SLIDING MODE CONTROL OF DISCRETE-TIME WEAKLY COUPLED SYSTEMS

by

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ABSTRACT OF THE THESIS

Sliding Mode Control of Discrete-time Weakly Coupled Systems

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Sliding mode control is a form of variable structure control which is a powerful tool to cope with external disturbances and uncertainty. There are many applications of sliding mode control of weakly coupled system to absorption columns, catalytic crackers, chemical plants, chemical reactors, helicopters, satellites, flexible beams, cold-rolling mills, power systems, electrical circuits, computer/communication networks, etc. In this thesis, the problem of sliding mode control for systems, which are composed of two weakly coupled subsystems, is addressed.

This thesis presents several methods to apply sliding mode control to linear discrete-time weakly-coupled systems and different approaches to decouple the sub-systems. The application of Utkin and Young's sliding mode control method on discrete-time weakly-coupled systems is studied in detail which is then compared with other control algorithms while emphasising the importance of the decoupling technique in each case. It also presents the possibility of integrating two or more control strategies for a single system; one for each sub-system, depending upon the respective requirements and constraints.

In this thesis, the effectiveness of the proposed methods is demonstrated through theory and simulation results.

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Table of Contents

Abstract ii

Acknowledgements iii

1. Discrete-Time Sliding Mode Control 1

1.1. Introduction 1

1.1.1. Constructing Sliding Surfaces of MIMO System 3

1.2. Sampled Data Control Stratergies 3

1.2.1. Multi-step Prediction Based Discrete-time Sliding Mode Control Algorithm (MSMPM) 8

1.2.2. Design of MSMPM 9

1.3. The Invariance Condition for Linear Systems with Exogenous Disturbances 11

2. Discrete-Time Weakly Coupled Systems 13

2.1. Introduction 13

2.2. Decoupling Transformation of Gajic and Shen 14

2.2.1. Decoupling Transformation of Qureshi 15

2.2.2. Decoupling Transformation for N Weakly Coupled Subsystems . 16

3. Sliding Mode Control of Linear Discrete-Time Weakly Coupled Systems 19

3.1. Introduction 19

3.2. Traditional Approach 19

3.2.1. Case Study 20

3.2.2. Observations 24

3.3. Decoupling Approach 25

3.3.1. Case Study 28
3.3.2. Observations 32
3.3.3. Analysis of Results 35
3.3.4. Other Control Methods 38
1. Bartolini et al's Control Algorithm 38
2. Relay Control Algorithm 47
3. Gao's Control Algorithm 53
3.4. Composite Control 56
3.4.1. Example 56

4. Conclusions and Future Work 59
4.1. Conclusions 59
4.2. Future Work 60

References 61
Vita 66

Chapter 1

Discrete-Time Sliding Mode Control

1.1 Introduction

Sliding mode control has been recognized as a robust control approach, which yields to reject disturbances and system uncertainties. The design of sliding mode control is achieved in two steps. Firstly, a sliding surface is described which ensures the system to remain on the surface after reaching it from any initial conditions in a finite time. Secondly, discontinuous control is designed to render a sliding mode.

Consider the following single input linear discrete-time system

$$\begin{bmatrix} x_{k+1}^1 \\ x_{k+1}^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \tag{1.1}$$

If $x_k^2 = \lambda x_k^1$, where $\mid \lambda \mid < 1$, then x_k^1 and x_k^2 are asymptotically stable because (1.1) yields $x_{k+1}^1 = \lambda x_k^1$. Define a line as follows

$$s_k = x_k^2 + \lambda x_k^1, \ \mid \lambda \mid < 1 \tag{1.2}$$

The control objectives are to design u_k to ensure that the system reaches the sliding line from any initial condition in a finite time and stay on the line after reaching it. The conditions to achieve these objectives are called reaching and sliding conditions. The reaching condition provides that the system state reaches the sliding surface in a finite time, whereas the sliding condition facilitates that the system state slides on the sliding line towards the origin.

The general reaching condition is given by

$$\|s_{k+1}\| < \|s_k\|$$

The general form of discontinuos control is given by

$$u_i(k) = \begin{cases} u_i^+(k), \text{ if } \ s_i(k) > 0 \\ u_i^-(k), \text{ if } \ s_i(k) < 0 \end{cases}, \forall i$$

The reaching condition described by Utkin and Young is (Utkin, 1977; Young, 1978)

$$s_{k+1} = -\delta \text{sgn}(s_k), \ \delta > 0 \tag{1.3}$$

where the signum function $\text{sgn}(s_k)$ is defined as follows

$$\text{sgn}(s_k) = \begin{cases} +1 \text{ if } \ s_k > 0 \\ \ \ 0 \text{ if } \ s_k = 0 \\ -1 \text{ if } \ s_k < 0 \end{cases} \tag{1.4}$$

Two different definitions of discrete-time sliding mode have been proposed for discrete-time systems Young *et al.* (1999). Both share the common base of using the concept of equivalent control. Drakunov and Utkin (1990) proposed a $u_k^{eq} = u(kT)$ which is the solution of

$$s_{k+1} = 0 \tag{1.5}$$

On the other hand, u_k^{eq} is defined in Furuta (1990) as the solution of

$$\Delta s_k = s_{k+1} - s_k = 0 \tag{1.6}$$

Note that (1.5) implies (1.6), however the converse is not true. Herein, the second definition shall be used since the magnitude of input required to achieve $s_{k+1} = 0$ would be considerably higher. Such high inputs might be hard to supply and, in the case of weakly-coupled systems, could cause instability due to the existence of residual external-input coupling even after decoupling techniques are applied (as will be studied in the following chapter).

From (1.2), we have

$$\Delta s_k = (\alpha - 1)x_k^1 + (\beta + \lambda)x_k^2, \ \mid \lambda \mid < 1 \tag{1.7}$$

For $\Delta s_k = 0$, it follows

$$u_{keq} = -(\alpha - 1)x_k^1 - (\beta + \lambda)x_k^2, \ \mid \lambda \mid < 1 \tag{1.8}$$

Therefore, the control law to satisfy the reaching condition (1.3) is

$$u_k = u_{keq} - \delta \mathrm{sgn}(s_k),\ \ \delta > 0 \tag{1.9}$$

1.1.1 Constructing Sliding Surfaces of MIMO System

1.2 Sampled Data Control Stratergies

If a discrete-time system without disturbances is modelled in the tracking error space (where the tracking error is given by $e(k) = r(k) - x(k)$) the resultant state-space structure is as follows

$$e(k+1) = A_d x(k) - b_d u(k) \tag{1.10}$$

$$A_d = e^{AT}; b_d = \int_0^T e^{A\tau} b d\tau;$$

where A, b are the state space matrices of the corresponding continuous time system.
All known state space based sliding mode design methods are special cases of the following general reaching law algorithm (Milosavljević *et al.*, 2006):

$$\Delta g(k) = g(k+1) - g(k) = -Tf(e(k), g(k)) \tag{1.11}$$

where T is the sampling period and

$$g(k) = c_d^T e(k) \tag{1.12}$$

The vector c_d has to be selected so that the motion has the desired dynamics.
LHS of equation (1.11) represents the first difference of the sliding function and RHS is a nonlinear function, f, of the tracking error and the sliding function. The value of control is determined by equaling the first difference of $g(k)$ with RHS of (1.11). By assuming $c_d^T b_d = 1$, without loss of generality, the following value of the control input is obtained

$$u(k) = u_{eq}(k) - g(k) + Tf(\bullet); \tag{1.13}$$

$$u_{eq}(k) = c_d^T A_d e(k).$$

Depending on the selection of the function $f(\bullet)$, various control algorithm were proposed (Milosavljević, 1985; Golo and Milosavljević, 2000; Milosavljević, 1982; Bučevać, 1985;

Salihbegović, 1985; Furuta, 1990; Chan, 1991; Pots and Yu, 1990; Yu and Pots, 1992; Gao *et al.*, 1995; Bartolini *et al.*, 1995; Bartoszewitz, 1998; Furuta and Pan, 2000; Milosavljević *et al.*, 2005). The most widely known are listed below.

1. Quasi-relay Control Algorithm

The algorithm is defined by the following representation (Milosavljević, 1985, 1982)

$$Tf(\bullet) = -u_{eq} + g(k) + \Sigma_{i=1}^{n-1} w_i \mid e_i(k) \mid \mathrm{sgn}(g(k));\ w > 0. \tag{1.14}$$

$$u(k) = \Sigma_{i=1}^{n-1} w_i \mid e_i(k) \mid \mathrm{sgn}(g(k)). \tag{1.15}$$

where w_i are the weights of the errors. The main feature of this control is the modulated amplitude of control. The control signal decays when the state approaches to zero and then the chattering disappears.

2. Relay Control Algorithm

In this algorithm, the function $f(\bullet)$ is given by

$$Tf(\bullet) = \delta \mathrm{sgn}(g(k)) + g(k) + u_{eq}(k);\ \delta > 0. \tag{1.16}$$

and the control input is

$$u(k) = \delta \mathrm{sgn}(g(k)). \tag{1.17}$$

This algorithm's main feature is the control signal switching between two large constant values. The disadvantage is significant chattering even in the steady state.

3. Bučevac-Salihbegović Control Algorithm

The function $f(\bullet)$ satisfies the following criterion (Bučevać, 1985; Salihbegović, 1985)

$$Tf(\bullet) = g(k) \tag{1.18}$$

and the equivalent control law u_{eq}, is given by

$$u(k) = u_{eq}(k),\ \forall g(k) \tag{1.19}$$

Regardless of the value of $g(k)$, control (1.2) will bring the system onto the sliding hypersurface (SHS) in one step if $g(k) \neq 0$ and then maintain the system state on the SHS $g(k)$=0 by the action of equivalent control. If the control input in the reaching phase is too high it may be moderated in the following way

$$u(k) = u_{eq}(k) - \alpha g(k),\ \alpha > 0 \tag{1.20}$$

4. Furuta's Control Algorithm

This algorithm is defined as follows (Furuta, 1990; Chan, 1991)

$$Tf(\bullet) = \psi^T e(k)\psi_0 g(k), \tag{1.21}$$

$$u(k) = u_{eq} - (1 - \psi_0)g(k) - \psi^T e(k), \tag{1.22}$$

$$\psi_i = \begin{cases} \psi \text{ if } e_i(k)g(k) < -\delta_i \\ 0 \text{ if } -\delta_i \ leq e_i(k)g(k) \leq \delta_i \\ -\psi \text{ if } e_i(k)g(k) > \delta_i \end{cases} \tag{1.23}$$

where $0 < \psi_0 < 1$

$$\delta_i = 0.5\psi(1 - \psi_0)^{-1} \mid e_i(k) \mid \sigma_{j=1}^n \mid e_j(k) \mid \tag{1.24}$$

Here the Quasi-Sliding Mode (QSM) stays within a domain that does not coincide with the sliding hyperplane but is very close to it. But, this method works only in the single input case.

5. Gao's Control Algorithm

In this algorithm the function $f(\bullet)$ satisfies

$$f(\bullet) = qg(k) + p\text{sgn}(g(k)) \tag{1.25}$$

where $q, p, 1 - q > 0$ (Gao *et al.*, 1995)

$$u(k) = [u_{eq}(k) - (1 - qT)g(k) + pT\text{sgn}(g(k))] \tag{1.26}$$

The choice of q and p defines reaching dynamics and the width of the QSM domain, which is $2\Delta = 2pT/(1 - qT)$ in a nominal system. For $p = 0$, $q = 1/T$ the width of the domain is zero and the algorithm reduces to (1.20).

6. Bartolini *etal*'s Control Algorithm

In this algorithm, the function $f(\bullet)$ is defined as follows (Bartolini *et al.*, 1995)

$$Tf(\bullet) = \begin{cases} \alpha\text{sgn}(u_{eq}(k)) - u_{eq}(k) + g(k) \text{ if } |u_{eq}| > \alpha \\ g(k) \text{ if } |u_{eq}(k)| \leq \alpha \end{cases} \tag{1.27}$$

and the control input is given by

$$u(k) = \begin{cases} \alpha \mathrm{sgn}(u_{eq}(k)) \text{ if } |u_{eq}| > \alpha \\ u_{eq}(k) \text{ if } |u_{eq}| \leq \alpha \end{cases} \tag{1.28}$$

This control strategy suitably takes into account the control magnitude limitation by the actuator in real systems. The relay component is active in the reaching regime, whereas u_{eq} acts in the QSM.

7. Bartoszewicz's Control Algorithm

Bartoszewicz introduced a desired hypersurface g_d (Bartoszewitz, 1998), which defines the control magnitude in the reaching phase as follows

$$Tf(\bullet) = g(k) - \lambda g_d(k+1), \tag{1.29}$$

where originally in (Bartoszewitz, 1998) λ=1, so that control is

$$u(k) = u_{eq} - g_d(k+1) \tag{1.30}$$

where

$$g_d(k) = \begin{cases} (1 - k^*/k)g(k) \text{ if } |g(k)| > 2\delta_d \\ 0 \text{ if } |g(k)| \leq 2\delta_d \end{cases} \tag{1.31}$$

where k^* is the desired number of sampling intervals needed to reach the SHS and δ_d is a function of lower and upper limits of disturbance. In order to improve the accuracy, Bartoszewicz also modifies the control strategy by introducing an additional integral action. The control input then becomes (h=1).

$$u(k) = u_{eq}(k) - \lambda g_d(k+1) + h\sigma_{1=0}^{k-1}(g(i) - g_d(i)). \tag{1.32}$$

8. Golo-Milosavljević Algorithm

This method proposes the following f($\bullet$) function (Golo and Milosavljević, 2000; Milosavljević *et al.*, 2005)

$$Tf(\bullet) = \begin{cases} \epsilon \mathrm{sgn}(g(k)) \text{ if } |g(k)| > \psi \\ g(k) \text{ if } |g(k)| \leq \psi \end{cases} \tag{1.33}$$

and the respective control law is given by

$$u(k) = \begin{cases} u_{eq} - g(k) + \psi \mathrm{sgn}(g(k)) \text{ if } |g(k)| > \psi \\ u_{eq} \text{ if } |g(k)| \leq \psi \end{cases} \tag{1.34}$$

9. Utkin and Young's Method

Consider a discrete-time linear system given by

$$x_{k+1} = Ax_k + Bu_k \tag{1.35}$$

where $x_k \in \mathbf{R}^n$, $u_k \in \mathbf{R}^m$, and A, B are constant matrices of appropriate dimensions, and B has full rank. There exists a similarity transformation defined by Utkin and Young (1978)

$$q_k = Hx_k \tag{1.36}$$

with

$$H = \begin{bmatrix} N & B \end{bmatrix}^T \tag{1.37}$$

and columns of the $n \times (n - m)$ matrix N composed of basis vectors in the null space of B^T, which puts (1.35) into the form

$$q_{k+1} = \bar{A}q_k + \bar{B}u_k \tag{1.38}$$

with $\bar{A} = HAH^{-1}$ and $\bar{B} = HB = \begin{bmatrix} 0 \\ \bar{B}_r \end{bmatrix}$. Equation (1.38) is decomposed as follows

$$\begin{bmatrix} q_{k+1}^1 \\ q_{k+1}^2 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} q_k^1 \\ q_k^2 \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{B}_r \end{bmatrix} u_k \tag{1.39}$$

where $q_k^1 \in \mathbf{R}^{n-m}$, $q_k^2 \in \mathbf{R}^m$, and $\bar{B}_r$ is an $m \times m$ nonsingular matrix.

Equation (1.39) yields

$$q_{k+1}^1 = \bar{A}_{11}q_k^1 + \bar{A}_{12}q_k^2 \tag{1.40}$$

and

$$q_{k+1}^2 = \bar{A}_{21}q_k^1 + \bar{A}_{22}q_k^2 + \bar{B}_r u_k \tag{1.41}$$

q_k^2 is treated as a control input to the system (1.40) and a state feedback gain K, which makes the system stable, is defined by

$$q_k^2 = -Kq_k^1. \tag{1.42}$$

For the system (1.40), Utkin and Young (1978) have shown that $(\bar{A}_{11}, \bar{A}_{12})$ is controllable if and only if (A, B) is controllable (see also Chen, 1999).

On the sliding surface, the system trajectory in the (q_k^1, q_k^2) coordinates is expressed as

$$\begin{bmatrix} K & I_m \end{bmatrix} \begin{bmatrix} q_k^1 \\ q_k^2 \end{bmatrix} = 0 \tag{1.43}$$

or

$$s_k = Gx_k = \begin{bmatrix} K & I_m \end{bmatrix} Hx_k = 0 \tag{1.44}$$

in the original coordinates.

Dote and Hoft (1980) firstly considered Discrete-time Sliding Mode Control (DSMC) and used a discrete-time reaching condition (DRC) to ensure the existence of the sliding mode. The DRC is obtained by slimply substituting the forward difference into the continuous-time reaching condition(CRC). Milosavljević *et al.* (1985), suggested the concept of the quasi-sliding mode, and pointed out that the DRC of (Dote and Hoft, 1980) is only a necessary condition and not a sufficient condition for the existence of such a quasi-sliding mode. Later a modified DRC given in the form of an inequality was presented in (Sarpturk *et al.*, 1987). Although the DRC can guarantee the magnitude of the sliding mode function value to be strictly decreased, to solve the corresponding inequality is uneasy. In (Furuta, 1990), Furuta proposed a DRC by the equivalent form of a Lyapunov-type of CRC, but it is difficult to extend to multi-input systems. In (Gao *et al.*, 1995), desired propertiies of DSMC systems, defined notions of the reaching condition, quasi-sliding mode and quasi-sliding mode band, were specified and used the so-called reaching law to approach DSMC algorithms. Since then, many investigations have been done on the basis of (Gao *et al.*, 1995), see (Xiao *et al.*, 2005; Yao *et al.*, 2001; He *et al.*, 2001; Zhai and Mwu, 2000; Mao *et al.*, 2001; Li, 2004)

1.2.1 Multi-step Prediction Based Discrete-time Sliding Mode Control Algorithm (MSMPM)

By creating a special multi-step sliding mode prediction model (Lingfei and Hongye, 2008) which includes the function of compensation for system parameter perturbations and external disturbances, the future information of the sliding mode can be used. The reachability of the sliding surface can be obtained by making the output value of MSMPM to track the sliding surface closely, even when matched or unmatched uncertainities influence the system; thus strong robustness is maintained.

1.2.2 Design of MSMPM

When an uncertainty does not appear, the nominal discrete-time system is represented by

$$x(k+1) = Ax(k) + bu(k) \tag{1.45}$$

Consider the following sliding mode function

$$s(k) = \sigma^T x(k), \text{where } \sigma^T = [\sigma_1 ... \sigma_n] \neq 0. \tag{1.46}$$

The sliding surface is $S = x|s(x) = 0$. The choice of $\sigma_i(i = 1, ..., n)$ should guarantee stability and dynamic performance of an ideal quasi sliding mode and $\sigma^T \neq 0$. For linear systems, the suitable σ_i can be obtained by eigenvalue placement. The sliding mode value at time k+i can be described as

$$s(k+i) = \sigma^T A^i x(k) + \Sigma_{j=1}^{i} \sigma^T A^{j-1} bu(k+i-j) \tag{1.47}$$

where $i, j, k \in Z$ are time instants. In the presence of uncertainty, the future value of sliding mode will not equal to (1.47). Therefore, we introduce the following sliding mode predictor

$$s(k+i) = \sigma^T A^i x(k) + \Sigma_{j=1}^{i} \sigma^T A^{j-1} bu(k+i-j) + \beta^i \text{sgn}(s(k)) \tag{1.48}$$

where β is a negative constant. The function of term $\beta^i\text{sgn}(s(k))$ is to make compensation for uncertainty. According to the receding horizon optimizaion approach in predictive control strategy, Xi (1993, 2000), in the control vector $U = [u(k), u(k+1), ...u(k+M-1)]^T$ only the first element of U, i.e.,$u(k)$ is transmitted to the process, other elements are not used for control, but serve as initial values for the next round of optimization. In order to decrease calculations, we let $u(k+i) = \alpha u(k+i-1)$, where

$0 < \alpha < 1$. Additionally, (Xi, 1993) points out that the control horizon should not be larger than the prediction horizon N, and $u(k+i) = u(k+i-1)$, $M \leq N$, therefore,

$$u(k+i) = \begin{cases} \alpha u(k+i-1), \text{ if } k+i < M \\ u(M-1), \text{ if } k+i \geq M \end{cases} \tag{1.49}$$

As a result, the multi-step sliding mode prediction model (MSMPM) is constructed as follows,

$$s_m(k+i) = \sigma^T A^i x(k) + \Sigma_{j=1}^{i} \sigma^T A^{j-1} \alpha^p b u(k) + \beta^i \text{sgn}(s(k)) \tag{1.50}$$

where

$$p = \begin{cases} i-j, i-j < M \\ M-1, i-j \geq M \end{cases} \tag{1.51}$$

The above S_m equation can be described in vector form as follows,

$$S_m = Fx(k) + GL_1 u(k) + L_2 \text{sgn}(s(k)) \tag{1.52}$$

where

$$S_m = [s_m(k+1) \ldots s_m(k+N)]^T,$$

$$F = [\sigma^T A \ldots \sigma^T A^N]^T,$$

$$L_1 = [1\alpha^1 \ldots \alpha^{M-1}]^T,$$

$$I_2 = [\beta \ldots \beta^N]^T,$$

$$G = \begin{bmatrix} \sigma^T & 0 & \ldots & \ldots & 0 \\ \sigma^T Ab & \sigma^T b & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma^T A^{(M-1)} b & \sigma^T A^{(M-2)} b & \ldots & \ldots & \sigma^T b \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma^T A^{(i-1)} b & \sigma^T A^{(i-2)} b & \ldots & \sigma^T A^{(i-M+1)} b & \Sigma_{j=1}^{i-M+1} \sigma^T A^{j-1} b \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma^T A^{(N-1)} b & \sigma^T A^{(N-2)} b & \ldots & \sigma^T A^{(N-M+1)} b & \Sigma_{j=1}^{N-M+1} \sigma^T A^{j-1} b \end{bmatrix}$$

where N is the prediction horizon and M is the control horizon.

Now, the performance index is defined as

$$J = \Sigma_{i=1}^{N} (s_m(k+i) - s_d)^2 + \lambda u^2(k) \tag{1.53}$$

where s_d is the sliding mode desired value, λ is a weight coefficient and $\lambda u^2(k)$ acts as a penalty factor for the control signal in the performance index.

According to SMC theory, (Utkin, 1992) states should be forced to the sliding surface and stay on it thereafter. Thus, the sliding mode desired value are zero, namely, $s_d = 0$. Therefore, (1.53) reduces to

$$J = \Sigma_{i=1}^{N} s_m^2(k+i) + \lambda u^2(k) = S_m^T S_m + \lambda u^2(k) \tag{1.54}$$

Substituiting (1.50) into (1.54), yields

$$\begin{aligned} J &= [Fx(k) + GL_1u(k) + L_2\text{sgn}(s(k))]^T[Fx(k) + GL_1u(k) + L_2\text{sgn}(s(k))] + \lambda u^2(k) \\ &= [H(k) + Qu(k)]^T[H(k) + Q(k)] + \lambda u^2(k) \end{aligned} \tag{1.55}$$

where $H(k) = Fx(k) + L_2\text{sgn}(s(k))$, $Q = GL_1u(k)$.

Optimizing (1.55), i.e., by taking $\frac{\partial J}{\partial u(k)}$ =0. The following sliding mode control law can be obtained.

$$u(k) = -\frac{Q^T H(k)}{Q^T Q + \lambda} \tag{1.56}$$

In the conventional SMC, a highly switched controlleris designed to realize the reachability of sliding surface. In the above algorithm, the reachability of the sliding surface is completed by making the future value of the sliding mode track $s(k) = 0$. Due to the future information of the sliding mode, control signal is able to adjust immediately to prevent system states cross the sliding surface, hence chattering can be avoided.

1.3 The Invariance Condition for Linear Systems with Exogenous Disturbances

Consider a multi input system with a disturbance d_k (Drazenovic, 1969)

$$x_{k+1} = Ax_k + Bu_k + Ed_k \tag{1.57}$$

where $x_k \in \mathbf{R}^n$, $u_k \in \mathbf{R}^m$, $d_k \in \mathbf{R}^l$ and A, B, E are constant matrices of appropriate dimensions, B and E have full rank. The sliding mode of (1.57) can be described as

$$s_k = Gx_k = 0, \tag{1.58}$$

where G_k is a $m \times n$ matrix. Equation (1.57) is invariant to d_k in the sliding mode if and only if

$$\text{rank}\begin{bmatrix} B & | & E \end{bmatrix} = \text{rank}\begin{bmatrix} B \end{bmatrix} \tag{1.59}$$

The above condition is known as Drazenovic's Invariance condition for linear systems with exogenous disturbances.

Chapter 2

Discrete-Time Weakly Coupled Systems

2.1 Introduction

Linear weakly coupled systems have been studied in different set-ups by many researchers since Kokotovic and his coworkers introduced them in 1969 (Kokotovic *et al.*, 1969; Gajić and Borno, 2000; Gajić *et al.*, 2009). Traditionally, solutions of weakly coupled systems were obtained in terms of Taylor series and power series expansions with respect to a small weak coupling parameter ϵ (Kokotovic *et al.*, 1969). In 1989, Gajić and Shen, under certain conditions, introduced a decoupling transformation which exactly decompose weakly coupled linear systems composed of two subsystems into independent two reduced-order subsystems. In (Qureshi, 1992), another version of the transformation was obtained.

The linear weakly coupled system composed of two subsystem is defined by (Kokotovic *et al.*, 1969)

$$\begin{aligned} x^1_{k+1} &= A_1 x^1_k + \epsilon A_2 x^2_k + B_1 u^1_k + \epsilon B_2 u^2_k \\ x^2_{k+1} &= \epsilon A_3 x^1_k + A_4 x^2_k + \epsilon B_3 u^1_k + B_4 u^2_k \end{aligned} \tag{2.1}$$

where ϵ is a small weak coupling parameter and $x^i_k \in \mathbf{R}^{n_i}$ are state space variables and, $u^i_k \in \mathbf{R}^{m_i}$ are subsystem controls. Two standard assumptions for weakly coupled linear system exist (Gajić *et al.*, 2009, pp. 98-100).

Assumption 2.1.1. *Matrices A_i, $i = 1, 2, 3, 4$, are constant and $O(1)$. In addition, magnitudes of all system eigenvalues are $O(1)$, that is, $|\lambda_j| = O(1)$, $j = 1, 2, \ldots, n$, which implies that the matrices A_1, A_4 are nonsingular with* $\det\{A_1\} = O(1)$ *and* $\det\{A_4\} = O(1)$.

Assumption 2.1.2. *Matrices A_1 and A_4 have no common eigenvalues.*

2.2 Decoupling Transformation of Gajic and Shen

Consider a linear weakly coupled system (Gajić and Shen, 1989; see also Gajić *et al.*, 2009)

$$x_{k+1}^1 = A_1 x_k^1 + \epsilon A_2 x_k^2 + B_1 u_k^1 + \epsilon B_2 u_k^2 \tag{2.2}$$

$$x_{k+1}^2 = \epsilon A_3 x_k^1 + A_4 x_k^2 + \epsilon B_3 u_k^1 + B_4 u_k^2 \tag{2.3}$$

where $x_k^1 \in \mathbf{R}^{n_1}$, $x_K^2 \in \mathbf{R}^{n_2}$, $n_1 + n_2 = n$, are subsystem states, $u_k^i \in \mathbf{R}^{m_i}$, $i = 1, 2$, are subsystem controls, and ϵ is a small coupling parameter. Introducing new variables η_1 and a matrix L_1 as follows

$$x_k^1 = \eta_k^1 + \epsilon L_1 x_k^2 \tag{2.4}$$

transforms (2.2) into

$$\eta_{k+1}^1 = A_{10}\eta_k^1 + \epsilon \Phi_1(L_1) x_k^2 + B_{10} u_k^1 + \epsilon B_{20} u_k^2 \tag{2.5}$$

where

$$\begin{aligned} A_{10} &= A_1 - \epsilon^2 L_1 A_3 \\ B_{10} &= B_1 - \epsilon^2 L_1 B_3 \\ B_{20} &= B_2 - L_1 B_4 \end{aligned} \tag{2.6}$$

and

$$\Phi_1(L_1) = A_1 L_1 - L_1 A_4 + A_2 - \epsilon^2 L_1 A_3 L_1 \tag{2.7}$$

If L_1 is chosen such that $\Phi_1(L) = 0$, (2.5) is completely decoupled subsystem

$$\eta_{k+1}^1 = A_{10}\eta_k^1 + B_{10} u_k^1 + \epsilon B_{20} u_k^2 \tag{2.8}$$

Introducing another change of variables as follows

$$\eta_k^2 = x_k^2 + \epsilon H_1 \eta_k^1 \tag{2.9}$$

we have from (2.3) and (2.8)

$$\eta_{k+1}^2 = \epsilon \Phi_1(H_1)\eta_k^1 + A_{40}\eta_k^2 + \epsilon B_{30} u_k^1 + B_{40} u_k^2 \tag{2.10}$$

where

$$\begin{aligned} A_{40} &= A_4 + \epsilon^2 A_3 L_1 \\ B_{30} &= B_3 + H_1 B_{10} \\ B_{40} &= B_4 + \epsilon^2 H_1 B_{20} \end{aligned} \tag{2.11}$$

and

$$\Phi_2(H_1) = H_1 A_{10} - A_{40} H_1 + A_3 \tag{2.12}$$

Assuming that matrix H_1 can be chosen such that $\Phi_2(H_1) = 0$, (2.10) represents another decoupled subsystem

$$\eta^2_{k+1} = A_{40}\eta^2_k + \epsilon B_{30} u^1_k + B_{40} u^2_k \tag{2.13}$$

The original system (2.2)-(2.3) is transformed into the decoupled subsystems using the similarity transformation

$$\begin{bmatrix} \eta^1_k \\ \eta^2_k \end{bmatrix} = \begin{bmatrix} I_{n_1} & -\epsilon L_1 \\ \epsilon H_1 & I_{n_2} - \epsilon^2 H_1 L_1 \end{bmatrix} \begin{bmatrix} x^1_k \\ x^2_k \end{bmatrix} = \mathbf{T}_1 \begin{bmatrix} x^1_k \\ x^2_k \end{bmatrix} \tag{2.14}$$

where

$$\mathbf{T}_1^{-1} = \begin{bmatrix} I_{n_1} - \epsilon^2 L_1 H_1 & \epsilon L_1 \\ -\epsilon H_1 & I_{n_2} \end{bmatrix}. \tag{2.15}$$

2.2.1 Decoupling Transformation of Qureshi

The difficulty of the decoupling transformation of Gajić and Shen is that computation must be done sequentially. Introducing the change of variables to overcome this difficulty (Qureshi, 1992; see also Gajić *et al.*, 2009, Chapter 5),

$$\begin{bmatrix} \eta^1_k \\ \eta^2_k \end{bmatrix} = \begin{bmatrix} I_{n_1} & -\epsilon L^2_k \\ \epsilon H^2_k & I_{n_2} - \epsilon^2 H^2_k L^2_k \end{bmatrix} \begin{bmatrix} x^1_k \\ x^2_k \end{bmatrix} = \mathbf{T}_2 \begin{bmatrix} x^1_k \\ x^2_k \end{bmatrix} \tag{2.16}$$

where

$$\mathbf{T}_2^{-1} = \begin{bmatrix} I_{n_1} - \epsilon^2 L^2_k M_k H^2_k & \epsilon L^2_k M_k \\ -\epsilon M_k H^2_k & M_k \end{bmatrix} \tag{2.17}$$

with $M_k = (I_{n_1} - \epsilon^2 H^2_k L^2_k)^{-1}$, the original system (2.1) is transformed into

$$\begin{aligned} \eta^1_{k+1} &= (A^1_k - \epsilon^2 L^2_k (A^3_k))\eta^1_k + B_{10} u^1_k + \epsilon B_{20} u^2_k \\ \eta^2_{k+1} &= (A^4_k - \epsilon^2 H^2_k A^2_k)\eta^2_k + \epsilon B_{30} u^1_k + B_{40} u^2_k \end{aligned} \tag{2.18}$$

where matrices L_k^2 and H_k^2 are obtained from

$$\begin{aligned}\Phi_3(L_k^2, L_{k+1}^2) = L_{k+1}^2 - A_k^1 L_k^2 + L_k^2 A_k^4 \\ - A_k^2 + \epsilon^2 L_k^2 A_k^3 L_k^2 = 0 \\ \Phi_4(H_k^2, H_k^2) = H_{k+1}^2 - A_k^4 H_k^2 + H_k^2 A_k^1 \\ A_k^3 + \epsilon^2 H_k^2 A_k^2 H_k^2 = 0\end{aligned} \tag{2.19}$$

with

$$\begin{aligned} B_{10} &= B_1 - \epsilon^2 L^1 B_3 \\ B_{20} &= B_2 - L^2 B_4 \\ B_{30} &= B_3 - H^2 B_1 \\ B_{40} &= B_4 - \epsilon^2 H^1 B_2 \end{aligned} \tag{2.20}$$

Note that equations for L_k^2 and H_k^2 are independent of each other.

2.2.2 Decoupling Transformation for N Weakly Coupled Subsystems

Consider a continuous-time systems consisting of n states represented by Gajić and Borno (2000); see also Gajić *et al.* (2009, Chapter 5),

$$x_{k+1} = Ax_k \tag{2.21}$$

where x_k is n-dimentional state vector partitioned consistently with N subsystems as $x_k = \begin{bmatrix} x_k^{1^T} & x_k^{2^T} & \dots & x_k^{N^T} \end{bmatrix}^T$, $x_k^i \in \mathbf{R}_i^n$, and constant matrix A is

$$A = \begin{bmatrix} A_{11} & \epsilon A_{12} & \dots & \epsilon A_{1N} \\ \epsilon A_{21} & A_{22} & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \epsilon A_{N1} & \epsilon A_{N2} & \dots & A_{NN} \end{bmatrix} \tag{2.22}$$

The similar assumptions as Assumption 1.2.1 and 1.2.2 of are imposed for N weakly coupled linear system (Gajić *et al.*, 2009, pp. 108-111).

Assumption 2.2.1. *All matrices A_{ij} are constant and $O(1)$, and magnitudes of all system eigenvalues are $O(1)$, that is, $|\lambda_j| = O(1)$, $j = 1, 2, \dots, n$, which implies that the matrices A_{ii}, $j = 1, 2, \dots, N$ are nonsingular with* $\det\{A_{ii}\} = O(1)$.

Assumption 2.2.2. *Matrices A_{jj} and A_{ii} have no eigenvalues in common for every $i, j, i \neq j$.*

The corresponding similarity transformation matrix is given by

$$\eta_k = \Gamma x_k \tag{2.23}$$

where

$$\Gamma(\epsilon) = \begin{bmatrix} I & \epsilon L_{12} & \dots & \epsilon L_{1N} \\ \epsilon L_{21} & I & \dots & \epsilon L_{2N} \\ \dots & \dots & \dots & \dots \\ \epsilon L_{N1} & \dots & \epsilon L_{N(N-1)} & I \end{bmatrix} \tag{2.24}$$

The original system (2.21) is decoupled into

$$\eta_{k+1}^i = \Omega_i \eta_k^i, \; i = 1, 2, \dots, N \tag{2.25}$$

with

$$\Omega_i = A_{ii} + \epsilon^2 \sum_{j=1, j\neq i}^{N} L_{ij} A_{ji}, \; j = 1, 2, \dots, N \tag{2.26}$$

where L_{ij} satisfies

$$\begin{aligned} \Omega_{ij}(L_{ij}, \epsilon) = L_{ij} A_{jj} - A_{ii} L_{ij} + A_{ij} + \epsilon (\sum_{r=1, r\neq i,j}^{N} L_{ir} A_{ri}) \\ - \epsilon^2 (\sum_{r=1, r\neq i}^{N} L_{ir} A_{ri}) L_{ij} = 0, \end{aligned} \tag{2.27}$$

$$i, j = \forall 1, 2, \dots, N, \; i \neq j$$

These equations can be solved iteratively by starting with

$$L_{ij}^{(0)} A_{jj} - A_{ii} L_{ij}^{(0)} + A_{ij} = 0 \tag{2.28}$$

and performing the following iteration

$$\begin{aligned} L_{ij}^{(m+1)} A_{jj} - A_{ii} L_{ij}^{(m+1)} + A_{ij} + \epsilon (\sum_{r=1, r\neq i,j}^{N} L_{ir}^{(m)} A_{ri}) \\ - \epsilon^2 (\sum_{r=1, r\neq i}^{N} L_{ir}^{(m)} A_{ri}) L_{ij}^{(m)} = 0, \end{aligned} \tag{2.29}$$

$$i, j = 1, 2, \dots, N, \; i \neq j; \; m = 0, 1, 2, \dots$$

This algorithm converges with the rate of $O(\epsilon)$, that

$$||L_{ij}^{(m)} - L_{ij}^{(0)}|| = O(\epsilon^i), m = 0, 1, 2, \ldots \tag{2.30}$$

Other methods, like the Newton method and eigen-value method (Gajić *et al.*, 2009), can be used to solve (2.27).

Chapter 3

Sliding Mode Control of Linear Discrete-Time Weakly Coupled Systems

3.1 Introduction

Weakly coupled systems have been traditionally controlled by assuming that the coupling between subsystems does not exist i.e., by setting the coupling parameter $\epsilon = 0$. This method is neither comprehensive nor accurate. The introduction of decoupling techniques has made it possible to handle each subsystem separately with much better precision. In this chapter, we address the problem of sliding mode control of a weakly coupled linear discrete-time system without external disturbance. Both the traditional approach and the novel decoupling approach will be studied. The Utkin and Young method of sliding mode control (Utkin and Young, 1978) is employed to achieve stability.

3.2 Traditional Approach

Consider the following discrete-time linear weakly coupled system

$$\begin{pmatrix} x_{k+1}^1 \\ x_{k+1}^2 \end{pmatrix} = \begin{pmatrix} A_1 & \epsilon A_2 \\ \epsilon A_3 & A_4 \end{pmatrix} \begin{pmatrix} x_k^1 \\ x_k^2 \end{pmatrix} + \begin{pmatrix} B_1 & \epsilon B_2 \\ \epsilon B_3 & B_4 \end{pmatrix} \begin{pmatrix} u_k^1 \\ u_k^2 \end{pmatrix} \tag{3.1}$$

The traditional method of decoupling this system is by setting $\epsilon = 0$. The system then reduces to the following two independent systems:

$$x_{k+1}^1 = A_1 x_k^1 + B_1 u_k^1 \tag{3.2}$$

$$x_{k+1}^2 = A_4 x_k^2 + B_4 u_k^2 \tag{3.3}$$

where $x_k^1 \in \mathbf{R}^{n_1}$, $x_k^2 \in \mathbf{R}^{n_2}$, $n_1 + n_2 = n$, are state variables, $u_k^i \in \mathbf{R}^{m_i}$, $i = 1, 2$, are control inputs and ϵ is a small weak coupling parameter. It is assumed that matrices A_1, A_4 are constant and $O(1)$. In addition, magnitudes of all system eigenvalues are $O(1)$, that is, $|\lambda_j| = O(1)$, $j = 1, 2, \ldots, n$, which implies that the matrices A_1, A_4 are nonsingular with $\det\{A_1\} = O(1)$ and $\det\{A_4\} = O(1)$. It is also assumed that matrices A_1 and A_4 have no common eigenvalues (see Assumption 2.1.1). A and B are constant matrices of appropriate dimensions.

3.2.1 Case Study

A physical example of a fifth-order distillation column control problem (Kautsky *et al.*, 1985) is used here to demonstrate the approach of section 3.2. The system matrices are as follows

$$A = 10^{-3} \begin{pmatrix} 989.5 & 5.6382 & 0.2589 & 0.0125 & 0.0006 \\ 117.25 & 814.5 & 76.038 & 5.5526 & 0.37 \\ 8.768 & 123.87 & 750.2 & 107.96 & 11.245 \\ 0.9108 & 17.991 & 183.81 & 668.34 & 150.78 \\ 0.0179 & 0.3172 & 1.6974 & 13.298 & 985.19 \end{pmatrix} = \begin{pmatrix} A_1 & \epsilon A_2 \\ \epsilon A_3 & A_4 \end{pmatrix}$$

$$B = 10^{-3} \begin{pmatrix} 0.0192 & -0.0013 \\ 6.0733 & -0.6192 \\ 8.2911 & -13.339 \\ 9.1965 & -18.442 \\ 0.7025 & -1.4252 \end{pmatrix} = \begin{pmatrix} B_1 & \epsilon B_2 \\ \epsilon B_3 & B_4 \end{pmatrix}$$

These matrices have been obtained from (Kautsky *et al.*, 1985) by performing discretization. The coupling parameter ϵ can be roughly estimated from the strongest coupled matrix - in this case matrix B. The strongest coupling is seen in the third row, which implies

$$\epsilon = \frac{b_{31}}{b_{32}} = \frac{8.2911}{13.339} \approx 0.62$$

Setting $\epsilon = 0$, we get

$$A_0 = \begin{pmatrix} 0.9895 & 0.0056 & 0.0003 & 0 & 0 \\ 0.1173 & 0.8145 & 0.0760 & 0 & 0 \\ 0.0088 & 0.1239 & 0.7502 & 0 & 0 \\ 0 & 0 & 0 & 0.6683 & 0.1508 \\ 0 & 0 & 0 & 0.0133 & 0.9852 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix}$$

$$B_0 = \begin{pmatrix} 0 & 0 \\ 0.0061 & 0 \\ 0.0083 & 0 \\ 0 & -0.0184 \\ 0 & -0.0014 \end{pmatrix} = \begin{pmatrix} B_1 & 0 \\ 0 & B_4 \end{pmatrix}$$

The resultant system is a concatenation of two completely independent sub-systems and each can be dealt with separately. Thus, two independent sliding surfaces are designed for them as follows.

Consider the first sub-system. Let

$$N_1 = \mathrm{null}(B_1') = \begin{pmatrix} -0.5909 & -0.8067 \\ 0.6515 & -0.4758 \\ -0.4758 & 0.3504 \end{pmatrix}$$

then

$$H_1 = [N_1 B_1]' = \begin{pmatrix} -0.5909 & 0.6515 & -0.4758 \\ -0.8067 & -0.4758 & 0.3504 \\ 0 & 0.0061 & 0.0083 \end{pmatrix}$$

Transferring to the new co-ordinates, we have

$$A_{1new} = H_1 A_1 H_1^{-1} = \begin{pmatrix} 0.7543 & 0.0828 & 2.6916 \\ 0.1679 & 0.9318 & -2.4260 \\ 0.0002 & -0.0011 & 0.8681 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$B_{1new} = H_1 B_1 = 10^{-3} \begin{pmatrix} 0 \\ 0 \\ 0.1056 \end{pmatrix} = \begin{pmatrix} 0 \\ B_{r1} \end{pmatrix}$$

The eigenvalues of A_{11} are at 0.6955 and 0.9906. The following sliding surface is designed to shift these eigenvalues to 0.5 and 0.6 respectively.

$$s_k^1 = G_1 x_k^1$$

where

$$G_1 = [K_1 \;\; 1] H_1$$

and K_1 is the feedback gain matrix for placing the eigenvalues of the system at the desired locations.

$$K_1 = \begin{pmatrix} -0.9165 & -1.2585 \end{pmatrix}$$

Therefore,

$$G_1 = \begin{pmatrix} 1.5569 & 0.0078 & 0.0034 \end{pmatrix}$$

The resultant feedback control law for this sub-system is defined by

$$u_k^1 = -(G_1 B_1)^{-1} G_1 (A_1 - I_{n1}) x_k^1 + \sigma \mathrm{sgn}(s_k^1) =$$
$$- \begin{pmatrix} -145.8167 & 73.4110 & 1.3452 \end{pmatrix} x_k^1 + \sigma \mathrm{sgn}(s_k^1)$$

Following a similar approach for the second sub-system, we obtain

$$N_2 = \mathrm{null}(B_4') = \begin{pmatrix} -0.0771 \\ 0.9970 \end{pmatrix}$$

$$H_2 = [N_2 B_4]' = \begin{pmatrix} -0.0771 & 0.9970 \\ -0.0184 & -0.0014 \end{pmatrix}$$

Transferring to the new co-ordinates

$$A_{4new} = H_2 A_4 H_2^{-1} = \begin{pmatrix} 0.9707 & -1.9822 \\ -0.0032 & 0.6828 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$B_{4new} = H_2 B_4 = 10^{-3} \begin{pmatrix} 0 \\ 0.3421 \end{pmatrix} = \begin{pmatrix} 0 \\ B_{r2} \end{pmatrix}$$

The eigenvalue of A_{11} is at 0.9707. To shift this eigenvalue to .55, the required K_2 is -0.2122. Consequently,

$$G_2 = [K_2 \;\; 1] H_2 = \begin{pmatrix} -0.0021 & -0.2130 \end{pmatrix}$$

The resultant feedback control law for this sub-system is defined in

$$u_k^2 = -(G_2 B_4)^{-1} G_2 (A_4 - I_{n2}) x_k^2 + \sigma \mathrm{sgn}(s_k^2) = -\begin{pmatrix} -6.2551 & 8.3009 \end{pmatrix} x_k^2 + \sigma \mathrm{sgn}(s_k^2)$$

Thus, the feedback law of the full-order system is given by

$$u_k = -\begin{pmatrix} u_k^1 & 0_{1\times 2} \\ 0_{1\times 3} & u_k^2 \end{pmatrix} =$$

$$-\begin{pmatrix} -145.8167 & 73.4110 & 1.3452 & 0 & 0 \\ 0 & 0 & 0 & -6.2551 & 8.3009 \end{pmatrix} \begin{pmatrix} x_k^1 \\ x_k^2 \end{pmatrix} + \sigma \begin{pmatrix} \mathrm{sgn}(s_k^1) \\ \mathrm{sgn}(s_k^2) \end{pmatrix}$$

3.2.2 Observations

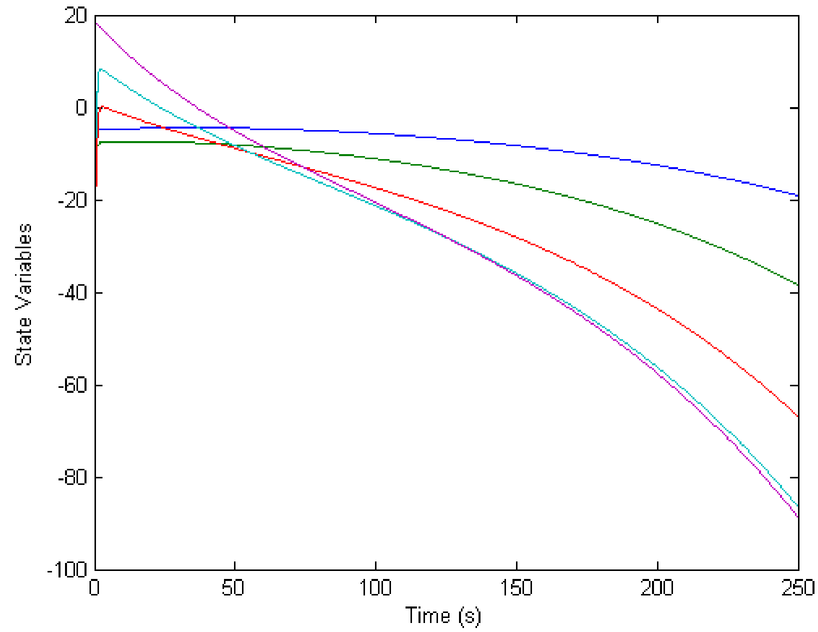

Figure 3.1: Divergence of state variables with $\sigma = 50$

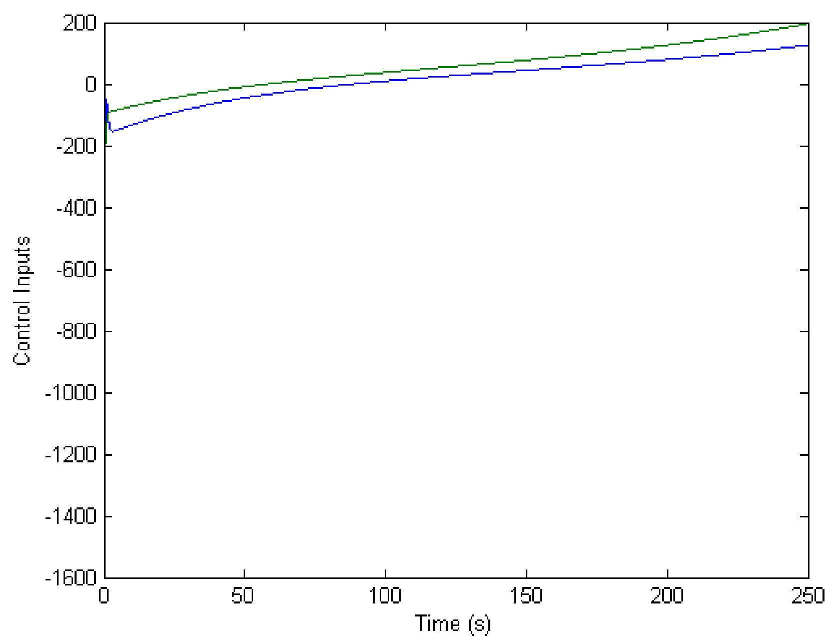

Figure 3.2: Control inputs

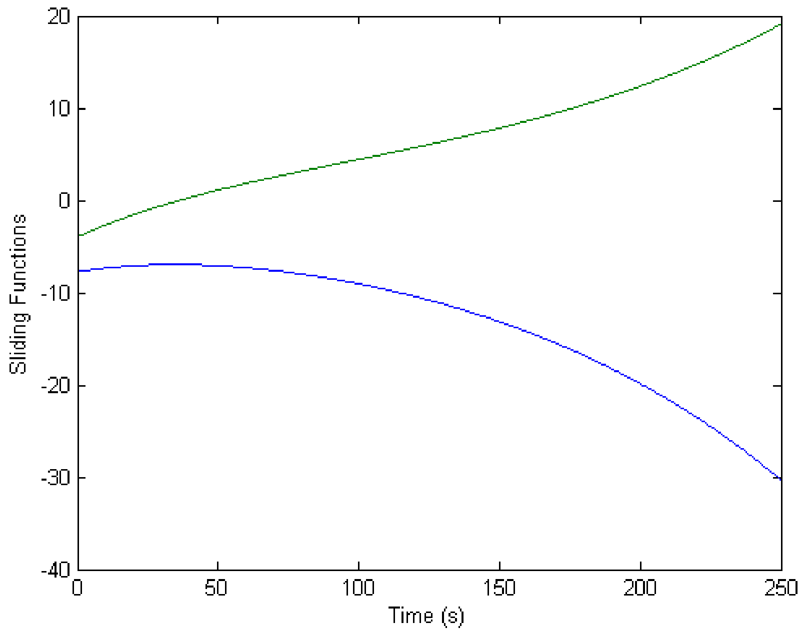

Figure 3.3: Sliding surfaces/ divergences

These observations clearly show that the traditional approach is insufficient in cases where ϵ is not very small. The interaction between the two subsytems and the alternate control inputs in this case upsets the system causing instability inspite of the controller design that put individual sub-system eigenvalues inside the unit circle.

3.3 Decoupling Approach

Here we use the decoupling transformaton proposed by Gajic and Shen (Gajić and Shen, 1989). This involves the introduction of a change of variables as follows. For the system

$$x_{k+1} = A_1 x_k + \epsilon A_2 z_k + B_1 u_k^1 + \epsilon B_2 u_k^2 \tag{3.4}$$

$$z_{k+1} = \epsilon A_3 x_k + A_4 z_k + \epsilon B_3 u_k^1 + B_4 u_k^2 \tag{3.5}$$

introducing

$$x_k = \eta_k + \epsilon L z_k \tag{3.6}$$

The original system is transformed into

$$\eta_{k+1} = A_{10}\eta_k + \epsilon F_1(L)z + B_{10}U_k^1 + \epsilon B_{20}U_k^2 \tag{3.7}$$

where

$$A_{10} = A_1 - \epsilon^2 LA_3 \tag{3.8}$$

$$B_{10} = B_1 - \epsilon^2 LB_3 \tag{3.9}$$

$$B_{20} = B_2 - LB_4 \tag{3.10}$$

$$F_1(L) = A_1L - LA_4 + A_2 - \epsilon^2 LA_3L \tag{3.11}$$

Assuming that a matrix L can be chosen such that $\mathrm{F}_1(L) = 0$, Equation (3.7) represents a completely independent (decoupled) subsystem

$$\eta_{k+1} = A_{10}\eta_k + B_{10}U_k^1 + \epsilon B_{20}U_k^2 \tag{3.12}$$

Introducing the second change of variables as

$$\zeta_k = z_k + \epsilon H\eta_k \tag{3.13}$$

Equation(3.5) becomes

$$\zeta_{k+1} = A_{40}\zeta_k + \epsilon F_2(H)\eta_k + \epsilon B_{30}u_k^1 + B_{40}u_k^2 \tag{3.14}$$

with

$$\begin{aligned} A_{40} &= A_4 + \epsilon^2 A_3 L \\ B_{30} &= B_3 + H B_{10} \\ B_{40} &= B_4 + \epsilon^2 H B_{20} \end{aligned}$$

and

$$F_2(H) = HA_{10} - A_{40}H + A_3 \tag{3.15}$$

In addition, if matrix H can be chosen such that $F_2(H)$=0, we have

$$\zeta_{k+1} = A_{40}\zeta_k + \epsilon B_{30}u_k^1 + B_{40}u_k^2 \tag{3.16}$$

Equations (3.12) and (3.16) represent two completely decoupled linear subsystems. Notice that the weakly coupled structure of the control inputs in (3.4) and (3.5) is preserved in the new coordinates. Also, the inverse transformation is applicable to the feedback structure. Thus, applying the non-singular transformation

$$\begin{bmatrix} \eta_k \\ \zeta_k \end{bmatrix} = \begin{bmatrix} I_{n_1} & -\epsilon L \\ \epsilon H & I_{n_2} - \epsilon^2 HL \end{bmatrix} \begin{bmatrix} x_k \\ z_k \end{bmatrix} = \mathbf{T_1} \begin{bmatrix} x_k \\ z_k \end{bmatrix} \tag{3.17}$$

with

$$\mathbf{T_1}^{\mathbf{-1}} = \begin{bmatrix} I_{n_1} - \epsilon^2 LH & \epsilon L \\ -\epsilon H & I_{n_1} \end{bmatrix} \tag{3.18}$$

Note that the transformation $\mathbf{T_1}$ is uniquely defined if the unique solutions of the following two algebraic equation exist:

$$A_1 L - LA_4 + A_2 - \epsilon^2 LA_3 L = 0 \tag{3.19}$$

$$H(A_1 - \epsilon^2 LA_3) - (A_4 + \epsilon^2 A_3 L)H + A_3 = 0 \tag{3.20}$$

It is important to notice that at $\epsilon = 0$ we have

$$A_1 L^{(0)} - L^{(0)} A_4 + A_2 = 0 \tag{3.21}$$

$$H^{(0)} A_1 - A_4 H^{(0)} + A_3 = 0 \tag{3.22}$$

so that

$$L = L^{(0)} + O(\epsilon^2) \tag{3.23}$$

$$H = H^{(0)} + O(\epsilon^2) \tag{3.24}$$

Equations (3.21) and (3.22) are Sylvester equations and their unique solutions exist if matrices A_1 and A_4 have no eigen values in common (Lancaster and Tismenetsky, 1985). Thus the presented results will be valid under the following Assumption 2.1.2.

3.3.1 Case Study

Let us consider the same distillation column system and apply the decoupling techniques on it. Applying Gajić and Shen's transformation we get the following tranformation matrix,

$$T = \begin{pmatrix} 1.0000 & 0 & 0 & 0.0033 & 0.0276 \\ 0 & 1.0000 & 0 & -0.1365 & 0.0745 \\ 0 & 0 & 1.0000 & 0.2850 & -0.1497 \\ -0.1726 & 0.3137 & -0.4038 & 0.8415 & 0.0791 \\ -0.1496 & -0.0155 & 0.0187 & 0.0070 & 0.9919 \end{pmatrix}$$

This matrix is used to transform the system as follows

$$A_n = TAT^{-1} = \begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix} = \begin{pmatrix} 0.9895 & 0.0057 & 0.0012 & 0 & 0 \\ 0.1172 & 0.8128 & 0.0581 & 0 & 0 \\ 0.0090 & 0.1289 & 0.8014 & 0 & 0 \\ 0 & 0 & 0 & 0.6186 & 0.1821 \\ 0 & 0 & 0 & 0.0129 & 0.9855 \end{pmatrix}$$

$$B_n = TB \begin{pmatrix} B_1 & 0 \\ 0 & B_4 \end{pmatrix} = \begin{pmatrix} 0.0001 & -0.0004 \\ 0.0052 & 0.0031 \\ 0.0107 & -0.0165 \\ 0 & -0.0078 \\ 0.0004 & -0.0017 \end{pmatrix}$$

The resultant system is a concatenation of two completely independent sub-systems and each can be dealt with separately. Thus, two independent sliding surfaces are designed for them as follows.

Consider the first sub-system. Let

$$N_1 = \text{null}(B_1') = \begin{pmatrix} -0.4376 & -0.8991 \\ 0.8105 & -0.3894 \\ -0.3894 & 0.2000 \end{pmatrix}$$

then

$$H_1 = [N_1 B_1]' = \begin{pmatrix} -0.5909 & 0.6515 & -0.4758 \\ -0.8067 & -0.4758 & 0.3504 \\ 0 & 0.0061 & 0.0083 \end{pmatrix}$$

Transferring to the new co-ordinates

$$A_{1new} = H_1 A_1 H_1^{-1} = \begin{pmatrix} 0.7440 & 0.0180 & 1.9162 \\ 0.1155 & 0.9818 & -1.3792 \\ 0.0007 & -0.0012 & 0.8778 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$B_{1new} = H_1 B_1 = 10^{-3} \begin{pmatrix} 0 \\ 0 \\ 0.1427 \end{pmatrix} = \begin{pmatrix} 0 \\ B_{r1} \end{pmatrix}$$

The eigenvalues of A_{11} are at 0.7356 and 0.9903. To shift these eigenvalues to .5 and .6 respectively, the following sliding surface is designed.

$$s_1 = G_1 x_k$$

where

$$G_1 = [K_1 \;\; 1] H_1$$

and K_1 is the feedback gain matrix for placing the eignvalues of a system with matices A_{11} and A_{12} at the desired locations.

$$K_1 = \begin{pmatrix} -0.7965 & -1.5604 \end{pmatrix}$$

Therefore,

$$G_1 = \begin{pmatrix} 1.7516 & -0.0327 & 0.0089 \end{pmatrix}$$

The resultant feedback control for this sub-system is

$$u_1 = -(G_1 B_1)^{-1} G_1 (A_1 - I_{n1}) x_k^1 + \sigma \mathrm{sgn}(s_k^1) =$$
$$-10^4 \begin{pmatrix} 155.0754 & -121.4321 & 10.4135 \end{pmatrix} x_k^1 + \sigma \mathrm{sgn}(s_k^1)$$

Following a similar approach for the second sub-system,

$$N_2 = \mathrm{null}(B_4') = \begin{pmatrix} -0.2069 \\ 0.9784 \end{pmatrix}$$

$$H_2 = [N_2 B_4]' = \begin{pmatrix} -0.2069 & 0.9784 \\ -0.0078 & -0.0017 \end{pmatrix}$$

Transferring to the new co-ordinates

$$A_{4new} = H_2 A_4 H_2^{-1} = \begin{pmatrix} 0.9303 & -9.8630 \\ -0.0020 & 0.6738 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$B_{4new} = H_2 B_4 = 10^{-4} \begin{pmatrix} 0 \\ 0.6379 \end{pmatrix} = \begin{pmatrix} 0 \\ B_{r2} \end{pmatrix}$$

The eigenvalue of A_{11} is at 0.9303. To shift this eigenvalue to .55, the required K_2 is -0.0386. Consequently,

$$G_2 = [K_2 \;\; 1]H_2 = \begin{pmatrix} 0.0002 & -0.0394 \end{pmatrix}$$

The resultant feedback control for this sub-system is

$$u_k^2 = -(G_2 B_4)^{-1} G_2 (A_4 - I_{n2}) x_k^2 + \sigma \text{sgn}(s_k^2) = -\begin{pmatrix} -8.9192 & 9.4422 \end{pmatrix} x_k^2 + \sigma \text{sgn}(s_k^2)$$

Thus, the feedback of the complete system is given by

$$u_k = -\begin{pmatrix} u_k^1 & 0_{1\times 2} \\ 0_{1\times 3} & u_k^2 \end{pmatrix} \times T^{-1} =$$

$$\begin{pmatrix} -136.7745 & 111.5125 & 7.3026 & 31.1710 & 3.4097 \\ 3.4112 & 3.1323 & -5.3822 & -8.9192 & 9.4422 \end{pmatrix} \begin{pmatrix} x_k^1 \\ x_k^2 \end{pmatrix} + \sigma \begin{pmatrix} \text{sgn}(s_k^1) \\ \text{sgn}(s_k^2) \end{pmatrix}$$

3.3.2 Observations

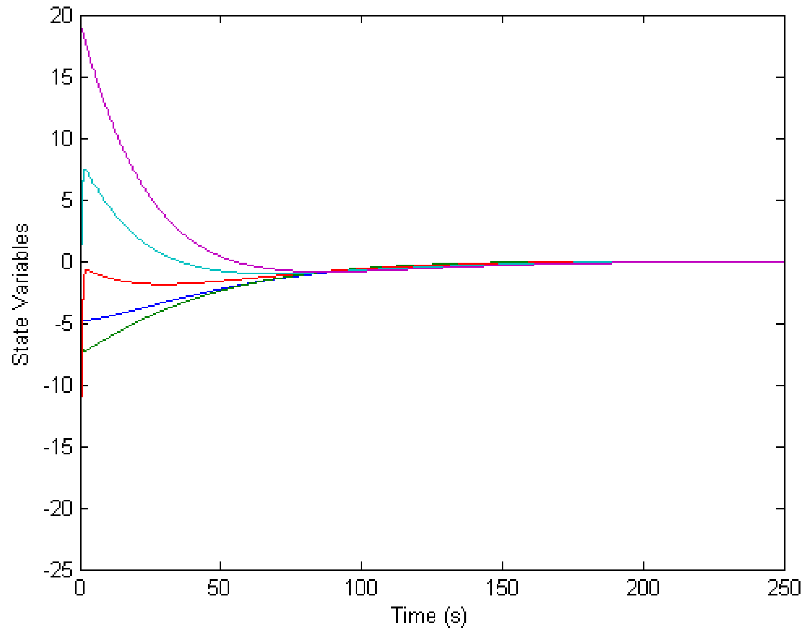

Figure 3.4: Convergence of state variables with $\sigma = 0$

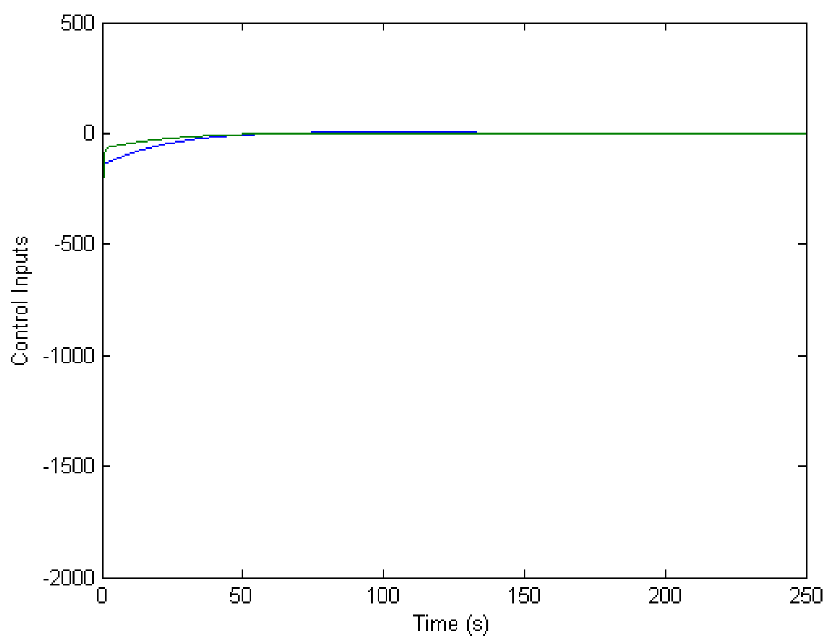

Figure 3.5: Control inputs

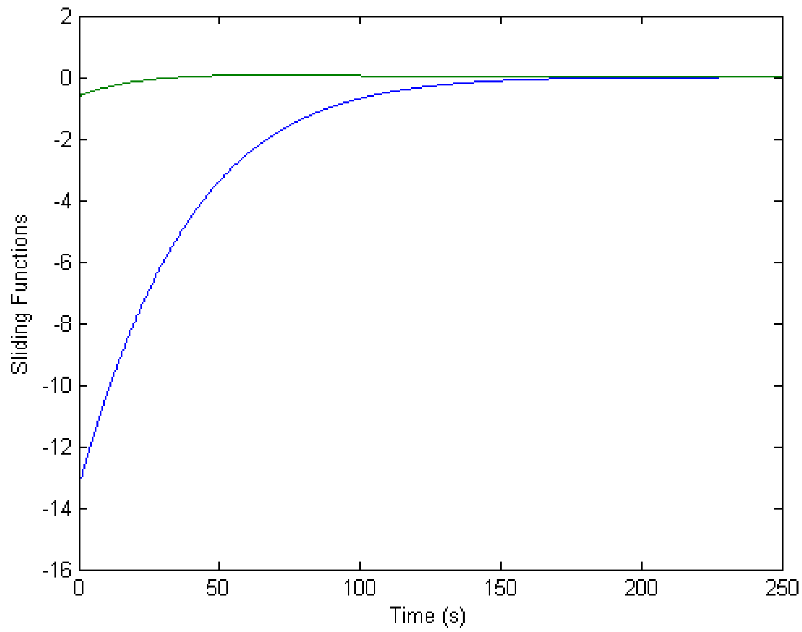

Figure 3.6: Sliding surfaces

For a better convergence, the feedback gain σ can be increased. But, this action causes chattering since there exists a trade-off between fast convergence and complete convergence. As σ is increased, the system converges faster but the chattering phenomenon escalates. The sign (sgn()) function can be replaced with the saturation (sat()) function to alleviate this side effect as illustrated below.

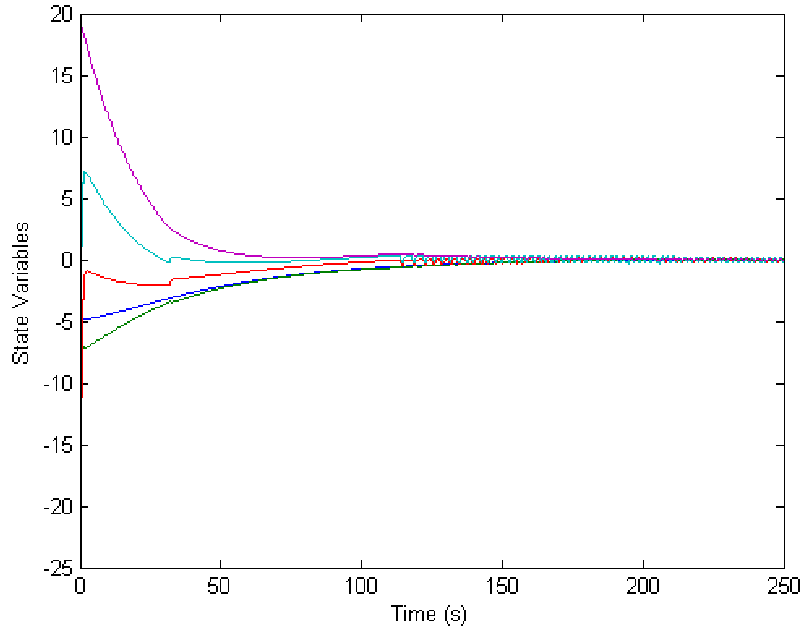

Figure 3.7: Convergence of state variables with $\sigma = 10$ and using sign function

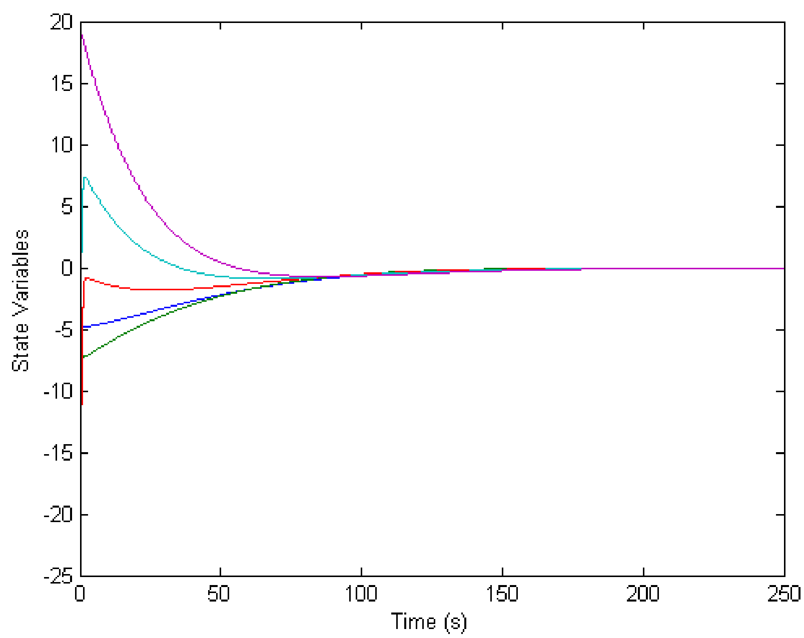

Figure 3.8: Convergence of state variables with $\sigma = 10$ and using saturation function

3.3.3 Analysis of Results

In order to control the reaching time of the subsystems to the sliding surfaces, the control input parameter available is σ. As σ is increased, the reaching time decreases. The effect of σ on the reaching time i.e. the time within which the sliding functions reach the neighborhood of zero (chosen here as -0.1 $\leq s_k \leq$ 0.1) is demonstrated as follows.

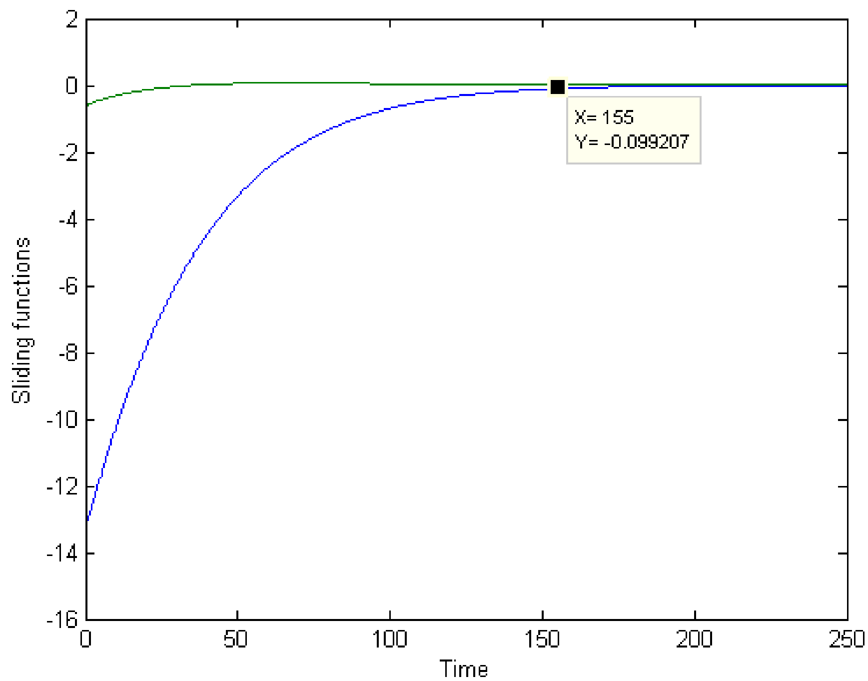

Figure 3.9: Reaching time when σ = 10

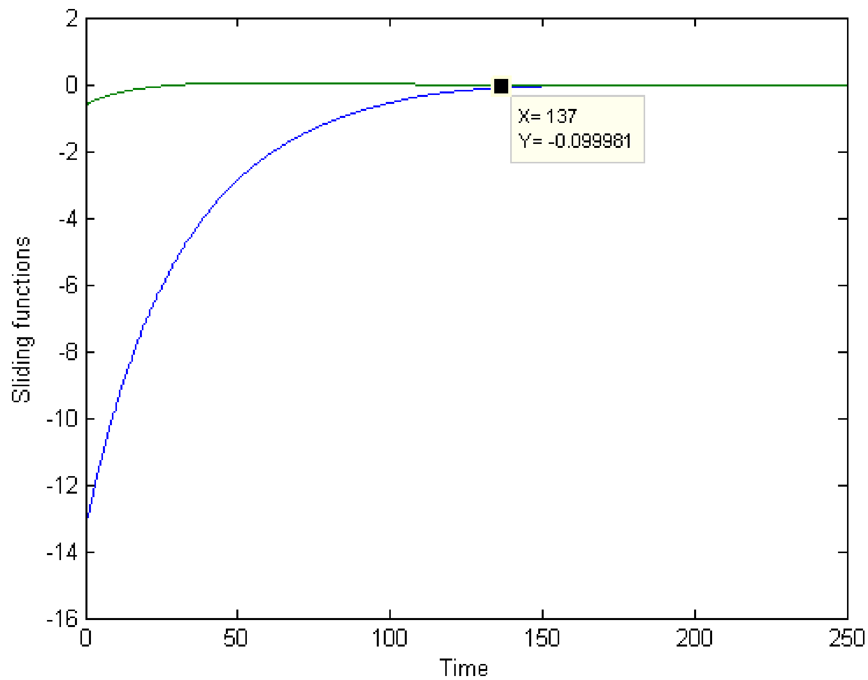

Figure 3.10: Reaching time when $\sigma = 100$

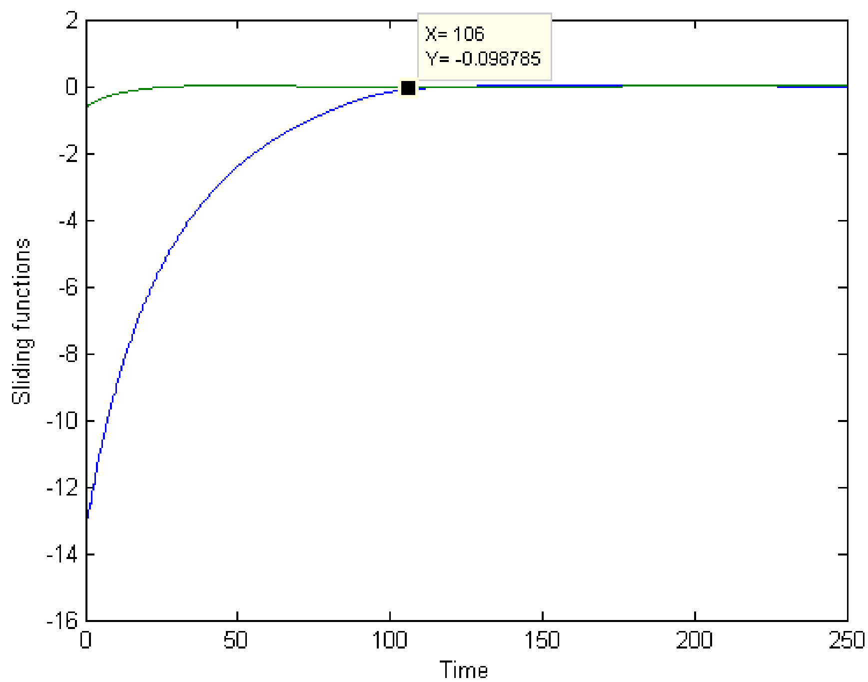

Figure 3.11: Reaching time when $\sigma = 200$

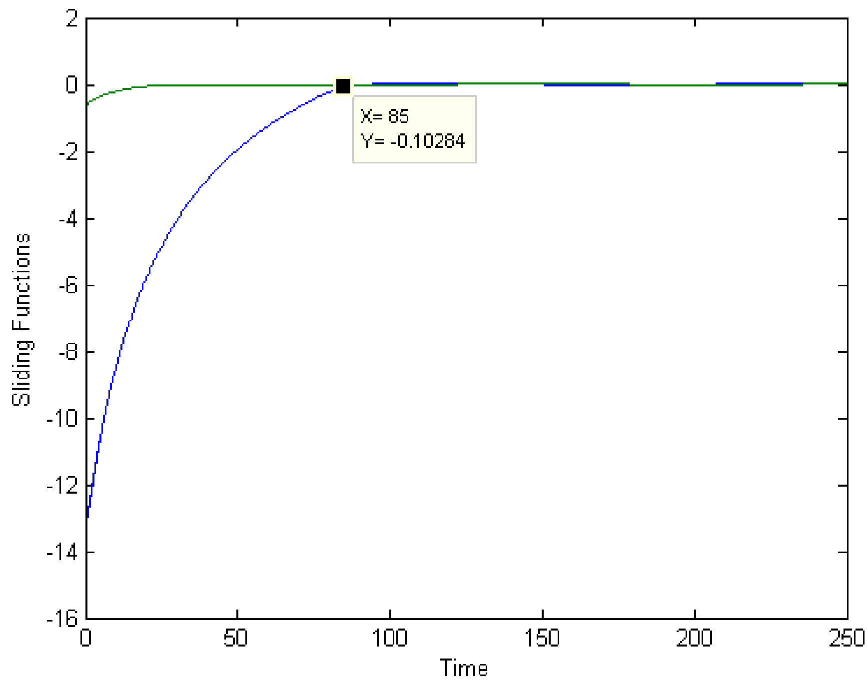

Figure 3.12: Reaching time when $\sigma = 300$

This indicates that the best way to achieve minimum reaching time under any conditions is by setting the feedback gain parameter σ to the maximum value feasible.

3.3.4 Other Control Methods

In this section, some of the discrete-time sliding mode control techniques introduced in Chapter 1 are applied to weakly-coupled linear systems.

1. Bartolini et al's Control Algorithm

The control law used in this approach is

$$u_k = \begin{cases} \alpha \text{sgn}(u_{keq}) \text{ if } |u_{keq}| > \alpha \\ u_{keq} \text{ if } |u_{keq}| \leq \alpha \end{cases} \tag{3.25}$$

This control method, used in sliding-mode control of sampled data systems, can also be applied to weakly-coupled discrete-time linear systems. The first step in this process is the decoupling of the weakly-coupled parts of the system. To decouple the subsystems, we can either employ the traditional approach or novel decoupling techniques.

Traditional Approach

Consider the following discrete-time linear weakly coupled system in the tracking error domain as described in Chapter 1

$$\begin{pmatrix} e^1_{k+1} \\ e^2_{k+1} \end{pmatrix} = \begin{pmatrix} A_1 & \epsilon A_2 \\ \epsilon A_3 & A_4 \end{pmatrix} \begin{pmatrix} e^1_k \\ e^2_k \end{pmatrix} - \begin{pmatrix} B_1 & \epsilon B_2 \\ \epsilon B_3 & B_4 \end{pmatrix} \begin{pmatrix} u^1_k \\ u^2_k \end{pmatrix} \tag{3.26}$$

The traditional method of decoupling this system is by setting $\epsilon = 0$. The system then reduces to the following two independent systems:

$$e^1_{k+1} = A_1 e^1_k - B_1 u^1_k \tag{3.27}$$

$$e^2_{k+1} = A_4 e^2_k - B_4 u^2_k \tag{3.28}$$

where $e_k^1 \in \mathbf{R}^{n_1}$, $e_k^2 \in \mathbf{R}^{n_2}$, $n_1 + n_2 = n$, are state variables, $u_k^i \in \mathbf{R}^{m_i}$, $i = 1, 2$, are control inputs and ϵ is a small weak coupling parameter. It is assumed that matrices A_1, A_4 are constant and $O(1)$. In addition, magnitudes of all system eigenvalues are $O(1)$, that is, $|\lambda_j| = O(1)$, $j = 1, 2, \ldots, n$, which implies that the matrices A_1, A_4 are nonsingular with $\det\{A_1\} = O(1)$ and $\det\{A_4\} = O(1)$. It is also assumed that matrices A_1 and A_4 have no common eigenvalues (see Assumption 2.1.1). A and B are constant matrices of appropriate dimensions.

Once the two subsystems are decoupled, a sliding surface is designed for each. The equivalent control input pertaining to each subsystem is calculated. The control law for the complete system is then described using Bartolini *et al*'s definition.

Numerical Example:

To study the application of this method, let us consider the same distillation column system introduced earlier. The system matrices in the tracking error domain (with reference $r(k)$ =0, $\forall k$) are as follows

$$A = -10^{-3}\begin{pmatrix} 989.5 & 5.6382 & 0.2589 & 0.0125 & 0.0006 \\ 117.25 & 814.5 & 76.038 & 5.5526 & 0.37 \\ 8.768 & 123.87 & 750.2 & 107.96 & 11.245 \\ 0.9108 & 17.991 & 183.81 & 668.34 & 150.78 \\ 0.0179 & 0.3172 & 1.6974 & 13.298 & 985.19 \end{pmatrix} = \begin{pmatrix} A_1 & \epsilon A_2 \\ \epsilon A_3 & A_4 \end{pmatrix}$$

$$B = -10^{-3}\begin{pmatrix} 0.0192 & -0.0013 \\ 6.0733 & -0.6192 \\ 8.2911 & -13.339 \\ 9.1965 & -18.442 \\ 0.7025 & -1.4252 \end{pmatrix} = \begin{pmatrix} B_1 & \epsilon B_2 \\ \epsilon B_3 & B_4 \end{pmatrix}$$

Setting $\epsilon = 0$, we get

$$A_0 = \begin{pmatrix} -0.9895 & -0.0056 & -0.0003 & 0 & 0 \\ -0.1173 & 0.8145 & -0.0760 & 0 & 0 \\ -0.0088 & 0.1239 & -0.7502 & 0 & 0 \\ 0 & 0 & 0 & -0.6683 & -0.1508 \\ 0 & 0 & 0 & -0.0133 & -0.9852 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix}$$

$$B_0 = \begin{pmatrix} 0 & 0 \\ -0.0061 & 0 \\ -0.0083 & 0 \\ 0 & 0.0184 \\ 0 & 0.0014 \end{pmatrix} = \begin{pmatrix} B_1 & 0 \\ 0 & B_4 \end{pmatrix}$$

The resultant system is composed of two completely independent sub-systems and each can be dealt with separately. Thus, two independent sliding surfaces are designed for them as follows.

Consider the first sub-system. Let

$$N_1 = \text{null}(B_1') = \begin{pmatrix} -0.5909 & -0.8067 \\ 0.6515 & -0.4758 \\ -0.4758 & 0.3504 \end{pmatrix}$$

then

$$H_1 = [N_1 B_1]' = \begin{pmatrix} -0.5909 & 0.6515 & -0.4758 \\ -0.8067 & -0.4758 & 0.3504 \\ 0 & -0.0061 & -0.0083 \end{pmatrix}$$

Transferring to the new co-ordinates,we have

$$A_{1new} = H_1 A_1 H_1^{-1} = \begin{pmatrix} -0.7543 & -0.0828 & 2.6916 \\ -0.1679 & -0.9318 & -2.4260 \\ 0.0002 & -0.0011 & -0.8681 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$B_{1new} = H_1 B_1 = 10^{-3} \begin{pmatrix} 0 \\ 0 \\ 0.1056 \end{pmatrix} = \begin{pmatrix} 0 \\ B_{r1} \end{pmatrix}$$

The eigenvalues of A_{11} are at - 0.6955 and - 0.9906. The following sliding surface is designed to shift these eigenvalues to - .5 and - .6 respectively.

$$g_k^1 = C_1 e_k^1$$

where

$$C_1 = [K_1 \ \ -1]H_1$$

and K_1 is the feedback gain matrix for placing the eigenvalues of the system at the desired locations.

$$K_1 = \begin{pmatrix} 0.9165 & 1.2585 \end{pmatrix}$$

Therefore,

$$C_1 = \begin{pmatrix} -1.5569 & -0.0078 & -0.0034 \end{pmatrix}$$

The resultant equivalent control for this sub-system is defined by

$$u_{eq}^1(k) = C_1 A e_k^1 = \begin{pmatrix} 1.5399 & 0.0036 & -0.0098 \end{pmatrix} e_k^1$$

Following a similar approach for the second sub-system,

$$N_2 = \text{null}(B_4') = \begin{pmatrix} -0.0771 \\ 0.9970 \end{pmatrix}$$

$$H_2 = [N_2 B_4]' = \begin{pmatrix} -0.0771 & 0.9970 \\ 0.0184 & 0.0014 \end{pmatrix}$$

Transferring to the new co-ordinates, we have

$$A_{4new} = H_2 A_4 H_2^{-1} = \begin{pmatrix} -0.9707 & -1.9822 \\ -0.0032 & -0.6828 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$B_{4new} = H_2 B_4 = 10^{-3} \begin{pmatrix} 0 \\ 0.3421 \end{pmatrix} = \begin{pmatrix} 0 \\ B_{r2} \end{pmatrix}$$

The eigenvalue of A_{11} is at - 0.9707. To shift this eigenvalue to -.55, the required K_2 is 0.2122. Consequently,

$$C_2 = [K_2 \;\; -1]H_2 = \begin{pmatrix} -0.0348 & 0.2102 \end{pmatrix}$$

The resultant equivalent control for this sub-system is defined in

$$u_{eq}^2(k) = C_2 A_2 e_k^2 = \begin{pmatrix} -0.0205 & -0.2018 \end{pmatrix} e_k^2$$

Thus, the equivalent control input of the full-order system is given by

$$u_{eq}(k) = \begin{pmatrix} u_{eq}^1(k) & 0_{1\times 2} \\ 0_{1\times 3} & u_{eq}^2(k) \end{pmatrix} =$$

$$\begin{pmatrix} 1.5399 & 0.0036 & -0.0098 & 0 & 0 \\ 0 & 0 & 0 & 0.0205 & -0.2018 \end{pmatrix} \begin{pmatrix} e_k^1 \\ e_k^2 \end{pmatrix}$$

The control law used in this approach is

$$u_k = \begin{cases} \alpha \mathrm{sgn}(u_{eq}(k)) \text{ if } |u_{eq}(k)| > \alpha \\ u_{eq}(k) \text{ if } |u_{eq}(k)| \leq \alpha \end{cases} \tag{3.29}$$

with $\alpha = 0.6$.

To simulate this method for the above example, the following Simulink model is constructed.

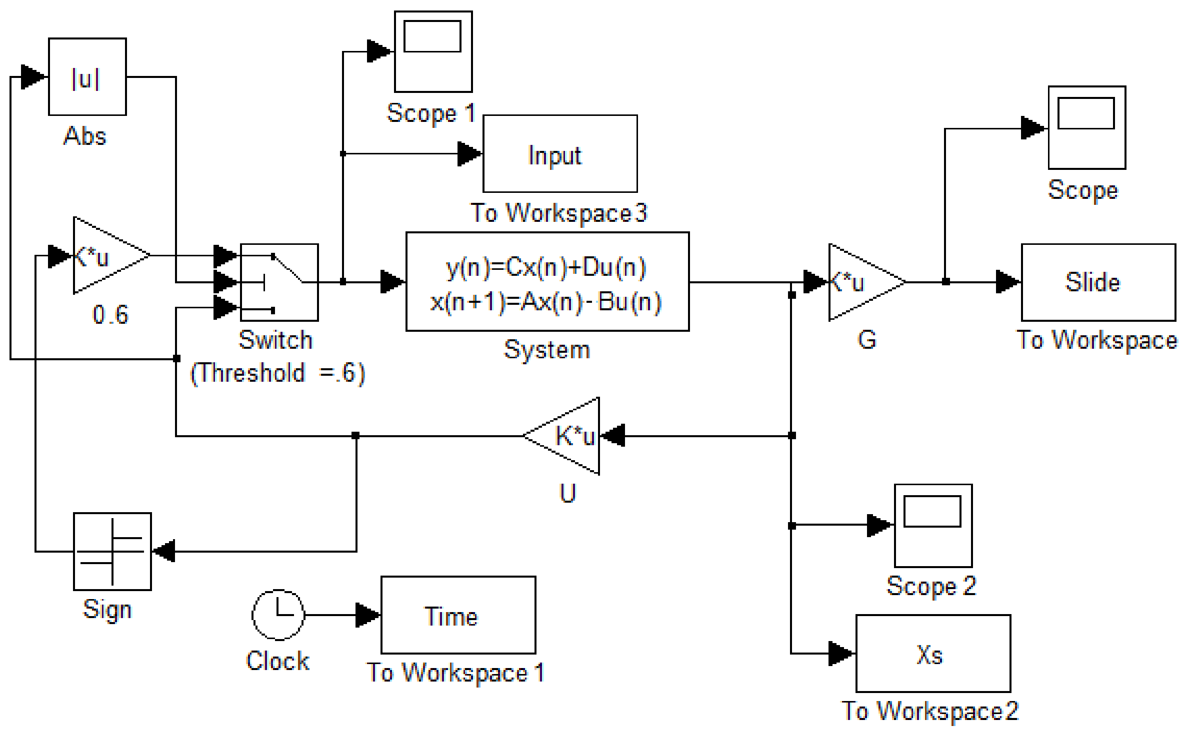

Figure 3.13: Bartolini's method

The resultant pattern of the state variables is given by the following graph.

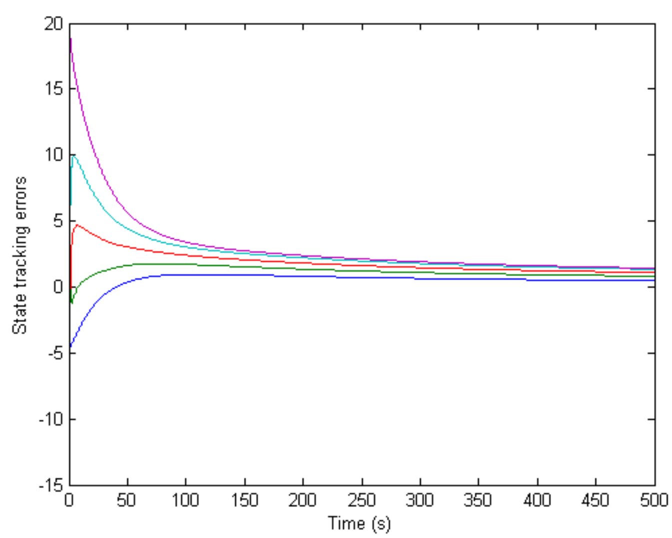

Figure 3.14: State tracking errors (with $\alpha = 0.6$)

From the above plot, it is clear that the traditional approach of setting the coupling parameter equal to zero does not present favourable results due to the fact that the coupling between the two systems is large that one system does not allow the other to settle to zero and vice versa. Therefore, there arises a need to apply alternative approaches to control the system.

Decoupling Approach

Another more comprehensive approach to decouple the subsystems would be to use the decoupling techniques. Applying Gajić and Shen's transformation on the above system, we get the following tranformation matrix,

$$T = \begin{pmatrix} 1.0000 & 0 & 0 & 0.0090 & 0.1832 \\ 0 & 1.0000 & 0 & -0.1972 & 0.1751 \\ 0 & 0 & 1.0000 & 0.5603 & -0.3584 \\ -0.4963 & 0.3213 & -0.5716 & 0.6119 & 0.1702 \\ -0.8301 & -0.0282 & 0.0300 & 0.0149 & 0.8322 \end{pmatrix}$$

This matrix is used to transform the system as follows

$$A_n = TAT^{-1} = \begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix} = \begin{pmatrix} -0.9895 & -0.0057 & -0.0012 & 0 & 0 \\ -0.1172 & -0.8128 & -0.0581 & 0 & 0 \\ -0.0090 & -0.1289 & -0.8014 & 0 & 0 \\ 0 & 0 & 0 & -0.6186 & -0.1821 \\ 0 & 0 & 0 & -0.0129 & -0.9855 \end{pmatrix}$$

$$B_n = TB \begin{pmatrix} B_1 & \epsilon B_2 \\ \epsilon B_3 & B_4 \end{pmatrix} = \begin{pmatrix} -0.0001 & 0.0004 \\ -0.0052 & -0.0031 \\ -0.0107 & 0.0165 \\ 0 & 0.0078 \\ -0.0004 & 0.0017 \end{pmatrix}$$

The resultant system has two decoupled sub-systems, which are weakly-coupled through the input matrix. Thus, two independent sliding surfaces are designed by neglecting

$O(\epsilon)$ coupling in the matrix B. Consider the first sub-system. Let

$$N_1 = \text{null}(B_1') = \begin{pmatrix} -0.4376 & -0.8991 \\ 0.8105 & -0.3894 \\ -0.3894 & 0.2000 \end{pmatrix}$$

then

$$H_1 = [N_1 B_1]' = \begin{pmatrix} -0.4376 & 0.8105 & -0.3894 \\ -0.8991 & -0.3894 & 0.2000 \\ -0.0001 & -0.0052 & -0.0107 \end{pmatrix}$$

Transferring to new co-ordinates

$$A_{1new} = H_1 A_1 H_1^{-1} = \begin{pmatrix} -0.7440 & -0.0180 & 1.9162 \\ -0.1155 & -0.9818 & -1.3792 \\ 0.0007 & -0.0012 & -0.8778 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$B_{1new} = H_1 B_1 = 10^{-3} \begin{pmatrix} 0 \\ 0 \\ 0.1427 \end{pmatrix} = \begin{pmatrix} 0 \\ B_{r1} \end{pmatrix}$$

The eigenvalues of A_{11} are at - 0.7356 and - 0.9903. To shift these eigenvalues to -.5 and -.6 respectively, the following sliding surface is designed.

$$s_1 = C_1 e_k$$

where

$$C_1 = [K_1 \ \ -1] H_1$$

and K_1 is the feedback gain matrix for placing the eigenvalues of a system with matices A_{11} and A_{12} at the desired locations.

$$K_1 = \begin{pmatrix} 0.7965 & 1.5604 \end{pmatrix}$$

Therefore,

$$C_1 = \begin{pmatrix} -1.7516 & 0.0432 & 0.0126 \end{pmatrix}$$

The resultant feedback equivalent control input for this sub-system is

$$u^1_{eq}(k) = C_1 A_1 e^1_k = \begin{pmatrix} 1.7278 & -0.0266 & -0.0105 \end{pmatrix} e^1_k$$

Following a similar approach for the second sub-system,

$$N_2 = \text{null}(B_4') = \begin{pmatrix} -0.2069 \\ 0.9784 \end{pmatrix}$$

$$H_2 = [N_2 B_4]' = \begin{pmatrix} -0.2069 & 0.9784 \\ 0.0078 & 0.0017 \end{pmatrix}$$

Transferring to new co-ordinates

$$A_{4new} = H_2 A_4 H_2^{-1} = \begin{pmatrix} -0.9303 & -9.8630 \\ -0.0020 & -0.6738 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$B_{4new} = H_2 B_4 = 10^{-4} \begin{pmatrix} 0 \\ 0.6379 \end{pmatrix} = \begin{pmatrix} 0 \\ B_{r2} \end{pmatrix}$$

The eigenvalues of A_{11} is at - 0.9303. To shift this eigenvalue to -.55, the required K_2 is 0.0386. Consequently,

$$C_2 = [K_2 \;\; -1] H_2 = \begin{pmatrix} -0.0158 & 0.0361 \end{pmatrix}$$

The resultant feedback equivalent control input for this sub-system is

$$u^2_{eq}(k) = C_2 A_4 e^2_k = -\begin{pmatrix} 0.0093 & -0.0327 \end{pmatrix} x^2_k$$

Thus, the equivalent control input of the complete system is given by

$$u_{eq}(k) = \begin{pmatrix} u^1_{eq}(k) & 0_{1\times 2} \\ 0_{1\times 3} & u^2_{eq}(k) \end{pmatrix} \times T^{-1} =$$

$$\begin{pmatrix} 1.2737 & -0.0536 & -0.0097 & -0.0207 & -0.4061 \\ -0.0270 & -0.0036 & 0.0061 & 0.0088 & -0.0325 \end{pmatrix} \begin{pmatrix} e^1_k \\ e^2_k \end{pmatrix}$$

The control law used in this approach is

$$u_k = \begin{cases} \alpha \mathrm{sgn}(u_{keq}) \text{ if } |u_{keq}| > \alpha \\ u_{keq} \text{ if } |u_{keq}| \leq \alpha \end{cases} \tag{3.30}$$

with $\alpha = 0.6$.

The resultant convergence pattern of the state variables is given by

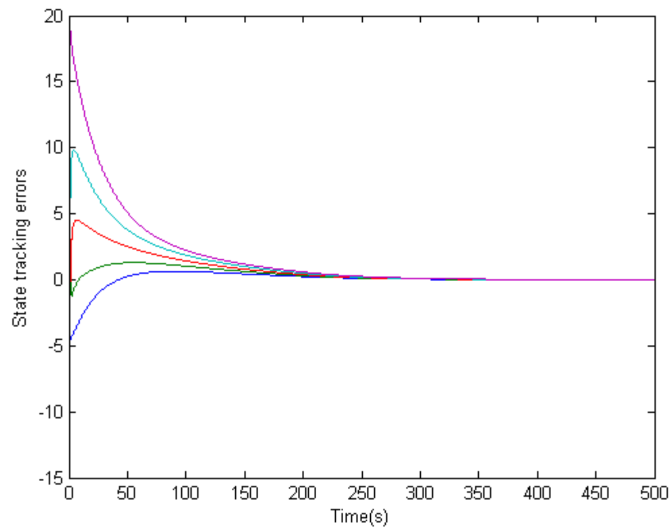

Figure 3.15: Tracking errors of state variables (with $\alpha = 0.6$)

Comparing results presented in Figures 3.15 and 3.14 it is obvious that the transformation approach outperforms the traditional approach. Note that in Figure 3.14 the plot does not converge to zero. This is due to the fact that the coupling between the sub-systems had been neglected. Comparatively, the plot in Figure 3.15 completely converges to zero due to the implementation of decoupling.

2. Relay Control Algorithm

The control law to be applied to achieve relay control (1.17) is

$$u(k) = \delta \mathrm{sgn}(g(k))$$

Both the traditional and transformation decoupling approaches are studied for the above introduced algortihm.

Traditional Approach

Considering the same distillation column system used earlier. Setting $\epsilon = 0$, in the system matrices we get

$$A = \begin{pmatrix} -0.9895 & -0.0056 & -0.0003 & 0 & 0 \\ -0.1173 & 0.8145 & -0.0760 & 0 & 0 \\ -0.0088 & 0.1239 & -0.7502 & 0 & 0 \\ 0 & 0 & 0 & -0.6683 & -0.1508 \\ 0 & 0 & 0 & -0.0133 & -0.9852 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 \\ -0.0061 & 0 \\ -0.0083 & 0 \\ 0 & 0.0184 \\ 0 & 0.0014 \end{pmatrix} = \begin{pmatrix} B_1 & 0 \\ 0 & B_4 \end{pmatrix}$$

The resultant system represents two completely independent sub-systems and each can be dealt with separately. A sliding surface is constructed for each of the two sub-systems and they are given by the following equations.

First sub-system:

$$g_k^1 = C_1 e_k^1$$

where

$$C_1 = [K_1 \ \ -1]H_1$$

and K_1 is the feedback gain matrix used for placing the eigenvalues of the system at the desired locations.

$$K_1 = \begin{pmatrix} 0.9165 & 1.2585 \end{pmatrix}$$

Therefore,

$$C_1 = \begin{pmatrix} -1.5569 & -0.0078 & -0.0034 \end{pmatrix}$$

Second sub-system: Similarly to the first sub-system take

$$g_k^2 = C_2 e_k^2$$

where

$$C_2 = [K_2 \;\; -1]H_2$$

Here K_2 is 0.2122. Consequently,

$$C_2 = \begin{pmatrix} -0.0348 & 0.2102 \end{pmatrix}$$

The control law is given by

$$u(k) = \delta \mathrm{sgn}(g(k))$$

To simulate this algorithm, the following simulink model is constructed.

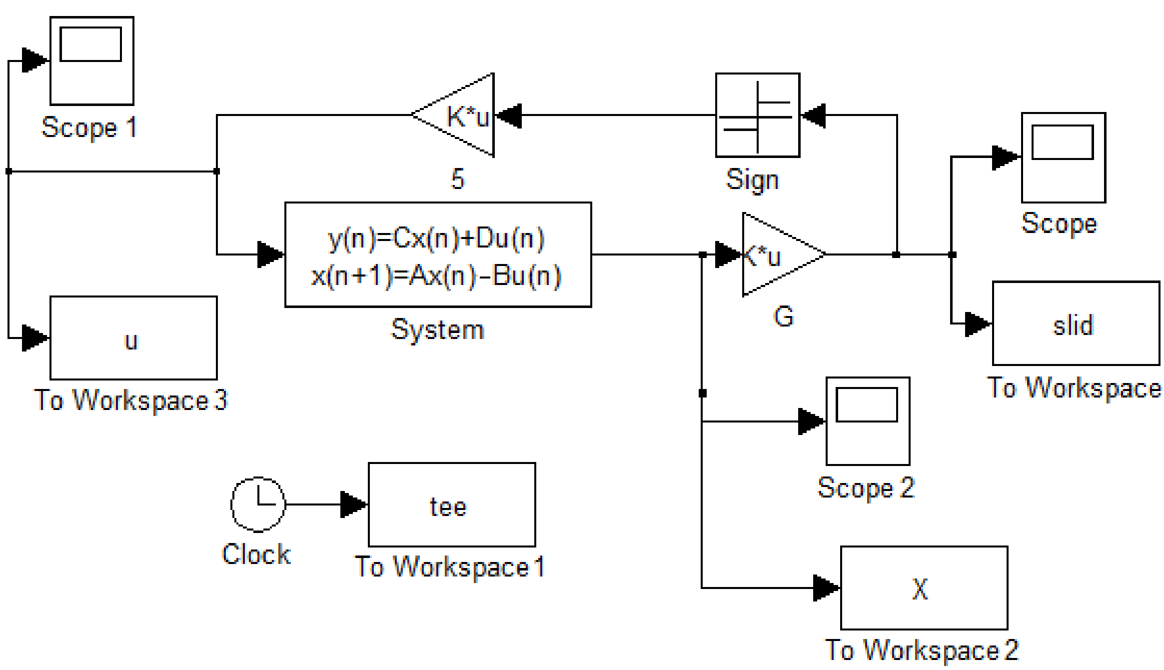

Figure 3.16: Relay control algorithm

The resultant pattern of the state variables is given by the following graph.

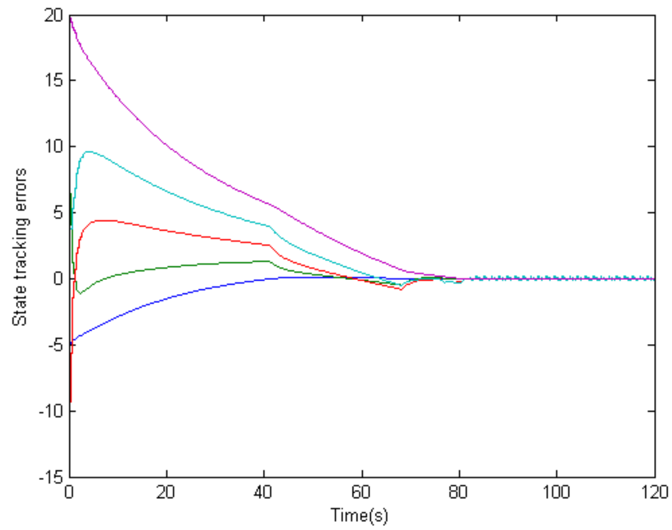

Figure 3.17: Behaviour of the state tracking errors with the feedback gain $\delta = 5$

Decoupling Approach

Applying Gajić and Shen's transformation on the above system, we get the following decoupled system matrices

$$A_n = \begin{pmatrix} -0.9895 & -0.0057 & -0.0012 & 0 & 0 \\ -0.1172 & -0.8128 & -0.0581 & 0 & 0 \\ -0.0090 & -0.1289 & -0.8014 & 0 & 0 \\ 0 & 0 & 0 & -0.6186 & -0.1821 \\ 0 & 0 & 0 & -0.0129 & -0.9855 \end{pmatrix}$$

$$B_n = \begin{pmatrix} -0.0001 & 0.0004 \\ -0.0052 & -0.0031 \\ -0.0107 & 0.0165 \\ 0 & 0.0078 \\ -0.0004 & 0.0017 \end{pmatrix}$$

Two independent sliding surfaces are designed for the two sub-systems respectively. This is done in the same manner as discussed in the previous algorithm.

Consider the first sub-system.

$$g_1 = C_1 e_k$$

where

$$C_1 = [K_1 \;\; -1]H_1$$

and K_1 is the feedback gain matrix for placing the eigenvalues of a system with matices A_{11} and A_{12} at the desired locations.

$$K_1 = \begin{pmatrix} 0.7965 & 1.5604 \end{pmatrix}$$

Therefore,

$$C_1 = \begin{pmatrix} -1.7516 & 0.0432 & 0.0126 \end{pmatrix}$$

Following a similar approach for the second sub-system,

The required K_2 is 0.0386. Consequently,

$$C_2 = [K_2 \;\; -1]H_2 = \begin{pmatrix} -0.0158 & 0.0361 \end{pmatrix}$$

The control law of this approach is

$$u(k) = \delta \mathrm{sgn}(g(k))$$

The resultant convergence pattern of the state variable tracking errors is presented in Figure 3.18

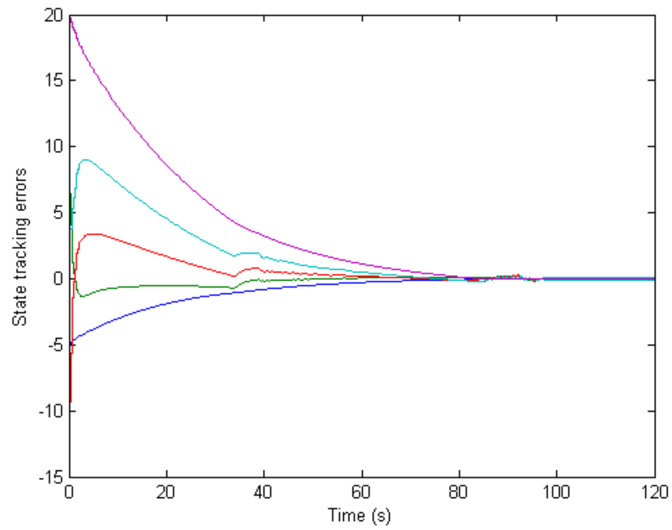

Figure 3.18: Behaviour of state tracking errors with feedback gain $\delta = 5$

In comparison to Utkin and Young's control approach, this method appears to work for both traditional and decoupling approaches since there appears no u_{eq} term in the control law. The major disadvantage though is chattering of the output and high speed swtching of the control input as shown in Figure 3.19.

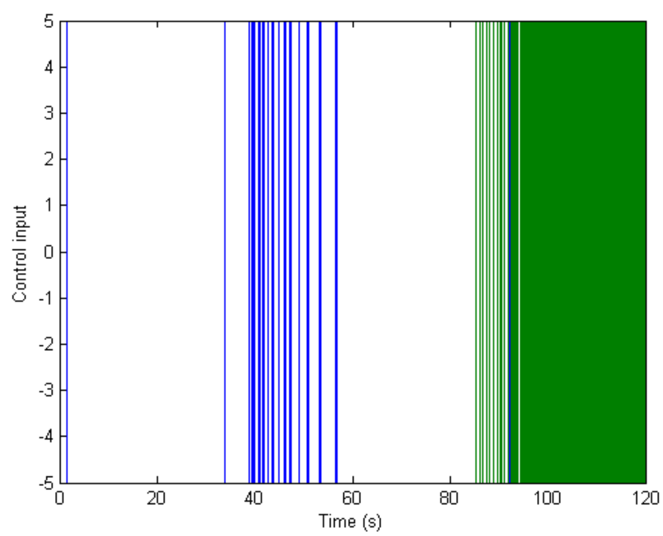

Figure 3.19: High speed switching of control inputs

3. Gao's Control Algorithm

This algorithm proposes the following control law

$$u(k) = [u_{eq}(k) - (1 - qT)g(k) + pT\text{sgn}(g(k))] \tag{3.31}$$

The values of q and p are chosen as 0.8 and 0.1 respectively. This gives a quasi-sliding mode domain (Δ) of 0.5 as follows

$$2\Delta = 2pT/(1 - qT) = 2 \times 0.1 \times 1/(1 - 0.8 \times 1) = 1$$

The application of this control to weakly-coupled systems is done using both the traditional and decoupling tehniques in the following sections with a numerical example.

Using the results of the distillation column example studied earlier, we can obtain u_{eq} for both the traditional and transformation decoulpling approaches. The following Simulink model is used to observe the behaviour of the state variables.

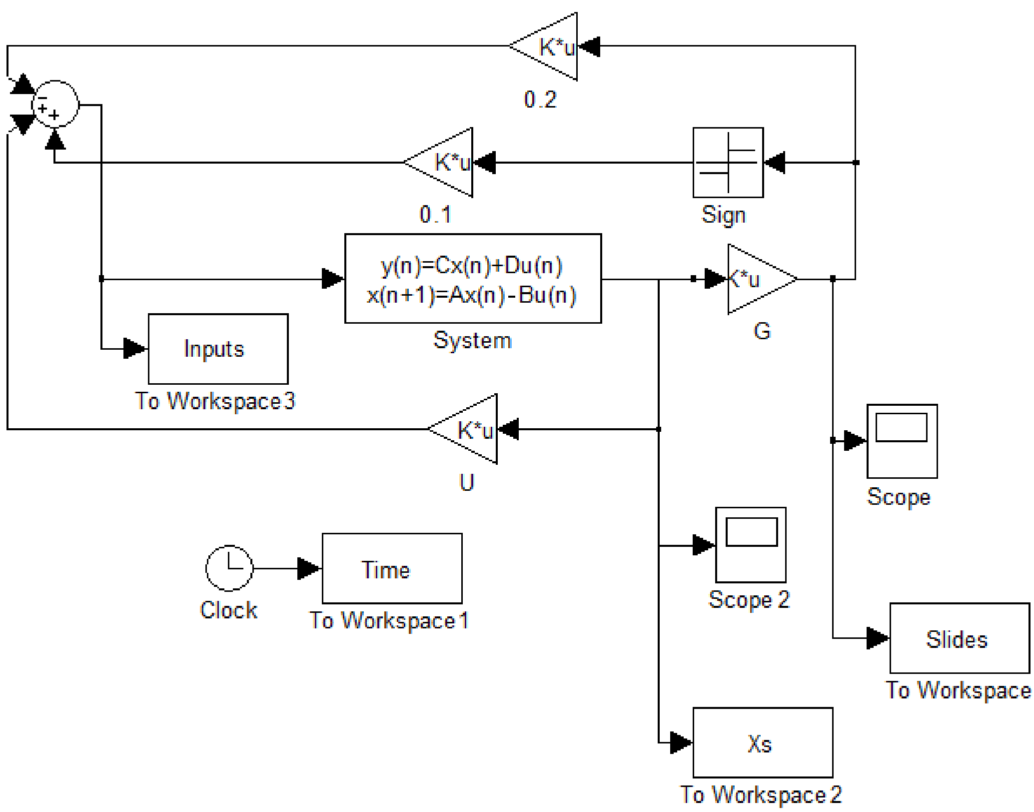

Figure 3.20: Gao's control algorithm

Traditional Approach

On setting the coupling parameter $\epsilon = 0$ and designing sliding surfaces for each sub-system, the following equivalent control was obtained to make the tracking errors stay on the sliding surfaces.

$$u_{eq}(k) = \begin{pmatrix} u_{eq}^1(k) & 0_{1\times 2} \\ 0_{1\times 3} & u_{eq}^2(k) \end{pmatrix} =$$
$$\begin{pmatrix} 1.5399 & 0.0036 & -0.0098 & 0 & 0 \\ 0 & 0 & 0 & 0.0205 & -0.2018 \end{pmatrix} \begin{pmatrix} e_k^1 \\ e_k^2 \end{pmatrix}$$

Substituting this in equation (3.31) and implementing it in the Simulink model the following plot is obtained.

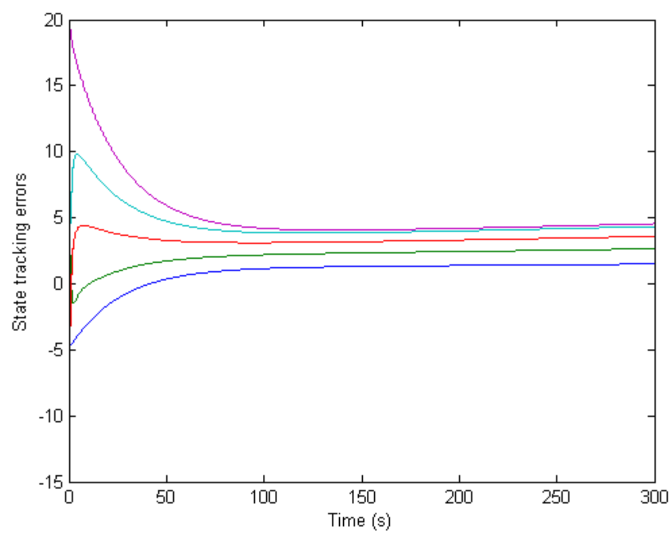

Figure 3.21: Behaviour of state variable tracking errors (with p =0.1 and q =.8)

It is observed that the states do not converge which implies the failure of the traditional approach of separating weakly coupled systems into their individual sub-units.

Decoupling Approach

Applying Gajić and Shen's transformation on the distillation column system, two decoupled sub-systems are obtained which are coupled only by the control inputs. Sliding surfaces are constructed for each sub-system and the equivalent control input is obtained. Obtained u_{eq} is given by

$$u_{eq}(k) = \begin{pmatrix} u_{eq}^1(k) & 0_{1\times 2} \\ 0_{1\times 3} & u_{eq}^2(k) \end{pmatrix} \times T^{-1} =$$

$$\begin{pmatrix} 1.2737 & -0.0536 & -0.0097 & -0.0207 & -0.4061 \\ -0.0270 & -0.0036 & 0.0061 & 0.0088 & -0.0325 \end{pmatrix} \begin{pmatrix} e_k^1 \\ e_k^2 \end{pmatrix}$$

Substituting this in (3.31) and implementing it in the simulink model the following plot is obtained.

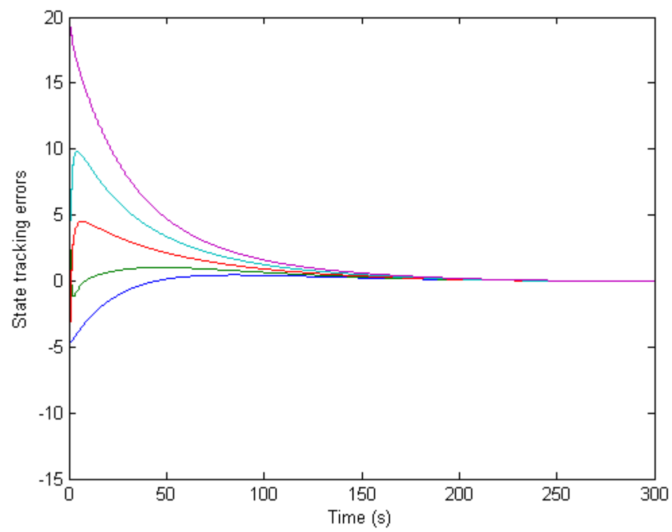

Figure 3.22: Convergence of state tracking errors (with p =0.1 and q =.8)

The above graph shows that this algorithm can be used for weakly-coupled systems provided decoupling is done prior to designing the sliding surfaces.

3.4 Composite Control

Composite control is the method of using two or more control approaches on a single system. In the case of weakly-coupled systems this refers to the use of different control algorithms for each subsystem. The above observations indicate that though the Utkin and Young method is an effective control scheme for all applications, other methods could be better suited for certain specific applications. For example, in an application where there is a definite threshold on the magnitude of the input signal, the relay control algorithm or Bartolini's method would be better suited. In some cases,a certain subsystem alone might have such a threshold in which case it would be best to use relay control on that particular subsystem alone while applying Utkin's method for the rest of the system.

3.4.1 Example

In the distillation column control problem, let us assume that the control input to the second subsystem is limited. Applying Utkin's control to the first subsystem and relay control to the second subsystem leads to the following results.

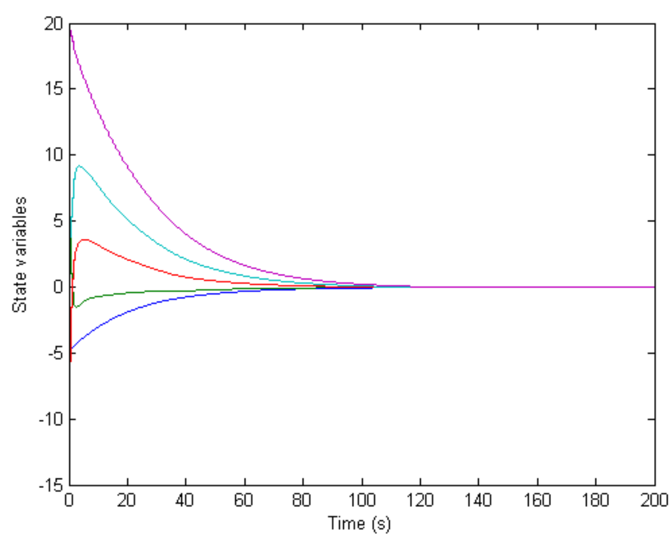

Figure 3.23: Convergence of state variables

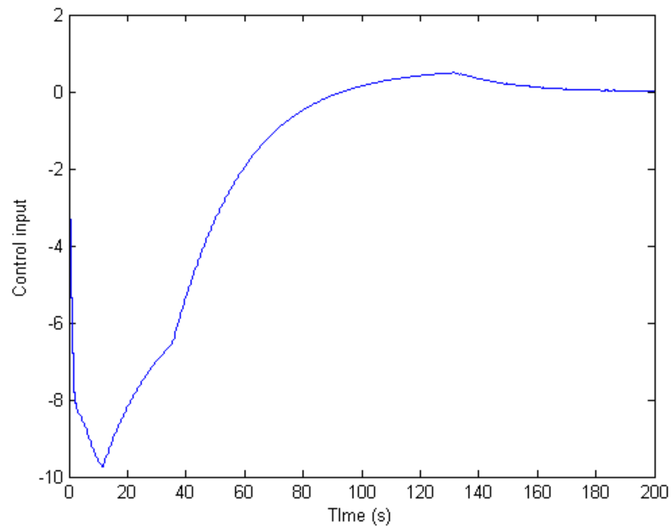

Figure 3.24: Control input of the first subsystem (U1)(Utkin & Young's)

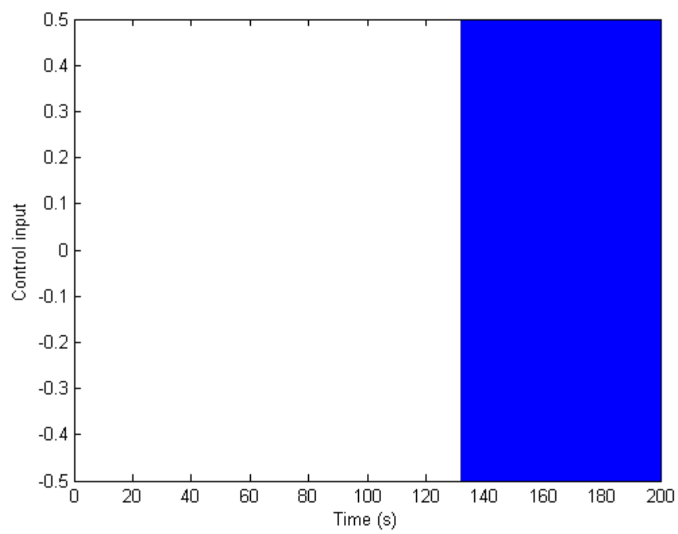

Figure 3.25: Control input of the second subsystem (U2)(relay control)

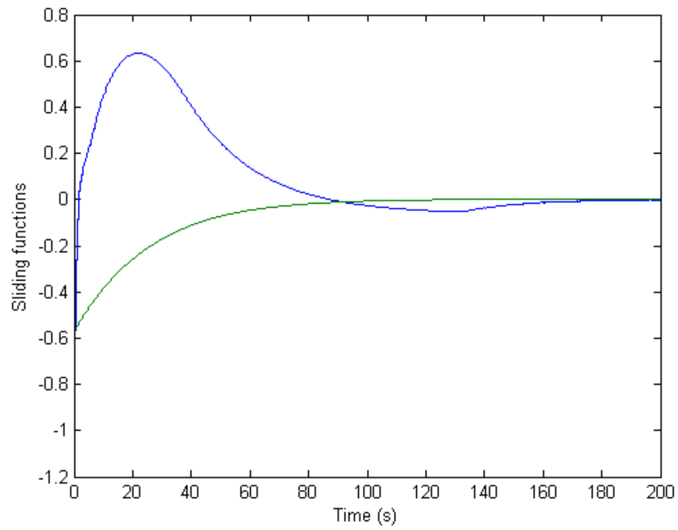

Figure 3.26: Sliding surfaces

Note that, the state convergence in this case takes around twice the time of that of a homogeneous Utkin and Young approach. But, the input requirement to the second subsystem remains well below the set threshold (ϵ =.5). By increasing this threshold value, it is possible to achieve faster convergence.

Chapter 4

Conclusions and Future Work

4.1 Conclusions

The study of sliding mode control for discrete-time weakly coupled systems is firstly introduced in this thesis. It is shown that using the decoupling transformation of weakly coupled systems, we can apply the sliding mode control technique to the individual subsystems. It is to be noted that fundamental choices like that of the reaching condition have several implications and have to be tailored to suit the type of the system being considered. Moreover, the Utkin and Young's method of designing sliding surfaces for continuos-time MIMO systems has been extended to discrete-time MIMO systems and has been proven to work well. Several other methods of discrete-time sliding mode control have been discussed and implemented. Though each method holds in its own importance in different applications and practical conditions, it can be concluded that the Utkin and Young's method is the most generally effective method considering the various trade-offs involved in these different techniques. Combining efficient decoupling and sliding mode control techniques, this thesis provides a novel and proficient control methodology for discrete-time weakly coupled systems. Moreover, the composite control approach provides for suitably adjusting the control based on presented constraints in any application. It also shows that the choice of control of one subsystem is completely independent from another though they are part of the same full-order system.

4.2 Future Work

We can extend the results of this thesis to the systems composed of N weakly coupled subsystems using the decoupling transformation for N weakly coupled subsystems. Also the multi-step prediction based discrete-time sliding mode control algorithm of Lingfei and Hongye (2008) can be applied on discrete-time weakly coupled systems to analyse its benefits. Furthermore, the sliding mode of a deterministic weakly coupled system could be developed with feedback of estimated states, and the optimal sliding Gaussian control of the weakly coupled system could be found for stochastic systems.

There is also scope to apply different mathetical techniques to try and decouple both the A and B matrices of a state-space weakly-coupled system at the same time so that we can achieve both internal and external decoupling which will further enable us to treat subsystems as completely separated systems.

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