Description: PUP Math – Towers of Hanoi **Location: Harding School – Kenilworth, NJ** **Researcher: Professor Carolyn Maher**	**Transcriber(s): Private Universe Project** **Verifier(s): Sigley, Robert, Sran, Kiranjeet** **Date Transcribed: Spring 2000** **Page: 1 of 15**

Line	**Time**	**Speaker**	**Transcript**	**Code**
1.		*Narrator*	*When the Kenilworth students were in the sixth grade, the late Robert B. Davis led a research session based on the classic game, Towers of Hanoi. The researchers were interested in finding out how the students would solve problems involving exponential functions, even though at this point, their formal knowledge of exponents was very limited.*	
2.		Robert Davis	You may know this puzzle. It's called The Tower of Hanoi. Do you know the story that goes with it?	
3.		Students	Yeah...	
4.		Robert Davis	They claim there was an order of monks in the City of Hanoi, who were religious men who lived by themselves. And they were concerned about when the world was going to end. And so they made a puzzle like this which has 100 disks in it. And they spent all of their time- plus they eat and sleep and things like that- but when they're not doing things like that, they spend all their time working to solve that puzzle. When they have it done, that's supposed to be when the world ends. Okay? And I thought it might be interesting to figure out when the world's going to end, so we'd know too.	
5.		Jeff	Yeah, but I'd be scared.	
6.		Robert Davis	Now let's agree on what the rules are. The rules are you can only move one disk at a time, and what else?	
7.		Ankur	You can't move a bigger onto a smaller one.	
8.		Robert Davis	You can never put a bigger one on top	

			of a smaller one. Okay. Now if we want to find out when the world is going to end- some safe way; we're not going to do a 100...	
9.		*Carolyn Maher*	*Bob Davis came to mathematics education as a mathematician. He had three degrees from MIT in mathematics, but decided that he was really interested in how people learned mathematics. He was really interested in thinking. And so he was one of our very first pioneers to come into this field and lead the way.*	
10.		Robert Davis	What could I do?	
11.		Ankur	Do you think if we get all hundred, the world will really end?	
12.		Mike	We'll probably be dead when you get it.	
13.		Jeff	Yeah. By the time we figure it out with a hundred, we'll be dead.	
14.		Robert Davis	I want you to do it.	
15.		*Carolyn Maher*	*For them, this was an unsolved problem. For them, like the mathematician who's working on a problem, they don't know the answer. And even though we know there is an answer, they know that we're not going to tell them what that answer is. So for them, the conditions are very much like mathematicians doing original mathematics.*	
16.		Amy-Lynn	We got the whole thing in....	
17.		*Fern Hunt*	*It's hard when you've got a lot of disks flying around, to try to find a way to move these disks without breaking those rules.*	
18.		Stephanie	...twenty two, twenty three, twenty four.	
19.		*Fern Hunt*	*I think if you were to start to actually carry this task out, one finds out pretty*	

			quickly that it can get pretty complicated.	
20.		Ankur	One, two..	
21.		Michelle	..Three, four..	
22.		Ankur, Michelle	..Five, six, seven.	
23.		Ankur	Three is seven.	
24.		*Fern Hunt*	*One of the first things that a mathematician often does is to simplify the situation. Rather than look at the problem in all its complexity, look at another problem. And that problem shares, perhaps, some of the characteristics of the original problem. But it has many fewer of the complexities. And you would work with that simpler problem to see what one could learn. Hopefully, what you learn in that situation, you can apply to the more complex situation.*	
25.		Robert Davis	Okay. I want somebody to come- Suppose we had just one disk. Somebody come and solve that puzzle.	
26.		*Fern Hunt*	*So a simplification is an important step.*	
27.		Student	It has to move at least once.	
28.		Robert Davis	Okay. And now I need to keep track. When there was one disk it took one move. Everybody agree with that?	
29.		Student	Yes.	
30.		Robert Davis	And that's what we've got here. Somebody told me if there was two disks, it would take three moves. Is that right?	
31.		Student	Yes.	
32.		Robert Davis	Somebody come and do that. Amy-Lynn, can you come do that?	
33.		Robert Davis	Great it took her three moves, okay? That looks all right. Is that okay? We	

			need somebody to come down and do it with three disks.	
34.		Jeff	I'll do it with three.	
35.		Robert Davis	Two, three, four, five, six, seven. Looks like it's right, huh? Okay. Now is that all right? Everybody happy with that? Now how about four moves? Milin, have you figured out what it will be with four moves?	
36.		Milin	That's nine.	
37.		Robert Davis	It's nine.	
38.		Michelle	I can do it in fifteen.	
39.		Jeff	Go Milin.	
40.		Mike	Oh it's times 2 + 1. Oh we got it. I know, I know... Can I tell everybody? Is it the number times 2 + 1?	
41.		Milin	Yes. The number times 2 + 1. It always works.	
42.		Robert Davis	Is that right?	
43.		Milin	Yes.	
44.		Jeff	Can we still play the game now?	
45.		Robert Davis	Okay, we need to try this, I think, because we've got some disagreements here.	
46.		Robert Davis	Fifteen. You did it in 15.	
47.		Stephanie	I did it in 15.	
48.		Robert Davis	Okay, Stephanie just did it in 15. Can anybody do it in less than 15?	
49.		Matt	I found a pattern. I found the pattern with it.	
50.		Robert Davis	You found the pattern?	
51.		Matt	Yes.	
52.		Robert Davis	What is the pattern?	
53.		Matt	It's like ...look at from this way, two times two, times one is three. Three times three plus one. Four times four minus one. Then it would go five times	

			five plus one..	
54.		Student	We noticed a pattern.	
55.		Robert Davis	You know the pattern too.	
56.		Matt	Six times six minus one.	
57.		Robert Davis	Don't do that too quickly here.	
58.		Michelle	Like one and one is three, and then you add one more, and then it's- three and three are six. And then you add one more, and then seven and seven are 14, and you add one, is 15, and 15 and 15 are 30-	
59.		Robert Davis	Michelle and Ankur have found something very clever, but we may not end the world today.	
60.		*Fern Hunt*	*Another thing that a mathematician does is look for patterns. They look at, perhaps, many instances. And from those classes of problems that the mathematician is solving, certain patterns may arise.* *The idea is in some sense to try to understand or somehow summarize what that pattern might be.*	
61.		Michelle	This is what we did. One plus one is two, and then one more is three. Three plus three is six, and then plus one is seven. And then ...	
62.		Brian	Wait. there's an easier way. See, there's two between there. It doubles becomes four? Four is between there, it doubles and becomes eight. Eight doubles..	
63.		Ankur	But that doesn't ...	
64.		Romina	Yeah, I know.	
65.		Brian	But it's the easiest way to figure it out.	
66.		Ankur	Oh please.	
67.		Robert Davis	Michelle- Could I get everybody's	

			attention, please, for just a minute, because Michelle has something interesting to say. Can you show everybody what you're doing?	
68.		Michelle	Well, one and one is three, and then you add one more, and then it's- Three and three are six, and then you add one more. And seven and seven are 14, plus one is 15. So then the next one would be 15 and 15 is 30, plus one is 31. And then so on, and so on, and so on...	
69.		Robert Davis	Thirty-one. Okay. And now what is the one we really care about? The one that counts...	
70.		Student	One hundred.	
71.		Robert Davis	.. is 100. So we want to know what number goes there.	
72.		Student	Oh my God.	
73.		Ankur	Maybe if we get ten, we can get like, 20, and then 30.	
74.		Stephanie	Ten is 1,023.	
75.		Ankur	Ten is 1,023.	
76.		Stephanie	I already got down to ten.	
77.		Ankur	Ten is..	
78.		Michelle	... 1,023. Want to work with us Steph? If you guys are just off in la-la land.	
79.		Stephanie	Matt, come on.	
80.		Brian	And then do ten times ten. Not ten times ten. Ten times 1,023.	
81.		Michelle	Very quickly here. We've got to catch on.	
82.		Ankur	Shelly, this is 2 to the tenth power.	
83.		Michelle	Oh my God. Duh, we had it right there.	
84.		Ankur	What's 2 to the 100th power? That's the answer.	
85.		Michelle	2 to he 100th power?...	
86.		Ankur	We got it. We got it.	

87.		Robert Davis	You've got it. Okay, can we get one discussion so everybody can hear? Who's going to do the talking about this problem?	
88.		Students	All of us.	
89.		Robert Davis	All of you?	
90.		Students	We all did it.	
91.		Robert Davis	All right. Can you sort of face the rest of the people and tell them what you got?	
92.		Ankur	We tried to figure out ten, right? And it was ..one hundred and twenty three ... so we found... that two to the tenth power also equals 123. So we figured that two to the hundredth power should equal the answer.	
93.		Robert Davis	Now, I'm not sure that I think two to the tenth is 1,023.	
94.		Michelle	We figured this out by going through the numbers.	
95.		Students	It's 1,024.	
96.		Stephanie	That's not right.	
97.		Students	It's 1,024.	
98.		Student	Because you can't have an odd number as the last number.	
99.		Robert Davis	Thank you very much. That's a cleaver idea.	
100.		Michelle	And then we realized since it would work for this to this, why wouldn't it work, oh excuse me, from 2 to the 100th power.	
101.		Student	Why can't it be 10,240?	
102.		Ankur	It's 2 times 2 times 2 and so on to a hundred.	
103.		Robert Davis	Yeah. Instead of multiplying, instead of writing ten twos and multiplying them, you have to write 100 twos and multiply them. That's more than	

			adding zeros.	
104.		Ankur	That's the equation but we didn't figure out the answer, yet.	
105.		Mike	I just saw something. What we're trying to do is 1 and 3, the difference is 2. Three and 7, the difference is 4. Seven and 15, the difference is eight.	
106.		Robert Davis	I'll write those numbers, too, if that helps.	
107.		Matt	Oh, I have it. I have it. You're multiplying everything by two. Two times 2 is 4 times 2 is 8 times 2 is sixteen, times two....	
108.		Robert Davis	Do we agree that we've got something very valuable here. Do we agree that that's a pretty good idea?	
109.		Students	Yes.	
110.		*Narrator*	*Almost seven years after this session, Matt, currently a freshman at Virginia Polytechnic Institute, watched and discussed his work as a sixth grader with Australian mathematician, Gary Davis, Professor of Education at the University of Southhampton, in southern England.*	
111.		Matt	You're multiplying everything by two.	
112.		Gary Davis	Have you got it?	
113.		Matt	Uh-huh.	
114.		Gary Davis	You got the pattern?	
115.		Matt	Uh-huh.	
116.		Garry Davis	Do you have any feeling when you look at that? Do you have a feeling of reconstructing what you were doing there? Because you're sitting there by yourself.	
117.		Matt	I just pretty much was sitting there, like, concentrating, just looking at	

			numbers, you know? Like with these and what we did here, it's a lot of just trying to look at patterns and looking at different kinds of patterns.	
118.		Garry Davis	That's right.	
119.		Matt	And if you just see, like, 2 and 4, you automatically say, "All right. That's either adding 2 or multiply by 2." And you say "Four times 2 is 8, times 2 is 16, times 2. No wait." And you see- I just pick up on things.	
120.		Garry Davis	So you just try? You're trying different things in your head, and-	
121.		Matt	Yeah. Pretty much. Yeah.	
122.		Garry Davis	So in a sense, what's on the board's really important to you-	
123.		Matt	Yeah.	
124.		Garry Davis	-because that's what you were looking at.	
125.		Matt	Yeah.	
126.		Robert Davis	We know one thing we could do is we could keep extending this table. All right, is that what you were doing or not? What would go here for six? What would go here for six if I used I think it was Michael's rule?	
127.		Brian	Sixty three.	
128.		Robert Davis	Right, 63. So one way, you could come down and find out what goes according to this rule. Who made up that rule? It's a neat rule. Who said "take this number and double it and subtract, no- add one; double this and add one"- who made up that rule?	
129.		Student	...not me.	
130.		Gary Davis	Well of course his original question was "how many moves would it take for 100 rings.?"	

131.		Mike	Probably a lot.	
132.		Gary Davis	A lot, it'll take a lot, there's no question about it. Is there any way you can figure that out from this?	
133.		Mike	Yeah, uhhmm, when moving the four you got this to a point where there was seven.	
134.		Gary Davis	There was seven.	
135.		Mike	When you move those three it took seven moves.	
136.		Gary Davis	It did. It did.	
137.		Mike	So then it took another eight to move the rest of it. I'm trying to think.	
138.		Gary Davis	Sorry, eight?	
139.		Mike	Yeah, it was fifteen, right?	
140.		Gary Davis	It was fifteen, yeah. yes it was.	
141.		Mike	So, maybe to move three, take seven.	
142.		Gary Davis	Yes.	
143.		Mike	Now you've got to move this guy somewhere.	
144.		Gary Davis	You do.	
145.		Mike	Eight. And to move those three again is another seven.	
146.		Gary Davis	Oh right. So the 15 is 7 plus 1 and 7?	
147.		Mike	Plus one, plus seven- That's a possibility. I don't know if that's-	
148.		Robert Davis	One of the questions is, I still don't know where this 2 to the 10th came from. What did you do to get the 2 to the 10th? But the other thing is, you're telling me 10 and you're telling me 100, and I'd like you to tell me how to do it with any number of disks. Okay? What would happen with 7 disks or 700 disks, or whatever? Because we really want to able to do it for any number. Okay?	

149.		Ankur	I got ten million with just 26 digits.	
150.		Matt	Guys, I ran out of room at about 20.	
151.		Robert Davis	Shell, do you get 2 times 50 on your calculator?	
152.			[simultaneous conversation]	
153.		Ankur	I'm just multiplying by 2 by hand.	
154.		Stephanie	Real smart. Okay, what number are we multiplying by?	
155.		Ankur	Seven, 7, 2, 8. This number.	
156.			[simultaneous conversation]	
157.		Matt	Thirty two... 64.	
158.		Robert Davis	Sixty four.	
159.		Matt	I figured it out.	
160.		Robert Davis	What?	
161.		Matt	I figured it out... plus 1 plus 1 plus 1 plus 1 plus 1 plus 1-	
162.		Robert Davis	That's a neat idea. Really neat idea.	
163.		Matt	Ooooohhhhhh.	
164.		Matt	I like to see things, kind of- more like a visual learner than sitting there and doing it in my head and saying, "Well, two times is this." Say, "Okay, put it down on a piece of paper. See what we have." Because then you can look for patterns.	
165.		Matt	That would be 127.	
166.		Robert Davis	That's what I get, too -127. See if you can make that table go through further than that.	
167.		Matt	Hey guys, I figured out the pattern.	
168.		Ankur	Is Matt's right? Does Matt have a pattern?	
169.		Matt	....Plus one, plus 1, plus 1.	
170.		Michelle	That's what we said before.	
171.		Matt	Just like this, not with all this. Just like this.	
172.		Michelle	Oh I get it. I see what you're doing.	

173.		Jeff	Oh these are going up.	
174.		Gary Davis	And as the end approaches, Ankur comes up and starts working with you on it, and Jeff comes over and starts working with you on it. It's like you're a magnet.	
175.		Matt	Yeah, maybe, I guess.	
176.		Gary Davis	Well, why does he come up to you?	
177.		Matt	I don't know. I guess he sees something, or he can see the same thing that I see- or sees something different than what I see, and can add on to what I've done.	
178.		Gary Davis	What allows him to do that?	
179.		Matt	Instead of just seeing a bunch of huge numbers on a paper, seeing more of a pattern to it, and seeing it written as, like, a pattern, instead of seeing it as 2 to the first, as 2 to the second- as this huge number, and keep on going in huge numbers- those that are easier to see a pattern between.	
180.			[simultaneous conversation]	
181.		Amy-Lynn	What are you doing?	
182.		Bobby	When I go home I'm going to write "times 2" 100 times to figure it out. I'm just going to keep on putting "times 2" in my calculator. I'm going to figure out the answer.	
183.		*Carolyn Maher*	*Fern outlined some of the things mathematicians do when they do mathematics. And I think it's very interesting to watch that the children and the tape do some of these same things. They do think of a simple problem, they do look for patterns, they look for finite differences- as you see. They notice the pattern and they notice that there's an exponential here. They*	

			posit two to the end. And that's what mathematicians do. They see these patterns, they pose a theory, that they have to go back and test it.	
184.		*Narrator*	*A few days later the students were still interested in finding out how long it would be before the world would end.* *Bobby reported that he knew how many moves it would take for 100 disks.*	
185.		Robert Davis	Okay, let me show you -Bobby wrote something here which I think several of you had last Thursday. If you had a hundred disks, he says it would take this many moves. Okay, let's assume - Bobby and Amy-Lynn worked pretty carefully on this and they think they've got the right number. So let's assume this right: 28 comma, 458 comma, 001 comma, 530 comma, 100 they say. Okay. Suppose it takes that many moves -and I don't really believe that story about the world ending, but let's pretend we did. Let's figure out when the world would end. If it takes that many moves, how long is that going to take?	
186.		Student	A long time.	
187.		Mike	It could take a day. It could take a day.	
188.		Jeff	... because if seven of them take ten minutes.	
189.		Robert Davis	Okay, I want somebody to come and solve the problem here with disks. Four disks. Milin will you time this carefully? Okay, Milin say go when you're ready... what?	
190.		Student	Now.	

191.		Robert Davis	Go.	
192.		*Narrator*	*The students performed a series of tests to find the average time per move.*	
193.		Student	Go Matthew.	
194.		Robert Davis	How long did it take?	
195.		Milin	Thirty one seconds.	
196.		Robert Davis	It took thirty one seconds.	
197.		Student	Yes, she's got it.	
198.		Robert Davis	How much time?	
199.		Milin	Two Minutes and fifty seconds.	
200.		Robert Davis	Two Minutes and fifty seconds.	
201.		Brian	Oh yeah.	
202.		Robert Davis	So it's about..	
203.		Ankur	... It's two seconds per move.	
204.		Robert Davis	So it's about twice as many seconds as there are moves. Right? If we assume that Bobby as the right number of moves here, Okay. He says that many moves. So how many seconds will that be?	
205.		Student	Oh boy.	
206.		Robert Davis	Well it's going to be twice as many. Would you all double this, multiply this by 2 and tell me what you get?	
207.		Narrator	Finally the students did a series of calculations to convert the units from seconds to years.	
208.		Robert Davis	So it's about, it's about that many years. Sot what is that? It's saying... 2 billion years. Isn't that what it's saying?	
209.		Student	Oh my god, it's going to take that many years to do that?	
210.		Robert Davis	Somebody once said if you really knew the world was going to end you wouldn't be able to get on the telephone, everybody would be busy calling somebody to say "I love you."	

211.		Student	I love you!	
212.		Mike	I love you Jeff.	
213.		Jeff	Really, what do you think you would be doing if you were going to die?....	
214.		*Narrator*	*We've seen students using a variety of problem solving strategies to approach the Towers of Hanoi problem. What strategies have you observed your students using to solve difficult problems?*	