A STUDY OF LARGE DEFLECTION OF BEAMS AND PLATES

BY VINESH V. NISHAWALA

A thesis submitted to the

Graduate School—New Brunswick

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements

for the degree of

Master of Science

Graduate Program in Mechanical and Aerospace Engineering

Written under the direction of

Dr. Haim Baruh

and approved by

New Brunswick, New Jersey

January, 2011

ABSTRACT OF THE THESIS

A Study of Large Deflection of Beams and Plates

by Vinesh V. Nishawala
Thesis Director: Dr. Haim Baruh

For a thin plate or beam, if the deformation is on the order of the thickness and remain elastic, linear theory may not produce accurate results as it does not predict the in plane movement of the member. Therefore, a geometrically nonlinear, large deformation theory is required to account for the inconsistencies. This thesis discusses nonlinear bending and vibrations of simply-supported beams and plates. Theoretical results are compared with other well-known solutions. The effects of geometric nonlinearities are discussed. The equation of motion for plates with 'stress-free' and 'immovable' edges are derived using modal analysis in conjunction with the expansion theorem. Theoretical results are compared with a finite element simulation for plates. 'Immovable' edges are studied for beams. For large bending of beams with 'stress-free' edges, a theory by Conway is presented. A brief introduction to Duffing's equation and Gaussian curvature is presented and their relevance to nonlinear deformations are discussed.

Nomenclature

$\bar{\delta}$	Dirac delta function
$\bar{\rho}$	Density - mass per unit volume
κ_{ij}	Curvature
$\mathcal{B}_{ij}$	A generic basis
∇^2	Biharmonic operator
ν	Poission's ratio
Ω	Excitation frequency
ω_{ij}	Natural frequency
ϕ	Airy's stress function
ρ	Plate: mass per unit area ($= \bar{\rho}h$). Beam: mass per unit length ($= \bar{\rho}bh$)
σ_{ij}	Stress
θ	Slope of deflection
A	Cross sectional area
a	Plate length
b	Beam or plate width
D	Flexural rigidity
E	Young's modulus

E_{ij}	Strain
h	Beam or plate thickness
I	Area moment of inertia
k	Bending stiffness for a beam ($= EI$)
L	Length
$M(x, y, t)$	Moment
$N(x, y, t)$	Membrane force
$Q(x, y, t)$	Shear force
$q(x, y, t)$	Applied transverse load
u, v, w	Displacements of the midplane in the x, y, and z directions, respectivley.
u_x, u_y, u_z	Displacements in the x, y, and z directions, respectivley.
$W(x, y, t)$	A modal function
$w_{mn}(t)$	Coefficient of the modal function

Acknowledgements

I would like to first thank Dr. Haim Baruh, my thesis advisor. His help throughout the years have made this work possible. Our conversations have, and always will be, of great value to me.

Also my committee members Dr. Haym Benaroya and Dr. Ellis H. Dill. Thank you for the time and effort in reading and reviewing my thesis. I would like to acknowledge Dr. William J. Bottega for his thorough teaching of the topics that have laid the foundation for this work.

Thank you to the SMART program for providing funding and allowing me to smoothly move beyond academia. I would also like to thank Mr. Kevin Behan, and Dr. Anthony Ruffa, my contacts at the Naval Undersea Warfare Center Division Newport, for helping me begin my professional career.

Rutgers University, thank you for letting me grow into the person I am today. Experiences as an undergraduate and a graduate student will be with me throughout my life. I would also like to acknowledge the Mechanical and Aerospace Engineering faculty at Rutgers as their interactions with me as an undergraduate motivated me to graduate studies.

I would like to acknowledge my colleagues, Dhaval D. Dadia, Jonathan Doyle, Adam J. Nagy, Tushar Saraf and Mark Seitel. I would especially like to thank Alexey S. Titovich, who has helped me tremendously in coursework as well as in my thesis.

Finally, I would like to thank my family for their unconditional support for my graduate studies. My parents, for their emotional support and understanding that a thesis is not completed within a day. My brother, who is the best brother one could ask for.

Dedication

To my father, Vijay Nishawala,

my mother, Niranjana Nishawala,

and my brother, Jatin Nishawala.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Motivation

When beams and plates are deflected beyond a certain magnitude, the linear theory loses its validity and produces incorrect results. Linear theory can predict that the deflection of the member may exceed the length of the member, which is unrealistic. In order for an accurate large deflection solution, one needs to include the coupling between axial and transverse motion, which is geometric nonlinearity. If the edges are allowed to move freely within the plane of the undeformed member, this boundary condition is called 'stress-free'. If the edges are restricted from moving, the edges require an equivalent axial load to prevent motion, which is called 'immovable' boundary conditions.

Nonlinear deflection theories also couple axial loads and transverse deflections. This becomes useful for buckling problems or axially loaded structures. An applied axial load can act as if it were stiffening or softening the member. This characteristic becomes important when the structure is rotated about an axis, like a helicopter blade or a compact disc, where the member is in tension, causing a stiffening of the member.

The modal solution used to solve the linear theory will be used to solve the nonlinear theory. Since we consider a simply-supported beam and plate, the solution is either a sine or double sine infinite sum, which simplifies the calculation tremendously. While in this study we only consider a simply-supported members it may be possible to extend this method of solution to other boundary conditions.

1.2 Literature Review

A short history of plate theory and nonlinear plate theory will be given below. Two highly cited literature reviews on nonlinear vibrations are by Chia [11] and Sathyamoorthy [39]. After Kirchhoff [25] established the classical linear plate theory, von Karman [48] developed his nonlinear plate theory. One of the first to study nonlinear plate dynamics were Chu and Herrmann [12], who began with vibrations of simply supported rectangular plates. The Reissner-Mindlin plate theory [31] took into account shear strains which is useful for thicker or composite plates. The Reissner-Mindlin plate theory is considered to be a 'first-order' shear theory.

Leung and Mao [27] compared the solution between movable and immovable edges of simply supported rectangular plates using Galerkin's Method. El Kadiri and Benamar [17] used the case of Chu and Herrmann [12] and created a simplified analytical model.

Berger [8] simplified nonlinear plate theory by ignoring terms in the strain energy. Prathap and Pandalai [35] incorporated rotary inertia and the correction for shear in their study of nonlinear plate theory. Yosibash and Kirby [52] compared three different versions of the geometric nonlinear plate theory. One version was neglecting the rotatory inertia term. The second simplified model ignored rotatory inertia as well as the time dependent terms of the plate in-plane coordinates. The reasoning to ignore the terms is that they are multiplied by the thickness squared, which is considered a small quantity. Other terms in the equation were not multiplied by the thickness at all. The last version included all of those terms. Amabili [3] considered several different boundary conditions and compared theoretical and experimental results.

Way [50] used Ritz energy method, Leung and Mao [27] used Galerkin's Method, Ribeiro [38] used finite element models and Wei-Zang and Kai-Yuan [51] utilized perturbation theory as an approximate method to solve the system. The double Fourier series was used by Levy [28] to investigate a simply supported plate under various conditions. Iyengar and Naqvi [24] used a combination of trigonometric and hyperbolic functions to approximate the defelction of clamped and simply-supported plates under immovable and stress-free edge conditions. Leissa [26] studied multiple boundary conditions of plates of different shapes for linear plate theory. Timoshenko's [45] book *Theory of Plates and Shells* is highly cited in the field and should be considered the first reference to consult for problems in plate theory.

1.3 Nonlinearities

Nonlinearities exist in an equation of motion when the products of variables, or their derivatives, exist. They can also exist when there are discontinuities or jumps in the system. There are several sources of nonlinear behavior.

One source is geometric nonlinearity. This characteristic is important to systems with large deformations, or systems that may fail due to buckling. In beams and plates, the nonlinearity is from the nonlinear strain equations, where the transverse displacement is coupled to the axial strains. As a result, mid-plane stretching of the beam or plate may occur. The von Kármán, or large deformation, theory of plates uses geometric nonlinearity in its derivation.

Nonlinear moment-curvature relationship become significant when we consider large deformations without stretching. This analysis does not consider the slope of the deflected middle surface to be small compared to unity. This analysis is usually done in terms of the slope of the beam.

Another cause of nonlinearity is material properties. These nonlinearities would render Hooke's law invalid because Hooke's law is a linear relationship between stress and strain. Hooke's law would have to be altered in order to account for the nonlinear relationship. In the elastic region of materials, we can define the slope of the linear region as the Young's modulus. However, this is just an approximation we use in order to simplify the system. Also, the material is considered isotropic, with the same material properties in all directions, but this too is an approximation of the material properties. It is important to note that no material has a perfectly linear elastic modulus or is perfectly isotropic, these are just approximations that are satisfactory for most situations. Models of materials with nonlinear elastic properties, like rubber, or anisotropic materials, like composites, end up in being nonlinear in the equations of motion. Material nonlinearities are not considered in this thesis.

Nonlinear systems are also caused by nonlinear boundary conditions. Examples to such a phenomenon include the use of a nonlinear spring or damper on the edge of a plate, or the case of a nonlinear spring in a mass-spring-damper system. Duffing's equation is a special case of a cubic nonlinear spring in a mass-spring-damper system.

The above list of nonlinearities is far from complete. There are many sources of nonlinear behavior and most linear behavior is an approximation. For certain cases, linearization has negligible effects. It is important to understand the system in terms of the material model, loading and expected response, in order to determine where a linear approximation is adequate and where the use of a nonlinear theory is needed.

1.4 Kirchhoff's Hypothesis

Kirchhoff, one of the developers of linear thin plate theory, used assumptions to develop linear plate theory that are known as Kirchhoff's Hypothesis. These assumptions provide great insight into Kirchhoff's plate theory. The first assumption is that the plate is made of material that is elastic, homogenous, and isotropic. The next assumption deals with the geometry of the plate. The plate is initially flat, and that the smallest lateral dimension of the plate is at least ten times larger than its thickness. The following assumptions deal with the geometry of plate deformation. The quotes below are taken from Szilard [43]:

'*The deformations are such that straight lines, initially normal to the middle surface, remain straight lines and normal to the middle surface*'. This means that for an initially flat plate, if we were to draw a line normal to the middle surface, through the thickness of the plate, and then deform the plate, and the line would remain straight and normal to the middle surface. This statement is equivalent in saying that there are no out-of-plane shear strains. Also the assumption that *the strains in the middle surface produced by in-plane forces can usually be neglected in comparison with strains due to bending*, would cause the undeformed line and the deformed line to have the same length. As a result of these assumptions, knowing the deformation of the middle surface of the plate is sufficient to find the deflection of every point on the plate. For example in figure (1.1),

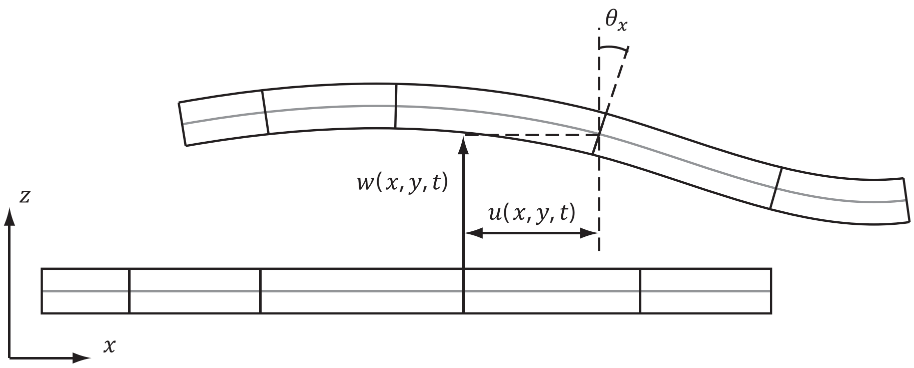


Figure 1.1: Deformation of Member

$$u_x(x, y, z) = u(x, y) - z\sin(\theta_x) \tag{1.1}$$

where θ_x is the angle between the plane of the undeformed midsurface and the tangent of deformed midsurface parallel to the x-axis at the point in question.

'*The slopes of the deflected middle surface are small compared to unity.*' This assumption allows the use of the small angle approximation in our studies. As a result, the sine of an angle can be approximated by the angle itself, $\sin(\theta_x) \approx \theta_x \approx \partial w/\partial x$. This allows our displacement equation to become

$$u_x(x, y, z) = u(x, y) - z\frac{\partial w}{\partial x} \tag{1.2}$$

'*The stresses normal to the middle surface are of a negligible order of magnitude*' allows the simplification of Hooke's law such that the terms with σ_z can be ignored.

The next two assumptions can be applied to small-deflection theory only.

- The deflections are small compared to the plate thickness. A maximum deflection from one tenth to one fifth of the thickness is considered as the limit for small-deflection theory.

- The deflection of the plate is produced by the displacement of points of the middle surface normal to its initial plate.

For large deflection theory, we begin by considering the bending and stretching of the plate. Stretching means that the in-plane strains are no longer zero and the deformed surface is a 'non-developable' surface, meaning that it no longer has zero Gaussian curvature. Also, the deflections are on the order of the plate thickness.

1.5 Outline

In all the cases studied, we consider a simply-supported beam or plate with either 'stress-free' or 'immovable edges'. Chapter 2 describes geometrically nonlinear theory of beams. The chapter begins with a derivation of the equation of motion followed by a solution procedure for linear theory of beams and then the nonlinear theory of beams. For static 'stress-free' edges, the theory by Conway is presented. Chapter 3 studies the geometrically nonlinear theory of plates. The chapter begins with the derivation and then presents the solution for linear theory of plates. That solution is followed by the static and dynamic solution to nonlinear theory of plates for both 'stress-free' and 'immovable edges'. A comparison between the theoretical results and a finite element simulation is presented. Conclusions and future work are presented in Chapter 4.

Chapter 2

Large Deflection of Beams

2.1 Introduction

In the traditional study of transverse motion of beams, coupling between the axial (membrane) forces and the transverse motion, known as geometric nonlinearity, is ignored. This assumption is widely used to help predict small deflection of beams. However, when the membrane force becomes significant, like in the case of buckling and large deformations, the linear theories become inaccurate and the need for a new model arises. The beam theory below takes into account the effect of the axial motion, as well as the membrane forces. As a result of the new 'degree of freedom' of the model, we need to define new boundary conditions that define the axial motion of the beam edges. This model, Euler-Bernoulli beam theory, does not consider the effect of rotatory inertia and the correction for transverse shear. The nonlinear beam theory below is valid for deflections on the order of magnitude of the beam's thickness. This particular large deflection model is used because it parallels to large deflection of plates as that it also requires axial boundary conditions.

Below, we will consider a pinned-pinned beam and compare the static and dynamic responses between the linear and nonlinear theories. For beams, it will be shown that geometric nonlinearity can only be applied to beams with immovable edges. When applied to stress-free edges, as it will be shown here, one loses the nonlinearity. Thus, a theory by Conway [13] will be presented to provide some insight into the large static bending of beams with stress-free edge conditions. Conway does not translate well to plates as it does not consider the stretch of the beam as well as the kinetics of the beam and solely concentrates on the geometry of deformation. Stretching in nonlinear plate theory plays a significant role. We present Conway's formulation here as a reference and an acknowledgement that other methods of solution exist. Sathyamoorthy [41] provides a literature review on the topic.

In modal analysis, the deflection of a beam is approximated by the sum of the beam modes. Each mode has a specific shape and corresponding frequency. When a beam is loaded, each mode is excited. Modal amplitudes are the influences of the particular modes on the overall deflection of the beam. It is important to include the influence of the modes with frequencies that are adjacent to the excitation frequency as they may have the significant influence on deflections. Usually, the lower frequencies have higher amplitudes. As a result of the mode separation in a beam, the sum of the first three modes usually is a sufficient approximation to the deflection of the beam.

2.2 Governing Equations

In order to better understand beams and their motion, it is wise to begin with the derivation of the equation of motion. The derivation is given below.

2.2.1 Elasticity

Beginning with Green's Strain, u_x, u_y and u_z are displacements of the member at any point in the x, y, and z direction, respectively. u, v, and w are displacements of the middle surface in the x, y, and z direction, respectively, as shown in figure (2.2).

$$E_{xx} = \frac{\partial u_x}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial x}\right)^2 + \left(\frac{\partial u_z}{\partial x}\right)^2\right] \tag{2.1}$$

Using the 'small strain, moderate rotation' approximation results in the geometrically nonlinear strain. Small strain, moderate rotation implies that

$$\left(\frac{\partial w}{\partial x}\right)^2 \sim \mathcal{O}\left(\frac{\partial u}{\partial x}\right) \tag{2.2}$$

which allows us to retain the nonlinear term of the strain. Let u, w be middle plane deflections of the beam. Hence, from Kirchhoff's Hypothesis we can relate the middle plane deflections to u_x, u_z, the deflections of any point on the beam, see figure (2.1).

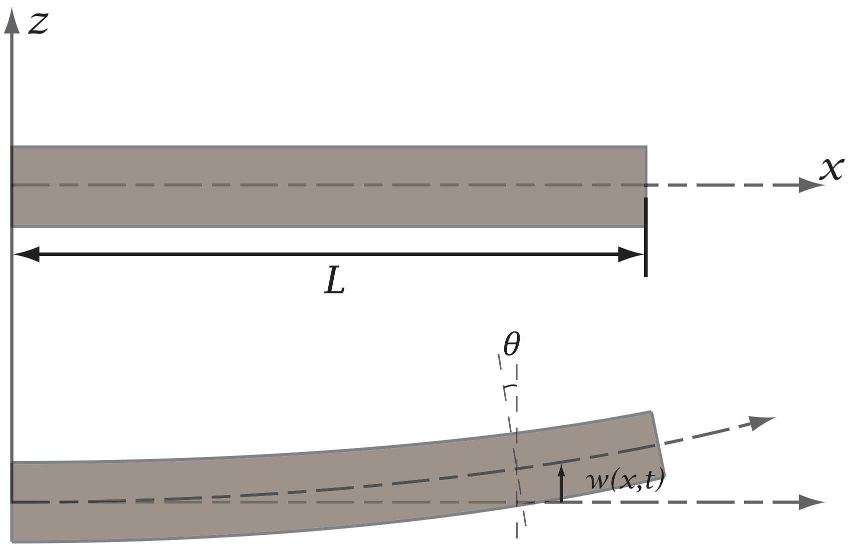


Figure 2.1: Undeformed, and Deformed Beam

$$u_x = u - z\frac{\partial w}{\partial x} \qquad u_y = v - z\frac{\partial w}{\partial y} \qquad u_z = w \tag{2.3}$$

$$\epsilon_{xx}(x,t) = \frac{\partial u_x}{\partial x} + \frac{1}{2}\Big(\frac{\partial u_z}{\partial x}\Big)^2 \tag{2.4}$$

$$\epsilon_{xx}(x,t) = \frac{\partial u}{\partial x} + \frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2 - z\kappa = \epsilon - z\kappa \tag{2.5}$$

$$\kappa = \frac{\partial \theta}{\partial x} \approx \frac{\partial^2 w}{\partial x^2} \tag{2.6}$$

where κ is the curvature and θ is the slope of the deformed beam. From Hooke's law we know $\sigma = E\epsilon_{xx}$. Note that we assume shear and normal stresses in the y or z direction are considered to be zero in magnitude. Next, we will find the resultant normal force and resultant moment of the beam below

$$N(x,t) = \int_A \sigma dA = \int_A E\epsilon_{xx} dA \tag{2.7}$$

$$= E\epsilon \int_A dA - E\kappa \int_A z dA \tag{2.8}$$

Since the x-axis passes through the centroid of the cross-section of the beam, $\int_A z dA = 0$. The resultant axial (membrane) force is

$$N(x,t) = EA\epsilon(x,t) = EA\Big(\frac{\partial u}{\partial x} + \frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2\Big) \tag{2.9}$$

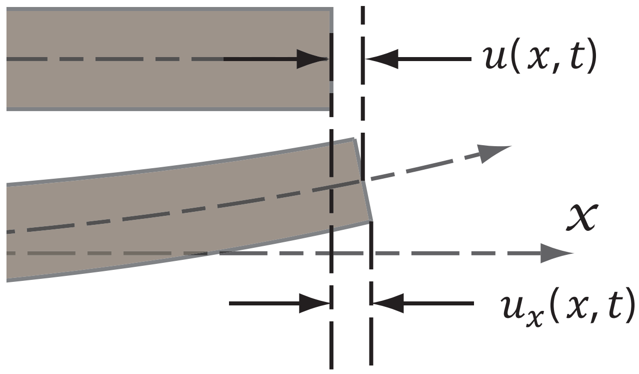


Figure 2.2: In Plane Displacement of Beam

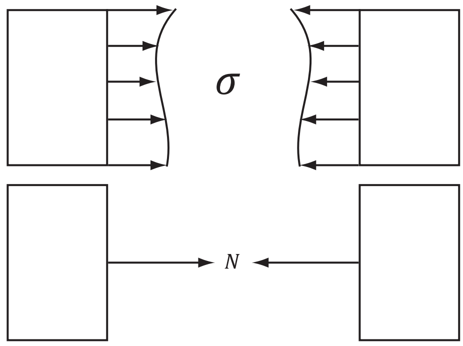


Figure 2.3: Axial Stress to Resultant Membrane Force

Calculating the bending moment

$$\begin{aligned} M(x,t) &= \int_A \sigma(x,z,t) z dA = E\epsilon(x,t)\int_A z dA - E\kappa \int_A z^2 dA \\ &= -EI\kappa(x,t) \end{aligned} \tag{2.10}$$

Where $I = \int_A z^2 dA$.

2.2.2 Derivation of Equation of Motion

In deriving the equation of motion for transverse motion of a beam there are several different levels of complexity we can add to the problem. Examples include the effects of rotatory inertia, shear distortion, body couples, and the effect of axial deformation and axial forces. Including all of these factors can result in a complex beam theory which would be cumbersome to solve and distract us from our true goal, the effect of axial deformation and force on transverse motion. Therefore, we will ignore these extra terms and concentrate only on the geometric nonlinearities. In plane stresses and displacements are critical in large deformation theory as it accounts for the stretching of the beam, an additional source of beam stiffness or buckling.

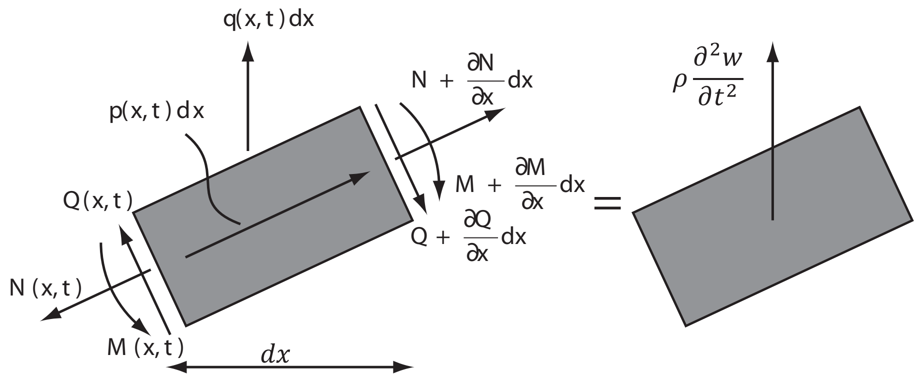


Figure 2.4: Beam Element

We begin with a deformed beam element of length dx, see figure (2.4). $q(x,t)$ is the applied transverse force, $p(x,t)$ is the applied axial force, $Q(x,t)$ is the internal shear force, $N(x,t)$ is the internal membrane force, $M(x,t)$ is the internal moment, ρ is the mass of the beam element and $w(x,t)$ is the vertical deflection of the beam. Summing forces in the vertical direction:

$$q(x,t)dx + p(x,t)dx\frac{\partial w}{\partial x} + \Big(Q(x,t) + \frac{\partial Q}{\partial x}dx\Big) - Q(x,t) = \rho(x)dx\frac{\partial^2 w}{\partial t^2} \tag{2.11}$$

$$q(x,t) + p(x,t)\frac{\partial w}{\partial x} + \frac{\partial Q}{\partial x} = \rho(x)\frac{\partial^2 w}{\partial t^2} \tag{2.12}$$

We need an equation for the shear force, $Q(x,t)$, which will come from the sum of moments about the center of the beam element.

$$\begin{aligned} &-\Big(M(x,t) + \frac{\partial M}{\partial x}dx\Big) + M(x,t) + \Big(Q(x,t) + \frac{\partial Q}{\partial x}dx\Big)\frac{dx}{2} + Q(x,t)\frac{dx}{2} \\ &-\Big(N(x,t) + \frac{\partial N}{\partial x}dx\Big)\left(\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}\,\mathrm{d}x\right)\frac{dx}{2} - N(x,t)\Big(\frac{\partial w}{\partial x}\frac{dx}{2}\Big) = 0 \end{aligned} \tag{2.13}$$

Right side of above equation is equal to zero as we are neglecting rotatory inertia.

Ignoring terms of order $(dx)^2$ and simplifying we get

$$Q(x,t) = \frac{\partial M}{\partial x} + N\frac{\partial w}{\partial x} \tag{2.14}$$

Inserting equation (2.14) into equation (2.12) results in

$$q(x,t) + p(x,t)\frac{\partial w}{\partial x} + \frac{\partial^2 M(x,t)}{\partial x^2} + \frac{\partial}{\partial x}N(x,t)\frac{\partial w}{\partial x} = \rho(x)\frac{\partial^2 w}{\partial t^2} \tag{2.15}$$

Noting that for this theory we can use the linear moment curvature relationship, equation (2.10).

$$\begin{gathered}\rho(x)\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}k(x)\frac{\partial^2 w}{\partial x^2} - N(x,t)\frac{\partial^2 w}{\partial x^2} \\ - \Big(\frac{\partial N}{\partial x} + p(x,t)\Big)\frac{\partial w}{\partial x} = q(x,t)\end{gathered} \tag{2.16}$$

We define $k(x) = EI(x)$ as the bending stiffness of the beam. For a beam under the same load, a larger k value would result in smaller deflections. From the equation of motion in the longitudinal direction

$$\rho(x)\frac{\partial^2 u}{\partial t^2} - \frac{\partial N}{\partial x} = p(x,t) \tag{2.17}$$

We will assume that the acceleration in the longitudinal direction is insignificant compared to the acceleration in the transverse direction. We do this by setting $\frac{\partial^2 u}{\partial t^2} \approx 0$, which results in

$$\frac{\partial N}{\partial x} \approx -p(x,t) \tag{2.18}$$

Substituting equation (2.18) into equation (2.16) we get

$$\boxed{\rho(x)\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}k(x)\frac{\partial^2 w}{\partial x^2} - N(x,t)\frac{\partial^2 w}{\partial x^2} = q(x,t)} \tag{2.19}$$

The above equation is known as geometrically nonlinear beam theory even though it is linear. It is considered a nonlinear equation when $N(x,t)$ is a function of w, which is when the beam in under large deflections and we cannot ignore axial characteristics. This equation is useful, along with equation (2.16), when the axial force is known as in the case of a rotating beam. However, in the case of large deformations, the axial force becomes a function of transverse displacement and another equation is needed to solve the system. This particular case will be discussed later.

The motion of a beam is defined as the sum of its modes. The modes of a particular structure are the fundamental movements that structure can make from its shape and boundary conditions. The shape of a pinned-pinned beam's first three modes are given in figure (2.5). Usually the first three modes would give sufficient information about the system. For a beam that is undergoing periodic excitation it is wise to include the modes that are associated with frequencies near the excitation frequencies.

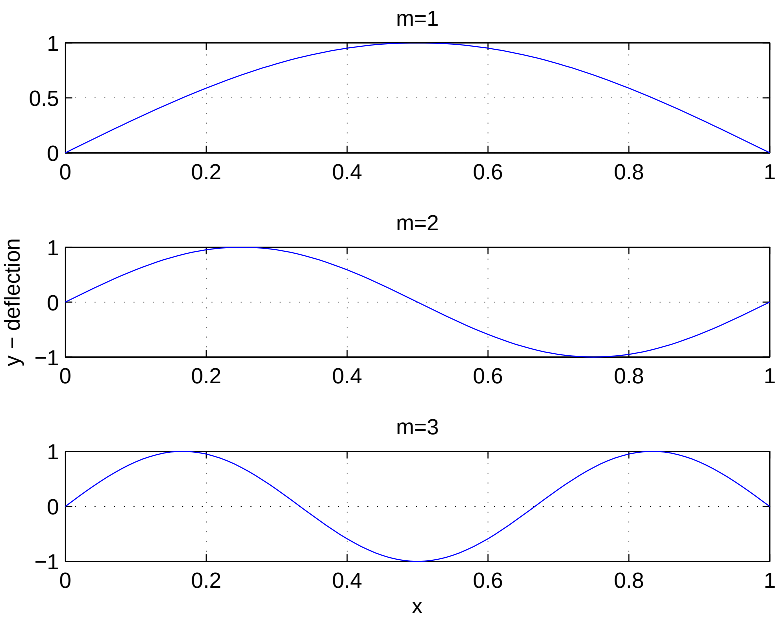


Figure 2.5: First Three Modes of a Simply-Supported Beam

2.2.3 Stretching

In large deflection beam theory, the beam begins to stretch. The variable s is defined as the length of the beam when deflected. When deflected, we can find the length of a beam element, $\mathrm{d}s$, by assuming it forms a right triangle. Total length of the beam, s, is the integral of local stretch expression over the length of the beam.

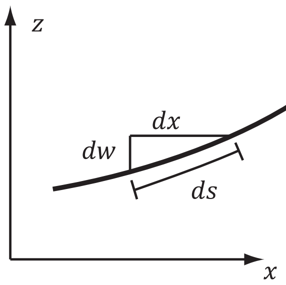


Figure 2.6: Change in Beam Length

$$
\begin{aligned}
s &= \int_0^L \sqrt{(dx)^2 + (dw)^2} = \int_0^L \sqrt{1 + \left(\frac{dw}{dx}\right)^2}\,\mathrm{d}x \\
&\approx \int_0^L 1 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \mathrm{d}x = L + \frac{1}{2}\int_0^L \left(\frac{dw}{dx}\right)^2 \mathrm{d}x
\end{aligned}
\tag{2.20}
$$

The change in axial length is $s - L$ and its ratio to the original length, L, is

$$
\frac{s-L}{L} = \frac{\Delta L}{L} = \frac{1}{2L}\int_0^L \left(\frac{dw}{dx}\right)^2 \mathrm{d}x \tag{2.21}
$$

We will see this term later on.

2.2.4 Expansion Theorem

The expansion theorem is an important tool in solving for beam deflections. The theorem allows us to expand any function over an orthogonal basis, an infinite sum, so that we can obtain the deformation. In order to show that the basis is orthogonal it must satisfy the condition below. Note that the symbol $\mathcal{B}$ will be used to signify a generic basis.

$$
\int_0^L \mathcal{B}_m(x)\mathcal{B}_n(x)\,\mathrm{d}x = \mathcal{C}\bar{\delta}_{(m,n)} \tag{2.22}
$$

Where $\bar{\delta}$ is the Dirac-delta function and $\mathcal{C}$ is an arbitrary constant. The basis is determined by boundary conditions of the beam. The basis for a particular set of boundary conditions is referred to as the modal function. For a beam simply supported on both sides, the modal function is

$$W_m(x) = C_m \sin(\frac{m\pi}{L}x) \tag{2.23}$$

In order to assign a value of C_m we normalize the modal function using the definition of inner product. The inner product is defined for two functions that are within angle brackets, $\langle$ and $\rangle$.

$$\begin{aligned} \langle W_m(x), W_m(x)\rangle_\rho &= \int_0^L \rho W_m(x) W_m(x)\, \mathrm{d}x \\ &= \int_0^L \rho C_m^2 \sin^2(\frac{m\pi}{L}x)\, \mathrm{d}x = 1 \end{aligned} \tag{2.24}$$

Solving for the coefficient

$$C_m = \sqrt{\frac{2}{\rho L}} \tag{2.25}$$

Our modal function becomes

$$W_m(x) = \sqrt{\frac{2}{\rho L}} \sin(\frac{m\pi}{L}x) \tag{2.26}$$

Our expansion theorem is

$$g(x,t) = \sum_{m=1}^{\infty} g_m(t) W_m(x) \tag{2.27}$$

$$g_m(t) = \frac{\langle g(x,t), W_m(x)\rangle}{\langle W_m(x), W_m(x)\rangle_\rho} \tag{2.28}$$

Since the modal functions are normalized, the denominator is equal to one. If we did not normalize our modal function the integral below would have a constant in front of it. It follows that since $\langle W_m, W_m\rangle_\rho = 1$ we have

$$g_m(t) = \int_0^L g(x,t) W_m(x)\, \mathrm{d}x \tag{2.29}$$

In the sections below we will take $C_m = 1$, as a result the corresponding expansion theorem becomes

$$g_m(t) = \frac{2}{\rho L} \int_0^L g(x,t) W_m(x)\, \mathrm{d}x \tag{2.30}$$

We shall use this form of expansion theorem because the maximum value of the modal function is one, which makes our calculations easier later on. Also note that the coefficient, C_m, is not a function of m which may not always be the case.

2.3 Linear Beam Theory

Linear beam theory is a simplification of the geometrically nonlinear theory. This theory is useful if the membrane force of the beam, N, is constant or it can be neglected, which is valid for beams for small deformations. Also, this theory does take into account the coupling between an axial load with a transverse displacement. As a result, linear beam theory does not predict buckling. This theory is also known as Euler-Bernoulli Beam Theory. As a result, we set the axial force equal to zero, $N = 0$, in equation (2.19). To simplify, we assume that the mass per unit length and the stiffness of the beam remains constant along the length of the beam.

$$\rho \frac{\partial^2 w}{\partial t^2} + k \frac{\partial^4 w}{\partial x^4} = q(x, t) \tag{2.31}$$

This model is a valid approximation for thin beams under small transverse deformations. As a good rule-of-thumb, 'small' is defined as deflections that are at least ten times smaller than beam thickness. This theory is useful when axial forces are insignificant to the problem.

2.3.1 Statics

In the study of static problems, the system is independent of time, the 'steady state' of the system. Therefore derivatives with respect to time are equal to zero. Furthermore, the defection and the loading function of the beam are no longer functions of time. Now that deflection is only a function of x, the partial derivative becomes a total derivative. Our equation of motion, the Euler-Bernoulli beam equation, becomes

$$k \frac{\mathrm{d}^4 w(x)}{\mathrm{d}x^4} = q(x) \tag{2.32}$$

While it is possible to solve the above equation directly by integration, direct integration does not translate to dynamics, or to the nonlinear theory of beams and plates. For that reason, we will solve the equation by expanding on an orthogonal basis, which we will call $W_m(x)$, and match coefficients of said basis. The orthogonal basis is also known as the modal function of the beam. The orthogonal basis on which to expand is defined by the boundary conditions of the beam. The modes of the beam are the natural shapes that a beam is able to produce under excitation [10].

A pinned-pinned beam is defined as no transverse deflection and zero moment at the edges. The corresponding boundary conditions are

$$-EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\bigg|_{x=0,L} = 0 \qquad\qquad w(0) = w(L) = 0 \tag{2.33}$$

The modal function for this set of boundary conditions is $W_m(x) = \sin(\alpha_m x)$. The equation of motion and boundary conditions are satisfied if the deflection takes the form of

$$w(x) = \sum_{m=1}^{\infty} w_m W_m(x) = \sum_{m=1}^{\infty} w_m \sin(\alpha_m x) \tag{2.34}$$

$$\alpha_m = \frac{m\pi}{L} \tag{2.35}$$

We can also expand our load function, $q(x)$, on our orthogonal basis.

$$q(x) = \sum_{m=1}^{\infty} q_m W_m(x) = \sum_{m=1}^{\infty} q_m \sin(\alpha_m x) \tag{2.36}$$

$$q_m = \frac{2}{L}\int_0^L q(x) W_m(x)\,\mathrm{d}x = \frac{2}{L}\int_0^L q(x)\sin(\alpha_m x)\,\mathrm{d}x \tag{2.37}$$

Substituting the above equations into equation (2.32) we get

$$k\alpha_m^4 w_m \sin(\alpha_m x) = q_m \sin(\alpha_m x) \tag{2.38}$$

Multiplying by W_n and integrating over the length of the beam, and utilizing the expansion theorem results in

$$k\alpha_m^4 w_m = q_m \tag{2.39}$$

$$\boxed{w_m = \frac{q_m}{k\alpha_m^4} = \hat{q}_m} \tag{2.40}$$

Using equation (2.40) along with equation (2.34) gives us the static deflection of an Euler-Bernoulli beam. For most applications the first three modes of the beam will give sufficient information of the system.

2.3.2 Dynamics

The mathematical model now includes the time derivative with respect of our deflection function $w(x,t)$. We need to assume a new form of solution in order to incorporate the time dependence on the deflection. However the modal function of the beam does not change. Note that in the study of a static beam w_m was a constant that depended on the loading function. In dynamic systems, the coefficient of the modal function is now a function of time, $w_m(t)$.

$$w(x,t) = \sum_{m=1}^{\infty} W_m(x) w_m(t) \tag{2.41}$$

$$W_m(x) = \sin(\alpha_m x) \tag{2.42}$$

In addition, loading function's coefficients are a function of time.

$$q(x,t) = \sum_{m=1}^{\infty} q_m(t) \sin(\alpha_m x) \tag{2.43}$$

$$q_m(t) = \frac{2}{L} \int_0^L q(x,t) \sin(\alpha_m x)\, \mathrm{d}x \tag{2.44}$$

Substituting equation (2.41) into equation (2.31), noting $\frac{\partial^4 W_m}{\partial x^4} = \alpha_m^4 W_m$, and using expansion theorem,

$$\rho W_m \ddot{w}_m + k W_m'''' w_m = q_m W_m \tag{2.45}$$

$$\ddot{w}_m + \frac{k\alpha_m^4}{m} w_m = \frac{q_m}{\rho} \tag{2.46}$$

Setting $\omega_m^2 = \frac{k\alpha_m^4}{\rho}$, where ω_m is the natural frequency of the system,

$$\ddot{w}_m(t) + \omega_m^2 w_m(t) = \frac{q_m(t)}{\rho} \tag{2.47}$$

The solution to the second order ordinary differential equation is below. Where $w_m(0)$ is related to the initial position and $\dot{w}_m(0)$ is related to the initial velocity of the beam.

$$w_m(t) = w_m(0)\cos(\omega_m t) + \frac{\dot{w}_m(0)}{\omega_m}\sin(\omega_m t) + \frac{1}{\omega_m}\int_0^t q_m(t-\tau)\sin(\omega_m \tau)\,\mathrm{d}\tau \tag{2.48}$$

$$w_m(0) = \frac{2}{L}\int_0^L w(x,0)\sin(\alpha_m x)\,\mathrm{d}x \tag{2.49}$$

$$\dot{w}_m(0) = \frac{2}{L}\int_0^L \dot{w}(x,0)\sin(\alpha_m x)\,\mathrm{d}x \tag{2.50}$$

2.4 Geometrically Nonlinear Beam Theory

In nonlinear beam theory we assume that the deflection has a significant effect on the boundaries of the beam. If we consider the edges 'immovable', the edges are fixed from moving in the lateral direction when the beam is deflected laterally. As a result, the beam's length increases, and a tensile force in the beam is produced. The equation of motion is below.

$$\rho(x)\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}k(x)\frac{\partial^2 w}{\partial x^2} - N(x,t)\frac{\partial^2 w}{\partial x^2} = q(x,t) \tag{2.51}$$

This equation has two unknowns, $w(x,t)$ and $N(x,t)$. In order to solve this equation we need an expression for $N(x,t)$ in terms of $w(x,t)$, which turns our equation into a nonlinear equation.

We know that there is no applied axial force, therefore $p(x,t) = 0$, as a result the equation of axial motion equation becomes

$$\frac{\partial N}{\partial x} \approx -p(x,t) = 0 \tag{2.52}$$

$$N(x,t) = N(t) = C_0(t) \tag{2.53}$$

Where $C_0(t)$ is some unknown function of t. Since we found that the membrane force is constant in the beam, not a function of x, the spatial variable, we bring our attention to the equation for the membrane force in terms of displacement, which will provide our necessary equation. Using the nonlinear strain equation we write

$$N(t) = EA\epsilon(t) = EA\Big(\frac{\partial u}{\partial x} + \frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2\Big) = EAC_0(t) \tag{2.54}$$

Since w is a function of x, we need to remove the dependence on x in the above equation by integrating over the length of the beam, where we will find an expression for $C_0(t)$.

$$C_0(t) = \frac{\partial u}{\partial x} + \frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2 \tag{2.55}$$

$$(C_0(t)x + k_0(t))_0^L = \int_0^L \frac{\partial u}{\partial x} + \frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2 \, \mathrm{d}x \tag{2.56}$$

$$C_0(t) = \left(\frac{u_L(t) - u_0(t)}{L} + \frac{1}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x\right) \tag{2.57}$$

where u_L is the axial deformation of the beam at $x = L$ and u_0 is the axial deformation of the beam at $x = 0$. The second term in the equation above is equivalent to equation (2.21).

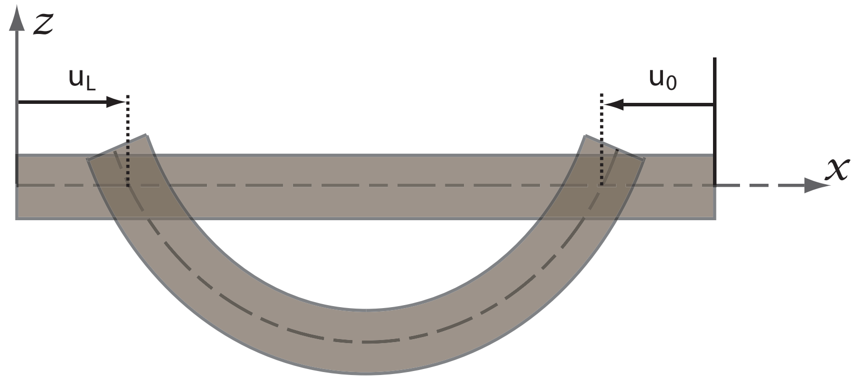


Figure 2.7: Axial Deformation of a Roller-Roller Beam

Plugging into equation (2.54) gives

$$N(t) = EA\epsilon(t) = EA\left(\frac{u_L(t) - u_0(t)}{L} + \frac{1}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x\right) \tag{2.58}$$

Inserting into our equation of motion

$$\boxed{\rho\frac{\partial^2 w}{\partial t^2} + k\frac{\partial^4 w}{\partial x^4} - \frac{EA}{L}\Big(u_L - u_0 + \frac{1}{2}\int_0^L \Big(\frac{\partial w}{\partial x}\Big)^2 \mathrm{d}x\Big)\frac{\partial^2 w}{\partial x^2} = q(x,t)} \tag{2.59}$$

Now we have one equation, however we still need to find a method to calculate u_L and u_0. If our boundaries dictate immovable edges, finding these values are quite easy because they are defined as zero. Stress free edges mean that at $x = 0, L$, $N = 0$. Since N is not a function of x we know that there is zero membrane force throughout the beam. We find from equation (2.54) that the equation of motion simplifies to linear beam theory. As a result, a different method is to find the solution is required. Conway's [13] formulation is presented as it provides a solution for the large deformation of beams. However, Conway does not consider the kinetics of the beam and only considers the geometry of deformation.

Note that for all examples we will use values of $h = 0.05$ for beam thickness, $b = 0.01$ for beam width and a value of 1 for L, the beam length. For other values see below

$$E = 1E8 \tag{2.60}$$

$$I = \frac{bh^3}{12} \tag{2.61}$$

$$A = bh \tag{2.62}$$

$$k = EI \tag{2.63}$$

2.4.1 Statics

As in previous sections, we remove the dependence on the time variable, t. The governing equations (2.59) and (2.58) become

$$k\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - N\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = q(x) \tag{2.64}$$

$$N = \frac{EA}{L}\Big(u_L - u_0 + \frac{1}{2}\int_0^L \Big(\frac{\mathrm{d}w}{\mathrm{d}x}\Big)^2 \mathrm{d}x\Big) \tag{2.65}$$

Immovable Edges

For immovable edges we assume $u_L = u_0 = 0$.

$$k\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - \frac{EA}{2L}\Big(\int_0^L \Big(\frac{\mathrm{d}w}{\mathrm{d}x}\Big)^2 \mathrm{d}x\Big)\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = q(x) \tag{2.66}$$

As in previous sections, we assume a modal function for a pinned-pinned beam and substitute.

$$w(x) = \sum_{m=1}^{\infty} w_m \sin(\alpha_m x) \tag{2.67}$$

Introducing equation (2.67) into the equation of motion we obtain

$$\begin{split} -\frac{EA}{2L}\Big(\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\int_0^L (\alpha_r w_r \cos(\alpha_r x))\,(\alpha_s w_s \cos(\alpha_s x))\ \mathrm{d}x\Big)(-\alpha_m^2 w_m \sin(\alpha_m x)) \\ + k\alpha_m^4 w_m \sin(\alpha_m x) = q_m \sin(\alpha_m x) \end{split} \tag{2.68}$$

Evaluating the integral

$$k\alpha_m^4 w_m \sin(\alpha_m x) - \frac{EA}{4}\Big(\sum_{r=1}^{\infty}\Big(\alpha_r w_r\Big)^2\Big)(-\alpha_m^2 w_m \sin(\alpha_m x)) = q_m \sin(\alpha_m x) \tag{2.69}$$

Using the expansion theorem to give us our final result for the coefficients of the infinite sum,

$$\boxed{EI\alpha_m^4 w_m + \frac{EA}{4}\Big(\sum_{r=1}^{\infty}\Big(\alpha_r w_r\Big)^2\Big)(\alpha_m^2 w_m) = q_m} \tag{2.70}$$

Example: Static, Immovable Edges - One-to-One Approximation

To give an idea on the motion of a nonlinear beam a one-to-one approximation is considered. This approximation ignores the coupling between modes. Hence, the system of equations are independent of each other which simplifies calculation. While this may affect the solution accuracy negatively, it may provide satisfactory information about the system. This assumption should be used when a particular mode is of interest or if a small set of modes are of interest. If we considered five modes, we would have a highly coupled system of five equations which would require computational efforts that may not be available. Also note that the influence of a mode will be directly affected by the magnitude of q_m. If q_m is much smaller for a particular mode, it is safe to assume that w_m will also be small. For example for a concentrated load in the center of the beam, if we only want to consider the first two modes, we know from our study of modes, the second mode does not contribute to the deflection. Hence, we can solve that mode independently of the others in order to save on computation time. Multiply equation (2.70) by our modal function and integrating, utilizing the expansion theorem:

$$EI\alpha_m^4 w_m + \frac{EA}{4}\alpha_m^4 w_m^3 = q_m \tag{2.71}$$

$$w_m + \frac{A}{4I}w_m^3 = \frac{q_m}{\alpha_m^4 EI} = \hat{q}_m \tag{2.72}$$

Note that $\hat{q}_m$ is the solution to the static-linear beam theory.

$$\boxed{\frac{A}{4I}w_m^3 + w_m - \hat{q}_m = 0} \tag{2.73}$$

For the first two modes of the system $m = 1, 2$, the system produces two equations and two unknowns, w_1 and w_2.

$$\frac{A}{4I}w_1^3 + w_1 - \hat{q}_1 = 0 \tag{2.74}$$

$$\frac{A}{4I}w_2^3 + w_2 - \hat{q}_2 = 0 \tag{2.75}$$

Example: Static, Immovable Edges - Two Term Approximation

For a more accurate solution, the mode coupling must be considered in the solution. In order to form a complete system for every term we take in the sum we need to add another equation in order to have an equal number of equations and unknowns. For a two term solution, we need two equations. In the example below, we take the first two terms in beam theory, $m = 1, 2$. In most cases, the first three modes of a beam are considered a sufficiently accurate solution.

$$EI\alpha_1^4 w_1 + \frac{EA}{4}\Big(\big(\alpha_1 w_1\big)^2 + \big(\alpha_2 w_2\big)^2\Big)(\alpha_1^2 w_1) = q_1 \tag{2.76}$$

$$EI\alpha_2^4 w_2 + \frac{EA}{4}\Big(\big(\alpha_1 w_1\big)^2 + \big(\alpha_2 w_2\big)^2\Big)(\alpha_2^2 w_2) = q_2 \tag{2.77}$$

These equations match with the solution given by Leung and Mao [27], who used Hamiltonian equations and Largrange's equations, after removing the time dependent terms. It is important to realize that the values of the coefficients are dependent on each other. This arises due to the nonlinearity of the system. Also note that setting $w_2 = 0$ results in the one term approximation.

Note that from figure (2.8), we measure w_1 as a response to a constant load at the middle of the beam. From the second mode's geometry we recognize that the second mode would have a zero magnitude for this load configuration. Therefore, the coupled solution breaks down to the uncoupled solution as shown in figure (2.8).

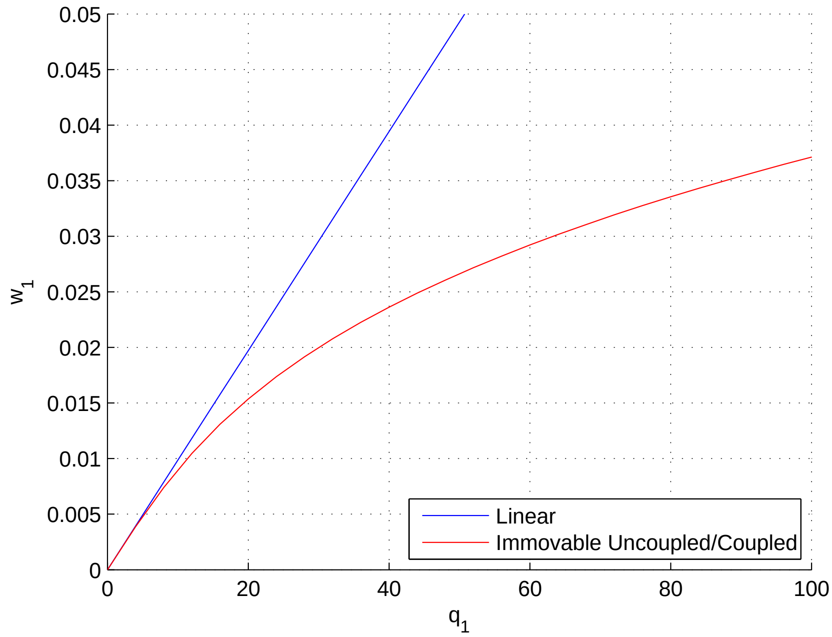
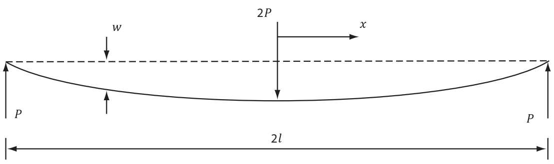


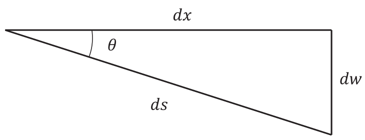
Figure 2.8: Static Bending of a Beam with Immovable Edges

Stress Free Edges

For beams with stress-free edges, we define the edges with zero axial stress. As a result, we must allow the edges to move in the axial direction. However, from equation (2.53) N is zero for the entire length of the beam. Therefore, the geometrically nonlinear beam theory reduces back to linear beam theory. From this, we conclude that beams with stress-free edge conditions behave similarly to a linear beam. However, this beam still has to follow the condition that the slope of deflection is small compared to unity. In order to relax this condition we present the theory from Conway [13], which uses beam geometry to find the deflection of the beam in terms of the slope. Note that this section derives its deflection using a very specific formulation. The formulation is specific for a given loading. If the loading were located at a different point, or if we had uniform loading, the slope-loading relationship would be different. Conway's formulation is presented in limited context to give the reader a solution to stress-free edges for a static, centrally loaded beam. Also, one should take into account that this derivation is not very flexible, in that it cannot be easily used for dynamic loading and



(a) Conway Beam



(b) Differential Element

Figure 2.9: Conway Beam Geometry

that it cannot be easily compared to other theories that derive a governing equation and its solution.

Conway's formulation is used for static deflection of a simply supported beams with the edges allowed to move in the axial direction. The assumption that there is no stretch in the beam is also made. As a result, the geometrically nonlinear beam theory is obsolete as it reduces to linear beam theory. Hence, we use the geometry of the deformed beam to calculate the deflection of beam. We begin by noting that the bending moment of a beam is proportional to the curvature, or the derivative of the slope, θ, of the beam. This coordinate system, with the origin at the middle of the beam, will only be used in this section.

$$M = k\frac{\mathrm{d}\theta}{\mathrm{d}s} = P(l - x) \tag{2.78}$$

M is the moment of the beam, P is half of the applied load at the center of the beam, l is half of the length of the beam, k is the beam stiffness, which is equal to the product of Young's modulus, E, and the moment of inertia of the beam's cross section, I.

Differentiation of each side with respect to s, and multiplying each side by $\frac{\mathrm{d}\theta}{\mathrm{d}s}$ results in

$$\frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = -\frac{P}{k}\frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s} \tag{2.79}$$

From the picture of the differential element we can see that $\cos(\theta) = \frac{\mathrm{d}x}{\mathrm{d}s}$. Then integrating each side with respect to s results in

$$\frac{1}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 = -\frac{P}{k}\sin(\theta) + C \tag{2.80}$$

where C is a constant of integration. Using the boundary condition that the curvature of the beam at the ends are zero allows us to find the constant. We also define θ_o as the slope of the beam at the ends.

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \sqrt{\frac{2P}{k}(\sin(\theta_o) - \sin(\theta))} \tag{2.81}$$

In order to simplify, we expand the left hand side of the above equation in order to have a function in terms of w, our displacement variable.

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{\mathrm{d}\theta}{\mathrm{d}w}\frac{\mathrm{d}w}{\mathrm{d}s} = \frac{\mathrm{d}\theta}{\mathrm{d}w}\sin(\theta) = \sqrt{\frac{2P}{k}(\sin(\theta_o) - \sin(\theta))} \tag{2.82}$$

Solving for $\mathrm{d}w$ and integrating results in

$$\boxed{w_{max} = \int_0^{\theta_o} \frac{\sin(\theta)}{\sqrt{\frac{2P}{k}(\sin(\theta_o) - \sin(\theta))}}\,\mathrm{d}\theta} \tag{2.83}$$

This equation is sufficient to find the maximum deflection of a simply-supported beam. However, we still need to find the value of θ_o. Equating equations (2.81) and (2.78) results in

$$\sqrt{\frac{2P}{k}(\sin(\theta_o) - \sin(\theta))} = \frac{P}{k}(l - x) \tag{2.84}$$

Using the relationship that $\theta = 0$ at $x = 0$ results in

$$\boxed{\theta_o = \arcsin\left(\frac{Pl^2}{2k}\right)} \tag{2.85}$$

At this point we have enough information to solve for the maximum deflection of the beam.

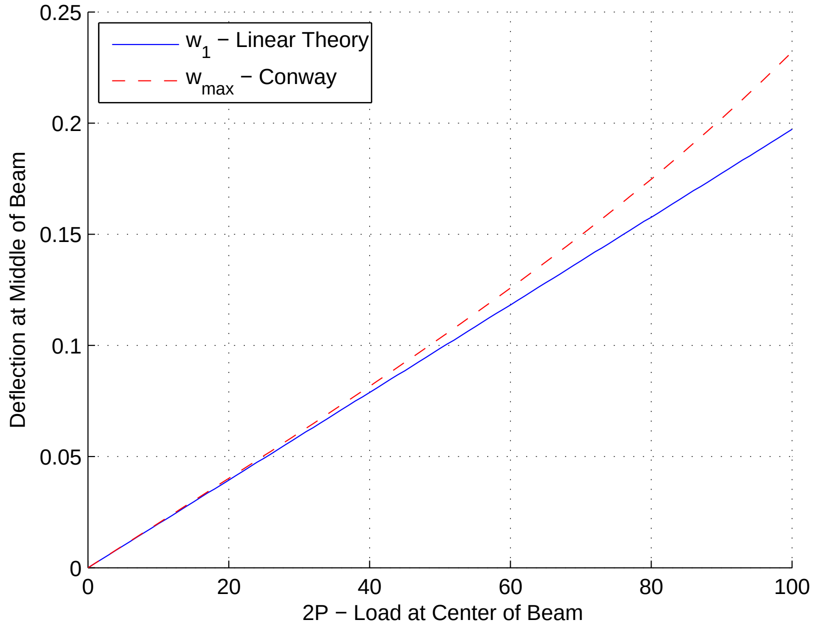


Figure 2.10: Deflection of a Beam using Conway's Formulation

From figure (2.10) we can see that the deflection of the beam is larger than predicted from linear theory. This results from allowing the beam to fold onto itself with no resistance. Physically, we know that the maximum deflection of a beam with roller edges, assuming no stretch, would be half the length of the beam. However, Conway's formulation theory does not take into account the resistance to the folding of the beam onto itself. As a result, this theory has a upper bound of deflection and values beyond that would be unrealistic. This upper bound has yet to be formally derived. One should use their best judgment whether a particular deflection is realistic.

Table (2.1) compares the coefficients and deflections of the four different cases presented, linear, immovable edges coupled and uncoupled, and the Conway formulation. Two different load cases were considered. Note that $q_1 = 2\text{Load}$. The magnitude of the smaller load was chosen as the linear theory's first mode predicts a deflection that is close in magnitude to the beam thickness. The magnitude of the larger load shows that the deflection of the immovable coefficients are close in value to the beam thickness, 0.05. For the smaller load, we can see reasonable agreement between Conway and linear formulations. However, the immovable edge condition is significantly different from linear theory. The larger load condition shows a greater difference between the immovable and linear theories. The difference between the immovable coupled and uncoupled conditions are negligible. This is a result from the loading condition. If the load were at a location where the third mode had a more significant contribution, the coupled and uncoupled coefficients would have a greater difference. But in our case, where the load was at the beam's center, the third mode's amplitude is about a hundred times smaller than the first mode's.

Beam Deflection								
	Load = 50				Load = 100			
	w_1	w_2	w_3	Total	w_1	w_2	w_3	Total
Linear	0.0986	0	-0.0012	0.0974	0.1971	0	-0.0024	0.1947
Immovable Uncoupled	0.0371	0	-0.0012	0.0359	0.0497	0	-0.0024	0.0473
Immovable Coupled	0.0371	0	-0.0010	0.0361	0.0496	0	-0.0018	0.0478
Stress-Free Conway	–	–	–	0.1033	–	–	–	0.2320

Table 2.1: Coefficients of a Beam

2.4.2 Dynamics

We retain the time derivative in our equation of motion and the coefficient of the modal function in our deflection equation, is now a function of time, $w_m(t)$. The method that Conway produced does not extend to dynamics as the formulation does not consider the time dependence of any of the terms.

Immovable Edges

As in the previous section, where we considered immovable edges, we assume $u_L = u_0 = 0$

$$\rho \frac{\partial^2 w}{\partial t^2} + k\frac{\partial^4 w}{\partial x^4} - \frac{EA}{2L}\Big(\int_0^L \Big(\frac{\partial w}{\partial x}\Big)^2 \,\mathrm{d}x\Big)\frac{\partial^2 w}{\partial x^2} = q(x,t) \tag{2.86}$$

We assume the same modal function, $W_m = \sin(\alpha_m x)$, and the same form of solution $w(x,t) = \sum_{m=1}^{\infty} w_m(t)W_m(x)$. We substitute this solution into our governing equation, and utilize expansion theorem.

$$\sum_{m=1}^{\infty} \big(\rho \sin(\alpha_m x)\ddot{w}_m(t) + EI\alpha_m^4 \sin(\alpha_m x) w_m(t)$$
$$-\frac{EA}{2L}\Big(\int_0^L \Big(\alpha_m w_m(t)\cos(\alpha_m x)\Big)^2 \mathrm{d}x\Big)(-\alpha_m^2 w_m(t)\sin(\alpha_m x))\Big) = \sum_{m=1}^{\infty} q_m(t)\sin(\alpha_m x)$$

Multiplying by $W_n(x)$ and integrating

$$\rho\ddot{w}_m(t) + EI\alpha_m^4 w_m(t)$$
$$-\frac{EA}{4}\Big(\sum_{r=1}^{\infty}\Big(\alpha_r w_r(t)\Big)^2\Big)(-\alpha_m^2 w_m(t)) = q_m(t)$$

Example: Dynamic, Immovable Edges - Two Term Approximation

We will use two terms $m = 1, 2$, we obtain:

$$\rho\ddot{w}_1(t) + EI\alpha_1^4 w_1(t) + \frac{EA}{4}\Big(\Big(\alpha_1 w_1(t)\Big)^2 + \Big(\alpha_2 w_2(t)\Big)^2\Big)(\alpha_1^2 w_1(t)) = q_1(t) \tag{2.87}$$

$$\rho\ddot{w}_2(t) + EI\alpha_2^4 w_2(t) + \frac{EA}{4}\Big(\Big(\alpha_1 w_1(t)\Big)^2 + \Big(\alpha_2 w_2(t)\Big)^2\Big)(\alpha_2^2 w_2(t)) = q_2(t) \tag{2.88}$$

These equations match with the solution given by Leung and Mao [27], who used Hamiltonian equations and Largrange's equations. Note that when we set $\ddot{w}_1 = \ddot{w}_2 = 0$ the static solution is recovered. Also, setting $w_2 = 0$ in equation (2.87) and $w_1 = 0$ in equation (2.88) results in the uncoupled approximation. Also note that the uncoupled approximation is in the form of Duffing's equation. Duffing's equation implies the existence of amplitude jumps for changes in excitation frequency for periodic loading.

Duffing's equation also predicts subharmonic and superharmonic resonance, chaos, and other phenomena. However, these topics are beyond the scope of this study. For the frequency response curves, we assumed a sinusoidal excitation with a constant magnitude and we chose $\epsilon = 0.01$. The Duffing equation is described in appendix B.

Figure (2.11) is the time response of a beam with a load suddenly placed at the center of the beam with magnitude of 25.

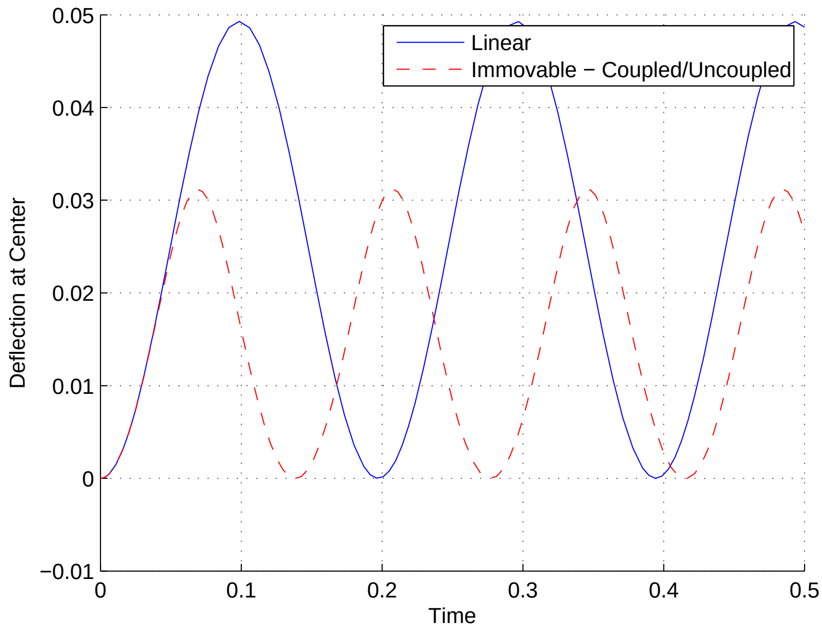


Figure 2.11: Dynamic Solution - Response to Constant Load at Center of Beam with Immovable Edges

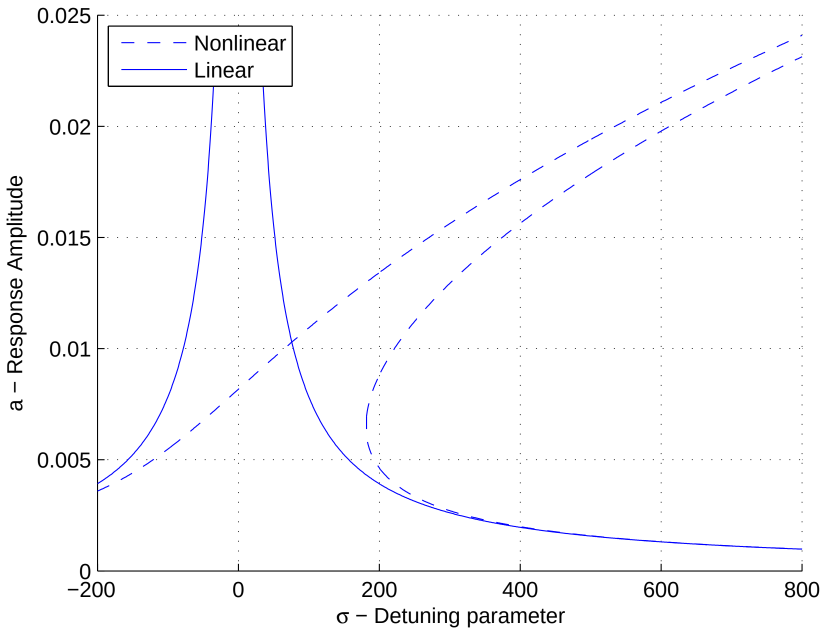


Figure 2.12: Frequency Response of Beam with Immovable Edges

Chapter 3

Large Deflection of Plates

3.1 Introduction

When a thin elastic plate undergoing small deformations, ($w < 0.1h$), where w is the transverse deflection and h is the plate thickness, is considered, it is reasonable to ignore geometric nonlinearities and use linear plate theory. However in larger deflections, ($w \approx \mathcal{O}(h)$), the middle surface of the plate begins to stretch or the in-plane motion of the plate edges become significant. When these effects become important one needs to consider geometrically nonlinear plate theory, which was first derived by von Kármán [48] in 1910.

This plate theory considers the effects of both bending and stretching of the middle surface of the plate. The method of solution is very similar to that of linear plate theory. Here we will assume a deflection (mode shapes) based on the boundary conditions of the plate and then utilize the expansion theorem. The nonlinear plate theory consists of two coupled nonlinear partial differential equations. Also, the use of a stress function will be required. The same method of solution will be used for the stress function. An assumption about the shape of the stress function will be based on the edge boundary conditions.

We will consider a flat, square plate with all pinned edges of length one. Pinned edges are used as they allow us to proceed analytically and without loss of generality. Other boundary conditions would necessitate the use of numerical approximations for finding mode shapes earlier in the derivation. As with nonlinear beam theory, an additional set of boundary conditions are required in order to describe edge effects of the plate. Either 'immovable' or 'stress-free' boundary conditions need to be defined. We will also compare static and dynamic responses between linear and nonlinear theories. The influence of rotatory inertia and the correction for shear are neglected, as they will make subsequent calculations cumbersome and may distract us from understanding the effect of geometric nonlinearities on the system. Rotatory inertia and shear are usually considered when the plate can no longer be considered 'thin', $h/min(a,b) < 1/10$ is a thin plate, or the plate undergoes a high frequency excitation where the wavelength approaches the plate thickness.

Beam theory showed that a beam has an infinite number of modes, each with unique amplitudes. If one considers a plate as a series of beams placed next to each other, we can see that the solution to plate theory requires a double infinite sum.

3.2 Governing Equations

To better understand plates and their motion the derivation of geometric nonlinear plate theory is below.

3.2.1 Elasticity

We begin with Green's Strain

$$E_{xx} = \frac{\partial u_x}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial x}\right)^2 + \left(\frac{\partial u_z}{\partial x}\right)^2\right] \tag{3.1a}$$

$$E_{yy} = \frac{\partial u_y}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial y}\right)^2 + \left(\frac{\partial u_y}{\partial y}\right)^2 + \left(\frac{\partial u_z}{\partial y}\right)^2\right] \tag{3.1b}$$

$$E_{xy} = \frac{1}{2}\left[\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} + \frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial x}\frac{\partial u_z}{\partial y}\right] \tag{3.1c}$$

In order to simplify the strain-displacement relation we use the 'small strain', 'moderate rotation' approximation. Following are the definitions used to define 'large' deflections. For small strains we have

$$\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} \ll \frac{\partial u_k}{\partial x_l} \qquad i, j, k, l = x, y \tag{3.2}$$

For moderate rotation we have

$$\frac{\partial u_z}{\partial x_i}\frac{\partial u_z}{\partial x_j} \sim \mathcal{O}\left(\frac{\partial u_i}{\partial x_j}\right) \qquad i,j = x,y \tag{3.3}$$

Using the two definitions for large deflections in equation (3.1) we obtain

$$E_{xx} = \frac{\partial u_x}{\partial x} + \frac{1}{2}\left(\frac{\partial u_z}{\partial x}\right)^2 \tag{3.4a}$$

$$E_{yy} = \frac{\partial u_y}{\partial y} + \frac{1}{2}\left(\frac{\partial u_z}{\partial y}\right)^2 \tag{3.4b}$$

$$E_{xy} = \frac{1}{2}\left[\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} + \frac{\partial u_z}{\partial x}\frac{\partial u_z}{\partial y}\right] \tag{3.4c}$$

The nonlinear terms above couple the transverse displacement with the axial displacement.

To relate stresses with strains, Hooke's Law for linearly elastic materials is used:

$$\sigma_x = \frac{E}{1-\nu^2}\left(E_{xx} + \nu E_{yy}\right) \tag{3.5a}$$

$$\sigma_y = \frac{E}{1-\nu^2}\left(E_{yy} + \nu E_{xx}\right) \tag{3.5b}$$

$$\sigma_{xy} = \frac{E}{(1+\nu)}E_{xy} \tag{3.5c}$$

The inverse relationship to relate the strains with stresses is

$$E_{xx} = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) \tag{3.6}$$

$$E_{yy} = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) \tag{3.7}$$

$$E_{xy} = \frac{1+\nu}{E}\sigma_{xy} \tag{3.8}$$

Since geometrically nonlinear plate theory deals with the relationship between axial stresses and the transverse displacement the equations above will become useful to calculate the axial strains, and axial displacements.

The Airy stress function, ϕ, will be used to represent the stresses.

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} \qquad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} \qquad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \tag{3.9}$$

It is also important to calculate the resultant membrane forces, as well as resultant moments, which are below.

$$N_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \, \mathrm{d}z = \sigma_x h \quad N_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y \, \mathrm{d}z = \sigma_y h \quad N_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xy} \, \mathrm{d}z = \sigma_{xy} h \tag{3.10}$$

$$M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_x \, \mathrm{d}z \qquad M_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_y \, \mathrm{d}z \qquad M_{xy} = -\int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_{xy} \, \mathrm{d}z \tag{3.11}$$

3.2.2 Derivation of Equation of Motion

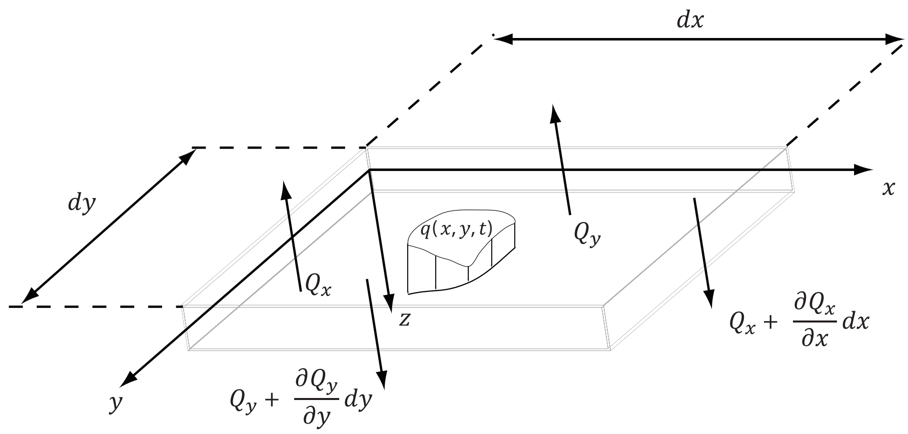


Figure 3.1: Shear Force Diagram on a Differential Plate Element

The origin of the coordinate system is selected to be at the corner of the plate on the midplane. The midplane is the 'middle', with respect to the thickness, of the plate, such that the top and bottom surfaces are at $z = h/2$ and $z = -h/2$, respectively. Let u, v, w be middle plane deflections of the plate. From Kirchhoff's hypothesis, the deflection varies linearly from the middle surface.

$$u_x = u - z\frac{\partial w}{\partial x} \qquad u_y = v - z\frac{\partial w}{\partial y} \qquad u_z = w \tag{3.12}$$

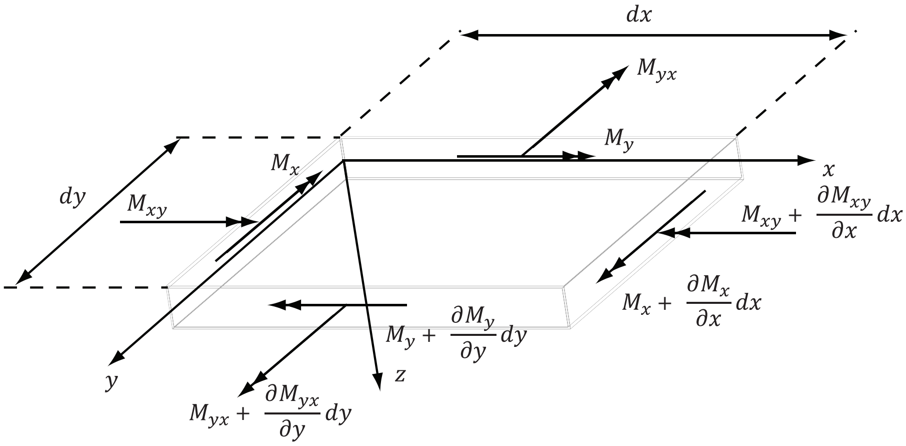


Figure 3.2: Moment Diagram on a Differential Plate Element

Substituting equation (3.12) into (3.4)

$$E_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2} \tag{3.13a}$$

$$E_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\frac{\partial^2 w}{\partial y^2} \tag{3.13b}$$

$$E_{xy} = \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] - z\frac{\partial^2 w}{\partial x \partial y} \tag{3.13c}$$

Next, we calculate moments by substituting equation (3.13) into (3.5) then into (3.11), which results in

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \tag{3.14a}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \tag{3.14b}$$

$$M_{xy} = D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} \tag{3.14c}$$

where D is flexural rigidity or bending stiffness of the plate, E is the elastic modulus, h is the plate thickness and ν is Poisson's ratio. We choose $E = 1E8$, $h = 0.05$ and $\nu = 0.316$ for the examples presented later. These quantities are related by

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{3.15}$$

Summing forces in the transverse direction, z, yields

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q(x,y,t) + q^*(x,y,t) = \rho \frac{\partial^2 w}{\partial t^2} \tag{3.16}$$

in which $q(x,y,t)$ is the applied transverse load and $q^*(x,y,t)$ is the resultant transverse force as a result of internal membrane forces caused by transverse deflections. $q^*(x,y,t)$ will be determined later.

The moment equations about the x and y axis are

$$\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} = Q_y \tag{3.17a}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} = Q_x \tag{3.17b}$$

Substitute equation (3.17) into (3.16)

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial y^2} + \frac{\partial^2 M_y}{\partial y^2} - \frac{\partial^2 M_{xy}}{\partial x^2} + q(x,y,t) + q^*(x,y,t) = \rho \frac{\partial^2 w}{\partial t^2} \tag{3.18}$$

Since $M_{xy} = -M_{yx}$ the above equation simplifies to

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial y^2} + \frac{\partial^2 M_y}{\partial y^2} + q(x,y,t) + q^*(x,y,t) = \rho \frac{\partial^2 w}{\partial t^2} \tag{3.19}$$

and using equation (3.14) along with (3.19), we obtain

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + \rho \frac{\partial^2 w}{\partial t^2} = q(x,y,t) + q^*(x,y,t) \tag{3.20}$$

recognizing the biharmonic operator as $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, the above equation becomes

$$D\nabla^4 w + \rho \frac{\partial^2 w}{\partial t^2} = q(x,y,t) + q^*(x,y,t) \tag{3.21}$$

Influence of Membrane Forces

From figure (3.3) we can see that there are components of the membrane force in the z direction. This is the source of the nonlinearity of the system because from

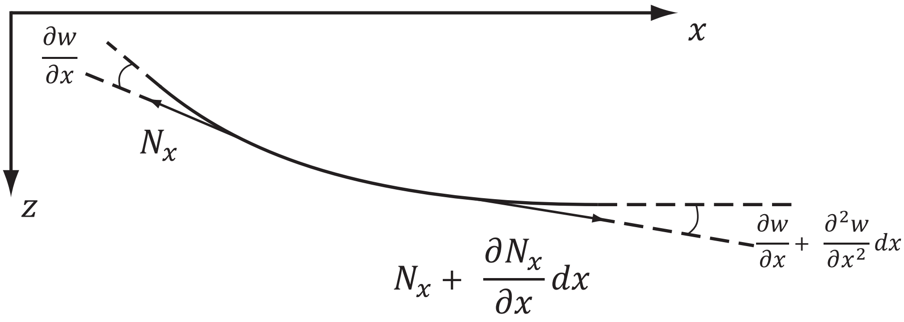


Figure 3.3: Projection of Membrane Forces

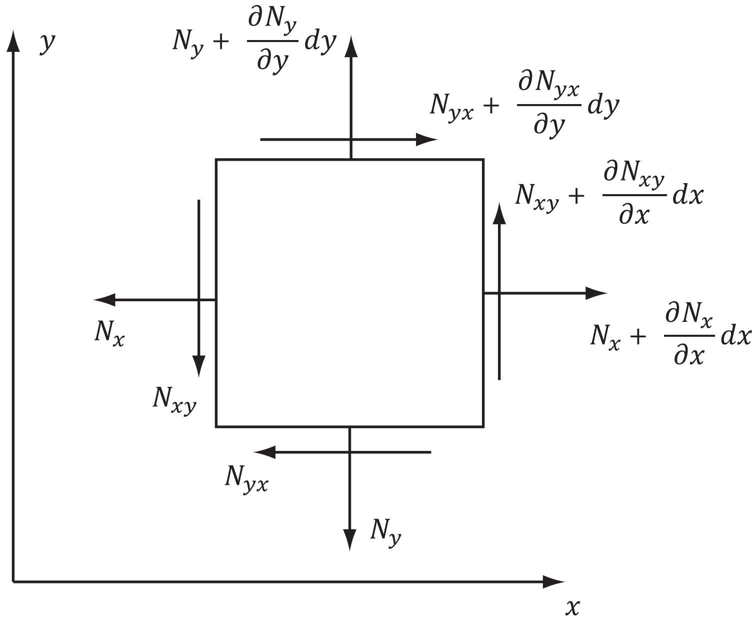


Figure 3.4: Differential Plate Element

the figure we can observe that as the deflection increases the transverse component of the membrane force increases as well. For simplicity, we will define $q^*(x, y, t)$ as the resultant 'transverse' force of the membrane forces. Therefore we need to find the sum of the projected forces from N_x, N_y and N_{xy}. Projection of N_x forces onto z plane gives

$$-N_x \, \mathrm{d}y \frac{\partial w}{\partial x} + \left(N_x + \frac{\partial N_x}{\partial x} \, \mathrm{d}x \right) \left(\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} \, \mathrm{d}x \right) \mathrm{d}y \tag{3.22}$$

Simplifying by ignoring terms of order $(\mathrm{d}x^2 \, \mathrm{d}y)$

$$N_x \frac{\partial^2 w}{\partial x^2} \, \mathrm{d}x \, \mathrm{d}y + \frac{\partial N_x}{\partial x} \frac{\partial w}{\partial x} \, \mathrm{d}x \, \mathrm{d}y \tag{3.23}$$

Similarly, projection of N_y forces onto z plane

$$-N_y \,\mathrm{d}x \frac{\partial w}{\partial y} + \left(N_y + \frac{\partial N_y}{\partial y}\,\mathrm{d}y\right)\left(\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial y^2}\,\mathrm{d}y\right)\mathrm{d}x \tag{3.24}$$

and similar simplification yields

$$N_y \frac{\partial^2 w}{\partial y^2}\,\mathrm{d}x\,\mathrm{d}y + \frac{\partial N_y}{\partial y}\frac{\partial w}{\partial y}\,\mathrm{d}x\,\mathrm{d}y \tag{3.25}$$

By the definition of the Airy's stress function, equation (3.9), $N_{xy} = N_{yx}$, and that the projection of N_{yx} onto the z plane take the same form.

$$N_{xy} \frac{\partial^2 w}{\partial x \partial y}\,\mathrm{d}x\,\mathrm{d}y + \frac{\partial N_{xy}}{\partial x}\frac{\partial w}{\partial y}\,\mathrm{d}x\,\mathrm{d}y \tag{3.26}$$

Summing the projections of the in plane shear forces, N_{xy}, and N_{yx} on the z plane results in

$$2N_{xy} \frac{\partial^2 w}{\partial x \partial y}\,\mathrm{d}x\,\mathrm{d}y + \frac{\partial N_{xy}}{\partial x}\frac{\partial w}{\partial y}\,\mathrm{d}x\,\mathrm{d}y + \frac{\partial N_{xy}}{\partial y}\frac{\partial w}{\partial x}\,\mathrm{d}x\,\mathrm{d}y \tag{3.27}$$

Summing the projections of all the forces on the z plane

$$\begin{aligned} q^*(x,y,t)\,\mathrm{d}x\,\mathrm{d}y =& 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}\,\mathrm{d}x\,\mathrm{d}y + \frac{\partial N_{xy}}{\partial x}\frac{\partial w}{\partial y}\,\mathrm{d}x\,\mathrm{d}y + \frac{\partial N_{xy}}{\partial y}\frac{\partial w}{\partial x}\,\mathrm{d}x\,\mathrm{d}y \\ &+ N_x \frac{\partial^2 w}{\partial x^2}\,\mathrm{d}x\,\mathrm{d}y + \frac{\partial N_x}{\partial x}\frac{\partial w}{\partial x}\,\mathrm{d}x\,\mathrm{d}y + N_y \frac{\partial^2 w}{\partial y^2}\,\mathrm{d}x\,\mathrm{d}y + \frac{\partial N_y}{\partial y}\frac{\partial w}{\partial y}\,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{3.28}$$

$$\begin{aligned} q^*(x,y,t)\,\mathrm{d}x\,\mathrm{d}y =& 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}\,\mathrm{d}x\,\mathrm{d}y + \frac{\partial w}{\partial y}\left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y \\ &+ \frac{\partial w}{\partial x}\left(\frac{\partial N_{xy}}{\partial y} + \frac{\partial N_x}{\partial x}\right)\mathrm{d}x\,\mathrm{d}y + N_x \frac{\partial^2 w}{\partial x^2}\,\mathrm{d}x\,\mathrm{d}y + N_y \frac{\partial^2 w}{\partial y^2}\,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{3.29}$$

With no body forces, we can use the equations of equilibrium, from figure (3.4), to simplify.

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \tag{3.30}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \tag{3.31}$$

which results in

$$q^*(x,y,t)\,\mathrm{d}x\,\mathrm{d}y = 2N_{xy}\frac{\partial^2 w}{\partial x \partial y}\,\mathrm{d}x\,\mathrm{d}y + N_x\frac{\partial^2 w}{\partial x^2}\,\mathrm{d}x\,\mathrm{d}y + N_y\frac{\partial^2 w}{\partial y^2}\,\mathrm{d}x\,\mathrm{d}y \tag{3.32}$$

Using the above equation with equation (3.21) and (3.10) results in

$$\rho\frac{\partial^2 w}{\partial t^2} + D\nabla^4 w(x,y,t) = q(x,y,t) + h\left(\sigma_x\frac{\partial^2 w}{\partial x^2} + \sigma_y\frac{\partial^2 w}{\partial y^2} + 2\sigma_{xy}\frac{\partial^2 w}{\partial x \partial y}\right) \tag{3.33}$$

Using our equation for Airy's stress function, equation (3.9)

$$\boxed{\rho\frac{\partial^2 w}{\partial t^2} + D\nabla^4 w(x,t) = q(x,t) + h\left(\frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right)} \tag{3.34}$$

The above equation is known as geometrically nonlinear plate theory. It was first derived by von Karman in 1910. This plate theory helps relate axial forces with transverse displacement. However in situations of large deformations, a transverse deformation may cause a significant membrane force. As a result, an equation that relates the amplitude of transverse deflection to the membrane forces is required. In order to find this equation we turn to St. Ventant's compatibility equations.

St. Venant's compatibility condition is satisfied for a particular strain field if the displacement field associated with those strains will be unique without gaps and overlaps. Compatibility ensures that the strain field is realistic. While there are six unique equations the only one equation is of significance for our case and is given below.

$$\frac{\partial^2 E_{xx}}{\partial y^2} + \frac{\partial^2 E_{yy}}{\partial x^2} = 2\frac{\partial^2 E_{xy}}{\partial x \partial y} \tag{3.35}$$

Substituting our strain displacement equations, equation (3.13), results in

$$\frac{\partial^2 E_{xx}}{\partial y^2} + \frac{\partial^2 E_{yy}}{\partial x^2} - 2\frac{\partial^2 E_{xy}}{\partial x \partial y} = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{3.36}$$

The right hand side of the above equation is called Gaussian curvature. The Gaussian curvature is defined as the product of the two principal curvatures of a surface. A developable surface is defined as a surface with zero Gaussian curvature. From the equation above we can say that for a developable surface all inplane strain, E_{xx}, E_{yy}, E_{xy}, are equal to zero. In other words, a developable surface is a surface that can be made into a flat plane without stretching or compressing the surface. The ratio between the area of a stretched surface to the unstretched surface is proportional to Gaussian curvature which is an interesting characteristic. For example, a cylinder can be unrolled onto a flat surface without stretching the material. Therefore, a cylinder is a developable surface. A cone, if cut along one edge to the tip can be flattened without any stretch, therefore it is a developable surface. While the value of Gaussian curvature may not have any direct consequences on the plate deformation, it is important to understand the meaning of Gaussian curvature. Appendix A provides more information on Gaussian curvature.

Using Hooke's law, equation (3.6), to relate the strains to stresses, as well as Airy stress function, equation (3.9), to relate the stresses to the Airy's stress function, ϕ, then substituting into equation (3.36) gives

$$\boxed{\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = \nabla^4 \phi = E\left(\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right)} \tag{3.37}$$

Equations (3.34) and (3.37) constitute a system of nonlinear partial of differential equations. This system is considered the large deflection theory of plates.

Here we use a square plate with all four edges pinned with a length of one. The modal functions for such a plate are given below, where W_{mn} is the modal function of the plate. Each modal function satisfies all of the boundary conditions. The modal functions are the fundamental shapes of motion of a plate and are orthogonal to each other. The motion of a plate can be described as the sum of these modes. Figure (3.5) plots the modal function for the simply supported plate that we are considering here.

$$w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn}(t) W_{mn}(x, y) \tag{3.38}$$

$$W_{mn}(x, y) = \sin(\alpha_m x) \sin(\gamma_n y) \tag{3.39}$$

Note that a and b are the edge lengths of the plate, which are equal to one.

$$\alpha_m = \frac{m\pi}{a} \qquad \gamma_n = \frac{n\pi}{b} \tag{3.40}$$

The stress function also takes the form of a double infinite sum. Its exact form is based on the boundary conditions.

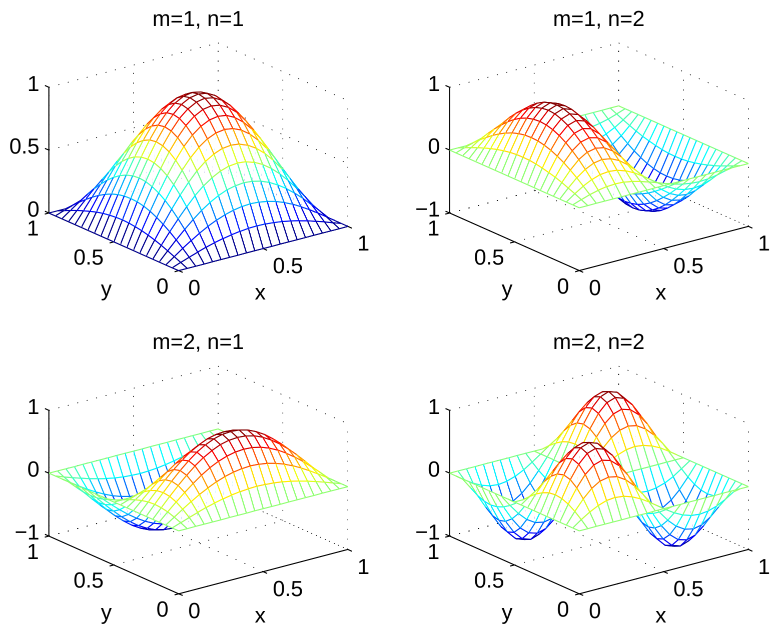


Figure 3.5: Modal Functions of a Plate

3.2.3 Expansion Theorem

The expansion theorem is an important tool in solving plate theory. The theorem allows us to expand any function over an orthogonal basis, an infinite sum, so that we can solve the system. For plates, we will expand over two bases, one for the x-axis and another for the y-axis. In order to show that the basis is orthogonal it must satisfy the condition below. Note that the symbol $\mathcal{B}$ will be used to signify a generic basis.

$$\int_0^b \int_0^a \mathcal{B}_{m,n}(x,y)\mathcal{B}_{r,s}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \mathcal{C}\bar{\delta}_{(m,r)}\bar{\delta}_{(n,s)} \tag{3.41}$$

For a plate simply supported on all sides, the modal function is

$$W_{mn}(x,y) = C_{mn}\sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y) \tag{3.42}$$

In order to assign a value to C_{mn} we normalize the modal function using the definition of inner product.

$$\begin{aligned}\langle W_{mn}(x,y), W_{mn}(x,y)\rangle_\rho &= \int_0^a \int_0^b \rho W_{mn}(x,y)W_{mn}(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^a \int_0^b \rho C_{mn}^2 \sin^2(\frac{m\pi}{a}x)\sin^2(\frac{n\pi}{b}y)\,\mathrm{d}y\,\mathrm{d}x = 1\end{aligned} \tag{3.43}$$

Solving for the coefficient C_{mn} we obtain, choosing the positive root,

$$C_{mn} = \frac{2}{\sqrt{\rho ab}} \tag{3.44}$$

Now our modal function becomes

$$W_{mn}(x,y) = \frac{2}{\sqrt{\rho ab}}\sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y) \tag{3.45}$$

The expansion theorem is

$$g(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} g_{mn}(t)W_{mn}(x,y) \tag{3.46}$$

$$g_{mn}(t) = \frac{\langle g(x,y,t), W_{mn}(x,y)\rangle}{\langle W_{mn}(x,y), W_{mn}(x,y)\rangle_\rho} \tag{3.47}$$

Since the modal functions are normalized, the denominator is equal to one. If we did not normalize our modal function the integral below would have a constant in front of it. It follows that since $\langle W_{mn}, W_{mn}\rangle_\rho = 1$ we have

$$g_{mn}(t) = \int_0^a \int_0^b g(x, y, t) W_{mn}(x, y) \, \mathrm{d}y \, \mathrm{d}x \tag{3.48}$$

In the sections below we choose C_{mn} to equal one, which results in the corresponding expansion theorem

$$g_{mn}(t) = \frac{4}{\rho ab} \int_0^a \int_0^b g(x, y, t) W_{mn}(x, y) \, \mathrm{d}y \, \mathrm{d}x \tag{3.49}$$

We shall use this form of expansion theorem because the maximum value of the modal function is one and this simplifies further calculation. This is important as the coefficient of the modal function gives the 'influence' each mode to the deflection. Also note that the coefficient, C_{mn}, is not a function of m or n which is not, in general, the case.

3.3 Linear Plate Theory

Linear plate theory assumes that the in-plane forces of the plate are negligible. As a result, we can set ϕ, the Airy's stress function and its derivatives equal to zero as it will result in a plate with zero membrane forces. This assumption will simplify the system to linear plate theory.

$$\rho \frac{\partial^2 w}{\partial t^2} + D \nabla^4 w(x, y, t) = q(x, y, t) \tag{3.50}$$

$$0 = \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \tag{3.51}$$

From equation (3.51) we can see that the Gaussian curvature is zero. As a result, we have a developable surface, which is an approximation for small deflections. Equation (3.50) is the classic, linear dynamic plate theory. The method of solution to this problem is similar to that of beams. We will assume a modal function, based on the boundary conditions, and then substitute into the governing equation. More information on modal functions of a plate can be found in Timoshenko [45]. Leissa [26] also studied a variety of boundary conditions and modal functions of plates.

A plate with all four edges pinned will be considered. This set of boundary conditions are considered the best way to proceed analytically.

3.3.1 Statics

The following equations define the static problem for a pinned-pinned simply supported plate.

$$D\nabla^4 w(x,y) = q(x,y) \tag{3.52}$$

$$w = 0|_{x=0,a}, \qquad M_x = 0|_{x=0,a} \tag{3.53}$$

$$w = 0|_{y=0,b}, \qquad M_y = 0|_{y=0,b} \tag{3.54}$$

We expand the solution in the form

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn} W_{mn}(x,y) \tag{3.55}$$

$$W_{mn}(x,y) = \sin(\alpha_m x)\sin(\gamma_n y) \tag{3.56}$$

$$q(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn} W_{mn}(x,y) \tag{3.57}$$

$$q_{mn} = \frac{4}{ab}\int_0^b\int_0^a q(x,y)W_{mn}(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{3.58}$$

Using equation (3.55) along with (3.56) in equation (3.52) allows us to find a form for the coefficients of the modal functions, w_{mn}. These coefficients are important as they give the 'weight' of the modal function on the overall deflection. For example, if several coefficients are on the order of 1, we can ignore the terms were the coefficients are on the order of 10^{-2} as they will not have very much significance on the final solution. Note that the summation notation is dropped and a relationship for coefficients are given. Also note the use of expansion theorem.

$$D\left(\alpha_m^2 + \gamma_n^2\right)^2 w_{mn} = q_{mn} \tag{3.59}$$

$$\boxed{w_{mn} = \frac{q_{mn}}{D\left(\alpha_m^2 + \gamma_n^2\right)^2} = \hat{q}_{mn}} \tag{3.60}$$

Example: Static - Four Term Approximation

For this approximation we assume that the first four terms of a plate would be a sufficient approximation of the plate's deflection.

$$w_{11} = \frac{q_{11}}{D\left(\alpha_1^2 + \gamma_1^2\right)^2} \tag{3.61}$$

$$w_{12} = \frac{q_{12}}{D\left(\alpha_1^2 + \gamma_2^2\right)^2} \tag{3.62}$$

$$w_{21} = \frac{q_{21}}{D\left(\alpha_2^2 + \gamma_1^2\right)^2} \tag{3.63}$$

$$w_{22} = \frac{q_{22}}{D\left(\alpha_2^2 + \gamma_2^2\right)^2} \tag{3.64}$$

After finding the four coefficients above we would place them in the sum below to get the approximate deflection of the plate.

$$w(x,y) = w_{11}W_{11}(x,y) + w_{12}W_{12}(x,y) + w_{21}W_{21}(x,y) + w_{22}W_{22}(x,y) \tag{3.65}$$

3.3.2 Dynamics

The equation motion for the plate is

$$\rho\frac{\partial^2 w}{\partial t^2} + D\nabla^4 w(x,y,t) = q(x,y,t) \tag{3.66}$$

Using an expansion of

$$w(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn}(t)W_{mn}(x,y) \tag{3.67}$$

we obtain the modal equation of motion

$$\ddot{w}_{mn} + \frac{D\left(\alpha_m^2 + \gamma_n^2\right)^2}{\rho} w_{mn} = \frac{1}{\rho} q_{mn} \tag{3.68}$$

Setting $\omega_{mn}^2 = \frac{D\left(\alpha_m^2+\gamma_n^2\right)^2}{\rho}$, where we consider ω_{mn} as the natural frequency of the system,

$$\ddot{w}_{mn} + \omega_{mn}^2 w_{mn} = \frac{1}{\rho} q_{mn} \tag{3.69}$$

The solution of the above equation takes the form of

$$w_{mn}(t) = w_{mn}(0)\cos(\omega_{mn}t) + \frac{\dot{w}_{mn}(0)}{\omega_{mn}}\sin(\omega_{mn}t) + \frac{1}{\rho\omega_{mn}}\int_0^t q_{mn}(t-\tau)\sin(\omega_{mn}\tau)\,\mathrm{d}\tau \tag{3.70}$$

and the initial conditions are

$$w_{mn}(0) = \frac{4}{\rho ab}\int_0^b\int_0^a w(x,y,0)\sin(\alpha_m x)\sin(\gamma_n y)\,\mathrm{d}x\,\mathrm{d}y \tag{3.71}$$

$$\dot{w}_{mn}(0) = \frac{4}{\rho ab}\int_0^b\int_0^a \dot{w}(x,y,0)\sin(\alpha_m x)\sin(\gamma_n y)\,\mathrm{d}x\,\mathrm{d}y \tag{3.72}$$

Example: Dynamic - Four Term Approximation

For this approximation we will use four terms of the infinite sum. Where $m, n = 1$ or 2.

$$\ddot{w}_{11} + \omega_{11}^2 w_{11} = \frac{1}{\rho} q_{11} \tag{3.73}$$

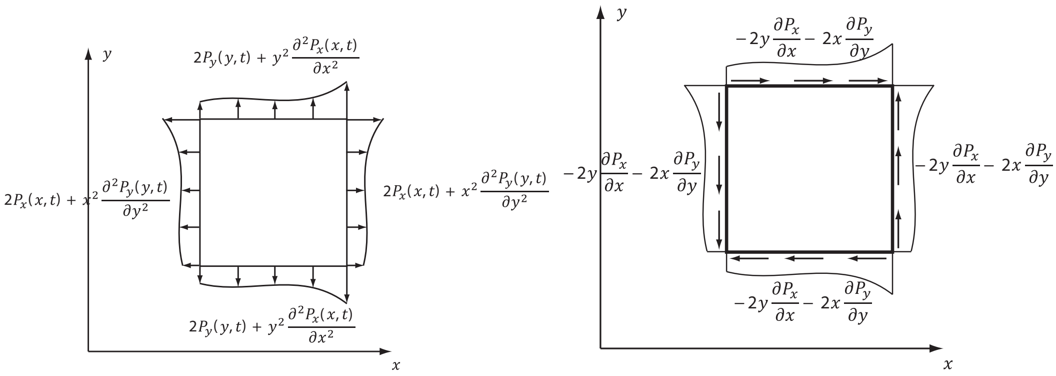
$$\ddot{w}_{12} + \omega_{12}^2 w_{12} = \frac{1}{\rho} q_{12} \tag{3.74}$$

$$\ddot{w}_{21} + \omega_{21}^2 w_{21} = \frac{1}{\rho} q_{21} \tag{3.75}$$

$$\ddot{w}_{22} + \omega_{22}^2 w_{22} = \frac{1}{\rho} q_{22} \tag{3.76}$$

The system above is a system of four independent differential equations. Each equation can be solved separately and placed back into our truncated sum in the expansion theorem.

$$w(x,y,t) = w_{11}(t)W_{11}(x,y) + w_{12}(t)W_{12}(x,y) + w_{21}(t)W_{21}(x,y) + w_{22}(t)W_{22}(x,y) \tag{3.77}$$



(a) Axial Loads - Normal (b) Axial Loads - Shear

Figure 3.6: Loads Caused by Immovable Edge Conditions

3.4 Geometrically Nonlinear Plate Theory

In this section, the geometrically nonlinear terms are retained as the membrane forces, ϕ and its derivatives, become significant when large deformations take place. As we add axial loads, we need to determine axial boundary conditions. We can either choose to have the edges restricted from moving, 'immovable edges', or edges that are free to move, 'stress-free' edges. These boundary conditions have a significant effect on the stiffness and response of the plate.

Based on the axial boundary conditions we will change the definition of the Airy stress function. Consider figure (3.6), P_x and P_y are the loads that prevent the edges from moving in the axial direction. For 'stress-free' edges these forces are equal to zero as the edges are allowed in move freely in the plane of the undeformed plate. Here we expand the stress function using a double sine series as it satisfies the condition that there are no axial stresses for stress-free edges. Other texts like Timoshenko [45] use a double cosine series in their expansion and integrate over the edge to show that there is a zero net force along the edge. If one were to use the double cosine series here one would find that the geometric nonlinear terms would drop out. This is because of the orthogonality relationship between sine and cosine. We write

$$\phi(x,y,t) = P_x y^2 + P_y x^2 + \sum \phi_{mn}(t) W_{mn}(x,y) \tag{3.78}$$

P_x and P_y are functions of the tensile loads at $x = 0, a$ and $y = 0, b$, respectively. For stress-free edges we would allow the edges to move in the axial direction and would set $P_x = P_y = 0$. For fixed edges, we would set the axial deformation to zero and P_x and P_y would not be zero. Another question of interest is the dependence of P_x and P_y on the spatial variables. For one case we will have $P_x(x)$ and $P_y(y)$. Another case will consider P_x and P_y to be a constant. As a basis for comparison we will use results from Levy [28], Iyengar and Naqvi [24], and Donnell [16].

In order to find the axial displacement, we first equate the strain-displacement relationship to the strain-stress relationship using Airy's stress function.

$$E_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 = \frac{1}{E}\left(\frac{\partial^2 \phi}{\partial y^2} - \nu \frac{\partial^2 \phi}{\partial x^2}\right) \tag{3.79}$$

$$E_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \frac{1}{E}\left(\frac{\partial^2 \phi}{\partial x^2} - \nu \frac{\partial^2 \phi}{\partial y^2}\right) \tag{3.80}$$

To find the axial displacement in x direction, defined by $\delta_x(y)$

$$\frac{\partial u}{\partial x} = \frac{1}{E}\left(\frac{\partial^2 \phi}{\partial y^2} - \nu \frac{\partial^2 \phi}{\partial x^2}\right) - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{3.81}$$

$$\delta_x(y) = \int_0^a \frac{\partial u}{\partial x}\,\mathrm{d}x = \int_0^a \left\{\frac{1}{E}\left(\frac{\partial^2 \phi}{\partial y^2} - \nu \frac{\partial^2 \phi}{\partial x^2}\right) - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right\}\mathrm{d}x \tag{3.82}$$

To find axial displacement in y direction, defined by $\delta_y(x)$

$$\frac{\partial v}{\partial y} = \frac{1}{E}\left(\frac{\partial^2 \phi}{\partial x^2} - \nu \frac{\partial^2 \phi}{\partial y^2}\right) - \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \tag{3.83}$$

$$\delta_y(x) = \int_0^b \frac{\partial v}{\partial y}\,\mathrm{d}y = \int_0^b \left\{\frac{1}{E}\left(\frac{\partial^2 \phi}{\partial x^2} - \nu \frac{\partial^2 \phi}{\partial y^2}\right) - \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right\}\mathrm{d}y \tag{3.84}$$

3.4.1 Statics

In the static case we remove the time dependent terms from equation (3.34).

$$D\nabla^4 w(x,y) = q(x,y) + h\left(\frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right) \tag{3.85}$$

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = \nabla^4 \phi = E\left(\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right) \tag{3.86}$$

Stress Free Edges

For stress free edges we set $P_x = P_y = 0$ so that the stress function would predict zero stress at the edges. Using

$$w(x, y) = \sum w_{mn} W_{mn}(x, y) \tag{3.87}$$

$$\phi(x, y) = \sum \phi_{mn} W_{mn}(x, y) \tag{3.88}$$

$$W_{mn} = \sin(\alpha_m x)\sin(\gamma_n y) \tag{3.89}$$

$$\Phi_{mn} = \cos(\alpha_m x)\cos(\gamma_n y) \tag{3.90}$$

Using summation notation in our equation of motion

$$\begin{aligned} D\sum(\alpha_m^2 + \gamma_n^2)^2 w_{mn} W_{mn} = \sum q_{mn} W_{mn} + h\Big\{&(\sum \alpha_a^2 \phi_{ab} W_{ab})(\sum \gamma_n^2 w_{mn} W_{mn}) \\ &+ (\sum \gamma_b^2 \phi_{ab} W_{ab})(\sum \alpha_m^2 w_{mn} W_{mn}) \\ &- 2(\sum \alpha_a \gamma_b \phi_{ab} \Phi_{ab})(\sum \alpha_m \gamma_n w_{mn} \Phi_{mn})\Big\} \end{aligned} \tag{3.91}$$

$$\begin{aligned} \sum(\alpha_m^2 + \gamma_n^2)^2 \phi_{mn} W_{mn} = E\Big\{&(\sum \alpha_a \gamma_b w_{ab} \Phi_{ab})(\sum \alpha_m \gamma_n w_{mn} \Phi_{mn}) \\ &- (\sum \alpha_a^2 w_{ab} W_{ab})(\sum \gamma_n^2 w_{mn} W_{mn})\Big\} \end{aligned} \tag{3.92}$$

Example: Static, Stress Free Edges - One-to-One Term Approximation

We make the following assumption initially, we ignore the coupling between modes. While this assumption may not result in an accurate solution, this procedure would give a reasonable estimate as to which modes contribute the most for a particular load case. If a large number of modes are required this assumption will reduce the computation time significantly. The coupled solution requires an addition equation for each additional mode, and all the equations increase in complexity as well.

$$D(\alpha_m^2+\gamma_n^2)^2 w_{mn} W_{mn} = q_{mn} W_{mn} + 2h\alpha_m^2\gamma_n^2\phi_{mn} w_{mn}\left(W_{mn}W_{mn} - \Phi_{mn}\Phi_{mn}\right) \quad (3.93)$$

$$(\alpha_m^2+\gamma_n^2)^2\phi_{mn} W_{mn} = E\alpha_m^2\gamma_n^2 w_{mn} w_{mn}(\Phi_{mn}\Phi_{mn} - W_{mn}W_{mn}) \quad (3.94)$$

We multiply by W_{ab} and integrate, taking note of the expansion theorem. Also, for ease of use, defining

$$\zeta_{mn} = \int_A \left(W_{mn}W_{mn} - \Phi_{mn}\Phi_{mn}\right) W_{mn}\, \mathrm{d}A \quad (3.95)$$

results in

$$\frac{D}{4}(\alpha_m^2+\gamma_n^2)^2 w_{mn} = \frac{q_{mn}}{4} + 2h\alpha_m^2\gamma_n^2\phi_{mn} w_{mn}\zeta_{mn} \quad (3.96)$$

$$\frac{1}{4}(\alpha_m^2+\gamma_n^2)^2\phi_{mn} = -E\alpha_m^2\gamma_n^2 w_{mn} w_{mn}\zeta_{mn} \quad (3.97)$$

Solving for ϕ_{mn}

$$\phi_{mn} = -4E\zeta_{mn}\frac{\alpha_m^2\gamma_n^2 w_{mn} w_{mn}}{(\alpha_m^2+\gamma_n^2)^2} \quad (3.98)$$

Using the above result in equation (3.96)

$$w_{mn} = \frac{q_{mn}}{D(\alpha_m^2+\gamma_n^2)^2} - 32Eh\frac{\alpha_m^4\gamma_n^4\zeta_{mn}^2}{D(\alpha_m^2+\gamma_n^2)^4} w_{mn}^3 \quad (3.99)$$

Simplifying, we obtain

$$\boxed{32\frac{Eh}{D}\frac{\alpha_m^4\gamma_n^4\zeta_{mn}^2}{(\alpha_m^2+\gamma_n^2)^4} w_{mn}^3 + w_{mn} = \frac{q_{mn}}{D(\alpha_m^2+\gamma_n^2)^2} = \hat{q}_{mn}} \quad (3.100)$$

We can see that the geometric nonlinear term is dependent on ζ_{mn}. If this term is zero, we recover the linear equation for that coefficient. Table (3.1) shows several values of ζ_{mn}. The equation above is of third order, meaning that it is possible to have

ζ_{mn}					
n \ m	1	2	3	4	5
1	$\frac{4}{3\pi^2}$	0	$\frac{4}{9\pi^2}$	0	$\frac{4}{15\pi^2}$
2	0	0	0	0	0
3	$\frac{4}{9\pi^2}$	0	$\frac{4}{27\pi^2}$	0	$\frac{4}{45\pi^2}$
4	0	0	0	0	0
5	$\frac{4}{15\pi^2}$	0	$\frac{4}{45\pi^2}$	0	$\frac{4}{75\pi^2}$

Table 3.1: Values of ζ_{mn}

up to three real solutions. However, with values we have chosen, the solution consists of one real value and two imaginary values.

For the first nine terms we get:

$$\frac{32}{9\pi^4}\frac{Eh}{D}w_{11}^3 + w_{11} = \hat{q}_{11}$$

$$w_{12} = \hat{q}_{12}$$

$$w_{21} = \hat{q}_{21}$$

$$w_{22} = \hat{q}_{22}$$

$$\frac{32}{625\pi^4}\frac{Eh}{D}w_{13}^3 + w_{13} = \hat{q}_{13}$$

$$\frac{32}{625\pi^4}\frac{Eh}{D}w_{31}^3 + w_{31} = \hat{q}_{31}$$

$$w_{23} = \hat{q}_{23}$$

$$w_{32} = \hat{q}_{32}$$

$$\frac{32}{729\pi^4}\frac{Eh}{D}w_{33}^3 + w_{33} = \hat{q}_{33}$$

From above we can see that several of the terms are not influenced by the geometric nonlinearity due to the value of ζ_{mn}. The coefficients of the w_{mn}^3 terms above are smaller as m and n grow larger. We can see that the coefficient of w_{13}^3 is more than sixty times smaller than the coefficient of the w_{11}^3 term. In other words, lower frequencies are more sensitive to the nonlinearity than higher frequencies. Also note that the equation for w_{11} matches the result given by Ventsel [47].

We can see that the one-to-one term approximation is useful in determining how several of the modes behave in a nonlinear system and how some modes are ignored completely. This approximation predicts that several of the modes are not changed by the nonlinearity, which, in general, is not the case. We next return to the coupled case.

Example: Static, Stress Free Edges - Four Coupled Terms

This example takes the first four terms of plate theory and solves the equations. It will be shown that this system results in eights equations and eight unknowns. The unknowns not only consist of w_{mn} but also the corresponding ϕ_{mn}. Expanding equations (3.91) and (3.92)

$$\begin{aligned} D(\alpha_1^2+\gamma_1^2)^2 w_{11}W_{11} &= q_{11}W_{11} \\ &+ h\Big\{(\gamma_1^2 w_{11}W_{11})(\alpha_1^2\phi_{11}W_{11}+\alpha_1^2\phi_{12}W_{12}+\alpha_2^2\phi_{21}W_{21}+\alpha_2^2\phi_{22}W_{22}) \\ &+ (\alpha_1^2 w_{11}W_{11})(\gamma_1^2\phi_{11}W_{11}+\gamma_2^2\phi_{12}W_{12}+\gamma_1^2\phi_{21}W_{21}+\gamma_2^2\phi_{22}W_{22}) \\ &- 2(\alpha_1\gamma_1\phi_{11}\Phi_{11})(\alpha_1\gamma_1 w_{11}\Phi_{11}+\alpha_1\gamma_2 w_{12}\Phi_{12}+\alpha_2\gamma_1 w_{21}\Phi_{21}+\alpha_2\gamma_2 w_{22}\Phi_{22})\Big\} \end{aligned} \tag{3.101}$$

$$\begin{aligned} D(\alpha_1^2+\gamma_2^2)^2 w_{12}W_{12} &= q_{12}W_{12} \\ &+ h\Big\{(\gamma_2^2 w_{12}W_{12})(\alpha_1^2\phi_{11}W_{11}+\alpha_1^2\phi_{12}W_{12}+\alpha_2^2\phi_{21}W_{21}+\alpha_2^2\phi_{22}W_{22}) \\ &+ (\alpha_1^2 w_{12}W_{12})(\gamma_1^2\phi_{11}W_{11}+\gamma_2^2\phi_{12}W_{12}+\gamma_1^2\phi_{21}W_{21}+\gamma_2^2\phi_{22}W_{22}) \\ &- 2(\alpha_1\gamma_2\phi_{12}\Phi_{12})(\alpha_1\gamma_1 w_{11}\Phi_{11}+\alpha_1\gamma_2 w_{12}\Phi_{12}+\alpha_2\gamma_1 w_{21}\Phi_{21}+\alpha_2\gamma_2 w_{22}\Phi_{22})\Big\} \end{aligned} \tag{3.102}$$

$$D(\alpha_2^2+\gamma_1^2)^2 w_{21}W_{21} = q_{21}W_{21}$$
$$+ h\Big\{(\gamma_1^2 w_{21}W_{21})(\alpha_1^2\phi_{11}W_{11}+\alpha_1^2\phi_{12}W_{12}+\alpha_2^2\phi_{21}W_{21}+\alpha_2^2\phi_{22}W_{22})$$
$$+ (\alpha_2^2 w_{21}W_{21})(\gamma_1^2\phi_{11}W_{11}+\gamma_2^2\phi_{12}W_{12}+\gamma_1^2\phi_{21}W_{21}+\gamma_2^2\phi_{22}W_{22})$$
$$- 2(\alpha_2\gamma_1\phi_{21}\Phi_{21})(\alpha_1\gamma_1 w_{11}\Phi_{11}+\alpha_1\gamma_2 w_{12}\Phi_{12}+\alpha_2\gamma_1 w_{21}\Phi_{21}+\alpha_2\gamma_2 w_{22}\Phi_{22})\Big\} \tag{3.103}$$

$$D(\alpha_2^2+\gamma_2^2)^2 w_{22}W_{22} = q_{22}W_{22}$$
$$+ h\Big\{(\gamma_2^2 w_{22}W_{22})(\alpha_1^2\phi_{11}W_{11}+\alpha_1^2\phi_{12}W_{12}+\alpha_2^2\phi_{21}W_{21}+\alpha_2^2\phi_{22}W_{22})$$
$$+ (\alpha_2^2 w_{22}W_{22})(\gamma_1^2\phi_{11}W_{11}+\gamma_2^2\phi_{12}W_{12}+\gamma_1^2\phi_{21}W_{21}+\gamma_2^2\phi_{22}W_{22})$$
$$- 2(\alpha_2\gamma_2\phi_{22}\Phi_{22})(\alpha_1\gamma_1 w_{11}\Phi_{11}+\alpha_1\gamma_2 w_{12}\Phi_{12}+\alpha_2\gamma_1 w_{21}\Phi_{21}+\alpha_2\gamma_2 w_{22}\Phi_{22})\Big\} \tag{3.104}$$

$$(\alpha_1^2+\gamma_1^2)^2\phi_{11}W_{11} =$$
$$E\Big\{(\alpha_1\gamma_1 w_{11}\Phi_{11})(\alpha_1\gamma_1 w_{11}\Phi_{11}+\alpha_1\gamma_2 w_{12}\Phi_{12}+\alpha_2\gamma_1 w_{21}\Phi_{21}+\alpha_2\gamma_2 w_{22}\Phi_{13}) \tag{3.105}$$
$$- (\alpha_1^2 w_{11}W_{11})(\gamma_1^2 w_{11}W_{11}+\gamma_2^2 w_{12}W_{12}+\gamma_1^2 w_{21}W_{21}+\gamma_2^2 w_{22}W_{22})\Big\}$$

$$(\alpha_1^2+\gamma_2^2)^2\phi_{12}W_{12} =$$
$$E\Big\{(\alpha_1\gamma_2 w_{12}\Phi_{12})(\alpha_1\gamma_1 w_{11}\Phi_{11}+\alpha_1\gamma_2 w_{12}\Phi_{12}+\alpha_2\gamma_1 w_{21}\Phi_{21}+\alpha_2\gamma_2 w_{22}\Phi_{22}) \tag{3.106}$$
$$- (\alpha_1^2 w_{12}W_{12})(\gamma_1^2 w_{11}W_{11}+\gamma_2^2 w_{12}W_{12}+\gamma_1^2 w_{21}W_{21}+\gamma_2^2 w_{22}W_{22})\Big\}$$

$$(\alpha_2^2+\gamma_1^2)^2\phi_{21}W_{21} =$$
$$E\Big\{(\alpha_2\gamma_1 w_{21}\Phi_{21})(\alpha_1\gamma_1 w_{11}\Phi_{11}+\alpha_1\gamma_2 w_{12}\Phi_{12}+\alpha_2\gamma_1 w_{21}\Phi_{21}+\alpha_2\gamma_2 w_{22}\Phi_{22}) \tag{3.107}$$
$$- (\alpha_2^2 w_{21}W_{21})(\gamma_1^2 w_{11}W_{11}+\gamma_2^2 w_{12}W_{12}+\gamma_1^2 w_{21}W_{21}+\gamma_2^2 w_{22}W_{22})\Big\}$$

$$(\alpha_2^2+\gamma_2^2)^2\phi_{22}W_{22} =$$
$$E\Big\{(\alpha_2\gamma_2 w_{22}\Phi_{22})(\alpha_1\gamma_1 w_{11}\Phi_{11}+\alpha_1\gamma_2 w_{12}\Phi_{12}+\alpha_2\gamma_1 w_{21}\Phi_{21}+\alpha_2\gamma_2 w_{22}\Phi_{22}) \tag{3.108}$$
$$- (\alpha_2^2 w_{22}W_{22})(\gamma_1^2 w_{11}W_{11}+\gamma_2^2 w_{12}W_{12}+\gamma_1^2 w_{21}W_{21}+\gamma_2^2 w_{22}W_{22})\Big\}$$

After integration by expansion theorem

$$\begin{aligned} D(\alpha_1^2+\gamma_1^2)^2 w_{11}\frac{1}{4} = q_{11}\frac{1}{4} + h\Big\{&(\gamma_1^2 w_{11})(\alpha_1^2\phi_{11}\frac{16}{9\pi^2}) \\ &+ (\alpha_1^2 w_{11})(\gamma_1^2\phi_{11}\frac{16}{9\pi^2}) - 2(\alpha_1\gamma_1\phi_{11})(\alpha_1\gamma_1 w_{11}\frac{4}{9\pi^2})\Big\} \end{aligned} \tag{3.109}$$

$$\begin{aligned} D(\alpha_1^2+\gamma_2^2)^2 w_{12}\frac{1}{4} = q_{12}\frac{1}{4} + h\Big\{&(\gamma_2^2 w_{12})(\alpha_1^2\phi_{11}\frac{64}{45\pi^2}) \\ &+ (\alpha_1^2 w_{12})(\gamma_1^2\phi_{11}\frac{64}{45\pi^2}) - 2(\alpha_1\gamma_2\phi_{12})(\alpha_1\gamma_1 w_{11}\frac{8}{45\pi^2})\Big\} \end{aligned} \tag{3.110}$$

$$\begin{aligned} D(\alpha_2^2+\gamma_1^2)^2 w_{21}\frac{1}{4} = q_{21}\frac{1}{4} + h\Big\{&(\gamma_1^2 w_{21})(\alpha_1^2\phi_{11}\frac{64}{45\pi^2}) \\ &+ (\alpha_2^2 w_{21})(\gamma_1^2\phi_{11}\frac{64}{45\pi^2}) - 2(\alpha_2\gamma_1\phi_{21})(\alpha_1\gamma_1 w_{11}\frac{8}{45\pi^2})\Big\} \end{aligned} \tag{3.111}$$

$$\begin{aligned} D(\alpha_2^2+\gamma_2^2)^2 w_{22}\frac{1}{4} = q_{22}\frac{1}{4} + h\Big\{&(\gamma_2^2 w_{22})(\alpha_1^2\phi_{11}\frac{256}{225\pi^2}) \\ &+ (\alpha_2^2 w_{22})(\gamma_1^2\phi_{11}\frac{256}{225\pi^2}) - 2(\alpha_2\gamma_2\phi_{22})(\alpha_1\gamma_1 w_{11}\frac{16}{225\pi^2})\Big\} \end{aligned} \tag{3.112}$$

$$(\alpha_1^2+\gamma_1^2)^2\phi_{11}\frac{1}{4} = E\Big\{(\alpha_1\gamma_1 w_{11})(\alpha_1\gamma_1 w_{11}\frac{4}{9\pi^2}) - (\alpha_1^2 w_{11})(\gamma_1^2 w_{11}\frac{16}{9\pi^2})\Big\} \tag{3.113}$$

$$(\alpha_1^2+\gamma_2^2)^2\phi_{12}\frac{1}{4} = E\Big\{(\alpha_1\gamma_2 w_{12})(\alpha_1\gamma_1 w_{11}\frac{8}{45\pi^2}) - (\alpha_1^2 w_{12})(\gamma_1^2 w_{11}\frac{64}{45\pi^2})\Big\} \tag{3.114}$$

$$(\alpha_2^2+\gamma_1^2)^2\phi_{21}\frac{1}{4} = E\Big\{(\alpha_2\gamma_1 w_{21})(\alpha_1\gamma_1 w_{11}\frac{8}{45\pi^2}) - (\alpha_2^2 w_{21})(\gamma_1^2 w_{11}\frac{64}{45\pi^2})\Big\} \tag{3.115}$$

$$(\alpha_2^2+\gamma_2^2)^2\phi_{22}\frac{1}{4} = E\Big\{(\alpha_2\gamma_2 w_{22})(\alpha_1\gamma_1 w_{11}\frac{16}{225\pi^2}) - (\alpha_2^2 w_{22})(\gamma_1^2 w_{11}\frac{256}{225\pi^2})\Big\} \tag{3.116}$$

By inspection we can see that the equations for w_{11} and ϕ_{11}, equations (3.109) and (3.113), form a closed system of two equations and two unknowns. We plug in the values of those equations below.

$$4\pi^4 D w_{11} = q_{11} + \frac{32}{3} h\pi^2 w_{11}\phi_{11} \tag{3.117}$$

$$\phi_{11} = -\frac{4}{15\pi^2} E w_{11}(5w_{11}) \tag{3.118}$$

Eliminating ϕ_{11} from our equations we get

$$w_{11} = \hat{q}_{11} + \frac{Eh}{D\pi^4}(-\frac{32}{9} w_{11}^3) \tag{3.119}$$

which is the same as the equation from the one-to-one term example. It should be noted that this is not always the case. If one more term were taken in the sum, say $(m, n) = (1, 3)$ we would see that this would not be the case.

The other equations become:

$$25d\pi^4 w_{12} = q_{12} + \frac{256}{9} h\pi^2 w_{12}\phi_{11} - \frac{128}{45} h\pi^2 \phi_{12} w_{11} \tag{3.120}$$

$$25d\pi^4 w_{21} = q_{21} + \frac{256}{9} h\pi^2 w_{21}\phi_{11} - \frac{9472}{225} h\pi^2 \phi_{21} w_{11} - \frac{128}{45} h\pi^2 \phi_{21} w_{11} \tag{3.121}$$

$$64d\pi^4 w_{22} = q_{22} + \frac{8192}{225} h\pi^2 w_{22}\phi_{11} - \frac{512}{225} h\pi^2 \phi_{22} w_{11} \tag{3.122}$$

$$\frac{25}{4}\pi^4 \phi_{12} = -\frac{16}{105} E\pi^2 w_{12}(7w_{11}) \tag{3.123}$$

$$\frac{25}{4}\pi^4 \phi_{21} = -\frac{16}{15} E\pi^2 w_{21}(5w_{11}) \tag{3.124}$$

$$16\pi^4 \phi_{22} = -\frac{64}{105} E\pi^2 w_{22}(7w_{11}) \tag{3.125}$$

Eliminating ϕ_{12}, ϕ_{21} and ϕ_{22} we get:

$$w_{12} = \hat{q}_{12} + \frac{Eh}{D\pi^4}(-\frac{631808}{421875} w_{12} w_{11}^2) \tag{3.126}$$

$$w_{21} = \hat{q}_{21} + \frac{Eh}{D\pi^4}(-\frac{13312}{9375}w_{21}w_{11}^2) \tag{3.127}$$

$$w_{22} = \hat{q}_{22} + \frac{Eh}{D\pi^4}(-\frac{2528}{3375}w_{22}w_{11}^2) \tag{3.128}$$

We can see that for w_{12}, w_{21}, and w_{22} the equations changed from the one-to-one example. Also note that if $q_{12} = q_{21}$, for a symmetric type loading, then $w_{12} \neq w_{21}$. This results from having a finite number of terms in the solution. As we take more terms we would find that w_{12} and w_{21} would begin to approach each other in value. The plot for w_{11} that is solved here is given in figure (3.7). The figure also shows the results given for Ventsel [47], Donnell [16], Iyengar [24], and Levy [28]. Levy's six term approximation assumed that there was a uniform load on the plate. We can see that there is good agreement of the results. Compared to linear theory, stress-free edges predict a smaller value. This is due to the fact that the inclusion of the geometric nonlinearity caused the axial stiffness to be an additional stiffness term in the equation of motion. If we look at the results in the form of

$$\left(\frac{32}{9\pi^4}\frac{Eh}{D}w_{11}^2 + 1\right)w_{11} = \hat{q}_{11} \tag{3.129}$$

$$D_{stress-free}(w_{11})w_{11} = \hat{q}_{11} \tag{3.130}$$

we can see that the nonlinear system would be 'stiffer' than the linear system.

Below is the function for w_{11} if we were to take the first nine terms in the series.

$$\begin{aligned} w_{11} = \hat{q_{11}} + \frac{Eh}{D\pi^4}(&-\frac{32}{9}\,{w_{11}}^3 + \frac{11584}{2625}\,{w_{11}}^2 w_{13} - \frac{1088}{675}\,{w_{11}}^2 w_{31} + \frac{15296}{2625}\,{w_{11}}^2 w_{33} \\ &+ \frac{992}{225}\,w_{11}{w_{13}}^2 + \frac{832}{7875}\,w_{11}w_{13}w_{31} - \frac{20288}{3375}\,w_{11}w_{13}w_{33} - \frac{32}{225}\,w_{11}{w_{31}}^2 \\ &+ \frac{3904}{2625}\,w_{11}w_{33}w_{31} - \frac{8416}{3375}\,w_{11}{w_{33}}^2) \end{aligned} \tag{3.131}$$

The complexity of the equation for w_{11} has grown significantly. However, it is still possible to recover our one-to-one term approximation.

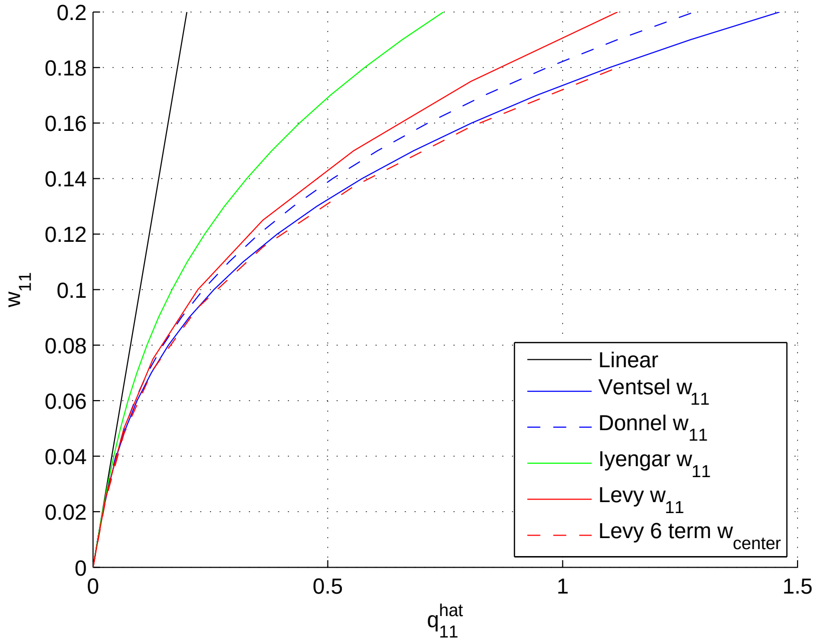


Figure 3.7: Comparing Stress-Free Boundary Condition

Immovable Edges

Immovable edges are defined by equations (3.82) and (3.84), which are set equal to zero. The resulting expressions are

$$\delta_x = 0 \qquad\qquad \delta_y = 0 \tag{3.132}$$

$$\phi = P_x(x)y^2 + P_y(y)x^2 + \sum \phi_{mn} W_{mn} \tag{3.133}$$

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 2P_x(x) + x^2 \frac{\partial^2 P_y}{\partial y^2} - \sum \gamma_n^2 \phi_{mn} W_{mn} \tag{3.134}$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 2P_y(y) + y^2 \frac{\partial^2 P_x}{\partial x^2} - \sum \alpha_m^2 \phi_{mn} W_{mn} \tag{3.135}$$

$$-\sigma_{xy} = \frac{\partial^2 \phi}{\partial x \partial y} = 2y\frac{\partial P_x}{\partial x} + 2x\frac{\partial P_y}{\partial y} + \sum \alpha_m \gamma_n \phi_{mn} \Phi_{mn} \tag{3.136}$$

Our equations of motion are reproduced below.

$$D\nabla^4 w(x,t) = q(x,t) + h\left(\frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right) \tag{3.137}$$

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = \nabla^4 \phi = E\left(\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right) \tag{3.138}$$

Expanding the terms

$$\begin{aligned}D\sum(\alpha_m^2+\gamma_n^2)^2 w_{mn}W_{mn} = \sum q_{mn}W_{mn} + h\Big\{&(\sum \alpha_a^2\phi_{ab}W_{ab})(\sum \gamma_n^2 w_{mn}W_{mn})\\ &+ (\sum \gamma_b^2\phi_{ab}W_{ab})(\sum \alpha_m^2 w_{mn}W_{mn})\\ &- 2(\sum \alpha_a\gamma_b\phi_{ab}\Phi_{ab})(\sum \alpha_m\gamma_n w_{mn}\Phi_{mn})\\ &- (2P_y(y) + y^2\frac{\partial^2 P_x}{\partial x^2})(\sum \gamma_n^2 w_{mn}W_{mn})\\ &- (2P_x(x) + x^2\frac{\partial^2 P_y}{\partial y^2})(\sum \alpha_m^2 w_{mn}W_{mn})\\ &- 2(2y\frac{\partial P_x}{\partial x} + 2x\frac{\partial P_y}{\partial y})(\sum \alpha_m\gamma_n w_{mn}\Phi_{mn})\Big\}\end{aligned} \tag{3.139}$$

$$\begin{aligned}\sum(\alpha_m^2+\gamma_n^2)^2\phi_{mn}W_{mn} = E\Big\{&(\sum \alpha_a\gamma_b w_{ab}\Phi_{ab})(\sum \alpha_m\gamma_n w_{mn}\Phi_{mn})\\ &- (\sum \alpha_a^2 w_{ab}W_{ab})(\sum \gamma_n^2 w_{mn}W_{mn})\Big\}\\ &+ -4\left(\frac{\partial^2 P_x}{\partial x^2} + \frac{\partial^2 P_y}{\partial y^2}\right) - y^2\frac{\partial^4 P_x}{\partial x^4} - x^2\frac{\partial^4 P_y}{\partial y^4}\end{aligned} \tag{3.140}$$

We need an expression for P_x and P_y in terms of w_{mn}, W_{mn} or ϕ_{mn}, Φ_{mn} then we can proceed with our expansion theorem.

Using these relationships

$$0 = \delta_x(y) = \int_0^1 \frac{1}{E}\left(\frac{\partial^2\phi}{\partial y^2} - \nu\frac{\partial^2\phi}{\partial x^2}\right) - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x \tag{3.141}$$

$$0 = \delta_y(x) = \int_0^1 \frac{1}{E}\left(\frac{\partial^2\phi}{\partial x^2} - \nu\frac{\partial^2\phi}{\partial y^2}\right) - \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \mathrm{d}y \tag{3.142}$$

$$\begin{aligned}0 = \delta_x(y) = \int_0^1 \frac{1}{E}\Big(&2P_x(x) + x^2\frac{\partial^2 P_y}{\partial y^2} - \sum \gamma_n^2\phi_{mn}W_{mn}\\ &-\nu(2P_y(y) + y^2\frac{\partial^2 P_x}{\partial x^2} - \sum \alpha_m^2\phi_{mn}W_{mn})\Big) - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x\end{aligned} \tag{3.143}$$

$$\begin{aligned} 0 = \delta_y(x) = \int_0^1 \frac{1}{E} \bigg(& 2P_y(y) + y^2 \frac{\partial^2 P_x}{\partial x^2} - \sum \alpha_m^2 \phi_{mn} W_{mn} \\ & -\nu(2P_x(x) + x^2 \frac{\partial^2 P_y}{\partial y^2} - \sum \gamma_n^2 \phi_{mn} W_{mn}) \bigg) - \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \mathrm{d}y \end{aligned} \tag{3.144}$$

$$\int_0^1 \left(\frac{\partial w}{\partial x} \right)^2 \mathrm{d}x = \sum \sum_{q=1}^{\infty} \frac{\alpha_m^2}{2} w_{mn} w_{mq} \sin(\gamma_n y) \sin(\gamma_q y) \tag{3.145}$$

$$\int_0^1 \left(\frac{\partial w}{\partial y} \right)^2 \mathrm{d}x = \sum \sum_{q=1}^{\infty} \frac{\gamma_n^2}{2} w_{mn} w_{mq} \sin(\gamma_n y) \sin(\gamma_q y) \tag{3.146}$$

P_x and P_y must satisfy equations (3.143) and (3.144). Therefore, we will expand the axial loads.

$$P_x(x) = \sum P_m^x \sin(\alpha_m x) \tag{3.147}$$

$$P_y(y) = \sum P_n^y \sin(\gamma_n y) \tag{3.148}$$

Example: Static, Immovable Edges - $P_x(x)$ and $P_y(y)$ - One Term

Because of the complicated nature of the problem, we will only consider a one term approximation, $m = n = 1$.

$$\begin{aligned} \delta_x = 0 = & \frac{4}{\pi} P^x - \frac{\pi^2}{3} P^y \sin(\pi y) - 2\pi \phi_{11} \sin(\pi y) \\ & - \nu(2P^y \sin(\pi y) - 2\pi y^2 P^x - 2\pi \phi_{11} \sin(\pi y)) - \frac{\pi^2 E}{4} w_{11}^2 \sin^2(\pi y) \end{aligned} \tag{3.149}$$

$$\begin{aligned} \delta_y = 0 = & \frac{4}{\pi} P^y - \frac{\pi^2}{3} P^x \sin(\pi x) - 2\pi \phi_{11} \sin(\pi x) \\ & - \nu(2P^x \sin(\pi x) - 2\pi x^2 P^y - 2\pi \phi_{11} \sin(\pi x)) - \frac{\pi^2 E}{4} w_{11}^2 \sin^2(\pi x) \end{aligned} \tag{3.150}$$

By expansion theorem

$$\begin{aligned} 0 = & \frac{4}{\pi} P^x - \frac{\pi^2}{6} P^y - \pi \phi_{11} \\ & - \nu(P^y - \frac{2\pi}{3} P^x - \pi \phi_{11}) - E w_{11}^2 \frac{\pi^2 - 4}{4\pi} \end{aligned} \tag{3.151}$$

$$\begin{aligned} 0 &= \frac{4}{\pi}P^y - \frac{\pi^2}{6}P^x - \pi\phi_{11} \\ &- \nu(P^x - \frac{2\pi}{3}P^y - \pi\phi_{11}) - Ew_{11}^2\frac{\pi^2-4}{4\pi} \end{aligned} \tag{3.152}$$

Collecting like terms

$$0 = P^x(\frac{4}{\pi} - \nu(-\frac{2\pi}{3})) + P^y(-\frac{\pi^2}{6} - \nu) - \pi\phi_{11}(1-\nu) - Ew_{11}^2\frac{\pi^2-4}{4\pi} \tag{3.153}$$

$$0 = P^x(-\frac{\pi^2}{6} - \nu) + P^y(\frac{4}{\pi} - \nu(-\frac{2\pi}{3})) - \pi\phi_{11}(1-\nu) - Ew_{11}^2\frac{\pi^2-4}{4\pi} \tag{3.154}$$

Noting like terms, we can use the form below

$$0 = P^x K + P^y \Upsilon - \Gamma \tag{3.155}$$

$$0 = P^x \Upsilon + P^y K - \Gamma \tag{3.156}$$

where

$$K = \frac{8}{\pi^2} + \nu\frac{2\pi^2-8}{\pi^2} \tag{3.157}$$

$$\Upsilon = -\frac{\pi^2}{6} - \nu \tag{3.158}$$

$$\Gamma = \pi\phi_{11}(1-\nu) + \frac{\pi E}{3}w_{11}^2 \tag{3.159}$$

Solving for P^x and P^y

$$P^x = \Gamma\frac{K-\Upsilon}{K^2-\Upsilon^2} = \Gamma\frac{1}{K+\Upsilon} \tag{3.160}$$

$$P^y = \Gamma\frac{1}{K+\Upsilon} \tag{3.161}$$

$$\tag{3.162}$$

We can see that P^x and P^y are equal in magnitude.

$$\bar{P} = P^x = P^y = \frac{-2\pi^3}{48(\nu-1)+\pi^2(\pi^2-6\nu)}(3(1-\nu)\phi_{11} + Ew_{11}^2) \tag{3.163}$$

From above we can see that P^x and P^y have the same magnitude this is a result of the symmetry of the first mode. Taking the one term, $(m,n) = (1,1)$, approximation for our equation of motion, equations (3.139) and (3.140).

$$\begin{aligned} 4D\pi^4 w_{11}W_{11} = q_{11}W_{11} + h\Big\{&2(\pi^4\phi_{11}w_{11}W_{11}^2) \\ &- 2(\pi^4\phi_{11}w_{11}\Phi_{11}^2) \\ &- (2P^y\sin(\pi y) - y^2\pi^2P^x\sin(\pi x))(\pi^2 w_{11}W_{11}) \\ &- (2P^x\sin(\pi x) - x^2\pi^2P^y\sin(\pi y))(\pi^2 w_{11}W_{11}) \\ &- 2(2y\pi P^x\cos(\pi x) + 2x\pi P^y\cos(\pi y))(\pi^2 w_{11}\Phi_{11})\Big\} \end{aligned} \tag{3.164}$$

$$\begin{aligned} 4\pi^4\phi_{11}W_{11} = E\Big\{&(\pi^4 w_{11}^2\Phi_{11}^2) - (\pi^4 w_{11}^2 W_{11}^2)\Big\} \\ &- 4\left(-\pi^2P^x\sin(\pi x) - \pi^2P^y\sin(\pi y)\right) \\ &- y^2\pi^4P^x\sin(\pi x) - x^2\pi^4P^y\sin(\pi y) \end{aligned} \tag{3.165}$$

Using expansion theorem

$$\begin{aligned} w_{11} = q_{11}\frac{1}{4\pi^4D} &+ \frac{2h}{D}\phi_{11}w_{11}\zeta_{11} \\ &+ \bar{P}h\frac{4\pi^2-18}{9D\pi^3}w_{11} \end{aligned} \tag{3.166}$$

$$\phi_{11} = -Ew_{11}^2\zeta_{11} + \frac{12-\pi^2}{\pi^3}\bar{P} \tag{3.167}$$

Using our expression for $\bar{P}$ and ϕ_{11}, equations (3.163) and (3.167). ϕ_{11} becomes

$$\phi_{11} = \frac{2}{3}\frac{Ew_{11}^2\left(96-96\nu+12\nu\pi^2+\pi^4-36\pi^2\right)}{\pi^2\left(24-24\nu+\pi^4-6\pi^2\right)} \tag{3.168}$$

Using ϕ_{11} in equation (3.166)

$$w_{11} = \frac{q_{11}}{4\pi^4D} + -\frac{4}{9}\frac{\left(2\pi^6-21\pi^4+8\nu\pi^4-84\nu\pi^2+180\pi^2-384+384\nu\right)hEw_{11}^3}{D\pi^4\left(24-24\nu+\pi^4-6\pi^2\right)} \tag{3.169}$$

Equation (3.169) is a complete system with one equation and one unknown, w_{11}. A plot is given after the next section so that we can compare the results of different assumptions of P_x and P_y.

Example: Static, Immovable Edges - P_x and P_y Constant - One Term

Following a similar procedure to above, ignoring all derivatives of P_x and P_y in equations (3.143) and (3.144), $\bar{P}$ becomes:

$$\bar{P} = \frac{\pi^3}{8-\nu\pi^2}\phi_{11}(1-\nu) + \frac{\pi^3 E}{3(8-\nu\pi^2)}w_{11}^2 \tag{3.170}$$

Substituting the above into our equation of motion, equations (3.139) and (3.140), noting that all derivatives of P_x and P_y are now zero, and using expansion theorem results in.

$$\begin{aligned} D\pi^4 w_{11} = q_{11}\frac{1}{4} &+ 2h(\pi^4\phi_{11}w_{11}\zeta_{11}) \\ &- h\bar{P}\pi^2 w_{11} \end{aligned} \tag{3.171}$$

$$\phi_{11} = -Ew_{11}^2\zeta_{11} \tag{3.172}$$

Using our expression for $\bar{P}$ in equation (3.170), equation (3.171) becomes

$$\begin{aligned} D\pi^4 w_{11} = q_{11}\frac{1}{4} &+ 2h(\pi^4\phi_{11}w_{11}\zeta_{11}) \\ &- h\left(\frac{\pi^3}{8-\nu\pi^2}\phi_{11}(1-\nu) + \frac{\pi^3 E}{3(8-\nu\pi^2)}w_{11}^2\right)\pi^2 w_{11} \end{aligned} \tag{3.173}$$

$$\phi_{11} = -Ew_{11}^2\zeta_{11} \tag{3.174}$$

Using equation (3.174) in conjunction with equation (3.173) results in

$$\begin{aligned} D\pi^4 w_{11} = q_{11}\frac{1}{4} &+ 2h(\pi^4(-Ew_{11}^2\zeta_{11})w_{11}\zeta_{11}) \\ &- h\left(\frac{\pi^3}{8-\nu\pi^2}(-Ew_{11}^2\zeta_{11})(1-\nu) + \frac{\pi^3 E}{3(8-\nu\pi^2)}w_{11}^2\right)\pi^2 w_{11} \end{aligned} \tag{3.175}$$

and simplifying, we obtain

$$\begin{aligned} w_{11} = & \frac{q_{11}}{4D\pi^4} - \frac{32Eh}{9\pi^4 D} w_{11}^3 \\ & - \left(\frac{-1}{8-\nu\pi^2} (\frac{4}{3\pi})(1-\nu) + \frac{\pi}{3(8-\nu\pi^2)} \right) \frac{Eh}{D} w_{11}^3 \end{aligned} \tag{3.176}$$

Using a more familiar form

$$\frac{32Eh}{9\pi^4 D} w_{11}^3 + \left(\frac{\pi}{3(8-\nu\pi^2)} - \frac{4}{3\pi(8-\nu\pi^2)}(1-\nu) \right) \frac{Eh}{D} w_{11}^3 + w_{11} = \frac{q_{11}}{4D\pi^4} \tag{3.177}$$

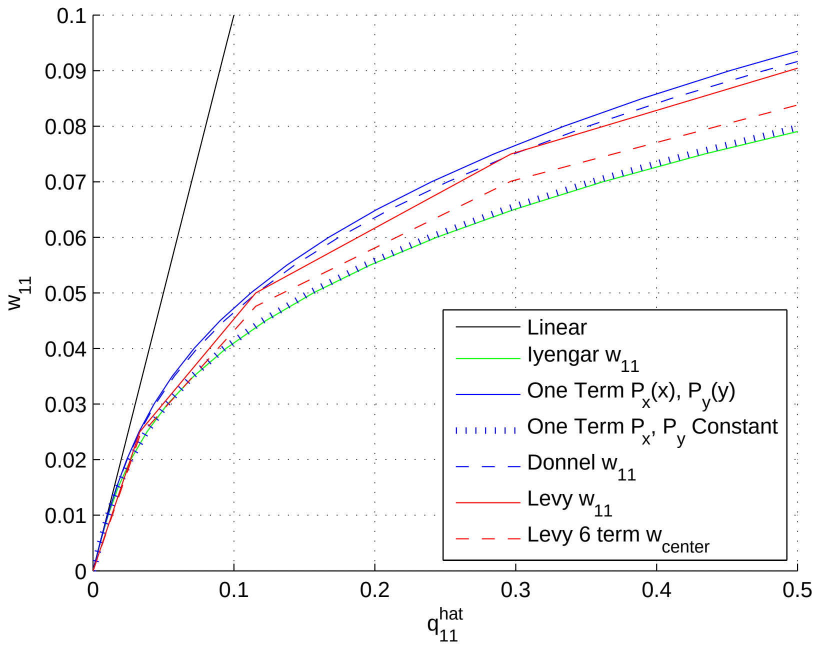


Figure 3.8: Comparing Immovable Edges Boundary Condition

Figure (3.8) shows that for the equation derived here and other theories we have a close agreement. Levy assumed that the plate was under a uniform load. However, the results of Donnell [16], Levy [28], and the one term approximation for $P_x(y)$ and $P_y(x)$ are very close in value. We also note that the deflections are smaller than linear theory, by using similar arguments that we used for equation (3.129) we can see that the nonlinear plate would be stiffer than the linear system.

Figure (3.9) compares the three different boundary conditions, linear, stress-free and immovable edges. We can see that the linear case predicts the largest deformations as the stiffness of the plate is only the resistance to bending. The nonlinear theories have a resistance to bending as well as stretching, which would create an overall stiffer plate. We can also see that the stress-free edges have a larger deformation than the immovable edges. This is because of the stiffness due to stretching. The nonlinear plates would pull in toward the center as it is being deformed. Immovable edges would resist that motion which is why the plate is stiffer than the stress-free edges which allows the movement. Also note that from equation (3.129) there is one additional term in the stiffness. From the immovable edge section we see that

$$\left(\frac{32Eh}{9\pi^4 D}w_{11}^2 + \left(\frac{\pi}{3(8-\nu\pi^2)} - \frac{4}{3\pi(8-\nu\pi^2)}(1-\nu)\right)\frac{Eh}{D}w_{11}^2 + 1\right)w_{11} = \frac{q_{11}}{4D\pi^4} \quad (3.178)$$

$$4\pi^4 D_{immovable}(w_{11})w_{11} = q_{11} \quad (3.179)$$

We can see that $D_{immovable} > D_{stress-free} > D$ for linear systems. Therefore for linear systems we have the largest deflections and for immovable edges we have the smallest deflections as shown in figure (3.9).

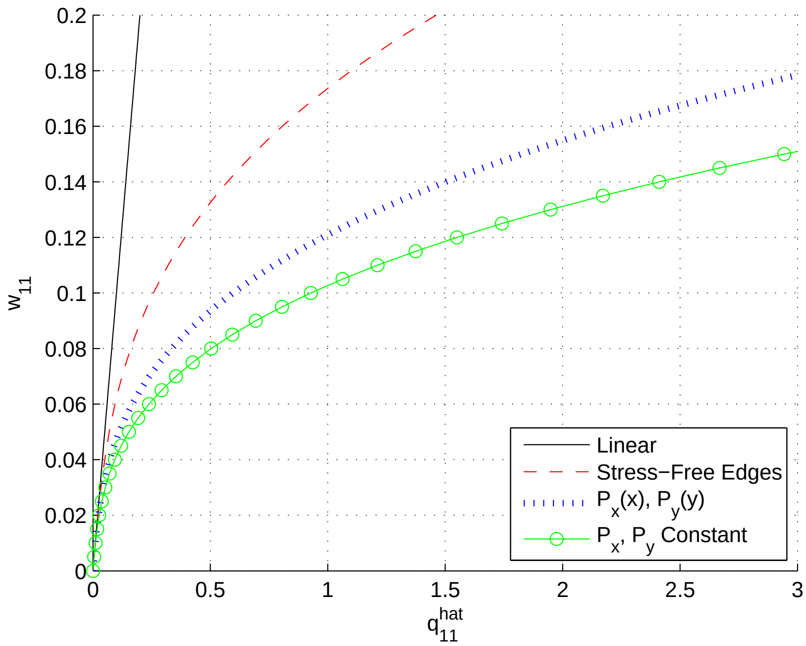


Figure 3.9: Comparing Boundary Conditions for Static Bending of Plates

Table (3.2) shows the coefficients for five different conditions, linear, stress free coupled and uncoupled, and immovable edges comparing the dependence of P_x and P_y on the spatial variables. The table only considers the coefficients w_{11} and w_{33}. We compared two different load cases, for Load = 1000, $q_{11} = 4000$ and $\hat{q}_{11} \approx 0.0089$. For Load = 8000, $q_{11} = 32000$ and $\hat{q}_{11} \approx 0.0710$. We can see that w_{33} contributes little to the total deflection. As a result, the stress free uncoupled, and coupled results are close in value. If a different load case were considered, such that the w_{33} coefficient was much larger, the coupled values would have a greater difference compared to the uncoupled values.

Plate Deflection						
	Load = 1000			Load = 8000		
	w_{11}	w_{33}	Total	w_{11}	w_{33}	Total
Linear	0.0089	0.0001	0.0090	0.0710	0.0009	0.0719
Stress-Free Uncoupled	0.0088	$3.7E-5$	0.0088	0.0506	0.0009	0.0515
Stress-Free Coupled	0.0088	0.0001	0.0089	0.0508	0.0008	0.0516
Immovable P_x, P_y Constant	0.0084	–	0.0084	0.0351	–	0.0351
Immovable $P_x(x)$, $P_y(y)$	0.0086	–	0.0086	0.0398	–	0.0398

Table 3.2: Coefficients of a Plate

3.4.2 Dynamics

In this section we include the time derivative of our displacement function $w(x, y, t)$ which is now a function of time. Many steps below have been omitted as they parallel steps from previous sections.

Below is equation (3.34) and equation (3.37), the coupled, nonlinear, partial differential equation that governs our system.

$$\rho \frac{\partial^2 w}{\partial t^2} + D\nabla^4 w(x, y, t) = q(x, y, t) + h\left(\frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right) \quad (3.180)$$

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = \nabla^4 \phi = E\left(\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right) \quad (3.181)$$

Stress Free Edges

For stress free edges we have

$$\sigma_x|_{x=0,a} = 0 \qquad\qquad \sigma_y|_{y=0,b} = 0 \tag{3.182}$$

Using

$$w(x,y,t) = \sum w_{mn}(t) W_{mn}(x,y) \tag{3.183}$$

$$\phi(x,y,t) = \sum \phi_{mn}(t) W_{mn}(x,y) \tag{3.184}$$

$$W_{mn} = \sin(\alpha_m x)\sin(\gamma_n y) \tag{3.185}$$

$$\Phi_{mn} = \cos(\alpha_m x)\cos(\gamma_n y) \tag{3.186}$$

Expanding our equation of motion

$$\begin{aligned}\rho\ddot{w}_{mn}W_{mn} + D\sum(\alpha_m^2+\gamma_n^2)^2 w_{mn}W_{mn} = \sum q_{mn}W_{mn}& \\ + h\Big\{(\sum\alpha_a^2\phi_{ab}W_{ab})(\sum\gamma_n^2 w_{mn}W_{mn}) + (\sum\gamma_b^2\phi_{ab}W_{ab})(\sum\alpha_m^2 w_{mn}W_{mn})& \\ - 2(\sum\alpha_a\gamma_b\phi_{ab}\Phi_{ab})(\sum\alpha_m\gamma_n w_{mn}\Phi_{mn})\Big\}&\end{aligned} \tag{3.187}$$

$$\begin{aligned}\sum(\alpha_m^2+\gamma_n^2)^2\phi_{mn}W_{mn} = E\Big\{(\sum\alpha_a\gamma_b w_{ab}\Phi_{ab})(\sum\alpha_m\gamma_n w_{mn}\Phi_{mn})& \\ - (\sum\alpha_a^2 w_{ab}W_{ab})(\sum\gamma_n^2 w_{mn}W_{mn})\Big\}&\end{aligned} \tag{3.188}$$

Example: Dynamic, Stress Free Edges - One-to-One Term Approximation

A simplification to the system is ignoring the coupling between modes. As discussed earlier, this may negatively affect the accuracy of our results, this approximation gives us some insight into the behavior of plates.

$$\rho\ddot{w}_{mn}W_{mn} + D(\alpha_m^2+\gamma_n^2)^2 w_{mn}W_{mn} = q_{mn}W_{mn} + 2h\alpha_m^2\gamma_n^2\phi_{mn}w_{mn}\left(W_{mn}W_{mn} - \Phi_{mn}\Phi_{mn}\right) \tag{3.189}$$

$$(\alpha_m^2 + \gamma_n^2)^2 \phi_{mn} W_{mn} = E\alpha_m^2 \gamma_n^2 w_{mn} w_{mn} (\Phi_{mn}\Phi_{mn} - W_{mn}W_{mn}) \tag{3.190}$$

Multiplying by W_{ab} and integrating, taking note of the expansion theorem. Also, for ease of use, noting that

$$\zeta_{mn} = \int_A (W_{mn}W_{mn} - \Phi_{mn}\Phi_{mn})\, W_{mn}\, \mathrm{d}A \tag{3.191}$$

results in

$$\frac{\rho}{4}\ddot{w}_{mn} + \frac{D}{4}(\alpha_m^2 + \gamma_n^2)^2 w_{mn} = \frac{q_{mn}}{4} + 2h\alpha_m^2\gamma_n^2\phi_{mn}w_{mn}\zeta_{mn} \tag{3.192}$$

$$\frac{1}{4}(\alpha_m^2 + \gamma_n^2)^2 \phi_{mn} = -E\alpha_m^2\gamma_n^2 w_{mn}w_{mn}\zeta_{mn} \tag{3.193}$$

Solving for ϕ_{mn}

$$\phi_{mn} = -4E\zeta_{mn}\frac{\alpha_m^2\gamma_n^2 w_{mn}w_{mn}}{(\alpha_m^2 + \gamma_n^2)^2} \tag{3.194}$$

Back into equation (3.192)

$$\ddot{w}_{mn} + \frac{D}{\rho}(\alpha_m^2 + \gamma_n^2)^2 w_{mn} = \frac{q_{mn}}{\rho} - 32Eh\frac{\alpha_m^4\gamma_n^4\zeta_{mn}^2}{\rho(\alpha_m^2 + \gamma_n^2)^2}w_{mn}^3 \tag{3.195}$$

Simplifying, and noting that $\omega_{mn}^2 = \frac{D}{\rho}(\alpha_m^2 + \gamma_n^2)^2$, where ω_{mn} is the natural frequency,

$$\boxed{\ddot{w}_{mn} + 32\frac{Eh}{\rho}\frac{\alpha_m^4\gamma_n^4\zeta_{mn}^2}{(\alpha_m^2 + \gamma_n^2)^2}w_{mn}^3 + \omega_{mn}^2 w_{mn} = \frac{q_{mn}}{\rho}} \tag{3.196}$$

Note that the equation above takes the form of the undamped Duffing's equation. Since the solution takes the form of Duffing's equation it provides greater insight into the problem. As a result, we can have jumps in amplitude as we change the excitation frequency. This characteristic was confirmed experimentally by Amabili [3]. We also have the presence of stable and unstable solutions. Duffing's equation also predicts subharmonic and superharmonic resonance and other phenomena. However, these topics are beyond the scope of this study. The Duffing equation is described in appendix B.

The first four terms results in the system of equations below. The four equations are uncoupled due to the assumptions made above and that some equations do not have any nonlinear terms. These observations are similar with the results of the static section.

$$\begin{aligned}
\ddot{w}_{11} + \frac{128}{9}\frac{Eh}{\rho}w_{11}^3 + \frac{4\pi^4 D}{\rho}w_{11} &= \frac{q_{11}}{\rho} \\
\ddot{w}_{12} + \frac{25\pi^4 D}{\rho}w_{12} &= \frac{q_{12}}{\rho} \\
\ddot{w}_{21} + \frac{25\pi^4 D}{\rho}w_{21} &= \frac{q_{21}}{\rho} \\
\ddot{w}_{22} + \frac{64\pi^4 D}{\rho}w_{22} &= \frac{q_{22}}{\rho}
\end{aligned}$$

For a coupled mode solution one would take similar steps to that of the 'four term' example from the static section. Figure (3.10) shows the time response of w_{11} for the uncoupled, stress free boundary condition. Figure (3.11) shows the frequency response for that system.

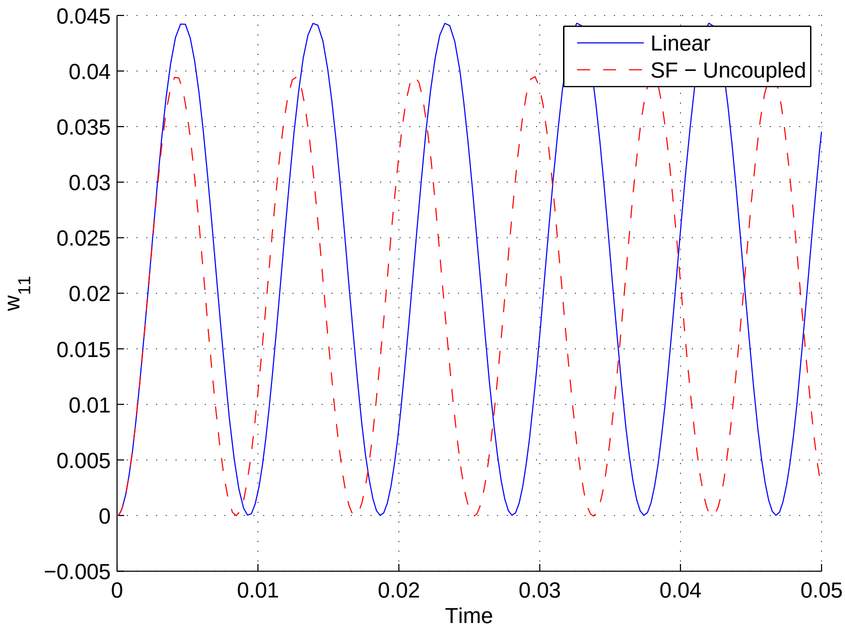


Figure 3.10: Comparing Linear and Nonlinear Time Response with Stress Free Edges

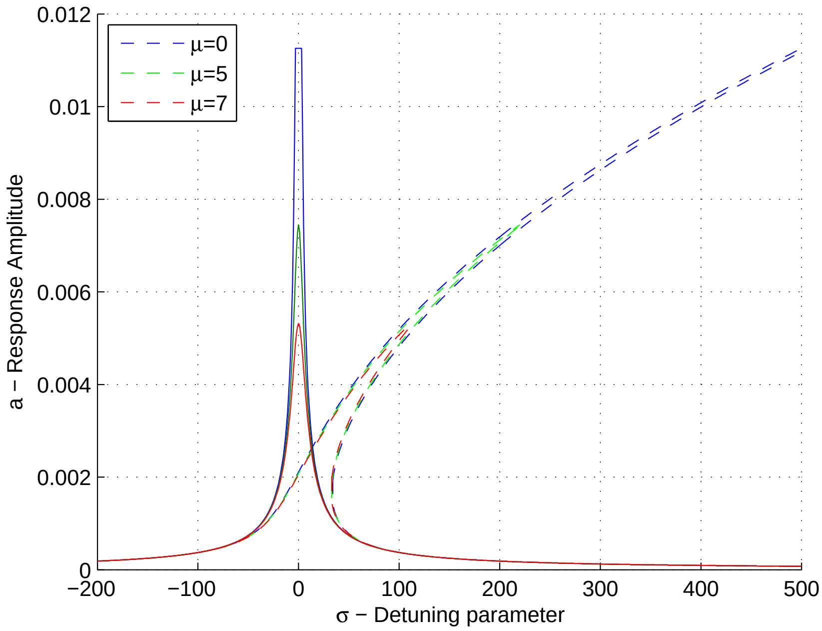


Figure 3.11: Comparing Linear and Nonlinear Frequency Response with Stress Free Edges. '–' Linear Response, '- -' Nonlinear Response. See Legend for Damping values for each Color.

Immovable Edges

Since the procedure for the solution is similar to that of the static section, only the solution will be presented below. Note that w_{11} is now a function of time, which results in ϕ_{11} and $\bar{P}$ also being functions of time.

Example: Dynamic, Immovable Edges - $P_x(x,t)$ and $P_y(y,t)$ - One Term

$$\bar{P}(t) = P^x(t) = P^y(t) = \frac{-2\pi^3}{48(\nu-1)+\pi^2(\pi^2-6\nu)}(3(1-\nu)\phi_{11}(t) + Ew_{11}(t)^2) \quad (3.197)$$

$$\phi_{11}(t) = \frac{2}{3}\frac{Ew_{11}(t)^2\left(96-96\nu+12\nu\pi^2+\pi^4-36\pi^2\right)}{\pi^2\left(24-24\nu+\pi^4-6\pi^2\right)} \quad (3.198)$$

$$\frac{\rho}{4\pi^4 D}\ddot{w}_{11}+w_{11}(t) = \frac{q_{11}}{4\pi^4 D}+-\frac{4}{9}\frac{\left(2\pi^6-21\pi^4+8\nu\pi^4-84\nu\pi^2+180\pi^2-384+384\nu\right)hEw_{11}^3}{D\pi^4\left(24-24\nu+\pi^4-6\pi^2\right)} \quad (3.199)$$

Note that the above equation is of the form of an undamped Duffing's equation. As stated earlier we can now predict the presence of amplitude jumps and other phenomena that are associated with Duffing's equation. Figure (3.12) shows the time response, and figure (3.13) shows the frequency response of the system.

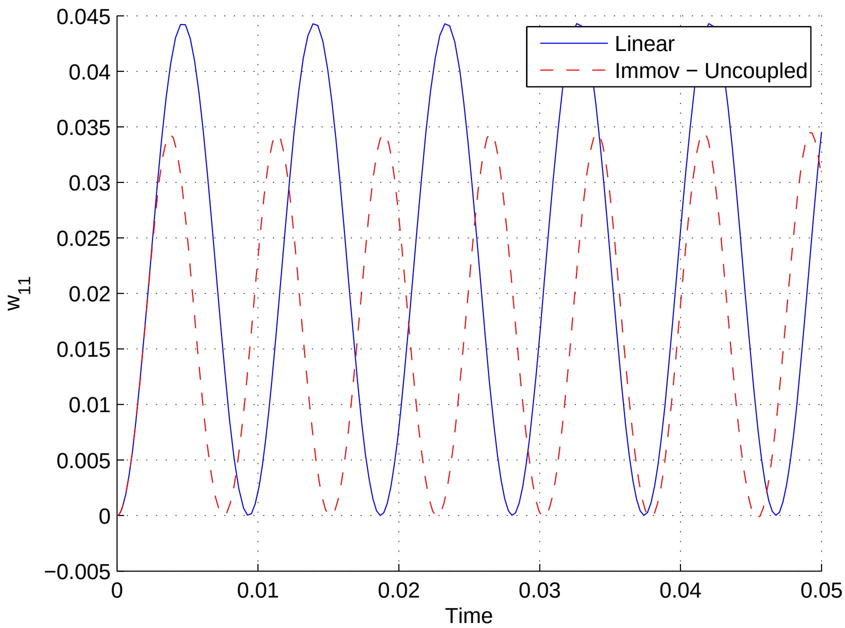


Figure 3.12: Comparing Linear and Nonlinear Time Response with Immovable Edges, P_x, P_y - Constant

Example: Dynamic, Immovable Edges - $P_x(t)$ and $P_y(t)$ - One Term

$$\bar{P}(t) = \frac{\pi^3}{8-\nu\pi^2}\phi_{11}(t)(1-\nu) + \frac{\pi^3 E}{3(8-\nu\pi^2)}w_{11}(t)^2 \tag{3.200}$$

$$\phi_{11}(t) = -Ew_{11}(t)^2\zeta_{11} \tag{3.201}$$

$$\frac{\rho}{4D\pi^4}\ddot{w}_{11} + \frac{32Eh}{9\pi^4 D}w_{11}^3 + \left(\frac{\pi}{3(8-\nu\pi^2)} - \frac{4}{3\pi(8-\nu\pi^2)}(1-\nu)\right)\frac{Eh}{D}w_{11}^3 + w_{11} = \frac{q_{11}}{4D\pi^4} \tag{3.202}$$

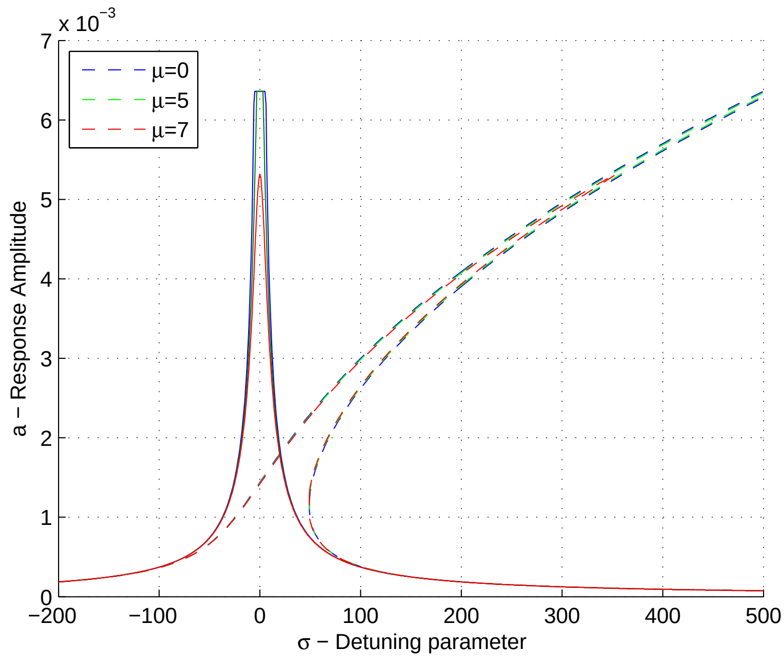


Figure 3.13: Comparing Linear and Nonlinear Frequency Response with Immovable Edges, P_x, P_y - Constant. '–' Linear Response, '- -' Nonlinear Response. See Legend for Damping values for each Color.

Note that the equation is of the form of Duffing's equation without damping which implies the presence of jumps in amplitude and other phenomena. Figure (3.14) shows the time response, and figure (3.15) shows the frequency response of the system.

Comparing the values for all the boundary conditions in figures (3.16) and (3.17). As in the static section, the stiffest plate predicted is with immovable edges. The least stiffest plate is linear plate theory. From the times response of the plate, we can see that the response frequency changes with the boundary conditions. This is a result of the boundary conditions of the nonlinear plate that stiffen the plate.

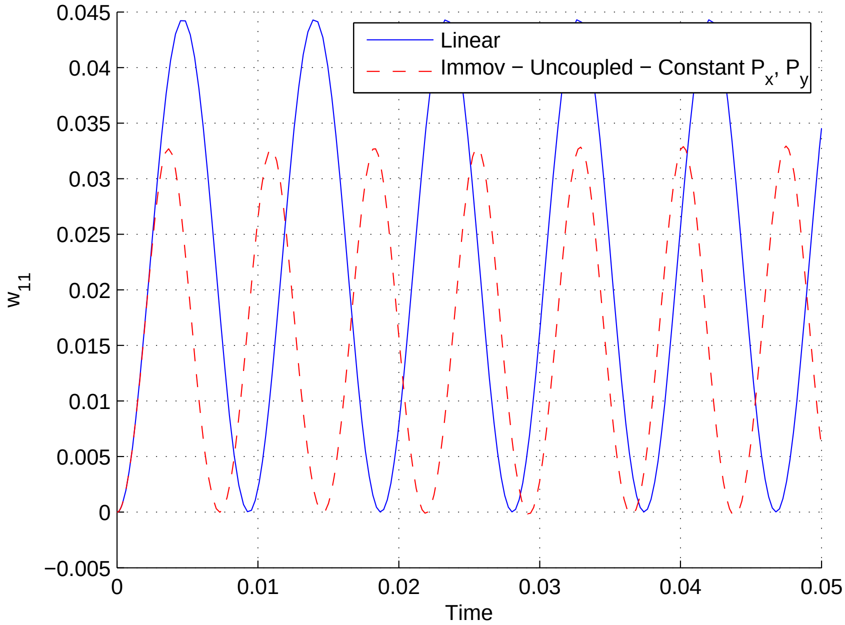


Figure 3.14: Comparing Linear and Nonlinear Time Response with Immovable Edges, $P_x(x)$, $P_y(y)$

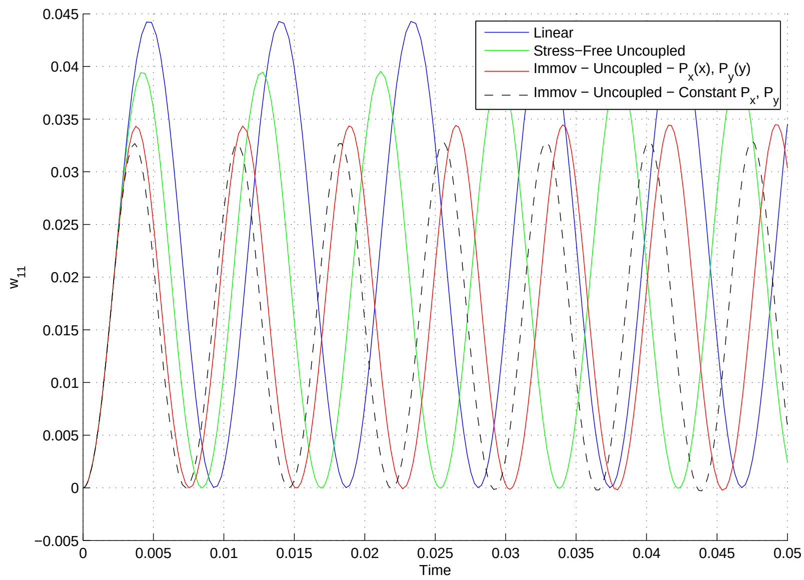


Figure 3.16: Comparing All Boundary Conditions - Time Response

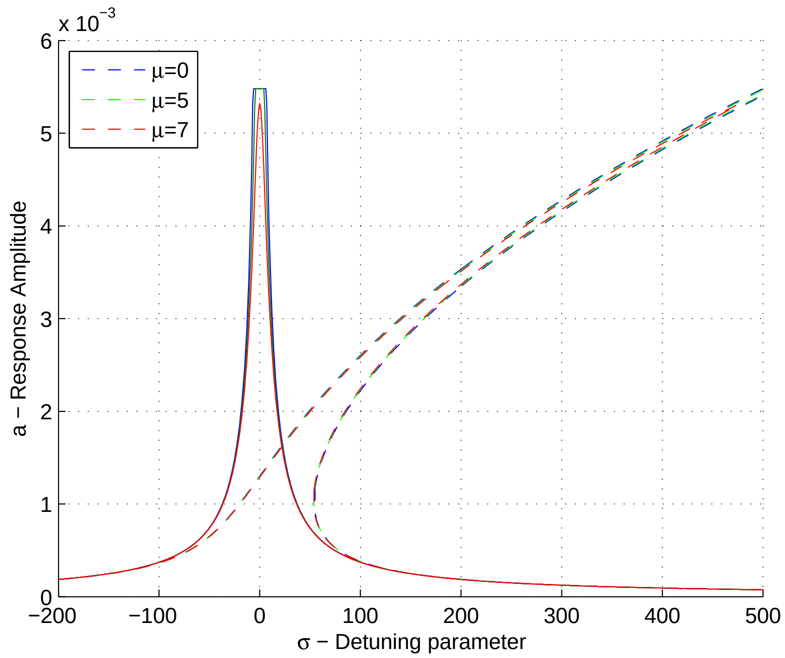


Figure 3.15: Comparing Linear and Nonlinear Frequency Response with Immovable Edges, $P_x(x)$, $P_y(y)$. '–' Linear Response, '- -' Nonlinear Response. See Legend for Damping values for each Color.

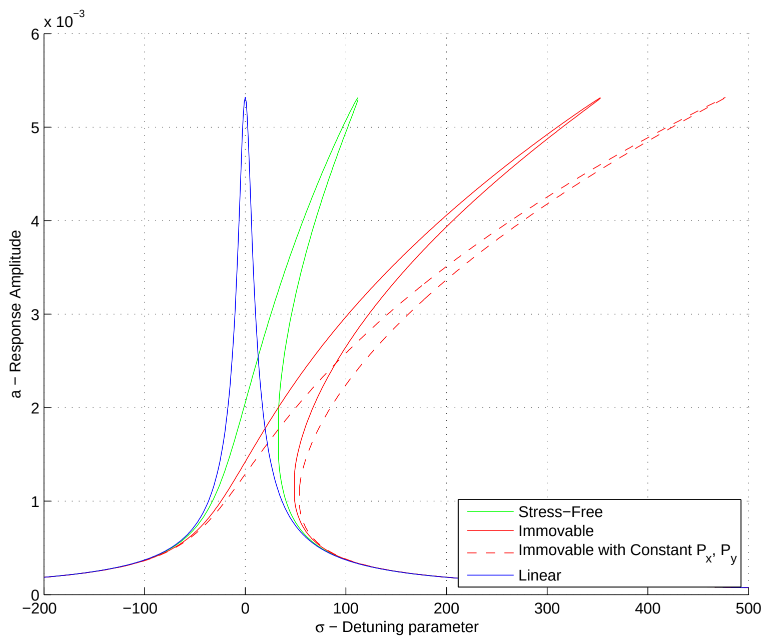


Figure 3.17: Comparing All Boundary Conditions - Frequency Response - $\mu \neq 0$

3.5 FEM Results

In order to gain some perspective on the theoretical results presented above, a finite element approach was used. Abaqus CAE 6.8-3 was used to solve the problem of a load suddenly placed at the center of a simply supported square plate. Using 400 linear, quadrilateral, S4R, shell elements, and 441 nodes, a flat shell with edge lengths of one and with a shell thickness of 0.05 was utilized. A S4R element is a conventional stress/displacement shell with 4 nodes and reduced integration. While the S4R element allows transverse shear, for a 'small' thickness it is not significant. These elements do not include rotatory inertia. Shell elements were used to model the middle surface of the plate and the load was placed on the middle surface. While we could have used solid elements to model the plate in a life like situation, we are only interested in the mid-plane response and used shell elements instead. A constant load was placed at the center of the plate and the edges were restrained from moving in the transverse direction and zero applied twisting moment, $M_{xy} = 0$, simply supported edges. For the dynamic case, two different magnitude loads were used in order to compare the change in edge conditions. Three different situations were compared: the linear, NLGEOM off, condition ,then the nonlinear, NLGEOM on, plate with edges not allowed to move in the plane of the undeformed plate, or immovable edges, the last condition was allowing the nonlinear plate to move in the plane of the undeformed plate, or stress-free edges. NLGEOM is the 'geometrically nonlinear' switch. When off, the problem is geometrically linear and when on, the problem is geometrically nonlinear. The NLGEOM parameter also accounts for the 'stress stiffening' of the structure.

For the static problems, Abaqus/Standard was used. Abaqus/Standard solves the nonlinear equilibrium equations by Newton's method. Newton's method is a root finding algorithm that uses Taylor series. For the dynamic problems, Abaqus/Explicit was used, which uses, by default, the lumped mass matrix. Abaqus/Explicit uses the explicit central difference integration rule as their method of solution. The lumped mass matrix assumes that the mass of an element is concentrated at the nodes of the element. Within the explicit central difference integration, the inverse of the mass matrix is required. With a lumped mass approximation, the mass matrix is diagonal, making it routine to invert.

The response is given alongside the theoretical solution. The theoretical solution is the solution derived in the earlier sections.

Figure (3.19) shows that the finite element method predicts a larger static bending value than the theoretical value. This is because the theoretical values presented are only the first term, and in some cases an additional term, in the infinite sum. If more terms were taken, the results would match closer to the finite element results. However, the relative values of linear, stress-free, and immovable edges agree with relative values of the theoretical results.

For the dynamic cases two different, constant, load magnitudes were used in order to compare a load where the nonlinearity would be insignificant and where it would become significant. Figure (3.20) has a load magnitude of 1000 and figure (3.21) has a load magnitude of 5000. From figure (3.20a) we can see that for all the boundary conditions tested, the finite element results are very similar in magnitude and response frequency. The frequencies match well between theoretical and finite element. On the other hand, the amplitudes are slightly under predicted by the theoretical results, just as with the static deflection.

From figure (3.21a) shows a significant change in response amplitude and frequency between the different boundary conditions in the finite element models. The linear and immovable edges match well with finite element models in terms of response frequency. The theoretical frequency of the stress-free edge condition is higher than finite element model. Just as with the other theories, the amplitude of the finite element model is higher than the theoretical models.

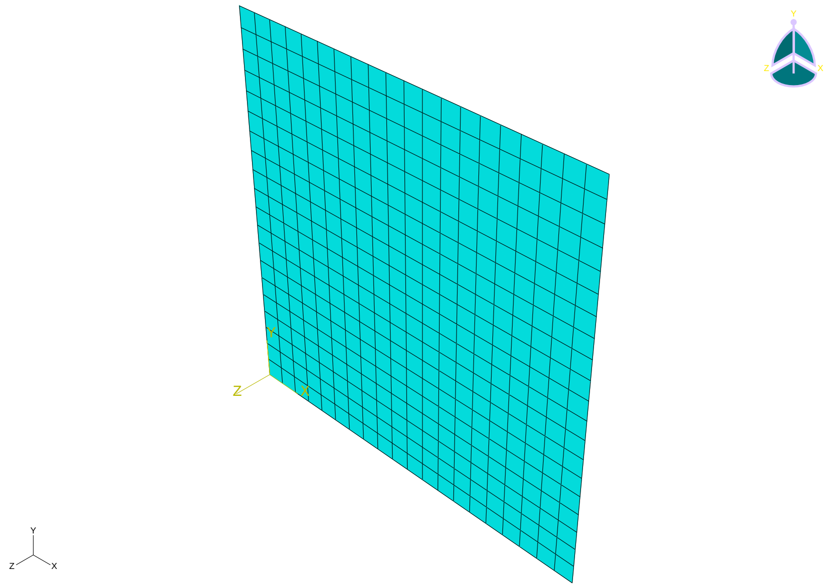


Figure 3.18: Mesh of Finite Element Model

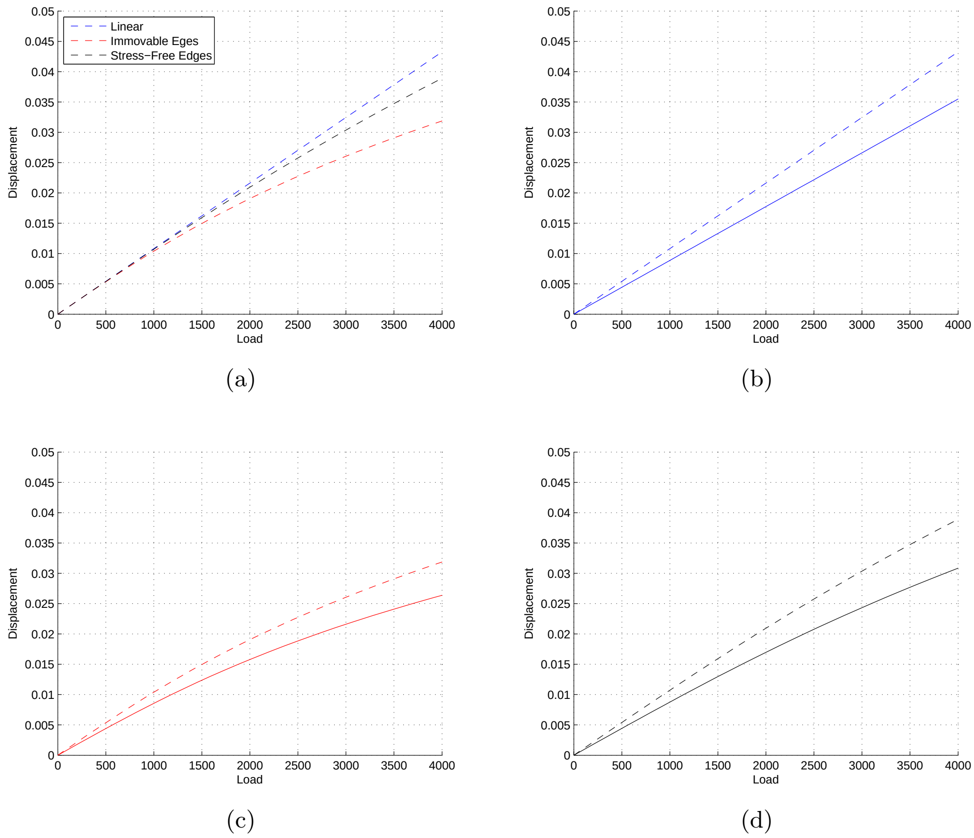
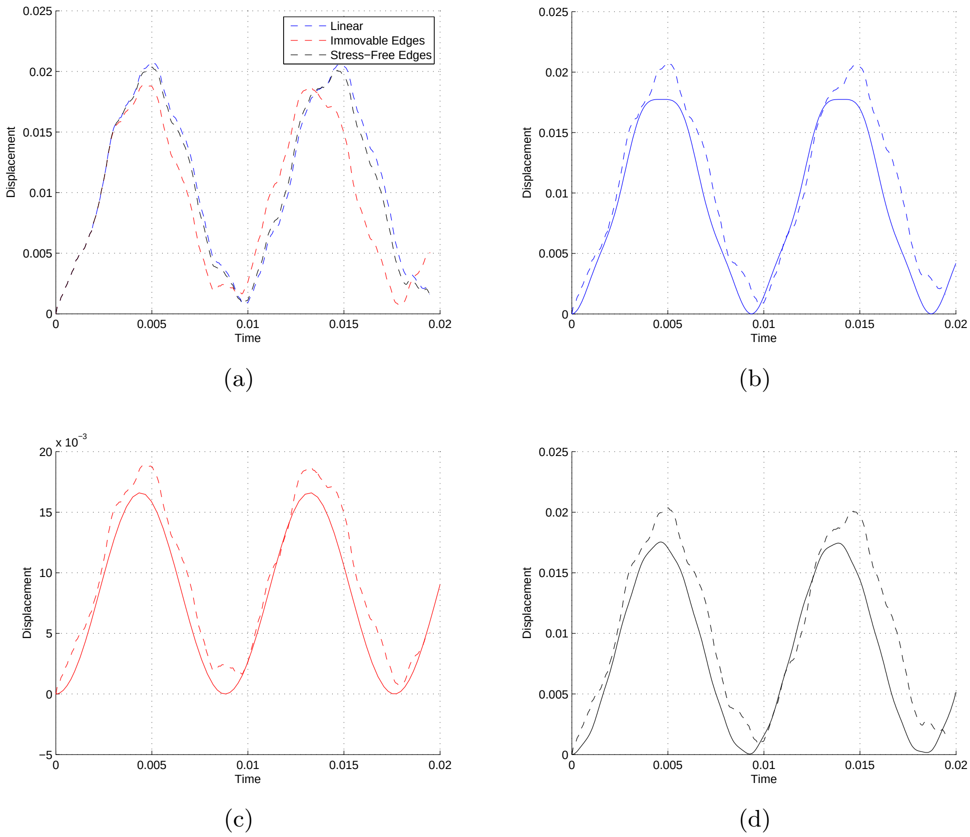


Figure 3.19: '–' Theoretical Response, w_{11}, '- -' FEM Response at Plate Center: (a) Comparing all Boundary Conditions; (b) Linear Response; (c) Immovable Edges Response; and, (d) Stress-Free Edges Response.



(a) (b) (c) (d)

Figure 3.20: Load = 1000, '–' Theoretical Response, '- -' FEM Response: (a) Comparing all Boundary Conditions; (b) Two Term Linear Response, w_{11}, w_{33}; (c) Immovable Edges Response; and, (d) Two Term Stress-Free Edges Response, w_{11}, w_{33}.

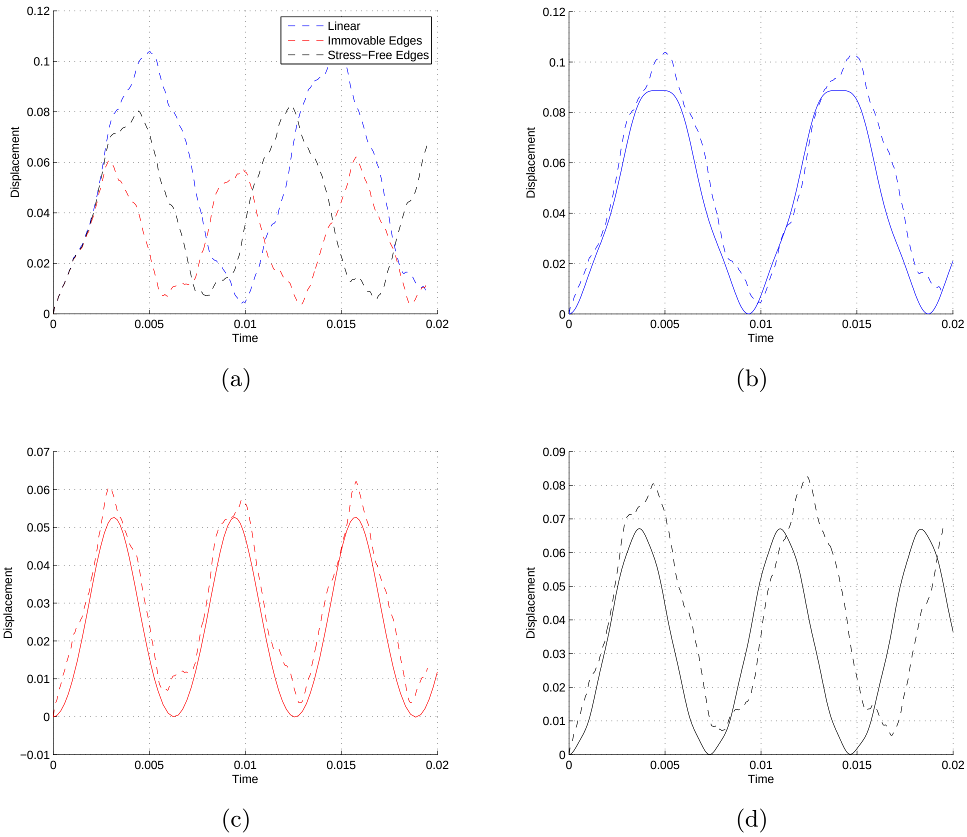


Figure 3.21: Load = 5000, '–' Theoretical Response, '- -' FEM Response: (a) Comparing all Boundary Conditions; (b) Two Term Linear Response, w_{11}, w_{33}; (c) Immovable Edges Response; and, (d) Two Term Stress-Free Edges Response, w_{11}, w_{33}.

Chapter 4

Conclusions and Future Work

4.1 Conclusion

From nonlinear beam theory we saw that immovable edges result in the nonlinear beam becoming stiffer than the linear beam, resulting in smaller deflections. We saw that stress-free beams simplifies to linear theory. As a result, a new approach was required, the theory by Conway was presented. The theory proposed by Conway ignores the kinetics of the problem and concentrates on the geometry of deflection. Conway's approach also does not take into account the resistance of the beam to folding onto itself as well as stretching of the beam. Thus, Conway's theory can result in unrealistic deflections.

For vibrations of beams we saw that the nonlinear beam with immovable edges resulted in the differential equation that took the form of Duffing's equation. Therefore, we were able to obtain the frequency response equation of the system. From that solution, we realized that we may have up to three different amplitudes for a given excitation frequency.

Considering a nonlinear plate would results in deflections smaller than in linear plate theory. We also see that a plate with immovable edges makes the plate to be stiffer than a plate with stress-free edges. Also, as the load grows, the deflection of the nonlinear plate becomes asymptotic. While this may not be realistic from a materials point of view, because the material would eventually become plastic, it is realistic in terms of the plate geometry. We also investigated Gaussian curvature, which quantifies the 'stretch' of a deflected plate.

For dynamic systems we saw that the solution takes the form of Duffing's equation.

Using perturbation technique we expanded the form of the solution, ignoring higher order terms, which allowed us to obtain a system of equations that we are able to solve. We found the frequency response equation of the system. From the frequency response plot we can see that jumps in response amplitude may exist, which are not predicted in linear theory. For nonlinear plates, as we slowly increase or decrease the excitation frequency we would see a large difference in response amplitudes.

4.2 Future Work

While only one boundary condition was considered, simply-supported, this method of solution may be extended to other boundary conditions, like clamped. However the use of a double sine series to represent the Airy's stress function may no longer be an accurate modal function. One may have to use the same modal function for a clamped plate as the modal function for Airy's stress function, which may or may not be valid.

Another future endeavor is other plate shapes. While this study can easily be extended to rectangular (not square) plates, circular plates require a bit more work. One question is how to handle the expansion theorem in polar coordinates because the integration can include a discontinuity at the plate's center.

Appendix A

Gaussian Curvature

The Gaussian curvature comes up in the study of nonlinear plate theory. The derivation of Gaussian curvature requires knowledge from differential geometry, specifically the *first fundamental form* and the *second fundamental form*. These topics are beyond the scope of this study. Pressley [37] and O'Neill [34] give the information on differential geometry theory that is required to derive Gaussian curvature. Here, the derivation of Gaussian curvature will be carried out by an alternate method, as this approach gives the reader a better physical understanding of Gaussian curvature and its application to plates.

Gaussian curvature, K, is defined as the product of principal curvatures, κ_1 and κ_2. In order to find the principal curvatures, we first find the curvatures of our x, y, z coordinate system.

A.1 Curvature - Displacement Relationship

A structural element, beam or plate, deformed in the $x-z$ plane would look like figure (A.1). Note that we consider $\mathrm{d}x$, $\mathrm{d}w$, and $\mathrm{d}s$ to form a right triangle.

From the figure above we can see that

$$R\,\mathrm{d}\theta = \mathrm{d}s \tag{A.1}$$

$$(\mathrm{d}s)^2 \approx (\mathrm{d}x)^2 + (\mathrm{d}w)^2 \tag{A.2}$$

$$\tan(\theta) = \frac{\mathrm{d}w}{\mathrm{d}x} \tag{A.3}$$

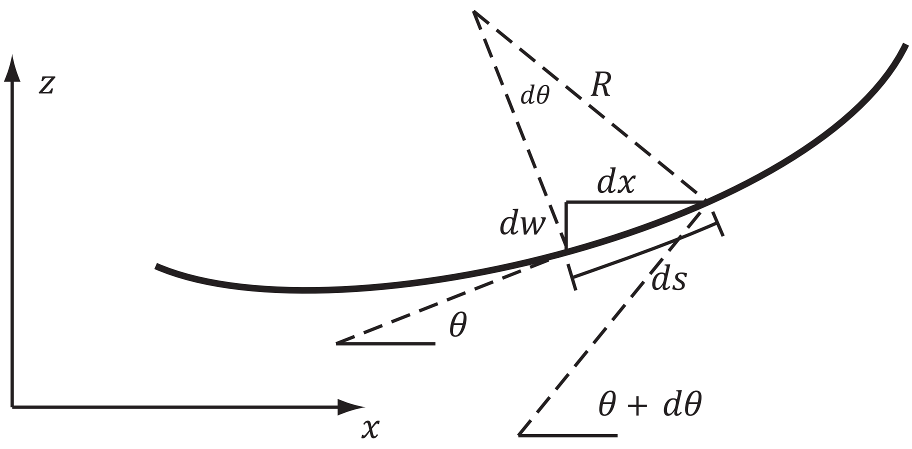


Figure A.1: Deformed Structural Element

We define curvature as the inverse of radius of curvature, R. Beginning with equation (A.3)

$$\tan(\theta) = \frac{\mathrm{d}w}{\mathrm{d}x} \tag{A.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan(\theta) = \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \tag{A.5}$$

$$\sec^2(\theta)\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \tag{A.6}$$

$$\sec^2(\theta)\,\mathrm{d}\theta = \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\,\mathrm{d}x \tag{A.7}$$

Using equation (A.1) and knowing that

$$\sec^2(\theta) = 1 + \tan^2(\theta) = 1 + \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \tag{A.8}$$

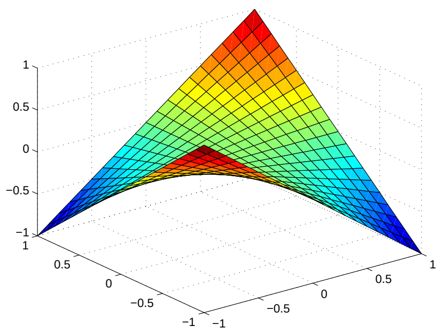
results in

$$\left(1 + \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right)\frac{1}{R}\frac{\mathrm{d}s}{\mathrm{d}x} = \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \tag{A.9}$$

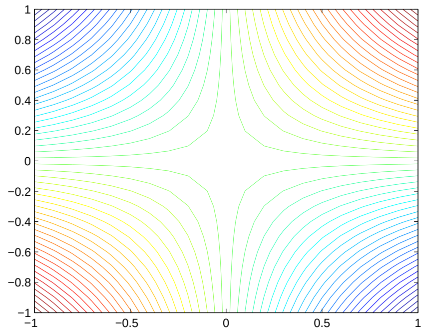
From equation (A.2) we can find that

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1 + \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2} \tag{A.10}$$

This results in our expression for curvature



(a) Surface Plot of Pure Twist



(b) Contour Plot of Pure Twist

Figure A.2: Pure Twist

$$\kappa_x = \frac{1}{R} = \frac{\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}}{\left(1 + \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right)^{\frac{3}{2}}} \tag{A.11}$$

But from Kirchhoff's hypothesis we know that '*the slopes of the deflected middle surface are small compared to unity*' which allows us to approximate the curvature with

$$\kappa_x \approx \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \tag{A.12}$$

When applicable, similar arguments produce the expressions for the curvature in the y direction, and the expression for twist, κ_{xy}. A plate with only nonzero curvature of κ_{xy} would be a surface resembling a saddle point, see figure (A.2), with lines of zero transverse displacement going from midpoint to midpoint of opposite edges. Also note that the expression for κ_x below will now have partial derivatives.

$$\kappa_x = \frac{\partial^2 w}{\partial x^2} \qquad \kappa_y = \frac{\partial^2 w}{\partial y^2} \qquad \kappa_{xy} = \frac{\partial^2 w}{\partial x \partial y} \tag{A.13}$$

From the above definition we see that $\kappa_{xy} = \kappa_{yx}$. We can construct the Hessian, or curvature matrix, $[H]$, and recognize that the principal curvatures are the solution to an eigenvalue problem. The eigenvectors are the directions of principal curvatures.

Principal curvatures are the largest and smallest value of curvature for given deflection, and there is also zero twist in the direction of principal curvature.

$$[H] = \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x \partial y} \\ \frac{\partial^2 w}{\partial x \partial y} & \frac{\partial^2 w}{\partial y^2} \end{bmatrix} = \begin{bmatrix} \kappa_x & \kappa_{xy} \\ \kappa_{xy} & \kappa_y \end{bmatrix} \tag{A.14}$$

$$\det \begin{bmatrix} \kappa_x - \kappa & \kappa_{xy} \\ \kappa_{xy} & \kappa_y - \kappa \end{bmatrix} = 0 \tag{A.15}$$

The characteristic equation is

$$\kappa^2 - (\kappa_x + \kappa_y)\kappa - \kappa_{xy}^2 + \kappa_x \kappa_y = 0 \tag{A.16}$$

The solution by quadratic formula gives two values, as expected, for principal curvature.

$$\kappa_1 = \frac{1}{2}\left[\kappa_x + \kappa_y + \sqrt{(\kappa_x - \kappa_y)^2 + 4\kappa_{xy}^2}\right] \tag{A.17}$$

$$\kappa_2 = \frac{1}{2}\left[\kappa_x + \kappa_y - \sqrt{(\kappa_x - \kappa_y)^2 + 4\kappa_{xy}^2}\right] \tag{A.18}$$

From our definition of Gaussian curvature

$$K = \kappa_1 \kappa_2 = \kappa_x \kappa_y - \kappa_{xy}^2 = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \tag{A.19}$$

Note that Gaussian curvature is equal to the determinant of the Hessian, which is an invariant. The Gaussian curvature at one point of the surface would allow us to have the value of the Gaussian curvature for the entire surface. A zero-Gaussian curvature surface is known as a *developable* surface. A characteristic of a developable surface is that one of the principal curvatures is equal to zero, like plate bending into a cylindrical shape. An easy way to think of it is that a developable surface is a surface made of an infinite number of straight lines. An example is a cone. If one were to take a straight line, fix one end and have the other end move in a circular motion we would have a cone. Any deflection of a simply supported plate would be non-developable. Linear plate theory ignores this property as it assumes that the small deflections produce zero stretch within the plate. Also, from St. Venant's Compatibility equations,

$$\frac{\partial^2 E_{xx}}{\partial y^2} - 2\frac{\partial^2 E_{xy}}{\partial x \partial y} - \frac{\partial^2 E_{yy}}{\partial x^2} = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{A.20}$$

A developable surface implies that all in-plane strains, E_{xx}, E_{yy},and E_{xy}, are equal to zero. Therefore, for a plate, we can say that there is no stretching of the middle surface. For example a cantilever, or clamped-free-free-free, plate or beam we would consider having no stretch of the middle surface as shown in figure (A.3). A half-sphere would not be a developable surface because if one were to try to flatten it, certain sections would compress and other sections would stretch.

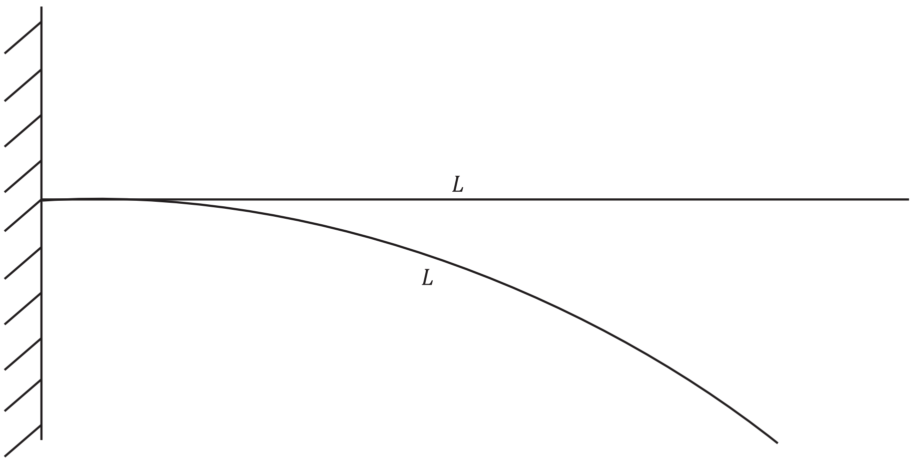


Figure A.3: Cantilever Member without Stretch

Appendix B

Duffing's Equation

Duffing's equation is a second-order ordinary differential equation with a cubic nonlinear term as given below.

$$\ddot{w} + \omega_o^2 w = -2\epsilon\mu\dot{w} - \epsilon\alpha w^3 + E(t) \tag{B.1}$$

Note the equation below also has an excitation term that is harmonic.

$$E(t) = K\sin(\Omega t) \tag{B.2}$$

In mechanical systems, the Duffing equation can used to describe a system with a nonlinear spring. For $\alpha > 0$ we have a stiffening spring and for $\alpha < 0$ we have a softening spring. However in our situation, this equation describes the amplitude of modes of a dynamic nonlinear plate or beam. We will always consider an α greater than or equal to zero. In order to solve the equation a perturbation method, known as multiple-scale analysis, will be used to approximate the solution to the system. Our perturbation variable, ϵ, is used to also define the coefficient of the excitation, $K = \epsilon k$. We also introduce σ as a detuning variable such that

$$\Omega = \omega_o + \epsilon\sigma \tag{B.3}$$

Where for an oscillatory system, ω_o is the natural frequency and Ω is the excitation frequency.

The method of multiple scales requires the use of a second time variable. Using the notation of Bender and Orszag [7] we choose:

$$\tau = \epsilon t \tag{B.4}$$

We also need to expand the solution in terms of t, ϵ and our new time variable, τ.

$$w(t;\epsilon) = w_0(t,\tau) + \epsilon w_1(t,\tau) + \cdots \tag{B.5}$$

$$E(t) = \epsilon k \sin(\omega_o t + \sigma\tau) \tag{B.6}$$

Since w is not solely a function of t anymore, we need to define the derivative of w with respect to t.

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \left(\frac{\partial w_0}{\partial t} + \frac{\partial w_0}{\partial \tau}\frac{\partial \tau}{\partial t}\right) + \epsilon\left(\frac{\partial w_1}{\partial t} + \frac{\partial w_1}{\partial \tau}\frac{\partial \tau}{\partial t}\right) + \cdots \tag{B.7}$$

Since $\frac{\partial\tau}{\partial t} = \epsilon$ the above equation becomes

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w_0}{\partial t} + \epsilon\left(\frac{\partial w_0}{\partial \tau} + \frac{\partial w_1}{\partial t}\right) + \mathcal{O}(\epsilon^2) \tag{B.8}$$

Next, finding the second derivative with respect to time results in

$$\frac{\mathrm{d}^2w}{\mathrm{d}t^2} = \frac{\partial^2 w_0}{\partial t^2} + \epsilon\left(2\frac{\partial^2 w_0}{\partial \tau\partial t} + \frac{\partial^2 w_1}{\partial t^2}\right) + \mathcal{O}(\epsilon^2) \tag{B.9}$$

Plugging back into our equation of motion and evaluating the ϵ^0 terms and the ϵ^1 terms

$$\frac{\partial^2 w_0}{\partial t^2} + \omega_o^2 w_0 = 0 \tag{B.10}$$

$$\frac{\partial^2 w_1}{\partial t^2} + \omega_o^2 w_1 = -2\frac{\partial^2 w_0}{\partial \tau\partial t} - 2\mu\frac{\partial w_0}{\partial t} - \alpha w_0^3 + k\sin(\omega_o t + \sigma\tau) \tag{B.11}$$

The general solution to equation (B.10) is

$$w_0 = A(\tau)e^{i\omega_o t} + \bar{A}(\tau)e^{-i\omega_o t} \tag{B.12}$$

Where $A(\tau)$ is yet to be determined and $\bar{A}(\tau)$ is its complex conjugate. Plugging that solution in equation (B.11), and utilizing the complex form of our forcing function, equation (B.13), yields

$$\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \qquad \sin(\theta) = \frac{-i}{2}\left(e^{i\theta} + e^{-i\theta}\right) \tag{B.13}$$

$$\frac{\partial^2 w_1}{\partial t^2} + \omega_o^2 w_1 = \left\{-\left[2i\omega_o(\frac{\mathrm{d}A}{\mathrm{d}\tau} + \mu A) + 3\alpha A^2\bar{A}\right]e^{i\omega_o t} - \alpha A^3 e^{3i\omega_o t} - i\frac{k}{2}e^{i\omega_o t}e^{i\sigma\tau}\right\} + cc \tag{B.14}$$

Where cc is the complex conjugate of the terms on the right hand side in the curly brackets. In order to remove 'secular' terms, or terms that cause a break down in perturbation theory, we choose A to be the solution of equation (B.15). 'Secular' terms are terms that could cause the solution to diverge. For an oscillatory system, secular terms would cause resonance, and the response amplitude would approach infinity. In our case, since the natural frequency of the system is ω_o, it is important to eliminate the terms that have sine or cosine of $\omega_o t$ in order to eliminate terms that would go off to infinity.

$$2i\omega_o(\frac{\mathrm{d}A}{\mathrm{d}\tau} + \mu A) + 3\alpha A^2\bar{A} + i\frac{k}{2}e^{i\sigma\tau} = 0 \tag{B.15}$$

Writing A in the form

$$A = \frac{a}{2}e^{i\beta} \tag{B.16}$$

where a and β are real values and plugging in our equation for A we obtain

$$i\omega_o\left(\frac{\mathrm{d}a}{\mathrm{d}\tau} + a\frac{\mathrm{d}\beta}{\mathrm{d}\tau}i + \mu a\right) + 3\alpha\frac{a^3}{8} + i\frac{k}{2}e^{i(\sigma\tau-\beta)} = 0 \tag{B.17}$$

Separating real and imaginary terms into two equations

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = -\mu a - \frac{k}{2\omega_o}\cos(\sigma\tau - \beta) \tag{B.18}$$

$$a\frac{\mathrm{d}\beta}{\mathrm{d}\tau} = \frac{3\alpha}{8\omega_o}a^3 - \frac{k}{2\omega_o}\sin(\sigma\tau - \beta) \tag{B.19}$$

Using $\gamma = \sigma\tau - \beta$ results in an equation where τ does not appear directly.

$$\boxed{\frac{\mathrm{d}a}{\mathrm{d}\tau} = -\mu a - \frac{k}{2\omega_o}\cos(\gamma)} \tag{B.20}$$

$$\boxed{a\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} = \sigma a - \frac{3\alpha}{8\omega_o}a^3 + \frac{k}{2\omega_o}\sin(\gamma)} \tag{B.21}$$

The system of equations above are the approximation to the solution to Duffing's equation. While it may seem that we increased the difficulty of our problem by exchanging a second order differential equation for two first order differential equations, we now have system of equations in a familiar form that have been well studied.

Now substituting equation (B.16) into equation (B.12) then into equation (B.5) results in an expression for w.

$$w = a\cos(\omega_o t + \beta) + \mathcal{O}(\epsilon) \tag{B.22}$$

Note that this expression is the 'general solution' to the differential equation, and the 'particular' or steady state solution will be given in the next section. However note that this general solution does not take into account the change in the natural frequency with load. We can factor our differential equation in such a form that we can obtain a new form for natural frequency

$$\ddot{w} + \left(\omega_o^2 + \epsilon\alpha w^2\right) w = -2\epsilon\mu\dot{w} + E(t) \tag{B.23}$$

$$\omega_{effective}^2 = \omega_o^2 + \epsilon\alpha w^2 \tag{B.24}$$

From the equation above we can see that for larger values of w we would see a higher natural frequency. As a plate deflects the tensile membrane force would grow. As a result, the plate would act as if it were stiffer than linear plate theory. As we know from past studies, a stiffer plate would have a higher natural frequency. Thus, we can use equation (B.22) as a approximation for the response but a numerical solution, like the Runge-Kutta method, would give a solution to the time response with varying natural frequencies.

B.1 Steady State Solution

For a steady-state, or equilibrium, solution we set $\frac{\mathrm{d}a}{\mathrm{d}\tau} = \frac{\mathrm{d}\beta}{\mathrm{d}\tau} = \frac{\mathrm{d}\gamma}{\mathrm{d}\tau} = 0$. These solutions are also known as finding the *critical*, or *singular points* of the system.

$$\mu a = -\frac{k}{2\omega_o}\cos(\gamma) \tag{B.25}$$

$$\sigma a - \frac{3\alpha}{8\omega_o}a^3 = -\frac{k}{2\omega_o}\sin(\gamma) \tag{B.26}$$

The sum of the squares of the above equation removes the dependence on γ and thus removing the time dependence. This results in the *frequency-response equation* for the Duffing equation.

$$(\mu a)^2 + \left(\sigma a - \frac{3\alpha}{8\omega_o}a^3\right)^2 = \frac{k^2}{4\omega_o^2} \tag{B.27}$$

In order to solve this equation we first simplify by expanding the square term.

$$(\mu a)^2 + (\sigma a)^2 + \left(\frac{3\alpha}{8\omega_o}\right)^2 a^6 - \frac{3\alpha\sigma}{4\omega_0}a^4 = \frac{k^2}{4\omega_o^2}$$

Which simplifies to

$$\frac{9\alpha^2}{64\omega_o^2}a^6 - \frac{3\alpha\sigma}{4\omega_0}a^4 + (\mu^2+\sigma^2)a^2 = \frac{k^2}{4\omega_o^2} \tag{B.28}$$

Set $p = a^2$

$$\frac{9\alpha^2}{64\omega_o^2}p^3 - \frac{3\alpha\sigma}{4\omega_0}p^2 + (\mu^2+\sigma^2)p = \frac{k^2}{4\omega_o^2} \tag{B.29}$$

This third-order algebraic equation relates the amplitude of the response a with the amplitude of the excitation k, and the excitation frequency, which is a function of σ. The only unknown in the above equation is p, which is related to the amplitude of the response. It is possible to have up to three, unique, real solutions for this problem. We will see that this equation predicts a unique solution for coefficients.

B.1.1 Example

Setting

$$\alpha = 1 \tag{B.30}$$

$$\omega_o = 1 \tag{B.31}$$

$$k = 1000 \tag{B.32}$$

results in the following frequency response, in figure (B.1), varying σ for different values of μ.

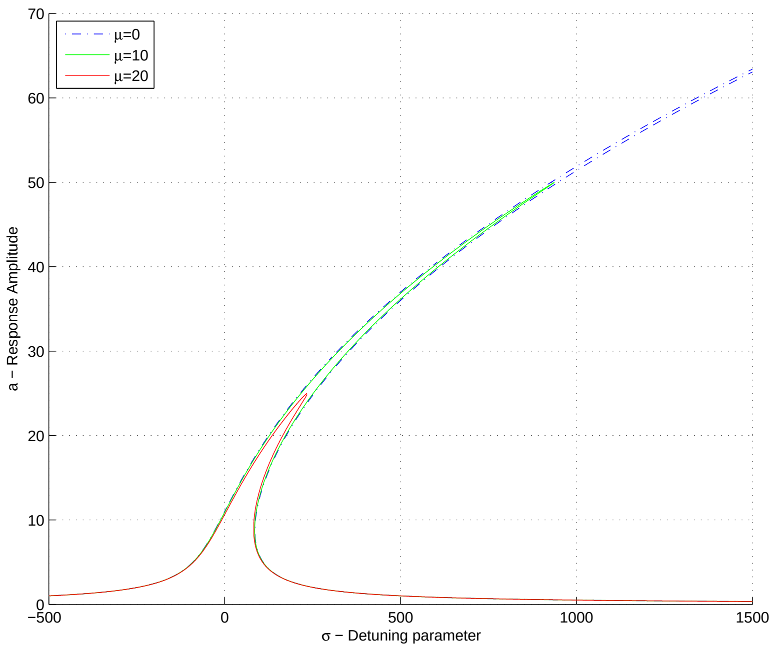


Figure B.1: Frequency Response

We can see from the figure above that for particular excitation frequencies we may observe up to three different amplitudes. However, from stability analysis in Nayfeh and Mook [32] we can see that two of the solutions are stable and one is unstable. As a result, we would have a 'jump in amplitude' as we change the excitation frequency, which will be discussed later.

From the figure it is easy to see that for a particular excitation frequency we may have up to three different amplitudes. However, these amplitudes depend on what direction one is coming from. If we began at a high frequency and gradually decreased the excitation frequency we would follow the lower most path. If we began at a low frequency and gradually increased the excitation frequency we would follow the upper most path. This was observed experimentally for plates by Amabili [3]. As a result, there is a region of the graph that is unused. As shown later, this region is known as the unstable region.

Now note that this solution is the steady state solution of the system.

$$w = a\cos(\Omega t - \gamma) + \mathcal{O}(\epsilon) \tag{B.33}$$

B.2 Stability

Since we have found the 'critical points', or the steady-state solution, of the system from equation (B.28) or (B.29) we can now investigate the stability of these points. In order to continue, we need to linearize the area around the critical point. By defining a generic critical point as (a_s, γ_s), where a_s and γ_s satisfy equation (B.28) and (B.29) we can continue by the method outlined in Greenberg [22].

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = -\mu a - \frac{k}{2\omega_o}\cos(\gamma) = P(a,\gamma) \tag{B.34}$$

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} = \sigma - \frac{3\alpha}{8\omega_o}a^2 + \frac{k}{2\omega_o}\frac{\sin(\gamma)}{a} = Q(a,\gamma) \tag{B.35}$$

Our linearized model becomes, note that $A = a - a_s$ and $G = \gamma - \gamma_s$

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = \frac{\partial P(a_s, \gamma_s)}{\partial a} A + \frac{\partial P(a_s, \gamma_s)}{\partial \gamma} G \tag{B.36}$$

$$\frac{\mathrm{d}G}{\mathrm{d}\tau} = \frac{\partial Q(a_s, \gamma_s)}{\partial a} A + \frac{\partial Q(a_s, \gamma_s)}{\partial \gamma} G \tag{B.37}$$

Substituting in the partial differentials

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = -\mu A + \frac{k}{2\omega_o} \sin(\gamma_s) G \tag{B.38}$$

$$\frac{\mathrm{d}G}{\mathrm{d}\tau} = \left(-\frac{6\alpha}{8\omega_o} a_s - \frac{k}{2\omega_o} \frac{\sin(\gamma_s)}{a_s^2} \right) A + \frac{k}{2\omega_o} \frac{\cos(\gamma_s)}{a_s} G \tag{B.39}$$

Using the relationships in equation (B.25) and (B.26) to eliminate γ_s

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}\tau} = -\mu A + \left(\frac{3\alpha}{8\omega_o} a_s^3 - \sigma a_s \right) G} \tag{B.40}$$

$$\boxed{\frac{\mathrm{d}G}{\mathrm{d}\tau} = \left(\frac{\sigma}{a_s} - \frac{9\alpha}{8\omega_o} a_s \right) A - \mu G} \tag{B.41}$$

In matrix form

$$\begin{bmatrix} \frac{\mathrm{d}a_1}{\mathrm{d}\tau} \\ \frac{\mathrm{d}\gamma_1}{\mathrm{d}\tau} \end{bmatrix} = \begin{bmatrix} -\mu & -\left(\sigma a_s - \frac{3\alpha}{8\omega_o} a_s^3 \right) \\ \left(\frac{\sigma}{a_s} - \frac{9 a_s \alpha}{8\omega_o} \right) & -\mu \end{bmatrix} \begin{bmatrix} a_1 \\ \gamma_1 \end{bmatrix} \tag{B.42}$$

The above is an eigenvalue problem with the characteristic equation of

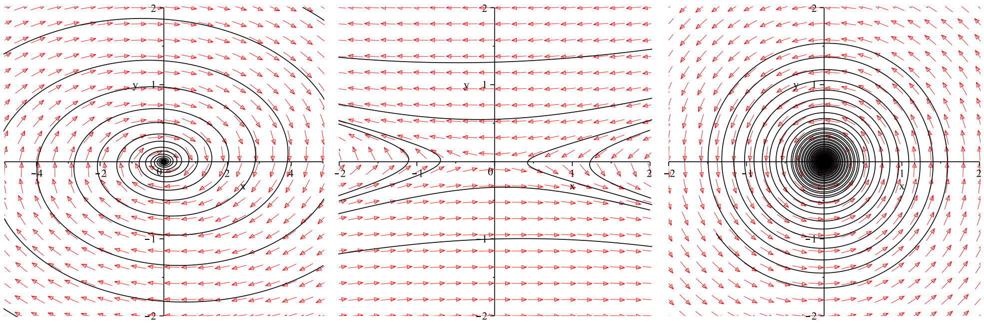
$$\lambda^2 + 2\mu\lambda + \mu^2 + \left(\frac{\sigma}{a_s} - \frac{9 a_s \alpha}{8\omega_o} \right) \left(\sigma a_s - \frac{3\alpha}{8\omega_o} a_s^3 \right) = 0 \tag{B.43}$$

which has a solution in the form

$$\lambda = -\mu \pm \sqrt{\left(\sigma - \frac{9\alpha}{8\omega_o} a_s^2 \right) \left(\frac{3\alpha}{8\omega_o} a_s^2 - \sigma \right)} \tag{B.44}$$

From Greenberg [22], we can classify the form of the stability based on the values of λ.

- Center: Purely imaginary roots



(a) Near a_1 (b) Near a_2 (c) Near a_3

Figure B.2: Phase Planes near the Critical Points

- Focus: Complex conjugate roots
- Node: Real roots of same sign
- Saddle: Real roots of opposite sign

B.2.1 Example

Using values of $\alpha = \omega_o = 1, \mu = 10, k = 1000$ (see figure (B.1)) we find the stability of the solutions at $\sigma = 500$.

At $\sigma = 500$ we find the values of a_s to be

$$a_1 = 36.8482 \tag{B.45}$$

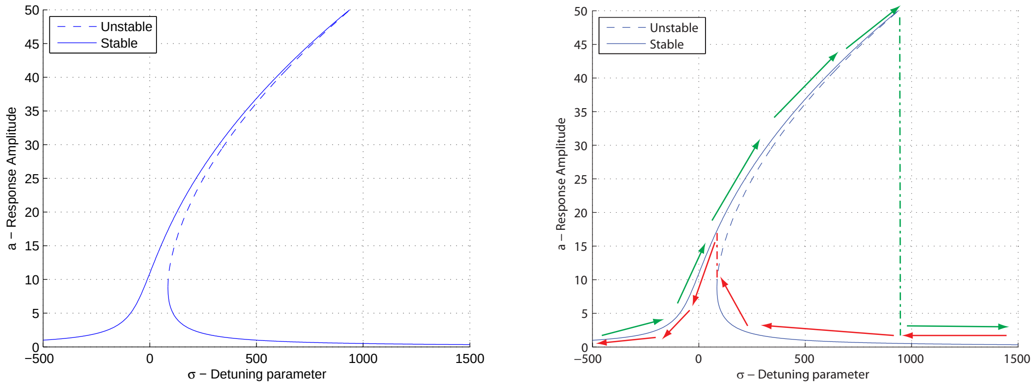
$$a_2 = 36.1645 \tag{B.46}$$

$$a_3 = 1.0005 \tag{B.47}$$

From figure (B.2) above we can see that the 'middle' solution is unstable. Our frequency response curve with a distinct unstable region is displayed in figure (B.3).

B.2.2 Jump Phenomena

From study of stability we can see that some values of the response amplitude do not exist. From left to right, a discontinuous curve for response amplitude is present, which results in jump phenomenon.



(a) Stable and Unstable Regions

(b) Amplitude Jumps

Figure B.3: Stability and Amplitude Jumps

If we were to start with a 'low' frequency and very slowly increase the excitation frequency until we reach the apex of the curve of figure (B.3a), then a small increase in excitation frequency would cause the amplitude to decrease significantly. Then, further increasing the frequency would cause a steady decline in amplitude. See figure (B.3b) and follow the green arrows, and the amplitude jump is marked be the green 'dash-dot' line. Starting with a high frequency and slowly decreasing the excitation frequency would cause a slight increase in amplitude until we reach the discontinuity where a slight decrease in excitation frequency would cause the amplitude to increase suddenly. Then decreasing the excitation frequency would cause the amplitude to decrease at a steady rate. This response is represented by the red arrows in figure (B.3b), also the amplitude jump is shown by the red 'dash-dot' line.

The discussion above was just a short introduction into Duffing's equation with many other topics, which may be of interest, omitted. For interest in subharmonic, and superharmonic, or overtone, resonance, when $\Omega \approx 3\omega_o$ and $\Omega \approx \omega_o/3$, respectively, Nayfeh [32] has a detailed explanation. Note that subharmonic and superharmonic frequencies arise when the response of the system resembles the superposition of two responses of the linear system. This is important if one is far away from the natural frequency, one may still have resonance because of their proximity to the subharmonic or superharmonic frequencies. In between large and small loading, a region exists where chaos, or non periodic response, may occur. While it is possible to predict chaotic regions, such analysis is beyond the scope of this study. As a result, for a particular system it is important to obtain a numerical solution in order to predict whether or not chaos will occur.

Appendix C

Solutions from other Authors

C.1 Stress-Free Edges

Taken from Levy [28] page 155, table 6. Note that p is the magnitude of the uniform load on a plate.

$\frac{pa^4}{Eh^4}$	$\frac{w_{11}}{h}$	$\frac{w_{center}}{h}$
0	0	0
12.1	0.5	0.486
29.4	1.000	0.962
56.9	1.500	1.424
99.4	2.000	1.870
161	2.500	2.307
247	3.000	2.742
358	3.500	3.174
497	4.000	3.600

Table C.1: Coefficients given by Levy for Stress-Free Edges

From Iyengar [24], page 115, equation (28). q is the magnitude of a uniform load, l is half of the plate edge's length.

$$\left(\frac{w_{11}}{h}\right)^3 + 5.8595\left(\frac{w_{11}}{h}\right) = 0.26335\frac{ql^4}{Eh^4} \tag{C.1}$$

From Ventsel [47], page 222, equation (7.91). Venstel uses f to represent w_{11} and p is the magnitude of a uniform load on the plate.

$$\frac{f}{h} + \frac{128(1-\nu^2)}{3\pi^4}\frac{f^3}{h^3} = \frac{4pa^4}{\pi^6 Dh} \tag{C.2}$$

From Donnell [16], page 232, equation (5.11). p_{11} is the magnitude of a uniform load on the plate.

$$\frac{12(1-\nu^2)a^4}{\pi^4 E h^4} p_{11} = \left(1 + \frac{a^2}{b^2}\right)^2 \frac{w_{11}}{h} + \frac{3(1-\nu^2)}{4} \left(1 + \frac{a^4}{b^4}\right) \left(\frac{w_{11}}{h}\right)^3 \tag{C.3}$$

C.2 Immovable Edges

Taken from Levy [28] page 156, table 9.

$\frac{pa^4}{Eh^4}$	$\frac{w_{11}}{h}$	$\frac{w_{center}}{h}$
0	0	0
14.78	0.5	0.485
51.4	1.000	0.952
132.0	1.500	1.402
278.5	2.000	1.846

Table C.2: Coefficients given by Levy for Immovable Edges

From Iyengar [24], page 121, equation (52). q is the magnitude of a uniform load, l is half of the plate edge's length.

$$\left(\frac{w_{11}}{h}\right)^3 + 0.47 \left(\frac{w_{11}}{h}\right) = 0.03375 \frac{ql^4}{Eh^4} \tag{C.4}$$

From Donnell [16], page 231, equation (5.10). p_{11} is the magnitude of a uniform load on the plate.

$$\frac{12(1-\nu^2)a^4}{\pi^4 E h^4} p_{11} = \left(1 + \frac{a^2}{b^2}\right)^2 \frac{w_{11}}{h} + \frac{3}{4} \left[(3-\nu^2) \left(1 + \frac{a^4}{b^4}\right) + \frac{4\nu a^2}{b^2}\right] \left(\frac{w_{11}}{h}\right)^3 \tag{C.5}$$

References

[1] *Abaqus Analysis User's Manual*, Dessault Systémes, Ver. 6.9, 2009.

[2] *Abaqus Theory Manual*, Dessault Systémes, Ver. 6.9, 2009.

[3] Amabili, M., "Nonlinear Vibrations of Rectangular Plates with Different Boundary Conditions: Theory and Experiments," *Computers & Structures*, Vol. 82, 2004, pp. 2587-2605.

[4] Baruh, H. *Analytical Dynamics*, McGraw-Hill, New York, 1999.

[5] Beléndez, T., Neipp, C., and Beléndez, A., "Large and Small Deflections of a Cantilever Beam," *European Journal of Physics*, Vol. 23, 2002, pp. 371-379.

[6] Benaroya, H. *Mechanical Vibration: Analysis, Uncertainties, and Control*, 2nd Edition, Marcel Dekker, New York, 2004

[7] Bender, C.M. and Orszag, S.A., *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill Book Company, New York, 1978.

[8] Berger, H.M., "A New Approach to the Analysis of Large Deflections of Plates," *Journal of Applied Mechanics*, Vol. 22, No. 465, 1955, pp. 465-472.

[9] Bisshopp, K.E., and Drucker, D.C., "Large Deflections of Cantilever Beams," *Quarterly of Applied Mathematics*, Vol. 3, No. 3, 1945, pp. 272-275.

[10] Bottega, W.J., *Engineering Vibrations*, Taylor & Francis Group, Boca Raton, FL, 2006.

[11] Chia, C.Y., *Nonlinear Analysis of Plates*, McGraw-Hill, New York, 1980.

[12] Chu, H., Herrmann, G., "Influence of Large Amplitudes on Free Flexural Vibrations of Rectangular Elastic Plates," *Journal of Applied Mechanics*, Vol. 23, 1956, pp. 532-540.

[13] Conway, H.D., "The Large Deflection of Simply Supported Beams," *Philosophical Magazine*, Vol. 38, 1947, pp. 905-911.

[14] Cook, R.D., Young, W.C., *Advanced Mechanics of Materials*, Macmillan Publishing Company, New York, 1985.

[15] Cook, R.D., Malkus, D.S., Plesha, M.E., Witt, R.J., *Concepts and Applications of Finite Element Analysis*, 4th Edition, John Wiley & Sons, Inc., New Jersey, 2002.

[16] Donnell, L.H., *Beams, Plates, and Shells*, McGraw-Hill, New York, 1976.

[17] El Kadiri, M., Benamar, R., "Improvement of the semi-analytical method, based on Hamilton's principle and spectral analysis, for determination of the geometrically non-linear response of thin straight structures. Part III: steady state periodic forced response of rectangular plates," *Journal of Sound and Vibration*, Vol. 264, 2003, pp. 1-35.

[18] Evensen, D.A., "Nonlinear Vibrations of Beams with Various Boundary Conditions," *AIAA Journal*, Vol. 6, No. 2, 1968, pp. 370-372.

[19] Fish, J., Belytschko, T., *A First Course in Finite Elements*, John Wiley & Sons, Inc., New Jersey, 2007.

[20] Fung, Y.C., *Foundations of Solid Mechanics*, Prentice Hall, New Jersey, 1965.

[21] Graff, K.F., *Wave Motion in Elastic Solids*, Dover Publications, Inc., New York, 1975.

[22] Greenberg, M.D., *Advanced Engineering Mathematics*, 2nd Edition, Prentice Hall, New Jersey, 1998.

[23] Howell, P., Kozyreff, G., and Ockendon, J.R., *Applied Solid Mechanics*, Cambridge University Press, New York, 2009.

[24] Iyengar, K.T.S.R., Naqvi, M.M., "Large Deflections of Rectangular Plates," *International Journal of Non-Linear Mechanics*, Vol. 1, No. 2, 1966, pp. 109-122.

[25] Kirchhoff, G.R., "Uber das gleichgewichi und die bewegung einer elasishem scheibe," *J. Fuer die Reine und Angewandte Mathematik*, Vol. 49, 1850, pp. 51-88.

[26] Leissa, A., *Vibration of Plates*, Government Printing Office, Washington, US, 1969.

[27] Leung, A.Y.T., and Mao, S.G., "A sympletic Galerkin Method for Non-Linear Vibration of Beams and Plates," *Journal of Sound and Vibration*, Vol. 183, No. 3, 1995, pp. 475-491.

[28] Levy, S., "Bending of Rectangular Plates with Large Deflections," *NACA, TR.*, 737, 1942.

[29] Mathews, J.H., Fink, K.D., *Numerical Methods Using Matlab*, 4th Edition, Pearson Prentice Hall, New Jersey, 2004.

[30] Mansfield, E.H., *The Bending & Streching of Plates*, 2nd Edition, Cambridge University Press, New York, 1989.

[31] Mindlin, R.D., "Influence of Rotatory Inertia and Shear on Flexural Motions of Isotropic Elastic Plates," *Journal of Applied Mechancis*, Vol. 18, 1951, pp. 31-38.

[32] Nayfeh, A.H., and Mook, D.T., *Nonlinear Oscillations*, John Wiley & Sons, INC., New York, 1979.

[33] Nayfeh, A.H., *Introduction to Perturbation Techniques*, John Wiley & Sons, INC., New York, 1981.

[34] O'Neill B., *Elementary Differential Geometry*, Revised 2nd Edition, Elsevier, New York, 2006.

[35] Prathap, G., and Pandalai, K.A.V., "Non-Linear Vibrations of Transversely Isotropic Rectangular Plates," *International Journal of Non-Linear Mechanics*, Vol. 13, 1977, pp. 285-294.

[36] Prescott, J., *Applied Elasticity*, Dover Publications, INC., New York, 1961.

[37] Pressley, A., *Elementary Differential Geometry*, 2nd Edition, Springer, London, 2010.

[38] Ribeiro, P., "Periodic Vibration of Plates with Large Displacements," *AIAA Journal*, Vol. 40, No. 1, 2001, pp. 185-188.

[39] Sathyamoorthy, M., *Nonlinear Analysis of Structures*, CRC Press LLC, Boca Raton, FL, 1998.

[40] Sathyamoorthy, M., "Nonlinear Vibration Analysis of Plates: A Review and Survey of Current Developments," *Applied Mechanics Reviews*, Vol. 40, 1987, pp. 1553-1561.

[41] Sathyamoorthy, M., "Nonlinear Analysis of Beams Part I: A Survey of Recent Advances," *Shock and Vibration Digest*, Vol. 14, No. 17, 1982, pp. 19-35.

[42] Singh, G., Sharma, A.K., and Rao, G.V., "Large-Amplitude Free Vibrations of Beams - A Discussion on Various Formulations and Assumptions," *Journal of Sound and Vibration*, Vol. 142, No. 1, 1990, pp. 77-85.

[43] Szilard, R., *Theory and Analysis of Plates: Classical and Numerical Methods*, Prentice-Hall, Englewood Cliffs, New Jersey, 1974.

[44] Woinowsky-Krieger, S., "The Effect of an Axial Force on the Vibration of Hinged Bars," *Journal of Applied Mechanics*, Vol. 17, No. 1, 1950, pp. 35-36.

[45] Timoshenko, S., and Woinowsky-Krieger, S., *Theory of Plates and Shells*, 2nd Edition, McGraw-Hill, New York, 1959.

[46] Tuttle, M.E., *Structural Analysis of Polymeric Composite Materials*, Marcel Dekker, New York, 2004.

[47] Ventsel, E., Krauthammer, T., *Thin Plates and Shells: Theory, Analysis, and Applications*, Marcel Dekker, New York, 2001.

[48] von Kármán, T., "Fesigkeitsprobleme in Maschinenbau," *Encycl de Math Wiss*, Vol. 4, 1910, pp. 348-351.

[49] Wang, T.M., "Non-Linear Bending of Beams with Uniformly Distributed Loads," *International Journal of Non-Linear Mechanics*, Vol. 4, 1969, pp. 389-395.

[50] Way, S., "Uniformly loaded clamped rectangular plates with large deflection," *Proc. 5th Int. Congr. on Applied Mechanics*, 1938, pp. 123-128.

[51] Wei-Zang, C., and Kai-Yuan, Y., "On the Large Deflection of Rectangular Plate," *Proceedings of IX International Congress of Applied Mechanics*, 1956, pp. 387-394.

[52] Yosibash, Z., and Kirby, R.M., "Dynamic response of various von-Kármán Non-Linear Plate Models and their 3-D Counterparts," *International Journal of Solid and Structures*, Vol. 42, 2005, pp. 2517-2531.

[53] Zill, D.G., Cullen, M.R., *Advanced Engineering Mathematics*, 3rd Edition, Jones and Bartlett Publishers, Mass., 2006.