INVESTIGATING STUDENT LEARNING AND BUILDING THE CONCEPT OF INVERSE FUNCTION

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ABSTRACT OF THE DISSERTATION

Investigating Student Learning and Building the Concept of Inverse Function

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Dissertation Chairperson Carolyn A. Maher, Ed. D

The concept of function is one of the most important ideas in the learning of mathematics (Dubinsky & Harel, 1992). Yet it is considered by many researchers to be the least understood by high-school and college students (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Sfard, 1992). Reforming early mathematics curricula in algebra, therefore, is justified. To this end, the National Council of Teachers of Mathematics (2000) called for a longitudinal view of algebra, from elementary to advanced mathematics education. As a strand in the Rutgers-Kenilworth longitudinal study in 1993, Robert B. Davis introduced early algebra ideas to eleven-year-old students during the sixth grade. Research prior to Davis’ intervention with the students showed how they built their understanding of linear, quadratic, and exponential functions (Spang, 2009; Giordano, 2008; Mayansky, 2007). Building on Davis’ approach to early algebra and the learning of function, Emily Dann designed a study to determine whether these students, now seventh graders, could extend their understanding to the concept of inverse function.

The present study analyzes videotaped work of seventh-grade students who were engaged in a series of activities that Dann had devised. The Guess-My-Rule activities, as they were called, were conducted over three consecutive days. Using the model that Powell, Francisco, and Maher (2003) described for analyzing videotaped data, this study examines in detail the students’ work as they collaborated in small groups to develop
rules for function and inverse; the study also investigates the obstacles students had experienced.

This research demonstrates that seventh-graders understood the idea of function by writing rules, symbolically, to describe the relationships of quantities. Understanding function as *action*, they progressed to the *process* concept when creating their own function tables and corresponding rules. Using inverse operations, students wrote inverse rules; however, due to difficulties with integer and fraction arithmetic, they needed to adjust their initial attempts in order to be successful.

This study maintains that having facility with function and inverse function concepts will permit students to learn the subject matter, to communicate ideas and solutions, and to interconnect mathematical ideas. In the process of exploring these related concepts, students will be encouraged to think independently and to devise original strategies in their work with function and inverse.

The results demonstrate to researchers and educators how students build the concepts of function and of inverse function through group work in a specific environment. Seventh-grade students can engage in activities, similar to those described above, that are essential to the study of algebra.
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Chapter 1 Introduction

1.1 Statement of the Problem

According to Dubinsky and Harel (1992), the concept of function is essential to mathematical learning on all levels, from kindergarten to graduate school. Carlson and Oehrtman (2005) assert that a strong understanding of the concept is especially important for students studying calculus. In The Principles and Standards for School Mathematics, the significance of the function concept is explicit: “instructional programs, from prekindergarten through grade 12, should enable all students to understand patterns, relations and functions” (NCTM, 2000, p. 37). Undoubtedly, it should occupy an important position in the mathematics curriculum.

The definition of understanding, as “the power to make experience intelligible by applying concepts and categories” (Understanding, 1971, p. 967), when considered in the context of mathematical learning must be more precise. According to Davis (1992b), mathematical thinking involves “representations of problem situations and of relevant knowledge,” as well as how to deal with this information. In this context, Davis uses the phrase, “the real essence,” to describe an intellectual process (p. 226) in which mental representations are, in effect, constructs of “previous experience” (p. 227). Based on the work of Minsky and Papert (1972), Davis (1992b) suggests that “the feeling of understanding” becomes evident when a new idea can be fitted into a larger framework of previously–assembled ideas” (p. 228). To describe this assemblage, he uses the analogy of a jig-saw puzzle in which “each new candidate piece, like each new idea, can be used only if it fits into the aggregate of pieces that have previously been assembled” (p. 228).
In the notion of assembled ideas drawn from experience, researchers include the possibility of creating new ideas. “Understanding the function concept,” as defined by Breidenbach et al. (1992), for example, involves more than merely applying intellectual concepts to mathematical experiences. It is the ability “to make sense out of a situation by constructing a mental process that transforms objects” (p. 1). These transformed constructs can then be utilized to “think about functions” (p. 1). The deeper meaning of understanding, as defined by Davis and Breidenbach et al., is directly applicative to this study of seventh-grade students working on function and inverse function activities. Students who understand the function and inverse concepts demonstrate the building of mental representations and functional thinking. The process of understanding, which is fundamental to mathematical learning, therefore, must be a pedagogical focus at all grade levels.

The concept of function, despite its inarguable importance in the curriculum, is least understood by college students, as claimed by Breidenbach, Dubinsky, Hawks and Nichols (1992). They maintain that students who have taken a fair amount of undergraduate mathematics, including the full calculus sequence, did not demonstrate an adequate understanding of the concept. At the beginning of the semester, these investigators posed a fundamental question to sixty-two college students. These sophomore and junior math majors who were preparing to be elementary, middle-school, or high-school teachers, were asked: What is a function? Forty percent of the student responses was categorized as pre-functional (i.e., students did not understand the function concept). Twenty-four percent who responded with an action concept understood function as an algebraic expression. According to these data, many of the participants did
not understand adequately the function concept as expected of college mathematics students.

The ongoing reform of the early mathematics curriculum in algebra is therefore justified. The Principles and Standards of School Mathematics (NCTM, 2000) advocates a longitudinal view of algebra as a strand of thinking and of problem solving, beginning in elementary school and extending throughout mathematics education. Algebraic instruction for grades K through 12 encompasses the relationships among quantities, the use of symbols, the modeling of phenomena and the mathematical study of change. In elementary grades, it includes elements of algebraic reasoning, although the word algebra is not used in their instruction. These algebraic experiences are important precursors to the more formalized study of algebra in the middle and secondary grades.

Carraher, Schliemann, and Schwartz (2008) point out that algebra is inherent in the early mathematics curriculum in the U.S., that is, in word problems, in addition, subtraction, multiplication, division, ratio and proportion, rational numbers and measurement. They note that it is also in representational systems, such as number lines, graphs, tables, written arithmetical notation, and explanatory structures. Teachers gradually introduce formal notation in early algebra, discuss student interpretations in class, and thereby provide opportunities through which students can adjust their understandings. Thus, these investigators stress that it is the role of the teacher to bring the algebraic character of the curriculum into public view.

Just as the function has served a unifying role in the history of mathematics, according to Carraher, Schliemann, and Schwartz (2008), the concept can also unite a wide range of isolated topics, including number operations, fractions, ratios and
proportions, formulas, and others. Introducing the concept of function to elementary school students, through grade-appropriate activities, can instill meaningful understandings of this important concept and can help set the stage for later learning.

The inverse function, like the function concept, requires attention in the classroom. The Principles and Standards for School Mathematics state that, in grades 9-12, all students should ”understand and perform transformations such as arithmetically combining, composing, and inverting commonly used functions, using technology to perform such operations on more complicated symbolic expressions” (NCTM, 2000, p. 295). Once students are able to work with functions, they should be introduced to the inverse. According to Benson and Buerman (2007) understanding inverse as a way of breaking something down by reversing the operations performed in the relationship, gives students the ability to wield mathematical power.

Students gain initiative when they are able to use mathematical terminology to communicate their ideas and solutions. The Connections Standard of the Principles and Standards for School Mathematics proposes that instructional programs, from prekindergarten through grade 12, should enable all students to “recognize and use connections among mathematical ideas and understand how mathematical ideas interconnect and build on one another to produce a coherent whole” (NCTM, 2000, p. 64). Once these concepts have been understood, they must then be communicated. The Communication Standard states that students should be able to “use the language of mathematics to express mathematical ideas precisely” (NCTM, 2000, p. 60).

When they are introduced to the inverse, students will be able to interrelate mathematical concepts. Benson and Buerman (2007) claim, for example, that students
will have a greater understanding of inverse if they discuss reciprocals as multiplicative inverses and opposites as additive inverses. The interconnection between these and other terms involving inverses will allow students to create a “coherent whole” which is the outcome the NCTM standards envisage.

Benson and Buerman (2007) believe that exploring the concept of inverse will promote individual thinking: students will be moved to defend, to refine, and to reinforce their ideas of what inverse is or does. In the process of working with inverses, students will recognize the importance in mathematics of identifying patterns, of asking questions, and of gathering information while reasoning their way towards solutions.

As discussed above, the pedagogical and curricular shortcomings in the teaching and learning of the concepts of function and of inverse are glaringly evident and remedies have been sought. The concept of function, important as it is, ironically is the least understood by college students. Reforming early mathematics curricula in algebra, therefore, is justified. To this end, the NCTM calls for a longitudinal view of algebra, from elementary to advanced mathematics education. One point of view holds that algebra is inherent in early mathematics curricula in the US: it is intrinsic in word problems and in related procedures, as well as in representational systems. The inverse function, similarly, requires attention from grades nine through twelve. Investigators argue that facility with functions, accompanied by an introduction to the inverse, is the basis, at an early age, for independent cognition. Having facility with both concepts will permit students to learn the subject matter, to communicate ideas and solutions, and to interconnect mathematical ideas. In the process of exploring these related concepts, students will be encouraged to think independently and to devise original strategies,
regarding their natures and functions. This study establishes, on the basis of classroom activities, that seventh-grade students can gain facility with both concepts.

1.2 Purpose of the Study

Understanding how students learn complex mathematical concepts is essential for teaching and defining curriculum and this is especially true on the elementary level. The purpose of the present study, a continuation of research conducted by Giordano (2008) of “Guess-My-Rule” activities, is to determine how students understand the concepts of function and inverse when presented with tables of values. Due to the complexity of the function concept the current study attempts to show that students demonstrated an understanding of function and its inverse. It focuses on the students’ demonstration of functional thinking in their work.

The present study investigates seventh-grade students at the Harding School in Kenilworth, New Jersey, who were participants in the longitudinal study conducted by researchers from Rutgers University. The longitudinal study began in 1984 with the students in first grade, and tracked their progress through high school and college. The purpose of the study was to observe and to analyze the development of mathematical ideas in children engaged in doing mathematics. The algebraic strand of the study, commenced with sixth graders, was developed by R. B. Davis. The seventh-grade sessions, a continuation of Davis’ work, were undertaken to track the development of algebraic thinking, specifically as students build the concept of function and extend it to the idea of inverse.
1.3 Research Questions

The students’ work on functions and on inverse functions led to the following research questions:

1. What evidence, if any, exists that students understand the idea of function?

2. How do students build the idea of inverse function?
   - What obstacles, if any, do they encounter?
   - If obstacles are encountered how are they overcome?

3. What evidence exists, if any, that students understand the idea of inverse function?
Chapter 2 Literature Review

2.1 History of Function

To understand the function concept in all of its complexity, one can study its historical development. For educators and researchers, this is especially important: familiarity with the concept’s history will assist them in analyzing how students, elementary to advanced, can comprehend it. Furthermore, being aware of past obstacles in the development of mathematics will illuminate the kinds of problems that students experience today (Liu, 2003). Researchers suggest that historical knowledge of this kind can help teachers to see the stages of learning in a clear light (Barbin, 2000). A historically-informed approach can have pedagogical implications: it can make teachers more acutely aware of the students’ experience—of the intellectual travails and of the time spent on mathematical concepts (Swetz, 1982). The historical development of the concept of function, surveyed below, can be useful in analyzing how students learn the concept and its inverse.

The modern concept of function, according to Kleiner (1993), originated in the early 18th century, but its roots stretch to antiquity. Kleiner (1989) claims that the mature concept could not emerge earlier due to the absence of algebraic prerequisites including realizing the continuum of real numbers and the development of symbolic notation. He also points out that the abstract concept was undefined due to lack of examples. Kleiner (1993) reasons that since mathematical concepts arise from mathematical needs, the precursors or anticipations of the concept, evident over 4000 years ago, were a response to the mathematical needs of ancient science. Over the past three centuries, the concept has indeed evolved and has become fundamental to all of mathematics (Rüthing, 1984).
2.1.1 As tables of values.

Scholars have surveyed the development of the concept of function in antiquity. Youschkevitch (1976) explains that, in 2000 B.C., Babylonian mathematicians used sexagesimal (base 60) tables of reciprocals, squares and square roots, cubes and cube roots, and other numerical tables for their calculations. In addition, he notes their use of the step-function and the linear-zigzag-function (as cited in Neugebauer, 1957) for the compilations of ephemerides of the sun, moon and planets. The tabulated functions, which were understood empirically, became the basis of astronomy. Youschkevitch asserts that these ancient calculations lacked the term function, as well as any reference to the abstract and general idea of the dependencies between quantities.

The Greeks did not contribute substantively to the concept’s evolution as can be determined from historical evidence. Bochner (1970) contends that the early Greeks did not have the concept of function in their thinking and that there is nothing in Greek mathematics that could be interpreted as the notion of function, as it has been understood from the 16th century to the present. He enumerates that although the Greeks were familiar with “correspondence,” “dependence,” “mapping,” and “binary relation” these categories alone did not make for the presence of functions. Bochner explains that something must also be “done” with those functions such as some kind of “mathematical” operation and that operational activity was missing from Greek thought.

According to Pedersen (1974) Ptolemy (85-165 A.D.), in the first-century A.D., contributed significantly to the development of the concept. He points out that Ptolemy in the Almagest (c.a. 150 A.D.) determined that the positions of the sun, moon and planets changed continuously and periodically in time and that these positions could be
determined by means of standard procedures and sometimes explained by numerical examples. In his standard procedures Ptolemy used several instances of up to three variables.

Pedersen notes that Ptolemy clearly represented a trigonometric function of one variable, known as the function of chords. The function calculates the length of a chord from an arc of x degrees, in a circle of radius R, as follows:

\[ ch(x) = 2R \sin \frac{x}{2} \]

The relationship was expressed in the form of a table with two columns where column 1 was the independent variable and column 2 the corresponding value of the dependent variable that had been used to determine the length of a chord from its arc. Pedersen adds that Ptolemy often used the table to find the arc from its chord, which represents the inverse function:

\[ x = 2 \arcsin \frac{ch(x)}{2R} \]

Pedersen infers syllogistically that if a function is a relation of elements of one set to elements of another set, then functions did in fact appear in the *Almagest*. He claims that even though the word “function” was missing, and even though the formula was not specified, the concept was implicit.

Bochner (1970) credits Nicole Oresme (1323-1382) as the most function-oriented of all medieval mathematicians. Oresme developed an early form of graphing and envisioned fractional exponentiation. Oresme developed the geometric theory of latitude of forms which uses general ideas about independent and dependent variable, quantities
which he called longitude and latitude. According to Youschevitch (1976) these quantities can be considered types of coordinates.

Oresme was the first to prove Merton’s theorem or the mean speed theorem. Oresme proved that uniform acceleration is equivalent to motion with uniform velocity which, in turn, is equal to the arithmetic mean of the initial and the final velocities (Baron, 1969). His proof consists of a geometric model showing equality of areas of certain rectilinear figures. A rectangle corresponds to uniform velocity and a triangle corresponds to uniform acceleration (Figure 1: Baron, 1969, p. 85).

Figure 1. Rectilinear areas.

Bochner notes the ironic fact that Oresme’s work, although known in the Middle Ages, was not influential at the time nor was it so in the 17th or 18th centuries. Not until the 19th century, during the Great Rehabilitation of the Middle Ages, was Oresme’s works revived and expanded upon.

The lack of necessary mathematics, in Butterfield’s view (1957), halted the development of mathematical physics in the Middle Ages. He contends that it was not until the Renaissance that the mathematics of the ancient world would be recovered. Butterfield asserts that in the 17th century the science of mathematics made remarkable
strides and if not for the achievements of the mathematicians of the period, the scientific revolution would have been impossible.

Throughout its gestation, the concept of function was intimately related to experimental physics. During the later 16\textsuperscript{th} and early 17\textsuperscript{th} century Galileo (1564-1642) used relationships between variables in his study of motion and developed the law of falling bodies in which the distances traveled from rest during accelerated motion are invariably proportional to the time squared. His mathematical interpretation of the free fall of heavenly bodies, as Youschkevitch (1976) observes, resembles Oresme’s interpretation of the Merton theorem, since the latter theorem appeared in nearly seventeen books printed in the 16\textsuperscript{th} century.

Butterfield notes that during this time arithmetic and algebra advanced. François Viéte (1540-1603) (also known as Vieta), in 1591, established the use of letters to represent numbers and, for the first time, made it possible to write algebraic equations and expressions using unknown quantities, and he was the first to use the term \textit{coefficient} (Youschkevitch, 1976). Contemporaries of Viéte made notable contributions to the concept’s development. Simon Stevin (1548-1620), in 1585, introduced the decimal system for representing fractions, and John Napier (1550-1617), in 1614, published a table of logarithms and their meaning and use (Smith, 1959).

\textbf{2.1.2 As geometric curves and expressions.}

Concurrent with Galileo’s work on the law of motion was René Descartes’ (1596-1650) application of algebra into geometry. In his \textit{La Géométrie} (1637), Descartes states that geometric curves bear a definite relation to all points of a straight line and this relation must be expressed by means of a single equation (Descartes, 1965).
Youschkevitch (1976) reports that Descartes clearly stated that an equation in two variables, $x$ and $y$, representing a curve, indicates a dependence between variable quantities. He credits Descartes with introducing the classification of geometric curves as lines described by equations of the second degree.

Youschkevitch (1976) recognizes that some basic principles of Descartes’ universal mathematics resemble Oresme’s theory of latitudes of forms. Specifically similar are the relationships among quantities represented by geometric forms and by segments of straight lines. Youschkevitch states that it is not known if Descartes actually read Oresme’s works; but indebtedness to Oresme is possible since, as Youschkevitch relates, a diary entry indicates that Descartes’ friend, I. Beeckman, was very familiar with Oresme’s ideas and with the Merton theorem.

Youschkevitch (1976) reports that at the same time as Descartes but independently, Pierre de Fermat (1601-1665), published the *Introduction to plane and solid loci* in 1679. In his publication, Fermat recorded equations of a straight line and of curves of the second order, using Viète’s notation, and the rectilinear coordinate system.

Youschkevitch (1976) notes that neither Descartes nor Fermat used the word function but their work implies the concept. He contends that the introduction of functions in the form of equations opened a new era in mathematics and “the use of analytical expressions, the operations with which are carried out according to strictly specified rules, imparted a feature of regular calculus to the study of functions, thus opening up entirely new horizons” (p. 53).

Isaac Newton (1642-1727) and Gottfried Leibniz (1646-1716) developed a calculus of geometric curves and methods for solving problems about curves, such as finding
tangents to curves, areas under curves and lengths of curves, and velocities of points moving along curves (Kleiner, 1989). It would take several more decades until calculus would be reshaped with the function concept as its centerpiece (Kleiner, 1993).

Hamley (1934) contends that, in Newton’s *Method of Fluxions*, the concept of variable is implicit. In his *Principia* of 1687 Newton uses the term “fluent” for variable and wrote that those quantities which gradually and indefinitely increase shall be called *fluents* or flowing quantities. Newton’s fluxion was the rate of change of a fluent. Hamley claims that today his theory of fluxions would be called the Theory of Continuous Functions. Youschkevitch (1976) identifies Newton as one of the first mathematicians to show how functions could be developed in infinite power series.

According to Hamley (1934) Leibniz introduced, in 1673, in his manuscript entitled “*The inverse method of tangents or about functions,*” the explicit concept and word “function” as a geometric concept as a tangent associated with a curve. In the manuscript the term function takes on the general meaning for different segments connected with a given curve. In published articles in 1692 and 1694, Leibniz calls functions any parts of straight lines. Hamley credits Leibniz to be the first to use the word “variable” in his introduction to his *Analyse* where he wrote that quantities which continually increase or decrease are called “variables” and those that remain the same are called “constants.”

Youschkevitch (1976) notes that, by the late 17th century, the function concept did not correspond to any broader analytical context. He adds that in 1694-1698 correspondence between Liebniz and Johann Bernoulli, there is evidence that each desired a general term to represent arbitrary quantities dependent on some variable. To this end, Johann Bernoulli (1667-1748) articulated the 1718 definition, “One calls here Function of a
variable a quantity composed in any manner whatever of this variable and of constants” (Rüthing, 1984).

A pupil of Johann Bernoulli, Leonhard Euler (1707-1793) claimed that mathematical analysis is the general science of variables and of their functions (Youschkevitch, 1976). In 1734, he introduced the $f(x)$ notation (Hamley, 1934) and in his 1748 *Introductio in Analysin Infinitorum* he presented “A function of a variable quantity is an analytical expression composed in any manner from that variable quantity and numbers or constant quantities” (Rüthing, 1984).

Sfard (1992) notes that Bernoulli’s and Euler’s definitions depended on the concept of variable but, because basic ideas were unclear, Euler redacted the original definition, in 1755, in his *Institutiones calculi differentialis*, to read “If, however, some quantities depend on others in such a way that if the latter are changed the former undergo changes themselves then the former quantities are called functions of the latter quantities” (Rüthing, 1984). Sfard notes that Euler’s new definition is explicitly operational. According to Kleiner (1989) Euler’s approach was entirely algebraic, and it was the first time that functions were discussed without geometry.

### 2.1.3 As discontinuous correspondences.

Subsequent developments of the concept came from mathematical physicists who found the existing mathematical tools inadequate for their purposes (Hamley, 1934). In the 18th century, the notion of function underwent many changes through the controversy over the problem of the vibrating string. The problem involves determining the function that describes the shape of an elastic string having fixed ends at a specific time, $t$, once it has been released to vibrate from some initial shape. Daniel Bernoulli (1700-1782), along
with Euler and d’Alembert (1717-1783), debated over the solution to this problem. The controversy led to increased thought and discourse about the definition of the term function (Kleiner, 1989).

Concerned with the problem of heat flow in material bodies, Joseph Fourier (1768-1830) considered temperature as a function of two variables, namely time and space. Continuing with Fourier’s work, Lejeune Dirichlet (1805-1859) needed to separate the concept of function from its analytical representation. In 1829, he defined it in these terms:

Let us suppose that \( a \) and \( b \) are two definite values and \( x \) is a variable quantity which to assume, gradually, all values located between \( a \) and \( b \). Now, if to each \( x \) there corresponds a unique, finite \( y \) in such a way that, as \( x \) continuously passes through the interval from \( a \) to \( b \), \( y = f(x) \) varies likewise gradually, then \( y \) is called a continuous function of \( x \) for this interval. It is, moreover, not at all necessary, that \( y \) depends on \( x \) in this whole interval according to the same law; indeed, it is not necessary to think of only relations that can be expressed by mathematical operations. (Rüthing, 1984, p. 74)

According to Kleiner (1989), Dirichlet’s definition was the first explicit example of a function that was not given by an analytic expression. Kleiner (1989) shows that Dirichlet demonstrated a function \( D(x) = c \), if \( x \) is rational and \( D(x) = d \), if \( x \) is irrational. This was the first example of a function that is discontinuous everywhere and it also illustrated the concept of function as an arbitrary pairing.
A group of mathematicians, known collectively as Nicolas Bourbaki, wrote a series of texts starting in 1935, with the goal of defining modern mathematics using set theory. In 1939, they defined function in terms of a set of ordered pairs as follows:

Let $E$ and $F$ be two sets, which may or may not be distinct. A relation between a variable element $x$ of $E$ and a variable element $y$ of $F$ is called a *functional relation* in $y$ if for all $x \in E$ there exists a unique $y \in F$ which is in the given relation with $x$. (Bourbaki, 1970, p. 351)

This definition has been accepted by many mathematicians as a comprehensive yet succinct way of understanding functions and is known as the Bourbaki definition.

The evolution of the function concept, from its tight connection to computational processes (Sfard, 1992), to its modern definition as a set of ordered pairs, demonstrates what Kleiner (1989) called a tug of war between the geometric, expressed in the form of a curve, and the algebraic, expressed as a formula. With the gradual abandonment of the geometric conception, Kleiner notes that the logical definition, which is expressed as a correspondence, was left to contend with the algebraic analytical conception.

Thompson (1994) also points out that the current definition highlights correspondence over variation. While the ordered pair definition solved many problems introduced historically, he found it can be meaningful only to those who recognize the problems that it solves. To students new to the idea of function, the modern abstract definition initially is so far beyond their understanding that it requires much time and motivation to learn (Sfard, 1992). According to Giordano (2008), the learning of the concept must be accessible through language students understand and through contexts and situations that have meaning to them.
2.2 Functional Concepts and Representations

Dubinsky and Harel (1992) defined the three basic concepts of function as action, process, and object. They note that the three concepts represent a progressive understanding of the abstract concept and are necessary in order to develop a conceptual image, aligned to the mathematical definition. The three concepts parallel the evolution of the function beginning from a table of values to its modern definition as a set of ordered pairs.

The three concepts and their representations are interrelated, and each has distinct characteristics. The representations of functions help in the understanding of the three basic concepts as well as the operational and computational aspects of functions. Akkoç and Tall (2002) introduced students to functional representations in six forms: a verbal form in everyday language, conveying a relationship between familiar items; a diagram consisting of two sets with arrows between them; a function machine, representing an input and output relationship; a set of ordered pairs; a numerical table of values; a geometrical graph; and a symbolic formula or algebraic equation. While each of these representations portrays an aspect of function, the core concept or the formal definition is not represented by any one of these (Thompson, 1994).

One goal for teaching functions to elementary-school students is to foster the understanding of function as action and as process, so that when students investigate it further in algebra and calculus, they will be able to attain the core-conceptual understanding. These are discussed in the sections that follow.
2.2.1 Action.

According to Breidenbach, Dubinsky, Hawks, and Nichols (1992) action is the most basic concept of function. They describe it as any repeatable mental or physical manipulation of objects. They indicate that a student who has an *action* understanding of function uses an algebraic expression or recipe to calculate an output. They report that function-as-action is a static concept because a student can only imagine a single value at a time as input or output. An example of this concept is an explicit rule that takes an input, transforms it by means of a specific algorithm, and then produces an output.

As reported by Selden and Selden (1992), the algebraic concept (e.g., expression, formula), although useful in pre-calculus and calculus, does not maintain that functions can have completely arbitrary pairings. They note that functions do not necessarily have to be represented by formulas, nor do they have to concern numbers.

Sfard (1992) claims that the action concept can lead to misunderstanding about the existence of discontinuous functions and about functions that are defined by two different formulas and over different domains. Sfard (1992) administered a three-item questionnaire (see Appendix A) to a group of 22- to 25-year-old students who were enrolled in a regular course on elementary mathematics which included the basics of set theory, algebra, and calculus. The questions were constructed to determine what concept of function the students held. Student responses to the first question indicted that 81% of students had an operational concept of function since they chose the answer that described the function concept as a computational process instead of a correspondence represented by a table of values. Responses to the second question indicated that 72% of students believed that functions are algorithmic and reasonably simple. The third question
was given to determine if students thought of functions as having two different formulas or arbitrary pairings. The choice of an arbitrary function was rejected by 83% of students.

The action concept can be represented by a table of values. A two-column table or t-chart is useful in identifying the mapping of input to output. This was demonstrated by Blanton and Kaput (2004) who examined how students in elementary grades were able to develop and to express functional relationships. The activities in this study were part of a 6-year professional development program for teachers called GEAAR (Generalizing to Extend Arithmetic to Algebraic Reasoning). The program was designed to help teachers apply instructional resources and practices to enhance algebraic reasoning. The participants, who were teachers and students in an urban school district from grades pre-K to grade 5, undertook the task of developing a functional relationship between an arbitrary number of dogs and the corresponding total number of eyes or the total number of eyes and tails (see Appendix B). Data were obtained in the form of students’ written work and from teacher interviews. Data were analyzed according to the types of representations students used at different grades, the progression of mathematical language in responses, how data was tracked and organized, the mathematical operations used, and how students expressed variation among quantities.

The pre-K class, consisting of students from age three to five, used paper cutouts of dogs and counted their eyes and tails. The activity was led by the classroom teacher who recorded, onto a t-chart, the number of dogs, eyes and tails. The students obtained their data from the cutouts for up to four dogs. Below each number, the teacher included the corresponding number of dots. The students counted the number of dots when the teacher asked for the total number of eyes and tails for a given number of dogs. They did not
make any predictions and there was no indication that the students looked for patterns. The significant result for these students was that they developed a correspondence between numeral and object and had organized data by using a function table (t-chart). This, in the researchers view, represented the early development of algebraic reasoning.

One kindergarten class described an additive relationship between the number of eyes and dogs and stated that every time you add a dog you add two more eyes. This was an indication that they were attending to the number of dogs and eyes simultaneously and were able to describe how these quantities co-varied.

Students in first grade were able to construct a t-chart of the data without the classroom teacher’s assistance. They described patterns when counting eyes and tails, stating that they were counting by threes, they saw the pattern of doubling for total eyes and tripling for total eyes and tails, and tried to predict the number of eyes for seven dogs. Second graders completed a t-chart for up to ten dogs, and using natural language, were able to give a multiplicative relationship. They were able to use their found relationship to predict the number of eyes for 100 dogs and then they predicted the number of eyes and tails for 100 dogs as 300.

Third, fourth and fifth graders, using t-charts, expressed the rule using multiplication both verbally and symbolically, and, using the rule, predicted the number of eyes or eyes and tails for 100 dogs. One third-grade class graphed their results comparing the number of eyes with dogs and then compared the number of eyes and tails with the number of dogs. Fourth and fifth graders were able to develop a function with less data, only going up to three dogs.
The results of this study indicate that pre-K through fifth-grade students can learn the action concept of function by engaging in the eyes and tails activity. Through guided instruction all grades demonstrated functional thinking by creating t-charts and graphs, and by recognizing patterns. Students in this study developed an understanding of the action concept of function, since for every number of dogs they obtained a number of eyes and tails. Fourth and fifth graders developed a process concept, by creating a function using only three input values to obtain outputs and began to generalize the changes from input to output. The process concept is discussed next.

2.2.2 Process.

Breidenbach et al. (1992) state that the process concept involves taking an object, transforming it, and producing a completely new object. It also has a repeatable characteristic similar to action. At the process level of understanding, vague transformations, instead of an explicit formula or rule, are functions. Reed (2007) provides $\cos x$, for example, as a function which has no explicit formula for evaluation. When a student moves from an action to a process understanding, the student has interiorized an action to form a process (Breidenbach et al., 1992). Function-as-process is a dynamic process because the student begins to see how one variable changes while imagining changes in the other. An example of this concept is the relationship between the height of water in a bottle and its volume.

As mentioned above, in the Eyes and Tails activity, the fourth and fifth-grade students began to generalize and did not need to perform a calculation to obtain every output value. They developed the rule for the function represented by the t-chart
demonstrating that their understanding of function was progressing to the process concept.

Selden and Selden (1992) suggest that using the idea of a function machine helps students view function as a process. The function machine accepts an input and produces an output. There is no need to know the contents of the machine. They note that it is useful in assisting younger students to understand function as a process, but it does not provide a complete notion of function.

McGowen, DeMarois, and Tall (2000) documented that using function machine representations helped undergraduate students, taking Introductory and Intermediate Algebra courses, form a rich, foundational concept of function. They measured students’ knowledge of functions using pre and post-course surveys. They measured growth of students’ understanding by analyzing students’ descriptions and explanations of their work in terms of an input/output process and in their improved ability to (i) interpret and use ambiguous function notation and (ii) translate between and among various function representations. They reported that 41% of the Introductory Algebra students and 71% of the Intermediate Algebra students were able to find output given input and 22% of the Introductory students and 46% of the Intermediate students were able to reverse the process at the end of the semester. The researchers reported that the Intermediate Algebra students showed improved flexibility of thought over the sixteen week course since the average change in correct responses was statistically significant. The results also indicated that least successful students did not make use of the function machine notation in their descriptions and that they showed little or no improvement in their ability to use
various functional representations. Thus using function machine representations proved to be effective for some students and ineffective for the lower performing students.

Breidenbach et al. (1992) have noted that students who attain a process understanding of function are able to combine it with other processes and even reverse it. By reversing the process students obtain the inverse. The Intermediate Algebra students in the McGowen et al. (2000) study obtained the inverse of functions using the function machine representation.

While graphs are the most recognizable functional representations, they are ordinarily introduced several years before functions are taught in the U.S. curriculum (Leinhardt, Zaslavsky, & Stein, 1990). Students learn graphs without learning about functions and are expected to connect them when functions are introduced. Graphs can help in the understanding of minima, of maxima, and of increasing and decreasing concepts but they are not all encompassing.

Tierney and Monk (2008) investigated activities of students (ages eight to ten). These students were engaged in a mathematics program for grades K-5, the curriculum of which contained a strand for the specific understanding of change over time. In an exploration of children’s’ understanding of the mathematics of change, the investigators selected episodes in which students told stories about how one variable changed over time. Students represented these stories in tables, in graphs, and in groups of additive changes.

Of particular interest to the present discussion is episode 3, in which fourth-grade students analyzed graphs of plant height over time, so as to compare changes in heights, as well as rates of change. The curriculum, from a series titled *Investigations in Numbers, Data, and Space*, had the children grow plants from seeds, and record and graph the
plants’ heights, each day for two weeks. The specific activity was for them to analyze qualitative graphs, in which no quantities were shown. Only the shapes of the graphs and the labels \textit{time} (on the horizontal axis) and \textit{height} (on the vertical axis) were visible (see Figure 1, Appendix C). All students interpreted a steeper graph as meaning the plant was growing more rapidly than others; and higher graphs as showing a taller plant. The students correctly observed that the dark line was growing slower and the light line faster over the same period of time. Some noted that the dark line was taller when the graph began. One even conjectured what the dark line would be \textit{before} the graph began (see Figure 2, Appendix C). The students made sense of the graph based on their knowledge of plant growth. Focusing on steepness or on quantitative information, they were able to interpret the functional representation of plant growth.

The students showed evidence of a process understanding of function since they neither had a specific algorithm to calculate plant height nor were they able to obtain any quantities from the graph. Through this activity, they were able to understand the change in plant height over time as a process. Tierney and Monk led fourth-grade students to the process concept through a familiar situation.

\textbf{2.2.3 Object.}

According to Dubinsky and Harel (1992) a function is considered an object when a process is transformed by some action. The object is constructed by encapsulating a process. Eisenberg and Dreyfus (1994) state that “the central feature of an object conception is the ability to conceive of a function as something on which an action such as differentiation, composition or a transformation can be performed” (p. 64).
The object concept is best represented by the ordered pair definition, cited earlier as the Bourbaki definition. The object concept defines a function as an infinite set of ordered pairs \((x,y)\), in which each \(x\)-coordinate is paired with only one \(y\)-coordinate.

The object concept and its representation accurately describe discontinuous functions and arbitrary pairings. The definition can be extended to account for functions the pairings of which are not numbers. One indication that a student understands the object concept is the ability to reason about operations on sets of functions (Thompson, 1994).

Many algebra and higher-level textbooks present the Bourbaki definition of function as the primary one. Researchers contend that due to the lack of a firm understanding of set theory, this definition is considered too abstract for middle- and high school students (Sfard, 1992).

### 2.2.4 Textbook definitions.

In the middle school text, *Mathematics, Applications and Concepts, Course 2* (Bailey, Day, Frey, Howard, Hutchens, McClain, et al., 2006) function is defined in chapter 4 in the section “Functions and Linear Equations,” as “a relationship where one thing depends on another. In a function, you start with an input number, perform one or more operations on it, and get an output number” (p. 177).

In the algebra text, *Algebra 1* (Larson, Boswell, Kanold, & Stiff, 2001) function is defined, in the chapter 1 section, “An Introduction to Functions,” as “a rule that establishes a relationship between two quantities, called the input and output. For each input, there is exactly one output. More than one input can have the same output” (p. 46).

In the algebra text, *Algebra 2* (Larson, Boswell, Kanold, & Stiff, 2001) function is defined in the first section of chapter 2, “Functions and Their Graphs,” as a relation
where there is exactly one output for each input. In addition the text states that a relation is not a function if at least one input has more than one output.

Although the textbook definitions may be superficially adequate, students may not be able to understand the concepts in depth. Nor will they be able to use them efficiently since aptitude in this area comes with time and with maturity. The cognitive, pedagogical, and curricular aspects of functional thinking are the foci of the following section.

2.3 Functional Thinking

The movement towards functional thinking, according to Hamley (1934), began in America in 1908. It was influenced by the Reform movement in Germany and by the work of Felix Klein (1849-1925). Klein claimed that the concept of function was not simply a mathematical method but rather the essence of mathematical thinking (Hamley, 1934).

The National Committee on Mathematical Requirements, a landmark in the history of American mathematical education at that time, reported on the fundamental importance of the functional relation, and that its general and abstract forms could become significant to the student, but only as a result of very considerable mathematical experience and training (Hamley, 1934). The Committee noted, in the chapter “Function Concept,” that in any rational treatment of formulas, equations, graphs, variation and proportion, congruence and similarity, the concept of functionality is implied. Within the purview of functionality, they contend, lay all of the subjects in the mathematics curriculum.

Hamley (1934) attributes to Georges (1929) the identification of three main abilities of functional thinking. The first is the ability to recognize mutual dependence between
variables and varying quantities; the second, the ability to determine the nature of the
dependence or relationship between variable quantities; and the third, the ability to
express and interpret quantitative relationships. Georges classifies six areas important to
expressing and interpreting functional relationships: (1) measurement, (2) representation,
(3) variation, (4) relationships, (5) transformations, and (6) generalization.

In *Problems in Teaching Secondary School Mathematics*, Breslich (1931) focused on
the placement of functional thinking in mathematics curricula (Hamley, 1934). He
discussed functional thinking not only in secondary school but also in elementary
arithmetic with tabular arrangements, correspondence and dependence. Carraher,
Schliemann, and Schwartz (2008), cited earlier, expressed ideas similar to those of the
National Committee, Georges, and Breslich by pointing out the algebraic and functional
nature of elementary arithmetic topics.

Fundamental to the understanding of functions is the study of mathematical change
(NCTM, 2000). Warren and Cooper (2005) discuss how the study of change aids in the
understanding of the processes of arithmetic. Early experiences with change involve
finding and describing attribute change (e.g., color, shape), qualitative change (e.g., I
grew taller), quantitative change (e.g., I grew 2 inches taller), relationships between these
changes (e.g., If everyone grew taller by the same amount and Jane’s height changed
from 52 inches to 54 inches, how much did the class grow by?) and using these
relationships to solve problems (e.g., If Alice’s height is now 51 inches, what height was
she before she grew?). They contend that these experiences are the beginning of
functional thinking.
Smith (2008) discussed the framework for representational and for functional thinking. Through representational thinking, elementary-grade students create meaningful representations, and, in so doing, build and express generalizations. He defines functional thinking as a type of representational thought focusing on the relationship between two or more varying quantities, specifically the kind of thinking that leads from specific relationships to generalizations of that relationship across instances.

Smith (2008) proposed six activities that underlie functional thinking and the construction of functions. The first two are related to the problem to be solved: the engagement in some type of physical or conceptual activity; and the identification of two or more quantities that vary in the course of the activity, the focus here being on the relationship between two variables. The third activity is to create a record of the corresponding values of the varying quantities by using a table or graph. The remaining three activities are: identifying patterns in the records created; coordinating the patterns with the actions involved in carrying out the activity; and using the coordination to create a representation of the identified pattern in the relationship. Smith (2008) contends that the six activities may not occur in the order presented above, nor do they necessarily have to be distinct; for example, the representation in the sixth activity may be identical to the record created in the third one.

Functional understanding, through quantitative reasoning, is illustrated by Ellis (2011) who argues that, for middle school students, an introduction to functions should be grounded on quantities, on relationships, and on meaningful situations. The results of her earlier investigations illustrated students’ functional thinking, as well as the difficulties they encountered.
In one teaching experiment, Ellis (2011) assigned to an eighth-grade student a problem (see Appendix D) consisting of a scalar diagram with 2 rolls of pennies on one side and 5 rolls of pennies on the other. The question proposed was: “if you were to compare the weights, what might you notice?” The eighth grader who made additive and multiplicative comparisons responded that the larger pile had 3 more rolls and weighed 2.5 times as the smaller pile. When presented with data in tabular form for the same problem (see Appendix D), however, the student could not develop an equation for the relationship between the number of rolls and the weight and believed that the data was not linear. Ellis points out that students had difficulty translating between tabular, graphical, and algebraic representations of functional relationships. They may only recognize a function as being linear if its tabular representation has uniformly-increasing x-values.

Ellis (2011) contends that students must be able to recognize change in order to understand functions. Her experiment with seven pre-algebra seventh-grade students who had not yet studied either linear functions or graphs investigated gear ratios. Her intention was to emphasize the activities of generalizing and of justifying and to do so through the use of quantitative referents. In the first eight of 15 sessions, they worked with physical gears to examine different gear ratios. In one session, two gears were spun at the same time: one had eight teeth and the other, 12. Students had to devise a way to keep track of the rotations of both gears. Placing masking tape on one of the teeth of each gear, the students were able to track the rotations and created a table of data (see Figure 2). The students considered the data carefully: for each complete small gear rotation the big gear rotated a two-thirds turn; and for each complete rotation of the larger gear the small one
rotated one and a half turns. This activity demonstrated that students were able to coordinate the rotations of each gear, and to formulate a covariational language to describe the nature of the coordinated quantities.

Figure 2. Students’ table of gear rotations.

Assigning a second gear problem with a table of data, Ellis asked students to determine whether the pairs of rotations came from the same gear pair or from different gears altogether and how could they make this differentiation (see Appendix D). Ellis felt that the situation was meaningful to students because it described a familiar situation that could be directly imagined. One student gave a verbal description stating: for every two-thirds of the teeth or eight teeth passed on the big gear, the small one turns once. Other students discussed the correlation further and determined, if the small gear rotated 3 times, then the big one would turn 2 times; and, if the small gear rotated 7 and a half times, the big gear would rotate 5. One student expressed the gear-ratio relationship by writing “s(2/3)=b”, where s represents the number of rotations of the small gear and b represents the number of rotations of the larger gear.

Ellis assigned another problem (see Appendix D), one in which the values were not well ordered. To determine the relationship between the two gears, students identified the difference between successive table entries and calculated, first, the difference between 1
and 4 to get 3 and, second, the difference between 5 \( \frac{3}{4} \) and 8 to get 2.25. Then they divided 2.25 by 3 to get \( \frac{3}{4} \). One student said that the big gear rotates \( \frac{3}{4} \) of what the small does and noticed that the big gear had already turned 5 times. Another student was able to express the relationship algebraically as \( \frac{3}{4} a + 5 = b \), where \( a \) is the number of rotations of the small gear and \( b \) is the number of rotations of the large gear. In sum, students were able to calculate the ratio of the change of one variable to the coordinated change in the other variable to determine the relationship between the gears.

Ellis contends that middle-school students can understand functional relationships through quantitative reasoning and that meaningful situations can facilitate student understanding of function as process. This approach, she posits, will allow for a more flexible view of function including the correspondence or object concept.

To think functionally, one needs considerable mathematical experience and training. It is reasonable to state, therefore, that students would benefit from early exposure to functional thinking, even at the elementary level. The next section discusses an approach which was designed to improve students’ learning of mathematics by making them active learners at any age.
Chapter 3 Theoretical Framework

The theoretical framework for the current study is based on Davis’ assimilation paradigm, mental representations, and metaphors. The sections that follow describe how Davis’ ideas helped to redirect mathematics education.

3.1 New Era in Mathematics

Schulman (2009) surveys an important era in mathematics education history which began in the 1950s. He contends that to achieve the goals of retaining its leadership role in the free world and of competing with the Soviet Union, the U.S. earmarked federal funds for educational reform in mathematics and science education. Of the many “New Math” programs beginning at this time, one was the Madison Project which received funding under the National Defense Education Act of 1958.

Schulman (2009) identifies several motivations that influenced mathematics education at that time. One was that public schools needed to prepare a workforce more knowledgeable in science and technology; another was to respond to declining enrollment in mathematics classes. All agreed that the talents of brighter students needed to be developed. The opportunity had arisen to rearrange mathematics curricula in junior high schools, in response to the contemporary committees, commissions, and critics. The revival of the “meaning” theory also offered new insights into how students learn.

According to Schulman (2009), the shift from progressive education to “meaning” theory occurred in the 1940s. This theory, which fostered meaning and readiness in mathematics education, resurfaced in the 1950s when it embraced understanding, problem solving and an earlier introduction to algebra. The theory emphasized the use of
meaningful experiences for students; and it stressed the importance of intermediate steps in learning.

3.2 Davis and the Madison Project

Schulman (2009) reports that Robert B. Davis started the Madison Project in 1957 by meshing preexisting educational philosophies, learning theories, and his own educational visions. The project, Schulman recalls, derived its name from the school in which it began, the Madison Junior High School in Syracuse, New York. Schulman contends that, as a reaction to the failings in mathematics education, Davis believed that mathematics had to be experienced by students in group discussions and through individualized instruction. Schulman believes that the Madison Project encouraged independent exploration, promoted abstract, rational, and analytical thinking. He describes the many objectives of the project, including those pertaining to curricular development, to student needs in the classroom, and to teacher training.

Davis’ (1992a) specific curriculum goals included the broadening of curriculum to include basic algebra and coordinate geometry topics in grades 2 to 8, mathematical logic and matrix algebra in grades 5 to 9, the greater use of physical materials in diverse activities, and the identification of basic, unifying concepts (e.g., variable, function, and isomorphism). He believed students should take a more active role in the learning of mathematics. The Project’s “learning by discovery” approach, Davis believed, made students responsible for inventing solutions to problems and experience mathematics as an open-ended activity. In Davis’ “Do-then-discuss” strategy, the instructor asked a question or showed a puzzle. Only after students had invented a method for finding an
answer, would class discussion of that kind of problem commence (Davis, 1985). He believed that curriculum should include “experience” lessons through which students could become familiar with some general area of mathematics in preparation for new work. This readiness-building, he maintained, was missing from the traditional mathematics curriculum.

Indebted to Piaget’s theory of assimilation and of accommodation, Davis developed his own assimilation paradigm (Davis, 1984). For Piaget, assimilation occurs when new learning is subsumed under a cognitive, operational structure but requires no change in that structure. Accommodation, on the other hand, occurs when new learning requires some modification of an operational structure (Piaget, 1980).

Giordano (2008) describes how Davis employed the assimilation paradigm to explain how an individual deals with unprocessed data. If the data matches an aspect of an individual’s own experience, then it is an assimilation paradigm. If no idea exists in the student’s mind that is similar to the new idea being taught, then the student should be provided the needed experience. Davis (1992a) created “paradigm-creating” lessons for the purpose of providing experiences that would parallel the proposed new mathematics idea. The lessons were carefully constructed, using models easy to understand. The Madison Project lesson “Pebbles-in-the-Bag” was developed to introduce the concept of negative whole numbers (Davis, 1984).

As Schulman (2009) notes, Davis understood the dual role of the teacher as one who moderated and encouraged student inquiry. Along with instruction, curricular development was also the job of the teacher. Implemented in many cities in the U.S. and
Canada, the Madison project included pre-service and in-service preparation programs for teachers.

Davis’ application of this new approach to the study of mathematical learning had a fourfold purpose: to help students understand the task or its goal; to give students the responsibility of inventing solutions; to help them to focus on mental representations; and to ensure that, with concrete materials and with “everyday” tasks, students will have sufficient and appropriate experience. Using videotaped data of task-based interviews, and of students working collaboratively on the same tasks, the new approach has enabled researchers to analyze student activities, comments, questions, revisions, and deliberations (Davis, 1985, 1992a).

Davis’ work with elementary schools in the 1960s and 1970s involved sustained and tailored interventions. It tracked students’ progress over many years. The Madison Project ran into the 1980s, and Davis continued his work at Rutgers University.

### 3.3 Davis and Functional Learning

Davis (1985) addressed two interrelated issues: the present state of knowledge and practice, in regard to teaching and learning of algebra in grades two through seven; and the major problems in creating and implementing appropriate curricula for algebra in these grades. He reported a number of observations on the algebraic thinking of students, from seven to twelve years of age. The first, on mathematics and the learning of mathematics, was that mathematical practice at that time involved following explicit instructions and knowing certain facts; and the second, that learning mathematics forms representations in the student’s mind. The practicing of mathematics, he conjectured,
would not benefit from early exposure to algebra as algorithms and facts. However, to learn mathematics, algebraic thinking is beneficial to students. The development of representations can be accomplished through constructivist and guided-discovery approaches starting in the early grades. Kaput (2008) credits Davis with setting the stage for subsequent research on the feasibility of early algebra learning.

Davis (1985) reported five recommendations for early algebraic instruction: to identify key ideas, such as variable, function, and open sentences; to provide an array of experiences from which younger students could build correct mental representations of these ideas; to encourage students to reflect on what they were doing; to individualize these experiences for each child; and to make the cumulative exercise both worthwhile, not merely as game playing, but as a stimulus to more advanced work.

Davis discussed the opportunities in grades two to eight when algebraic ideas, namely that of the function, are introduced. According to Davis (1992a), basic to the Madison Project was the introduction of the concept of function as early as grade four. Davis utilized a method, “guessing functions,” that W. Warwick Sawyer had devised. Students, working in groups of three, were given the task of creating a rule so that, when they were told a number, they would be able to apply the rule to produce another number. Others in the class then had to guess what the rule was. An implementation of the “Guess-My-Rule” activity, in 1993, was analyzed by Giordano (2008) and is described below.

Davis (1985) describes a related activity, including pictures of spiders. Students were asked to count the number of spiders and their legs. Built upon students understanding of $S$ number of spiders with eight legs (L) each, the activity had as its initial goal to create
experiences clearly illustrating the logical relationship between the spider pictures and the equation $8 \times S = L$.

### 3.4 Guess My Rule

Researchers from Rutgers University conducted a twenty-year-long longitudinal research project to follow the development of children’s mathematical thinking and the development of proof. The videotaped sessions of students, starting from first grade, showed students at work on open-ended tasks from combinatorics, algebra, geometry and probability. Students who participated in this project, starting from first grade, were from a public school in Kenilworth, N.J. Giordano (2008) examined their work in sixth grade, as they worked on functional activities in the fall of 1993.

Two days prior to working on functional activities, the students participated in several activities, including “Pebbles in the Bag,” “Open Sentences,” and “Finding Truth Sets” all of which were conducted by Davis (Spang, 2009). These activities laid the groundwork for subsequent ones. “Pebbles in the Bag” provided a concrete way for students to think about signed numbers. “Open Sentences” used placeholders, boxes or triangles, for the placement of numbers. It introduced the concept of variable because students had to fill in the open box or triangle with a number. In “Finding-Truth- Sets,” students were presented with equations written using boxes and triangles for which they had to provide values to make each equation true. For the study discussed here, the students were engaged in the “Guess-My-Rule” activities for two sessions over two days, also led by Davis. Giordano’s research questions were: what evidence was there that
students built an understanding of the concept of function? and how did the teacher introduce and develop the idea of function?

Davis introduced the activity by distributing a handout to the students (see Appendix E). The nine problems presented in table format (see Appendix E) contained two columns: the first column heading was a box, and the second was a triangle. He then asked them what they thought they should be doing. One student responded that they should be finding equations. Davis emphasized that the task was to write equations using box and triangle. Davis added a tenth table which was not on the handout.

The students worked both in pairs and individually. At first they found patterns in the tables without using box and triangle notation and used them to find the next number in the triangle column. For the linear relationships, problems 1 through 5, students found the constant difference between triangle values, which was the slope of each relationship. Some students found this value by subtracting while others described the difference as the amount the triangle values went “up by.” Although students did not specifically mention y-intercept, many of them referred to “the plus number” and used it in their equations. After working out the slopes and y-intercepts for each table, the students proceeded to write the equations using boxes and triangles. Once they found an equation for one of the linear relationships, they replicated their findings to write the equations for the other linear examples.

Students worked on the next set of problems which described quadratic relationships. They identified second differences, that is, the differences in the differences in the triangle values. In problem 6, for example, the differences in the triangle values were one, three, five, seven, and nine, and the second differences were all two. After much
discussion, students realized that they needed to use the number in the box more than once. They also identified the y-intercept in each relationship; and, for problems 6 and 7, the y-intercept was the triangle value when the box value was 0. They tried similar strategies and adjusted what they did in problems 6 and 7, to find the equations for the remaining tables.

Davis facilitated students’ comments, discourse and arguments. He guided the discussion and encouraged students to continue working and thinking about their discoveries. He often requested that students not divulge what they had found when forming their equations, and often asked them to clarify their explanations, and to present their solutions to the entire group. He believed that students should work independently, with minimal input from the teacher, so as to minimize teacher interruption. Davis’ approach as “guided discovery” was similar to those described in Brousseau’s (2008) studies. Students found similarities between problems and developed new representations when they could not fit new information into their existing representations.

For this activity, Davis defined a function as a rule that tells you what to do with the number in the box, in order to produce the number in the triangle which is the action concept of function. Students exhibited Vergnaud’s (1996) theorem-in-action, when developing the equations: using the scalar approach, they found differences in the triangle values to determine the slope of each linear relationship; using the functional approach, they recognized the relationship between the box and triangle to get the y-intercept needed to complete the equation.
Chapter 4 Methodology

4.1 Background of Study

In 1984, Carolyn Maher and other researchers from Rutgers University initiated a longitudinal study at the Harding School, a public K-8 school located in Kenilworth, Union County, New Jersey. The study, begun when students were in the first grade, analyzed the development of students’ mathematical ideas while engaged in a variety of activities.

Three NSF grants supported the longitudinal study: the first NSF grant, MDR-9053597 (directed by R. B. Davis and C. A. Maher), expanded the original plan to include two other New Jersey school districts, in New Brunswick and Colts Neck. It supported the research in all three districts when the students were in grades four to six (Spang, 2009). The second NSF grant, REC-9814846 (directed by C. A. Maher), extended the original Kenilworth study and followed the students’ mathematical learning from seventh grade through their senior year in high school (Spang, 2009).

The original study included four strands: 1) combinatorics and counting, 2) probability, 3) algebra, and 4) pre-calculus/calculus. The sessions on algebra included activities related to the concept of function, such as “Finding Truth Sets” (Spang, 2009), “Guess My Rule” (Giordano, 2008), and “Towers of Hanoi” (Mayansky, 2007).

Spang (2009) recalls that during the first year of the study, students were visited by the Rutgers researchers four to six times a year. Each visit would last three consecutive school days and the students were videotaped, in their classroom, for about an hour and a half as they worked on unique mathematical activities. These activities were separate from their school curriculum.
The current study analyzes data taken from the videotaped sessions of a seventh-grade class from the Harding School in the spring of 1995. The seventh-grade sessions were undertaken to follow the development of algebraic thinking specifically when working on inverse functions. The majority of the seventh graders also participated in the algebra sessions conducted from September 1993 to November 1993 (see Table 1).

Table 1

*Earlier algebra sessions conducted by R.B. Davis*

<table>
<thead>
<tr>
<th>Dissertation</th>
<th>Session Dates</th>
<th>Description</th>
<th>Number of CDs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kathleen Spang (2009)</td>
<td>9/30/93</td>
<td>Variable, open sentence, truth set notation, solving linear equations in one variable, solving quadratic equations, solving linear equations in two variables and Pebbles in a Bag. 6th grade</td>
<td>4</td>
</tr>
<tr>
<td>Patricia Giordano (2008)</td>
<td>9/30/93 (last 10 minutes)</td>
<td>Beginning of Guess My Rule 6th grade</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>10/1/93</td>
<td>Guess My Rule 6th grade</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>11/1/93</td>
<td>Solving Towers of Hanoi 6th grade</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>11/12/93</td>
<td>Solving Towers of Hanoi 6th grade</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>11/15/93</td>
<td>Solving Towers of Hanoi 6th grade</td>
<td>4</td>
</tr>
</tbody>
</table>
4.2 Data

The videotaped sessions analyzed in this study were conducted over three days at the end of the seventh grade. On the first two days three videographers each taped a different group of students working at separate tables (Bobby’s, Jeff’s and Stephanie’s tables); and three other individuals assisted with audio equipment, one at each table.

The data used in this study consisted of 10 CDs copied from the original videotapes (see Table 2). The first day’s activities, recorded on 6 videotapes on May 16, 1995, were copied to 6 CDs, two for each group of students. The session on day two, May 17, 1995, was copied onto three CDs, one for each group. On day three, May 18, 1995, one videographer, on a single videotape, recorded all three groups with a roaming camera; due to its size, it was copied to a DVD. Despite the occasional audio fade-out and unfocused footage of students’ work, the quality of the videotapes is satisfactory.

Written work from the students was collected after each session. At the time of the research, however, students’ written work and researchers’ debriefing materials were not located and, as a result, were unavailable. In the present text, a selection of students’ work from day 3 (replicated from Bellisio [1999]) appears in the corresponding results section.

The thirteen students present on the first day were divided into three groups: Bobby’s, Jeff’s and Stephanie’s tables, respectively. Bobby, Michelle I. (Shelly), Amy-Lynn and Magdalena (Magda/Maggie) sat at Bobby’s table; Jeff, Michael, Sarah, and Angela sat at Jeff’s table; and Stephanie, Romina (Erin), Michelle R., Brian, and Ankur sat at Stephanie’s table. The members of each group remained on all three days, except for
Michelle R. who was not present on the second day. Jeff worked with Stephanie’s group on occasion.

Table 2

*Videotaped data analyzed in this study*

<table>
<thead>
<tr>
<th>Date of taping/ Brief description</th>
<th>Name of CD</th>
<th>Transcript Reference</th>
<th>Length (HH:MM:SS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>05/16/1995 (6 CDs)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Students worked on finding the rule and inverse for seven different t-charts. Third-grade student presented her algorithm for multiplication by 5.</td>
<td>Bobby’s Table (1 of 2)</td>
<td>B1A</td>
<td>00:55:41</td>
</tr>
<tr>
<td></td>
<td>Bobby’s Table (2 of 2)</td>
<td>B1B</td>
<td>00:36:45</td>
</tr>
<tr>
<td></td>
<td>Jeff’s Table (1 of 2)</td>
<td>J1A</td>
<td>00:46:11</td>
</tr>
<tr>
<td></td>
<td>Jeff’s Table (2 of 2)</td>
<td>J1B</td>
<td>00:46:55</td>
</tr>
<tr>
<td></td>
<td>Stephanie’s Table (1 of 2)</td>
<td>S1A</td>
<td>00:57:38</td>
</tr>
<tr>
<td></td>
<td>Stephanie’s Table (2 of 2)</td>
<td>S1B</td>
<td>00:39:49</td>
</tr>
<tr>
<td>05/17/1995 (3 CDs)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Students reviewed properties of the number system and then worked on finding the rule and inverse.</td>
<td>Bobby’s Table</td>
<td>B2</td>
<td>00:51:47</td>
</tr>
<tr>
<td></td>
<td>Jeff’s Table</td>
<td>J2</td>
<td>00:52:14</td>
</tr>
<tr>
<td></td>
<td>Stephanie’s Table</td>
<td>S2</td>
<td>00:46:52</td>
</tr>
<tr>
<td>05/18/1995 (1 DVD)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Students worked on finding the rule and inverse for two sets of data.</td>
<td>All students (DVD)</td>
<td>A3</td>
<td>01:21:57</td>
</tr>
</tbody>
</table>

Researcher Dr. Emily Dann (henceforth: Researcher 1 [or R1]) facilitated on the first two days; Dr. Carolyn Maher (henceforth: Researcher 2 [or R2]) assisted and presented materials on days one and two. Included on day one with Dr. Maher was Lindsey, a third-grade student who presented an algorithm to the class. Mr. Miller (henceforth: Researcher 3 [or R3]) observed activities on day one and participated in student activities on days...
two and three. Researcher 4 [R4], who was unidentified, asked students questions at the end of the session on day 2. Day three was facilitated by the classroom teacher, Ms. Toye (henceforth: Teacher 1 [or T1]), and was assisted by R3. Others, seated in the back of the classroom as observers, had minimal interaction with students.

4.3 Method of Analysis

The videotapes were examined and analyzed, based on the model described by Powell, Francisco and Maher (2003). The model utilizes a sequence of seven interacting, non-linear phases.

The transcription of a videotaped session began, first, with a viewing to familiarize this researcher with the overall activities. The researcher viewed the session, again, while taking detailed notes of the tasks presented to the students. Using pause, rewind, and repeat play functions, at 1 minute intervals, the reviewer carefully transcribed each individual’s dialogue. Indecipherable dialogue was temporarily held in abeyance. Difficulties occurred mostly when more than one person was talking, as in the case when the facilitator and a student worked on a problem, or when several students were talking about a problem together. To overcome audio difficulties, the transcriber paid special attention to each speaker’s words, replaying segments where necessary.

Using transcripts and videotapes, this researcher wrote narratives of each days’ activities. Careful analysis of both transcripts and videotapes identified critical events which were referenced to support this research.

Screen shots from the overhead projector were added first. Screen shots of students’ work from the video were added if clearly visible and if work was relevant to the task at
hand. In the case of relevant student activity, videographers at times required several viewings to attain a clear picture. On occasion, videographers captured the clearest picture several minutes after the data were initially written.

4.4 Transcript Information

As is noted above, researchers and teachers are referenced simply as Researcher 1 (R1), Researcher 2 (R2), and Teacher 1 (T1); and students, by first name (with last-name initial, if needed). Included also are time codes, noted at five-minute intervals or less. Each transcript was typed as a MS Word document with the line number option enabled, and each transcript starts with line number 1.

Daily videotapes present overlapping footage of activity and dialogue occurring in two locations: in front of the class and at the overhead projector. At these times researchers and students presented written work and researchers questioned the entire class. Like the videotaped footage, the transcripts overlap one another.

Transcripts include screen shots from the overhead projector, as presented by both researcher and students. These pictures were cropped to show only the section containing the most recent writing; therefore, they are not uniform in size. Cropping was done to minimize the size of each transcript file. Screen shots of overhead slides were taken from the videotapes of Stephanie’s table, the only location where these data were captured; hence, the transcript of each group contains the same screen shots of the overhead projector data. Indistinct screen shots were sharpened through the use of free software, GNU Image Manipulation Program (GIMP) 2.6.10. Each screen shot is identified by author, whose identity is noted parenthetically. Each shot is bordered on both sides by blank lines, each line being assigned a number.
The ten transcripts, included in Appendix G of this document, are referenced by code (see Table 2), and by corresponding line number(s). The movements and gestures of participants (both researchers and students) are recorded parenthetically.

During two summer practica, graduate students transcribed seven of ten transcripts. Two were transcribed by this researcher. The tenth (the data of day three) was retyped from a document used by Bellisio (1999) who had analyzed students’ use of notation and variables on algebraic tasks. Graduate students and this researcher verified dialogue transcriptions and screen shot accuracy.
Chapter 5 Results

5.1 Introduction

This section discusses critical events arising as students demonstrated their understanding of function and inverse. Activities related to function and inverse are analyzed exclusively here, as students were presented each t-chart (see Appendix F). Evidence is presented with references to students’ dialogue and to written work. The section is organized chronologically. The questions guiding this research are as follows:

1. What evidence, if any, exists that students understand the idea of function?

2. How do students build the idea of inverse function?
   - What obstacles, if any, do they encounter?
   - If obstacles are encountered, how are they overcome?

3. What evidence exists, if any, that students understand the idea of inverse function?

5.2 Day one: May 16, 1995

R1 asked the class if they remembered working with Dr. Davis on Guess My Rule activities two years ago. The only audible response was from Michelle I. who said that she remembered the sheets (B1A, 12), referring to the worksheets of blank t-charts given to each student. R1 first presented a simple rule on the overhead projector, using a two-column table, the left column heading a box, and the right heading a triangle. The rule was box plus 3 equals triangle, and R1 started with positive numbers in the box column. R1 tried to get students to offer another choice of numbers for the box (e.g., negative numbers). Then R1 asked them to find the box value with a ten in the triangle. Students calculated the box to be seven, by following the inverse of the rule: triangle minus 3
equals box. In response to the question of how he got the seven, Brian said, “Ten minus three” (see Appendix F, S1A, line 52). R1 added student responses to the table shown on the overhead projector (see Figure 3).

R1 discussed what inverse operations are and then asked what *five squared* means. Amy-Lynn responded, “five times five” (B1A, 70). Then R1 asked, “Does anybody know what you would call it if you went back the other direction?” (B1A, 71-72), to which Ankur responded, “The square root” (J1A, 70). R1 inquired if the group knew how to write square root. Angela came to the overhead projector and using correct notation, she wrote the square root of 25 (J1A, 75-79) (see Figure 3). R1 confirmed that the students knew what inverse operations are (e.g., addition and subtraction, multiplication and division, square and square root), and that they knew the notation for square root.

![Figure 3. Written work for plus three and square root.](image)

After completing the first task, R1 reiterated the earlier question pertaining to working with Dr. Davis on *Guess my Rule* activities. There was no audible response from the class.
5.2.1 *Times one-third.*

Presenting another table to the class but this time without the rule, R1 asked the students to find a rule that would work on every line (see Figure 4).

![Figure 4. R1’s table for times one-third.](image)

5.2.1.1 *Bobby’s table finds differences.*

Students at Bobby’s table calculated the difference between the box value and the triangle value for each row of the table. Michelle I. wrote: ‘2, $+1\frac{1}{3}, +\frac{2}{3}, 0, -\frac{2}{3}, -1\frac{1}{3}, -2$’ (B1A, 105-116) (see Figure 5).

![Figure 5. Michelle I.’s paper showing differences.](image)
Magdalena, looking at the differences, stated, “When it’s positive, it goes down by two-thirds; it goes down by two-thirds, and here it goes up by two-thirds, and then, here, it goes up by two-thirds, yes. From the zero it goes down by two-thirds…” (B1A, 130-132).

5.2.1.2 Jeff’s table finds box times one-third.

At Jeff’s table, Michael and Angela noticed a relationship between box and triangle. Angela said, “It looks like whatever the number [it is] is divided by three, like over three” (J1A, 96-97) and Michael said “The box times one-third” (J1A, 98).

5.2.1.3 Stephanie’s table finds one-third.

Ankur, who was talking at Stephanie’s table said, “Each number represents one-third” (S1A, 97). Stephanie posed the question: what does each number mean (S1A, 99)? Ankur responded, “Like one equals negative one-third. Negative two equals negative two. Negative two equals negative two-thirds” (S1A, 100-101). Also relating the box to the triangle, Brian agreed with Ankur. Ankur, who correlated the box values to triangle values, continued with, “Positive two equals two-thirds. Three equals one. Four equals…” (S1A, 103). As Brian said, “One-third or, four over”, Ankur interjected: “One and one-third” (S1A, 105-106). The table written on the overhead slide ended at box equals three, and Ankur added a row for box value of four to his paper.

R1 said that this was a hard one and that they would go back to it.

5.2.2 Times four.

R1 then presented another t-chart to the class (see Figure 6).
Students reacted to the table. Bobby and Amy-Lynn both said “Times four” (B1A, 142-143); Bobby, Amy-Lynn and Michelle I. raised their hands ready to supply the rule for this table. Angela, while working at her table, said, “Times four. Because a negative times a negative equals a positive” (J1A, 138). R1 called on Angela to present the rule and she responded, “It’s box times four equal triangle” (J1A, 146) and wrote it on the overhead (J1A, 149-151) (see Figure 7).

Continuing with this t-chart, R1 then wrote 60 in the triangle column and asked what the box value would be. Bobby replied 15, to which R1 asked how he had arrived at that number. Bobby’s response was: “Sixty divided by four” (B1A, 165-173).

Acknowledging Bobby’s use of the inverse, R1 stated, “Sixty divided by four so he’s using the idea of the inverses” (S1A, 158).
5.2.3 *Times* one-third revisited.

R1 returned to the earlier t-chart (see Figure 4). Michael, seated at Jeff’s table, had figured out the rule. He wrote: \[ \square \times \frac{1}{3} = \triangle \] (J1A, 185-188) (see Figure 8).

![Figure 8. Michael’s rule for times one-third.](image)

R1 asked Michael to show why his rule works for values of box and triangle. Michael explained his reasoning using multiplication and wrote calculations for his rule (J1A, 207-212) (see Figure 9).

![Figure 9. Michael’s calculations for times one-third.](image)

R1 had an additional rule, \[ \square / 3 = \triangle \], written on the overhead (S1A, 200-209) (see Figure 10).

![Figure 10. R1’s rule for times one-third.](image)
R1 asked the class if they thought that it, too, would work. They concluded that there could be two rules for one table. R1 pondered, “So you think you could talk about inverse [in] two different ways in this case?” (S1A, 218-219). R1 then asked what the box would be if the triangle were two and Amy-Lynn stated, “Six” (B1A, 257).

5.2.4 Times four plus one.

5.2.4.1 Amy-Lynn’s rule: times four plus one.

R1 wrote another table at the overhead projector (see Figure 11). Bobby noticed that one side goes down by fours and the other side goes down by one (B1A, 267). Michelle I. realized that they had found a pattern but now needed a rule (B1A, 268). Amy-Lynn stated “Wait I got something. Look. This times four plus one, times four plus one, times four plus one, times four plus one….” (B1A, 270-271). She described this pattern as she pointed from line to line on the table. She wrote on her paper: ‘x 4 + 1’ (B1A, 272). Magdalena, however, did not think it worked on the first row. Bobby offered an alternative and said it should be times four minus one (B1A, 273-275). Michelle I. had two rules, one for positive numbers and one for negative numbers and said that, “When it’s negative, it’s times four minus one. When it’s positive, it’s times four plus one” (B1A, 277-278). Bobby said it should be negative (B1A, 280), while Magdalena said, “It can’t be plus, or minus one: it has to be plus negative one” (B1A, 283). Michelle I. observed, “Plus negative one, it is the same thing, plus negative one and minus one” (B1A, 284-285). Using the first line of the table, Magdalena realized that, if you get a negative twelve and plus one, the result is negative. She stated “No, [be]cause if you plus one you [are] like minus-ing from it, [be]cause if you go negative three times four plus one, so it’s a negative. Oh that’s it. That’s it” (B1A, 286-288).
Figure 11. R1’s table for *times four plus one*.

Bobby also realized the first line was correct. He provided an analogy to communicate this idea of adding one to negative twelve: “Say you owe someone twelve dollars and you gave him a dollar then you only owe him eleven” (B1A, 315-316).

5.2.4.2 Angela adds four.

Angela, who was working at Jeff’s table said, “Look, it looks like on this side, the triangle side, it keeps adding four” (J1A, 254-255).

5.2.4.3 Ankur’s rule: box times four plus one.

Brian whispered “x minus, difference in between them, going down by 3” (S1A, 239-240). Erin (actual name: Romina) continued the table, on her paper, using the pattern she has recognized vertically in the box column and in the triangle column. She wrote ‘4 17’ and ‘5 21’ (S1A, 243-248). Ankur stated “No, it's box times four plus one. It works” (S1A, 291) Then Erin worked out the triangle value when box is four using Ankur’s rule and found that it worked (S1A, 298).

5.2.4.4 Amy-Lynn presents her rule.

R1 recognized Amy-Lynn and invited her to present her rule on the overhead projector (see Figure 12).
Figure 12. Amy-Lynn’s rule for *times four plus one*.

Writing ‘49’ (see Figure 13) in the triangle column, R1 asked what the value should be in the box.

Figure 13. R1’s 49 in the triangle column.

Michelle I. replied “twenty-four”; and Bobby followed with “twelve” (B1A, 341, 343). Other students thought the value was twelve and then Michelle I. recognized that she had made an error in her calculation (B1A, 347). Angela at Jeff’s table stated the inverse calculation: that she subtracted one from forty-nine and then divided by four to get twelve (J1A, 321-322); and she wrote the inverse on the overhead (J1A, 333-336) (see Figure 14, top equation). R1 stated the inverse rule as a fraction: the numerator as *triangle minus one*; and the denominator as *four*. (see Figure 14, bottom equation).
5.2.5 *Times two plus six.*

5.2.5.1 *Bobby’s rule: times two plus six.*

R1 placed another table at the overhead (see Figure 15). Michelle I. said *plus three* works for the first two lines but Magdalena corrected her and stated that it only works for the first line (B1A, 393-396). Recognizing the difference between box and triangle across each row, Bobby said: “Plus three, plus four, plus five, plus six, plus seven, plus nine” (B1A, 398). As in the previous task, Michelle I. reiterated that they have to find a rule (B1A, 402, 404), to which Bobby responded “Times two plus six” (B1A, 407).

*Figure 14. Angela and R1’s inverse rule for times four plus one.*

*Figure 15. R1’s table for times two plus six.*
5.2.5.2 Jeff’s table rule: times two plus six.

Observing the pattern in the triangle column, Angela stated, “It’s zero, two, four, six, eight” (J1A, 365). Seeing the relationship between the box and triangle values, Jeff exclaimed: “If three, twelve equals goes up by nine, two ten, goes up by eight” (J1A, 366). Then Angela said “Times two plus six” (J1A, 367). Jeff explained his findings to Angela, and they conversed about the rule:

Jeff (Pointing to Angela’s paper, [see Figure 16].): Zero plus six is six. One plus seven is eight. Two plus eight is ten. Three, it’s going up by one.

Angela Yeah, but you have to have the same exact rule for each one.

Jeff Well it’s gonna come up eventually. I'll figure out something for there, but right now, this looks nice. (J1A, 385-393)

Figure 16. Angela’s paper for times two plus six.

Stephanie also identified the relationship between box and triangle: “Add three, add four, add five, add six, add seven, add eight” (S1A, 379).

Michael who had been working on the task got the rule: “Box times two plus six. It works, just believe me” (J1A, 402). Jeff asked him how he arrived at that conclusion, to which Michael replied “I don’t know” (J1A, 407).

5.2.5.3 Magdalena presents Bobby’s rule.

Bobby found the rule, and Magdalena went to the overhead to write it for the class (B1A, 422-425) (see Figure 17).
R1 asked Magdalena to place a good number in the triangle column so the class could find the number in the box. Magdalena wrote ‘56’ in the triangle column (B1A, 446-447). Michael responded with twenty-five, (J1A, 432), and wrote the rule on the overhead (J1A, 437-441) (see Figure 18).

5.2.5.4 R1’s rule for times two plus six.

R1 presented another rule for the same table: \( 2 \times (\Box + 3) = \Delta \), (S1A, 429-435) (see Figure 19). R1 asked the class if just two rules work for the table and to explain why it works. Bobby provided an explanation:

You have to do the work in the parenthesis first and like, then, if you add three to negative three, it’s zero, and, if you times two by it, it like stays the same cause it’s zero, and it like works on every one. (B1A, 484-487)

Ankur from Stephanie’s table also contributed an explanation: “Two outside the parenthesis so it becomes sort of like a six” (S1A, 449); and followed with: “The two outside the parenthesis doubles whatever's inside” (S1A, 451).
5.2.6 **Box times box.**

Presenting another table (see Figure 20), R1 commented that it might be too easy for the class.

5.2.6.1 **Amy-Lynn’s times its number.**

Students at Bobby’s table copied the table to their papers. Amy-Lynn stated “It’s times its number” (B1A, 521). Michelle I. corroborated her statement “Times its number” (B1A, 522). Bobby offered a rejoinder: “No [be]cause [if] you times three by negative three, it is negative nine” (B1A, 523).

5.2.6.2 **Jeff’s box times box.**

Thinking, at first, that it was three squared, Michael then reconsidered (J1A, 497). Jeff posed the question: “Why isn’t it box times box equals triangle?” (J1A, 498).
Michael and Angela agreed that the rule was box times box (J1A, 506-507). R1 asked Jeff to write his rule, □ x □ = △, on the overhead (J1A, 509-512) (see Figure 21).

Figure 21. Jeff’s rule for box times box.

5.2.6.3 Bobby’s table debates.

Bobby, thinking the rule was incorrect, raised his hand (B1A, 538). Michelle I. thought it was incorrect and stated “That does not work [be]cause, if it’s negative three times negative three, it’s negative nine” (B1A, 539-540). Michelle I. explained that those at her table confused her and said: ”I was just repeating what they said, and I disagreed with it, and then they tell me that I am wrong. So it is their fault that I was wrong” (B1A, 546-548). R1 acknowledged what was happening at Bobby’s table and stated: “So they are having a discussion at this table. They feel that some people were led astray about negative three times negative three being negative nine or positive nine” (S1A, 507-509). Amy-Lynn and Michelle I. then concurred that the answer was nine (B1A, 558-559).

5.2.6.4 Michael’s notation for box times box.

Michael, at Jeff’s table, offered an alternative formula: “There is another way to write it. It’s box squared equals triangle” (J1A, 537-538) (see Figure 22). Going up to the overhead, he wrote the rule another way: □^2 = △, (J1A, 549-552).

Figure 22. Michael’s rule for box squared.
5.2.6.5 *Square root inverse.*

R1 asked if someone mentioned square root. Angela said: “No, but if the number is triangle, then to find what box is, you have to find the square root of the triangle” (J1A, 557-558). R1 wrote ‘25’ in the triangle column, and that one would have to find the square root to get the box value.

Then R1 mentioned that, when the triangle is 9, then box is three and box could also be negative three. R1 asked what would box be if the triangle value is -36 (S1A, 529-536). Brian answered: “It is either a positive or a negative six. I’m not sure” (S1A, 537). Magdalena said “It’s a negative” (B1A, 595). Michelle I. said “I am not saying anything [be]cause you people always make me look wrong” (B1A, 596-597). R1 called on Bobby, and he responded: “You can’t put a negative there. [Be]Cause to get a negative number you have to times like a positive and a negative. So you get a negative” (B1A, 599-600). Then Michelle I. said “So that’s a negative times a negative and that’s a positive” (B1A, 601). R1 said “They’re saying over here that they don’t think I should put a negative over here [and] that you wouldn’t be able to get an answer” (S1A, 547-548). R1 then stated “We shouldn’t use this negative thirty-six. Okay” (S1A, 552).

5.2.7 *Plus two.*

Placing another table at the overhead (see Figure 23), R1 said that maybe it is a little bit harder (S1A, 555-556). Bobby, Magdalena, and Michelle I. agreed on *plus two* for the difference between the box and triangle values (B1A, 631-633). R1 found an error in the last three lines of the table and corrected them (see Figure 24) (S1A, 562-563). Bobby and Michelle I. maintained that they were correct in saying *plus two* (B1A, 646-649).
5.2.7.1 Bobby’s table rule for plus two.

R1 let Michelle I. write her rule and inverse rule on the overhead (B1A, 650-655). She originally wrote the rule as *box plus two equals triangle* and the inverse as *triangle divided by two equals box*. While working on the subsequent rule, Angela noticed an error in Michelle I.’s inverse rule, and that it should be *triangle minus 2 equals box* not *triangle divided by 2 equals box*. Raising her hand, Angela brought the error to R1’s attention. Michelle I. seeing the error, corrected it on the overhead (see Figure 25).
5.2.7.2 R1’s other rule for plus two.

R1 thought of another rule for plus two and said: “I had three times the number in the box, plus six divided by three is equal to the number in the triangle. Do you think that would work? Do you think that would work?” (S1A, 581-583). R1 wrote the rule on the overhead (see Figure 26).

![Figure 26. R1’s other rule for plus two.](image)

Bobby replied: “No, [be]cause like negative two times three is negative six plus six is twelve, and, if you divide it by three, it will be four not zero” (B1A, 679-680). R1 confirmed that Bobby had, indeed, gotten a different answer with R1’s rule (B1A, 681-683).

Magdalena worked it out on her paper: “It is negative three and when you divide by three it is zero...” (B1A, 687).

Continuing with the question about the alternative rule (see Figure 26), R1 asked if it worked for every line. Bobby tried to rework it, and then Amy-Lynn, Michelle I. and Magdalena, working together, calculated it: “Negative three times three is negative nine plus six is negative three divided by three is zero” (B1A, 713-715). Bobby then said: “Three divided by three is one” (B1A, 717). The girls immediately replied, “It works. It works. We just kinda [of] miscalculated again” (B1A, 721-722).
Angela, working at Jeff’s table, initially miscalculated the rule because she had not seen the division by three: “Look three, look, look. Three times zero is zero. Plus six. Oh, it does work. Duh, I didn’t see the divided by three, I’m blind” (J1B, 78-79).

Stephanie and Romina, while working out R1’s rule for box equal to one, figured out the triangle value but did not think that the rule worked (S1A, 640-642). Continuing to work on it, they realized their error (S1A, 653-655).

5.2.8 Two rules for one table.

R1 then asked the class to make up a rule that could work two ways.

5.2.8.1 Rules at Bobby’s table.

Bobby started by thinking of a word problem using $mc^2$ and the speed of a car on a highway (B1B, 14-15). Michelle I., Magdalena and Amy-Lynn thought of a rule starting with box squared and then considered subtracting a number. Magdalena wanted to use plus a minus number and Amy-Lynn and Michelle I. thought of using thirteen (B1B, 11-13, 20-21). Magdalena’s paper showed: $\Box^2 + -13 = \triangle$ (B1B, 29). They calculated triangle values for box values of two, four, six and zero. Bobby wrote down the box and triangle values and stated that the rule appeared to be box minus eleven equals triangle (B1B, 91-95).

Magdalena and Michelle I. decided not to place a zero in their table since they felt that this would give their rule away. They also decided to make the rule harder by dividing their rule by negative one-third (B1B, 147-151, 174). They attempted to recalculate the triangle values with their new rule and needed the aid of a calculator (B1B, 154-162).
5.2.8.2 Rules at Jeff’s table.

Filling in a table of values where all values in the triangle column are one, Michael wrote the rule box divided by box equals triangle (J1B, 224, 239). Researcher 2 [R2] had a different rule and asked those at Jeff’s table if the rule five box divided by five box equals triangle worked (J1B, 295). R2 also asked them if the rule million box divided by million box equals triangle worked (J1B, 302).

Jeff had a rule, box plus one equals triangle, and he also stated that it worked the other way: triangle minus box equals one (J1B, 323-326). Jeff then corrected his inverse rule to be triangle minus one equals box (J1B, 330-331); however, Angela dismissed his rule as being too easy (J1B, 335). R1 reviewed the group’s work and commented on Jeff’s rule and inverse (J1B, 338), while R2 reviewed Jeff’s rules. Jeff stated that his rule “works two ways and it’s very simple and it lets you express your point very easily because it can show you how the inverse pairs work” (J1B, 442-444).

R2 wrote another rule on Jeff’s paper as five box plus five divided by five equals triangle (J1B, 452-456) (see Figure 27) and asked them if it worked. Angela figured out the rule for box equal to one and, after correcting her addition, stated it worked (J1B, 470-472). R2 asked if it worked for four (J1B, 475). Angela and Jeff agreed that it did (J1B, 476-477). R2 wrote the rule ten times box plus ten divided by ten equals triangle (J1B, 478-482) and asked if it worked. Jeff replied that it did because it was the same thing that Mike had said (J1B, 485). Angela observed that, because it was a multiple of five, it worked (J1B, 486).
5.2.8.3 **Rules at Stephanie’s table.**

Romina and Ankur devised a rule and a table of values for *box plus seven equals triangle*. They also provided an alternative: *five times two, minus three plus box equals triangle* (S1B, 87-90) (see Figure 28). R2 had two other ways to write the same rule: one was *the quantity eight times box plus fifty-six divided by eight equals triangle*; and the other, *the quantity one hundred times box plus seven hundred divided by one hundred equals triangle* (S1B, 135-144). Ankur wrote the four rules on his paper and then on the overhead projector, as he presented them to the class (S1B, 249-252). Researchers 1 and 2 asked Ankur why the rules gave the same set of ordered pairs (S1B, 254-255).

![Figure 27. R2’s rule and Jeff’s function and inverse rules.](image)

![Figure 28. Romina and Ankur’s rules for box plus seven.](image)

5.2.9 **Summary of day one activities.**

For each t-chart presented, students found patterns going down the columns and found relationships between box and triangle values. They readily knew that they needed to find a rule since many first found a pattern and then worked to find the rule to
represent the relationship between the box and triangle. They also found the rule to represent the inverse function.

At times, integer arithmetic especially addition and multiplication of numbers with mixed signs created an obstacle which they did however overcome.

5.3 Day two: May 17, 1995

After R1 reviewed the properties of arithmetic, a discussion developed over Lindsey’s algorithm for multiplying a number by five that had been presented on the previous day. Then the class worked on finding the rule and inverse of a table where triangle values were fractions (see Figure 29).

![Figure 29. R1’s data for triangle equals box/(box+1).](image)

5.3.1 Bobby’s table rule: box/(box + 1).

Michelle I. whispered to herself what she wrote on her paper as she copied the table presented by R1 (see Figure 30) (B2, 382-383). She quickly got the value of the question mark in the box column as eleven (B2, 385-386). Bobby and Amy-Lynn both agreed with Michelle I., and they discussed that the number in the box was the numerator of the value
of the triangle (B2, 395-399). Bobby stated: “It goes one two, two three, three four, four five; it’s a pattern” (B2, 400-401).

Michelle I. acknowledged that the numbers on the left side or box column were present in the top of the number in the triangle column, and got the pattern (B2, 403-406). Along with Magdalena, she then asked what the rule was (B2, 411-412). Michelle I. then responded “It’s that with the number that you’re timesing…the box times, times number equals… box over box plus one. Box plus whatever, you know, plus number whatever” (B2, 413-415). Magdalena said she did not understand it (B2, 416). Bobby stated “I found another pattern it goes times, times two, like, see? Times one, times two, times three, times four, times five, times six” (B2, 419-420). Magdalena responded that was a pattern, not a rule (B2, 421). Bobby replied that he was not trying to find a rule, just patterns (B2, 422). The conversation continued with Michelle I., Magdalena, and Bobby discussing how to find the rule:

Michelle I. You do box… it’s box like plus or minus or whatever the number equals box over box plus one.
Magdalena Yeah but what’s the number?
Michelle I. That’s what we have to figure out (both girls giggle).
Bobby That’s like mine. (B2, 425-429)

Figure 30. Michelle I.’s paper for triangle equals box/(box+1).
Magdalena calculated a result when *box is one* and stated:

It’s easy ‘cause if, if it’s the box, the box and the box, hold on (points to Michelle I.’s work) (see Figure 30). So it’s, for example one and then one plus one it’s two and if you divide one by two…get the answer and then you take the…. (B2, 438-441)

Bobby replied, “Wait, so it’s box over box plus one” (B2, 445). Then Michelle I. went a step further: “Yeah. So triangle equals box over box plus one. Yes!” (B2, 446). They discussed how the rule worked. Amy-Lynn stated that when box is one the triangle equals one over one plus one (B2, 464) and that equals one-half (B2, 469). She also explained the calculation of triangle when *box is two*: “That’s a triangle. So, two-thirds equals two over two plus one. It still works, two-thirds” (B2, 482-483). Michelle I. and Amy-Lynn recognized that it is the same rule as Bobby’s (B2, 492-493).

All agreed that they must find the inverse (B2, 510, 515).

5.3.2 Bobby’s table inverse rule for triangle equals box/(box +1).

Michelle I. reiterated the rule “triangle equals” to which Amy-Lynn added “box over box plus one” (B2, 516-517). Michelle I. stated “Box times box minus one” as the inverse (B2, 520). Writing on her paper, Magdalena stated “Um, you minus one from the box and then you divide that box by that box and you get the answer” (B2, 540-541). She tried to work it out for one-half but it failed (B2, 555-558). She then stated that it should involve multiplication rather than division (B2, 564). Amy-Lynn tried another unsuccessful approach for the inverse (B2, 576-577). Magdalena devised another rule “How ‘bout [about] how ‘bout [about] triangle minus something equals” (B2, 592).

While figuring on her paper, Amy-Lynn proposed that “Box equals triangle over triangle minus one” (B2, 629), and she continued to find the box when triangle was one-
half. Michelle I. and Magdalena worked on calculating the inverse using a rule but got the value three-fourths which did not work out (B2, 637-646).

Amy-Lynn stated that she was close to getting the inverse (B2, 663) and concluded: “Box equals triangle… times triangle plus one” (B2, 667-670). Using a box value of two and a triangle value of two-thirds to test her rule, she calculated two-thirds times two-thirds as four-thirds then added one and got two and one-third (B2, 674-678).

Michelle I. and Magdalena adopted Amy-Lynn’s rule. Michelle I. claimed to have understood it added, “you put parentheses around these things” (B2, 682). The inverse rule that they derived was box equals triangle times parentheses box plus one or □=△x(□+1) (B2, 683-685). They proceeded to check the inverse rule with box value of three and five and were successful (B2, 696-708). Amy-Lynn stated the inverse rule as: “Box equals triangle times parentheses box plus one parentheses” (B2, 719).

R3 challenged the group to use their inverse rule to find the box if they did not know the original rule, and asked them to find the box if the triangle was forty-nine fiftieths (B2, 727). R3 pointed out to them that their rule used triangle and box on the right side and that they had two things on the right side (B2, 759-764).

After Michael and Angela (see 5.3.4 below) presented their findings, R1 asked Michelle I. to present both the rule and inverse rule to the class at the overhead projector.

5.3.3 Angela’s rule: box/(box +1).

Looking at the table, and finding a pattern between the box and triangle columns, Angela stated that the question mark was eleven (J2, 401-406). Michael discussed the table with Angela and stated that box equals box over box plus one (J2, 409); however, he thought it was not a rule (J2, 411-415). Angela questioned why it was not a rule (J2, 417).
She proceeded to write on her paper *box equals box divided by box plus one* (J2, 432). After discussing the table of values, Angela and Michael agreed that the rule should be *triangle equals box divided by box plus one* (J2, 436-437). Angela replaced the box on her paper with a triangle (see Figure 31) (J2, 440-441).

![Figure 31. Angela’s paper for triangle equals box/(box+1).](image)

**5.3.4 Michael and Angela’s inverse rule for box/(box +1).**

Angela tried to work out the inverse and considered the possibility of *triangle times triangle minus one* (J2, 467). After pondering this for a moment, she said, “No it’s not gonna [going to] work because you can’t multiply to have so many fractions. Or can you multiply?” (J2, 483-484). Michael collaborated with her to calculate the box value when *triangle equals one-half*. They determined that *one-half times one-half was one over four*. But then Michael said “No, it’s not the inverse. Forget that” (J2, 497). Angela then conjectured that they should *minus one* somewhere (J2, 499). Michael then thought that denominator or numerator should be in the inverse rule (J2, 507-508).

Angela and Michael tried to revise the inverse. Michael demonstrated to Angela that the inverse rule could be written as *box equals triangle times the denominator of the triangle*. He calculated box with triangle value of three-fourths as *three-fourths times four* which is three (J2, 528-541). Sarah contributed to the discussion, saying to Jeff “Box equals triangle times the denominator” (J2, 546).
R1 checked the work at Jeff’s table, asked if all were convinced that their rule worked, and asked about the inverse (J2, 581). Michael presented the inverse rule and calculated box using *three-fourths times four equal three* (J2, 590-591). R1 responded that the denominator itself has a box in it (J2, 592-593).

Discussing what it meant for the denominator not to have a box, Michael and Angela confirmed that, if they start with a value for triangle, then they needed a rule to find the box (J2, 598-606). Although they calculated the box value when the triangle was five-sixths, they still used the denominator (J2, 616-617). In response, R1 inquired if there was another way that they could write that (J2, 622). Angela observed that they, at this point, had one way to get the box and one to get the triangle (J2, 629-630). She proceeded to break the triangle into numerator and denominator, since the triangle was the whole fraction (J2, 635-637). R1 interjected that this was the problem they needed to address (J2, 641-642). Angela responded “So wouldn’t it have to be like numerator times denominator” (J2, 643)? Both she and Michael spent several minutes using their rule, *numerator times denominator divided by denominator*, to arrive at a value for the box when the fraction or triangle value is eleven-twelfths. Their discussion centered on how to multiply a fraction by an integer (J2, 713-742).

R1 asked Michael and Angela to present their inverse rule, which was similar to Michelle I.’s (see Figure 32). Michael answered that the box equals *the triangle times the denominator over one* (J2, 803-804).

*Figure 32.* Overhead slide showing Michael/Angela’s and Michelle I.’s work.
After the bell rang to end their session, the researchers directed several questions to Angela and Michael. R3 requested that Michael and Angela write the denominator in terms of the triangle (J2, 837-840); R4 asked them how the triangle is related to the denominator (J2, 863-864); and R1 posed a new approach to Angela: if she had a fraction and wanted to multiply it by its denominator, was one minus the fraction a possibility (J2, 887-890)?

5.3.5 Romina’s rule: box/(box +1).

While looking at the t-chart, Romina, Stephanie and Brian agreed that the box value for eleven-twelfths was eleven (S2, 379-388). Romina remarked, “There’s got to be some sort of rule to it” (SB, 390); and she wondered how one can get “thirteen-fourteenths?” (S2, 426). Stephanie pondered “…From twelve, how can you get twelve-thirteenths? Why don’t we start at the top though? Because they’re smaller” (S2, 427-429). Romina stated, “I know. Okay. Add a one” (S2, 430); Ankur stated that it’s multiplying by something (S2, 432). R1 asked them what the pattern was and Stephanie responded “..say we have twelve, it would be twelve-thirteenths, and we have thirteen and thirteen-fourteenths” (S2, 448-449). R1 asked how could they say that using boxes or x’s or something (S2, 450-451). Romina stated that box equals box over triangle. Ankur questioned it and then they both agreed on box equals box over box plus one (S2, 459-461). Stephanie asked “So, where’s your triangle for that?” (S2, 467). Romina responded “the square plus one, that would equal the triangle. Does that work?” (see Figure 33) (S2, 475-476). Ankur hesitantly agreed: “That’s the rule, I guess” (S2, 488).
Romina asserted that it is a rule and then asked what the inverse rule was (J2, 489). Romina thought that she got the inverse rule as $\text{square equals square over square plus two, plus one}$. R1 asked the class that if they got the rule then they should also find the inverse rule (S2, 527-528). Brian and Ankur continued to work to find the inverse rule and Brian stated “One minus box” (S2, 558). Brian added “times box equals box” (S2, 562). When R1 asked if they had the rule, all replied affirmatively, but Brian replied “We just can’t get the inverse” (S2, 565-567). R1 asked them to work a little harder (S2, 571).

After some off-task conversation, Stephanie’s group came back to the work at hand. R3 asked about their findings (S2, 676), to which Romina replied: “That box equals box over box plus one” (S2, 677). R3 immediately challenged: if you had triangles and you tried to get squares, what would you do? (S2, 696-697). Since there was no response from the group, R3 supplied a clue: “Do the opposite. What’s the opposite…?” (S2, 699). After a long silence, Romina conjectured: “Box minus one” (S2, 701); and she suggested: “ Wouldn’t it be box equals box over triangle minus one?” (S2, 703-704). Ankur replied “No, wait, you have to have, like, triangle at the beginning [be]cause that’s what you know” (S2, 710-711); Romina’s paper showed $\text{triangle blank space box over box minus}$
one (S2, 714-715). R3 guided the inquiry with the question: what do you want to find? (S2, 717-718), to which Romina responded: “So, it’s box equals” (S2, 721). R3 queried: “So, if you have box here, and you want to find it out... you have a box over here. Can you do that?” (S2, 724-725). Ankur’s solution was to insert a triangle (S2, 729). R3 interrupted at this point, suggesting that they try numbers to see what happens (S2, 730). Romina substituted numbers in the rule with box equal to one and triangle equal to two but decided it did not work (S2, 732-733).

After surveying the room, R1 stated that everybody was pretty convinced about the rule (S2, 758-759). After some tangential conversation, Ankur spoke up “Why don't we do this, in the triangle, why don't we divide it, like, top part of the triangle is another shape, and the bottom is another shape, and the entire thing is the triangle” (S2, 817-819).

5.3.6 Summary of day two activities.

Day two started with R1 reviewing the properties of arithmetic with the class. Students worked to find the rule and inverse of a t-chart that contained multiple fractions. They were able to find the rule by focusing on the relationship between the box and the triangle. Students from all groups wrote the rule as triangle equals box over box plus one. Finding the inverse rule appeared to be more difficult. Most students could calculate the inverse from the data in the chart. Some students broke the triangle into its numerator and denominator and used them in their inverse rule. Some used both triangle and box on the left side of the rule.
5.4 Day three: May 18, 1995

Day three commenced with T1 presenting a table of values to the class (see Figure 34). T1 asked the class to find the rule and the inverse rule. Students worked in the same groups as on days one and two. One camera filmed the entire class, and it went from table to table. The section is presented by group, rather than chronologically, as seen on the videotape.

\[
\begin{array}{cc}
\square & \triangle \\
\frac{1}{2} & \frac{1}{2} \\
9 & -8 \\
1 & 0 \\
2 & -1 \\
\frac{1}{3} & \frac{2}{3} \\
0 & 1 \\
\frac{2}{3} & \frac{1}{3} \\
-1 & 2
\end{array}
\]

*Figure 34. T-chart presented by T1.*

5.4.1 Rules for triangle equals one minus box.

5.4.1.1 Bobby’s table activities.

At the outset, Bobby determined a relationship between box and triangle. For the first row and seventh row, he stated: “Two minus one is one. Keep the two. Three minus two is one. Keep the three” (A3, 9-10). Michelle I. asked him what about regular numbers, that is, the integers in the table (A3, 12). Both he and Amy-Lynn instructed Michelle I. to
put a one under it (A3, 14). Bobby proceeded to calculate the triangle with the box value of two, using two over one: “Yeah, look, one minus two is negative one” (A3, 15). The dialogue continued, with Michelle I. asking what the rule would be (A3, 16). Bobby responded: “The denominator minus the numerator” (A3, 21). Using a box value of one, Amy-Lynn calculated the triangle: “Here, one minus one equals zero (A3, 23). Bobby added “Subtract the numerator from the denominator. The denominator minus the numerator” (A3, 29-30). As Michelle I. wrote the rule the others at her table said: “Denominator minus the numerator equals the numerator of triangle, and denominator stays the same” (see Figure 35: rpt. from Bellisio, 1999, p. 163) (A3, 31-35).

Figure 35. Michelle I.’s rule expressed as a verbal expression.

The researchers posed two questions: R3, about the rule working for the row when box is one and triangle is zero; and T1, about the rule for box value of zero and a triangle value of one (A3, 67-68, 71). To these questions, Amy-Lynn answered: “Subtract the numerator from the denominator” (A3, 76). R3 followed with another question about the rule when box is eight and triangle is nine, and when box is minus eight and the triangle is nine (A3, 80). Michelle I. stated that they couldn’t get the rule to work with negatives (A3, 81) and she wrote negative eight over one on her paper. The interchange between R3, Michelle I. and Bobby continued:

Researcher 3 Subtract, minus eight from that one.
Michelle I. No, it’s supposed to be one minus negative eight.
Researcher 3: So that would be nine over one, right. Does anybody know?
Bobby: Yeah, but can you have a negative fraction?
Michelle I.: But it’s negative nine. Does anyone know? And it’s regular (positive) nine (in the problem) so (A3, 87-92).

Michelle I. asked T1 for another negative entry in the chart for a box (A3, 95-96). R3 wrote two more rows on the chart: -8 and 9, and -7 and 8 (A3, 109-112).

Bobby then created a word problem relating the negative eight to one and he, along with Michelle I. and Magdalena, continued:

Bobby: Taking away the negative eight so it turns into a positive, so it would be nine. Cause, say you have like eight dollars, say you owed eight dollars and you had a dollar and got eight dollars, you would have nine, not negative nine.
Michelle I.: But it’s one minus eight.
Bobby: You’re minus-ing a negative and the opposite of a negative is addition.
Magdalena: You’re right, Bobby.
Michelle I.: Okay, so let’s try with negative seven. Negative seven over one. One minus negative seven is eight. Okay, we’re done. (A3, 115-124).

Once they had expressed the rule using denominator and numerator, Bobby suggested using a circle to represent the denominator of the box value and a box to represent the numerator of the box value (A3, 179-180). Following Bobby’s lead, Michelle I. rewrote her equation using circles (see Figure 36, rpt. from Bellisio, 1999, p. 165) (A3, 247) (A3, 180-184). However, T1 stated that they introduced another variable here (A3, 186).

\[
\APTER \text{Denominator} \\
\Box \text{Numerator} \\
\Delta \text{Answer’s denominator numerator} \\
\text{\fbox{\text{Box}}} - \Box = \Delta
\]

Figure 36. Michelle I.’s rule using circles.
Placing the value, *negative one-fourth*, in the chart for box, T1 asked the class what the triangle value would be (A3, 190-198) and asked Michelle I. specifically, to use her rule: she got *one and a fourth* (A3, 208-210). When T1 said she was correct, Michelle I., Bobby, and Amy-Lynn slapped hands in victory (A3, 213).

R3 asked them to try to write the rule without separating the numerator and the denominator and encouraged them to start again (A3, 235-239). Bobby wrote the rule using box to represent the next highest number and he numbered each square placing a one inside the first square in the denominator and a two inside the second square in the numerator (see Figure 37) (A3, 250-252).

![Figure 37. Bobby’s rule using box.](image)

T1 instructed Michelle I. to use a rule to get the triangle value when box is two sevenths (A3, 300-302). Along with Amy-Lynn, they calculated *seven minus two equals five* and got *five over seven* (A3, 303-305). Michelle I. then confirmed that they had used three variables in their calculation (A3, 306). Acknowledging that they had calculated the correct value for triangle, T1 indicated that Michelle I.’s thinking was good but that the rule was not *triangle over box* and that the triangle had to be the answer and encouraged her to work her way out of it (A3, 318-322). T1 hinted that the rule she had herself
devised was not complicated, for it had one operation (A3, 327, 329). To Amy-Lynn’s question about how many variables there were, T1 responded, two (A3, 330-331).

Having overheard one of the other groups, Bobby suggested a rule: “I have a good one. Listen. Box minus one and switch from positive to negative” (A3, 332-333); and he wrote it on his paper (see Figure 38) (A3, 336).

Bobby used his rule to find triangle when box is one-third: “And two-thirds minus one is negative one-third, and you switch it to a positive. It’s one-third” (A3, 343-344). Since T1 had said that they were on the right track, Michelle I. wanted to stay with her rule (A3, 345-346).

5.4.1.2 Work at Stephanie’s table.

Jeff, working with Stephanie’s group, discussed the t-table with Ankur, Brian and Romina. Brian stated: “If it’s positive you subtract one and that number turns into a negative” (A3, 43-44). Ankur noticed: “That’s the thing that’s weird. A half equals a half” (A3, 49). Romina commented that for this one you added a zero and multiplied by one (A3, 50-51). Jeff then stated: “Subtract one and add a negative” (A3, 52).

The camera returned to their table about three minutes later as Romina calculated a triangle value of positive six when box equals negative five and a triangle value of
negative six when box equals positive five (A3, 99-100). Brian corrected Romina’s values and stated: “Positive five will be a negative four” (A3, 105).

Someone from Stephanie’s table stated “Box minus one, one-half minus one equals” (A3, 274). Jeff and Ankur both agreed the switch from positive to negative and vice versa (A3, 275-281). Romina calculated the triangle value as one and one-fourth when box was negative one-fourth. She wrote a rule on her paper as box minus one and switch from positive to negative or negative to positive equals triangle (see Figure 39, rpt. from Bellisio, 1999, p. 169) (A3, 290-295).

![Figure 39. Romina’s rule.](image)

The camera returned to their table when R3 asked them if their rule worked (A3, 351). Ankur explained “And multiply the number by negative one” (A3, 352). R3 prompted them to write the equation (A3, 363). Ankur wrote on his paper

\[ (\square \cdot -1) \times -1 = \triangle \]  (see Figure 40) (A3, 367).

![Figure 40. Ankur’s rule.](image)
R3 asked them to check their rule to see if it worked for all values in the table (A3, 371). They checked it for box value of one-half. Their interchange follows:

<table>
<thead>
<tr>
<th>Romina</th>
<th>Use a half. Half minus one.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ankur</td>
<td>Put it in.</td>
</tr>
<tr>
<td>Romina</td>
<td>Half minus one would be negative one and one-half.</td>
</tr>
<tr>
<td>Ankur</td>
<td>Negative one-half.</td>
</tr>
<tr>
<td>Romina</td>
<td>Times negative one that would be one and one-half or zero.</td>
</tr>
<tr>
<td>Ankur</td>
<td>Half minus one equals negative half.</td>
</tr>
<tr>
<td>Jeff</td>
<td>Negative half times negative one makes it a positive half.</td>
</tr>
<tr>
<td>Ankur</td>
<td>So it works. (A3, 374-382)</td>
</tr>
</tbody>
</table>

5.4.1.3 Michael and Angela’s rule.

Michael, Angela and Sarah had been working to find the rule, as Jeff left his table to work with Stephanie’s table. On her paper, Angela wrote box minus one equals triangle (see Figure 41) (A3, 226).

![Figure 41. Angela’s first rule.](image)

The camera did not capture more of their work. When T1 brought the class together to present their findings she asked Michael, Angela and Sarah to present their rule. On the overhead slide Angela wrote two equations and then made a correction to the second one. Her rule for the function was as follows: $(\square \cdot 1) - 1 = \triangle$. For the inverse, she wrote: $\triangle \cdot 1 + 1 = \square$ (see Figure 42).
Figure 42. Function and inverse rules at Jeff’s table.

Explaining how the rule worked when box was a half, Michael worked with Angela to calculate the triangle value correctly. Their interchange follows:

Michael  A half. Box minus one is one and a half.  
(Michael writes $1 - 1\frac{1}{2}$ on the overhead and Angela corrects him below:)

Angela  Minus one is negative a half.

Michael  Negative a half.  
(Michael crossed out the 1 so it is $-\frac{1}{2}$ as shown in Figure 43)  
Plus, and then it’s times a negative one which brings it to a regular half. When you times something by a negative one if it’s positive, if the number you’re dividing, times-ing.

Angela  A negative times a negative turns into a positive. (A3, 447-459)

Figure 43. Michael and Angela’s work.

Magdalena did not understand the triangle calculation above, so Angela suggested that they calculate triangle when box is two, and this alternative was written on the overhead slide (see Figure 44) (A3, 461-467).
To clarify their notation for the rule, T1 enclosed negative one within parentheses (see Figure 45) (A3, 469-477).

Angela recalculated, using the rule with parentheses for box equal one-half: “A half minus one equals negative one-half. Okay, then multiply it by negative one and that equals positive one-half [be]cause negative times a negative equals a positive” (see Figure 46) (A3, 484-486).

Amy-Lynn suggested that they calculate triangle when box is negative one, and Angela proceeded to perform the calculation. Their exchange follows (see Figure 47):

Amy-Lynn: Can you do negative one?
Angela: Negative one? All right. Negative one minus one equals zero.
Sarah: It doesn’t work.
Angela: Times negative one. (A3, 499-502)
Figure 47. Angela’s work for box equal negative one.

T1 asked if anybody saw a problem with the reasoning here, to which Michelle I. responded: “Isn’t negative one minus one negative two or is it just me?” (A3, 507-510). Angela corrected her work (see Figure 48) (A3, 521).

Figure 48. Angela’s correction for box equal negative one.

They went on to calculate triangle when box equals negative three-fifths, and Michael showed the calculation: “Negative three-fifths minus one equals, equals negative one and three-fifths” (see Figure 49) (A3, 533-534).

Figure 49. Michael’s work for box equal negative three-fifths.

Michael and Angela completed the calculation of triangle by multiplying negative one and three-fifths by negative one. Michael then asked the class for the value of negative one and three-fifths times negative one (A3, 575-577), but offered his own answer: “It equals one and three-fifths” (A3, 579). To Jeff’s request for an explanation (A3, 583), Angela responded: “Because a negative times a negative equals a positive” (A3, 584).
Michelle I. queried: “So you are saying your answer for negative three-fifths is one and three-fifths?” (A3, 586-587), to which Angela answered affirmatively (see Figure 50) (A3, 588).

![Figure 50. Michael’s results for box equal negative three-fifths.](image)

Michelle I. replied: “I want to check this....” (A3, 589) and then did so using her rule (see Figure 51). She confirmed that their result was correct (A3, 595).

![Figure 51. Michelle I.’s results for box equal negative three-fifths.](image)

5.4.1.4 Michelle and Amy-Lynn’s rule.

Michelle I. and Amy-Lynn demonstrated their rule to the class by performing their calculation when box equals two-thirds. Michelle I. stated: “See, you do the denominator minus the numerator, which is one, and you keep the denominator, three. So that’s your answer” (see Figure 52) (A3, 614-615).

![Figure 52. Michelle I.’s results using her rule for box equal to two-thirds.](image)
Students asked if their rule worked for other values. To this question, Michelle I. responded affirmatively. To demonstrate this, she calculated the triangle value for box equal to one-half and for box equal to negative eight. (A3, 624-641). She wrote her rule on the overhead slide using circles as she had written on her paper (see Figure 36).

5.4.1.5 T1’s rule.

T1 asked Stephanie’s table to present their findings but they responded that their rule was the same as Michael’s (A3, 661-664). At this point, T1 presented her rule which differed from the two rules presented: *one minus box equals triangle* (A3, 668-669). When T1 referred to Bobby’s table rule as one that works with certain kinds of numbers, Michelle I. and Amy-Lynn responded that instead of whole numbers they used fractions (A3, 676-682).

R3 asked how Michael and Angela’s rule could be equated to T1’s rule (A3, 687). Writing the word *distribute* on the overhead slide T1 assisted them in this process (A3, 710). Multiplying *negative one by box and negative one times negative one*, Amy-Lynn got *negative box plus one equals triangle* (A3, 715-717). After reviewing these calculations, the students agreed that through the use of the commutative property they could switch the terms to get *one minus box equals triangle*. T1 wrote commutative on the overhead slide and applied the property to the rule (see Figure 53) (A3, 733, 740-743).
5.4.2 Inverse rule for triangle equals box/(box +1).

T1 directed the class to revisit the t-table they worked on the day before. R3 asked who remembered the rule (A3, 778). Bobby replied: “Square divided by the next highest number” (A3, 779) and Michelle I. and Amy-Lynn both responded: “It’s square divided by square plus one” (A3, 780-781). Amy-Lynn wrote the rule on her paper as square divided by square plus one equals triangle but then added parentheses around the square plus one as directed by Bobby (see Figure 54) (A3, 783-787).

Figure 54. Amy-Lynn’s rule.
Michelle I. explained their rule to T1 and T1 wrote it on the overhead as a fraction (see Figure 55) (A3, 790-797).

5.4.2.1 Bobby’s group finds inverse.

T1 asked the class to find the inverse expression (A3, 799-804). Ankur stated as a question: “Triangle minus one equals box?” (A3, 805). Bobby stated: “It’s triangle times square plus one” and wrote it on his paper (see Figure 56) (A3, 807).

Michelle I. responded to this: “But you don’t know what square is. It’s like you’re turning it around so you don’t know what square is” (A3, 808-809). R3 reminded them that their primary goal was to find an equation with box equals something (A3, 810-811). Using his inverse rule, Bobby tried to calculate triangle when triangle is six-sevenths and box is seven but was unsuccessful (A3, 816-827).

Several minutes later, the camera returned to Bobby’s table, where Bobby had written on his paper a different inverse rule, separating the denominator (see Figure 57) (A3,
1068-1089). Stating that his notation did not make sense, T1 suggested that Bobby give it another variable name (A3, 1072-1076).

Figure 57. Bobby’s new inverse rule.

Shortly thereafter, Michelle I. experienced difficulty with dividing by a fraction. She had been dividing one by one-third and got three (A3, 1203-1205). Magdalena pointed out to Michelle I. that the inverse rule worked for two-thirds: “Shelly, look. Two-thirds over one-third. And when you divide that it’s six over three and that equals two and that’s the answer” (A3, 1211-1212). Testing their rule using five-sixths, Michelle I. stated: “Five-sixths divided by four-fifths, no, divided by one-sixth. What is that? Wouldn’t you, like, switch those around? Five times six over six. It works!” (A3, 1217-1220). Michelle I. then stated the rule verbally to Magdalena:

Look let’s say it’s was with five-sixths, well to make it whole, let’s just say you do the thing to make it whole, divided by one-sixth. And then that’s thirty over six, which is five, which is the answer. But now we have to realize how to write it. (A3, 1227-1231)

T1 acknowledged that they have the same problem with trying to write the rule using triangle.

Bobby, who had been working, showed a new inverse rule (see Figure 58) (A3, 1239-1242).
As they worked with T1, Michelle I. and Magdalena, realized that the inverse rule should be triangle over something. Michelle I. stated that the denominator should be triangle minus one and that the sign should change from negative to positive. She wrote a new inverse rule on her paper (see Figure 59) (A3, 1256-1262).

T1 reminded them that no mathematical rule permitted the switching of signs, as they had noted; consequently they had to state this in their rule in some manner (A3, 1264-1269). Michelle I. borrowed Bobby’s paper for a second, but he retrieved it (A3, 1270-1271).

To R3’s question as to their progress, Bobby answered that they had several versions of the inverse rule (A3, 1283-1284). Pointing to Bobby’s rule, R3 asked if it worked (A3, 1288-1289). He replied affirmatively and, working it out for triangle value of one-half (see Figure 60), commented that: “Yeah. It works. A half equals negative, a half equals negative one times negative one equals one” (A3, 1295-1296).
Michelle I. noticed the similarity of Bobby’s rule to hers and Magdalena’s (A3, 1294). Michelle I. adjusted their rule by changing the triangle in the numerator to a negative triangle and stated: “And times it by a negative one. It’s this. Negative triangle over triangle minus one equals square” (A3, 1301-1302). R3 asked Bobby how he had gotten the rule (A3, 1307). He replied the he had changed the square to a triangle and then subtracted “something” (A3, 1308-1311). Michelle I. and Magdalena commented that although their rule was the same as Bobby’s, they derived it on their own (A3, 1313-1314). Bobby responded “And like the other problem where they times-ed it by a negative one to get a positive, well I did that” (A3, 1316-1317).

5.4.2.2 Stephanie’s group finds inverse.

After writing the rule on her paper, Romina questioned whether “Triangle over triangle minus one?” or “Box over triangle minus one?” were the inverse (A3, 844). Ankur initially stated: “The top one is box, that’s all I know” (A3, 849) and Romina followed with the comment: “Then wouldn’t the bottom be triangle minus one?” (A3, 850). Stephanie chose a triangle value of zero to calculate the box value using the inverse rule triangle over triangle minus one: “Let’s see. Box, wait, triangle over triangle minus one. If the triangle is zero, zero divided by zero equals zero” (A3, 851-852). Then Stephanie correctly calculated box when triangle equals zero: “Hold on. Can I? Triangle
over triangle. If the triangle is zero and the box is zero because it’s zero and zero. Zero minus one equals negative one. And then zero times anything is zero. So it equals zero” (A3, 860-863). Ankur asked if they should write the inverse out (A3, 864), at which point Romina queried: “Now you think it’s this minus one?” (A3, 866); she proceeded to write on her paper, \( box \ equals \ triangle \ over \ triangle \ minus \ one \) (see Figure 61) (A3, 870).

![Figure 61. Romina’s inverse rule.](image)

Encouraging the students not to be concerned if their initial procedure failed to result in the correct box value, R3 instructed them to test their inverse rule for triangle equal to one-half, two-thirds and three-fourths (A3, 904-909). Romina had difficulty calculating box when triangle equaled one-half, so R3 worked with all at the table to calculate box. Their interchange follows:

<table>
<thead>
<tr>
<th>Romina</th>
<th>That would be one-half over one-half minus one. That would be negative one and one-half, and that wouldn’t come out.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Researcher 3</td>
<td>No, you have one-half and this down here, one-half minus one is what?</td>
</tr>
<tr>
<td>Romina</td>
<td>Negative one and one-half?</td>
</tr>
<tr>
<td>Researcher 3</td>
<td>Negative one-half, okay, so you get negative one-half, one-half over negative one-half, and what does that equal?</td>
</tr>
<tr>
<td>Stephanie</td>
<td>So we’re dividing a negative one-half by a positive one-half. So it’s a negative, isn’t it?</td>
</tr>
<tr>
<td>Ankur</td>
<td>Equals one over…no.</td>
</tr>
<tr>
<td>Researcher 3</td>
<td>What’s one-half over a negative one-half?</td>
</tr>
<tr>
<td>Ankur</td>
<td>One-half over negative one-half. One.</td>
</tr>
<tr>
<td>Researcher 3</td>
<td>One?</td>
</tr>
<tr>
<td>Ankur</td>
<td>One-half over negative one-half. One. Zero.</td>
</tr>
<tr>
<td>Jeff</td>
<td>One, zero?</td>
</tr>
<tr>
<td>Researcher 3</td>
<td>Keep trying. You’ll hit it eventually (he laughs).</td>
</tr>
<tr>
<td>Ankur</td>
<td>Negative one?</td>
</tr>
</tbody>
</table>
Researchers 3

Stephanie wrote the calculation on her paper (see Figure 62).

![Figure 62](image)

Figure 62. Stephanie’s calculation of $\frac{1}{2}$ divided by $-\frac{1}{2}$.

For several minutes, R3 assisted Stephanie’s table with division by a negative number using examples such as one divided by minus one, two over minus one, and one-half over minus one-half (A3, 939-993).

Then R3 suggested that they calculate box when triangle is two-thirds (A3, 999-1001). At first, Stephanie said, “Wouldn’t it just be negative two-thirds?” (A3, 1012). Instead of providing an explicit answer, R3 guided the students towards a solution and asked: “What’s two-thirds minus one?” (A3, 1013). Brian replied: “Negative one-third” (A3, 1014) to which Ankur responded, “The answer is negative two” (A3, 1015). R3 moved to the next phase of the calculation and asked what is two-thirds over negative one-third (A3, 1020-1021), to which Ankur correctly replied: “Two-thirds over negative one-third. Two. Negative two” (A3, 1022).
Stephanie, Romina and Ankur, in trying to get the correct inverse rule, wanted to add one instead of subtracting one (A3, 1028-1033). R3 wanted the students to see where and how their calculations were awry. Their interchange follows:

- Researcher 3: What’s wrong, you have to look at what’s wrong with the answer and then figure out how to change your formula to make the answer right.
- Brian: It’s not supposed to be negative.
- Researcher 3: Okay, then how do you make something that’s negative, positive?
- Romina: The last problem we just did, we multiplied by negative one.
- Stephanie: All right. We’ll, try it.
- Romina: Multiply this by negative one. It’s positive two.
- Ankur: All right. Could we do that to every problem?
- Romina: I don’t see why not. See, triangle over triangle minus one times negative one. (A3, 1041-1051)

The students discussed the inverse rule Romina had written on her paper (see Figure 63) (A3, 1057). Ankur asked if it should be simplified but Stephanie and Romina wanted to leave it as is (A3, 1052-1054).

![Figure 63. Romina’s revised inverse rule (1st version).](image)

T1 asked Romina to explain the rule they had developed. She explained that they needed to multiply the final answer, not just the denominator, by negative one to get the correct box value (A3, 1095-1108). Romina’s paper showed the inverse rule (see Figure 64).
T1 suggested that they clean up the inverse rule (A3, 1118-1120) since it is not doing what they said it was doing. Romina proposed *triangle over triangle minus one equals square times negative one* (A3, 1121-1122) and wrote that on her paper (see Figure 65).

**Figure 65.** Romina’s adjusted inverse rule.

T1 pointed out that square should be isolated in order for all the action to take place on the left side of the equation (A3, 1130-1132). Stephanie offered a change to the rule: “Why don’t we do triangle over triangle minus one, and then the whole thing negative one, instead of just the denominator” (A3, 1133-1134). Ankur asked if there was a way to simplify the rule (A3, 1138), and Romina replied that they could write *triangle over triangle minus one* and multiply the whole thing times negative one. She then wrote on it on her paper (see Figure 66) (A3, 1153-1157).
T1 asked them to test it and if possible simplify their rule (A3, 1159, 1166). Stephanie and Romina, however, did not want to change it (A3, 1167-1168) but suggested that Ankur try to do so (A3, 1179-1182). Ankur said he had asked if they could simplify it, not that he wanted to do it (A3, 1184).

When the camera panned back to their table several minutes later, R3 asked them if they had something that worked (A3, 1324). Romina replied: “It’s now basically this. Negative triangle over triangle minus one equals square” (see Figure 67) (A3, 1326-1327).

R3 asked if the inverse worked (A3, 1332). While calculating box when triangle is thirteen-fourteenths, Ankur encountered difficulty with the negative thirteen-fourteenths divided by negative one-fourteenth. T1 instructed him to work out the division of these fractions (A3, 1345-1355), and he wrote the calculation on his paper (see Figure 68) (A3, 1356-1360).
Others at the table were unsure if the result was thirteen over one or fourteen over one.
To answer the question, Romina performed the calculation on Ankur’s paper (see Figure 69), obtaining a result of thirteen, and then she subtracted one to get twelve. (A3, 1379-1380). T1 questioned why she subtracted one (A3, 1381), to which she replied that she did not know. Ankur’s calculated the result to be thirteen over one (A3, 1383).

![Figure 68. Ankur’s calculation for triangle equal thirteen-fourteenths.](image)

**5.4.2.3 Michael and Angela’s inverse rule.**

The camera captured little of the work at Jeff’s table. After testing their rule for triangle equal to three-fourths (see Figure 70), six-sevenths and four-fifths, respectively, Michael and Angela established the inverse rule: *one over one minus triangle minus one equals box*. Michael had written the rule on his paper with the work for triangle equal to three-fourths (see Figure 70) (A3, 1387-1393).
5.4.2.4 Presenting inverses.

Michelle I. and Magdalena presented their inverse rule (see Figure 71).

R3 asked the class to comment on their inverse rule (A3, 1433), to which Ankur and Brian stated that they have a better one and it is simplified (A3, 1434-1435). They presented their inverse rule to the class (see Figure 72).

In response to R3’s question of the difference between the two inverse rules (A3, 1451-1453), Michelle I. stated: “Instead of times-ing it by negative one they put the negative triangle, so they would have to, like, do that, but it is the same thing” (A3, 1454-1456).
Angela, Sarah and Michael presented their inverse rule as R3 introduced it as something completely different (see Figure 73) (A3, 1458). T1 asked for a demonstration of their rule and R3 agreed to let Angela use a triangle value of four-fifths (A3, 1473-1474).

![Image](image.png)

Figure 73. Angela, Michael, and Sarah’s inverse rule.

5.4.2.5 Summary of day three activities.

On day 3, students, led by the classroom teacher, T1, worked to find function and inverse rules for two t-charts containing integers and fractions. Students were successful in finding the function rule for the first t-chart. Then T1 redirected them to find the inverse rule for a t-chart worked on the previous day for which they were unsuccessful in finding the inverse. Resolving difficulties with integer and fraction arithmetic and guided by R3, students were able to write the inverse rule for the second t-chart.
Chapter 6 Conclusions

This chapter, organized into four sections, addresses the following research questions:

1. What evidence, if any, exists that students understand the idea of function?
2. How do students build the idea of inverse function?
   - What obstacles, if any, do they encounter?
   - If obstacles are encountered, how are they overcome?
3. What evidence exists, if any, that students understand the idea of inverse function?

The first section shows that students understood the idea of function by creating function rules for data presented as t-charts. Difficulties students experienced as they formulated these rules are also discussed. Through analyses of students’ work, the second section shows how students constructed inverse function rules. As in section one, the obstacles students encountered when creating inverse rules, and their methods for overcoming them, are described. The third section presents evidence that students understood the idea of inverse. The chapter concludes with a fourth section discussing the limitations of this study.

6.1 Understanding Function

6.1.1 Function tables.

One of the earliest forms of function, a table of values, was utilized in the sessions investigated in this study. The data presented to the students in the form of a t-chart or function table (see Appendix F) required that they focus on the relationship between the two columns. Students had to identify patterns, to coordinate patterns with actions, and to use this coordination to represent the identified pattern in a relationship (Smith, 2008).
Ultimately, students represented the relationships as rules for both functions and inverse functions.

Students were presented with t-charts with the left column labeled ‘□’ and the right column labeled ‘△’ (see Appendix F). Although there was little recall of working with Dr. Davis, as mentioned in section 5.2, they were, nevertheless, familiar with the tasks presented to them. From previous experience with t-charts during the “Guess My Rule” sessions with Davis (Giordano, 2008), students knew that they had to create a relationship between the box and the triangle. They were familiar with the form of the equation needed to represent the t-chart: an equation with triangle on one side is equal to an expression using box on the other side.

Students found function rules using the methods coded in table 3. Table 4 lists each function rule with the methods employed in the process of their creation.

Table 3

Methods used for function rules

<table>
<thead>
<tr>
<th>Activity</th>
<th>Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>Guess and Check</td>
<td>FG</td>
</tr>
<tr>
<td>Pattern recognition of triangle values or finding slope</td>
<td>FPT</td>
</tr>
<tr>
<td>Pattern recognition of box values</td>
<td>FPB</td>
</tr>
<tr>
<td>Finding y-intercept (the plus number)</td>
<td>FY</td>
</tr>
<tr>
<td>Verbal description</td>
<td>FV</td>
</tr>
<tr>
<td>Finding patterns between box and triangle for each row</td>
<td>FP</td>
</tr>
<tr>
<td>Other</td>
<td>FO</td>
</tr>
</tbody>
</table>
Table 4

Methods used to find each function rule

<table>
<thead>
<tr>
<th>Function Rule, △ =</th>
<th>Method(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 □ + 3</td>
<td>N/A</td>
</tr>
<tr>
<td></td>
<td>Rule provided by R1</td>
</tr>
<tr>
<td>2 □ + 3</td>
<td>FP</td>
</tr>
<tr>
<td>3 □ x 4</td>
<td>FP</td>
</tr>
<tr>
<td>4 □ x 4 + 1</td>
<td>FPT, FPB</td>
</tr>
<tr>
<td>5 □ x 2 + 6</td>
<td>FP, FPT, FG</td>
</tr>
<tr>
<td>6 □^2</td>
<td>FP</td>
</tr>
<tr>
<td>7 □ + 2</td>
<td>FP</td>
</tr>
<tr>
<td>8 □ + 1</td>
<td>FP</td>
</tr>
<tr>
<td>9 1 - □</td>
<td>FP, FO</td>
</tr>
</tbody>
</table>

6.1.2 Finding patterns down columns.

According to Vergnaud’s (1996) scalar theorem-in-action, students often work down a column in a function table and calculate a value based on the previous number in the column utilizing a procedure known as recursion. The participants often used recursion to find the next triangle value or next box value in a column. While working on the t-chart for the rule times four plus one, Bobby, in 5.2.4.1, recognized that the triangle column went down by four, and the box column went up by one. Like Bobby, Angela in 5.2.4.2 saw that, on the triangle side, the value increased by four. When working on the rule for times two plus six, through recursion, Angela in 5.2.5.2 stated that the triangle values increased by two.
After finding patterns through recursion, students continued their work to create function rules. No evidence indicated that students thought they had completed their tasks. Michelle I. stated more than once during these sessions that they had found a pattern and now needed to find a rule (B1A, 268, B1A, 402, 404). Magdalena, working on day 2, stated that her group had found a pattern, not a rule (B2, 421).

Recursion was not employed by students to find rules for the t-charts presented on day 2 and on day 3. A possible reason for its absence could have been that the triangle values in the t-charts presented (see Appendix F) contained both integer values and fractions, which may have made pattern recognition more difficult.

6.1.3 Finding patterns across rows.

In 5.2.1, Michelle I. found the differences between the box and triangle for each row but did not write the rule. Michael and Angela, in 5.2.1.2, noticed the differences between box and triangle. Michael said the rule was “box times one third” and wrote it in section 5.2.3 as box times one-third equals triangle (J1A, 185-188). In 5.2.1.3, Ankur observed that each number represented one-third, and he related each box value to its corresponding triangle value, using the relationship of one-third (S1A, 103-106).

In 5.2.2, Bobby, Amy-Lynn, Michelle I. and Angela found the rule for triangle equals box times four quite easily. They were able to see the relationship between the box and triangle values in the t-chart that R1 had presented to them; and the rule consisted of one operation.

In 5.2.5.1, recognizing the differences in each row between box and triangle allowed Bobby to state the rule as times two plus six. For the same t-chart, in 5.2.5.2, Jeff noticed that the box plus a number equals triangle and that for each subsequent row, the number
increases by one. Angela reminded him that they needed to find one rule that would work for each row. Jeff added that he would find the rule eventually. Michael found the rule, but, when Jeff asked him how he had found it, he replied that he did not know. Michael was not able to explain his findings, which implied that he had guessed or that his experience with similar tasks had given him the knowledge to find the rule.

In 5.2.6, R1 presented the only quadratic rule to the class. Amy-Lynn and Jeff recognized it quickly as times its number or box times box equals triangle. Michael simplified the rule to box squared. Again, students recognized the relationship between the box and triangle values across rows.

Students worked across the rows of a table to find a pattern, translating a pattern to a relationship and then to a rule. The ability to find a pattern and to generate a rule indicated that the students understood function as action (Breidenbach et al., 1992). Students generalized by writing a rule symbolically, using box and triangle. This was a task familiar to them since they had worked with t-charts, two years earlier with Davis (Giordano, 2008). Once they found patterns they continued working to create rules. They used one or more operations to create relationships between the box and the triangle. They had experienced similar tasks and knew how to go about finding rules. This was evidence of an assimilation paradigm (Davis, 1984, pp. 314-315).

The students started with a table of values representing the action concept, created a symbolic representation, and generalized from it. Students described the function using a rule, which represented the same transformation as that of the table of values. Students were able to describe, reflect, and reverse the steps of a transformation which indicated
that they had been able to move from an action to a process understanding of function (Reed, 2007).

### 6.1.4 Writing their own rules.

In 5.2.8, R1 asked the class to write a rule that could be written in two ways. In this exercise, the students were able to demonstrate their functional thinking by creating rules without the presence of a table of values. First, they wrote rules and then substituted box values to calculate triangle values. These values were then placed in a t-chart which would be presented to others so as to find a rule. That students were able to write rules representing the relationship between two quantities demonstrated their understanding of the process concept.

In 5.2.8.1, Michelle I. and Magdalena did not place a zero as a box value since they felt that it would reveal their rule, thus making it obvious (B1B, 147-151, 174). This suggested that they knew a relationship existed between the box when zero and triangle value. The triangle value is the plus number or the y-intercept when the box is zero. Their rule was box squared plus negative thirteen equals triangle. In 5.2.8.1, Bobby verbalized a physics problem as a rule using \( mc^2 \) and the speed of a car which implied his understanding of function as process.

In 5.2.8.2, Jeff jotted down the rule box plus one equals triangle, and, in 5.2.8.3, Romina and Ankur created the rule and table of values for box plus seven equals triangle.

On day 1, the students demonstrated their ability to write rules for the function tables that had been presented to them. These students, while in the sixth grade, had been participants in similar activities, conducted by Davis. Those paradigm-creating lessons helped them to acquire facility in interpreting and writing rules. Their work also showed
that their understanding of the action concept was essential to their understanding of function and to their gradual understanding of the process concept.

### 6.1.5 Function tables with fractions.

The more challenging function tables, on which students spent more time, were presented on the second and third day. The t-chart on day 2 (see Appendix F), representing a non-linear function, contained triangle values of which all were fractions. In 5.3.1, Bobby noticed that the numerator of the triangle values increased by one and that the denominator of the triangle was one more than the numerator. Bobby’s group quickly recognized that the missing box value was 11. In 5.3.3, Michael and Angela worked to find the same rule, and in 5.3.5, Stephanie’s group also found the rule. All students realized that the numerator in the triangle value was equal to the box value for every row and that the denominator was one more than the numerator. Using the relationship between box values and corresponding triangle values, they wrote the rule as $\text{triangle equals box over box plus one}$.

The function table on day 3 (see Appendix F) consisted of integers and fractions for both box and triangle values. This array of values, however, posed additional challenges for the students.

In 5.4.1, Bobby found a relationship between box and triangle values. He converted the integers in the box column to fractions by placing them over one. Breaking up both box and triangle values into numerator and denominator, he and others at his table wrote the rule as follows: the denominator of the box value minus the numerator of the box value equals the numerator of the triangle and the denominator of the triangle is the same.
as that of the box value. This was expressed verbally and then symbolically using triangle, box and circle.

Noticing that the sign of the triangle would be the opposite, in 5.4.1.2, Romina wrote the rule as *box minus one* and then said, “and switch from positive to negative or negative to positive”. Ankur recognized that this would be accomplished by multiplying *box minus one* by *negative one*: \((\Box - 1) \times -1 = \triangle\). In 5.4.1.3, Michael and Angela had written their rule as *box minus one equals triangle*, but they included multiplication by negative one when they presented it to the class on the overhead.

### 6.1.6 Obstacles to finding function rules.

Students were not certain of rules for multiplication of integers. A discrepancy occurred at Bobby’s table when working on the rule *box squared*. In 5.2.6.3, Michelle I. stated that she thought the rule, *triangle equals box squared*, was wrong because negative three times negative three equals negative nine. She explained that she originally thought that negative three times negative three equals nine but that those at her table told her that she was incorrect. She and Amy-Lynn resolved the conflict and agreed that negative three times negative three equals nine.

Students had difficulty in performing calculations with fractions. In 5.2.7.2, R1 presented the rule *quantity three times box plus six divided by three equals triangle* (see Figure 74) and asked the class if that worked for the table of values representing the rule *box plus two equals triangle*.

\[
\frac{3 \times \Box + 6}{3} = \triangle
\]

*Figure 74. R1’s rule for box plus two equals triangle.*
Bobby calculated the triangle value to be four when box is negative two. He multiplied three by negative two and got negative six. Then he stated “… negative six plus six is twelve” (B1A, 679-680). This indicated that he made an error in integer arithmetic. Herscovics and Linchevski (1994) attributed this error to students’ ignoring the minus sign preceding the number six, and they found this omission to be a common occurrence in their study of seventh-graders’ algebra potential. Magdalena calculated negative three divided by three to be zero. Amy-Lynn, Michelle I. and Magdalena calculated a triangle of zero when box was negative three. Bobby finally made the correction and calculated three divided by three to be one and not zero.

Difficulty with fractions and signed numbers is detectable on day 3, in 5.4.1.2, when Romina tests the function rule, 

\[(\square - 1) \times -1 = \Delta,\]

by substituting one-half for box. She incorrectly stated that one-half minus one was negative one and one-half. Ankur corrected her calculation to be negative one-half. At this point, Romina continued with the calculation, multiplying negative one-half by negative one; her result was one and one-half or zero. Then Ankur reiterated the calculation as “half minus one equals negative half”, and Jeff completed the calculation: “negative half times negative one makes it a positive half”. After making the corrections, all at Stephanie’s table confirmed that the inverse rule worked.

Some students had difficulty with arithmetic when box was a fraction. Angela and Michael, in 5.4.1.3, explained to the class how their rule, (i.e., \([\square - 1] \times -1 = \Delta)\) worked when box was one-half. Magdalena did not follow their explanation, so Angela suggested that they calculate triangle for box equal to two. T1 clarified the rule by adding parentheses to the rule \((\square - 1) (-1) = \Delta,\) and then Angela reworked the calculation for
box equal to one-half. Amy-Lynn then suggested that they calculate triangle when box is negative one, and Angela calculated triangle to be zero by \((^-1 - 1) = 0\), then \(0 \times (-1)\).

Angela’s error in ignoring the minus sign preceding the number supports the Herscovics and Linchevski claim. T1 asked the class if there was any problem with reasoning here and, in 5.4.1.3, Michelle I. responded, “Isn’t negative one minus one negative two?” (A3, 510). Angela subsequently corrected her calculation of triangle: \((^-1 - 1) = -2\), then \(-2 \times -1 = 2\) (A3, 520).

6.1.7 Summary of understanding function.

The t-charts under discussion facilitated the understanding of the action concept of function. Students understood function as action and demonstrated that their understanding had progressed to the process concept. Clearly, these students had engaged in functional thinking (Smith, 2008), as evidenced by their focus on relationships between two quantities and, as Breidenbach et al. (1992) and Davis (1992b) had postulated, by their ability to create rules, symbolically, to represent the relationships. Moreover, they capably formulated their own rules without t-charts, further evidencing their understanding of function as process.

Because students participated in various activities with Davis in the longitudinal study (Spang, 2009; Giordano, 2008), they were able to build upon their knowledge to find patterns and relationships between two quantities and to formulate function rules.

6.2 Building the Idea of Inverse

For the t-tables presented to them on day one (see Table 5), students were able to calculate inverse values and to find inverse rules. The inverse rules, it is important to
mention, were not always written symbolically. Students performed mental calculations for the simpler, earlier tasks and were able to reverse calculations, using reciprocals for many of the function rules. This meant that the understanding of function was progressing from action to process.

Table 5

*Function and inverse rules*

<table>
<thead>
<tr>
<th>Function Rule, △ =</th>
<th>Inverse rule, □ =</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 □ + 3</td>
<td>△ - 3</td>
</tr>
<tr>
<td>2 □/3</td>
<td>△ x 3</td>
</tr>
<tr>
<td>3 □ x 4</td>
<td>△/4</td>
</tr>
<tr>
<td>4 □ x 4 + 1</td>
<td>△ - 1/4</td>
</tr>
<tr>
<td>5 □ x 2 + 6</td>
<td>△ - 6/2</td>
</tr>
<tr>
<td>6 □²</td>
<td>√△</td>
</tr>
<tr>
<td>7 □ + 2</td>
<td>△/2</td>
</tr>
<tr>
<td>8 □/□ + 1</td>
<td>△/1 - △</td>
</tr>
<tr>
<td>9 1 - □</td>
<td>1 - △</td>
</tr>
</tbody>
</table>

6.2.1 Using inverse operations.

As indicated in section 5.2, R1 began the session on the first day by presenting students with the rule *box plus 3 equals triangle* and a t-chart with corresponding values. R1 asked students to find the box value when the triangle value was ten. To this request, Brian replied “seven” and explained that ten minus three is seven. He demonstrated his ability to calculate the inverse of the rule using the inverse of addition. Then R1 asked the
class for a term that could be used for the inverse of five squared, to which Ankur responded, square root. Further questioning confirmed for R1 that students knew what inverse operations were.

In 5.2.2, R1 asked what the box value would be if the triangle were 60 for the rule \( \text{box times four equals triangle} \). Bobby replied 15 and recalled how, using the inverse of multiplication, he had calculated it by dividing 60 by four. Table 6 lists the different methods used by students when creating inverse rules.

Table 6

Methods used for inverse rules

<table>
<thead>
<tr>
<th>Activity</th>
<th>Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>Guess and Check</td>
<td>IG</td>
</tr>
<tr>
<td>Pattern recognition of box values</td>
<td>IPB</td>
</tr>
<tr>
<td>Finding y-intercept (the plus number)</td>
<td>IY</td>
</tr>
<tr>
<td>Verbal description</td>
<td>IV</td>
</tr>
<tr>
<td>Finding differences between triangle and box for each row</td>
<td>ID</td>
</tr>
<tr>
<td>Using inverse operation</td>
<td>IO</td>
</tr>
</tbody>
</table>

In 5.2.3, students found the rule, \( \text{triangle equals box times one-third} \), and Amy-Lynn responded to R1’s question as to what the box would be if the triangle were two. Without writing the inverse rule, she calculated the box value using mental math by simply multiplying two by three.

In 5.2.4.4, R1 asked what the box value would be if the triangle value was 49 for the rule \( \text{box times four plus one equals triangle} \). Michelle I. replied it was 24; and Bobby, that it was twelve. At this point, Michelle I. realized her mistake. Angela wrote the
inverse as triangle minus one divided by four equals box, $\triangle - 1 \div 4 = \Box$, an equation she had derived by using the inverse of the operations utilized in the function rule. She explained her calculation: subtract one from forty-nine and then divide by four to get twelve. R1 rewrote the inverse rule as a fraction, with triangle minus one as the numerator and four in the denominator. There was no mention, however, of the order of operations in Angela’s inverse rule. Angela calculated the inverse using her rule and, rather than divide first, performed the subtraction. To change the order of operations, her rule should have contained parentheses: $(\triangle - 1) \div 4 = \Box$.

In 5.2.5.3, Magdalena presented the rule box times two plus six equals triangle and, in response to R1’s request to provide a number in the triangle, wrote 56 in the triangle column. Michael found the box value of 25 and wrote the inverse rule, with box equal to the fraction: triangle minus six in the numerator and two in the denominator. This further demonstrated that, to create the inverse rule, students used the inverse of the operations in the function rule.

In 5.2.6.5, R1 asked about the inverse rule for box squared equals triangle, and Angela replied that it is the square root of the triangle equals box. A discussion of the square root of a negative number led to the decision that they should not use a negative number for the triangle.

In 5.2.7.1, Michelle I. wrote the rule box plus two equals triangle and the inverse rule as triangle minus two equals box. As in the previous work, this showed that the inverse operation was utilized in the inverse rule. Table 7 lists the methods employed when finding the inverse rule for each function rule.
In 5.2.8.2, R1 asked the class to write a rule in two different ways. Jeff quickly created a rule (box plus one equals triangle) and its inverse (triangle minus one equals box), but Angela dismissed it as being too easy. Jeff defended his rule as being very simple and showed how the inverse pairs worked (J1B, 442-444). The other students worked to create rules but did not work on the inverse rules.

Table 7

Inverse rules and methods

<table>
<thead>
<tr>
<th>Inverse rule, □ =</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 △ - 3</td>
<td>IV, IO</td>
</tr>
<tr>
<td>2 △ x 3</td>
<td>IV, IO</td>
</tr>
<tr>
<td>3 △ / 4</td>
<td>IV, IO</td>
</tr>
<tr>
<td>4 △ - 1/4</td>
<td>IV, IO</td>
</tr>
<tr>
<td>5 △ - 6/2</td>
<td>IV, IO</td>
</tr>
<tr>
<td>6 △ / 2</td>
<td>IV, IO</td>
</tr>
<tr>
<td>7 △ / 1 - △</td>
<td>IG, IO</td>
</tr>
<tr>
<td>9 1 + △</td>
<td>IG</td>
</tr>
</tbody>
</table>

6.2.2 Adjusting inverse rules.

For the function rule, triangle equals box over box plus one, Magdalena, in 5.3.2, on day 2, tried the inverse rule box minus one divided by box. She probably meant to write the inverse using triangle which would have been box equals triangle minus one divided...
by triangle. When she substituted one-half for the triangle value, however, she did not get
the expected value of one for box. Then she stated that the inverse should contain
multiplication instead of division.

Amy-Lynn proposed box equals triangle over triangle minus one and, calculating box
when triangle is one-half, derived a box value of negative one, not one as expected. She
stated she was getting close to a solution and revised her inverse rule to be box equals
triangle times triangle plus one. Using a triangle value of two-thirds, she tested the
inverse rule. Multiplying two-thirds times two-thirds, she arrived at four thirds, to which
she added one for a box value of two and one-third. Her calculation, however, was
incorrect since box should have been one and four-ninths proving, as well, that the
inverse rule was not correct.

Magdalena and Michelle I. revised Amy-Lynn’s inverse, writing it as:

□ = △ x (□ + 1). This rule worked for the t-chart values of box and triangle, but it did
not represent the inverse rule. As R3 pointed out, their rule used triangle and box on the
right side, meaning that they had two things on the right side. Since the previous inverse
rule had the correct notation, the latter result was unanticipated.

Michael and Angela, in 5.3.4, wrote the inverse as box equals triangle times triangle
minus one and tested it with triangle equal to one-half. Their inverse rule contained
inverse operations of the function rule. Calculating one-half times one-half equals one-
fourth, they decided it did not work, for the box value should have been one.

Consequently, they revised their inverse rule to be box equals triangle times the
denominator of the triangle. For a triangle value of three-fourths, the box value was
three-fourths times four or three. Although their inverse rule worked, R1 wanted them to
express it without referring to the denominator of the triangle and that they should be able to write the rule using only triangles. Though they used inverse operations, they failed to express the inverse strictly in terms of the triangle.

Romina and Ankur, in 5.3.5, tried to find the inverse rule and, with some help from R3, knew that it should contain the inverse operation of subtraction. Others at Stephanie’s table had some difficulty finding the inverse and wrote the rule \( \text{box equals box over box minus one} \). As the session came to a close, R3 reminded the students that they could not have box on both sides. Their continued attempts to find the inverse using inverse operations were unsuccessful.

In 5.4.1.3, on day 3, Angela found the inverse rule for the function rule,

\[(□-1) x -1 = Δ, \text{ as } \frac{Δ}{-1} + 1 = □.\]

Since one camera was filming on day 3, it omitted Angela’s work and all of the work at Jeff’s table while they were working on the first t-chart. Angela found the inverse rule; but, in this session, no discussion of the inverse rule ensued for this t-chart.

6.2.3 Multiplying by negative one.

On day 3, after having worked on the rule for the first t-chart, T1 redirected the class to the t-chart of the previous day and asked for the inverse expression for the rule, \( \text{triangle equals box over box plus one} \). It so happened that students had failed, on the previous day, to find the inverse for this data.

In 5.4.2.1, Bobby tried the inverse rule, \( Δ = Δ \times (□+1) \), for a triangle value of six-sevenths and a box value of seven. Although he employed the inverse operation of multiplication, he failed to state the inverse as \( \text{box equals} \). Aware of this, Bobby revised
his rule (see Figure 75). He explained that he had changed the square to a triangle and then subtracted “something” to get his revised rule. Despite the vagueness of his explanation, Bobby correctly calculated the box to be one, using a triangle value of one-half.

\[
\frac{\Delta}{\Delta - 1} \times (-1) = \Box
\]

*Figure 75. Bobby’s revised inverse rule.*

Michelle I. and Magdalena wrote their inverse (see Figure 76) and pointed to a switching of signs for the box value.

\[
\frac{\Delta}{(\Delta - 1)} = \pm \Box
\]

*Figure 76. Michelle I.’s new inverse rule showing change of sign.*

T1 reminded them that there was no mathematical rule permitting them to write their inverse using a switching of signs. Borrowing Bobby’s paper for some help, Michelle I. changed the triangle in the numerator to a negative triangle, indicating that she multiplied the right side by negative one (see Figure 77).

\[
\frac{-\Delta}{(\Delta - 1)} = \Box
\]

*Figure 77. Michelle I.’s revised inverse rule showing negative triangle.*

Romina had two possible inverse rules in mind, both of which used the inverse of addition in the function rule. In 5.4.2.2, she considered *box equals triangle over triangle minus one* and *box equals box over triangle minus one.* Ankur wanted the top to be box. Using the inverse rule *triangle over triangle minus one,* Stephanie substituted a triangle
value of zero and initially calculated box to be zero divided by zero. She then corrected her calculation to be zero over minus one equals zero. Students were encouraged by R3 to test their inverse rule with different values for triangle.

Difficulty arose while working with fractions. Romina, for example, had difficulty calculating the box value when triangle was one-half. Ankur and Jeff were also unsure. Stephanie resolved the problem by dividing one-half by negative one-half (which is equal to one-half times negative two). Then she calculated negative two over two to be negative one.

R3 pointed out to Stephanie’s group that their inverse rule did not work and asked what was wrong with the answers that they had obtained. Brian realized that their answers were negative and that the box values should be positive. R3 asked them how they could change a number from negative to positive, to which Romina answered: multiply by negative one. Romina revised her rule, accordingly, to be box equals triangle over triangle minus one times negative one. After working a few minutes on their inverse rule, the students had a final version: box equals negative triangle over triangle minus one. This was the same as Michelle I.’s revised inverse rule (see Figure 77).

In 5.4.2.3, Michael and Angela had written the inverse rule as one over one minus triangle minus one equals box (see Figure 78).

\[
\frac{1}{(1 - \triangle)} - 1 = \square
\]

*Figure 78. Michael and Angela's inverse rule.*

Students, demonstrating that they were able to find the inverse rules by reversing the function rules, attained the process understanding of function as noted by Breidenbach et
al. (1992). Students utilized inverse operations when creating inverse rules, tested their inverse rules with different triangle values, and made adjustments as needed.

6.2.4 Obstacles to finding inverse rules.

Students had difficulty with integer arithmetic and division of fractions, as discussed above, when they worked on finding function rules and inverse rules. When Brian was asked what the square root of -36 was, he was unsure if the square root would be negative or positive. Bobby’s table clarified the question, stating that you cannot take the square root of a negative number.

In 5.3.2, Amy-Lynn tested out her inverse rule (*triangle times triangle plus one*), multiplying two-thirds by two-thirds to get four-thirds, instead of four-ninths. No one corrected her error, however, and those at her table proceeded to adjust her inverse rule.

Romina, in 5.4.2.2, had difficulty calculating triangle when box was one-half in the inverse rule, *triangle over triangle minus one*. R3 worked with Stephanie’s group to test their inverse rule. Romina stated that one-half minus one was negative one and one-half. R3 corrected the calculation to be negative one-half and then asked the group to calculate one-half over negative one-half. Stephanie responded that the result would be negative, and Ankur stated that the result was one. After R3 asked the question, what is one-half over negative one-half again, Ankur replied one or zero. R3 asked them to choose one, to which Stephanie replied “negative one”.

Then R3 guided the group to calculate box when triangle was two-thirds and they were successful in performing the calculation using fractions.
6.3 Understanding Inverse

Students demonstrated their understanding of inverse by creating rules using the inverse of the function operations. When given the triangle values in t-charts, they knew how to reverse the calculations to arrive at the box values; and, when an inverse rule did not work, they made adjustments. Their understanding of inverse progressed from the action to the process concept.

Initially, as shown in 6.2.1, students calculated box from triangle values in the t-charts by reversing operations in the function rule. Using the inverse of operations presented in the function rules, they created inverse rules. In 6.2.2, they adjusted their rules by replacing symbols and multiplying by negative one. These results confirmed the claims of Breidenbach et al. (1992) and Davis (1992b) that students built and then transformed mental representations; as a result, students formulated new concepts.

6.4 Limitations of Study

This study was limited, at the outset, because some student work was omitted by the videographers. On days 1 and 2, for example, one camera for each group attempted to record students’ work, but, at times, the video was either obscure or incomplete. Undoubtedly, these technical oversights inhibited the interpretation of the learning process to some extent. Gaps were evident, for instance, on day 3, when one roaming camera did not capture the work of Michael and Angela who were working at Jeff’s table. Though they were successful in finding function rules and inverse rules, the process through which these results were obtained could not be analyzed. Similarly, on day 3, since only one camera was in use, only some of the students’ work was captured. Had
these technical anomalies not been present, an understanding of their work may have been more revealing.

A third factor complicating interpretation was the unavailability of researchers’ notes and students’ written work. The analysis, therefore, was restricted to videotaped footage, portions of which, as mentioned above, were not entirely visible.
Chapter 7 Implications

7.1 Early Algebra

The results reported in this research indicate that students benefited from exposure to algebra in the seventh grade. Not only did they comprehend the concept of function, but they were also able to understand, and to build upon, the concept of inverse function; moreover, the students could connect the relationship between the concept of function and its inverse.

The Connections Standard of the Principles and Standards for School Mathematics recommends that instructional programs prepare students to be familiar with, and to use, connections between mathematical ideas. It further states that students, having interconnected mathematical ideas, should work to build these concepts into coherent wholes (NCTM, 2000). The students built function rules and, using multiplicative inverses, additive inverses and pattern recognition, formed inverse function rules. They built linear function and inverse function rules and progressed to non-linear functions and their inverses. The findings show support for the NCTM Connections Standard. Also, the students worked in groups to create rules to represent functions and their inverses, which were expressed in table form. Their collaboration and the results thus obtained met the NCTM Communication Standard.

Once these concepts were understood, students needed to communicate them clearly. The Communication Standard explicitly states that students should be able to explain their ideas precisely, using the language of mathematics (NCTM, 2000). Evidence from this research shows that students communicated mathematically with teachers and researchers, as well as with peers. They were able to write their rules, verbally and
symbolically, and to adjust them as experience suggested after discussion with others. Working collaboratively, students compared their findings and resolved difficulties through dialogue and independent work.

Paradigm-creating lessons and activities, facilitated earlier by Davis, provided students with the opportunity to discover patterns and to identify slope and y-intercept. They wrote equations or function rules representing the relationships between box and triangle values in the t-charts assigned (Giordano, 2008). In the subsequent sessions analyzed in this research, students demonstrated familiarity with the assigned tasks, and they worked with relationships between two quantities to build function and inverse function rules. This research demonstrates that seventh-grade students were able to build inverse function rules, and they did so prior to their formal study of algebra.

On the basis of the present research, it is reasonable to conclude that elementary mathematics curricula could successfully incorporate activities instilling important algebraic concepts. As Carraher et al. (2008) suggest, algebra is already present in early mathematics curricula. Thus, in domains where function and inverse can be incorporated, educators can devise appropriate strategies (i.e., lessons and activities) that are conducive to the understanding and to the use of these concepts.

7.2 Improving Prerequisite Skills

The results of this research suggest that mastery of the prerequisite skills of fraction and integer arithmetic is essential, if students are to be successful in their creation of function and inverse function rules. Deficiencies in students’ proficiency with multiplying and dividing fractions and of adding mixed sign numbers rendered the formulation of rules more challenging. While it may have been expected that students
would have learned fraction arithmetic in earlier grades and, more importantly, that they had been introduced to integers in their regular mathematics class prior to engaging in these activities, this was not the case. Many of the participants in this study were handicapped in their reasoning by lack of mastery of fraction multiplication and division, as well as of integer arithmetic. This implies, at the very least, that these skills should be mastered and periodically revisited in the mathematics classroom until the principles and procedures become part of students’ everyday knowledge. These prerequisite skills, if mastered, might have made the process of finding inverse rules easier.

Another key observation is that students resolved their differences without the use of calculators, even though one student, Michelle I., requested its use on the first day when working with her group on the creation of their own rule. Significantly, when working on fractions on the second and third day, R3 guided the groups through their difficulties without the use of a calculator. Their success, in this regard, reinforces the claim that such skills may be improved without the use of calculators and, with practice, honed further. This research implies, further, that the development of skills necessary for the learning of algebra, with special emphasis placed on acquiring proficiency with fraction and integer arithmetic, may need to be prioritized in the middle-school mathematics curricula.

### 7.3 Learning Environment

As did Davis in earlier algebra sessions (Spang, 2009; Giordano, 2008), the researchers in the present study provided an amicable and relaxed environment in which students could express their ideas uninhibitedly and construct algebraic ideas more
confidently. Providing sufficient time over a three-day period for developing solutions was an important factor helping to de-pressurize the classroom atmosphere.

The seventh-grade participants in this study were given the opportunity to investigate problems that dealt with the idea of function and inverse. Class time was extended beyond the standard 45-minute period for these investigations, allowing for introspection, for the resolution of difficulties, and for the refinement of solutions. Protraction of the time allowed teachers, researchers, and students to address tasks systematically and to re-visit them. This was possible over the three-day period. Significantly, extra time allowed students to resolve differences in dialogue with their peers, and this was particularly evident while they worked with integer and fraction arithmetic. In light of these accommodations, students’ level of understanding was raised, providing a starting-point for more complex analyses in future lessons.

7.4 Future Research

One focus for future research might be the investigation of group dynamics and how collaborative work influences learning. After students work in groups, activities or tasks could be assigned on an individual basis in order to assess how learning is transferred from a group setting to an individual.

Another area of interest might be the analysis of proficiency levels in prerequisite skills and how proficiency in rudimentary skills affects student outcomes on higher levels. Such a study could assess prerequisite skills before assigning an activity or problem. Subsequently, students could be assigned a problem to work on individually. Their written and verbal solutions could be analyzed to determine if, and to what degree, their prerequisite skills might have affected the outcomes.
A third area of focus could be the introduction of a computer-assisted program which students could utilize to improve or build prerequisite skills, such as fraction and integer arithmetic. Students could use the application until it is determined that they have attained an adequate proficiency level in a particular skill, at which point a problem would be assigned. Once again, the solutions could be analyzed to determine the extent to which improvements in students’ skills affect the outcomes. This research could also explore the usefulness of one or more computer algebra applications to ameliorate students’ deficiencies. Through this means, the applications that yield the greatest improvement for specific student populations could be identified.
References


Bellisio, C.W. (1999). Elementary students’ ability to work with algebraic notation and variables. Unpublished dissertation, Rutgers, the State University of New Jersey, New Brunswick, N.J.


Schulman, S. M. (2009). *What was the Madison Project?* Unpublished dissertation, Rutgers, the State University of New Jersey, New Brunswick, N.J.


Appendix A Questionnaire

The Questionnaire

1. Which one of the following sentences is, in your opinion, a better description of the concept of function?
   A. Function is a computational process which produces some value of one variable (y) from any given value of another variable (x).
   B. Function is a kind of (possibly infinite) table in which to each value of one variable corresponds a certain value of another variable.

2. True or False?
   A. Every function expresses a certain regularity (the values of x and y cannot be matched in a completely arbitrary manner).
   B. Every function can be expressed by a certain computational formula (e.g., \( y=2x+1 \) or \( y = 3\sin(\pi +x) \)).

3. Which of the following propositions describe functions? (x and y are natural numbers)
   A. If \( x \) is an even number then \( y = 2x + 5 \); Otherwise (\( x \) is an odd number) \( y = 1-3x \).
   B. If \( x = 0 \) then \( y = 3 \).
   \( \text{If } x > 0 \text{ then to find the corresponding value of } y \text{ we add } 2 \text{ to the value of } y \text{ corresponding to } x-1. \)
   C. For every value of \( x \) we choose the corresponding value of \( y \) in an arbitrary way (e.g., by throwing a dice).
Appendix B Eyes and Tails

Eyes and Tails
Suppose you were at a dog shelter and you wanted to count all the dog eyes you saw. If there was one dog, how many eyes would there be? What if there were two dogs? Three dogs? 100 dogs? Do you see a relationship between the number of dogs and the total number of eyes? How would you describe this relationship? How do you know this works?

Suppose you wanted to find out how many eyes and tails there were all together. How many eyes and tails are there for one dog? Two dogs? Three dogs? 100 dogs? How would you describe the relationship between the number of dogs and the total number of eyes and tails? How do you know this works?
Appendix C Height vs. Time graph

Figure 1: Plant height vs. time

Figure 2: Plant height vs. time with earlier growth shown
Appendix D Quantitative Problems

Penny-roll Problem: Say you have a pile with 2 rolls of pennies and a pile with 5 rolls of pennies. If you were to compare their weights, what might you notice?

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<th>Weight</th>
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<td>5</td>
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<td>12</td>
<td>54 oz.</td>
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<td>16</td>
<td>72 oz.</td>
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Table of numbers of rolls and weight values

First Gear Data Problem: The following table contains pairs of rotations for a small and a big gear. Did all these entries come from the same gear pair, or did some of them come from different gears altogether? How can you tell?

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Table of gear pairs

Second Gear Data Problem: The following table contains pairs of rotations for a big and small gear. What is the relationship between the two gears?

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Table of gear pairs representing a $y = mx + b$ situation
Appendix E Guess My Rule Tables

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Appendix F Tables presented to students

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Day 2: May 17, 1995

\[
\begin{array}{cc}
0 & 0 \\
1 & 1/2 \\
2 & 2/3 \\
3 & 3/4 \\
4 & 4/5 \\
5 & 5/6 \\
? & 11/12 \\
\end{array}
\]

Day 3: May 18, 1995

\[
\begin{array}{cc}
1/2 & 1/2 \\
9 & -8 \\
1 & 0 \\
2 & -1 \\
1/3 & 2/3 \\
0 & 1 \\
2/3 & 1/3 \\
-1 & 2 \\
\end{array}
\]

\[
\begin{array}{cc}
0 & 0 \\
1 & 1/2 \\
2 & 2/3 \\
3 & 3/4 \\
4 & 4/5 \\
5 & 5/6 \\
? & 11/12 \\
\end{array}
\]
### Appendix G Transcripts

Bobby’s table 1 of 2: 5/16/1995 (B1A)

Camera View: Bobby’s Table 1 of 2 (Bobby, Amy-Lynn, Magdalena, Michelle I.)  
Date of filming: 05/16/1995  
Harding public school, Kenilworth NJ, Grade 7  
Advanced Guess my Rule (AGMR): Inverse: problem  
Transcribed by: Andrea DePaolo  
Date of transcription: June 2008  
Verified by: Eman Aboelnaga  
Date of verification: September 2008  
Length of session: 00:55:41

<table>
<thead>
<tr>
<th>Time</th>
<th>Speaker</th>
<th>Transcription</th>
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</thead>
<tbody>
<tr>
<td>00:00</td>
<td>(Videographers and facilitators set-up.)</td>
<td></td>
</tr>
<tr>
<td>05:52</td>
<td>(Students enter classroom and sit at tables)</td>
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<tr>
<td>11:48</td>
<td>(Bobby, Amy-Lynn, Michelle I, and Magdalena test speaker and prepare for the session by getting paper and pens)</td>
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<tr>
<td>11:48</td>
<td>Researcher 1</td>
<td>Are you ready to start? I’m Dr. Dann and I work with these people at Rutgers. Dr. Maher who comes in a lot of times and you probably, some of you, have already guessed what we are going to do today by those sheets that are on the table. No, no. Do you remember doing something called Guess My Rule about two years ago with Dr. Davis? Do you remember something about that? Let’s look at an example.</td>
</tr>
<tr>
<td>11:48</td>
<td>Michelle I</td>
<td>I remember the sheets but not …</td>
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<tr>
<td>11:48</td>
<td>Researcher 1</td>
<td>Suppose that, can you see it, that I am thinking of a rule like this, and what I am going to do is add three to some numbers, and suppose that I put a zero in the box, what would I have to put in the triangle?</td>
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<tr>
<td>11:48</td>
<td>(Researcher 1 writes on the overhead as shown below.)</td>
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<tr>
<td>11:48</td>
<td>Michelle I</td>
<td>A three.</td>
</tr>
<tr>
<td>11:48</td>
<td>Researcher 1</td>
<td>A three, okay. Suppose I put a one in the box, what would I have to put in the triangle?</td>
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<tr>
<td>11:48</td>
<td>Student from another table: Four</td>
<td></td>
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<tr>
<td>11:48</td>
<td>Researcher 1</td>
<td>A four, all right, okay, so this is pretty easy, right.</td>
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</table>
(Researcher 1 has entered three entries for this rule on the overhead as shown below.)

So far we have limited ourselves to what kind of numbers? What kind of numbers are we using there? We are using positive numbers, does anybody know another name we could call these ever? What other kinds of numbers do you think we could use?

Student? Negative numbers

Researcher 1 We could use negative numbers. What about if we put in negative one what would we have to put in the triangle?

Bobby two

Researcher 1 And if we put negative two we would have to put one in the triangle.

(Researcher 1 fills in two more entries in the table as shown below.)
Researcher 1: A seven, and how did you get the seven?

Brian: Ten minus three.

Researcher 1: Okay, so the rule that we started off with said that we were adding three but when you were doing it backwards you subtracted three. So addition and subtraction are inverse operations, one undoes the other. Do you know any other operations that work like that?

Bobby: Multiplication and division.

Researcher 1: Multiplication and division work like that, one undoes the other. They could be called inverses. Do you know of any other kind of operations that work like that? That may be all you’ve had so far. What if you square a number? What if I start with the number five and square it? What do you get?

Student?: Twenty-five.

Researcher 1: What does five squared mean?

Amy-Lynn: Five times five.

Researcher 1: It means five times five and so you get twenty-five. Does anybody know what you would call it if you went back the other direction?

Ankur: The square root.

Researcher 1: The square root. Have you talked about that some, the square root, and does anybody know how to write it? The square root, could you come up and write that for us and do the square root of twenty-five?

(Angela, from Jeff’s table, comes to the overhead and writes $\sqrt{25}$ as shown below.)
Researchers 1: Okay, would be five, okay, so do you think squaring and square root are also inverse operations right? What I want to do is I'm gonna think of a rule and I am going to put down some numbers that work in the rule and see if you can guess the rule. Do you remember doing that now with Dr. Davis? One day maybe? What do you think? My rule says that if I put a one in the box, that in the triangle I'd have to put one-third and if I put a two I would get two-thirds and... Would you talk to some of your friends there and see if you can think of a rule that would work?

(Researchers 1 writes on the overhead as shown below.)
Michelle I.  The first one would be plus two and the last one would be minus two.

Researcher 1  We want a rule that will work on every line.

Amy-Lynn  And then they just go, you see how it’s minus two, plus two.

Magdalena  It goes by two then one and one-third, then two-thirds, going down.

Bobby  Up by two-thirds...

Magdalena  Down.

Bobby  Going down...minus two-thirds, minus one-third, minus two.

Michelle I./Amy-Lynn/Bobby/Magdalena (together respond while Michelle I. writes the numbers on her paper)  Plus two, plus one and one-third, plus two-thirds, plus zero.

(Michelle I. writes: ‘+2, +1 1/3, +2/3’, see below)

Amy-Lynn  And then you are just starting with negatives.

Michelle I./Amy-Lynn/Bobby/Magdalena (continue while Michelle I. writes the numbers on her paper)  minus two-thirds, minus one and one-third, minus two.

(Michelle I. writes: ‘-2/3, -1 1/3, -2’ as shown below.)

Michelle I.  Wow, that was quick.

Magdalena  You are going down by two-thirds then you are going up by two-thirds. Are we supposed to write that down?

Michelle I.  No

(Conversation turns to a Spanish Test)

Bobby  I wonder if we are going to get a test in Spanish today, vocab test...

Amy-Lynn  With the department and everything, patio el la planto...
I think of sardines.

Researcher 1 Oh you did do something. Okay, all right.

Student We found out the pattern. We are done.

Researcher 1 But what about a rule, is there something you can do that would work on every line?

Magdalena When it’s positive, it goes down by two-thirds; it goes down by two-thirds, and here it goes up by two-thirds, and then, here, it goes up by two-thirds, yes. From the zero it goes down by two-thirds…

Michelle I. …

Researcher 1 I started with a really hard one didn’t I. I’d like to come back to that one. Could we come back to that one later? Let’s try another one. I think that one may not be a rule.

Michelle I. It is not a rule. You have to have a rule for every single one. What if you put minus twenty?

Researcher 1 Suppose that when I put in say negative three in the box, then what I get is negative twelve and when I put negative two I get negative eight.

Bobby/Amy-Lynn Times four.

Researcher 1 and so forth like this. What do you think of that rule?

(Researcher 1 writes on the overhead as shown below.)

(Bobby, Amy-Lynn, and Michelle I. have raised their hands ready to answer.)

Researcher 1 Angela you think you know that one already?

Angela It’s box times four equals triangle.

Researcher 1 She thinks it is box times four is equal triangle. Anyone else agree with it? Could you come up and write that for us?
Researcher 1: Could you show us how some of the ordered pairs work there. Like how it is that one and four. Could you show us some of that?

Angela: One times four equals four.

Researcher 1: Okay, two times four equal eight. Is everybody seeing that? All right, thank you Angela.

Researcher 1: Um, what about if I put something here like sixty in the triangle. Could you tell me what would have to be in the box? Bobby?

(Researcher 1 adds ‘60’ to the triangle column as shown below.)

Bobby: Fifteen.
Researcher 1: Fifteen, do you think so? And how did you do that Bobby?
Bobby: Sixty divided by four.

(Researcher 1 adds ‘15’ to the box column as shown below.)

Researcher 1: Sixty divided by four so he’s using the idea of the inverses.

Somehow I think you ought to be able to do this one. (R1 goes back to the previous slide at the overhead.) Want to try again? Did you notice, you know that one that fast already.

(Overhead slide is shown below.)

Students?: We already knew it.

Researcher 1: Oh you knew it. All right, Michael come up and see if, what other people think about yours.

Michael: (comes to the overhead) Uh, write that rule? Just write it down.

Researcher 1: Yeah and see if some other people agree with you.
Researcher 1: What do you think? Did some other people get that rule or if you didn’t get that rule would you see if you think it works? Michael, would you mind coming back and convincing some of them, show why you think it works.

Michael: Two times one-third.

Researcher 1: Can you write some of those out?

Michael: Two times one-third equals two-thirds and two times one-third is one-third plus one-third and that equals two-thirds.

Researcher 1: All right. Are you convinced or you want to see him do some of the other pieces. I think they’d like to see you do some of the other pairs. They may be convinced that two and two-thirds work, that this row works, what about some of the others?
At 22:02 Michelle I. has written on her paper as shown below.

Michael

Three times one-third equals one, because there’s three of the one-thirds and that means that three of the one-thirds equal three thirds, which three-thirds equal one.

(Michael writes ‘$3 \times \frac{1}{3} = 1$’ on the overhead, as shown below.)
Researcher 1: Who is convinced that his rule works? All right Michael, thank you. Do you know what though it is a little strange to me because I had another rule in mind and I think it works too. Um, here’s what I was thinking about. I was thinking about the box divided by three. The number in the box divided by three is equal to the number in the triangle. Do you think that would work too? Do you think that would work?

(Researcher 1 writes on the overhead)

[Diagram]

\[
\frac{\square}{3} = \triangle
\]

as shown below.

Students?

Yes.

(Michelle I. added \[\frac{\square}{3} = \triangle\] to her paper as shown below.)
Researcher 1: So you think that there could be two rules for some, some tables?

So are you telling me that if you divide a number by three that you get the same answer as if you say one-third times that number?

Would that always work? Do you think that works for every number? I’m not sure whether you agree with that or not.

Student?: inverse operations …

Researcher 1: So you think you could talk about inverse two different ways in this case. You could say a third, all right. What if I gave you in the triangle the number two, what number would have to be in the box? Okay, Amy-Lynn.

Amy-Lynn: Six.

Researcher 1: Does everybody think six will work? All right, all right. Very good you’re off to a good start. What about, maybe this one is a little harder for you.

(Researcher 1 writes on the overhead as shown below.)

(Bobby’s table works on the rule. Each student copies the table onto their own paper.)

Bobby: This side goes down by four. This goes down by one.

Michelle I.: What’s the rule? We have a pattern but now we need the rule.

(Students continue to work on the problem)

Amy-Lynn: Wait I got something. Look. This times four plus one, times four plus one, times four plus one, times four plus one.

(Amy-Lynn writes ‘x 4 + 1’ on her paper.)

Magdalena: It won’t work. No, it won’t work. No. It doesn’t work on the first one.

Bobby: It should be times four minus one.

Amy-Lynn: Minus one.

Michelle I.: When it’s negative, it’s times four minus one. When it’s positive, it’s times four plus one.

(Amy-Lynn and Michelle I. hi five)
Bobby: It should be negative. It should be like negative one, negative five.

Michelle I.: When it’s negative, it’s times four minus one, when it’s positive it’s…

Magdalena: It can’t be plus, or minus one, it has to be plus negative one.

Michelle I.: Plus negative one, it is the same thing, plus negative one and minus one.

Magdalena: No, cause if you plus one you like minus-ing from it, cause if you go negative three times four plus one, so it’s a negative. Oh that’s it. That’s it.

Amy-Lynn: Yeah, like when it’s negative like these, it’s times four minus one, cause and then when it’s positive you do times four plus one.

Researcher 1: Amy-Lynn thinks she has a rule. Does anybody else? We will give them another minute or two? Then we can let you put that up and see what they think?

Michelle I.: So, it’s always times four plus one.

Amy-Lynn: Always times four plus one?

Bobby: Yeah, cause when you add it to a negative number it's.

Michelle I.: Okay.

Amy-Lynn: Times four plus one.

Michelle I.: It’s always times four plus one.

Magdalena: Yeah that’s it.

(They all write that down on their paper.)

(At 29:32 Michelle I.’s paper is shown below.)

Bobby: It is box times four plus one equals triangle. Cause the box times four.

Amy-Lynn: Then what would this be negative three times four…

Magdalena: Equals negative eleven plus one like minus-ing so it’s negative eleven. Get me.
Amy-Lynn  No.  
Magdalena  Cause if you plus the positive number to a negative number it goes down.  
Bobby  Say you owe someone twelve dollars and you gave him a dollar then you only owe him eleven.  
(All laugh)  
Researcher 1  Does anybody else think they have it yet? The rule? How are you doing over there?  
Student from another table  We got one.  
Researcher 1  You got one? Let’s let Amy-Lynn put hers up and see if you agree with that one. Amy would you do that for us.  
Amy-Lynn  Should I just do it in the thing, cause this is what we came up with, like times four plus one, that was like our rule, to do with everything in the box.  
(Amy-Lynn writes  
\( \boxed{x \times 4 + 1} = \Delta \) on the overhead, as shown below. )

Researcher 1  What do you think of Amy-Lynn’s rule? That same rule you got? Or you couldn’t? What about over there at the far table? Do you agree with it?  
Students  (whisper) Yes.  
Researcher 1  Do you want her to convince you or do you agree with her? Do you agree? You Agree? Thank you Amy-Lynn that’s really nice. What if I put forty-nine in the triangle? Could you tell me what would have to be in the box? Already? Michelle.  
Michelle I.  Twenty-four.  
Researcher 1  Michelle thinks twenty-four.  
Bobby  Twelve.  
31:07  Researcher 1  And you think twelve.  
Magdalena  It’s twelve now.  
Researcher 1  Lots of people are thinking twelve.  
Michelle I.  I did it wrong. I calculated wrong.  
Researcher 1  Do you agree with that twelve? What did you do? Does everybody agree that twelve works? You had twelve times four. Angela?
Angela: Forty-nine minus one divided by four. Forty-nine minus one is forty-eight and you divide by four.

Researcher 1: So Angela is saying that she took the forty-nine and subtracted one and then divided by Four.

Angela: Four. Do you think you could write that for us where we had triangle equal to something with the box?

Researcher 1: (Researcher 1 writes on the overhead)

= Δ as shown below.

(Angela goes to overhead)

Angela: I can do it like backwards?

(Angela writes on the overhead)

Δ – 1 ÷ 4 = □ as shown below.)
Can you do that? Do it either way you want to. All right. Now I heard you say something different from that before, you took forty-nine minus one and you got forty-eight and then what did you do?

You divided forty-eight by four.

Oh and you have division there. Okay, I see what you did. Okay, all right that’s good. I read that as a plus sign. Does everybody know that you could also write this, this way if you want to.

It is just a different way of writing. That’s really nice. Do you want to try another one, see if we can stump you?

We are child prodigies. That guy at Wendy’s believed Mike too.

Let’s say that when we do this we get zero and when we put in negative two that we get two.

(Researcher 1 writes a new table at the overhead as shown below.)
(Students at Bobby’s table copy the table to their paper.)

33:26 Michelle I.  Plus three. It worked for the first two didn’t it?

Magdalena For the first one

Michelle I.  The first one negative three plus three is zero.

Magdalena For the next one is plus four.

Michelle I.  Oh, sorry.

Bobby Plus three, plus four, plus five, plus six, plus seven, plus nine

Michelle I.  Shut up Bobby.

Bobby No, look really, plus three, plus four, plus five, plus six, plus seven, plus nine.

Michelle I.  But you have to find the rule.

Bobby Messed up. No, really. It goes from seven to nine.

Michelle I.  But we have to find a rule.

Magdalena I know.

Michelle I.  So do I.

Bobby Times two plus six,

Amy-Lynn What about for this?

Bobby …two times two, negative two times two is negative four plus 6

equals two, negative one times one is negative two plus six is four.

Amy-Lynn Cool Bobby. Way to go Bobby

Michelle I.  It has to be the same rule every time

Magdalena Yeah, it is.

Michelle I.  I thought he said like negative.

Magdalena No look, it’s times two plus six.

Michelle I.  So two times two plus six equals triangle, ten.

Researcher I Who thinks they have the rule?

Student?  I do.

Researcher I Oh, Okay. Magdalena you want to put the rule that you have up

and see if they agree with you?

Amy-Lynn Bobby
Magdalena goes to overhead and writes: \( \square \times 2 + 6 = \triangle \) as shown below.

Bobby: This is not fair, I never get any credit.
Michelle I.: It is on camera Bobby, don’t worry.
Amy-Lynn: Hopefully she could hear you.

(Laughter)

Researcher 1: Is that what the rest of you got or somebody have a different rule?
Everybody had that same rule?

35:59 Researcher 1: Magdalena, can you give them a number to put in the triangle and see if they can find the number in the box. Can you think of a good number to put in the triangle?
Michelle I.: Box divided by two minus six equals.

Bobby: What about the second one?
Amy-Lynn: Let Bobby go up.

Bobby: No.
Amy-Lynn: Let Bobby go up.
Michelle I.: I have a different one.
Amy-Lynn: What?
Michelle I.: It’s triangle divided by two minus six equals box.
Amy-Lynn: Does it work?
Michelle I.: It’s the same thing but backwards.
Magdalena: I’ll put fifty-six.
(Magdalena writes ‘56’ in the triangle column.)

Bobby: Just put this on everybody’s paper so they know it’s mine.

Michael: Twenty-five.
Researcher 1: Twenty-five and everybody agrees with that. And how did you get it Michael?
Michael: Minus six, fifty-six minus six and I divided by two.
Michelle I.: Isn’t it..
Researcher 1: Could you write that as a rule?
(Michael goes to the overhead and writes: \( \triangle - 6 = \square \) as shown below.)
Michelle I. Divided by two minus six.
Magdalena It is times two plus six.
Michelle I. Don’t mind me. I came from gym and I am still a little sloppy.
Researcher 1 All right that’s nice. But you know what I had another, a different rule that I think works. I had this. Tell me if you think that this would have to work too. I had two times box plus three is equal to triangle.

(Researcher 1 writes on the overhead)

\[ 2 \times (\Box + 3) = \triangle \]

as shown below.)

Michelle I. Wow, that’s heavy stuff. No.
Researcher 1 Do you think that works?
Michelle I. Yes, it definitely does.
Researcher 1 Well do you think there are just two rules that work? Do you think that one works, that both of those work? Do you think there are any other rules that work?
Amy-Lynn There might be but I don’t feel like working on them.
Researcher 1 Can anybody tell me why that works? It seems a little strange to me. Bobby?
Bobby You have to do the work in the parenthesis first and like, then, if you add three to negative three, it’s zero, and, if you times two by it, it like stays the same cause it’s zero, and it like works on every one.
Researcher 1 Did you understand what Bobby said? Does anybody think it’s odd
that you have a three down there you have a box plus three but
when you have the rule the other way you have a six and that you
get the same number? What do you think Ankur?
Ankur Two outside the parenthesis so it becomes sort of like a six.
Researcher 1 Would you go up and explain what you are saying?
Ankur The two outside of the parenthesis doubles what ever's inside.
Researcher 1 You mean it doubles everything that's inside.
Ankur Yes.
Researcher 1 All right, what do the rest of you think?
Bobby? Two squared?
Researcher 1 Think that will always be true, Jeff?
Jeff Yes
Researcher 1 Jeff thinks yes.
Student? I do too.
Researcher 1 You think yes too. All right, all right. Let’s try another one.
Amy-Lynn She’s getting mad at us. We are knowing, we are getting these too
fast for her.
Michelle I. ??
Researcher 1 Maybe this is too easy.
Michelle I. ???
Amy-Lynn What?
Michelle I. (Laughter) ?
Michelle I. I am still slow from gym and you people.
Magdalena or Amy-Lynn
I should be slow I had Health.
39:57 Researcher 1 This one is probably too easy for you.
(Researcher 1 writes a new table at the overhead as shown below.)

(Students at Bobby’s table copy the table to their paper.)
Magdalena It um times um.
Amy-Lynn It’s times its number.
Michelle I. Times its number.
Bobby No, cause you times three by negative three, it is negative nine.
Amy-Lynn No it ain’t.
Bobby: Yes it is.

Researcher 1: Jeff thinks he has a rule. Anybody else? Jeff do you want to come put up the rule and see if they agree with you?

Bobby: Box times the square root of fifty-two adding negative nine plus two you get it.

Michelle I.: Yeah, Bobby. whatever.

Amy-Lynn: Yeah, that is what we had.

(Jeff comes to the overhead and writes $\square \times \square = \triangle$ as shown below.)

Jeff: Box times box is triangle.

Bobby: (raises his hand). Ur that’s wrong.

Michelle I.: That does not work cause if it’s negative three times negative three it’s negative nine.

Amy-Lynn: It’s positive nine.

Michelle I.: I was just repeating what they said, and I disagreed with it, and then they tell me that I am wrong. So it is their fault that I was wrong.

Michelle I.: It’s on the camera. I’m telling you.

Amy-Lynn: I said it before. Shelly (Michelle I.) and I said it and you (meaning Magdalena) said we were wrong.

Magdalena: No I didn’t. Bobby said it.

Amy-Lynn: Bobby said you were wrong.
Michelle I. Bobby the supposedly child prodigy over there.
Researcher 1 So they are having a discussion at this table. They feel that some people were led astray about negative three times negative three being negative nine or positive nine?
Amy-Lynn/Michelle I. It’s nine.
Bobby You times the whole number …?
Researcher 1 Do you want to write that for us Michael? Michael has another way of writing it.
Student Sarah wants to.
Researcher 1 Sarah.
(Laughter at Bobby’s table)
(Michael comes to the overhead and writes
\( □^2 = △ \) as shown below.)
Researcher 1: What would you get on the other side? Brian?
Brian: It is either a positive or a negative six. I’m not sure.
Researcher 1: Hey, Brian thinks it’s either a positive six or a negative six. What did you think, Joe? (talking to Jeff)
Joe (Jeff): Nothing.
Magdalena: It’s a negative.
Michelle I.: I am not saying anything cause you people always make me look wrong.
Researcher 1: Bobby?
Bobby: You can’t put a negative there. Cause to get a negative number you have to times like a positive and a negative. So you get a negative.
Michelle I.: So that’s a negative times a negative and that’s a positive. Which Bobby got wrong before and made me look dumb.
Researcher 1: They’re saying over here that they don’t think I should put a negative over here that you wouldn’t be able to get an answer.
Michelle I.: No.
Bobby: Yes, that’s what I’m saying.
Researcher 1: Do you think that’s right?
Amy-Lynn: Bobby’s saying it, not us.
Researcher 1: We shouldn’t use this negative thirty-six over here. Okay.
Amy-Lynn/Michelle I.: We are saying it because it is right.
Researcher 1: All right. You’re doing great. You told me you would.
Amy-Lynn: Bobby’s on a roll.
Michelle I.: I am going to solve this next one. My brain is in full gear.
Amy-Lynn: Don’t get emotional.
Researcher 1: Maybe this one will be a little bit harder.
Michelle I.: Don’t make me look dumb.
Magdalena: …
Michelle I.: And you agreed with him too.
Amy-Lynn: No you didn’t.
... You made yourself look like an idiot.

I don’t think it is plus two anymore.

No, we just messed up the whole box.

This should be three. This should be four. This should be five.

(Researcher 1 corrects the table on the overhead as shown below.)

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(Students at Bobby’s table correct their paper.)

Bobby
For some reason, I was right.

Michelle I.
We all said that Bobby, don’t try to take the credit.

Researcher 1
Michelle has a rule already.

Michelle I.
Plus two.

Researcher 1
Want to come and write the rule for us?

Amy-Lynn
Yeah, Shelly. She got one, her brain is in gear. It’s on camera.

Gonna play the tape back for Shelly.

(Michelle I. goes to the overhead and writes as shown below.)

Researcher 1
That turned out pretty easy didn’t it. And here I thought I’d made a harder rule. Do you know what my rule was?

Amy-Lynn
Plus two.

Bobby
Times four minus two.

Researcher 1
I had three times the number in the box plus six divided by three is equal to the number in the triangle. Do you think that would work?

Do you think that would work?

(Researcher 1 writes on the overhead

\[
\frac{3 \times \Box + 6}{\Delta} = \Box
\]

as shown below.)
Magdalena/Amy-Lynn

…complicated, why can’t you do it the easy way?

Student? I guess so.

Researcher 1 Would you tell me if it works?

Student? It does.

Researcher 1 So what do you think Stephanie why does it, that work out to be
the same? Think that might work out to be the same?

46:45 Bobby? Divide by three’s and zero.

Researcher 1 Think why that might work out to be the same.

Bobby No, cause like negative two times three is negative six plus six is
twelve, and, if you divide it by three, it will be four not zero.

Researcher 1 So are you, I wasn’t sure what you are saying Bobby, do you get
the same answer or a different answer?

Bobby A different answer.

Magdalena Three times negative two….

Researcher 1 Bobby is getting a different answer. Are some other people getting
a different answer too? Angela?

Magdalena ….It is negative three and when you divide by three it is zero….

Angela Shelly made a mistake up there. She put box plus two then she put
triangle divided by two.

Michelle I. It’s the same thing, Angela. Oh, sorry, it’s supposed to be minus
not divide. Sorry about that.

Researcher 1 All right. Thank you Angela, I hadn’t noticed it. That’s good.

Magdalena So if you times three times negative three.

Researcher 1 Do you want to change that who put it up?

(Michelle I. goes to the overhead and changes the division sign to a
minus sign,

\[ \triangle - 2 = \square \] as shown below.)
Magdalena: Cause it you go three times negative three it’s negative nine plus six is negative three.

Researcher 1: What I want to know if three times the number in the box plus six divided by three does that work for all of them?

Magdalena: It doesn’t.

Bobby: Negative three times two is six.

Amy/Lynn: Negative six.

Bobby: Plus six is...

(Bell rings)

Amy/Lynn: It doesn’t, cause look at the next one.

Magdalena/Bobby: Look at the first one.

Amy/Lynn/Michelle I./Magdalena: Negative three times three is negative nine plus six is negative three divided by three is zero.

Michelle I.: We know the answer.

Bobby: Three divided by three is one.

Researcher 1: It doesn’t work?

Michelle I./Amy-Lynn/Magdalena: It works. It works. We just kinda miscalculated again.

Michelle I.: I can’t agree with you people any more you always make me look dumb.

Researcher 1: Some people think it works and some people think it doesn’t work.

Magdalena: I think it works.

Michelle I./Amy-Lynn: It works.

Bobby: I messed up.

Amy-Lynn: We miscalculated.

Michelle I.: Bobby you messed up again.

Researcher 1: What do you think Michael. It works?

Michelle I./Amy-Lynn: It works.

Bobby: Three divided by three is really zero.

Michelle I./Amy-Lynn: It works.
Researcher 1: Now, it would be a whole lot easier for me to write it as box plus two equals triangle if I knew that was the same. Is there any way I could know that it was the same?

Student?: Test it?

Researcher 1: To do?

(Bobby’s table is working)

Magdalena: What are we doing?

Amy-Lynn: I don’t know.

Researcher 1: Does anybody have an idea why that could be the same?

(Bell rings)

Michelle I.: Maybe there’s two rules for every one.

Magdalena: Look six divided by three is two so it might work. I don’t know I just thought of it.

Michelle I.: Six divided by two. Whoa.

Magdalena: Is it two?

Bobby: What?

Magdalena: Six divided by three.

Bobby/Amy-Lynn: Yea. (laughter)

Amy-Lynn: Like I was just making sure.

Michelle I.: I was just making sure.

Amy-Lynn: Then because if you say something… I can blame Bobby.

Michelle I.: Do you know what one plus one is?

Magdalena: Is it two?

Amy-Lynn: No that’s one divided by one.

Researcher 1: Want to come try? Michael just wants to stay at his desk. He doesn’t want to come up.

Michael: Six on the top divided by the three at the top equal two. Then the two threes, on the bottom and the top, cross each other out, so it’s box plus two.

Magdalena: That’s what I got.

Amy-Lynn: Way to go Magda.

Researcher 1: What do you think, did you hear what Michael said?

Michelle I.: Yes I did.

Researcher 1: What do you think about that?

Michelle I.: I think it’s true.

Researcher 1: I don’t know that’s sounds a little dangerous this crossing things out. I wonder if it’s anything like trying to find the one you did before?

Michelle I.: Multiplication. I just said that for the heck of saying.

Amy-Lynn: Scooby Dooby Doo.

Researcher 1: We were looking at something else where you had two times the number in the box plus six, I think it was, and you told me that that would be the same as this. And I wonder if there is anything about this one that’s like that one?

(Researcher 1 writes on the overhead:
Because if you have a variable in the parenthesis, you can divide, I mean times each number by itself. Like two times the box is two box, and two times three is six, so it’s the two times the .. I’ll write it down. 
(Michael goes to the overhead and explains how two times box plus three is equal to two times box plus two times three using distribution)

If you need anymore here…

Okay, well, here if you don’t know how, box plus three, there is the variable here, you can times two times the box which is two box, plus two times three is six which is the same thing as this.

We had the same thing…

Okay, what does that have to do, don’t go away. Is there any thing about this one that’s like that? …Okay.

cold

I keep my chin up.

I don’t wanna..

I got the answer. (shows paper to ..

Three times the box can be written as three box plus six. They are similar to each other, three box plus six and two box plus six.

(Michael writes on the overhead:

\[ 3 \Box + 6 = \triangle \text{ as shown below.} \]
813
814  Researcher 1  Instead of multiplying by two
815  Michael   the top and then dividing it, multiplying....
816  53:48  Researcher 1  Would it hurt anything if I put, don’t go away, would it hurt
817  anything if I put parenthesis right here?
818  (Researcher 1 adds parenthesis to the overhead:
819     (3 x □ + 6) = △ as shown below.)
820  3
821
822
823
824
825  Michelle I.  No
826  Michael   No, it wouldn’t hurt.
827  Researcher 1  Would it help any?
Amy-Lynn: If would be three box plus nine using the distributive. It would be 
three box plus nine.
Michelle I.: ..
Bobby: What did I do
Michelle I.: You have the answer (shows Bobby’s paper to the camera)
Researcher 1: So you’re saying this one looks like this one.
Researcher 1: That is except for the division…..
Researcher 1: That instead of multiplying you are dividing it.
Bobby: Who cares, it is the right answer.
Researcher 1: So what do you think about what Michael is saying? Do you think 
that might be the reason that?
Researcher 1: Could somebody make up another one that you think would work 
two ways? See if you can make up one that would work two ways.
Amy-Lynn/Bobby: Okay.
Michelle I.: Zero, two
Magdalena: What are you making up?
Michelle I.: I don’t know.
Amy-Lynn: She’s making up numbers.
Michelle I.: Let’s do it so it’s...
Bobby: Let’s use this equation (Bobby points to the equation on his paper)
Michelle I.: Let’s use this one, negative three times two plus one. Let’s use 
square root to make it really hard. That would be fun.
Magdalena: And you get lost again
Michelle I.: Get lost in your own problem. Wait a second what did I do here?
Bobby: I am going to figure this out.
Michelle I.: Okay, so…
Time  Speaker  Transcription
1 00:01  Michelle I.  Bobby, work. (Takes away extra paper Bobby is writing on.)
2  Don’t worry about m and c you know, and squared equals mc squared and stuff. Okay. So-
3 4  Magdalena  I’m so messed up.
5  Michelle I.  We’ll do something- We’ll do like the rule is s- box
6  Magdalena  Squared.
7  Michelle I.  Squared
8  Magdalena  Plus.
9  Michelle I.  Plus.
10  Magdalena  Wait no. It’s the minus.
11  Michelle I.  Minus… negative something. No.
12  Magdalena  No if it was plus… Oh, how about plus negative something?
13  Michelle I.  It’s the same as minus.
14  Bobby  A hundred and thirty two times the speed a normal car’s going on
15  the interstate highway.
16  Amy-Lynn  (mumbles) Minus (unintelligible)
17  Magdalena  But let’s make it… confusing! (Amy-Lynn and Michelle I. giggle)
18 00:46  Michelle I.  Okay, plus negative uh, six. No-
19  Magdalena  No it’s not six. Uh how ‘bout uh, eleven? No-
20  Amy-Lynn  (With Michelle I.) Thirteen.
21  Michelle I.  (With Amy-Lynn) Thirteen.
22  Magdalena  Yeah.
23  Michelle I.  Thirteen equals… triangle.
24  Magdalena?  Triangle.
25 1:02  Michelle I.  So the numbers would be- Let’s try- What would- If two was in the
26  box- so the square root of two is four plus-
27  (Magdalena’s paper shows:)
31 Bobby It’d be times two-
32 Magdalena Minus thirteen is negative-
33 Michelle I. Nine.
34 Magdalena Negative nine.
35 Michelle I. No wait hold on. Negative nine.
36 Bobby Wow.
37 1:24 Michelle I. So it’s two and negative nine. (sings chanting a cheer) Sorry. Okay now it- if it was-
38 Bobby What’s the (?) cheer?
39 Magdalena How ‘bout four-
40 Michelle I. Four. Oh (giggles). Okay.
41 Bobby (During Amy-Lynn below) Negative five.
42 Amy-Lynn So it’s sixteen. Sixteen! ‘Cause it’s four times four. Sixteen.
43 Magdalena Six and minus thirteen-
44 Michelle I. (During Magdalena above) Four times four is sixteen minus thirteen.
45 Amy-Lynn Oh. (giggles with Michelle I.)
46 Michelle I. Minus thirteen is-
47 Magdalena Negative…
48 Michelle I. Negative…
49 Magdalena Six- uh, four times four it’s eight and minus that is negative seven.
50 2:02 Michelle I. Negative seven yeah ‘cause it’s just (two more?) Now if we put zero the answer is negative thirteen.
51 Amy-Lynn But would that one be too easy for them? Then they would be able to figure it out.
52 Magdalena Don’t think of them like that.
53 Amy-Lynn Yeah.
54 Michelle I. But then they would just think- how could they figure it out like (snaps) that then?
55 Magdalena Yeah if it’s a zero like that they-
56 Amy-Lynn Yeah but then they won’t- but then they’ll figure out maybe the.. exponent.
57 02:24 Michelle I. But maybe they won’t realize that it’s zero t- times two.
58 Bobby Yeah but from my understanding ten feet away from-
59 Magdalena Zero times zero was zero. And zero times two is zero.
Michelle I. Yeah I know.
Magdalena So it’s- (pause) Okay let’s put that down.
Bobby If we manage to ???? (mumbles a lot to himself, unintelligible)
Michelle I. But would it give it away to them?
Magdalena Yeah it sort of does.
Michelle I. Well then let’s put
Magdalena Equals six and negative-
Michelle I. No that’s the same thing. It’s going down by two.
Magdalena Yeah. Let’s go-
Michelle I. So five,
Magdalena And then eight, three-
Michelle I. Three, ten, negative one. Now let’s do like negative…two…that would be four.
Magdalena How did you get negative two here? Oh, negative two, okay. It’s supposed to be on the top.
Bobby Um, this is the same thing as negative eleven. It’s minus eleven from everything.
Magdalena (During Bobby above) And negative…
Michelle I. What if we did this…
Magdalena And negative thirteen.
Michelle I. …divided by something?
Bobby Wait.
Michelle I. and Magdalena Whooooooaaa.
Michelle I. That’s intense.
Bobby This one’s kinda easy ‘cause look, just minus eleven from everything.
(Bobby’s paper shows:)

<table>
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</tbody>
</table>
Michelle I. But that’s a different thing. What if we divided it by something?
Whooooaaa.
Bobby Divided from the s-like a man stood thirty feet away from a brick
wall. He turns on the flashlight. How long will it take for the light
to get there?
(table giggles)
Michelle I. Bobby. You’re scaring me.
Bobby It’s true. (During Michelle below) I’m gonna work on that.
Magdalena And we gotta make something similar to that.
Michelle I. Divided by- five thousand. No.
Magdalena Five thousand go- So plus negative thirteen.
Michelle I. Divided by-
Magdalena Um.
Michelle I. Divided by, um,
Magdalena Wait um, how ‘bout two
Magdalena Four.
Amy-Lynn We need something, uh, that’ll be easy to divide into the numbers.
Magdalena How ‘bout by three?
Michelle I. Okay.
Magdalena No but it won’t divide. Every number won’t divide by three.
Doesn’t work.
Amy-Lynn What about- we need like an even or something.
Michelle I. So? We’ll put point something something something. We’ll put
four and a third or something like that to get ‘em confused.
Amy-Lynn Yeah.
Michelle I. and Magdalena
Whooooaaa.
Amy-Lynn Whooooaaa. Cool.
Bobby Wait, I’m working on something.
Michelle I. So, if it was two over here that’d be negative nine divided by three.
Bobby Divided by thirty nanometers.
Amy-Lynn So that would be- negative nine divided by three. (To Bobby)
Negative nine divided by three is negative three, right?
Michelle I. We could put divided by n-point three.
Amy-Lynn It’s negative- huh?
All girls Whooooaaa.
Michelle I. No.
Magdalena That’s too confusing, man.
Amy-Lynn So? These kids are supposed to be brainiacs. Kid prodigies.
Especially Mike.
Bobby No I am. Remember I asked- told the guy that.
Michelle I. Okay. Bobby.
Michelle I. (During Amy-Lynn and Bobby above) Divided by point three. No.
Let’s just put- Let’s just put three.
So it’s four plus negative three is negative nine.

Magdalena: No hold on divided by one third.

Michelle I.: Whoa.

Amy-Lynn: Whoa.

Michelle I.: So it’s negative nine divided by a third. Negative one third. Or no, divided by a third.

Magdalena: Negative one third. Let’s make it even confusing. We can’t figure it out so they won’t.

Michelle I.: (During Magdalena above) Negative one third. Oh, man.

Amy-Lynn: (During girls giggles) Got it. Got it.

Michelle I.: (During girls giggles) Oh, man.

Researcher I: Well- I don’t know where they are.

Amy-Lynn: Do you know where they’re kept?

Michelle I.: Yes I do. (Gets up from seat)

Researcher 2: Are you not gonna let me see? (Referring to Bobby’s work)

Amy-Lynn: No that’s not it. This is it.

Michelle I.: We messed up over there.

Researcher 2: Oh, what do you have here?

Bobby: Um, we’re working on it… (Covering up his work)

Researcher 2: What do you have here?

Amy-Lynn: (During Michelle I. below) This rule.

Michelle I.: We’re making a rule and then we have to find the problems.

(Michelle I.’s paper shows:)

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<td><strong>B:</strong></td>
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<td><strong>3:</strong></td>
<td><strong>9</strong></td>
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</tbody>
</table>

Researcher 2: Cool.
That’s (his?)

That looks very complicated I think I’ll go to the next table.

(Students and Researcher 1 laugh)

Hmm.

We gotta figure it out ourselves.

That looks very complicated I think I’ll go to the next table.

(Students and Researcher 1 laugh)

Hmm.

We gotta figure it out ourselves.

Oh my goodness. Is it-

How about we go with the same um, numbers-

Uh, no it won’t turn on. (referring to calculator)

Shelly, how ‘bout we go with the same numbers. Shelly?

(To Amy-Lynn) This is supposed to give me this? Is that what

you’re telling me?

Oh, no, this is- that was from this. But we’re making it even more

intense so we have to figure out the numbers.

No how about- no how about we keep the same numbers?

Of course I would take a broken calculator with my luck.

I’ll get it. I got one. (Gets up from seat)

Today is not gonna be a good day.

How fast is one billionth of a second?

(Takes Bobby’s papers) Get the papers away from him.

No come on. I have the- I’ve been working on them. (Magdalena

hands him back a paper). No that’s not mine. The one on the

bottom (points to stack of papers).

Bobby stop working on this stupid-

E equals M C squared-

It’s energy equals mass times speed of sunlight to the second

power. (Magdalena and Michelle I. giggle) I’m so smart.

So wait, if it’s two-

How about let’s keep all the numbers from the square and just

figure out the answers.

Yeah, I was gonna say that.

(To herself as she’s writing) Two, four, six, eight, ten, negative

two. (Aloud) We sort of, like, skip around ‘cause it goes from ten

to negative two.

That’s good. Makes them more confused.

(Arriving back to table) Wait, now what’d you get?

We’re using the same numbers on the side.

I don’t think that’s what she asks us to do. Didn’t she ask us to it

like (what are the things about them??)

Yeah.

To make a problem, right?

That’s what we’re doing. Can we use the man standing thirty feet

away from a brick wall?

(Using calculator) So it’s two times two minus thirteen-

It’s not times two, it’s times the number. Oh.

It is two.

(Gasps) We could do that! Box… (whispers to herself) box plus…
Magdalena and Amy-Lynn

What about box to the triangle power?

Amy-Lynn

No we can’t do that ‘cause then how could you find out the triangle?

Bobby

Who knows?

Amy-Lynn

It is a never-ending problem.

Magdalena

No it’s not a-

Michelle I.

So what if it’s two? So it’s two times two minus thirteen.

Amy-Lynn and Magdalena (with Michelle I. above)

…times two minus thirteen

(Bobby)

Let’s put the Pythagorean theorem in there.

Magdalena (giggles)

How can you do that?

Amy-Lynn

Sine, cosine, tan.

Bobby

You square the sides of this, you get the triangle.

Magdalena

Bobby.

Michelle I.

No wait it sh- it should actually be-No wait- Okay. Two times two minus thirteen equals. A hundred nine. Oops

Michelle I. (punches numbers into calculator)

Two times two minus negative one third.

Bobby

Wait if you square the sides of this you can get the cube.

Magdalena

Okay. Can I (two unintelligible words)?

Michelle I.

Hold on where’s the dash sign?

Magdalena (Pointing to Michelle’s calculator) Dash sign? What are you talking about? Negative?

Michelle I.

Ohhhh.

Bobby

Wait. If you square the sides of the triangle you can get a cube.

Michelle I.

Two times two (punches numbers into calculator)

Amy-Lynn (To Michelle I.) You turned it off. (giggles)

Michelle I. (giggles)

Two times two minus one third- oops.

Magdalena

Come on Shelly.

Amy-Lynn (During Michelle I below, to herself or Bobby)

We need to get the sharps. Do you see any of the sharps? Do you see where the sharpies are?

Michelle I.

I can do this.Two times two plus negative one third divided by-

Magdalena

by negative three.

Michelle I.

Negative-

9:27 Magdalena

How do you get decimals on there?

Michelle I.

One- it’s one of these buttons here. Um-

Magdalena

How will you get, uh-

Bobby (To Amy-Lynn) I don’t know.

Magdalena (To Michelle I.) How do you get fractions there?
Amy Lynn: Get the book.

Magdalena: It would be nice if we knew how to use it.

Michelle I.: The sharpie (?) calculators are right back there somewhere.

Amy Lynn: Where are they? Sharpie calculators are weird.

Bobby: We’re not that complex.

Magdalena: Minus-

Amy Lynn: Wait would it be, wait-

Magdalena: No defi- what- th-

Amy Lynn: We have to do, da da da da da- We have to do a fraction.

10:05 Researcher I: You’re doing fancy things here.

Amy Lynn: Yeah, but-

Bobby: I had like a more fancy thing but they wouldn’t listen to me.

(others giggle as Researcher I reads Bobby’s work). What’s the problem? That’s a nice answer. See these- no look it’s easy. These two cross each other out. Except for the mass part. (To himself) Plus two, and that’s minus three. Minus mass equals triangle.

Magdalena: (During Bobby above) I thought they were in the filing cabinet but they’re not. How did-?

10:41 Bobby: See you can simplify it. So it like, makes it easier.

Amy Lynn: These ones may not able to fractions.

Michelle I.: Bobby’s doing something weird. We’re doing something besides what he’s doing because-

Bobby: And see since the mass is nothing, it’s just triangle minus three.

(Bobby’s paper shows the following:)

Magdalena: Bobby. Go (unintelligible word- asks him to find calculators)

Amy Lynn: Maybe they’re in her closet.

Bobby: No I wasn’t gonna use them. I was just gonna use this one.

Michelle I.: (To Amy-Lynn) Wanna go check?

Magdalena: It’s stuck (referring to Michelle I.’s calculator).

(To Michelle I.) Figure out how-

Bobby: A man is standing thirty feet away from a brick wall. (Starts whispering) How long will it take the (?) to get to the wall?

11:13 Michelle I.: It’s negative nine (using calculator).
Magdalena: Hold on. (Looking up something in calculator handbook). Oh, here. Negative number. No, that’s not it. Um, the practice negatives.

Bobby: You missed, uh, the appendix.

Magdalena: I used it, it says one through eight.

Bobby: Yeah, but still, ya know. What’s an index?

Michelle I.: (To herself) Negative nine divided by-

Bobby: It’s the appendix.

Michelle I.: One. One of these buttons here? (Referring to calculator)

Bobby: It might be on second function.

Magdalena: You see that-

Bobby: We can’t find the negative sign.

Magdalena: We found the negative, we can’t find the fraction.

Bobby: Oh, don’t you push the A-B-? Here (leans toward Michelle)

Michelle I.: Ohhh.

Bobby: You push the, um, A-B-C button.

Michelle I.: Where’s the A-B-C button?

Bobby: I don’t know.

Magdalena: It’s on the other calculators, the A-B-C?

Bobby: No. It’ll be on this one, too.

Michelle I.: There’s A-S.

Magdalena: Look on her desk, maybe it’s there. (To Amy-Lynn)

Bobby: We’re just gonna, like, destroy her desk.

Michelle I.: It’s already destroyed.

Magdalena: How ‘bout divided by thr- point three. ‘Cause it’s the same thing.

Michelle I.: Divided by three three three three three three. Point one three three- no point three three three three three three.

Michelle I.: Negative thirty.

Magdalena: Whoa.

Michelle I.: So, four times four, is four…

Magdalena: (To Amy) Amy, never mind.

Michelle I.: … plus negative thirteen, is three divided by point three, is ten.

Researcher 1: (During students above) As I’ve gone around to the tables, I see you doing some really interesting things. Um, Ankur you want to show us what you have? Would you come show everybody what you have…at your table?

Magdalena: (To Michelle I.) Ten or negative ten?

Michelle I.: Ten.

Amy-Lynn: Is it negative point three?

Magdalena: Yeah.

Michelle I.: Oh wait. I didn’t put negative point three so it’s negative ten.

Magdalena: That’s what I thought.

Amy-Lynn: What is it? Point what? Two and negative thirty

Bobby: (Singing) A never-ending story.

Michelle I.: (Whispering) Magda, are you chewing gum?
353 Magdalena No. (giggles)
354 Michelle I. You know that’s not allowed.
355 Bobby Yea it’s like illegal in forty eight states including Hawaii and Alaska.
356 Magdalena Shut up, Bobby.
357 Bobby It’s legal in Montana and Arizona.
358 Michelle I. (giggles)
359 Bobby Well, it is.
360 Michelle I. You know that you look really weird to the people that are gonna be watching this.
361 Bobby (Shrugs)
362 Amy-Lynn They’ll edit it out. Hopefully. For Bobby’s sake.
363 Michelle I. But then how could you get anything in on us?
364 Amy-Lynn (Giggles)
365 14:09 Michelle I. What about the sixth one?
366 Amy-Lynn He’s doing a problem.
367 Michelle I. Six times six is thirty six.
368 Bobby Yeah but they edit these videos.
369 Michelle I. Minus thirteen.
370 Magdalena I got some weird answer.
371 Amy-Lynn Let me check.
372 14:20 Michelle I. Let me see. Let’s see. Oh, whoa. Let’s hope (microphone cuts out) it’s not right.
373 (NOT MIKED, so may be slightly inaccurate transcript)
374 Amy-Lynn? Thirteen... twenty three... divided by
375 Magdalena? Divided by...
376 14:41 Michelle I. It’s really weird, it must not be right.
377 Magdalena No, let’s look at- no that’s two
378 Michelle I.? Seventy (or thirty?) six... sixteen
379 14:55 Magdalena That’s sort of like..?
380 Amy-Lynn ??? ...nightmare.
381 (Students may be talking during Researcher 1 below, but it is inaudible.)
382 15:02 Researcher 1 Well, uh, Ankur could you explain to them now why, he thinks all of those would give you the same set of ordered pairs. Does everybody agree with that? I know you’ve been working on similar things. Could you tell if they are right? I think your friends at this table agree. Could you go through them and tell us why you think they are alike?
383 (Ankur has written on the overhead.)
Well this is like the small-shortest way to say that’s like the way.
most people would think of the pattern.

Okay, it’s just complicated people who would think of it the other
way.

This just gets more complex and see the first, behind the equal sign
this side equals seven this part equals seven, behind the equal sign.

Okay
And so does this side.
Do you know- I don’t see a box in that second one.
Yeah.
Oops.
You forgot to put it. You put a box after the three.
It’s minus three plus box. That’s behind the equal sign.
(Ankur corrects the equation as shown below.)
Okay, could you explain why you think the next one works? I think there is some confusion about that.

This one?

Uh-huh.

This one. It’s just, you can divide the hundred box by a hundred so that turns down to one box.

We’re not quite clear is why one hundred times the number in the box, divided by a hundred would be the same as the number in the box. Could you explain that? And tell us why that works? Maybe write it over to the side or something. Just that part, a hundred times the number in the box divided by a hundred.

That equals one. A hundred divided by a hundred equals one. And fifty-six—no I mean seven hundred divided by one hundred equals seven. So that’s one box plus seven equals triangle.

Can you write that down, those pieces that you just said?

Those pieces.

That tells me a lot.

If I write the things...

Just the two pieces you said were really nice.

She wants you to say a hundred...

I know but if I write it on the overhead.

Take another piece of paper.

Okay.

Can you find another clean one under there?

Rewrite that piece of it.

I can help you find one, they’re under here, the clean ones.

(As Ankur works on the overhead, the microphone comes back. Bobby’s table was chatting and working during Ankur and Researcher 1 above, but it was inaudible until now)

(Ankur added to the slide as shown below.)
Bobby: ...two and negative thirty. (Inaudible) two equals negative three.

Three equals ten. So four equals one.

Michelle I.: ...six six(??)

Amy-Lynn: (To Bobby) We took off the last number.

Bobby: Yeah.

Amy-Lynn: (To Michelle I.) We took off the last number, it compares.

Bobby’s (inaudible) thing.

Bobby: (inaudible during Amy-Lynn above) ‘Cause look. Two equals negative thirty, minus two equals number three. Four equals ten, minus four equals one. Six equals seventy six point six six - move the decimal, equals seven point six six.

Amy-Lynn: (Writing to herself) Moving...

Bobby: It moves the decimal place over one and like it cuts it one number.

Researcher 2: The people back there said to me...

Bobby: I don’t know how I thought of that.

Researcher 2: You see this piece here, are you all looking? Right, which Ankur wrote like that. What they said to me was that they could write that top part like this.

(Researcher 2 rewrites one of Ankur’s equation on the overhead as shown below.)

Bobby: (During Researcher above) Isn’t that weird?

Amy-Lynn: It’s cool.

Bobby: No, it’s scary. It’s like dividing by zero. Whoa.

Researcher 2: You can. Could you explain that to me I don’t understand that?

Bobby: (During Ankur explaining to the class) It’s divide by zero, so it messes the whole thing up.
Ankur: Because one hundred is outside the parenthesis and one hundred times box equals one hundred box and one hundred times seven equals seven hundred so that’s one hundred box plus seven hundred.

Researcher 2: How do you get, how do you end up getting what you finally got? How do you go from there to get this result here.

Ankur: You simplify.

Researcher 2: I’m sorry I didn’t want to mess that one up. Can you put the one back in? Thanks. Can you show me how we go from here to here and why that works?

Ankur: From here you can go

Researcher 2: To yours, yeah, how do you simplify that?

Ankur: From here you go to here

Researcher 2: No, the one under.

Ankur: This one? How do you simplify?

Researcher 2: How do you go from here to here? Can you show us?

Magdalena: How about we square root number two?

Ankur: Okay, one hundred is outside the parenthesis so that equals one hundred box that’s right here.

(Microphone turns back on)

Magdalena: (During Ankur above, pointing to Michelle I.’s work) And- and that minus root thirteen equals- will never give this, uh, you know- no- divided by two.

Bobby: Last time were you dividing by one third or point three?

Michelle I.: Shhh.

Bobby: (Whispering) Last time you divided by one third or point three?

Amy-Lynn: (Whispering to Bobby) Point three.

Bobby: (Whispering) That’s why.

Amy-Lynn: (Whispering) Whoa.

Bobby: It’s the same.

Michelle I.: It’s negative four.

Magdalena: (Crosses out her previous work)

Researcher 2: You’ve all seen what Ankur’s doing here. What do you think Angela?

Angela: That’s like what we were doing like crossing out the zeros and you asked us why and we want to know if Ankur knows why.

Researcher 2: Now, can you tell me why it works? I was confused.

Angela: We don’t know why. We want to know if Ankur knows why.

Researcher 2: You want to know if Ankur knows why, well

Michelle I.: (Whispering) Magda. Magda we have to pay attention to Ankur.

Magdalena: We’re supposed to- we’re supposed to (inaudible)

Researcher 1: Ankur your explanation didn’t convince her that anybody knows why. Can you try, can you think of any other way to say it?

(Ankur explains the work on the overhead as shown below.)
Ankur: The one hundred cancels out the one hundred.
Researcher 2: What do you mean by cancels out?
Ankur: Because it’s on both sides so you can eliminate it.
Researcher 2: That’s where you lost me.
Ankur: Then all you have left is box plus seven. That’s one box plus seven equals triangle.
Researcher 2: I don’t understand when you say cancels out. I see one hundred divided by one hundred here.
Ankur: That equals one.
Researcher 2: I can see that that’s one. I’ll buy that that is another name for one.
Ankur: So that equals one box and seven. It’s still seven. So one box equals something.
Researcher 1: Ankur do you know what worries me about cancelling? Is that sometimes people see things you know and they just mark them out because they look alike. Could you show us an example with numbers where you have a hundred times something divided by a hundred and to be sure that it makes sense in numbers or something.
Ankur: How should I do that?
Researcher 1: What if you have a hundred times five.
Ankur: Five hundred.
Researcher 1: …divided by a hundred. Let’s say will you write that down and see? You have a hundred multiplied times five and then that divided by a hundred. Let’s see if you can tell us what happens.
Ankur: So this is on both sides of the equation.
(Ankur added 100 x 5 divided by 100 on the overhead as shown below.)
Researcher 2  That’s not an equation, Ankur. There is no equation there. There is no equation. Look at that that’s just a fraction.

Ankur  Then the fraction.

Researcher 2  Do you all see that? That’s where you confuse me. I see a hundred times five divided by a hundred. That looks like a fraction to me.

Ankur  Okay, then you can put-

Researcher 2  You guys were talking about equations even at the other table and I don’t see an equation.

Ankur  Then this equals five hundred and you can cancel out the two zeros.

Researcher 2  I don’t understand what you do when you cancel out zeros.

(Ankur has crossed out zeros as shown on the overhead below.)

Ankur  Because they’re on both sides. You can do anything on one side. Whatever you do to one side you can do on the other side.

Researcher 2  Um, Angela and Stephanie (both have raised their arms)

Angela  I finally figured out why you, uhm, cancel each other out cause they’re inverse operations. Like if you had one and you plus one and you subtracted one they cancel each other out. If you, had I don’t know it just cancels each other out. And you don’t even need the one up there because one box is just box so it would be box plus seven.

Ankur  Or you could do it like this. Five hundred over one hundred that simplifies to five over one. That’s five.
Researcher 1  But why? (Yet why?)
Ankur  What does why mean?
Researcher 2  Stephanie you were going to say something?
Stephanie  Oh no I was just going to say kinda what she said. It’s just what it
is you have to whatever if you take away a zero from one side
because you can do that. Because it’s I can’t it’s like whatever you
do to the one side you have to be able to do to the other side
otherwise you can’t do it.
Researcher 2  Okay, I think you guys are going to get into a lot of trouble when
you study algebra unless we really work this out. Because first of
all you are saying things about side and this is a fraction. There’s a
number on the top and a number on the bottom, that’s a fraction.
You are giving me examples that have to do with equations and
balances and that’s something else. So I really think we are going
to have to work this out otherwise you may fall into some trouble.
We have all the algebra teachers in this room, in horror, worrying
about what’s going to happen to you guys down the line. So we are
gonna maybe take a time out here. The question I want to leave
you with- Bobby you want to make a comment before I leave you
with a question? Go ahead Bobby.
Bobby  Oh, ‘cause like five hundred divided by one hundred is five.
Researcher 2  Very nice.
Bobby  Even if you cross the hundreds out it is still going to be five no
matter what.
Researcher 1  But she’s looking at it now as division with everything.
Researcher 2  Okay I’ll- that I’ll buy Bobby. I’ll buy that. Five hundred divided
by hundred is five. I am going to ask you something else here.
(Michael has raised his hand) Yes Michael?
(Researcher 2 places the slide shown below on the overhead.)
Well the time before you could not only write it as a fraction, you can also write it as an equation, as an equation. So when you write an equation then you can cross out things.

Well that’s where I get a little confused because this is an equation. We’ve written this as an equation. But you see when you talk about crossing out things this is one side of an equation and this is another. This is the numerator of a fraction and this is a denominator and I get very confused when you’re crossing things from the fraction and then you are talking about an equation.

Then change it into an equation and then you can understand it more.

(Whispering to table) Wait I have an idea. On the bottom of it put (five equals six…five?)

Okay, but we need to work on that. I think our time is out. But we have a visitor here for just one minute. Tell me your name?

Lindsey?

Lindsey, um, Lindsey sent me a wonderful problem. Lindsey’s a third grader and you know how I am about why. Don’t you all know how I am about why? And Lindsey, um, come say hi to this class, Lindsey. Lindsey- Lindsey discovered a- a- an interesting rule. And my question to all of you is to help, uh, Lindsey tell me why. If I can find a blank overhead.

(Loud laughter)

Lindsey’s- now these- You guys used to do this to me when you were third graders. Are we out of paper? Overheads? (Loud laughter)
Researcher 1: There should be some under there.
Researcher 2: I don’t see any more.
Researcher: Here you go.
Researcher 2: One piece is all I need. Okay what Lindsey said is the following.

Lindsey do you want to say it or do you want me to? Do you want to say it? This is what you wrote me. Remember?
Lindsey: Yeah.
Researcher 2: Go ahead. Why don’t you tell me and I’ll write- write what you want me to write. Tell the class.

Lindsey: Five times an even number you, um, split the even number in half and you add a zero and you get your answer.
Researcher 2: So five times an even number- and you said, for example, six, right?
Lindsey: Mm hmm (nods)
Researcher 2: She says you split the even number in half. Spitting six in half gives you…

Students: Three
Researcher 2: Three. She says then you add-
Lindsey: A zero.

(Researcher 2 writes down the example using Lindsey’s rule.)

Zero. And that’s how you get five times six equals thirty. Right?
Now let’s try it with another even number. Five times- give me another even number.

Student: Forty-six.
Researcher 2: Forty-six. You split the even number in half, right? Split forty-six Jeff and what do you get?

Students: Twenty-three.
Researcher 2: Twenty-three, right? So five times forty-six is going to be- adding a zero-

Students: Two thirty.
Researcher 2: Which will be two thirty. Does that work?

(Researcher 2 has added the second example to the overhead.)
Michelle I. Cool.
Bobby Whoa.
Reseacher 2 Right? So you can- you can try some oth- now Lindsey has
another one for you. (Class laughs) ‘Cause Lindsey does not want
to limit her theories to just even numbers. She also wants odd
numbers, right? So what’s your rule Lindsey about odd? This is for
the even, right?
Jeff I wish I learned this. (Class laughs)
Reseacher 2 Jeff, Lindsey invented this. Okay, let’s go on.
Lindsey Five times an odd number you, um, make the odd number, um, the
next smallest number. Then you split that in half again and then
you add a five.
Jeff Yeah so five times three would be fifteen.
Reseacher 2 Obviously. (Class laughs) You got that? I had to think. What is it-
when you took, you- you took the odd number again, Lindsey.
You reduced it by…
(Researcher 2 has written this example on the overhead.)
Lindsey Um, the next even number.
Reseacher 2 And then you-
Lindsey Add a five.
Reseacher 2 And you- when she said add a five, you append a five, right? A
five at the end. Let’s try another one. Yes?
Jeff (To Lindsey) What were you doing when you thought this up?
Lindsey I was just doing some problems and I just figured it out.
Student Wow.
Bobby: Geez.
Researcher 2: Angela you have a question?
Angela: It’s the same thing, how could you figure this out?
Researcher 2: Oh no, that’s your problem. (Class laughs) Lindsey has the
(inaudible) up here, Angela. This is the problem. The theory is,
let’s write this down, five- let’s write the first one down, Lindsey.
Lindsey: Five times an even number, right?
Researcher 2: Mm hmm. You split it-
Lindsey: How you obtain it- you split it- you split the even number in half,
right? And what do you do?
Lindsey: You add- you um, put a zero at the end.
Researcher 2: And, um, append, alright? Append a zero. Right? You’re not
adding, like an addition operation, okay? And the next one is five
times an odd number, right?
Lindsey: Yeah.
Researcher 2: You-reduced it by
Lindsey: You, um, make it the next even number-
Researcher 2: Okay, so you make the next even- smaller or larger?
Lindsey: The even smaller.
Researcher 2: Okay. You subtract by one is that okay?
Lindsey: The even smaller.
Researcher 2: Okay. You subtract by one is that okay?
Michelle I.: Yeah that’s that’s like what you’re doing.
Lindsey: Yeah.
Michelle I.: That’s- that’s exactly what you’re doing- subtracting by one.
Researcher 2: So you go to the next smallest even- next smallest number. Even, right?
Lindsey: Yes.
Michelle I.: (Whispering) You subtract one.
Researcher 2: And then what do you do?
Lindsey: You add- you put, um, a five at the end.
Researcher 2: And? Subtract a one, don’t you? (Pauses)
Michelle I.: (During Researcher 2) Whoa. We leave in five minutes those two
periods went by fast.
Amy-Lynn: Thank god.
Michelle I.: Thank god (inaudible)
Amy-Lynn: Yeah.
Bobby: Do we still have one more period?
Michelle I.: Yeah. But then we’ll be in science, and we’re not allowed to stay.
I want to stay, though.
Amy-Lynn: So stay (?)
Researcher 2: (To Lindsey) What’s the divide number? And split it, and split it,
Researcher 2 writes Lindsey’s rule in words on the overhead as
shown below.)
Researcher 2: So your job is, why does this work? Because you know you could- you could make - I’m gonna tea - all the math you’re telling me oh, you just cross out zeros or you just- or you just add this. You have to know why things work because if- and does it always work? I want you to write up why it works. Lindsey, would you like to know why this works? Do you know why it works? No. So she’s discovered it, so now she needs some help. And maybe she can come back tomorrow when you all can tell her. I mean she looks to the wisdom of the seventh grade- almost eighth grade class to guide her in her future. Let’s give her a round of applause. Let’s give her a round of applause. (Class claps). Thank you very much Lindsey. This was lovely. Any theories? Okay, thank you, Lindsey.

31:38  Bobby: I know, I know why.

Researcher 2: Bobby knows why?

Bobby: Yeah.

Researcher 2: Why?

Bobby: Because, like, with an even number, it has to have a zero at the end. (Researcher 2 begins to speak during Bobby’s table’s conversation below) Five times any number has zero. Five times an odd, had a five.

Michelle I.: It’s true.

Amy-Lynn: It is true.

Bobby: (partially inaudible)... always pick me.

Michelle I.: (laughs and looks at camera)

Amy-Lynn: Way to go, Bobby.

Bobby: Thank you, Amy.

Michelle I.: Convincing me that you’re smart.

Bobby: I know it.

(During Bobby, Michelle I, Amy-Lynn above)
31:47 Researcher 2  My final- my final question to you before you all disappear- if you
could just look at this for a moment. Yes, Michael.
(Researcher 2 presents the slide shown below on the overhead.)

779 32:00 Michael  Well you said we’re crossing zeros without an explanation. Well
we are - we do have an explanation. We are taking off a number- a
certain number off each side and- to balance it out. Like, if we take
off a number from each side and it’s the same thing as crossing it
out- if you take off a zero off each side- you take off a- you divide
by- (writes on paper) you divide a hundred by ten on top and that
equals ten. And you do it each number, you’re not just crossing it
out, you’re dividing it. When you take out a zero you’re dividing a
hundred by ten equals ten, it’s like crossing a zero out at the end…
a shortcut…

32:48 Researcher 2  So what you’re say- what I’m hearing you say, then, partially is
that, um, it’s okay to divide- not saying each side- divide the top of
a fraction- the numerator by a number- as long as you divide the
denominator by the same number. So in other words if I had a
fraction four eights, right?

32:48 Researcher 2  So that’s the same- I could divide both the numerator and
denominator by the same number.

32:50 Michael  That’d be saying four over eight (?)

32:50 Researcher 2  It would be a half.

32:50 Researcher 2  So I could say, for instance, I could divide them both by four,
right? And why does that work? Why does that give you one half?

32:50 Researcher 2  What does it mean to divide the numerator by four and the
denominator by four?

32:50 Student  (inaudible)
Right but what am I really doing?

You just-

Why does that work?

(Researcher 2 writes the fraction work as shown below.)

Because it’s the same thing as a half except that it’s in smaller pieces. So all you gotta do is add the four eighths, like four eighths they equal two eighths. One half because it’s just like one half cut into little pieces.

Anybody else have any ideas? Michael had a- an explanation.

You’re just making four eighths, which is one half, more complicated.

When you write it as four eighths it’s a more complicated version of one half?

‘Cause like four times one and four- over four times two is just the same as four eighths, it’s just more complicated.

What made it more complicated?

Like adding times one and times two.

Okay, anybody else have any thoughts on that? Anybody else have any thoughts? That’s it? I think our time is out. Okay well it’s something to think about for tomorrow. Don’t forget Lindsey’s problem. Isn’t that neat for a third grader? Were you all impressed?

(To Amy-Lynn) Do we have to write up something about that?

No. I don’t think so. We just have to think about it.

(Students exit)
Camera View: Jeff’s Table 1 of 2(Jeff, Michael, Angela, Sarah)
Date of filming: 05/16/1995
Harding public school, Kenilworth NJ, Grade 7
Advanced Guess my Rule (AGMR): Inverse: problem
Transcribed by: Vanessa I. Bell
Date of transcription: 6/2009
Verified by: Andrea DePaolo
Date of verification: August 2010
Length of session: 00:46:11

<table>
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<th>Time</th>
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<td>08:09</td>
<td>(Students enter classroom and sit at tables)</td>
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<td>14:03</td>
<td>Jeff, Michael, Angela and Sarah test speaker and prepare for the session by getting paper and pens)</td>
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| 14:03 | Researcher 1 | Are you ready to start? I’m Dr. Dann and I work with these people at Rutgers. Dr. Maher who comes in a lot of times and you probably, some of you, have already guessed what we are going to do today by those sheets that are on the table. No, no? Do you remember doing something called Guess My Rule about two years ago with Dr. Davis? Do you remember something about that? Let’s look at an example. Suppose that, can you see it, that I am thinking of a rule like this, and what I am going to do is add three to some numbers, and suppose, that umm, I put a zero in the box, then what will I have to put in the triangle? (Researcher 1 writes on the overhead:)

| 17 | Michelle I. | A three. |
| 19 | Researcher 1 | A three, okay, suppose I put a one in the box, what would I have to put in the triangle? |
| 20 | Student from another table: Four | |
| 22 | Researcher 1 | A four, all right, okay, so this is pretty easy, right. (Researcher 1 has entered three entries for this rule on the overhead.) |
Researcher 1 So far we’ve limited ourselves to what kind of numbers? What kind of numbers are we using there? We are using positive numbers, does anybody know another name we could call those ever? What other kinds of numbers do you think we could use?

Student Negative numbers.

Researcher 1 We could use negative numbers. Okay. What about if we put in negative one what would we have to put in the triangle?

Student from another table Two

Researcher 1 Two. And if we put negative two we would have to put one in the triangle.

(Researcher 1 continues and fills in two more entries in the table.)

Researcher 1 Okay, so what, let me ask you a little bit harder question. This is probably a very easy rule for you but what if I put a ten in the triangle. What would I have to put in the box? Brian?

Brian Seven

(Researcher 1 writes ‘10’ in the triangle column and then enters ‘7’, from Brian’s response, in the box column.)
Researcher 1: A seven, and how did you get the seven?
Brian: Because ten minus three is seven.
Researcher 1: Okay, so the rule that we started off with said that we were adding three but when you were doing it backwards you subtracted three. So addition and subtraction are inverse operations, one undoes the other. Do you know any other operations that work like that?
Bobby: Multiplication and division.
Researcher 1: Multiplication and division work like that, one undoes the other. They could be called inverses. Do you know of any other kind of operations that work like that? That may be all you’ve had so far. What if you square a number? What if I start with the number five and square it? What do you get?
Angela: Twenty-five.
Researcher 1: What does five squared mean?
Amy-Lynn: Five times five.
Researcher 1: It means five times five and so you get twenty-five. Does anybody know what you would call it if you went back the other direction?
Ankur: The square root.
Researcher 1: The square root. Have you talked about that some, the square root, and does anybody know how to write it? The square root, could you come and write that for us and do the square root of twenty-five?
(Angela, from Jeff’s table, comes to the overhead and writes:)
\[ \sqrt{25} = 5 \]
Okay, would be five, okay, so do you think squaring and square root are also inverse operations right? What I want to do is I’m gonna think of a rule and I am going to put down some numbers that work in the rule and see if you can guess the rule. Do you remember doing that now with Dr. Davis? One day maybe? What do you think? My rule says that if I put a one in the box, that in the triangle I’d have to put one-third and if I put a two I would get two-thirds and… Would you talk to some of your friends there and see if you can think of a rule that would work?

(Overhead shows:)
18:34 Angela Okay.

95 Michael The box means one-third.

96 Angela It looks like whatever the number is divided by three, like over

97 three.

98 Michael The box times one-third.

99 Researcher 1 We want a rule that will work on every line.

100 Angela Let’s check it again. The box times one-third equals the triangle,

101 right?

102 Sarah (Asks something inaudible to Angela.)

103 Angela What? What the rule is Sarah.

104 Sarah What?

105 Angela We’re trying to figure out what the rule is.

106 Sarah What Rule?

107 Angela That.

108 Sarah Oh. The box times… (looks puzzled.)

109 Angela Times one-third equals the triangle.

110 Sarah One-third?

111 Angela Yes.

112 Jeff Just look, zero and zero and one and one-third equals zero and one-third.

114 Angela (Laughs) Three times one-third equals one Sarah because it’s

115 three-thirds. (spoken in a condescending tone)

116 Sarah I know that.

117 Angela (laughs) God. (mutters to herself while fixing her name tag and

118 shaking her head.)

119 Angela Michael, what are you doing?

120 Michael (no response given)

121 Michael Are we done?

122 Angela I don’t know, I guess.

123 Sarah Angela, you know what my name is spelled backwards?

124 Angela Haras, haras, harass.

125 Sarah (Laughing) I didn’t know that.

126 Researcher 1 Okay, alright. But what about a rule, is there something you can

127 do that would work on every line?

128 Angela That’s not good. Sarah, go and sing into the microphone this time.

129 Angela Mike, do you know what her name spelled backwards is? Harass.

130 Researcher 1 I started with a really hard one didn’t I?

131 Angela That was easy. That was easy, we got it.

132 Researcher 1 I’d like to come back to that one. Could we come back to that one

133 later?

134 Angela Yeah.

135 Researcher 1 Let’s try another one. I think that one may not be a rule. Suppose

136 that when I put in say negative three in the box, then what I get is

137 negative twelve and when I put negative two I get negative eight.

138 Angela Times four. Because a negative times a negative equals a positive.

139 Researcher 1 and so forth like this. What do you think of that rule?
(Researcher 1 writes the table below on the overhead)

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140
141

Angela Oh that’s easy. ( Raises hand.)

Researcher 1 Angela you think you know that one already?

Angela It’s box times four equals triangle.

Researcher 1 She thinks it is box times four is equal triangle. Anyone else agree with it? Could you come up and write that for us?

(Angela goes to the overhead and writes $\Box \times 4 = \triangle$.)

142
143

144 Angela
145 Researcher 1
146 Angela
147 Researcher 1
148
149
150

151
152

153 Researcher 1
154
155 Angela

Could you show us how some of the ordered pairs work there?

Like how it is that one and four. Could you show us some of that?

One times four equals four.
Researcher 1: Okay, two times four equal eight. Is everybody seeing that? All right, thank you Angela.

Dr. Dann: Um, what about if I put something here like sixty in the triangle. Could you tell me what would have to be in the box? Bobby?

(Researcher 1 writes ‘60’ in the triangle column.)

Bobby: Fifteen.

Researcher 1: Fifteen, do you think so? And how did you do that Bobby?

Bobby: Sixty divided by four.

(Researcher 1 writes ‘15’ in the box column.)
Researcher 1  Sixty divided by four so he’s using the idea of the inverses.
Researcher 1  Somehow I think you ought to be able to do this one. (she goes
Researcher 1  back to the previous sheet at the overhead) Want to try again? Did
Researcher 1  you notice, you know that one that fast already.
Researcher 1  (Overhead shows:)

---

Michael  We already knew it.
Angela  We knew it.
Researcher 1  Oh you knew it. All right, Michael come up and see if, what other
Researcher 1  people think about yours.
Michael  (comes to the overhead) Uh, write that rule? Just write it down.
Researcher 1  Yeah and see if some other people agree with you.
Researcher 1  (Michael writes on the overhead: $\square \times \frac{1}{3} = \triangle$ as shown
Researcher 1  below:)

---
23:39 Researcher 1: What do you think? Did some other people get that rule or if you didn’t get that rule would you see if you think it works? Michael would you mind coming back and convincing some of them, show why you think it works.

Michael: Two times one-third.

Researcher 1: Can you write some of those out?

Michael: Two times one-third equals two-thirds and two times one-third is one-third plus one-third and that equals two-thirds.

(Michael writes on the overhead 2 \times \frac{1}{3} = \frac{2}{3} and \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.)

Researcher 1: All right. Are you convinced or you want to see him do some of the other pieces. I think they’d like to see you do some of the other pairs. They may be convinced that two and two-thirds work, that this row works, what about some of the others?

Michael: Three times one-third equals one, because there’s three of the one-thirds and that means that three of the one-thirds equal three thirds, which three-thirds equal one.

(Michael writes on the overhead 3 \times \frac{1}{3} = 1 and \frac{3}{3} = 1.)

Researcher 1: Who is convinced that his rule works? All right Michael, thank you. Do you know what though it is a little strange to me because I
had another rule in mind and I think it works too. Um, here’s what
I was thinking about. I was thinking about the box divided by
three. The number in the box divided by three is equal to the
number in the triangle. Do you think that would work too? Do you
think that would work?
(Researcher 1 writes on the overhead:
\[ \Box \div 3 = \triangle \].)

Students? Yes.
Jeff Possibly. Because they’re inverse op...(cut off)
Researcher 1 So you think that there could be two rules for some, some tables?
So are you telling me that if you divide a number by three that you
get the same answer as if you say one-third times that number?
Would that always work? Do you think that works for every
number? I’m not sure whether you agree with that or not.
Jeff Inverse operations, wouldn’t that work?
Researcher 1 So you think you could talk about inverse two different ways in
this case. You could say a third?
Jeff Yeah.
Researcher 1 All right. What if I gave you in the triangle the number two, what
number would have to be in the box? Okay, Amy-Lynn.
Amy-Lynn Six.
Researcher 1 Does everybody think six will work? All right, all right. Very good
you’re off to a good start. What about, maybe this one is a little
harder for you.
(Researcher 1 writes on the overhead:)
(Jeff’s table looks at the new table on the overhead, trying to figure out the rule. Then, Michael, Jeff, Angela, and Sarah copy the table on their own paper.)

28:47 Angela Hey Mike, this side keeps adding four.
Michael This side.
Angela It’s like consecutive numbers. (Pointing to Jeff’s paper.) Look, it looks like on this side, the triangle side, it keeps adding four.
Jeff Three and one, that sounds like a four.
Angela Negative three.
Jeff Positive four.
Angela No you can’t add four Sarah because it’s just on this side. It doesn’t work for this side. So what’s our new rule?
Sarah This is too difficult. (Hits hand on head.)
Angela Negative three plus four is not negative eleven.
Jeff It’s close.
Angela It’s negative seven, which is down here.
Sarah Negative three plus four is one.
Angela This is plus eight here.

30:16 Researcher 1 Amy-Lynn thinks she has a rule. Does anybody else?
Angela No.
Researcher 1 We will give them another minute or two? Then we can let you put that up and see what they think?
Jeff They did exactly what we thought.
Angela (To Jeff) Are you by any chance tired?
Jeff Me? Very. I didn’t sleep last night.
Angela This is plus, minus five triangle. Three doesn’t go into eleven.
Jeff I was so looking forward to a nice easy math today.
Angela Yeah, things are never easy.
Jeff If we worked on our paper, it would have been all good. Did you finish your paper?
Angela Sarah did. We were working together to figure some of it and then she wrote the letter.
Jeff  Sarah you’re good.
Angela  Do you know what Sarah’s name spelled backwards is? Haras.
Jeff  Haras? You have too much time on your hands.
Angela  No, no, she thought…
Researcher 1  Does anybody else think they have it yet? The rule? How are you doing over there?
Student from another table  We got one.
Researcher 1  You got one? Let’s let Amy-Lynn put hers up and see if you agree with that one. Amy would you do that for us.
Amy-Lynn  Should I just do it in the thing, cause this is what we came up with, like times four plus one, that was like our rule, to do with everything in the box.
(Amy-Lynn writes on the overhead:
\[ \square x 4 + 1 = \triangle . \])
Researcher 1  What do you think of Amy-Lynn’s rule? That same rule you got? Or you couldn’t? What about over there at the far table? Do you agree with it?
Students  (whisper) Yes.
Researcher 1  Do you want her to convince you or do you agree with her?
Michael  I agree.
Researcher 1  Do you agree? You Agree?
Michael  Yes, I do.
Researcher 1  Thank you Amy-Lynn that’s really nice. What if I put forty-nine in the triangle? Could you tell me what would have to be in the box?
(Michael and Sarah have raised their hand with the answer.)
Researcher 1  Already? Michelle.
Michelle I.  Twenty-four.
Researcher 1  Michelle thinks twenty-four.
Bobby  Twelve.
Researcher 1  And you think twelve.
Magdalena  It’s twelve now.
Researcher 1  Lots of people are thinking twelve.
Michelle I.  I did it wrong. I calculated wrong.
Researcher 1: Do you agree with that twelve? What did you do? Does everybody agree that twelve works? You had twelve times four. Angela?

Angela: Forty-nine minus one divided by four. Forty-nine minus one is forty-eight and you divide by four.

Researcher 1: So Angela is saying that she took the forty-nine and subtracted one and then divided by...

Angela: Four.

Researcher 1: Four. Do you think you could write that for us where we had triangle equal to something with the box?

(Researcher 1 writes on the overhead:  
\[ \triangle = \square \]

(Angela goes to overhead)

Angela: I would have to do it backwards?

Researcher 1: Can you do that? Do it either way you want to.

(Angela writes on the overhead)

\[ \triangle - 1 ÷ 4 = \square \]

Researcher 1: All right. Now I heard you say something different from that before, you took forty-nine minus one and you got forty-eight and then what did you do?

Angela: Then I divided forty-eight by four.

Researcher 1: Oh and you have division there. I see what you did. Okay, all right that’s good. I read that as a plus sign.

Angela: Sorry.

Researcher 1: Does everybody know that you could also write this, this way if you want to.

(Researcher 1 writes on the overhead:  
\[ \triangle - 1 ÷ 4 = \square \] )
Researcher 1: It is just a different way of writing. That’s really nice.
Angela: Sure.
Researcher 1: Do you want to try another one, see if we can stump you?
Angela: I went up twice.
Michael: I went up once.
Researcher 1: Let’s say that when we do this we get zero and when we put in
negative two that we get two.
(Overhead shows:)

Jeff: (Whispers to Ankur at Stephanie’s table.)
Angela: (Angela copies the table to her paper.)
Angela: It’s zero, two, four, six, eight
Jeff: If three, twelve equals goes up by nine, two ten, goes up by eight.
Angela: Times two plus six.
Jeff: What?
Michael: Three times two.

Angela: Times two plus six?

Jeff: Why can’t you just say that you add the number it goes up by?

Michael: Two times three is six.

Angela: (While working on her paper.) Times two plus six. (Speaking to the group) Let’s just use our paper.

Jeff: Nah, listen to me!

Angela: Go ahead.

Jeff: From zero, there’s six then add one: seven.

Michael: (Excited) No, I got it, got it, got it!

Jeff: Add two: eight. I’m adding one.

Angela: Uh-huh.

Jeff: Look.

Angela: I’m not understanding what you are trying to say.

Jeff: Zero.

Angela: No, do it on the paper.

Jeff: (Pointing to Angela’s paper.) Zero plus six is six. One plus seven is eight. Two plus eight is ten. Three, it’s going up by one.

(Angela’s paper shown below:)

Angela: Yeah, but you have to have the same exact rule for each one.

Jeff: Well it’s gonna come up eventually. I’ll figure out something for there, but right now, this looks nice.

Michael: I figured it out.

Angela: Yeah, but Mike has a rule.

Researcher 1: Who thinks they have the rule?

Michael: I do.

Researcher 1: Oh, Okay.

Jeff: What is it?

Researcher 1: Magdalena you want to put the rule that you have up and see if they agree with you?

Michael: Box times two plus six. It works, just believe me.

Sarah: I agree with you.

Jeff: But how?

Sarah: But it works.

Jeff: Where’d you get that from?

Michael: I don’t know. (Looking at the board)
Angela: It works.

Michael: I was right.

(Magdalena goes to overhead and writes:

\[ \square \times 2 + 6 = \triangle . \])

Researcher 1: Is that what the rest of you got or somebody have a different rule?

Michael: Yes, I got it.

Researcher 1: Everybody had that same rule?

Jeff: I came up with one for half of it.

Angela: Except for Jeff.

Michael: (Speaking to Jeff). The next one get right or Sarah is going up.

Sarah: We don’t want him to get jealous. Not so easy to write them. I’m not comfortable writing them.

Jeff: (Turns and talks to Stephanie at another table.)

Jeff’s table: (Talks inaudibly among themselves.)

Researcher 1: Magdalena, can you give them a number to put in the triangle and see if they can find the number in the box. Can you think of a good number to put in the triangle?

Magdalena: I’ll put fifty-six.

(Magdalena writes 56 in the triangle column.)


Michael: Twenty-five.

Researcher 1: Twenty-five and everybody agrees with that. And how did you get it Michael?

Michael: Minus fifty-six, I mean fifty-six minus six and I divided by two.

Researcher 1: Could you write that as a rule?

(Michael goes to the overhead and writes:

\[ \frac{\square - 6}{2} = \triangle . \])
Researcher 1: All right that’s nice. But you know what I had another, a different rule that I think works. I had this. Tell me if you think that this would have to work too. I had two times box plus three is equal to triangle. (Researcher 1 writes on the overhead:

\[ 2 \times (\square + 3) = \triangle \]

Michelle I.: Wow, that’s heavy stuff. No.

Researcher 1: Do you think that work?

Michelle I.: Yes, it definitely does.

Researcher 1: Well do you think there are just two rules that work? Do you think that one works, that both of those work? Do you think there are any other rules that work?

Researcher 1: Can anybody tell me why that works? It seems a little strange to me. Bobby?

Bobby: You have to do the work in the parenthesis first and like, then if you add three to negative three it’s zero and if you times two by it, it like stays the same cause it’s zero and it like works on every one.

Researcher 1: Did you understand what Bobby said?

Jeff: (Whispering) Sarah, could you get your paper for me.

Sarah: Hands Jeff her paper.

Jeff: Thanks.

Researcher 1: Does anybody think it’s odd that you have a three down there you have a box plus three but when you have the rule the other way
you have a six and that you get the same number? What do you think Ankur?

Ankur Two outside the parenthesis so it becomes sort of like a six.

Researcher 1 Would you go up and explain what you are saying?

Ankur The two outside of the parenthesis doubles what ever's inside.

Researcher 1 You mean it doubles everything that’s inside.

Ankur Yes.

Researcher 1 All right, what do the rest of you think?

Researcher 1 Think that will always be true, Jeff?

Jeff Yes

Researcher 1 Jeff thinks yes.

Michael I do too.

Researcher 1 You think yes too.

Jeff That’s my partner.

Michael/Jeff (Shake hands)

Researcher 1 All right, all right. Let’s try another one.

Angela Genius.

Angela I can’t believe the people at Wendy’s believe you.

Jeff Hey, that’s because they’re working at Wendy’s.

Michael Don’t talk because…

Jeff Yeah, but that’s why they’re working at Wendy’s.

Researcher 1 Maybe this is too easy.

(Researcher 1 is writing on the overhead)

Researcher 1 This one is probably too easy for you.

(Overhead shows:)

42:38 Michael Three squared, no.

Jeff Why isn’t it box times box equals triangle?

Michael Box times box.

Researcher 1 Jeff thinks he has a rule.

Jeff Yeah, I do.

Angela Yeah, it is.

Researcher 1 Anybody else? Jeff do you want to come put up the rule and see if they agree with you?
Oh, Okay.
Box, squared squared.
No, box times box.
(Slams pen to table.) Yeah.
(Jeff comes to the overhead and writes:
\[ \square \times \square = \triangle . \])

Box times box equals triangle.
(raises his hand). Ur that’s wrong.
That does not work
Yes it does, it has to work.
Because if it’s negative three times negative three it’s negative
nine.
Yes!
No, you people, you people told me that.
He’s right.
It’s a positive times a positive is another positive.
A negative times a negative equals a positive.
I was just repeating what they said and I disagreed with it and then
they tell me that I am wrong. So it is their fault that I was wrong.
We know Michelle.
It’s on camera, I’m telling you.
(Raising and waiving hand.) I know another way.
Sarah is next time.
(Talking to Sarah) So go box is squared equals triangle. Hey, you
want to go? (Talking about going up to the overhead.)
(Shakes head to indicate ‘no’.)
So they are having a discussion at this table. They feel that some people were led astray about negative three times negative three being negative nine or positive nine?

Michael (raises his hand) There is another way to write it. It’s box squared equals triangle.

Researcher 1 Do you want to write that for us Michael? Michael has another way of writing it.

Angela Sarah wants to.

Researcher 1 Sarah.

Sarah (Giggles and turns away from the front of the classroom, then starts to get up.)

Michael (Gets up, but doesn’t see Sarah rising.)

Angela Mike, she was going up.

Michael I’ll just write it fast.

(Laughter at Jeff’s table)

(Michael comes to the overhead and writes the rule another way)

\[ \square^2 = \triangle . \]

All right. Thank you. That’s great. Now, did I hear someone talking about square roots? Were you talking about square root?

Researcher 1 Were you talking about the square root?

Angela No, but if the number is triangle, then to find what box is, you have to find the square root of the triangle.

Researcher 1 Angela is saying that if I gave you a number over here like twenty-five, that you would’ve to find the square root. (Researcher 1 adds ‘25’ to the triangle column.) There is a little confusion about that. Look, when we had nine on this side, and we went backwards we could get a three. We also worked that nine would give us a negative three. So that’s a little different isn’t it? What if I put a negative thirty-six here, (Researcher 1 adds ‘-36’ to the triangle column). What would you get on the other side? Brian?

(Overhead shows:)
Brian: It is either a positive or a negative six. I’m not sure.

Researcher 1: Hey, Brian thinks it’s either a positive six or a negative six. What did you think Jeff?

Jeff: Nothing.

Researcher 1: Bobby?

Bobby: You can’t put a negative there. Cause to get a negative number you have to times like a positive and a negative. So you get a negative.

Michelle I.: So that’s a negative times a negative and that’s a positive. Which Bobby got wrong before and made me look dumb.

Researcher 1: They’re saying over here that they don’t think I should put a negative over here that you wouldn’t be able to get an answer.
Jeff’s table 2 of 2: 5/16/1995 (J1B)

Camera View: Jeff’s Table 2 of 2(Jeff, Michael, Angela, Sarah)
Date of filming: 05/16/1995
Harding public school, Kenilworth NJ, Grade 7
Advanced Guess my Rule (AGMR): Inverse: problem
Transcribed by: Vanessa I. Bell, Andrea DePaolo
Verified by: Elijah Brookes
Date of verification: 9/2010
Length of session: 00:46:11

<table>
<thead>
<tr>
<th>Time</th>
<th>Speaker</th>
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<tbody>
<tr>
<td>1</td>
<td>Researcher 1</td>
<td>Do you think that’s right?</td>
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<td>2</td>
<td>Angela</td>
<td>Yes, it’s obvious.</td>
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<td>3</td>
<td>Researcher 1</td>
<td>We shouldn’t use this negative thirty-six over here. Okay.</td>
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<td>4</td>
<td>Angela</td>
<td>Plain positive numbers.</td>
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<td>5</td>
<td>Researcher 1</td>
<td>All right. You’re doing great. You told me you would. Maybe this one will be a little bit harder.</td>
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<td>(Researcher 1 writes on the overhead:)</td>
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<td>11</td>
<td>Jeff</td>
<td>You’re going up next. (to Sarah)</td>
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<td>12</td>
<td>Angela</td>
<td>Sarah, you’re going up next, okay? No matter whose idea it is.</td>
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<td>13</td>
<td>Michael</td>
<td>Allegeda?</td>
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<tr>
<td>14</td>
<td>Angela</td>
<td>Alega? Sarah has the funniest name.</td>
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<td>15</td>
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<td>(Group spends time spelling each other’s first and last names backwards instead of working on the problem.).</td>
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</table>
Researcher 1: No, I did that wrong. I did that wrong. This should be three. This should be four. This should be five.

(Researcher 1 corrects the table on the overhead:)

[Image of a corrected table]

Group is still focused on names instead of math problem.

1:37 Researcher 1: Michelle has a rule already. Want to come and write the rule for us?

Michelle: Uh, I do.

(Michelle I. goes to the overhead and writes:)

[Image of a written rule on the overhead]

Researcher 1: That turned out pretty easy didn’t it. And here I thought I’d made a harder rule. Do you know what my rule was? I had, uh, three times the number in the box plus six divided by three is equal to the number in the triangle.

Angela: We liked the easier one.

Researcher 1: Do you think that would work? Do you think that would work?

(Researcher 1 writes on the overhead:)

\[
3 \times \square + 6 = \triangle
\]
Michael I guess so, if you make the…
Researcher 1 Would you tell me if it works?
Student? It does.
Researcher 1 So what do you think Stephanie why does it, why does that work out to be the same? Does anybody think why that might work out to be the same?

Bobby Well, no, cause like negative two times three is negative six plus six is twelve and if you divide it by three it will be four not zero.

Angela (to Michael) Shouldn’t that be (inaudible) plus two and then divided by two?

Researcher 1 So are you, I wasn’t sure what you are saying Bobby, do you get the same answer or a different answer?

Bobby A different answer.

Researcher 1 Bobby is getting a different answer. Are some other people getting a different answer too? Angela?

Angela I just, like, wanted to say something. Shelly made a mistake up there. She put box plus two then she put triangle divided by two.

Michelle 3:35 It’s the same thing, Angela.

Angela No, it would be like triangle minus.

Michelle Oh, sorry, it’s supposed to be minus not divide. Sorry about that.

Researcher 1 All right. Thank you Angela, I hadn’t noticed it. That’s good.

Do you want to change that who put it up?

(Michelle I. goes to the overhead and changes the division sign to a minus sign

\[ \triangle - 2 = \square. \)
Researcher 1: What I want to know is three times the number in the box plus six divided by three does that work for all of them?

Angela: Three times zero, plus zero, plus six, no it doesn’t work.

Michael: Two times three, three times two, plus six times six, I mean divided by twelve is negative...(cut off by Angela)

Angela: Look three, look, look. Three times zero is zero. Plus six. Six. Oh, it does work. Duh, I didn’t see the divided by three, I’m blind.

Michael: Three times two plus six times six divided by three is twelve.

Angela: Three times three is nine plus six… It doesn’t work.

Researcher 1: Does anybody have an idea why that could be the same?

Michael: Ah, the six on top gets divided by the three on the side.

Sarah: Huh?

Researcher 1: Does anybody have an idea why that could be the same?
Sarah: Triangle times three.

Angela: Why don’t you just like, stop (the six?).

Sarah: Minus six.

Michael: But see then, if you cross the two threes out it’ll be box plus six, I just made that up-

Sarah: Divided by box. Divided by three.

Michael: That’s why you gotta divide six by three on that side.

Angela: They could possibly cancel each other out.

Researcher 1: Want to come try?

Michael: No, I’ll just say it.

Researcher 1: Michael just wants to stay at his desk. He doesn’t want to come up.

Jeff: Sarah come up.

Michael: Yeah, Sarah come up. Six on top divided by the three on the top, too equals two. And then the two threes, on the bottom and the top, cross each other out, so it’s box plus two.

Researcher 1: What do you think, did you hear what Michael said? What do you think about that?

Angela: He was right.

Researcher 1: I don’t know that sounds a little dangerous this crossing things out. I wonder if it’s anything like, I’m trying to find the one you did before?

Angela: It’s right.

Michael: I know. No one listen to the master.

Angela: It’s right. Divide both by three, that’s one times…

Researcher 1: Do you know, you, we were looking at something else where you had two times the number in the box plus six, I think it was, and you told me that that would be the same as this. And I wonder if there is anything about this one that’s like that one?

(Researcher 1 writes on the overhead:

\[ 2 \square + 6 = \triangle \]

\[ 2 \left( \square + 3 \right) = \triangle \] )
(Maybe?) because, cause the two, the parenthesis things, if you don't have, if you have a variable in the parenthesis, you can divide, I mean times each number by itself. Like two, two times the box is two box, and two times three is six, so it's the two times the box. .. I’ll write it down.Um.

(Michael goes to the overhead and explains how two times box plus three is equal to two times box plus two times three using distribution)

In fact, if you need anymore of this, here is some.

Okay. Well, here if you don’t have-if you don't know how, like what box plus three is, cause there is like a variable here, you can times two times the box which is two box, plus two times three is six which is the same thing as this. You can like, multiply this.

Okay, what does that have to do, don’t go away. Is there any thing about this one that’s like that?

I don’t know.

Okay.

(laughs)

Wait, I think I see something. (Some students laugh) Three times the box can be written as three box plus six. They are similar to each other, three box plus six and two box plus six.

(Michael writes on the overhead:

\[
\begin{align*}
3\square + 6 &= \triangle \\
2\square + 3 &= \triangle \\
3 \\
\end{align*}
\]
Researcher 1: But, instead of multiplying it by two-
Michael: And then dividing by three. But the top, like, you can take out the multiply, this times thing and that means three-
Researcher 1: Would it hurt anything if I put, don’t go away, would it hurt anything if I put parenthesis right here?
(Researcher 1 adds parenthesis to the overhead:)

Michael: No. It wouldn’t do. It wouldn't.
Researcher 1: Would it help any?
Michael: This all, like, this all means you do this first thing—you divide it by three.
OK. So you’re saying that this is a little like this one.

Yeah, except for the division……

That instead of multiplying you are dividing each one.

See, this is, is three box plus six divided by three, and the rule for this one was two box plus six without any division. So maybe this is like, kind of similar.

So what do you think about what Michael is saying? Do you think that might be the reason that?

Could somebody make up another one that you think would work two ways? See if you can make up one that you think would work two ways.

Mike? Mike, for this one I got another one right. Three parenthesis box plus two equals triangle. You see that?

You’re not sending me up there again. Send Sarah!

You see that? Just look at that. It works the same way. Three box plus six.

You want to spend another hour on this problem? We can do science.

Can you pick this up please?

No, we have a double period here today.

This works either way. Zero plus one equals one. One plus one is two. Two plus two, I mean two plus one, is three.

That’s a little too easy.

But it works each way. One plus one-

Four plus one is six?
Jeff: That's a five you doof. Five plus…(cut off by Michael)

Michael: (inaudible) you missed a big step.

Jeff: Are you okay Sarah?

Sarah: Yes, I’m okay.

Angela: I need more space.

Jeff: This is a perfect problem.

Angela: This-

Michael: Box divided by box equals triangle.

Jeff: What about?

Michael: There’s no number in here.

Angela: That’s still too easy.

Michael: That won’t work out.

Angela: Yes they will.

(Talking over each other.)

Michael: No they won’t.

Sarah: One.

Angela: Let’s see if that’s easy to figure out.

Sarah: I don't know what the answer.

Michael: See, see. They all try to do it with minuses and stuff. Let’s just try this.

(Michael’s paper @ 12:36)

Jeff: Well, somebody tried to treat it like gold. You see, but there’s two answers to that. Minus zero, two minus one.

Angela: You have to do one thing. You have to have the same exact…(cut off by Jeff)

Jeff: Three minus two, four minus…(cut off by Angela)

Angela: Look, look Jeff, Jeff. You have to have the same pattern for each

Michael: Each.
Angela: Each one. Each row has to be the same exact thing.
Michael: OK, I’m going to go put this up.
Jeff: This is good. No, this is good.
Angela: (To Michael) I’ll put it up, you went up too many times.
Michael: Sarah puts it up.
Angela: No, Sarah doesn’t want to go up. I want to go up.
Michael: Jeff do you want to put it up?
Jeff: I want to put mine up. Sarah’s putting it up.
Angela: I want to put it up.
(Jeff hits Angela’s arm)
Michael: Sarah puts it up.

Angela: I want to look smart.
Sarah: Alegna.
Michael: Alegna. (Saying names backwards)
Jeff: Magda’s is aneladgam. (reading Magdalena’s name backwards)
Sarah: Anelad.
Jeff: Magda is a worm.
Angela: Sarah is alge.
Sarah: No I’m not, I'm a rat or her.
Michael: Let’s put mine up. Sarah puts it up. She never gets to put anything up.
Angela: But, I want to put it up. Sarah can put the next thing up. Even though we keep wanting Sarah to put one up I want to put this up cause it’s cool.
Sarah: No, too bad I want to put it up.
Angela: These people are too stupid to figure this out.
Michael: Let me fix it up. I'll make it look better.
Angela: You're going to give it to her.

14:05 (Researcher 2 approaches their table.)
Angela: Look I know what it says,
Researcher 2: What do you have there?
Angela: Mike's-Mike’s fixing it. It’s like alright, one, two, three, … no it’s (inaudible 2 words) one, two, three, four, five, blah, blah, blah, alright and all the answers are one. Alright, now. (fills out the numbers in her table)
Researcher 2: They’re all one.
Angela: Yeah, the answers are all one.
Researcher 2: OK.
Angela: So it's box divided by box, wait, no.
Michael: It's pretty easy. Yeah.
Angela: Box divided by box equals-
Michael: Triangle.
Angela: Triangle.
Researcher 2: OK. I had a different rule. Mine was five box divided by five box equals triangle.
Michael: What box divided by?
Jeff: Mine is easy but-and it works both ways.
Researcher 2: Five box divided by five box equals triangle. Does that work?
Angela: It's still, the same thing. It's the same thing.
Researcher 2: Mine is a thousand box-
Angela: No, cause you're like sort of cancelling each other out.
Researcher 2: What do you mean by cancelling out?
Angela: Like you're dividing this by this-can always equal one.
Michael: If you do do five plus-
Researcher 2: Can I put a million box divided by a million box equals triangle?
Angela: Yeah. It will still be one.
Researcher 2: Why?
Michael: If you put five plus ten equals uh, um.
Angela: It's like, um, even if you put like five, wait no (inaudible word), five hundred box or something, it will still be this after, because it's like you are dividing it by itself so it's going to be one.
Researcher 2: I see. I found all these other rules for this problem, box plus two equals triangle (shows the transparency with table and the rules). My question is do all these other rules work and why?
(That) I found all these other rules, now do they work for this and why? ?
Michael: Mike, you don't have-
15:54 Jeff: Mine works fine. You have to get something really complicated.
Angela: It's not complicated. It's easy.
Jeff: It works, this is easier.
Angela: No it's not. It is not a great rule.
You have to have the same rule for each and every row.
Jeff: This is the easiest you could possibly do.
Jeff: This works! Dam!
Angela: Look zero plus one,… do you have one plus one
Jeff: Add plus one. Box plus one equals triangle.
Angela: It's too easy.
Jeff: No wait and then it works the other way. Triangle minus box equals one.
Angela: No.
Jeff: Why not? Dah!
Angela: wouldn't it be triangle minus one…
Jeff: (corrects his inverse and erases). Triangle minus one equals box.
This is how you go-now you got me confused.
Angela: Though it could be like that but you want to figure out what box is.
Right.
Jeff: That's so perfect.
Angela: But Jeff, it's too easy
Jeff: Easy or not that's a good, isn't that a good problem? It might be easy but it works. It's easy but it works fine.
16:54 Researcher 1: Oh, that's neat and you did the inverse.
(Jeff's paper shows the rule and inverse.)
Jeff: Yeah, it was very easy, very simple. You ask the girl with the sixty minus four plus eight equals two squared equals five.

Michael: Take one of these (inaudible word). Take one of these.

Sarah: Hey, (inaudible word).

Angela: Wait, where is that problem that she had? (looking at the transparency from Researcher 2) Right, this one. So, wait.

Sarah: Stop it. All right, do all of these work?

Angela: A hundred box times … (stop?) Box plus two.

Jeff: What is that? (using a pencil to put dots on his paper)

(inaudible Michael and Jeff about the dots)

Angela: So wait, three hundred plus five hundred divided by hundred is five. This, this one works. This one works (pointing to the transparency).

Michael: It works. Every, every one works. Everything works.

Angela: It works. They all work.

(Angela looks to the other side of the room.)

Sarah: Dinosaur. (at Jeff's paper)

Jeff: That's what I was thinking, it's a dinosaur, a brontosaurus or it's a cool looking M.

Angela: These three things … Mike, these three are the same thing cause you just got to take away the zeroes.

Michael: But they don’t know what crossing out is.

Angela: It's like, dare not cross out.

Michael: I don’t understand what you mean by crossing out.

Researcher 2: Did you figure out why?

Angela: Yeah. They all work.

Researcher 2: Why now? The challenge to you is why do all those rules work.

Angela: Alright, well, these two are the same as this one, because you just gotta like cross out the zeroes and it’s gonna be the same.

Researcher 2: I don’t understand what you mean cross out zeroes? You just don’t cross out things.

Angela: Just get rid of them, because they are meaningless.

Researcher 2: Get rid of them

Angela: I don’t know, they just hold a place making it hundreds, ten-

Michael: See, cause you're (inaudible word) you're dividing this side by ten, you divide this side by ten too-
Angela: You have to take out-
Michael: You divide both sides by ten to make it equal.
Researcher 2: OK, so you are telling me you divide both sides, and then what do you mean by sides, you mean the-
Michael: The top.
Researcher 2: Top, the numerator and the denominator you are dividing each by ten.
(Researcher 2 points to the transparency showing the rules.)

Angela: Yeah.
Researcher 2: Now, why does that work?
Angela: (laughs). We didn’t get that far.
Researcher 2: You didn’t get that far.
Angela: Uh, like-
Michael: You can do a division problem. Fifty divided by six hundred and if you want to make it easier-
Angela: Just cross out that zero
Michael: You can cross out the zero.
Researcher 2: Why does that work?
Michael: And, and that means divide this side by-
Angela: What you do with one number is what you do to the other one.
Michael: You divide it by a number-
Researcher 2: Do you know why that works?
Michael: Cause you’re, OK. (draws on his paper) If you have a scale, the and little weight things…

Researcher 2: Right, okay.

Michael: and you have 5 grams in this one, and 10 grams in this one and you have like 5, 5 and 5. If you want to figure out.. no these are question marks, question marks, 5, and 10. So you have to find the unknown weight. So if you take this 5 out--you take the 5 grams off each scale that means and these will still be equal because you take away the same amount off each side, so that means that these two equal ten.

Researcher 2: I can see that scale and I can see that like an equation but I see this is a fraction, I don’t know why it works? It’s a fraction.

Michael: Each side balances out if you take away the same amount out.

Angela: If you take zero away from this one you have to take it away from this one and this one too (pointing to the paper) Cause, like, you have to do it to all three numbers that have like zero at the end (inaudible word).

(Michael’s paper showing his scale.)
Researcher 2: Do you have an opinion Jeff? Did you look at my others?
Jeff: Um, yeah. This is a very easy problem and they won’t let me do it.
Angela: Jeff still likes his problem.
Researcher 2: Let me see it.
Jeff: It works two ways and it’s very simple and it lets you express your point very easily because it can show you how the inverse pairs work.
Michael?: (I can't do this)?
Angela: (Laugh) Tell him to do our problem too.
Researcher 2: Now is this another rule for this?
Jeff: Like-
Researcher 2: Five box plus five divided by five. Can I write on your paper here? Can I write on your paper here?
Jeff: Yeah, sure.
Researcher 2: Is this one five box plus five divided by five equals triangle. Does that one work?
(Researcher 2 has written her rule on Jeff’s paper as shown below.)

Jeff: (works on the problem). Five ….
Angela?: All you're going to do is like, box plus five hundred minus five.
Michael?: We'll just say five.
Angela?: Eek, that’s triangle,
Michael?: Probability would be five...
Angela: Wait, Mike, Mike look at it. This is a rule.
Jeff: No it doesn’t work. Here, Angela, Angela, look.
Angela: Yes.
Jeff: That doesn’t work for that, for this problem.
Angela: Five, wait, wait, five times zero plus five, fi-divided by five. Yes it does. For this one it does (pointing to the table on Jeff’s paper).
Jeff: Alright, well then try the next one.
Angela: Five times one is five plus five is seven. Okay. No dur. Five plus five. Dur, it does work. Yes it works. I was adding wrong. I was looking at the two. (Angela laughs)
Jeff: Well, then it does work.
Angela: It does work.
Researcher 2: Does it work for four?
Four times five is twenty plus four is twenty-four. Yeah.

Ahum.

What about this one: ten times box plus ten divided by ten equals triangle?

(Researcher 2 writes another rule on Jeff’s paper.)

It works.

Yeah because you are doing the same thing that Mike said.

It’s a like a multiple of five.

What did Mike say?

He was talking about the scale and how he takes some off from each side so they remain even.

It works.

That’s what you are doing.

Scale I see is what you do to this side, you do to this side (pointing to paper). I am doing the numerator and denominator of fractions so I don’t see how the scale doesn’t helps this.

Turn the scale this way, yeah.

I don’t understand that. I mean you’re telling me that this always has to equals to this.

Ah, well see it’s right here. It’s like inverse operations. Divide by ten, eh-like multiply by ten, divide by ten. See it’s like adding and subtracting. I got an idea and she’s not listening.

You notice how annoying. As soon as like you make a big break-through then she turns around and walks away. It makes you wanna like scream. I just found a cure for!…. AIDS.

(is laughing)

OK. Well, that’s pathetic (looking at Sarah’s paper).

It’s a shark…

It doesn’t look like Sharkey.

So.

Yes it does (takes a pen and starts to make points on Sarah’s paper).
Angela: You can't see.

Michael: Except for the feet.

Sarah: Well, ah they are deformed feet. It's his deformed brother.

Angela: It's Sarah. It's deformed brother.

Jeff: Sarah gasa.

Angela: Sarah gasa.

Jeff: It's a dinosaur eating.

Angela: Anchor.

Sarah: It's an amoeba.

Jeff: It's a, it's a, it's a dinosaur, yeah.

Angela: Why can't you cancel it out.

Jeff: Children, it's having children. No, it's a lake.

Sarah: Swollen gland

Jeff: It's a lake.

Angela: No what is not fair. Mrs. Toye teaches us to cancel out and then they come and snatch it away from you because it's easier.

Michael: No it's binary.

Angela: I couldn't stand the binary system.

Jeff: What are you talking about. I have a calculator that adds in binary.

Michael: Yeah, mine, (inaudible) binary, ten,

Angela: Why is it that when Mrs. Toye teaches us an easier way, uh, we are not allowed to do it here. Sara will you stop it (in reaction to Sarah's drawings). Let's take away all of the pens (tries to take a pen from Sarah).

Jeff: (laughing)

Sarah: (laughing)

(Jeff and Sarah find something that she has drawn very funny)

Angela: Sarah will you stop it. Sarah

Jeff: (laughing)

Angela: Might as well sing now. Alright.

Sarah: What do you want me to sing? Sarah I don't want you to sing. Now Jeff, why is it, why can't we cancel it out?

Jeff: (using a pen writes dots on Sarah's paper)

Angela: Okay, she understands the scale thing but-

Jeff: Compose yourself Sarah there's people around. What is that?

Angela: (gently taps her head on the table) It's a snail.

Sarah: It's an amoeba.

Angela: It's a really messed up snail.

Sarah: It's a piece of clay.

Angela: No you should have …
Researcher 1: Well, Ankur could you explain to them how, he thinks all of
those would give you the same set of ordered pairs. Does
everybody agree with that? I know you’ve been working on similar
things. Can you tell if they are right?

27:20 Researcher 1: (I think?) your friends at this table agree. Could you go through
them and tell us why you think they are alike?

Ankur: Well this is like the shortest way to say that’s like the way most
people would think of the pattern.

Researcher 1: Okay, it’s just complicated people who would think of it the other
way.

Ankur: And this just gets more complex and see the first, behind the equal
sign this side equals seven this part equals seven, behind the equal
sign.

(Jeff and Sarah write on their papers)

(Researchers 1, 2, and Ankur discuss and write on their papers, captured in the
dialog to follow which often
occurs at the same time as dialog by Researchers 1, 2, and Ankur.)

Sarah: It's a gun.

Angela: Let me go.

Michael: I know, I know. I’ll tell you what to say.

Now does that mean … box. You tell them this but you have to
explain to them the crossing out. It’s crazy. So tell them the little
scale thing, like this scale (working on his paper). There’s box and
box, five, you take the times five out.

Angela: And this side is going to weigh the same, yes, but she is going to
ask me about the bottom. You don’t even have to divide by one,
times by hundred, or do you? Sarah.

Researcher 1: Okay.

Ankur: And so does this side.

Researcher 1: You know I don’t see a box in that second one.

Student: Yeah.

Ankur: Oops.

Student: You forgot to put it. You put a box after the three.

Ankur: It’s minus three plus box. That’s behind the equal sign.

Researcher 1: Okay, could you explain why you think the next one works? I
think there is some confusion about that.

Ankur: This one.

Researcher 1: Uhun.

Ankur: This one. It’s just, you can divide the hundred box by a hundred so
that turns down to one box.

Researcher 1: We’re not quite clear why one hundred times the number in the
box, divided by a hundred would be the same as the number in the
box. Could you explain that and tell us why that works? Maybe
write it over to the side or something. Just that part, a hundred
times the number in the box divided by a hundred.
Ankur: That equals one. A hundred divided by a hundred equals one. And fifty-six no I mean seven hundred divided by one hundred equals seven. So that’s one box plus seven equals triangle.

Jeff: (to Sarah) It’s a fetus.

Researcher 2: Can you write that down, those pieces that you just said?

Ankur: Those pieces.

Researcher 2: Cause that tells me a lot.

Ankur: If I write the things..

Jeff: (to Sarah) I thought it was a fetus.

Researcher 2: Just the two pieces you said were really nice.

Researcher 1: She wants you to say a hundred…

Ankur: I know but if I write it on the overhead-

Researcher 2: Take another piece of paper.

Ankur: Okay.

Researcher 1: Can you find another clean one under there?

Researcher 2: Rewrite that piece of it.

Researcher 1: I can help you find one, they’re under here, the clean ones.

Angela: Like, should I write this and this too.

Michael: Do one and (inaudible) ... box times one.

Angela: Should, wait, wait, say she asked us what would happen if you added five.

Michael: (inaudible)

Angela: You’re so stupid they are taping us when we are saying this.

Michael: I don’t care right now. (inaudible)

Angela: (Ankur?) I used to hate him.

Michael: Just don’t make fun of people (inaudible)

If we whisper…

Researcher 1: All right so does everyone see what Ankur did up there? He said a hundred times the number in the box divided by a hundred would just be one times the number in the box.

Researcher 2: Yeah, Ankur, the people back here said something to me that was different and I’m confused. Can you help me? The people back there said to me, you see this piece here, are you all looking, right, which Ankur wrote like that. What they said to me is that they could write that top part like this.

Ankur: (whispers one hundred?). You can.

Researcher 2: You can. Could you explain that to me I don’t understand that?

Ankur: Because one hundred is outside the parenthesis and one hundred times box equals one hundred box and one hundred times seven equals seven hundred so that’s one hundred box plus seven hundred.

Researcher 2: Well, how do you get, how do you end up getting what you finally got? How do you go from there to get this result here.

Ankur: You simplify.
I’m sorry I didn’t want to mess that one up. Can you put the one back in? Thanks. Can you show me how we go from here to here and why that works?

You’ve been trying to say simplify it. (to Michael?)

From here you can go to-

To yours, yeah, how do you simplify that?

From here you go to here-

No, the one under.

This one? How do you simplify?

How do you go from here to here? Can you show us?

(inaudible)

He probably has an explanation.

Okay, one hundred is outside the parenthesis so that equals one hundred box that’s right here. And one hundred box divided by one hundred equals one box so that piece is now, that piece is one box and one hundred plus seven, one hundred times seven equals seven hundred so that’s seven hundred box divided by one hundred equals seven. So then you have one box plus seven equals triangle.

He probably has an explanation of it.

Yeah.

(continues to write/draw on her paper)

What do you think what is?

Ankur your explanation didn’t convince her that anybody knows why. Can you try, can you think of any other way to say it?

Because the-The one hundred cancels out the one hundred.

What do you mean by cancels out?

Because it’s on both sides so you can eliminate it.

That’s where you lost me.

Then all you have left is box plus seven. That’s one box plus seven equals triangle.

I don’t understand when you say cancels out. I see one hundred divided by one hundred here.

That equals one.
Reasearcher 2: I can see that that’s one. I’ll buy that that is another name for one.

Ankur: So that equals one box and seven. It’s still seven. So one box equals

Michael: One box…

Ankur: Because they’re inverse operations (raises her arm)

Researcher 1: Ankur do you know what worries me about cancelling? Is that

sometimes people see things you know and they just mark them

out because they look alike. Could you show us an example with

numbers where you have a hundred times something divided by a

hundred and to be sure that it makes sense in numbers or

something.

Angela: Mike, they’re inverse operations. If you add one and then you

subtract one it cancels it out and makes it a zero.

Michael: Okay…

Ankur: How should I do that? Like how?

Angela: You don’t even have to do one plus seven all you have to do is box

plus seven. You know what I am saying.

Researcher 1: What if you have a hundred times five.

Ankur: A hundred.

Researcher 1: divided by a hundred. Let’s say will you write that down and see?

Ankur: A hundred.

Researcher 1: You have a hundred multiplied times five and then that divided by

a hundred. Let’s see if you can tell us what happens.

Angela: Mike, Mike, should I say it or did Ankur already say it?

Ankur: This is on both sides of the equation?

Michael: I don’t know.

Researcher 2: That’s not an equation, Ankur. There is no equation there.

Angela: Sarah, did Ankur already say that you can’t cross each other out

because they’re inverse operations?

Ankur: Equation, fine then the sides.

Sarah: What (and then she gestures toward Jeff)

Angela: Jeff, will you stop it. (takes Jeff’s paper)

Jeff: (laughs)

Researcher 2: There is no equation. Look at that that’s just a fraction.

Ankur: Then the fraction.

Researcher 2: Do you all see that? That’s where you confuse me. I see a hundred

times five divided by a hundred. That looks like a fraction to me.

Ankur: Okay, then you can put-

Researcher 2: You guys were talking about equations even at the other table and I

don’t see an equation.

Ankur: Then this equals five hundred and you can cancel out the two

zeros.

Michael: It is the same…. (starts to play drums on the table)
Researcher 2: I don’t understand what you do when you cancel out zeros.

Ankur: Because they’re on both sides. You can do anything on one side.

whatever you do to one side you can do on the other side.

Researcher 2: Angela and Stephanie. (both have raised their arms)

Angela: I finally figured out why you, uhm, cancel each other out cause they’re inverse operations. Like if you had one and you plus one and you subtracted one they cancel each other out. If you had, like, I don’t know it just cancels each other out. And you don’t even need the one up there because it's one box is just box so it would be box plus seven.

Ankur: Or you could do it like this. Five hundred over one hundred that simplifies to five over one. That’s five.

Researcher 1: Yeah, why?

Ankur: What does why mean?

Stephanie you were going to say something?

Stephanie: Oh no I was just going to say kinda what she said. It’s just what it is you have to whatever if you take away a zero from one side because you can do that. Because it’s, you’re, I can’t it’s like, whatever you do to the one side you have to be able to do to the other side otherwise you can’t do it.

Researcher 2: Okay, I think you guys are going to get into a lot of trouble when you study algebra unless we really work this out. Because first of all you are saying things about side and this is a fraction. There’s a number on the top and a number on the bottom, that’s a fraction.

And you're giving me examples that have to do with equations and balances and that’s something else. So I really think we are going to have to work this out otherwise you may fall into some trouble.

We have all the algebra teachers in this room, in horror, worrying about what’s going to happen to you guys down the line. So we are gonna maybe take a time out here. The question I want to leave you with Bobby you want to make a comment before I leave you with a question? Go ahead Bobby.

Bobby: Is it--Five hundred divided by one hundred is five.

Researcher 2: Very nice.

Bobby: Even if you cross the hundreds out it is still going to be five no matter what.

Researcher 1: But she’s looking at it now as division with everything.

Researcher 2: That I’ll buy Bobby. I’ll buy that. Five hundred divided by hundred is five. I am going to ask you something else here. (Michael has raised his hand) Yes Michael?

Michael: Well the time before you could not only write it as a fraction, you can also write it as an equation, as an equation. So when you write an equation then you can cross out things.

Researcher 2: Well that’s where I get a little confused because this is an equation. We’ve written this as an equation. But you see when you talk about crossing out things this is one side of an equation and this is...
another. This is the numerator of a fraction and this is a
denominator and I get very confused when you’re crossing things
from the fraction and then you are talking about an equation.

Michael Then change it into an equation and then you can understand it
more, like...

Researcher 2 Okay, but we need to work on that it seems. I think our time is out.
But we have, we have a visitor here for just one minute. Tell me
your name?

Lindsey Lindsey.

Researcher 2 Lindsey? Lindsey, um, Lindsey sent me a wonderful problem.
Lindsey’s a third grader and you know how I am about why. Don’t
you all know how I am about why? And Lindsey, um, come say hi
to this- this class, Lindsey. Lindsey- Lindsey discovered a- a- an
interesting, um, rule. And my question to all of you is to help, uh,
Lindsey tell me why. If I can find a blank overhead.

Student (Justice?) (Class laughs)

Researcher 2 Lindsey’s- now these- You guys used to do this to me when you
were third graders. Are we out of paper? Overheads? (Class
laughs)

Researcher 1 There should be some under there.
Researcher 2 I don’t see any more.
?

Researcher 2 One piece is all I need. Okay, what Lindsey said is the following.
Lindsey do you want to say it or do you want me to? Do you want
to say it? This is what you wrote me. Remember?

Lindsey Yeah.

Researcher 2 Go ahead. Why don’t you tell me and I’ll write- write what you
want me to write. Tell the class.

Lindsey Five times an even number you, um, split the even number in half
and you add a zero and you get your answer.

Researcher 2 So five times an even number- and you said, for example, six,
right?

Lindsey Yeah. (nods)

Researcher 2 She says you split the even number in half. Splitting six in half
gives you...

Students Three

Researcher 2 Three. She says then you add-

Lindsey A zero.

Researcher 2 Zero. And that’s how you get five times six equals thirty. Right?

Now let’s try it with another even number. Five times- give me
another even number.

(Researcher 2 has written on the overhead)
Student Forty-six. Forty-six. You split the even number in half, right? Split forty-six Jeff and you get? Twenty-three. Twenty-three, right? So five times forty-six is going to be- adding a zero- Two thirty. Which will be two thirty. Does that work? Yes.

(Researcher 2 has added the second example to the overhead)

Jeff Can I ask you something? Where did you come up with this? Right? So you can- you can try some oth- now Lindsey has another one for you. (Class laughs) ‘Cause Lindsey does not want to limit her theories to just even numbers. She also wants odd numbers, right? So what’s your rule Lindsey about odd? This is for the even, right?

Jeff I wish I learned this. (Class laughs)

Lindsey Five times an odd number you, um, make the odd number, um, the next smallest number. Then you split that in half again and then you add a five.

Student Plus five.

Jeff Yeah so five times three would be fifteen.

Lindsey

Jeff

Researcher 2 Obviously. (Class laughs) (Jeff raises his hand) You got that? I had to think. What is it- when you took, you- you took the odd number again, Lindsey. You reduced it by…
(Researcher 2 has written this example on the overhead)

Lindsey: Um, the next even number.
Researcher 2: And then you-
Lindsey: Add a five.
Researcher 2: You- when she said add a five, you append a five, right? A five at the end. Let’s try another one. Yes?

(Jeff lowers his hand and calls out)
Jeff: (To Lindsey) What were you doing when you thought this up?
Lindsey: I was just doing some problems and I just figured it out.
Jeff: I, wouldn't-Whoa-ho.
Researcher 2: Angela you have a question?
Angela: It’s the same thing, how did you figure this out?
Researcher 2: Oh no, that’s your problem. (Class laughs) Lindsey has come back here, Angela. This is the problem. The theory is, let’s write this down, five- let’s write the first one down, Lindsey. Five times an even number, right?
Lindsey: Mm hmm. You split it in half.
Researcher 2: How you obtain it- you split it- you split the even number in half, right? And what do you do?
Lindsey: You add- you um, put a zero at the end.
Researcher 2: And, um, append, alright? Append a zero. Right? You’re not adding, like an addition operation, okay? And the next one is five times an odd number, right?
Lindsey: Yes.
Researcher 2: You-
Lindsey: You, um, make it the next even number-
Jeff: (talks to Ankur at Stephanie’s table)
Researcher 2: Okay, so you make the next even- smaller or larger?
Lindsey: The even smaller.
Angela: You just subtract by one.
Researcher 2: Okay. You subtract by one is that okay?
Michelle I.: Yeah that’s that’s like what you’re doing.
Lindsey: Yeah.
Michelle I.: That’s- that’s exactly what you’re doing- subtracting by one.
Angela: Subtract one.

Researcher 2: So you go to the next smallest even- next smallest number, even, right?

Lindsey: Yes.

Researcher 2: And then what do you do?

Lindsey: You add, uh, you put, um, a five to the end.

Researcher 2: And? Subtract a one, don’t you? (Pauses)

Angela: Supposed to, like, divide it, divide it by two.

Researcher 2: No this is the odd number one.

Angela: Split it in half.

Lindsey: You put a five at the end.

Researcher 2: (To Lindsey) Then-And split it, and split it, right?

Lindsey: Yes.

Researcher 2: And append what?

Lindsey: A five.

(Researcher 2 has written the rule on the overhead)

Researcher 2: A five. Okay. So your job is, why does this work?

Brian?: (to Jeff, who laughs) I told ya.

Researcher 2: Because you know you just- you could make- I’m gonna tea- all the math you’re telling me oh, you just cross out zeros or you just- or you just add this. You have to know why things work because if-

Student?: I think I know why the even works?

Researcher 2: - and does it always work? I want you to write up why it works.

Lindsey, would you like to know why this works? Do you know why it works? No. See she has discovered this, so now she needs some help.

Student?: If she doesn’t know how are we supposed to know?
And maybe she can come back tomorrow when you all can tell her.
I mean she looks to the wisdom of the seventh grade- almost eighth grade class to guide her in her future.

Student? She shouldn’t.

Do you have any of these for Social Studies?

Let’s give her a round of applause. Let’s give her a round of applause. (Class claps). Thank you very much Lindsey. This was lovely. Any theories? Okay, thank you Lindsey.

You were thinking a little too much on that math homework.

Ankur found out why. OK, we can go now.

I want to look smart. Third grader is smarter than I am.

My final, uh, my final question to you before you all disappear-if you could just look at this for a moment. Yes, Michael.

Well you said we’re crossing zeros without an explanation. Well we are - we do have an explanation. We are taking off a number- a certain number off each side and- to balance it out. Like, if we take off a number from each side and it’s the same thing as crossing it out- cause it's not- if you take off a zero off each side- like you take off a- you divide it by- (writes on paper) you divide the hundred by ten on the top and that equals ten. And you do it to each number, you’re not just crossing it out, you’re dividing it.

When you take out a zero you’re dividing a hundred by ten equals ten, it’s like crossing a zero it’s the same thing. It’s a shortcut… So what you’re say- what I’m hearing you say, then, partially is that, um, it’s okay to divide- not saying each side- divide the top of a fraction- the numerator by a number- as long as you divide the denominator by the same number. So in other words if I had a fraction four eights, right?
Researcher 2: So that’s the same thing as if I divide both the numerator and denominator by the same number.

Researcher 2: Right but what am I really doing?

Student: You just-

Researcher 2: Why does that work?

(Researcher 2 has written the division on the overhead slide)

Michael: Because it’s the same thing as a half except that it's in--it's in smaller pieces. So all you gotta do is, like, add the four eighths, like four eighths they equal two eighths. One half because it’s just in like one half cut into little pieces.

Researcher 2: Anybody else have any ideas? Michael had a-- an explanation.

Angela: Making four-eighths, which is one-half, more complicated.

Researcher 2: When you write it as four eighths it’s a more complicated version of one half?

Angela: No, it's like four times one and four- over four times two is just the same as four eighths, it’s just more complicated.

Researcher 2: What made it more complicated?

Angela: Like adding times one and times two.

Ankur: The numbers are larger.

Researcher 2: Okay, anybody else have any thoughts on that? Anybody else have any thoughts? That’s it? I think our time is out. Okay well it’s something to think about for tomorrow. Don’t forget Lindsey’s problem. Isn’t that neat for a third grader? Were you all impressed?

(Students exit)
Time | Speaker | Transcription
--- | --- | ---
00:00 | (Videographers and facilitators set-up.) |
0x.xx | (Students enter classroom and sit at tables) |
9:53 | (Stephanie intermittently slaps the hands of Ankur, who is picking at the name tag on the table) |
13:56 | Researcher 1 | Are you ready to start? I'm Dr. Dann 1 and I work with these people at Rutgers. Dr. Maher who comes in a lot of times and you probably, some of you, have already guessed what we are going to do today by those sheets that are on the table. No, no. Do you remember doing something called Guess My Rule about two years ago with Dr. Davis? Do you remember something about that? Let's look at an example. |
13:56 | Researcher 1 | Suppose that, can you see it, that I am thinking of a rule like this, and what I am going to do is add three to some numbers, and suppose that I put a zero in the box, then what would I have to put in the triangle? |
13:56 | Researcher 1 | (Researcher 1 writes on the overhead as shown below.) |
13:56 | Researcher 1 | A three, okay, suppose I put a one in the box.
A four, all right, okay, so this is pretty easy, right.
(Researcher 1 has entered three entries for this rule on the overhead.)

Researcher 1  So far we have limited ourselves to what kind of numbers? What kind of numbers are we using there? We are using positive numbers, does anybody know another name we could call these ever? What other kinds of numbers do you think we could use?

Student?  Negative numbers

Researcher 1  We could use negative numbers. What about if we put in negative one what would we have to put in the triangle?

Two and if we put negative two we would have to put one in the triangle.
(Researcher 1 continues and fills in two more entries in the table.)

Researcher 1  Okay, so what, let me ask you a little bit harder question. This is probably a very easy rule for you but what if I put a ten in the triangle. What would I have to put in the box? Brian?

Brian  Seven
(Researcher 1 writes ‘10’ in the triangle column and then enters ‘7’, from Brian’s response, in the box column.)
Researcher 1: A seven, and how did you get the seven?
Brian: Ten minus three.
Researcher 1: Okay, so the rule that we started off with said that we were adding three but when you were doing it backwards you subtracted a three. So addition and subtraction are inverse operations, one undoes the other. Do you know any other operations that work like that? Bobby?
Bobby: Multiplication and division.
Researcher 1: Multiplication and division work like that, one undoes the other. They could be called inverses. Do you know of any other kind of operations that work like that? That may be all you’ve had so far. What if you square a number? What if I start with the number five and square it? What do you get?
Student?: Twenty-five.
Researcher 1: What does five squared mean?
Amy-Lynn: Five times five.
Researcher 1: It means five times five and so you get twenty-five. Does anybody know what you would call it if you went back the other direction?
Ankur: The square root.
Researcher 1: The square root. Have you talked about that some, the square root, and does anybody know how to write it? The square root, could you come and write that for us and do the square root of twenty-five?
(Angela, from Jeff’s table, comes to the overhead and writes:)
Okay, would be five, okay, so do you think squaring and square root are also inverse operations right? What I want to do is I’m gonna think of a rule and I am going to put down some numbers that work in the rule and see if you can guess the rule. Do you remember doing that now with Dr. Davis? One day maybe? What do you think? My rule says that if I put a one in the box, that in the triangle I’d have to put one-third and if I put a two I would get two-thirds and… Would you talk to some of your friends there and see if you can think of a rule that would work?

(Overhead shows:)

(indistinct whispering)
Stephanie: We want to begin, you want to know what that equation is... Erin?

Erin?: What?

Ankur: (laughs)

Stephanie: Forget it. No, because... (holds hand to face, and taps pencil)

Ankur: Each number represents one third.

Researcher 1: We want a rule that will work on every line.

Stephanie: Each number meaning?

Ankur: Like one equals negative one-third. Negative two equals negative two. Negative two equals negative two thirds.

Brian: Okay.

Ankur: Positive two equals two thirds. Three equals one. Four equals.. (smiles and gestures for Brian to respond)

Brian: (indistinct, rocks chair)

Ankur: One and one third.

Brian: (indistinct, rocks chair)

Ankur: That's what I said.

19:08 Jeff from the other table: This is stupid.

Ankur: Yeah, I know.

Stephanie: (indistinct, whispers to Erin who is Romina, who nods)

(Further whispering, Ankur, Brian, and Jeff, from the other table, begin to talk about basketball.)

19:41 Erin: No basketball.

Brian: (describes a basketball game)

19:53 Ankur: We found the answer.

(Stephanie’s table talk about basketball and baseball, not related to task at hand.)

20:55 Researcher 1: I started with a really hard one didn’t I. I’d like to come back to that one. Could we come back to that one later? Let’s try another one. I think that one may not be a rule.

(Students are shaking heads no, whispering)

Researcher 1: Suppose that when I put in say negative three in the box, then what I get is negative twelve and when I put negative two I get negative eight. And so forth like this. What do you think of that rule?

Student?: This is really hard. (pencil tapping)

(Overhead shows:)
Researcher 1: Angela you think you know that one already?
Angela: It’s box times four equals triangle.
Researcher 1: She thinks it is box times four is equal triangle. Anyone else agree with it? Could you come up and write that for us?
(Angela goes to overhead and writes:)

Researcher 1: Could you show us how some of the ordered pairs work there. Like how it is that one and four. Could you show us some of that?
Angela: One times four equals four.
Researcher 1: Okay, two times four equal eight. Is everybody seeing that? All right, thank you Angela.
Researcher 1: Um, what about if I put something here like sixty in the triangle. Could you tell me what would have to be in the box? Bobby?

(Researcher 1 wrote ‘60’ in the triangle column.)

Bobby: Fifteen.

Researcher 1: Fifteen, do you think so? And how did you do that Bobby?

(Researcher 1 wrote Bobby’s response ‘15’ in the box column.)
156 Bobby       Sixty divided by four.
157 Researcher 1 Sixty divided by four so he’s using the idea of the inverses.
158            Somehow I think you ought to be able to do this one. (she goes
159            back to the previous sheet at the overhead) Want to try again? Did
160            you notice, you know that one that fast already.
161            (Overhead shows:)
162
163 Students? We already knew it.
164 Researcher 1 Oh you knew it. All right, Michael come up and see if, what other
165            people think about yours.
166 Michael    (comes to the overhead) Uh, write that rule? Just write it down.
167 Researcher 1 Yeah and see if some other people agree with you.
168 Michael    (Michael writes on the overhead:)
169
Researcher 1: What do you think? Did some other people get that rule or if you didn’t get that rule would you see if you think it works?

Michael would you mind coming back and convincing some of them, show why you think it works.

Ankur: That's what I meant. (whispered, camera is at board)

(table is talking during Researcher 1, indistinguishable)

Michael: Two times one-third.

Researcher 1: Can you write some of those out?

Michael: Two times one-third equals two-thirds and two times one-third is one-third plus one-third and that equals two-thirds.

(Michael writes on the overhead:)

---

Researcher 1: All right. Are you convinced or you want to see him do some of the other pieces. I think they’d like to see you do some of the other pairs. They may be convinced that two and two-thirds work, that this row works, what about some of the others?

Michael: Three times one-third equals one, cause there’s three of the one-thirds and that means that three of the one-thirds equal three thirds, because three-thirds equal one.

(Michael writes on the overhead:)

---
Researcher 1: Who is convinced that his rule works? All right Michael, thank you. Do you know what though it is a little strange to me because I had another rule in mind and I think it works too. Um, here’s what I was thinking about. I was thinking about the box divided by three. The number in the box divided by three is equal to the number in the triangle. Do you think that would work too? Do you think that would work?

(Researcher 1 writes on the overhead:)

Students? Yes.

Researcher 1: So you think that there could be two rules for some, some tables? So are you telling me that if you divide a number by three that you get the same answer as if you say a third times that number?
Would that always work? Do you think that works for every number? I’m not sure whether you agree with that or not.

Jeff

Researcher 1 So you think you could talk about inverse two different ways in this case. You could say a third, all right. What if I gave you in the triangle the number two, what number would have to be in the box? Okay, Amy-Lynn.

(Researcher 1 writes ‘2’ in the triangle box:)

Amy-Lynn Six.

Researcher 1 Six, does everybody think six will work? All right, all right. Very good you’re off to a good start. What about, maybe this one is a little harder for you. Let’s see.

(Researcher 1 writes on the overhead:)

(camera remains on board until 27:26)

Erin Let's discuss the new rule. (indistinguishable whispering)

Researcher 1 (walks to table with paper)

If you want to, there's paper you can use. There's either side of it.

Brian (whispers, "x minus, difference in between them, going down by 3," indistinguishable)

Ankur So is this the next one?

Erin What, no ... (louder?)
(camera zooms in on Erin's paper until 28:52, during that time she writes a "-8," to the right of the first line, then erases it, and writes "4    17" on one line to the right and writes "5    21" on the next line underneath as shown below:)

(camera zooms in on Erin's paper until 28:52, during that time she writes a "-8," to the right of the first line, then erases it, and writes "4    17" on one line to the right and writes "5    21" on the next line underneath as shown below:)

243 244 245 246 247

Brian Yeah, it is, it is.

Ankur What is the next one?

Brian So... (indistinguishable)

Ankur Why not?

(indistinguishable whispering)

Brian Goes by 22. Goes before the..

Ankur Right there, why?

Brian It's up by thirteen (thirty?), no okay, you're right, you're right.

Erin? Hold on a second.

Stephanie? Oh, you know, what is it?

Brian It goes four, squared, seventeen then back; five, squared, then one in the triangle.

Stephanie? No, how did you get that.

28:52 Ankur He doesn't even get what it's for.

Stephanie The difference is not..

Ankur Between each one on the right side, (smiles as he says) look.

Stephanie Oh.

(Stephanie slaps him in the back of the head with her pencil)

Erin (indistinguishable, but Ankur responds)
Ankur: It'll be quicker, you had no idea.
Brian: You're not doing nothing, you're just sitting there.
Stephanie: (indistinguishable, then drops her pencil and puts her chin on the desk)
Ankur and Erin begin scribbling on name tags and tape
Erin: How come you didn't get a new name tag? (to Ankur, who shrugs)
Stephanie: I don't like you Ankur. (she sits up, picks up her pencil and puts it next to him)
Brian: (indistinguishable)
Ankur: You're just, you're just playing with your...
Stephanie: Um, yeah, but see they already explained that one.
Erin?: Oh.
Researcher 1: Amy-Lynn thinks she has a rule. Does anybody else? We will give them another minute or two? Then we can let you put that up and see what they think?
(Ankur whispers to Brian, and Stephanie whispers amongst the girls, off topic over Researcher 1)
Ankur: This is what it is. It's box.
Erin: Plus four.
Ankur: No, it's box times four plus one. (looks around) It works. Box times four plus one.
(camera focuses on Ankur writing "box times four plus 1 equals triangle" as shown below:)
Erin: Four times four, sixteen, okay.
(Ankur whispers to Brian, and Stephanie whispers amongst the girls, off topic over Researcher 1)
Researcher 1: Does anybody else think they have it yet? The rule?
Stephanie: Um, we have it, Ankur has it (pokes Ankur with pencil).
Researcher 1: How are you doing over there?
Ankur: We have ...
Brian: We got one.
Ankur: Mhmm.
Researcher 1: You got one? Let’s let Amy-Lynn put hers up and see if you agree with that one. Amy would you do that for us?
Amy-Lynn: Should I, like, just do it in the thing, cause this is what we came up with, like times four plus one, that was like our rule, to do with everything in the box.
(Amy-Lynn writes on the overhead:)

Researcher 1: What do you think of Amy-Lynn’s rule? That same rule you got? Or you couldn’t? What about over there at the far table? Do you agree with it?
Students: (whisper) Yes.
Researcher 1: Do you want her to convince you or do you agree with her? Do you agree? What if I put forty-nine in the triangle? Could you tell me what would have to be in the box? Already? Michelle.
Brian?: Twelve. (whispered)
Michelle I.: Twenty-four.
Researcher 1: Michelle thinks twenty-four.
Bobby and others: Twelve.
Researcher 1: And you think twelve. Lots of people are thinking twelve.
Brian: Cause that’s the answer.
Researcher 1: Do you agree with that twelve? What did you do? Does everybody agree that twelve works? You had twelve times four. Angela?
Brian: I can’t take this.
Angela: Forty-nine minus one divided by four. Forty-nine minus one is forty-eight and you divide by four.
Researcher 1: So Angela is saying that she took the forty-nine and subtracted one and then divided by
Angela: Four.
Researcher 1: Four. Do you think you could write that for us where we had triangle equal to something with the box?
(Researcher 1 writes on the overhead:)

(Researcher 1 goes to overhead)

Angela I can do it like backwards?

(Researcher 1 writes on the overhead:)

(Researcher 1 goes to overhead)

Angela You divided forty-eight by four.

Researcher 1 Oh and you have division there. Okay, I see what you did. Okay, all right that’s good. I read that as a plus sign. Does everybody know that you could also write this, this way if you want to.
It is just a different way of writing. That’s really nice. Do you want to try another one, see if we can stump you?

Let’s say that when we do this we get zero and when we put in negative two that we get two.

Student? Seriously, we’re winning.

(Overhead shows:)

(It appears Brian and Erin were the ones whispering, still too low)

It goes up by how m...
375  36:24  Erin    Six.
376    Stephanie  No.
377     Erin        No, I just like ...
378      Brian  (interrupting Erin at the beginning) What are you doing?
379        Stephanie  Add three, add four, add five, add six, add seven, add eight.
380         Brian  (unintelligible at first) I'm a car dryer.
381     Stephanie  Are you really, is it official?
382      Brian  Yes, I am. I'm one of them stupid people, that dry your cars after
383                  the car wash.
384      Stephanie  I'll come get my car washed just so I can make fun of you.
385     Erin        We'll go together.
386      Stephanie  And then we'll get Chinese food and watch Heather's.
387      Brian  Better tip me.
388      Stephanie  Um, depends on how good you dry our car.
389     Erin        We'll think about it.
390  36:45      Brian  I won't dry it, your, I'll just be sitting..
391    Stephanie  Well, then we're not going to tip you.
392      Erin        Where?
393    Stephanie  Here, we should make a math problem. (Brian begins to say
394                  something) If Brian washes...
395      Erin        Where is it?
396      Brian  It's either in Westfield or Garwood.
397     Erin        There's a shop in Westfield.
398    Stephanie  Oh, is it by my doctor's office? Cause there's a car wash right by
399                  my doctor's office in Westfield.
400  37:06      Researcher 1  Who thinks they have the rule?
401      Brian  It had better not be.
402    Stephanie  Every time I go to the doctors I'll come, get my car washed.
403      Student?  I do.
404   Researcher 1  Oh, okay. Magdalena you want to put the rule that you have up and
405                       see if they agree with you?
406  37:44      Stephanie  I know what it does, I just can't figure out how to get a rule.
407                    (Magdalena goes to overhead and writes:)
408

409
410
411    Researcher 1  Is that what the rest of you got or somebody have a different rule?
412                    Everybody had that same rule?
413  38:08    Researcher 1  Magdalena, can you give them a number to put in the triangle and
414                   see if they can find the number in the box. Can you think of a good
415                   number to put in the triangle?
416                    (unintelligible and off topic whispers at table)
(Magdalena writes ‘56’ in the triangle column)

417  Researcher 1  Magdalena put fifty-six in the triangle. Could you tell her what you
418  had in the box? Okay, Michael.
419  Michael  Twenty-five.
420  Researcher 1  Twenty-five and everybody agrees with that. And how did you get
421  it Michael?
422  Michael  Minus six, fifty-six minus six and I divided by two.
423  Researcher 1  Could you write that as a rule?
424  (Michael goes to the overhead and writes:)

425
426

427
428
429  Researcher 1  All right that’s nice. But you know what I had another, a different
430  rule that I think works. I had this. Tell me if you think that this
431  would have to work too. I had two times box plus three is equal to
432  triangle.
433  (Researcher 1 writes on the overhead:)
434
435
436
437  Researcher 1  Do you think that works? Well do you think there are just two
438  rules that work? Do you think that one works, that both of those
439  work? Do you think there are any other rules that work?
440  40:15  Researcher 1  Can anybody tell me why that works? It seems a little strange to
441  me. Bobby?
442  40:20  Bobby  You have to do the work in the parenthesis first and like, then if
443  you add three to negative three it’s zero and if you times two by it,
444  it like stays the same cause it’s zero and it like works on every one.
Researcher 1: Did you understand what Bobby said? Does anybody think it’s odd that you have a three down there you have a box plus three but when you have the rule the other way you have a six and that you get the same number? What do you think Ankur?

Ankur: Two outside the parenthesis so it becomes sort of like a six.

Researcher 1: Would you go up and explain what you are saying?

Ankur: The two outside of the parenthesis doubles what ever's inside.

Researcher 1: You mean it doubles everything that's inside.

Ankur: Yes.

Researcher 1: All right, what do the rest of you think? Think that will always be true, Jeff?

Jeff: Yes.

Researcher 1: Jeff thinks yes.

Michael: I do too.

Researcher 1: You think yes too. All right, all right. Let’s try another one.

( unintelligible whispering )

Researcher 1: Maybe this is too easy.

Student: But it's not like they're filming you or anything.

Student: I feel like evidence.

Researcher 1: This one is probably too easy for you.

( Overhead shows:)

Stephanie: I have a question. If you multiply a negative times a positive, do you get a negative, or a positive?

Ankur: Negative.

Stephanie: If you multiply a negative times a positive?

Brian: Yeah, you get a negative.

Stephanie: So, if you multiply a negative times a negative, what do you get?

Ankur: I get, two negatives is positive.

Stephanie: How do you remember this?

Ankur: I don't know.

Stephanie: I went searching the other day, like, when we started this, for my notes, for Mr. Po's class last year, cause me and Miller wrote up like this big, huge, thesis thing and like...
Researcher 1: Jeff thinks he has a rule. Anybody else? Jeff do you want to come put up the rule and see if they agree with you?

Ankur: Box squared. Box squared (unintelligible). Box times box is triangle.

Stephanie: You're not going to write this all down, are you, Ankur?

(Jeff comes to the overhead and writes:)

Jeff: Box times box is triangle.

Student: Ah, that does not work.

Jeff: Yes it does, it has to work.

Researcher 1: Okay, Michelle says.

Michelle I.: That does not work cause if it's negative three times negative three it’s negative nine.

Student: It's a positive times a positive is a positive.

Michelle I.: No, you people, you people should have told me that.

Amy-Lynn: Yeah .. I said it wasn’t right.

Michelle I.: I was just repeating what they said and I disagreed with it and then they tell me that I am wrong. So it is their fault that I was wrong.

Michelle I.: It’s on the camera. I’m telling you.

Amy-Lynn: I said it before. Shelly (Michelle I.) and I said it and you (meaning Magdalena) said we were wrong.

Magdalena: No I didn’t. Bobby said it.

Amy-Lynn: Bobby said you were wrong.

Michelle I.: Bobby the supposedly child prodigy over there.
Researcher 1: So they are having a discussion at this table. They feel that some people were led astray about negative three times negative three being negative nine or positive nine?

Stephanie: (during Researcher 1) Oh I don't know where my books are, forget.

Jeff?: There's a camera right on you.

Amy-Lynn/Michelle I.: It's nine.

Stephanie: They did.

Researcher 1: Do you want to write that for us Michael? Michael has another way of writing it.

Student: That's okay, another way to write it.

Researcher 1: Sarah wants to.

Researcher 1: (Laughter at Bobby’s table)

(Laughter at Bobby’s table)

(Michael comes to the overhead and writes:)

Researcher 1: All right. Thank you. That’s great. Now, did I hear someone talking about square roots? Were you talking about square root?

44:20 Stephanie: No, I hate square roots.

Researcher 1: Angela is saying that if I gave you a number over here like twenty-five, that you would've to find the square root. (Researcher 1 adds ‘25’ to the triangle column.) There is a little confusion about that. Look, when we had nine on this side, and we went backwards we could get a three. We also worked that nine would give us a negative three. So that’s a little different isn’t it. What if I put a negative thirty-six here, (Researcher 1 adds ‘-36’ to the triangle column). What would you get on the other side? Brian?

Brian: It is either a positive or a negative six. I’m not sure.

Researcher 1: Hey, Brian thinks it’s either a positive six or a negative six. What did you think Joe?

Joe: Nothing. (Jeff changed his nametag to Joe)
Bobby? You can’t put a negative there. Cause to get a negative number you have to times like a positive and a negative. So you get a negative. So that’s a negative times a negative and that’s a positive. Which Bobby got wrong before and made me look dumb. Oh wow. They’re saying over here that they don’t think I should put a negative over here that you wouldn’t be able to get an answer. They’re right. No. Do you think that’s right? We shouldn’t use this negative thirty-six. Okay. We are saying it because it is right. All right. You’re doing great. You told me you would. Maybe this one will be a little bit harder. (students at table discuss watching something on TV last night.) (Researcher 1 writes on the overhead:)

No, I did that wrong. I did that wrong. This should be three. This should be four. This should be five. (Researcher 1 corrects the table on the overhead:)
Jeff? What's awkward mean backwards?
Researcher 1 Michelle has a rule already.
Michelle I. Plus two.
Researcher 1 Want to come and write the rule for us?
Stephanie Wait, I want to figure, I want to see her rule.
Ankur? We've got a rule already, I didn't even get the problem.
Stephanie (reply to something) After you saw it.
(Michelle I. goes to the overhead and writes:)

That turned out pretty easy didn’t it. And here I thought I’d made a harder rule. Do you know what my rule was?
I had three times the number in the box plus six divided by three is equal to the number in the triangle. Do you think that would work?
You know, do you think that would work?
(during Researcher 1) You sure?
I think they are trying to prove to us there are two different ways.
(Researcher 1 writes on the overhead:)

\[
\square + a = \triangle \\
\triangle : a = \square
\]
Stephane: Why would you do that when it's a lot easier to do the other thing though?

Ankur: (unintelligible)

Stephanie: But why would you, why would you do that?

Researcher 1: Would you tell me if it works?

Student?: It does.

Researcher 1: So what do you think Stephanie why does it, that work out to be the same? Does anybody, think that might work out to be the same?

Bobby: No, cause like negative two times three is negative six plus six is twelve and if you divide it by three it will be four not zero.

Researcher 1: So are you, I wasn’t sure what you are saying Bobby, do you get the same answer or a different answer?

Bobby: A different answer.

Researcher 1: Bobby is getting a different answer. Are some other people getting a different answer too? Angela?

Angela: Shelly made a mistake up there. She put box plus two then she put triangle divided by two.

Michelle I.: It’s the same thing, Angela. Oh, sorry, it’s supposed to be minus not divide. Sorry about that.

Researcher 1: All right. Thank you Angela, I hadn’t noticed it. That’s good.

Stephanie: Do you want to change that who put it up?

Stephanie: Want to hang out?

(Michelle I. goes to the overhead and changes the division sign to a minus sign.)
Researcher 1: What I want to know if three times the number in the box plus six divided by three does that work for all of them?

(Brian, Ankur, and Jeff all lean in together)

(Bell rings)

Stephanie: I can't believe we missed it.

Stephanie: (interrupts Jeff, who is still leaning toward their table talking) You know, they could get us to talk more if they gave us the problem first, and then let us get into the problem, and then asked us the questions, instead of just putting it on the overhead and expecting us to answer. We go so much into it. They should've had it on a piece of paper and asked us to solve them.

Jeff: Really?

Michelle I./Amy-Lynn/Magdalena: It works. It works. We just kinda miscalculated again.

Jeff: (continues whispering to Ankur and Brian)

Michelle I.: I can't agree with you people any more you always make me look dumb.

Researcher 1: Some people think it works and some people think it doesn’t work.

Michelle I./Amy-Lynn: It works.

Stephanie: Class six is six. Divided by three is 20. I don't think theirs works.

No, do it with the one and the three.

Stephanie: Is seven divided by three.

Student?: Oh, it works, Okay.

Jeff: Where does it not work? Where does it not work?

Erin: Okay, three times one equals three, plus six ...

Stephanie: Is seven divided by three.

Erin: Is six.

Stephanie: Is seven.

Erin: Three times one is three.

Stephanie: Oh, geez.

Researcher 1: Now, it would be a whole lot easier for me to write it as box plus two equals triangle if I knew that was the same. Is there any way I could know that it was the same?

Stephanie: No, seven divided by three. Seven divided by three isn't three.

Erin: No, no, no, three times one.

Stephanie: Oh, god.
(camera remains on board, students begin whispering)

52:10 Stephanie Ankur, can I draw on your hand? I'll make a pretty picture.

Researcher 1 Does anybody have an idea why that could be the same?

(Bell rings)

(table is talking about writing names, as anagrams?)

Researcher 1 Want to come try? Michael just wants to stay at his desk. He doesn't want to come up.

Jeff Sarah wants to come up.

Michael Six on the top divided by the three at the top equal two. Then the two threes, on the bottom and the top, cross each other out, so it's box plus two.

Researcher 1 What do you think, did you hear what Michael said?

What do you think about that? I don't know that's sounds a little dangerous this crossing things out. I wonder if it's anything like trying to find the one you did before?

Student? That sucks, are you threatening me?

Researcher 1 We were looking at something else where you had two times the number in the box plus six, I think it was, and you told me that that would be the same as this. And I wonder if there is anything about this one that's like that one?

(Researcher 1 writes on the overhead:)

\[
\frac{3 \times 0 + 6}{\Delta} = \Delta \\
\frac{2 \times 0 + 6}{\Delta} = \Delta \\
2(\text{top} + \text{bottom}) = \Delta
\]

54:12 Michael Because if you have a variable in the parenthesis, you can divide, I mean times each number by itself. Like two times the box is two box, and two times three is six, so it's the two times the .. I'll write it down.
(Michael goes to the overhead and explains how two times box plus three is equal to two times box plus two times three using distribution)

(table is whispering, indistinguishable)

Researcher 1
If you need anymore here...

Okay, now what does that have to do, don’t go away. Is there anything about this one that’s like that? ... Okay

55:31  Michael
Wait, I think I see something. Three times the box can be written as three box plus six. They are similar to each other, three box plus six and two box plus six.

(Michael writes on the overhead:)

30 + 6 = \Delta
\frac{3 \times 10 + 6}{3} = \Delta

20 + 6 = \Delta

Researcher 1
But instead of multiplying it by two, and the, and then divide it. The top, and like ... you have to multiply this (points to the three in the numerator).

55:58  Researcher 1
Would it hurt anything if I put, don’t go away, would it hurt anything if I put parenthesis right here?

(Researcher 1 adds parenthesis to the overhead:)

20 + 6 = \Delta
No. It wouldn’t hurt.

Would it help any?

So you do this first thing divided by three.

So you’re saying this one looks like this one.

Yeah, except for the division…..

That instead of multiplying you are dividing it.

(indistinguishable) ... I guess so. (Indistinguishable, something about six divided by three)

So what do you think about what Michael is saying? Do you think that might be the reason that?

Could somebody make up another one that you think would work two ways? See if you can make up one that would work two ways.

No, but see it's going to... Why would you make it work two ways if you can just get the easy way?

Cause the one is like, three times the square root of 91, divided by seven.

For what?

And all you have to do is like, it's like three plus two, plus box.

You know what I mean?
Stephanie’s table 2 of 2: 5/16/1995 (S1B)

Camera View: Stephanie’s Table 2 of 2 (Stephanie (center), Romina (right-close), Ankur (left-far), Brian (left-close), Michelle R. (right-far))

Date of filming: 05/16/1995
Harding public school, Kenilworth NJ, Grade 7
Advanced Guess my Rule (AGMR): Inverse: Problem
Transcribed by: Jenn Forbes
Date of transcription: July 2010
Verified by: Andrea DePaolo
Date of verification: August 2010
Length of session: 00:39:49

<table>
<thead>
<tr>
<th>Time</th>
<th>Speaker</th>
<th>Transcription</th>
</tr>
</thead>
<tbody>
<tr>
<td>00:04</td>
<td>Romina</td>
<td>Just make a pattern…</td>
</tr>
<tr>
<td>00:33</td>
<td>Stephanie</td>
<td>There’s no point to it if they’re trying to prove there’s more than one way to do a problem, they do it…But there’s no point.</td>
</tr>
<tr>
<td>00:38</td>
<td>Romina</td>
<td>It’s the same thing, just more complicated.</td>
</tr>
<tr>
<td>00:47</td>
<td>Stephanie</td>
<td>But why would you want to make it complicated?</td>
</tr>
<tr>
<td>00:54</td>
<td>Romina</td>
<td>(inaudible)</td>
</tr>
<tr>
<td>00:54</td>
<td>Stephanie</td>
<td>Oh um…(unintelligible) is gonna ask her instead…because Patrick is coming over today.</td>
</tr>
<tr>
<td>1:02</td>
<td>Ankur</td>
<td>Is Gina (inaudible)…</td>
</tr>
<tr>
<td>1:03</td>
<td>Romina</td>
<td>(inaudible)</td>
</tr>
<tr>
<td>1:04</td>
<td>Stephanie</td>
<td>She told me.</td>
</tr>
<tr>
<td>1:07</td>
<td>Brian</td>
<td>What</td>
</tr>
<tr>
<td>1:13</td>
<td>Stephanie</td>
<td>She knows what she’s (something unintelligible) I bet.</td>
</tr>
<tr>
<td>1:16</td>
<td>Brian</td>
<td>What’s the bet….Can you tell me the bet?</td>
</tr>
<tr>
<td>1:19</td>
<td>Romina</td>
<td>No.</td>
</tr>
<tr>
<td>1:22</td>
<td>Brian</td>
<td>Why not?</td>
</tr>
<tr>
<td>1:27</td>
<td>Romina</td>
<td>(inaudible)</td>
</tr>
<tr>
<td>1:29</td>
<td>Brian</td>
<td>Michelle, Hi. (waves to Michelle)</td>
</tr>
<tr>
<td>1:29</td>
<td>Stephanie</td>
<td>Um, alright.</td>
</tr>
<tr>
<td>1:30</td>
<td>Romina</td>
<td>Five times two minus three</td>
</tr>
<tr>
<td>1:33</td>
<td>Ankur</td>
<td>Wow…ten, seven, ten….equals seven</td>
</tr>
<tr>
<td>1:36</td>
<td>Romina</td>
<td>Squared….Equals..</td>
</tr>
<tr>
<td>1:38</td>
<td>Romina</td>
<td>(Brian gets up from table and leaves room)</td>
</tr>
<tr>
<td>1:40</td>
<td>Ankur</td>
<td>(Ankur laughs)</td>
</tr>
<tr>
<td>1:40</td>
<td>Stephanie</td>
<td>(to Romina) Can I just put that brackets around these things.</td>
</tr>
<tr>
<td>1:43</td>
<td>Ankur</td>
<td>How is that going to work?</td>
</tr>
<tr>
<td>1:45</td>
<td>Stephanie</td>
<td>I’m sorry, Romina.</td>
</tr>
<tr>
<td>1:48</td>
<td>Brian</td>
<td>(returns to table)</td>
</tr>
<tr>
<td>1:49</td>
<td>Brian</td>
<td>I got me a (packet?) Yes.</td>
</tr>
<tr>
<td>1:51</td>
<td>Romina</td>
<td>Gum?</td>
</tr>
</tbody>
</table>
Brian: Yes
Stephanie: Why?
Romina: (inaudible)
Stephanie: What, what flavor?
Brian: 1:37 Watermelon.

Stephanie: Watermelon is so good, do you have a piece?
Brian: No.
Stephanie: Ankur...

(Romina and Brian laugh)
Romina: Hold on now let's, okay, let's make this one....

Brian: What are you doing
Romina: ...Five times two, that's ten minus three is seven plus one is eight.

(Brian whispers something to Ankur and they both laugh.)
Ankur: One, eight, next one?

1:59 Jeff: I came up with the most perfect one and they wouldn’t let me do it.
Student?: Let’s see Jeff.
Brian: Oh ya, what a pen.

(holds up two pens stuck together end to end)
Jeff: (writes something down) It works either way
(Researcher 1 comes over to group to listen in)
Brian: What is it?
Jeff: It’s easy but it works either way.
Brian: What is it?
Ankur: Since I found it, I got four, five, six,
Romina: And this one could be...hold on this one, you guys, this one could be five times two minus three plus square or just plus seven.

2:20 Jeff: (laughing) Could what?

(table laughs)
Brian: Five times two minus three plus seven plus four.
Jeff: I-I like the traditional easy one.
(Researcher 1 walks away from group)
Romina: Well ours is easy.
Stephanie: Oh ya it’s real easy. Five times three plus one minus two equals seven times five times one....

Jeff: Watch this Steph. One plus one.... No zero plus one is one. One plus one is two. Two plus one is three.

Stephanie: There’s probably two ways to do it
Ankur: (to Romina during Stephanie and Jeff above) Wait...isn’t the next one going to be ten? Isn’t three gonna be ten? Ya but look at the answers... one – eight , two – nine, three – ten. Watch three is going to be ten.
Romina: (to Ankur) It is. Either five times two minus three plus square or square plus seven. It doesn’t matter.
Brian: (something unintelligible about his pen to Stephanie)
Jeff: Or this could work either box plus triangle.
Stephanie: I just lost my eraser mate so I had to get one out of my locker.

Brian: Oh cool.

Stephanie: It doesn’t have a cap so it bothers me.

Jeff: (writing) Check that out. That’s perfect.

(Shown below is what Jeff wrote on Ankur’s paper.)

Brian: What? (laughs) What is it?

3:01 Ankur: Check Romina’s out..uhhh….Five minus two minus one minus seven minus six times square root of seven …lllluuullll (makes weird noise)

Romina: Five times two minus three plus square, or square plus seven.

Stephanie: (puts head in her hands) Didn’t we just go over this whole big we don’t want a complicated problem thing.

Romina: Ya they want us to complicate it. That’s the whole point.

Stephanie: Alright, ya but see this is nice and easy.

Ankur: We have to make more complicated ones simple. Like this, watch.

3:33 Romina: (interrupting above) One equals eight, two equals nine, three equals ten.

Brian: That’s, oh my God.

Romina: It’s plus seven, it’s not that hard.

Ankur: So that equals, it’s either square plus seven...

Romina: Square plus seven, or...

Ankur: …Equals triangle

Romina: ..Or five times two minus parenthesis three plus square equals triangle.

Ankur: (repeating what Romina said as he writes) or five times two minus three plus square...

(Romina’s paper shown below.)
Brian (looking at Romina’s paper) Why did you put five times [two]?
Romina To make it look longer, more complicated, okay?
Brian Ten minus three.
Romina Is seven.
4:06 Stephanie When you think about it you can change all the numbers in the parenthesis.
Brian That’s true. Okay, we got one.
Stephanie As long as you can get it down to seven, ya.
Romina Seven times seven, divided by seven.
(Ankur laughs at Jeff’s table conversation about Jeff’s rule)
Brian (looking at Jeff’s table) You’ll still trying to fight your case.
4:48 Brian (to Stephanie) Why are you coloring in the box? Now we can’t put a number in it.
Romina You don’t want to put a number in it.
(Researcher 2 comes to table and points at Ankur’s paper)
Brian It works.
Romina It works.
5:02 Researcher 2 Can you find another one?
Ankur Another one that equals the same thing as this?
Researcher 2 Uh – huh. I have another one. Mine is different mine is eight box plus fifty-six.
Ankur Eight box…
Researcher 2 Oh wait a minute…divided by eight.
Ankur Eight box…
Researcher 2 …plus fift- six divided by eight. It’s my rule. Do you have a piece of blank paper? Okay, my rule is for yours, eight box plus fifty-six divided by eight, equals triangle.
(Researcher 2 writes a rule shown below.)
And I know that uh, I know that I’m going to try another rule.

(Researcher 2 writes another rule shown below.)

Brian  
Ahhhh…

Ankur  
(pointing to the rule) Those two cancel out.

Romina  
No just...(unintelligible) One plus seven is eight…

Stephanie  
Can I see? It works.

Ankur  
What about this one?

Stephanie  
Eight, it does look. Put a one in there because an eight would go over there.

(Stephanie writes a one in the box and an eight in the triangle).

Then it’s eight plus fifty six that’s sixty four. Sixty four divided by eight is eight.

Brian  
(Really, really?) Okay, okay, okay

Ankur  
So these two cancel out.

Brian  
Ah, okay. Does that work for every one though?

Ankur  
So we can add these two.

(Ankur begins to write down the new rules on his paper)

Stephanie  
Okay, I can make your pen balance too.

Michelle can I borrow your pen?

Stephanie  
Twice, and you still didn’t get that one.

Romina  
(trying to balance her pen) It’s slanted. See now it’s going to fall this way towards me, see?

Stephanie  
Mines gonna fall this way towards me.

Romina  
No you – (something unintelligible)

Stephanie  
(laughs) I’m kidding Romina.
(grabs Romina’s pen) I’m holding the pen.

(long silence at the table)

8:44 Stephanie (yawns) Oh my goodness, I’m tired.

(another long silence at table)

9:10 Romina (mumbles something to Brian about the camera)

Brian No they don’t

Romina Yes they do, look (who?) the camera is pointing at.

Brian (I’ll keep it?) under the table.

Stephanie I don’t understand why you people are still amazed at that.

Brian What? At what?

9:25 Stephanie Seven years. Seven years. And you’re still amazed at the fact that

the camera is pointed at you when you’re doing strange things.

Brian (laughs) But (no one?) sees this.

Brian (singing) I’ve been working on the railroad….

9:48 Stephanie (laughs again) Oh my god. No the secret is (that they?) send it in to

America’s funniest home videos.

Stephanie is hitting her pencil on the paper)

Brian flips over his paper and is writing something in big letters)

10:19 Romina What are you trying to do?

Stephanie I’m trying to hit the point. Make a point on your paper, then try

and hit it.

(Romina laughs and tries to do the same thing as Stephanie.)

10:58 Brian What are you doing? You’re making holes in the table

11:09 Researcher I So, (unintelligible, 2 words) Oh, lets see what you got here.

Ankur Where is it?

Brian Where is the other one?

Stephanie I have it. Here.

Brian I got these.

Ankur And the dotted line separates them.

Researcher I Alright, now which ones are equivalent?

Ankur All of them.

(Ankur’s paper is shown below.)
Researcher 1 All of them are. Oh goodness, looks like… That’s neat. That’s neat. Really, really good. That’s great. And can you explain why to them?

Ankur ‘Cause these just simplify and you eventually get down to this. (Romina mumbles something and the group laughs)

Stephanie Ankur…Oh my god, you know?

(Ankur writes something inappropriate about Brian on his paper in big letters and holds it up to the camera)

(Brian and Ankur laugh)

Stephanie (to Romina) And we associate with them

Romina I know

Brian There’s nothing else (you can?) do.

(Brian and Ankur laugh)

Researcher 1 As I’ve gone around to the tables, I see you doing some really interesting things. Um, Ankur you want to show us what you have? Would you come show everybody what you have…at your table?

Brian (during Researcher 1 above) C’mon Ankur.

(Ankur walks to the front of the room and begins writing on the overhead as shown below.)

13:25 Ankur Um, I guess I’ll go to the overhead.

(Ankur walks to the front of the room and begins writing on the overhead as shown below.)
(During Ankur’s writing there is quiet unintelligible chatter among the group, below)

Romina: You wouldn’t want to be (copying this at other tables?)

Brian: I am (the table?)

Brian: Want to see if I can make my finger (something unintelligible).

We got to use erasable pen now. Do it on the other side. Alright just, (something unintelligible). And you got to (color?) your finger in.

Romina: It doesn’t work.

Brian: What the—

(Ankur has written four equations on the overhead as shown below.)

15:12 Researcher 1: Well, uh, Ankur could you explain to them now why, he thinks all
of those would give you the same set of ordered pairs. Does everybody agree with that? I know you’ve been working on similar things. Could you tell if they are right? I think your friends at this table agree. Could you go through them and tell us why you think they are alike?

Ankur: Well this is like the small- shortest way to say that’s like the way most people would think of the pattern.

Researcher 1: Okay, it’s just complicated people who would think of it the other way.

Ankur: This just gets more complex and see the first, behind the equal sign this side equals seven this part equals seven, behind the equal sign.

Researcher 1: Okay

Ankur: And so does this side.

Researcher 1: Do you know- I don’t see a box in that second one.

Student: Yeah.

Ankur: Oops.

Student: You forgot to put it. You put a box after the three.

Ankur: It’s minus three plus box. That’s behind the equal sign.

(Ankur added + □ to his work on the overhead as shown below.)

Researcher 1: Okay. Could you explain why you think the next one works? I think there is some confusion about that.

Ankur: This one? (points to the third one down)

Researcher 1: Uh- huh.

Ankur: This one. It’s just, you can divide the hundred box by a hundred so that turns down to one box.

Researcher 1: We’re not quite clear is why one hundred times the number in the box, divided by a hundred would be the same as the number in the
box. Could you explain that? And tell us why that works? Maybe write it over to the side or something. Just that part, a hundred times the number in the box divided by a hundred.

17:09 Ankur That equals one. A hundred divided by a hundred equals one. And fifty-six- no I mean seven hundred divided by one hundred equals seven. So that’s one box plus seven equals triangle.

Researcher 2 Can you write that down, those pieces that you just said?
Ankur Those pieces.
Researcher 2 That tells me a lot.
Ankur If I write the things that’s here…
Researcher 2 Just the two pieces you said were really nice.
Researcher 1 She wants you to say a hundred…
Ankur I know but if I write it on the overhead
Researcher Take another piece of paper
Ankur Okay.

Researcher 1 Can you find another clean one under there?
Researcher 2 Rewrite that piece of it.

17:47 Researcher 1 I can help you find one, they’re under here, the clean ones.
(Ankur begins writing again and the quiet chatter in the group resumes.)

18:01 Jeff Why is he laughing so hard?
Stephanie I don’t know, I don’t know. Romina, why is he laughing?
Romina I don’t know. Why is he so (something unintelligible)

(Ankur’s work on the overhead shown below.)

18:36 Researcher 1 Alright, so does everyone see what Ankur did up there?
Student Uh – huh.
Researcher 1 So a hundred times the number in the box divided by one hundred would just be one times the number in the box?

18:50 Researcher 2 Ankur, people back here said something to me that is different and I’m confused. Can you help me? May I work with you? The people back there said to me…
(Researcher goes to overhead and begins to write)

Researcher 2: You see this piece here, are you all looking? Right, which Ankur wrote like that. What they said to me was that they could write that top part like this.

(Researcher 2 rewrites one of Ankur’s equation on the overhead as shown below.)

Ankur: You can.

Researcher 2: You can? Could you explain that to me - I don’t understand that?

Ankur: Because one hundred is outside the parenthesis and one hundred times box equals one hundred box and one hundred times seven equals seven hundred so that’s one hundred box plus seven hundred.

Researcher 2: How do you get, how do you end up getting what you finally got? How do you go from there to get this result here.

(Researcher 2 points to □ + 7 = △.)

Ankur: You simplify.

Researcher 2: I’m sorry I didn’t want to mess that one up. Can you put the one back in? Thanks. Can you show me how we go from here to here and why that works?

Ankur: From here you can go

Researcher 2: To yours, yeah, how do you simplify that?

Ankur: From here you go to here

Researcher 2: No, the one under.

Ankur: This one? How do you simplify?

Researcher 2: How do you go from here to here? Can you show us?
Okay, this one hundred is outside the parenthesis so that equals one hundred box, is right here. And one hundred box divided by one hundred equals one box. So that’s how we go… so one hundred times seven equals seven hundred, so seven hundred box divided by one hundred equals seven so then you get one box plus seven equals seven.

What do you think?

You’ve all seen what Ankur’s doing here. What do you think Angela?

That’s like what we were doing like crossing out the zeros and you asked us why and we want to know if Ankur knows why.

Now, can you tell me why it works? I was confused.

We don’t know why. We want to know if Ankur knows why.

You want to know if Ankur knows why, well (He just gave us the answer?)

Ankur your explanation didn’t convince her that anybody knows why. Can you try, can you think of any other way to say it?

The one hundred cancels out the one hundred.

What do you mean by cancels out?

Because it’s on both sides so you can eliminate it.

That’s where you lost me.

Then all you have left is box plus seven. That’s one box plus seven equals triangle.

I don’t understand when you say cancels out. I see one hundred divided by one hundred here.

That equals one.

I can see that that’s one. I’ll buy that that is another name for one.

So that equals one box and seven. It’s still seven. So one box equals

Ankur do you know what worries me about cancelling? Is that sometimes people see things you know and they just mark them out because they look alike. Could you show us an example with numbers where you have a hundred times something divided by a
hundred and to be sure that it makes sense in numbers or

something.

22:16 Ankur How should I do that?

Researcher 1 What if you have a hundred times five

Ankur Five hundred.

Researcher 1 …divided by a hundred. Let’s say will you write that down and
see? You have a hundred multiplied times five and then that

divided by a hundred. Let’s see if you can tell us what happens.

Ankur So this is on both sides of the equation.

(Ankur added 100 x 5 divided by 100 on the overhead as shown
below.)

Researcher 2 That’s not an equation, Ankur. There is no equation there. There is
no equation. Look at that that’s just a fraction.

22:57 Ankur Then the fraction.

Researcher 2 Do you all see that? That’s where you confuse me. I see a hundred
times five divided by a hundred. That looks like a fraction to me.

Ankur Okay, then you can put-

Researcher 2 You guys were talking about equations even at the other table and I
don’t see an equation.

Ankur Then this equals five hundred and you can cancel out the two
zeros.

Researcher 2 I don’t understand what you do when you cancel out zeros.

(Ankur has crossed out zeros as shown on the overhead below.)
Ankur: Because they’re on both sides. You can do anything on one side. Whatever you do to one side you can do on the other side.

Researcher 2: Um, Angela and Stephanie

Angela: I finally figured out why you, uhm, cancel each other out cause they’re inverse operations. Like if you had one and you like plus one and you subtracted one they cancel each other out. If you, had like I don’t know it just cancels each other out. And you don’t even need the one up there because one box is just box so it would be box plus seven.

Ankur: Or you could do it like this. Five hundred over one hundred that simplifies to five over one. That’s five.

Researcher 1: But why? (Ya why?)

Ankur: What does why mean?

Researcher 2: Stephanie you were going to say something?

Stephanie: Oh no. I was just going to say kinda what she said. It’s just what it is you have to whatever if you take away a zero from one side because you can do that. Because it’s I can’t it’s like whatever you do to the one side you have to be able to do to the other side otherwise you can’t do it.

Researcher 2: Okay, I think you guys are going to get into a lot of trouble when you study algebra unless we really work this out. Because first of all you are saying things about side and this is a fraction. There’s a number on the top and a number on the bottom, that’s a fraction.

You are giving me examples that have to do with equations and balances and that’s something else. So I really think we are going to have to work this out otherwise you may fall into some trouble. We have all the algebra teachers in this room, in horror, worrying about what’s going to happen to you guys down the line. So we are gonna maybe take a time out here. The question I want to leave you with- Bobby you want to make a comment before I leave you with a question? Go ahead Bobby.

Bobby: Oh, ‘cause like five hundred divided by one hundred is five.

Researcher 2: Very nice.

Bobby: Even if you cross the hundreds out it is still going to be five no matter what.

Researcher 1: Okay I’ll- that I’ll buy Bobby. I’ll buy that. Five hundred divided by hundred is five. I am going to ask you something else here.

Researcher 2: (Researcher 2 places the slide shown below on the overhead.)
Well the time before you could not only write it as a fraction, you can also write it as an equation. So when you write an equation then you can cross out things.

Well that’s where I get a little confused because this is an equation. We’ve written this as an equation. But you see when you talk about crossing out things this is one side of an equation and this is another. This is the numerator of a fraction and this is a denominator and I get very confused when you’re crossing things from the- from the fraction and then you are talking about an equation.

Then change it into an equation and then you can understand it more.

Okay, but we need to work on that it seems. I think our time is out. We have a visitor here for just one minute. Tell me your name?

Lindsey? Lindsey, um, Lindsey sent me a wonderful problem. Lindsey’s a third grader and you know how am about whys. Don’t you all know how I am about why? And Lindsey, um, come say hi to this- this class, Lindsey. Lindsey- Lindsey discovered a- a- an interesting um rule. And my question to all of you is to help, uh, Lindsey tell me why. If I can find a blank overhead.

(Justice?) (Class laughs)
Researcher 2: Lindsey—now these—You guys used to do this to me when you were third graders. Are we out of paper? Overheads? (Class laughs)

Researcher 1: There should be some under there.

Researcher 2: I don’t see any more.

Researcher?: Here you go.

Researcher 2: One piece is all I need. Okay what Lindsey said is the following.

Lindsey do you want to say it or do you want me to? Do you want to say it? This is what you wrote me. Remember?

Lindsey: Yeah.

Researcher 2: Go ahead. Why don’t you tell me and I’ll write—write what you want me to write. Tell the class.

Lindsey: Five times an even number you, um, split the even number in half and you add a zero and you get your answer.

Researcher 2: So five times an even number— and you said, for example, six, right?

Lindsey: Mm hmm (nods)

Researcher 2: She says you split the even number in half. Spitting six in half gives you…

Students: Three

Researcher 2: Three. She says then you add—

Lindsey: A zero.

Researcher 2: Zero. And that’s how you get five times six equals thirty. Right?

Brian: Oh my god.

(Researcher 2 writes down the example using Lindsey’s rule.)

Researcher 2: Now let’s try it with another even number. Five times—give me another even number.

Student: Forty-six.

Researcher 2: Forty-six. You split the even number in half, right? Split forty-six Jeff and what do you get?

Students: Twenty-three.

Researcher 2: Twenty-three, right? So five times forty-six is going to be—adding a zero—

Students: Two thirty.

Researcher 2: Which will be two thirty. Does that work?

Brian: Ya

Students: Cool. Whoa.
(Researcher 2 has added the second example to the overhead.)

Jeff: Where did you come up with this?

Researcher 2: Right? So you can - you can try some oth- now Lindsey has another one for you. (Class laughs)

Student: Another one?

Researcher 2: ‘Cause Lindsey does not want to limit her theories to just even numbers. She also wants odd numbers, right? So what’s your rule Lindsey about odd? This is for the even, right?

Jeff: This is awesome – how does she do this?

Researcher 2: Jeff, Lindsey invented this. Okay, let’s go on.

Lindsey: Five times an odd number you, um, make the odd number, um, the next smallest number. Then you split that in half again and then you add a five.

Jeff: Yeah so five times three would be fifteen.

Researcher 2: Obviously. Class laughs) You got that? I had to think. What is it- when you took, you- you took the odd number again, Lindsey.

You reduced it by…

(Researcher 2 has written this example on the overhead.)

Lindsey: Um, the next even number.

Researcher 2: And then you-

Lindsey: Add a five.

Researcher 2: And you- when she said add a five, you append a five, right? A five at the end. Let’s try another one. Yes?

Jeff: (To Lindsey) What were you doing when you thought this up?

Lindsey: I was just doing some problems and I just figured it out.
Student: Wow.

Brian: Gosh!

Researcher 2: Angela you have a question?

Angela: It’s the same thing, how could you figure this out?

Researcher 2: Oh no, that’s your problem. (Class laughs) Lindsey has the (inaudible) up here, Angela. This is the problem. The theory is, let’s write this down, five- let’s write the first one down, Lindsey.

Lindsey: Mm hmm. You split it- right? And what do you do?

Researcher 2: How you obtain it- you split it- you split the even number in half, right? And what do you do?

Lindsey: You add- you um, put a zero at the end.

Brian: Oh my god.

Romina?: (whispering) How would she think to do this?

Researcher 2: And, um, append, alright? Append a zero. Right? You’re not adding, like an addition operation, okay? And the next one is five times an odd number, right?

Lindsey: Yeah.

Researcher 2: You-

Lindsey: You, um, make it the next even number-

Researcher 2: Okay, so you make the next even- smaller or larger?

Lindsey: The even smaller.

Researcher 2: Okay. You subtract by one is that okay?

Michelle I.: Yeah that’s that’s like what you’re doing.

Lindsey: Yeah.

Michelle I.: That’s- that’s exactly what you’re doing- subtracting by one.

Researcher 2: So you go to the next smallest even- next smallest number. Even, right?

Lindsey: Yes.

Researcher 2: And then what do?

Lindsey: You add- you put, um, a five at the end.

Researcher 2: And? Subtract a one, don’t you? (Pauses)

Student: I would never be able to come up with this.

Brian?: If someone like showed this to me I would laugh. Crazy. Hitting the (crack of the book?) too hard lately?

Researcher 2: (To Lindsey) What’s the divide number? And split it, and split it, right? And append what? A five. Okay. So your job is, why does this work?

(Researcher 2 writes Lindsey’s rule in words on the overhead as shown below.)
592  Brian  Told ya.
593  Researcher 2  Because you know you could- you could make- I’m gonna tea- all
594  the math you’re telling me oh, you just cross out zeros or you just-
595  or you just add this. You have to know why things work because
596  if- and does it always work? I want you to write up why it works.
597  Lindsey, would you like to know why this works? Do you know
598  why it works? No. So she’s discovered it, so now she needs some
599  help.
600  Brian  If she doesn’t know, how are we supposed to know?
601  Ankur  Now you know why …(something unintelligible)
602  Researcher 2  And maybe she can come back tomorrow when you all can tell her.
603  I mean she looks to the wisdom of the seventh grade- almost eighth
604  grade class to guide her in her-
605  Jeff  Do you have any of these for like Social Studies? (Class laughs)
606  31:33 Researcher 2  Let’s give her a round of applause. Let’s give her a round of
607  applause. (Class claps). This was lovely. Any theories? Okay,
608  609  610  thank you, Lindsey.
611  Jeff  Everything a little too much in there
612  Stephanie  And see, and see if she doesn’t know it then how are we going to
613  know it?
614  31:57 Researcher 2  My final- my final question to you before you all disappear- if you
615  could just look at this for a moment. Yes, Michael.
616  (Researcher 2 presents the slide shown below on the overhead.)
Well you said we’re crossing zeros without an explanation. Well we are - we do have an explanation. We are taking off a number- a certain number off each side and- to balance it out. Like, if we take off a number from each side and it’s the same thing as crossing it out- if you take off a zero off each side- you take off a- you divide by- you divide a hundred by ten on top and that equals ten. And you do it each number, you’re not just crossing it out, you’re dividing it. When you take out a zero you’re dividing a hundred by ten equals ten, it’s like crossing a zero out at the end… a shortcut…

(during Michael above) Is it like multiplication? Where you have a double number like one hundred and fifty-six and divide just (inaudible)…instead of just doing that you just…

(in response to Stephanie during Michael above) Ya but…. it’s going to be wrong and then you didn’t even get a chance to do it

So what you’re say- what I’m hearing you say, then, partially is that, um, it’s okay to divide- not saying each side- divide the top of a fraction- the numerator by a number- as long as you divide the denominator by the same number. So in other words if I had a fraction four eights, right?

That’d be saying forty over eighty.

So that’s the same- I could divide both the numerator and denominator by the same number.

It would be a half.

So I could say, for instance, I could divide them both by four, right? And why does that work? Why does that give you one half?

What does it mean to divide the numerator by four and the denominator by four?
Student: (Simplify?)

Researcher 2: Right but what am I really doing?

Student: You just-

Researcher 2: Why does that work?

(Researcher 2 writes the fraction work as shown below.)

\[ \frac{4}{8} = \frac{4 \times 1}{4 \times 2} = \frac{1}{2} \]

Student: Because it’s the same thing as a half except that it- it’s in smaller pieces. So all you gotta do is add the four eighths, like four eighths they equal two eighths. One half because it’s just like one half cut into little pieces.

Researcher 2: Anybody else have any ideas? Michael had a- an explanation.

Angela: You’re just making four eighths, which is one half, more complicated.

Researcher 2: When you write it as four eighths it’s a more complicated version of one half?

Angela: ‘Cause like four times one and four- over four times two is just the same as four eighths, it’s just more complicated.

Researcher 2: What made it more complicated?

Angela: Like adding times one and times two.

Ankur?: The numbers are larger.

Researcher 2: Okay, anybody else have any thoughts on that? Anybody else have any thoughts? That’s it? I think our time is out. Okay well it’s something to think about for tomorrow. Don’t forget Lindsey’s problem. Isn’t that neat for a third grader? Were you all impressed?

(Students exit)
Bobby’s table: 5/17/1995 (B2)

Camera View: Bobby’s Table (Bobby, Amy-Lynn, Michelle I., Magdalena )
Date of filming: 05/17/1995
Harding public school, Kenilworth, NJ, Grade 7
Advanced Guess my Rule (AGMR): Inverse Function
Transcribed by: Morgan Sawin
Date of transcription: June 2010
Verified by: Andrea DePaolo
Date of verification: June 2010
Length of session: 00:51:47

Time  Speaker  Transcription
1  00:00  (Students are entering and preparing for class)
2  7:12  Bobby  Do you, like, know what I explained yesterday?
3  Bobby  What?
4  Michelle I.  Do you, like, know what I explained yesterday?
5  Michelle I.  Yeah.
6  Bobby  Okay.
7  Michelle I.  You add a zero with the even numbers because, you know, if you
times, uh, five times the even, it’s, zero’s always at the end. And
when it’s with the five, there’s always, when it’s with the odd,
there’s always a (divided by?). So, it hurts! (referring to her
finger)
8  Bobby  Oh I know why you split it in half with the evens. You split in half
because, like, there’s two- let’s say you have two point five. Then
like, you know, 6 divided by…
9  Michelle I.  I thought, I didn’t know you split it in half. I thought you just, um,
put it to the one under it. If it was four, no never mind you split it
in half. Sorry about that. Bobby…
10  Bobby  (conversation about gum)
11  Michelle I.  Why do we have to figure out what the third grader did? I mean
she’s the genius let her figure it out.
12  Amy-Lynn  I know.
13  Bobby  If we find out, and it become a multimillion dollar deal,
14  Amy-Lynn  Yeah but she…
15  Bobby  …like we get part of it.
16  Amy-Lynn  Yeah, one dollar.
17  Bobby  So? Better than nothing.
18  Magdalena  And the work all this time…
19  Amy-Lynn  (Reads the board) What stops a why and how to get a why off
your back.
20  Michelle I.  What?
21  Bobby  Huh.
22  Michelle I.  Okay…
I noticed at the end of class yesterday, we, um, we got kind of hung up with some whys. And I thought maybe you might like to discuss this morning, um, ways in which you could stop a why. Um, for example, just an every day example, what about, if, uh, you have a situation where you are going to the movie one Saturday, and you ask your mother, tell your mother that you need five dollars, and she says why. So what stops a conversation like that? What do you have to do?

Answer a question?

How much does it cost you to get in the movie?

Six-fifty. Six fifty, so five dollars is way too low already, so you probably say to her, "I need ten dollars to go to the movie," right? And she says, "why?" And then, you have to say, the movie is six fifty and whatever else you needed, maybe you can establish with her the why because these are factual things, right? That you have to pay six fifty to get into the movie. Um, there are some things in mathematics that are just accepted by everybody as properties of the number system, and there was one in particular that was involved with some of the whys yesterday, and, um, what that one is in particular is the property of multiplying by one. Are you familiar with that? That if you have any number in the number system, and you multiply it by one, then what happens?

It equals, like, the number that you, um…

You just get back the number that you started with, right? And, uh, that's called the identity for, uh, multiplication. So, if, if Dr. Maher or somebody else is asking you why on something, and you want to multiply by one, that you're saying that you're multiplying by one, and that's why, and that's a property of the number system, and mathematicians accept that that's true. Nobody can really prove it. Now you were giving some good arguments yesterday as part of what you were saying, and, and one of the things you were saying, I think Michael in particular was answering this question, that if you had a hundred over a hundred that that's equal to what?

That's equal to one, isn't it? So, we could go back to this, we say, well it's a property of the number system that any number multiplied by one, you just get back that number. It's equivalent. So, certainly if you multiply a number by a hundred over a
hundred, you should get back to the number that you started with. And he gave, I think it was Michael that gave a nice explanation of why a hundred divided by a hundred would be equal to one. Did you say, do you remember what you said to that?

Michael Is it like, a hundred, divided by ten, or whatever, is equal, n-like, simplify the problem to make it just ten over ten, or one over one. By, it's dividing each side by a hundred, you can make it one over one.

Researcher 1 Alright. Yesterday you might have said something about inverses, too, you know. Ho-, Alex, what would the rest of you say? What would somebody say? Why does a hundred divided by a hundred equal to one?

Bobby Well, if like, you had a hundred, like, I don’t know, candy bars, and you put them in a hundred groups, there’d be one in each group, so that’s like, a hundred candy bars.

Researcher 1 Alright, and say again, try to say again, Michael, what your view of it was.

Michael You can simplify it by making it ten over ten or one over one by dividing each side by the same number.

Researcher 1 Okay, so you could think of dividing both the numerator and denominator by one hundred.

Michael Just like when you simplify fractions (unintelligible, 3 words)

Researcher 1 Alright. Um, probably it’s good enough right there to say that a hundred divided by a hundred is equal to one by definition. Or by, um, what do you think? What would you say at that stage?

Student Me?

Researcher 1 Anybody, anybody, (really?). (student laughs)

Student I don't know.
Now, what about this? This was another step that you were using.

You said in your discussion yesterday that a hundred over a hundred, if you want to multiply that times something like, uh, box plus two, that you can write that this way: a hundred times box plus two over a hundred. What's the answer to that why, why can you do that?

(Makes confused face)

That happens to just be a definition in mathematics too, that whenever you have two fractions, can, is it alright if I write it $a$ over $b$? Times, say, $c$ over $d$? That just the definition of multiplication of two fractions is that you can multiply $a$ times $c$ over $b$ times $d$. And that's what you do here isn't it? (points to second line from bottom) So, if we back up from that, and we have a hundred times a number plus two divided by a hundred, and we want to say that's equivalent (points to third line from bottom) to this, it's by this definition of, in, uh, the number system. The definition of multiplication. Do you know any other things that are properties of number systems? (removes slide)

That you could stop and just say, well that's true because all mathematicians, everybody agrees. Just some very simple things.

One of them that's so easy that we take it for granted that if you're adding two numbers together, what if I were to say, well, I know that the answer to two plus five is the same thing if I added five plus two. Sounds pretty dumb, doesn't it? Cause we all know that works. So, that's one of the real fundamental properties of the number system. You've probably even heard the name for that. A word, do you remember what that's called by mathematicians. It
always works, doesn't it? It's not anything you have to prove. What do you think?

Stephanie? Who, me? I think it's the commutative, whatever.

Researcher 1 You're right. She's right. Isn't she? That's the commutative property is the name that it's given. Can you think of any other property of the number system that you don't have to prove?

16:55 Amy-Lynn Um, distributive?

Researcher 1 The distributive property. And in fact, we used that yesterday, didn't we? So that if you have a number, say, multiplied by the sum of two numbers, suppose I do, symbolize that this way, you get what? What do you have to do?

Amy-Lynn Triangle. Um, it'd be like that, like the box times the triangle, and then the box times the oval or circle.

Researcher 1 Ahuh. And then, why did we say that's true?

Amy-Lynn It's distributive property?

Researcher 1 It's just the distributive property. It's not anything that we have to add answer why to further than that. It's a property of the number system. That's just assumed by mathematicians. Nobody can really prove it, or they haven't found a way to prove it. Everybody just sort of tries to convince themselves that it's true, and then we start with that assumption. So what you were using that places yesterday we had one, didn't we, that was something like, three times the number in the box plus two, and you were telling me that was equivalent to three times the number in the box plus ..

Amy-Lynn (Whispers) Six.

Ankur? The box plus six.
Researcher 1 Plus six. Okay. Good. You think of any other properties?
Amy-Lynn Associative?
Researcher 1 The associative property. That's really good.
Amy-Lynn (whispers to Michelle I.) I don’t know what that is, though.
Researcher 1 that's really good. You must have been talking about this thing.
Michelle I. (whispers to Amy-Lynn) I know the name but not what it is.
Amy-Lynn I don’t know what it is.
Researcher 1 Look at, look at Misses Poise, beaming back there. Can you write
for us what the associative property is?
(everyone laughs)
Amy-Lynn (Smiles and shakes her head no)
Researcher 1 Can anybody write it? I believe Amy-Lynn can write it, she doesn't
want to.
Amy-Lynn No. I just know the name. (giggles)
Michelle I. She doesn’t know it. (giggles)
Researcher 1 You can’t? You just remembered the name. What is the associative
property? Can anybody remember that?
Brian What is it?
Michael? (unclear over table)(I might happen to know the right now.?) Three
times one equals three, or something.
(class laughs)
Researcher 1 That may be the identity property we talked about up here.
Michael? Okay.
Researcher 1 That sounds kind of like that one, doesn't it?

Researcher 1 What if I'm adding this column of numbers, and I have a four here,
and maybe a three here, and a six here, and the way I add numbers,
I look at the six and the four, and see that I could add those,
and that makes it a lot, nicer, and what I said was not a very good
example. Suppose I had three plus four plus six. Instead of adding
three plus four first, I might add four plus six first, and this is a
way to talk about that. Cause you could add three plus four first,
and get seven, and seven and six is thirteen. But instead of that, if
you decide to add the four plus six first, and get three plus ten, it's
also thirteen. And that's the associative property: the fact that you
can regroup 'em, cause you can only really add two things at once.
You think of any other things that are definitions? Or properties of
the number system that would stop a why? If there's an identity for
the operation of multiplication, if we take a number 'a,' and we say
if we multiply it by one, and we get back the 'a,' is there an identity
ah, for addition?

20:20 Bobby Um, if you add or subtract zero from something, it stays the same?
Researcher 1 So you think zero is the identity for addition? So, if you take any
number, and you add zero, you get back the same number. So that's
the identity problem for addition. That's really nice. Now, I'm
wondering, if you thought of the problem that we had. What was
the third grader's name yesterday?
Michelle I. (Whispers) Lindsey.
Researcher 1 Lindsey. Did you think about, uh, the problem she said, that if she
multiplied, wanted to multiply a number by five, how did she say
she did that?
Student [unclear](Multiply by) an even number?
Researcher 1 A, well, an even number by five, you're right.
Student And, um, then you split it in half, and then the number that
(unclear word) with it, and (then?) you added zero.
Researcher 1 So, if you started with an even number, and you wanted to multiply
it by five,
Michelle I. (Whispers) You split it in half.
Researcher 1 Instead of that, you could take one-half of the number,
which would give you sixteen in this case, and add a zero. I don't
know how to say that exactly. This, tha-that's not saying it quite
right. But, (n?) ... Alright? So we know that she was getting this,
right? Do you know why that works? Anybody figure it out?
Bobby? Okay, cause like five times one, that's an odd number, and since
like, it goes five, zero, five, zero, all the way through, then every
odd number has to be five. And with even you have zero, it's like,
five times two, and then ten, and then like, every other one would
be a zero.
21:55 Researcher 1 Can you come up and show us that set sounds really interesting.
Bobby: Well, wait, five times one equals five. So every odd number will be like, fifteen, twenty-five, thirty-five. (to self) And five times two equals ten. And then that would be like every other one would be like ten, twenty, and then thirty. And since there's like two numbers, like five, ten, you have to like cut it in half, so like, you can put it in there. And you can't like, cut an odd in half, so like the one over there, so you have to minus one so you get an even number, and then you divide it in half.

Researcher 1: Would you go a little further with your list? I think would help me.

You have five times one is five, and five times two is ten. Could you put a few more, like five times three ...

Bobby: Five times three equals fifteen. Five times four equals twenty. See, it's going to keep on going like that.

Researcher 1: Okay.

Bobby: So then I'm like, that's why you have to add zero to all the evens, it’s gonna end in zero no matter what. And for the (fives?) you have all the odds because it has to end in a five or else it’d be the wrong answer. And the reason why you divide in half works is because, like, there's two in, like, every one. Like, to get the ten you need like two numbers. So, like, when you're putting the whole thing in, you're like, it's like double the real answer. So, like, you have to cut it in half. And then you have to find the number. Since it ends in, like, it can't end in six.
23:27  Researcher 1  Well that's really an interesting explanation. I bet she thought of that sort in that same way, seeing that pattern. What do you think?
274  Did you understand what Bobby is saying, everybody? Do you have any questions for Bobby? Like why?
276  Brian  Ankur has one.
277  Researcher 1  Ankur? A- somebody over at this table, uh, you have another one that you were doing?
279  23:53  Michelle I.  (Whispers to Amy-Lynn?) You had the same way as him.
280  Brian  (during Researcher 1) He has one.
281  Researcher 1  Ankur, did you have a (way you would do it?)?
282  Ankur  Yeah, I think that since five is like half of ten, she splits the even number in half so you're really multiplying by ten, so it's like, five is half of ten, so really like, you split the even number in half, and then, if you multiply by ten, it'll be the same as multiplying by five.
286  Researcher 1  Wow. Did you write that?
288  Ankur  No.
289  (class laughs)
290  Researcher 1  It reminds me of something we were doing with th-, with identities and fractions up here right at the beginning. That would say what you said. You said, "five is equivalent to ..." or the same as ...
293  Ankur  Like when she split the number in half, and then she multiplied by ten.
295  Researcher 1  Right.
296  Ankur  That's the same as multiplying by five with the original number.
297  Researcher 1  Are you saying this?
298  24:53  Michelle I.  Is, is he saying that, like, the number that you times five by, you would, you split it in half because it's like timesing like the number by ten. And put like a zero at the end.
Researcher 1: So, if we had a number times five, are you telling me that if you had the number, and somehow, you multiplied it by ten, and you divided by -

Michelle I.: By two.

Researcher 1: By two.

Student: You'd get the same answer.

Researcher 1: Which could be written maybe this way, couldn't it?

Student: Mhmm.

Michelle I.: (Nods)

Researcher 1: That's interesting. Do you see how that kind of uses what we were talking about in the beginning, too? That's beautiful. Anybody have anything else you'd like to add to that discussion? Yes, Michael?

Michael: The box times ten over two, that would be bo--box times ten, over two. It's not, cause you wrote that. It would be box times-

Researcher 1: Are you saying this is equivalent to this? Is it equivalent to this?

Michael: I don't know.

(class laughs)

Researcher 1: What do you think? What do the rest of you think? Is it equivalent to that?

Michael: Cause (there wasn't one?) before box times ten divided by two with the division sign?

Researcher 1: Mhmm.

Michael: It's like, that means you take the box times ten and the answer to that, divide it by two.

Researcher 1: Are you saying that you think it should be written this way?

Michael: Or if you want, if you want to write it like box times ten over two, then you have to put the bo- the ten divided by two in parenthesis on the one above it.

Researcher 1: Come write it for me. For us.

Michael: The way you wrote it here, you took the box times five. The one...

Researcher 1: Okay.

Michael: Over here you wrote it right where you, there, this was number six. Six times ten is thirty, divided by two is, um, fifteen. But, that means you took the answer of these two (points to box times ten) and divided by two.

Researcher 1: I see what you're saying. Ahuh.
So, if you're going to write it like that, that this is the answer of these two divided by two. (circles the algebraic equivalent)

Okay. Alright. Now, the question I have now is those three rows right there, are all of those pieces equivalent? Is it alright if we put equals between all of those? What do you think back there at the back table? Do you think that all the pieces are equivalent?

Yeah. Ah, yeah.

It's important. These are answers to whys, right here.

(Whispers to table) But see you can also think it as five…

(Whispers to table) Do have a problem, like, staring at the camera?

Yea

Do you agree that those are all equivalent, or what do you think?

(same who was speaking last) Yes, yes I agree.

Because it's just the definition of multiplying fractions, isn't it?

That those are equivalent.

Because you can say this times this is this, and one times two is two, or you can do it this way, and could you say it this way? Ten times the number in the box divided by two.

(in background, during Researcher 1 above) You get the same answer.

There are all sorts of ways you could say it, aren't there?

It's the same.

It's the same. It's the same thing as like the one up there except the ten times the box is switched around.

And why? Why are they all the same?

Commutative? 'Cause, like, if you switch the two...

Yeah, the-there's a lot of the properties in here, aren't there? Commutative property, and the definition of multiplying
fractions. That's, that's really super. That's really super. When we were doing 'Guess My Rule' yesterday, Mr. Miller made up a very hard problem for you that I'm sure it will just take you a few minutes to get.

Michelle I. Oh, thanks. (giggles) We need, like, paper.

Researcher 1 And maybe you need, uh, you want a little of this, paper.

Researcher 2 They need ...

Researcher 1 Yeah.

Michelle I. Thank you (to Researcher).

Researcher 1 I don't know if this is the best paper. You can use the back, the back of it or something if you don't like it.

Michelle I. (Whispers to herself as she copies from the board) Two, two thirds, three, three fourths, four, four fifths, five, five sixths…

Researcher 1 Did everybody get a piece of paper?

Michelle I. It's eleven. The question mark is eleven. I'm quick on things, aren't I.

Amy-Lynn Way to go.

Michelle I. (Giggles)

Bobby Yeah but you have to guess what property.

Michelle I. I don’t know the property I just know the answer.

Bobby I know because all the top fractions…

Michelle I. Yeah because the top, because the number in the box goes on the top the…

Amy-Lynn Numerator

Michelle I. Numerator, the numerator. Yeah I didn’t know if it was the numerator or the denominator.

Bobby It goes one two, two three, three four, four five, it’s a pattern (smiles).

Amy-Lynn Ooooo.
Michelle I. Yea but look at the numbers on the left side it’s because it goes one, and then one is on top and then the next highest number is underneath and then it’s two, and then two thirds, and then three, and then three fourths, and four and you know, you get the pattern. And if it was like twenty over-

Amy-Lynn Twenty-one.

Michelle I. Twenty-one. Yes. We’re quick, aren’t we?

Amy-Lynn (Giggles)

Michelle I. Uh, what’s the rule?

Magdalena (In unison with Michelle I.) What’s the rule? (Giggles)

Michelle I. It’s that with the number that you’re timesing…the box times, times number equals…box over box plus one. Box plus whatever, you know, plus number whatever.

Magdalena I don’t getcha.

Michelle I. I don’t get you either. No.

Magdalena I wasn’t saying anything.

Bobby I found another pattern it goes times, times two, like, see?

Times one, times two, times three, times four, times five, times six,

Magdalena But that’s a pattern, it’s not a…a rule

Bobby I know but I’m just trying to find patterns

Michelle I. See look…

Bobby And this keeps on going down to the next fraction

Michelle I. You do box… it’s box like plus or minus or whatever the number equals box over box plus one.

Magdalena Yeah but what’s the number?

Michelle I. That’s what we have to figure out. (both girls giggle)

Bobby That’s like mine.

Researcher 2 Michelle, move your hand. (Michelle’s paper shown below)

Michelle I. Oh, Okay.
Magdalena: Hmm. Wait so this is…
Michelle I.: So now we have to figure out what the number is.
Amy-Lynn: Zero times…

Magdalena: It’s easy ‘cause if, if it’s the box, the box and the box, hold on.
(points to Michelle I.’s work) So it’s, for example one and then one
plus one it’s two and if you divide one by two…get the answer and
then you take the…

Researcher: That’s right.
Michelle I.: (giggles) You’re confusing me, Magda.
Magdalena: I-I know what I’m doing.
Bobby: Wait so it’s box over box plus one.
Michelle I.: Yes. So triangle equals box over box plus one. Yes!
Amy-Lynn: Now what is- now what is the rule?
Bobby: That is the rule.
Michelle I.: No it isn’t. You have to find-
Bobby: Oh yeah, that…
Michelle: …it’s like, plus something… you know.
Amy-Lynn: It could be anything.

Michelle I.: So, we have to figure out what n is.
Magdalena: Hold on. I think-
Amy-Lynn: Okay, well let’s try and put, let’s try and put half. Half, a half
equals one over one plus one.
Bobby: n is the square root of…three…
Michelle I.: n is the speed of light as Bobby would say.
Bobby: No. (unintelligible) One, the quanity of three. (Now to Amy-
Lynn)What is that saying?
Amy-Lynn: Right is that what you’re saying? That there? You’re saying it’s
one and a half. And the triangle equals a half.
Bobby: It can’t be…

Amy-Lynn: And the box equals one over one plus one.
Bobby: No.

Amy-Lynn: That’s right Bobby.
Bobby: So it would suck. (?)
Amy-Lynn: The half equals a half. One over two.
Michelle I.: We have to find out like, when you do the box you leave- you’re
adding-
Magdalena: No that’s it, that’s the thing. See?
Amy-Lynn: (Showing Michelle) That’s it because if you use a half, the triangle
equals a half, and then one, and then one over one, and then it’s
one, and two.
Bobby: Yea ‘cause one over two is half.

Students (at table) Wow. (all giggle)
Michelle I.: So wait a sec, so you put one on top of here?

Amy-Lynn: Yea ‘cause that’s a box.
Michelle I. Oh...kay. So- but then- (during Amy-Lynn) so if you put two over one plus one-
Amy-Lynn (During Michelle I.) That’s a triangle. So, two thirds equals two over two plus one. It still works, two thirds.
Michelle I. Oh, so you put two plus one? Wow.
Amy-Lynn Cool. Way to go Bobby.
Bobby What’d I do?
Amy-Lynn Didn’t you put this up? (points to her paper)
Michelle I. See but it’s still the same things, it’s the same rule. It’s the same rule as Bobby’s and Bobby’s is like the same rule as mine. And you said we were wrong Magda.
Magdalena Yea but, if you-
Michelle I. 33:39 It’s the same rule!
Amy-Lynn It’s the same rule!
Michelle I. That’s how Amy got it.
Bobby Magda thinks they’re beautiful.
Michelle I. Woooowww. (Amy-Lynn and Michelle I. giggle)
Amy-Lynn Magda’s like…
Magdalena I’m-
Bobby It’s Magda’s turn to contribute a rule.
Amy-Lynn That’s okay, Magda you did your job. You did- You (??)
Michelle I. So all you do is two- Magda’s turn to -
Amy-Lynn See the triangle-
Bobby Wait, now it says “What is the inverse rule?”
Michelle I. So you do, triangle-
Amy-Lynn Which is, say, two- say, three fourths. And then that equals the box, which is three over three plus one so that equals three fourths.
Michelle I. 34:09 Yea, but what is the inverse rule?
Michelle I. (Aloud) We’re done.
Bobby No we didn’t answer-
Magdalena What’s the inverse rule? The inverse rule…
Michelle I. Uh oh.
Bobby We need a dictionary. (Amy-Lynn, Michelle I., and Magda giggle)
Michelle I. What?
Michelle I. So…
Bobby We do. We need to know what the inverse rule is.
Michelle I. So wait, it would be triangle equals-
Amy-Lynn Box over box plus one.
Michelle I. Box…
Amy-Lynn Over box plus one.
Michelle I. 34:35 Box times box minus one.
Magdalena What? (Amy-Lynn and Michelle I. giggle.)
Michelle I. Um…
Researcher I Did you have a question about inverse?
Amy-Lynn Yea
Michelle I. He did (points to Bobby)
Amy-Lynn: He did (points to Bobby)
Michelle I.: He doesn’t know what it means.
Bobby: No I know what it-
Magdalena: No, I think it’s, uh-
Bobby: (Whispers to Amy-Lynn and Michelle I. during Magda) It’s where
you (??)
Michelle I.: Good, Bobby.
Amy-Lynn: Way to go, Bobby.
Magdalena: I think it’s-
Amy-Lynn: Okay, so this is one of them. This is-
Magdalena: Fine, don’t listen to me!
Amy-Lynn: I’m just writing this, sorry. I’m listening now.
Magdalena: No you’re not, your writing.
Michelle I.: Yes we are, we’re listening don’t worry. (giggles)
Magdalena: Um, you minus one from the box and then you divide that box by

Amy-Lynn: Huh?
Michelle: No wait, let’s check this out.
Amy-Lynn: What did you say-
Michelle I.: Okay. Let’s say this is one half.
Bobby: Wait what rule was this?
Michelle I.: And let’s say you have- No wait I’m doing my own thing.
Amy-Lynn: Wait what’d you say?
Michelle I.: Hold on a second.
Magdalena: Okay, look. If you- for example you have one-
Amy-Lynn: Write it down. (During Magda below) Is that the triangle?
Magdalena: One min- You have minus one and then over one and you want-
When you divide that, hold on, it’s gonna be a zero and when you
divide that it’s one so it equals- No it doesn’t work! (Amy-Lynn
giggles)
Michelle I.: Oh my god!
Amy-Lynn: (To herself)Box equals…
Magdalena: I thought it did…ahhh.
Michelle I.  No wait. I think I have something going here. If you- No wait
nevermind.
Magdalena  It has to be multiplying. I was doing division.
Amy-Lynn  (To herself during Michelle I. and Madga above) times box minus
one.. half… one minus one.  Minus one… one minus one. One
minus…. Zero? Gosh darnit!
Magdalena  It doesn’t work.
Amy-Lynn  It’s minus a half. (To herself, during Bobby below)  That works.
Point five. One times one minus point five.  So let’s see if that
works.
Bobby  I looked up the inverse rule. The inverse rule is when you (??) a
number to get the same answer.
Magdalena  You didn’t have to find it, Bobby.
Bobby  No, it says what is.
Amy-Lynn  (To herself) Two thirds equals box. Two times two minus point
five. Four… Nope don’t work.
Magdalena  Oh, minus that...
Bobby  You’re confusing me Amy.
Amy-Lynn  I was just trying something. It didn’t work, though.
Michelle I.  Okay. What if you put box equals- I mean triangle equals box…
no wait… box…
Amy-Lynn  I already tried box times box minus-
Magdalena  That’s what I was talking about but it was??
Michelle I.  This doesn’t work though because one minus one is zero times one
is one- no zero.
Magdalena  (With Michelle I.) No, zero.
Michelle I.  (Giggles) I’m getting into those bad calculation habits again now.
Amy-Lynn  Box- triangle equals- Maybe it’s-
Michelle I.  Box times box minus one.
Magdalena  How ‘bout how ‘bout triangle minus something equals-
Michelle I.  Whoa… (sighs in frustration) Why don’t we do- Tri- Triangle
Amy-Lynn  (Working by herself, during Michelle) One? Damn (erases work)
Michelle I.  Box minus triangle. What’s that?
Amy-Lynn  Box minus triangle?
Michelle I.  LIKE see if (mumbles)- Three, minus three fourths equals one and a
fourth.
Magdalena  (During Michelle I. above) So that’s two and-
Michelle I.  One and a fourth.
Magdalena  Two and a fourth
Michelle I.  Oh, two and a fourth.
Amy-Lynn  Two and a fourth.
Michelle I.  Two and a fourth. Now I gotta find out a way to do this. I don’t
know what I’m doing so don’t mind me.
Magdalena  Um- two and one-
Michelle I.  Plus one and a fourth! No- one and two fourths. No, nevermind.
Amy-Lynn: Yea but will that work for everyone else.
Magdalena: How about (pointing to Michelle’s paper) that minus that and plus something and that would equal the triangle.
Amy-Lynn: Wait (begins writing).
Magdalena: I don’t-
Amy-Lynn: How ‘bout box equals triangle- triangle.. let’s try plus one. Try something I don’t know (???)
Michelle I.: Oooo go Amy.
Amy-Lynn: (Giggles) I don’t know…
Magdalena: I-
Bobby: Minus two (??)
Michelle I.: Let’s see minus one half divided by one half is one. Plus one is two. Box is not two.
Amy-Lynn: Darn.
Bobby: I have a good one.
Magdalena: I think it’s something with the triangle.
Bobby: Minus-
Amy-Lynn: What if you try-
Michelle I.: That’s what we’ve been trying to do.
Bobby: Minus two and a fourth- no-
Amy-Lynn: (During Michelle I. above) Wait wait wait wait wait. (To herself)Box equals triangle over triangle minus one…. Equals one half (??) over
Magdalena: Something’s gonna work
Bobby: (During Amy-Lynn, Magda, and Michelle I.) no- box minus bo-
Michelle I.: (During Amy-Lynn, to herself) Box times box- box times box
Magdalena: It won’t work, see? See you’re going to get negative numbers.
Bobby: So?
Michelle I.: (Gasp) I think I have it.
Amy-Lynn: What?
Michelle I.: At least- At least it works for one and a half. You do box- I mean triangle which is one half times… triangle plus one… what’s one half plus one. That’s one and a half times one half what’s that?
Bobby: Um...(giggles) Three over two-
Magdalena: So that’s one and one-
Michelle I.: …times one half. That’s
Magdalena: Three over four-
Michelle I.: Three over two times t- Nevermind. Doesn’t work.
Amy-Lynn: What if we had-
Magdalena: Three over four, isn’t it?
Michelle I.: (Gasps) Whoa!
Amy-Lynn: (To herself, during Magda and Michelle I. below) Box equals
triangle times triangle times one.
Magdalena: Three over four, isn’t it?
Michelle I.: Yea.
Magdalena: Wow.
Michelle I.: It doesn’t work anyway.
Magdalena: That’s too confusing.
Amy-Lynn: (Still to herself) Two equals one… okay, so that’s four- four thirds
equals one and a third plus one.
Michelle I.: We need the inverse rule.
Amy-Lynn: I am so close!
Amy-Lynn: Look at this. Look box-
Michelle I.: Box (girls laugh)
Amy-Lynn: Box equals- Wait let me show you something else. Box equals
triangle-
Magdalena: Triangle-
Amy-Lynn: …times triangle plus one.
Michelle I.: Triangle- What’s that? Times?
Amy-Lynn: Times.
Michelle I.: Times..
Amy-Lynn: Two equals two thirds times two thirds-
Magdalena: I’ll try it with the four.
Amy-Lynn: Four.. One and a third. That’s two and a third. Before when I had
minus one it came out to a third and then all we need is one more
third. (Amy-Lynn’s paper shown below)
Michelle I. (Gasps) I got it! You put parentheses around these things. So you go- triangle, which is four, equals, I mean square, equals triangle times, in parentheses that (draws box) plus one. So that would be four fifths times, um-

Michelle I. Four

Michelle I. Fi- Four. Plus one is five, over one equals twenty over five, which is four. (Michelle I.’s paper shown below)

Michelle I. (To Magda) Sorry. Did I hit you? Sorry.

Amy-Lynn Box equals triangle times six…
Bobby (During Amy-Lynn above) Wait, I thought- square minus triangle times five divided by five plus triangle equals square.

Amy-Lynn and Michelle I.

(Saying the numbers they are writing down- both checking the same numbers)

Michelle I. Yes it works! We have the inverse rule.

Amy-Lynn Funky.

Researcher 3 Did you get it?

Amy-Lynn and Michelle I.

Yeah.

Michelle I. We- The regular rule we put is-

Amy-Lynn It works.

Michelle I. Triangle equals box over box minus one.

Amy-Lynn (During Michelle I.) Box over box plus one.

(Corrects Michelle I.) Plus one.

Michelle I. The inverse rule- plus one- The inverse rule is… hold on.

Amy-Lynn Box equals triangle times parentheses box plus one parentheses.

Michelle I. (Copies down inverse rule and mumbles it to herself as she writes)

And it works, too. See? I’m actually thinking today.

Amy-Lynn Way to go Shelly.

Researcher 3 So if you never saw a number- If I gave you a triangle.

Michelle I. Mm hmm.

Researcher 3 And- and you didn’t know about any of this up here-

Michelle I. Mm hmm.

Researcher 3 Okay. If I gave you a triangle, say um, forty-nine fiftyths how would you plug that triangle into here and come out with a box?

Michelle I. Uh, I don’t know. Actually-

Amy-Lynn If you didn’t know any of this?

Michelle I. If you didn’t know-

Magdalena No if you knew the triangle-

Michelle I. If you didn’t know the triangle and you didn’t know that it was like this, you didn’t like know any patterns-

Researcher 3 You didn’t know any patterns you just-

Michelle I. And, you could just say, it’s, um- That would be like- (To researcher) What did you say this was?

Researcher 3 I don’t know, twenty four twenty fifths.

Michelle I. So, twenty four over twenty five.

Researcher 3 (With Michelle I.) ..over twenty five.

Michelle I. So this would be- (To Amy-Lynn) What’s the top one? The nomiter-

Amy-Lynn Numerator.

Michelle I. The numerator. So it would be the numerator- the numerator plus one-

Researcher 3 How do you know that? I mean you- you’re- you’re making a- assumption based on what you know.

Michelle I. Oh.
Researcher 3: ‘Cause, you don’t know that this is num- this has anything to do with the numerator, right? Right?
Magdalena: So we’re wrong.
Amy-Lynn: But what’s these-
Researcher 3: Somehow- What’s, what’s characteristic of this rule? (points to Michelle I.’s paper) What do you have on the left hand side?
Michelle I.: The triangle.
Researcher 3: Triangle. And on the right hand side? (Pauses) Squares right?
Michelle I.: Yeah and then plus one.
Researcher 3: But over here you have on the left hand side-
Amy-Lynn: Box.
Michelle I.: Box and over here-
Researcher 3: And on the right hand side you have two things.
Michelle I.: Okay.
Amy-Lynn: Triangle and box.
Researcher 3: So it’s hard to- hard to do that. Right?
Michelle I.: So you have to keep like the two unknown things on one side.
Amy-Lynn: Okay.
Magdalena: Shelly didn’t get it.
Amy-Lynn: It was a good thought though Shelly. It does work.
Researcher 3: What if you just forgot all about this and just looked at this and tried to get that? (Points to Michelle I.’s paper)
Michelle I.: Okay.
Researcher 1: (To table) We only have like about two more minutes. Is there somebody over here who would like to report on how far you’ve gotten on your thinking?
Researcher 3: (During Researcher 1) Okay, go ahead.
Michelle I.: (During Researcher 1) Is there (?) science?
Amy-Lynn: We got the rule.
Researcher 1: (To Michelle I.) Are you willing to do that? Michelle?
Michelle I.: (Nods ‘yes’ to Researcher 1)
Alright.
Wait. I got it. Lo. Square minus triangle times five divided by five plus triangle equals square.
(Writing, to herself) Triangle… Probably- Probably doesn’t work.
Yes it does.
What is two thirds minus-
Watch. One minus- (keeps working silently during Researcher 1)
I think there are two tables here that are sort of on the verge of getting this. And um, I’d like for- Michelle has agreed that she will tell you, uh, where their table has gotten. And then if you people can talk kind of quietly at that table I know you’re in the midst of wonderful problem solving so, Michelle- You might want to hear a little of what Michelle says.
And then maybe you can finish it before tomorrow.
So that's the rule. And, like, I can sort of, well, I can prove it because then, like, if you have the triangle is one that equals one divided by one plus one, and that's one half. Okay. It works for every one. So, like, a five...
So, Michelle, eh, you're saying that you can demonstrate for each row that we have--that that rule works. Okay.
(whispering at board, followed by) And then, we found out, like, this other rule, but it's not really, like, the inverse rule. And- and it works, but, like, you're not supposed to use it this way.
(Researcher 1 laughs; Michelle continues to write silently)
So, it's like, box, which is, five, mmm, equals, five sixths times... so then you do, um, box plus one, which is six. And then you do it over one, which is thirty, over six, and thirty divided by six is five.
That's great Michelle, thanks. Since we just have like, another, half minute, A-, Angela and Michael want to come up together and, uhm, they're thinking along the same lines, Michelle, as, as your group was, and they think they have words that will add to this.
(during Researcher 1 and Michelle I. above)


Amy-Lynn What? But box (???) remember?

Bobby Equals triangle.

Amy-Lynn Really?

Bobby Yeah watch. One divided by two...two divided by three equals two thirds (writes)

46:59 Amy-Lynn (To Michelle I.) Look at this. Look what he got. (Shows Michelle I. Bobby’s paper) Box divided by next highest number equals triangle. One divided by two equals half. Two divided by three equals two thirds.

Somehow I thought of that.

Magdalena ??

Amy-Lynn I have a new discovery (raises hand)

47:22 Michelle I. This is another rule. (raises hand)

Amy-Lynn Okay Magda it’s your turn.

Bobby I’ll go.
Magdalena: I don’t wanna go up.

Michelle I. (Referring to Michael’s work on overhead) That’s like the same thing we did.

Researcher 1: Yeah, it's, it is the same thing. Uh, he said it a little different.

Michelle I.: It is except we didn’t put the number.

Bobby: (To table) Can I tell ’em? It was my discovery?

Michelle I.: Yes you can. We just want to point out that we have it.

Amy-Lynn: Talk louder (referring to Angela at overhead)

(Bell Rings)

Amy-Lynn: Wait, keep this.

Michelle I.: Write all our names on this.

Bobby: No, this is my idea. (Amy-Lynn giggles)

Michelle I.: So?

Bobby: ?? (mumbles to Amy-Lynn)

Amy-Lynn: You got it right there.

Researcher 1: (interrupts Angela) Misses Toye said she'll continue this tomorrow morning and I am very disappointed that I can’t hear the end. They're the same, right? (unintelligible) That's two, that's two, that equals two.

Researcher 3: Michael. Michael, can you write the denominator in terms of (the?) triangle?

Michael: Like, the number? The triangle.

Researcher 3: Write the denominator in terms of the triangle.

Michael: Of her rule? Of her rule?

Researcher 3: In terms of the triangle. See right now you wrote the word denominator up there.

Michael? Denominator (of what?).

Researcher 3: If you could, if you could write that denominator in terms of triangles and numbers, then you'd, I think you'd have a real good thing.

Michael: Box equals the, the... what do you mean in terms of triangles?

Researcher 3: (unintelligible word, problem?).

Michael: I just, I wrote(de?)nominator, like, triangle times...

Researcher ?: Yes you do, you just start checking (unintelligible, 2 words?), that's all.

Michael: Triangle times the triangle's denominator (over one?).
Researcher 3: Michael, in formulas we usually don't write denominator. Numerator. So, you have to somehow, so you have to somehow come up with a way of extracting that from the fraction.

Researcher 3: (Then?) she's writing numerator and denominator--

Researcher 3: (interrupts Michael above) So, think about it. I mean, you don't, you don't have to...

Researcher 4: What, I think what he's asking you is, is how is this triangle related to that denominator.

Michael: Um.

Researcher 3: Okay, basically, the idea is to just, when you write formulas, (bell rings)

Researcher 3: you don't write denomina--you can't--it's not fair. Somehow, it's not part of the rules (for) a denominator. So, what you have to do is, kind of, see if you can't come up with a way of writing exactly what you wrote...

Michael: Okay.

Researcher 3: In terms of the triangle instead of denominator. Cause you're right, you want the square on one side and the triangles on the other side, but now you need, somehow--

Researcher 3: Right, so that's not fair either.

Michael: And, but she, eh--her ferm--her formula has "keeping the same denominator in your answer."(read from screen)

Researcher 3: That's right.

Michael: You need, like, numbers and...

Researcher 3: Right. So just, look, just play around with it, and I'll bet you can do it. (Okay?)

Researcher 1: (Conversation with Angela unintelligible)
Researcher 1: Is that when you were saying (the?) five sixths, for example, and you wanted to multiply it times the denominator, I'm just messing around with the five sixths, and I thought, "well, what's one minus five sixths."

(Angela smiles)

Researcher 1: If you take one, which would be six over six, minus five sixths, would give you... one sixth?

Is it?.

Angela (Is it?).

Researcher 1: That somehow, that gets you closer to having that six by itself. You know what I mean?
Time   Speaker                  Transcription

00:00   (Students are entering and preparing for class)

3:08   Researcher 1 I noticed at the end of class yesterday, we, um, we got kind of
       hung up with some whys. And I thought maybe you like- might
       like to discuss this morning, um, ways in which you could stop a
       why. Um, for example, just an every day example, what about, if,
       uh, you have a situation where you are going to the movie one
       Saturday, and you ask your mother, tell your mother that you need
       five dollars, and she says why. So what stops a conversation like
       that? What do you have to do?

3:44   (inaudible class response)

3:44   Researcher 1 Could you answer a question?

3:44   Michelle I. Say that you need five dollars for the movies?

3:44   (The class laughs)

3:44   Researcher 1 How much does it cost you to get in the movie?

3:44   Jeff Six-fifty.

3:44   Students Six-fifty.

4:02   Researcher 1 Six fifty, so five dollars is way too low already, so you probably
       say to her, "I need ten dollars to go to the movie," right? Then she
       says, "why?" And then, you have to say, the movie is six fifty and
       whatever else you needed, maybe you can establish with her the
       why because these are factual things, right? That you have to pay
       six fifty to get into the movie. Um, there are some things in
       mathematics that are just accepted by everybody as properties of
       the number system, and there was one in particular that was
       involved with some of the whys yesterday, and, um, what that one
       is in particular is the property of multiplying by one. Are you
       familiar with that? That if you have any number in the number
system, and you multiply it by one, then what happens?
(Angela raises her hand)
Student  You just get back the number that ...
(Jeff is writing on Sarah’s nametag)
Researcher 1 You just get back the number that you started with, right? And, uh,
that’s called the identity for, uh, multiplication. So, if, if Dr. Maher
or somebody else is asking you why on something, and you want
to multiply by one, that you’re saying that you’re multiplying by
one, and that’s why, and that’s a property of the number system,
and mathematicians accept that that’s true. Nobody can really
prove it. Now you were giving some good arguments yesterday as
part of what you were saying, and, and one of the things you were
saying, I think Michael in particular was answering this question,
that if you had a hundred over a hundred that that’s equal to one.
(Researcher 1
5:59 Michael (mumbles) One over one is …
Researcher 1 That’s equal to one, isn’t it? So, we could go back to this, we say,
well it’s a property of the number system that any number
multiplied by one, you just get back that number. It’s equivalent.
So, certainly if you multiply a number by a hundred over a
hundred, you should get back to the number that you started with.
And he gave, I think it was Michael that gave a nice explanation of
why a hundred divided by a hundred would be equal to one. Did
you say, do you remember what you said to that?
(Researcher 1
5:59
Michael Is it like, a hundred, divided by ten, or whatever, is equ-you cann-
like, simplify the problem to make it just ten over ten, or one over
one. By dividing each side by a hundred, you can make it one over
one so it’s equal.
Alright. Yesterday you might have said something about inverses, too, you know. Ho-, Alex, what would the rest of you say? What would somebody say? Why does a hundred divided by a hundred equal to one?

Angela? Oh.

(Researcher 1 laughs)

(Jeff is talking to Stephanie)

Michael? (whisper)Anybody get an answer?

Student Well, if like (you have?) a hundred like, I don't know ten works, put them in a hundred groups, maybe one in each group. So that's a (unintelligible)

Researcher 1 Alright, and say again, try to say again, Michael, what your view of it was.

(Jeff is still talking to Stephanie)

Michael You can simplify it by making it ten over ten or one over one by dividing each side by the same number.

Researcher 1 Okay, so you could think of dividing both the numerator and denominator by one hundred.

7:18 Researcher 1 Alright, and say again, try to say again, Michael, what your view of it was.

Researcher 1 Alright. Um, probably it's good enough right there to say that a hundred divided by a hundred is equal to one by definition. (Michael nods yes)

Or by, um, what do you think? What would you say at that stage?

Ankur? Me?

Researcher 1 Anybo, anybody, but-

(Student laughs)

Stephanie? I don't know.

(Jeff hands something to Ankur)

8:08 Researcher 1 Now, what about this? This was another step that you were using. You said in your discussion yesterday that a hundred over a hundred, if you want to multiply that times something like, (Ankur passes something back to Jeff)

uh, box plus two, that you can write that this way: a hundred times box plus two over a hundred. What's the answer to that why, why can you do that?

(Jeff leaves table and exits the room)

Researcher 1 And that happens to just be a definition in mathematics too, that whenever you have two fractions, can, is it alright if I write it a over b? Times, say, c over d? That just the definition of multiplication of two fractions is that you can multiply a times c over b times d. And that's what you're doing here isn't it? So, if we back up from that, and we have a hundred times a number plus two divided by a hundred, and we want to say that's equivalent to this, it's by this definition of, in, uh, the number system. The definition of multiplication. Do you know any other things that are properties of number systems?
That you could stop and just say, well that's true because all mathematicians, everybody agrees. Just some very simple things. One of them that's so easy that we take it for granted that if you're adding two numbers together, what if I were to say, well, I know that the answer to two plus five is the same thing if I added five plus two.

(Jeff returns to table)

It sounds pretty dumb, doesn't it? Cause we all know that works. So, that's one of the real fundamental properties of the number system. You probably even heard the name for that. A word, do you remember what that's called by mathematicians. It always works, doesn't it? It's not anything you have to prove. What do you think?

Who, me? I think it's the commutative, whatever.

You're right. She's right. Isn't she? That's the commutative property is the name that it's given. Can you think of any other properties of the number system that you don't have to prove?

Um, distributive.

The distributive property. And in fact, we used that yesterday, didn't we? So that if you have a number, say, multiplied by the sum of two numbers, suppose I do, symbolize that this way, you get what? What do you have to do?

Um. Um, I think like that, the box times the triangle, and then the box times the circle, but only the circle.

Ahuh. And then, why did we say that's true?

It's distributive?
It's just the distributive property. It's not anything that we have to add answer why to further than that. It's a property of the number system. That's just assumed by mathematicians. Nobody can really prove it, or they haven't found a way to prove it. Everybody just sort of tries to convince themselves that it's true, and then we start with that assumption. So what you were using that places yesterday we had one, didn't we, that was something like, three times the number in the box plus two -

(Michael nods yes)

and you were telling me that was equivalent to three times the number in the box plus ..

Three box plus six.

Plus six. Okay. Good. You think of any other properties?

Amy-Lynn? Associative?

The associative property. That's really good, that's really good.

You must have been talking about this thing. Look at, look at Ms. Poise, beaming back there. Can, can you write for us what the associative property is?

(Ira writes on Sarah's nametag)

Can anybody write it? I believe Amy-Lynn can write it, she doesn't want to.

No.

You can't? You just remembered the name. What is the associative property? Can anybody remember that?

(Michael raises his hand)

I think I might-

(student chatter) (Jeff writes on Sarah's nametag)

I might have an idea, but, like--Three times one equals three, or
something.
(class laughs)
That's not-
Researcher 1 That may be the identity property we talked about up here.
Michael Oh, okay. (mumbles something)
Researcher 1 That sounds kind of like that one, doesn't it?

Researcher 1 What if I'm adding this column of numbers, and I have a four here, and maybe a three here, and a six here, and the way I add numbers, I look at the six and the four, and see that I could add those, and that makes it a lot, nicer, and what I said was not a very good example.
(Jeff is laughing with someone across the room and mouthing to them)
Suppose I had three plus four plus six. Instead of adding three plus four first, I might add four plus six first, and this is a way you talk about that. Cause you could add three plus four first, and get seven, and seven and six is thirteen. But instead of that, if you decide to add the four plus six first, and get three plus ten, it's also thirteen.
(Jeff writes on his own nametag)
And that's the associative property: the fact that you can regroup 'em, cause you can only really add two things at once. You think of any other things that are definitions? Or properties of the number system that would stop a why? If there's an identity for the operation of multiplication, if we take a number 'a,' and we say if we multiply it by one, that we get back the 'a,' is there an identity, ah, for addition? Bobby?
Bobby? If you add that zero times something.
Researcher 1 So you think zero is the identity for addition?
So, if you take any number, and you add zero, you get back the same number. So that's the identity problem for addition. That's really nice. Now, I'm wondering, if you thought of the problem that we had. What was the third grader's name yesterday?

Angela Lindsey.

Researcher 1 Lindsey. Did you think about, uh, the problem she said, that if she multiplied, wanted to multiply a number by five, how did she say she did that?

(Jeff is still writing on and playing with his nametag)

14:46 Angela Wasn't it an even number?

Researcher 1 A, well, an even number by five, you're right.

Student Then, um, then you'd split it in half, and like the number that has like the thing, and then add a zero at the end.

Researcher 1 So, if you started with an even number, and you wanted to multiply it by five,-

Angela Split thirty-two in half.

Researcher 1 -instead of that, you could take one-half of the number, which would give you sixteen in this case, and add a zero. I don't know how to say that exactly. This, tha-that's not saying it quite right. But, (n?) ... Alright? So we know that she was getting this, right? Do you know why that works? Anybody figure it out?

Bobby Okay, like five times one, that's an odd number, and since like, it goes five, zero, five, zero, like all the way through-

Researcher 1 Ahuh.

Bobby -so then every odd number has to be five, and with even you have zero, it's like, five times two, and then ten, and then like, every other one would be a zero.

Researcher 1 Can you come up and show us that set sounds really interesting.

(whispering unintelligibly-Angela and Michael?)
(Jeff writes something on nametag between Stephanie and Ankur)
Bobby Well, like, five times one, is five. So every odd number will be
like, fifteen, twenty-five, thirty-five. (to self) (And five ... ---ls ten.)
And then that would be like, what would be another one, would be
like ten, twenty, then thirty. And since there's like two numbers,
like five, ten, you have to like cut it in half, so like, you can put it
in there. And you can't like, cut an odd in half, so like the one over
there, so you have to minus one 'til you get an even number, and
then you divide it in half.
16:23 Researcher I Would you go a little further with your list? I think would help me.
You have five times one is five, and five times two is ten. Could
you put a few more, like five times three ...
(Bobby writes something again, Stephanie laughs, and Ankur writes
something back)
Bobby (Ah?) Five times three equals fifteen. Five times four equals
twenty. See, it's going to keep on going like that.
Researcher I Okay.
Bobby So then I'm like, that's why you have to add zero to all the evens,
it's going to end in zero no matter what. And for the fives you have
all the odds because this has to end in five- 'less it be the wrong
answer. And the reason like, where you divide in half works is
because, like, there's two in, like, every one. Like, to get the ten
here, the two numbers- So, like, when you're putting the whole
thing in, you're like, it's like double the real answer. So, like, you
have to cut it in half. And then you have to find the number. Since
it ends in, like, it can't end in six.
(Jeff writes something again on nametag between Stephanie and
Ankur and then Ankur scribbles on it.)
Well that's really an interesting explanation. I bet she thought about it sort of in that same way, seeing that pattern. What do you think? Did you understand what Bobby is saying, everybody? Do you have any questions for Bobby? Like why?

Brian? Ankur has one.

Ankur? A- somebody over at this table, uh, you have another one that you were doing?

Ankur, did you have a way you would do it?

Yeah, I think that since five is like half of ten, she splits the even number in half so you're really multiplying by ten, so it's like, five is half of ten, so really like, you split the even number in half, and then, if you multiply by ten, it'll be the same as multiplying by five.

Wow. Did you write that?

No.

(class laughs)

It reminds me of something we were doing with th-, with identities and fractions up here right at the beginning. That would say what you said. You said, "five is equivalent to ..." or the same as ...

Like when she split the number in half, and then she multiplied by ten.

Right.

That's the same as multiplying by five with the original number.

(student whispering)

Are you saying this?

Is, is he saying that, like, the number that you times five by, you would, you split it in half? Because it's like timesing like the number by ten? And put like a zero at the end.

So, if we had a number times five, are you telling me that if you had the number, and somehow, you multiplied it by ten, and then you divided by-

By two.

You'd get the same answer.

(Jeff is laughing looking at a student across the room.)

Which could be written maybe this way, couldn't it.

Mhmm.

That's interesting. Do you see how that kind of uses what we were talking about in the beginning, too? That's beautiful. Anybody have anything else you'd like to add to that discussion? Yes, Michael?

With the box times ten over two, that would be bo--box times ten, over two. It's not, cause you wrote that. It would be box times... Are you saying this is equivalent to this? I don't know.

Is it equivalent to this?
Michael: I don't know. [mumbles something unclear]
(class laughs)
Researcher 1: What do you think? What do the rest of you think? Is it equivalent to that?
(Sarah is doodling on her nametag)
19:48 Michael: Cause the way you wrote it before box times ten divided by two with the division sign?
(Sarah is still writing on her nametag.)
Researcher 1: Mhmm.
Michael: It's like, it's like that means you take the box times ten and the answer to that, divide it by two.
Researcher 1: Are you saying that you think it should be written this way?
Michael: Or if you want, if you want to write it like box times ten over two, then you'd have to put the box the ten divided by two in parenthesis on the one above it.
(Sarah is still writing on her nametag.)
20:12 Researcher 1: Come write it for me. For us.
Michael: The way you wrote it here, uh, you took um the box times bo-. The one...
Researcher 1: Okay.
Michael: Over here you wrote it right where you, there, this was number six. Six times ten is thirty, divided by two is, um, fifteen. But, that means you took the answer of these two and divided by two.
Researcher 1: I see what you're saying. Auh.
(Ankur is elbowing Jeff’s hand to try and stop him from writing on his name tag)

Researcher 1 | What do you think back there at the back table? Do you think that all the pieces are equivalent?

Stephanie | Yeah. Ah, yeah.

(Ankur and Jeff seem to be quietly arguing over something on Ankur's name tag)

Researcher 1 | It's important. These are answers to why, right here.

Angela | (whispers to Michael) If you used five, five times five is twenty-five, if you use ten and five-

Michael | (It's simple?) Alright. It is the same.

Angela | It’s the same! It’s equal.

Michael | I know.

Researcher 1 | Do you agree that those are all equivalent, or what do you think?

Angela | They’re all equivalent.

Researcher 1 | Because it's just the definition of multiplying fractions, isn’t it?

Angela | (mumbles) You get the same answer all the time.

Researcher 1 | And could you say it this way? Ten times the number in the box divided by two. There are all sorts of ways you could say it, aren't there?

Angela | (mumbles) You get the same answer all the time.

Researcher 1 | Because it's just the definition of multiplying fractions, isn’t it?

Angela | That those are equivalent. Because you can say this times this is this, and one times two is two, or you can do it this way.

Researcher 1 | And could you say it this way? Ten times the number in the box divided by two. There are all sorts of ways you could say it, aren't there?

Students | It's the same.

Angela | It’s the same. All different ways. They’re all equivalent.

Researcher 1 | And why? Why are they all the same?

Student | Commutative because, like, if you switch the two ...

Researcher 1 | (interrupts) Yeah, the-there's a lot of the properties in here, aren't there? Commutative property, and the definition of multiplying fractions. That's, that's really super. That's really super. When we were doing 'Guess My Rule' yesterday, Mr. Miller made up a very hard problem for you that I'm sure it will just take you a few minutes to get here.

Jeff | Do you have any pens? (Indian accent) Sure, plenty of them.

Researcher 1 | And maybe you need, uh, you want a little of this, paper.

Researcher 2 | They need ...

Researcher 1 | Yeah.

Researcher 1 | (hands out paper to table) I don't know if this is the best paper.

You can use the back, the back of it or something if you don't like it.

(The class is tapping pencils loudly)

Researcher 1 | Did everybody get a piece of paper?

(Jeff, Sarah, and Michael flip their paper to the back and begin writing and doodling on it)

Angela | Guys…Sarah...

Michael | (Shows his paper to the group) I have a dog.

Jeff | (mumbles something unintelligible)
I have a dog.

(Students flip paper back to the front)

(writing on front) 1 and 1 like-

(to Michael) No but wait, just look at number … I just …I realize
it’s weird…that the last one would be…um…eleven..

(pointing to the front of room)

See the one where you have to figure out where that is? It’s going
to be eleven.

(Michael erases what he started to write on paper).

(Jeff and Sarah are writing on Sarah’s nametag)

(two thirds, it’s like… do you see that? That little pattern or … I
don’t think of the rules I think of patterns. I don’t know why. See
the question mark is gonna be eleven. Because it’s five, is five
over six , four is four over five, three is three over four, two is two
over three.

Do you see that Mike?

(Angela laughs)

Box equals box over box times, plus one.

Box equals what?

I don't know. (writing) Box equals box (over?) box plus one. Isn’t
it?

Ya.

But that’s not a rule. I don't know. It's time to do the numbers
already.

(Sarah taps her pencil)

Why can’t it be a rule? Why not? Why isn’t it a rule?

I don't know, it doesn’t make sense.

Yes it does.

Okay. Then you say it. I’m not saying it. Go on up there.
Angela: Yeah, but what would be the answers for it.

Michael: (sounds like-I don't know).

Angela: Wouldn’t it be like box minus one times box? No?

Michael: No you (just take a box?) There is no inverse! It’s not even a problem.

(Jeff and Sarah are writing and their papers and quietly talking.

25:01 Angela: (laughs) Why, why can’t it be a rule?

Michael: I don’t know. K, you made it up, okay?

Angela: It’s true. And then, then the question mark would be eleven.

(Angela’s paper is shown below)

Angela: See they have the same thing, they have the same thing.

Michael: It’s triangle. It's triangle.

Angela: Yeah triangle equals.

(Angela erases the box and draws a triangle in its place as shown below.)
Angela: So then, to get this it would be triangle.

(Angela draws a triangle then erases it)

Wai—what would be the inverse? Don’t do that anymore guys?

(Sarah has drawn an arrow)

Jeff?: What is that? That’s a one way sign.

Michael: Box over box plus one equals (inaudible).

Sarah?: It’s a dancer…It’s a flower…it’s…..

Angela: (pointing to Sarah’s paper) This isn't a chair

Jeff: (also pointing to Sarah’s paper) This is interesting (mumbles something)

Sarah: Those look like stockings.

Jeff: No...(mumbles something)

Angela: 26:22 What.

Michael: Or maybe socks on a line or something with a bucket here.

Angela: (Sarah and Angela laugh)

Sarah: That's what it is. This is an eighth note.

Angela: Why would you use a triangle though?

(unintelligible chatter)

Oh, times.

Now, now, alright, whatever.

It looks like a mouse. A really big mouse.

You ruined it. You can’t do this.

It does look like a mouse.

(laughter)

It's a new species of man slash aardvark.

Alright, um…. Yeah, but, alright, let’s see…

(Researcher 1 comes over to table and whispers something to Jeff.

Jeff laughs and puts his head down)

No it’s not gonna work because you can’t multiply to have so many fractions. Or can you multiply?

(Jeff and Sarah are laughing)

Half…alright, let’s try this one. Half times half plus one. One and a half plus one.

O-K…
Michael: That's one and a half-
Angela: Oh you're right, never mind. Okay, that, because there's a fraction. Forget it.
Ankur?: We're done.
Researcher 1: You have the rule, do you also have the inverse?
Angela: Oh you're right, never mind. Okay, that, because there's a fraction. Forget that.
Michael: I don't know.
Angela: We have it, yes! Is it right?
Michael: No, it's not the inverse. Forget that.
(Researcher 1 is talking to the other group)
Angela: (erasing her paper) Maybe we have the minus one somewhere wrong.
Michael: How bout this.
Jeff: Mike (over?) when you're done.
Angela: El even because eleven over twelve. See, one over two, two over three…
Michael: Three times three?
Angela: No. But wh-what we have to figure out how to get triangle out.
Michael: (to Angela) Yeah. Is this- is this the denominator and that's the numerator right?
Angela: Yeah. Triangle…
J2 335

30:00 Michael  Equals triangle times the denominator.
30:03 Angela  But wait the whole the whole, the whole fraction is triangle.
30:06 Sarah (looking up at the board) Oh, it, it does work. It does work.
30:09 Michael I know watch. Two times three times…
30:12 Sarah (interrupting above) Two thirds times three is….
30:15 Angela Yes. We just said.
30:18 Michael …times three equals six thirds. So three is the denominator.
30:21 Angela Twelve times (inaudible word)….it's right, it’s right.
30:24 Michael (raises his hand) It’s right.
30:27 Sarah Triangle times denominator.
30:30 Jeff What?
30:33 Sarah (to Jeff) Box equals triangle times the denominator.
30:36 (Jeff grabs Sarah’s nametag)
30:39 Angela Gimme! (Sarah grabs nametag back)
30:42 Jeff (Going?) to the other table again?
30:32 Michael (still raising his hand) I’ve got the answer. Of course I have, I figured out both of them. So, I'm both of them so I'm very talented.
30:35 Jeff Sarah has to go over the plan.
30:38 Michael Yeah.
30:41 Jeff It’s her turn.
30:44 Michael No.. it's you, you didn't go up.
30:47 Jeff I went up on-
30:49 Angela? I figured out who's really (weird?) on here.
30:52 Sarah I'll look smart.
30:55 Angela Sarah, tell them we have two theories.
30:58 Sarah Uh, what’s the other one?
30:49 Jeff We have Angela’s pathetic one which she insists on acting smart for the camera. Even though it’s going to be wrong and she’s going to feel stupid. And then there’s….
30:54 Sarah? (interrupting Jeff) What is it?
30:57 Angela Box, no triangle equals box over box plus one.
30:59 Jeff Angela, are you trying to make (unclear word) look smart?
30:55 Angela (laughing) No.
31:04 Sarah OK, now how does this work? Triangle is, let’s see, one-half equals two fourths…
31:07 Angela It, it does work. No, look Sarah look.
31:10 Sarah Oh it does work. Okay, I get it, I get it, I get it, I get it.
31:13 Michael (Michael who was still raising his hand finally gets Researcher 1’s attention and she comes over to the group).
31:16 Jeff Yeah, but what about-
31:19 Angela We don’t have the inverse.
31:22 Michael Actually I, I have something.
31:25 Angela I have something too.
31:23 Michael (to Researcher 1) For the, it’s box, triangle equals box over box plus one.
Researcher 1: So, everybody’s convinced that rule works?

Michael: Yeah.

Researcher 1: Okay.

Michael: Did anyone find the inverse?

Angela: No.

Michael: Box is say, say a half okay, times the denominators…two two, two, two times one half is one so equals triangle times the denominator equals box. It works.

Researcher 1: Would it work for every one of them?

Michael: Yeah cause look. Three fourths times four is twelve fourths and twelve fourths equals three.

Researcher 1: The trouble is that the denominator itself has a box in it. If you really wrote what you have for that.

(Michael’s paper shown below.)

But we can’t write box plus one because the denominator can’t have a box. But if we know the triangle, cause we’re trying to find the box, then, if cause if you’re gonna find, if you’re gonna find the box you have to know the triangle. If you don’t know both the numbers you’re not going to find anything. So if you want to find the box, if you have the, if you have the triangle…

Angela? If you want to find box.

Michael: -and then, and then all you need is the, and then you have the both numbers the denominator and the numerator.

Angela: Numerator. (correcting Michaels pronunciation of the word)

Michael: Numerator. So if you have the…

Angela: Denominator and numerator

Michael: ….five sixths, you take the denominator. Five sixths times the denominator equals the answer, is thirty, thirty six.

Researcher 1: Say it again. Can you say it just the same way?

Michael: Yeah. Five sixths, if you have the triangle and you want to find the box.

Researcher 1: Mmhmm.
Michael: Five sixths, times six and that equals thirty, thirty over six. And that equals five. Cause...
(Michael’s work shown below.)

Researcher 1: Now let’s think. Is there another way we can say that?
Michael: Unless we give the numbers, like the square-
Angela: It’s like three…
Researcher 1: The box is actually the five sixths.
Michael: No, the box, we’re trying to find the box.
Researcher 1: You see the rule is writing it now in terms of the boxes and triangles that have-
Angela: You see we have the…we have one way to get the box and one way to get the triangle.
Researcher 1: I think I’d like you to tell everybody that when you do it that there’s further we can go with it somehow to write it as a, what, you know to write it as box is equal to triangle and something else with a triangle in it and then maybe that would give us the answer.
Angela: See I don’t think we could use triangle times the denominator.
We’d have to use like num- like numerator times denominator because like triangle is the whole answer the whole fraction.
Jeff?: What?
Angela: See triangle is the whole answer like two thirds is the whole answer so…
Researcher 1: It is and that’s…you’re right. That that’s the problem that we need to address right now.
Angela: So wouldn’t it have to be like numerator times denominator?
(Angela erases the triangle and writes the word ‘numerator’
Researcher 1: Well you have to find something that will work.
Michael: But see for trying to find the square. You, this is where you try to find the square right?
Angela: What?
Is this where you try to find the square?

Yeah. Try to find box. Alright.

If you have both. If you need...the numerator....

But wait, would you look a second?

You need triangle to find box. If you have triangle and you know what the denominator is.

Yeah like here. Like eleven twelfths. You have to multiply the numerator times the denominator not the whole, the whole thing times twelve. Right?

No, it’s the denominator times numerator.

Yeah but you had triangle times denominator. That would be this whole fraction times twelve

(Michael’s paper shown below.)

I know. That’s what I mean.

But, you don't, you don’t – you don’t multiply the whole fraction, you multiply twelve times eleven.

I know. When we do it like this that means eleven twelfths, not elev—If we’re going to do one half and we times it by 2, we don’t times this by two...we only times...

(interrupting above) Yeah I know. But I-but you're, I'm say—you had it like that. You had it like that.

No.

You have triangle. Triangle is this whole fraction

I know. I’m –you---not , not the whole the fraction, not the whole like both numbers the fraction.

Look , look. Don’t you have to multiply twelve times eleven?

Correct.

(interrupts and writes on paper) You know what this, you can just (inaudible). You can just get rid of this one.

Then you could just multiply numerator times the denominator

But it can also be given as...
...the - the numerator times twelve, the numerator times twelve is you don’t when you times a fraction by a number you don’t times the, um, the denominator, you just times the numerator.

I- I know what you’re saying.

K, let’s just do what you said.

No but just listen. If you’re saying triangle times the denominator, that’s this.

(Michael’s work shown below.)

‘Cause twelve is the denominator.

I know. That’s twelve, twelve elevenths.

That, that’s what you want to say?

Yes. Twelve, twelve elevenths.

No, look. Like that’s, that’s exactly what you want? Twelve times eleven twelfths?

(interrupting above) But when you do this you don’t times twelve by this. It’s twelve over one. (during Angela below) It’s twelve-

But that’s what you want to happen?

You put twelve over one, twelve, is twelve by…I don’t know.

OK.

(laughs) It’s one third

Okay, so.

No wait one third no, one twenty three I think, yeah. That’s what it is. Ah. Does that look like a two?

One twenty three, you-you put this whole thing about triangle times denominator over one?

Ah. But you want to multiply this whole fraction, I mean don’t you want to just do twelve times eleven

But you—but this--but the-
Angela: But wait, wait, wait, look isn’t this what you wanna do? Just twelve times eleven?

Michael: But if you put this like that it looks better.

Angela: But it’s easier – twelve times eleven.

Michael: It's, the answer is one twenty three, okay, and then over twelve.

Angela: Yeah. But the same thing is…Let’s use a better number so we can multiply.

Angela: (laughs)

Michael: So the denominator-

Angela: Yes.

Michael: Over one. Two, twos equals one. Yeah, and yeah equals one.

Angela: Yeah but you can also do it this way. Write one over two, one times two is two. No, that’s messed up.

Michael: (laughs and erases what she just wrote)

Angela: But this is an easier way instead of writing the denominator…

Angela: (laughing) What did I just do over here?

Michael: …The numerator times the denominator

Angela: I know, see it would be one over two is two over one, no two over two.

Michael: K, tell me what you're gonna--tell me what you think.

Angela: Alright look, like this here. I’ll use this because I have it down already. Wait, wait, wait. Eleven twelfths, you multiply eleven times twelve, right? And the answer…you keep the denominator.

And you got one twenty three. Twelve goes into one twenty three eleven times.

(Michael and Angela’s work shown below.)
Michael (interrupting above) But wait… wha... (inaudible). So this is what you’re saying. You times the…

Unknown I don't know what they did.

Researcher 1 I think there are two tables here that are sort of on the verge of getting this. And I’d like for Michelle has agreed that she will tell you um, where their table has gotten. And then, if you people can talk kind of quietly at that table, I know you're in the midst of wonderful problem solving, so... Michelle, you might want to hear a little of what Michelle says.

Researcher 1 (interrupts Michelle) You do numerator times the denominator and then you get an answer but you don’t have, you don't have a denominator yet so you have to write and then you’re gonna write and then put it under the denominator.

Angela You have the- (have?) the denominator – its just keep the original. (Angela writes twelve as the denominator and circles the original denominator of twelve.)

Michael These two (inaudible word). I’m just saying. Mine’s better… I’ll write mine up, when we go up.

Researcher 1 (interrupts Michelle) And then, maybe you can finish it before tomorrow.

Michelle? So that's--

Researcher 1 (interrupts Michelle) And then, maybe you can finish it before tomorrow.

Michelle? So that's the rule. And, like, I can sort of, well, I can prove it because then, like, if you have the triangle is one half equals one divided by one plus one, and that's one half. Okay. It works for every one. So, like, a five...

Researcher 1 So, Michelle, eh, you're saying that you can demonstrate for each row that we have--that that rule works. Okay.

Angela (whispers) I said like (inaudible). (raises hand)

During Michelle below) We can go up together.

Michelle? (whispering at board, followed by) And then, we found out, like, this other rule, but it's not really, like, the inverse rule. And does, it works, but, but you're not supposed to use it this way.

Researcher 1 laughs)

Michelle? So, it's like, box, this is, five, umm, equals, five sixths times... so then you do, um, box plus one, which is six. And then you do it over one, which is thirty over six, and thirty divided by six is five.

Michael? (During Michelle above) I have a good-I have a better rule.

Angela (to Researcher 1 who comes over to table during Michelle talking) Alright, will we be able to go up together. Cause we have the same rule, but two different ways of doing it?

Jeff Your way is horrible. (inaudible)

(Right begins whispering)

Researcher 1 (nods yes to Angela) That's great Michelle, thanks. Since we just have like, another, half minute, A-, Angela and Michael want to come up together and, uh, they're thinking along the same lines,
Michelle, as, as your group was, and they think they have words that will add to this.

(Michael and Angela get up and go to board)

Jeff (to Ankur during Researcher 1 above) (-Savreid?-) just wants to, I mean Angela just wants to see what she wants.

40:48 Michael Well first. To the inverse, we have the box--

Researcher 1 Yeah, we know, we are, we're all pretty convinced about the rule, I guess.

(Jeff, Ankur, and Sarah and grabbing each others nametags and papers and are not paying attention)

Michael This finding the triangle, but the box is box equals the triangle time, times denominator over one. And then, it works for the (unintelligible word) box equals one, and the triangle equals, um, one-half, when... when you want to find the box equals one--

41:22 Researcher 1 Yeah, it's, it is the same thing. Uh, he said it a little differently.

Michael (Continues over Researcher 1 at 'Uh') The one-half, um, over the denominator, [which] is two over one. So that means that equals two [over] two, which equals one. So, the box equals the one.

Student? (during Michael) Is that the same thing you did?

41:45 Angela Wait. Box equals numerator times denominator. Ah, I don't know, if it's (unintelligible, about spelling 'denominator'). Um, but, of the triangle.

(Bell rings)

Angela So, it'd be, like, over one-half. You have one times two... two, and you, like, keep the same denominators.

Michael But then, how are you going to write that cause you just have the numerator times denominator, which only equals two. And box doesn't equal two.

42:04 Angela Yeah, it's the same. The same denominator. But you don't have it written down that way.

Researcher 1 (interrupts Angela) Misses Troy said she'd continue this tomorrow morning and we're very disappointed that I won't be here at the end.

(Most of the students exit the classroom)

Michael They're the same, right? (unintelligible) That's two, that's two, that equals two.

Angela But it's the it's always the same thing it’s just that same...

42:35 Researcher 3 Michael. Michael, can you write the denominator in terms of (the?) triangle?
839  Michael  Like, the number? The triangle.
840  Researcher 3  Write the denominator in terms of the triangle.
841  Michael  Of her rule? Of her rule?
842  Researcher 3  In terms of the triangle. See right now you wrote the word denominator up there.
843  Michael  Denominator of triangle.
844  Researcher 3  If you could, if you could write that denominator in terms of triangles and numbers, then you'd, I think you'd have a real good thing.
848  Michael  Box equals the, the... what do you mean in terms of triangles?
849  Researcher 3 (unintelligible word, problem?).
850  Michael  I just, I wrote [de]nominator, like, triangle times...
851  Researcher 4  Yes you do, you just start checking (unintelligible, 2 words?), that's all.
853  Angela  You know what. (Stop me (inaudible)?) One second, Shelly go. (picks up a marker at overhead).
855  Michael  The triangle times-Triangle times the triangle's denominator (over one?).
857  43:28 Researcher 3  Michael, in, in formulas we usually don't write denominator.
858  Researcher 3  Or numerator. So, you have to somehow, so you have to somehow come up with a way of extracting that from the fraction.
860  Michael  Then she's writing numerator and denominator--
861  Researcher 3 (interrupts Michael above) So, think about it. I mean, you don't, you don't have to...
863  43:47 Researcher 4  What, I think what he's asking you is, is how is this triangle related to that denominator.
865  Michael  Um.
866  Researcher 3  Okay, basically, the idea is to just, when you write formulas,
867  44:00 (bell rings)  (bells rings)
868  Researcher 3  you don't write denomina--you can't--it's not fair. Somehow, it's not part of the rules to write denominator. So, what you have to do is, kind of, see if you can't come up with a way of writing exactly what you wrote...
872  Michael  Okay.
873  44:12 Researcher 3  In terms of the triangle instead of denominator. Cause you're right, you want the square on one side and the triangles on the other side, but now you need, somehow--
876  Michael  Well--
877  Researcher 3  To come up with a denominator--
878  Michael  And, but she, eh--her ferm--her formula has "keeping the same denominator in your answer." (read from screen)
880  Researcher 3  Right, so that's not fair either.
881  Michael  (during Researcher 3 above) That's, that's very long.
882  Researcher 3  That's right.
883  Michael  You need, like, numbers and...
Researcher 3: Right. So just, look, just play around with it, and I'll bet you can do it. Oh-

Researcher 1: (Conversation with Angela unintelligible)

Researcher 1: Is that when you were saying the five sixths, for example, and you wanted to multiply it times the denominator, I was messing around with the five sixths, and I thought, "well, what's one minus five sixths."

(Angela smiles)

Researcher 1: If you take one, which would be six sixths, minus five sixths, would give you... one sixth?

Angela: Is it?

Researcher 1: That somehow, that gets you closer to having that six by itself.

Do you know what I mean?

Angela: Yeah.

Researcher 1: I know, it's not good help at all (unintelligible word) you.

(The two laugh, and seem to say goodbye)

??: Ah, excuse me Miss Abby, I'm very late.
Stephanie’s table: 5/17/1995 (S2)

Camera View: Stephanie’s Table (Stephanie(center), Romina(right-far), Ankur(left-far), Brian(left-close), Jeff(Behind Stephanie and Ankur))

Date of filming: 05/17/1995
Harding public school, Kenilworth NJ, Grade 7
Advanced Guess my Rule (AGMR): Inverse Function
Transcribed by: Elijah Brookes
Date of transcription: January 2010
Verified by: Andrea DePaolo
Date of verification: March 2010
Length of session: 00:46:52

<table>
<thead>
<tr>
<th>Time</th>
<th>Speaker</th>
<th>Transcription</th>
</tr>
</thead>
<tbody>
<tr>
<td>00:00</td>
<td>(Students are entering and preparing for class)</td>
<td></td>
</tr>
<tr>
<td>3:30</td>
<td>Romina</td>
<td>(Sits in chair on end, then switches name tags with Michelle and sits next to Stephanie)</td>
</tr>
<tr>
<td>3:35</td>
<td>Jeff</td>
<td>Michelle's not here? Can I sit where Michelle sits?</td>
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<tr>
<td>3:35</td>
<td>Student ?</td>
<td>Yeah.</td>
</tr>
<tr>
<td>3:35</td>
<td>Romina</td>
<td>(Hands Michelle's name tag to Jeff at the adjacent table, Stephanie puts it back)</td>
</tr>
<tr>
<td>3:40</td>
<td>Jeff</td>
<td>I don't know if that's legal, I have to ask. Can I sit where Michelle usually sits? She's not here today.</td>
</tr>
<tr>
<td>3:45</td>
<td>unknown</td>
<td>No moving around.</td>
</tr>
<tr>
<td>3:51</td>
<td>Jeff</td>
<td>Cause, like, um...</td>
</tr>
<tr>
<td>4:34</td>
<td>Researcher 1</td>
<td>I noticed at the end of class yesterday, we, um, we got kind of hung up with some whys. And I thought maybe you might like to discuss this morning, um, ways in which you could stop a why. Um, for example, just an every day example, what about, if, uh, you have a situation where you are going to the movie one Saturday, and you ask your mother, tell your mother that you need five dollars, and she says why. So what stops a conversation like that? What do you have to do?</td>
</tr>
<tr>
<td>5:10</td>
<td>(inaudible class response)</td>
<td></td>
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<tr>
<td>5:14</td>
<td>Researcher 1</td>
<td>How do you, how would you answer a question?</td>
</tr>
<tr>
<td>5:14</td>
<td>Michelle I.</td>
<td>Say that you need five dollars for the movies?</td>
</tr>
<tr>
<td>5:14</td>
<td>(The class laughs)</td>
<td></td>
</tr>
<tr>
<td>5:14</td>
<td>Researcher 1</td>
<td>How much does it cost you to get in the movie?</td>
</tr>
<tr>
<td>5:29</td>
<td>Jeff</td>
<td>Say why not.</td>
</tr>
<tr>
<td>5:29</td>
<td>(Everyone laughs again)</td>
<td></td>
</tr>
<tr>
<td>5:29</td>
<td>Researcher 1</td>
<td>Why not, alright. Some people that doesn't satisfy too well, does it. What about, how much does it cost you to get in the movie?</td>
</tr>
</tbody>
</table>

| 5:29 | Jeff | Six fifty, and then... |
Six fifty, so five dollars is way too low already, so you probably say to her, "I need ten dollars to go to the movie," right? And she says, "why?" And then, you have to say, the movie is six fifty and whatever else you needed, maybe you can establish with her the why because these are factual things, right? That you have to pay six fifty to get into the movie. Um, there are some things in mathematics that are just accepted by everybody as properties of the number system, and there was one in particular that was involved with some of the whys yesterday, and, um, what that one is in particular is the property of multiplying by one. Are you familiar with that? That if you have any number in the number system, and you multiply it by one, then what happens?

You just get back the number that ...

Michelle I. You just get back the number that you started with, right? And, uh, that's called the identity for, uh, multiplication. So, if, if Dr. Maher or somebody else is asking you why on something, and you want to multiply by one, that you're saying that you're multiplying by one, and that's why, and that's a property of the number system, and mathematicians accept that that's true. Nobody can really prove it. Now you were giving some good arguments yesterday as part of what you were saying, and, and one of the things you were saying, I think Michael in particular was answering this question, that if you had a hundred over a hundred that that's equal to what?

Jeff? One over one is...

That's equal to one, isn't it? So, we could go back to this, we say, well it's a property of the number system that any number multiplied by one, you just get back that number. It's equivalent. So, certainly if you multiply a number by a hundred over a hundred, you should get back to the number that you started with. And he gave, I think it was Michael that gave a nice explanation of why a hundred divided by a hundred would be equal to one. Did you say, do you remember what you said to that?
Michael: Is it like, a hundred, divided by ten, or whatever, is equal, n-like, simplify the problem to make it just ten over ten, or one over one. By, it's dividing each side by a hundred, you can make it one over one.

Researcher 1: Alright. Yesterday you might have said something about inverses, too, you know. Ho-, Alex, what would the rest of you say? What would somebody say? Why does a hundred divided by a hundred equal to one?

(Jeff writes on Ankur's name tag, Stephanie laughs at it, then Ankur writes something back.)

Stephanie: Oh.

(Researcher 1 and Stephanie laugh)

Brian: (unintelligible)

Stephanie: (whispers to Brian) No, I didn't hear what she said.

Researcher 1: Alright, and say again, try to say again, Michael, what your view of it was.

(Ankur writes on his name tag again while Brian and Jeff whisper)

Michael: You can simplify it by making it ten over ten or one over one by dividing each side by the same number.

Researcher 1: Okay, so you could think of dividing both the numerator and denominator by one hundred.

Michael: Just like when you simplify fractions (unintelligible, 3 words)

Researcher 1: Alright. Um, probably it's good enough right there to say that a hundred divided by a hundred is equal to one by definition. Or by, um, what do you think? What would you say at that stage?

Student: Me?

Researcher 1: Anybody, anybody, (really?).

(Student laughs)

Student: I don't know.
Now, what about this? This was another step that you were using. You said in your discussion yesterday that a hundred over a hundred, if you want to multiply that times something like, uh, box plus two, that you can write that this way: a hundred times box plus two over a hundred. What's the answer to that why, why can you do that? That happens to just be a definition in mathematics too, that whenever you have two fractions, can, is it alright if I write it $a$ over $b$? Times, say, $c$ over $d$? That just the definition of multiplication of two fractions is that you can multiply $a$ times $c$ over $b$ times $d$. And that's what you do here isn't it? (points to second line from bottom) So, if we back up from that, and we have a hundred times a number plus two divided by a hundred, and we want to say that's equivalent (points to third line from bottom) to this, it's by this definition of, in, uh, the number system. The definition of multiplication. Do you know any other things that are properties of number systems? (removes slide)

That you could stop and just say, well that's true because all mathematicians, everybody agrees. Just some very simple things. One of them that's so easy that we take it for granted that if you're adding two numbers together, what if I were to say, well, I know that the answer to two plus five is the same thing if I added five plus two. Sounds pretty dumb, doesn't it? Cause we all know that works. So, that's one of the real fundamental properties of the number system. You probably even heard the name for that. A word, do you remember what that's called by mathematicians? It always works, doesn't it? It's not anything you have to prove. What do you think?
Stephanie? Who, me? I think it's the commutative, whatever.

Researcher 1 You're right. She's right. Isn't she? That's the commutative property is the name that it's given. Can you think of any other properties of the number system that you don't have to prove?

Student Um, distributive.

Researcher 1 The distributive property. And in fact, we used that yesterday, didn't we? So that if you have a number, say, multiplied by the sum of two numbers, suppose I do, symbolize that this way, you get what? What do you have to do?

Student Um. Um, Put in (plug in?) that, the box times the triangle, and then the box times the circle, (but only the circle?).

Researcher 1 Ahuh. And then, why did we say that's true?

Student It's distributive?

Researcher 1 It's just the distributive property. It's not anything that we have to add answer why to further than that. It's a property of the number system. That's just assumed by mathematicians. Nobody can really prove it, or they haven't found a way to prove it. Everybody just sort of tries to convince themselves that it's true, and then we start with that assumption. So what you were using that places yesterday we had one, didn't we, that was something like, three times the number in the box plus two, and you were telling me that was equivalent to three times the number in the box plus ..

Ankur? The box plus six.

Researcher 1 Plus six. Okay. Good. You think of any other properties?

Student Associative?

Researcher 1 The associative property. That's really good, that's really good. You must have been talking about this thing. Look at, look at
Misses Poise, beaming back there. Can you write for us what the
associative property is?
(everyone laughs)
Researcher 1 Can anybody write it? I believe Amy-Lynn can write it, she doesn't
want to (do it?).
Amy-Lynn? No.
Researcher 1 You can't? You just remembered the name. What is the associative
property? Can anybody remember that?
(during Researcher 1)
Romina Is that even (drowned by class noise)?
Stephanie Maybe, I don't know. (And?) try it, (and?) see if it's right.
Brian What is it?
(in response to Researcher 1)
Michael? (unclear over table)(I might happen to know the right now.?) Three
times one equals three, or something.
(class laughs)
Researcher 1 That may be the identity property we talked about up here.
Michael? Okay.
Researcher 1 That sounds kind of like that one, doesn't it?
Stephanie (during Researcher 1 above, to Romina) Don't want to listen to
(it/you), you're right.
Romina I'm okay.

Researcher 1 What if I'm adding this column of numbers, and I have a four here,
and maybe a three here, and a six here, and the way I add numbers,
I look at the six and the four, and see that I could add those,
and that makes it a lot, nicer, and what I said was not a very good
example. Suppose I had three plus four plus six. Instead of adding
three plus four first, I might add four plus six first, and this is a
way to talk about that. Cause you could add three plus four first, and get seven, and seven and six is thirteen. But instead of that, if you decide to add the four plus six first, and get three plus ten, it's also thirteen. And that's the associative property: the fact that you can regroup 'em, cause you can only really add two things at once.

You think of any other things that are definitions? Or properties of a number system that would stop a why? If there's an identity for the operation of multiplication, if we take a number 'a,' and we say if we multiply it by one, and we get back the 'a,' is there an identity, ah, for addition?

If you add [difficult to hear] (that to yours or something?)

Researcher 1 So you think zero is the identity for addition? So, if you take any number, and you add zero, you get back the same number. So that's the identity problem for addition. That's really nice. Now, I'm wondering, if you thought of the problem that we had. What was the third grader's name yesterday?

Researcher 1 Lindsey. Did you think about, uh, the problem she said, that if she multiplied, wanted to multiply a number by five, how did she say she did that?

Student [unclear](Multiply by) an even number?

Researcher 1 A, well, an even number by five, you're right.

Student And, um, then you split it in half, and then the number that (unclear word) with it, and (then?) you added zero.

Researcher 1 So, if you started with an even number, and you wanted to multiply it by five, instead of that, you could take one-half of the number, which would give you sixteen in this case, and add a zero. I don't know how to say that exactly. This, tha-that's not saying it quite right. But, (n?) ... Alright? So we know that she was getting this right? Do you know why that works? Anybody figure it out?

Bobby

Okay, like five times one, that's an odd number, and since like, it goes five, zero, five, zero, all the way through, then every odd number has to be five, and even (you have?) zero, it's like, five times two, (then?) ten, and then like, every other one would be zero.

Researcher 1 Can you come up and show us that set sounds really wonderful.

(Student and Stephanie are whispering unintelligibly)
Well, wait, five times one, five. So every odd number will be like, fifteen, twenty-five, thirty-five. (to self)(And five ...) ---Is ten. And then that would be like (what would be another one?) would be like ten, twenty, (and?) then thirty. And since there's like two numbers, like five, ten, you have to like cut it in half, so like, you can put it in there. And you can't like, cut an odd in half, so like the one over there, so you have to minus one 'til you get an even number, (and?) then you divide it in half.

Would you go a little further with your list? I think would help me. You have five times one is five, and five times two is ten. Could you put a few more, like five times three ...

(Ah?) Five times three equals fifteen. Five times four equals twenty. See, (and?) it's going to keep on going like that.

So then I'm like, that's why you have to add zero to all the evens, so they end in zero no matter what. And for the (fives?) you have all the odds because the (fives?) that end in five must (not?) be the wrong answer. And then you can, like, where you divide in half (um?), it's because, like, there's two in, like, every one. It gets kind of (two unintelligible words) at first. So, like, when you're putting the whole thing in, you're like, it's like double the real answer. So, like, you have to cut it in half. And then you have to find the number. Since it ends in, like, it can't end in six.

Well that's really an interesting explanation. I bet she thought of that sort in that same way, seeing that pattern. What do you think? Did you understand what Bobby is saying, everybody? Do you have any questions for Bobby? Like why?
Brian has one.

Researcher 1: Ankur? A- somebody over at this table, uh, you have another one that you were doing?

Brian: (during Researcher 1) He has one.

Researcher 1: Ankur, did you have a (way you would do it?)?

19:10 Ankur: Yeah, I think that since five is like half of ten, she splits the even number in half so you're really multiplying by ten, so it's like, five is half of ten, so really like, you split the even number in half, and then, if you multiply by ten, it'll be the same as multiplying by five.

19:30 Researcher 1: Wow. Did you write that?

Ankur: No. (class laughs)

Researcher 1: It reminds me of something we were doing with th-, with identities and fractions up here right at the beginning. That would say what you said. You said, "five is equivalent to ..." or the same as ...

Ankur: Like when she split the number in half, and then she multiplied by ten.

Researcher 1: Right.

Ankur: That's the same as multiplying by five with the original number.

(Brian whispers something to Ankur)

Researcher 1: Are you saying this?

Student: Is, is he saying that, like, the number that you times five by, you would, you split it in half? Because it's like timesing like the number by ten? And put like a zero at the end.

Researcher 1: So, if we had a number times five, are you telling me that if you had the number, and somehow, you multiplied it by ten, and you divided by-

Student: By two.

20:29 Researcher 1: By two.
You'd get the same answer.

Which could be written maybe this way, couldn't it.

Mhmm.

That's interesting. Do you see how that kind of uses what we were talking about in the beginning, too? That's beautiful. Anybody have anything else you'd like to add to that discussion? Yes, Michael?

The box times ten over two, that would be bo--box times ten, over two. It's not, cause you wrote that. It would be box times...

Are you saying this is equivalent to this? Is it equivalent to this?

I don't know.

(class laughs)

What do you think? What do the rest of you think? Is it equivalent to that?

Cause (there wasn't one?) before box times ten divided by two with the division sign?

Mhmm.

It's like, that means you take the box times ten and the answer to that, divide it by two.

Are you saying that you think it should be written this way?

Or if you want, if you want to write it like box times ten over two, then you have to put the bo- the ten divided by two in parenthesis on the one above it.

(Stephanie or Romina is whispering during the explanation, Brian can be heard soon after)

Come write it for me. For us.

The way you wrote it here, you took the box times (ba?). The one...

Okay.

Over here you wrote it right where you, there, this was number six. Six times ten is thirty, divided by two is, um, fifteen. But, that means you took the answer of these two (points to box times ten) and divided by two.

I see what you're saying. Ahuh.
Michael: So, if you're going to write it like that, that this is the answer of these two divided by two. (circles the algebraic equivalent)

Researcher 1: Okay. Alright. Now, the question I have now is those three rows right there, are all of those equivalent? Is it alright if we put equals between all of those? What do you think back there at the back table? Do you think that all the pieces are equivalent? (the camera shifts to show Brian and Ankur fighting over things written on Ankur's name tag)

Stephanie: Yeah. Ah, yeah.

(Students seem to be quietly arguing over something on Ankur's name tag)

Researcher 1: It's important. These are answers to why, right here.

(A student says something unintelligible)

Researcher 1: Do you agree that those are all equivalent, or what do you think?

Student: (same who was speaking last) Yes, yes I agree.

Researcher 1: Because it's just the definition of multiplying fractions, isn't it? That those are equivalent. (Brian is looking over the name tag, Jeff speaks to him)

Researcher 1: Because you can say this times this is this, and one times two is two, or you can do it this way, and could you say it this way? Ten times the number in the box divided by two.

Student: (in background, during Researcher 1 above) You get the same answer.

Researcher 1: There are all sorts of ways you could say it, aren't there?

Students: It's the same.

Researcher 1: And why? Why are they all the same?

Student: Commutative because, like, if you switch the two...

Researcher 1: (interrupts) Yeah, the-there's a lot of the properties in here, aren't there? Commutative property, and the definition of multiplying fractions. That's, that's really super. That's really super. When we were doing 'Guess My Rule' yesterday, Mr. Miller made up a very
hard problem for you that I'm sure it will just take you a few
minutes to get.
Jeff Do you have any pens? (Indian accent) Sure, plenty of them.
Researcher 1 And maybe you need, uh, you want a little of this, paper.
Researcher 2 They need ...
Researcher 1 Yeah.
Researcher 1 I don't know if this is the best paper. You can use the back, the
back of it or something if you don't like it.
Stephanie? Um. Oh.
Romina Well, let me just get a pencil.
(The class is tapping pencils loudly)
Researcher 1 Did everybody get a piece of paper?
Brian (Whispers to Stephanie) Did you get the microphone?
Stephanie Hmm? (Brian points) Um, it looks like the number on the, um,
triangle, like what you were... Oh, wait no, I'm looking at it, hmm.
Romina? Okay, I got (unintelligible over banging noise)
Stephanie? Hmm?
Romina? I have a question with the eleven. It's going to be twelve, and
twelve over thirteen. (Then?) see how like it's a five, and it has five
over (unintelligible word, interrupted) right after.
Jeff? It's going to be eleven.
Stephanie Yeah, it is.
Brian? Eleven and eleven twelfths, then half that.
Romina? Eleven twelfths, and twelve thirteenths, and thirteen fourteenths,
but what's it multiplying.
Stephanie (echoing with Romina) Twelve thirteenths, thirteen fourteenths,
fourteen, yeah.
Brian? I don't know.
Romina There's got to be some sort of rule to it.
(Stephanie takes (her?) paper from Brian, and he whispers something)

Stephanie I totally understood that Brian.
Romina Brian, just say it.
Stephanie Oh, I understand. I understand. I heard what you said. I got it. No. (Brian leans over to write on her paper, Stephanie pushes his hand back)
Brian (He?) told me to spit all over the, thing. It wasn't that (points).
Stephanie Oh, I understand. I understand. I got it. No.
Ankur What?
Romina Remember, we were talking? (To Ankur)
Stephanie I don't spit.
Romina Neither do I. (Okay?)
Ankur I wasn't even looking this way.
Brian It's not really spit. (This?)
Romina (very low) Alright, what would be number (twenty?).
Ankur But it's not really spit, just water, right?
Stephanie Um, yeah, but figure out the difference. How do you get that?
Romina I don't know.
(Stephanie and Romina laugh)
Romina Hey, I know (two unintelligible words) rule.
Stephanie Mmm. Figures.
Romina So, we could take that one.
Stephanie Oh, yeah right.
unknown Are you kidding, box times (thirteen?), that's it.
Romina Well, thirteen...
Stephanie I can't even figure out (unintelligible, two words)
Researcher 2 (in background) That's not good.
Stephanie We're sitting there, we're like (she makes a face).
Romina: Well, this could happen. One plus one is, what am I doing?

Stephanie: (writes on paper) What's my name?

Romina: Well, how can thirteen-how can you get thirteen fourteenthths?

Stephanie: How can you get twelve-From twelve, how can you get twelve thirteenthths? Why don't we start at the top though? Because eh-the pro-they're smaller.

Romina: I know. Okay. Add a one.

Brian? It goes up a hundred.

Ankur: It's multiplying by something.

unknown: Don't multiply it by over a fraction.

Romina? The one. One is divided by ha-one divided by, two, would it be like a half?

Brian: What?

Stephanie: Um, wait, can I just say something?

Brian: (to Romina) No. No, can't.

Romina: I know.

Ankur? I was just...

Brian: The first one's right though. (It?) could be like a fourth or something.

Ankur?: We know what the pattern is. It's not on the board, but-

Researcher 1: What is the pattern?

Stephanie?: Like, say we have twelve (interrupted by Brian)

Brian: Could you imagine (if we?) had that (unintelligible word) lady?

Researcher 1: (Laughs at Brian's remark)

Stephanie?: No, say we have twelve, it would be twelve thirteenthths, and we have thirteen and thirteen fourteenthths, but...

Researcher 1: Ahuh. How do-how could you say that with boxes, or X's, or something?

Brian: (starts scribbling, is ignoring Stehpanie) Lala, lala, la, lala, lala, and over, and over, and over.

Stephanie: Um(laughs), go ahead Romina.

Romina: The box equals box over triangle. Maybe?

Ankur: How can that work?/(That cannot work?)

Romina: I don't know.

Stephanie: Box equals--

Ankur: Box equals box over box plus--

Romina?: One.

Ankur: Box plus one.

Romina: (background box over box plus one)

Ankur: (Was?) box over box plus one.

Romina: Ah, that's it (though?).

Stephanie: Um...

Stephanie?: So, where's your triangle for that.

Ankur: Where's your triangle?

(Stephanie laughs)
Stephanie? Does it--
Brian How many bo(xes?)...
Romina Does it, and then, the plus one, that equals triangle.
Stephanie Triangle--
Brian Good job.
Romina The square plus one, that would equal the triangle. Does that work?
Brian Does that work?
Stephanie I don't know.
Ankur Yeah, it does.
Brian Really?
Stephanie (laughs) I don't know.
Romina? Yeah, and what's the rule.

Ankur That's the rule, I guess.
Romina Well, that is a rule, so... What's (the?) inverse rule.
Brian I don't know.
Romina Square equals square over square times two, minus one?
Brian (during Romina above) Triangle minus one--
(after Romina) You're no(t)...
(Romina puts her head on the table, laughing?)
Stephanie What? Ahhh, Okay.
??? We got an answer (3 unintelligible words)
Romina Well, I think we got the rule. Now (we?) think of the inverse rule.
(Brian and Ankur are writing on name tags again)
I got it. Square equals square over square plus two, plus one.
Stephanie Oh, yeah.
Romina Yeah.
Brian: I have Niles.
Romina: What?
Brian: Animal.
Romina: Is that white (wool/wolf)?
Brian: Animal, or...
Romina: What's yours?
Brian: (L, I hit you?).
Romina: (Nair?).
Ankur: Nile, Niles.
Stephanie: Mine's like so long, I don't even remember.
Ankur: (Sketching?)'s too long.
Romina: What's yours?
Stephanie: Mine's like so long, I don't even remember. (I'm?) seriously. I'm serious.
Romina: So, we have a rule. Should we tell them the rule?
Ankur: Brian has to do it.
Romina: Brian, you can work with Michelle, go ahead.
Stephanie: Okay, Brian.
Brian: Awww, Go-
Romina: (points to paper) Hey, me and Stephanie figured out that and the rule, so you guys figure out the inverse.
Stephanie: We did it all yesterday, you were just sitting there.
Brian: No, I did it all yesterday.
Stephanie: Ahuh, (we?) did it all (yesterday?). So, go ahead Brian.
Romina: Oh, yeah. So then Brian. No, Brian.
Brian: No, I did everything at this table. Michelle drew--
Stephanie: Brian has to do it.
Ankur: Brian, you can work with Michelle, go ahead.
Romina: Brian, you can work with Michelle, go ahead.
(No noise)
Jeff: 'Sup, you suck.
Stephanie: (Brian?) are you deaf?
Jeff: Are you (defending?) me?
Stephanie: No.
Jeff: (noises)
(Jeff and Stephanie make faces at each other and turn back)

549  Brian (Fine/Why?-- I don't even like doing this. (Why should I help you?).
551  Ankur (unintelligible, 1-2 words)
552  (Brian reacts physically as if 'I don't know')
553  Negative?
554  Brian One minus box, divided--
555  Ankur Is it negatives?
556  Brian One minus box--
557  Romina (unintelligible word) box over box plus two, minus one.
558  Brian One minus box--
559  Romina Well, they're not the same thing.
560  Brian One minus box--
561  Romina Hey.
562  30:06  Brian Times box equals box.
563  Romina Yeah Bri(an?).
564  Brian Is that (an?) equals right there?
565  Researcher 1 (to this group?) You think you have a rule over here?
566  Romina Yeah.
567  Brian We just can't get the inverse.
568  Researcher 1 Okay, that's a, that's a wicked problem. There's (unintelligible word, Dan B--?).
569  (Brian waves, Romina and Stephanie look).
570  So all you have to do is work a little harder.
571  30:28  Stephanie (3 unintelligible words?), yeah.
572  (Brian whispers to Ankur, Ankur replies something about the camera)
573  Stephanie Poor Ankur, his whole name tag is, like, ruined.
574  Romina Mine (two unintelligible words)
575  Stephanie? Mine is too.
576  Jeff Just cause Ankur works at Speedy Mart. (in Indian accent) I'd like a slush please.
578  Stephanie/ Romina Hmm?
579  (Jeff taps Ankur's shoulder, he seems to be bearing the joke well.)
580  Romina (Not my black secret, ?) you.
582  30:58  Stephanie (to Romina) What did I do?
583  Brian Ankur, I'm doing it again. (Ankur crosses out something on his name tag)
584  Ankur (unintelligible 1-2 word), now, huh?
585  Brian I was holding it up to the camera.
586  Ankur You did it again?
588  Romina (unintelligible response)
589  Stephanie Oh, yeah.
590  Jeff Brian, Brian, regular, or unleaded?
Stephanie: Oh, yeah, well, well, my paper's messy because of you.

Ankur: (2 unintelligible words), now she's not looking.

Romina: Because you misunderstood me?

Stephanie: No, (because of your sit there. ?)

Romina: Well, it gotta be square equals square over square plus two, minus one.

Brian?: (Same thing?) with you.

Ankur: (laughs) (unintelligible response, ends in 'nah'?)

Stephanie: Why would you want to do that?

Romina: (laughs) I (don't?) know.

Brian: (Probably?) funny if she does it (again?).

Stephanie: Eh, you probably sit at home with this. I swear to God. Last night, this is what you were doing. This is why--

Brian: And she sits on the (?ns?)

Stephanie: She's sitting here-- (laughs)

Brian: I always do that. I (had/have) the--

Romina: I'm bored.

Brian: How much do you weigh?

Romina: As long as you weigh more than me.

Stephanie: As long... fine.

Ankur: He weighs more than you?

Romina: Yeah.

Brian: She weighs one-twenty-four.

Ankur: Seriously?

Romina: Yeah, I probably weight one-twenty-four and...

Brian: Fifteen, fifteen ounces. Fifteen point nine, nine, nine, nine, nine--

Romina: Like, ninety-nine hundredths.

Brian: I made a guess between one-ten and one-fifteen.

Romina: Who, me?

Brian: No, him.

Romina: Oh.

Brian: Well, after your ninety six--

Stephanie: (during Brian above) Ankur, what do you weigh?

Ankur: Like, one-o-five.

Stephanie: He weighs, wait, Romina, did you hear him?
Brian He weighs less than you.
Stephanie He weighs one-o-five.
Romina I know, I know. I knew it.
Brian One-ten.
Stephanie What are you?
Romina I don't know. I really haven't weighed myself.
Brian Then, what'd you weigh last time?
Romina Last time, like, when last time?
Brian Like, last time you weighed yourself.
Stephanie Like, the last time you weighed yourself, duh.
Romina I don't--cause I have to go to my doctor again, he had to take blood from me.
Brian Well, when was the last time?
Romina Brian, if I remembered, I would tell you.
(Stephanie laughs)
Brian What do you think you are?
Romina (Fa-), what do I think I am?
Ankur Did you get weighed when you went to the hospital?
Romina What?
Ankur Did you get weighed in--
Romina What I weighed in the hospital, when I was in the hospital, I weighed--
Brian They have scales in every room at the hospital.
Student? I know.
Stephanie Those things are fun.
Brian Where my dad was, they had, like, a hundred scales.
Romina When I used to take my walks in the hospital, we (went/weighed?).
Brian I used to get lost.
Ankur (laughs, says something unintelligible, lost?)
Romina (4 unintelligible words) where the scale is, just right around it.
Brian Wait, where my dad (unintelligible word). I used to think I'd be walking through an elevator, but there was this nurse's station, wait, right in the middle, of like, think, I just kept on walking around it, and I ended up back at my dad's room, and then I got lost. I (went/ran?) down the stairs, and then I went up to the top floor.
Researcher 3 You guys, what did you guys find out here?
Romina That box equals box over box plus one. And then the box (unintelligible word) the triangle.
Researcher 3 And how do you work the other way?
Brian I don't even know yet.
Romina We don't know yet.
Researcher 3 How do you get the inverse of it?
Brian Ah--
Stephanie Um...
Brian That's what I (can do?). I did it.
Ankur: Yeah, you didn't do anything.
Brian: I came up with (it?).
Ankur: Did not. Deny--
Romina?: Yeah.
Brian: My suggestion. I did.
Researcher 3: This, this works though, huh?
Stephanie?: Mhmm.
Romina?: Yeah.
Researcher 3: So what the, the question mark is what?
Romina: It's, um, (whatever's in the box?).
33:52 Researcher 3: (unintelligible word), Okay. But the, uh, but if you had, if you had
the triangles, and you tried to get the squares, what would you do?
Researcher 3: (no response)
Romina: Do the opposite. What's the opposite, of...?
Romina?: Box minus one.
Brian?: It's got to be the inverse.
34:21 Romina: Wouldn't it be triangle, hold on. Wouldn't it be. Box equals box
over triangle minus one?
Jeff?: Maybe.
Ankur?: Mmm, yess.
Jeff?: I told ya.
Researcher 3: Box equals box--
Ankur: No, wait, you have to have, like, triangle at the beginning cause
that's what you know.
Stephanie?: No, you have triangle there.
Ankur?: (That?) equals.

(Romina's work reads: triangle (blank space) box over box minus
one)
34:40 Researcher 3: What did you--what do you, what do you want to, you want to find
box, right? You have triangle.
Ankur?: Triangle equals box--
Researcher 3: But, you want to say box equals, right?
Romina: So, it's box equals.
Ankur: Box equals...

(Background: Whoa! ... It's not the box we're trying to find though)
Researcher 3: So, if you have box here, and you want to find it out... you have a box over here. Can you do that?

Ankur: No.
Researcher 3: Hmm.
Student?: Put triangles.
Ankur: (What?) put triangle.
Researcher 3: (Bat?) some numbers, see what you, see what happens.
Stephanie: Let's see.
Romina: Okay, hold on, let's try the, the one, one. Okay, sorry. I have box with a one, and it equals two over two minus, never mind.
Brian: (to himself, during Romina) Darn it.
Brian?: Rokna, come on Rokna.
Romina: I'm trying.

Brian?: It's on you, Rokna.
Jeff: Ankur, put your number, your name plate back to where
( unintelligible, 3 words? )
Brian?: I messed up so bad. (Brian stands up)
Jeff: On what?
Stephanie: Showing no equal. But see the triangle squared.
Stephanie: Well, I don't want to do all ( unintelligible, 2 words? ) I'd rather do ( unintelligible, 3 words? ).
( Apparently Romina has been speaking softly to Stephanie, there is no audio, and her head is turned from the camera, but Stephanie is responding)
Stephanie: Yes we are. We have to do that stupid frog ( sounds like Prague, but is frog ) thing.
Brian: ( during Stephanie above ) La, la...
Romina? I wonder if I can plan on doing it Friday?
Brian: ( during Romina above ) What?
Stephanie: No, the paper frog, Romina. It's so stupid, if I have to do it, I'm going to...
Romina: Yeah, I know.
( Researcher 1 walks between Romina and Stephanie)
Romina: We have box equals box over box plus one.

Brian: ( very deep ) La, la, la...
Stephanie: Um, whoever my partner is, what we're going to do is ( unintelligible trail off/ interrupted)
Brian: I'm going to get ( severed? ) for the frog dissection.
( Everyone laughs)

Brian: You know I am.
Stephanie: Mmm, stab 'em and say it was Nick's.

Ankur: (during Stephanie) He'd lik-he'd be taking the scalpel...

Brian: He'd, he'd be, like, poking himself.

Ankur: (He'd?) know you did it.

Stephanie: (Be?) eating (it?).

Ankur: That means--

(Brian and Ankur start laughing)

Stephanie: 36:34 I've had (Sevried?), and I've done everything.

Brian: (during Stephanie) He'd be sticking the eyes.

Brian: He'll be pretending like the frog's alive; start making it move (two unintelligible words, trails off with Ankur under Stephanie).

Stephanie: No, but it (drives/dries?) me my time. It's like that one day in social studies, I had him in group work, right; I was so mad because everybody else was done, and I couldn't take on the homework because (Sevried?) dragged me behind, like, twenty problems.

Romina: I know because I have to do it in (fast?).

Jeff?: Really?

Stephanie: I would have been done with, like--

Ankur: (back on task) Wait, this (is)...

Stephanie: --before the, um...

Romina: Before, like, the first bell would have rang?

Stephanie: Before, like, um, I don't know. I would have had, like, twenty minutes to spare because it wasn't hard.

Romina?: (Wow/I know?).

Jeff?: What?

Stephanie: The one day I got stuck with (Sevried?), in group work.

Jeff: Ah. You were on, like, question three.

Stephanie: (Jeff, Romina, and Stephanie laugh)

Stephanie: I don't--

Jeff: We were done. Seems like...

Brian: Were you by yourself, or were you alone?

Stephanie: No, I was with (Sevreid?). But he asks--

Brian: Like, were you and (Sevreid?), or were you with (Sevreid?) and somebody else.

Stephanie: No, me and (Sevreid?).

Romina: (And?) he asks questions about everything. Why question: I know I'm right. Just... (no?)

Stephanie: 37:27 And then he ripped his pages in half. He was, like, trying to turn the pages and--

Ankur: When was this?

Stephanie: One day in, um, my social studies class.

Ankur: Well, Mike is dumb, when he had to work with him in science, ah, when they worked with the graph, Mike did his graph--

Jeff: (during Ankur) Oh, I'm going to get him on the frog dissection.
Ankur and then, Se-, Seve-, (Sevreid?)'s like, "do my graph," so Mike does his graph. (Jeff laughs)

Ankur But he does it all wrong.

Brian You know if you do this the, um, plastic gets hot? (he is bending his plastic pen (cap?).)

Ankur Why don't we do this, in the triangle, why don't we divide it, like, top part of the triangle is another shape, and the bottom is another shape, and the entire thing is the triangle.

Jeff (over Ankur, to Brian) Um, I talked her into letting me go to the after school thing tomorrow.

Brian I know.

Jeff (3 unintelligible words) I totally forgot I didn't know (unintelligible, 2 words?). Let me see that Bri-?

Romina Don't you have a game tomorrow?

Jeff No.

Romina When's your next game?

Stephanie Today.

Brian (simultaneously with Stephanie) Today.

Romina I know, after today.

Jeff Monday.

Stephanie Monday. Home or away?

Jeff Home.

Stephanie Home.

Romina When's your next Monday's game?

Stephanie Geez, Romina.

Romina No, I already gave you (unintelligible, 3 words?)

Jeff There's one after that. Brian, can I see your pen?

Romina (Brian tosses his pen)

Romina (8 unintelligible words?)

Jeff (Weston?) doesn't get hot.

Romina No, I want the same thing as. (she begins writing)

Stephanie Danielle comes up to me, and she's like, "write me a note." I always have to write her a note, like, every day. She's like, starting to write; I'm like, "no, I think you owe me, like, twenty."

Romina (Stephanie is watching Romina work)

Romina What are you doing?

Ankur (to Jeff and Brian) Ah, you guys are...

Stephanie Oh, you're trying to do that. (Stephanie begins writing)

Romina? Oh, no.

Brian? (Dai--) is the coolest guy in the world.

Brian? Can you see it?

Jeff See what?

Romina? What is the (sum?) question, why does this work?

(Stephanie is watching Stephanie drawing triangles and a pentagon)

Ankur Where'd you put it?
Brian? What?
Ankur My name tag.
Stephanie (That/mine)'s not working.
Romina Mine's (just?) working. (unintelligible, 14 words?)
(sound of Jeff laughing?)
Romina (unintelligible)
Stephanie Because there's a method. That you can use to get it done. End of story.
Ankur Is it my name tag?
Jeff No.
Ankur Is it Sarah?
(Sarah shrugs)
Ankur Come on. (laughs) Hey, Mike. (3 unintelligible words) give it back.
Mike? I don't know what it is.
Jeff Nothing.
Researcher 1 I think there are two tables here that are sort of on the verge of getting this.
Jeff (laughs) And they're (no-?)--(that isn't you?).
Researcher 1 I('d?) like that Michelle has agreed that she will tell you um, where their table has gotten. And then, if you people can talk kind of quietly at that table, I know you're in the midst of wonderful problem solving, so... Michelle, you might want to hear a little of what Michelle says.
Michelle? So that's--
Researcher 1 (interrupts Michelle) And then, maybe you can finish it before tomorrow.
Michelle? So that's the rule. And, like, I can show that, well, I can prove it because then, like, if you have the triangle is one that equals one divided by one plus one, and that's one half. Okay. It works for every one. So, like, a five...
Researcher 1 So, Michelle, eh, you're saying that you can demonstrate for each row that we have--that that rule works. Okay.
Michelle? (whispering at board, followed by) And then, we found out, like, this other rule, but it's not really, like, the inverse rule. And (unintelligible word), it works, but, but you're not supposed to use it this way.
(Researcher 1 laughs; Michelle continues to write silently)
Michelle? So, it's like, box, this is, five, mmm, equals, five sixths times... so then you do, um, box plus one, which is six. And then you do it over one, which is thirty over six, and thirty divided by six is five.
(Class begins whispering)
Researcher 1 That's great Michelle, thanks. Since we just have like, another, half minute, A-, Angela and Michael want to come up together and, uhm, they're thinking along the same lines, Michelle, as, as
your group was, and they think they have words that will add to
this.

(during Researcher 1 above)

(-Savreid?) just wants to, I mean Angela just wants to see what
she wants.

Well first...

No, he does(n't?) look like Hitler.

To the inverse, we have the box--

Yeah, we know, we are, we're all pretty convinced about the rule, I
guess.

Is this finding the triangle, but the box is box equals (the/a?)
triangle time, times denominator over one. And then, it works for
the (unintelligible word) box equals one, and the triangle equals,
um, one-half, when... (if?) you want to find the box equals one--
(during Michael) Is that the same thing you did?

(Continues over Researcher 1 at 'Uh') The one-half, um, over the
denominator, [which] is two over one. So that means that equals
two [over] two, which equals one. So, the box equals the one.
I like(d?) (unintelligible word), somewhat the same, but they're a
little different.
I have, um...
Somewhat the same, but a little different...
Wait. Box equals numerator times denominator. Ah, I don't know,
exactly (unintelligible, about spelling 'denominator'). Um, but, of
the triangle.
(Bell rings)
So, it'd be, like, (over?) one-half. You have one times two... two,
and you, like, keep the same denominators.
Michael? But then, how are you going to write that cause you just have the numerator times denominator, which only equals two. And box doesn't equal two.

Angela? Yeah, it's the same. The same denominator. But you don't have it written down that way.

Researcher 1 (interrupts Angela) Misses Troy said she'll continue this tomorrow morning and I am very disappointed that I can't hear the end. (Angela and Michael are talking during Researcher 1)

Michael? They're the same, right? (unintelligible) That's two, that's two, that equals two.

Researcher 3 Michael. Michael, can you write the denominator in terms of (the?) triangle?

Michael Like, the number? The triangle.

Researcher 3 Write the denominator in terms of the triangle.

Michael Of her rule? Of her rule?

Researcher 3 In terms of the triangle. See right now you wrote the word denominator up there.

Michael? Denominator (of what?).

Researcher 3 If you could, if you could write that denominator in terms of triangles and numbers, then you'd, I think you'd have a real good thing.

Michael Box equals the, the... what do you mean in terms of triangles?

Researcher 3 (unintelligible word, problem?).

Michael I just, I wrote(de?)nominator, like, triangle times...

Researcher ? Yes you do, you just start checking (unintelligible, 2 words?), that's all.

Michael Triangle times the triangle's denominator (over one?).

Researcher 3 Michael, in, in formulas we usually don't write denominator. Numerator. So, you have to somehow, so you have to somehow come up with a way of extracting that from the fraction.

Michael (Then?) she's writing numerator and denominator--

Researcher 3 (interrupts Michael above) So, think about it. I mean, you don't, you don't have to...

Researcher 4 What, I think what he's asking you is, is how is this triangle related to that denominator.

Michael Um.

Researcher 3 Okay, basically, the idea is to just, when you write formulas,
(bell rings)

Researcher 3: you don't write denomina--you can't--it's not fair. Somehow, it's not part of the rules (for) a denominator. So, what you have to do is, kind of, see if you can't come up with a way of writing exactly what you wrote...

Michael?: Okay.

Researcher 3: In terms of the triangle instead of denominator. Cause you're right, you want the square on one side and the triangles on the other side, but now you need, somehow--

Michael?: Well--

Researcher 3: To come up with a denominator--

Michael: And, but she, eh--her ferm--her formula has "keeping the same denominator in your answer." (read from screen)

Researcher 3: Right, so that's not fair either.

Michael: (during Researcher 3 above) That's, that's very long.

Researcher 3: That's right.

Michael: You need, like, numbers and...

Researcher 3: Right. So just, look, just play around with it, and I'll bet you can do it. (Okay?)

Researcher 1: (Conversation with Angela unintelligible)

Researcher 1: Is that when you were saying (the?) five sixths, for example, and you wanted to multiply it times the denominator, I'm just messing around with the five sixths, and I thought, "well, what's one minus five sixths."

Angela: (Is it?).

Researcher 1: That somehow, that gets you closer to having that six by itself.

Angela: You know what I mean?
<table>
<thead>
<tr>
<th>Time</th>
<th>Speaker</th>
<th>Text</th>
</tr>
</thead>
<tbody>
<tr>
<td>1007</td>
<td>Angela</td>
<td>Yeah.</td>
</tr>
<tr>
<td>1008</td>
<td>Researcher 1</td>
<td>I know, it's (no big?) help at all (unintelligible word) you.</td>
</tr>
<tr>
<td>1009</td>
<td>??</td>
<td>(The two laugh, and seem to say goodbye)</td>
</tr>
<tr>
<td>1010</td>
<td>??</td>
<td>Ah, excuse me Miss Abby, I'm very late.</td>
</tr>
</tbody>
</table>
The class opened with the teacher at the overhead projector with a list of paired values labeled square and triangle as shown below:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>9</td>
<td>-8</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
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<tr>
<td>1</td>
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</tr>
<tr>
<td>1/3</td>
<td>2/3</td>
</tr>
</tbody>
</table>

1. 00:00 (Teacher 1 and students: set up and sit at tables)
2. 00:24 Teacher 1 Do you have enough information?
3. Ankur Does negative eight equal nine?
4. Teacher 1 Does negative eight equal nine? Now if you’re at the point where you can predict an outcome, you should be able to tell me what the rule is. Let me give you a little time. I’ll leave these up here. See if you can come up with the rule.
5. Bobby Two minus one is one. Keep the two. Three minus two is one.
6. Keep the three.
7. Magdalena (inaudible)
8. Michelle I. What about the regular numbers? (whole numbers)
9. Bobby/Amy-Lynn
But you put a one under it.

Yeah, look, one minus two is negative one.

Then what would be the rule?

That’s just like saying with all of that with the fractions. Because if you just have this side, say it’s three fourths, how are you going to find out what the triangle is? That’s what we have to figure out.

You don’t know.

The denominator minus the numerator.

And you get one-fourth

Here, one minus one equals zero.

One minus one is one.

You put the one in the bottom. You go one minus two

So it’s like four over three. All you do is...

Subtract the numerator from the denominator.

Subtract the numerator from the denominator. The denominator minus the numerator.

The denominator minus the numerator equals triangle.

Rule is Denominator minus the numerator equals the numerator of triangle and denominator stays the same.

The denominator minus the numerator equals triangle.

(Demichelle I. writes what the others said:

Rule is Denominator minus the numerator equals the numerator of triangle and denominator stays the same.

looks at Michelle I.’s paper as she writes the rule). Check out these problems with the rule.

(Camera moves to Stephanie’s table)

Does three equal negative two? (to teacher)

Excuse me.

Does three equal negative two?

That’s what I’m saying. If it’s positive you subtract one and that number turns into a negative.

But the rule is you have to put it in a rule form such that the triangle is equal to something.

And a negative… it’s one… it’s positive

How does it equal a positive?

That’s the thing that’s weird. A half equals a half.

This one you have to add a zero and multiply by one. What do you have to do with this? (we have to do this)

Subtract one and add a negative.

( Camera changes to Bobby’s table)

How’d you do?

We can’t figure it out for the negative, but for the positive and for fractions so with a third you minus the denominator from the numerator and you get this numerator and then you keep the third the same and that works there. And if you have like a whole
number… You put two and one, and one minus two is negative one.

(Michelle I. wrote on her paper as shown below:)

---

Teacher 1 What makes you think that this one is like one you’ve had before?
Researcher 3 How about the one-zero one? Does that work? What would you do for the one-zero one? Is that working out?
Michelle I. Well, one minus one is zero

(Michelle I. wrote on her paper: 1/1 0/1)

Teacher 1 How about the zero one?

(Michelle I. wrote on her paper: 0/1 1/1 as shown below:)

---

Amy-Lynn Subtract the numerator from the denominator.
Researcher 3 And then put the
Michelle I. Actually we subtract the denominator from the numerator.
Actually it’s not. (she corrects herself)

Researcher 3 How about eight and nine, minus eight and nine?
Michelle I. See, we can’t get it to work with the negatives.
Researcher 3  Minus eight, nine. Then what did you do? You subtract. What do you do then?
Michael  We’re supposed to subtract this (she pointed to -8) from that. (she pointed to 1)
(Michelle I. has written -8/1)
Researcher 3  Subtract, minus eight from that one.
Michelle I.  No, it’s supposed to be one minus negative eight.
Researcher 3  So that would be nine over one, right. Does anybody know?
Bobby  Yeah, but can you have a negative fraction?
Michelle I.  But it’s negative nine. Does anyone know? And it’s regular (positive) nine (in the problem) so
Researcher 3  I see what you are saying. It should be negative nine in your instance. So that didn’t quite make it.

06:42  Michelle I.  Teacher1, Teacher1 could you give us like another negative for the square?

(Camera moves to Stephanie’s table.)

06:59  Romina  So a negative five would be a positive six. A positive five would be a negative six.
Brian  A positive five would be a negative four.
(Researcher 3 is at the overhead.)
Ankur  Negative seven equals.
Researcher 3  What does it equal? Hmmm
Brian  Positive five will be a negative four O.
Brian  Positive eight.
Ankur  Eight
Researcher 3  Eight.
(Researcher 3 has written two more rows on the overhead slide for this rule -8 and 9 and -7 and 8 as shown below:)

(Camera moves back to Bobby’s table.)
Taking away the negative eight so it turns into a positive, so it would be nine. Cause, say you have like eight dollars, say you owed eight dollars and you had a dollar and got eight dollars, you would have nine, not negative nine.

But it’s one minus eight.

You’re minusing a negative and the opposite of a negative is addition.

You’re right, Bobby.

Okay, so let’s try with negative seven. Negative seven over one. One minus negative seven is eight. Okay, we’re done.

That’s the second problem I got in a row!

(Michelle I.’s written rule: “denominator – the numerator equals the denominator of the triangle and the denominator stays the same”)
Amy-Lynn (says something about the inverse)

Bobby This should like get me into Harvard.

Michelle I. We got the rule.

Bobby’s table (inaudible conversation among table about their work)

Bobby And the last one I got it right too.

Michelle I. (hand raised) Ah, Mrs. Toye. We got it. Our theory works.

Teacher 1 Your theory works?

Michelle I. Even with negatives.

Bobby I proved it.

Researcher 3 It works with negatives?

Michelle I. (points to paper) Yes, cause if you have negative eight and you

minus one… One minus negative eight and it’s minusing a

negative so it’s like adding it. So it’s like one plus eight.

Researcher 3 Right.

Michelle I. So (writes on paper) it’s one plus eight.

Researcher 3 That’s nine… over one. Okay, now can you write? So you have this

numerator denominator problem, on how you write this equation.

It becomes difficult to write the equation.

(Bobby writes an equation involving box and triangle fractional

notation as shown below)

Researcher 3 Hmm. (He covers part of Bobby’s equation with his hand and

points to the box over box part) What is that equal to?

Bobby This? I don’t know hold on… So it’s equal to box over triangle,

triangle over box..

Researcher 3 So you’re trying to define this as two different numbers over here,

instead of just one. That’s the problem they ran into with the other

problem you’re left with from yesterday.

Bobby Oh yeah but we found that one out.

Researcher 3 Did you? Great well I look forward to hearing about that in a

second here but uh…

(Speaks to Teacher 1 as she walks over.) They kind of got

something that works. Does it work on seven minus seven and

eight too? I think so.

Amy-Lynn Uh-huh.

Researcher 3 But it’s hard to express.

Teacher 1 It is hard to express, okay.
Michelle I. It’s sort of like this. (Michelle I. points to what she has written) It’s like the denominator minus that, minus the numerator equals this numerator, but I don’t know how to write that. This one stays the same, this one, and then that one. Actually these are two different numbers so they can’t be represented the same way.

11:01 Bobby Use a circle.

Michelle I. And then the circle goes here.

(Michelle I. rewrites her equation using circle as shown on her paper below:)

Teacher 1 Ah, so you’re throwing in another variable?

Researcher 3 They’re trying to set up the square as two variables; one square is the numerator, a square numerator and a square denominator.

Michelle I. And they are not the same.

Teacher 1 How about if I give you a negative fraction.

(Speaking to the class) I am going to give you a negative fraction. If the box is equal to negative one-fourth, now think, if you can possibly predict these outcomes that were coming out here and some of you were (she points to the overhead with the list of corresponding box and triangle values.) If I’m going to input a negative one fourth? What do you think?

(Teacher 1 writes ‘-1/4’ on the overhead as shown below:)

(174  Michelle I. It’s sort of like this. (Michelle I. points to what she has written) It’s like the denominator minus that, minus the numerator equals this numerator, but I don’t know how to write that. This one stays the same, this one, and then that one. Actually these are two different numbers so they can’t be represented the same way.

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(Teacher 1 writes ‘-1/4’ on the overhead as shown below:)}
Students (unidentified students call out possible answers)

Positive three-fourths, negative one,

Students Five, negative three-fourths?

Teacher 1 No.

Brian Five?

Teacher 1 Five?

Michael No, not five, has to be a negative.

Teacher 1 Michelle, use your rule on it and see if you can come up with a number.

Michelle I. It would be one and a fourth.

Teacher 1 You’re right.

Michelle I. Yes! See our rule does work.

Teacher 1 I’ll take any rule that’s going to produce these over here.

Brian That’s dumb. That’s dumb!

Bobby Could we put our rule up?

Teacher 1 If you have it in a form, in some kind of a

Michael It’s like this. Box times negative two minus box.

Angela Do you see a similarity?

Sarah Yes, we do see a similarity.

Angela If you had negative eight, nine, negative nine, eight. Maybe if there were a negative nine it would be eight. Mrs. Toye, could you give us negative nine?

(Angela has written on her paper shown below:)

\[
\begin{array}{c}
\frac{2}{3} \quad \frac{1}{3} \\
-1 \quad 2 \\
-8 \quad 9 \\
-\frac{1}{4}
\end{array}
\]
Sarah: Yeah it’s up there Angela.
Angela: Negative nine?
Sarah: Oh, no.
Angela: Mrs. Toye could you give us negative nine?
Sarah: Negative nine would be ten.

(Camera returns to Bobby’s table)

14:14 Researcher 3: It’s hard to write if you were trying to separate the numerator and the denominator, right? Try and keep the value without using, without separating the numerators and the denominators. It’s tough, but you got the rule. And it might be easiest to do that if you just start anew. Just try to think of it again.
Michelle I.: Okay, hmm
Amy-Lynn: We could use different variables. It’s only way because, what if you did.
Michelle I.: This would work, but I don’t think he wants us to separate these, like they have to stay on top.

(Michelle I. has written shown below:)

\[ \frac{A}{B} = \text{Answer's denominator} \]
\[ \frac{O}{\Delta} = \text{Numerator} \]

\[ A = \text{Answer} \]
\[ \Delta = 4 \]
Michelle I.  One minus one equals…
Bobby  Put a one inside one of the squares and put a two inside one of the squares. Wait, how about this? Two variables. If it’s like that it’s a one, if it’s a fraction.

(Bobby’s paper is shown below:)

(Bobby shows his paper to the girls)

(Taking Bobby’s paper) Hold on, I’m looking at these dollars. Do you mean this?
(She points to top of paper where Bobby had written: 8 dollars)

No, at the bottom. (she points to the equation)

I think he wants, like these things still on top.

How about this? This triangle minus the square, triangle…

(Amy-Lynn wrote on her paper shown below:)

(Bobby shows his paper to the girls)

-denominator minus the numerator equals the answer.
Michelle I.: But you don’t know what the triangle is.

Amy-Lynn: This could be the numerator.

Michelle I.: You’re trying to find out, like, what the triangle is.

(Camera moves to Stephanie’s table)

16:17 Student?: Box minus one, one half minus one equals

Romina: The negatives and the positives.

Jeff: Then switch from what it is to the other thing. Sounds right.

Ankur: Positive to negative, negative to positive.

Romina: See, these are both positives.

Jeff: I know. That’s supposed to be a negative and you switch it like we did all the other ones. You get it?

Romina: One fourth minus one

Ankur: One fourth. One fourth minus one equals. What does it equal?

Romina: It’s negative one fourth

(Romina wrote on her paper shown below:)

Ankur: Yeah, okay. Minus one

Romina: Has to be negative one fourth and one and one fourth. Okay it works, and switch the signs.

(Romina wrote ‘☐ -1 and switch from positive to negative’ on her paper as shown below:)

(Camera moves to Bobby’s table.)
Michelle I. Cause when you have a whole number you have to put one underneath, so we’re stuck with the fractions, the denominator.

Teacher 1 Okay, put down over here, box is two sevenths, two-sevenths. All right, now use a rule on there that you think is going to get the right answer.

Michelle I. Seven minus two equals.

Amy-Lynn Five over seven

Michelle I. Equals five and then you just put the seven and so it’s just like…Could we use like three variables, three things? Cause this one checks?

(In the background you hear, “Teacher1, we’ve got the answer.”)

Teacher 1 Okay, you got the triangle. That’s what it would be if you used my rule.

Michelle I. Cause if you go like this, and then you have the circle here, then the circle represents a two. And all you have to put is this minus circle equals triangle over.

(Michelle I. has written on her paper as shown below:)

Teacher 1 The triangle has to be the answer, Michelle. So when you come up with this, with the triangle over the box, it’s really not the answer. However, your thinking is very, very, very good there. I know exactly what you just developed. I know exactly what you just did and you can work your way out of it. I know you can.

Amy-Lynn What did we develop? What did we develop then we can we work our way out of it. (laughing)

Teacher 1 I think I know what she developed. I’ll point it out a later.

Amy-Lynn We did it by ourselves. (said to other group of students)

Teacher 1 My rule is not that complicated.

Amy-Lynn Can we use three variables though, or just two?

Teacher 1 My rule has only one operation.

Amy-Lynn Two or three variables?

Teacher 1 Just two.
Bobby I have a good one. Listen. Box minus one and switch from positive to negative.

(Bobby wrote on his paper shown below:)

Amy-Lynn What if we switched negatives, positive to negative?

(She pointed to the + and – in parenthesis as shown above.)

Michelle I. What if there are no negatives? I don’t know.

It would be like one minus one equals

Amy-Lynn negative one half and we switch it to a positive.

Bobby And two-thirds minus one is negative one-third and you switch it to a positive. It’s one third.

Michelle I. Well, that is not like what we’re doing and she said we’re on the right track.

Bobby I know… (inaudible words as Michelle I. laughs)… If that doesn’t work ….

Researchers 3

Does it work here?

Ankur And multiply the number by negative one.

Researcher 3 Does it work up here?

Ankur Uh hum.

Researcher 3 So is that what you mean here?

(Researcher 3 has written and lightly circled words on Ankur’s paper as shown below. The paper shows: -1 and switches from positive to negative or negative to positive = △.)
Okay, so now try to write the equation. Honest to God.

Times negative one.

(Ankur wrote on his paper shown below:)

I found a bug the other day and it had the hardest shell in the world...

Okay, now check it out. See if that will work on everything.

What do you want to put in there?

Put in a few numbers.

Use a half. Half minus one.

Put it in.

Half minus one would be negative one and one half

Negative one half

Times negative one that would be one and one half or zero.
379  Brian  Where did Jeff go?
380  Ankur  Half minus one equals negative half.
381  Jeff  Negative half times negative one makes it a positive half.
382  Ankur  So it works.
383  Jeff  For someone who wasn’t even supposed to be at the table, I came…
384  Brian  You lent us your pen.
385  Ankur  Brian just sat here.
386  Romina  He did more work than you.
387  Brian  No.
388  Ankur  You didn’t do anything. You’re just sleeping (to Brian).
389  Brian  I came up with the multiplier.
390  Jeff  No you didn’t.
391  Ankur  You just said multiplier. You didn’t even find out the number.
392  Jeff  And I said by negative one.
393  Ankur  No he said negative one.
394  Brian  I did.
395  Ankur  He did.
396  (Jeff and Ankur go back and forth with ‘I did’ and ‘He did’.)
397  Brian/Ankur  (pointing to Jeff) He did.
398  Jeff  I did.
399  Brian/Ankur  He did.
400  Ankur  I am (not) talking to that guy.
401  Romina  Let’s write all of our names
402  Brian/Ankur  He did.
403  22:05  (Camera moves to Bobby’s table)
404  Teacher 1 That’s right Michelle. That’s right.
405  Researcher 3 So you were actually the first ones to get it but it’s
406  Michelle I.  We can’t like write it out.
407  Researcher 3 It’s hard to write it out, yes.
408  Teacher 1  Where is the rule?
409  Michelle I.  (points to her paper, specifically to the \[-\frac{2}{7} \ 1^2/7\] section,
410  as shown below:)
Hm. It’s like…

You need kind of instructions to tell people what to do.

Think Magda, think.

(Video moves to overhead projector where Sarah, Angela, and Michael show their rule.)

Mike want to explain what you think your rule is?

I’ll write it I have nice handwriting.

Actually I did it all.

(Angela wrote on the overhead:

\[ (\sqrt{-1})^{-1} = \Delta \]

\[ \frac{\Delta}{1} + (-1) = 0 \]

Plus one

Plus one?

Plus regular one.

(Angela then changes the second equation by removing the parenthesis around -1 and removing the negative sign:)

Okay, what do you have underneath there? What do you have underneath there, Angela?

What do you mean?
Sarah: The inverse
Angela: That’s the inverse.
Teacher 1: Now, who can explain this?
Michael: Which one should I explain?
Teacher 1: Well, explain my rule first.
Michael: The first one?
Teacher 1: Ahuh.
Michael: Okay. Box minus one. Which one should we do?
Angela: Do…
Michael: A half. Box minus one is one and a half.
Teacher 1: (Michael writes $1 - 1\frac{1}{2}$ on the overhead and Angela corrects him below:)
Michae: Minus one is negative a half.
Teacher 1: Negative a half.
Michael: (Michael crossed out the 1 so it is $-\frac{1}{2}$ as shown below:)

Plus, and then it’s times a negative one which brings it to a regular half. When you times something by a negative one if it’s positive, if the number you’re dividing, times-ing.
Angela: A negative times a negative turns into a positive.
Michael: A negative times a positive turns and then so…Any questions?
Magdalena: I don’t get it.
Michael: Okay…
Angela: Try it with two. Two minus one is one, okay? Times negative one .
(Angela shows the work when box is ‘2’ and wrote on the overhead:)}
Okay, I think you’re having a little trouble with your notation here. What you have over here, perhaps what we wanted to do, instead of making it look like, perhaps this might be a negative. If we did that it would look like what we wanted to multiply by negative one. (Teacher 1 goes to overhead and adds parenthesis around the negative one in the rule, as shown below:)

Oh, I’m sorry.

Now go over this one again because Magda is having a lot of trouble. You know how she is. Go ahead. (laughter)

A half minus one equals negative one-half. Okay, then multiply it by negative one and that equals positive one-half ‘cause negative times a negative equals a positive. (Angela wrote the work for box = ‘½’ as shown below.)
Michelle I. Can you do it with some other numbers besides one-half?

Do nine. Amy wants to do nine.

Angela Nine minus one equals eight, times negative one equals negative eight.

(Angela wrote the work for box = ‘9’ as shown below.)

Amy-Lynn Can you do negative one?

Angela Negative one? All right. Negative one minus one equals zero

Sarah It doesn’t work.

Angela Times negative one

(Angela wrote the work for box = ‘-1’ as shown below.)

Teacher 1 Does anybody see any kind of problem with the reasoning here?

Michael Negative one times negative one, no minus one.

Sarah No, it’s times negative one.

Michelle I. Isn’t negative one minus one negative two or is it just me?

Michael It’s two times

(Angela corrected her work as shown below.)
Researcher 3: All this looks pretty confusing to me.

Teacher 1: Wait, it turns out negative first.

Michael: Take out a negative. It’s negative two, (Angela put a negative sign in front of the 2 as shown below.)

Researcher 3: I’m totally confused here.

Teacher 1: Did they convince you, Magda, or what?

Magdalena: No

Ankur: Why does it work?

Angela: Why does it work? Ankur, we don’t like the word “why.”

Student: Negative three-fifths...

Michael: Negative three-fifths minus...

How does it work?

Michelle I.: Well we do cause we don’t understand.

We want to know how does it work.

Michael: Negative three-fifths minus one equals, equals negative one and three-fifths.

(Michael wrote work for box = ‘-3/5’ shown below.)

Jeff: Why does three fifths minus one equal negative one and three-fifths?
Michael: Because like subtract a negative from a negative it goes up you know the number line.

Jeff: Why? I want to understand you on that.

Michael: Like if you do negative one minus two equals negative three.

Brian: I’m kind of confused here!

Michael: Okay, negative three fifths minus one equals negative one and three fifths.

Researcher 3: But I don’t see the equals.

(Michael added an equal sign as shown below.)

Michael: Okay, you agree on that.

Michelle I.: Wait. No, no it’s two fifths.

Michael: Minus one, it’s not minus one fifth.

Ankur: It’s minus one.

Researcher 3: Hold up, does everybody agree with what that equal sign up there?

That’s okay, right.

Researcher 3: Okay, you got that. Now what do you do?

Michael: Then you multiply it by negative one.

(Michael added ‘x – ‘1’ as shown below.)

Researcher 3: But wait, wait, wait, you’re multiplying one side of the equation by negative one and the other side is not being… Don’t you have to do the same thing to both sides of an equation, of an equal sign?

Michael: Oh, let’s cross this out.

(Michael crosses out what he just wrote, ‘x – 1’ and it is shown below.)
Okay, the answer is negative one and three-fifths times negative one.

Do you know what that equals, don’t you?

No.

It equals one and three fifths.

Why?

Because a negative times a negative equals a positive.

And one times one and three fifths equals one and three fifths.

So you are saying your answer for negative three-fifths is one and three fifths?

Yes.

Hold on a second. I want to check this. (to herself she says) Eight over five, that’s one. Five minus…

(Michelle I. checked the results on her paper as shown below.)
Michelle I. All right. You’re right.
Amy-Lynn You’re right.
Michelle I. Your thing works. Oh sorry. It works. I understand.
Amy-Lynn Can we show ours now?
Bobby I am not going up to explain though cause I thought of it. I have to sit here and think.
Michelle I. Can we go up now?
(Michelle I. and Amy-Lynn went to the overhead projector.)
Researcher 3 Does everybody agree with that? That that works okay?
Angela Any questions? Anybody?
Researcher 3 We have more sheets.
Teacher 1 You might want to preserve this on a sheet.
Researcher 3 This (referring to Michelle I. and Amy Lynn’s work) is different from what you have done, but it works. See if you can figure out why.
Michelle I. We have an answer but we don’t have, like, how to write it out. So I’ll just give you an example.
Teacher 1 Take a look at it and maybe we can examine what their thinking was.
Michelle I. See you do the denominator minus the numerator, which is one, and you keep the denominator, three. So that’s your answer.
(Michelle I. wrote out the results for box equal to 2/3 as she explained, as shown below.)

Student Does it work for anything else?
Michelle I. Yes it does
Student Does it work with one half?
Michelle I. Yes, it does. One half, two minus one is one. You take the denominator, two. (Michelle I. wrote on the overhead, shown below, as she explained.)

Now like if you put negative eight, negative eight over nine, it’s nine minus negative eight, so it’s like you’re adding eight, so shouldn’t it be a one up there.

Sorry about that. (Michelle I. corrects negative eight-ninths to negative eight over one.) So it’s one minus negative eight so it’s like one plus eight, so that’s a nine. That’s over one. So the answer’s nine. (Michelle I. wrote as she explained for box equals ‘-8’ as shown below.)

Yesterday I just thought about this, the denominator minus the numerator and you said I couldn’t write it down ‘cuz the denominator, so you can’t write it down like that, the denominator in the equation.

How about D and N?

That’s right. She has no way of writing it as an equation like you folks did with your example, yet it’s right, it works every time.

If we could use three variables we could write it out, but we’re supposed to use three variables, ‘cuz if could use three variables, it’s like circle minus box

She’s using that circle as a variable not a zero. (Michelle I. wrote on the overhead shown below.)
Michelle I.  It’s like, you know, it’s like representing them as a numerator, I mean denominator.

Teacher 1  Okay, now how about a rule from this table? (table 2)

Researcher 3  I think they had the same thing as Michael

Jeff  We got the same thing as Mike.

Teacher 1  Aren’t you curious as to what my rule is? Neither one of those is the rule that I had.

Student  Neither?

Student  What is your rule?

Teacher 1  I told you I used only one operation on my rule. My rule was one minus box equals triangle.

(Teacher 1 wrote her rule on the overhead.)

Student  That was it?

(Ankur looks at his paper)

Teacher 1  Now I do believe that the analysis that that table gave right there is some kind of rule that we use when we’re working with certain kinds of numbers. Michelle.

Michelle I.  but I had, like, box as the numerator, so I didn’t put like you know…

Amy-Lynn  We used it as a fraction instead of a whole number.

Michelle I.  Oh, that was good (as she slaps her hand on her forehead).

Teacher 1  So, I didn’t expect it to take you that long to get to it. I really didn’t. I thought somebody would see the rule before that. But

Researcher 3  Is that rule, Michael, is that rule the same as your rule?

Michael  No

Researcher 3  How do you get from your rule to her rule?

Student  You simplify probably.

Michael  Minus

Angela  See we have like the box.
Michael  Box minus one times negative. (Bell rings)
(teacher 1 wrote on the overhead:)

Angela  Yeah and then you had to multiply.
It’s sort of like it but not really. It’s more complicated.

Teacher 1  What do you think?

Angela  You know how some people make everything complicated. (not sure of dialogue here)

Teacher 1  Ah huh. What would make this rule kind of like this rule over here?

Angela  If you switched the box and the one, like in the parenthesis there, but like kept the minus sign there?

Researcher 3  So it’s minus one plus box?

Angela  No, one minus box.

Researcher 3  But you have to somehow get those two things to look alike.

Teacher 1  Let’s go back to a property, all right? Are you ready for this one?
How about this?
(Teacher 1 writes the word “distribute” below her rule.)

Teacher 1  All right how does the distributive property apply to this? Amy?

Amy-Lynn  Negative one times box is negative one box, negative box, and then negative one times negative one is one, so it’s plus one equals triangle.
(Teacher 1 wrote on the overhead as Amy-Lynn spoke.)
Now we have only two terms on the left. Now, when you are trying to get to this rule over here (she points to the rule \(- \Box + 1 = \triangle\), and you want to write it exactly this way (meaning \(1 - \Box = \triangle\)) what property can I incorporate to allow this left side to look exactly like that left side?

Teacher 1: Just switch it around.

Student: Commutative?

Teacher 1: Commutative. Okay

Michael: So it would be positive one, minus, plus negative box and

Teacher 1: So it would be positive one plus negative box. (Bell rings)

(Teacher 1 has now written on overhead, another line starting with “Commutative”)

And we can switch the (\(?\)), plus something plus a negative. No one says plus a negative box, you change it to just minus box.
(Teacher 1 wrote the equation using the commutative property as shown below.)

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Teacher 1: Okay, so you think that it’s perfectly legal now to draw this conclusion from here?

Michael: Yes.

Teacher 1: Why? What do we know?

Amy-Lynn: You can take the sign out after the one. Because you can just change it to one minus box. And then you just minus box you pretend is negative (?), you just minus it.

Teacher 1: Because I think we agreed that adding the negative is (pause) Is it equivalent to this?

(Teacher 1 points to the lines:

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Teacher 1: Okay, so actually your rule wasn’t too far off, except for a little distributive change here and a little commutative change here. Right. Not too bad. I know exactly what went into that group over there but I don’t think I am going to share. Now that rule yesterday that you were going to turn into, you were supposed to turn it in so that it was the inverse. Right? And Robert would like to share.

Bobby: I didn’t think of anything.

Teacher 1: Oh, you didn’t think of anything? I thought you did.

Amy-Lynn: He’s camera shy.

Teacher 1: Is he?

Bobby: I like think of the stuff and they like explain it.
Camera shifts to Bobby’s table where Amy Lynn is writing.

Well you better convince somebody else of it who can explain it.

Explain it to them.

Explain it to them and maybe they will do it for you.

What was the rule anyway? I wasn’t here for the very end of the class because I was concerned about the other class. What was the rule you were using anyway? What rule did you come up with for it?

Who remembers the rule?

Square divided by the next highest number.

It’s square divided by square plus one.

Put that in parenthesis.

Equals triangle.

Who remembers the rule?

He explained it to us. It’s square divided by, and then in parentheses

Divided by and parentheses under

Well, actually you don’t have to put them in parentheses if it’s underneath…square plus one equals triangle.

(Michelle I. wrote on the overhead as shown below.)

Is that easy? And you are now looking for an expression that’ll give you the box is equal to…the other way.
Did anybody give this some thought yesterday between the time that you had and the time that you came back today? We’re looking for the inverse expression so that you can find the box if you’re given the triangle.

Ankur: Triangle minus one equals box?

Teacher 1: I don’t know. I wasn’t working on this rule.

Bobby: It’s triangle times square plus one? Six sevenths times like…

Michelle I.: But you don’t know what square is. It’s like you’re turning it around so you don’t know what square is.

Researcher 3: Write it down. Now remember what we want now is box equals something, something in terms of triangle.

Bobby: (Bobby wrote this on his paper.)

Say this is like six sevenths. Six sevenths times square plus one so it’s seven equals twenty two sevenths (screenshot was illegible, Bobby uses $\frac{6}{7}$ for triangle and 7 for square and wrote on his paper $\frac{6}{7} \times \frac{7}{1} = \frac{42}{7} = \frac{6}{1}$. Then he crosses out $\frac{6}{6}$ and writes $\frac{7}{1} = 7 -1$.)

Michelle I.: It’s like you’re turning it around. Before we didn’t know what the square was. Now we don’t know what the triangle is. That may work but you don’t know what box is.

Bobby: So the other one is like better.

Magdalena: It’s six. You got seven. It’s supposed to be six.

Bobby: I know I said minus one.

Magdalena: Oh.

Michelle I.: So.

Teacher 1: We are now looking for the rule that will now give you the box if you have the triangle.

Ankur: We need a third variable.

Teacher 1: That’s exactly what Michelle thought but as you found out from Michael you didn’t really need the third variable.

Ankur: For this one do you?

Teacher 1: Well, if you think you need it. You’ll have to convince me.
(Stephanie is flipping back her hair. Romina begins to write.)

(Romina wrote the rule on her paper shown below.)

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\frac{\Delta}{\triangle - 1} = \frac{\square}{\triangle - 1}
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Romina
Triangle over triangle minus one? Box over triangle minus one?

Brian
No.

Stephanie
Why would you put box in the answer if it’s box over triangle minus one? Why don’t we use numbers. Maybe it will make it easier.

Stephanie
Let’s see. Box, wait, triangle over triangle minus one. If the triangle is zero, zero divided by zero equals zero.

Ankur
Triangle minus one will work.

Romina
Hold one. Say the box is five. Then five and then

Ankur
Five over there.

Stephanie
Why don’t you just use zero?

Romina
And then minus one would be like adding one so wouldn’t it be six minus one equals five? Hold on, never mind.

Stephanie
Hold on. Can I? Triangle over triangle. If the triangle is zero and the box is zero because it’s zero and zero. Zero minus one equals negative one. And then zero times anything is zero. So it equals zero.

Ankur
So, can you write it out?

Stephanie
Me? No.

Romina
Now you think it’s this minus one?

(Romina wrote box equals triangle over triangle minus one as shown below)

\[
\frac{\triangle}{\triangle - 1} = \frac{\square}{\triangle - 1}
\]

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Stephanie
I don’t know. I have to test it with others.

Romina
Okay. Let’s test it.
Stephanie: One. If the one is the box, so the answer is one, then the triangle is what, a half.

Romina: A half minus one.

Stephanie: Is negative one half.

Brian: I thought it was equal zero.

Stephanie/Romina: Times one. Times positive one.

Stephanie: We have to get one for an answer.

Brian: What do we have to get? A half?

Stephanie: No, we have to get one as our answer because triangle times triangle minus one.

Romina: Isn’t that what you just said?

45:00 (Camera shifts to Bobby’s table for a few seconds)

45:08 (Camera is back at Stephanie’s table)

Researcher 3: Why don’t you put down what you get for this.

(He points to Romina’s paper shown below.)

Romina: Is it right?
Researcher 3: Well, try it out. You don’t really know. Do a bunch of them and see what your table would come out with if you start with triangle as one-half, two thirds, three-fourths. Just try it out for a bunch of them and then put down the corresponding thing over here. I mean don’t be discouraged if the first one comes out wrong. Just do a bunch, see what happens.

Romina: Hold on. (Romina crosses out something on her paper)

Researcher 3: Okay, just try it out. This is your… You guessed that, okay. If you had one-half here what would you get?

Romina: That would be one-half over one-half minus one. That would be negative one and one-half, and that wouldn’t come out.

Researcher 3: No, you have one-half and this down here, one-half minus one is what?

Romina: Negative one and one-half?

Researcher 3: Negative one half, okay, so you get negative one-half one half over negative one-half, and what does that equal?

Stephanie: So we’re dividing a negative one-half by a positive one-half. So it’s a negative, isn’t it?

Ankur: Equals one over,…no.

Researcher 3: What’s one half over a negative one-half?

Ankur: One-half over negative one-half. One.

Researcher 3: One?

Ankur: One-half over negative one-half. One. Zero.

Jeff: One, zero?

Researcher 3: Keep trying. You’ll hit it eventually. (he laughs)

Ankur: Negative one?

Researcher 3: Well which one is it? Pick one.

Stephanie: Negative one. ‘Cuz it comes out two over two but one of them is negative.

(Stephanie wrote on her paper shown below which shows multiplication of one-half and two over one which is equal to two over two which is equal to one and the paper shows negative one.)
Researcher 3: What is one over minus one?

Ankur: One over minus one.

Researcher 3: What is that? What does that equal?

Ankur: It equals one over minus, negative one.

Researcher 3: Can you make it a simpler?

Ankur: No.

Researcher 3: What’s two over minus one, what’s two over one?

Ankur: Two over one, two.

Researcher 3: Okay. What’s two over minus one?

Ankur: Three.

Researcher 3: Two over a minus one?

Ankur/Brian: One.

Researcher 3: I have this. Two over minus one. What does that equal?

(Researcher 3 wrote:)

Ankur: A half.

Romina: One. One. Negative one.

Stephanie?: No negative one.

Researcher 3: What am I doing here?

Ankur: You have two divided by negative one.

Romina: How many times does one go into two?

(Romina points to Researcher 3’s paper.)

Researcher 3: How many times? Two times. But there’s a minus sign there.

All: Negative two.

Researcher 3: So that equals minus two?

(Researcher 3 added -2 to above equation.)

Ankur: Ah hmm.

Researcher 3: And so one over minus one equals

Voices: Negative one. (Researcher 3 added ‘-1’ to his paper.)

Researcher 3: Okay, and one-half over minus one-half?
(Researcher 3 wrote the fraction on his paper shown below.)

Ankur Equals negative half.

(Researcher 3’s work shown below.)

Researcher 3 Equals…

Voices Negative a half. (pause) Negative a half. (pause)

Negative one? Negative one?

(Researcher 3’s added ‘-1’ as shown below.)

Researcher 3 How many times does one-half go in one-half?

Romina One.

Researcher 3 Okay. So now we’ve got this and you said that this was one-half

over one-half minus a half, which is… one-half minus one-half is

negative one-half. So we have one-half over minus one-half, and

we know that equals minus one, right? So we now have one-half

and we have a minus one.

Stephanie You have to plus a positive two to get a positive one.

48:22 Researcher 3 So now let’s try some other before we jump to any conclusions,

here, right. What’s another number up here? Try a few more of the

triangles and see what you get.

Romina One over one minus one, that would be..
Researcher 3  Well we don’t have that triangle up there that says one, so we
1000 won’t do that one. Try one of the other ones. Two-thirds over two-
1001 thirds minus one.
1002 ((Romina wrote one over one minus one on her paper and then
1003 erased it, and wrote two-thirds over two-thirds minus one on her
1004 paper as shown below.)

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1008 Researcher 3  Okay, see if you can figure out what that comes out to.
1009 Ankur  Use your theory to figure it out.
1010 Romina  My theory?
1011 Researcher 3  You can figure it out. You guys know how to do this, right?
1012 Stephanie  Wouldn’t it just be negative two-thirds.
1013 Researcher 3  What’s two-thirds minus one?
1014 Brian  Negative one-third.
1015 Ankur  It’s negative two. The answer’s negative two.
1016 Researcher 3  Two-thirds, oh the whole answer? Well, just do it in stages for me
1017 because I can’t work that fast. Two-thirds minus one is?
1018 Ankur  Two-thirds minus one? Negative a third.
1019 Romina  Negative one-third?
1020 Researcher 3  Negative one-third? Okay so now you have two-thirds over
1021 negative one-third.
1022 Ankur  Two-thirds over negative one-third. Two. Negative two.
1023 Researcher 3  Negative two, okay.
1024 Stephanie  What if you added the one instead of subtracted it, would that
1025 make it positive?
1026 Researcher 3  Okay, you got two-thirds and now you got minus two. Try one
1027 more and you might, you know, you might become…
1028 Stephanie  What if you added the one instead of subtracting it, would that
1029 make it positive?
1030 Researcher 3  So this is delta so now that is square so something’s not quite right,
1031 right?
1032 Romina  What if we do do that?
1033 50:00 Ankur  What if we add one instead of minus one?
1034 Romina  Okay, two-thirds, two-thirds, that would be one and one-third
1035 plus… No one-third, right? Negative one-third, no, it’s so
1036 confusing. Two thirds plus one, two thirds plus one is
Romina wrote on her paper:

\[
\frac{2}{3} + \frac{1}{3}
\]

Researcher 3: What’s wrong, you have to look at what’s wrong with the answer and then figure out how to change your formula to make the answer right.

Brian: It’s not supposed to be negative.

Researcher 3: Okay, then how do you make something that’s negative, positive?

Romina: The last problem we just did, we multiplied by negative one.

Stephanie: All right. We’ll, try it.

Romina: Multiply this by negative one. It’s positive two.

Ankur: All right. Could we do that to every problem?

Romina: I don’t see why not. See, triangle over triangle minus one times negative one.

Ankur: Can we simplify that?

Stephanie: No, let’s not try. Let’s just leave it because…

Romina: I like it like this.

(Romina wrote the inverse on her paper as shown below.)

Ankur: If we can’t simplify

Brian: Shut up.

Ankur: (inaudible)

Romina: Brian shush.

(Camera shifts to Bobby’s table)

(Bobby has written on his paper as shown below.)
Teacher 1: ...using the triangle to split, to only represent some of what you said the triangle was here.

Michelle I.: Yeah, it’s like a law.

Teacher 1: Ah ha. What you might want to do is give it another variable name or incorporate something else in here so you can do something like that. Because right now this is not going to make sense. Your notation doesn’t make sense. Your idea is fine. Your notation is not.

Bobby: What is two-thirds times three?

Teacher 1: See this right here (Teacher 1 pointed to ‘(2/3 x 3)’)

is not being represented by this down here. That’s the difference, okay. This is a good idea. This is not representing it.

(Teacher 1 pointed to ‘(Δ· —)’.

Michelle I.: That doesn’t work for six-sevenths.

Teacher 1: Well, let him try it, let him try it.

(Bobby wrote on his paper as shown below.)

Michelle I.: I already did.

Teacher 1: Wait a minute. Show me.
Triangle over triangle minus one. We get negative. That’s why we had to multiply by a negative one to get a positive.

All right. Where are you multiplying by a negative, just the denominator or the whole final answer?

(Romina has written triangle over triangle minus one times negative one as shown below.)

The final answer.

Oh, the whole final answer. Are you using a whole negative one then? A whole negative one.

Yeah, like for this one, because we got for one-half, we got negative one as our answer, and all you have to do is multiply it by negative one and you get a positive, same thing happened here, we kept on getting negatives.

Okay, so if you input one-half in the rule that you originally had you get a negative. But then if you use this idea…

You get a positive.

Ah, something that was borrowed perhaps but was good, right. Why reinvent the wheel right.

Borrowed?

Yeah, like we multiplied by one-half to get the number from the negative…

We had it for the last problem.

Oh, you had it for the last problem. Now maybe you want to clean this up a little, because it’s not doing exactly what you said it was going to do, I don’t think.

How about triangle over triangle minus one equals square times negative one?

(Romina wrote on her paper as shown below.)
Brian: Oh no.
Ankur: That sounds better.
Teacher 1: Oh but I thought we wanted the square to be isolated so that all the action takes place over here. (She pointed to the left side of the equation.)
Stephanie: Why don’t we do triangle over triangle minus one, and then the whole thing negative one, instead of just the denominator.
Teacher 1: Check it see if it works.
Researcher 3: Check it out. See if it works.
Stephanie: Oh box, I like couldn’t figure it out.
Ankur: Is there any way we could simplify this?
Teacher 1: That was a good question Ankur. I think you might be able to simplify that. I don’t know if you have enough rules
(Several talk at once)
Romina: We want to have action on both sides of the scale. That’s the way we want it to be.
Teacher 1: Now wait. Take a look at that rule over there. All the action takes place on the left to get the triangle.
Romina: But we want action on both sides. We think it will be even for both triangles.
Stephanie: See, it works.
Teacher 1: Suppose I give you a job to do, and I say you’re going to take care of this whole job, and then you come back to me and say no you’ve got to do this part. See? Did you take care of the whole job? No. See?
Romina: Then we’ll just do this. Triangle over triangle minus one, and then we’ll times that whole thing by a negative one, equals square.
(Romina wrote on her paper as shown below.)
Teacher 1: Now all you have to do is you have to test it and see if in fact that’s what’s going on.
Romina: It works. We tested it.
Teacher 1: You tested it? Okay. That’s a good form.
Romina: Write really big on a piece of paper for everything.
Teacher 1: The only thing that Ankur asked me is
Romina: No we can’t simplify it.
Teacher 1: is some way could this be simplified in some way?
Romina: No, we don’t want to simplify it.
Stephanie: No, we like it like that.
Teacher 1: But Ankur doesn’t. He like his life a little simpler than that.
Stephanie: Get out Ankur, go away.
Romina: Well, if he likes his life so simple, just leave it the way it is?
Teacher 1: Let’s just take a look at what we have been able to do in other situations. If you think this could be simplified, perhaps you want to take a look at it with some numbers in here. Okay? Put some numbers that you like to work, with the ones that you can change because this is a variable, put some numbers that you like to work with and see if you can make a simpler expression out of that.
Romina: Five-sixths. And that has to equal five. Have a blast, Ankur.
Stephanie: But remember, the times negative one is on the other side. Here, I have it. Five over six equals five. Have fun. C’mom Ankur, we want you to simplify it.
Ankur: I’m thinking.
Stephanie: If you hadn’t opened your mouth!
Ankur: I just asked if we could. I didn’t say I wanted to.

56:35  (Camera shifts to Bobby’s table)
Michelle I.: Oh, wait a second. You have, if you do, if it’s something like two-thirds. If…three minus two is one and you have a third. It’s like that thing
Teacher 1: Yeah?
Michelle I.: The… and then a third divided by one, I mean one third over one is three.
(Michelle I.’s paper is shown below.)
(Bobby’s paper is shown below.)

Teacher 1: What kind of notation is this right here? What is this telling you to do?
Michelle I.: Divide a third by… But see you had to get to a third so now that we know how to get to a third, see but now we don’t know how to write it out again.
Teacher 1: (softly). This isn’t too hard for you, is it?
Michelle I.: No.
Teacher 1: You should be able to do this. Okay, this is a good idea here. No.
How does that relate to that if you could use that idea?
Michelle I.: It’s dividing again. Wow.
Magdalena: Shelly, look. Two-thirds over one-third. And when you divide that it’s six over three and that equals two and that’s the answer.
(Magdalena’s paper is shown below.)
A1 415

Michelle I.: Whoa! Try that with the other ones. Six-sevenths, no five-sixths.

Five-sixths divided by four-fifths, no, divided by one-sixth. What is that? Wouldn’t you, like, switch those around. Five times six over six. It works!

Teacher 1 (working with Bobby) Times numerator. It does. I agree with you.

It works.

Michelle I.: Mrs. Toye, we got something. Magda thought of it.

Teacher 1 Really?

Magdalena Oh my God.

Michelle She’s starting to think.

(working with Magdalena) Look let’s say it’s was with five-sixths, well to make it whole, let’s just say you do the thing to make it whole, divided by one-sixth. And then that’s thirty over six, which is five, which is the answer. But now we have to realize how to write it.

(Michelle I.’s paper shown below.)

Teacher 1 Again, the same problem.

Michelle I.: We have a lot of rules but we don’t know how to write it.

Teacher 1 I know your rules are fabulous. This rule writing isn’t that easy.

(Bobby’s paper at 59:46 is shown below.)
Michelle I. Maybe it’s like triangle
Magdalena No it’s going to be triangle over something.
Michelle I. Wait, wait, triangle divided by one. Wait, no triangle minus isn’t it?
Magdalena Triangle like that and then…
59:56 Teacher 1 Aha.
Michelle I. See, it’s triangle over something but you have to figure out how to represent that.
Magdalena It’s probably something so easy.
Well, it’s not that easy. If it were that easy you would…
Michelle I. If you divided a negative by a positive do you get a negative or a positive? A positive? A negative?
Teacher 1 A negative.
Michelle I. A negative shucks, I have it. You do this divided by triangle minus negative square, so then you just change it to positive (she changed the sign from negative to positive)
1:00:39 (Michelle I.’s paper shown below.)

Teacher 1 Aha. But in your notation you’re going to have to show that. See?
The way you just did that right now? There’s no rule in math that allows you to do that. You can’t just say “Well you just do this.” So you’re going to have to come up with some kind of a device that’s going to allow you to do that, something that you know about.
Michelle I. Can I see your paper a second? (Michelle I. takes Bobby’s paper but Bobby takes it back.)

Bobby Hold on. (Bobby finishes what he is writing.)

(Bobby’s paper is shown below.)

Amy-Lynn has been writing on her paper and at 1:01:24 it is shown below.

Researcher 3 How are you guys doing?

Bobby I have a couple of different versions (inaudible).

Researcher 3 Ah, yeah.

Michelle I. We have a couple of ways to do it, but we don’t know how to write it out into a rule.

Researcher 3 How about this rule (pointing to the rule on Bobby’s paper as shown below)? Does that work?
Bobby: Yeah.

Michelle I.: He has something like we have Magda. (meaning Bobby’s work).

Bobby: Yeah. It works. A half equals negative a half equals negative one times negative one equals one.

(Bobby shows his work when triangle is ½ as shown below.)

Michelle I.: Yes, that works. And times it by a negative one. It’s this. Negative triangle over triangle minus one equals square.

Researcher 3: So we have a half equals one. Yeah. That’s it.

Michelle I.: Mrs. Toye, we have it.

Magdalena: Oh my God.

Michelle I.: We have the same thing he does.

Researcher 3: How did you get it?

Bobby: Messed around like I did that. I just changed the square to a triangle.

Researcher 3: You what?

Bobby: I changed the square to a triangle and like I minused (inaudible).

Researcher 3: Ah.

Michelle I.: We got the same thing, but we thought of it on our own. Actually Magda did and I helped her along. Mrs. Toye, we got it.

Teacher 1: (Standing near the other table) I knew you were working at it.

Bobby: And like the other problem where they times-ed it by a negative one to get a positive, well I did that.

Magdalena: Oh my God, it feels good.
Teacher 1 (to Researcher 3) They have a simplified version that might be working.

Researcher 3 You guys have something that works.

Romina We think we simplified it. It’s not this (turning over the paper), but it’s this. It’s now basically this. Negative triangle over triangle minus one equals square.

Researcher 3 Okay, does that work?

Romina Yes, it works. We just can’t prove it.

Researcher 3 Well, if you can show it for everything up there, isn’t that, that’s as much proof as you need, I guess. Right?

Teacher 1 What will the denominator turn out to be?

Ankur For this, fourteen, for just this part, like this minus one, that will equal negative one-fourteenth.
Teacher 1: Okay, that’s going to be negative one-fourteenth. All right and so you’re going to have negative thirteen-fourteenths divided by negative one-fourteenth and how can you show me that that’s going to be thirteen?

Ankur: I don’t know.

Teacher 1: Well, write it out. See. Negative thirteen-fourteenths.

Ankur: Should I write under it?

Teacher 1: However you like to do it, divided by negative one-fourteenth.

Ankur: (repeating after the teacher) Negative thirteen-fourteenths divided by negative one-fourteenth.

Teacher 1: Okay. Now, what do you know that’s going to figure this out.

Stephanie: How do you divide fractions?

Brian: I don’t know.

Stephanie: It’ll be thirteen over one.

Romina: It’ll be fourteen over one.

Teacher 1: Which one?

Stephanie: This one (points to Ankur’s paper).

Teacher 1: Ah.

Brian: Thirteen over one.

Stephanie: It’ll be fourteen over one.

Ankur: Over one, one-thirteenth

Romina: One-thirteenth

Ankur: No, wait. Thirteen’s on top.

Teacher 1: (Romina’s paper is shown below.)

Ankur: (Romina’s paper is shown below.)

Romina: Thirteen over one, that is thirteen, and you subtract one, and that’s twelve.
Teacher 1: Wait a minute. Why would you be subtracting one?
Romina: I don’t know.
Ankur: It’s thirteen over one.

(Camera shifts to Jeff’s table)
Researcher 3: One over one minus one half, right?
Michael: Three-fourths. One minus three-fourths equals one-fourth, and that means one over one-fourth, and that equals four. Do we agree?

Minus one is equal to three and that’s the answer.
(Romina’s paper, showing inverse rule and work for triangle equal three-fourths, is shown below.)

Angela: Yeah, okay, it works.
Researcher 3: Try another one up there. Try six-sevenths. See if that works.
Michael: It works. We already tried it.
Angela: Yeah, we tried that already.
Researcher 3: Oh, you tried that one. That works.
Angela: Mrs. Toye
Micha: Four-fifths.
(Michael wrote his work for four-fifths as shown below.)

Researcher 3: You got it? Okay, so write out the formula. Oh, you have it written out over there? Okay, so you got it. Good.
Researcher 3: And you guys got it (at Bobby’s table).
Michelle I.: Yes we do.

(Camera shifts to Stephanie’s table)
Researcher 3: You guys had it too, right?
Teacher 1: We’re working on a division of fractions idea here. So this is going to work? So if I say three-fifths divided by one-fifth, I’ll get three.
If I say three-fifths divided by two-fifths, what will I get?
Brian: One and a half.
Teacher 1: One and a half. Good. I’m not even going to ask you what you’re doing, okay? Suppose I have four-fifths divided by three-fifths. What will I get?

Brian: One and a third.

Teacher 1: Good, Brian, you got it, didn’t you?

(Teacher 1’s work with students shown below.)

Researcher 3: Okay. Here’s a solution.

(Michelle I. and Magdalena at the overhead projector and Michelle I. wrote the inverse rule on the overhead as shown below.)

Researcher 3: Aah, so how’s that one?

Ankur: We have a better one.

Brian: Simplified.

Angela: Do they have the same one as us?

Researcher 3: They have a different one.

Brian: We’ll bring it up there and we’ll go show them.

Researcher 3: Okay, you’ve tried that out and it works. Okay. Brian, are you going to show us?

Brian: Is there a sheet up here or no?

Teacher 1: Do you have any more blanks?

Researcher 3: Let me give you a sheet
Put your names on your work guys, all your scratch paper.

(Researcher 3 places a blank slide on the overhead projector)

(Brian and Ankur go to overhead projector and wrote as shown below.)

[Image]

Researcher 3 So how’s that compared to this one? (Researcher 3 places Michelle’s inverse underneath Brian’s) What did you do differently from what they did?

Michelle I. Instead of times-ing it by negative one they put the negative triangle, so they would have to, like, do that, but it is the same thing.

Ankur Same thing.

Researcher 3 Okay, now for something completely different.

(Angela, Michael and Sarah come to the overhead projector)

Teacher 1 We need one more… wait don’t write on the glass. We will get another overhead.

Researcher 3 We may have another overhead. Let’s see. Yeap we do. You’ve done a good job cause this is the last one.

(Angela wrote on the overhead as shown below.)

[Image]

Researcher 3 So, what do you think, guys?

Bobby Is that multiplied by one?

Researcher 3 No, subtract one.
Brian: Ours is better.

Teacher 1: I would like to see a demonstration. Suppose the triangle is...

Angela: Can we use four-fifths?

Researcher 3: Would you like to use four-fifths? Okay, four-fifths is all right.

Michael: One minus four-fifths is one-fifth.

Angela: One over one fifth equals five.

Angela: Five minus one equals four.

(Angela wrote the work for triangle equals four-fifths as shown below.)

Teacher 1: This table is claiming that theirs’ is easier.

Michael: Yeah, but theirs’ is the same thing as that. Ours is easier. Ours is different. Why would you want to complicate your life? They had an extra step.

Michelle I.: All of them are sort of the same. Ours has one less step.

Researcher 3: I think this is just as easy as yours. The calculation is very easy.

Brian: Why does this work? Why is this like yours? What happened here?

Ankur: They have another step.

Brian: Ours has one less step. It’s easier.

Teacher 1: You know what I don’t get? I don’t get how if you subtract one it’s kind of like multiplying by the negative one over there and here by using the negative in the numerator. I don’t get it. I thought subtracting and multiplying were a lot different. That’s what I thought. And here’s I’m looking at this, and they just subtract one and they’re all right. And you just use a negative in the numerator...
and you’re okay. And you people just multiply by a negative one. And I’ll tell you what. If I went and subtracted one every single time somebody wanted me to multiply by a negative one, I don’t think I would get the same answer. I’ve had quite a few years’ experience. How could all these be the same?

Michael We’ll use one-fifth as the triangle.

Researcher 3 No wait, your triangle was four-fifths. Four-fifths was your triangle.

(Michael wrote the work when the triangle is four-fifths using Stephanie’s table inverse rule)

Teacher 1 It has to be positive, Mike.

Ankur Ours is the same thing.

Teacher 1 Do you want to explain why it equals four?

Ankur Isn’t negative four-fifths divided by one-fifth four? I mean, how many times does one-fifth go into four-fifths?

Michael Ours is better.

Ankur Why? Isn’t our simpler? Because you’re wrong. No, ours is simpler.

Michelle I. Do you remember how she said there can be more than one rule for each thing?

Researcher 3 When I look at theirs’, I see that I don’t have a fraction on top of a fraction which is kind of tough for me to figure out. theirs’ just has one over a fraction.

Michael Fractions are tricky.

Teacher 1 I have something for them to work on that is a change of pace.

(students play with their papers by making noise and crumpling up some and throwing them in the waste basket)

(Teacher 1 hands out a geometry problem to students to work on and their work is not recorded on this videotape).