on your birthday
 and then you wait 30 days and then you get your license
 1138 Robert I don't want to drive though
 1139 Stephanie See I don't want to drive either and everyone's like what do you
 mean you don't want to drive? And I'm like I like to be the
 passenger.

1140		Robert	Too boring
1141		Stephanie	And I don't pay attention enough either
1142		Amy	(quietly) the two comes from if I have.. I don't know how to explain it in words.
1143		Shelly	You could explain it in like paragraph form not like in sentence-
1144	92:54	Stephanie	Lets not hehehe..but you know how to explain it right?
1145		Amy	Yeah
1146		Stephanie	So
1147		Amy	SO the two comes from
1148		Robert	Each number can go two ways
1149		Shelly	Yeah put form each topping you would be able to go two ways
1150		Stephanie	I mean you were the one who figured out the two part out so I wouldn't think that you wouldn't.

Appendix J

Date of Session: 11-13-1998

Author: Anoop Ahluwalia

Verified by: Kristen Lew

Date of transcript: 5-30-2010

Line	Time	Speaker	Transcript
1.	0:05	Alice	You did too, didn't you Magda? Did you join in the middle of the year or did you join early?
2.		Angela	I joined in the middle.
3.		Magda	What? I came in the middle.
4.		Alice	That's what I thought cause I remember all of a sudden, you all came in.
5.		Angela	It was like the first marking period or something like that.
6.		Girls	Whisper then laugh.
7.		Alice	Okay let's talk about that a little bit because I wanna know where all of you came from. And Robert, you and Michelle were with us from earlier than that. What, from first grade?
8.		Michelle	I was from first.
9.		Alice	You were from first, okay.
10.		Bobby	Yeah it was somewhere around there.
11.		Alice	And you from?
12.		Bobby	Like from second.
13.		Alice	Like from second grade.
14.		All	(Chatter.)
15.		Alice	I remember Mrs. Marinez, Marinez is that her name? (Michelle Nods) Was the first year that we had this group.
16.		Michelle	Right.
17.		Alice	And then in the second grade we were in... I can remember being in 2 or 3 different rooms. And then in the third grade with...
18.		Michele	Mr. Bender
19.		Alice	(Laughs and nods) Mr. Bender with the pianos.
20.		Michelle	Yeah.
21.		Girls in corner	(Whisper)
22.		Alice	Are we on?
23.		Guy on Camera	We've been on.
24.		Alice	Oh okay. Yeah. What I would like to know is sort of a little bit more of the conversation we were just saying. Bobby you say you... Michelle you say you remember since from the first grade.
25.		Michelle	Mmmhmm.
26.		Alice	And Bobby from about second. (Bobby nods) And Angela and Magda from

27. Angela Sixth
28. Alice Did you move to Kenilworth at that time? Or had you been..
29. Angela I went to St. Teresa's so sixth grade then I went to Harden
30. Alice Yeah. Now what about you?
31. Magda Umm, I moved from Rundleport.
32. Alice In sixth grade?
33. Magda Yeah.
34. 1:59 Alice Yeah. And then joined this group that we were doing sort of after the first marking period?
35. Magda Yeah, kind of.
36. Alice And then Ashley? You're brand new.
37. Ashley I just move here.
38. Alice Just moved here?
39. Ashley Well not just but 2 years ago.
40. Alice 2 years ago. And so came here pretty early when it opened.
41. Ashley Yeah
42. Alice Where were you from?
43. Ashley Elizabeth.
44. Alice In Elizabeth? (Ashley nods) (To Sherly) And Sherly? (She nods) From where?
45. Sherly I went to St. Teresa's since sixth grade.
46. Alice From sixth grade, and then came to Burley last year?
47. Sherly Yeah.
48. Alice What do you all remember about anything that we did?
49. Michelle The cubes. The cube problems.
50. Alice You remember the cubes? Do you all know what they're talking about? (Ashley and Sherly shake their heads) The cube problems?
51. Michelle What are they called? (To Bobby) Do you know what they're called? (Bobby shakes his head no.) (To other side of table? Do you know the cube? And kinda like those yeah. And doing all the combination problems.
52. Alice Mmhmm. What do you mean the combination problems?
53. Michelle Like if you had 2 colors and you could make stacks of 4 how many different combinations could you make.
54. Alice Do you remember those? (Michelle nods) What about you Bobby? What are your memories?
55. Bobby There was this garage door problem like Ms. O'Brian she couldn't get her car in the garage because she forgot her combination and the garage door opener.
56. Angela I remember that one
57. Bobby And we had to write out all the combinations and see how many possibilities there were.
58. Alice Ohh for the numbers that you have on the garage door?
59. Bobby Yeah and there were like special rules like. I don't think you could have 2 numbers in a row. Like you couldn't have 1 1 2.

60. Alice Ehhhha, ehha. Did you do the cube thing too?
61. Bobby Yeah.
62. Alice You remember that one too? (Loud noise) What about you Angela? You remember the garage door?
63. Angela Yeah I remember the garage door and the thing was how many ways could you make 7 with like 2 dice, 2 die or whatever. It was something like that.
64. 4:01 Alice Ohhh for throwing. It was some game.
65. Angela Yeah it was just like how many ways you could make 7 or something.
66. Alice Do you remember how many?
67. Angela I have no clue. I don't remember
68. Alice If you were throwing 2 dice... can anybody think that in their head? How many ways could you make 7?
69. Bobby Like 6.
70. Michelle 6.
71. Alice Here everybody want a pen in case you have to think of these things. (Passes out pen and paper.)
72. Angela 6.
73. Magda 6.
74. Alice Yeah. What would they be?
75. Angela 1 and 6
76. Alice I mean I'm not saying yeah but is that what you're thinking? So there might be 6 ways?
77. Angela Yeah cause 1 and 6, 6 and 1, 2 and 5, 5 and 2, 3 and 4, and 4 and 3. (They solve the problem on paper and whisper.) Is that 6 or 8? I don't know.
(Girls are talking amongst themselves.)
78. Alice What were you saying Magda? What do you mean backwards?
79. Magda No like, one die will say 6 the other will say 1 and then like switch them. And like the one that says 6 will say 1 and the other one will say 6.
80. Alice Okay what because there are 2 dice, 2 die it makes a difference.
81. Magda Yeah cause they, they have different numbers on them. (Ashley whispers something) No. (Laughs)
82. Anglea Yeah.
83. Alice And so all of you agree and you remember that from 6th grade?
84. Angela Yeah it was either 6th or 7th grade.
85. Alice And you Magda what do you remember from when you came in here?
86. Magda I have no clue. I don't remember anything. I know we did like number problems like how many different combinations could you get. We did a lot of those the dice with different numbers and stuff like that.
87. 6:10 Alice Yeah. And you 2 have you done that kinds of problem before?
88. Ashley I think in Mr. Pandoli's class.

89. Alice Like what? What kinda of?
90. Ashley I didn't do problems like that but I did problems like a bouncing ball problem.
91. Alice A what?
92. Ashley A bouncing ball problem. And like we had to make a graph of how it bounced.
93. Alice Yeah.
94. Ashley Like the way it bounced.
95. Alice Yeah like when it comes down. That's really good.
96. Ashley Yeah and umm we did a balloon problem. We did (Loud noise drowns her out) We did like cube problems.
97. Alice Out of these things?
98. Ashley Yeah. We did something like. You know the big round cube that has a lot of different colors. And he was like trying to tell us if you go like that what colors do you think would be there.
99. Alice Ohhh. I'm having a hard time imagining. Can somebody else help me imagine what that kind of cube would be?
100. Ashley It's like a big cube and there's a lot of different colors on all sides and it has 4 sides.
101. Michelle Rubik's Cube?
(?)
102. Alice Like a Rubik's Cube.
103. Ashley Yeah something like that I think.
104. Alice I happened to imagine... Have you ever done any with this kinda of cube? (Ashley shakes her head.) So this is new for you. (She nods.) What do you remember Sherly?
105. Sherly Nothing. I didn't do any of this.
106. Alice Yeah. And so you're sorta coming in to this, do you hear them talking about these things ever or is this just sort of the first time you've heard about this?
107. Sherly No I knew they were doing problems before but I didn't know exactly what they were doing.
108. Alice What kinds of problems.
109. Sherly No I didn't know that. Yeah
110. Alice Yeah. (To the others) And so you remember these probably more than some other things.
111. 8:03 Michelle Yeah cause we worked with them like every year. I remember doing something with them every year from first grade on.
112. Alice Yeah I remember in Ms. Marinara's room. What did you do in the first grade? Do you remember?
113. Michelle I think they were just the tower problems, I'm not really sure.
114. Alice What was the tower problem can you, can you name it?
115. Michelle Umm. It was like, we were given two different colors like blue and yellow or something.
116. Alice (takes out blocks) Yeah, I'm gonna pull up some two different colors. Okay.

117. Michelle Okay and we were given a set amount like 3 blocks high or whatever.
118. Alice Yeah, let's say four, let's make it more interesting. And so 4 blocks high.
119. Michelle Right, and we had to like mix the colors and make...
120. Alice Well you got 2 colors to choose from is that right?
121. Michelle Mmmhmm. And see how many combinations we could make. Like it would be blue on top and 3 yellow, or...
122. Alice Do you remember that all of them had to go this way? You couldn't have a tower, this is for you guys, you couldn't have a tower going this way, cause it would fall over. Okay? So towers are this way. And the reason that I'm saying that is that sometimes that might make a difference. But any way they're 4 tall and you've got...
123. Michelle Like 2 colors.
124. Alice 2 colors to choose among, to choose from.
125. Michelle And you lay the colors differently to find as many combinations as you can.
126. Alice Okay. Just so that we all understand and can remember, I'm gonna ask us to do that again. And I'm gonna have you 2 work together to see, you know, what you remember and how you come up with it. And I'd like you 2 to work together, is that okay? And you 2 to work together. Now, can you? Let's see... I'm gonna give everybody and I've got tons of these things under here. (Gives out the blocks) Maybe Bobby and Michelle for you all. And if that's not enough please ask for more. Here. Okay and so the task here is, just as Michelle just said, is that if this is a tower...
- 10:09 Can everyone imagine one that's also 4 tall and you can select from the 2 colors here that would look different from this? What comes to your mind? (To Ashley) Can you build another?
127. Ashley Mumbles softly and builds.
128. Alice Well that would look different don't you think? Okay help me with what you're thinking.
129. Ashley (Builds) If you put this here.
130. Alice Uhh okay so you all would agree that these 2 are the same? Okay so your job with your partner is to come up with all possible different towers, okay? That are four tall. And the other part of it is, and I'm sure that Michelle and Bobby can remember that this is a part of what we're saying, is that not only to find all possible combinations or all possible towers, but to come up with your partner a way to convince yourself and convince the rest of us, that you have all of them. And you don't have any duplicates. Can you work on that for just a minute? (They work in groups)

131. Magda We should group the yellow ones. There we go... I don't know what I'm doing.
132. Angela We already have that one.
133. Magda Where? Yeah but it's different, look.
134. 11:59 Angela Oh I thought you meant that one... nevermind. I thought you meant...
135. Magda Idiot. (Laughter) How about we just do that like that one. Let's switch that one, let's switch that one now.
136. Angela We have that.
137. Magda Oh, well 2 in the middle now. Blue?
138. Angela Magda, you messed up my system, now it's all chaos.
139. Magda Do we have that one? How about 3 blue and a yellow, like that one right here.
140. Angela Give me a blue.
141. Magda Hold on. Are we missing 2 beginning?
142. Angela Put this there so I can actually see. It's a little... You know.
143. Magda We have that one.
144. Angela Do we? Nope.
145. Magda Forget it. I lied.
146. Angela Wait we got...
147. Magda Oh then this one should go here.
148. Angela And this, hold on. This pattern goes here. Is that it?
149. 13:56 Magda If you have 1 blue over there. Here you have 3s here you have 2s. Get it? (She rearranges)
150. Alice What were you thinking, Magda, when you organized the blocks?
151. Magda There's one blue in these, there's 2 blues in each of these, 3.
152. Angela 3 then 4.
153. Alice 4 in that yellow one?
154. Angela Yeah. Well 4 and 4.
155. Alice Yeah, uhhh, okay help me to understand. You have just one blue in the first part?
156. Angela Yeah 1 blue in the first group and 3 yellows, then 2 and 2, then 1 yellow 3 blues, then 4 of each.
157. Alice And how did you decide you had all of them in the one group?
158. Angela Well, you kinda just gotta... It moved down. I dunno.
159. Magda Yeah the blue's in a different position each time.
160. Angela Each possible position. Considering there's only 4 spots.
161. Alice And did you use that same for this 3 one?
162. Angela Yeah, with the yellows.
163. Alice How can you be sure you have all that are exactly 2?
164. Angela I don't know how to explain it there's just no other possibilities for it. There's only four places, you have yellow on top blue on bottom, then blue on top yellow on bottom, then blue on top and bottom, then yellow on top and bottom.
165. Alice I guess I'm convinced with these two, I'm having a hard time

- being sure about the 2 and 2.
166. Angela Oh, wait.
167. Magda We missed it.
168. 15:57 Angela Yeah we missed it. Ok well we got 1 more, sorry 2 more.
169. Micelle (To Bobby) Where's the other 1 for this 1.
170. Bobby Right here. I think it looked better the first time.
171. Michelle Well I like them in 2s. Is that okay?
172. Alice (To Angela and Magda) So you found 2 more? You think there might be some more?
173. Michelle Okay so fix it. You want to fix it?
174. Bobby Well then we can group these.
175. Michelle Do whatever you want. Does it really matter the way they're set up.
176. Alice (To Michelle and Bobby) What do you think?
177. Michelle We think we have them all.
178. Alice Why?
179. Michelle Why?
180. Alice Mmmhmm.
181. Michelle I don't know. Cause we can't think of any others. (Laughter)
182. Alice How did you go at it.
183. Michelle I just went at it like whatever.
184. Bobby We started...
185. Michelle I didn't have any set plan like all nice and neat or anything.
186. Alice These are pretty nice.
187. Bobby We started with 4 yellow and 0 blue. Then we did 3 yellow and 1 blue.
188. Alice Right, this group over here.
189. Bobby Yeah, then we 2 yellow and 2 blue which is over here. Then we did 1 yellow and 3 blue. Then we did 4 blue and no yellow.
190. Alice Those two. No blue or all blue.
191. Bobby Yeah.
192. Alice And how did you figure out that you had them all for 2 and 2? Or do you think you have them all? Do you think there might be more?
193. Bobby We started with 2 here. Then we moved this down and it was like this. Then we moved this one down and had that. And then we moved the top one down to the second position so we got this. And moved the bottom one down so we got that. And then we moved them all down one more position and that was the only one left.
194. Alice Okay I have to understand what you said. Say that again.
195. Bobby All right we started with one on the top.
196. Alice You mean like blue.
197. 17:57 Bobby Yeah like 1 blue on top then we took the other one and moved it to each position. Like down, like here 2, 3, 4.
198. Alice Ahha. Show me.

199. Bobby Then we moved the top one down to 2. Uhh right here. And put this one below it.
200. Alice Oh and so where's the... I see. Does that make sense to you Michelle?
201. Michelle Mmhhh.
202. Bobby Then we moved the second one down to 3.
203. Alice Which one is that?
204. Bobby Oh this one we had at 2 then we moved it down one. To 4, to 3.
205. Alice That's this one ok.
206. Bobby Yeah. Then since it couldn't go down more, there were no more combinations with blue in the second, we moved it down to 3.
207. Alice That's this one.
208. Bobby Yeah.
209. Alice Moved which one down to three?
210. Bobby Well this top one which went to here, then again to here.
211. Alice Ohh.
212. Bobby So this one's like always staying in the same place. Until there's no more combinations. Then when there's no more it moves down one.
213. Alice I see. So there's 6. Is that right are there six of them?
214. Bobby Yeah.
215. Alice That are 2 and 2.
216. Bobby Yeah.
217. Alice I see.
218. Bobby I think that's all of them. I think there's like 16 all together.
219. Alice If, let me ask you this one talking to the others. If there were 3 total how many do you think there'd be? How many do you have all together?
220. Bobby 16. I don't know 6 or 8, I don't know.
221. Student (?) Maybe 9.
222. Bobby Maybe 9.
223. Alice (Laughter) Those are all good ideas. What do you think? Why 6, 8, or 9?
224. Bobby Don't know.
225. Alice What if there were 5? What do you think there? Would there be more or would there be fewer?
226. Bobby You take out this one.
227. Alice What are you talking about, for 3? What are you saying?
228. 19:58 Bobby Yeah just take out, like the ones that have blues above 4. So take these out. You know what I'm saying? Then take like one of these... I don't know.
229. Alice But you're not sure. What do you think Michelle?
230. Michelle Umm, I have no idea. (Laughter)
231. Alice Would there be, let's...
232. Bobby I think there'd be 8.
233. Michelle If there were 3 total?

234. Bobby Yeah.
 235. Michelle I say 9. I don't think there's gonna be 8. Why don't we build them?
 236. Bobby No, I'm telling you.
 237. Michelle Here why don't we do it.
 238. Bobby You can use this.
 239. Michelle Yeah here why don't we move this. Maybe she's wrong, you think she's always right?
 240. Bobby Now where are the yellow?
 241. Alice (To Ashley and Sherly, rest of the room discusses if there were 3 blocks.) And there couldn't be any more with just one in it?
 242. Sherly No.
 243. Alice Why not.
 244. Sherly Cause it doesn't....
 245. Alice No more places. Yeah and I'm okay with these. How did you decide about the 2 and 2s?
 246. Sherly Well its just the opposite color so we put them together.
 247. Alice Are you sure there are only 6 that are 2 yellows and 2 blues.
 248. Sherly I'm not sure its just...
 249. Ashley If you put the 2 yellows here then the 2 blues over here. And here we just switched them.
 250. Alice Yeah, and so you think there's 16. (They agree) How many do you think there would be that are just 3 tall.
 251. Ashley 3?
 252. Alice Mmhmm. If you made towers only 3 tall instead of 4 tall. Do you think there'd be more or do you think there'd be fewer.
 253. Sherly I think there'd be fewer.
 254. Ashley Less.
 255. Alice Why?
 256. Sherly Cause you have less blocks to build with.
 257. Ashley Yeah.
 258. Alice How many do you think there'd be.
 259. 22:10 Ashley Maybe like 9.
 260. Alice Why? That's a good guess why? What made you think 9?
 261. Ashley Well 4 would be 16. So I'm think 3 squared, I mean cubed is 9 and 4 cubed is 16.
 262. Alice Yeah 3 times 3 gave you 9. And what would that mean for 5.
 263. Sherly 25.
 264. Alice It might be 25.
 265. Ashley Yeah.
 266. Alice I think what I'd like you to do is build the 3 tall towers. And see what you think about that, if you think you can find those out. Magda and Angela, what did you think? You sort of... How did you decide about the 2 and 2s.
 267. Magda What that?
 268. Alice Mmhmm. Cause I think everybody used the 3 and 1s. What did

269. Angela you do with the 2 2s, how did you decide you had them all.
 270. Alice We didn't have any fancy little method like Bobby over there.
 271. Angela Yeah sort of at the end you're like gosh I might take that.
 272. Magda But I mean there's no other way to put it.
 Okay, you know you have to have 2 yellows and 2 blues and you
 just figure out different stuff. You just switch the colors. You
 have blue here so you put the yellow here.
 273. Alice Yeah you sort of have the opposites of each other.
 274. Angela Yeah.
 275. Alice Is that what you mean by this one and this one and this one and
 this one.
 276. Angela and Yeah.
 Magda
 277. Alice Okay.
 278. Angela They're pretty much all opposites of each other cause you could
 do it like that. (She rearranges the towers.)
 279. Alice Yeah you could. But you didn't use that quite as much except
 for those.
 280. Angela Yeah the 2 and 2. The 3 and 1 it was just a lot easier to see it
 this way. (She rearranges again.)
 281. Alice With that pattern going down.
 282. 24:00 Angela Yeah.
 283. Alice Now what do you think if they were 3 tall? How many do you
 think there'd be?
 284. Angela 8.
 285. Alice Why?
 286. Angela Here we figured it out. (Shows Alice the paper she wrote on.)
 287. Alice (Chuckles) Oh, you figured it out. I see.
 288. Angela Cause we were just sort of talking about that and trying to see if
 you could figure out a way without doing all that, like actually
 building it. So I'm trying to figure out all the combinations for 5.
 289. Alice Is that what you're working on now?
 290. Bobby We think we have a way.
 291. Michelle Don't tell her yet.
 292. Alice (to Bobby) Yeah? Let's let other people work on it. My question
 to you, as you're thinking about that, you had six that are exactly
 2 blues and 2 yellows?
 293. Bobby Yeah.
 294. Alice When you had the 4s.
 295. Bobby Yeah, they're right here.
 296. Alice Yeah are those the guys?
 297. Bobby Yeah.
 298. Alice If they were 5 tall, how many do you think there'd be?
 299. Bobby 32.
 300. Michelle 32.
 301. Alice Yeah from what you were building over here? Okay, what I'd

302. Michelle like you to think about how many of those 5 tall, if that's the truth, would have exactly 2 blues in it? Can you figure that out? Should we build it so it looks like we're doing something? Or you can sit and figure it out cause we obviously have some kind of math like genius over here.
303. Bobby How many do you think it's gonna be? Just guess now so we can see if we're right.
304. Michelle Ok.
305. Bobby So 2 high there's 2.
306. Michelle Wait.
307. Bobby 3 high is 4. 4 high is 6.
308. Michelle Wait, what color are we doing, blue?
309. Bobby Yup we're doing 5 so I think there's gonna be 8.
310. Michelle Wait, no. I don't get it; if it's 3 high its 4?
311. Bobby Look see, 3 high is 2, oh that's the wrong number its 3. Is that right?
312. Alice (to Ashley and Sherly) Why do you think there aren't 9?
313. Michelle Right. Yeah. Okay yeah, yeah.
314. Bobby So 4 high is 6.
315. Magda (To Angela) Good thing you have good hand-writing cause I don't understand this. (Giggles)
316. Alice (to Ashley and Sherly) Couldn't it be 9?
317. Michelle Could like, stop writing over your old handwriting? Because I don't understand this. I don't understand.
318. Bobby Do you wanna do the 5? Because we can just take these apart.
319. Michelle Yeah, yeah I wanna do the 5.
320. Bobby Alright
321. Michelle So, we don't need the 3's anymore do we?
322. Bobby We don't need the 4's either, I don't think
323. Michelle Yeah, but...
324. Bobby We're gonna run out of cubes.
325. Magda No, I don't think. We can't have an odd number cause you have to have an opposite. You know what I'm saying?
326. Alice Okay, so it needs to be even.
327. Magda Yeah, that's what I said.
328. Alice So you all, on your paper, figured out 8. Is that right?
329. Magda Mhmm
330. Alice So you agreed with him. How many do you think there'd be for 5? (Loud noise.)
331. Angela 32
332. Magda 32
333. Alice Why?
334. Angela I'll let Magda explain.
335. Ashley I think there's 24.
336. Alice You think 24? Why?
337. Magda I don't know.

338. Angela You explained it 2 seconds ago.
 339. Magda (To Ashley and Sherly) (laughing) Yeah but it could be 48.
 340. Alice And your theory before...
 341. Ashley And the theory before, since this was 8.
 342. Alice Well I don't understand, what were you thinking? Where did the 8...
 343. Bobby (To Michelle) We don't need to do all of them, we could just do the 2, 3.
 344. Michelle Yeah. Wait, we're doing 2 blue? Are we on the same wavelength here?
 345. Bobby Yeah, then we have to do 2 yellow right?
 346. Michelle Well yeah.
 347. Bobby We could just multiply it by 2. (Knocks over towers) Ohh.
 348. Michelle You're a moron.
 349. 28:10 Bobby There's gonna be a lot more.
 350. Alice (to Magda) You are doing it by drawing?
 351. Magda Yeah, you want me to build them?
 352. Alice Build them if you want, if that would help you. Since there seem to be two really different ideas here. That two of you are, some of you are going 32 and some of you are going 24.
 353. Michelle I just did that. Did you just do the same one as I did?
 354. Bobby No, I started opposite way.
 355. Alice Okay, are you working on my question? It is how many that had exactly 2?
 356. Bobby Yeah. We took a guess but we're trying it out.
 357. Alice What was your guess?
 358. Bobby I guessed 10 she said 8.
 359. Michelle I said 9.
 360. Bobby 9. I think they're wrong though.
 361. Alice You think she's wrong, you think you're wrong?
 362. Bobby Yeah.
 363. Alice You think you're both wrong?
 364. Bobby Well I think I'm wrong. I don't know if she is.
 365. Alice You think there might be more or less?
 366. Bobby Maybe more.
 367. Michelle I think more. (Loud noise.) Is that right?
 368. Bobby Yeah I just gotta bring in this one. It doesn't look right but it is.
 369. Michelle Are you sure?
 370. Bobby Yeah.
 371. Michelle Very complicated here. Oh wait I see one that's missing. You're probably doing it right now.
 372. Bobby This one?
 373. Michelle No.
 374. Angela Yeah Bobby.
 375. Michelle We got that one, don't we?
 376. Bobby Oh.

377. Michelle Wait, you just messed up my train of thought.
 378. Bobby What about 2 yellows 2 blues, then a yellow.
 379. Michelle Is it this one maybe? Do we have this one?
 380. Bobby Yeah that's what I was saying. I mean, is that all of them?
 381. Angela Magda I'm too confused. I'm getting doubles and it's just not working. (She counts)
 382. Bobby (To Michelle) That was a lucky guess.
 383. Michelle So that's 10. Alright so there's 2 yellows in everything, right? We should be writing these down.
 384. Bobby Wait isn't that all of them? So there's 10.
 385. Michelle No, then there's 20. If you times by two then, then you..
 386. Angela 22.
 387. Michelle 22.
 388. Bobby Why do you times by two?
 389. Michelle Cause you gotta do the opposite if there's 2 yellow and 3 blue.
 390. Bobby So let's start writing them down.
 391. Michelle Okay I wanna do it.
 392. Bobby Well I'll write it for myself.
 393. Michelle No tell me cause I can't read that.
 394. Bobby There was 2, right?
 395. Michelle 2 what? Of 2 colors?
 396. Bobby Yeah.
 397. Michelle Okay.
 398. Bobby Then there's 3, 3 right?
 399. Michelle I don't know. I don't remember.
 400. Bobby Yeah. 4 high was 6.
 401. Michelle I don't, I guess so.
 402. Bobby Well we have them, let's just check. There's 20.
 403. Angela There's 24.
 404. Magda There's 24?
 405. Angela How many did you get?
 406. Magda I didn't finish it yet, calm down.
 407. 31:58 Angela Oh wait, there's more than 24, never mind. I lied.
 408. Alice (To Bobby) Are you convinced? Are you still working on the...
 409. Bobby We think we got it.
 410. Michelle I don't know why.
 411. Alice Not quite sure why? Well it might be interesting to think about 6, 9 or 3, 9 whatever it is.
 412. Bobby I looked at 6 times 3 plus 2 is 20. And 3 times 2 is 6.
 413. Michelle That's not a pattern. Why don't we do 6 and see if that works.
 414. Bobby All right.
 415. Michelle Well, I mean we don't have to; I don't want to twist your arm.
 416. Bobby No, I'll do it. You just have to add a blue to the bottom of each one.
 417. Michelle Why do we have to add a blue?
 418. Angela (to Magda) Now I have 28.

419. Bobby Cause. Cause there's gotta be another one, see?
 420. Michelle But if we add a blue there's gonna be 3 blue.
 421. Bobby Ok.
 422. Michelle Maybe I'm wrong.
 423. Bobby Maybe I'm wrong.
 424. Magda (to Angela) I have 20.
 425. Angela You have 22.
 426. Magda Oh yeah, I have 22.
 427. Angela I don't know I might have doubles, I got a little confused in the drawings.
 428. Bobby (to Michelle) What about this and 1 on the bottom.
 429. Magda (to Angela) Hold on, cause two here.
 430. Michelle (to Bobby) What?
 431. Bobby We need um, blue in 2nd and blue in 6th. With the one here and one on the bottom.
 432. Magda Two blues I shifted down, then I shifted that down. And they're like the opposites.
 433. Bobby See that's what I was talking about.
 434. Michelle Okay.
 435. Bobby And we need one here too.
 436. Magda And then when I shift that thing down, then it's the same as that.
 437. 34:03 Michelle I think we're gonna need more blocks.
 438. Bobby There's gonna be a lot more.
 439. Angela (to Magda) You can shift this down so you have yellow...
 440. Magda I know, I don't have three in a row on the top with the blue.
 441. Bobby There's gonna be a whole another row too.
 442. Angela Yes you do, right here.
 443. Michelle Don't we have it there? Oh no.
 444. Magda Yes I do.
 445. Bobby Yeah, you see? Add 3 more.
 446. Angela Umm, you don't have three in a row and then yellow, yellow. Do you?
 447. Magda No. And we can do the opposite for that.
 448. Michelle Ma'am, can you give me a yellow you have over there? (Alice passes blocks over) I have some yellow.
 449. Bobby You can have those.
 450. Michelle Oh, thank you.
 451. Magda (to Angela) When I shift it down. (Angela interrupts) Hold one, but when I shift it down. Hold, now this is what I'm thinking. Shift the whole 3 things...
 452. Angela No, (laughs).
 453. Magda Do I have that one down?
 454. Bobby (to Michelle) Do you have more yellow?
 455. Michelle We need 1 more right.
 456. Magda (to Angela) We don't have this one right?
 457. Bobby Yeah. Forgot this one. We got 3, 6, 9, 12, 15, 30. That's only 5

up. Is that all of them. It looks like it.
 458. Sherly (to Magda and Angela) How are you guys doing it? (Chuckles)
 459. Magda So we're shifting,
 460. Angela We shifted those down.
 461. Sherly Yeah.
 462. Bobby That's all of them?
 463. Michelle Yeah, I think so.
 464. Angela We did groups of 2; did you do groups of 2? Yeah.
 465. Ashley (inaudible)
 466. Angela Well what about mixed up funk like this?
 467. Bobby (to Michelle) So six high there's 30. How the bricks get here? (?)
 468. Michelle How?
 469. Bobby Yeah how?
 470. Michelle I don't know I just do it. I like reading my thing.
 471. 36:03 Angela (To Magda) What about blue, yellow, blue, blue, yellow. Do you
 have that already?
 472. Magda Blue, yellow, blue, blue, yellow.
 473. Angela Blue, yellow, blue, blue, yellow. (together)
 474. Angela (laughs) This is too confusing.
 475. Magda Blue, yellow, blue, blue, yellow. I should have that.
 476. Angela Do you? I see it not.
 477. Magda No cause...I didn't... Did I shift that?
 478. Angela No cause you gotta have a blue on top. So yellow, blue, yellow,
 yellow, blue.
 479. Alice And there were how many?
 480. Michelle 30, because. Well there are only 15 here cause you gotta
 multiply it by 2.
 481. Alice Oh cause the other side.
 482. Michelle Right
 483. Alice If you're doing it for 2 blue, you gotta do it for...
 484. Michelle Right, do it for 2 yellow.
 485. Alice Okay so in terms of building it up, what with this be? What does
 that say, what did you find for 5? I can't remember.
 486. Bobby Ahh, read hers.
 487. Michelle 20. Mine you can actually read. There's 20.
 488. Alice Yeah but of 1 column.
 489. Michelle Yeah. What do you mean?
 490. Alice Is it 10 and 10 or?
 491. Michelle Yeah.
 492. Alice Can you draw it down so it can remember?
 493. Michelle It was 10 and 10 here. This one was 15 and 15.
 494. Alice Okay and what was it before?
 495. Michelle Wouldn't it be 3 and 3.
 496. Bobby Yeah.
 497. Alice You skipped the 4s. What were the 4s.
 498. Michelle This is the 4.

499. Alice 4 high was?
500. Michelle 6
and Bobby
501. Michelle Did we times that by 2?
502. Bobby No I think we did them all cause its 4 so you'd have 2 yellow
anyway. You know what I'm saying? If you have 2 blue you
have 2 yellow.
503. Michelle Right. So 3 and 3. No we couldn't have done it cause that can't
be possible.
504. 37:57 Bobby Yeah cause there's 3 so it'd be... we're missing some.
505. Michelle You think?
506. Bobby Let's redo the 3 then.
507. Michelle Yeah let's redo 3.
508. Alice I don't get that 3, 3 thing.
509. Michelle Yeah we're gonna do that one over.
510. Bobby Take off the top.
511. Michelle Can't we just do it normal?
512. Bobby Just take 3 off the top of that one.
513. Michelle Take 3 off here? No you wanna take 2 off.
514. Bobby 3.
515. Michelle But no, shouldn't we do 4 over?
516. Bobby Let's just do 3 first.
517. Michelle Okay, whatever. I just wanna have some organization.
518. Angela (To Magda) You think we have them all? I don't know. We have
no logic.
519. Magda Yeah we're just doing it. Then I started with 2 blues.
520. Angela Wait so there?
521. Ashley That one.
522. Magda That one. 2 blues together. That's what I did.
523. Angela That's 2 blues together.
524. Ashley (to Sherly) We have 28.
525. Sherly Do we have any same?
526. Angela Do it this way, hold on. Go like this.
527. Magda Oops. See, I'm not that dumb.
528. Angela 2 blues, do we have another 2 blues, do we have this 1? So
yellow, yellow, blue, blue.
529. Magda How about 2 blues?
530. Angela Yeah, that's what I was saying. You know, it doesn't matter
when I say it though.
531. Sherly (to Ashley) I'm not seeing a pattern here...
532. Angela What about 3 now. Do you have 3s anywhere? Wait the 3s, 3 in
a row. Oh, well there the same thing they're just backwards,
right? 2 and 3 are pretty much the same thing, right?
533. Magda Yeah.
534. Angela Okay. This is our mixed up ones.
535. Alice (To Bobby and Michelle) In terms of exactly 2 yellows, or 2

blues whichever one you're focusing on?

536. Bobby Yeah.

537. Alice How many were there?

538. 39:59 Bobby I think like 30.

539. Michelle 6.

540. Bobby Yeah there's 6 so 3 and 2.

541. Michelle That's 4.

542. Alice Uh, 6 with exactly 2 yellows?

543. Bobby That's what we got.

544. Michelle For 3 we have 6, for 3 high we have 6.

545. Alice I would like to know if you have 6 with exactly 2 yellow.

546. Bobby For which one 3 or 4?

547. Alice For 3. You just showed me.

548. Bobby Oh, we just had 3. Do the yellows now.

549. Michelle What do we need?

550. Bobby Yellow blue yellow.

551. Alice Okay and so what did you say?

552. Michelle For 3 high its 6.

553. Bobby Yeah and for 4 high its 6 too.

554. Alice 6 what?

555. Bobby Oh, 6 different combinations. (Someone asks him a question)
Yeah.

556. Alice Okay I'm concentrating on this one block.

557. Bobby Oh, this one's here.

558. Alice Oh you had it a minute ago, you took it away.

559. Michelle Where the one we made before that goes in between those 2.

560. Alice Oh don't kill that.

561. Bobby Oh we already killed most of it.

562. Alice Well there's plenty more. (She brings them more blocks)

41:57 My confusion, my question to you is if, if you're focusing on one color. Okay? You're telling me there are exactly, how many that area 3 tall that have 3 blues?

563. Michelle 6?

564. Bobby No 3, and there's 3 tall that have 2 yellows.

565. Alice Oh so you could have done it on either thing. And then for 4 tall?

566. Bobby There's 6 of each.

567. Alice You mean there's 12?

568. Bobby Yeah. No there's 6 with 2 blues and there's 6 with 2 yellows.

569. Alice And they're different?

570. Bobby Yeah.

571. Alice They're different towers?

572. Bobby Oh no they're the same towers but...

573. Alice But you could have focused on either color.

574. Bobby Yeah.

575. Alice Is that true when they were 3, are they the same towers?
576. Bobby No they're different. They're different towers here but, because there's only 4, then it's like.
577. Michelle There's only 2 colors.
578. Bobby Yeah there's only 2 colors and 2 of each, that's why.
579. Alice 4's kinda of a special case.
580. Bobby Yeah
581. Alice And so for 5, if you're focusing on just the one color, the blue?
582. Bobby 10.
583. Alice What do you think the next one would be? You said it was 3, then it was 6, then you said it was what?
584. Bobby I think 21.
585. Alice Okay, you said it was 3, then it was 6, then what was the next number that you gave me?
586. Bobby 10.
587. Alice And then?
588. Bobby 15.
589. Alice And then?
590. Bobby It would be 21.
591. Alice Why?
592. Bobby Cause well, when there were 2 there was 1, right, of each, then there was 3 of each then there was 6 of each, then 10, 15. That's like plus 2, plus 3, plus 4, plus 5. So keep going in order it'd be plus 6 and 21 then 28.
593. Alice You've gotta prove that. It's a wonderful theory but you know...
594. 44:00 Michelle Oh my god, I cannot read that! What number is this there?
595. Bobby All right see when we had 2 high.
596. Michelle Alright, okay. Can we write this down in my handwriting?
597. Bobby You wanna make a chart?
598. Michelle No I just wanna write it down. If there's 2 high.
599. Bobby There was 1 of each.
600. Michelle 1 of each. Then it's 3 high
601. Bobby 3 of each.
602. Michelle 3 of each. Then it's 4 high.
603. Bobby 6
604. Michelle Then it's 5 high.
605. Bobby 10.
606. Michelle 6, 15.
607. Bobby Yeah and when there's 1 it's 0. Put that down. So see, plus 1, plus 2, plus 3, plus 4, plus 5, plus 6, plus 7, plus 8, plus 9, 10.
608. Michelle Oh yeah. I understand.
609. Bobby Is that, is this good then? And then what about the other when, how many possible there are?
610. Michelle Like... Out of everything?
611. Bobby Yeah, we had umm. 2 to the first power is what 1 or is that 2?
612. Michelle Is it 1?

613. Bobby Yeah I think so right? I always forget. So that wouldn't work though, for that. You know what I'm saying? When there's 3 high there's 8 combinations? When there's 4, there's 16?

614. Michelle Want me to write it down?

615. Bobby Yeah, you wanna write it down?

616. Michelle Total combinations.

617. Bobby For 2 separate colors, I guess.

618. Alice What did you do? What did you find?

619. Sherly 30, but Magda's thing says its 32.

620. Ashley You said its 32?

621. 46:03 Bobby So let's just start when it's 2 high. There's 2.

622. Michelle There's 2. When it's 3 there's 8?

623. Bobby Yeah.

624. Michelle 4?

625. Alice (to Ashley, Sherly, Magda, and Angela) Why don't you all talk to each other? And figure it out as you go?

626. Bobby 16.

627. Michelle 5?

628. Alice Ohh.

629. Bobby Umm... 32. That's all we got to do.

630. Alice Yeah, you can't have them both. But now you're back to 30?

631. Ashley Yeah.

632. Bobby But see it doesn't work for the first 2 but then when it gets to 3, 4, 5 it works and then on it probably works too.

633. Alice But, Sherly help me to understand your organization.

634. Sherly I can't talk.

635. Angela Okay, these two are just like, you know, one of each color. This is two blues

636. Bobby We should try it with 3 separate colors. No I'm serious...

637. Alice Exactly two blues.

638. Bobby See if it works then. It wouldn't work.

639. Michelle Then it's gonna get too confusing. If you really wanna do it.

640. Angela Yeah. Two blues in each, this is three blues in each. And they're just like you know, the opposites. So this is you know two yellows and this is three, or...

641. Bobby No, no. I don't think its gonna work for that.

642. Alice Yeah, here's some more?

643. Bobby Cause you're gonna have less combinations and 3 is a higher number.

644. Angela Yeah (inaudible), these are what we call (inaudible) because we there's no organization.

645. Magda Yeah, you see, like these are two together

646. Angela Yeah, everything like moves down

647. Alice Oh, I see. And so over here, how come. you can't have two of them? That's the same thing as that.

648. Bobby I doubt when it's like 4 high you're gonna have 81

- combinations, for 3.
649. Alice This one's the same one as that one too. You've gotta get rid of them.
650. Angela Okay.
651. Bobby So its not gonna work for that one.
652. Alice Okay, but anyway. Okay, so what you did. That's interesting. And then you took one off the bottom and put it on the top? That's cute. Okay and you did the same thing here?
653. Angela Yeah
654. Magda Yeah, we just did (inaudible)
655. Bobby Do you notice anything?
656. Alice Okay, so that was really sort of taking them down. And then, and then when you put this one back up it started over, didn't it? Okay, let's look at these.
657. Bobby Do you wanna do 7, but 21 I don't think it will work.
658. Alice Okay, here's these.
659. Michelle Why don't we try?
660. Bobby Oh yeah, yeah it will.
661. Alice And those have exactly...
662. Michelle If you wanna try it we can. Whatever.
663. Alice I don't understand.
664. Angela There's no organization to this.
665. 47:57 Bobby See you should have made a table like this. Like, um... size...
666. Alice Some of them have two blues and some of them have three?
667. Michelle You want me to do that? Let's do that. If it will make you happy, whatever makes you happy.
668. Bobby 2 blues, 2 yellow, total. That would be nice.
669. Michelle Ok hold on.
670. Bobby So it will be 1, 0, 0, 0. 2, 1, 1, 2.
671. Michelle Hold on, this one here is size, height.
672. Bobby 4, 6, 6, 6.
673. Michelle 2 blues,
674. Bobby See it's a lot more organized.
675. Michelle There has to be another column.
676. Bobby No there doesn't.
677. Michelle Do you know what combination this is?
678. Bobby Yeah.
679. Michelle So the height is 1...
680. Bobby 0, 0, 0.
681. Michelle It's not 0, its 1.
682. Bobby If the heights 1 how can you have 2 blues.
683. Michelle No I mean the total combinations.
684. Bobby Total, with 2 in it. Total Combinations with 2.
685. Michelle Oh, so this doesn't really mean total combinations,
686. Bobby No total like adding this to this. You wanna do total combinations as another column?

687. Michelle Yeah I would rather do that. Because it's a little more.
688. Bobby Make it right there, since you already messed up.
689. Michelle Okay, 0 and this is 1.
690. Bobby No there's 2.
691. Michelle No, oh yeah.
692. Bobby 1 high...
693. Michelle Okay, okay. 2, 1, 1, 2, and what was this one? 8, 2?
694. 50:00 Bobby No I think that's 4 total. Oh yeah 4 total.
695. Michelle So this is 4?
696. Bobby Yeah. It's 4, It's 4.
697. Michelle But it can't be 4 because right here we have...
698. Bobby 2 high. You wanna see 2 high? I'll take apart the 3. 4.
699. Michelle Oh. Why didn't you have that before?
700. Bobby I don't know I guess I couldn't read my handwriting.
701. Sherly (to Ashley) We have that over here.
702. Michelle Yeah. Okay what's 3? 3 and 3?
703. Sherly (to Ashley) Do we have three?
704. Bobby It was 3, 3, 6, then
705. Michelle 8?
706. Sherly (to Ashley) Yeah, we have this.
707. Bobby Yeah, 8. And then 4 was 6, 6, and then another 6, then 16.
708. Sherly (to Ashley) Oh man...
709. Bobby Because remember that's the odd one. Put an asterisk next to 4.
710. Michelle Suzy snowflake.
711. Bobby Yeah I remember Suzy snowflake.
712. Michelle Yeah it was Ms. Emerson's class. 5, 10 and 10, then it was 20.
713. Bobby Yeah.
714. Michelle Hold it, shouldn't this be 12.
715. Bobby No because that was the special one.
716. Michelle Oh yeah, that was the special one.
717. Bobby That's why the Suzy snowflake's there. Then it was 32.
718. Michelle Okay so 6.
719. Bobby 15, 15, 30, then 64.
720. Michelle We didn't do that. Would that be 64?
721. Bobby Well if our thing works.
722. Michelle Okay so 2 to the... (Loud noise)
723. Bobby 2 to the h for height. The only thing that wouldn't work is this one but...
724. Michelle Yeah it's an exception. We're about half way right?
725. 52:03 Bobby Uhh yeah, like an hour. So this is the only exception to the rule.
726. Magda (to Angela) Okay so now, we going, this and move this one on top. And moving that one on top with two blues.
727. Sherly (to Ashley) This goes with this. These go with these, I just haven't figured out how. There are two blues here and there are two blues here.
728. Bobby I thought we could take 7 like that. It could be 7, 21, 28, 42, 48.

729. Michelle You wanna see if that works?
730. Bobby No cause if there's gonna be 128, I don't wanna make 128 things.
731. Michelle That's true.
732. Bobby That'd take so long.
733. Magda So you doubled.
734. Alice (to Magda and Angela) How many do you have?
735. Angela We have, what,
736. Magda 32.
737. Alice (laughs) Like you said.
738. Bobby We destroyed 6, if we left 6 it would be easier.
739. Michelle Yeah.
740. Magda No cause like I thought...
741. Bobby Oh wait we can destroy 6. Do you wanna do it?
742. Magda ...that cause when you have a thing like with 2 height you can make umm...
743. Michelle No that's okay.
744. Magda ...8 of them, then you have three and you can make 16
745. Angela It sorta doubles
746. Bobby We'll just destroy the beginning.
747. Magda Yeah sixteen, it doubles then we have 5 and make 32.
748. Angela Making it like
749. Bobby Oh well.
750. Magda And we have 5 makes 32, so probably like 6 will make 64
751. Alice Why does that work?
752. Michelle I don't think they're doing the same thing.
753. Magda I dunno it doubles.
754. Bobby Oh we couldn't test this one though, cause then we'd have 128.
755. Angela I don't know; I'm confused. No 3 was 8, I'm a little retarded today.
756. Bobby If we just wanted to test this one, we could make 42 but that's too much work.
757. Alice Wait you sure you have them all? Are you sure you have all of this one?
758. Angela Yeah, because if you move this one up, like that.
759. Michelle Yeah. Nice chart.
760. Bobby Thanks, it was my idea too.
761. Alice Mhmm.
762. Bobby Look at that. It would be so much easier if we just did that in the beginning.
763. Alice Okay, and you did it.
764. Bobby Instead of looking at my notes.
765. Alice You used the same idea for that, didn't you?
766. Michelle Yeah, well your notes are a little too messy.
767. Alice Could you have used the same idea for those?
768. Bobby These are better than my old ones. (?)

769. Michelle You have to start writing neater.
 770. Bobby I can if I wanted to.
 771. Alice That moving up thing?
 772. Michelle Well, you should. It's important to be able to read someone's writing.
 773. Bobby Not in school.
 774. Michelle Yes it is.
 775. Bobby I don't know how (Drowned out by other students)
 776. 53:57 Michelle I hope, are we supposed to get report cards this weekend?
 777. Bobby Probably not.
 778. Alice Okay, and so what you're telling me is that when you get to the end of the group; that you know you're done. Is that true? Is that right?
 779. Magda Yeah.
 780. Angela Yeah.
 781. Alice How do you know there isn't any more?
 782. Angela I don't know.
 783. Bobby Are you convinced we got them all?
 784. Alice (to Sherly and Ashley) How're we coming?
 785. Sherly and (laughter)
 Ashley
 786. Michelle Yeah I'm convinced because you said so.
 787. Bobby Do you wanna do 7?
 788. Michelle No, I don't wanna do 7.
 789. Alice You have a (inaudible) pretty much.
 790. Bobby You can do it yourself.
 791. Michelle I don't want to.
 792. Alice And then you have those and then you have these.
 793. Bobby Let's do the 4,098 one.
 794. Michelle No, that's okay.
 795. Sherly I think we do, too.
 796. Alice Can you do that? In there though.
 797. Sherly We don't have those done?
 798. Alice I don't get this.
 799. Sherly Like, here one moves up, one yellow moves up
 800. Bobby 6, 7, 8, 9, 10, 11, 12. It would be the 12 high. Its 2 to the 12th.
 801. Alice Oh I see.
 802. Magda But,
 803. Alice But
 804. Angela But, is it, are they these, you mean.
 805. Magda Yeah, that's what I'm like looking at, because if you turn this one over...
 806. Alice Oh, I see. And if you kept going up then you get this one, but, over there.
 807. Bobby 3 is different.
 808. Sherly I dunno we were just like, we don't know. We kind of made it

- up
809. Bobby So this only works for 2 colors, we should write that down.
810. Michelle Right it's only 2.
811. Alice No, I was just trying to see if you left out anything
812. Bobby Maybe we should write it like this
813. Alice And this one, oh they can still stand next to each other.
814. Bobby Formula for total combinations with 2 colors is 2^h .
815. Alice Yeah. Mhmm. And so they go up?
816. Sherly Yeah. Oh that helped.
817. Alice It helped me a little bit. How many do you have?
818. Sherly and (laughter)
Ashley
819. Bobby All right. Now, $C = 2^h$, like, less c is total, umm..
820. Alice That was 12 of those and then 8 of those,
821. Bobby You didn't know who did this?
822. Alice And then 22, 24, 26, 28, 30. How come you don't have a partner?
823. Michelle Mumbles.
824. Bobby We write like, 'let..'
825. Angela We were working on that. (laughter)
826. Michelle We weren't.
827. Bobby No we were. We're working on it.
828. Michelle We don't really need that.
829. Ashley We need a blue, yellow, yellow, blue, yellow.
830. Magda Did you get 64?
831. Michelle It's understandable... (inaudible)
832. Angela Yeah they got 64, Magda.
833. 56:56 Bobby (To Michelle) It's understandable to you, but to someone who never saw this problem before. You see that's what you have to keep in mind.
834. Michelle You don't have to write 'let c equal'.
835. Angela They'll understand.
836. Bobby Yeah but what if someone doesn't understand then?
837. Angela Oh who cares?
838. Michelle Whatever.
839. Angela Oh, just let him...
840. Bobby I'll just put $c = 2$ high. I should put that, I should put it only works for 2 high ones.
841. Angela I'm stealing my paper Magda.
842. Bobby We should think of something like... What if you had the 4, how would you find the 3 and the 5 without looking at it? You know what I'm saying? No, cause what if someone just did this problem for the first time and they came up with how many 2 yellow for 15 and they wanted to find out 14 without doing it. How would they do that? (Michelle shrugges) You know what I'm saying?

843. Michelle Yeah, I don't know.
 844. Angela You're too smart for your own good.
 845. Bobby No, I'm serious.
 846. Michelle I know you're serious.
 847. Angela He makes the rest of us look bad.
 848. Bobby Because if they do 2 to the whatever, 200 high. There not gonna sit out and write 1 plus 2 plus 3 plus 4 plus 5 all the way up to 200. You know what I'm saying? That's your job. That looks nice.
849. 57:59 Michelle Yeah that's cause they're neat, they're legible.
 850. Alice (To Angela and Magda) Do you still have your rotation thing?
 851. Angela Yes.
 852. Alice And you know where they start. Okay we'll just take a minute then we'll talk about what each person's doing.
- (To Bobby and Michelle) Did 21 work out?
853. Bobby Oh, we didn't try it.
 854. Alice Hmm?
 855. Bobby We didn't try it.
 856. Alice But you think it would.
 857. Bobby Yeah.
 858. Alice Why?
 859. Bobby Cause we figured out how it goes. We also, I just also saw this right now but I don't know if it's right because I didn't have a chance. But 1 times 0 is 0. 2 times a half is 1, 3 times 1 is 3, 4 times 1.5 is 6.
860. Alice Help me with that. What did you say?
 861. Bobby All right umm, I guess you start here like 1 times 0 is 0.
 862. Michelle Oh.
 863. Bobby And then keep going up by a half. 2 times a half is 1, and 3 times 1 is 3.
864. Alice Now where does this half thing come from?
 865. Bobby I don't know I just looked at it.
 866. Alice No, no, I'm just asking. I'm just trying to understand what you're saying. You're taking this number and you're multiplying it by that number?
867. Bobby No, no we just started with 0 then .5 then 1 then 1.5. Like 1 times 0. Like this number doesn't even count. Let's say we didn't know was this number was. Take 1 time 0 and you get 0. Take 2 times .5 and you get 1, take 3 times 1 you get 3. Add another .5 and 4 times 1.5 is 6. You know what I'm saying? And 5 times 2 is 10 and 6 times 2.5 is 15. And I guess 7 times 3 is 21. You know what I'm saying?
868. Alice Okay, you're taking a half of the one before it?
 869. 59:59 Bobby I guess.
 870. Alice Where does this half thing... 2 times a half is 1

871. Bobby No, no. Like 1 time 0, you start with 0 and then 1 times 0 is 0 and that's how many 2 blue there are.
872. Alice Okay.
873. Bobby And then for some reason, I don't know why, I just looked at it and went up .5 and so 2 times .5 is 1. That's how many 2 blue there are.
874. Alice Yeah, that's where this number came from.
875. Bobby Yeah.
876. Alice 2 times .5 is 1.
877. Bobby And then you go up...
878. Alice Wait, I need for you to write (gives paper to Michelle). All right. So you said...
879. Bobby Okay start with 0 then you go up .5
880. Michelle 0 times .5 is 0?
881. Bobby No do like 1...
882. Magda What's .5?
883. Bobby (to Magda) We're working.
884. Bobby Do like 1 times 0.
885. Michelle 1 times 0 is 0.
886. Bobby Yeah. And that's how many 2 blue there are, right?
887. Michelle Right.
888. Bobby Then you go up
889. Alice No 1 times 0. Okay.
890. Bobby Put equals. Then you do 2 and its .5.
891. Michelle Oh I understand, is 1.
892. Bobby And then 3 times 1
893. Michelle And 3 times by 1 is 3.
894. Alice And the 1 came from half of this? Or you're just adding .5 to itself.
895. Bobby We're just adding .5.
896. Alice I get it.
897. Bobby 4 times by 1.5 is 6. (Michelle says it with him) Cause 4 and a half is 6.
898. Michelle 5 times by 2 is 10. 6 times 2.5 is
899. Bobby 15.
900. Michelle 15. 7 times by 3 is gonna equal 21.
901. Alice I see because it proves to you what it was. That's very, very interesting.
902. Bobby Yeah.
903. Alice Why does that work?
904. Bobby No idea. I was just... because we came up with the number thing like a... what was it? Oh, like if you add 1 then you add 2 then you add 3. I was thinking what if you told us you wanted 200 high, we're not gonna go adding 1 2 so we tried to figure an easier way out.
905. Alice Oh. And so then, how are you gonna get that number? You

multiply 6 times 2.5

906. 1:02:01 Bobby Divide by 2.

907. Alice 7 times 3.

908. Bobby I guess you could divide by 2 and minus by .5 because 1 divided by 2 is .5, minus .5 that's 0. And then 2 divided by 2 is, or half of 2 is 1 minus .5 you get .5.

909. Alice What? That's really interesting. What's 8 gonna be?

910. Bobby Umm let me see 8. Yeah, it be 24 plus...

911. Michelle 28.

912. Bobby It'd be 28. I guess because half of 8 is 4. You minus .5 you get 3.5, that's what you're gonna multiply it by. If you take 7, divided that by 2, you get 3.5 divide that by...

913. Alice Can you do the next one so I can remember that one?

914. Michelle 8?

915. Alice Mmhmm. Times 3.5 is?

916. Michelle 28?

917. Bobby Yeah.

918. Alice Okay and you were saying that there was some sort of adding pattern that you saw also?

919. Bobby Oh yeah, you have 0 plus 1 plus 2 plus 3 plus 4 plus 5.

920. Alice Can you maybe?

921. Bobby I wrote it here but I don't know if you can read it.

922. Alice Well yeah. I'm really fascinated to see if that's gonna keep working. Does it have to?

923. Bobby Well I guess we did enough examples that...

924. Alice And so 9 would be? Let's do one more.

925. Bobby 36. 9 times 4.

926. Alice Okay and so your rule was?

927. Bobby Oh. I guess we could have a formula for this like h. Put in parenthesis h.

928. Michelle Where?

929. Bobby On the back of this. Here want me to write it?

930. Michelle Yeah. Why don't you try and write it first?

931. Bobby All right. H divided by 2 minus .5. I'll put that in parenthesis, you don't have to, equals... What number are we finding? The 2, number of 2 colors I guess, I don't know how to write that.

932. 1:04:03 Alice It'd be exactly 2 colors.

933. Bobby Yeah. And h is how high the cube is.

934. Alice Yeah. Okay.

935. Bobby H equals height of tower. And that's...

936. Alice Uh huh, and that's the number... That's really fascinating, make sure you, yeah. (Michelle writes it down.)

937. Michelle I'll write it down, nice and neat.

938. Bobby Don't write it like that.

939. Michelle Oh god, would you rather it like this?

940. Bobby Yeah. Put it in parenthesis too.

941. Michelle Does this have to have a thingy?
942. Bobby Yeah close that, minus .5, equals number of 2 colors. I don't know how to write that. How would you write that?
943. Michelle Number of 2 colors, they'll understand what we're doing.
944. Bobby Yeah, I guess. I guess we're done.
945. Michelle This is H, right?
946. Bobby Yeah. We knew it worked for 2 but does it work for 3. You know what I'm saying, does it work for 3? Would you just take h divided by 3?
947. Michelle I don't know.
948. Bobby Do you wanna test a couple?
949. Michelle We can't because we don't have 3 colors.
950. Bobby Just take these apart.
951. Michelle Yeah but we need 3 colors.
952. Bobby No, oh yeah.
953. Alice What are we saying?
954. Bobby We were just wondering if it would work if you had 3 different colors. Or 4 different, do you know what I'm saying? If it only works if you have 2 colors. Because the one we figured out for how many total there are, only works for 2, it doesn't work for 3.
955. Alice What would it be for 3?
956. Bobby For t.. For total combinations?
957. Alice Yeah, I mean if you had 3 different colors and they were 2 tall. If they were 1 tall what would they be?
958. Bobby Ah, 0 right? Or 3. If they were 2 tall I guess it would be 9 total. We didn't test it out.
959. 1:06:06 Alice So you'd have to go back and revise your thing.
960. Bobby No, yeah, I guess, I don't know.
961. Angela We got it. It works because 2 tall is 9, 3 squared is 9. So if you, if x is the number of colors and n is the height, so its just x to the n.
962. Alice What if you had to convince somebody why that works?
963. Angela I don't know.
964. Alice But it does seem to work.
965. Bobby What about me. Like Mr. (Something) or a, he's a teacher so he thinks up with a problem, doesn't know the answer but he'll find it out, just not right away. You know what I'm saying?
966. Alice Robert and Michelle were sort of saying they thought it would work the same way.
967. Bobby What?
968. Alice If you added another color.
969. Bobby Oh yeah, yeah. We were, we didn't test it so we don't know if it's right.
970. Alice But, I mean, not for your exactly 2s, that was a different thing that you'd probably have to work on. How many all together is what they were...

971. Bobby Oh yeah
 972. Angela 9, 10, 11, 12, 13, 14, 15, 16.
 973. Dr. Maher What's this? Tell me about this.
 974. Bobby So we have 2 equations, or 2 formulas for both them. Yeah that one. Oh let me explain this. You figure out total combinations, we just figure you take the total number of colors, like your using 2. And you raise it to the height. The only thing I don't know if it works for is I don't know if 2 to the 1st is 1 or 2.
 975. Michelle No umm, cause 2 to the 0 is 1.
 976. 1:08:00 Bobby Yeah so we didn't figure that out yet.
 977. Alice 2 to the zero... So is if it was no height, you'd have to have 1.
 978. Bobby Yeah, I know. But it works for everything else.
 979. Alice 2 to the 1st power is?
 980. Bobby 2 and 2 to the 2nd is 4 and 2 to the 3rd is 8 and that's why with 12 combinations there were.
 981. Alice Why does that work?
 982. Bobby I don't know, I just saw the pattern. So I just wrote it down.
 983. Alice Show me what it would be for 2 to the 1st. What would the towers be?
 984. Bobby Oh, this and this.
 985. Alice This is 2 tall.
 986. Michelle No that's just 1, so it would be 1 and 1.
 987. Bobby Yeah total combinations.
 988. Alice 2 to the 1st would be what? Show me.
 989. Bobby Right here.
 990. Michelle No, that's wrong. It's these 2.
 991. Alice So 2 to the 1st would be these guys. And 2 to the 2nd according to your theory would be how much?
 992. Bobby Oh, right here. 4. We got them right here.
 993. Alice Oh okay. How did you get from here to here?
 994. Bobby Umm we just built them.
 995. Alice Show me.
 996. Bobby We put one up here, then one on top of here. Then we just moved this on top and this on bottom and we just put this on top and this on bottom. We switched them.
 997. Alice But if you started with these 2 you said you could have gotten these 2? If you started with those 2 could you have gotten those 2?
 998. Bobby Yeah probably, just put the opposite color on top.
 999. Alice Show me.
 1000. Bobby Just take this and put it on top, then take a yellow and put it on top.
 1001. Alice So your telling me then from those 2 originals, we actually don't need that many.
 1002. Bobby Yeah I know.
 1003. 1:09:58 Alice Yeah. From this one building up which could you get to?

1004. Bobby Umm this.

1005. Alice No leave this one so I can see. You got them right here. I want to know which ones you can get to.

1006. Bobby This one and this one.

1007. Alice What about from here?

1008. Bobby These 2.

1009. Alice Okay is there any way to get from those to the 3 ones or not.

1010. Bobby I guess if you take this one and you build on it, you can get this one and uh we don't have this one.

1011. Alice Here build it, show me so I can see what you're talking about.

1012. Bobby You could have this one. And I guess we could build on this one we could build on this one and we could build on this one.

1013. Alice Tell me what you'd get. For instance where would this one come from?

1014. Bobby Umm, right here. And you could build on this one with...

1015. Alice Don't kill your babies.

1016. Bobby Oh, alright.

1017. Alice Leave it.

1018. Bobby So we'll go with this one.

1019. Alice Do you see what he's doing?

1020. Michelle Mhmm.

1021. Bobby Yeah, yeah, yeah. I understand now. You could go like this and then this. Then you could go like this.

1022. Alice How many did you say you had for this one?

1023. Bobby 8. Then this and this.

1024. Alice And did you build the 8?

1025. Bobby Yeah there's 8. I'll just spread them out.

1026. 1:12:01 Alice Is there any other that you could have built?

1027. Bobby No, I don't think so.

1028. Alice Why?

1029. Bobby Oh cause the bottom, see you have all possibilities here on the bottom and then here you could have all that could be on the bottom. You're just adding to the top I guess. The if we take this and add a blue and a yellow to the top of this one, then take this and add a blue and a yellow to the top of this and all the way through. You know what I'm saying?

1030. Alice Does that make sense?

1031. Bobby Yeah it does.

1032. Alice Does it keep going?

1033. Bobby Yeah I guess it'll keep going forever, that's why that thing works too though. Because you're just adding an extra set of two. Like... I dunno.

1034. Alice Michelle, do you see what he's saying? Do you agree with him?

1035. Michelle Mmhhh.

1036. Alice How would it work for 3?

1037. Michelle Oh, okay.

1038. Alice Don't kill those.

1039. Michelle No I'm just moving them. If there were 3 then you'd just have 1 more here and then 3 would come off of that.

1040. Alice That's really interesting. (To Angela and Magda) And you all really believe that this would happen?

1041. Angela and Magda Yes.

1042. Alice Did you hear what they were saying?

1043. Angela No.

1044. Alice Can you all show that to these and see if they agree or not because they think it should double as well and it should be something to the n?

1045. Bobby You can explain it.

1046. Michelle Why don't you do it. You do it so well.

1047. Angela Michelle, explain it to me.

1048. Alice (inaudible) (loud noise)

1049. Michelle You want me to? (Angela nods) Okay why don't we move it the other way so they can see.

1050. Bobby All right.

1051. Angela I can see, it's okay.

1052. Bobby Start over here. (They move the towers.)

1053. 1:14:06 Angela Oh, we got a little pyramid going.

1054. Bobby No I think that goes over here. You know what I'm saying?

1055. Michelle Right.

1056. Bobby And then this goes over here. That goes in the corner. This goes here.

1057. Michelle You see?

1058. Magda You kind of add that one on top of the other?

1059. Bobby We just took like this and added a yellow and blue, then we took this and added a yellow and blue, then we took this and added a yellow and blue. And we did that for all of them.

1060. Angela Oh.

1061. Bobby So you just take this, add a yellow and blue, do that for all of them. Then they grow another row and you just keep going adding a yellow and blue to each one. That's it. It looks nice.

1062. Angela It's very lovely.

1063. Bobby She said it branches.

1064. Michelle It branches.

1065. Angela Yeah because it does. I thought of factor tree.

1066. Bobby Oh yeah, we used to do that with the bunny rabbits.

1067. Alice Would that work if you had 3 different colors as well? Angela and magda, what do you think? How would it look if you had 3 colors.

1068. Angela Well it'd be like 3 of these little thingies branching out. It'd be a lot bigger.

1069. Bobby Yeah.

1070. Alice Would it prove your theory or not?
 1071. Angela I don't know.
 1072. Bobby Yeah.
 1073. Angela (To Bobby) Would you like to see our theory?
 1074. Bobby Yeah I would.
 1075. Angela It's x to the n . It's Angela's towers.
 1076. Alice Okay, because you guys each had the same theory, would you put your names on the papers, okay? To make sure?
 1077. Bobby Change that 2 to x then in this.
 1078. Michelle What 2?
 1079. Bobby Because remember we were doing this for 2s.
 1080. Michelle Well, all right.
 1081. Bobby We'll make it separate.
 1082. Michelle We'll change 2 to the h to colors, x which is the colors.
 1083. Bobby Yeah, just write that in the corner.
 1084. 1:16:00 Angela Well this is what we have.
 1085. Bobby Put like the Law of Towerism or something.
 1086. Angela It's Mag's and Angela's Law of Towers.
 1087. Michelle It's x to the h , right? Where x is equal to the number of colors, right?
 1088. Bobby Yeah, yeah and h is the height. Put it in like a box. Name it too.
 1089. Michelle We don't have to name it.
 1090. Angela I named ours.
 1091. Bobby How are they supposed to know what it is? If you just write x to the h , they're not gonna know what it is. You have to tell what it does.
 1092. Michelle (Writes and says) Tells possible combinations.
 1093. Bobby All right.
 1094. Michelle See, it says it right there.
 1095. Bobby Yeah but... They might be trying to teach this to someone who doesn't know what they're doing.
 1096. Angela God, Michelle
 1097. Magda They'll explain the problem.
 1098. Bobby Mr. Durchill (?) forgets like half my name, he leaves out the e and the r .
 1099. Michelle Are you serious? (?)
 1100. Bobby Yeah.
 1101. Michelle They don't need that, they can't read your chicken scratch anyway.
 1102. Bobby He capitalizes the t too.
 1103. Angela So its like...
 1104. Michelle Don't write on our work.
 1105. Bobby Here write on this. Yeah. He even has it in his marking book like that. He sends stuff home like that. Because we had a substitute and she was like is RobT here? (They laugh) She had to fix it in his book because he spelled it wrong.

1106. 1:18:03 Angela That's so funny.
 1107. Magda Why aren't you in our class anymore?
 1108. Bobby I'm taking the same class again.
 1109. Magda, (laughter)
 Angela,
 and
 Michelle

1110. Bobby I'm serious.
 1111. Angela Why were you like absent every day last year?
 1112. Bobby I was sick.
 1113. Angela Every day?
 1114. Bobby Yeah. We're at the revolution, we finished.
 1115. Angela Finished what?
 1116. Bobby The revolution, we're going on to the civil war.
 1117. Angela Oh, you mean in history. I was so confused, like the revolution is 1776.

1118. Bobby (To Michelle) If they make us explain this, you're doing all the explaining.
 1119. Michelle I don't get it.
 1120. Angela (To Magda) If they make us explain this, you're doing all the explaining.

1121. Bobby No wait, if they make us write. You do the writing.
 1122. Michelle I'll do the writing.
 1123. Bobby I'll talk.
 1124. Magda (To Angela) I can write, you talk.
 1125. Angela No, I don't like talking. I like talking but not about that.
 1126. Bobby I'll talk, I don't mind.
 1127. Angela Bobby, you can talk about ours too.
 1128. Bobby All right.
 1129. Angela Do you mind if I call you Bobby or do you prefer Robert.
 1130. Bobby It doesn't matter, you're not gonna stop.
 1131. Angela You want me to call you Robert. Okay I call you Robert. Sorry, I'm just used to saying Bobby.

1132. 1:19:20 (Magda, Angela, Michelle, and Bobby have a conversation about television shows.)

1133. 1:20:56 Alice Dr. Maher you remember, Dr. Carolyn, do you remember all these people? You remember, you know Robert.

1134. Dr. Maher Yup.
 1135. Alice And you remember Michelle.
 1136. Dr. Maher Absolutely.
 1137. Alice And Angela.
 1138. Dr. Maher Absolutely.
 1139. Alice And Magda.
 1140. Dr. Maher Absolutely.
 1141. Alice They were saying they remember making 7s out of dice. (laughs) (inaudible) 6th grade when the came. And I saw you talking to

- Ashley and Sherly. But because we do have to close up, and people were doing such different things, I thought you all could exchange.
1142. Dr. Maher I'm still interested in what Ashley and Sherly did with the 2 of a color. Yeah. Are you convinced that you have all of the 2 that aren't together.
1143. Sherly Yeah
1144. Dr. Maher Why don't you tell them the question I was asking you?
1145. Sherly What was it again?
1146. Dr. Maher Do you remember what I was asking you?
1147. Ashley Mumbles.
1148. Dr. Maher Are you all- Do you all hear what Ashley's saying here?
1149. Bobby (Whispers to Michelle)
1150. Alice Bobby.
1151. Dr. Maher Go ahead Ashley.
1152. 1:21:59 Ashley (Talks very softly) We couldn't make any more combinations other than these 4 right here with 2 blue.
1153. Sherly Well, you know how in this one we had the 2 blues together, well in this one the 2 blues are separated by a yellow. Then this one we had 2 blues separated by 2 yellows. Then this one we had 2 blues separated by 3 yellows.
1154. Dr. Maher Oh that's interesting, so here we had 2 blues separated by
1155. Sherly No, it's just together.
1156. Dr. Maher Oh, no yellows.
1157. Sherly Yeah.
1158. Dr. Maher And these blues are separated by one yellow. (Sherly agrees) Okay how do you know you have all the 2 blues separated by one yellow? Can you convince me of that? (Sherly laughs) Cause Angela was really wondering about that. Weren't you Angela?
1159. Sherly What were you wondering Angela?
1160. Magda Didn't you just shift them down one?
1161. Sherly Yeah.
1162. Magda See what I'm saying?
1163. Dr. Maher No.
1164. Magda Okay, this is what we did. It's like us taking this and putting it on top, and you did the same thing.
1165. Angela Then you take the one on the bottom and put it on top. And then you take the one on the bottom and put that on top.
1166. Dr. Purid Oh so you actually had a system.
(?)
1167. Angela Like we did here it basically the same idea. Well if you took the one on the bottom and put it on top then you'd have that one. I don't want to mess up our little thing. Then if you took the one on the bottom and put it on top you'd have that one. Then if you took the bottom of this one and put it on top and so on, you'd

1168. Alice have this one so you know there's no more in that group.
 How many groups did you have?
 1169. Angela 1, 2, 3, 4, 5, 6. Or did I count something else twice.
 1170. Magda 6.
 1171. Dr. Maher So you have 6 groups?
 (?)
 1172. 1:24:01 Magda Yeah plus this little thing here.
 1173. Dr. Maher Okay, I wanna hear about that but did you follow what they were
 saying, Ashley and—
 1174. Sherly Yeah
 1175. Dr. Maher But you weren't thinking about it that way were you?
 1176. Sherly No.
 1177. Dr. Maher Tell them how you were thinking about it cause it's a little
 different.
 1178. Sherly We just had a little sequence going down. You know how
 there's that and then... I don't know. Yeah, you can just kinda
 look at it and.
 1179. Ashley This one right here is 2 blues on top, then in the middle. This 2
 blues is separated by a yellow. Then in the middle is 2 blues
 separated by a yellow. Then on the bottom is 2 blues separated
 by a yellow and it keeps going like that. And these right here are
 just the opposite of these. There's 2 yellows separated by a blue,
 then in the middle are 2 yellows separated by a blue, then right
 here again right here are 2 yellows separated by a blue. It's just
 the opposite.
 1180. Alice And then what did you do?
 1181. Ashley I don't know.
 1182. Sherly No then we had like 2 blues separated by 2 yellows.
 1183. Dr. Purid Okay. And that's all there are?
 (?)
 1184. Sherly Yeah.
 1185. Dr. Purid Why?
 (?)
 1186. Sherly Cause you can't do anything else, you can see it, though. Like
 you can kinda see it. Like you can't have 2 blues separated by
 what was it? 4.
 1187. Ashley We each had to make it and we kept getting the same thing so
 we just stopped.
 1188. Alice That's sort of like what Angela was saying. And then you told
 me you couldn't have them separated by 4?
 1189. Sherly No.
 1190. Alice Why?
 1191. Sherly Cause there'd be 6 then.
 1192. Alice Oh, how many did you all come up with?
 1193. Ashley 32.
 1194. Alice You had predicted?

1195. Ashley 24.
1196. Alice Yeah sure.
1197. Dr. Maher Okay are you really convinced you have them all and you accounted for them all?
1198. Ashley No, I'm not. I'm not convinced. I don't know what we're missing but...
1199. Dr. Maher Okay, is there anybody who's convinced that they have them all? Of any of the towers any particular height.
1200. 1:26:01 Bobby Yeah, we think we do.
1201. Dr. Maher You think you're convinced?
1202. Bobby Yeah
1203. Dr. Maher Michelle, let's hear about it.
1204. Michelle Okay.
1205. Dr. Maher They think they're convinced so let's listen to them.
1206. Bobby Yeah and then show them what you showed them.
1207. Michelle Oh, that branchy thing.
1208. Bobby Yeah.
1209. Michelle Okay, well for the total number of combinations we had x which would be the total number of colors raised to h which would be the height. And it works for like if you had 2 high it would be like 2 to the 2nd and you get 4. You wanna explain it cause you're better?
1210. Dr. Maher No, go on Michelle I follow you.
1211. Michelle And then if you have 3 it would be 2 to the 3rd, which would be 8 and so on. Like if you had 4 it would be 2 to the 4th which is uh, 16. (Giggles) Shut up, oh my god. And then what was this one, total combination?
1212. Bobby Yeah.
1213. Michelle Okay so, this was another way of doing it.
1214. Dr. Maher Oh my, this looks different.
1215. Michelle (Assembles the branches) Give me a second.
1216. Dr. Maher So with your rule, what would you predict if you were making them 10 tall?
1217. Michelle Umm, if it was 10 tall then it would like be 2 to the 2nd. If it were like...
1218. Bobby 2 to the 10th.
1219. Michelle 2 to the 10th. If it were like 2 colors.
1220. Dr. Maher So why do you think that works?
1221. Michelle I don't know. (To Bobby) Do you know why it works?
1222. Bobby Uh, no.
1223. Dr. Maher You don't know why it works; what did you do there?
1224. Alice Can you explain what you did there? You were explaining it to Angela and Magda a while ago.
1225. Bobby Okay umm, when we had 1, like, everyone knows that, its plain to see that these are the only 2 combinations if you had 1 high, like 2 separated colors. And we built on that by putting a blue on

- top and a yellow on top of each one. So you know like, you have each combination for that. And now we elaborated by keep going out and we put a blue on top of this and a yellow on top of this. Like, blue yellow, blue yellow, blue yellow.
1226. 1:28:17 Dr. Maher So do you understand what he's talking about Magda, Angela?
1227. Bobby And she helped us with this 'cause she started it and we figured it out. And then, I guess it could work for like any color. If you had 4 colors, say you had red, blue, green, and yellow. You put a red, blue, green, and yellow and you put one of each on top. Like you put a red on top of a yellow, green on top of a yellow, blue on top of a yellow, yellow on top of yellow. And you'd have like- The things will be bigger, the branches, but you'll still have all the combinations.
1228. Dr. Maher Okay, what are you showing me there? That looks very different then what I'm seeing here so I'm confused. Does that have anything to do with this?
1229. Bobby Well it does. It's like
1230. Michelle It's just another way of
1231. Bobby Of starting it. To prove that we have them all, I guess. Just a way. Because you start out with this and you keep adding on to it. And when you get up to five, if you keep adding on to it when you get to five, and there'll be 32 and we have proof that that's it.
1232. Dr. Purid Show me how you'd go to the next level from where you are
(?) now.
1233. Bobby Do you wanna do it? Alright.
1234. Michelle Whatever.
1235. Bobby Well see there's one here with 3 on the bottom. You take two 3 on the bottoms and you take a yellow on top of one.
1236. Michelle No this one too...
1237. Bobby No we're starting over here. You add a yellow on top of one part and a blue over here. Then you take 2 bases that look exactly like this, 2 blue on the bottom and one yellow on top, and then you add a blue and a yellow to it. Then you keep doing that for all of them, and then... This one you'll take a blue on the bottom and 2 yellow on top and add a blue on top and a yellow on top. Then you'll take this one and make 2 exactly like it and add a blue and a yellow. You just do that for all of them and then when you do the next thing you take 4 blue on the bottom and you add a yellow and blue on top. And then keep going until you wanna stop I guess.
1238. 1:30:26 Dr. Purid Are you sure you've got all of them by doing that?
(?)
1239. Bobby Yeah cause if you're sure you've got all of this, and there's only like 2 ways you can change this and that's by putting one on top of each. Then there's only 2 ways of changing this, by putting a

1240. Dr. Purid (?) yellow and a blue on top. Couldn't I put on the bottom?
1241. Bobby Uhh. But that would be the same thing as something over here. So you just always add on to the top and you keep going. Then this is all that's possible for 2, then if you just add one to the top of each of them you'd have all that's possible for three. Then do that for 3 until you get all for 4, etcetera.
1242. Dr. Maher So what do you think about that? The rest of you, does it make any sense?
1243. Sherly Can you explain it again Bobby?
1244. Ashley (Talks inaudibly soft) Then you add one and every time you add one your making the same pattern. (Talking extremely soft) You just keep adding one for every stack, like 4. These are 4s then you just start again, you just add a blue and add a yellow and keep doing that for all of them. I think you do get all of them that way.
1245. Dr. Maher Okay, is there anything you wanna ask each other about what you did?
1246. 1:32:00 Alice I would really like if Bobby and Michelle ex-, or just sort of showed that other thing you worked on in the beginning because that's still a mystery to me as to how it works.
1247. Bobby We figured out how there's 2 blue and 2 yellow things.
1248. Alice Yeah it was the question you were exploring at first. Which is that I asked you, asked you to build 4 high, and I asked you how many there would be 5 high had exactly 2 blue.
1249. Bobby Yeah. We figured it out.
1250. Alice And you said there would be?
1251. Bobby 10.
1252. Alice And they said there would be 10.
1253. Dr. Maher Do you agree? 5 high with exactly 2 blue, do you all agree that there'd be 10?
1254. Alice You all have it in front of you, you can prove it.
1255. Dr. Maher Let's ask them, Alice.
1256. Angela So wait, 5 high and 2 blue.
1257. Alice Mmhmm. With exactly 2 blue. There's five of them are there any more?
1258. Angela (?) 10.
1259. Dr. Maher Okay, but it's a different question. You have 10, how do you know you have them all?
1260. Angela We did the same idea we took one of the bottom and put it on top.
1261. Magda Wait is that it?
1262. Angela It's not exactly in order because I mixed some of them up before. Yeah well backwards. We took that one off the bottom and put it on top, we took that one off the bottom and put it on

- top of this, take this one off the bottom and put it on top of this. And then if you took this off the bottom and put it on top, you'd have this, so you'd have doubles then. You know. We went through the whole little cycle thing.
1263. Dr. Maher So you're convinced by the same reasoning that you used to generate them. Is that true for you also, Magda? (to Sherly and Ashley) What about you, you're convinced you have all of them, 10.
1264. 1:34:02 Sherly and Ashley Yeah.
1265. Dr. Maher But your argument is a little bit different than them, right? You weren't taking it off and moving them.
1266. Sherly Yeah, we were just kinda like... We saw the sequence. Where was it? Like in this one you can kind of just see it, you can see it was going down a little bit.
1267. Dr. Maher We talked about that, but your argument is now different than theirs. Wow, let's hear it, do you wanna hear it? Do you all wanna hear their argument?
1268. Class Yeah.
1269. Dr. Maher Okay, go ahead. Alice says she's confused.
1270. Bobby Okay, first we started doing this then we did it up to 5 and we figured out how many 2 blue and 2 yellow. And we made a nice little chart right here.
1271. Dr. Maher Show the camera. Okay, thank you.
1272. Bobby Okay, and the chart just tells.
1273. Michelle You can put it down.
1274. 1:34:56 Bobby Okay. How high the tower is and how many 2 blues and 2 yellows and how many total there are, and how many total combinations. For the one it was 1 height and it was 0 2 blues, 0 2 yellows, 0 2 blue 2 yellow total, and 2 total combinations. We did all the way up to 5. Then we looked at the numbers and we figured out if you take the height and divide it by 2 and minus that by .5. Like take 2 divided by 2 is 1 minus .5 would be half. Then you multiply, you get the number of color towers. So, I guess, yeah go ahead.
1275. Michelle Like if you had 4, you would divide it by 2 which is 2 minus .5 is 1.5.
1276. Bobby Then you multiply the height times that number and you get how many 2 blue towers there would be. That's it right.
1277. Dr. Maher Does that work for?
1278. Bobby Yeah we did it for every one.
1279. Dr. Maher And it worked for all of them?
1280. Bobby Then we did 6 and we guessed ahead of time what we thought it would be, based on our 2 things we thought of, formulas or whatever. And what we did came out, exactly what we thought it would be.

1281. 1:36:05 Alice Which was what?
1282. Bobby 15 2 blues, 15 2 yellows, 30 2 total, then 64 total combinations.
1283. Dr. Purid Do you have any idea why that interesting formula works?
(?)
1284. Bobby Not yet, we didn't have a chance to look at it.
1285. Alice They were gonna work a bit more.
1286. Dr. Maher If you want to think about that. Maybe you can share that and you could all think about that.
1287. Alice You had an addition thing that worked too. How did that work?
Oh...
1288. Michelle Yeah that, like.
1289. Bobby Do you wanna explain it.
1290. Michelle No you can explain it, that's fine.
1291. Bobby Well the first thing we did was looked at the numbers and if you take the total, like for 2 blue, (waits for bell to stop) and you added 1 to the first number it would be 1, and if you added 2 it'd be 3, and if you added 3 it'd be 6. Like we just took out the difference and it was plus 1, plus 2, plus 3, plus 4, plus 5, and I guess it would keep going. But then we came up with the second one because what if someone wanted to come up with a tower that was 100 high and you wanted to find out how many 2 yellow were in it. We didn't want to go plus 1, plus 2 so we looked at all the numbers and we figured out that thing.
1292. Alice What would it be, according to your formula?
1293. Bobby What? Oh, um 50, I guess 495, I don't know whatever 49.5 times 100 is.
1294. Alice 49.5 times...
1295. Bobby 100. So 4950, I don't know.
1296. Dr. Purid I'm fascinated where the .5 comes in. There's no such thing as a
(?) .5 high tower. How did you get into that?
1297. Bobby I really don't know, we were just looking at numbers.
1298. Dr. Purid You were looking at numbers.
(?)
1299. Bobby And it seemed to work so, just stuck with it. 'Cause we were looking at the differences and notice, like, when you take 1, started at 0 the 1 times 0 is 0 and that's how many 2 there is. But then we looked at 2 and at first we were going up by 1, but 2 times 1 is 2, and there's only 1 blue. So we tried to see what gets there and that's half. Then we went to 3 and kept going up by half and 3 times 1 is 3 and that's how many there are. And 4 times 1.5 is 6. And then 5 times 2 is 10. They keep going on. I guess it's just by trial and error we figured it out.
1300. 1:38:25 Dr. Maher (Bobby writes down equation) You were all great to come today, on your day off. Are you gonna come back? You all gonna come back? Some of you are coming Monday, but if you haven't signed up and you still want to come, we'd like to talk to you

- individually. You can still sign up, even to the later shift, I know some of you are working. But we're bringing our class here Monday night. We have class that week and since some of the folks here are in that class, and others aren't. And so if you would like to give us a chance to chat with you and you haven't signed up to the list and you want to go with the class, you can still do that. Or if not we'll do it another time. What I would really, really like you to do--would you mind writing up what you did today? Would you mind writing up?
1301. Michelle It's written already.
1302. Dr. Maher Okay. We're gonna collect that and put it aside and try to rethink it, rethink what went on what made sense to you. I mean you worked with your partner, right? I mean you can talk to each other if you want to but would you mind writing up your own and bringing it Monday? And even if your not staying after school, we'll catch you sometime. I know it's sort of... It really helps us a lot if we have it and it will help you later as we move, to be able to look at some of this, we think. So if you could take the time while it's sort of fresh in you mind and write up your ideas and have a chance to rethink them. If you wanna borrow some of these to think about, you're welcome to. Okay Linda. If you really would like to take some and think about it some more over the weekend. Or explain it to someone. You know someone might ask you what were you doing, and that will teach them to ask you, right? (Laughter) You might be able to show them what you were doing, you can take them as long as you promise to bring the cubes back on Monday or I'll be in big trouble. Okay? So you can decide that. Do you have anything you want to ask us? Anything you want to ask our visitors?
1303. 1:40:40 Alice Did you meet our visitors?
1304. Dr. Maher We have 2 professors' here from out of the country. And do you wanna ask any question? Do you wanna ask Professor Han (?) any questions? She studies proof in students, that's sort of her special specialty, proof. And she has no questions to ask, do you have any questions you want to ask her? This is your very last chance. They go home this weekend.
1305. Alice She (Dr. Purid) teaches at the University of Toronto in Canada and at the University of British Columbia, which is on the other side of Canada.
1306. Dr. Maher Does she sound like she's from Canada? Do you wanna know where she's really from? Anyone want to guess? Talk some more so they can.
1307. Dr. Purid Let me say thank you very much for letting me be here.
(?)
1308. Ashley England
1309. Dr. Maher Well done Ashley. Where? Where do you think she's from in

- England?
1310. Ashley London? (?) I don't know.
1311. Angela No idea.
1312. Dr. Maher She's from Oxford. Have any of you, you know of Oxford right?
1313. Dr. Purid At the University.
1314. Dr. Maher Yes at the University there.
1315. Alice Before you all go, Linda whispered to me that we really need your papers and if you can remember, looking at them, and on the back of them put the numbering of the order in which you wrote them.
1316. 1:42:00 Students (They work in groups to do this and all talk at once.)
1317. Alice Write your names and then any order, especially you all (to Bobby and Michelle) because this is something (mumbles).
1318. Bobby We don't, do we need this one cause we didn't really use it.
1319. Alice Oh sure, just put your names on it and your scrap work cause we need it too.
1320. Bobby All right.
1321. Michelle It's just scratches.
1322. Alice Yes but that's where you came up with that.
1323. Bobby No because she doesn't like my handwriting so I wrote it down first.
1324. Alice That's all right. Thank you all so much.

Appendix K

Date of Session: 12-13-1994

Author: Anoop Ahluwalia and Kathleen Dougherty

Verified by: Kiranjeet K. Sran

Date of transcript: 6-15-2010

Line	Time	Speaker	Transcript
1.	0:16	Amy-Lynn	No one will come to our meeting.
2.		Bobby	I didn't do my homework, my science.
3.		Amy-Lynn	What did we have?
4.		Bobby	Page 98, all.
5.		Amy-Lynn	(At the same time) Oh yeah, page 98.
6.		Bobby	Then I'll answer those finished questions we started. It's my first zero.
7.		Amy-Lynn	(Mumbles) I think nobody's coming today. Did you start the test, for gym?
8.		Bobby	Yeah. Only 8 people went though.
9.		Amy-Lynn	Good then maybe if only eight people went I might not have to go tomorrow.
10.		Bobby	No, we're going to start at the beginning cause not a lot of people passed.
11.		Amy-Lynn	I know, but umm. (mumbling)
12.		Bobby	Yeah. Okay. Okay. Yeah.
13.		Amy-Lynne	The half-day is the 23 rd .
14.		Bobby	Yeah. The day before Christmas Eve. No come, come to school Friday. Then we go from Saturday until like next Monday or Tuesday or something.
15.		Amy-Lynn	Yeah, I think so. (Magdalena arrives) (To Magdalena) Hi. I didn't know they we're coming
16.		Magdalena	What?
17.		Amy-Lynn	I didn't know they we're coming.
18.	1:58	Magdalena	Me neither.
19.		Bobby	(Someone outside the groups asks a question.) Yes, he is. (Knocks on the table).
20.		Brian	Hello people sitting at the table. (Whispers to Bobby)
21.		Bobby	Just don't tilt the table please. You'll like break the microphone and tip the table. (Girls whisper to each other).
22.		Brian	(Girl 1 begins to talk but is interrupted by someone outside the group who asks Brian a question) True, I hope it's the lightest one.
23.		All	(Mumbling)
24.	2:57	Teacher	Umm. I want your attention for a few minutes. I looked over all of the responses you made to the candle problem that I gave yesterday. First of all I would like to know, because I wasn't here, what kind of set up you had. (Bobby raises his hand) For example,

- Brian where were you sitting and what was your group like when you were doing the problem?
25. Brian I was sitting with Jeff and Ankur and we were working.
26. Teacher You were on task?
27. Brian We were on task the whole day.
28. Teacher Ok good. Umm okay. I have a few other questions to ask you about it but okay. How about Romina. Where were you sitting and what was your group like?
29. Romina I was sitting with Stephanie and we basically tried. (giggles)
30. Teacher Okay. And how about Michelle?
31. Michelle I was sitting with Megan and Sarah. I was basically just me and Megan working, Sarah was working on her own.
32. 3:58 Teacher Okay. Now what I wanted to say was the results were fantastic. I really enjoyed reading all your comments on how you got to them. And umm, Michelle. How many did you come up with?
33. Michelle 30.
-
34. Teacher Okay. How many candles did we need?
35. Michelle 48.
36. Teacher We needed 48. Well arrangements we needed, ok. Now how many candles were we supposed to have in each window Stephanie?
37. Stephanie Five.
38. Teacher Five. Okay and what was the restriction we put on the colors, Amy?
39. Amy-Lynn Red and Green. No red and gold.
40. Teacher Yeah. So there was a restriction on colors, a restriction on number of candles, and the question was exactly what Sarah?
41. Sarah How many combinations or arrangements the candles could go in.
42. Teacher Okay did everybody understand the problem pretty much?
43. All (Mumbling in agreement)
44. Teacher Okay now Michelle said she came up with 30 arrangements. Michelle could you explain how you came up with thirty arrangements?
45. Michelle Well, you know the binary system thing so we took that out and used that when r equals 0 or g equals 1. (Bobby whispers to Brian "I made that up" and Brian whispers something back) It would be 32 but since there had to be different colors in each window. (Amy-Lynn whisper to the boys) All gold and all red could be in one window.
46. Teacher Okay excuse me hun.
47. Michelle All gold and all red could be in one window.
48. Teacher Umm. I'm not saying yes or no on that. Here's how the problem

- went. He thinks a design of 5 candles in 2 different colors. Red candles or gold colors would be most appropriate in each window. How many different five candle arrangement can he make? So Michelle decides that she's definitely going to use 2 colors in each arrangement and Mike what did you picture?
49. 5:58 Mike I put the thirty in all windows in all reds and all golds.
50. Teacher Okay Michelle could you explain about the binary thing that you talked about?
51. Michelle You know the chart you gave in class?
52. Teacher Which chart?
53. Michelle It went like 1s, 1s and the 1, 0 and so on and so on. So like when it was 1 I put like 0s in front of it and (Drawing on the chalkboard) then the 0s equal red and then gold was 1. So like on my paper I just put like 4 0s and then a 1 and it was mainly just the binary system thing with like 5 of each but like (mumbles).
54. Teacher Okay the reason why I wanted to talk about it a little bit was because I didn't expect it to take the direction that it took. Alright I expected an entirely different worksheet from you people and I was very surprised, but I thought it was good that you linked that particular idea. Umm why? (Amy-Lynn and Bobby raise their hands.) Why did you think of binary for this particular thing?
55. Bobby Bobby? Well I had my paper out and I was looking at it and like I saw all these five digit numbers so I thought maybe you should just change it cause there's only two different number in it and its base 2 so that means you wanna use like 0, 1 and there's 2 different colors red and gold. And so after you did it you switch it around cause you can switch places.
56. Teacher Ah ha
57. Bobby Like 0 could be gold and 1 could be red.
58. Teacher Okay. So I heard the word places. What made you think of places?
59. 7:53 Bobby Well like there was five windows, and you could use them as numbers. Say there was like 5 0s or like five 1s and you use that as like you know just... Well like you just say that instead of candles they were numbers. And say you can use them 1 and 0 they turn out the same cause the binary system, they turn out the same as numbers.
60. Teacher Ah ha. Okay what answer did you put down Bobby?
61. Bobby 32
62. Teacher 32. Okay so you must have counted... what things did you have to count?
63. Bobby The all reds and the all golds.
64. Teacher Okay so what should we have as an agreement for the answer? You know how sometimes they have a blank at the end, we didn't put blanks on this, but there's a blank and you have to put an answer down. What would you think is the right answer if, if

- somebody were marking this would be? What do you think, Angela?
65. Angela 32
66. Teacher 32 because why?
67. Angela Like there is like 16 one way and because of the binary we have like gold is one (Amy-Lynn raises her hand) as the 0 and like 16 and 16 is 32.
68. Teacher Ah ha. What do you think Jeff?
69. Jeff (Mumbles)
70. Teacher How Many do you think would be right?
71. Jeff 30
72. Teacher 30 Because?
73. Jeff Cause It wouldn't look right (inaudible).
74. Teacher Okay so that's a subjective argument right? (Amy-Lynn and bobby raise their hands.) Amy?
75. Amy-Lynn But still it asks for all the combinations that could be put and for red and gold that is a combination that could be put. Just all red and all gold is a combination.
76. Teacher Okay it says how many different? And explain how you know you have found all possible, you right the word all was used here. What do you think Michelle?
77. Michelle But like it says to use different colors so like you have to eliminate all gold and all red. (Bobby taps on the table.)
78. 9:58 Teacher Okay. So could we say that there could be 2 answers depending on interpretation?
79. Brian (Nods his head in agreement.) Yeah.
80. Teacher Hmm. Your ready to give that up like that, 2 answers depending on interpretation. (Bobby raises his hand.) Bobby.
81. Bobby Well you could say that you could only use all solid colors and there'd be a different number, if it says you could use any possible and there'd be a different one, if it says you could use only changing ones so there'd be like a red gold red gold then there'd be a different answer. It matters how they phrase the words.
82. Teacher Okay. We didn't hear too much from anybody else here. Anybody have any other opinions they want to express, like Michelle R?
83. Michelle R I was absent so I don't know what we're doing.
84. Teacher Oh okay. How about Romina?
85. Romina I think it's 32 because it didn't ask you if it would look right, it asked you how many possibilities were.
86. Jeff But it said in the thing different colors in each window.
87. Romina They wanted to know how many combinations were there.
88. Jeff But that's not a combination it's all one color.
89. Student There's still a variable across, you should have specified.
90. Teacher Did it remind anybody of anything else you've done in the past? Mike?
91. Mike The tower problem.

92. Teacher Well, Mr. Poe and I left it a little open to interpretation. We didn't word it so specifically. We could have put another sentence in to word it very specifically... but we left it open just to see what you were going to do with the problem. And I really liked the ideas I thought you did a really fine job. I couldn't believe it because we weren't here and you really worked it through. You wrote down all the possibilities and I think it was a very, very complete job for where we're at right now and that's what I wanted to talk about. Are you ready for... Yes Jack.
93. 11:59 Jack I have a question. Did Mr. Poe write that?
94. Teacher Umm... Let's say we both did.
95. Amy-Lynn Is that what you were doing at the Mac Lab Friday?
96. Jack Is that, is that graded as a quiz?
97. Teacher Yes.
98. Jack What was right?
99. Teacher Well that was why I was asking for your opinions. Whether if somebody put down 30 or if somebody put down 32, what do you think should happen with the points?
- 100 Jack Obviously the majority of the class would have one together.
- 101 Teacher Ah ha so do you think that when we're taking a look at them we should say that all the ones that are 30 are absolutely wrong?
- 102 Brian No
- 103 Student No because each person could interpret it differently. (Amy-Lynn raises her hand.)
- 104 Teacher Exactly, exactly. That's why I was asking for a little bit more input. Okay so I think that thirty is a fair answer based on the fact that if you see two different colors you might think that it should include at least one color that's different from the other four. So in being fair Mr. Poe and I want to assign some points to explanation and to seeing your work, which is really important. So it was really nice. I'll tell you I was pleasantly surprised. Very pleasantly surprised. I couldn't believe it. Okay are you ready for a challenge?
- 105 Bobby No. (Class mutters in background, Brian shakes his head no.)
- 106 Teacher Are you ready? Now you're gonna have to give it some time and some thought alright. One of your favorite people's involved in this problem. (Class looks up and mumbles.) Everyone will have their own personal copy.
- 107 Student Remote control gadget?
- 108 Teacher Do you know anything about remote controls?
- 109 Student Not really. (Bobby and Brain whisper on the side.)
- 110 All (Teacher passes out the papers and students talk amongst themselves.)
- 111 14:09 Bobby Why can't we just lift it up ourselves?
- 112 Amy-Lynn (Class reads the handout) (Whispers) This is supposed to be for grade 6. (Points on Bobby's paper Brian examines the markers and

- draws on Bobby's name tag.)
- 113 Teacher Okay, do you see this as a possibility? Could it happen in real life?
- 114 Amy-Lynn Mmmhmm. Has anything like this happened to you? Amy?
- Sort of because my mom keeps the garage door opener in her car and when she stops short or something it always falls out and one time it got so banged up she couldn't open it.
- 115 Teacher Haha. Bobby.
- 116 Bobby When I first like... before I moved here we lived in Hillside and we had this alarm on our house and my mom forgot and we went on vacation and when we got back she didn't remember the thing so she tried all these codes and the door wouldn't open and she had to finally call the company and they had to get it open by using something.
- 117 Teacher Yes Mike.
- 118 Mike My dad like when it was broken and I had to like climb up because the switches were really small and we could see if it was on or off. It was really hard.
- 119 Teacher Okay did you think of it as a mathematical problem when you were playing around with it?
- 120 Mike No.
- 121 Teacher Well guess what... it's your math problem.
- 122 16:05 Bobby Why doesn't she just open the door herself?
- 123 Brian Maybe cause she's a girl.
- 124 Bobby My sister can open the garage.
- 125 Amy-Lynn (Mumbles)
- 126 Bobby My sister can open the garage cause I don't feel like it sometimes.
- 127 Amy-Lynn Maybe it doesn't have a handle to open it. Mine doesn't.
- 128 Bobby So go under the door and pull up.
- 129 Amy-Lynn Maybe you can't.
- 130 Bobby So just park the car outside.
- 131 Brian So you don't park the car in the garage. Takes the stupid thing out (interrupted by Amy-Lynn). Rewires the whole thing, puts a new one in, gets a new remote, and you're set.
- 132 Amy-Lynn Except she doesn't know the combination. She doesn't know the combination.
- 133 Brian Rip the whole system out.
- 134 Amy-Lynn Yeah but she doesn't even know. She forgot the original garage door opener combination.
- 135 Brian Yeah but usually there's a door that leads from the garage to the inside.
- 136 Bobby I know what I'm gonna do.
- 137 Amy-Lynn Yeah but if she wants to get a new one she couldn't program it because she doesn't remember the original garage door opener combination. (Bobby whispers to Brian.)
- 138 Brian Rip the whole system, cause you have to have something inside.
- 139 Amy-Lynn Yeah, I know but then she's gonna need a new one.

140 Brian Yeah so what. Just rip it out and put a whole new system in and she's set. She doesn't have to spend like 6 hours.

141 Bobby I got the answer. There will be 9 different combinations for it. And you put low high and medium. So it'd be 1395.

142 Brian What?

143 Bobby Look. You could do watch. Low medium high.

144 Brian And you get like a thousand combinations

145 Bobby No you can get 9 because you can only use 1 and 0. What's the problem?

146 Amy-Lynn How come everybody doesn't know?

147 Bobby I don't know.

148 Brian With nine switches.

149 Amy-Lynn You could just do...

150 17:57 Bobby Oh that's not fair. See if there was only one wrong number which means you could only use one code. And you could only use 3 numbers it'd be easy because see (writes on paper).

151 Amy-Lynn Yeah but then how do you get that there's only 2 numbers.

152 Bobby I don't know.

153 Magdalena What are you doing?

154 Bobby Well just imagine there were 2 numbers.

155 Brian It would be so hard for her to find out the combination. Why couldn't they just make like some stupid... like replacement TV remotes? Where you just like buy it and it's already programmed and boom. You're set.

156 Amy-Lynn We have a lot of them in our car we have 4 of them. Two already don't work.

157 Brian On your car?

158 Amy-Lynn Yeah.

159 Brian On 1 car or 2 cars?

160 Amy-Lynn 2 cars.

161 Brian Okay I was gonna say.

162 Amy-Lynn I think there's four for each though.

163 Magdalena What is he doing? (Talking about Bobby)

164 Bobby I'm just imagining if there's 2 numbers. (Bobby and Amy-Lynn laugh.)

165 Magdalena How many numbers does the code have?

166 Brian (Mumbles)

167 Bobby It never says that.

168 Amy-Lynn It's gotta be nine because umm there's nine switches each of which can be set to low medium or high. With the switches in each code it's gotta be nine because...

169 Brian There's more than nine.

170 Bobby Well there's a lot of possibilities

171 Magdalena 1 2 3 4 5 6 7 8 9 10. 1, 2...

172 Bobby See there's more than that. There's low medium or high and there's nine different switches. So there could be nine possibilities,

- there could be a lot of possibilities where the switches could be.
- 173 Magdalena Yeah.
- 174 Bobby So the low medium high is 3 and the 9 switches so it's 3 to the 9th power.
- 175 Magdalena What? (Bobby and Brian whisper.)
- 176 Amy-Lynn It's 3 to the 9th power cause look 3 9 switches 3 to the 9th power.
- 177 Bobby See you could take low medium high and and it be like the switches on the bottom. Then you take the 9 switches and put it on the top since there's like different possibilities. So you end up with 3 to the 9th power.
- 178 Amy-Lynn You're forgetting about Magdalena she has no idea what you're doing.
- 179 20:01 Brian Can you draw a picture?
- 180 Bobby Sure (draws on the paper).
- 181 Amy-Lynn There is nine switches and it's low medium high. You put that on top somehow.
- 182 Class (Chatters very loud)
- 183 Bobby You could do that for all of them. You could turn them all to low medium or high. So instead of saying the trouble you could say that there's 3 possibilities so that would be the bottom and there's nine switches. Am I right?
- 184 Brian You're right.
- 185 Magdalena Are we supposed to figure out the combinations of the thing or...?
- 186 Bobby So that means that there's 81 different possibilities.
- 187 Brian Okay.
- 188 Amy-Lynn Do we even have to find out how many possibilities there are?
- 189 Brian No what do you think happened?
- 190 Alice There's two things that I understand. If this is a solution that you can convince me and everybody else makes sense then you can go tell the others. Do you understand what he's saying?
- 191 Brian Yeah.
- 192 Magdalena No.
- 193 Alice All right let's try it again.
- 194 Bobby All right, there's three possibilities that you can switch each switch on. If there was one switch it'd be low medium or high it'd be 3 to the 1 there'd be only 3 possibilities. But there's nine switches so you gotta put 3 to the 9.
- 195 Alice (Draws) If there's on switch here would be high, down here would be medium, down here would be low.
- 196 Bobby So you put 3 to the 1 switch. # for the high medium and low. There's 1 switch so to the 1 power equals 3. So it's 3 possibilities. So it's nine switches so instead of 1 on the top you put nine. (Group agrees with Bobby.)
- 197 Magdalena I got it.
- 198 21:59 Brian Yeah. Yeah.
- 199 Amy-Lynn I think it's a good explanation.

200 Alice That's a pretty good explanation. Do you understand what that would be for 2 switches?

201 Bobby Well then you make another switch like this (he draws). High medium low and you can have high medium and low. Se then there's, there's still 3 switches but then there's 2, I mean there's 2 high medium low but there's still 2 switches so there would be to the 2nd power which would be 6 and there's 6 possibilities.

202 Alice Alright show me what those possibilities would be. And does everyone understand what he's saying? (Group nods their heads yes.) Okay so what would be, tell me exactly what the possibilities of the switches are.

203 Bobby High high, high medium, high low... high medium low high medium low.

204 Alice Why don't you draw it or something cause I'm not sure I can picture this. (Bobby draws it, Brian whispers something to him.)

205 Bobby Here. It looks the same as here. You see this is like the same thing as this one except it's reversed.

206 Brian Hold on. You're only using 2 switches, you're using three here.

207 Bobby Where?

208 Brian I think you...

209 Alice You're saying. You're saying that (mumbles).

210 Bobby Oh that's right. And then it's just medium medium, medium low, and low low. I actually thought I had a good idea though. (Amy-Lynn explaining something to Magdalena.)

211 24:17 Magdalena How many do you have? Nine, nine.

212 Amy-Lynn But he only has six.

213 Magdalena I know.

214 Bobby It's simple.

215 Amy-Lynn Yeah buy if we can fine out what power it is, then maybe we can figure out the difference.

216 Alice What is three to the second?

217 Bobby Nine.

218 Amy-Lynn Nine.

219 Bobby So then it's right.

220 Bobby (Looks at Amy-Lynn's drawing.) Everybody makes mistakes sooner or later. So then nine. If it was three switches... Then it should be 3 to the 3rd, 27. Yeah but we gotta do 3 to the third we gotta go from 1 switch to 2 switches.

221 Alice What would be 3 switches?

222 Bobby Then it should be 3 to the 3rd which is 27. Yeah but we gotta do 3 to the third we gotta go from 1 switch to 2 switches...(Bobby and Am-Lynn draw.)

223 Brian What's he doing?

224 Amy-Lynn You gotta do every one up to nine.

225 Magdalena What are we doing now? Amy?

226 Amy-Lynn Yeah.

227 Magdalena What are we doing now?
 228 Amy-Lynn We gotta draw these things.
 229 26:03 Magdalena What are you doing? I don't even understand.
 230 Bobby It's so easy.
 231 Brian I don't know what your doing. Oh well.
 232 Amy-Lynn You have to draw the combinations to turn the switches and the ones that come out of it and then do the combinations.
 233 Bobby Yeah but keep it... Keep it quiet so the other groups don't hear.
 234 Magdalena (Accidentally throws the marker cap.) Oops.
 235 Brian What'd you do?
 236 Bobby (To the Alice) (Amy-Lynn and Magdalena talk on the side.) Like we're doing 3 2 1 2. We're making 1 with three branches and then listing the possibilities. For the one high medium low. We're doing the second one with two branches high medium low high medium low and listing the possibilities. Then we'll make three branches with high medium low and make all the possibilities.
 237 Alice Oh okay what are you saying for 3 to the 2nd, this one and this one I'm confused?
 238 Bobby No there's a difference. See these lines are boundaries.
 239 Brian (To Magdalena) Do you know what their doing?
 240 Magdalena No.
 241 Brian What are you drawing?
 242 Amy-Lynn We're drawing the switches 3 to the 1st. So there three... low medium or high. And there's 1 switch which is 1. And you're coming off, you're making the low medium and high come off that 1 switch. (Brian leaves)
 243 Bobby (To Alice) If 3 branches combine to make 1 they're saying they combine like 3 all together... kinda like a branch. So they combine 1, 2, 3.
 244 Alice Okay so this is the first switch and this is the second switch?
 245 Bobby This is if there was only 1 switch on it and this is if there were 2 switches on it. And if there were three switches in it, it would look like this (he draws).
 246 Magdalena (To Amy-Lynn) I have no idea what you're doing.
 247 Bobby And then you draw the boundaries... you write the possibilities like high high high, high high medium, high high low, and then it's like high...
 248 28:00 Alice Oh I see and so you're gonna make a list?
 249 Bobby Yeah.
 250 Alice So you think you figured it out. So you think... how many do you think there's gonna be for 3?
 251 Bobby For three there'd be...(he draws)
 252 Amy-Lynn It's not working (talking about the marker).
 253 Alice (To Magdalena) Okay h m h, h m l. What other h m that you found? You have h h h, h h l, and h l m
 254 Magdalena H h m.

255 Alice Mmhmm.

256 Magdalena (Talks quietly to herself) I have 27.

257 Alice Okay so there were nine before. The nine that started with h.

258 Magdalena And then the nine that started with m and the nine that started with l.

259 Alice Is that gonna work?

260 Magdalena I think so.

261 Alice It sounds good to me. So for each, so for each first switch there were nine that started with h, there were nine with then m, and nine with the l. Maybe you better go ahead and keep working on that to prove it.

262 29:59 Amy-Lynn (Counts her number of possibilities.)

263 Bobby Is this the right answer? This is how many different possibilities there are.

264 Alice (Looks at his paper) Wow.

265 Bobby So I did it, I did it this way. I made a line of nine 3's. Then I took these two and multiplied it by 9 times 3 times eight 3's. Then these two 27 times seven 3's. Then 81 times and so on.

266 Alice Yeah okay so you're saying that this was 3 times 3...

267 Bobby 3 to the 9. Then there's 9...

268 Alice Why is it that there's so many possibilities? (inaudible)

269 Bobby Well yeah cause there's 3 branches and there's 1 switch so it's 3. Then there's 2 branches and there's 2 sets of those so instead of saying 3 to the... if you want here you can do 3 times 3 times 3. But then you can use scientific notation to make it easier. Like if you had a number like 10,000.

270 Magdalena What are you doing?

271 Bobby I'm explaining it to her. You can use scientific notation to make it like 10 to the 10th or something. 10 to the 100th or something.

272 Alice Yeah true, I understand what you're saying. That it's 3 to the nine. Yeah and I guess what Magda, Magda just did was to show me... Do you think it's gonna be 27?

273 Magdalena Yeah cause like.

274 Alice She has 3 to the 3rd to prove. (To Magdalena) Can I have that? So why don't you explain to Bobby and see if the two of you agree.

275 Bobby (Taps Amy-Lynn and whispers in her ear.)

276 31:54 All (Whisper and giggle.)

277 Alice (To Bobby about Dr. Carolyn Maher) Would you explain to Dr. Maher?

278 Bobby Okay. Well Okay, see there 3 branches come to form as 1 so like there's 3 so there'd be 3 and 1 branch so 1. And see here there's still 3 and there's 2 of them so it'd be 3 to the 2nd. And then we get 3 to the 9th. Instead of writing you could just say like here. There's 3 on this branch so it'd be 3 times 3 times 3. You can use scientific notation to make it easier so like 3 to the 2nd or 3 to the 3rd. And for 9 it'd be 3 to the 9th.

279 Caroline Okay so can you write that up, what you just told me.
 Maher

280 Bobby Yeah.

281 Caroline I'd like you to describe it in words, all of you. Then maybe you
 Maher can put it on an overhead. Bobby, Bobby you agree with me
 Bobby's slim needs, Brian. You okay Brian, Brian I fell too on my
 knee, so I'm sort of empathizing with you.

282 Bobby He fell off the stall bars in gym.

283 Caroline That's pretty bad.
 Maher

284 Brian Then I hurt my right knee last night.

285 Caroline Have you had it x-rayed?
 Maher

286 Brian I had the left one not the last one. But the left one, I mean the right
 one I heard something snap last night. And there's like this big
 lump.

287 Caroline Maybe you should have it looked at. Did the nurse see it?
 Maher

288 Brian No.

289 Caroline Maybe that's something that you can fit in today? A trip to the
 Maher nurse after this class.

290 Brian Yeah.

291 Caroline (To Bobby) Would you write this up? Cause maybe we'll put this
 Maher on the overhead.

292 Bobby Sure. (Bobby and Brian whisper) You have to finish the question
 from yesterday though. The one that she gave us in class.

293 34:01 Magdalena I hope she's not collecting cause I didn't like... (Amy-Lynn
 explains to Magdalena).

294 Bobby (To Brian) No the only thing is I did the questions before. I did that
 before.

295 Brian I didn't.

296 Bobby You just had to do the top through. It's somewhere in my
 notebook.

297 Brian Oh well.

298 Bobby My notebook in the garbage.

299 Brian In the what?

300 Bobby In the garbage, I threw it out yesterday.

301 Brian That was yours?

302 Bobby Yeah.

303 Brian What are we doing here?

304 Bobby They're gonna get an overhead for me, cause I'm special.

305 Brian Yo, let me finish. What let me see it (Grabs marker from Amy-
 Lynn). That's because the tip's fading.

306 Amy-Lynn Mine's faded.

307 Bobby Here use this one.

308 Amy-Lynn I gotta good one.

309 Bobby Brian can make a hypothesis on it. It's garbage.
 310 Magdalena That's how you do it. First you do it with h's in the beginning.
 Then you do the m's in the beginning. Then you do the l's in the
 beginning and you get the answer.
 311 Amy-Lynn What's the difference between that and maybe if we do another
 one, maybe...
 312 Magdalena But that's not the question. That's not...
 313 Amy-Lynn I know but we have to explain it so that people understand it.
 314 Magdalena What do you think is gonna happen?
 315 Amy-Lynn They're gonna ask questions about it.
 316 Bobby (To teacher) I figured it out.
 317 Teacher I'm gonna tell you something, you guys. You get right to it.
 318 Bobby And I figured out the binary system like yesterday.
 319 Amy-Lynn Yeah he was the one who figured it out.
 320 Teacher And what did you do?
 321 Amy-Lynn I was doing it the long way but then he told me.
 322 Teacher Who said that your way's wrong?
 323 Amy-Lynn No I said that I was doing it the long way.
 324 Bobby Long way.
 325 Teacher Ahh okay.
 326 Amy-Lynn I was doing combinations and I did it the long way. I forgot about
 the binary system.
 327 Bobby I was smart cause I was throwing my notebook away and I changed
 it and I saw binary so 5 numbers so.
 328 35:39 Teacher So what happened is, is that it triggered something. Okay so can
 you get it on paper?
 329 Bobby Okay
 330 Teacher You really think that there's that many.
 331 Bobby Yeah cause look...
 332 Brian All she has to do is rip out the system and put a new one in.
 333 Amy-Lynn He has to tell you a secret anyway.
 334 Teacher Okay nine right.
 335 Bobby So there's high, medium, and low and there's 3 in each branch, like
 for each switch. So for the first one, if there was only 1 switch, it
 be 3 to the 1, there's 3 different combinations. If there were 2
 switches, it'd be 3 to the 2nd so it'd be 9.
 336 Student (To teacher) We need you're help when you get a chance.
 337 Teacher Okay.
 338 Bobby Hold on, back away we don't want you to hear this. And then it'd
 be 3 to the 3rd so it'd be 9 combinations. And then 3 to the 9th
 would be 19,683 combinations.
 339 Teacher I am so impressed.
 340 Brian When did you do this calculation.
 341 Teacher I am so impressed. (Carolyn Maher gives Bobby an overhead
 sheet.)
 342 Brian We still have more than another period left in here.

343 Bobby You know what I'm talking about.
 344 Brian Magda shoot me I cannot stand this.
 345 Amy-Lynn Here give me that.
 346 Bobby (Magdalena and Brian whisper). Well you can put on the thing
 about how, how or draw like pictures.
 347 Magdalena Ahh this thing then write... He doesn't have to write it, why don't
 you write it.
 348 Bobby (To Amy-Lynn) Just draw pictures.
 349 Alice (Gives Magdalena an overhead sheet) For the 3...
 350 Brian They got it.
 351 Bobby Yeah.
 352 Alice It seems to me that in order to understand that you need to know
 what you did here.
 353 Brian Big combination here. (To Magdalena) They put your whole name
 there.
 354 Alice Bobby one thing I'd like to know is if you and Magda were
 thinking of the same idea.
 355 Bobby Yeah cause she just wrote hers out, all the possibilities out.
 356 Alice But is it really the same thing as yours.
 357 Bobby Yeah.
 358 37:58 Amy-Lynn Yeah. Yeah same thing cause 27 would be 3 to the 3rd.
 359 Alice Is that just a coincidence?
 360 Amy-Lynn No. Cause the other ones work too. That's 6.
 361 Bobby Nine. Nine.
 362 Brian We should have just used the calculator. Save marker.
 363 Alice Do you trust his multiplying?
 364 Brian I do. I studied it.
 365 Alice Did you check it with a calculator?
 366 Brian No.
 367 Alice Are there any?
 368 Bobby They're over there. In that umm. See where the TV.
 369 Brian Under the TV in the cabinet.
 370 Bobby See where the camera is. All the way below the flag.
 371 Brian I gotta check it so you can give it to me.
 372 Magdalena Bobby, what are you doing?
 373 Bobby I'm writing my hypothesis.
 374 Magdalena Yeah but she's writing...
 375 Bobby She's drawing it.
 376 Amy-Lynn I'm drawing.
 377 Magdalena We can write it over here.
 378 All (Whisper)
 379 Amy-Lynn Can I have the combinations?
 380 Bobby There's 19,683.
 381 Amy-Lynn Can I have the combinations?
 382 Magdalena Okay, starting with h, there's h h h, then h h l, h h m. And then h m
 h.

383 Amy-Lynn What hold on. H h m
 384 40:08 Brian I need this multiplication thing.
 385 Magdalena H m h, h m l, and h m m. And h l h, h l l, and h l m. And then
 386 M m m. M m m, m m l
 387 Bobby We can't go to the computer room today. It was Harkens class
 today.
 388 Brian Why?
 389 Bobby Cause we have that winter concert.
 390 Amy-Lynn Hey, I'm in it. I'm in it.
 391 Magdalena (Bobby and Brian talk.) M m h, m h l, mh m, m h h, m l h, m l m, m
 l l. And now starting with l! L l l, l l m, l l h, l m h. Oh you have
 that already?
 392 Amy-Lynn No.
 393 Magdalena L m h, l m l, l m m, l h l, l h m, l h h.
 394 Brian I don't wanna go to the game tonight.
 395 Bobby What game?
 396 Brian Basketball game today. You got blown out by Winfield Park.
 397 Magdalena So what the girl is like 6 feet tall. She scored like all their baskets
 almost.
 398 Brian (Mocking) All the Baskets.
 399 Magdalena Maybe 10 of them were scored by other players.
 400 Brian 60 to 35.
 401 Bobby But the girl scored 50 points.
 402 41:57 Magdalena I scored, I was second on the... I scored 8 (Loud beeping noise)
 and Jen scored like 10.
 403 Brian It was the highest the girls ever scored. I remember the one game
 they won like 8 to 6. They're poor. We're gonna win. How big is
 the rec, you had to play at the rec, the rec center right?
 404 Magdalena It's so small, your feet are in bounds...
 405 Brian When you're sitting on the bench.
 406 Magdalena Yes. You can't save the ball cause there are chairs all over the
 sidelines, you can't save the ball.
 407 Bobby That's cool.
 408 Brian Haha. He's sitting on his golf cart. (To Carolyn Maher who just
 walked over) Okay we checked it.
 409 Bobby My, my addition is right. It was 19,683. (Brian shows her on the
 calculator.)
 410 Carolyn I'm more interested in the reason for it, you know, how do you
 Maher know? I'm not so much interested in the answer except to get and
 estimate on how long it would take her if she had gone through all
 of this.
 411 Bobby A long time.
 412 Brian It would take her...
 413 Carolyn Look it says here, how long do you think it would take her to find
 Maher the code.
 414 Brian I think she got through about 100 combinations and then...

415 Carolyn She sat there for half an hour and entered different codes. Is that
 Maher reasonable a half an hour?
 416 Bobby No.
 417 Carolyn What do you think is reasonable? You see what I'm saying?
 Maher
 418 Bobby A couple of days because...
 419 Carolyn Why don't you tell me why you think it.
 Maher
 420 Bobby Well because there's like 19,683 combinations (all talk at once).
 421 Carolyn Is it one day, is it two day, is it three days? And why you know?
 Maher
 422 Brian She'd have to write it won cause if she didn't she'd probably have
 the same combination like once or twice or a couple times.
 423 Amy-Lynn Yeah it would take a lot of time because..
 424 Carolyn What do you think, Magda? What's a long time though? Is she
 Maher working constantly without resting?
 425 Brian A couple months.
 426 Carolyn Well why?
 Maher
 427 Amy-Lynn Yeah but she has to sleep or something.
 428 Carolyn All right well I want to know exactly why. Think about that and
 Maher talk about that.
 429 Brian If she had to write every single combination down, and there's no
 way that she could have every combination without repeating 1 or
 2.
 430 Magdalena Are you on the basketball team?
 431 Brian Yes I am. I made it yesterday. Thank you very much.
 432 Magdalena You did?
 433 43:59 Brian I can't play though. (Bell rings.)
 434 Magdlena That was a long curve. (Brian whispers to Bobby)
 435 Brian (Points) That. It does.
 436 Amy-Lynn Do I have to do 3 to the 4th and 3 to the 5th?
 437 Magdalena (Talks softly)
 438 Amy-Lynn Do I have to do 3 to the 4th and 3 to the 5th?
 439 Brian (Does it in calculator) 3 to the 4th is 81.
 440 Bobby Just go to the ninth.
 441 Amy-Lynn No thank you.
 442 Bobby Just go to the ninth.
 443 Amy-Lynn I have to do 19683 combinations?
 444 Bobby No, just put 3 to the 9th is that.
 445 Brian 3 to the 5th is 243. 3 to the 9th is... (Shows Amy-Lynn the
 calculator.)
 446 Magdalena (Amy-Lynn sobs) That's okay we don't have to do it.
 447 Amy-Lynn My little mousy is telling me not to.
 448 Brian So figure...
 449 Magdalena My sister's chorus stinks.

450 Brian Let's see...

451 Amy-Lynn Wait, wait. Add them up all together. Add 19, 683... 3 to the 1, 3 to the 2, 3 to the 3, 3 to the 4, 3 to the 5, 3 to the 6, 3 to the 7, 3 to the 8... Add it all together.

452 Brian So figure, each of us do three thousand nine hundred thirty six point six combinations and you get them

453 Amy Ok, alright. Let's start today, we'll get done next month

454 Brian Dude this guy is so stupid, this one guy is trying to count up to billion on his calculator, he's only up to thirty million and he went through fifteen calculators already

455 Bobby Wasn't he just like...

456 Amy You are still out of that curve

457 Bobby I don't care. This is the way I write.

458 Amy You got a problem with it? Bobby? What should I do? Should I write out the combinations before ...

459 Bobby It's only three copies

460 Amy Yeah, but we don't have the combinations before

461 Bobby We don't need one

462 Magda Do you know how long it will take us to write all the combinations?

463 Bobby Yeah, I know You want to recopy this or do you want me to just erase it

464 (Researcher walks up to Bobby, he hands her the paper)

465 Bobby I wrote it, so it's a little messy

466 Amy You think we should write the combinations for

467 Brian No, we are not going to write twenty thousand combinations

468 Amy You are not going to write them up, I am

469 (Researcher walks up to Amy)

470 Brian My knee is affecting all of my body (Laughs) It's cool though, cause there's not much space between the knee cap and like and then it push down and there's this line and it comes like, comes like this

471 Researcher Amy

472 Amy What did I do?

473 Researcher I know, I never got to the question you asked me before Amy, remember the question you asked me before? Just about a minute ago.

474 Amy Yeah

475 Researcher Ok. Um what do you think?

476 Amy I don't think we should list them all

477 Researcher Because why

478 Amy It will take a million years

479 Brian Because it will take a long time

480 47:05 Researcher Ok

481 Brian Unless she writes every combination down she'll repeat like a couple like five or six times because she would not know that she

repeated

482 Researcher Uh-huh

483 Brian And even if she wrote it down, she'd probably repeat it cause every time she'd have to write one down, she'd have to go through the list to see if she had it so it will take a long time

484 Amy Inaudible

485 Researcher Ok, now. Do you think that this is going to give the right idea of what the ??? is gonna look like

486 Amy I think so

487 Researcher You think so,

488 Amy (inaudible)

489 Researcher Exactly. That's the whole thing, to explain it right. That's what I think you are gonna have to do here. Explain

490 Amy Ok

491 Researcher Why you think this is a pattern and how you think it's gonna continue. Ok?

492 47:47 Bobby I can explain it

493 (Researcher walks away)

494 Brian I should not get any more homework for the rest of the year, I have not exploded yet

495 Amy Way to go. Oh, wow

496 Brian I am holding my (inaudible)

497 Amy You're not gonna be holding that tomorrow for health.

498 Brian Why?

499 Amy I can't say nothing

500 Brian Why?

501 Bobby Is there test tomorrow

502 Brian There is?

503 Amy Quiz

504 Brian Ah, god.

505 Bobby Thanks, now I got to study

506 Brian This is this thinking is blowing me away you guys

507 Bobby Isn't it amazing how I found that out and the binary system

508 Researcher How is it amazing?

509 Bobby I gotta go to the high school

510 Researcher You do have to go to high school and you're gonna make it too. Now, you have to put your name of some of this stuff too. Ok

511 Bobby I wanna go to Harvard

512 Amy I wanna go to the you know

513 Bobby Next Robert is a hero

514 Brian I just want to get a scholarship in sports.

515 Amy My mom Wants me to go to Myrtle beach cause that's where she wants to retire.

516 Magda Me too

517 Researcher Ha ha

518 Brian My aunt was in there... that's where she retired

519 Bobby I want to be a lawyer, they just make all this money, they just sit in court all day.

520 Brian Robert should be (inaudible), he's making like twenty million dollars for a movie

521 Maher Did you convince bobby

522 Brian No, cause I did it on the calculator and it came out with the right answer so, we were just thinking about how long it will take

523 Amy That's the question

524 Maher Have any idea how long it is going to take Brian?

525 Brian I would say at least like a year maybe more

526 Maher Have a basis for thinking it, when you really think, what's your basis? How many what's reasonable for a person to work on it? Think of it in a reasonable way that not just about why about one year out of the two years, or year and a half or three months or a weekend you understand what I am saying?

527 Brian Everyday...

528 Maher It's good that you are thinking about it but try to bring yourself down

529 Brian Every day she wakes up she takes eight hours of the day to she's got to eat so that takes she works up until
(Camera moves away from the table to another student standing and talking to another researcher)

530 Amy You count till twenty days

531 Brian What?

532 49:48 Amy If she tries a little bit of everything and then stops(inaudible)

533 Maher How did you figure that out

534 Amy If you did twenty four

535 Bobby So you get (Bobby is talking to Maher)

536 Brian Ok, that's it, we just estimated She wakes up at eight o'clock

537 Amy Doesn't sleep, eat or do anything

538 Brian Yeah, so she starts at

539 Maher Maybe what it is reasonable time that she had to eat some, and sleep some so why don't you think of a practical

540 Brian Works until lunch which is three hours

541 Maher I don't know. What do you think? Why don't you talk about it and decide what's reasonable

542 Bobby But what about?

543 Maher Yeah, that's a good way to that's certainly a way

544 Bobby Cause she has to how many

545 Maher Ok, so you could say twelve hours in a day if you want to, that's a pretty full day, she's determined. Talk to Brian now probably works twelve hours a day, yeah, he's a doctor now. Do you know that? He didn't tell you? Next time you see him, you say hi Dr. Brian and surprise him
(Bobby smiles, Maher leaves the table)

546

547 Magda That many combinations and it will take her that many minutes

548 50:57 Brian It would take here one thousand seven hundred eighty nine days to get done

549 Magda So, how many hours, how many hours, how many days is it?

550 Brian It'll take her that many days (shows the calculator to Magda, Bobby raises his hand and calls out to a researcher)

551 Bobby I have a, I think it will take her twenty seven days

552 Researcher That's it? Why

553 Bobby But that's like if she changed it once every minute

554 Brian It would take her seventy four hours

555 Bobby If she only worked like twelve hours a day

556 Researcher She'll need some sleep

557 Bobby Yeah, And she's a doctor you know

558 Researcher That's true

559 Brian She's a doctor now?

560 Amy Does she

561 Researcher Oh watch

562 Bobby If she does it twelve hours a day

563 Researcher Right

564 Bobby And switches it every minute

565 Researcher Uh-huh

566 Bobby It would take her twenty seven days about to That's she doesn't do any repeat and like every minute she switches twelve hours a day

567 Amy Estimate a month

568 Bobby Yeah

569 Researcher Ok

570 Amy So, if she does make repeat, it would say add the four days

571 Brian Well, that is if she works the whole time

572 52:01 Researcher She may take a few days off here and there

573 Brian There's all that

574 Amy She'll get tired... she'll get cross-eyed!

575 Bobby That's what I said! If she did like once every minute you know and she switched every minute for straight twelve hours then it would be twenty seven or else it'd be more

576 Researcher Ok, how did you figure that out?

577 Bobby Well, I did nineteen thousand six hundred and eighty three divided by... or nineteen thousand six hundred and eighty three divided by sixty, sixty minutes in an hour

578 Researcher Yeah

579 Bobby And I got three hundred twenty eight

580 Researcher Hours

581 Bobby Yeah, she can't work all day cause she needs food, water, she has to go to the bathroom, sleep and all that stuff has to do all that person stuff, so you divide by twelve say she works half the day and then round it to twenty six, twenty seven

582 52:37 Researcher What

583 Brian Twenty seven Point three, three seven five

584 Amy You just give me nice stuff
 585 Amy I would say about a month
 586 Bobby That's what that's if she worked twelve hours a day a like a minute
 each day
 587 Magda I got fourteen days without like non-stop
 588 Brian Unless she has, is she married?
 589 Researcher Yeah and she has a dog
 590 Brian Unless she had a husband working with her then they could have
 gotten twice as many
 591 Bobby But do you think her husband would do anything?
 592 Amy What kind? I like beagles.
 593 Maher Oh you think he would take over for her, when she stops
 594 Amy I like beagles.
 595 Brian Well if he uses the garage too
 596 Bobby No
 597 Amy I like beagles.
 598 Brian He has to use the garage too or maybe you'd want it open and you
 would help her... so divide that in half
 599 Amy German shepherd could chew up the papers. My dog would eat em.
 My dog would eat em. My cat like, we have two dogs.
 600 Bobby We had to bring them to like a vet and we have a couch and my
 grandma, and my mom has like sheets over them so it doesn't get
 dirty and the cats like grabbing on to the sheets, he pulled all the
 sheets off the couch and like he wouldn't go in the animal kettle in
 the corner there and it had to get neutered or something cause
 sometimes he goes out sometimes but he's real bad now he just
 hisses and stuff whenever you go near him he goes hsss
 601 Brian What is the number, what did you divide that by
 602 Bobby Twelve
 603 Brian Twelve
 604 Magda Hold on, can I explain what I think?
 605 Amy Yes
 606 Magda It would take fourteen days if she's working non-stop
 607 Amy If she's working non-stop. She can't work non-stop, she would die
 608 Magda I am (inadudible)
 609 54:29 Researcher (pointing to a sheet of paper on the table) Now, you're gonna have
 to alter your idea here perhaps. What did you say here and what did
 you really find out? Oh that's the good paper?
 610 Bobby Yeah, it's the good one That's cause I can't write straight
 611 Brian So, they would each work thirteen point six six eight seven five
 hours.
 612 Bobby Women gotta go to bathroom more
 613 Brian That's true
 614 Bobby She might have urinary problems
 615 Brian Prostate
 616 Amy (talking to Magda) Yeah, but she's not really working non-stop

617 Magda At least eight hours
 618 ?? No
 619 Bobby I am just being practical
 620 Brian Just twelve hours, just tell her twelve hours
 621 Amy It would be about a month
 622 Magda So, twenty eight days
 623 Amy Yeah, so about a month, cause say she skips a couple of days, like
 vacation days wait that would give her like thirty, thirty one days
 624 Brian It would take her about a month and if she takes a vacation a little
 bit more
 625 Bobby But she might be more aggressive over the weekends though
 626 Magda It will take her about a month
 627 Brian Plus she's gotta clean the house and cook for her husband
 628 Bobby Hey, my daddy used to cook
 629 55:44 (Brian goofing off, Amy exchanges pen with Bobby, group talk
 to about raking leaves, sneakers, shoes, talk about TV commercials,
 57:40 Amy and Magda work on paper, Bobby fills in his name tag with
 marker)
 630 Amy Can we put one combination per minute?
 631 Brian Yes, that's how we figured it out
 632 Bobby For twelve hours a day
 633 Amy Yeah we put twelve hours
 634 Brian Cool, her dress is red in that picture
 635 Bobby Who? (Brian points to someone)
 636 Bobby They are making us look smart
 637 58:06 (talk about a Nike commercial while Amy and Magda talk about
 to the problem. Amy writes on paper, Magda watches)
 59:24
 638 Bobby If they worked one hour and they switched it every minute for
 twelve hours, it'd be...
 639 Brian If they did a combination per minute
 640 59:29 Amy Twelve hours a day, it'd be around a month
 641 (inaudible)
 642 Maher No days off, you gave her vacation? She's gonna appreciate that
 643 Amy Like four days
 644 Bobby Twenty seven days
 645 Amy Makes it thirty, thirty one days
 646 Maher So makes about a month and you wrote all that down and reasons
 why
 647 (Somebody Can I explain that, Do you mind?
 not on the
 able)
 648 Brian It would have been better to just spend a hundred bucks and get a
 whole new system
 649 Bobby No cause
 650 1:00:00 (Bobby, Brian, Amy talk about quarter backs and play offs teams of

to
1:02:27 Super bowl, and about Game betting)

651 Bobby Put our names on it
652 Brian Put our names on it that's perjurism or whatever that is
653 (Brian makes a singing sound, Amy writes on paper, Brian says something)
654 Brian I am lucky I'm not smart cause I don't have to go to the interviews and stuff
655 Bobby I hate them
656 Amy I never do
657 Bobby It's actually fun sometimes though
658 Brian In here?
659 Bobby Yeah
660 Brian Cause they always ask why, why, why
661 Bobby Not really, they just listen to you
662 Brian I am shocked they didn't kick me out today, thought they would still be mad from last night
663 Bobby I thought you were gonna go crazy, like kick the cameras
664 Brian Oh, I came close I don't know why she gets us to write it out when sixty other people are doing it
665 1:03:44 (All talk about a Spanish quiz and students in the Spanish class, to
1:11:36 play with markers, talk about other classes they share, other teachers, etc)
666 1:11:37 Maher So, you have written up the solutions to your first problem?
667 Brian Yeah
668 Maher Ok, I guess you have done the second problem, how about now thinking about the third and the fourth problem?
669 Brian There's a third and a fourth problem? Just give me the fourth.
670 Maher Would you make sure your names are on all of everything you have done and you could bring everything back with the third and the fourth, we are gonna collect this, can you put your name on this Amy? Your own work, would you put your names on it, we're going to collect you stuff so that you can talk about it tomorrow. You can do that now
671 Bobby This one sucks cause its combinations too. Two thousand one hundred and eighty seven
672 Brian How do you know?
673 Bobby Because if the first one is high, first one's low,
674 Amy We answered this, we answered this
675 Brian What?
676 Amy how many combinations might she try and how long it might take
677 Bobby Well, it's got part four we got more paper. I got part three. I figured out part three, two thousand one hundred and eighty seven, yeah because if you take away two switches, it'd be the same thing as there were seven so it's two thousand one hundred and eighty seven
678 1:12:48 Brian You take away two switches, is that what it says?

679 Bobby No, just says she knows the position of two switches
 680 Amy How would that help?
 681 Bobby Cause that means there's only seven switches to figure out so
 makes it less possibilities
 682 Amy That's true
 683 Bobby two thousand one hundred and eighty seven different combinations
 684 Brian No, eight hundred thirty three
 685 Bobby Two to the... three to the seventh, I figured, I figured out part
 three(calling out to a researcher)
 686 Brian We did number two, number two we finished. She said we have to
 now find out how many combinations there is and how long it will
 take and that's what part two was
 687 Amy Why don't you
 688 Brian Yeah, Ok, here's number two
 689 Bobby I figured out part three... Miss Toy (Robert pointing to his paper,
 trying to get researcher's attention)
 690 Researcher That's the over head, over here sign this, oh wait
 691 Amy That's the bad one
 692 Bobby I figured this out
 693 Researcher Ok
 694 Bobby So you just take away two of them so
 695 Brian They make everything in math low pain
 696 Researcher You want this to be part of your uh
 697 Brian My knee is killing me
 698 Researcher Now what?
 699 Brian Can I go down after this class? Nah, I won't go down
 700 Researcher Ok, now wait a second, ok, uh, Amy, Amy, you'll still have to say
 something about this pattern, it's not going to stand by itself. OK
 701 Bobby We can explain it
 702 Researcher Yeah, but write yourself a few notes now so that tomorrow when
 you have to explain it, you have your notes. Lots going on between
 now and tomorrow, I am telling you
 703 Bobby I figured out the third one too
 704 Amy How many you gave, how many
 705 Brian What, these people are coming? Two more days?
 706 Bobby Two periods tomorrow,
 707 Amy So, Friday, Friday will be our last day
 708 Researcher Thursday
 709 Bobby Are we gonna see our green candle sheets?
 710 Researcher Yes
 711 Brian They were so good
 712 Amy Are we gonna get grades?
 713 Researcher Yes, yes, we are evaluating both that and your idea that you that
 you submitted
 714 1:15:25 (talk about scoring brownie points, Researcher asks Bobby to help
 to Amy on the problem)

1:15:46

715 Researcher She needs to make a statement that says that she is sure that that's the pattern you know like you gotta base it on what you see here so that it's convincing.

716 Amy One switch

717 Brian Yea, aah, My knee

718 Amy What did you do?

719 Brian I jumped up and heard like a, like a click I don't know it was like a space, (talks about his knee injury to Magda while Amy and Bobby go back to the problem)

720 Bobby Figure out how many branches are, cause if it's two switches, then it's two

721 Bobby No, that's counting how many switches there are but see there are two switches, it would be how many different branches there are

722 Amy How many switches,

723 ??? Or how many switches

724 Brian (inaudible)

725 Bobby I don't care, It's my first (inaudible)

726 Bobby I think I did it too, I am not sure

727 Amy I'll do it

728 Bobby Just put a, b, c, d All I have to do is the top

729 Amy No ...you have to do the whole page

730 Brian What page is this on?

731 Magda Why are you so smart? Why are you so smart? (to Amy)

732 Amy I am not

733 Bobby I am smart, I am smarter than her. See I am smarter than her in math. But she's smarter than me in everything else

734 Amy (Laughs)

735 (All listen to a classroom conversation, Amy gets back to working on paper)

736 1:18:28 (All Talk about teachers grading their work harshly, games, etc)
– 1: 19:
42

737 1:19:41 Alice Tell me what you came up with

738 Bobby Uh, how long it would take?

739 Alice No, how many, how many things it was gonna

740 Bobby Oh, I was right

741 Alice You were? I bet you were

742 Alice Did Brian or somebody check you

743 Brian Yeah, I checked it

744 Alice I could not have done that and come out with the right answer. So tell me where's, where's your paper? Where did you do the multiplying yourself?

745 Bobby It's on the other side

746 Alice Can I see it?

747 Bobby I saw it like (Magda looks for Bobby's paper)

748 Alice Ok, help me again because for the next time you did a big thing,
what did you do?

749 Bobby I did three times three times three and then I did the nine times then
I did the first two which is nine times three which is eight threes
and then twenty seven divided by seven threes, it's eighty one by
six, two

750 Alice Oh, so you really did all this multiplying

751 Brian He didn't use the calculator either

752 Alice Seven twenty nine ...

753 1:20:42 Bobby by three, two thousand one hundred and eighty seven (inaudible)
followed by one zero and that's it

754 Alice Alright, alright

755 Alice So, you all agree on this one? This is

756 Amy This all the of the drawings cause we did it two to the second, two
to the third, it would take too much time, till

757 Alice And these were what? Magda is gonna go this time

758 Amy All the list

759 Alice All the list of things and after that you decided to list them, so this
would be eighty one and going there

760 Bobby This also like helps us out for like part three you know why? Cause
see it says that she knows what two of the switches were, so you
just move out, cross out this one this one and you get two thousand
one hundred and eighty seven combinations if there's seven
switches because part three she found out that she found out two of
the switches, so that means

761 Alice Oh, so she knew two of the switches

762 Bobby Yeah, she found that she remembered two of them were, leaving
seven

763 Alice Did she know which two they were

764 Bobby No, don't know which one they are

765 Alice oh

766 Bobby They are like

767 Alice I see, so then it threw out this one and this one and that one and that
one

768 Bobby Yeah. Cause then she only has to figure out seven combinations

769 Alice It is good to have all those numbers. Yeah, quite a good job.

770 1:22:09 (All chat about bands, Brian talks to Romina about science class,
to Brian sings silly songs)
1:23:40

771 Amy We're done with number two

772 Brian We did part two

773 Amy That's what I am saying, Robert only figured out one, two and three

774 Bobby Oh,

775 1:23:51 Magda We could have done that too

776 (Too noisy as Brian hits marker on the table. All talk about
homework. Bobby talks to Ankur across the room but it is

inaudible. They tell Ankur that there are four parts to the problem)

777 1:24:53 Brian How many combinations are there
 778 Group Eight hundred and
 779 Brian How many hours would it take?
 780 Group Too many

781 1:25:15 Bobby Come on we are on part four, we are so advanced,
 782 Bobby You have to make your own problem. You know now that I figured out most of this... Miss Toy wants to send me to high school, I want to go.

783 Bobby It's on tape too
 784 Brian It was on the monitor
 785 Amy Can we have the paper back now? Tell us your, tell us your story
 786 Bobby My mom forgot the combination when we got back from vacation and it's like a nine digit or a ten digit number and there's like... it's kinda like a telephone keypad... and she already knew the first two numbers

787 Amy Ok
 788 Magda Where's part four... my mom
 789 Bobby No, Mrs. Sigley,
 790 Amy Ok, Mrs. Sigley, Mrs. Anonymous,
 791 Bobby Yeah, Mrs. X
 792 Amy Mrs X left on vacation and when she came back,
 793 Bobby she forgot her alarm combination
 794 Amy she forgot her alarm combination (Amy dictates as Magda writes the problem down)

795 Brian (Bell rings) Good bye
 796 Bobby Maybe we should stay here and work on it
 797 Amy It was a nine digit number
 798 Bobby It was like a telephone keypad... No, it looked like twelve cause they add the pound sign, start key and a zero

799 Amy It's a twelve digits (Amy dictates and Magda writes)
 800 Amy Now, we should answer the question?
 801 Bobby Yeah
 802 Amy How many combinations are there?
 803 Bobby (Make a joke and laughs)
 804 Amy Um, how many combinations would she have to go through to find the right one?

805 Bobby This would be different though cause this will be like twelve to the twelfth power

806 (The Researchers calls every one's attention and asks students to make some notes on their papers as these ideas would be discussed next day. Most of the students are on part two at least. Students begin to leave.)

Appendix L

Date of Session: 12-14-1994

Author: Anoop Ahluwalia

Verified by: Kiranjeet K. Sran

Date of transcript: 5-30-2010

Line	Time	Speaker	Transcript
1.	0:30		(Bobby sits at the table alone when his teacher calls him.)
2.	1:40 – 6:35		(Magda and Amy-Lynn sit down at the table) (Brian arrives and they discuss the new arrangement and school. Bobby returns at 5:54.)
3.		Dr. Maher	I guess before we discuss the problem that we worked on yesterday, I was really very interested in the problem you had worked on 2 days ago that you were talking about in the beginning of yesterday's period, Christmas tree light problem. Now Michelle, you weren't here for the problem that was the day you were absent. Can someone state the problem so Michelle knows what the problem is? Anyone?
4.		Student	(Speaks very softly) He had to decorate a school. He had to put 5 candles in each window and there were 2 different colors red and gold. And you had to find how many he could have. (?)
5.	7:34	Dr. Maher	The different arrangements. Do you understand that Michelle? Okay and there were some solutions, I had a chance to look at your work for some of these. Just randomly picking some of them up. This is one example of the 30 solutions, I don't even know whose this is. (Bell sounds) What I was curious about is that you all had either 32 or 30. The discussion was about whether you were going to all them all to be the same color, right? Some of you thought not, your instincts for the arrangements all can't be the same color. Some of you thought you had to had different combinations. But the thing is you were all pretty convinced that there were 32 possibilities. I didn't hear any dissention on that idea, right? Now how did you know 32. How did you know that maybe you left something out? How were you convinced that there were 32 minus the 2 you could discard for 30? Do you understand my question? That maybe if you kept looking there might be some more, and how do you know you didn't duplicate something? So how did you really know that? Okay Angela?
6.		Angela	Well I used the binary system and I think everybody else did.
7.		Dr. Maher	Well tell me, you didn't do this one clearly. That wasn't yours, right?
8.		Angela	No.
9.		Dr. Maher	So tell me about that. Is this yours? Is that what you need?
10.		Angela	Yeah it's that. And its was 2 and Micheal helped me a little bit and

- so did Bobby. And showed me that gold is 1 and red is 0 or red is 1 and gold is 0. And there's 5 candles and we had this chart and we went over like 5 cases and then counted all the ones that had that thing in it.
11. Dr. Maher I'm not sure I understand how you know you had them all. Bobby?
12. 9:57 Bobby Well base 2 uses 2 different numbers 0 and 1 and when you're using the candles you're using 2 different colors red and green, so it's the same and then the base 2 uses all the numbers within all the five digit numbers within 16 and 32 and you can just switch those to candles and it would be the same.
13. Dr. Maher How do you know you've accounted for all possibilities?
14. Bobby Well if the system is right and that's all the numbers then all the candles would be right, the same amount.
15. Dr. Maher Did anybody think about it differently? Think about the first one, this one. Whose is this on the board?
16. Student That was mine but that's not how I did it.
17. Dr. Maher Well how did you do it?
18. Student Well actually I did it like in the binary system. 0 is red and 1 is gold or the other way around. If you do it this way it's like the same thing just the g is gold and the red is r and instead of 1 and 0 there's g and r.(?)
19. Dr. Maher How do you know you have them all?
20. Student We used the binary so...
21. Dr. Maher You think keeping track here is like keeping track here? They look different to me. Do you understand what I'm saying? Michael?
22. Michael For the binary chart we just put the 1 and g and 0 as r and then did it.
23. Dr. Maher This looks like g g g g g, five gs.
24. Student2 It's r. Yeah it's r g.
25. Dr. Maher Oh r, so the next one is...
26. Student R g ...
27. Dr. Maher That doesn't look like this one, that one looks different. I don't understand how this one looks like binary.
28. 12:02 Student2 Someone did that the other way, they didn't do that binary.
29. Dr. Maher This doesn't look like binary to me. Now this one, the first one is a 1 here followed by four 0s and then moved the 1 here. And then you have two 1s. I don't know how you are keeping track. I'm having trouble understanding how do you know you have them all. Do you understand my question?
30. Student2 That one on the left isn't binary, it's just like here. They put it upside down...
31. Dr. Maher But I don't even see in this one how you know you have them all... you understand why... If I turn it this way?
32. Student That chart there.
33. Student I think it's mine.
34. Dr. Maher It looked like it was supposed to be this way. (They discuss the

- direction of the paper inaudibly.) Maybe I had flipped it wrong, but even if I had flipped it wrong could you help me see what your pattern is. I honestly don't see it.
35. Michael I think the one on the left is mine
36. Dr. Maher Okay, will you come up and explain it to me Michael. Do I have this one? I think I'm reading it correctly g equals gold r equals red. I mean I'm concerned, suppose we had 6 lights or 7 lights you were putting in these windows and you really needed to know what they all were. Do you know how many there would be if you were working with 7 lights?
37. Michael (?) Yeah.
38. Dr. Maher Can you explain that?
39. Michael You would have to use the binary system.
40. 13:50 Dr. Maher Do you know how many there would be before you start? How many of you think you know how many there would be if you were arranging 2 color Christmas lights and you had 7 positions for those lights? (Bobby's hand is raised) Only 1 person thinks he knows. (Brian whispers to Magda) Want to think about that for a minute?
41. Bobby (To Brian) It's 256. Oh yeah I was thinking it had to go between cause 64.
42. Brian Yeah it's 128.
43. Bobby It's on my paper right here, so. (To Dr. Maher) This is the original binary system see.
44. Dr. Maher Do if there were 7 what would it be?
45. Bobby 1, 2, 3, 4, 5
46. Dr. Maher What if there were 7?
47. Bobby 6 uhh 128.
48. Dr. Maher Why?
49. Brian Because the first row down stands for 1 one, then next one stands for 2, the next one it just doubles, the next one stands for 4, the next one stands for 8, and 16 and 32 and 64.
50. Dr. Maher Is there a pattern?
51. Bobby and Brian Yeah it just doubles.
52. Brian Every space you move to the left the number of like... like... digit... I don't know, this doubles.
53. Bobby It's gonna be like 32, 64, 128, 256, 512, 1024, 2048... 4096.
54. 15:30 – 16:15 (Brian, Magda, Amy-Lynn, and Bobby discuss basketball games.)
55. Carolyn My question to you is, while they're thinking about it, have you thought about it yet? Suppose I had 3 color lights and 5 positions?
56. Amy-Lynn You do big 3.
57. Dr. Maher Can you think about that for a minute, talk about that.
58. Bobby That'd be 3, 9, 27, 81.
59. Brian Yeah it would be tripled.
60. Bobby That was the thing the problem was based on yesterday was 3.

- Cause there was 3 switches high medium and low in each. And 3 to the 9th is 19683, we figured that out.
61. Dr. Maher What if we had 4 lights?
62. Bobby It's 4, 16
63. Brian And then you do that times 4.
64. Bobby 64, or something like that.
65. 17:02 –
18:15 (Amy-Lynn looks through her notebook while the other 3 discuss school)
66. Dr. Maher Suppose now you had 7 positions for your lights, alright? But you wanted them to be... Can you make a sketch for the 7 positions of the lights? Make 7 positions. Draw a little line or something where the positions would be. Okay now suppose, let's see suppose I wanted these 2 to be the same color and this to be I don't really care and this to be the same color as this. Suppose I wanted all of these to be the same color. It would look prettier, right? How would it look to have 1 color here and all colors here? Two color lights right? Do you understand my question?
67. Brian So it's like red, gold, gold, red, gold, gold, red.
68. Dr. Maher Yeah or gold, red, red, gold, red, red, gold. Right so do you understand my question?
69. Brian Yeah.
70. Dr. Maher How do you think that would work?
71. Bobby Well it would be 9 to the 2nd power.
72. Dr. Maher Think about it now, think about it.
73. Bobby 9 to the 2nd.
74. Amy-Lynn Why are we working on this, we're skipping from one thing to the other.
75. Bobby I'm working on 2 powers. 2 to the 21. (Brian sings O Christmas Tree) (Magda whispers to Brian).
76. Brian We have to see how many combinations we can get out of that.
77. Bobby 81.
78. Brian If 3 is like red and 4 are gold.
79. Bobby 81.
80. 19:45 –
20:25 (Brian sings O Christmas Tree then they talk about chorus)
81. Amy-Lynn (Brian continues to sing) (To Bobby) How many are there, 81?
82. Bobby Yes.
83. Amy-Lynn How did you get that?
84. Bobby 9 to the 2nd.
85. 20:27 –
22:22 (Brian sings then they discuss the microphone, then they discuss Magda.)
86. Dr. Maher Okay I think we're ready to talk, I'd like to talk about it for a few minutes. Okay can we have a discussion? As I walked around I heard some differences of opinions. I would like to discuss those differences, let's go around. This table here was talking about 2 colors, 7 positions for the lights. And this table decided how many?

87. Student 127.
88. Dr. Maher They decided 127. Do you wanna try to say why?
89. Amy-Lynn Uh-oh.
90. Michelle We used the binary chart, we used 1 for red and 0 for gold. On the binary chart you just add zeros until there's 7. Then you just count up, you count up to 7 cause there's 7 places and that's where you stop adding zeros in front of it. Then at the end of that you use the counting chart and we have the number right here and you get 127.
91. Dr. Maher How many of you follow what Michelle said? Do you agree that there's 127 lights? How many of you agree with what this table came up with?
92. 24:07 Brian Did we come up with an answer?
93. Bobby 81.
94. Dr. Maher Now Amy you said you followed what she said?
95. Amy-Lynn Yeah. I followed what she said but that's not what we got for an answer. I understood what she said.
96. Dr. Maher But you don't agree. (Amy-Lynn shakes her head) Okay. All right let's go on to the next table. (They mumble) I can't hear you, I'm sorry.
97. Student We got 128.
98. Dr. Maher You got 28?
99. Student 128.
100. Dr. Maher You got 128. Okay we have a difference of opinion here so it's really important that we listen to each other so that we understand what that difference is. Who's gonna explain from this table?
101. Student I'm not doing it.
102. Dr. Maher You mean you don't know? (Multiple students talk at the same time). Michelle do you want to tell us?
103. Michelle Well I wasn't really following what they were doing so (mumbles).
104. Dr. Maher So this table has a couple of different solutions it sounds like. Want to give it a try Ankur?
105. Ankur Okay.
106. Amy-Lynn (To Bobby) How did we get that?
107. Bobby 9 different lights, 2 different colors.
108. Brian 81? (Amy-Lynn whispers to Brian) Huh?
109. Amy-Lynn 9 different lights, 2 different colors.
110. Student (?) (To Magda) What did you get?
111. Magda 128.
112. Bobby No we didn't we got 81.
113. Amy-Lynn We got 81.
114. Brian Yeah.
115. Magda We're talking about the first problem she gave us.
116. Amy-Lynn How did we get 128?
117. Brian Yeah we got 128.
118. 25:55 Dr. Maher Let's suppose for now we can have all reds and all golds, just for discussion. We don't have any explanation from this group here.

- 119 Student Okay for the... five digit numbers we got 32.
 120 Dr. Maher What do you mean five digit numbers?
 121 Student For the 5 candles.
 122 Dr. Maher The 5 candles okay.
 123 Student We got 32 because 32 in binary because the binary count goes in order from 1 0 and 1 0 and it counts in order and 128 is 7 candles.
 124 Dr. Maher How do you know there are 128 from 7? I'm really confused. You expect me to believe you just because you say it and it don't really. She said 127 a minute ago and expected me to believe her as if it were absolutely right. Then you changed your mind from 128 126 and I'm still supposed to believe her and now I have all these different answers floating around and I'm not in my head for sure that I'm convinced yet. Stephanie you want to try and explain it?
 125 Stephanie All right, the binary system, the way it goes, if you want to make a binary system they go 1 2 4 8 16 32 64 128.
 126 Dr. Maher Why?
 127 Stephanie What?
 128 Dr. Maher Why?
 129 Stephanie Cause that's their pattern. So...
 130 27:59 Dr. Maher Wait you're not going to answer me why? Do you understand my question? All ready right now I have a question. What makes the binary system go 1 2 4 16...
 131 Stephanie That's how many there are. If you're going to make a binary system what it is is you're going to put out 1 1 and like the second one is 2. It would be 1 0 cause there's 2 numbers. And then you go 1 1... and the next one would be 4.
 132 Dr. Maher Why does it work that way?
 133 Stephanie It keeps on alternating. (Amy-Lynn raises her hand)
 134 Dr. Maher Maybe Amy-Lynn can help you. Before you go forward Stephanie I really need to understand that works.
 135 Amy-Lynn Well like when we were going through all the numbers trying to do the binary system it went by a pattern like in the column like it went 1 0 1 0 all ther way through. Another one went, like some columns, like the 4 column or something it went by like 4 1s and then you went to another column and it did something. Like it sort of had a pattern to it, each column. And that's how it's done.
 136 Dr. Maher And what does that tell us?
 137 Amy-Lynn I don't remember each column but some of the columns it went 1 0 1 0 1 0. (Magda passes her a paper) Okay, like the first column it went 1 0 1 0 1 0 etc., and in the second column it went two 1s two 0s two 1s two 0s.
 138 Dr. Maher Yeah that's sort of what Michelle was saying before. Michelle explained it to me earlier before that kind of pattern. Maybe we'll hold that question aside and come back to it as Stephanie continues. You haven't convinced me, as a group yet, why that pattern is that way and how one gets that pattern. I'm kind of curious suppose I

- wanted to have 12 lights. I don't want to have to think about all these patterns. I can't do this all in my head. Do you understand what I'm saying? Because I have 12 positions for the candles. I want to be able to imagine how I can think of all the possibilities. I'm sorry Stephanie why don't you start again explaining to me.
- 139 30:10 Stephanie All right well we started with the 5 numbers 1 stands for red and 0 stands for gold. There's 32, there's 32 numbers for 5 and so.
- 140 Dr. Maher 5 places?
- 141 Stephanie Yeah with 5 places and so what it would be if that's 32 if you count all gold and all red for 5 then the next number is 6 and there's 64 places. And so the next number is 7 and 7 has 128 places, 128 numbers for 7 places and that's how...
- 142 Dr. Maher Dr. Alston seems very confused do you have a question?
- 143 Dr. Alston I do, a very, very confusing thing to me is (continues but is hard to hear because Bobby starts talking)
- 144 Bobby (To Amy-Lynn) It's 128.
- 145 Amy-Lynn Why?
- 146 Bobby Because 2 4 6 8 16 32. It's between 64 and 128. 64 times 2 so it would be 128.
- 147 Amy-Lynn How did you do that?
- 148 Bobby Look, 5 is 32 then its 64 times 2 is 128. 7 digits is gonna be 1111111 right? So they need to switch them around. So it would be 64 combinations for each color. (?) Like see there would be 1 0 0 0 0 0 0 and 0 1 1 1 1 1 like gold gold gold gold gold gold red red red red red red and there's 64 here and you just double it to switch it around.
- 149 32:18 Student It's everything up to that.
- 150 Dr. Alston But you told me that...
- 151 Student We don't use 32 numbers like Stephanie said, we use 60 and you double that.
- 152 Bobby (To Magda) It's 128.
- 153 Student But we don't count 32. 32 isn't one of our numbers.
- 154 Bobby But I got an explanation.
- 155 Magda What?
- 156 Bobby In between 32 and 64 there's 6 digit.
- 157 Magda (Whispers something)
- 158 Bobby Wait. I messed up yeah.
- 159 Amy-Lynn Raise your hand.
- 160 Bobby Why?
- 161 Amy-Lynn (Whispers to him. He shrugs.)
- 162 Student It shows like place values and stuff. And when you get up to 32, and that's the first number would be in the 32 column.
- 163 Dr. Alston Sure and you have to have that before you have 32 and Michelle what did you say the number is?
- 164 Amy-Lynn (To Bobby) Why don't you answer?
- 165 Bobby I'm waiting until she calls my name.

166 Amy-Lynn Bobby you have to answer.
 167 Bobby Why?
 168 Amy-Lynn You have the explanation.
 169 Bobby Duh, 2 to the 7th equals 128. There's 7 candles, 2 different colors.
 170 Amy-Lynn Tell that. (Bobby raises his hand.)
 171 34:06 Dr. Maher Say that again more slowly. Michael, why don't you come up and write it. Why don't you take a blank overhead.
 172 Bobby (Whispers to his table) You have to ask us questions too, you know.
 173 Michael The first 0 in all the numbers. First this one, it's like the first column in the chart 2 to the 0 that equals 1, 2 to the 1st (mumbles) well this is the second cause in our number system its based on 10(?). Well this is 2 to the 1st, 2 to the 2nd equals 4. 2 to the 3rd equals 8. And so if you had this one you put them in there. It'd be 2 to the 4th, 2 to the 5th and that's 32. You put the numbers in these charts, there's none in here so there's no more. There's none in this column, there's none in this column, there's none in this column, there's none in this column. So if there's one 32 that means this number equals 32.
 174 36:18 Dr. Maher Can you tell me what that 32 is? What you wrote about that, can we just check that for a minute?
 175 Student Explain what everything on the top is.
 176 Student You skipped 2 to the 5th.
 177 Michael Okay so 2 to the 5th equals 32. 2 to the 4th equals 16. (Drowned out) 10 to the 1st is 10. 10 to the 2nd is 100 then thousands and millions.
 178 Dr. Maher What is it you mean by 2 to the 5th?
 (?)
 179 Michael 2 to the 5th power.
 180 Dr. Maher What does that mean?
 181 Michael 2 times 2 times 2 times 2 times 2. 2 times itself 5 times.
 182 Dr. Alston Can you write that out for me?
 (?)
 183 Michael 2 times 2 times 2 times 2 times 2.
 184 Dr. Altman That helped me some.
 185 Dr. Maher Bobby has something to say.
 186 Bobby Well in 2 to the fourth, there's 4 numbers in that system in that part that's at. Like 60... there's 4 numbers right before it. So 2 to the 7th there would be 7 numbers there. And that's 128 so there's 128 possible chances to get the candles to fit into the window.
 187 Dr. Maher I'm a little confused. 2 to the 0 how many numbers are there?
 188 Bobby 0
 189 Amy-Lynn 1
 190 Bobby 1, darn.
 191 Dr. Maher What, what are they? Is it 2 or just 1?
 192 38:01 Bobby 1
 193 Dr. Maher What is it?
 194 Bobby 1... 0

195	Dr. Maher	Do you understand my question?
196	Bobby	0
197	Dr. Maher	The 2 to the 0 column how many numbers can we put in that column if you're in base 2? What can that be in base 2?
198	Bobby	0. You put 1 number that's 0.
199	Dr. Maher	You can put 1 number and what's that number?
200	Bobby	0.
201	Dr. Maher	Do you agree? What do you put in the 0 column? Can you put 1 number in the 0 column? Can you put 1 in that column? How many of you think you can put 1 in that column?
202	Bobby	It matters what the number is. If it's 1 or 0 then it matters.
203	Dr. Maher	I think our time is just about up. But I think that, let's try and raise some questions that I want you to think about for tomorrow. Dr. Alston do you wanna ask your question?
204	Dr. Alston	Yeah.
205	Dr. Maher	Why don't you ask your question?
206	Dr. Alston	My question is still over by the table I was standing today were explained to me how you counted all the way up and you got, you got 32 is that what you were telling me? But then he told me that when he got to your 32 it was... and of course I did it wrong.
207	Dr. Maher	Rewrite it.
208 39:43	Dr. Alston	I put too many 0s. You tell me that this was equal, this was the same in binary as 32 and explain that to me. Is that right? 1 0 0 0. Okay you said this was 2 to the 0, 2 to the 1, 2 to the 2, 2 to the 3, 2 to the 4, 2 to the 5. This is what Mike just explained to me and everybody agreed. And you said 2 to the this meant 2 times 2 times 2 times 2, 2, 3, 4, times 2. Is that right? And I agree that that certainly is 32. And so this is 32. My question is though, and the other thing you told me is you just went back and assigned a color of a candle to the 1 and the other color of the candle to the 0, and you're fine. And so since you were counting, that was fine. What my problem is, if this one is 32 it takes 6 candles and so I want to understand how many is left.
209	Dr. Maher	I'd like to also raise another question after you figure that one out. Do you understand? How many of you understand the question? There are 6 positions there right? Or candles right? So I'm imagining that could be yellow and the rest could all be red. Where the 0s are. Or it could be red and the rest could all be 0s. Clearly there are 6 positions so I am confused, right? Do you understand that Michael? Okay maybe you could think about that. I have a second question. Does this help you, once you figure that out, can that help you deciding if you have more positions than candles? Does that help you in deciding if you wanted to have more of bulb colors that are positions of candles? Do you understand what I'm saying? Not for 2 color bulbs but for more than 2. These are the things I want you to think about between today and tomorrow and I

hope to talk to you guys about it. In fact I would like you to write something as a group write that during lunch, you can do it. Anyways see you tomorrow.

(The students leave)

210 42:25

Appendix M

Date of Session: 12-15-1994

Author: Anoop Ahluwalia, Kathleen Dougherty, Kiranjeet K. Sran

Verified by: Anoop Ahluwalia, Kathleen Dougherty, Kiranjeet K. Sran

Date of transcript: 5-26-2010

Line	Time	Speaker	Transcript
1.	0:00		[Students are talking about a school test]
2.	0:58	R1 (Maher)	OK. I guess we should start. Um... Good morning.
3.	1:05	Bobby	Good morning.
4.		R1	How many of you have seen um the inside mechanism of a garage door opener before? [Children look around, no one responds] How many of you saw it for the first time today? [All raise hands] Yeah, that's kind of neat, isn't it? Did you like that? So, what do you think we do when we run out of codes? Like more people -
5.		Michael	Make another switch.
6.		R1	Make another switch? What do you mean, Michael?
7.		Brian	Like, add another switch on if you run out of codes. Then you have a whole... like a lot of codes.
8.		R1	OK, then you can figure out, um, how many people we can give codes to?
9.		Brian	I guess so.
10.		R1	Do you think they ever give the same code to two people?
11.		Brian	Probably.
12.			[Robert and Angela say yes]
13.		R1	Do you think they mean to?
14.			[Robert disagrees]
15.		Brian	Yeah. Probably. No.
16.		Bobby	There are just not enough combinations so they can get, you know
17.		R1	Because, uh
18.		Brian	[As R1 is speaking] What if somebody went around from house to house and used their remote
19.	2:02	R1	Mrs. Megan was saying she went she was in uh parking lot in another state opening her car and two other cars opened. You know, how they have those remote-controlled locks for cars? I don't think I would be very happy if somebody were was uh going through a parking lot opening car doors and my car door opened too, would you? Put your door inside shopping. OK anyway, um who can, um, who can, ah, state for us the question that Dr. Alston left us with yesterday at the end of the class? That you were supposed to think about and write about? Who can remember what that, what the dilemma was, uh, we were discussing yesterday? [Children are thinking] How many of you

- remember? OK, anyone [Child raises hand] One.
20. 3:01 Bobby Oh yeah! [raises hand]
21. R1 OK two people? [Amy-Lynn raises hand] Three
22. Bobby [Whispering to Amy-Lynn] Is it that thing with two to the sixteen and there's a zero?
23. R1 Think some more... You may want to whisper at your table to help your classmates figure it out...
24. Magda [To Robert] No its when she wrote um
25. Bobby two to the six
26. Magda That there were six numbers but there was two to the five power or something?
27. Amy-Lynn There were six different candles and two different colors.
28. Bobby [As Amy-Lynn is speaking] Two to seven, cause there were six numbers before it...
29. Amy-Lynn [To Michael, as Robert finishes speaking] But that's because there was six candles... I know what the question was. I know what the -
30. Magda OK!
31. Amy-Lynn I know what the question was!
32. Bobby two to the zero... two to the one...
33. Brian [Whispering nonsense]
34. Magda Shut up Brian.
35. R1 OK?
36. Brian I know what it was about...
37. Bobby two to the zero, two to the one
38. Brian Oh OK...
39. 3:51 R1 So you remember now Brian?
40. Brian Yes I do.
41. R1 OK this table, how many of you remember at this table what the issue was? [All four students raise hands] OK this whole table remembers the issue. [R1 goes to other table] Um, what about this table here? How many remember what the issue was yesterday?
42. Bobby [Talking to group] I heard about two. Give me a pen.
43. Amy-Lynn There was six candles in two pillars. You have to find out the answer and you write why.
44. [Inaudible conversation between group, R1 talking to other students to see if they remember the problem from previous day]
45. 4:24 R1 Who's going to tell us what the issue was yesterday? Ok hey Amy, I think that would be appropriate.
46. Amy-Lynn Well, like, the question that, um that Ms Alston raised was like, if there was like, if there was six candles or spaces, um and there's only two colors like how many could you make and then like why that was so? And like we were talking about like if it was um like with the candles if it was 128 or 126 or 127 then we were going through everybody's um like all of everybody's

- answers and trying to figure out to prove what answer it was.
47. 5:03 Bobby [Robert raises hand]
 48. R1 Bobby?
 49. Bobby Well, we also did that thing like where it was two to the zero and ah there was one thing when there was with the zero on the when there was supposed to be two places, 1 in the zero places, and then two to the one there was supposed to be one place but there's two and to the second there was three places when there is supposed to be two?
50. R2 (Alice) Tell me, what was that?
 51. Bobby Well if we said that like to the zero you said there's supposed to be no place for that but there was really one and that place was zero. And two to the 1st it was really it was supposed to be one but there's really two. one and zero. And then two -
52. R2 Can you show me up here?
 53. 5:39 Bobby [Bobby gets up to show on projector and writes] Since this is like the first, there's supposed to be one here, like this is the first number, there's only supposed to be one but, it, it says that there's two.
54. [Girl raises hand]
 55. R2 Which one and which one?
 56. Girl Bobby? Bobby you messed up. You have to put like
 57. Boy You put two to the second, then you did two to the fourth. It's really two to the third and two to the fourth.
58. R2 Ok. That helped me. And, If you put a one right here, what does that represent?
59. Bobby Well, that's, it could keep on going on and you could stop it here. Like here's six. And then there's like one, two, three, four, five, six, seven numbers in. Here's five and here's six... four here's five and then three and there's four and two and there's three, two, one and there's two and two there's a one. So its cause, if it was really like... cause they were saying that this can be on the different number there are, but it can't be 'cause its two to the second, two to the first, there's two numbers and really how many different numbers there were. There was only one. 0 there wouldn't be any places, or two to the sixth there'd be six places, not seven.
60. 7:28 R2 That was part of the dilemma that we were talking about a little bit.
61. R1 I'm still confused.
 62. R2 I'm still confused though.
 63. R1 Can someone help – Can you say that one more time for me Bobby, I'm still not sure...
64. Bobby Ok well, two to the zero. People like we were saying that zero means there's going to be zero places in it. But there's really one. That's zero. It takes a one place. And then two to the first.

- It's supposed to be a one digit number but its really... like say this is a one... one, zero. And two to the second that was a one, it would be one, zero, zero. And you're supposed to test two so there should be two places. And if it's two to the third then there's four places here and two to the fourth there's five places, two to the fifth there's six... one, two, three, four, five, six... and two to the sixth there's seven.
65. 8:09 R1 Ok so you're saying I think... someone help me summarize what Bobby's saying something very important here can someone help me summarize what Bobby is saying? I think I heard what Bobby said but I'd like to hear from somebody else to be sure I'm hearing it correctly? Bobby don't go away, cause we need we... Bobby one more time so someone can help summarize it
66. Bobby Ok. two to the zero like, then using the binary it's supposed to be like you know how this is supposed to be the... it shows that the digits, like with the candles how there's like thirty two and we're using the binary and it was like if you use scientific notation you're supposed to show how many there is but when you do like two to the sixth, right it there's supposed to be six numbers in it but there's really seven. If you use binary. So there's one, two, three, four, five, six, seven.
67. 9:02 R1 So if I remember, Michael was saying something like that to me about two days ago. Right Michael?
68. Michael Yeah
69. R1 You remember? Ok so can you help summarize what Bobby is saying?
70. Michael Uh, is it two to the, is that a six? Yeah... six equals thirty two and if there's a one there that means you get one thirty two.
71. R1 Two to the sixth means one thirty two?
72. Michael No no two to the sixth means thirty two and if you have a one under that column that means there's... that one, zero, zero... no wait six... isn't supposed to be five zeros? Did you mess up...
73. 9:43 Bobby No because see like this one there's one place and this one there's two places but this is two to the one so it's showing you one place and for this one it should be zero but the binary system it's like different cause there's one more and more...
74. Michael That's not thirty two. It's sixty four. Sixty four.
75. Bobby This is zero, one, two, three, four, five, six.
76. Michael But that's sixty four.
77. Bobby Where?
78. Michael Two to the six is sixty four.
79. Bobby Yeah, but you told me to put two to the sixth. But in the binary system you put a one at the last one and zero following. So like in this one it's supposed to if it was like real number, to the first, the zero... this one wouldn't be here... and this would be here. And then two numbers, three numbers, four numbers, five

- numbers.
80. R1 I think there are two ideas floating around and I'd like to sort of separate them. The number that's represented up there right now... Ok? Where he has two to the zero, two to the one, two to the two, two to the three, two to the four, two to the fifth, and two to the sixth... Where he has the one under the two to the sixth column and zeros every place else. How many of you think you know or have an opinion about what number is represented there now? Only three people? Know what number Bobby has represented there? four, five, six? Alright so how many of you aren't sure? Raise your hand if you're not sure. Ok. So the people who seem to know what number has, have to help the people who aren't sure. Bobby, what number do you claim is up there?
81. 11:18 Bobby Well, it was... it's...
82. R1 Right now? one with six zeros? one followed by six zeros?
83. Bobby sixty four.
84. R1 Bobby says it's sixty four. How many of you say it's sixty four raise your hand. How many of you believe it's something else? Ok. How many of you still don't know? Ok. Those of you who aren't sure... do you need an explanation? The "aren't sure" people... would you raise your hand? Or maybe you believe it's sixty four but it was your confusion you thought Bobby was saying thirty two?
85. Michael I thought he meant thirty two.
86. R1 Bobby was saying sixty four. That wasn't Bobby's point. But I think that's a point of confusion. Ok. How many of you now agree that the number Bobby has up there which I think Bobby was saying earlier is sixty four? Raise your hand if you believe that. Ok. Now this is were... that's not Bobby's point so please listen. To what Bobby's point is so that you can summarize it.
87. 12:24 Bobby Well, see two to the sixth. See this means like either ...regularly it would be like it's supposed to mean there's like six numbers but instead it's six zeros, six numbers that follow it. And two to the fifth there would be five numbers but there's really five numbers following it. And it's like if it was really it's like different because... it would sit there be like the same cause like down cause if you add two to the zero it will mess the whole thing up because you skip two to the zero, cross that one out, two to the first it would be just one number. two to the second there would be two. two to the third there would be three. But then when you add two to the zero, it like adds an extra number at the end.
88. R2 Ok. Would somebody please try to say what they believe Bobby is saying? Someone...
89. Ankur He's saying that two to the sixth does not have six numbers it

- has one more seven because of two to the zero.
90. Bobby Yeah. But if you take away the two to the zero then it would be normal because two to the zero messes up the whole thing.
91. 13:29 R1 So what does that have to do with the problem you had to solve?
92. Bobby Well, because...
93. R1 What does all this have to do with the problem you were asked to solve the day before we came and perhaps even... does that have anything to do with the problem you were solving here about the garage door opener
94. Bobby If you cross out two to the zero then you'd have to be thirty two because two, four, ... um two, four, eight, sixteen... or no it would be like one, two, four, eight, sixteen, thirty two if you cross out two to the zero, but if you kept it in there it would be sixty four because it's an extra place.
95. R1 So... you're saying then if we have six positions for candles, it can be represented by two to the sixth or it can't?
96. 14:12 Bobby If... it matters because if you like do it with the two to the zero in it, it can't. But if you do it with two to the sixth... If you don't do it with two to the zero then you can. But then if you did it with two to the zero, then you just have to go up to five. It would be the same.
97. R1 What do you think the rest of you? So Bobby, you're suggesting that if you have six positions for candles you can't represent the number of positions by two to the sixth.
98. Bobby Mhmm. Because if you like multiply anything by zero, it's zero. So if you add anything to zero it's the same. If you minus anything from the zero it's the same. If you divide anything by zero, it's zero. So...
99. R1 I think you just lost me, but...
100. Bobby Well like zero, it's not really a whole number. It's like if you multiply two times zero
101. R1 Where does multiplying by zero have anything to do with it?
102. Bobby Well, just say that like zero, it's not like a whole number, so if you add it to anything, it's just...
103. R1 I believe that, but what does that have to do with this?
104. Bobby Well like, if you take out this two to the zero then there'd be six numbers... alright? But if you add it it's not real whole number it just adds an extra place
105. R1 This is where I get confused at this part here.
106. 15:14 Bobby Well it makes it hard if you get a zero because like it makes more places and if you like do two times zero and two divided by zero, it just makes it zero mess the whole thing up?
107. R1 But I don't see zero I see a two to the zero.
108. Bobby Well, yeah like the zero on the top is a number and
109. R1 That's zero? two to the zero?
110. Student No it's one so you need it.

111. Bobby Yeah, but if you just did two to the one which would be two, you skip it.
112. R1 So when you're using a binary code can you skip one of them? To represent your possibility?
113. Michael You can't because you need the two to the zero to make odd numbers
114. Bobby Yeah but there's like if you're doing then if you do two to the sixth then you know there's seven numbers followed by six zeros.
115. R1 I see the problem of um... you see the problem that we have here? That two to the sixth isn't going to give you six positions. How many of you agree with that? Two to the sixth isn't going to give you six positions? But the question is it helpful, yes or no, and what do we do about that? Dr. Olsten?
116. R2 I'm I... I need to be caught up... on the first thing is that I'm trying to remember where we where we started out, where we left yesterday, and it would help me to go back to that one and then we could deal with the same issue. Was the number that I left you all with sixty four? Yesterday?
117. Student Yes.
118. Michael Yesterday you used thirty two.
119. 16:40 R2 It was thirty two... Now this... can it... how would you represent... this is representing sixty four. How would you represent thirty two?
120. Bobby Well, you'd cross this one off and put zero and a one in the five spot.
121. R2 Ok. Are you agreeing Angela?
122. Angela Yeah.
123. R2 Ok. So let's do this one is sixty four. Could you come up and show me thirty two, Angela? Underneath it?
124. Bobby Can I sit down?
125. R2 Yeah. For right now. I might bring you back in a minute. Ok don't do it on the same thing. Do it down underneath.
126. [Angela writing on overhead]
127. R2 And can you say what that's equal to?
128. Angela Thirty two.
129. R2 It is... does everybody agree about those two representations? Ok now... I, I do think that Bobby that all of this has been talking to the point, to the question that I left. Because I was really confused. Because each table yesterday said that there were thirty two window combinations? Is that right?
130. Class Yes.
131. R2 Of five candles? Is that right? And all of you said that it somehow could be represented by this binary thing? To get thirty two? And then all of a sudden, if it was going with the positions, do you understand what I'm saying... remember what

- we were saying? That you were saying that the binary chart or whatever you were talking about was how you proved there were thirty two different combinations? And that all you had to do was to put in red and green and whatever it was instead of zero and one? Is that what everybody said to me? Don't you remember that? Ok, and then, you got up to thirty two, and said this is what... this is how you proved it. And it left me really confused because that looks like six candles to me. Yeah. And so now we need to understand this first
132. R1 Is there a way, that you could try to test this? I mean I... we're dealing with lots of possibilities here to write out. Is there a way you can test your theory? We have a problem here. How many of you see what the problem is? Raise your hand if you see what the dilemma is. So, at least someone at each table seems to see that the problem is. My question is if that's the case, then we are having trouble understanding with six candles, five candles, six candles... somewhere in a larger candle category, is there a way to try and figure out your thinking, before you start thinking of the five and six candle category? Any ideas? (waits for students to respond, no body speaks up) You got some ideas Michelle?
133. 20:29 Michelle Well, I'm not sure yet... This is like the garage opener thing not with like the candles.
134. R1 Are they connected in any way?
135. Michelle I guess, but like that's what I'm trying to figure out now.
136. R1 Is that what you're thinking about ... Any other ideas?
137. Michael You can use the argument with that...you use base in both of these like...In the garage door opener you use base three and in this you use base two...so you use base in them.
138. R1 So unless you work out this dilemma, you may have to deal with both problems and rethink it. Do you have any ideas? You have any strategies you use when you get to a problem like this? Strategies would be very important at this point. What makes this problem hard? What makes a problem like this difficult?
139. Student I don't think we have to go into binary system.
140. R1 Jeff?
141. Jeff I think that when most of us started we were thinking about the mathematical part of the binary just the fact that five spaces and each one was different so you looked at it as just as a physical point of view and that we got lost when we were getting into it.
142. R1 Ok. Amy, you had a comment? What makes it difficult?
143. Amy-Lynn Well, like because I guess this dilemma, like with the... two to the fifth like its giving like it's supposed t give five spaces and its giving actually six. And like you are trying to get thirty two and your and you need six spaces... five spaces whatever and you're getting six. You have like the answer but you don't have like the right spaces. So you can't get the so you really don't

- have the answer yet and you have to figure out um how to get the spaces.
144. 22:20 R1 How many of you would a hint? How many of you don't want a hint? Ok this is a hint that not only applies to this problem but one you can use the rest of your life. You have to remember I'm giving you... do I usually give you hints? (Students say no) Ok so this is a hint that you can think of now and the rest of your life. You're in college doing mathematics or if your in high school, if you can't solve a hard problem, think of a simpler problem that's like it. You understand? And if it works for the simpler problem, test it on a problem a little bit more difficult in between, until you work your way to your hard problem. Is that a good strategy? Do you understand my hint? Raise your hand if you understand the hint. It's a very important what we call strategy when you're doing mathematics. If you're dealing with a hard problem, put the problem aside for a while and think of a simpler problem that is like it. Jeff?
145. Jeff When we were...that first day when we were totally focused on that garage door opener problem, um, I, to figure out all the different combinations we started out with um one switch with high, medium, low. And that gave us three. Next one kept on multiplying by three and that's how we got our conclusion that it was nineteen thousand six hundred and eighty three
146. R1 Ok. Michael?
147. 23:46 Michael Um well that... I've known that strategy before because when we were like littler, and we would do like hard um ah division problems or, or like, like it's mixed in with subtraction and division and so they taught us to make the numbers smaller so it's easier than, than really big numbers.
148. R1 Ok. Sometimes we can't think of the strategy, and sometimes solving the problem is thinking of the strategy. Right? Once you think of the strategy to approach it, that's really the big inside to begin to solve that problem. And thinking of the appropriate simple problem, right, is also the test. Because you might think of a simple problem that doesn't relate to the problem you/re trying to solve. So you might want to discuss in your group what might be appropriate simple problems. I sort of remember looking at all the tapes by the way, seeing you use that strategy. And I wonder if you can remember anytime in any problems that we've done where you've used that strategy of thinking of a simple problem to solve a harder problem.
149. 25:04 R2 Jeff was saying an example just then. Of how you said that not yesterday but the day before yesterday, and I think I watched Stephanie, and Ankur and Angela and Romina. Michelle I guess wasn't there doing the same thing, when you were uh arguing the uh garage door, that when you, when you used a smaller

- example, not one switch but maybe two.
150. R1 Would you like some time to go back to your group now? Is that a reasonable thing to do? How many of you would like to go back to your group?
151. R2 I think you should, but I would like also one more time to hear... we've talked about so many interesting but somewhat confusing things that I would like to know what you think the problem is, what do you think this hard... how would you state it if you had to... if you were Mrs. Toye and would have to write it up for a homework problem? How would you state the problem that you are trying to solve? I think that's part of the problem. What do you think you are trying to figure out? Does anybody know? Yeah, Michael?
152. Michael The combinations we can make.
153. R2 You're trying to find out how many combinations there can be for
154. Michael The candles
155. R2 For five candles? Because you came up with thirty two, and the thirty two that you said was related to the binary seems to take six candles. Does everybody understand that? So what you're trying to do, is to figure out how there can be this many with five positions. And Bobby gave us one idea, which was to throw away the first position. Now, if that's a possibility, you have got to be able to prove that it works. And makes really good sense. Does everybody understand? Ok then, work it out, figure it out. (Students go back to working their groups)
156. 27:26 Bobby I can prove it because look... zero isn't a whole number it doesn't really represent a place. Cause there's no 0th place. But there's a first place, there's a second place. Or you could put a decimal point there.
157. 27:57 Bobby Or you can put a decimal point in between one and zero.
158. 28:07 Bobby You can like put a decimal point in between the one and the zero. Instead of throwing it away.
159. Michelle What does this have to do with... What does the candle problem have to do with? What would you use as the number? Would you use two as the number or five? And what would you use for the number on top?
160. Magda I think it's five... five candles and... it's five candles, um... what's the two for?
161. Michelle Cause that's how many different you can use.
162. Magda So, two to the fifth. Cause two different colors and two different... Because there are five candles but if you like multiply there's still gonna be two colors, but six candles.
163. Bobby Um, something minus
164. Brian You wouldn't understand, I would explain it to you in a second
165. Amy-Lynn This is nothing and this is counting as something, these two

166. Someone Now, wait,(inaudible)
167. Amy-Lynn These two, these two, these two are counting as nothing, and these two are
168. Bobby Are deducting
169. Brian How can you be so stupid and not know how to open a calculator
170. Amy-Lynn And that equal something, see
171. Bobby Yeah
172. Amy-Lynn These have to equal something, even if you decided, all you have to do is to have two plus you said, you have to have a space, you don't have to have anything in there, two plus two
173. Bobby Yeah, plus you do what else, look, do two to the There's like a first place, right
174. 30:25 Amy-Lynn Two to the what?
175. Bobby Just, two to the zero, right, there's like no such thing as a zeroth place but there is the first place, a second place, a third place, so a zeroth place is usually represented by decimal
176. Amy-Lynn Yeah, but it equals one
177. Bobby Remember like last year
178. Amy-Lynn But see, you need that because that equals one
179. Bobby Remember last year Mr. Powell, where he meant, he put zero, he divide something by zero
180. Amy-Lynn Two plus one equals three, two divided by zero, no one, and two times one, two, two minus one equals one so it's counting as something
181. Bobby So then use it like as a decimal point or something cause, or you can do two to the fifth and there's five numbers following it, like, see two to the fifth is a number and there's five numbers after it and there's two the fourth, there's four numbers after it and two to the zero, there's zero numbers after it.
182. Amy-Lynn Yeah but, see but one counts as something, you always use one number in counting, if you use it like two plus, just a space, that's nothing, its two... and if you use, two divided by nothing, equals zero, you just count, these are like invisible zeros, zero's like counts as nothing here but they count as something here
183. Bobby They count as nothing in our system but they count as something in binary system
184. 32:11 Amy-Lynn Plus and minus equals nothing cause you don't even have to have a space there
185. Bobby Yeah if you are doing multiplying and dividing, it counts as something, it takes
186. Amy-Lynn But if you say everything is zero, one is used in everything so, you can't take away one
187. Michael I'm in on this.
188. Bobby Just don't tip the table please?
189. Michael I won't. Trust me. Nah, I wasn't paying attention Oh well

190. Amy-Lynn Look here, I'll show you. Look. You want to get in on it? You want to get in on it? Look,
191. Guy Just put me in on it
192. Amy-Lynn I can at least show you what we were doing
193. Guy Two plus zero equals two, two divided by zero equals zero, two times zero equals zero, two minus zero equals two (reading Amy-Lynn's notes)
194. Bobby So, zero only counts sometimes, like in the binary system when you are multiplying and dividing, it would count
195. Amy-Lynn But two to the zero equals one and you always use one, two plus one is three, two divided by one, one Two times one two, two minus one is one
196. Bobby Yeah remember, last year when Mr. Powell, Mr. Powell messed up the whole system cause he divided something by zero and then like five is equal to zero, five was equal to
197. Michael Ahhhh
198. Amy-Lynn Yeah, that's what I said, when you divide by something. But if you take away two the zero, that equals one and then you'll
199. Bobby Yeah, cause look, if you didn't use that then everything would always be equal
200. Amy-Lynn I don't think everything works
201. Bobby No, cause if you did two like divided by zero and if you took out that zero, like and you did something else, like five divided by nothing it would all be equal because nothing will be equal to nothing and then everything will turn out the same cause if nothing was like a one nothing will be a zero like it is, nothing was like a negative number, so you can't really tell
202. Amy-Lynn But it is a zero
203. Bobby Yeah, but if we didn't have zeroes and you just put nothing, you would not know what nothing is.
204. Amy-Lynn Can you come here? If you wanna take out two different zeros, two different zeros equal one,
205. Bobby Yeah, I know
206. Amy-Lynn it does not equal zero
207. Bobby That's not good. That's called cheating. How can that be? Two times nothing?
208. Amy-Lynn No, it's kinda like
209. Bobby It's like two times half or something
210. Amy-Lynn See, but if you use one, see you are going get higher, you're always getting an answer, you are not really getting a zero, you get a one
211. Bobby Yeah, but it like you can take zeros out and two plus zero is just going to be the same , two minus zero
212. Michael Bob, what is your number
213. (everyone gives him home phone numbers)
214. 35:50 Amy-Lynn You can't take away two to zero

215. Bobby So, you can't yeah, you can't take it away cause then you have to take away zero from everything and you mess up our system too
216. Amy-Lynn But that doesn't work, (inaudible) does not work
217. Bobby So, we just have to do like two to the one, there's one number following it, two to the zero, there's zero numbers following it
218. R1 Did you test your theory Bobby?
219. Bobby I am not sure
220. R1 What did you do?
221. Bobby Well, she did it, well we worked on it together, she can explain it to you
222. R1 Amy-Lynn?
223. Amy-Lynn I like... You can like do two plus zero, like it's two, like its three and um and two minus zero is like two, so you can just take away the zero, like you have two minus and this is spacer, so that right there, zero does not count as anything
224. R1 I don't understand what that has to do with the problem, I am very confused
225. 36:39 Bobby Wait, you will see, cause then like, then this if there's nothing here, then it'd be all messed up, because like see if there was two divided by and there's a blank space, cause as there was no such thing as a zero equals one, now if we took out of the binary system zero, we would have to take it out of our system too
226. R1 I don't understand. First of all, I question that these are all true
227. Bobby They're all true
228. R1 I question that. Cause one of them that I believe is not true
229. Bobby This one right?
230. R1 Uh-huh
231. Bobby It should be two, right?
232. R1 Well, if you think of zero, if you think of zero, um two divided by zero is something, let's suppose we don't know
233. Bobby Because like you have two
234. R1 Well, well, let's just write two divided by zero, right equals something. Let's suppose we don't know what it is, let's just put a box, right, OK? Now, if we were doing two divided by let's say one, what is that?
235. Bobby Two
236. R1 Two divided by one half, what would that be
237. Bobby Four
238. R1 Two divided by one third, that would be
239. Bobby Nine, nine... oh no, six
240. R1 Six, Ok. And you can check that right? Can you check... How do you check division?
241. 38:04 Bobby This times this, equals this
242. R1 That all works, one here times the four is two, one third times six is two. Now, You say two divided by zero is zero, I am gonna

- say it's three. And it works, if you, does that work?
243. Bobby No, because, zero times three equals zero
244. R1 So, if you are gonna put zero there, does that work?
245. Bobby No
246. R1 No, it doesn't check. We got a problem here
247. Bobby Because zero is like, oh Mr.
248. R1 But, Wait a minute, it doesn't check
249. Bobby Yeah, I know because Mr. P... last year, where he put zero in here divided by five and then everything is equal to something, cause it messed up the whole system because five can't go into zero
250. R1 Ok. But this is zero divided by five, Ok, zero divided by five I claim that is zero
251. Bobby No, the other way around,
252. R1 But five divided by zero is the same category as two divided zero, see how you will have trouble, that doesn't check does it? You're telling me it is zero, and if you think about it, two divided by zero is a problem OK. And you notice, what's happening as you are dividing with the two, not by zero, you are getting a number ...is one half closer to zero than one
253. Bobby It's closer to zero, yeah
254. R1 Is one third closer to zero than half?
255. Bobby Yes
256. R1 Ok, what is two divided by one tenth?
257. Bobby Twenty
258. R1 What's two divided by one hundredth
259. Bobby Two hundred
260. R1 What is two divided by one millionth
261. Bobby Two million
262. R1 Now, what's happening as you are getting closer and closer to zero
263. Bobby It goes up
264. R1 It goes up, so you're telling me zero and I am saying as you are getting closer and closer to zero, the number you are dividing by is getting bigger and bigger and not smaller and smaller so that's another argument, why I would say, two divided by zero, does not seem likely it going to be zero. See what I am saying, Amy-Lynn?
265. Bobby Yeah. Cause this zero, like it, shouldn't be here because like can't take zero, can't take two from zero if you don't have it
266. R1 But you are not taking it, you are dividing, you know
267. Bobby I can't divide two with zero people
268. Amy-Lynn Like if you have nothing, and you want to divide it by two, you can't cause there is nothing to divide by
269. R1 Well, you can divide nothing by two, everybody gets nothing but we are talking about two divide by nothing and that's the

- difference
270. Bobby Yeah, because nobody has it, so you can't divide it
271. R1 Ok. well, think of these ideas, you see I had trouble with this same thing here, and I don't follow, I don't see how this connects to your argument, I don't see the connections you want to work on that a little bit and then persuade me? cause I don't see the connections
272. Bobby OK , should I just spill...like two to zero that equals one, but then we convert our system if you took away zero in this one, you compare to ours you'd have to take away the zero on this, our the regular system because they are like the same
273. R1 But I don't see that because in our regular system where you have one, ten, one hundred, right? Or ten to the zero, right, ten to the one, ten to the second, right? In our regular system this is one, zero has nothing to do with, it's just a question of notation, just for consistency we called, we called the number to zero power one, just so that it looks nice and pretty and the pattern holds, it's just a convention, now it maybe is a confusing convention, but this ten to the zero, has nothing to do with zero, which is it's useful we say any number to the power of zero by definition is one it's a thing we make a definition of, that's not really logical or anything, it's convenient except zero itself, that's another troublesome one by the way, that's not one
274. Bobby It's zero
275. 41:36 R1 Well, let's just say that's bothersome like this one is and that involves more higher mathematical ideas like this one, so but with the exception of zero, we have said and if you take a number other than zero and raise it to the zero power, by definition that's one but if you take a number and divide it by zero, right? We see that we don't know what to call that, but we know what happens if you get closer and closer to zero, your answer gets bigger and bigger and so this is kind of like a limit problem and this is
276. Bobby But, if you like you do ten over zero and then wouldn't ten over zero be equal to two over zero and then like a hundred over zero be equal to two over zero and then over zero
277. R1 Well think about it ten over zero, go through the same reasoning Bobby, divide by number that is close to zero, divide by ten, ten over ten is
278. Bobby Zero, ten over, one
279. R1 Ok, divide it by a number little closer than ten, two. Ten over two is
280. Bobby Five
281. R1 Ten over one is
282. Bobby Ten
283. R1 Ten over one tenth is

284.	Bobby	Hundred
285.	R1	So what happens as you divide by a number that gets closer and closer to zero?
286.	Bobby	It gets to like fractions and then
287.	R1	But what happens to your answer?
288.	Bobby	It gets father away, like there's more it's more
289.	R1	The number gets bigger and bigger right? And see that's the problem, don't think of this as zero because it's zeros, because if I take ten and divided by a billionth, right? The number is getting, one over billion, you see, you get the idea
290.	Bobby	Ten billion
291.	R1	No, if I divide it by one, one billionth, right? So, these are trouble some things and try not to let this piece confuse you. In our place number system, the reason this one is here, it tells you how many ones, and you could have up to how many?
292.	Bobby	One
293.	R1	No you can, up to how many ones can you put in ones' column? You can put one ones, two ones, three ones,
294.	Bobby	Nine
295.	R1	Yeah, you can put no ones, right?
296.	Bobby	So, ten
297.	R1	So, you can put ten possible digits here, so don't let that zero confuse you, so think about your problem some more
298.	43:39 Bobby	OK, so the zero can be useful be in some parts, but it can be useful in some others cause it is not useful here, but it's useful here
299.	R1	Well, this, I fail to see what this is connected with in our original problem
300.	Bobby	Well, it has zero, so you count zero
301.	R1	Yeah, but a lot of things have zero
302.	Bobby	Well, two minus zero is same and two plus zero is same but when it gets to multiplying and dividing it changes around
303.	R1	But see the division is tricky, isn't it?
304.	Bobby	Hmm hmmm OK ... Two times zero to the hundred
305.	Amy-Lynn	Let me get the calculator
306.	Bobby	Yeah, put two divided by zero in that
307.	Amy-Lynn	Error two
308.	Bobby	Oh darn, it is an error.
309.		Amy-Lynn tries again
310.	Amy-Lynn	Error two
311.	Bobby	That's not fair. Let's see. Let's try zero divided by two
312.	Amy-Lynn	Zero. So, let's put two divided by zero
313.	Bobby	Error two, darn
314.	Amy-Lynn	Maybe, we should
315.	44:46 Bobby	Nah, it's impossible like see there's no such number cause if you do like, if you multiply anything with zero, look, cause what's

your thing up here, if you multiply anything with zero, it equals zero, say if you put a million in this box, zero times million still equals a zero ... darn! So, like, it's a question mark

316. Michael Why do you do it the hard way? Just explain it like that and it makes it easier?

317. Bobby We're doing see because two over zero can't be done. So, we...

318. Amy-Lynn Inaudible

319. Bobby Maybe you put the zero on the top

320. Amy-Lynn But that's gonna give you zero divided by two. It's not gonna give you zero

321. Bobby So, you can just reverse it around, zero times two is zero and zero minus two is

322. Amy-Lynn Well negative two

323. Bobby Zero plus two is positive two (pause) Tough choice

324. Amy-Lynn So, wait, won't you go

325. Bobby No, she did. She just had the same idea

326. (Bobby looks at paper)

327. 46:36 Bobby You prove it wrong on one problem too, one problem

328. Amy-Lynn Everything else is fine, go ahead, right there

329. Bobby Hopefully, she would, maybe she sees the time and skips this one and get to the next one

330. Amy-Lynn (Looks around)

331. Bobby (Writes on the paper)

332. 47:12 R3 I don't see anybody around the table, must mean you convinced everyone? That you got it?

333. Bobby No

334. R3 Oh

335. Bobby Cause, look, you can't do this two divided by zero, cause, two to the zero anything you multiply by zero is just zero, not two

336. R3 Two divided by zero is something we can't do right?

337. Bobby Yeah

338. R3 So, what are you trying, you are trying to divide by two, by zero

339. Bobby Oh, cause dividing, Like we are proving, showing how like zero can be useful some ways but can't be useful, like here it's not useful here because two plus zero is two, two minus so it's like two, then it can be useful in times, two can equal zero, then you are multiplying something you don't have when you are dividing

340. R3 So, how's that relevant to what we are trying to figure out here

341. Bobby Well, zero can be helpful because

342. R3 You are trying to figure out two to the zero?

343. Bobby No, it's one but we're just seeing if two to the zero be useful in the binary system, if you take it out

344. R3 If you take it out?

345. Bobby Then it is messes the whole thing, like

346. R3 What if you took out ten to the zero from the decimal system?

347. Bobby Then you mess up the whole thing up, then there'd be like two... there's no zero and you are stuck with the same thing, it would be like two times blank equals two, oh, zero since you can't, since if you took zero out of our system, it would be like two times blank equals zero
348. R3 And if you did it in the binary system?
349. Bobby Then it be two, one then it will be zero and mess the whole thing up two, three, instead of two to the zero crossed out
350. R3 I'm really confused too when, in the ...you had that up there you had the place holder did you, do you have that written down there?
351. Bobby No, (writes on paper)
352. R3 Ok, That's fair enough... Ok... now... why are you filling in zeros here?
353. 49:35 Bobby Well, just it could be a one here or something zero or one, but this one has to be a one since it is the last number
354. R3 Alright, but now you are saying you get rid of this
355. Bobby Yeah, If you get rid of this then it'd be like two to the first, there's one place there so that's right
356. R3 If you get rid of this, what happens to this one?
357. Bobby Disappears too and so then makes it a smaller binary system like having instead of having five digits for sixteen or thirty two, there'd be four digits cause you took out one number
358. R3 What if you take out the next one?
359. Bobby Then it'd go down by two digits, so instead of being five, it would be three
360. R3 But if you... so you just keep taking away digits?
361. Bobby If you keep on taking away digits, it just gets smaller and smaller and smaller, if you take away these three, then two to the third would equal one
362. R3 Hmm
363. 50:28 Bobby Or like here, but see, cause now two to the first, there's two digits here
364. R3 How would I count in the... this system of yours
365. Bobby One ten hundred, hundred and one, hundred and ten, hundred eleven , so on
366. R3 Yeah, I mean, but is there a, so...
367. Bobby It's base two
368. R3 Ok
369. R3 So, the first number is
370. Bobby Base two, One ten eleven
371. R3 Ok, now that's good, now what would happen to the list if I eliminated this
372. Bobby That'll be ...if you eliminate
373. R3 If I eliminated this two to the zero

374. Bobby Then it'd be take out these Ok, let me try, it would be zero, one, ten, eleven, hundred, so on, if you eliminate, it would take away the first one and move it down a place, so then two this would equal two to the first and then this would be
375. R3 So, is that the same?
376. Bobby No
377. R3 Oh, it's different, so if you eliminate, you lose something, if you eliminate right?
378. Bobby Yes
379. 51:36 R3 Something would be different
380. Bobby And then
381. R3 So, guess that's why I wonder why you want to eliminate it cause it's completely different
382. Bobby So, if you eliminated two to the first, it would be zero, zero, one, ten, eleven, and this would equal to the second, this would equal to the third and if you eliminate this, this one goes zero (inaudible) and so on
383. R3 Ok, as I eliminate things, I lose numbers basically, right?
384. Bobby Yes
385. R3 So, if thisOK
386. Bobby And, if I took away two to the third, then it would be four zeros and one would equal two to the fourth, ten would equal two and so on
387. R3 Ok, now, why is it in this candle problem, why is it you want to eliminate, eliminate some counts?
388. Bobby Well, two to the third so then it'd be easier cause then there's only three digits in it but now there's its four, so you have to do this number and then three digits following it
389. R3 Ok, so, but then, is it the candle analogy still hold if you throw this one away?
390. Bobby No, (inaudible) zero, so then you have to, like it would probably be less cause you took away a number
391. R3 It will be less
392. Bobby Because moving everything down one space, say the last number in the binary system is one and O, O, O, O, O, O, O ... and you took the away and one and moved it down a place and put a zero in last number you get at a one O, O, O, O, O, O, O one
393. R3 Ok, but now, you see now where I am falling behind is, is you told me that you are going to represent the candles by, by zeros and ones, right
394. Bobby Yes
395. R3 And then you said, well, there's thirty two candles, so we need to go six spaces to get that thirty two, that's one zero, zero, well one followed by five zeros right? and yet, then, then of those six spaces, which ones, I mean, then we have the dilemma that Dr. Alston was running into is that you get six spaces and five

- candles, right? But now you seem to want to correct it by getting rid of that space
396. Bobby Yeah, then when you get rid of it
397. R3 But isn't that space representing a candle or not
398. Bobby Yeah so, like
399. R3 So, how can you get rid of it?
400. Bobby Well, you can get rid of one candle, that would not really make a difference, it would all be the same, but then after the candles, you can't really take it out
401. R3 So, You can or can't take it out?
402. Bobby Can
403. R3 You can take it out
404. Bobby If you take it out here, you have to minus candles, but if you take it out after, it would not matter, cause then it would just affect the numbers after it,
405. R3 Ahh, right
406. Bobby Say this was five candles, here, these two, it would affect this one right, it would affect everything here on and nothing in this way, it only affects stuff this way and not this way, cause this
407. R3 Oh OK, so, I kind of,
408. Amy-Lynn Jeff, I think this would be easier
409. R3 It's confusing, but if you can, can you show me with a list of the possible candle arrangements and a list of the numbers how this, how this corresponds to what you want it to correspond to, would that make sense?
410. Bobby Yeah
411. R3 That might convince Dr. Alston and myself.
412. 55:00 Bobby Cause you can't like take out, you can't take it out because it would mess up the whole thing, right? But then if you, you can't take out something past the five candle numbers cause it wouldn't affect it. It wouldn't make any difference with the zero because it would only affect the stuff past it. If you take out here (pointing at diagram he made) where you take it out, say you take out two to the first, it would affect two to the first and two to the zero and like two to the negative one, two to the negative two and so on.
413. R3 Ok, well why don't you try and show me how, show me the correlation and why, you know and even if, you wanna, well, yeah five candles isn't too many to do, I think. And then show the correlation. And then that may convince us so of why they can get rid of that number.
414. Bobby Ok. (Robert works on a new diagram to try to convince everyone why they can get rid of the number).
415. Brian I'm so tired. I wanted to go to the (inaudible) too.
416. Bobby Go to the rec center. Last week, last week we were playing

cards.

417. Brian What was that?

418. Jeff Can I ask you a question?

419. Bobby Yeah.

420. 56:26 Jeff When you did you do this problem originally, did you (someone calls him). What? I know, but they're gonna find out anyways. Were you thinking anything that's (mumbles).

421. Bobby Huh?

422. Jeff You used binary.

423. Bobby Yeah.

424. Jeff Were you thinking about the mathematical aspect about it or were you just thinking it's easier for us to write it down and get down with it.

425. Bobby Hey, it was an accident. I was the one who thought of it.

426. Jeff Ok, you're smart.

427. Male (Whispers) I (inaudible)

428. Bobby Good for you.

429. Brian What?

430. Bobby Good for you.

431. Brian I ain't doin (inaudible)

432. Amy- Lynn Cause you don't know.

433. Brian Awwww, I don't even know what we're talking about. Neither does Jeff. Neither does Mike. They don't care.

434. All (Mumbling)

435. Amy- Lynn Oh look. Look what I did. (playing with calculator)

436. Brian What? Stop. How'd you do that.

437. Amy- Lynn I don't know. I was just pushing buttons.

438. Brian I know how to get to some, like, little like money up here. Too dollars and fifty cents.

439. 57:53 Amy- Lynn We already knew that.

440. Brian No you didn't.

441. Amy- Lynn I did too.

442. Brian Today there was fifty cents. And then there's four dollars and fifty cents.

443. Amy- Lynn That's a lot of money.
(Amy-Lynn to Bobby who is still making his diagram): What are you doing? Two to the zero, two to the one, to two to the fifth. What are writing out, all the candle combinations?

444. Bobby No. He said I just have to show him like write these numbers (pointing at his diagram) and prove that if you take something out of here it will mess it up and if you take something out past it, like two to the sixth, it would only mess up the rest of the binary system. So..
(Pause)
I wrote my explanation. See if you take something out of these numbers, it would affect, um, it would affect them. If you take out two to the forth, right here, then, then two to the fifth, two to the sixth would be messed up. If you take out two to the zero the whole binary system would be messed up, so....so, it's like, and if you took out like two to the millionth or something, two to the millionth until infinity would be messed up.
445. Amy- Lynn So you're saying if you... (starts writing) did two to the zero, two to the one, two to the two, two to three, two to the four, two to the five, and if you took out this (pointing at two to the zero), from here (pointing at two to the one, all the way to two to the fifth), here, here, here, here and on is messed up.
446. Bobby Yeah, and if you take out from like here (pointing at two to the three)
447. Amy- Lynn Then here (pointing at two to the fourth, then fifth), here, and on is messed up.
448. Bobby And this is ok (pointing at two to the two and one).
449. Amy- Lynn Yeah, ok.
450. Amy- Lynn Bobby's got an easy explanation (talking to someone else at their table). Bobby's got another easy explanation. Bobby's got an easy explanation.
You didn't even do that yet?
451. Bobby You didn't?
452. Amy- Lynn He didn't even finish his second sheet.
453. Brian I did the fifty.
454. 1:00:11 Bobby Brian, you didn't do your homework.
455. Amy- Lynn You're supposed to classify them though.
456. Brian Classify them, non-living, (Inaudible), living.
457. All (Mumbling about science homework.)
458. Brian Bob, were you invited to Sam's party?
459. Bobby Yeah.
460. Brian What are you wearing?
461. Bobby I don't know. Aren't we supposed to wear something nice?
462. Amy- Lynn Apparently it's formal.
463. Bobby I'll just wear black jeans, a nice shirt and a tie.
464. Brian That's stupid though. (inaudible)
465. Bobby I'm telling you, I'm gonna wear just a nice shirt and a tie.
466. Brian We should go in all flannels and stuff.
467. Bobby Yeah, I'll wear a flannel and a tie and jeans.
468. Brian (Inaudible) It's sorta like a reddish kind of flannel.

469. Bobby I'll just wear a flannel shirt and a tie.
 470. Brian I'm wearing sneakers and that's as informal as you're gonna get.
 471. ALL (Mumbling about party.)
 472. Brian It didn't say on the invitation.
 473. Bobby Yeah. And it said 6 not 5.
 474. Brian That's good though. Cause all the parents are gonna be in the room, so you really can't do nothing bad. If you don't like it, it's an hour shorter. And if it's good, (inaudible) you wanna stay a little longer.
475. Bobby Last time I went...
 476. Amy- Lynn I have a feeling Monday's gonna be a bad day, after this.
 477. Brian Eh, I don't really care.
 478. Amy- Lynn Cause then I have to go out caroling from 6 to 8.
 479. Brian If you (mumbles).
 480. Bobby I'll wear flannel. Nah, I'll just wear a button up shirt and a tie.
 481. 1:01:46 Amy- Lynn I bought a dress from JC Penny's. That ain't fair (inaudible)
 482. Brian Nah, but the flannel's 30 bucks.
 483. Bobby Yeah. Oh yeah, my flannel was like 20 dollars.
 484. Amy- Lynn I got a dress...
 485. Brian And I got this shirt that goes underneath that was like 20 bucks (??) which is good cause I usually freeze.
486. Amy- Lynn I got a dress that was about 60 bucks.
 487. Bobby And jeans.
 488. Brian My whole outfit cost about like 80 bucks.
 489. Bobby And jeans cost about 30 dollars.
 490. Brian My jeans cost 30 bucks. The shirt cost 30 bucks and the shirt underneath cost 20 bucks.
491. Amy- Lynn These other girls better be wearing dresses or I'm gonna puke.
 492. Brian All the girls are wearing dresses. They're gonna be wearing like body suits. Who cares? Guys will be wearing flannel, girls will be wearing body suits. Who cares?
493. Bobby Clip on tie.
 494. Bobby & (Attempt to call Jeff over, he doesn't come, they sit quiet for a bit. Ryan gets up.)
 495. Bobby But it kinda makes sense, though, because if you take out two to the sixth, (starts drawing again) two to the sixth.
496. Amy- Lynn It does not affect you and you've done ...
 497. R2 Explain to me what you guys are doing.
 498. Bobby & Ok.
 Amy- Lynn
499. R2 What did you decide about your two to the zero thing?
 500. Bobby It can't work, cause...
 501. R2 What do you mean?
 502. Bobby Cause you can multiply anything by zero to equal zero and you can't equal two. Even if it's like a million, a billion, a trillion, or anything.

503. R2 Yeah but two to the zero can't ever equal two.
504. Bobby I know. That's why it's so messed up. Cause it just equals question mark. But like this thing with the binary
505. R2 Ok. Help me with it.
506. Bobby (Pointing at old work) Like zero. Anything you multiply by zero equals zero.
507. 1:03:31 R2 Sure.
508. Bobby So it can't equal two, cause to check it you have to do zero times a number, or divide this denominator times the answer, should equal this. But you can't do that, cause it's always zero.
509. R2 So what is it?
510. Bobby It's, it's question mark.
511. R2 Oh, but what's this?
512. Bobby Oh, this? Ok. See, if you take out, like a number, if you take out a number it would affect the whole system. Say if you take out two to the four, right?
513. R2 Mmhmm.
514. Bobby Then it would, it would mess up every number past it. Cause you'd add an extra zero in it. (Pointing at diagram) So it would be two to the fourth, two to the fifth, two to the sixth, it would be all messed up. So if you take out the zero, you mess up the whole binary system by adding all these zeros in it.
515. R2 Yeah, I don't, but...
516. Bobby Here, cause, two to the zero, if you took that out, right, it would be zero, one, and one would have to equal two to the one, right? Cause that's the first number. And two to the zero. And then ten would have to equal two to the zero. If you took out two to the second, it would be zero, zero, and you have to...
517. R2 Ok, so what you're saying is that this position (pointing at two to the zero) is important.
518. Bobby Yeah
519. R2 What does it mean?
520. Bobby Well, it's the first number of the whole system. And see if I took out two to the first...
521. R2 What is the first number of the other system?
522. Bobby One. See if you took out the last number of the binary system, it wouldn't matter cause all you'd have to do is add the zero, but it doesn't add. But say you take out two to the three...
523. R2 Ok, but, but then how are you coming to terms with my question? Cause that was the way you were getting that 32 thing.
524. Bobby Yeah cause, see, if you... Two to the zero is important because it would add all these zeros, like infinity zeros, and then start the number. So, like two to the infinity one, would equal one, and two to the infinity second...
525. R2 But really, I still, my question is really a simple one. I wanna

- know how you're going to get 32 combinations with your binary thing when it looks to me like you needed six candles to get up to 32.
526. Bobby Two to the fifth, you can do like, if that's the last number, the first number would just be one so the five can be how many zeros or how many numbers that are following the first number. So then, if you want seven, six of them, you do two to the fifth is one, two, three, four, five. There's the one number followed by five zeros.
527. R2 But yeah, but you're still not answering my question. (Starts writing) It was this, this was just zero, two one, two two, two to the third, two to the fourth, two to the fifth, and so you just said that was 32. Yeah, I wanna know how you get 32 window combinations out of the five.
528. 1:06:14 Bobby This equals one, right (pointing at two to the zero), this equals two (pointing at two to the first), 4, 8, 16, 32 (keeps pointing at the corresponding powers of two). Cause two to the fifth is 32.
529. R2 What does that have to do with the windows?
530. Bobby Well, because, we made this binary sheet. I think I still have it but I don't know where it is, though. And like it's based on numbers two, 1, 2, 4, 8, it keeps on doubling. Like 16, 32, 64.
531. R2 Ok so two to the zero is one (writes this). Two to the one is two (Bobby joins in). Two to the second is four. Is this what you're telling me?
532. Bobby Yes.
533. R2 Two to the third, (Bobby joins in again) eight. Two to the fourth, sixteen. Two to the fifth is thirty-two. Ok. What does that have to do with the windows?
534. Bobby Well then you can just do two to the fifth. And you can see the last number's two to the fifth, right? Like see this two, if this was just the only number, you know the answer would be one. Cause that's the only number. If there's these two, then you know the answer would be two.
535. R2 Oh, you mean if it was this (circles two to the zero equals one), there would be one combination?
536. Bobby Yeah.
537. R2 And if there was this.. (pointing at two to the first equals two)
538. Bobby There would be two combinations.
539. R2 Ok, so there would be two combinations.
540. Bobby And then four combinations (pointing at two to the second equals four), 8, 16, 32, 64.
541. 1:07:36 R2 But I need to see them. I don't understand them.
542. Bobby Well like, (inaudible)
543. R2 If there was one, if there was one candle in the window, how many, and what you had red, and you had green, how many ways could you have your window candles?

544. Bobby Two.
545. R2 But you just told me one.
546. Bobby (Bobby pulls out sheet of paper from his backpack) See this like thing shows the binary system from like 1 to 64. And it shows that like, it goes up by, like 16 and 32, there's sixteen numbers here but then you can switch them around. You can do 011 if you were doing...
547. R2 Ok, help me understand this. Is that the way you do this?
548. Bobby Yeah.
549. R2 Ok, help me to understand what you're saying.
550. Bobby Well we found the numbers between 16 and 32, right? There's 16. But since...
551. R2 Wait a minute, 16? And 32? Or 16 up to 31? Which one?
552. Bobby 16 to 32. So there's 16 in between.
553. 1:08:33 R2 Show me the 16.
554. Bobby Right here, there's 16.
555. R2 Count them. Just because I'm...
556. Bobby 1 (as he points to the binary number), 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16.
557. R2 Ok, so there's 16 starting here and going down to here.
558. Bobby Yes.
559. R2 That have the five digits. Ok.
560. Bobby And then 1 can equal gold (write this down). 0 can equal red. Then after you do all of them, right, then, 1 can equal red, 0 can equal gold.
561. R2 Oohh. So it doesn't have anything to do with that.
562. Bobby No, you can just cross that out or something.
563. R2 Oohh. Because all along, I thought you were going to be counting up in the binary starting with one.
564. Bobby It's just between these last two numbers and you times them by two because. But if there's three colors, then there'd be ummm, if there's three different colors then there'd be 16, 32, uhh 48. So then there'd be 48 combinations if there's three.
565. R2 Ohh, help me with that.
566. Bobby Well whatever's in between the last two numbers, like 16, you multiply the colors or how many different like things you have. Like if there's four different colors, 16 times 4. If there's ten different colors, 16 times 10.
567. R2 Ok, what about, what about if you had, had only, only four candles in the window.
568. Bobby Then there'd be...
569. R2 For two colors what would it be?
570. Bobby 8. 16. Because then there's 8 and then you can switch it around.
571. R2 Ok. What if there were three candles.
572. Bobby 8.
573. R2 It'd be 4 and 4? (Bobby nods.) What if there were two candles?

574. Bobby 4.
 575. R2 And if there's 1?
 576. Bobby 2.
 577. R2 I see. Now let me ask you another thing. You were talking about three colors.
 578. 1:10:24 Bobby If there's three colors..
 579. R2 If there was three colors and you only had one candle, how many would there be?
 580. Bobby 3.
 581. R2 And if there were two candles?
 582. Bobby Uhh, 6.
 583. R2 Show me what they would be. Suppose we had red and green and black. (Starts writing with different colored markers). Is this blue?
 Ok, I'm gonna ask you all to do this. Ok, suppose you had only two candles in your window and you had a red (mumbling) we don't have a red. We have a black, green, and blue. Ok, now you can have two in the window. What are your combinations gonna be?
 584. 1:11:32 Bobby Blue blue.
 585. R2 We gotta get somebody to draw, or make a, you can do whatever (passes the paper and markers to Amy-Lynn). Ok. You just told me there were gonna be 6, is that...
 586. Bobby You're going by two in a window?
 587. R2 Mmhmm.
 588. Bobby Ok. Blue blue. Blue green. Uh..
 589. Amy-Lynn Blue black.
 590. Bobby Yeah, blue black. Green green, green black, so that's five. So then there's green blue. Oh wait you can't do that cause see you already did blue green.
 591. R2 Oh but that's a different combination
 592. Bobby Oh ok, green blue. And then there's black black. Green black. Black green. Black black. So there'd be nine.
 593. R2 So there'd be nine for two, how about for three?
 594. Bobby Oh, I know why it doesn't work. That's base two. But there's three different colors so you have to go to base three. So this would be like you can use 0, 1, or 2. Because, you see, and if you had four different colors it would be base four see cause this is only base two.
 595. R2 Ok. So how many would there be if you had three colors with three in the window?
 596. Bobby So then, you'd have to make a whole new system, like, um, you can use like 1, 1 and 2, uh, I can't draw it though. It's like you can use 1, um 1 and 2, or 1 and 1 cause then you add a new number to the system.
 597. R2 Oh, ok, so here it would be (starts writing)

598. Bobby 3, 0.
 599. R2 Three to the zero?
 600. Bobby Three to the first.
 601. R2 Ok, that was (inaudible)
 602. Bobby Yeah, and then three to the second. Three to the third.
 603. R2 And so forth.
 604. Bobby Yeah
 605. R2 Ok, now tell me what the number that could go under here would be?
 606. Bobby 0, 1, or 2.
 607. R2 So you would have 0 (Bobby joins in), 1 or 2.
 608. Bobby And all of them
 609. R2 Ok and you could have...
 610. Bobby Oh, and then here you could have (points) 0, 1, 2.
 611. R2 You could have (write 1 0) zero, 1 1,
 612. Bobby Yeah, 1 2. It's kinda, it's kinda hard. See then you could just do like, see this is like three to the three, three to the two, three to the...
 613. R2 I'm just figuring out how you count. You'd go 0,1, 2, then you'd go 1 0, 1 1, 1 2..
 614. 1:14:10 Bobby Something like that.
 615. R2 I think that's right. And then what would happen? What would come next?
 616. Bobby Like. You don't start with the zero, you start with the 1.
 617. R2 Oh, you don't have zeros in these numbers?
 618. Bobby No, it doesn't start with a zero.
 619. R2 Why? Can't you have a zero in base two? You mean there's no such thing as zero?
 620. Bobby No, it doesn't start. It always starts off with 1.
 621. R2 Why?
 622. Bobby Cause three to the zero is one, not zero.
 623. R2 Oh, you can't have a zero unless you're in base 10. In our system can't you have a zero?
 624. Amy-Lynn You just can't start with it. Cause like three to the zero is one and that's what you're starting with if you start with the zero you don't have a number that starts like that. 0 1 (writes 0 1 0 0 0) you can't have a number that starts with zero you always have one that starts with one (writes 1 0 0 1 0) something.
 625. R2 Oh, so what you're saying is that only in base 10. If we're counting in base 10 it's 10 to the 0, 10 to the one, 10 squared. Is that what you're saying?
 626. Bobby Yeah.
 627. R2 Ok.
 628. Bobby But then like here, you're still multiplying like three times three times three is twenty-seven. Three times three is nine.
 629. R2 Sure, isn't that the same thing here? Ten times ten is 100. And

- so forth.
630. Bobby Yeah.
631. R2 But you're telling me that there's no such thing as zero. There's no zero in any number system.
632. Bobby You can't start with zero cause look, you can't like... When you count you go 1, 2, 3, 4, 5, 6. And you don't go 0 most of the time because you can't like. 10 times 0 is 1. The other person proved that to use because see, you can't do anything by zero because it equals, when you multiply something by zero you can't do it because look. If you put even a million here, a million times zero is still two (referring to paper from before).
633. 1:15:49 R2 What's that got to do with zero?
634. Bobby Well you can't start with it. Because if you did that...
635. R2 So you can never put a zero just in here to begin with.
636. Bobby Yes.
637. R2 Who says?
638. Bobby Well, see, we tried doing that before, see like...
639. R2 No I'm not saying that this, that three to the zero is going to be zero. I'm saying how many three to the zeros do I have?
640. Bobby Cause see, if you did three over zero, right?
641. R2 No, I understand that.
642. Bobby Yeah, ok.
643. R2 But I thought this was telling me how many of them I have (pointing at the powers of 2).
644. Bobby Well, the other woman, she said that...
645. R2 Dr. Maher?
646. Bobby Yeah. She said that, she said that it's mathematically proven that you have to start with three to the, something to the zero power always equals one because they like proved that. Cause, see, she wrote that.
647. 1:16:45 R2 Oh sure, ok, three to the zero is equal to one, three to the one (Bobby joins in) is equal to three. Three squared is nine. Three to the third is twenty-seven. But what does it mean what you're doing down here?
648. Bobby Well, the number in the binary system, it could be like zero, it can't be zero, but it can be like 0 0 0 1, that would be 1,000 then...
649. R2 Ok, so you're saying the first number you can possibly have is this (1).
650. Bobby Or you could have a 0.
651. R2 Or zero that would be you don't have anything
652. Bobby Or 2.
653. R2 You told me you couldn't have a zero there.
654. Bobby Nooo. This couldn't equal zero, but...
655. R2 Oh, I know that, but you couldn't have zero of...
656. Bobby It can't, no you can't have zero

657. R2 Uhh, Dr. Maher, Dr. Maher, we are really confused over here and we need your expert advice for just a second.
658. 1:17:42 Bobby No, cause look, this is bigger than zero cause you can't start with adding zero and...
659. R2 So binary numbers, you can't, you can't have a zero.
660. Bobby No, you can't start counting with a zero cause, it'd be like 1, 2, then it would be 10, 11, but skip this (pointing at 0), this is the end of a number. Because you could have a zero at the end of a number, cause it would be like 1 0 0 0.
661. R2 So even when you're counting in our own base, with 10 to the zero (Dr. Maher interrupts with "You can't have a zero there?"), you can't have a zero there?
662. R1 You can't have the number 10?
663. R2 You can have the number one zero but you can't have the number zero by itself. You can't have a zero.
664. Bobby You can't count, you can't count with it, cause see, then you have to take it out. Cause like if you take it out, it'd be the same thing as putting the zero in.
665. R1 Ok, let's talk about that separately.
666. R1 (To class) Ok, we're ready to start. And I think we had a request from Michelle to go first. And after Michelle, it's gonna be Sara or Jeff, we'll go in order. (Pause, background chatter.) Ok can we... this is what your assignment is going to be that we're going to want to collect next week so I want you to listen very carefully. We're going to collect it before you go on this wonderful break. Um, not only do I want to know your thinking about it, but I want you to pay attention to how other people are thinking about it so you can tell me what about what they are saying that makes sense or doesn't make sense. Do you understand? So you're going to have to write about your own solution but you're also going to write about what other people say that makes sense or doesn't. And you're also going to have to write about your own process, what didn't make sense, what became sensible, and how it became sensible. Sort of think about how you changed the way of thinking right now. Ok, so it's gonna be like a little essay. And then we'll write a book and we'll publish it and send it out to all the other students in the world who are working on this problem (inaudible).
667. 1:19:56 Michael Are you gonna put our names in it?
668. R1 Absolutely. You want your names in it?
669. Michael Yeah.
670. R1 Ok. So you're going to have this writing assignment to help us. And we're going to begin with this first group. We've combined groups. Is that what's happened here? Michelle and (inaudible)?
671. Michelle (Michelle presenting to the class using an overhead) This is like

- the, um, garage door opener thing but I have to explain this first to explain the other thing. Ok. We noticed like a pattern. That there are three choices, high, medium and low. And like there's one switch, shown here. And then so you have three to the power of one, so there's like three choices. Do you understand? Do you understand? (Someone responds yes.) So then like when there's two switches, um, three, then it's like three to the power of two because it's like whatever how many switches, the three always stays the same because that's how many choices you have of high, medium, and low. And then the, the, the, exponent changes when like, the exponent is how many switches there are. So like this with two switches, it's three to the second, which is nine choices. Ok. And then there's like three and then since there's three switches and three choices it's three to the third. So then there's like 27 choices. Ok, so you see the pattern. The base is three because there are three choices, and that never changes. The number of switches is the exponent so to get the exponent you use three to the power of nine. The answer is nine, one, well 19,683. Do you understand that? Cause you have to understand that to...
672. 1:21:37 R1 Um, can you move back a little bit, put that back up, Michelle.
673. Michelle Who me?
674. R1 Yeah, put your overhead back up, please.
675. Michelle Oh.
676. R1 Thank you. Now, now tell me what your exponent represents in this, (Michelle begins to answer) and what your base represents, and what your answer represents. But very slowly, and very quietly so we can all process what you're saying, very slowly.
677. Michelle Ok. The base, this number, well yeah, the base which is the bottom number, but you know that though, the base is three and it will always be three because there's three choices, so yeah...
678. R1 Ok, so, let's write that somewhere on another view graph. You're saying that the base is the number of choices? Is that what I heard you say?
679. Michelle Yeah, the number of choices. Like how many there are.
680. R1 Oh, ok, three choices. Ok I see that now.
681. Michelle The three choices of high, medium, and low.
682. R1 And that's what the three is. Ok, I see it now.
683. Michelle And then the one up here is one because that's how many switches there are. The exponent is how many switches there are.
684. R1 I don't understand. Isn't one of those threes a switch? What's a switch and what's the three? The language is getting me mixed up.
685. Michelle You know like this thing, one little thing like here, the box with high, medium, and low in it, like the switch?

686. R1 Maybe we need to agree on some language.
687. Michelle Like the switch, with high, medium, and low on it. The little thing.
688. R1 Ok, so that one might be, can we call it the setting?
689. Michelle Ok, the setting. The number of settings is the exponent.
690. R1 That's where I'm confused. You just told me there were three settings and now you said the number of settings is the exponent.
691. Michelle There's like...
692. R1 Do you see where I'm confused, Michelle?
693. 1:23:37 Michelle No, cause like the setting is like the whole box, the whole entire thing, the whole thing.
694. R1 Ok, so can we call that a position?
695. Michelle K, position.
696. R1 You have one position, then you have another position, then you have another position, you're thinking of what like Mr. Miller showed us, that garage door opener.
697. Michelle Yeah.
698. R1 And I saw nine positions and I saw three settings for each position.
699. Michelle Yeah.
700. R1 Can we agree on that language? Is that language clear to everybody? That there are nine positions, because that kind of works with candles too, doesn't it? Positions for candles, positions, and then there are settings. I want to be sure that the language makes sense. And I want to agree on it. Is that clear to you? You can stay by the way. You don't have to leave.
701. Michelle The sections, the sections are those little boxes and the settings are the number of choices you have inside the little settings, ok. Settings. The settings are the things you have, are the choices you have outside like the boxes or the sections, ok? So then like, in like this one, you have like two sections so like and since there's like three um, there's like three settings, ok I keep on forgetting what it's called. There's three settings so like that's the base. And since there's two um since there's two sections, then that's the number of exponents. So let's say three to the power of two, which is nine. So, like, that's how we got the answer. So like when there's, the three is always the base number because there's high, medium, and low in this problem like the whole time. So, like there's three settings so like that's the settings, the number of settings is like the exponent. And since there's three, that's like (inaudible). Since there's three settings in each thing, the thing is the to get the answer you use, wait hold on, you use three to the ninth. And three is the little like high, medium, and low, the settings, and then the nine is the number of switches. So like, it's three to the ninth and like we used a calculator and that's 19,683.

702. 1:26:13 R1 Ok. Any comments, questions? (Pause) How do you follow what they presented? Does anyone have a question? Anyone disagree? How many of you agree? (Some students raise hands including Bobby.) How many of you disagree? How many of you aren't sure? (Some students raise hands.) There's a couple of us that aren't sure. We have three aren't sures.
703. Michelle Pay attention. Sara you have to pay attention, ok? The, the, the, it's not the switch anymore it's the section. It's the section. So like the little box, Sara and Sara and Michelle, is the section so then and there's three choices in each section, the settings. Do you follow me? (Sara nods.) Ok. And the three settings is the base number. Do you understand that? (She nods again.) And then the the the the exponent, the number up there, is like the box or the section. Do you understand? (She nods again.) So like in this one, when there's two switches, the two switches is the exponent, so there's two, so the exponent is two and there's three choices still so it's three to the second which is nine choices. Ok? (Sara nods again.) And then it's the same thing with the third one. Do you understand? (She nods.) So then you use three, since it's since there's, for the final one cause like there's nine switches and there's three choices high, medium, and low. Then you use like the high, medium, and low as the base. And the nine as the how many switches. So you just times that and we did that on the calculator.
704. 1:28:18 R1 (Someone calls on the intercom for the classroom teacher.) She's not here.
705. Michelle And so we got 19,683. Do you understand now? Sara, do you understand? (She nods.) Do you understand Michelle? Ok. We had to show that to you first to understand this stuff. Now this is with the candles. Ok. This is like the, that's a real lousy drawing but that's like the base and those little blue ones, those blue and green things, those are the candles. Ok. (Whispers to her partner: Do you want to explain this or should I?) Ok. So there's, there are five candles, there are five candles, which is the number, wait first, the base number, ok, the base number is the number of colors that you have to choose from, that you can change them, ok? And then the number of candles that you have is the exponent. Ok? So, you just like times that, you know, two to the fifth power, which is 32, which is like 32 choices to switch the candles around. So then, like to put it to start from the beginning so that you can understand it, when there's one candle, like you know that there's two choices when there's two colors. There's green and, green and blue here. So there's two choices, like green and blue. So there's one candle which stands for the exponent and there's two colors to choose from. So there's, so that's the base. So then, two to the first power equals

- two which is how many choices you have to use on the candles. So then, in this one, there's two candles in one little thing. There's two colors, there's two colors so that's, so the base and there's two candles so that's the exponent. So it's really two to the second which is four. Does everybody follow? Ok. And then, so then like back to here since I explained this, this. There's two colors, green and blue, so then there's five candles, which is um, so the five candles is like the exponent and the number of colors is the base so it's two to the fifth, which is 32 choices. And then the pattern is the same as the garage door pattern. There are two colors so that the base, wait. The pattern is the same as the garage door pattern. There are two colors so that is the base number. The number of the candles is the exponents number. So the answer is 32 choices. Does everybody understand? (Someone says "I understand.") Sara, do you understand? (Sara nods.) Are there any questions or anything? Do I have to keep this up?
706. 1:31:11 R1 Ok. Well thank you very much. Does anybody have a question? We'll have a chance to have general questions in a couple minutes. Does anybody want to volunteer to go up next? (Chatter amongst students.) Michael, are you going up next?
707. Michael Yeah, I'll be up in a second.
708. (Group gets set up, students chat in background.)
709. R1 Ok, let's go.
710. Michael Ok, this, this is, don't look at this right now. Just look at this. (Pointing toward his transparency.) (Background chatter.)
711. R1 Can you move that down a little bit. That's fine.
712. Michael Is this good?
713. R1 Yes.
714. 1:32:21 Michael Ok. Ok, from 16, that is five digits. That's cause if you were to convert, if you were to have one, what color would you want one to be? Ok, gold. (Class laughs.) And zero will be red, alright red. Ok, so from 16 to 31 is all the ones that start with gold. You see that, right? But where's the ones that start with red? They are right here (class laughs again) because, what? Stop laughing. Well, see, if you were to put zeros in front of each one to make them have five digits, (pause)
715. Jeff? You'd get the ones on the other side.
716. Michael Yeah, there'd be, cause since you're dealing with numbers, if you were to put a zero in front of a ten, it'd still equal ten, you'd just take the zero off, but..
717. Jeff? (Someone is confused.) He's saying in the regular decimal system.
718. Michael Yeah see, watch, in the regular decimal system. This is a ten, right? That's a ten, right? Ok, if you were to put a zero in front of it, it'd still equal ten, you'd just take the zero off. The zero

- doesn't matter. So same thing you do with here, when you have when you have three zeros and two ones, you take the zeros off cause the zeros have like no value. Like, they don't mean anything. But, but, if you're dealing with candles, they do mean something. They, they mean that there's, there's four right here there's four red ones and one green one, so like when you're dealing with numbers you don't you eliminate the zero in front . So that means there's only 31 possibilities. What happened to all the red ones? They're right here. Zero is a number so all you have to do is add all, put all zeros and that's and then you got all your 32 combinations. This is not a combination. This is only if you had, had six candles. Cause it starts from 0 to 31, that's 32 numbers and that's, these are all the combinations. If you were to switch the one to a green and the zero to a red. Does anyone not understand? (class laughs)
719. 1:34:45 R1 They should all be able to write that up for me for next week. So this table may have to end up being consultants.
720. Jeff Consultants?
721. Michael Hey, Jeff, you can come up now.
722. Jeff Ok, this is my section. (Chatter.) What my section basically explains is, wait, is that most of the people when they started this, didn't realize the mathematical background to it. See, if you read it, it basically says that. Because what happened was everyone figured, well, the teacher ain't here, the easiest way is to go to the binary because Robert thought of it, go to the binary system. Realize that everything is ok, that there's 32 different combinations, they're all different, so it must be working, and that's how the whole problem started. Because everybody just didn't think of the mathematical background to the whole situation. And that's how we started with the binary system.
723. Class-mate Can we read your part?
724. Jeff Yes, you can read it. It's right there.
725. Michael I'll read it. After extensive studying, the group consisted of Jeff..
726. Jeff Consisting.
727. Michael Consisting of Jeff, Michael, Sara, and (Jeff joins in too) Michelle, found that nobody who did this problem
728. R1 (Over the intercom, someone calls for the teacher.) She's not here, she's in her next class.
729. Michael Nobody
730. R1 Start the whole thing again, please.
731. 1:36:01 Jeff I'll read it. After extensive studying, the group consisting of Jeff, Michael, Sara, and Michelle found that nobody who did this problem originally looked at the mathematical aspects of it. The only reason they are showing it now is because, you, you brought that to our attention. He put that in.

732. Michael You told me to.
733. Jeff You brought that to our attention and cause if that wouldn't have happened, we would have never have gotten this far to the problem.
734. Michael Going back to my thing, does anyone really understand this? Raise your hand if you understand. (Dr. Maher is the first to raise her hand, followed by many of the students.) Good. Thank you. Who doesn't understand?
735. Class-mate Mike, we understand it.
736. Michael Ok.
737. R1 I can't wait to see your written essays for next week.
738. Michael Ok, thank you very, very much.
739. (Background noise as Dr. Maher encourages the next group and all its members to make their way to the front of the classroom.)
740. 1:37:00 Romina Ours was nothing like Michael's.
741. Michael Mine was better, right?
742. Stephanie Ok, the thing about this is, we didn't use the number 32. Everybody thought that in the binary system that we used 32. That like 32 was one of the number's in the binary system we used. But we didn't. Thirty-two was just our answer. What we did was we used the number 16 to 31 and it says so right there.
743. Romina We used 16 through 31. And those are the only numbers that contain 5 spaces. Cause we have 5 candles.
744. Stephanie (Whispers) I was just gonna say that. Anyways, so what happened is that we came up with 16 combinations for the numbers in the binary system but then we reversed them. So that what it was was like, it was like all the numbers with red standing for 1, so you got like red green green green green; red green green green red; red green green green no – red green green red; red green red red red, you know. And then, but after we got, after we got all the green, we reversed it so it was green red red red red; green red red red green, you know what I mean?
745. Michelle How were you sure you had them all?
746. Michael Yeah, how were you sure?
747. Class (Whole class talks loudly.)
748. Stephanie Binary counts in order.
749. Jeff Yeah, but that's not a suitable answer cause we had to say that.
750. Stephanie In our system, (keeps getting interrupted) in our system, you know how our numbers go 1, 2, 3, 4, 5, 6, 7, 8, 9? Alright, well their numbers, that's all the numbers except for 0, but I'm not counting 0 because I don't feel like it. Um, those are all the numbers in their system
751. Michelle He just said don't call on me.
752. Stephanie Those are all the numbers in the system, (gets interrupted again) those are all the numbers in the system that have one digit in them. And those are all the numbers in the binary system that

- have 5 digits in them. If you want to find another one, you can sit there and chart the whole thing out, make new ones, and look like an idiot because they match the old ones.
753. Michelle But what if you reversed them wrong and there were more but you reversed them.
754. Student No we already did it.
755. Jeff Don't flip the table like the binary thing? (inaudible)
756. 1:39:16 Michelle But what if you reversed them wrong?
757. Student See we didn't reverse them wrong. This one we messed up so I put them down here.
758. Stephanie Alright, anyways, here's how we'll prove it. (Whispers to group member: do you want to explain the number of positions?) Our group decided x is the number of positions. Like candles, we had 5 candles. Then $x - 1$ to the power of 2 is the answer. Do you understand what I'm saying?
759. R1 $x - 1$ to the power of 2?
760. Stephanie Yeah.
761. R1 I'm confused.
762. Stephanie Alright. Because if our problem is 5, then it's 2 to the – it's four to the second power will give you the answer, times 2. Because it will only give you the answer for the one color in front. It won't give you the answer for both colors in front or something?
763. R1 You need to think about what you're saying. Maybe write it down because I'm not really quite understanding what you're saying.
764. Stephanie Well, what it is, it's mess up, because the number is because the 0 in the in the um, ok, 2 to the 0 messes everything up because... what happens is... This all goes back to what we were saying in the beginning. Because 1 is 2 to the 0 and 2 is 2 to the 1. And 3 is 2 to the...like the numbers of positions you can get with 3 is 2 to the second because 0. So, the number of combinations you can get is x is like if you number is x , minus 1 to the second power, times 2.
765. R1 That's not what you have written.
766. Student I don't understand.
767. Michelle I understand this now.
768. Michael I don't comprehend.
769. R1 You don't have $x - 1$ to the second power written there.
770. Student No, we have it's x , x is the number of positions,
771. Student It's 2 to the $x - 1$ power. Two to the x minus 1 power.
772. R1 You see the difference Steph to what you were saying? You were saying it backwards.
773. Stephanie Ok.
774. R1 You were saying the x minus 1 was to the second power.
775. Michael It's 2 to the x minus 1 power.
776. R1 K, but that's not what you have here, and I'm not so sure people

- are following your chart and how you got your chart and how it relates to the binary system. Is someone going to be able to explain the chart?
777. Ankur The first one says number of positions, that's like the number of candles. That's what it means. Then it says 2 to the 0 power equals 1. That's the combinations. If you multiply it by 2 then you'll have all the combinations. 1 times 2 is 2. For one candle there's 2 combinations.
778. R2 Say that again for me one more time, Ankur?
779. 1:42:39 Ankur Number of positions stands for the candles. If there's 1 candle, that's 2 to the 0 power which equals 1. And if you multiply it by 2 because you can have red or green, then you'll have the number of combinations, 2.
780. R2 Ok. And so if there were 2?
781. Ankur One candle is 2, there's 2 combinations, red or green.
782. R2 (To another student) Please don't do that, I'm trying to listen. Ok, so tell me what would happen if it's 2. Help me understand what you're saying.
783. Ankur It's 2 to the first power which equals 2, then you can change the first one, which equals 2 times 2, that's 4. So if you have 2 candles there's 4 different...
784. R2 So what you're saying is something about the number of positions and then you took one away from that? Is that what you're saying? From the exponent?
785. Ankur Yes because when you have, yeah because we took 1 away because 2 to the 0 messes everything up.
786. R2 Ok. So you had to take away an exponent.
787. Ankur One. Yeah.
788. R2 Ok, so it's the number of positions minus 1 times to that...
789. Ankur Times
790. R2 And then you double it?
791. R1 Why do you double it?
792. Ankur Because when you start, cause you can start, it starts, binary counts it always starts with 1. So 1 means, 1 stands for red. The first one's always going to be red. So double it because you can change the first one to green or gold instead.
793. R2 So this doubling has to do with switching the colors of the candles?
794. Ankur Mhm. Yes.
795. R1 And you're going to be able to tell us why you get one set of candles and why you had to switch, you don't have that written in your overhead but you have it in your notes I saw?
796. Ankur Yeah, somewhere.
797. R1 Ok. I'm afraid we have to stop because we've run out of time. We have about 3 minutes, can you do it in 3 minutes, Bobby?
798. Bobby Yes.

799.		R1	Ok, c'mon guys.
800.	1:45:04	Bobby	Ok, well the thing with 2 to the 0, you can take it out
801.		R1	(Class is very noisy.) Ok, can we all listen, please.
802.		Bobby	Well, the thing with 2 to the 0, you really can't take it out, cause if you take a number out of the binary system, it would affect all the numbers after it. If you like took out 2 to the fourth, all the like 2 to the fifth, 2 to the sixth, 2 to the seventh, and such on would be messed up cause like then you'd have to put 0's in there. And if you took out 2 to the 0, you'd mess up the whole binary system by putting in all 0's instead of 2 to the first...
803.		R1	So, your conclusion is therefore?
804.		Bobby	You can't take out any numbers out of the binary system unless you mess it up.
805.		R2	Then how do you get it to be 32?
806.		Michael	Wait, didn't you say you're going to take the 0 out? Now you're...
807.		R1	He changed his mind.
808.		Bobby	You can't. Cause it'd mess up the whole thing.
809.		Student	You were wrong, you wouldn't have noticed. (inaudible)
810.		R1	Who said "To be great is to contradict yourself"? Was it Henry David Thoreau. It's ok to change your mind. Emerson.
811.		R2	Most real mathematicians do. So did you get up to 32 somehow also or?
812.		Bobby	Yes.
813.		R2	How?
814.	1:46:13	Bobby	That's like you count, cause now with the 0, it counts as 1 so it makes you get to um like 32 without using as many numbers. For the candle thing, it's the numbers in between the last two numbers that you're counting. And you have to times it by 2. It would be 16 and 32 and in between those numbers and you times it by 2.
815.		R2	Which is a little bit like before what Ankur was saying?
816.		Bobby	Yeah.
817.		R2	Are you agreeing with him? Do you want to show us that?
818.		Bobby	Just forget the top, it's messed – it's from the other problem. There are 16 numbers between 16 and 31 and like after you get to all of them, you have to like switch them to different colors.
819.		R2	Dr Maher?
820.		R1	I think that we have to stop.
821.		R2	Ok, everyone knows that your assignment is to write up as carefully as you can the theories that you have just presented.
822.		R1	I want everyone to write their own because I want one for everybody. With your name on it. Every person. You can put any supporting materials. We'll get your papers back to your teacher. So that you'll have your own packet.
823.		Stephanie	So when is this due?

824. R1 Before you go on vacation. Ok?

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Kalsi, A. & O'Leary, D. (2006). Fast algorithm for structured least square and total least square problems. *Journal of Research for National Institute of Standards and Technology*, 111,113-119.

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