TASK ANALYSIS - THE INHERENT MATHEMATICAL STRUCTURES

IN STUDENTS' PROBLEM-SOLVING PROCESSES

by

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Abstract

Research Questions: One way students may develop conceptual understanding is through working on strands of related mathematical tasks and thus developing and refining their understanding of the underlying mathematical concepts contained in the tasks. The purpose of this study is to illuminate this process by detailing the inherent mathematical structures in such a strand and discuss what aspects of it facilitated student learning. The research questions addressed are: (1) *What mathematical structures can be uncovered by exploring/engaging with the combinatorics tasks used in the Rutgers longitudinal study?* (2) *In what ways are these mathematical structures revealed during students' problem-solving processes?*

Methodology: Ten tasks from the combinatorics/counting strand are selected from the Rutgers longitudinal project for this qualitative study. The data available for analysis are in the form of digitized video tapes, verified transcripts, and students' written work. The analysis focuses on decoding students' solutions into formal mathematical definitions and theorems. Concept maps are used to illustrate the overall hierarchy of the presented mathematical structures.

Findings: There are a total of sixty-three inherent mathematical structures extracted from the formal solutions of ten selected combinatorics tasks. These structures are categorized as definitions, notations, axioms, properties, formulas, and theorems. When classified with respect to the seven relevant sub-domains of mathematics, these structures pertain to: set theory, enumerative combinatorics, graph theory, sequences & sets, general algebraic system, probability theory, and geometry. The analysis suggests that the participating students uncovered many of these mathematical structures primarily in the following ways: (1) Manipulating a concrete model, (2) Listing all possible combinations, (3) Inventing different representations, (4) Seeking patterns, and (5) Making connections.

Conclusion and Suggestions: These findings support the following suggestions for practice: (1) Teachers may benefit from studying the underlying structures of a task thoroughly before assigning the task to students, (2) In determining the order of related tasks within a strand, teachers need to consider the sophistication level and the coherence of the underlying structures across tasks, (3) Using concrete models can help students to both develop and verify solutions to complex problems, and (4) Tasks whose inherent structures belong to a variety of mathematical sub-domains can help students build an increasingly interconnected view of mathematics.

Significance: This study outlined a method of extracting inherent mathematical structures from mathematical tasks. The results suggest that students have natural abilities to uncover these structures by themselves. It is hoped that this will motivate mathematics teachers to improve the way they think about using problem solving in their teaching.

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Chapter 1: Introduction

1.1 The Importance of Mathematical Structures in Task Analysis

According to the goals of mathematics education reform, "students are expected to apply academic knowledge to real-world contexts, communicate effectively and work collaboratively" (Ormerod & Ridgway, 1999, p. 401). To achieve these goals, problem solving is listed in the 2008 New Jersey Mathematics Core Curriculum Content Standards for Mathematics (NJMCCCS, 2008) as the first strand of the standard of mathematical processes, and school teachers are encouraged to allocate part of their instructional time to problem solving. According to the National Council of Teachers of Mathematics (NCTM, 2000), "most mathematical concepts or generalizations can be effectively introduced using a problem situation," a claim which is supported by a significant number of studies that investigate approaches to instruction focused on problem-solving.

One of the important aspects of problem solving is to design "worthwhile mathematical tasks" (NCTM, 2000, pp. 18-19) through which students can learn important mathematical concepts (Ormerod & Ridgway, 1999). The question is: what characterizes a worthwhile task? Researchers are seeking answers through formal task analysis that usually examines a variety of student responses to a particular mathematical task. More specifically, some researchers classify the task based on different level of students' *mathematical behaviors* (i.e. students' actions) (Resnick, Wang, & Kaplan, 1973), based on students' explanations (Hadas & Hershkowitz, 2002), or based on cognitive demand (Stein et al., 2000). Other researchers do not categorize mathematical tasks but examine the task outcomes thoroughly from the perspective of students' reasoning (Powell, 2003), from the perspective of task design (Ainley and Pratt, 2005), and from the perspective of underlying mathematical structures (Torkildsen, 2006). In this study, tasks will be analyzed based on underlying *mathematical structures*. Mathematical structures are defined by the author as a hierarchy of interconnected mathematical objects building on one another to produce a coherent whole.

It is important to note that the analysis of underlying mathematical structures may help teachers to see connections between particular tasks and the mathematical ideas presented in the curriculum. Seeing different mathematical objects as a coherent whole may help teachers to "move away from teaching unconnected, isolated topics and toward teaching mathematical concepts and ideas" (Stein & Kim, 2006, p. 17). Students as well as their teachers will benefit from organizing mathematical concepts as a "knowledge package" (Ma, 1999, pp. 17-19) rather than collecting those mathematical ideas as discrete pieces. Evidence shows that "building on connections can make mathematics a challenging, engaging, and exciting domain of study" (NCTM, 2000, p. 204). Integrated mathematical knowledge not only helps students to remove the burden of memorizing too many unconnected concepts and skills but also helps students to apply mathematics to more complex and practical situations across disciplines and of the real world.

For all of these reasons, it is important and necessary to conduct task analyses with regard to mathematical structures so that mathematics can be learned in the processes of solving these task problems.

1.2 Statement of the Problem

It has been established (NCTM, 2000) that a problem solving approach to teaching can reflect the creative nature of mathematics and give students opportunities to discover important mathematical ideas themselves. However, not all mathematical tasks require the same level of cognitive ability (Stein et al., 2000). A task of high level cognitive demand probably will not engage the group of students whose cognitive ability is low. Conversely, a task of low level cognitive demand may fail to engage those students whose cognitive ability is high. Presenting students with tasks whose level of cognitive demand matches their cognitive ability is crucial for a successful problem-solving-based approach to instruction.

Research has shown great benefits from using cooperative learning in mathematical problem solving in K-12 mathematics. It is believed that group work can promote students' creative thinking, mathematical reasoning, and their social relations (NTCM, 2000). However, this cooperative approach may not always be an ideal way of learning subject contents (Gillies, 2003; Manouchehri & Goodman, 2000; Whicker, Bol, & Nunnery, 1997). From a student's perspective, the difficulty level and the design of the task determine how group members interact. A task problem can be either well-defined or ill-defined. According to Ormerod (2005), a well-defined problem is one in which many features such as the starting point, the ending state, the procedures, and the constraints are prescribed. An ill-defined problem has some or all of these features missing from in the problem description. An ill-defined problem is often referred to as "open-ended" because it allows the problem solver to experiment with a variety of solution paths. Gillies (2003) found that with a well-structured (i.e. well-defined) task, student interactions tended to be limited to exchanging information, providing explanations, or requesting assistance. When the task was ill-structured (i.e., ill-defined), students showed high levels of cooperation as they discussed how they would proceed as a group and shared ideas and information (Gillies, 2003). To make cooperative learning more productive, teachers should be aware of the following. First, teachers need to realize that a performance-based mathematical task is an enriched activity that has multiple entry points and pathways leading to a solution. Such a task requires students to use integrated subject knowledge and a variety of strategies during the problem solving process (Thomas, Williams, & Gardner, 2007). Second, teachers ought to choose tasks that accommodate students' abilities and the intended learning goal for students (NCTM, 2000, p. 53), because "well-chosen problems can be valuable in developing or deepening students' understanding of important mathematical ideas" (NCTM, 2000, p. 257).

Nowadays, mathematics tasks can be found in many resources outside the mathematics textbooks, such as newspapers, magazines, and online resources. However, choosing a good task that "integrates multiple mathematical topics and involves significant mathematical ideas" (NCTM, 2000, p. 52) is not as easy as it may seem. Teachers need a good understanding both of the mathematical objects and the relationship among those objects involved in a task and of their students' abilities in order to determine how appropriate a task is for a particular classroom or group of students. Therefore, understanding underlying mathematical structures is crucial to the success of selecting appropriate tasks that can promote learning.

There is more to be considered if problem solving is to be used as a means to create new mathematical knowledge for students. Because most mathematics curricula are organized and taught by topics, not by similar methods of solving problems, introducing students to a set of coherent mathematical objects requires not only one task, but a strand of tasks and the students' long term involvement with the strand of tasks. Researchers at Rutgers have found that students in a longitudinal study had constructed "mathematical problem-solving schema" (Weber, Maher, & Powell, 2006) over time, which helped them to relate task problems to one another based on the principles or modes of reasoning applicable to each problem rather than the phrasing of each task, and that also helped them to solve challenging problems through sophisticated representations and modes of reasoning. Therefore, the questions about which task should be used with which topic in the curriculum, and how to sequencing a strand of tasks, deserve a close investigation.

The Rutgers longitudinal project has been continuing for more than 17 years (a detailed description of the project can be found in chapter 4 - Methodology). Many studies have been done in regard to this longitudinal project. Students' reasoning, explanations, the use of representations, and learning of standard notations have been analyzed by many Rutgers researchers. However, the topic of the underlying mathematical structures of a task or a strand of tasks has not yet been fully explored. My proposed study aims to address, at least in part of, this gap.

1.3 The Purpose of the Study

The purpose of this study is to explore the mathematical structures uncovered by the high school participants as they worked on open-ended tasks that involved combinatorial and probability concepts. The problem-solving sessions took place set in settings in which sufficient time was allotted for collaborative learning. Data available for analysis are in the form of digitized video tapes, verified transcripts, and students' written work. Using an adapted methodology introduced by Torkildsen (2006), the analysis will focus on decoding students' solutions into formal mathematical definitions or theorems that are either mathematical objects or structures. A diagram adapted from concept mapping (Novak & Canas, 2008) will be used to illustrate the overall hierarchy of the presented mathematical objects and structures within a task and among tasks. Hope that the resulting report will serve as a guide to mathematics teachers interested in incorporating open-ended tasks in their teaching.

1.4 Research Questions

The present study is guided by the following two research questions for the strand of tasks used in Rutgers longitudinal project and involve combinatorial and probability concepts:

1. *What mathematical structures can be uncovered by exploring/engaging with the combinatorics tasks used in the Rutgers longitudinal study?*
2. *In what ways are these mathematical structures revealed during students' problem-solving processes?*

Given the qualitative nature of the proposed study and the informal after-school setting of the Rutgers longitudinal study, the results of this study may not be generally applicable to a regular mathematics classroom. Mathematics teachers may have to tailor the tasks to fit their instructional goals. However, this study provides an in-depth exploration of mathematical structures and their connections to student learning as well as to New Jersey curriculum standards that can assist teachers in their efforts to design instruction focused on problem-solving.

Chapter 2: Literature Review

2.1 Task Analysis

Outside of a specific domain, the term ***Task*** is generally understood to refer to a piece of assigned work that may be hard and that has to be done within a certain time frame (Merriam-Webster). For mathematics educators and researchers, a mathematical *task* may be synonymous to a mathematical *activity* or a mathematical *problem* that gives students a chance to explore to certain mathematical topics in the curriculum (Ormerod & Ridgway, 1999). In this study, all these three phrases (i.e. mathematical task, activity, or problem) will be used interchangeably.

Mathematical task investigations are often conducted in a cooperative learning environment over an extended period of time because the scope and complexity of an open-ended task go beyond those of a regular class exercise. Collaborative learning and sufficient time allotted for the task are important factors to the success of task implementation that produces quality solutions (Francisco & Maher, 2005; Powell, 2003; Uptegrove, 2004; Torkildsen, 2006). Given the little precious class time, how much time should be allocated to open-ended mathematical activities? Is spending time on mathematical activities truly worthwhile? What do students learn from working on these mathematical tasks? What kind of mathematical tasks promote student learning? To answer these questions, task analysis has been tackled by many researches. Subsequently, task design has received a lot of attention.

Resnick, Wang, and Kaplan (1973) conducted a series of task analyses in the context of children's learning of the concept of number. Their goal was to identify and propose optimal sequences of learning objectives. In their study, a *task* was defined as a series of mathematical behaviors. For example, under the curriculum objective "Counting and One-to-One Correspondence," task C was guided by the sub-objective "Given a fixed ordered set of objects, the child can count the objects". To conduct task C in the classroom, a teacher may give children some concrete objects such as a pile of tiles or paper strips and ask children to count them. Analyzing Task C meant to identify the *component behaviors* and the *prerequisite behaviors* required to accomplish the objective. The following figure 2.1-1 summarizes the analysis of task C:

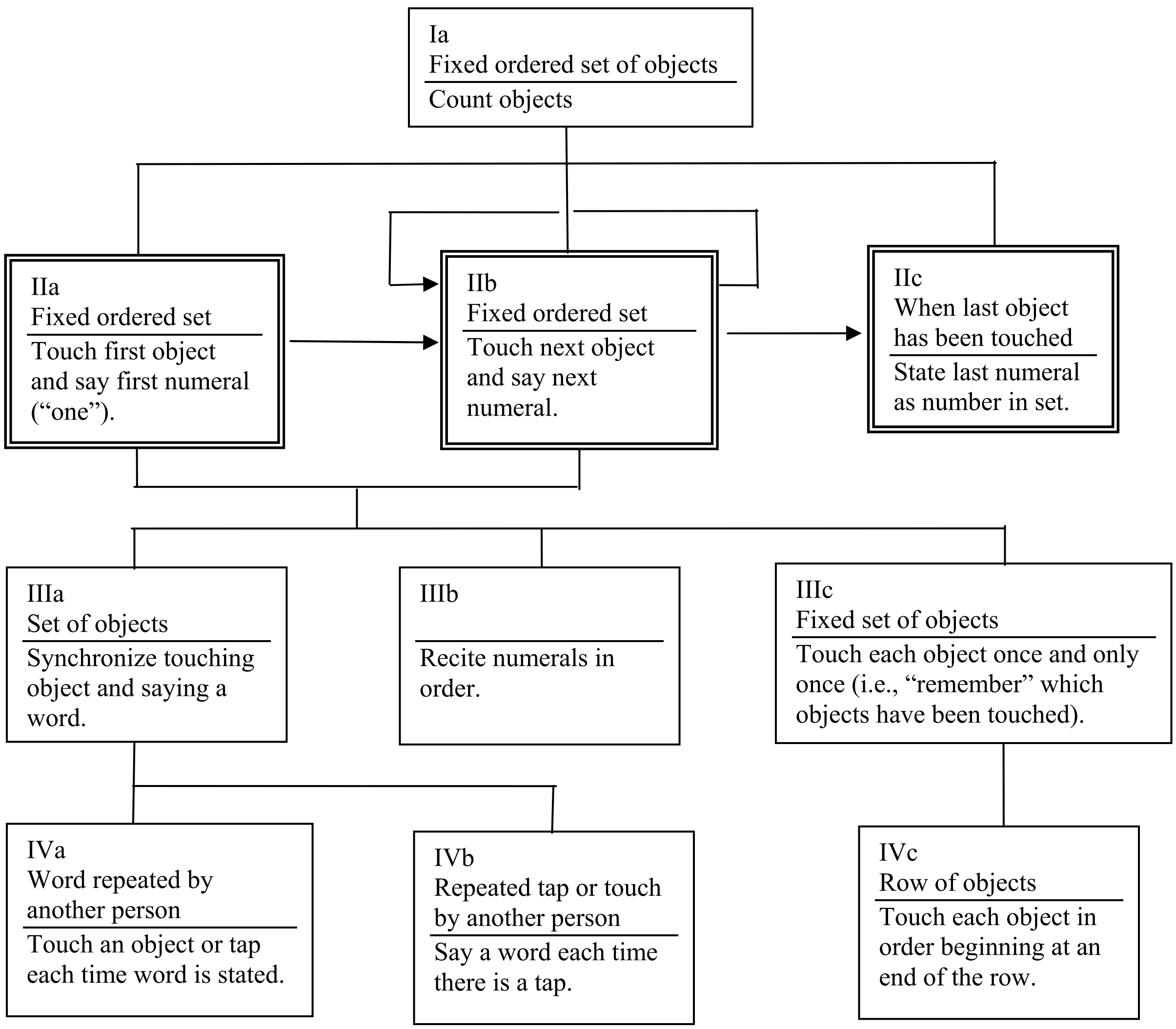

Figure 2.1-1. *Example of analysis of mathematical behaviors* (Resnick, Wang, & Kaplan, 1973, p.688)

The *component behaviors* were those steps that needed to be actually performed. The *prerequisite behaviors* were simpler behaviors not actually performed but which the learner must have had the ability to perform before engaging in the present task. For task C above, three component behaviors were "Touch first object and say first numeral ('one')," "Touch next object and say next numeral," and "When last object has been touched, state last numeral as number in the set." Examples of prerequisite behaviors could be "Recite numerals in order," or "Synchronize touching object and saying a word." There could be more than one level of prerequisites behaviors. To perform "Synchronize touching object and saying a word," a child must be able to "touch an object or tap each time the word is stated by another person." Resnick, Wang, & Kaplan (1973) applied prerequisite analysis first to the curriculum objectives and then to the highest level of tasks within each objective. The resulting hierarchy suggested a complete structure of mathematical behaviors and optimal teaching and learning sequencing that provided a practical basis for curriculum design.

In a review of prior literature on task analysis and design, Resnick (1976) pointed out that associationists and behaviorists had found that sequencing tasks according to difficulty levels and letting students work from easier to more complicated tasks optimized learning. The Gestalt school suggested that the key to effective teaching was to let students understand the internal structure of the problem and discover the solution by themselves. Finally, Piagetian approaches focused on the differences between novices and experts when they approach certain tasks. Resnick (1976) evaluated these different approaches through four criteria: *Instructional relevance*, *psychological formulation*, *instructability*, and *recognition of stages of competence*. She indicated that Piaget's work contributed primarily to the last criteria and led to an *information-processing* task analysis model, which can be characterized as *rational* analysis and *empirical* analysis.

Rational task analysis, according to Resnick (1976), focused on mathematical behaviors that were supposed to "succeed in responding to task demands" (p. 22). The goal was to "specify processes or procedures that would be used" in the tasks (p. 23). It was "derived from the structure of the subject matter" (p. 23) and "prescribing what to teach" (p. 24). The analysis of the simple number-sense tasks - task C (Resnick, Wang, & Kaplan, 1973) described previously are examples of this type. When the analysis was completed, a hierarchy of ideal mathematical behaviors was built, which would help teachers to see the logical sequence of learning in performing the task.

Do these ideal performances actually occur in the classroom? Empirical task analyses (based on children's actual performances during task sessions) are meant to explore this question. The goal of the researchers was to develop all possible performing models that could be used to describe children's reactions and thinking processes when they worked on specific mathematical tasks. Resnick (1976) had given an example in the context of children developing skills of single-digit *addition*. First, an addition algorithm of counting blocks was directly taught by teachers. After children mastered this physical model, the blocks were removed and children were forced to find other ways to compute the sum of two numbers. Counting fingers was a typical choice at the beginning. Gradually, most children switched to mental processing and "invented" a *sum* model and a *min* (minimum) model; both models were simpler, improved versions of the original taught algorithm.

The evidence of children "inventing" simplified models through problem solving should have significant pedagogical impact. However, that does not mean the rational task analyses were unnecessary, or initial teaching of algorithms should be omitted entirely. Resnick (1976) argued that the structure of the subject matter, teaching routines (i.e. pedagogical choices), and children's performances were intimately intertwined. Teaching basic skills actually provided children with an entry point to develop their discovery and invention abilities. What should be taught directly? What should be left for children to discover? Resnick (1976) suggested that more empirical analyses of specific tasks should be conducted in order for educators to design appropriate teaching routines that would help children efficiently acquire underlying structure of the subject matter.

Many researchers responded to Resnick's (1976) call for empirical task analyses. Students' performances during task sessions were examined carefully. Robert B. Davis and Carolyn A. Maher directed and conducted a Rutgers longitudinal study beginning in the 1980's. In the context of this Rutgers project, there are numerous studies that could be classified as empirical task analyses. The present dissertation is based on the data collected for this longitudinal project that is thoroughly described in chapter 4 Methodology.

In studying mathematics instructional tasks, Stein et al. (2000) developed a Task Analysis Guide based on four levels of task cognitive demands. *Memorization* tasks and *Procedures without Connections* tasks were classified as being of lower cognitive level, which utilized facts, rules, definitions, or previously learned formulas. *Procedures with Connections* tasks and *Doing Mathematics* tasks were classified in the higher cognitive level category, these being the tasks through which students explored a variety of situations and built deeper understanding of mathematical concepts and relationships. The characteristics of these tasks were described in detail (pp. 12-23). For example, students' prior knowledge was needed for solving any type of tasks. But memorization tasks "involve exact reproduction of previously seen material," while doing mathematics tasks "require students to access relevant knowledge and experiences and make appropriate use of them" (p. 16).

Stein et al.'s (2000) task analysis guide provided teachers with a useful tool in matching tasks with their instructional goals to promote student learning. However, the cognitive demand level for a particular task was not easily agreed upon by educators. That was because the cognitive level could be easily changed during the task set up or implementation phases. Through analyzing six task sessions conducted by different teachers with different students on different mathematics topics, Stein et al. (2000) presented six instances of changing of cognitive demand level in the classroom. Four of these cases started with high-level cognitive demands and degenerated into low-level mathematical activities. Various factors contributed to these declines. Inappropriate task selection, too much or too little time, and inappropriate teacher interventions were the main reasons. The other two cases successfully maintained the intended cognitive demand levels. These two cases presented similar characteristics, such as the "task building on students' prior knowledge, high-level performance was modeled, appropriate amount of time and teacher scaffolding, and sustained pressure for explanation and meaning" (p.31). Stein et al. (2000) further encouraged all mathematics teachers to reflect on these six cases and on their own teaching practices based on the mathematical tasks framework introduced above. By doing so, teachers could gain thorough understanding of the tasks and of students' thinking, both of which are necessary factors for effective teaching and learning.

Because nowadays technological tools such as Geometer's Sketchpad and computer algebra systems play an important role in mathematic classrooms, Heid et al. (2002) developed a Mathematics Task Coding Instrument (MaTCI) for their study of students working on conceptually oriented mathematical tasks in the context of technology. They, like Stein and her colleagues, categorized tasks by the cognitive demands. However, unlike Stein et al., their focus was "primarily on the goal state or the end product of the activity," and not on the intricacies of students' performances. Heid et al. (2002) first divided tasks into three major classes: *Concept, Product, and Reasoning*. The goal of a concept task was to characterize a concept; in the case of a product task, it was to generate a mathematical object; for reasoning tasks, it was to give a rationale for a conclusion. Under each major class, there were several subclasses that were ordered by cognitive demands from easier to more difficult. Detailed explanation and examples for codes under each subclass were provided. The following table 2.1-1 summarizes the overall structure of Heid et al.'s (2002) MaTCI:

Category	Subcategory	Code
Concept	Identify	Identify object
	Describe	Observation or procedure
	Elaborate	Compare/explain/describe phenomenon
Product	Produce	Produce a value or an output given an input value; Input value given an output value; Produce a graph
	Generate	Function specifics; A procedure
	Predict	Predict
	Generalize	Generalize
Reasoning	Corroborate	A procedure or a generalization
	Justify	Justify

Table 2.1-1 *MaTCI - Mathematics Task Coding Instrument* (Heid et al., 2002)

A task that requires students to explain why something is true or false would probably be classified as a Reasoning task by Heid et al.'s (2002) MaTCI, and as Doing Mathematics by Stein et al.'s (2000) Task Analysis Guide. In analyzing a geometric task, Hadas & Hershkowitz (2002) further classified students' explanations into five categories: *No* explanation, *Inductive* explanation, *Partial Deductive* explanation, *Visual* explanation, and *Deductive* explanation. About 12% (4 out of total 34 explanations) were classified as visual explanations and 23% (8 out of 34) as inductive. Both visual and inductive explanations were typical choices for students working in a learning environment such as Dynamic Geometry computer software used in this study. Most students' explanations fell into partial deductive (41%) and deductive (12%). Hadas & Hershkowitz (2002) argued that the study result showed that "students tend to rely on their geometrical knowledge and use deductive strategies to explain their conclusions" (p. 55).

Analyzing students' explanations was the last stage of task analysis in Hadas & Hershkowitz's (2002) study. There were three analyses before the explanation analysis. The *epistemological* analysis explored "all possible investigation paths" that students might take during the activity. The *didactical characteristics* analysis reflected task designers' prediction on students' actions and subsequently developing "favorable conditions" to lead students toward the goal of the activity. The third one, *conjectures* analysis focused on students' initial conjectures, examining how these conjectures were altered during the study and to what extent they contradicted the findings. In the context of Resnick's (1976) broad category of information-processing analyses, Hadas & Hershkowitz's (2002) first two analyses, epistemological and didactical characteristics, are rational analyses. The other two (conjectures and explanations analyses) can be categorized as empirical analyses.

Choppin's (2006) analysis of two similar tasks from two reform curricula was purely rational. These two tasks had almost identical goals but were designed differently. Choppin (2006) compared the structure and sequence of these two task designs and claimed that one of them "explicitly elicits and builds from multiple interpretations or strategies" (p. 6), while the other one, more structured with highly constrained subtasks, would reduce "the opportunities for student engagement and interactivity" (p. 7). For task implementation, Choppin (2006) pointed out that the more structured task required less teacher expertise. However, he advised teachers to adapt such tasks for greater student interactions, which he claimed may lead to deeper understanding. To accomplish this, the necessary steps include "teachers make conjectures based on prior experience and reading of research, test out new adaptations, note how students react to them, reflect on the task design, and revise the task for the next iteration" (p. 7). Choppin (2006) concluded that teachers can improve their understandings of student thinking and of the underlying mathematics ideas and develop complex practices through this kind of task design experiments.

One year before Choppin's study, a group of researchers discussed different perspectives on task design and how it could affect student learning at the 29th PME (Conference of the International Group for the Psychology of Mathematics Education). They chose the topic "proportional reasoning" as examples in their studies. Gravemeijer, van Galen, and Keijzer (2005) introduced three task design heuristics in the Realistic Mathematics Education instructional design: *guided reinvention*, *didactical* *phenomenology*, and *emergent modeling*. Guided reinvention means that the task should guide students' experiences through a similar process as that through which the mathematics was invented. Didactical phenomenology involves real-world applications that challenged students to find situation-specific solutions. The emergent modeling involves a series of similar tasks that initially encourages students to create informal context-specific models through making drawings, diagrams, tables, or developing notations to solve the problem; then, as students have more experience with similar tasks, the student-constructed models become more object-oriented, which lay a foundation for more formal mathematical reasoning. Gravemeijer, van Galen, and Keijzer (2005) further argued that the emergent modeling heuristic should be the main focus in the instructional task design because it fosters student reasoning in a long-term learning processes.

Friedlander and Arcavi (2005), the other group of researchers at the 29th PME, talked about a particular technology-based activity designed for students to use Excel spreadsheets as a tool that enabled students to create emergent models in the sense mentioned by Gravemeijer, van Galen, and Keijzer. Through observing and interpreting results generated by the spreadsheet, students were able to see the general pattern and to make initial predictions at a very early stage, surprising the task designers.

Ainley and Pratt (2005) confirmed the benefit of using an Excel spreadsheet as a tool through their study of task design based on the aspects of *purpose* and *utility*. They pointed out that many real-world activities did not make much sense to pupils who could not get a meaningful outcome thereafter. Ainley and Pratt (2005) stated that "the *purpose* of a task is not the 'target knowledge'" but "creates the necessity for the learner to use the target knowledge in order to complete the task." Ainley and Pratt defined the *utility* of a task as "knowing how, when and why that idea is useful"; the *utility* "is seen as an intrinsic, but frequently unacknowledged, facet of the concept itself" (p. 115). In Ainley and Pratt's (2005) study, two boys showed the interplay of these two key features when they worked on the task. Recognizing the purpose of the task created needs for inventing possible utilities that led the boys to the next step of the path toward the solution. "Such exploration enabled the boys to construct meanings" (p. 118) behind mathematical calculations (i.e. utilities). On the other hand, two girls in the study asked teachers for more guidance. The girls stuck with what they were told to do and "went down a much narrower predictable pathway" (p. 119). Ainley and Pratt (2005) concluded that because "no task can offer rich pathways for all children," (p. 119) issues of task design should be considered and explored more.

De Bock, Van Dooren and Verschaffel's (2005) reported on a small-scale study that required four 11-year old students to examine and sort ten given word problems into groups. Participating students were asked to explain their reasoning for such groupings, and to provide different ways of grouping if they could. The purpose of this task was to see if learners could distinguish between proportional and non-proportional situations (that is, between word problems that could or could not be solved through the means of a proportion). For example, in the following three out of the ten given word problems, problem C is proportional while problems B and F are non-proportional.

> Problem B: Mama put 3 towels on the clothesline. After 12 hours they were dry. The neighbor put 6 towels on the clothesline. How long did it take them to dry?
>
> Problem C: Mama buys 2 trays of apples. She then has 8 apples. Grandma buys 10 trays of apples. How many apples does she have?

Problem F: Today, Bert becomes 2 years old and Lies becomes 6 years old. When Bert is 12 years old, how old will Lies be?

The study results were "rather disappointing" (p. 102), as none of the four participants came up with what the researchers considered to be the correct grouping. One student grouped B, C, and F together, another student grouped C and F together with other problems. De Bock, Van Dooren and Verschaffel (2005) claimed that participants "mainly looked for linguistic or other superficial differences between the problem formulations and not for an underlying mathematical structure." They further argued that "the word-problem format is inadequate or insufficient to meaningfully contextualize mathematics in the mathematics classroom" (p. 102).

Indeed, not many studies can be found that focus on students' understanding about underlying mathematical structure. Torkildsen (2006) is one of the few. In his dissertation study, he examined what mathematics students could or could not do when they were solving open-ended problems. He focused on uncovering the mathematical structures that were inherent in students' solutions. He examined six tasks associated with number theory and combinatorics. For each task, he first provided solutions in every possible approach, found in mathematical textbooks or papers, with mathematical structures identified formally and explicitly. This part could be classified as Resnick's (1976) rational analysis. Next, he analyzed students' solutions produced in different class sessions. Then, he discussed the mathematical structures found in students' solutions. This later part matched Resnick's (1976) description of empirical analysis.

The present dissertation study will build on Torkildsen's (2006) efforts in examining mathematical structures embedded in the process of solving mathematical tasks. The goal of the study is to see what mathematical structures can be used to solve certain task problems, and to learn how those mathematics structures are emergent when students work on the tasks.

2.2 Mathematical Structure

The word ***structure*** means "something that is constructed or organized such that its parts are dominated by the general character of the whole" (Merriam-Webster, 2008). The term ***mathematical structure*** was not used by mathematicians until 1930 (Corry, 1990), and the gradual development of the concept to which it refers changed how mathematics was perceived by mathematicians (Corry, 1990, 1992, 2001). The birth of Van der Waerden's textbook *Modern Algebra* (1970) significantly changed the content of classical algebra and led to "a new consensus as to what algebra as a discipline henceforth would be" (Reed, 2000, p. 182). To examine what is meant by mathematical structure, the next two subsections will (1) review its definition and historical evolution, and (2) consider its implication for teaching and learning.

2.2.1. Historical Evolution

Before discussing the evolution of mathematical structure, Corry (1990, 2001) defined and distinguished between two terms he used to describe the knowledge of a scientific discipline:

- **Body of knowledge:** theories, facts, methods, and claims that address questions directly related to the subject matter.
- **Images of knowledge:** claims that address questions which express knowledge about that discipline such as: What is the legitimate methodology of the discipline? What is a good theory? What is the most efficient technique to solve a certain kind of problem in the discipline? What is the relationship among entities in the discipline? What are the burning issues of the discipline?

According to Corry, the idea of a mathematical structure is “a classical example of an *image of mathematics*” (Corry, 2001, p. 1), which had the potential to cause real changes in the *body of mathematics*. This is because of the unusual characteristics of the mathematics as a discipline:

> In most scientific disciplines, facts and theories are continually added to and deleted from the body of scientific knowledge, while the images of knowledge are affected by these and by a wide variety of other factors. But in contrast, claims that enter the body of mathematics through proof are seldom if ever rejected. As a rule, new theorems and new proofs of old theorems do not falsified old theorems and proofs. Still, the process of mathematical change is not one of linear accumulation. (Corry, 1990, p. 383)

Corry continued to describe the nature of non-linear change in mathematics:

> It is the images of knowledge (which are determined by social and philosophical factors, by the interaction with other sciences, etc.) that determine the way in which a new item will be integrated to the existing picture of knowledge; Eventual changes in the images of knowledge may later transform the status of existing pieces of knowledge and produce a different overall picture of mathematics. (Corry, 1990, p. 383)

Van der Waerden

This image of mathematics appeared for the first time in *van der Waerden*’s *Modern Algebra* (Corry, 2001; Dold-Samplonius, 1997; Mac Lane, 1997; Reed, 2000) which was considered to be “the watershed event in the rise of mathematical structure” and to have significantly “changed the content of algebra” (Reed, 2000, p. 182). According to Mac Lane (1997) and Corry (1990, 2001), this book consolidated algebraic research and resulting theories from various mathematical branches developed in the previous two decades. The ideas presented in the book such as *groups*, *fields*, *ideals*, and *rings* had been influenced by the work of Emmy Noether, Artin, Gottingen, Hamburg, Ernst Steinitz, and others. For example, the idea of *groups* was included in the mainstream textbook *Serret’s Cours* in 1866. The *fields* and *ideals* were introduced by Dedekind in 1871. Additionally, David Hilbert “pioneered in finding connections (i.e.

structural properties) between different area of mathematics, drawing upon ideas from separate and not obviously related disciplines" (Reed, 2000, p. 184). Also, Hilbert's postulate analysis inspired an axiomatic approach to mathematical theories (i.e. mathematical statements are logically derived from self-evident axioms). However, Hilbert and his students neither carried out this structural image of algebra "in its completed form, nor suggested that it should be adopted in algebra" (Corry, 2001, p.16).

It was van der Waerden who attempted to adopt a unified and systematic way to define and study each and all of the algebraic branches and "attempted to fully elucidate their structure" (Corry, 1990, p.386). The attempt to define structure was informal. Van der Waerden never explicitly asked questions such as: What is it meant by structure? What must be known of a certain algebraic domain in order to claim that its structure is known? But the discussion of the various algebraic domains in his book implicitly provided "an account of the essence of the structural image of algebra" (Corry, 2001, p.6); and the *modern axiomatic method* formulated in the book was considered "the most essential feature of the *structural approach*" (p.9).

In reviewing Corry's book *Modern Algebra and the Rise of Mathematical Structures*, Reed (2000) pointed out that after van der Waerden, there were three early attempts of defining the notion of structure formally. They are: (1) Oystein Ore's *Lattice-theoretic approach*, (2) Nicolas Bourbaki's *Theory of structures*, and (3) Samuel Eilenberg and Saunders Mac Lane's *Category theory*.

<u>*Oystein Ore*</u>

The concept of **lattice** was first studied by Richard Dedekind and Ernst Schröder. In 1930, Oystein Ore, a Norwegian mathematician, collaborated with Emmy Noether to edit and publish the *Collected Works of Richard Dedekind*. Since then, Ore's research interest had turned to *lattice theory* (Roman, 2008, p. vii; O'Connor & Robertson, 2005). He started a research program at Yale in 1935 with the goal "to develop a general foundation for all abstract algebra based on the notion of *lattice*, which he denoted with the term **Structure**" (Corry, 2004, p. 259). Ore believed that since many theorems "had recurrently appeared in different algebraic domains," there must be "a single, general concept from which equivalent theorems could be derived simultaneously valid in all those domains" (p. 268). He suggested that, in studying the *structure* of the algebraic domain, one should overlook the elements and operations of these domains and focus on the relationships among certain distinguished sub-domains. Ore's own work on *lattice* led him to the study of *equivalence* and *closure relations*, *Galois connections*, and finally to *graph theory* (O'Connor & Robertson, 2005). However, Ore's project only focused on a limited framework of abstract algebra. For this reason it was not given much attention among researchers and soon the spotlight moved to Nicolas Bourbaki's concept of *structure* that incorporated a much larger domain of pure mathematics.

<u>*Nicolas Bourbaki*</u>

Bourbaki's name is inseparable from the notion of *structure* in modern mathematics (Cartan, 1980; Corry, 2001). Bourbaki extended van der Waerden's achievement "in presenting the whole of algebra as a hierarchy of *structures*" and "succeeded in presenting much larger portions of mathematics in a similar way" (Corry, 2004, p. 306). In the book *Elements of Mathematics: Theory of Sets*, Bourbaki (1970) explained that the *axiomatic method* is nothing but a systematic instrument to write and read mathematical text unambiguously with rules of syntax of a formalized language so that once a general theorem has been established it may be applied to many different contexts with different meanings attached to the words and symbols of the theorem (p.8). In regard to the concept of *structure*, Bourbaki described it through the lens of the axiomatic method as follows:

> The axiomatic method allows us, when we are concerned with complex mathematical objects, to separate their properties and regroup them around a small number of concepts: that is to say, using a word which will receive a precise definition later, to classify them according to the *structures* to which they belong. (Of course, the same *structure* can arise in connection with various different mathematical objects.) For example, some of the properties of the sphere are topological, others are algebraic, others again can be considered as belonging to differential geometry or the theory of Lie groups. (Bourbaki, 1970, p. 9)

In the article *The Architecture of Mathematics*, Bourbaki (1950) had identified three **mother-structures**: *order* structures, *algebraic* structures, and *topological* structures, which Bourbaki claimed were the nucleus of the universe of mathematics. Within each mother-structure there is a most general structure with the smallest number of axioms. By adding supplementary axioms, this most general structure is enriched and new subsequent structures are defined, which possess particular characteristics other than the original ones. In the case of order structures, an *ordered set* can be either *partially ordered* or *totally ordered*. In general, the notation $x\mathrm{R}y$ is used to represent a relation R between two elements x and y. A partial ordering has to satisfy three axioms: (1) *Reflexivity*: for every element x, $x\mathrm{R}x$; (2) *Antisymmetry*: if $x\mathrm{R}y$ and $y\mathrm{R}x$, then $x = y$; and (3) *Transitivity*: if $x\mathrm{R}y$ and $y\mathrm{R}z$, then $x\mathrm{R}z$. Besides these three axioms, a total ordering has to satisfy an additional axiom - *Comparability*: for every pair of elements x and y, either $x\mathrm{R}y$ or $y\mathrm{R}x$ holds. If every subset of a totally ordered set has a "least" element, then the set is said to be *well-ordered* (Bourbaki, 1950, p.226-229; Artin, 1991, p.588; Hungerford, 1997, p.420-421). Thus, a vertical hierarchy is constructed: ordered set →

partially ordered set → totally ordered set → well-ordered set. Under this structure, every well-ordered set is totally ordered and every totally ordered set is partially ordered. However, the converse statements are not true.

To illustrate through particular examples, the set of integers under the relation "divides" is partially ordered. Here, "*x* divides *y*" means "*x* is a divisor of *y*" or "*x* is a factor of *y*" (Hungerford, 1997, p. 7). The set is partially ordered because (1) for every integer x, x divides x; (2) if x divides y and y divides x, then $x = y$; and (3) if x divides y and y divides z, then x divides z. However, this set is not totally ordered because, for instance, 3 and 5 are not comparable (i.e., neither "3 divides 5" nor "5 divides 3" holds). On the other hand, the set of real numbers under the relation "≤" is totally ordered but not well-ordered because many subsets of the real numbers do not have a least element; the set of positive real numbers is such a subset. Nevertheless, the set of positive integers with the relation "≤" is a well-ordered set because each of its subsets has a least element.

A hierarchy can also be constructed horizontally. Any two or more of these mother-structures and constructed-structures might be "combined organically by one or more axioms which set up a connection between them" to form another layer called **multiple structures** (Bourbaki, 1950, p. 229). For examples, the theory of *divisibility* that determines whether a number is divisible by other numbers is constructed by combining order structures and algebraic structures; the theory of *integration* (the act or operations of finding integrals) is constructed by combining order structures, algebraic structures, and topological structures. Bourbaki used the phase "combined organically" (Bourbaki, 1950, p. 229) because most mathematical objects were developed independently from one another. Structural combinations were not explicitly examined until Bourbaki tried to organize all mathematical objects into a large hierarchical system.

Bourbaki claimed that the construction of *multiple structures* could keep going until reaching **particular theories**, in which the structures were individually characterized. Thus, the hierarchy of *mathematical structures* was "going from the simple to the complex, from the general to the particular" (Bourbaki, 1950, p. 228). This sketch, Bourbaki continued, was a "very rough approximation of the actual state of mathematics" and was "schematic, idealized as well as frozen (i.e. static)" (Bourbaki, 1950, p. 229). Bourbaki admitted that his *structures* were "not immutable" and expected that "future development of mathematics may increase the number of fundamental structures, revealing the fruitfulness of new axioms or of new combinations of axioms" (Bourbaki, 1950, p. 230).

Bourbaki was later criticized for abandoning the formally defined structures outlined in his *Theory of Sets* (Corry, 2001) and using only informal *ad hoc* definitions of basic concepts in the subsequent volumes of his book *Eléments* (Bourbaki, 1970). Corry pointed out that, in *Eléments*, "the only theorems proven in terms of *structures* are the most immediate ones, such as the first and second theorems of *isomorphism*" (Corry, 2004, p. 323). Corry further pointed that "no new theorem is obtained through the structural approach" (Corry, 2004, p. 324) and that many of Bourbaki's assertions "belong strictly to Bourbaki's non-formal images of mathematics" (Corry, 2004, p. 334). Nevertheless, Bourbaki's concepts of mother structures and structural schemes were commonly accepted among mathematicians and researches, "implicitly or explicitly, as results obtained in the framework of a standard mathematical discipline", which "was a main force in shaping mathematical activity all over the world for several decades after its emergence" (Corry, 2001). One such outgrowth is the current research in **Model theory** (Corry, 2004, p. 330, footnote 90). In Corry's opinion:

> Neither Bourbaki's theory nor any similar attempt have succeeded to the present day in fully elucidating the idea of mathematical structure. However, since mathematical knowledge is usually granted special status of unquestioned certainty, beyond that of other forms of human knowledge, the very existence of formal theories dealing with structures has often been interpreted at face-value, by mathematicians and non-mathematics alike, as though this concept would be unequivocally understood within mathematics (Corry, 1990, p. 388).

Samuel Eilenberg and Saunders Mac Lane

Saunders Mac Lane had studied algebra at Yale University in 1929-30 under Oystein Ore's supervision (Corry, 2004, p. 349; Mac Lane, 2005, p. 32). In 1943, Mac Lane started working on homological algebra with Samuel Eilenberg (Corry, 2004, p. 339; Mac Lane, 2005, p. 73, 98, 100) who was a Bourbaki member (Corry, 2004, p. 255). To answer certain questions in homology theory, Mac Lane and Eilenberg introduced **category theory** in 1945, which is "the general theory of Natural Equivalences" (Mac Lane, 2005, p. 209; Reed, 2000). A **category** is a fundamental and abstract way to describe mathematical entities and their relationships (Marquis, 2007). Ore proposed to concentrate study on the relationship between a given algebraic domain and its sub-domains in order to understand the essence of its structural nature (Corry, 2004, p. 348). This approach was passed down toward the study of *category theory*, which emphasizes the study of *morphisms* (*structure-preserving mappings* or simply *functions*), between mathematical objects (Marquis, 2007).

On the other hand, Bourbaki's study was primarily focused on the structures of algebraic objects. There was "a debate within the Bourbaki group over whether *categories* should play an explicit role in the *Eléments*." Unfortunately, Bourbaki "never really acknowledged the greater power of *category theory* in formalizing one of the most important aspects of the structural approach, namely its exploitation of *mappings* between structures to reveal their properties" (Reed, 2000, p. 189).

Mac Lane is well-known for his work on the *coherence theorem* and the use of *diagram* (McLarty, 2005). In category theory, a *diagram* (*graph*) consists of *objects* (*structures*, *vertices*) and *morphisms* (*mappings*, *functions*, *arrows*, *edges*). It is believed that this diagrammatic approach had a great impact on contemporary mathematics. The *coherence theorem* describes a condition when two or more *morphisms* between two given *objects* are equal. The following figure 2.2-1 illustrates a generic *commutative diagram* that shows a *coherence condition* existing between objects A and D because $h \circ f = k \circ g$ by function compositions.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

Figure 2.2-1. *Commutative Diagram showing coherence condition between A and D*

The category of **Sets** is the most basic and commonly used category, in which the *objects* are sets, the *morphisms* are all functions between sets, and *composition* is the usual function composition. Nowadays, *categories* can be found in most branches of mathematics, in some areas of theoretical computer science where they are called *types*, and in mathematical physics where they are used to describe *vector spaces*. *Category* *theory* provides a formal way to unify notion and terminology for these disciplines (Marquis, 2007; McLarty, 2005).

Reed (2000) argued that most mathematicians "would probably agree that *category theory* has a decided advantage over" Ore's and Bourbaki's attempt to define mathematical structures formally. However, "Corry does not seem to feel that any formal definition of structure could do justice to the use of the concept of structures in actual mathematical practice" (Reed, 2000, p. 189). Corry (2004) pointed out that Mac Lane once claimed, "The mathematical reality is much more varied than category theory can exhaust" (p. 368). Therefore, Corry (2004) reminded us that Ore's *Lattice theory* "played an important role in the consolidation of the images of mathematics" (p. 353) and Bourbaki's influence on changing the image of mathematics cannot be contested although his formalizing of a structural approach based on an axiomatic method was "forced and unnatural" (p. 330).

Many aspects of the structural approach can hardly be defined in formal terms (Corry, 1990). For the same reason, neither *lattice theory* nor *category theory* "succeeded in exhaustively encompassing the meaning of the non-formal idea of mathematical structure" (p. 386). The meaning of mathematical structure "significantly varies from author to author and even among different texts of the same author" (p. 387), which is important to keep in mind as the discussion of mathematical structures and their implications for the teaching and learning of mathematics are discussed in the next section.

2.2.2 In Learning and Teaching

As the focus on structures changed the image of algebra in mathematics discipline, school mathematics also shifted its focal points. “Most of the arithmetic and algebraic procedures long viewed as the heart of the school mathematics curriculum can now be performed with handheld calculators. Thus, more attention can be given to understanding the number concepts and the modeling procedures used in solving problems” (NCTM, 2000, p. 20). The New Jersey Mathematics Core Curriculum Content Standards (NJMCCCS, 2008) contain similar statements, stressing that students should learn important mathematical concepts rather than simply memorize and practice procedures. As new technology becomes available, less time is required for lengthy computational processes, and “more effort should be devoted to the development of number sense, spatial sense, and estimation skills” (p. 2).

How can one best learn about mathematical concepts? Constructivism argues that mathematical concepts *cannot* be passed directly from teacher to students, but that with the teacher’s help, students themselves must construct concepts and build concept images. Therefore, Problem Solving was recommended by many mathematics educators as one of the process standards. It appears that there is a conflict between problem-based instruction and the axiomatic approach discussed in the previous section. Teachers and students may experience difficulties in attempting to extract formal mathematical definitions and theorems from problem solving activities. This makes the construct of mathematical structures very important because when students start glimpsing the relationships across mathematical content areas, they can view and learn mathematics as an integrated whole (NCTM, 2000, p. 355).

In the following, the discussion of mathematical structures will be organized under four subsections: curriculum reform, teacher knowledge, impact on student learning, and structure sense.

Curriculum Reform

One of the reasons that students may struggle to learn mathematics is because "the curriculum offered does not engage them" (NCTM, 2000, p. 5). Many students feel the traditional curriculum is boring and useless because they can rarely make connections between mathematics and the real world. Additionally, some students think they are lacking innate ability to be successful in studying mathematics (NJMCCCS, 2008, p. 1). To improve the quality of learning, the mathematical curriculum reform started its long journey as early as the 1950's (Kilpatrick, 2009). Many reform proposals, projects, discussions and debates have been carried on continuously over the past three decades.

Howard Fehr's Unified Mathematics Curriculum was part of these reforms efforts. According to Fehr (1976), the study of algebra had evolved from the nineteenth century. By 1910, modern algebra was defined as the study of structures. These algebraic structures had gradually become "a unifying thread which has extended into all branches of mathematics" (p. 26) and created "a need to reorganize the mathematics of the traditionally separated branches into a unified single study" (p.4). Fehr (1976) reported that the unified mathematics curricula made significant progress from 1950 to 1965. The four novel concepts of *sets*, *relations*, *functions*, and *operations* had become common fundamental topics for all mathematics branches.

Robert B. Davis also contributed significantly to the mathematics curriculum reform. According to Maher (1998), Davis' Madison Project began at the Madison

School in Syracuse, New York. There were “over 20,000 teachers in New York, 1,800 teachers in Chicago, and large groups of teachers in other American cities involved in this project” (Mayansky, 2007, p. ix). In the 1950s and 19060s, Davis led participating teachers and researchers to create the Madison project curriculum that covered subject matters such as elementary algebra, functions, and graphs, with a concern for axiomatic development. (Davis, 1980). To study students’ mathematical thinking, Davis (1984) used the term KRS (Knowledge Representation Structure) to represent the knowledge stored in children’s memory and which was retrievable from the memory. Davis argued that these KRSs helped learners to recognize patterns by “matching up input information with an appropriate previously-created representation structure” (Davis, 1984, p.125), which could lead to the creation of a new KRS. Thus, ways to enlarge the database of KRSs had been considered during the development of the Madison Project curriculum.

The most current reform efforts that originated in the 1980s resulted in the *Principles and Standards for School Mathematics* published by National Council of Teachers of Mathematics (NCTM, 2000). In this latest version, the goal of reform is to achieve equity and excellence in student learning (p. 5) by means of implementing “a coherent curriculum” that “effectively organizes and integrates important mathematical ideas so that students can see how the ideas build on, or connect with, other ideas, thus enabling them to develop new understandings and skills” (p. 15).

The construct of *structure* is not directly addressed in the NCTM standards or the *New Jersey Mathematics Core Curriculum Content Standards* (NJMCCCS, 2008). However, the idea of *mathematical structures* is relevant to many aspects of the curriculum standards. In NCTM 2000, mathematics is described as “a discipline that is highly interconnected" (p. 30). The Standards are organized into two big categories, *Process Standards* and *Content Standards*, which are "overlapped and integrated" (p. 30). The *Content Standards* cover five areas in mathematics: number and operations, algebra, geometry, measurement, and data analysis and probability. *Process Standards* address the processes of problem solving, reasoning and proof, connections, communication, and representation (p. 7). The connection between these two set of standards is so rich that "processes can be learned within the Content Standards, and content can be learned within the Process Standards" (p. 30).

The intention of the NTCM Standards is clear. If connections can be made between mathematical entities, between mathematical content and processes, and between mathematics and other disciplines, then students can learn mathematics as a coherent whole rather than disconnected individual pieces. Such an approach to learning has the potential to engage students more successfully. However, to implement mathematics reform is not all that simple. "Sometimes the changes made in the name of standards have been superficial or incomplete" (NCTM, 2000, p. 5). The challenges are complex. Being in the position to bring new curricula to life, mathematics teachers are usually held responsible for the success of the implementation of mathematical reform.

Responding to reform efforts, many teachers incorporated more problem based activities in their instructions. Students were often asked to work on designated tasks in groups. More task time was allocated. Pre- and post- discussions were conducted. However, in spite of all these efforts, authentic mathematical discourse did not always take place. Bonnie, a 7^{th} grade mathematics teacher who participated in Manouchehri and Goodman's (2000) study, stated that her students were initially excited to be doing mathematical activities but lost interest later on because the problem solving program seemed to drag everyone in different directions and was going nowhere. Bonnie claimed that her students preferred more structured instructions like those presented in the traditional curriculum:

> They wanted structure -- They kept asking why they were doing these things -- when they were going to do what they were supposed to do -- They did not see what we were doing as doing mathematics ……I felt my instruction was disjointed -- I was doing different things with different kids and this was not good for them -- Some of them felt that if they worked on the new curriculum it meant that they could not do the regular math stuff and those that wanted more traditional stuff would not working with others. (Manouchehri & Goodman, 2000, p. 8)

Manouchehri and Goodman (2000) had carefully observed Bonnie's classes as well as the classes of the other teacher (Gina). They found that the richness of the teacher's content knowledge had a great impact on how problem based instruction was conducted. Bonnie's lack of ability to recognize the connections among mathematical activities and mathematical structures directly contributed to the unsuccessful outcome of her problem solving sessions. Bonnie was somewhat puzzled when several students came up with unexpected solutions, and she was not able to answer students' questions promptly and confidently. The questions were as simple as "Do you see what I have done?" or "Why can all these answers be right?" or "Which answers should I write down?" (p. 18).

In the post-observation interview, Bonnie admitted that "she did not really know what the problem was all about when she casually assigned it to groups," but she felt "happy that she had managed to engage nearly all students in discussion" sharing their thoughts (Manouchehri and Goodman, 2000, p. 18). Nevertheless, the discomfort situation caused by insufficient understanding of the mathematics involved prevented Bonnie from using the activity again: "I am not going to do this with my next class though -- It takes too much of the class time and I think I should get through my lesson" (p. 18).

Bonnie's reflection was understandable. It is typical that teachers complain about time constraints if they see mathematical activities as separated pieces departing from the mathematical topics they are supposed to cover. Additionally, teachers find themselves in a more vulnerable position when engaging students in open-ended mathematical activities, as they need to be able to handle student comments or questions that they may not necessarily foresee at the beginning of the activity. However, "teaching is itself a problem-solving activity" (NCTM, 2000, p. 341). With adequate professional development, teachers may become good problem solvers in mathematics as well as in teaching mathematics. There is consensus among researchers that the focus of this professional training should be enriching teacher knowledge (Anders, 1995; Darline-Hammond, 2002; Ma, 1999; Manouchehri & Goodman, 2000; Mullens and Murnane, 1996; NCTM, 2000, p.17; Shulman, 1986, 1987; Wilson, Fernandez, and Hadaway, 1993). The following is a brief review of this body of research.

Teacher Knowledge

According to Wilson, Fernandez, and Hadaway (1993), "To become a good problem solver in mathematics, one must develop a base of mathematics knowledge. How effective one is in organizing that knowledge also contributes to successful problem solving" (p. 6). In studying problem solvers' perception of specific mathematics problems and their knowledge structure, Schoenfeld and Herrmann (1982) found that novices tended to attend to surface features of the problems such as the specific words or objects described in the problem statement, whereas experts possessed more richly structured knowledge bases in which relationships among parts were structurally built. These findings reinforce the importance of exploring mathematical structures inherent in the process of a problem-based activity.

Shulman (1986) suggested that teachers needed to possess three kinds of knowledge, including *subject matter content knowledge*, *pedagogical content knowledge*, and *curricular knowledge*. He proposed that these three kinds of knowledge could be organized and represented in forms of propositions, cases, and strategies. Later, Shulman (1987) added four more to the previous list of types of teacher knowledge -- *general pedagogical knowledge*, *knowledge of learners*, *knowledge of educational contexts*, and *knowledge of educational ends, purposes, and values*. He claimed that an effective teacher requires a knowledge base that contains these seven types of knowledge.

Examples of these categories of knowledge can also be found in Anders' (1995) study. A teacher's classroom knowledge was analyzed "based on the proposal that knowledge is stored in a schematic structure called a *script* " (Anders, 1995, p.311). The *curriculum script* was defined as the combination of Shulman's subject matter content knowledge and curricula knowledge; the *classroom script* was equivalent to Shulman's knowledge of learners and knowledge of educational context. Anders (1995) found that every teacher possessed a unique classroom script that helped him/her in planning, predicting, interpreting, and reflecting on his/her teaching practices. Three points were drawn as conclusion: (1) teachers' content knowledge is embedded in classroom events involving students; (2) the components of a teacher's script are interconnected and significantly affect each other; and (3) a teacher's script is the effect of the teacher's previous experiences with students on current events.

Ma's (1999) study focused on teachers' subject matter knowledge; nevertheless, she could not avoid talking about teachers' pedagogical content knowledge. She emphasized the importance of longitudinal coherence throughout a mathematics curriculum. She found that, when preparing to teach a math topic, Chinese teachers developed a *knowledge package* that is "a network of procedural and conceptual topics supporting or supported by the learning of the target topic" (p. 124). These well-developed, interconnected knowledge packages enable teachers to form a solid network that is supported by the structure of the subject. Ma (1995) introduced the term *Profound Understanding of Fundamental Mathematics* (*PUFM*) in arguing that the Chinese teachers' understanding of the content is broad, deep, and thorough. She claimed the teaching of a teacher with PUFM has four properties: (1) making connections among mathematical concepts and procedures, (2) using multiple approaches to solving a problem, (3) Re-visiting and reinforcing basic ideas, and (4) Longitudinal coherence of the whole elementary mathematics curriculum. Here, the first and the fourth properties are directly related to mathematical structures.

"Effective teaching requires knowing and understanding mathematics, students as learners, and pedagogical strategies" (NTCM, 2000, p. 17). Research has shown that teacher knowledge can help teachers to make curricular judgments, select and interpret textbook contents, answer students' questions, and make decisions and take actions in the classroom, all of which have a considerable impact on student learning.

<u>Impact on Student Learning</u>

Different teachers teach differently. It is often the case that "children learn more from some teachers than from others" (Mullens, Murnane, & Willett, 1996, p. 141). In school years, all students probably had, at one point or another, great teachers who opened their eyes, polished their minds, motivated their spirits, and inspired them to learn. On the other hand, students also had some teachers who made students fall asleep, employed mostly teacher-centered instruction, scared students through homework and tests, and failed students with biased grades. What makes a student hungry for learning? Darling-Hammond (2002) found that teacher quality and student achievement were highly correlated. Teachers' subject matter knowledge had "positive influence on student outcome up to some level of basic competence in the subject" (p. 52).

Similar to the story of mathematics teacher Bonnie in Manouchehri and Goodman's (2000) study, Shulman (1987) gave an example of an English teacher Colleen whose teaching behavior varied during classroom instruction, depending on her comfort level with different topics:

> When teaching a piece of literature, Colleen performed in a highly interactive manner, drawing out student ideas about a phrase or line, accepting multiple competing interpretations as long as the student could offer a defense of the construction by reference to the text itself. Student participation was active and hearty in these sessions (Shulman, 1987).

However, Colleen was uncertain about the content of prescriptive grammar. When she taught that unit,

> Her interactive style evaporated. In its place was a highly didactic, teacher-directed, swiftly paced combination of lecture and tightly-controlled recitation…. After the session, she confessed to the observer that she had actively avoided making eye contact with one particular student in the front row because that youngster always had good questions or ideas in this particular lesson (Shulman, 1987).

As educators, what can be said to the students in Bonnie's and Colleen's classes besides 'sorry' and 'good luck'? Nevertheless, it is not realistic to expect every teacher to be all-knowing either. A more reasonable expectation is for teachers to continuously work to improve their subject matter knowledge.

Interested in the issue of teacher effectiveness, Mullens et al. (1996) conducted a two-stage quantitative study on teaching effectiveness of third grade teachers in Belize. Student learning was documented through their pre-test and post-test scores. Teaching effectiveness was analyzed with the use of three characteristics of teachers - teacher's pedagogical training, teacher's formal education, and teacher's math content knowledge (i.e. represented by teacher’s test scores on a mathematics subject exam). The study’s result suggested that *none* of these three teacher characteristics had a significant impact on student learning of *basic* concepts. However, for the learning of *advanced* concepts, the teacher’s mathematical content knowledge was critical.

One aspect of this critical content knowledge is the ability to make connections. The end products of making connections among mathematical objects are mathematical structures. In Ma’s (1999) studies, very few of the U.S. teachers indicated any significant PUFM. She related this finding to U.S. students’ unsatisfactory mathematics achievement in the International Mathematics Studies. She argued that most U.S. teachers focused more on procedures and algorithms, rather than on conceptual understanding, particularly in division by fractions. She claimed that the U.S. teachers’ subject matter knowledge was fragmented and this fragmentation was an effect of the fragmentation of the mathematics curriculum and mathematics teaching in U.S, which directly affects student learning (p. 144-146). Although mathematical structure was not explicitly mentioned, Ma’s assertion implied that finding a way to organize these fragmented pieces of knowledge was necessary for success in mathematics education.

Mulligan, Mitchelmore, and Prescott’s (2006) study provided evidence that “early mathematics achievement is strongly linked with the child’s development of mathematical structure" (p. 210). Theoretically, there are four stages of structural development: *pre-structural*, *emergent*, *partial*, and *structural*. However, not every child's structural sense would be fully developed. Mulligan et al. found that "children at a pre-structural level may not necessarily progress to an emergent stage because they do not perceive some structural features with which to construct new ideas" (p. 214). This finding supported their initial hypothesis that "the more that a child's internal representational system has developed structurally, the more coherent, well-organized, and stable in its structural aspects will be their external representations, and the more mathematically competent the child will be" (p. 214). They also found that "young students can be taught to seek and recognize mathematical patterns and structures" (pp. 214-215).

To further explore student learning, some researchers looked into students' structural sense in learning certain mathematical topics.

Structure Sense

According to Hoch and Dreyfus (2004), *structure* in mathematics is defined as the system of relationships between the component parts of a mathematical entity. *Algebraic structures*, defined in terms of *shape* and *order*, can be represented by algebraic expressions or sentences. The *shape* is the external appearance of the structures. For example, the *structure* of quadratic equations has the standard *shape* $ax^2 + bx + c = 0$, where a, b, and c are real numbers. The internal *order* is "determined by the connections between the quantities and operations that are the component parts of the structure" (p. 50). The internal order may be revealed while transforming an equation into the standard form.

On the other hand, **structure sense**, first introduced by Linchevski and Livneh (1999) in describing students' understanding of structural notions in arithmetic, was described, by Hoch and Dreyfus (2004), as a collection of abilities such as the ability to see a mathematical entity, the ability to recognize a previously known structure, the ability to divide an entity into sub-structures, the ability to know which manipulations to perform next, the ability to make connection among structures, etc. There may be different interpretations of the same structures. For example, the expression $(3x-1)(2x+5)$ is the product of two linear factors, which is equivalent to the quadratic expression $6x^2+13x-5$. Students will need to possess strong algebraic structure sense in order to recognize that the first expression can be transformed into the same quadratic structure as the second one.

Novotná and Hoch (2008) refined the definition of *structure sense* which was considered "to be an extension of *symbol sense*, which is an extension of *number sense*" (p. 94). They provided the following examples to illustrate the relationship between a mathematical structure and students' structure sense about that structure.

Structure of Difference of Squares: a^2-b^2

Structure sense 1: Recognize that x^2-81 is a structure of difference of square.

Structure sense 2: Recognize that $(x-1)^2-(x+3)^4$ is a structure of difference of square, where $(x-1)$ and $(x+3)^2$ act as the single entities *a* and *b* in the described structure.

Structure sense 3: Recognize that $24x^6y^4 - 150z^8$ has the possibility to be a structure of difference of squares. By factoring, the expression is equal to $6[(2x^3y^2)^2 - (5z^4)^2]$, where $(2x^3y^2)$ and $(5z^4)$ act as the single entities *a* and *b* in the original structure.

Novotná and Hoch (2008) declared that structure sense in High School Algebra was based on the use of symbols, while structure sense in college-level Algebra was based on formal definitions and proof. They suggested that more emphasis placed on algebraic structure would help students make the transition from high school mathematics to university mathematics. They continued to argue that teachers' unawareness of conceptual structure would lead to teaching mathematics by emphasizing memorization, and that would cause students to struggle with problem solving.

The research discussed above indicates that learning individual mathematical ideas is no longer considered enough. Making connections is emphasized as a curriculum standard because through mathematical structures teachers and students alike can effectively organize isolated mathematical objects into an easily-accessible system of mathematics. Although the research reviewed uses the term "structures" in several different contexts related to student reasoning, this study is specifically concerned with structure in mathematics itself. The next section discusses the theoretical framework for this study.

Chapter 3: Theoretical Framework

The theoretical framework for the proposed study consists of two main categories of research literature. The first category informs the nature of conceptual and procedural knowledge and the second category describes how to use concept mapping in organizing these two types of knowledge. The following sections discuss these two categories in detail.

3.1 Conceptual and Procedural Knowledge

In recent reform efforts, a great deal of attention is put on learning mathematics with understanding because "conceptual understanding is an important component of proficiency" (NCTM, 2000, p. 20). The distinction between procedural and conceptual knowledge needs to be noted here. *Procedural knowledge* refers to the knowledge of formal language or symbolic representations (i.e., simple mathematical objects or structures), of rules, algorithms, and procedures. *Conceptual knowledge* is the knowledge rich in relationships and understanding (i.e. multiple structures). It is a connected network in which the linking relationships are as conspicuous as its discrete parts of information (Haapasalo, 2003; Rittle-Johson, Kalchman, Czarnocha, & Baker, 2002). To gain conceptual understanding, students should, under the teacher's guidance, be "actively building new knowledge from experience and prior knowledge." (NCTM, 2000, p. 20) Researchers are interested in further exploring the relationship between procedural and conceptual knowledge.

Tall (2007) described the long-term development of mathematical thinking as the mind operating in three distinct "worlds of mathematics": (1) *conceptual embodiment*, (2) *proceptual* (i.e. process and conceptual) *symbolism*, and (3) *axiomatic formalism*. These three worlds are "intertwined" and rooted in "the use of language to *compress* a complex phenomenon into a *thinkable concept* whose meaning can be refined by experience and discussion and connected to other thinkable concepts in rich cognitive schemas" (p. 1-2). The first world, *conceptual embodiment* is "based on perception of and reflection on properties of objects" (p. 2). For example, a triangle is conceptually embodied as a figure consisting of three line-segments. *Proceptual symbolism* uses symbols flexibly as thinkable concepts in a process to be carried out. For example, $m\angle A + m\angle B$ represents the sum of the measurement of an angle with vertex *A* and the measurement of an angle with vertex *B*. Third, *Axiomatic formalism* formalizes concepts based on formal definitions and proofs. For example, in any ΔABC, $m\angle A + m\angle B + m\angle C = 180^\circ$.

Tall (2007) continued to argue that by shifting among these three worlds an individual's mathematical thinking may become more sophisticated and mature. In embodiment and symbolism, definitions are constructed on known concepts that "act as a foundation for ideas that are formalized in the formal-axiomatic world" (p. 16) and the formally defined axiomatic structure leads to more sophisticated embodiment and symbolism. Tall (2007) also made the following points:

- The sophisticated level of symbolic compression goes from *pre-procedure*, *procedure* (step-by-step action), *multi-procedure*, *process* (flexible alternative choice of actions), to *procept* (process and thinkable concept).
- When the focus shifts from the *steps* of a procedure to the *effect* of the procedure, the sophistication level upgrades from procedure to process.

- In the formalism world, a schema may be described as a list of axioms which can be an object in a higher theoretical framework.
- Naming a schema (i.e. "the use of language" (p. 1)) may compress the schema into a thinkable concept.

Unfortunately, not every child can successfully make the transition between embodiment and symbolism, which explains why some students have difficulties with the shift from arithmetic to algebra. Therefore, symbolizing mathematical concepts and further formalizing them are important, necessary steps towards bringing one's mathematical thinking to more sophisticated levels (Tall, 2007). This implies that procedural knowledge may enhance conceptual understanding if it is learned properly. Most mathematics educators probably will not agree on learning procedures solely through memorization. A more efficient tool is needed to help learners organize and reflect on what they already know because one's prior knowledge plays an important role in conceptual understanding (Hasemann & Mansfield, 1995; Novak & Cañas, 2008). To this end, the method of *concept mapping* is recommended by many educators and researchers (Brinkmann, 2005; Hasemann & Mansfield, 1995; Leou & Liu, 2004; Mwakapenda, 2003; Novak & Cañas, 2008; Woolfolk, 2001).

3.2 Concept Mapping

In building a mathematical structure, Brinkmann (2005) recommends the use of *knowledge maps* that include *mind maps* and *concept maps* to represent mathematical knowledge in the form of a graphical network. *Mind mapping* refers to "the natural thinking process, which goes on randomly and in a nonlinear way" (p. 4). A *mind map* has an open structure in which the topic is placed in the center and related main ideas are lines branching out from the topic and denoted by keywords written on the lines. Then secondary ideas can be added to the main branches and so on. Although the principle is: "from the abstract to the concrete, from the general to the special" (p. 2), every idea should flow freely and be added without any mental effort.

On the other hand, a *concept map* is in a top-down hierarchical form in which knowledge is organized into categories and sub-categories so that retrieval is easy. (Brinkmann, 2005). *Concept mapping* was first introduced by Joseph D. Novak as a research tool for science education in the 1970s (Hasemann & Mansfield, 1995). This technique can be used to monitor students' conceptual understanding and to trace students' difficulties with specific concepts over a long period of time. As children's cognitive development and mathematical knowledge improve, student-constructed concept maps can change remarkably. Hasemann & Mansfield (1995) used four characteristics to evaluate concept maps: (1) *context-oriented* - concepts and figures from real-life situations are grouped, (2) *domain-oriented* - concepts and figures from the mathematical realm are grouped, (3) the degree of *structure* - the number of relationships shown, and (4) the *reference to actions* - to what extent students indicate there is something to perform. Hasemann & Mansfield (1995) found that most fourth graders' concept maps were *context-oriented* while sixth graders' concept maps were mostly *domain-oriented*. This orientation change in concept maps "might be regarded as an effect of their cognitive development" and it might be caused by "a certain kind of teaching" in which "older students are expected to master concepts, rules and procedures" and "informal procedures are banned" (pp. 67-68).

McGowen and Tall (1999) were also interested in how students' concept maps may change over time. In their study, college participants were asked to build and extend their concept maps for the concept of function at five-week intervals during a sixteen-week algebra course. In analyzing and comparing students' successive maps, McGowen and Tall (1999) used *schematic diagrams* to highlight items (i.e. concepts) from the previous concept map, items moved to other positions, and new items. They found that the lower achievers' sequence of maps "revealed few stable items" and that "no basic structure was retained throughout" (p. 7); on the other hand, higher achievers were able to relate new concepts to previous knowledge so that their consecutive maps retained a basic structure that "gradually increased in complexity and richness" (p. 1).

The concept mapping approach not only facilitates meaningful learning but also helps teachers in designing their instruction. Ma (1999) described that most Chinese teachers were concerned with how to organize pieces of knowledge into a *knowledge package* of related ideas, which is somewhat similar to concept mapping. The Chinese teachers had different opinions on which and how many knowledge pieces should be included in a particular package, but they all agreed that the teacher should be aware of a knowledge package while teaching a piece of it. Teaching will be much more effective if the teacher knows the role of the present piece in the package, and knows the relationship between the present piece and other ideas or procedures in the package (p. 18).

Leou & Liu's (2004) case study of an experienced junior-high school teacher further confirmed the benefit of using concept mapping in professional development. The participating teacher's belief about mathematics teaching has been significantly changed after she integrated concept mapping strategy in her geometry class for five months. The teacher gained a higher level of self-confidence, moved away from citing mathematical procedures and rules, and became a facilitator who inspired students to conduct meaningful learning.

This does not mean that direct teaching is worthless. According to Novak and Gowin (1984), "both direct presentation and discovery teaching methods can lead to highly rote or highly meaningful learning by the learner, depending on the disposition of the learner and the organization of the instructional materials" (p. 4). Brinkmann (2005) also noted that "maps with a great degree of complexity seem to be rather confusing than helpful" because "a productive degree of complexity is dependent on the individual, or at least on the achievement level of a learning group" (p. 7).

In evaluating the complexity of a concept map, Novak and Gowin (1984) suggested scoring criteria based on the number of (1) meaningful and valid *propositions* (i.e. relationships represented by links and linking words), (2) valid levels of *hierarchy*, (3) meaningful *cross links*, and (4) specific events or objects shown as *examples*. To construct a good concept map, Novak & Cañas (2008) emphasized forming a *Focus Question* first, which refers to a problem or issue that the concept map aims to resolve. A Focus Question defines the context, limits the domain, and determines the hierarchical structure of the map. Next, it is necessary to identify and build a list of 15 to 25 key concepts in this defined domain from the most general, inclusive concept at the top to the most specific, least general concept at the bottom. This ordered list of individual concepts is called a *parking lot*. Then, a *preliminary concept map* should be constructed by drawing a line to connect any two individual concepts if a relationship is observed between them. Linking words should be added to each line. If an individual concept cannot be linked to any other concepts, it should stay in the parking lot. The last step is to seek and add *cross-links* to the map, which are links between concepts in different domains or sub-domains.

Figure 3.2-1 below contains a concept map on the topic of “Concept Maps” constructed by Coffey & Hoffman (2003).

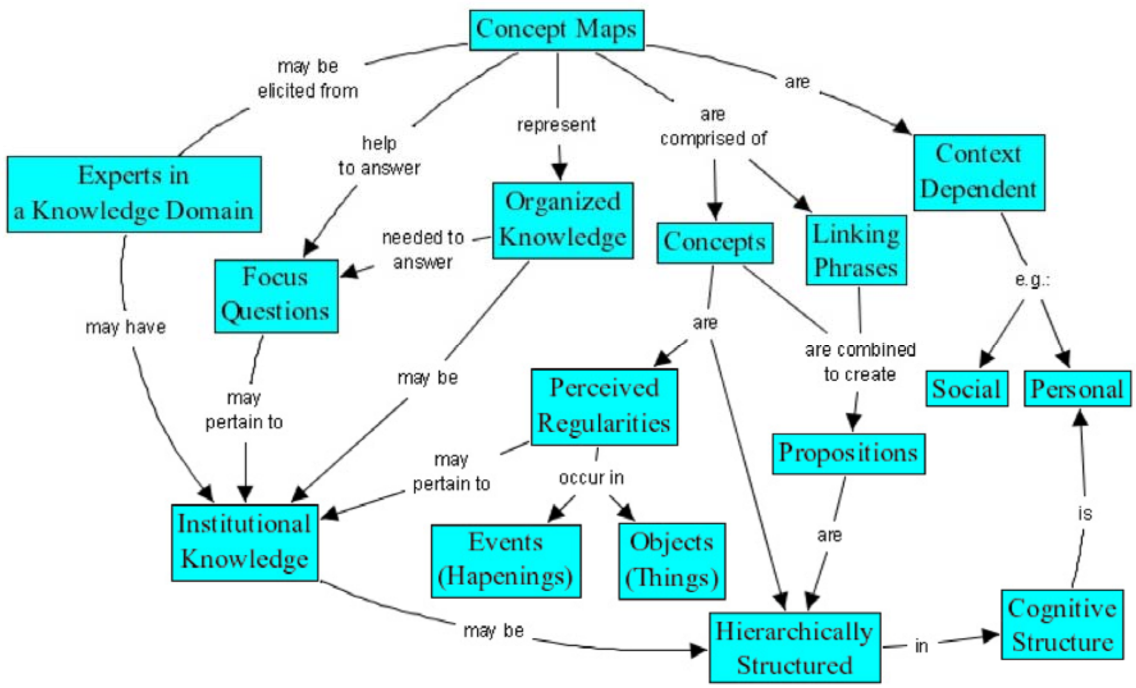

Figure 3.2-1. *A concept map of “Concept Maps”* (Coffey & Hoffman, 2003)

Novak & Cañas (2008) made a few more important points about the concept map as follows:

(1) The construction of a concept map is never completed. Three or more revisions are usually required for a good map.

(2) Generally speaking, all concepts are related to one another in some way. It is better to choose the most representative links that clearly serve the goal of the concept map.

(3) Avoid constructing a *string map* that only shows a linear hierarchical relationship from the top to the bottom.

(4) The most challenging task is to find proper linking words that precisely describe the relationship between concepts. These linking words show the level of the mapmaker's conceptual understanding about the relationships.

(5) Use an *expert skeleton* concept map for difficult topics as scaffold.

An *expert skeleton* concept map is a map with a small number of the most general key concepts and accurate relationships between these concepts built by an expert on the topic. Then, students or teachers can expand this skeleton map by adding concepts sitting in the parking lot (Novak & Cañas, 2008).

Concept maps can help learners to organize their mathematical knowledge, especially when these concept maps are constructed by the learners themselves (Brinkmann, 2005). It is established that the concept mapping approach promotes conceptual learning and students' problem solving ability (Brinkmann, 2005; Hasemann & Mansfield, 1995; Novak & Cañas, 2008; Woolfolk, 2001).

In the present study, a concept map will be constructed for each task, with the purpose of identifying the mathematical concepts and the relationships among them involved in solving the task problem. An overall concept map will also be built for the totality of the tasks in the study. Although these concept maps only represent the author's point of view, they can help teachers engage in reflection on the mathematical structures associated with these tasks before bringing the tasks to the students.

Chapter 4: Methodology

4.1 Setting

The proposed qualitative study is based on data collected from the Rutgers University longitudinal study initiated in 1984. The latter is documented in a number of research articles and dissertations (Francisco & Maher, 2005; Kiczek, 2000; Maher, 2005; Maher and Martino, 1996; Maher, 2002; Martino, 1992; Muter, 1999; Powell, 2003; Tarlow, 2004 and Uptegrove, 2004). In the longitudinal study, a group of eighteen first-graders was chosen from Harding Elementary School in Kenilworth New Jersey. In the beginning, researchers met with participating students in their regular classes about four times a year, for two to three days at a time, in sessions lasting from one to three hours. Later, when the students were in grades 10 through 12, the meetings took place after school. Some of these students continued to meet informally with the Rutgers researchers when they were in college. During the sixteen year span of the study, some students left and some students joined the group of participating students. There were seven students who participated in the study for its whole duration.

The longitudinal study made use of four strands of mathematical tasks: (1) combinatorics and counting, (2) probability, (3) algebra, (4) pre-calculus and calculus. The proposed study will analyze selected tasks from the combinatorics and counting strand. The students started to work on open-ended counting problems in second grade. Gradually, more challenging combinatorics and probability tasks were given to the participants through their high school years. During the problem-solving sessions, the students usually worked in pairs or in small groups and were encouraged to share their thoughts and to incorporate others' ideas. When an agreeable solution and justification were reached within the group, the students were asked to present their solution and reasoning to the whole class for further discussion. The researchers acted as facilitators, keeping intervention to a minimum. There were no set time constraints for any of the tasks and the students were encouraged to pursue any direction they deemed relevant to the problem at hand. The students had a chance to revisit problems and to justify their reasoning during individual follow-up interviews.

4.2 Data

The data for the present study comes from three main sources. The first is the digital video recording of every task session. Each video recording has at least two camera views. One recorded the students and the other recorded their work. The second type of data comes from transcripts of these task sessions, which have been verified and used by other researchers working on the Rutgers longitudinal project. The third source of data is the collection of students' work during the taped sessions.

The following table lists the selected tasks and data sources for this proposed study:

Task	Problem Title	Video Date and Students' Grade	Transcripts and students' work used in
1	Shirts and Jeans	May 1990, Grade 2 October 1990, Grade 3	Martino, 1992
2	Towers 4-tall with 2 colors	October 1990, Grade 3 December 1992, Grade 5 November 1998, Grade 10	Martino, 1992 Tarlow, 2004
3	Towers 4-tall with 3 colors	March 1992, Grades 4 January 1998, Grade 10	Sran, 2010 Muter, 1999
4	Tower of Hanoi	October 1993, Grade 6	Mayansky, 2007
5	Pizza with Halves	March 1993, Grade 5	Muter, 1999

6	a. 4-topping Pizza b. 4-topping Pizza with 2 crusts c. 4-topping Pizza with Halves and 2 crusts	April 1993, Grade 5 March 1999, Grade 11	Muter, 1999 Tarlow, 2004
7	Ankur's Challenge (i.e. another Towers 4-tall with 3 colors)	January 1998, Grade 10	Muter, 1999
8	World Series	January 1999, Grade 11	Kiczek, 2000
9	Points	February 1999, Grade 11	Kiczek, 2000
10	Taxicab	May 2002, Grade 12	Powell, 2003

Table 4.2-1 *Selected tasks and data resource for the present study*

The complete text of each task problem can be found in Appendix A.

4.3 Method of Analysis

Using an adapted methodology introduced by Torkildsen (2006), the analysis will focus on decoding students' solutions into formal mathematical definitions or theorems that are either mathematical objects or structures. A diagram adapted from concept mapping (Novak & Canas, 2008) will be used to illustrate the overall hierarchy of the presented mathematical objects and structures within a task and among tasks. To ensure the progression of the overall concept map from the most general concepts toward the more specific ones, the analysis will not follow the tasks in the order listed in Table 4.2-1. Instead, the tasks with solutions containing the most general structures will be analyzed before the tasks with solutions containing particular instances of those structures. The following describes the analysis plan for selected tasks.

1. The problem will first be solved formally by the researcher, using definitions and theorems. The mathematical structures embedded in the solution will be identified. This step addresses the first research question of this study.
2. Students' solutions and the way they reached the solutions will be examined and decoded into the mathematical structures found in step 1. During problem solving processes, students might or might not "recognize" these structures. They might simply "uncover" these structures through making connections and other strategies. This step addresses the second research question.
3. Based on the results from step 1 and step 2, the research will make conclusions for the task selection and suggest possible research topics for future studies.

4.4 Validity

To ensure the validity of the results, the following three steps will be taken throughout the process of analysis:

1. Triangulation of data with the use of video recordings, verified transcripts, and student work will validate the accuracy of the storyline that is constructed.
2. The resulting report will contain a rich, detail description of the data, thus providing the reader with enough information in order to be able to decide whether the interpretations and conclusions are supported by the data.
3. Formal definitions and theorems will be drawn from published textbooks written by mathematicians or from published papers written by mathematical educators. This will ensure the accuracy of the mathematical structures identified by the researcher in the solutions given to each of the problems.

4. The concept maps produced by the researcher will be verified by third-party mathematics education scholars for legitimacy.

4.5 Significance

It is the hope that the proposed study will bring significant implications to mathematical education and research. This study will investigate the way that the participating students uncover elements related to underlying structures of the tasks during the longitudinal study. The results of the proposed study may contribute to a larger effort aiming to help teachers improve their subject matter knowledge, students to effectively organize the mathematical objects, and researchers to examine the relationship among task design, mathematical structures, and the implementation of mathematics curriculum reform.

4.6 Limitations

The qualitative nature of the study reduces the generalizability of the findings. However, the insight gained from closely documenting the development of the participating students' structural sense will inform educators with respect to students' potential to construct formal mathematical structures through informal problem solving processes.

Chapter 5: Analysis and Discussion

5.1 Introduction

In this chapter, ten selected tasks from the combinatorics and counting strand are analyzed (see Table 4.2-1 for the complete list of tasks). Section 5.2 focuses on identifying embedded mathematical structures in the researcher's solutions to these problems. The tasks are solved by the researcher following the order in which they were presented to the students. The embedded mathematical structures are described in terms of definitions, axioms, and theorems. After each task is analyzed, section 5.3 consolidates all the embedded mathematical structures found in section 5.2 into a summative table. Then, an overall concept map is built to highlight the relationships among the structures summarized. In section 5.4, student solutions are then examined, decoded, and compared with the mathematical structures identified in the previous sections.

5.2 Formal Solutions and Mathematical Structures

In this study, mathematical structures are defined as "a hierarchy of interconnected mathematical objects building on one another to produce a coherent whole." (see section 1.1, p.2). The terms "mathematical structure" and "mathematical object" have been used in a number of different ways by various researchers. In this study, a mathematical object is seen as an abstract "thing" or "entity" arising in mathematics. Geometric objects such as points, lines, triangles, and circles can be easily recognized. Algebraic objects may include groups, rings, or fields, for example. Other commonly recognized objects may be numbers, matrices, sets, functions, and relations.

In the case of most of these objects, each can be seen as a hierarchy of other interconnected mathematical objects. In this sense, the objects that are hierarchies of other objects can be thought of as structures built of the relationships among the objects in the hierarchies. For this reason, definitions, axioms, and theorems are considered mathematical structures in this study while notations and representations are not. When analyzing the solution to a problem, the focus should not be on trying to differentiate between objects and structures, but rather on identifying relationships among their components.

There are many ways to solve a problem task. In this section, one or more methods are used for each task in order to collect enough objects and structures that the students' solutions will be compared against in section 5.5. Each of these solution methods solves the problem independently by its own way that may or may not relate to other methods. Keep in mind that these selected methods do not cover all the possible ways to solve the problem. Further, a method used in solving one task may also be used in solving other tasks. In such a case, different representations for the same mathematical structures are introduced.

5.2.1 Task 1: Shirts and Jeans (Grades 2 and 3)

Stephen has a white shirt, a blue shirt, and a yellow shirt. He has a pair of blue jeans and a pair of white jeans. How many different outfits can he make? Convince us that you found them all.

Solution – Method 1

Let W = White, B = Blue, and Y = Yellow.

Then, the colors of the shirts can be represented by the *set* {W, B, Y} and the colors of jeans can be represented by the *set* {B, W}.

An outfit is represented by the *ordered pair*: (color of the shirt, color of the jeans).

Then, all possible distinct outfits are the *Cartesian product*: {(W, B), (W, W), (B, B), (B, W), (Y, B), (Y, W)}.

There are six ordered pairs in total; therefore, there are six different outfits. All possible outfits are found because there is no other ordered pair that is not already listed.

Mathematical Structure

This solution involves the concepts of *set*, *ordered pair*, and *Cartesian product*. The formal definitions of these terms are following.

(T1.1) Definition. A ***set*** is a well-defined collection of objects, and these objects are called the elements of the set (Morash, 1991, p. 4; Stewart, Redlin, & Watson, 2002, p. 8).

(T1.2) Definition. A set can be described by the ***roster method*** that lists the names of the elements, separated by commas, with the full list enclosed in braces. Example: {W, B, Y} (Morash, 1991, p. 4).

(T1.3) Definition. A set *S* can be described by the ***set-builder*** notation that is in the form *S* = {*x*| *x* satisfies some property or properties} (Morash, 1991, p. 5; Stewart, Redlin, & Watson, 2002, p. 8).

(T1.4) Definition. An ***ordered pair*** is a pair of elements *x* and *y* from a set, written as **(*x*, *y*)**, where *x* is distinguished as the first element and *y* the second (Morash, 1991, p. 21).

Hence, the ordered pairs (W, B) and (B, W) represent different two outfits. (W, B) is white shirt with blue jeans, while (B, W) is blue shirt with white jeans.

(T1.5) Definition. Given sets *A* and *B*, the ***Cartesian product A × B*** (also called ***cross product***) is the set $\{(a,b) \mid a \in A, b \in B\}$ (i.e. *A* × *B* consists of all possible distinct ordered pairs whose first elements come from *A* and whose second elements come from *B*.) (Morash, 1991, 21).

The formal representation of the solution is:

Let *A* = {W, B, Y} and *B* = {B, W}, then

$A \times B = \{(a,b) \mid a \in A, b \in B\}$ = {(W, B), (W, W), (B, B), (B, W), (Y, B), (Y, W)}.

By definition 5.5, these are all possible distinct ordered pairs (i.e. six distinct outfits).

Solution – Method 2

There are three ways to choose a shirt and two ways to choose a pair of jeans. Based on the *fundamental counting principle*, there are 3 × 2 = 6 ways to choose an outfit. Therefore, there are six different outfits.

Mathematical Structure

To solve a simple counting problem like this task, the easiest way is to systematically list and count every possible outcome. However, when the number of outcomes is too large to be completely listed out, then a formula is useful.

> (T1.6) Theorem. **The Fundamental Counting Principle (Product Rule)**: Suppose that two events occur in order. If the first can occur in ***m*** ways and the second in ***n*** ways (after the first has occurred), the two events can occur in order in ***m* × *n*** ways Further, if $E_1, E_2, E_3, ..., E_k$ are events that occur in order and if E_1 can occur in n_1 ways, E_2 can occur in n_2 ways, and so on, then the all events can occur in order in $n_1 \times n_2 \times n_3 \times ... \times n_k$ ways. (Larson & Hostetler, 2004, p. 663; Morash, 1991, p. 42; Roberts & Tesman, 2005, p. 16; Ross, 1998, p. 2-3; Stewart, Redlin, & Watson, 2002, p. 867).

Solution – Method 3

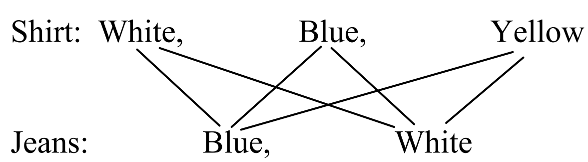

Figure 5.2.1-1. *A simple bipartite graph showing six different outfits.*

In the figure (i.e., *graph*) above, each line (i.e., *edge*) represents one different outfit, consisting of the shirt and jeans specified by the *endpoints* (i.e., *vertices*) of the line. There are six lines in total, representing six different outfits. These are all the possible outfits because no other lines can be drawn connecting a shirt color and a jeans color without repeating an already existing line. Therefore, there are six distinct outfits.

Mathematical Structure

This solution involves many concepts in *graph theory*.

(T1.7) Definition. An ***unordered pair*** (i.e. not an ordered pair) is a pair of elements *x* and *y* from a set, that can be written as **{*x*, *y*}** (defined by the researcher based on the (5.4) definition of ordered pair).

(T1.8) Definition. A ***graph*** *G* (*V*, *E*) consists of a set *V* of vertices, a set *E* of edges, and a mapping associating to each edge an *unordered pair* of vertices called the ***endpoints*** (Roberts & Tesman, 2005, p.124-126; Van Lint & Wilson, p.1).

(T1.9) Definition. A graph is ***simple*** when it has no loops and no two distinct edges have exactly the same pair of endpoints. (Van Lint & Wilson, 2001, p.2).

(T1.10) Definition. In a simple graph, the unordered pair **{*x*, *y*}** represents the ***edge*** that joins vertices *x* and *y* (Roberts & Tesman, 2005, p. 126; Van Lint & Wilson, 2001, p. 2).

(T1.11) Definition A graph is ***bipartite*** if and only if the vertices can be partitioned into two classes so that all edges in the graph join vertices in the two different classes (Roberts & Tesman, 2005, p.155; Van Lint & Wilson, 2001, p.24).

Thus, there are five *vertices* and six *edges* in the solution graph *G* (*V, E*), where

V = {white shirt, blue shirt, yellow shirt, blue jeans, white jeans}.

E = {{white shirt, blue jeans}, {white shirt, white jeans}, {blue shirt, blue jeans}, {blue shirt, white jeans}, {yellow shirt, blue jeans}, {yellow shirt, white jeans}}

Note that *G* is *simple* because there is no loop and every edge joining distinct two vertices. Also the graph is *bipartite* because the five vertices are split into two classes ("shirts colors" and "jeans colors") and all edges (outfits) join vertices in these two classes. The edges are represented by *unordered pairs*, meaning that the pairs {white shirt, blue jeans} and {blue jeans, white shirt} are the same outfit.

5.2.2 Task 2: Towers 4-tall with 2 colors (Grades 3 and 5)

Your group has Unifix cubes of two colors. Work together and make as many different towers four cubes tall as is possible when selecting from two colors. See if you and your partner can plan a good way to find all the towers four cubes tall.

Solution – Method 1

Let W = "White Cube" and B = "Black Cube". Under this notation, the choices of colors of towers can be illustrated by a *tree* diagram:

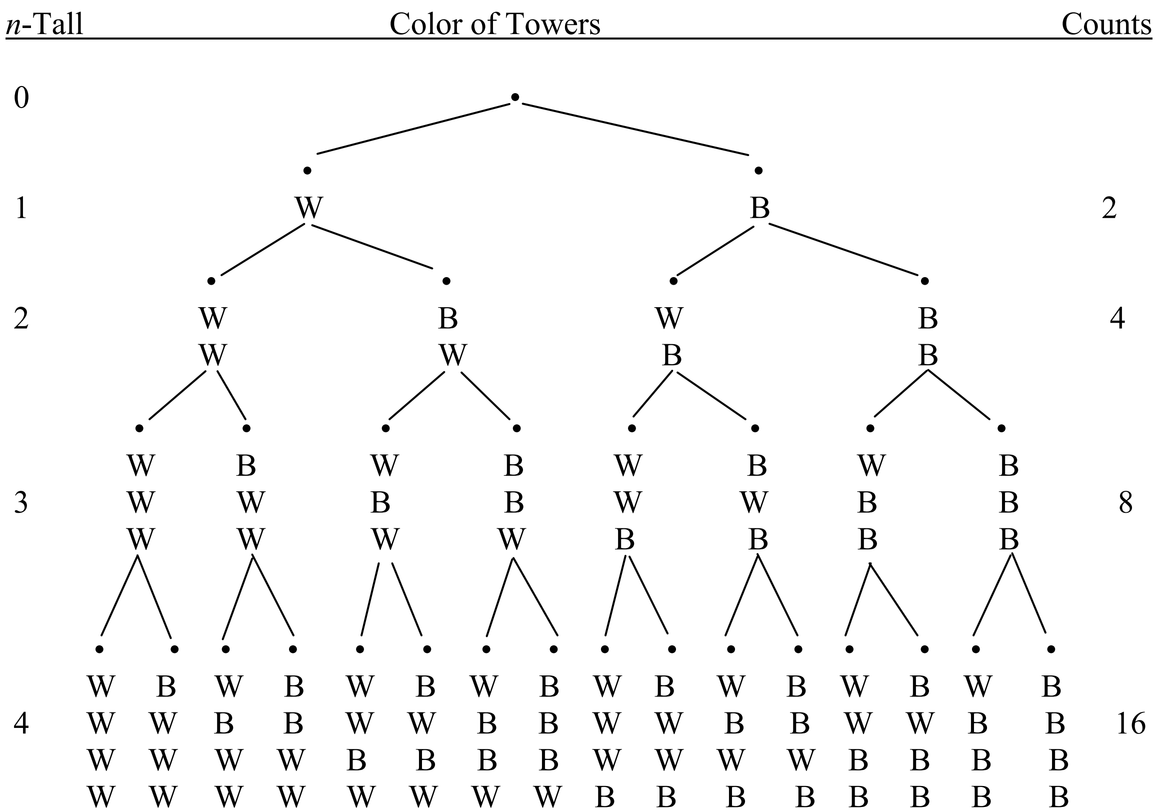

Figure 5.2.2-1. *Tree diagram for 4-tall towers when selecting from 2 colors*

The tree "grows" from a root node branching into two distinct nodes – W and B, which represent two distinct 1-tall towers. This binary branching continues for each node in the tree until the end nodes have four-letter-high labels, which represent 4-tall towers. The number of final nodes (16) represents the number of distinct 4-tall towers.

Mathematical Structure

This solution involves a *tree* structure, which is commonly studied in graph theory courses. Before formally defining a *tree*, it is necessary to define a few other related terms.

(T2.1) Definition. A ***walk*** in a graph *G* is a sequence $u_0, e_1, u_1, e_2, u_2, ..., u_{k-1}, e_k, u_k$, where u_i is a vertex ($i = 0,1,2,...,k$) and e_i is the edge { u_{i-1} , u_i } ($i = 1,2,...,k$).
The walk is ***simple*** if all the vertices $u_0, u_1, u_2, ..., u_k$ are distinct.
The walk is ***closed*** if $u_0 = u_k$.
The walk is called a ***path*** from u_0 to u_k if all the edges $e_1, e_2, ..., e_k$ are distinct. (Roberts & Tesman, 2005, p. 135; Van Lint & Wilson, 2001, p. 4-5).

(T2.2) Definition. A graph is ***connected*** if between every pair of vertices *u* and *v* there is a path (Roberts & Tesman, 2005, p.136; Van Lint & Wilson, 2001, p. 5).

(T2.3) Definition. A ***tree*** is a graph *T* that is connected and contains no simple closed paths (Roberts & Tesman, 2005, p. 185; Van Lint & Wilson, 2001, p. 6).

Solution – Method 2

The binary branching of the tree in the previous method gives a so called *complete binary tree* (see figure 5.2.2-2 below). Instead of labeling the vertices, the edges of the tree can be labeled with the two colors of the cubes. For example, for each vertex, the edge branching to the left is labeled "W" and the edge branching to the right is labeled

"B." Then, the same 16 distinct towers can be found by a procedure commonly known as *backward traverse*. The following shows the solution method using these representations.

Let W = "White Cube" and B = "Black Cube"; the *complete binary tree* is:

Level Binary Branching # of vertices

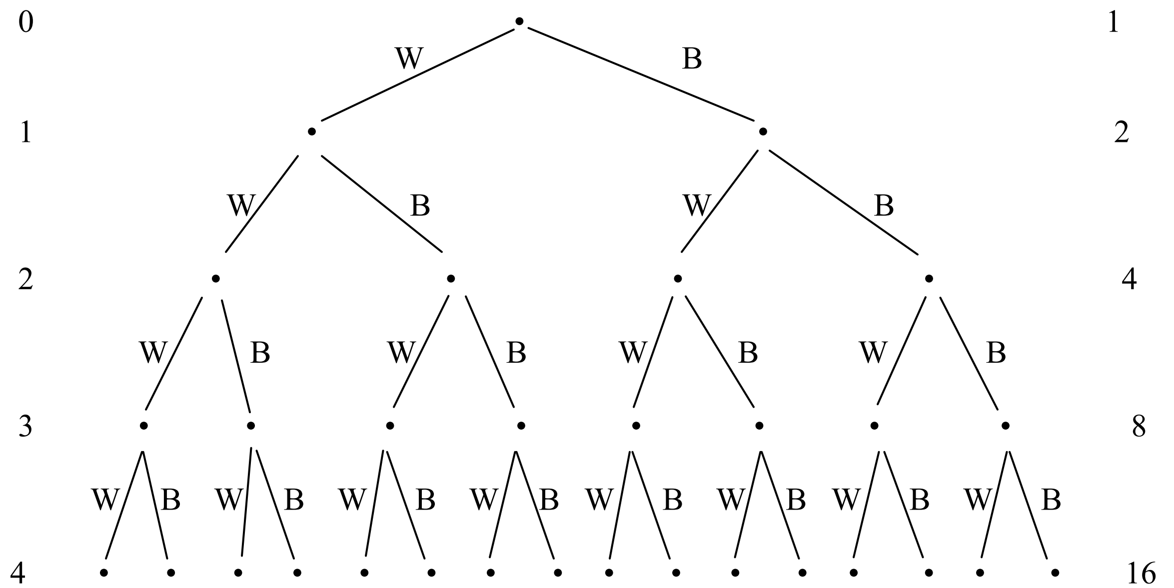

Figure 5.2.2-2. *Complete Binary Tree for 4-tall towers selecting from 2 colors*

The 16 towers are outputs from performing *backward traverse* through the tree, which are represented by the following *tower strings*:

WWWW, BWWW, WBWW, BBWW,
WWBW, BWBW, WBBW, BBBW,
WWWB, BWWB, WBWB, BBWB,
WWBB, BWBB, WBBB, BBBB

Note that the sequence of output towers is exactly in the same order obtained by method 1. All the 16 distinct towers are presented.

Mathematical Structure

The solution tree can be described as a *complete binary tree* of four *levels* with 16 nodes of *degree* one (the *leaves*), the *root* vertex has *degree* two, and all other intermediate vertices have *degree* three. Related structures are:

> (T2.4) Definition. The ***degree of vertex*** *u* of graph *G*, denoted as **deg(*u*)**, counts the number of edges incident with *u*. (Roberts & Tesman, 2005, p. 127; Van Lint & Wilson, 2001, p. 4).
>
> (T2.5) Definition. Each vertex in a tree is associated with a ***level***, 0, 1, 2, …, *k*. The number *k* is called the ***height*** of the tree. (Roberts & Tesman, 2005, p. 202).
>
> (T2.6) Definition. A tree is called a ***rooted tree*** if there is a distinguished vertex, the ***root***, at level 0; all adjacent vertices differ by exactly one level; and each vertex at level *i* + 1 is adjacent to exactly one vertex at level *i* (Roberts & Tesman, 2005, p. 202; Van Lint & Wilson, 2001, p. 19).
>
> (T2.7) Definition. In a rooted tree with root v_0, the ***ancestors*** of vertex *u* are the vertices traversed by the unique path from *u* to the root v_0. The first vertex other than *u* on that path is the ***parent*** of *u* (*u* is a ***child*** of this vertex). If vertex *u* is an ancestor of *v*, then *v* is a ***descendant*** of *u* (Roberts & Tesman, 2005, p. 202; Van Lint & Wilson, 2001, p. 20). If *u* has no children, then *u* is called a ***leaf*** (Roberts & Tesman, 2005, p. 196).
>
> (T2.8) Definition. A rooted tree is called a ***binary tree*** if every vertex has two or fewer children (Roberts & Tesman, 2005, p. 202).
>
> (T2.9) Definition. A rooted tree is called ***complete*** if every vertex has either no children or two children. (Roberts & Tesman, 2005, p. 202).

Solution – Method 3

Although the binary tree configurations presented in method 1 and method 2 look a little different, they are actually the same structures with different representations. The solution to this task can also be obtained inductively by using another kind of representation: the two-way tables. For example, in the 2 × 2 table below, starting from each side as 1-tall towers one can build four 2-tall towers as follows:

		1-tall tower	
		W	B
1-tall towers	W	WW	WB
	B	BW	BB

Similarly, 3-tall towers can be represented using either a 2 × 4 or a 4 × 2 table. Below is a 2 × 4 table showing eight 3-tall towers, which can be seen as obtained by placing 1-tall towers on top of 2-tall towers:

		2-tall towers			
		WW	WB	BW	BB
1-tall towers	W	WWW	WWB	WBW	WBB
	B	BWW	BWB	BBW	BBB

Finally, sixteen 4-tall towers can be obtained from a 2 × 8, 8 × 2, or 4 × 4 table. Below is an example of a 4 × 4 table, whose output towers are built by placing 2-tall towers on top of 2-tall towers:

		2-tall towers			
		WW	WB	BW	BB
2-tall towers	WW	WWWW	WWWB	WWBW	WWBB
	WB	WBWW	WBWB	WBBW	WBBB
	BW	BWWW	BWWB	BWBW	BWBB
	BB	BBWW	BBWB	BBBW	BBBB

5.2.3 Task 3: Towers 4-tall with 3 colors (Grade 4)

Your group has Unifix cubes of three different colors. Work together and make as many different towers of four cubes tall as is possible when selecting from three colors. See if you and your partner can plan a good way to find all the towers four cubes tall. Convince us that you have found them all.

Solution – Method 1

The tree structure is relevant here as well. This time every vertex has three incident edges linking to three adjacent vertices that are labeled with the designated tower colors. The solution is represented by a *complete 3-ary tree* with 81 *leaves* (see Figure 5.2.3-1 on the next page). Thus, there are 81 different 4-tall towers when choosing from three colors. Using the procedure *backward traverse* (5.22), each of these 81 towers can be listed out uniquely by following a path from a leaf to the root. To illustrate how the procedure works, the following list gives the first nine 4-tall towers represented by the *tower strings*, where the leaves are underlined (W = "While", B = "Black", R = "Red"):

  <u>W</u>WWW, <u>B</u>WWW, <u>R</u>WWW,
  <u>W</u>BWW, <u>B</u>BWW, <u>R</u>BWW,
  <u>W</u>RWW, <u>B</u>RWW, <u>R</u>RWW, ……

<u>Mathematical Structure</u>

Most terms used in the solution of this task have been introduced in section 5.2.2. There is only one more definition here.

> <u>(T3.1) Definition.</u> A rooted tree is called ***m-ary*** if every vertex has *m* or fewer children. (Roberts & Tesman, 2005, p. 202).

Therefore, a binary tree is also called a *2-ary* tree. In this task, the tree is a *complete 3-ary tree* because every vertex has three children except for the leaves.

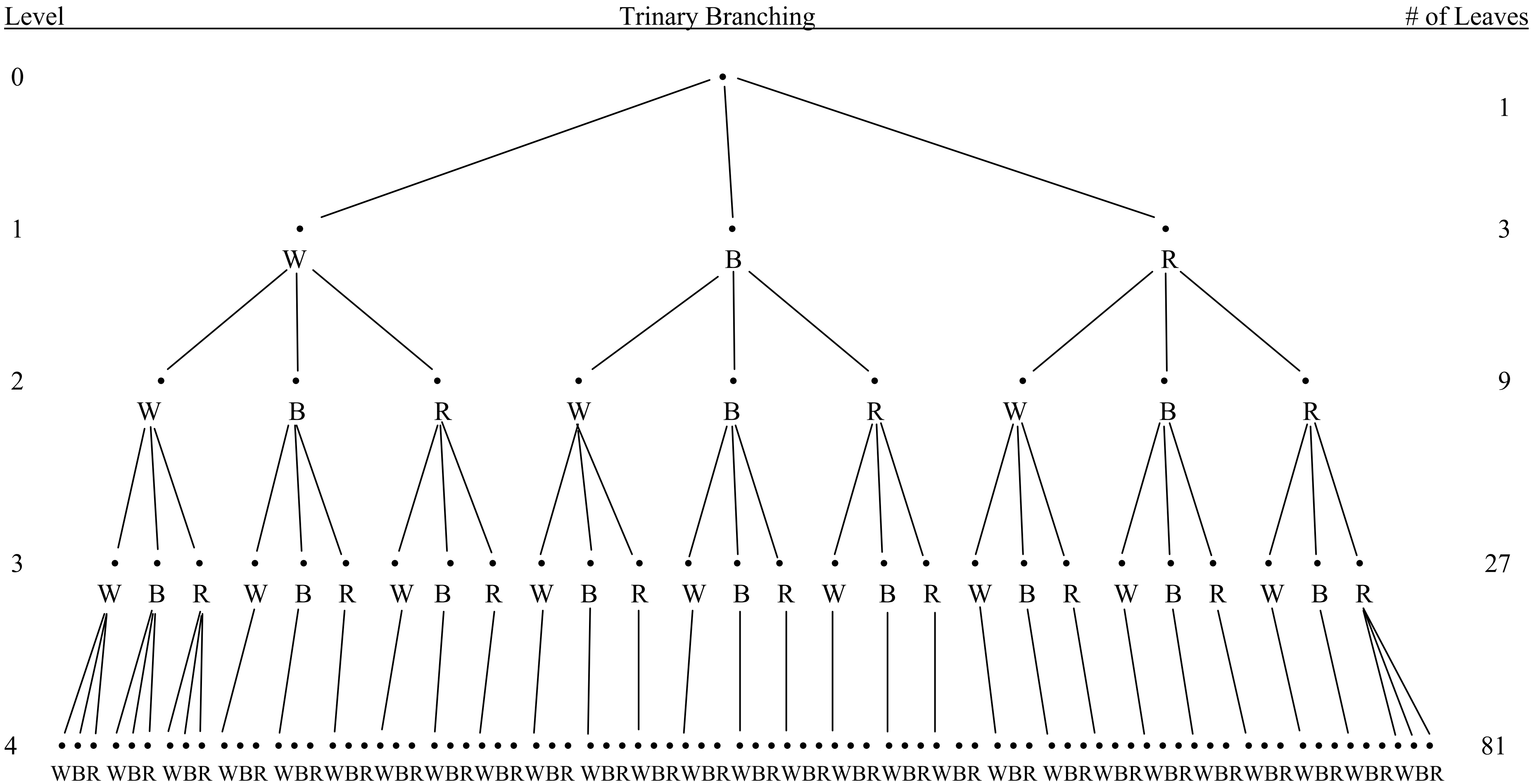

Figure 5.2.3-1. *Complete 3-ary tree for 4-tall towers when selecting from 3 colors* (not all edges are shown for level 4)

Solution – Method 2

In solving a problem, the method of "Divide into Cases" is often used. According to Morash (1991, p. 164-166), the cases must be mutually exclusive (categories are non-overlapping) and exhaustive (categories include all possibilities). For this task, the possible *combinations* of tower colors can be enumerated under three cases:

Case 1: Towers with only 1 color

There are ${}_3C_1 = 3$ ways to choose one color out of three to build towers.

Case 2: Towers with exactly 2 colors

There are ${}_3C_2 = 3$ ways to choose two colors out of three to build towers.

Sub-case 1: There are ${}_4C_1 = 4$ ways to position one cube of the first color, then three cubes of the second color complete the tower.

Sub-case 2: There are ${}_4C_2 = 6$ ways to position two cubes of the first color, then two cubes of the second color complete the tower.

Sub-case 3: There are ${}_4C_3 = 4$ ways to position three cubes of the first color, then one cube of the second color completes the tower.

Therefore, the total number of towers built with two colors is:

$${}_3C_1({}_4C_1 + {}_4C_2 + {}_4C_3) = 3(4 + 6 + 4) = 42$$

Case 3: Towers with exactly 3 colors

Because the tower is 4 cubes tall, one of the three colors must be used twice.

There are ${}_3C_1 = 3$ ways to choose one "repeating" color.

There are ${}_4C_2 = 6$ ways to choose two positions for the "repeating" color cubes.

There are ${}_2C_1 = 2$ ways to choose positions for the remaining two "non-repeating" color cubes.

So, there are $({}_3C_1)({}_4C_2)({}_2C_1) = (3)(6)(2) = 36$ ways to build 4-tall towers with exactly 3 colors.

(Note that Case 3 also provides a solution to the task in 5.2.7, Ankur's Challenge.)

Summing up the results from the 3 cases presented above, we have $3 + 42 + 36 = 81$ ways to build towers selecting from three different colors.

Mathematical Structure

This solution involves the mathematical structure of *combinations* and the problem-solving heuristic *dividing into cases*. The terms used in the explanation above are formally defined as follows.

(T3.2) Definition. If *n* is a positive integer, ***n* factorial** is denoted by $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$. As a special case, $0! = 1$ (Larson & Hostetler, 2004, p. 616; Stewart, Redlin, & Watson, 2002, p. 851).

(T3.3) Definition. A ***combination*** of *r* elements of a set is any subset of *r* elements from the set without regard to order. If the set has *n* elements, then the number of combinations of *r* elements is denoted by ${}_nC_r$ or C(*n*, *r*) or $\binom{n}{r}$, and is called the ***number of combinations of n elements taken r at a time*** (Larson & Hostetler, 2004, p. 667; Roberts & Tesman, 2005, p. 35; Ross, 1998, p. 5-6; Stewart, Redlin, & Watson, 2002, p. 875).

(T3.4) Theorem. ${}_nC_r = \dfrac{n!}{r!(n-r)!}$, where *n*, *r* are nonnegative integers and $r \le n$ (Larson & Hostetler, 2004, p. 667; Roberts & Tesman, 2005, p. 35; Ross, 1998, p. 5-6; Stewart, Redlin, & Watson, 2002, p. 876).

Solution – Method 3

According to Polya (1945/1985), there is "very little logical connection" (p. 114) between the processes of *induction* and *mathematical induction*. *Induction* is the process of discovering general laws by the observation and combination of particular instances. It is used in all sciences, including in mathematics. On the other hand, *mathematical induction* is a method only used in mathematics to prove a mathematical statement *P*(*n*), which is depending on the nature number *n*, to be true (Abbott, 2001, p. 10; Artin, 1991, p. 348; Larson & Hostetler, 2004, p. 644-645; Morash, 1991, p. 171; Stewart, Redlin, & Watson, 2002, p. 842-843). Nevertheless, these two methods are often used together.

From the solution tree (figure 5.2.3-1) in method 1, it is not hard to see a pattern in the number of nodes at first three levels – 3^0 nodes at level 0, 3^1 nodes at level 1, 3^2 nodes at level 2. The pattern, discovered by the process of *Induction*, suggests the following claim: there are a total of 3^n different *n*-tall towers when choosing from three different colors. Below is a proof by *mathematical induction* of this claim:

Basic step:

When $n = 1$, there are $3^1 = 3$ distinct 1-tall towers.

Induction step:

Assume that when $n = k$, there are 3^k distinct *k*-tall towers.

Need to show that when $n = k + 1$, there are 3^{k+1} (*k*+1)-tall distinct towers.

This is true because $3^{k+1} = 3^k \times 3$, where 3^k is the number of distinct *k*-tall towers, which we multiply by 3 because of we have three color choices ("W", "B", or "R") for the cube to be added on top of a *k*-tall tower to make it (*k*+1)-tall (end of proof).

Therefore, according to the claim that has been proved, there are a total of 3^4 (81) different *4*-tall towers when choosing from three different colors.

Furthermore, taking into account the result from the task "Towers 4-tall choosing from two colors" (discussed in 5.2.2), we can conjecture that the total number of different *n*-tall towers when choosing from *m* different colors is $\boldsymbol{m^n}$. This is indeed true – the proof is similar the mathematical induction one described for the case of three colors.

5.2.4 Task 4: Tower of Hanoi (Grades 6)

Figure A.A-1 below shows a puzzle with three posts, and there are seven disks stacked as a tower on one of the three posts. You have to move all the disks from the post to another post. The rule is: you can only move one disk at a time and you can never put a bigger disk onto a smaller disk. How many moves do you need to complete the task? If this is a 100-disk tower, how many moves do you need?

Figure A-1. *Tower of Hanoi Puzzle*

Solution – Method 1

For someone who has never played this game, it may be a good idea to approach the problem by experimenting with a smaller number of disks in order to see what is happening. The results obtained in the first five cases (the number of disks = 1, 2, 3, 4, and 5) are summarized below:

# of disks	# of moves
1	1
2	3
3	7
4	15
5	31

Table 5.2.4-1. *The number of moves for one through five disks.*

Observing the numbers in the right hand column (number of moves), they are very close to numbers of powers of two. A pattern can be found as following:

# of disks	# of moves
1	1 $(= 2^1 - 1)$
2	3 $(= 2^2 - 1)$
3	7 $(= 2^3 - 1)$
4	15 $(= 2^4 - 1)$
5	31 $(= 2^5 - 1)$

Table 5.2.4-2. *The number of moves for one through five disks*.

In general, if number of disks is n, then the number of moves required is $2^n - 1$. Thus, when $n = 100$, $2^{100} - 1$ moves are needed.

Solution – Method 2

Using table 5.2.4-1 again, calculating the difference between the numbers in any two consecutive rows of this column, a pattern becomes apparent.

# of disks	# of moves	Difference in # of moves
1	1	(2^0)
2	3	$3 - 1 = 2$ $(= 2^1)$
3	7	$7 - 3 = 4$ $(= 2^2)$
4	15	$15 - 7 = 8$ $(= 2^3)$
5	31	$31 - 15 = 16$ $(= 2^4)$

Table 5.2.4-3. *The differences between numbers of moves form a geometric sequence*.

This suggests that, if the number of disks is n, where n is any positive integer greater then 1, then the difference of moves between the n^{th} and $(n - 1)^{\text{th}}$ cases is the *geometric sequence* $\{2^{n-1}\}$ (this claim can be proved to be true by mathematical induction). The number of moves for the first five cases can now be rewritten as follows:

# of disks	# of moves	
1	1	$(= 2^0)$
2	3	$(= 2^0 + 2^1)$
3	7	$(= 2^0 + 2^1 + 2^2)$
4	15	$(= 2^0 + 2^1 + 2^2 + 2^3)$
5	31	$(= 2^0 + 2^1 + 2^2 + 2^3 + 2^4)$

Table 5.2.4-4. *The numbers of moves form geometric series*.

This suggests that the number of moves for *n* disks is the sum of the number of moves for $(n - 1)$ disks and the difference of moves between the n^{th} and $(n - 1)^{\text{th}}$ disks. Hence, if the number of disks is *n*, where *n* is a positive integer, then the number of moves needed is $2^0 + 2^1 + 2^2 + ... + 2^{n-1}$, which can be written in *summation* (*sigma*) *notation*: $\sum_{k=0}^{n-1} 2^k$. The following is the *induction* proof for this claim:

When $n = 1$, the number of moves needed is $2^0 = 1$. Assume that when $n = k$, the number of moves needed is $2^0 + 2^1 + 2^2 + ... + 2^{k-1}$. Because the difference of moves between the k^{th} and $(k + 1)^{\text{th}}$ disks is 2^k, when $n = k + 1$, the number of moves needed is $(2^0 + 2^1 + 2^2 + ... + 2^{k-1}) + 2^k$, which fulfill the *induction* step.

Thus, when there are seven disks ($n = 7$), the number of moves needed to move all disks is $\sum_{k=0}^{6} 2^k = 1 + 2 + 4 + 8 + 16 + 32 + 64 = 127$. When there are 100 disks, the number of moves needed is $\sum_{k=0}^{99} 2^k$, which is equal to $2^0 + \sum_{k=1}^{99} 2^k = 1 + \sum_{k=1}^{99} 2^k$.

Note that $\sum_{k=1}^{99} 2^k = S_{99}$ is the *99th partial sum* of the *geometric sequence* $\{2^{n-1}\}$ (or the *99th partial sum of the geometric series* $\sum_{k=1}^{\infty} 2^k$) with the first term *a* = 1 and the common ratio *r* = 2. Use the formula $S_n = \frac{a(1-r^n)}{1-r}$ to find the value of $\sum_{k=1}^{99} 2^k$ as follows:

$$\sum_{k=1}^{99} 2^k = s_{99} = \frac{1(1-2^{99})}{1-2} = 2^{99} - 1 = 633{,}825{,}300{,}114{,}114{,}700{,}748{,}351{,}602{,}688$$

To get an idea how big this number is, it may be helpful to try to estimate how long it would take a person to complete the moves in the case of 100 disks. Suppose a person can make one move per second, it will take 20,098,468,420,665,737,593,491 years to move the 100-disk tower to another post.

Mathematical Structure

Through the heuristic *pattern recognition*, the solution is found by adding one to the n^{th} *partial sum* of a *geometric series*. Involved structures are:

(T4.1) Definition. A ***sequence*** is a set of numbers written in a specific order: $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$. It can be seen as a function *f* whose domain is the set of natural numbers. The values *f*(1), *f*(2), *f*(3),… are called the ***terms*** of the sequence (Larson & Hostetler, 2004, p. 614; Stewart, Redlin, & Watson, 2002, p. 807).

(T4.2) Definition. A ***sequence*** $\{a_1, a_2, a_3, a_4, \ldots, a_n, \ldots\}$ is also denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$ (Stewart, 1999, p. 693).

(T4.3) Definition. A ***geometric sequence*** is a sequence of the form $a, ar, ar^2, ar^3, ar^4, \ldots$. The number *a* is the ***first term***, and *r* is the ***common ratio*** of the sequence. The ***n*th *term*** of a geometric sequence is given by $a_n = ar^{n-1}$ (Larson & Hostetler, 2004, p. 635; Stewart, Redlin, & Watson, 2002, p. 824).

(T4.4) Definition. For the sequence $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$, the ***partial sums*** are

$S_1 = a_1$ (the 1st partial sum)

$S_2 = a_1 + a_2$ (the 2nd partial sum)

$S_3 = a_1 + a_2 + a_3$ (the 3rd partial sum)
$S_4 = a_1 + a_2 + a_3 + a_4$ (the 4th partial sum)
…
$S_n = a_1 + a_2 + a_3 + a_4 + ... + a_n$ (the *n*th partial sum)
…
The sequence $S_1, S_2, S_3, S_4, ..., S_n, ...$ is called the ***sequence of partial sums*** (Stewart, Redlin, & Watson, 2002, p. 812).

(T4.5) Theorem. For the geometric sequence $a_n = ar^{n-1}$, the *n*th partial sum is given by $S_n = \dfrac{a(1-r^n)}{1-r}$ (Larson & Hostetler, 2004, p. 637; Stewart, 1999, p. 706; Stewart, Redlin, & Watson, 2002, p. 826).

(T4.6) Definition. ***Summation (Sigma) notation*** derives its name from the Greek letter ∑ (“sum”): $\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + ... + a_n$, where *k* is the index of summation (Larson & Hostetler, 2004, p. 618; Stewart, Redlin, & Watson, 2002, p. 813).

(T4.7) Definition. Given an infinite sequence $\{a_n\}_{n=1}^{\infty}$, the ***infinite series*** (or just a ***series***) is $\sum_{n=1}^{\infty} a_n$ (or just $\sum a_n$) (Larson & Hostetler, 2004, p. 619; Stewart, 1999, p. 704).

(T4.8) Definition. A ***geometric series*** is a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + ar^4 + ... + ar^{n-1} + ...$$

If |*r*| < 1, the geometric series is convergent and $\sum_{n=1}^{\infty} ar^{n-1} = \dfrac{a}{1-r}$

If |*r*| ≥ 1, the geometric series is divergent (Larson & Hostetler, 2004, p. 635, 638; Stewart, 1999, p. 706; Stewart, Redlin, & Watson, 2002, p. 829).

Solution – Method 3

When exploring the first five cases, it may be helpful to consider the additional moves that one has to make when the number of disks is increased by one. For example, suppose there are four disks stacked on the middle post at the beginning and the number to move a 3-disk tower to another post is known to be seven. Assume that these seven moves have already been made and the top three disks have already been moved to the leftmost post. The next move must be moving the biggest (fourth) disk from the middle post to the rightmost post. Then, another seven moves are necessary to move the three disks from the leftmost post to the rightmost post on top of the biggest (fourth) disk. Therefore, when the number of disks is four, the number of moves needed must be 7 + 1 + 7 = 15. The number of moves necessary in the other cases can be rewritten as follows:

# of disks	# of moves
1	1
2	3 = 1 + 1 + 1 = 2 × 1 + 1
3	7 = 3 + 1 + 3 = 2 × 3 + 1
4	15 = 7 + 1 + 7 = 2 × 7 + 1
5	31 = 15 + 1 + 15 = 2 × 15 + 1

Table 5.2.4-5. *The numbers of moves have a recursive relationship*.

Thus, using the relationship between consecutive terms, the sequence {1, 3, 7, 15, 31, ….} (the number of moves) can be *defined recursively* as: $a_1 = 1$ and $a_n = 2a_{n-1} + 1$ for all $n > 1$, where n is the number of disks.

So, the number of moves needed to move a 7-disk tower is:

$$a_7 = 2a_6 + 1 = 2(2a_5 + 1) + 1 = 2(2 \times 31 + 1) + 1 = 127$$

It takes a little longer to find a_{100}, but it is not impossible:

$$a_{100} = 2a_{99} + 1 = 2(2a_{98} + 1) + 1 = 2^2 a_{98} + 2 + 1$$

$$= 2^2(2a_{97} + 1) + 2 + 1 = 2^3 a_{97} + 2^2 + 2 + 1$$

$$= 2^3(2a_{96} + 1) + 2^2 + 2 + 1 = 2^4 a_{96} + 2^3 + 2^2 + 2 + 1$$

$$= \ldots\ldots..$$

$$= 2^{98}(2a_1 + 1) + 2^{97} + \ldots + 2^2 + 2 + 1 = 2^{99} a_1 + 2^{98} + 2^{97} + \ldots + 2^3 + 2^2 + 2 + 1$$

$$= 1 + \sum_{k=1}^{99} 2^k$$ (This is the same result as found in solution - method 2.)

Mathematical Structure

> (T4.9) Definition. A ***recursively defined sequence*** is a sequence defined in a way such that the n^{th} term of the sequence (a_n) depends on some or all of the terms preceding it (Larson & Hostetler, 2004, p. 616; Stewart, Redlin, & Watson, 2002, p. 810).

5.2.5 Task 5: Pizza with Halves (Grades 5)

A local pizza shop has asked us to help them design a form to keep track of certain pizza sales. Their standard "plain" pizza contains cheese. On this cheese pizza, one or two toppings could be added to either half of the plain pizza or the whole pie. How many choices do customers have if they could choose from two different toppings (sausage and pepperoni) that could be placed on either the whole pizza or half of a cheese pizza? List all possibilities. Show your plan for determining these choices. Convince us that you have accounted for all possibilities and that there could be no more.

Solution

Let C = Plain (Cheese only); S = Sausage;

P = Pepperoni; M = Mixed (Sausage and Pepperoni).

Then, the choices one has for each pizza half are contained by the set {C, S, P, M}. Considering that each pizza is made up of two halves, each pizza can be described in one of following two cases:

Case 1: Two halves with the same topping(s).

There are four of this kind: {CC, SS, PP, MM}.

Case 2: Two halves have different topping(s).

This is *the number of combinations of 4 choices taken 2 at a time*. Calculate by formula (T3.4): $C(4,2)=\dfrac{4!}{2!(4-2)!}=\dfrac{4\times3\times2\times1}{2\times1(2\times1)}=6$

Thus, there are six pizzas of this kind: {CS, CP, SP, CM, SM, PM}

Therefore, a total of 4 + 6 =10 different types of pizza can be obtained under the conditions of this problem. The following two-dimensional table can be used to display all the possibilities.

	C	S	P	M
C	CC	CS	CP	CM
S	SC	SS	SP	SM
P	PC	PS	PP	PM
M	MC	MS	MP	MM

Table 5.2.5-1. *Permutations of 4 choices of toppings taken 2* (2 *halves*) *at a time*.

Note that the table has 12 different entries. The main diagonal of the table (shaded in grey) contains the four choices for which the two halves have the same toppings (Case 1). The six choices of pizza below the main diagonal (shaded in yellow) are {SC, PC, PS, MC, MS, MP}, and the six choices of pizza above the main diagonal (shaded in blue) are {CS, CP, SP, CM, SM, PM} (Case 2). The number 12 is *the number of permutations of 4 choices taken 2 at a time*, which is given by the formula:

$$P(4,2) = \frac{4!}{(4-2)!} = 4 \times 3 = 12.$$

The difference between *permutations* and *combinations* is that in the case of *permutations*, ordered subsets are counted, whereas *combinations* are concerned with unordered subsets. In this task, SC is considered the same as CS as both refer to a pizza which has cheese only on one half and sausage on the other half. For this reason, *combinations* (not *permutations*) represent the mathematical structure relevant to this problem.

Mathematical Structure

There is no new structure involved in the solution of this task. However, given that the two-dimensional table displays the list of possible 2-permutations of the set of four possible toppings, it seems appropriate to discuss *permutations* further.

(T5.1) Definition. A ***permutation*** of a set of distinct elements is an arrangement of these elements. (Larson & Hostetler, 2004, p. 664; Roberts & Tesman, 2005, p. 25; Ross, 1998, p. 3; Stewart, Redlin, & Watson, 2002, p. 872).

(T5.2) Theorem. The number of ***permutations*** of *n* elements is ***n*!** (Larson & Hostetler, 2004, p. 664; Roberts & Tesman, 2005, p. 26; Ross, 1998, p. 4; Stewart, Redlin, & Watson, 2002, p. 872).

(T5.3) Definition. ***P*(*n*, *r*)** denotes ***the number of permutations of n elements taken r at a time***. It is also called ***r-permutation of the n-set*** (Larson & Hostetler, 2004, p. 665; Roberts & Tesman, 2005, p. 32; Stewart, Redlin, & Watson, 2002, p. 873).

(T5.4) Theorem. $P(n,r) = \frac{n!}{(n-r)!} = n(n-1)(n-2)\cdots(n-r+1)$ (Larson & Hostetler, 2004, p. 665; Roberts & Tesman, 2005, p. 32; Stewart, Redlin, & Watson, 2002, p. 873).

5.2.6 Task 6a: 4-Topping Pizza (Grades 5 and 11)

A local pizza shop has asked us to help them design a form to keep track of certain pizza choices. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many different choices for pizza does a customer have? List all the possible choices. Convince us that you have them all.

Solution (Task 6a) – Method 1

Let P = Peppers; S = Sausage; M = Mushrooms; R = Pepperoni;

One can systematically list all the choices a customer has for pizza by listing all the possibilities in each of the 0-topping, 1-topping, 2-topping, 3-topping, and 4-topping categories. The left of table 5.2.6-1 below shows all 16 possible choices of pizza obtained this way. The corresponding number of choices for each type of pizzas is shown on the right of the table.

<table>
<tr><th></th><th></th><th>P</th><th>S</th><th>M</th><th>R</th><th>Number of Different Choices</th></tr>
<tr><td>1</td><td>Plain</td><td></td><td></td><td></td><td></td><td>1</td></tr>
<tr><td>2</td><td rowspan="4">1-topping</td><td>X</td><td></td><td></td><td></td><td rowspan="4">4</td></tr>
<tr><td>3</td><td></td><td>X</td><td></td><td></td></tr>
<tr><td>4</td><td></td><td></td><td>X</td><td></td></tr>
<tr><td>5</td><td></td><td></td><td></td><td>X</td></tr>
<tr><td>6</td><td rowspan="6">2-topping</td><td>X</td><td>X</td><td></td><td></td><td rowspan="6">6</td></tr>
<tr><td>7</td><td>X</td><td></td><td>X</td><td></td></tr>
<tr><td>8</td><td>X</td><td></td><td></td><td>X</td></tr>
<tr><td>9</td><td></td><td>X</td><td>X</td><td></td></tr>
<tr><td>10</td><td></td><td>X</td><td></td><td>X</td></tr>
<tr><td>11</td><td></td><td></td><td>X</td><td>X</td></tr>
<tr><td>12</td><td rowspan="4">3-topping</td><td>X</td><td>X</td><td>X</td><td></td><td rowspan="4">4</td></tr>
<tr><td>13</td><td>X</td><td>X</td><td></td><td>X</td></tr>
<tr><td>14</td><td>X</td><td></td><td>X</td><td>X</td></tr>
<tr><td>15</td><td></td><td>X</td><td>X</td><td>X</td></tr>
<tr><td>16</td><td>4-topping</td><td>X</td><td>X</td><td>X</td><td>X</td><td>1</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td>Total</td><td>1 + 4 + 6 + 4 + 1 = 16</td></tr>
</table>

Table 5.2.6-1. *Sixteen choices of pizza when selecting from four toppings*.

Solution (Task 6a) – Method 2

Similarly to Method 1, one can consider how many choices the customer has within each category (i.e. for a fixed number of toppings), but instead of listing them all one can use combinations to determine the number of choices associated with each category. The combinations under each type of pizza are:

${}_4C_0 = 1$ is the number of choices of pizza with no topping (plain).

${}_4C_1 = 4$ is the number of choices of pizza with 1 topping.

${}_4C_2 = 6$ is the number of choices of pizza with 2 toppings.

${}_4C_3 = 4$ is the number of choices of pizza with 3 toppings.

${}_4C_4 = 1$ is the number of choices of pizza with 4 toppings.

Notice that the numbers of different choices, namely 1, 4, 6, 4, and 1, match the enumeration from Method 1, and they are the numbers in the 4th row of *Pascal's Triangle*:

0th row	1
1st row	1 1
2nd row	1 2 1
3rd row	1 3 3 1
4th row	1 4 6 4 1
5th row	1 5 10 10 5 1
Etc.	1 …………………………………….. 1

Figure 5.2.6-1. *Pascal's Triangle* (*in numbers*)

The French mathematician Blaise Pascal (1623-1662) found that the numbers in the *n*th row of this triangle are exactly the same as the *coefficients of binomial expansions* of $(a+b)^n$ (Larson & Hostetler, 2004, p. 656; Roberts & Tesman, 2005, p. 71-73; Stewart, Redlin, & Watson, 2002, p. 848-852). Therefore, *Pascal's Triangle* can also be written in the form of *binomial coefficients* that are defined using the combination notation as follows:

0th row	${}_0C_0$
1st row	${}_1C_0$ ${}_1C_1$
2nd row	${}_2C_0$ ${}_2C_1$ ${}_2C_2$
3rd row	${}_3C_0$ ${}_3C_1$ ${}_3C_2$ ${}_3C_3$
4th row	${}_4C_0$ ${}_4C_1$ ${}_4C_2$ ${}_4C_3$ ${}_4C_4$
5th row	${}_5C_0$ ${}_5C_1$ ${}_5C_2$ ${}_5C_3$ ${}_5C_4$ ${}_5C_5$
Etc.	1 ………………………………………………………….. 1

Figure 5.2.6-2. *Pascal's Triangle* (in *binomial coefficients*)

In general, the sum of the numbers in the *n*th row (starting from $n = 0$) of *Pascal's Triangle* is the number of all possible choices of pizzas for a customer if there are n toppings to choose from. For $n = 4$, by theorem T6.5, ${}_4C_0 + {}_4C_1 + ... + {}_4C_4 = 2^4 = 16$.

Mathematical Structure

This solution involves a commonly known mathematical object, *Pascal's Triangle*, and its relationship with the *Binomial Theorem*:

(T6.1) Definition. ***Pascal's Triangle*** is named after the French mathematician Blaise Pascal (1623-1662) who rediscovered the triangle that previously appeared in a Chinese document titled "The Old Method Chart of the Seven Multiplying Powers" by Chu Shi-kie, dated 1303. The triangle has the form shown in Figure 5.2.6-1. The first and last numbers in each row are 1. Every entry other than a "1" is the sum of the two entries diagonally above it. (Larson & Hostetler, 2004, p. 656-657; Roberts & Tesm:an, 2005, p. 39; Stewart, Redlin, & Watson, 2002, p. 849).

(T6.2) Definition. The ***binomial coefficient*** is ${}_nC_r = \frac{n!}{r!(n-r)!}$, where n and r are nonnegative integers with $r \le n$ (Stewart, Redlin, & Watson, 2002, p. 851).

(T6.3) Theorem. The key ***property of Pascal's triangle*** in terms of ***binomial coefficients***: for any nonnegative integers r and k with $r \le n$, ${}_kC_{r-1} + {}_kC_r = {}_{k+1}C_r$ (Roberts & Tesman, 2005, p. 38; Stewart, Redlin, & Watson, 2002, p. 851).

(T6.4) Theorm. ***Binomial Theorem*** (or ***Binomial Expansion***): For $n \ge 0$, $(a+b)^n = \sum_{k=0}^{n} {}_nC_r a^k b^{n-k}$ (Larson & Hostetler, 2004, p. 654, 695; Roberts & Tesman, 2005, p. 71; Ross, 1998, p. 8; Stewart, Redlin, & Watson, 2002, p. 853).

(T6.5) Theorem. ${}_nC_0 + {}_nC_1 + {}_nC_2 + ... + {}_nC_n = 2^n$ (Roberts & Tesman, 2005, p. 72; Stewart, Redlin, & Watson, 2002, p. 877).

Solution (Task 6a) – Method 3

Alternatively, check for the presence of each topping. Each different topping can be either added to or not added to a pizza. That is two different choices for each topping.

By (T1.6) the *Product Rule* (applied repeatedly), for four toppings, there are a total of $2 \times 2 \times 2 \times 2 = 2^4 = 16$ choices.

Task 6b: 4-topping Pizza with 2 Crusts (Grade 5)

The pizza shop was so pleased with your help on the first problem that they have asked us to continue our work. Remember that they offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. The pizza shop now wants to offer a choice of crusts: regular (thin) or Sicilian (thick). How many choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possible choices

Solution (Task 6b)

After selecting one of 16 possible choices of topping combinations (see Task 6a), customers have to select one of two types of crusts. Thus, by the Product Rule (T1.6), there are a total $16 \times 2 = 32$ choices of pizza for this problem.

Task 6c: 4-topping Pizza with Halves and 2 Crusts (Grade 5)

At customer request, the pizza shop has agreed to fill orders with different choices for each half of a pizza. Remember that they offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushroom, and pepperoni. There is a choice of crusts: regular (thin) and Sicilian (thick). How many different choices for pizza does a customer have? List all the possible choices. Find a way to convince each other than you have accounted for all possible choices.

Solution (Task 6c)

Let   C = Plain (with cheese and tomato sauce only),

P = Peppers; S = Sausage; M = Mushrooms; R = Pepperoni;

Then, from the solution to Task 6a, there are 16 topping combinations to choose from for each of the two halves of a whole pizza. These 16 choices are: {C, P, S, M, R, PS, PM, PR, SM, SR, MR, PSM, PSR, PMR, SMR, PSMR}. Hence, there are one choice for no topping (plain) or 4-topping, four choices for 1-topping or 3-topping, and six choices for 2-topping pizzas.

Using the two-dimensional table that has been introduced in Task 5 (see Table 5.2.5-1), put the 16 different choices on the leftmost column to represent the choices for the first half of pizza, and put the 16 different choices in the top row to represent the choices for the second half of the pizza. Then, the number of choices for both halves is the number of entries in either the upper triangle (including diagonal entries) or the lower triangle (including diagonal entries) of the table 5.2.6-2 (on the page after the next page). As explained in Task 5, entries in the upper triangle are correspondent to entries in the lower triangle with reversed ordering of toppings. This task does not concern about the ordering of toppings. Therefore, only the number of unordered pairs is counted.

The number of entries of table 5.2.6-2 is 16 × 16 = 256; the number of diagonal entries is 16; so the number of entries under (or above) the diagonal is (256 – 16)/2 = 120. Thus, the number of choices for both halves is 16 + 120 = 136. Lastly, don't forget the two choices of crust! Therefore, there are a total of 136 × 2 = 272 choices for pizza with halves when selecting from 4 toppings and 2 crusts. Listing all 272 choices may be a boring and meaningless task. However, as shown in Table 5.2.6-2 (on the next page), it can be done systematically.

	C	P	S	M	R	PS	PM	PR	SM	SR	MR	PSM	PSR	PMR	SMR	PSMR
C	C|C	C|P	C|S	…	…	…	…	…	…	…	…	…	…	…	…	C|PSMR
P	P|C	P|P	P|S													P|PSMR
S	S|C	S|P	S|S	S|M												S|PSMR
M	M|C	M|P	M|S	M|M												…
R	…	…	…	R|M	R|R											…
PS	…				…	PS|PS										…
PM	…					…	…									…
PR	…							…								…
SM	…								…							…
SR	…									…						…
MR	…										…					…
PSM	…											PSM| PSM				…
PSR	…												PSR| PSR			…
PMR	PMR|C													PMR| PMR		PMR|PSMR
SMR	SMR|C														SMR| SMR	SMR|PSMR
PSMR	PSMR|C	PSMR|P	…	…	…	…	…	…	…	…	…	…	…	…	…	PSMR|PSMR

Table 5.2.6-2. *Permutations of 16 choices of toppings taken 2* (2 *halves*) *at a time* (Incomplete entries).

Further checking on the result from the table, it is easy to see that the number of entries in the lower triangle including the main diagonal is equal to 16 + 15 + 14 + … + 3 + 2 + 1, which is 136.

5.2.7 Task 7: Ankur's Challenge (Grades 10)

Find all possible towers that are four cubes tall, selecting from cubes available in three different colors, so that the resulting towers contain at least one of each color. Convince us that you have found them all.

Solution – Method 1

This task is the same as "Case 3: Towers with exactly 3 colors" in the solution method 2 of "Task 3: 4-tall Towers when choosing from 3 colors" (section 5.2.3). Here's a copy of the solution presented there:

Because the tower is 4 cubes tall, one of the three colors must be used twice.

There are ${}_3C_1 = 3$ ways to choose one "repeating" color.

There are ${}_4C_2 = 6$ ways to choose two positions for "repeating" colored cubes.

There are ${}_2C_1 = 2$ ways to choose positions for the remaining two "non-repeating" colored cubes.

So, there are $({}_3C_1)({}_4C_2)({}_2C_1) = (3)(6)(2) = 36$ ways to build 4-tall towers with exactly 3 colors.

Solution – Method 2

If the four cubes of a tower are divided into different color groups, there must be three groups: two cubes of color-1, one cube of color-2, and one cube of color-3. By formula T7.2 (discussed below), there are $\frac{4!}{2!1!1!} = 12$ *distinguishable combinations* (once color-1 is fiexed). Further, there are ${}_3C_1 = 3$ ways to choose a color to be the "repeating" color-1. So, there are a total of $3 \times 12 = 36$ ways to build 4-tall towers containing at least one of each color.

Mathematical Structure

Note that solution method 1 focuses on the positions of different color cubes, while solution method 2 focuses on the grouping of different color cubes.

(T7.1) Definition. If a set of n objects consists of k different kinds of objects with n_1 objects of the first kind, n_2 objects of the second kind, n_3 objects of the third kind, and so on, where $n_1 + n_2 + ... + n_k = n$, then the number of ***distinguishable combinations*** of these objects is denoted as $C(n; n_1, n_2, ..., n_k)$, which is also called the ***multinomial coefficient*** (Larson & Hostetler, 2004, p. 666; Roberts & Tesman, 2005, p. 59; Ross, 1998, p. 10-11; Stewart, Redlin, & Watson, 2002, p. 874).

(T7.2) Theorem. $C(n; n_1, n_2, ..., n_k) = \dfrac{n!}{n_1! n_2! n_3! ... n_k!}$ (Roberts & Tesman, 2005, p. 61; Ross, 1998, p.5; Ross, 1998, p. 11; Stewart, Redlin, & Watson, 2002, p. 874).

(T7.3) Theorem. $P(n; n_1, n_2, ..., n_k) = C(n; n_1, n_2, ..., n_k)$ (Larson & Hostetler, 2004, p. 666; Roberts & Tesman, 2005, p. 63).

(T7.4) Theorem. ***Multinomial Theorem***:
For $n, n_1, n_2, ..., n_k \geq 0$ and $n_1 + n_2 + ... + n_k = n$,

$(a_1 + a_2 + ... + a_k)^n = \sum_{(n_1, n_2, ... n_k)} C(n; n_1, n_2, ..., n_k) a_1^{n_1} a_2^{n_2} ... a_k^{n_k}$ (Ross, 1998, p. 12; Roberts & Tesman, 2005, p. 196).

Solution – Method 3

Alternatively, when choosing from three colors:

The number of all possible 4-tall towers built with three or less colors is: 3^4

The number of all possible 4-tall towers built with exactly one color is: 3

Next, consider the following in order to find the number of all possible 4-tall towers built with exactly two colors: there are 3 ways to choose two colors out of three colors; there are 2^4 ways to build all possible 4-tall towers with the selected two colors; among these 2-color 4-tall towers, two towers are single colored. Therefore, the number of all possible 4-tall towers built with exactly two colors is: $3(2^4 - 2)$.

Therefore, all 4-tall towers built with exactly three colors can be computed as:

$$3^4 - [3(2^4 - 2)] - 3 = 81 - 42 - 3 = 36$$

5.2.8 Task 8: World Series (Grades 11)

In a World Series, two teams play each other in at least four and at most seven games. The first team to win four games is the winner of the World Series. Assuming that both teams are equally matched, what is the probability that a World Series will be won: (a) In four games? (b) In five games? (c) In six games? (d) In seven games?

Solution – Method 1

Let the two teams be team *A* and team *B*.

Let *E* be the *event* that team *A* wins the World Series.

Let *F* be the *event* that team *B* wins the World Series.

Then *E* and *F* are *mutually exclusive events*.

Focus on the number of ways that team *A* can win the World Series, this can be denoted as $n(E)$, the number of outcomes in *E*. The following table explains how to find $n(E)$:

Series Ended in	Necessary Condition	$n(E)$
4 games	team A won 3 games in the first 3 games and won the 4th game	${}_3C_3 = 1$
5 games	team A won 3 games in the first 4 games and won the 5th game	${}_4C_3 = 4$

6 games	team A won 3 games in the first 5 games and won the 6th game	${}_5C_3 = 10$
7 games	team A won 3 games in the first 6 games and won the 7th game	${}_6C_3 = 20$

Table 5.2.8-1. *The numbers of ways that team A can win the World Series*.

The number of elements in the *sample space*, denoted by $n(S)$ and representing the number of all possible ways that the Series ended in n games (where $4 \le n \le 7$), is 2^n (this is because each game can ends in two ways: either A wins or B wins). By definition T8.2, the *probability* $P(E)$ of team A to win the Series in n games can be computed as in the following table:

n	$n(E)$	$n(S)$	$P(E) = n(E) / n(S)$
4	1	$2^4 = 16$	1/16 = 0.0625
5	4	$2^5 = 32$	4/32 = 0.125
6	10	$2^6 = 64$	10/64 = 0.15625
7	20	$2^7 = 128$	20/128 = 0.15625

Table 5.2.8-2. *The probability that team A wins the World Series in n games*.

Similarly, team *B* can win the World Series in *n* games exactly like team *A* does. Therefore, $n(F)$ is the same as $n(E)$, $n(S)$ does not changed, and the $P(F)$, the *probability* of team *B* to win the Series in *n* games, is the same as $P(E)$. Assuming there is no tie in any game, one of the two teams must win the World Series. Therefore, event *F* is a*n complement* of *E* (denoted by E^c), and $P(F) = P(E^c)$. Because both teams are equally matched, it must be true that both $P(A$ wins the World Series) and $P(B$ wins the World Series) are both equal to 0.5.

Because events E and F are *mutually exclusive*, by theorem T8.6, the *probability* of either team A or team B winning the Series in n games (the World Series ends in n games) is $P(E \text{ or } F) = P(E \cup F) = P(E) + P(F)$:

n	$P(E)$	$P(F)$	$P(E$ or $F)$
4	0.0625	0.0625	0.125
5	0.125	0.125	0.25
6	0.15625	0.15625	0.3125
7	0.15625	0.15625	0.3125
Sum:	0.5	0.5	1

Table 5.2.8-3. *The probability of the World Series ending in n games*.

Note that every *probability* calculated is between 0 and 1 and that, as expected, the probability that team A (or team B) wins the World Series is 0.5.

Mathematical Structure

The solution of this task involves probability-related concepts and terminology:

(T8.1) Definition. An ***experiment*** is a process that gives definite results, called the ***outcomes*** of the experiment. The ***sample space*** *S* of an experiment is the set of all possible outcomes. An ***event*** is any subset of the *sample space S* (Larson & Hostetler, 2004, p. 672; Roberts & Tesman, 2005, p. 42; Ross, 1998, p. 25; Stewart, Redlin, & Watson, 2002, p. 882-883).

(T8.2) Definition. Let *S* be the sample space of an experiment in which all outcomes are equally likely, and let *E* be an event. The ***probability of E*** is $P(E) = \frac{n(E)}{n(S)} = \frac{number.of.elements.in.E}{number.of.elements.in.S}$ (Larson & Hostetler, 2004, p. 673; Roberts & Tesman, 2005, p. 42; Ross, 1998, p. 36; Stewart, Redlin, & Watson, 2002, p. 883).

(T8.3) Axiom. $0 \le P(E) \le 1$ (Larson & Hostetler, 2004, p. 673; Roberts & Tesman, 2005, p. 42; Ross, 1998, p. 30; Stewart, Redlin, & Watson, 2002, p. 883).

(T8.4) Axiom. $P(S) = 1$ (Ross, 1998, p. 31; Stewart, Redlin, & Watson, 2002, p. 883).

(T8.5) Axiom. The probability that event *E* does not occur, which denoted as E^c the ***complement of E***, is $P(E^c) = 1 - P(E)$ (Larson & Hostetler, 2004, p. 679; Roberts & Tesman, 2005, p. 44; Ross, 1998, p. 32; Stewart, Redlin, & Watson, 2002, p. 883).

(T8.6) Theorem. If *E* and *F* are ***mutually exclusive events*** in a sample space *S*, then the ***probability of E or F*** is $P(E \cup F) = P(E) + P(F)$ (Larson & Hostetler, 2004, p. 676; Roberts & Tesman, 2005, p. 44; Ross, 1998, p. 31; Stewart, Redlin, & Watson, 2002, p. 886).

Solution – Method 2

In table 5.2.8-1, *n*(*E*) are written in terms of binomial coefficients (combinations). With knowledge acquired through work on previous tasks (4-tall towers), the solution can be refined by using Pascal's Triangle:

Series Ended in *n* games	*n*(*E*) (red and framed)	(1/2)*n*(*S*) (Sum of the row)	*P*(*E*)
1 (impossible)	1	1	0
2 (impossible)	1 1	2	0
3 (impossible)	1 2 1	4	0
4	1 3 3 $\boxed{1}$	8	1/16 = 0.0625
5	1 4 6 $\boxed{4}$ 1	16	4/32 = 0.125
6	1 5 10 $\boxed{10}$ 5 1	32	10/64 = 0.15625
7	1 6 15 $\boxed{20}$ 15 6 1	64	20/128 = 0.15625

$_3C_3$ $_4C_3$

$_6C_3$ $_5C_3$

Table 5.2.8-4. *The probability that team A wins the World Series in n games.* (*using Pascal's Triangle*).

The sum of each row of Pascal's Triangle represents only half of the number of elements in the *sample space*, because the entries in the Triangle includes only the combinations that team A wins under different conditions. The other half of the elements are those entries in another Pascal's Triangle that includes the combinations that team *B* wins. With the same *P*(*E*) as found in solution method 1, *P*(*E* or *F*) must also be matched.

5.2.9 Task 9: Points (Grades 11)

Pascal and Fermat are sitting in a café in Paris and decide to play a game of flipping a coin. If the coin comes up heads, Fermat gets a point. If it comes up tails, Pascal gets a point. The first to get ten points wins. They each ante up fifty francs, making the total pot worth one hundred francs. They are, of course, playing "winner takes all". But then a strange thing happens. Fermat is winning, eight points to seven, when he receives an urgent message that his child is sick and he must rush to his home in Toulouse. The carriage man who delivered the message offers to take him, but only if they leave immediately. Of course, Pascal understands, but later, in correspondence, the problem arises: how should the hundred francs be divided?

Solution – Method 1

Let H = Head (i.e. Fermat wins a point) and T = Tail (i.e. Pascal wins a point).

When Fermat got the emergency message, Pascal had 7 points and Fermat had 8 points already. This means they had already played a total of 15 rounds of coin flipping. To win the game, Pascal needed 3 more tails, or Fermat needed 2 more heads. The following table examines how many more rounds are needed to produce a winner:

1 more round	H or T	No winner
2 more rounds	HH	Fermat wins
	HT, TH, TT	No winner
3 more rounds	HTH, THH	Fermat wins
	TTT	Pascal wins
	HTT, THT, TTH	No winner
4 more rounds	HTTH, THTH, TTHH	Fermat wins
	HTTT, THTT, TTHT	Pascal wins

Table 5.2.9-1. *Number of rounds and corresponding outcomes to produce a winner.*

From the analysis above, given the preexisting situation that Fermat has 8 points and Pascal has 7 points (abbreviated as "8H7T"), Fermat would win in two, three, or four "*more*" rounds. The word "*more*" here indicates that the given situation has been taken into account. This is called *conditional probability* (T9.2), and *P*(Fermat wins the game | 8H7T) is used as standard notation for "the probability that Fermat wins the game given that currently Fermat has 8 points and Pascal has 7 points".

However, the probability to get an "H" in any round of flipping a fair coin is 0.5, which is not dependent on 8H7T or any other preexisting situations. Therefore, the *conditional probability* *P*(HH|8H7T) is the same as the simple probability *P*(HH). This is also the case for the rest of the scenarios under which Fermat wins. This fact makes the calculations easier. Using the information in table 5.2.9-1, the probability of each case that Fermat wins are:

2 more rounds: $P(HH)=\frac{1}{2^2}=\frac{1}{4}$

3 more rounds: $P(HTH \cup THH)=P(HTH)+P(THH)=\frac{1}{2^3}+\frac{1}{2^3}=\frac{1}{4}$

4 more rounds:
$$\begin{aligned}&P(HTTH \cup THTH \cup TTHH)\\&=P(HTTH)+P(THTH)+P(TTHH)\\&=\frac{1}{2^4}+\frac{1}{2^4}+\frac{1}{2^4}=\frac{3}{16}\end{aligned}$$

Therefore, *P*(Fermat wins the game | 8H7T)

= *P*(Fermat wins in 2 *more*, 3 *more*, or 4 *more* rounds)

$=\frac{1}{4}+\frac{1}{4}+\frac{3}{16}=\frac{11}{16}=0.6875$

Note that because of the *H* is an independent event, by theorem T9.1, $P(HH)=P(H \cap H)=P(H)P(H)$. Because events HTH and THH are mutually exclusive in the sample space of outcomes of three tosses, by theorem T8.6, $P(HTH \cup THH) = P(HTH) + P(THH)$. The same reason applies in the case of "4 more rounds".

It is impossible to have a tie in any round. The only two ways for the game to end are either "Pascal wins" or "Fermat wins". Therefore, the event "Pascal wins" is the *complement* (T.8.5) of "Fermat wins". So,

P (Pascal wins the game | 8H7T) = 1 – *P*(Fermat wins the game | 8H7T)

= 1 – 0.6875 = 0.3125

Hence, the hundred francs should be divided according to the *conditional probability* of each person winning the game if the game continued from the given situation:

Fermat: $100 \times 0.6875 = 68.75$ francs

Pascal: $100 \times 0.3125 = 31.25$ francs

Mathematical Structure

The concept of *conditional probability* is involved in the solution:

(T9.1) Theorem. If *E* and *F* are ***independent events*** in a sample space *S*, then the ***probability of E and F*** is $P(E \cap F) = P(E)P(F)$ (Larson & Hostetler, 2004, p. 678; Roberts & Tesman, 2005, p. 44-45; Ross, 1998, p. 84; Stewart, Redlin, & Watson, 2002, p. 889). [this theorem is not relevant to the solution…]

(T9.2) Definition. A ***conditional probability P(E|F)*** (read as "the probability of *E* given *F*") gives the probability of event *E* under the condition that event *F* has occurred. If $P(F) > 0$, then $P(E \mid F) = \dfrac{P(E \cap F)}{P(F)}$ (Bock, Velleman, & De Veaux, 2007, p. 349; Moore & McCabe, 1999, p. 350, 352; Ross, 1998, p. 67-68).

(T9.3) Definition. Two events *E* and *F* are ***independent*** if $P(E \mid F) = P(E)$ (Bock, Velleman, & De Veaux, 2007, p. 351; Moore & McCabe, 1999, p. 357; Ross, 1998, p. 84).

Solution – Method 2

Because a fair coin can come up with either Head or Tail in every round, a binary tree structure (T2.9) can be used to show the possible ways in which the game can end: start with 8 heads (Fermat got 8 points) and 7 Tails (Pascal got 7 points) as the root, which has probability 1 because it happened already. Each of the two branches (edges) from the root has probability of 0.5 and produces no winner. That means the resulting nodes have to branch out again. The next level of the tree has four nodes. The leftmost node 8H7THH represents "Fermat wins the games (10 Hs) in two more rounds" (the 2nd level of the tree). The other three nodes continued to branch out for the third round. The process continues until every end node (leaf) represents a winner. Note that each edge from the binary branching is labeled with the probability of 0.5 because it is equally likely to get a Head or a Tail at each toss.

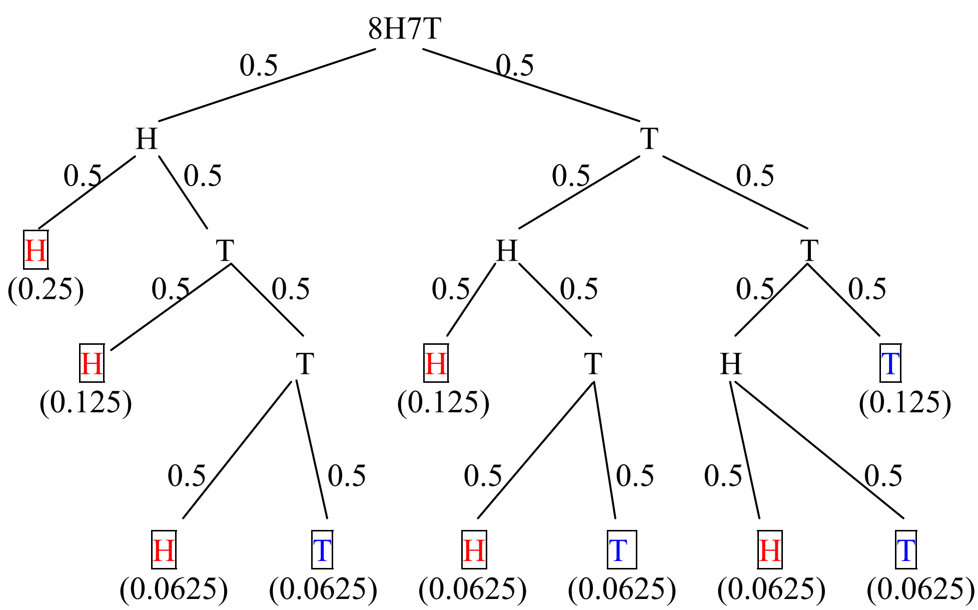

Figure 5.2.9-1. *The binary tree with conditional probability of the winner.*

In this tree, there are six $\boxed{H}$s (i.e. Fermat wins the game) and four $\boxed{T}$s (i.e. Pascal wins the game). The number in "( )" below each winning node is the *probability* of each winning case based on the *general multiplication rule* (T9.4). For example, the leftmost winning node on the forth level is HTTH (abbreviation from 8H7THTTH for better readability), and P(HTTH) = P(H)P(T|H)P(T|HT)P(H|HTT) = 0.5×0.5×0.5×0.5 = 0.0625. Therefore, P(Fermat wins the game|8H7T) = 0.25 + 2(0.125) + 3(0.0625) = 0.6875. This matches the calculation shown in method 1.

Mathematical Structure

(T9.4) Definition. A ***General Multiplication Rule*** for compound events E_1, E_2, E_3, …, E_n that does not require the events to be independent is:

$$P(E_1E_2E_3...E_n) = P(E_1)P(E_2 \mid E_1)P(E_3 \mid E_1E_2)...P(E_n \mid E_1E_2...E_{n-1})$$

(Bock, Velleman, & De Veaux, 2007, p. 350; Moore & McCabe, 1999, p. 352; Ross, 1998, p. 71).

Solution – Method 3

Because the edges of each branching in figure 5.2.9-1 are labeled with probability of 0.5 (equal chance to get a Head or a Tail in every coin toss), a simpler solution method is suggested. Build a complete binary tree and count the number of outcomes of Fermat wins (represented by $\boxed{\text{H}}$):

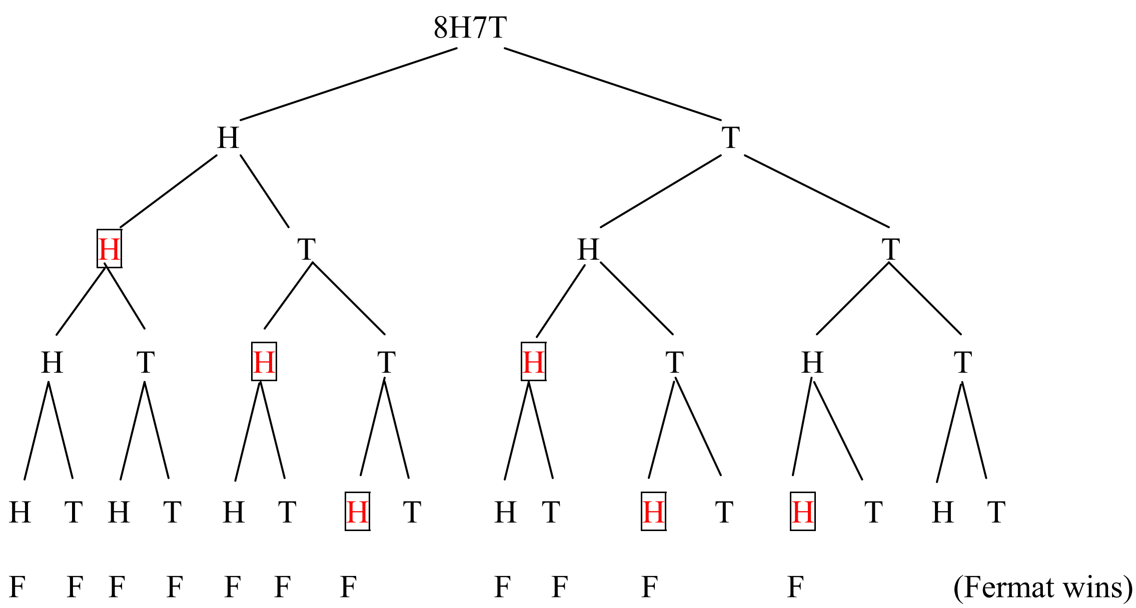

Figure 5.2.9-2. *The complete binary tree representation for Fermat winning.*

In this tree, total number of outcomes (leaves) is 16. If $\boxed{H}$ appears at any level of the tree, the resulting outcomes (leaves) are marked as Fermat winning. There are 11 leaves marked "F". So P(Fermat wins) = $\frac{11}{16}$.

5.2.10 Task 10: Taxicab (Grade 12)

A taxi driver is given a specific territory of a town, represented by the grid in the diagram below. All trips originate at the taxi stand, the point in the top left corner of the grid. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated by the other points on the grid. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route. What is the shortest route from the taxi stand to each of three different destination points? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answers.

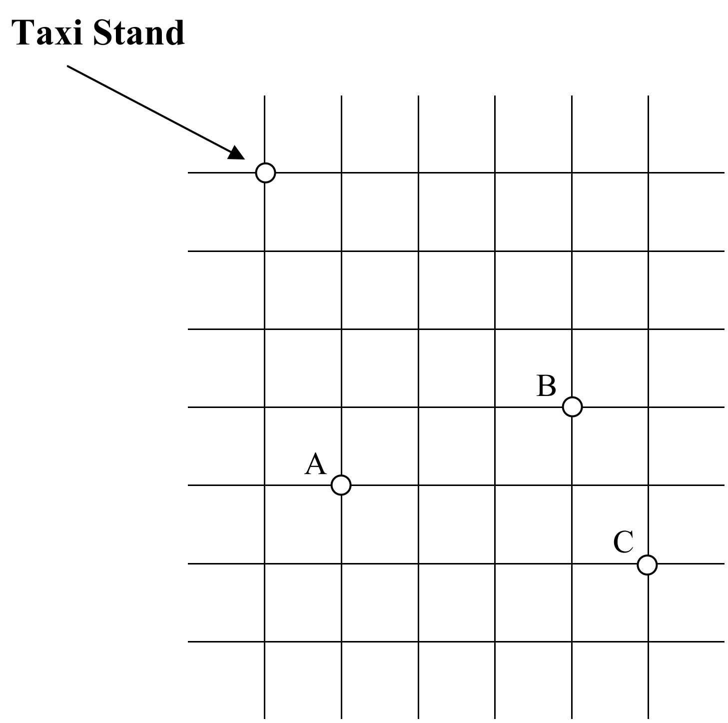

Figure A-2. *The map of the town and the taxi stand*

Solution

This problem was first proposed by Hermann Minkowski in the beginning of the 20^{th} century (Gardner, 2007, p. 160). The term "taxicab" was not used until 1952 by an Austrian mathematician Karl Menger (Powell, 2003, p. 5-6). According to Krause (1975/1986), Minkowski's concept is a good model in the development of a particular non-Euclidean geometry named Taxicab geometry that differs from Euclidean geometry by just one axiom – the *distance function* (p. 2).

In general, on the *Cartesian* (*coordinate*) *plane* (T10.1), a shortest route from the taxi stand with *coordinates* (x_1, y_1) to the destination point with *coordinates* (x_2, y_2) can be found in this way: start at the taxi stand and travel $|y_2 - y_1|$ (T10.2) blocks vertically to the turning point $T(x_1, y_2)$, which lies on the same horizontal line as the destination; then, turn the taxicab 90˚ heading toward the direction of the destination point and go $|x_2 - x_1|$ blocks horizontally. The *Taxicab distance* (T10.4) of this shortest route is $|x_2 - x_1| + |y_2 - y_1|$ blocks.

These routes are the shortest because (1) the taxi stand and the turning point are on the same vertical line, (2) the turning point and the destination are on the same horizontal line, and (3) the shortest route between two points is a line connecting them.

However, this shortest route is <u>not unique</u> because at each intersection along the route, the taxicab can choose to either go southward or eastward but still toward the destination point, resulting in a route of equal length with the one described above. It is important to note that this is true only in a city where all the streets run either straight northward and southward or straight eastward and westward and all the streets are equidistant from one another.

Figure 5.2.10-1 below places the Taxi Stand at the *origin* of the *Cartesian* (*coordinate*) *plane*. The *coordinates* of each pick-up point are identified. The light blue route is a shortest path from Taxi Stand (0, 0) to point *A* (1, −4), turning direction at point (0, −4). The red route is a shortest path from Taxi Stand to point *B* (4, −3) changing direction at point (0, −3). The light green route is a shortest path from Taxi Stand to point *C* (5, −5) passing through the turning point (0, −5).

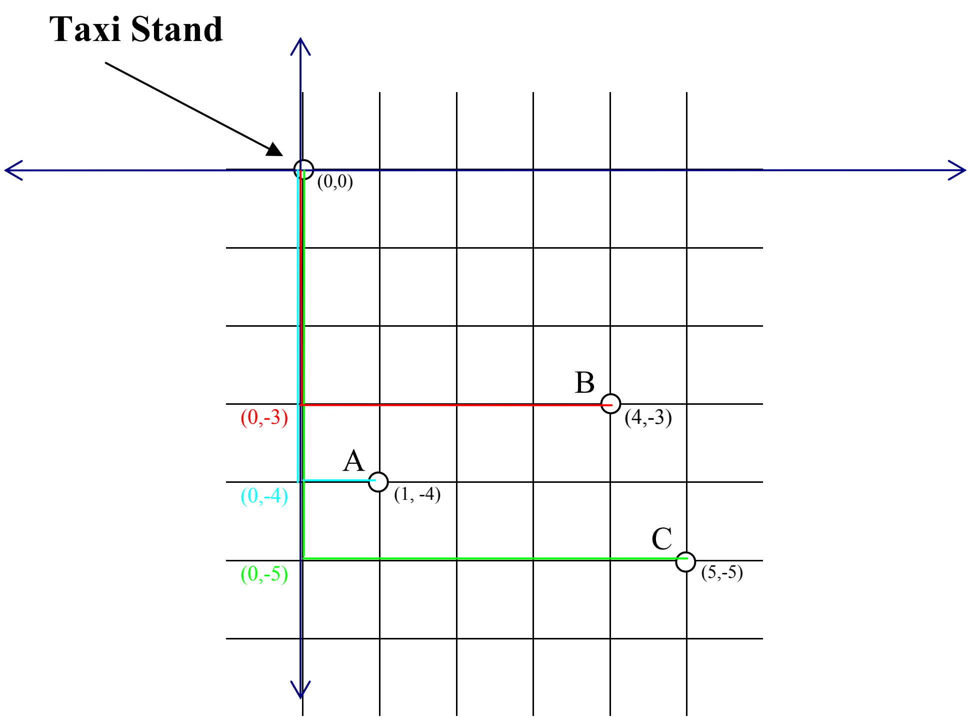

Figure 5.2.10-1. *Three shortest routes on the Cartesian Plane*

Taxicab distances can be found by counting the number of blocks along a shortest route. In the case of a huge city with hundreds of streets, one can use the formula described in T10.4 to compute:

d_T (Taxi Stand, *A*) = $|1-0|+|-4-0|=5$

d_T (Taxi Stand, *B*) = $|4-0|+|-3-0|=7$

d_T (Taxi Stand, *C*) = $|5-0|+|-5-0|=10$

Note that the *taxicab distance* is usually longer than the corresponding *Euclidean distance* unless the destination point is located on the same street as Taxi Stand. This is because the *taxicab distance* (T10.4) is the sum of the lengths of two legs while the *Euclidean distance* (T10.5) is the length of the hypotenuse of the right triangles whose vertices are the Taxi Stand, the destination point, and the turning point.

The last sub-question of this task is: *how many shortest routes are there?*

Because all three pick-up points are located southeast of the taxi stand, to find other shortest paths, always go either south or east, but never north or west, at each intersection. Let S = "going south" and E = "going east", then from the taxi stand to point A, the taxicab must choose one of five intersections to turn east. Thus, there are ${}_5C_1$ = 5 shortest routes: SSSSE, SSSES, SSESS, SESSS, and ESSSS.

Similarly, point B is four blocks east and three blocks south from the Taxi Stand. Therefore the taxicab must choose four of seven intersections to go eastward. Thus, there are ${}_7C_4$ = 35 shortest routes (Alternately, in stead of going eastward, the taxicab can choose at which three intersections to go southward. The result is the same because ${}_7C_4 = {}_7C_3 = 35$ (T10.7)).

Here is a strategy to list all 35 routes systematically: there are 3 "S"s and 4 "E"s in the letter strings representing a shortest path. Hence, there are 5 slots between and around the 4 "E"s : (1) E(2) E (3) E (4) E (5) . From these 5 slots, choose where to place 3 "S"s. The 35 shortest routes can be separately listed by four cases:

Case 1: SSS is put into one of 5 slots. There are ${}_5C_1$ = 5 ways to do this:

SSSEEEE, ESSSEEE, EESSSEE, EEESSSE, EEEESSS

Case 2: SS_S is put into two of 5 slots. There are ${}_5C_2$ = 10 ways to do this:

SSESEEE, SSEESEE, SSEEESE, SSEEEES,
ESSESEE, ESSEESE, ESSEEES,
EESSESE, EESSEES,
EEESSES

Case 3: S_SS is put into two of 5 slots. There are ${}_5C_2 = 10$ ways to do this:

SESSEEE, SEESSEE, SEEESSE, SEEEESS,
ESESSEE, ESEESSE, ESEEESS,
EESESSE, EESEESS,
EEESESS

Case 4: S_S_S is put into three of 5 slots. There are ${}_5C_3 = 10$ ways to do this:

SESESEE, SESEESE, SESEEES, SEESESE, SEEESES, SEESEES,
ESESESE, EESESES, ESESEES, ESEESES

By the same reasoning, the number of shortest paths from Taxi Stand to station *C* is ${}_{10}C_5 = 252$. Because point C is five blocks east and five blocks south from Taxi Stand, the taxicab must choose five of ten intersections to turn east. (The complete list of these routes is omitted because it is very time consuming.)

Further, the binary branching resulting from having the option to go either east or south at each intersection suggests that the binary coefficients and Pascal's Triangle that appear in many of the previously discussed tasks (e.g., Towers, Pizzas, World Series, and Points) may be relevant to this problem as well. By labeling each intersection on the grid with the number of shortest routes one can go from the Taxi Stand to that particular point and toward destination points A, B, and C, Pascal's Triangle can be found "lying" on the diagonal of the grid as shown in the following figure:

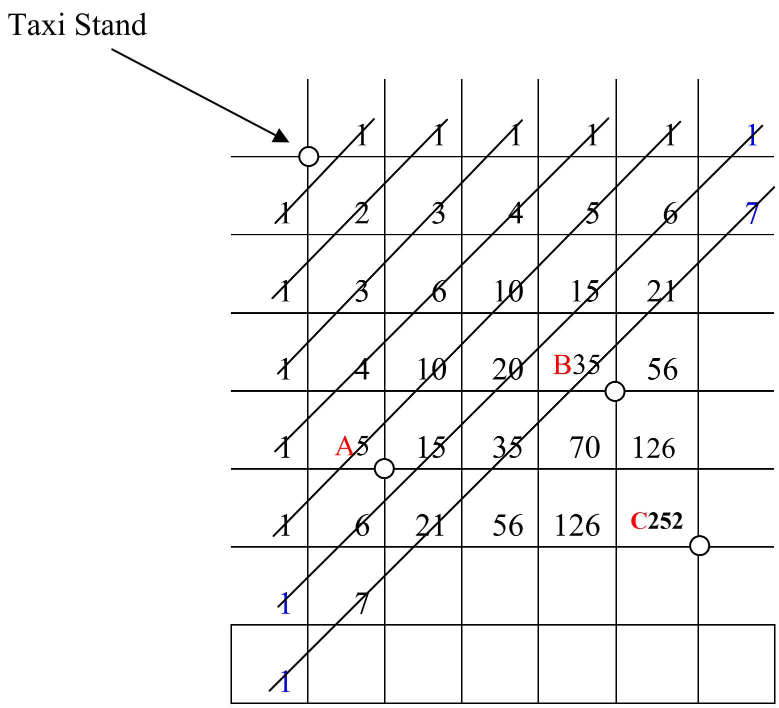

Figure 5.2.10-2. *Embedded Pascal's Triangle*.

From this diagram, a general process by which to find the shortest routes from the Taxi Stand to any point on the grid is evident.

Mathematical Structure

Some elementary terms of the coordinate system are mentioned in the discussion of the solution:

(T10.1) Definition. The ***Cartesian (coordinate) plane*** is represented by drawing two intersecting perpendicular real lines called ***x-axis*** (horizontal line) and the ***y-axis*** (vertical line). Any ***point*** *P* in the plane can be identified by an ordered pair (*a*, *b*) of real numbers. The first number *a* is called ***x-coordinate*** of *P*, and the second number *b* is called the ***y-coordinate*** of *P*. The intersection of the *x*-axis and the *y*-axis, the point (0,0), is called the ***Origin*** (Larson & Hostetler, 2004, p. A78; Stewart, Redlin, & Watson, 2002, p. 89-90).

(T10.2) Definition. If *a* is a number, then the ***absolute value*** of *a* is $|a| = a$ if $a \geq 0$ and $|a| = -a$ if $a < 0$ (Larson & Hostetler, 2004, p. A4; Stewart, Redlin, & Watson, 2002, p. 10).

(T10.3) Definition. If *a* and *b* are real numbers, then the ***distance*** between the points *a* and *b* on the real line is $d(a,b) = |b - a|$ (Stewart, Redlin, & Watson, 2002, p.11).

Comparing Taxicab geometry with Euclidean geometry, the *distance functions* must be clearly defined:

<u>(T10.4) Definition.</u> Given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the ***Taxicab distance*** between P and Q, denoted as $d_T(P, Q)$, is the count of blocks (does not have to be an integer) that a taxicab chooses to travel in either horizontal or vertical direction at each intersection it passes through along the way and $d_T(P,Q) = |x_2 - x_1| + |y_2 - y_1|$ (Krause (1975/1986, p. 4).

<u>(T10.5) Definition.</u> Given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the ***Euclidean distance*** between P and Q, denoted as $d_E(P, Q)$, is derived from the *Pythagorean Theorem* and $d_E(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (Krause, 1975/1986, p. 4; Larson & Hostetler, 2004, p. A80; Stewart, Redlin, & Watson, 2002, p. 91).

<u>(T10.6) Theorem.</u> In a right triangle with a and b as the lengths of two legs and c the length of the hypotenuse, the ***Pythagorean Theorem*** states: $a^2 + b^2 = c^2$ (Larson & Hostetler, 2004, p. 349).

Note that unlike the *taxicab distance*, the *Euclidean distance* corresponds to a unique route.

In addition, a fact has shown in Pascal's triangle and Binomial theorem but never formally defined before:

<u>(T10.7) Theorem.</u> ${}_nC_r = {}_nC_{n-r}$ (Roberts & Tesman, 2005, p. 36).

5.3 Overall Concept Map

In this study, mathematical concepts are meant to be abstract and unorganized mathematical ideas (knowledge) residing in our mind. Mathematical structures, on the other hand, are characterized by the authors as constructed images of mathematics that describe and organize mathematical concepts into a hierarchy of interconnected entities that are building on one another to produce a coherent whole. The main focus of this section is to organize mathematical structures identified in section 5.2 into a hierarchy. Concept mapping is used to construct this hierarchy. Novak & Cañas (2008) suggested that each construction of the concept map should include only 15 to 25 key concepts at a time. However, there are many more structures found from the ten tasks discussed in the previous section. Therefore, these mathematical structures are categorized into seven broad, general, and overlapping sub-domains: *Set Theory*, *Enumerative Combinatorics*, *Graph Theory*, *Sequences & Sets*, *General Algebraic System*, *Probability Theory*, and *Geometry*. A concept map is constructed for each sub-domain. Cross links are added to connect related concepts among these sub-domains.

One may ask, "What is the hierarchy of these seven sub-domains?" Based on the Mathematics Subject Classification 2000 (MSC2000), the highest level concept map, called the global map, is structured as follows:

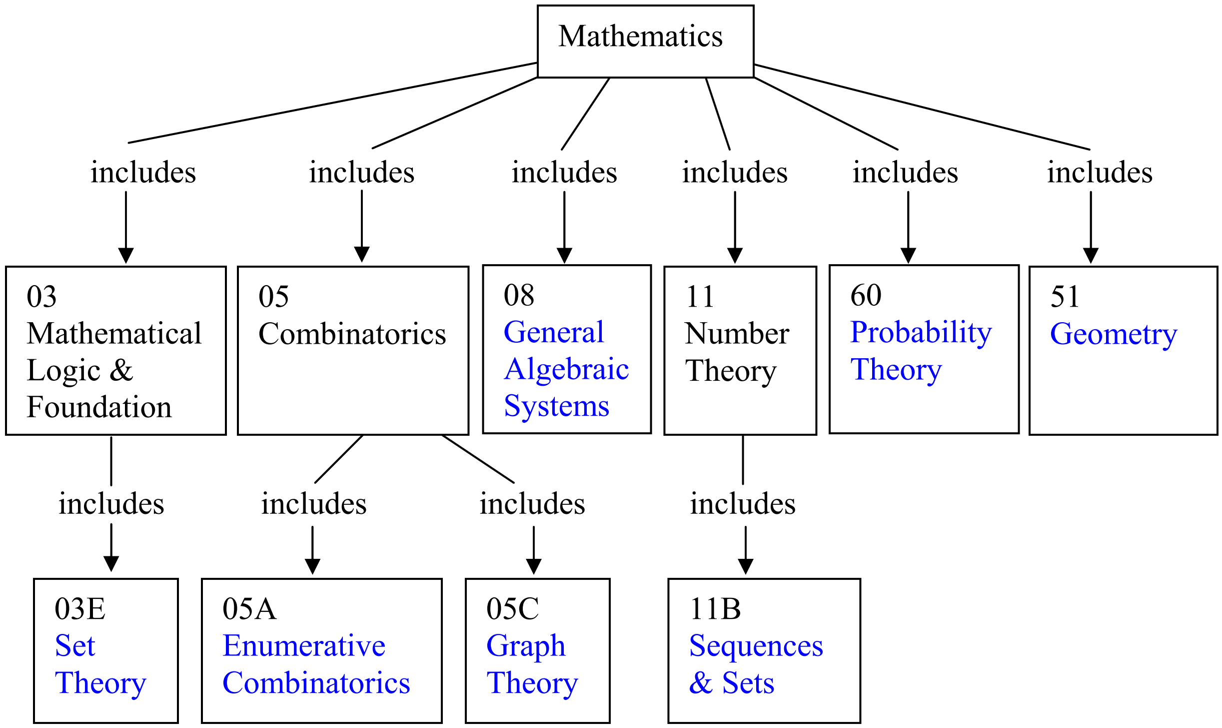

Figure 5.3-0. *The concept map covers seven sub-domains* (in blue).

For easy referencing, the number showing on the first line of each box is the section number originally used by MSC2000. There are many sub-domains in the mathematics field; this initial concept map only includes mathematical subjects related to the seven designated sub-domains.

For each sub-domain, a list of related structures is collected from section 5.2, and then a concept map is constructed. The structure code (e.g., T1.1 meaning the first structure in task 1) is attached to each entry in the structure list and in each rectangular box of the concept maps in order to ease back referencing to the description of each mathematical structure. Some boxes do not contain a structure code because these concepts or structures are not embedded in the solutions of the ten tasks. However, it is necessary to add them to the maps as they are commonly used and understood by everyone who has studied mathematics. For example, there are boxes like “Euclidean

Geometry" and "Triangle" in the concept map of Geometry. On the other hand, some structures are broken down and put into several boxes in order to increase the clarity and understandability. The structures of each sub-domain are listed in the order of their appearance in section 5.2. Some structures can be listed in more than one sub-domain. These are overlapping structures that serve as bridges, connecting pieces of mathematical knowledge into a whole. These structures are listed in italic blue in their primary sub-domain, and are listed in italic purple when they appear in other sub-domains. The boxes of overlapping concepts are framed with double lines.

Bear in mind that the purpose of constructing concept maps is to retrieve knowledge more easily and they represent only the author's point of view. The look of these concept maps may be quite different if they are constructed by other researchers, teachers, or students. Also, because every mathematical concept is somehow related to other concepts, only the most representative links are shown.

(Continued on the next page.)

Set Theory

Set structures are introduced in Task 1, Shirts and Jeans (5.2.1). These fundamental structures are used in many sub-domains and subsequent task problems. For example, a sequence $\{a_n\}$ is defined using the set notation. Similarly, the set $\{x, y\}$ may represent an edge between vertices x and y. For this reason, T1.10 and T4.2 are also included in the table.

T1.1	*Definition*	*Set*
T1.1	Definition	Object
T1.1	*Definition*	*Elements*
T1.2	*Definition*	*Roster Method*
T1.3	Definition	Set-Builder
T1.4	Definition	Ordered pair (x, y)
T1.5	Definition	Cartesian Product $A \times B$
T1.5	Definition	Cross Product
T1.5	Definition	$\{(a,b) \mid a \in A, b \in B\}$
T1.7	*Definition*	*Unordered pair* $\{x, y\}$
T1.8	*Definition*	*Graph* $G(V,E)$
T1.8	*Definition*	*V: Vertices*
T1.10	*Definition*	*Edge* $\{x, y\}$
T4.2	*Definition*	$\{a_n\}$
T10.1	*Definition*	*point* (x, y), *x-coordinate, y-coordinate*

Table 5.3-1. *List of identified structures of Set Theory.*

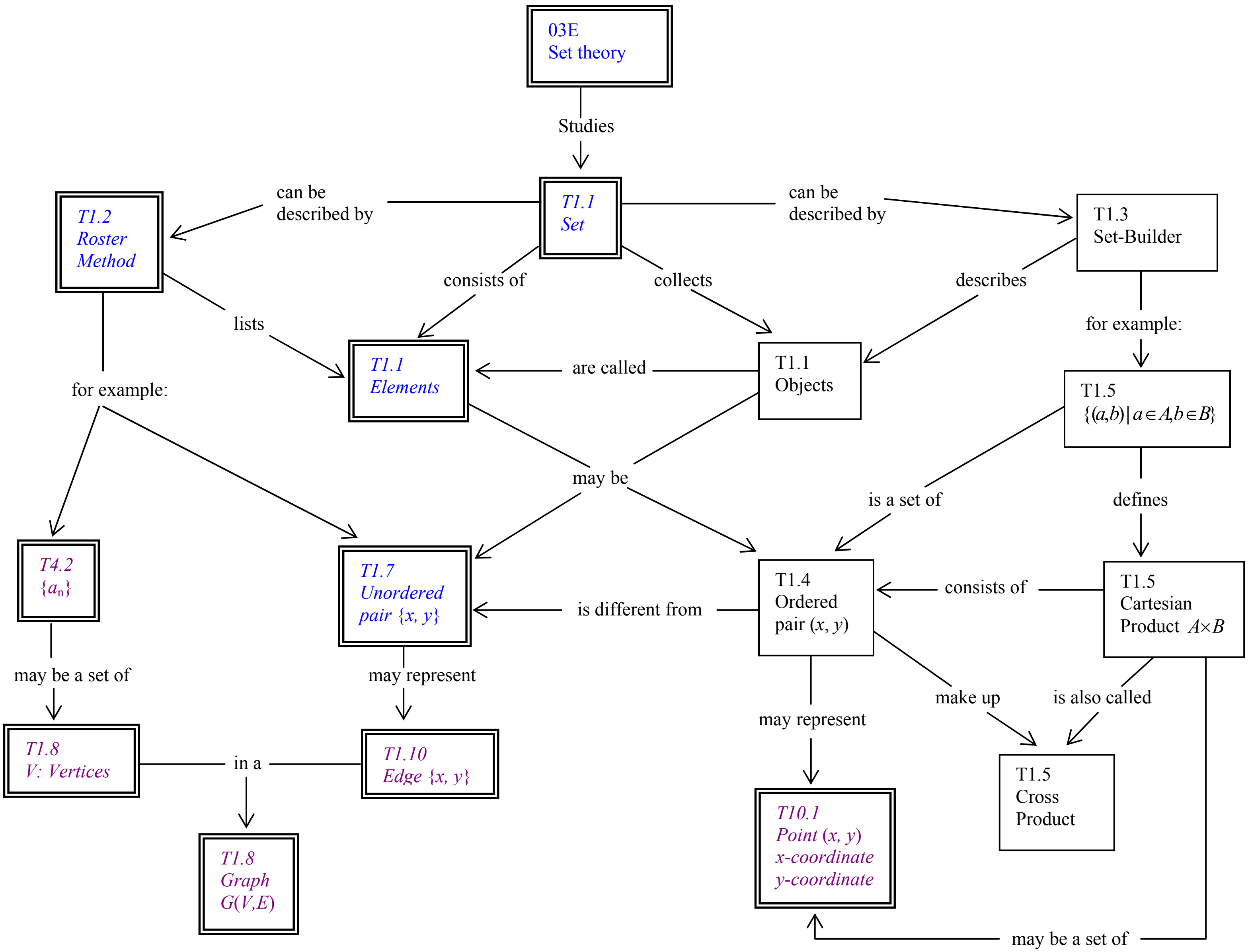

Figure 5.3-1. *Concept map of Set theory*.

Enumerative Combinatorics

Because of the isomorphism among the ten tasks of this study, every task can be solved by theorems or formulas in this sub-domain. T6.4 Binomial Theorem and T7.4 Multinomial Theorem from General Algebraic Systems are counted as overlapping structures because binomial coefficients and multinomial coefficients are represented by combination numbers.

T3.3	Definition	Then number of combinations of n elements taken r at a time: nCr or $C(n, r)$
T3.4	Theorem	${}_nC_r = \frac{n!}{r!(n-r)!}$
T5.1	Definition	Permutation
T5.2	Theorem	The number of permutations of n elements is $n!$
T5.3	Definition	The number of permutations of n elements taken r at a time: nPr or $P(n, r)$
T5.4	Theorem	$P(n,r) = \frac{n!}{(n-r)!} = n(n-1)(n-2)\cdots(n-r+1)$
T6.1	*Definition*	*Pascal's Triangle*
T6.2	Definition	Binomial Coefficients are ${}_nC_r$
T6.3	Theorem	${}_kC_{r-1} + {}_kC_r = {}_{k+1}C_r$
T6.5	Theorem	${}_nC_0 + {}_nC_1 + {}_nC_2 + \ldots + {}_nC_n = 2^n$
T7.1	Definition	The number of distinguishable combinations (Multinomial Coefficients): $C(n; n_1, n_2, \ldots, n_k)$
T7.2	Theorem	$C(n; n_1, n_2, \ldots, n_k) = \frac{n!}{n_1!n_2!n_3!\ldots n_k!}$
T7.3	Theorem	Distinguishable permutations: $P(n; n_1, n_2, \ldots, n_k) = C(n; n_1, n_2, \ldots, n_k)$
T10.7	Theorem	${}_nC_r = {}_nC_{n-r}$
T1.1	*Definition*	*Elements*
T1.6	*Theorem*	*Fundamental counting Principle (Product Rule)*
T3.2	*Definition*	$n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$
T6.4	*Theorem*	*Binomial Theorem (Binomial Expansion)*
T7.4	*Theorem*	*Multinomial Theorem*

Table 5.3-2. *List of identified structures of Enumerative Combinatorics*.

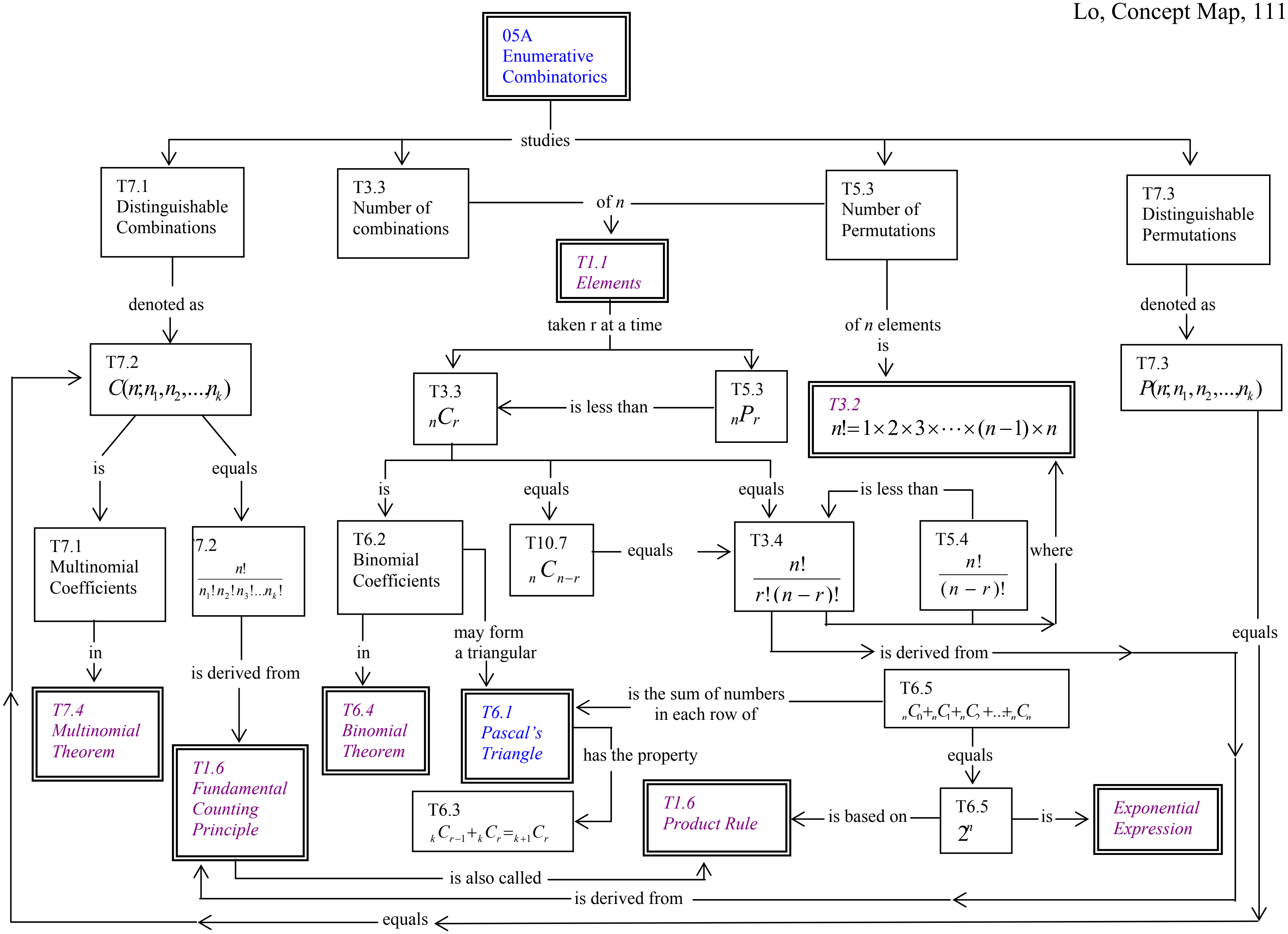

Figure 5.3-2. *Concept map of Enumerative Combinatorics.*

Graph Theory

Tree structures are described in tasks 1, 2, and 3. As it was the case earlier with Enumerative Combinatorics, every one of the tasks can also be solved by using tree structures. The use of set notations is considered to an overlapping structure.

T1.8	*Definition*	*Graph G(V, E)*
T1.8	Definition	*V* a set of vertices
T1.8	Definition	*E* a set of edges
T1.8	Definition	Endpoints
T1.9	Definition	Simple graph
T1.10	*Definition*	*Edge {x, y}*
T1.11	Definition	Bipartite graph
T2.1	Definition	Walk
T2.1	Definition	Simple walk
T2.1	Definition	Closed walk
T2.1	Definition	Path
T2.2	Definition	Connected graph
T2.3	Definition	Tree
T2.4	Definition	Degree of vertex u: deg(u)
T2.5	Definition	Level of a tree
T2.5	Definition	Height of a tree
T2.6	Definition	Rooted tree
T2.6	Definition	Root
T2.7	Definition	Ancestors of a vertex
T2.7	Definition	Parent of a vertex
T2.7	Definition	Descendent of a vertex
T2.7	Definition	Leaf of a tree
T2.8	Definition	Binary tree
T2.9	Definition	Complete tree
T3.1	Definition	m-ary tree
T1.1	*Definition*	*Set*
T1.7	*Definition*	*Unordered pair {x, y}*
T4.1	*Definition*	*Sequence* $a_1, a_2, a_3, \ldots, a_n, \ldots$

Table 5.3-3. *List of identified structures of Graph theory*.

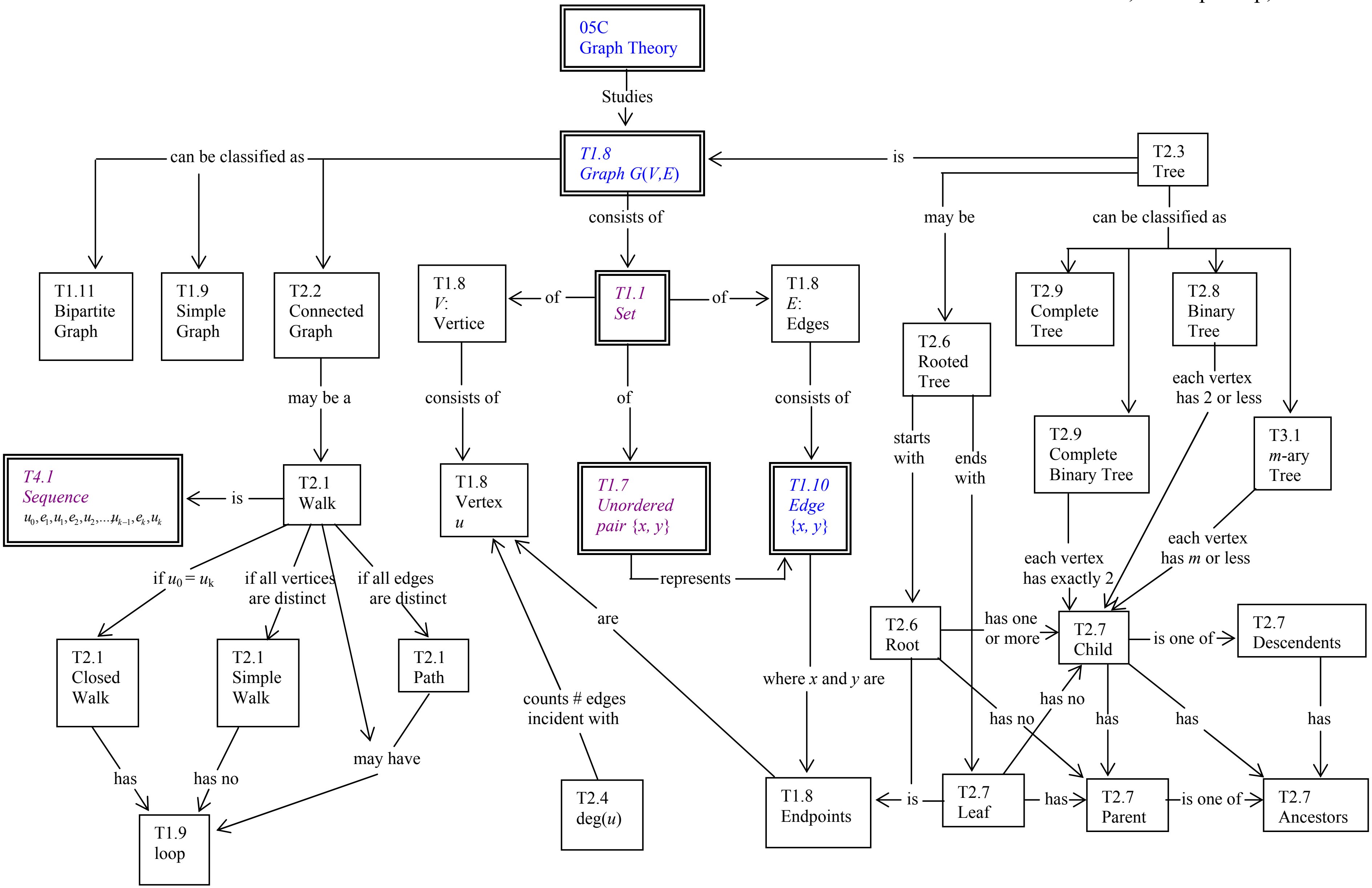

Figure 5.3-3. *Concept map of Graph Theory.*

General Algebraic Systems

General Algebraic Systems covers operation rules of variables and numbers. Therefore, some formulas listed in Enumerative Combinatorics and series are also included in this sub-domain as overlapping structures because these formulas are most likely derived from algebraic operations.

T3.2	Definition	$n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$
T4.6	Definition	Sigma: $\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots + a_n$
T6.4	Theorem	Binomial Theorem (Binomial Expansion)
T6.4	Theorem	$(a+b)^n = \sum_{k=0}^{n} {}_nC_r a^k b^{n-k}$
T7.4	Theorem	Multinomial Theorem
T7.4	Theorem	$(a_1 + a_2 + \ldots + a_k)^n = \sum_{(n_1, n_2, \ldots n_k)} C(n; n_1, n_2, \ldots, n_k) a_1^{n_1} a_2^{n_2} \ldots a_k^{n_k}$
T10.1	Definition	Cartesian (coordinate) Plane
T10.1	Definition	x-axis, y-axis
T10.1	Definition	point (x, y), x-coordinate, y-coordinate
T10.1	Definition	Origin (0, 0)
T10.2	Definition	Absolute value
T10.2	Definition	$\|a\| = a$ if $a \geq 0$, $\|a\| = -a$ if $a < 0$
T10.3	Theorem	Distance between a and b on real line
T10.3	Theorem	$d(a,b) = \|b-a\|$

Table 5.3-4. *List of identified structures of General Algebra Systems.*

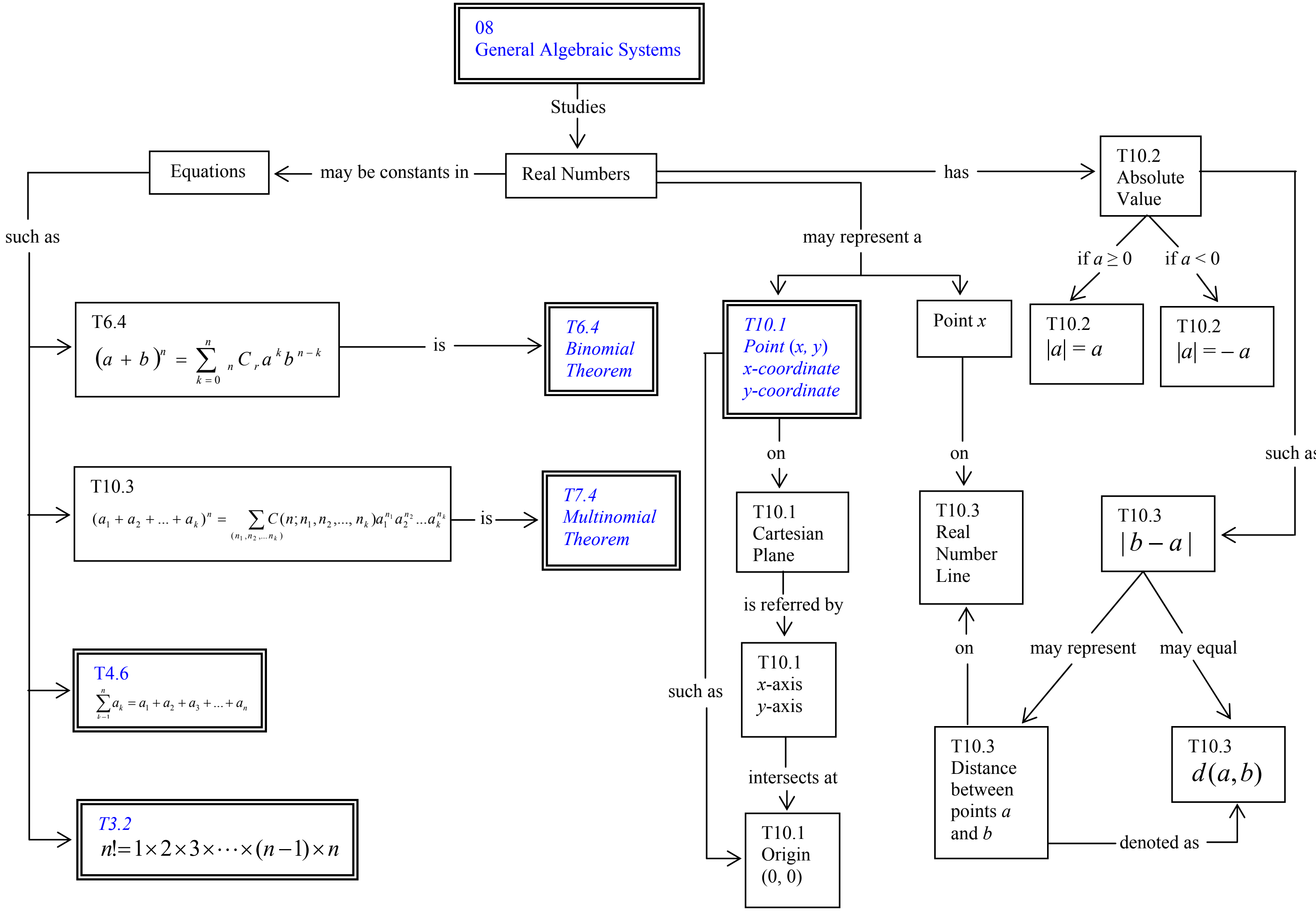

Figure 5.3-4. *Concept map of General Algebra System.*

Sequences & Sets

The roster method of set notation is often used to define a sequence. Sigma notation is used to define the *n*th partial sum and series. The structures in this sub-domain are primarily extracted from 5.2.4 task 4.

T4.1	*Definition*	*Sequence* $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$
T4.2	*Definition*	$\{a_n\}$
T4.3	Definition	Geometric Sequence $a, ar, ar^2, ar^3, ar^4, \ldots$
T4.3	Definition	The *n*th term of geometric sequence: $a_n = ar^{n-1}$
T4.4	Definition	The *n*th partial sum of a sequence: S_n
T4.4	Definition	$S_n = a_1 + a_2 + a_3 + a_4 + \ldots + a_n$
T4.4	Definition	Sequence of partial sums
T4.4	Definition	$S_1, S_2, S_3, S_4, \ldots, S_n, \ldots$
T4.5	Theorem	For a geometric sequence, $S_n = \frac{a(1-r^n)}{1-r}$
T4.7	Definition	Infinite Series $\sum_{n=1}^{\infty} a_n$
T4.8	Definition	Geometric Series
T4.8	Theorem	$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ if $\|r\| < 1$
T4.8	Definition	$\sum^{\infty} ar^{n-1}$ divergence if $\|r\| \geq 1$
T4.9	Definition	Recursively defined sequence
T4.6	*Definition*	*Sigma:* $\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots + a_n$

Table 5.3-5. *List of identified structures of Sequences & Sets*.

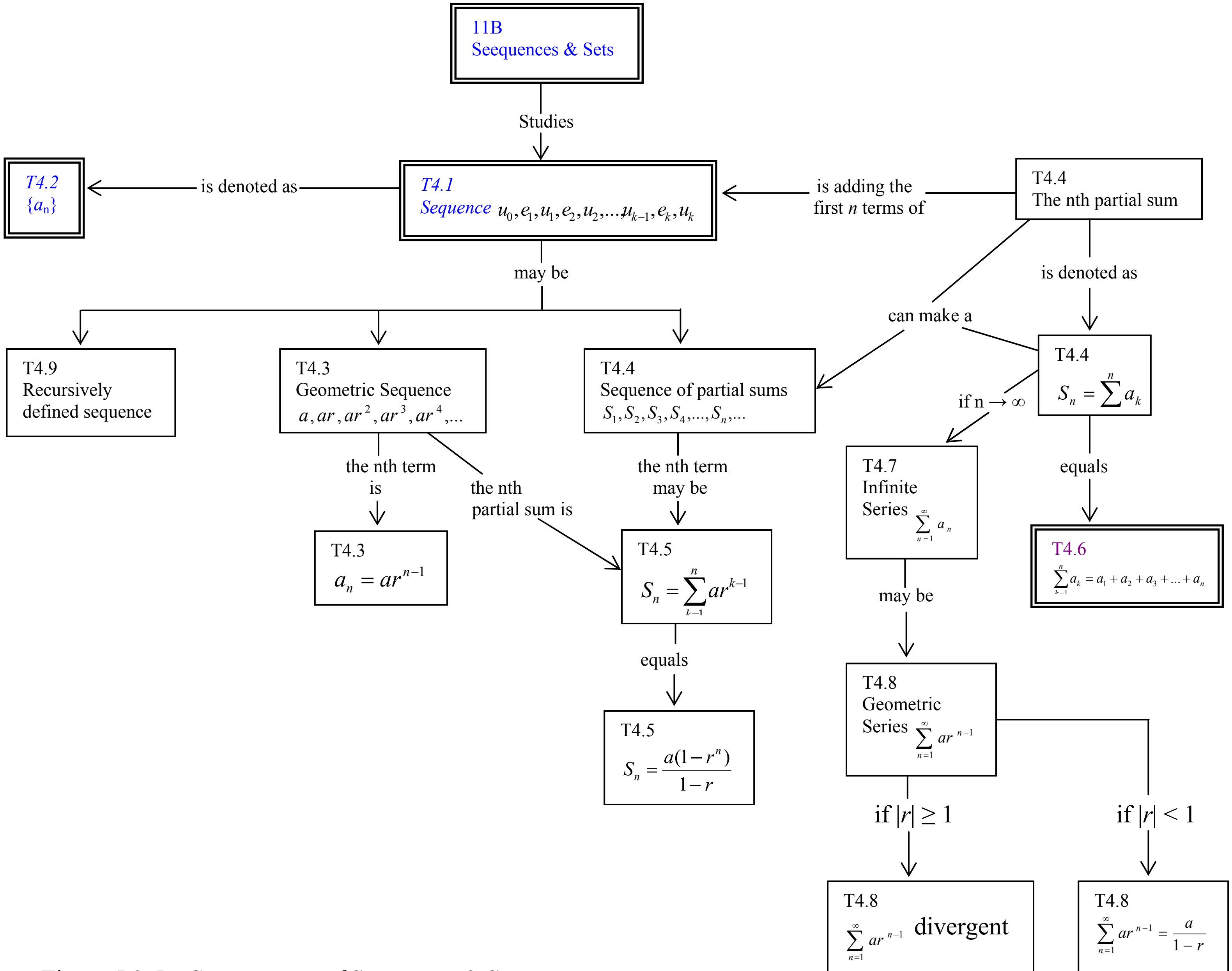

Figure 5.3-5. *Concept map of Sequences & Sets.*

Probability Theory

Fundamental counting principle (Product Rule), already listed in Enumerative Combinatorics, primarily resides in this sub-domain, and it is often used in finding the number of elements in the set of events or in the set of sample space.

T1.6	*Theorem*	*Fundamental Counting Principle (Product Rule)*
T8.1	Definition	Experiment
T8.1	Definition	Outcomes
T8.1	Definition	Sample space
T8.1	Definition	Event
T8.2	Definition	Probability of event *E*: $P(E) = \frac{n(E)}{n(S)} = \frac{number.of.elements.in.E}{number.of.elements.in.S}$
T8.3	Axiom	$0 \le P(E) \le 1$
T8.4	Axiom	$P(S) = 1$
T8.5	Definition	E^c is the complement of event E
T8.5	Axiom	$P(E^c) = 1 - P(E)$
T8.6	Theorem	For mutually exclusive events: $P(E \cup F) = P(E) + P(F)$
T9.1	Theorem	For independent events: $P(E \cap F) = P(E)P(F)$
T9.2	Definition	Conditional Probability: $P(E \mid F) = \frac{P(E \cap F)}{P(F)}$
T9.3	Definition	Events E and F are independent if $P(E \mid F) = P(E)$
T9.4	Definition	General Multiplication Rule: $P(E_1E_2E_3..E_n) = P(E_1)P(E_2 \mid E_1)P(E_3 \mid E_1E_2)..P(E_n \mid E_1E_2..E_{n-1})$
T1.1	*Definition*	*Set*

Table 5.3-6. *List of identified structures of Probability Theory.*

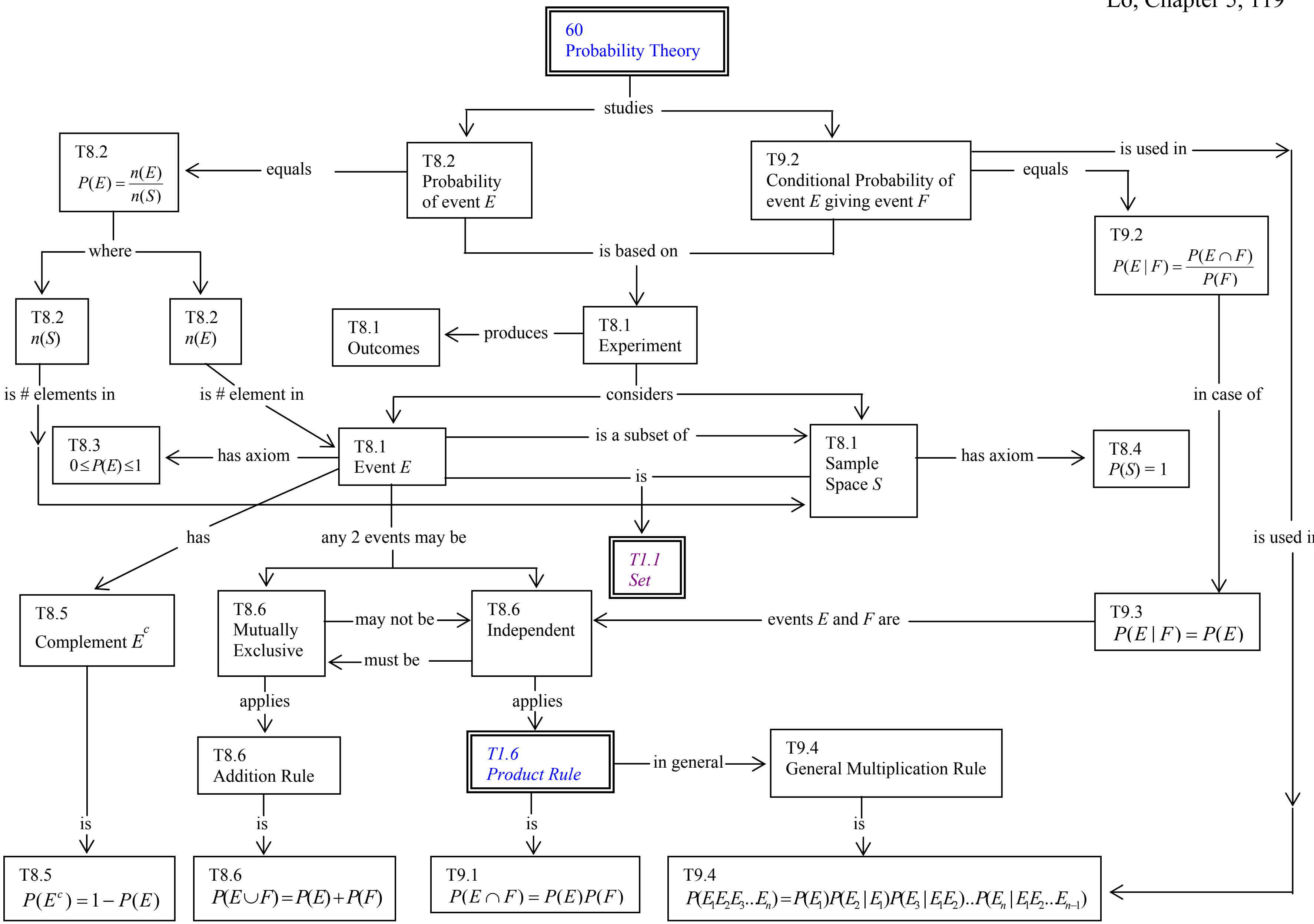

Figure 5.3-6. *Concept map of Probability Theory*.

Geometry

Very few geometry entities are identified and they are all in section 5.2.10 Taxicab. Pascal's triangle is not related to Geometry at first sight. However, its name reflects the fact that binomial coefficients can be arranged to form a triangular array, which connects and is used as a counter example to a geometry figure "triangle" in the concept map.

T10.4	Definition	Taxicab distance
T10.4	Definition	$d_T(P,Q)$
T10.4	Definition	$d_T(P,Q) = \lvert x_2 - x_1 \rvert + \lvert y_2 - y_1 \rvert$
T10.5	Definition	Euclidean distance
T10.5	Definition	$d_E(P,Q)$
T10.5	Definition	$d_E(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
T10.6	Theorem	Pythagorean Theorem
T10.6	Theorem	Right triangle
T10.6	Theorem	Legs *a* and *b* and hypotenuse *c*
T10.6	Theorem	$a^2 + b^2 = c^2$
T6.1	*Definition*	*Pascal's Triangle*
T10.1	*Definition*	*point (x, y), x-coordinate, y-coordinate*
T10.2	*Definition*	*Absolute value*
T10.3	*Definition*	$d(a,b) = \lvert b - a \rvert$

Table 5.3-7. *List of identified structures of Geometry*.

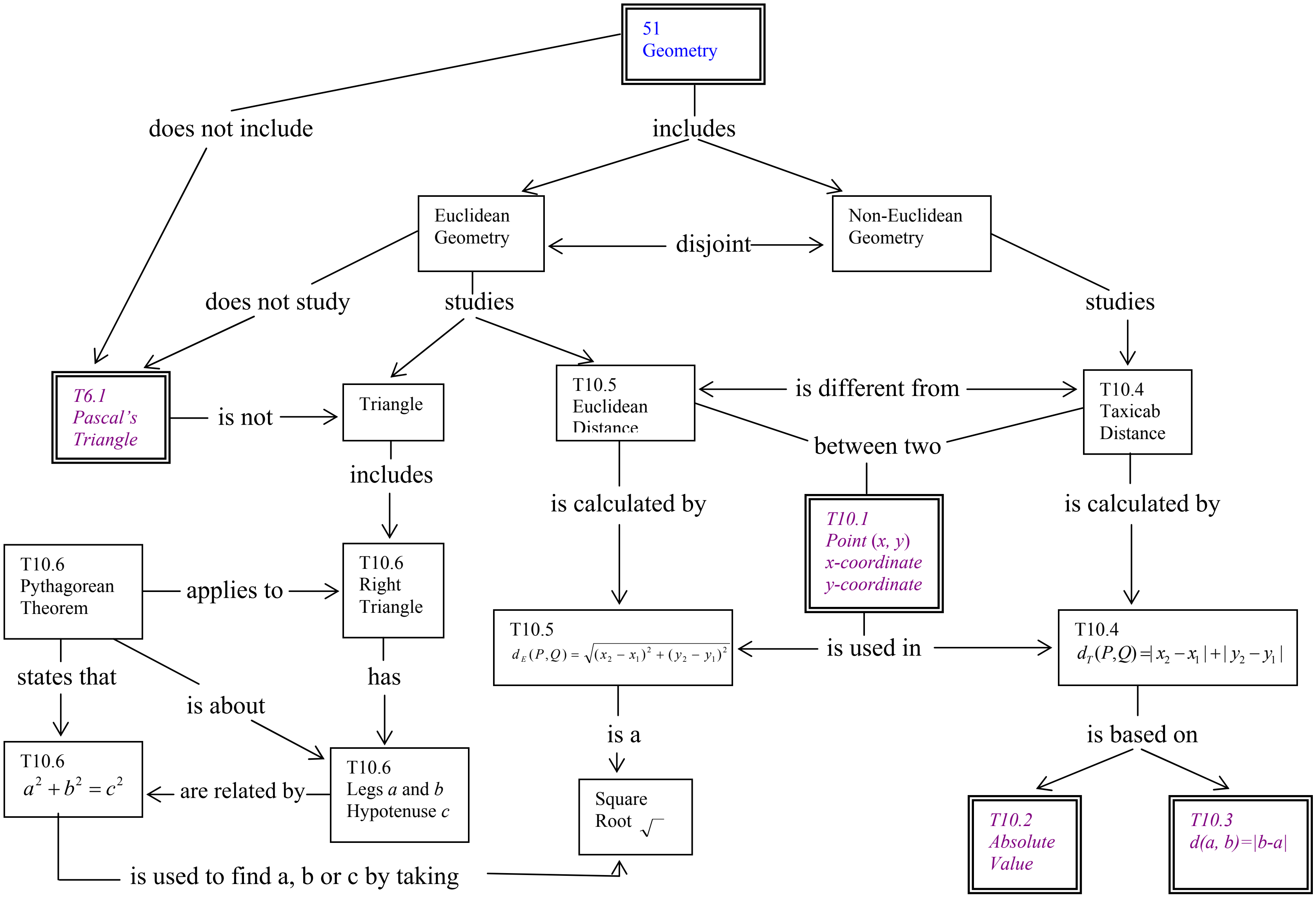

Figure 5.3-7. *Concept map of Geometry*.

Theoretically, it is not impossible to put all these seven concept maps together and make a big overall concept map on one sheet of paper. In that way, the connections among different sub-domains can be seen all at once. However, a giant concept map with more than fifty or sixty concepts may not be the most effective way of displaying this information. Therefore, in this section, a method of macro/micro maps used by Novak & Cañas (2008) is applied. Figure 5.3-0 can be seen as a global "macro map" displaying the main mathematical areas relevant to the analyzed tasks. Then, seven rather more specific "micro maps" are showing mathematical structures in specific sub-domains. The relationships among different sub-domains are shown by the overlapping boxes and the cross links connecting these boxes to others.

5.4 Student Solutions and Mathematical Structures

In this section, students' solutions to each task are examined with the goal of matching inherent mathematical structures with those found and discussed in sections 5.2 and 5.3. As mentioned in chapter 4, video transcripts and students' written work are extracted from Rutgers researchers' papers and dissertation studies. Because the main focus of this analysis is on mathematical structures, only critical events relevant to this topic are discussed.

5.4.1 Task 1: Shirts and Jeans (see Appendix A for task description)

As early as grade 2, Stephanie and Dana had demonstrated some understanding about ideas from *set theory* (03E). Working on the task "Shirts and Jeans", they drew pictures to represent the problem data (see Appendix B, Figure B-1, B-2, B-3). Dana clearly drew the *set* (T1.1) of shirts apart from the *set* (T1.1) of jeans. She put three shirts in the top row and two pairs of jeans in the bottom row. Stephanie did not draw shirts and jeans into two separate clusters; instead, she listed all three shirts followed by two pairs of jeans.

All three students used the capital letters *W*, *B*, *Y* as labels for the shirts and jeans. This is similar to using the *Roster Method* (T1.2) to list all *elements* (T1.1) in the set. Then, Stephanie and Dana working together used *ordered pairs* (T1.4), written vertically without the "( )", to list the elements in the *Cartesian product* (T1.5). Although Stephanie did not list these elements in a systematic way, she would have had all six outfits if she had broken her fifth $\begin{matrix}Y\\W\\B\end{matrix}$ into $\begin{matrix}Y\\B\end{matrix}$ and $\begin{matrix}W\\B\end{matrix}$. This mistake may have been caused by not clearly separating the shirt set from the jeans set, which made the pairing more difficult.

Besides writing down the pairs, Dana was the first one to draw a line to connect a shirt with a pair of jeans, thus representing a possible outfit. She had a question regarding the third outfit, "yellow shirts and white jeans". Dana said, "*It can't…yellow can't go with the white.*" Stephanie seemed more mathematically mature about making valid outfits in an abstract mathematical context. She firmly addressed Dana, "*No, how many outfits can it make? It doesn't matter if it doesn't match as long as it can make outfits. It doesn't have to go with each other, Dana* (Martino, 1992, p. 49-50)". Dana appeared not convinced. There was no connecting line between yellow shirt and white jeans in her drawing.

Michael was working alone for the most of time in grade 2. He drew and labeled shirts and jeans for the three pairs of outfits where the colors were the same. Perhaps his understanding of the problem was different than that of Dana and Stephanie. His work did not show any evidence of understanding ideas for *set* structures.

In grade 3, Stephanie picked up Dana's 2nd grade separate-row strategy to distinguish the shirt set from the jeans set. Michael did not bother to draw shirts and jeans at all. He worked with the phrases "white shirt", "blue shirt", and "yellow shirt" printed on the top row, and "blue jeans", "white jeans" printed on the second row of the original task statements and formed a tree diagram with the words (see Appendix B, Figure B-4, B-5, B-6).

All three students used Dana's 2nd grade connecting line strategy to show their outfits. The diagrams they drew look almost identical to the *simple bipartite graph* (T1.9

and T1.11) shown in Figure 5.2.1-1 of the solution method 3 in section 5.2.1. Without formally studying *graph theory* (05C), these young students seemed to possess the ability to construct a *graph* $G(V, E)$ (T1.8) with two sets of *vertices* (T1.8) and a set of *edges* (T1.8). They knew how to represent an outfit by drawing an *edge* (a line) (T1.10) to connect two *endpoints* (T1.8), one from the shirt set and the other from the jeans set. To find the answer to the problem, they counted the number of outfits by counting the number of edges. In grade 3, Stephanie and Dana did not write down ordered pairs of outfits at all. Michael listed all six ordered pairs correctly. However, it appears that he still did not prefer to list outfits systematically.

Stephanie further demonstrated her ability to use a *distribution method* in finding all the elements in the Cartesian product of two sets. This happened when the instructor added another pair of black jeans to the original problem and asked Dana and Stephanie to find the number of possible outfits. Stephanie used a connecting line strategy again. She systematically distributed three pairs of jeans to each of the three shirts and quickly determined that the answer was nine possible outfits (Martino, 1992, p. 64):

> Stephanie: *It's nine.*
> Instructor: *How did you get nine?*
> Stephanie: *See we drew the shirts and since each one of them could go to three pairs of jeans… three* [She pointed to the yellow shirt and characterized three outfits, the yellow shirt with each pair of jeans.]; *six* [She pointed to the blue shirt as she referred to six outfits, the three with the yellow shirt and the three with the blue shirt]; *nine.* [She continued in the same manner as she now pointed to the white shirt.]
> Instructor: *Oh, three, six, nine.*
> Stephanie: *Because there's three pairs of shirts and three pairs of pants.*

This kind of *distribution* – fixing one element in one set and distributing it to all elements in the other set, guaranteed that she would include all possible ordered pairs in the Cartesian product, and avoided any redundancy. It also laid a solid foundation for understanding the *Fundamental Counting Principle* (*Product Rule*) (T1.6). When the instructor asked, *"What do you think would happen if I had four pairs of jeans?*" Dana stopped Stephanie from drawing a new diagram and said, "*You'd just have to do like*…3, 6, 9, 12! *It'd be* 12! 3, 6, 9, 12!" (Martino, 1992, p. 65) This time, with or without realizing, Dana seemed to fix each pair of jeans and then distributing it to three different shirts. Nevertheless, she reached the correct answer without writing anything down.

5.4.2 Task 2: Towers 4-tall with 2 colors (see Appendix A for task description)

When a class of eighteen students was given this task in grade 3, they immediately began to manipulate colored cubes, a *set* (T1.1) of red cubes and a *set* (T1.1) of blue cubes, to build different 4-tall towers. According to Martino (1992, p. 91), it was not until several towers were built that the young students started to feel the need to have a system to check for new or duplicate towers. For example, Stephanie and Dana were in a group:

Stephanie: *I think I have that one …*
Dana: *No you don't.*
Stephanie: *Hold on. Let me check. I think… oh no I had it the other way. Everything we make we have to check. Let's make a deal… everything we make we have to check…*
Dana: *I'll always make it and you'll always check it.*
Stephanie: *Okay, you make it and I'll check it.*

This concern about duplicates seemed to play a significant role in the students' rearranging their already built towers into pairs of "opposite colors". Another group, Michael and Jaime:

Michael: *Did you do red, blue, blue, red?*
Jaime: *Red, blue, blue, red… yeah, red, blue…*

Michael: *We don't have it.*
Jaime: *Oh, thank you.*
Michael: *And then you could do blue…, red, red, blue. See! I'm thinking!*

Thus, one example of opposite pairs would be "red, blue, blue, red" and "blue, red, red, blue". Martino (1992, p. 107) reported that five out of nine groups of students organized their sixteen 4-tall towers into eight pairs of opposites, like Michael and Jaime had done. Two groups had some towers in pairs of opposites, and the remaining two groups had totally random arrangements.

In constructing these pairs of opposite colored towers, students seemed to realize that there were two color choices for each position of the tower (${}_2C_1 = 2$ (T3.3, T3.4)). This realization may have contributed significantly to the later development of solutions to generalizations of the *fundamental counting principle* (T1.6).

This task was given to participating students again (on November 13, 1998) when they were in grade 11, with an extended goal to make a generalization to towers *n*-tall. According to Tarlow (2004, p. 48-49), Michelle and Robert had experiences working on this Tower problem in earlier grades. The other four students, Angela, Magda, Ali, and Sherly were new to this task. Six students were paired up: Michelle with Robert, Angela with Magda, and Ali with Sherly.

All three groups found, in a very short time, the correct solution for 4-tall choosing from 2 colors: 16 different towers. Magda and Angela built their towers by cases based on how many cubes were blue: one blue, two blues, three blues, four blues, and four yellows. Robert used the same "cases" strategy, only he named the "four yellows" case "zero blue". Michelle built her towers in random fashion then organized them in "two's", which meant she paired two towers together if they had "opposite colors" for every position. Ali and Sherly also arranged their towers in "opposite colors".

Using the "cases" strategy (this strategy was used in solution method 2 in section 5.2.3), Magda, Angela, and Robert could better explain why they were sure that they had built all the possible towers. For the case of one blue, Magda and Angela said that they "had moved the one blue cube down into each of the four possible positions (Tarlow, 2004, p. 55-56)." This essentially means choosing one out of four (${}_4C_1$ (T3.3)) positions to place the one blue cube. Hence, they found ${}_4C_1 = 4$ (T3.4) different towers with one blue and three yellow cubes. Magda and Angela also did this for the case of three blues, by moving the yellow cube.

To explain the case of two blues, Robert "held the upper blue cube in a fixed position beginning at the top, while he moved the lower blue cube down one position in each tower. Each time that the lower blue cube had been moved down to all of the possible positions, the upper blue cube was moved down one position (Tarlow, 2004, p. 58)." Without using the formal terminology of combinations ${}_nC_r$ (T3.3), what Robert had done was systematically showing the instances of ${}_4C_2 = 6$ (T3.4) different combination of towers containing exactly two blue cubes. Tarlow's analysis suggests that Robert was the first student who provided complete justification for the case of two blues (p. 59).

The remaining time of the session was spent on finding the total number of different 3-tall towers and 5-tall towers. Searching for a pattern, Robert and Michelle put the number of towers in a chart from 1-tall up to 5-tall. Then Robert predicted that there would be sixty-four 6-tall towers. To avoid actually building towers for different heights, Robert looked for and eventually found a formula for calculating the total number of towers for any given height h, choosing from any number of color choices x, to be x^h.

None of the *tree* structures introduced in section 5.2.2 are found in any of the students' solutions. However, students used the same letter coding scheme (as used in section 5.2.2) to represent towers; they justified the solution "by cases" (as done in section 5.2.3) and by "opposite colors" strategy; they also found a formula for the generalization of the problem.

This task was briefly discussed again when another cohort of students (Michelle was one of them) worked on the Tower of Hanoi with Dr. Davis. See section 5.4.4 for further discussion.

5.4.3 Task 3: Towers 4-tall with 3 colors (see Appendix A for task description)

There was no problem solving session on this task. Sran (2010) described a task-based interview conducted on March 6, 1992 when the participating student Milin was in grade 4. According to Sran (2010, p. 82), by working on towers 1-tall and 2-tall in previous sessions, Milin had found that if choosing from two colors, there were two 1-tall towers; multiplied this two by two (*product rule* (T1.6)) could get four 2-tall towers. If choosing from three colors, there were three 1-tall towers; multiplied this three by three (*product rule* (T1.6)) could get nine 3-tall towers (i.e., *Induction* on the height of the tower). Therefore, when this task was presented to Milin on the date of interview, he had no problem to find the answer "three" for 1-tall and "nine" for 2-tall towers choosing from three colors. However, he suggested that this tripling pattern would not work out (p. 100). Even after Milin himself built three 3-tall towers from one of the nine 2-tall towers, and the researcher also built three 3-tall towers from another 2-tall tower, Milin still insisted that this tripling pattern must “breaks up somewhere” (p. 101). He did not have time to do 4-tall towers choosing from three colors.

When the students worked on Ankur’s Challenge (see section 5.4.7) in grade 10, they investigated the total number of 4-tall towers choosing from three colors. Michael and Ankur used the numbers 1, 2, and 3 to represent red, yellow, and blue. They listed out all 81 combinations (i.e., using the *Roster Method* (T1.2) to list all *elements* (T1.1) in the solution *set* (T.1)) (see Appendix B, Figure B-12).

5.4.4 Task 4: Tower of Hanoi (see Appendix A for task description)

Dr. Robert B. Davis conducted four problem-solving sessions on this task in 1993 when the participants were in grade six. According to Mayansky (2007, p. 50-54), two previous sessions before these four occurred about one month earlier, when the students worked on Guess My Rule problems, which had great influence on how the students solved the Tower of Hanoi task.

On the Day 1 (10/29/1993), Dr. Davis prepared the students for the new task by reviewing three Guess My Rule problems so that students had a chance to practice recognizing patterns, generalizing relationships among numbers, and finding a formula. Then, he told students an intriguing story about Tower of Hanoi to motivate them to find a solution. Dr. Davis suggested that students play with the puzzle disks and solve the problem starting with one, two, and three disks. He reminded students to record the results of their experiments and drew the table used in the Guess My Rule problems to record the first three results as follows (Mayansky 2007, p. 58):

□	Δ
1	1
2	3
3	7

Table 5.4.4-1. *The number of moves for one through three disks*.

Dr. Davis and the students used the symbol □ to represent the number of disks and Δ for the number of corresponding moves required. When the students found the numbers to be listed under Δ (1, 3, 7, …), they had actually found the first three terms of a *sequence* (T4.1) (although the term "sequence" was not used in their discussion).

Table 5.4.4-1 seemed to remind Michael of the formula $(\square \times 2) + 1 = \Delta$ they found for problem 1 of Guess My Rule at the beginning of the session: "*I know what it is, we found it. Is it the number times two, plus one?* (Mayansky 2007, p. 58)" Although Michael did not clearly defined what he meant by "*the number*", he seemed to recognize the *recursive* relationship between the consecutive numbers of moves at this early stage.

Dr. Davis invited students to verify Michael's idea by working with four disks. After trying to solve the puzzle with four disks for a while, students agreed that a minimum of fifteen moves were required. With one more entry (*term*) added to the table (the *sequence* became {1, 3, 7, 15, …}), Michelle and Ankur recognized the *recursive* pattern that Michael claimed earlier. This time, Michelle explained the idea to the whole class in a very clear and confident voice, "*OK… You have one and one is two, plus one is three; three and three is six, plus one is seven; seven and seven is fourteen plus one is fifteen; So then the next one will be fifteen and fifteen is thirty, plus one thirty-one, and so on…*" (Mayansky 2007, p. 60).

Michelle successfully described a way to find the number of moves for *n* disks to be twice of the number of moves for (*n* – 1) disks plus one. Using this *recursively defined sequence* (T4.9), Michelle was able to predict the next table entry – thirty-one moves for five disks. Dr. Davis asked students to verify Michelle's prediction and then reminded everyone that the number of moves for 100 disks was what the task problem originally was asking for.

Continuing with Michelle's pattern, Stephanie expanded the table up to ten disks, "*Ten* (disks*) is 1023* (moves). *I already got down to ten* (disks)" (Mayansky 2007, Transcription, line 519). Students recognized another pattern (lines 521-524):

> Ankur: *Shelly, this is two to the tenth power*.
> Michelle: *Oh my God! Duh, we had it right there*.
> Romina: *What's two to one hundredth power*?
> Ankur: *That's the answer*!

Students were excited about what they found. From the relationship between the number of disks (ten) and the number of moves (1023), they figured out that 100 disks required 2^{100} moves. They did not subtract one from it; otherwise this solution was the same as solution method 1 discussed in section 5.2.4. Not until later, Stephanie pointed out that two to the tenth power is 1024, not 1023. However, in Day 1 session, no one had a chance to further investigate this small difference.

The correct answer fell into place in the Day 3 (11/12/1993) session when Dr. Davis guided students to compare the task "Towers of Hanoi" with the task "Towers selecting from two colors". He asked students to make a table to record how many different towers they could build for different heights of towers, selecting from two colors. When Ankur said, "*Seven is one hundred fifty eight, eight is two hundred fifty six, ten is one thousand twenty four*", Matt was intrigued, "*Hey, aren't those the same* *numbers we got for the Tower of Hanoi? They're the same numbers*!" Ankur noticed the "1" difference this time, "*No, they're not, it's minus one, it's one less. No this is one more.*" Michelle confirmed, "*It's one less every time.*" (Transcription, lines 1493-1498). Students then filled the table up to 10-tall towers. They also wrote the number of different towers (terms of a *sequence*) as powers of 2:

How tall	How many
1	$2 = 2^1$
2	$4 = 2^2$
3	$8 = 2^3$
4	$16 = 2^4$
5	$32 = 2^5$
6	$64 = 2^6$
7	$128 = 2^7$
8	$256 = 2^8$
9	$512 = 2^9$
10	$1024 = 2^{10}$

Table 5.4.4-2. *The number of different towers choosing from two colors*.

Comparing this table with the table for the Tower of Hanoi task, students once again confirmed the isomorphism between these two tasks. Ankur was very sure about the difference, "*The Tower of Hanoi is one less number than building blocks* (line 1572)." When Dr. Davis asked how many 100-tall towers could be made selecting from two colors, Stephanie went to the board and wrote "2^{100}" without hesitation. How many moves it would take to move one hundred disks? Matt wrote down "$2^{100} – 1$".

Then, Dr. Davis guided students to use powers of ten to approximate the numerical value of 2^{100}. This was a different approach from the method 2 shown in section 5.2.4. However, students had a chance to work with very big numbers and learn some laws of exponents through this activity.

5.4.5 Task 5: Pizza with Halves (see appendix A for task description)

Seven fifth graders, Michael, Ankur, Romina, Brian, Bobby, AmyLynn, and Michelle, participated in three thirty-five minute sessions working on this task over three consecutive school days – March 1, 2, and 3, 1993. On day 1, students spent some time trying to find appropriate representations to record possible combinations of pizzas. Ankur led the group work and found six combinations initially. However, at the end of the session, students came up with varying answers. According to Muter (1999, p. 58), "they were sure that some of the lists contained duplicates", but they did not have enough time to address the conflicting answers.

On the second day, the teacher/researcher reminded students that they better had some way to organize the combinations they found. So, the students set up a system to work on the problem: one person read the combination and the others checked for duplicates. They discussed the validity of some combinations. For example, is a pizza with one half pepperoni and one half a pepperoni/sausage mixture acceptable? Soon they agreed that the solution to the problem was "ten pizzas". In order to justify this answer, the students classified the ten pizzas into three categories (Muter, 1999, p. 61; also see Appendix B, Figure B-7):

<u>Whole</u>
1 plain
1 sausage
1 pepperoni
1 mixed (pepperoni and sausage)

<u>Half</u>
½ pepperoni ½ plain
½ sausage ½ plain
½ pepperoni ½ sausage

<u>Mixed</u>

½ plain	½ sausage & pepperoni Mixed
½ pepperoni	½ pepperoni & sausage Mixed
½ sausage	½ pepperoni & sausage Mixed

Figure 5.4.5-1. *Students' classification of the ten pizzas into three categories.*

The students did not use code words for the pizza toppings. Their "Whole" category was equivalent to case 1 of the solution in section 5.2.5. Their "Half" and "Mixed" categories covered the case 2 of the solution in section 5.2.5. The students did not indicate that they knew the formula ${}_nC_r = \dfrac{n!}{r!(n-r)!}$ (T3.4) to find the combinations. However, they correctly found all ten possible pizzas with halves.

5.4.6 Task 6a: The 4-topping Pizza (see Appendix A for task description)

This task was given to the students on April 2, 1993 when they were in grade 5, after they had reviewed and discussed Pizza with Halves. According to Muter (1999, p. 66), the group of Ankur, Brain, Jeff, and Romina solved the problem in "approximately fifteen minutes" using a letter coding scheme to list all possible combinations. Muter stated that students generated 16 pizzas randomly. However, when Ankur provided a justification, he organized these 16 pizzas into categories (see Appendix B, Figure B-8). Five whole (i.e., whose two halves were the same) pizzas were *P* (peppers), *S* (sausage), *M* (mushroom), *PE* (pepperoni), and *PL* (Plain). Mixed (i.e., having two different halves) pizzas included two toppings (*P/S*, *P/M*, *P/PE*, *S/M*, *S/PE,* and *M/PE*), three toppings (*P/S/M*, *PE/M/S*, *P/M/PE*, and *PE/S/P*), and four toppings (*P/M/S/PE*).

Using this representation, students decided that the *order* of toppings (in the alphabetic coding) did not make different pizzas (it is *combinations* (T3.3), not *permutations* (T5.3)). When the teacher researcher asked, "*So why is it you can't go M* *with P?*" Ankur pointed to "*P/M*" and said, "*Because you already have i*t (Muter, 1999, p. 67)."

Task 6b: The 4-topping Pizza with 2 Crusts

Students found the answer, 32 pizzas, "within forty-five seconds, without writing down a single notation" (Muter, 1999, p. 68). Michael gave the reasoning, "*Since there's sixteen to make with those toppings, you put a Sicilian crust on it. That's sixteen. Plus then you put a regular on it and that thirty-two. Sixteen and sixteen*." Although Michael sounded as if he were adding 16 to 16, this operation was equivalent to doubling the 16 combinations, which might be viewed as applying the *Fundamental Principal of Counting* (*product rule*) (T1.6).

Task 6c: The 4-topping Pizza with Halves and 2 Crusts

Matt proposed a method that Ankur used to verify the solution of the first 4-topping pizza problem (Task 6a). First, Matt chose plain (cheese only) for one pizza half and linked it to all sixteen combinations that he found for the first pizza problem to represent the other pizza halves (see Appendix B, Figure B-9). Next, he used the second topping *PR* (peppers) on one pizza half and linked it to the remaining fifteen combinations to represent the other possible pizza halves pizzas (see Figure B-10). Then, Matt seemed to recognize the pattern. So, on the next work sheet, Matt tried to compute 16 + 15 + 14 + 13 + … + 3 + 2 + 1 and multiplied the sum by two to account for the two crust choices (see Figure B-11). This method was the same as adding the number of entries in the lower triangle (including the main diagonal) of the permutation table (Table 5.2.6-2) used in the solution of Task 6c in section 5.2.6. Matt did not get the correct answer because he made computational mistakes.

No other group of students reached the solution with confidence that they had found all combinations. They finally listened and agreed on Matt's argument after neglecting it several times before.

Tarlow (2004, p. 116) reported that the students were given the Task 6a again when there were in 11th grade (on March 1, 1999). The eight participants were divided into two groups. Robert, Stephanie, Shelly, and Amy-Lynn sat at Table A; Angela, Magda, Michelle, and Sherly sat at Table B. Both groups found the correct answers pretty quickly. There were a total of sixteen different pizzas when choosing from four available toppings; and thirty-two for five available toppings. The students drew tree diagrams (similar to what they used in the "Shirts and Jeans" task) to show and count the number of different pizzas. They also listed out these combinations using an alphabetic coding scheme.

Both groups of students connected their solutions to Pascal's triangle. At Table A, Shelly and Stephanie grouped pizzas by cases according to the number of toppings. In Pizza with four available toppings problem, they found "1 4 6 4 1" for five cases: plain, 1-topping, 2-topping, 3-topping, and 4-topping. In the Pizza with five available toppings problem, these numbers were "1 5 10 10 5 1" for six cases. Then, Shelly seemed to remember something:

> Shelly: *One, four, wait a minute. One, four, six, four, one, so the next one will be one,… This is the …*
> Stephanie: *The triangle.*
> Shelly: *The triangle.*
> Stephanie: *Yeah.* [leans over to Shelly's paper.] *So the next one is one, five, ten, five, one.*
> Shelly: *We're done.* [laughs] (Tarlow, 2004, p.125)

However, this was not enough for Stephanie:

Stephanie: *But what does that mean?*
Shelly: *I don't know.*
Stephanie: *What does that mean to me?*
Shelly: *I don't know, but that's the answer*. [laughs] *Um*.
Stephanie: *But what it, like, what does one, four, six, four, one. That means nothing to me.*
Shelly: *It means nothing to me either, but it's the pattern we saw.*
Stephnie: *Oh dear Lord… Oh, so we have a pattern, but how do we apply it to getting sixteen pizzas?* (Tarlow, 2004, p. 125-126)

Seeking the meaning of these numbers in Pascal's triangle, Stephanie described these numbers in terms of pizza combinations, "*So, well, okay, let's figure this is saying that we have one plain pizza…. Okay. So we have four pizzas with one topping… we have six pizzas with two toppings, four pizzas with three toppings*." Shelly added, "*And one pizza with four toppings*." (Tarlow, 2004, p. 126-127)

Stephanie, Shelly, and Robert had struggled for a while when the researcher asked if they could explain the Addition Property (${}_kC_{r-1} + {}_kC_r = {}_{k+1}C_r$ (T6.3)) on Pascal's triangle in terms of pizzas. Stephanie said to the researcher, "*We have no idea how one pizza and three pizza make a whole new category of four pizzas* (i.e., 1 + 3 (in "1 3 3 1") = 4 (in "1 4 6 4 1")). *Because this is one plain pizza, right? Like this one right here is plain, and these three, pizzas with one topping…. If you add one plain pizza to three pizzas with one topping, you get like one pizza with…no topping and three pizzas with one topping…, but like in reality you don't get four… So I don't know how to answer the question* (Tarlow, 2004, p. 136)."

The researcher asked them to consider what would happen when a new available topping was added. After discussing with Shelly, Stephanie provided an explanation, "… *Okay, this, to get four pizzas with one topping, you already have three pizzas with one topping. And the plain pizza becomes the pizza with the new topping* (Tarlow, 2004, p.

138)." With this understanding, Stephanie was able to explain 3 + 3 (in "1 3 3 1") = 6 (in "1 4 6 4 1"), "*So then here, um, you have six pizzas with two toppings. Now you already have three pizzas with two toppings. So these three pizzas with one topping get an extra topping added on* (p. 138)." Stephanie continued to explain 3 + 1 = 4 with the same reasoning.

Robert's written work (see Appendix B, Figure B-13) showed that he found that the sum of numbers in each row of Pascal's triangle was "two to the number of available toppings", the total number of pizza combinations. He and Stephnie also connected the numbers "1 3 3 1" to towers three high with two available color choices. Robert added that the height of towers was the same as the number of toppings. When the researcher asked how many pizza combinations there were for *n* available toppings, Robert and Stephanie answered: two to the *n* (Tarlow, 2004, p. 150).

Before discovering that Pascal's triangle might be involved, the students at Table B found that the number of pizza combinations would be "doubled" by adding one available topping. While they did not further generalize this idea, they were able to explain the Addition property of Pascal's triangle in terms of placing toppings on pizzas. Angela said that the Pizza problem did not relate to Towers because "pepperoni, mushroom" and "mushroom, pepperoni" were the same thing on the pizza but "yellow, red" and "red, yellow" were two different towers. Sherly agreed. There was no further investigation of this argument.

5.4.7 Task 7: Ankur's Challenge (see Appendix A for task description)

On January 9, 1998 when participating students were in grade 10, Ankur proposed this problem after he and four other students (Brian, Jeff, Michael, and Romina) had provided the solution and justification to the problem of 5-tall towers when choosing from two colors.

According to Muter (1999, p. 93-94), the five students were working in two groups at first. From the previously determined total number of towers 5-tall when choosing from two colors (2^5), Michael and Ankur determined that the total number of towers 4-tall when choosing from two colors should be 2^4, and that the total number of towers 4-tall when choosing from three colors should be 3^4. Then they began to list combinations using the numbers 1, 2, and 3 to represent red, yellow, and blue cubes. However, they soon found that the list contained many duplicated towers (see Appendix B, Figure B-12). Jeff, Brian, and Romina joined the discussion around this time. Ankur and Jeff explained to Romina why there were eighty one 4-tall towers when choosing from three colors (Muter, 1999, p. 95):

> Ankur: *It's three to the fourth* (power).
> Jeff: *To the fourth.*
> Ankur: *'Cause look…*
> Jeff: *Three times three is nine times three is twenty-seven,…*
> Jeff & Ankur: *times three is eighty-one.*
> Ankur: *You want to know why we multiplied it like that.* [*pause*] *'Cause, look, you have four spaces (positions). In the first space you have three. In the second space you can have three* [*pause*]

Ankur and Jeff's explanation provides evidence regarding their ability to apply the *Fundamental Counting Principle* (T1.6). Romina also gave evidence of understanding. Then, Jeff said that thirty-six might be the solution to the problem that Ankur proposed because he had found thirty-seven towers and that he was sure that he had a duplicate. Romina gave her consent, "*You know, it might be thirty-six. 'Cause I'm working with "sixes" now. And okay you put them, like you pair 'em up. 'Cause you're only gonna have …okay…* (Muter, 1999, p. 96)"

What Romina referred to as "*sixes*" was the six ways to choose two of four positions for placing the "repeating" (or "duplicate" as she called them) colored cubes. She used "1" to represent the duplicate colored cubes, and used "0" and "x" to represent the other two colors in a tower. She had explained her ideas to the teacher/researcher and other students several times. Her final and best version of the solution that she wrote on the chalkboard for Michael is shown in the following figure (Muter, 1999, p. 107):

1	1	0 x	x 0
1	x 0	1	0 x
1	x 0	0 x	1
x 0	1	1	0 x
x 0	0 X	1	1
x 0	1	0 x	1

$$6 \cdot 2 = 12 \cdot 3 = 36$$

Table 5.4.7-1. *Romina's final version of her solution to Ankur's problem.*

Romina pointed out that in order to use exact three colors in four positions, there must be two cubes of the same color, which was represented by "1". She first chose two positions to put "1" in and she found six possible combinations. Subsequently, she filled the remaining two positions with "x" and "0", and with "0" and "x". Thus, for each fixed "1" combination, there were two ways to place the other two different colored cubes.

Therefore, there were $6 \cdot 2 = 12$ towers for any one duplicate color "1". However, this "1" could be any one of the three colors. So, there were a total of $12 \cdot 3 = 36$ towers that were 4-tall and had exact three colors.

Although Romina did not use any terminology of *the number of combinations* (T3.3), her explanation showed that her reasoning was very similar to the solution method 1 presented in section 5.2.7. Her method of fixing repeating colors then filling in with the other two colors was a version of splitting four cubes into three different color groups described in solution method 2 in section 5.2.7. In other word, her $6 \cdot 2 = 12$ could be alternatively gotten by using the formula for finding the number of *distinguishable combinations* (T7.1) : $C(4;2,1,1) = \frac{4!}{2!1!1!} = 12$ (T7.2).

Michael and Ankur were interested in finding the *complement* of Ankur's problem, which meant finding the number of 4-tall towers that did not have exactly three colors. This idea was similar to the approach shown in solution method 3 of section 5.2.7 except that Michael had considered his cases in a slightly different manner. Michael's first case was "three cubes of one color and one cube of another color". He used the numbers 1, 2, and 3 to represent the colors red, blue, and yellow, respectively, while the number 0 represented a color other than the color already presented. Romina, serving as the recorder, wrote 12 combinations on the chalkboard as indicated in the table below (Muter, 1999, p. 110):

1	2	3	1	2	3	1	2	3	0	0	0
1	2	3	1	2	3	0	0	0	1	2	3
1	2	3	0	0	0	1	2	3	1	2	3
0	0	0	1	2	3	1	2	3	1	2	3

Table 5.4.7-2. *The first case of the complement of Ankur's problem: 3 cubes of one color and 1 cube of another color.*

For each combination listed above, there were two color choices for “0”. This brought the total number of towers up to 24. Then Ankur added 3 one-color towers and said, “*red, red, red, red. Yellow, yellow, yellow, yellow. And blue, blue, blue, blue. So that’s twenty-seven* (Muter, 1999, p. 110).” The second case of the complement was towers with two cubes of one color and two cubes of another color. There were 18 of this kind as indicated in the table below (p. 111):

1	1	2	2	3	3	1	1	2	2	3	3	1	1	2	2	3	3
1	1	2	2	3	3	2	3	1	3	1	2	2	3	1	3	1	2
2	3	1	3	1	2	2	3	1	3	1	2	1	1	2	2	3	3
2	3	1	3	1	2	1	1	2	2	3	3	2	3	1	3	1	2

Table 5.4.7-3. *The second case of the complement of Ankur’s problem: 2 cubes of one color and 2 cubes of another color.*

From this, Michael had proved that the answer to the *complement* of Ankur’s problem was 27+18 = 45, and thus that the number of 4-tall towers with exactly three colors was 81 – 45 = 36.

5.4.8 Task 8: The World Series Problem (see Appendix A for task description)

Kiczek (2000) examined how students’ probabilistic thinking developed when they worked on the World Series task and the follow-up Problem of Points task in the 11th grade. On January 22, 1999, five participants (Ankur, Brain, Jeff, Michael, and Romina) worked on this new problem and found the correct solution. The progress they made was parallel to the two solution methods presented in section 5.2.8.

They interpreted the problem as a collection of four separate cases (one for each number of games by which the World Series can be won). For the case of a win in four games, they solved it quickly. Romina wrote AAAA and BBBB to represent the only two combinations (*outcomes* (T8.1)) in this case. Brian pointed out that the *probability* (T8.2) for (the *event* (T8.1)) "team A wins in four games" or (the *event* (T8.1)) "team B wins in four games" was "*the odds of winning one game, times the odds of winning one game, times the odds of winning one game*". Ankur responded, "*Look, it's a fifty percent chance of winning the first game*". Then Brian stated that it was "*like flipping a coin*" and Romina concluded that "*Yeah, that's how you do it: a half times a half times a half times a half*" (Kiczek, 2000, p. 39-40). Hence, all five students agreed that the probability of a team to win the first four games was $\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\left(=\frac{1}{16}\right)$ (For *independent events E* and *F*, $P(E \cap F) = P(E)P(F)$ (T9.1)).

To determine the number of total possible combinations (the *sample space* (T8.1)), Jeff and Romina related it to the generalization of the towers and the pizza problems. Jeff conjectured that the total number of possible game combinations in a world series would be 2^7, because in the past they had decided that the total number of pizzas with *n* available toppings was 2^n; and the total number of *n*-tall towers when selecting from two colors was also 2^n. This idea was refined by Ankur when later he pointed out that the probability for a four-game world series was $\frac{2}{2^4}$

($P(E) = \frac{n(E)}{n(S)}$ (T8.2)) (Kiczek, 2000, p 43):

> Ankur: *It's two over two to the fourth.*
> Jeff: *Why is it two to the fourth*?
> Ankur: *Because that's the total, like – two to the fourth will give you the total possibilities of four things… you know what I mean? It has to be over – do you know what I'm talking about or not?*
> Jeff: *Yeah, it's gotta be over two to the fourth – four spaces.*

Then, Ankur and Jeff confidently claimed that the total number of ways a world series could be won in five games was 2^5 (= 32); in six games, 64 (= 2^6); in seven games, 128 (= 2^7) (Kiczek, 2000, p. 44).

Ankur had listed out all possible outcomes for team A to win in four, five, six, or seven games (see Appendix B, Figure B-14) to be 1, 4, 10, and 20 respectively. Then he doubled the number of outcomes in order to account for the possibility that team B won in each case. He realized that team A had to win the last game in order to win the World Series. Ankur used the five game series as an example to explain his idea to Jeff (Kiczek, 2000, p. 42):

> Ankur: *It would be eight, for the five…*
> Jeff: *Yeah, but you're not going to know if you have them all, though. How are you gonna know?*
> Ankur: *To have – this will be, five games will be eight.*
> Jeff: *Think so?*
> Ankur: *'Cause like it'd be these four, 'cause look, you can only lose one game, right? … You can't lose the last one, or, 'cause, or they already won four, you know what I mean?*
> Jeff: *Yeah, yeah, yeah, yeah.*

Based on Ankur's list of team A's winning outcomes and the number of total outcomes, the students correctly found the correspondent probability of a four, five, and six game World Series to be $\frac{1}{8}$, $\frac{1}{4}$, and $\frac{5}{16}$. When working on the case of seven games, Jeff suggested finding the *complement* (T8.5): subtract the sum of the other probabilities from one ($P(E^c) = 1 - P(E)$ (T8.5)). After adding two missed winning outcomes to the list for a six game series, Ankur found the probability for a seven game series to be also $\frac{5}{16}$. Jeff had a doubt that the six and the seven games in the series had the same probability, because in the earlier discussion the students expressed their intuition that "it should be easier to win when more games are played" (Kiczek, 2000, p. 40).

Michael worked alone for the first half hour. He used the binary coding scheme, "1" represented a win and "0" a loss. Similarly to what Ankur had done, Michael fixed the last place of each coding string with "1" to show that the team in question must win the last game in order to win the series. Michael also noticed the numbers of possible outcomes (1, 4, 10, 20) for a team to win were embedded in *Pascal's Triangle* (T6.1) (see Appendix B, Figure B-15). He stated what he discovered, "*I just found, like, if you take the fourth number in each one* (row)*, that way, if you double each number, 'cause you have two teams, you can get the possibilities of four games. Four games, um, equals two, right? You got eight, twenty and forty, like they said* (Kiczek, 2000, p. 48)."

Michael continued to explain the entries "1 3 3 1" in Pascal's triangle in terms of the number of pizzas with three choices of toppings. There were "1" plain, "3" one-topping, "3" two-topping, and "1" three-topping pizzas. With respect to the World Series, Michael and Ankur jointly explained that the first "1" represented the probability, $\frac{1}{8}$, of a team, say team A, winning three games in a row. In other words, there were "1" way to get three A's, "3" ways to get two A's and one B, "3" ways to get one A and two B's, and "1" way to get three B's.

Later, on August 31, 1999, Michael and Robert were given this problem again. Robert was presented with this problem for the first time. Michael remembered that in an earlier after-school session (on 5/12/1999), the students had created the formula for ${}_nC_r$, and that Robert had proposed to use it in solving combinatorics problems. However, they both forgot the formula. So Michael and Robert spent some time on recreating the formula for "choose" ($_nC_r = \frac{n!}{r!(n-r)!}$ (T3.4)). They then tested the formula for the five-game and six-game series and found that the results matched those obtained by the method of listing all outcomes. They found that the formula worked for the seven-game series too.

5.4.9 Task 9: Points (see Appendix A for task description)

On February 5, 1999, after solving the World Series problem, Ankur, Jeff, Michael, and Romina were given this follow-up task. By writing letter string (W and L) or binary strings (0 and 1) to represent coin flips, the students found ten possible ways (*outcomes* (T8.1)) that the game could be ended: Fermat had six ways to win and Pascal four ways. Then they wanted to decide the *sample space* (T8.1) (Kiczek, 2000, p.89):

> Ankur: *It should be sixty-forty.*
> Romina: *Yeah. Mike, what are the total possible*?
> Michael: *Sixteen.*
> Romina: *Sixteen. So it's four out of sixteen and eight, six out of sixteen*?
> Jeff: *So what happens to the other six things*?
> ……
> Romina: *We can just add these up and it'd be ten, and it'd be forty-sixty.*
> Jeff: *That would probably be the six that go to nowhere.*

Although Jeff raised a question about the missing "six things" (i.e. 16 – 4 – 6 = 6), the students did not tried to resolve this issue. They tried to find another way to reason that the sample space was sixteen. Realizing that the winner would be determined in four or fewer rounds of the coin flips, Ankur and Jeff suggested reducing the problem to "five coin tosses" with the first flap always a head, because Fermat was one point ahead of Pascal (Kiczek, 2000, p. 90):

> Ankur: *So there'd be five places and you put Fermat in the first one. Then you get the sixteen….*
> Jeff: *No, it's more than sixteen.*
> Ankur: *No, you don't count the first one 'cause you already won it.*

Jeff: *Exactly. Then sixteen.*

In spite of what they agreed on (that the total number of possible ways should be sixteen), the students reached a consensus that the money should be split "sixty-forty" (using ten as the sample space). Jeff explained, "*Six out of the sixteen get crossed out. That leaves you with ten…* (Kiczek, 2000, p. 89)." Michael stated that he crossed out six combinations from sixteen because "they represented games that would not have occurred (p. 92)".

Later, Jeff suggested an alternative solution when he and Romina "compared the situation to that of the World Series problem (Kiczek, 2000, p.92)":

> Jeff: *In the World Series, they won four, they wouldn't have gone to the next three games. … If we would have said they were gonna flip out, no matter what, they were going to keep flipping until the end, then it'd just be all the ways for Fermat – it'd be eleven to five.*

The teacher/researcher pointed out that in the World Series problem, the probabilities would be the same, no matter if the series ended earlier or continued for all seven games. Two different solutions to the Points problem motivated the students to further investigate the initial conditions. Ankur suggested that the second solution "eleven to five" resulted from the unequally likely outcomes that might be caused by Fermat's one point lead at the beginning of the reduced problem "five coin tosses" (Kiczek, 2000, p. 92-93):

> Ankur: *Because – it's different because those aren't all equally as likely to happen.*
> T/R3: *They're not?*
> Ankur: *'Cause the first one, look, it's Pascal winning three in a row and that's supposedly the same chance of happening as Fermat winning two in a row.*
> …
> Ankur: *'Cause one had the advantage – of course, there will be more ways Fermat can win – he'll win quicker, therefore there'll be*

more games not played, so all these games not played, these possibilities will go to Fermat.

Michael: *Because Fermat had the advantage at the beginning.*
Jeff: *That's what I'm saying, because no one had the advantage in the World Series, that's why.*
Ankur: *That's why. They started off even, so – that goes back to Fermat winning the first game.*

Thus, the students realized the existence of the uneven preexisting condition and indirectly found the *conditional probability* (T9.2) of "Fermat wins" to be $\frac{11}{16}$ and for Pascal, $\frac{5}{16}$. However, the first solution (sixty-forty) still made sense to the students. They did not know which solution was the correct one. Therefore, the investigation continued. After a while, the students noticed that each of the winning outcomes had different probabilities. Ankur (without knowing the terminology) applied the *General Multiplication Rule* (T9.4) to calculate the probability of each possible outcome by multiplying the individual probabilities (similar to what has been done in solution method 2 of section 5.2.9). Then he found the total of these individual probabilities was equal to one ($P(S) = 1$ (T8.4)).

Although Ankur's computational results sounded promising, not all the students were convinced. The session ended as follows (Kiczek, Appendix E, lines 4336-4343):

Michael: *So you're saying eleven and five is right?*
Ankur: *Yeah.*
Michael: *I disagree with that.*
T/R2: *You –*
Michael: *Disagree.*
T/R2: *You disagree with that.*
Ankur: *Each one doesn't make sense in its own way – and each one does make sense in its own way.*

Six months later, on August 31, 1999, Michael had a chance to revisit this task with Robert after they solved the World Series problem. It was Michael who suggested that "*It's just kind of, like, coming back to the World Series one. Like, all the different chances of coming up…* (Kiczek, 2000, p. 98)." Robert agreed and, like Ankur did before, he calculated the probability of possible outcomes by multiplying the individual probabilities. The answer was $\frac{11}{16}$ for Fermat to win and $\frac{5}{16}$ for Pascal to win. Michael said, "*That's what we came up with before – another way, a different way of doing it, though. I remember eleven and five* (Kiczek, 2000, p. 98)." This time, Michael accepted this solution without any objection.

5.4.10 Task 10: Taxicab (see Appendix A for task description)

Powell (2003, p. 68) reported that four students (Brian, Jeff, Michael, and Romina) joined the problem solving session conducted on May 5, 2000 when they were in the 12th grade. On their grid paper, station A, station B, and station C (Figure A-2) were labeled as blue dot, red dot, and green dot respectively. The students' first attempt was drawing routes on the grid and counting the paths. Brian quickly declared that there were five shortest routes to go to the blue dot (station A). Romina and Jeff confirmed Brian's claim. Romina also found that each shortest route contained five blocks. Jeff asked, "*Why is it the same every time*? (Powell, 2003, Appendix C, Transcript, Line 74)" Romina answered, "*Ours is a four by one, right*? (Line 76)" Michael explained that "*You can't get around going four down and right one* (Line 87)" ($d_T(P,Q) = |x_2 - x_1| + |y_2 - y_1|$ (T10.4)) because "*you can't go diagonal* (Line 81)" or "*backward* (Line 77)". Romina wanted to devise a "four by one" grid as an area. Jeff and Michael disagreed stating, "*It's not area… It's the perimeter* (Lines 100-102)." Later, Michael followed up on this "perimeter" idea several times. He often reminded others that the *length of the shortest route* (*Taxicab distance* (T10.4)) to a certain pick-up station was equals to one half of the perimeter of the rectangle having the taxi stand as the top left vertex and the pick-up point as the bottom right vertex.

Brian observed that the number of shortest routes and the number of blocks to the blue dot were both five. He asked other students to check if this was true for the red dot and the green dot. However, by drawing different routes on the same grid, the students soon lost count. Romina said that she wanted to devise a method to do this. After several times of losing track of what they were doing, Romina said, "*Okay, we can't count. Like we need a – can't we – can't we do towers on this?* (Transcript, Line 159)" Then she spent some time discussing her idea (of using towers) with Michael. They also looked into the relationships between the number of blocks and the number of shortest routes from the taxi stand to each pick-up points.

Making no significant progress, the students tried to look into the problem from different angles that they could think of. After struggling for a while, Romina suggested, "*I think we're going to have to break it apart and draw as many* (routes) *as possible*." Jeff said, "*…why don't we do easier ones*? (Transcript, Lines 293, 295)"

Thus, Romina and Jeff found a way to record the number of shortest routes from the taxi stand to each intersection on the grid. They used one grid sheet for recording the numbers of shortest routes, and used many sub-grid rectangular boxes for drawing shortest routes. In each sub-grid rectangular box, the top left point represented the taxi stand, and the most bottom right point represented the target intersection. For each target intersection, they drew all possible shortest routes in one sub-grid rectangular box with colored markers, counted the number of routes, and wrote this number in the corresponding box on the recording sheet. The numbers on the recording sheet looked like the entries in an array. Romina filled this recording sheet starting from 1×1, 2×2, 2×3 (see Powell, 2003, p. 81), 3×3 (p. 82), to 3×4 array (p. 84). Powell (2003) pointed out that Romina and Jeff perceived a symmetrical relationship along the diagonal when they had this array (p. 81):

2	3	4	5	
3	6	9		
4	9			

Figure 5.4.10-1. *The incomplete 2×3 array from which Romina and Jeff perceived a symmetrical relationship.*

Some entries in these arrays were incorrect because some routes were missed while drawing. Brian and Michael came to join Romina and Jeff. Brian helped to correct the mistakes of entry "9" in row two and row three. Then the array looked like this (Powell, 2003, p. 87):

2	3	4	5	
3	6	10	12	
4	10	15		
5				

Figure 5.4.10-2. *Romina and Jeff's taxicab grid with incorrect entries 12 and 15.*

Right away, Romina announced, "*All right. It's, um, - it's Pascal's triangle* (Transcript, Line 778)." Soon she saw the entry "12" and said, "*No, it's not. It doesn't work out* (Line 787)." Jeff noted the other error and said, "*That* ("12") *should be a 15 … that* ("15") *should be a 20* (Line 790)." So the students tried to resolve this issue. After a while Brian found the number of the fourth entry in the second row, "*It's fifteen*", he said.

Michael also found and confirmed that the third entry in the third row should be "*twenty*". So, the recording sheet (the taxicab grid) became this (Powell, 2003, p. 91) :

2	3	4	5	
3	6	10	15	
4	10	20		
5	15			

Figure 5.4.10-3. *A portion of taxicab grid containing the number of shortest routes to each intersection.*

Michael suggested, "*It should be ones on all the sides* (Transcript, Line 922)." Jeff wrote them. Later, Romina wrote a five-row Pascal's triangle in the form of top-to-bottom. For the rest of the time in this the session, the students tried to make sense of the entries in Pascal's triangle by relating them to the Towers and the Pizza problems. Romina used the fourth row of Pascal's triangle "1 4 6 4 1" as an example. She explained that in the Towers problem, "1 4 6 4 1" represented the number of different 4-tall towers when choosing from two colors. In the Taxicab problem, "1 4 6 4 1" represented the number of different shortest routes from the taxi stand to pick-up points that were four blocks apart, and at each intersection the taxi could choose to either go across (east) or go down (south). The first and the last "1" were towers with all four blocks of one color (or routes going either four across or four down). The first "4" was towers with one color-1 and three color-2 (or routes going one across and three down). The "6" was towers with two color-1 and two color-2 (or routes going two across and two down). The second "4" was towers with three color-1 and one color-2 (or routes going three across and one down).

Michael explained "1 4 6 4 1" in terms of the number of pizzas with four available toppings. The first "1" was the plain pizza, the first "4" one topping pizzas, "6" two topping pizzas, the second “4” three topping pizzas, and the last “1” four topping pizzas. Michael stated that he related the number of toppings “*to the number of times going across* (Transcript, Line 1420).”

To explore further, Romina interpreted “1 4 6 4 1” in terms of *x’s* and *y’s* (*x* as going across and *y* going down) and Michael represented “1 4 6 4 1” in terms of 0’s and 1’s. At the end of the session, the students reached a general conclusion that the *r*th row of Pascal’s triangle would represent the number of shortest routes with *r* moves away from the taxi stand.

Chapter 6: Conclusion

6.1 Introduction

Problem-solving-based teaching and cooperative learning provide opportunities for students to discover mathematical ideas by themselves (NCTM, 2000). However, the characteristics of task problems significantly influence the mathematical insight that students may gain from working on the tasks. Teachers have the responsibility to choose rich open-ended tasks that require students to integrate knowledge of multiple mathematical topics and to practice a variety of strategies (Thomas, Williams, & Gardner, 2007). There is a need for teachers to understand the mathematical structures of tasks in order to select or tailor the task problem appropriately for classroom use. In this study, ten tasks are selected from the Rutgers longitudinal project for a close investigation of the underlying mathematical structures. The following sections will summarize the answers to the two research questions of this study. Section 6.2 discusses inherent mathematical structures in these ten tasks. Section 6.3 discusses the ways students uncovered these mathematical structures. Section 6.4 provides conclusions and suggestions to teachers. Section 6.5 addresses the implications of this study and suggests topics for future researches.

6.2 Inherent Mathematical Structures

Recall that the first research question of this study is: *What mathematical structures can be uncovered by exploring/engaging with the combinatorics tasks used in the Rutgers longitudinal study?* In section 5.2, this question has been addressed in detail. The inherent mathematical structures are grouped in three major categories: definitions, axioms, or theorems. One may ask, “How can all of these be called mathematical structures?” To answer this question, first of all, the term “*mathematical structure*” is not yet clearly or consensually defined among mathematicians and researchers. However, most people should not object to the idea that a wooden cross, a wooden frame, a wooden box, a wooden toy, a wooden desk, or a wooden house can all be called a “wooden structure.” Similarly, considering the meaning of the word “*structure*”, definitions, axioms, or theorems are all mathematical structures in this study because they are all constructed by “pieces of mathematics”.

Usually, high school textbooks, such as Larson and Hostetler (2004) and Stewart, Redlin, and Watson (2002), do not classify these “pieces of mathematics” into distinct structure types. In textbooks for college students, the classification of these “pieces of mathematics” is somewhat subjective. In this study, a structure type is assigned to one of the three aforementioned categories either based on the original structure type used in the textbook or according to the following characterizations formulated by the author:

- Definition: a statement that describes the meaning or the essential nature of a mathematical term.
- Axiom: a mathematical statement assumed to be true without proof.
- Theorem: a mathematical statement that is deduced from prior definitions, axioms, or theorems.

To summarize the number of inherent mathematical structures identified in section 5.2, the following table provides a break-down in each structure type:

Task	# of Definitions	# of Axioms	# of Theorems	# of Structures
1	10		1	11
2	9			9
3	3		1	4
4	8		1	9
5	2		2	4
6	2		3	5
7	1		3	4
8	2	3	1	6
9	3		1	4
10	5		2	7
Total	45	3	15	63

Table 6-1. *The number of mathematical structures identified under each structure type.*

Remember that these inherent mathematical structures are not all that can be found. There may be more inherent structures that are not referenced by the limited number of solution methods used in section 5.2. Some of the presented methods can also be used to solve other task problems. This means that some of the identified structures may also be present in tasks other than the one under which they were discussed in chapter 5. However, in section 5.2, mathematical structures are described only once in the first task embedding it. In order to include as many structures as possible, for the subsequent tasks different methods are offered to solve the problems. Bearing all these limitations in mind, some descriptive statistics can be drawn from table 6-1:

- Out of a total of 63 identified structures, 71.4% (45/63) are categorized as *definitions,* 4.8% (3/63) as *axioms,* and 23.8% (15/63) as *theorems*.
- The structure type *Definition* is the largest. Every task has extracted one or more definitions.
- Except for task 2 (Tower 4-tall choosing from 2 colors), every task has extracted one or more theorems.

- All three *axioms* are extracted from task 8 (World Series) and belong to probability theory.

When constructing the concept maps in section 5.3, these inherent structures are categorized into seven broad and overlapping sub-domains: *Set Theory, Enumerative Combinatorics, Graph Theory, Sequences & Sets, General Algebraic System, Probability Theory, and Geometry*. Again this classification is somewhat subjective. Each structure is assigned to one primary sub-domain, and may be included in other sub-domain(s) if it can be linked to structures in that other sub-domain(s). Remember that, in general, every entity in mathematics is related to other entities in some way. Therefore, only the key relationships (links) are shown in the concept maps. The following table displays the mathematical structures identified in section 5.2 and the primary sub-domain of each:

Task	Set Theory	Enumerative Combinatorics	Graph Theory	General Algebraic Systems	Sequences & Sets	Probability Theory	Geometry
1	T1.1 to T1.5 and T1.7		T1.8 to T1.11			T1.6	
2			T2.1 to T2.9				
3		T3.3 T3.4	T3.1	T3.2			
4				T4.6	T4.1 to T4.9 except T4.6		
5		T5.1 to T5.4					
6		T6.1 to T6.5 except T6.4		T6.4			
7		T7.1 to T7.3		T7.4			
8						T8.1 to T8.6	
9						T9.1 to T9.4	
10		T10.7		T10.1 to T10.3			T10.4 to T10.6

Table 6-2. *The identified mathematical structures in each sub-domain*

Table 6-3 below lists the number of identified mathematical structures in each sub-domain:

Task	Set Theory	Enumerative Combinatorics	Graph Theory	General Algebraic Systems	Sequences & Sets	Probability Theory	Geometry	Total
1	6		4			1		11
2			9					9
3		2	1	1				4
4				1	8			9
5		4						4
6		4		1				5
7		3		1				4
8						6		6
9						4		4
10		1		3			3	7
Total	6	14	14	7	8	11	3	63

Table 6-3. *The number of identified mathematical structures in each sub-domain*

Notice that the links between and among different sub-domains are not shown in this table (see section 5.3 for these links.) All the limitations described before also affect the values of the information shown in tables 6-2 and 6-3. Nevertheless, descriptive statistics are useful in the task analysis:

- The distribution of inherent structures is: 9.5% (6/63) *Set Theory*; 22.2% (14/63) *Enumerative Combinatorics*; 22.2% (14/63) *Graph Theory*; 11.1% (7/63) *Sequences & Sets*; 12.7% (8/63) *General Algebraic System*; 17.5% (11/63) *Probability Theory*; and 4.8% (3/63) *Geometry*.
- A little less than half of the identified structures are categorized as *Enumerative Combinatorics* or *Graph Theory*, both of which belong to the domain *Combinatorics*.
- All three structures of *Geometry* are extracted from task 10 (Taxicab).
- All six structures of *Set Theory* are extracted from task 1 (Shirts and Jeans).

- All eight structures of *Sequences & Sets* are extracted from task 4 (Tower of Hanoi).
- Structures embedded in task 2 (Tower 4-tall choosing from 2 colors) are primarily categorized as *Graph Theory*.
- Structures embedded in tasks 5 and 6 (Pizza problems), are primarily categorized as *Enumerative Combinatorics*.
- Structures embedded in task 8 (World Series) and task 9 (The Points) are primarily categorized as *Probability Theory*.

6.3 Mathematical Structures Uncovered

Recall that the second research question of this study is: *In what ways are these mathematical structures revealed during students' problem-solving processes*? This question has been explored in section 5.4. Students' written work and the ways they approached their solutions are discussed in details. In summary, the inherent mathematical structures were uncovered by students primarily in the following ways:

(1) Manipulating the concrete model: This approach was used when the concrete model was available. In the Towers problems, the students used two *sets* (T1.1) of colored cubes to build towers (finding different *combinations* (T3.3)). They constructed towers in systematic ways. Strategies like "by cases" or "by induction on the height" were used to verify and explain the solutions. The students realized that there are two color choices for each position in the tower (${}_2C_1 = 2$ (T3.3, T3.4)). For an already-built tower, they chose an opposite color for each position to build another tower. This laid a solid base for understanding the *fundamental counting principle* (T1.6).

In the task Tower of Hanoi, the students also began to play with the puzzle (a concrete model) in order to get the first few terms of a *sequence* (T4.1).

(2) Listing all possible combinations: When a concrete model was unavailable, the students would begin the task by making a list of all possible *outcomes* (T8.1) of a certain *event* (T8.1). In the task Shirts and Jeans, the students demonstrated a natural ability to collect similar *objects* (T1.1) and placed them into a *set* (T1.1). Then, without knowing the terminology, they applied the *Roster Method* (T1.2) and listed all possible outfits (*elements* (T1.1) in the *Cartesian product* (T1.5)) by literally drawing them.

In all other tasks, the students used many different coding schemes to list combinations they found for the purpose of recording outcomes, checking for duplicates, verifying for completeness, or explaining results to others. Younger students tended to use whole words or letter codes. High school students seemed to prefer the binary code 0 and 1 since Michael had first introduced it.

(3) Inventing different representations: This approach was usually driven by the need to organize information for a better understanding or a better controlling of problem situations. In the Taxicab task, when the students lost track of counts for the shortest routes they drew, Romina wanted to devise a method to do it. She invented a grid rectangular box and clearly defined what each position on the grid meant. In this way, she was able to find and record the number of shortest routes for each intersection on the grid, one by one. The counts were much better represented and arranged in this rectangular box. Before long, these number of counts strongly suggested *Pascal's triangle* (T6.1).

(4) Seeking patterns: The students were able to see patterns as they were working from the simplest case and increasing the number of the control variable by one each time. In the Tower of Hanoi, Michelle and Ankur recognized the *recursive* pattern after they played the concrete puzzle for one, two, three, and four disks. After they found the number of moves were {1, 3, 7, 15,…} (*recursively defined sequence* (T4.9)) respectively, Michelle was able to predict that the moves required for five disks would be 31.

Building towers from 1-tall, 2-tall, 3-tall, … also enabled the students to see patterns and eventually generalize the total number of different n-tall towers choosing from m colors to be n^m.

(5) Making connections: This approach was extremely powerful when the students were facing unfamiliar situations raised by a new task they had not worked on before. Relating the World Series task to the Towers and Pizza problems, Jeff and Romina were able to determine that the total number of possible games played (the *sample space* (T8.1)) in an n-game World Series was 2^n because the total number of pizzas with n available toppings was 2^n; and the total number of n-tall towers selecting from two colors was also 2^n.

Making connection also helped the students better understand the Addition property (${}_kC_{r-1} + {}_kC_r = {}_{k+1}C_r$ (T6.3)) in *Pascal's triangle* (T6.1). Stephanie explained why in Pascal's triangle 1 + 3 (in "1 3 3 1") = 4 (in "1 4 6 4 1") in terms of pizzas, and Robert explained this in terms of towers (see section 5.4.6). This suggests that the relationships among the structures involved in these ten tasks may have been an instrumental factor in creating an environment that encourages students to capitalize on observed structural similarities among prior and new tasks in order to make progress on the new tasks.

6.4 Conclusion and Suggestion

Dr. Maher once indicated that students may not have a language to describe what they are thinking, but they possess the mathematical ideas (Mayansky, 2007, Appendix 1, Transcript, Lines 497-506). To prepare students with such a (formal or mathematical) language, teachers should deepen their understanding of the underlying mathematical structures for each task they are going to use in the classroom. From the above summary along with the detailed discussion in chapter five, conclusions and suggestions are made as follows:

(1) Mathematical structures are inherent in each of the ten selected tasks analyzed in this work from the counting strand of the longitudinal study. Teachers may benefit from studying the underlying mathematical structures of a task thoroughly before assigning the task to students.

(2) The students in this study often found solutions based on their previous knowledge and experiences. In determining the order of related tasks within a strand, teachers need to consider the sophistication level and the coherence of the underlying mathematical structures across tasks.

(3) Formally defined mathematical structures make more sense to students if there is meaning attached to them. Using concrete models can help students to both develop and verify solutions to complex problems. Teachers should ask students to

verify their solutions by making connections, when appropriate, to a concrete model or a real-world model.

(4) A strand of tasks whose inherent mathematical structures belong to a variety of mathematical sub-domains can help students build an increasingly interconnected view of mathematics. Teachers should encourage students to solve problems by different approaches and share their thoughts with one another in order to explore whether other mathematical structures might be uncovered while working on the tasks.

6.5 Concluding Remarks and Future Research

The Rutgers longitudinal study is ongoing and currently is in its 23rd year now. Recently, Rutgers researchers have traced the development of mathematical ideas and the ways of reasoning for participating students' beyond their high school years. Pantozzi's (2009) work examines how the students make sense of the Fundamental Theorem of Calculus when revisiting it three years later. Steffero's (2010) case study explores Romina's mathematical beliefs and behaviors for the development of mathematical ideas and reasoning based on interviews from her high school, college, and post-college career, spanning seventeen years. Ahluwalia's (2010) dissertation, based on the student's fifth grade problem-solving sessions and post-graduate interviews, analyzes how Robert built connections among mathematical ideas, while exploring Pascal's Pyramid and connecting and structure of its solution to problems solved in the counting/combinatorial strand.

This study has demonstrated a way to extract and analyze inherent mathematical structures while engaging in the solution of a collection of task problems. The analysis, it is hoped, will motivate mathematics teachers to better understand the underlying mathematical structures of tasks and thus improve the way they prepare students for a problem-solving session. Further investigations on exploring the mathematical structures that might be elicited in working on the tasks are strongly recommended. For example, researchers may want to explore the range of definitions offered for mathematical structure. They may also wish to find analysis tools for assessing the value of a task in term of the mathematical structures that could be uncovered. Questions that may arise are: How might students be made aware of the underlying mathematical structures in a task? Are there preferable ways to help students uncover connections among mathematical structures? If so, what implications does this have for teachers' classroom practices?

Another direction for future study may concern the order of the tasks in a particular strand used in the Rutgers longitudinal study. The ten selected tasks from the combinatorics/counting strand (i.e., the tasks discussed in this study) were arranged and presented to the students in a seemingly logical order. Does this sequence of the tasks fit students' mathematical background? How appropriate is the study of these tasks for pre- and in-service professional development? Are there other sequences of the tasks that might be effective in building similar mathematical understandings? The Rutgers longitudinal study provides a collection of rich data that has inspired researchers to come up with interesting and valuable research questions. Others are offered for future study.

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Appendix A: The Task Problems

Task 1: Shirts and Jeans

Stephen has a white shirt, a blue shirt, and a yellow shirt. He has a pair of blue jeans and a pair of white jeans. How many different outfits can he make? Convince us that you found them all.

Task 2: Towers 4-tall with Two Colors

Your group has two colors of Unifix cubes. Work together and make as many different towers four cubes tall as is possible when selecting from two colors. See if you and your partner can plan a good way to find all the towers four cubes tall.

Task 3: Towers 4-tall with Three Colors

Your group has three colors of Unifix cubes. Work together and make as many different towers four cubes tall as is possible when selecting from three colors. See if you and your partner can plan a good way to find all the towers four cubes tall.

Task 4: Tower of Hanoi

Figure A-1 below shows a puzzle with three posts, and there are seven disks stacked as a tower on one of the three posts. You have to move all the disks from the post to another post. The rule is: you can only move one disk at a time and you can never put a smaller disk onto a bigger disk. How many moves do you need to complete the task? If this is a 100-disk tower, how many moves do you need?

Figure A-1. *Tower of Hanoi Puzzle*

Task 5: Pizza with Halves

A local pizza shop has asked us to help them design a form to keep track of certain pizza sales. Their standard "plain" pizza contains cheese. On this cheese pizza, one or two toppings could be added to either half of the plain pizza or the whole pie. How many choices do customers have if they could choose from two different toppings (sausage and pepperoni) that could be placed on either the whole pizza or half of a cheese pizza? List all possibilities. Show your plan for determining these choices. Convince us that you have accounted for all possibilities and that there could be no more.

Task 6a: 4-topping Pizza

A local pizza shop has asked us to help design a form to keep track of certain pizza choices. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many different choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possible choices.

Task 6b: 4-topping Pizza with 2 Crusts

The pizza shop was so pleased with your help on the first problem that they have asked us to continue our work. Remember that they offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. The pizza shop now wants to offer a choice of crusts: regular (thin) or Sicilian (thick). How many choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possible choices.

Task 6c: 4-topping Pizza with Halves and 2 Crusts

At customer request, the pizza shop has agreed to fill orders with different choices for each half of a pizza. Remember that they offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushroom, and pepperoni. There is a choice of crusts: regular (thin) and Sicilian (thick). How many different choices for pizza does a customer have? List all the possible choices. Find a way to convince each other than you have accounted for all possible choices.

Task 7: Ankur's Challenge

Find all possible towers that are 4 cubes tall, selecting from cubes available in three different colors, so that the resulting towers contain at least one of each color. Convince us that you have found them all.

Task 8: World Series

In a World Series two teams play each other in at least four and at most seven games. The first team to win four games is the winner of the World Series. Assuming that the teams are equally matched, what is the probability that a World Series will be won: a) in four games? b) in five games? c) in six games? d) in seven games?

Task 9: The Problem of Points

Pascal and Fermat are sitting in a café in Paris and decide to play a game of flipping a coin. If the coin comes up heads, Fermat gets a point. If it comes up tails, Pascal gets a point. The first to get ten points wins. They each ante up fifty francs, making the total pot worth one hundred francs. They are, of course, playing "winner takes all." But then a strange thing happens. Fermat is winning, 8 points to 7, when he receives an urgent message that his child is sick and he must rush to his home in Toulouse. They carriage man who delivered the message offers to take him, but only if they leave immediately. Of course, Pascal understands, but later, in correspondence, the problem arises: how should the 100 francs be divided?

Task 10: Taxicab

A taxi driver is given a specific territory of a town, represented by the grid in Figure A-2 below. All trips originate at the taxi stand, the point in the top left corner of the grid. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated by the other points on the grid. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route. What is the shortest route from the taxi stand to each of three different destination points? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answers.

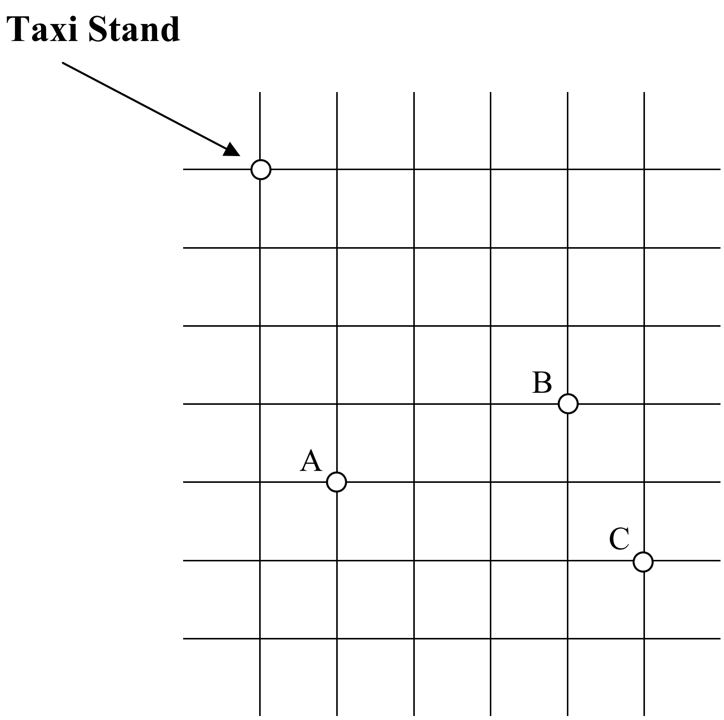


Figure A-2. *The map of the town and the taxi stand*

Appendix B

Students' Written Works

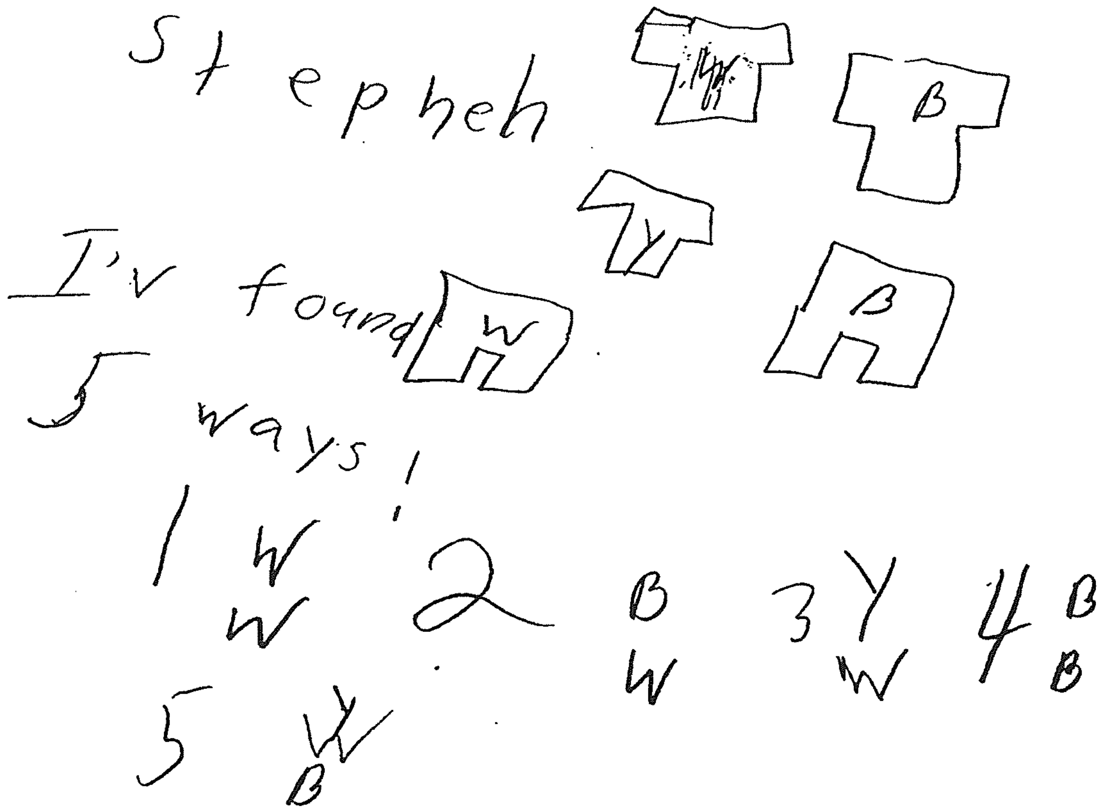


Figure B-1. *Stephanie's written work for Shirts and Jeans* (grade 2).

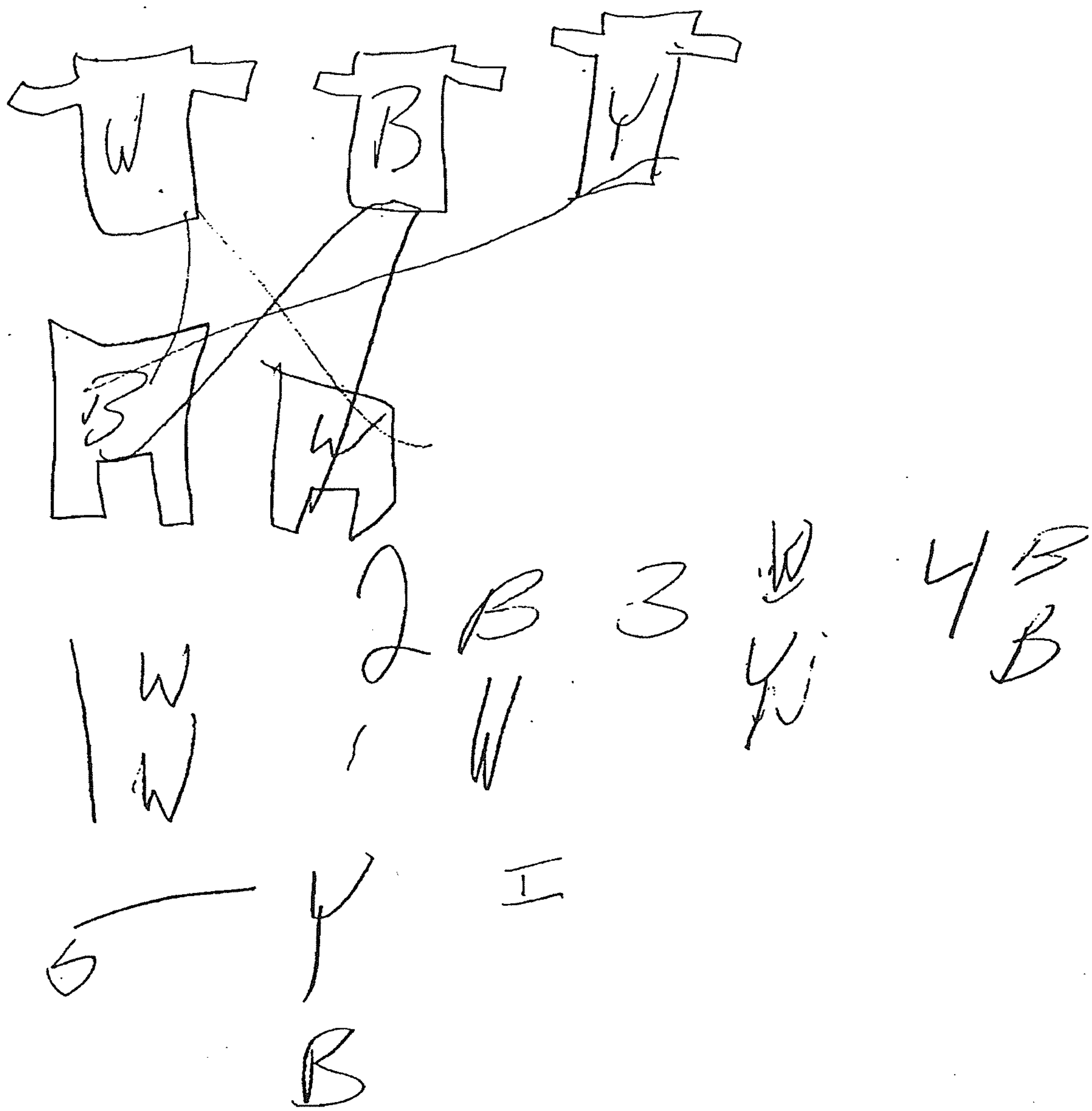


Figure B-2. *Dana's written work for Shirts and Jeans* (grade 2).

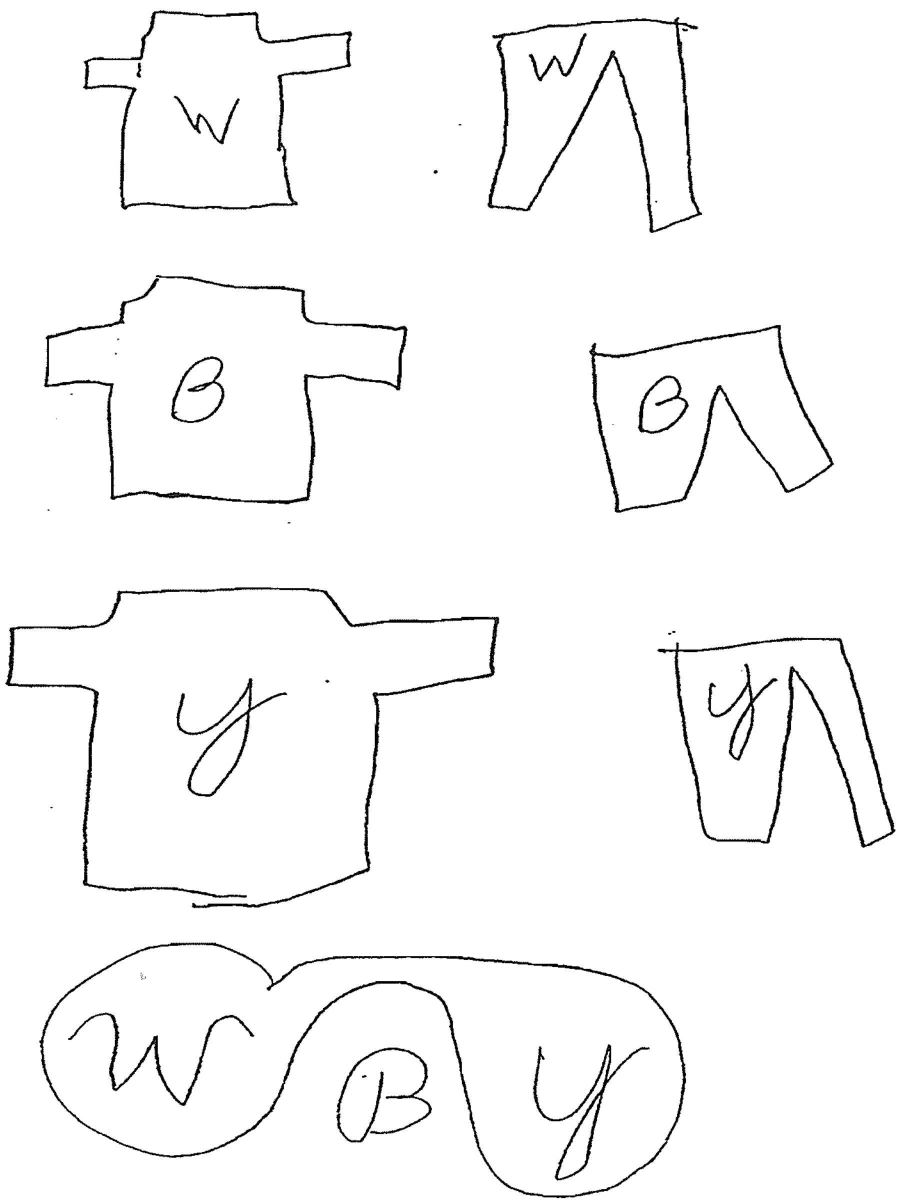


Figure B-3. *Michael's written work for Shirts and Jeans* (grade 2).

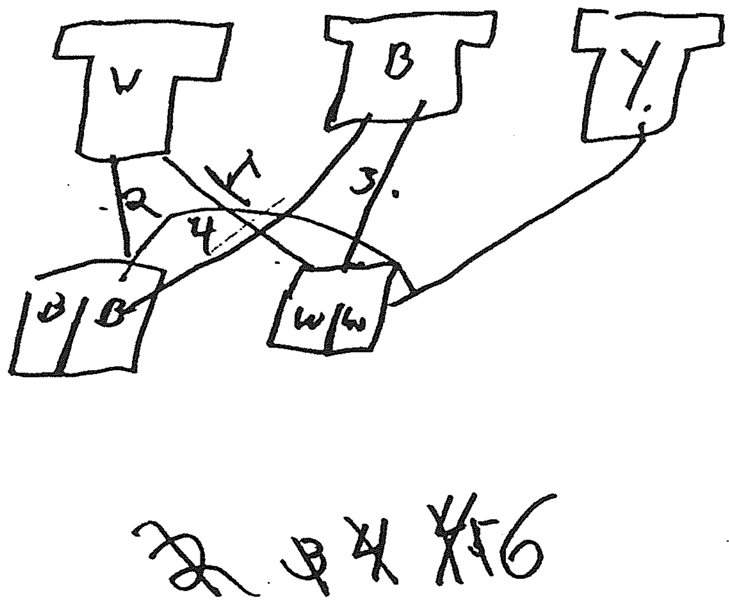


Figure B-4. *Stephanie's written work for Shirts and Jeans* (grade 3).

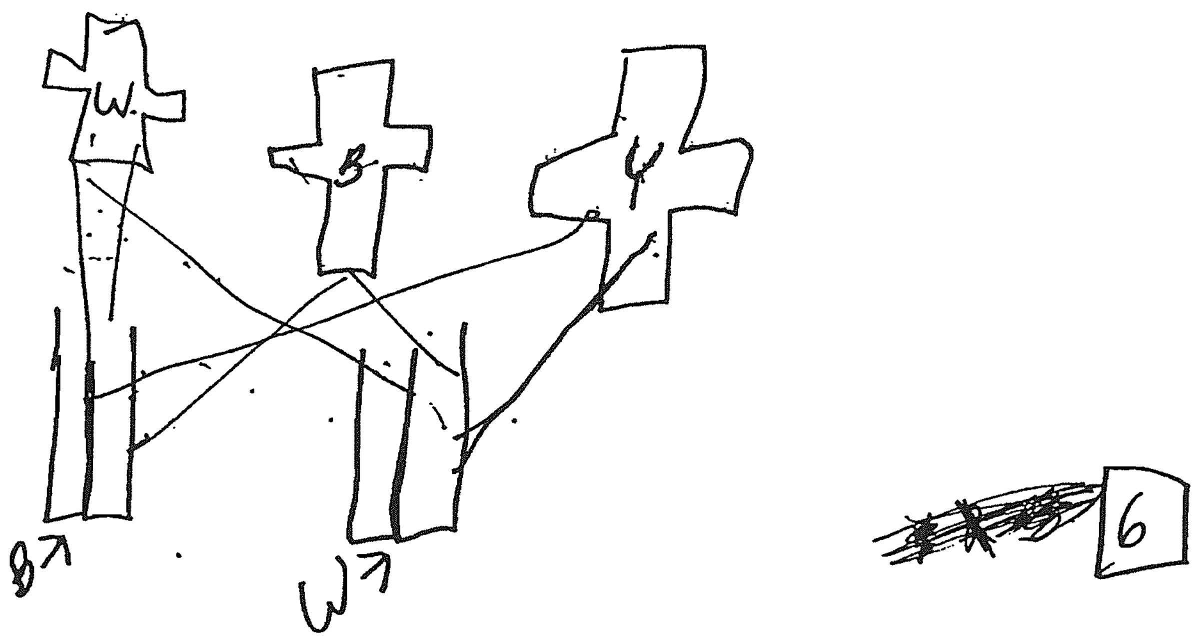


Figure B-5. *Dana's written work for Shirts and Jeans* (grade 3).

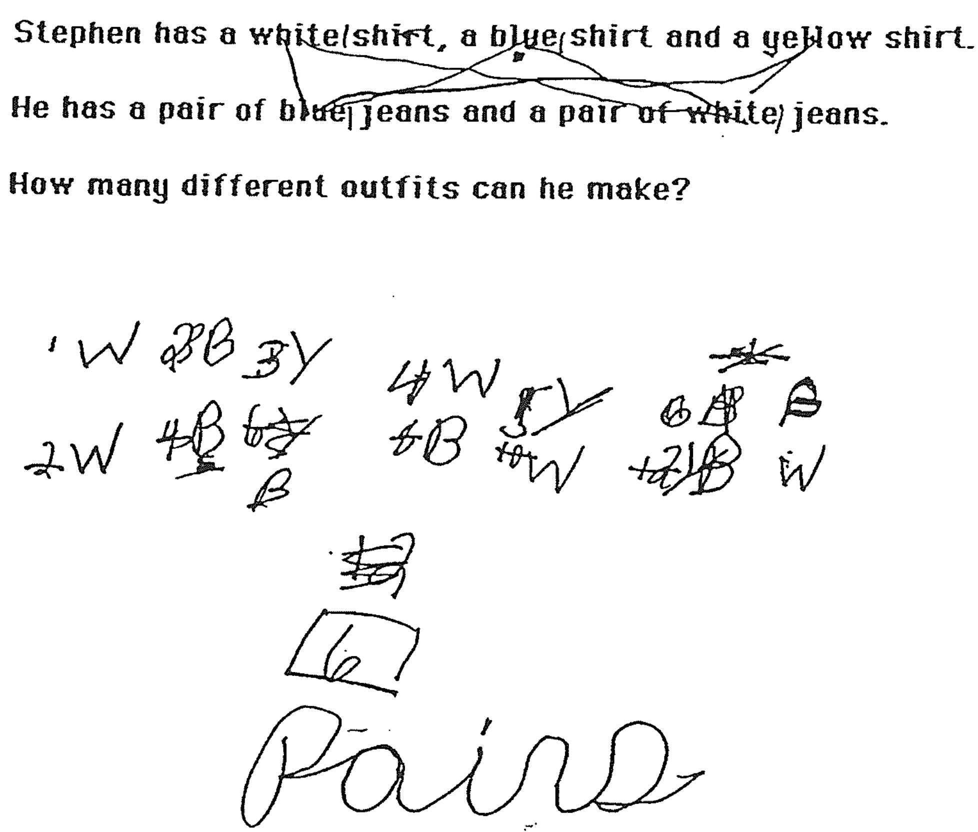


Figure B-6. *Michael's written work for Shirts and Jeans* (grade 3).

Whole

1 plain

1 sausage

1 pepperoni

1 Mixed (pepperoni & sausage)

Half

$\frac{1}{2}$ pepperoni $\frac{1}{2}$ plain

$\frac{1}{2}$ sausage $\frac{1}{2}$ plain

$\frac{1}{2}$ peperoni $\frac{1}{2}$ sausage

$\frac{1}{2}$ plain $\frac{1}{2}$ sausage & pepperoni MIXED

$\frac{1}{2}$ pepperoni $\frac{1}{2}$ pepperoni + sausage MIXED

$\frac{1}{2}$ sausage $\frac{1}{2}$ pepperoni + sausage MIXED

Figure B-7. *The students' final collaboration for Pizza with Halves* (grade 5).

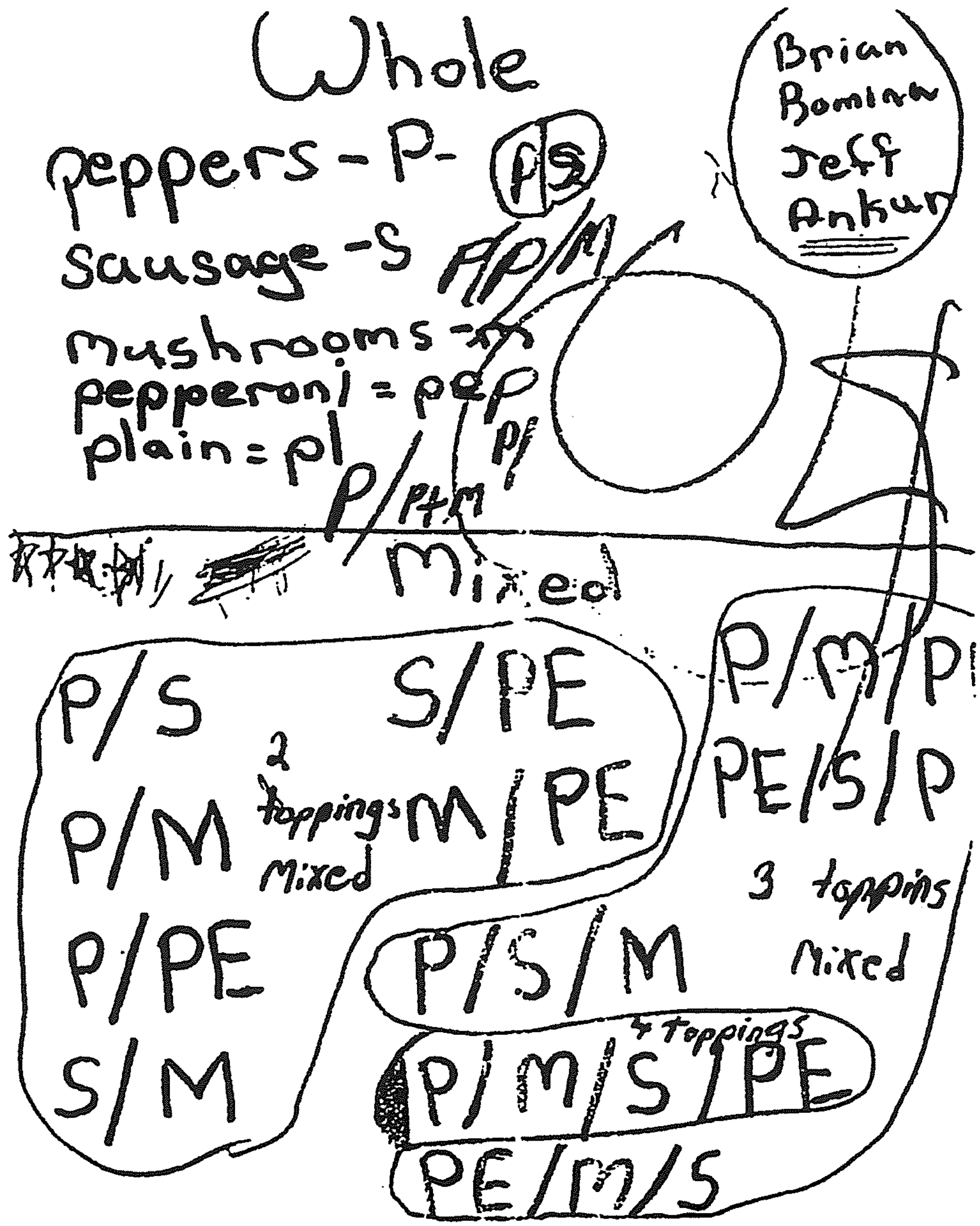


Figure B-8. *Ankur, Brian, Jeff, and Romina's written work for Pizza choosing from four toppings* (grade 5).

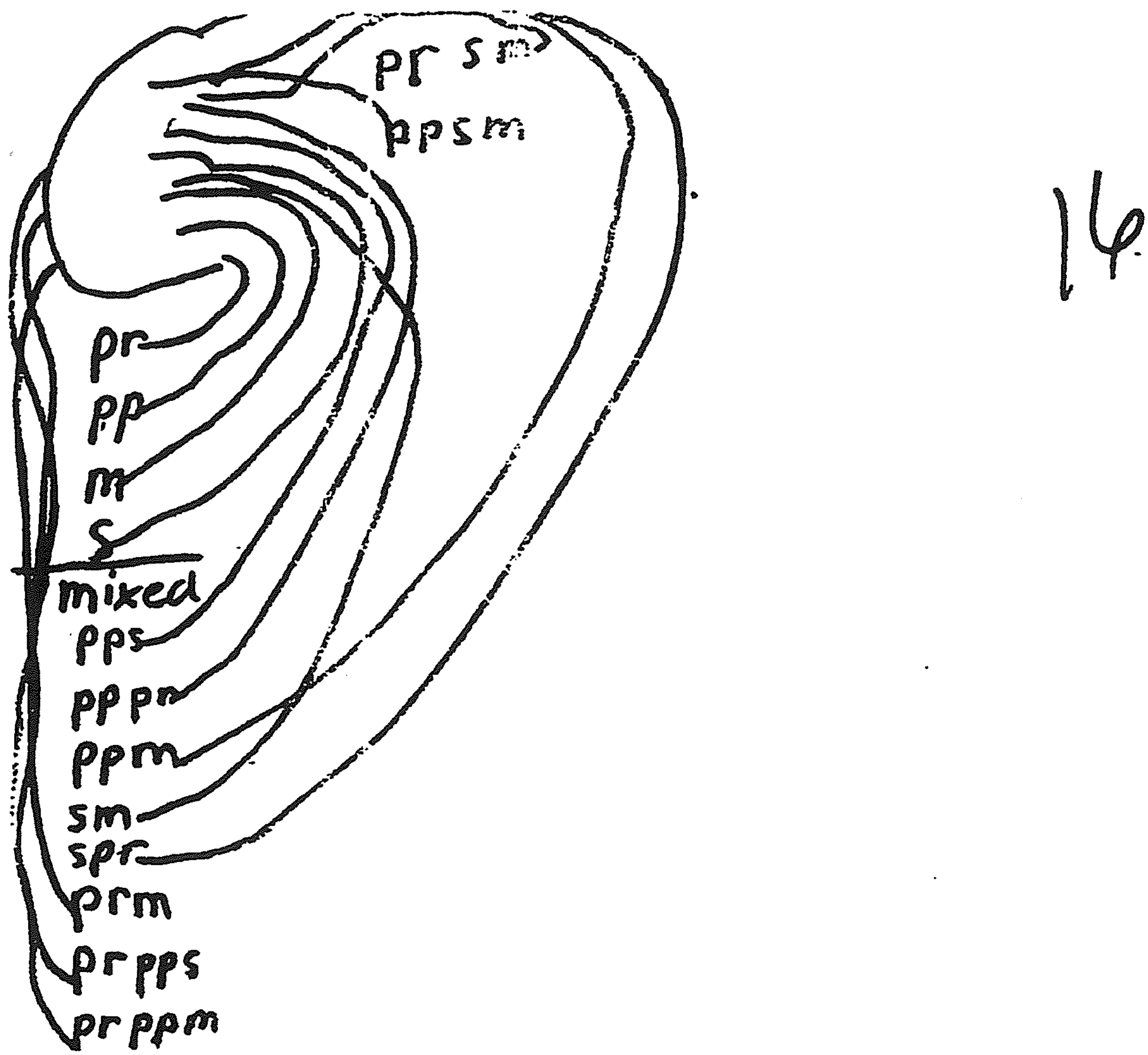


Figure B-9. *Matt's written work for Pizza with Halves choosing from four toppings and two crusts* (grade 5) – page 1.

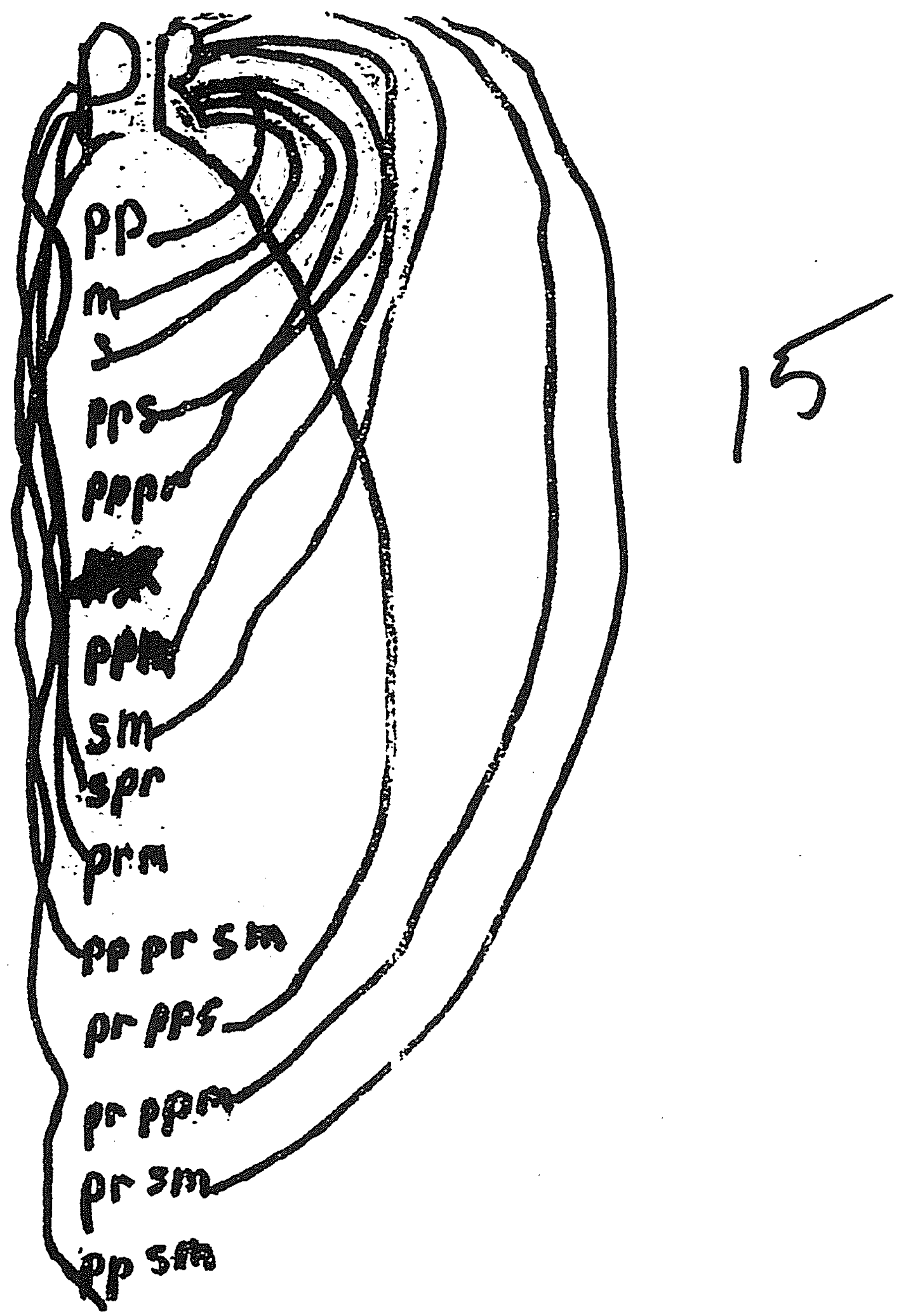


Figure B-10. *Matt's written work for Pizza with Halves choosing from four toppings and two crusts* (grade 5) – page 2.

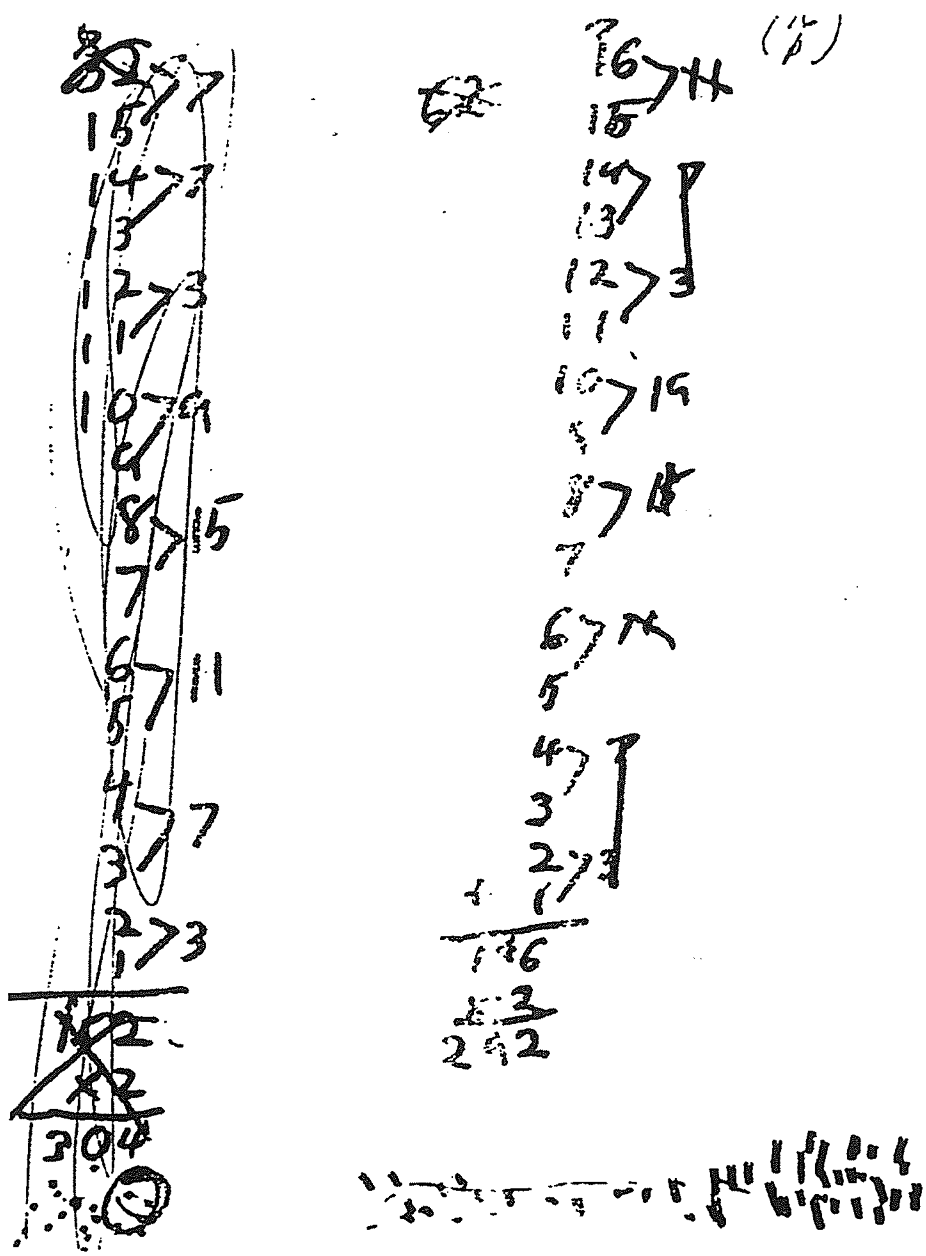


Figure B-11. *Matt's written work for Pizza with Halves choosing from four toppings and two crusts*(grade 5) – page 3.

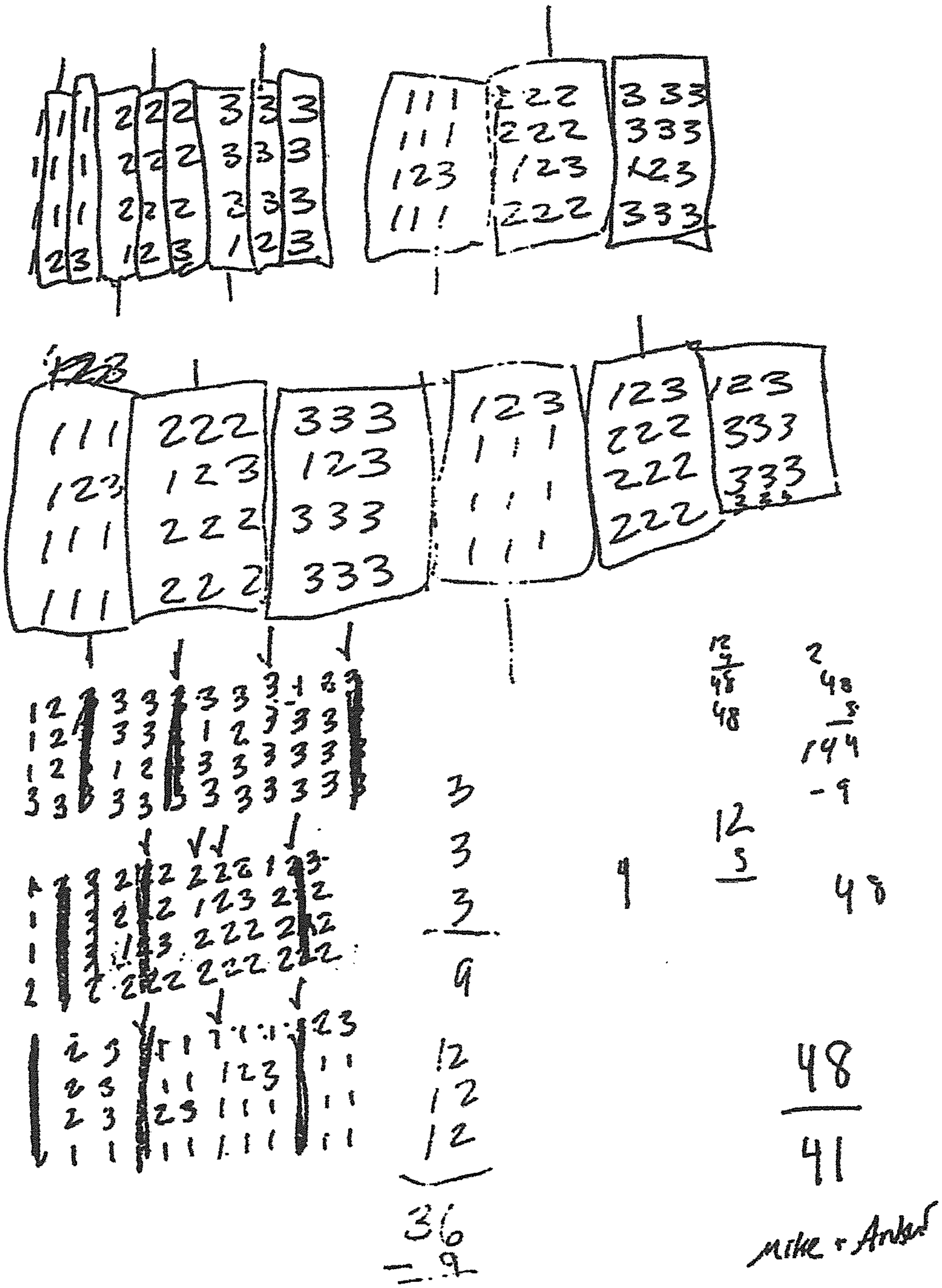
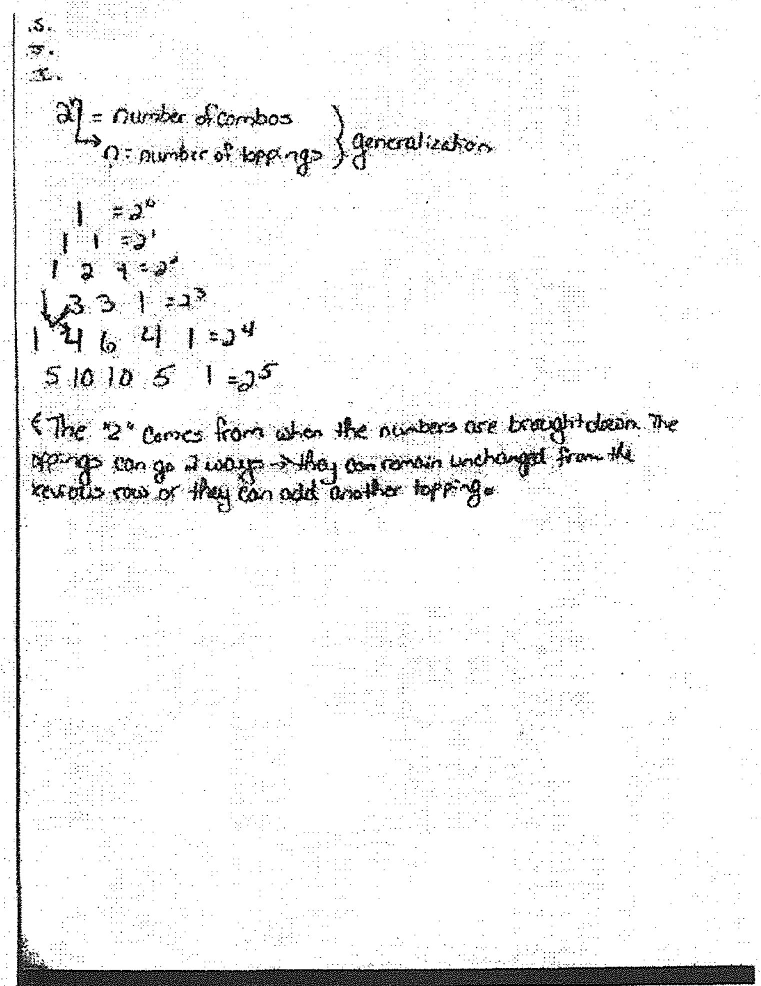


Figure B-12. Ankur and *Michael's written work for 81 4-tall Towers choosing from three colors* (grade 10).



.5.
7.
I.

2^n = number of combos
n = number of toppings } generalization

1 $=2^0$
1 1 $=2^1$
1 2 1 $=2^2$
1 3 3 1 $=2^3$
1 4 6 4 1 $=2^4$
5 10 10 5 1 $=2^5$

← The "2" comes from when the numbers are brought down. The ppings can go 2 ways → they can remain unchanged from the revious row or they can add another topping.

Figure B-13. *Robert's written work for Pizza choosing from four toppings* (grade 11).

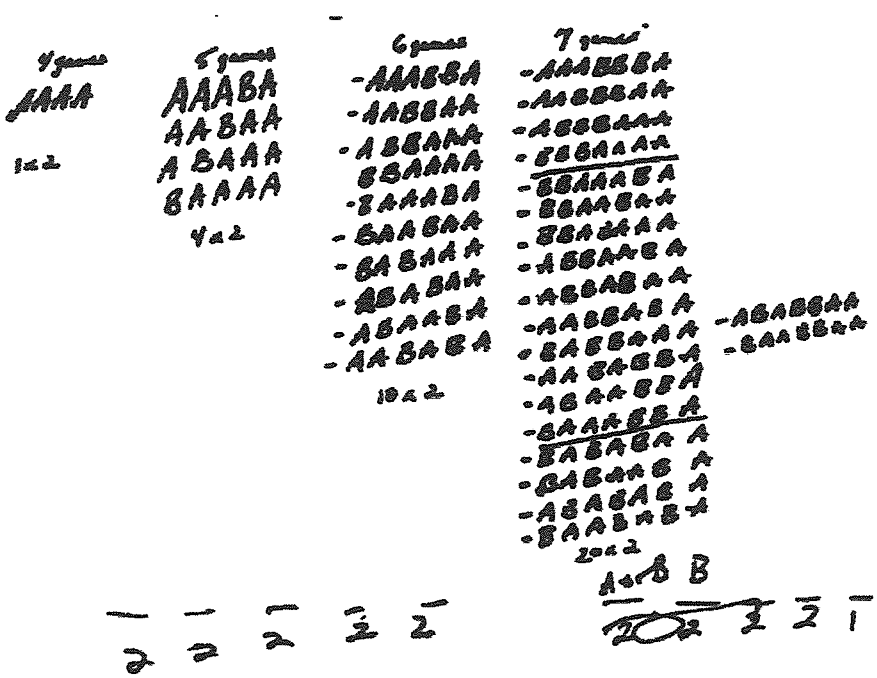


Figure B-14. *Ankur's written work for World Series* (grade 11).

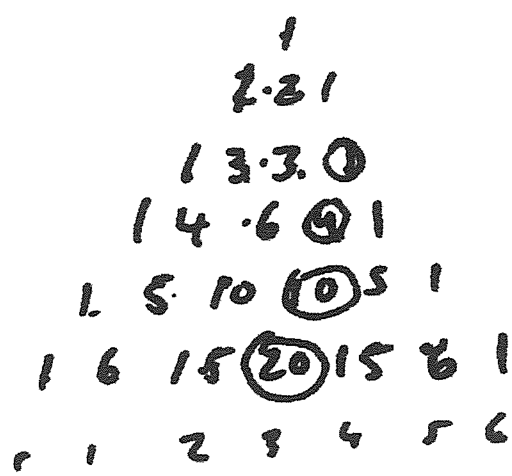


Figure B-15. *Michael's representation of Pascal's Triangle for World Series* (grade 11).