HAUSDORFF DIMENSION OF INVARIANT SETS
AND POSITIVE LINEAR OPERATORS

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In this thesis we obtain theorems which give the Hausdorff dimension of the invariant set for a family of contraction mappings on a complete, perfect metric space. We obtain results for the graph-directed systems similar to that of Mauldin and Williams [17] but for more general contraction mappings which are “infinitesimal similitudes” rather than just “similitudes”. Also the underlying spaces are not assumed to be finite dimensional. For finite graph-directed systems our results are valid on any bounded, complete, perfect metric space. We use the theory of positive linear operators and generalizations of the Krein-Rutman theorem to characterize the Hausdorff dimension as the unique value of $\sigma > 0$ for which $r(L_\sigma) = 1$, where $L_\sigma$, $\sigma > 0$, is a naturally associated family of positive linear operators and $r(L_\sigma)$ denotes the spectral radius of $L_\sigma$. We also obtain theorems for infinite graph-directed systems (with finitely many vertices and countably many edges) on compact, perfect metric spaces. In this case too we have a family of positive linear operators $L_\sigma$ for $\sigma > \sigma_0 > 0$, and the Hausdorff dimension of the invariant set is given by the infimum of $\sigma > \sigma_0$ for which $r(L_\sigma) < 1$. We discuss an infinite iterated function system given by complex continued fractions and obtain lower bound for the Hausdorff dimension of the invariant set. Our estimate improves the lower bound to $1.787$ from $1.2484$ obtained in [16]. Finally we also give a theorem
proving the continuity of the Hausdorff dimension with respect to the contractions in
the graph-directed systems.
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Dedication

To my parents
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Chapter 1

Introduction

The thesis is organized as follows. In Chapter 2 we provide some preliminaries. We recall the definitions of Hausdorff measure and Hausdorff dimension on a metric space. We discuss iterated function systems (both finite and infinite cases), and prove the existence of an invariant set. Finally, in Section 2.4 we talk about self-similar sets. In Chapter 3 we introduce the generalized graph-directed systems and prove the existence of an invariant set list in this context. Chapter 4 contains the study of Perron-Frobenius operators related to the graph-directed systems. We use generalizations of the Krein-Rutman theorem to prove the existence of a positive eigenvector with eigenvalue equal to the spectral radius of the operator. Chapter 5 contains our main theorems giving formulas for the Hausdorff dimension of the invariant set list. In Section 5.1 we introduce the concept of an infinitesimal similitude and learn its basic properties. Section 5.2 lists the assumptions needed for our theorems, and in Section 5.3 we prove theorems to obtain the upper and lower bounds for the Hausdorff dimension of the invariant set list. Finally, in Section 5.4 we discuss the importance of choosing an appropriate metric. Chapter 6 contains results for infinite graph-directed systems. We discuss an interesting example of an infinite iterated function system given by the complex continued fraction expansions. In Chapter 7 we give a numerical result which improves the previous known lower bound for the Hausdorff dimension of the invariant set for the complex continued fraction example. Finally, Chapter 8 gives a continuity theorem for the Hausdorff dimensions of graph-directed systems. A substantial portion of this thesis is to appear in a joint paper with Nussbaum and Verduyn Lunel [24].
Chapter 2

Preliminaries

2.1 Hausdorff Measure and Hausdorff Dimension

Let \((X, d)\) be a metric space and let \(\text{diam}(A)\) denote the diameter of a subset \(A\). Let \(E\) be a subset of \(X\) and \(s \geq 0\). For \(\delta > 0\), we define

\[
H^s_\delta(E) = \inf \left\{ \sum_{k=1}^{\infty} (\text{diam}(E_k))^s : E \subset \bigcup_{k=1}^{\infty} E_k, \text{diam}(E_k) < \delta \right\}.
\]

For \(s = 0\), we use the following interpretation of the 0th power when \(\text{diam}(E_k) = 0\):

\[(\text{diam}(E_k))^0 = 1 \text{ if } E_k \text{ is a single point, but } (\text{diam}(E_k))^0 = 0 \text{ if } E_k = \emptyset.\]

It is easy to verify that \(H^s_\delta\) is an outer measure on \(X\). For a given \(s \geq 0\), the function \(\delta \mapsto H^s_\delta\) is decreasing and we define

\[
H^s(E) = \lim_{\delta \to 0^+} H^s_\delta(E) = \sup_{\delta > 0} H^s_\delta(E).
\]

\(H^s\) is easily seen to be an outer measure, but it is also a metric outer measure. It follows that \(H^s\) is a Borel measure (see Chapter 1 of [5]) and is called \(s\)-dimensional Hausdorff measure. It is not hard to prove that there is a unique number \(s_0 \geq 0\) such that \(H^s(E) = \infty\) for \(0 \leq s < s_0\) and \(H^s(E) = 0\) for \(s > s_0\). The number \(s_0\) is called the Hausdorff dimension of \(E\). Thus the Hausdorff dimension of a set \(E\) may be thought of as the number \(s\) at which \(H^s(E)\) jumps from \(\infty\) to 0. The measure \(H^s(E)\) can be zero or infinite when \(s\) is the Hausdorff dimension of \(E\). But in the case that \(H^s(E)\) is positive and finite, \(E\) is called an \(s\)-set. We refer the reader to [5] and [15] for the basic properties of Hausdorff measure.
2.2 The Hausdorff Metric

Let \((S, d)\) be a complete metric space. If \(A \subset S\), we define the diameter of \(A\) by

\[
\text{diam}(A) = \sup \{d(s, t) : s, t \in A\}.
\]

We shall say that \(A \subset S\) is \textit{bounded} if \(\text{diam}(A) < \infty\). For \(A \subset S\) and \(s \in S\) we define

\[
d(s, A) = \inf \{d(s, a) : a \in A\}.
\]

If \(A \subset S\), we say that \(A\) is \textit{totally bounded} if given any \(\epsilon > 0\), \(A\) can be covered by a finitely many balls of radius \(\epsilon\) with centers in \(S\).

If \(A \subset S\) and \(\delta > 0\), we define \(N_\delta(A)\) by

\[
N_\delta(A) = \{s \in S : d(s, A) < \delta\}.
\]

If \(A\) and \(B\) are nonempty, closed, bounded subsets of \(S\), we define

\[
D(A, B) = \inf \{\delta > 0 : A \subset N_\delta(B) \text{ and } B \subset N_\delta(A)\}.
\]

If \(\mathcal{B}(S)\) denotes the collection of nonempty, closed, bounded subsets of \(S\), then it follows that \((\mathcal{B}(S), D)\) is a metric space. The metric \(D\) is called the \textit{Hausdorff metric}. It is known (see [20], Exercise 7, pages 280-281) that if \((S, d)\) is a complete metric space, then \((\mathcal{B}(S), D)\) is also a complete metric space.

2.3 Iterated Function Systems and Invariant Sets

Let \((S, d)\) be a complete metric space. A map \(\theta : S \to S\) is called a \textit{contraction} if there exists a \(c < 1\) such that \(d(\theta(x), \theta(y)) \leq cd(x, y)\) for all \(x, y \in S\). Let \(I\) be a countable set and \(\{\theta_i : i \in I\}\) be a collection of contractions on \(S\) for which there exists a \(c < 1\) such that \(d(\theta_i(x), \theta_i(y)) \leq cd(x, y)\) for all \(x, y \in S\) and for all \(i \in I\). Such a collection is called an \textit{iterated function system}. If \(I\) is a finite set, we call it a \textit{finite iterated function system}; otherwise it is called an \textit{infinite iterated function system}.

A nonempty compact set \(E\) is said to be an \textit{invariant set} for the iterated function system \(\{\theta_i : i \in I\}\) if \(E = \bigcup_{i \in I} \theta_i(E)\). The following theorem proves the existence and
uniqueness of invariant set for a finite iterated function system on a complete metric space.

**Theorem 2.3.1.** Let \((S, d)\) be a complete metric space and \(\{\theta_i : i = 1, 2, \ldots, N\}\) be a finite iterated function system. Then there exists a unique nonempty, closed and bounded set \(E\) such that 
\[E = \bigcup_{i=1}^{N} \theta_i(E).\]
Furthermore, \(E\) is indeed compact, so we may remove the closure in the above equation.

**Proof.** Let \(\mathcal{B}(S)\) denote the collection of nonempty, closed, bounded subsets of \(S\) with the Hausdorff metric \(D\). Then since \((S, d)\) is a complete metric space, we know that \((\mathcal{B}(S), D)\) is a complete metric space. Define the map \(\Theta : \mathcal{B}(S) \to \mathcal{B}(S)\) by
\[\Theta(A) = \bigcup_{i=1}^{N} \theta_i(A).\]

Note that for any \(i\), \(\theta_i(A)\) is bounded for any bounded set \(A\) because \(\theta_i\) is a contraction. Hence \(\bigcup_{i=1}^{N} \theta_i(A)\) is bounded, being a finite union of bounded sets. Thus the map \(\Theta\) is well defined.

We claim that \(\Theta\) is a contraction map. Let \(A, B \in \mathcal{B}(S)\) and let \(\delta > D(A, B)\). Let \(1 \leq i \leq N\) and \(a \in A\). Since \(D(A, B) < \delta\), there exists \(b \in B\) such that \(d(a, b) < \delta\). Then \(d(\theta_i(a), \theta_i(b)) \leq cd(a, b) < c\delta\). This shows that \(\bigcup_{i=1}^{N} \theta_i(A)\) is contained in a \(c\delta\)-neighborhood of \(\bigcup_{i=1}^{N} \theta_i(B)\). Similarly we can prove that \(\bigcup_{i=1}^{N} \theta_i(B)\) is contained in a \(c\delta\)-neighborhood of \(\bigcup_{i=1}^{N} \theta_i(A)\). This shows that \(D(\Theta(A), \Theta(B)) \leq c\delta\). Since \(\delta > D(A, B)\) was arbitrary, we have proved that \(D(\Theta(A), \Theta(B)) \leq cD(A, B)\). Hence the claim.

So we have a contraction map \(\Theta\) on a complete metric space \(\mathcal{B}(S)\). By the contraction mapping theorem, \(\Theta\) has a unique fixed point \(E\). Thus we have \(E = \Theta(E) = \bigcup_{i=1}^{N} \theta_i(E)\).

To see that \(E\) is compact, let us restrict the map \(\Theta\) to \(\mathcal{K}(S)\), where \(\mathcal{K}(S)\) denotes the collection of nonempty compact subsets of \(S\) with the Hausdorff metric \(D\). We claim that \(\mathcal{K}(S)\) is a closed subset of the complete metric space \(\mathcal{B}(S)\), which implies that \(\mathcal{K}(S)\) is itself a complete metric space. To see the claim, let \(A \in \mathcal{B}(S)\) be in the closure of \(\mathcal{K}(S)\). Then there exist a sequence \(\{A_n\}_{n \geq 1}\) in \(\mathcal{K}(S)\) such that \(D(A_n, A) \to 0\)
as $n \to \infty$. To show that $A \in \mathcal{K}(S)$ we only need to show that $A$ is compact. Note that $A$ is complete because it is a closed subset of the complete metric space $S$. So to show that $A$ is compact it is enough to show that $A$ is totally bounded. If $\epsilon > 0$ is given, we can find $n_0$ such $A \subseteq N_{\epsilon/2}(A_n)$ for all $n \geq n_0$ because $D(A_n, A) \to 0$ as $n \to \infty$. Also since $A_{n_0}$ is totally bounded, it can be covered by a finitely many balls of radius $\epsilon/2$. It follows that $A$ can be covered a finitely many balls of radius $\epsilon$. This proves that $A$ is totally bounded and hence we have proved our claim. Therefore, $\Theta$ maps the complete metric space $\mathcal{K}(S)$ into itself, and is a contraction map as seen above. Thus $\Theta$ must have a unique fixed point in $\mathcal{K}(S)$. The fixed point must be the same as $E$, since otherwise the original map $\Theta$ would have two fixed points. Therefore $E$ is compact, and the theorem is proved. 

We have the following theorem in the case of an infinite iterated function system on a compact metric space. Note that we cannot remove the closure in the following theorem because an infinite union of compact sets need not be closed.

**Theorem 2.3.2.** Let $(S, d)$ be a compact metric space and $\{\theta_i\}_{i=1}^{\infty}$ be an infinite iterated function system. Then there exists a unique nonempty compact set $E$ such that $E = \bigcup_{i=1}^{\infty} \theta_i(E)$.

**Proof.** If $A$ is a compact subset of $S$, $\bigcup_{i=1}^{\infty} \theta_i(A)$ is a closed subset of the compact metric space $S$, and hence is itself compact. Thus the map $\Theta : \mathcal{K}(S) \to \mathcal{K}(S)$ given by $\Theta(A) = \bigcup_{i=1}^{\infty} \theta_i(A)$ is well defined, where $\mathcal{K}(S)$ denotes the collection of nonempty, compact subsets of $S$ with the Hausdorff metric. The map $\Theta$ is seen to be a contraction map as in the proof of the previous theorem. By the contraction mapping theorem, $\Theta$ has a unique fixed point $E$. Thus we have proved the theorem. 

**Remark 2.3.3.** Assume that either $\{\theta_i : i \in I\}$ is a finite iterated function system on a complete metric space or an infinite iterated function system on a compact metric space. For $i \in I$, let $x_i$ be the unique fixed point of the contraction $\theta_i$ on the complete metric space $S$. Then the invariant set $E$ given by Theorem 2.3.1 in the case of finite iterated function system or by Theorem 2.3.2 in the case of infinite iterated function
system must contain closure \( \overline{\{ x_i : i \in I \}} \) of all the fixed points. To see this let us denote by \( A \) the set \( \{ x_i : i \in I \} \). Then by the contraction mapping theorem \( \Theta^n(A) \) converges in the Hausdorff metric \( D \) to the fixed point \( E \) of \( \Theta \). Since \( A \) is contained in \( \Theta^n(A) \) for any \( n \geq 1 \), it follows that \( A \) is contained in \( E \).

**Remark 2.3.4.** If we only assume that the metric space \((S, d)\) is complete and bounded but not necessarily compact in Theorem 2.3.2, we can still prove that there exists a unique nonempty, closed set \( E \subset S \) such that \( E = \bigcup_{i=1}^{\infty} \theta_i(E) \) but in this generality \( E \) need not be compact. To see this we consider the map \( \Theta : B(S) \to B(S) \) given by \( \Theta(A) = \bigcup_{i=1}^{\infty} \theta_i(A) \), where \( B(S) \) denotes the collection of nonempty, closed subsets of \( S \) with the Hausdorff metric. Then as in the proof of Theorem 2.3.1, we see that \( \Theta \) is a contraction map on the complete metric space \( B(S) \). By the contraction mapping theorem, \( \Theta \) has a unique fixed point \( E \) in \( B(S) \). Note that the sets in \( B(S) \) are automatically bounded because we have assumed that \( S \) is bounded. To see that \( E \) need not be compact, let \( S \) be the closed unit ball in an infinite dimensional Hilbert space \((X, \langle \cdot, \cdot \rangle)\) and let \( \{ e_i : i \geq 1 \} \) be an orthonormal set in \( X \). For \( i \geq 1 \), define \( \theta_i : S \to S \) by \( \theta_i(x) = e_i + \frac{1}{2^i}(x - e_i) \). Then \( \theta_i \) is a contraction map with contraction ratio \( 2^{-i} \) which is less than or equal to \( 1/2 \) for all \( i \geq 1 \). Also \( e_i \) is the unique fixed point of \( \theta_i \) for \( i \geq 1 \). Clearly \( \{ e_i : i \geq 1 \} \) is not totally bounded because \( ||e_i - e_j|| = \sqrt{2} \) for all \( i \neq j \).

Arguing as in remark 2.3.3, we see that the invariant set \( E \) must contain \( \{ e_i : i \geq 1 \} \). It follows that \( E \) cannot be compact because otherwise any subset of \( E \) has to be totally bounded.

### 2.4 Self-similar Sets

Let \((S, d)\) be a complete metric space. A map \( \theta : S \to S \) is said to be a *similitude* if there exists an \( r > 0 \) such that \( d(\theta(x), \theta(y)) = rd(x, y) \) for all \( x, y \in S \). We shall call \( r \) the ratio for the similitude \( \theta \). If \( S \) is a normed linear space with metric \( d \) derived from the norm on \( S \) and if the similitude \( \theta \) is an onto map (which is necessarily true if \( S \) is finite dimensional), then a theorem of Mazur and Ulam (see [18] or [28]) implies that \( \theta \) is an affine linear map. Suppose that \( \theta_i, 1 \leq i \leq N \), are similitudes on \( S \) with ratios
\( r_i \), where \( 0 < r_i < 1 \). Let \( E \) be the invariant set for iterated function system \( \{\theta_i\}_{i=1}^N \).

Moran [19] and Hutchinson [10] have studied the Hausdorff dimension of \( E \) when \( S \) is a finite dimensional normed linear space (i.e., \( S = \mathbb{R}^n \)). The maps \( \{\theta_i\}_{i=1}^N \) are said to satisfy the open set condition if there exists a nonempty open set \( U \) such that the sets \( \theta_i(U) \), \( 1 \leq i \leq N \), are pairwise disjoint and are contained in \( U \). If \( \{\theta_i\}_{i=1}^N \) are similitudes with ratios \( r_i < 1 \) and the open set condition is satisfied, then the Hausdorff dimension of \( E \) is given by the unique number \( s \) such that \( \sum_{i=1}^N r_i^s = 1 \). Furthermore, \( E \) is indeed an \( s \)-set, i.e., the measure \( \mathcal{H}^s(E) \) is strictly positive and finite.

As an example, the Cantor set is the invariant set for the iterated function system given by two similitudes \( \theta_1(x) = x/3 \) and \( \theta_2(x) = 2/3 + x/3 \) on the real line. The ratios are \( r_1 = r_2 = 1/3 \). Also the open set condition is satisfied by taking \( U \) to be the open interval \((0,1)\). Hence the Hausdorff dimension is given by \( 2(1/3)^s = 1 \), which gives \( s = \log 2 / \log 3 \). Similarly one can calculate the Hausdorff dimension of other self-similar sets like the snowflake curve, the Sierpiński gasket, etc (see [4, 5], for example).

Schief [27] has shown that on a complete metric space the Hausdorff dimension of the invariant set is still given by the equation \( \sum_{i=1}^N r_i^s = 1 \), if one replaces the open set condition by the strong open set condition, which requires, in addition to the open set condition, that the open set \( U \) must intersect the invariant set \( E \). He has also shown that in this generality \( E \) need not be an \( s \)-set.

It is of considerable interest to allow maps \( \theta_i \) which may not be affine linear in our iterated function systems. For example, in studying subsets of \( \mathbb{R} \) defined by properties of their continued fraction expansions, one is led to maps \( \theta_i : [0,1] \to [0,1] \) defined by \( \theta_i(x) = (x + m_i)^{-1} \), \( m_i \) a positive integer (see [2, 3, 9, 11]). The most common approach to dealing with such nonlinear maps and studying the Hausdorff dimension of their associated invariant set has been the “thermodynamic formalism” (see [6, 11]).

Our intent in later chapters is to derive a basic formula in its proper generality for the Hausdorff dimension of the invariant set and to emphasize the utility of the theory of positive linear operators in this setting.
Chapter 3

Generalized Graph-Directed Systems

3.1 The Generalized Graph-Directed Set-up

Let $V$ and $E$ be finite sets and for each $v \in V$, let $(S_v, d_v)$ be a complete metric space. We shall call $V$ the set of vertices and $E$ the set of edges. Let $\Gamma$ be a subset of $V \times E$, and $\alpha : \Gamma \to V$. For each $(v, e) \in \Gamma$, let $\theta(v, e) : S_v \to S_\alpha(v, e)$ be a Lipschitz map with $\text{Lip}(\theta(v, e)) \leq c < 1$. Recall that a map $\psi : (S_1, d_1) \to (S_2, d_2)$ is said to be Lipschitz if there is a constant $c$ such that $d_2(\psi(s), \psi(t)) \leq c d_1(s, t)$ for all $s, t \in S_1$, and

$$\text{Lip}(\psi) := \sup \left\{ \frac{d_2(\psi(s), \psi(t))}{d_1(s, t)} : s, t \in S_1, s \neq t \right\}.$$ 

We shall keep in mind two important particular cases.

**Example 3.1.1. (The Mauldin-Williams graph)** Let $V$ be the set of vertices and $E$ be the set of edges of a directed multigraph. Let $i(e)$ and $t(e)$ denote the initial and terminal vertices of edge $e \in E$. The set $\Gamma$ is defined by $(v, e) \in \Gamma$ if and only if $v = t(e)$. The map $\alpha$ in this case is $\alpha(v, e) = i(e)$. See Chapter 4.3 in [4] for a discussion of the Mauldin-Williams graph.

**Example 3.1.2.** Let $(T, d)$ be a bounded complete metric space. Assume that $T = \bigcup_{k=1}^{p} T_k$, where each $T_k$ is a closed subset of $T$ and $T_k \cap T_l = \emptyset$ for $k \neq l$. For $1 \leq i \leq m$, let $\theta_i : T \to T$ be a continuous map such that $\theta_i(T_k) \subset T_{\nu(i, k)}$, $1 \leq k \leq p$, where $\nu(i, k) \in \{1, 2, \ldots, p\}$, and $\text{Lip}(\theta_i|_{T_k}) \leq c < 1$. In this case, we take $V = \{k : 1 \leq k \leq p\}$, $S_k = T_k$ for $1 \leq k \leq p$, $E = \{i : 1 \leq i \leq m\}$, $\Gamma = V \times E$. The map $\alpha(k, i) = \nu(i, k), 1 \leq k \leq p, 1 \leq i \leq m$, and $\theta_i|_{T_k}$.

Note that a finite iterated function system $\{\theta_i : 1 \leq i \leq N\}$ on a complete metric space $(S, d)$ is a special case of the general set-up discussed above. To see this, let
\[ V = \{1\}, \mathcal{E} = \{1, 2, \ldots, N\}, \Gamma = V \times \mathcal{E} \text{ and } \alpha(v, e) = v \text{ in the terminology above.} \]

The examples in [17] show that the graph-directed set-up covers cases which do not fall within the framework of iterated function systems.

For \( u \in V \), define
\[ \Gamma_u = \{(v, e) \in \Gamma : \alpha(v, e) = u\} \]
and
\[ E_u = \{e \in \mathcal{E} : (u, e) \in \Gamma\}. \]

For \( n \geq 1 \), define
\[ \Gamma^{(n)} = \left\{ [(v_1, e_1), \ldots, (v_n, e_n)] : (v_i, e_i) \in \Gamma, \alpha(v_{i+1}, e_{i+1}) = v_i, 1 \leq i \leq n - 1 \right\}. \]

For \( u \in V, n \geq 1 \), define
\[ \Gamma_u^{(n)} = \{ [(v_1, e_1), \ldots, (v_n, e_n)] \in \Gamma^{(n)} : \alpha(v_1, e_1) = u \}. \]

Define \( V_\infty = \{ u \in V : \Gamma_u^{(n)} \neq \emptyset \quad \forall n \geq 1 \}. \)

The notations and assumptions will be as in the preceding paragraph for the remainder of the chapter.

### 3.2 The Existence of an Invariant Set List

The following theorem proves the existence of a list of invariant compact sets in the context of a graph-directed system. This generalizes Theorem 2.3.1, which gave the existence of an invariant set for a finite iterated function system.

**Theorem 3.2.1.** Let \( V, \mathcal{E}, \Gamma, \alpha \) be given as before, and \( \theta_{(v, e)} : S_v \to S_{\alpha(v, e)} \) be a Lipschitz map with \( \text{Lip}(\theta_{(v, e)}) \leq c < 1 \) for all \( (v, e) \in \Gamma \). Assume \( \Gamma_u \neq \emptyset \) for all \( u \in V \). Then there exists a unique list \( (C_v)_{v \in V} \) of nonempty closed bounded sets \( C_v \subset S_v \) such that
\[
C_u = \bigcup_{(v, e) \in \Gamma_u} \theta_{(v, e)}(C_v) \tag{3.1}
\]
for all \( u \in V \). Furthermore, \( C_v \) is indeed compact for all \( v \in V \), so we may remove the closure in the above equation.
Proof. Let $\mathcal{B}(S_v)$ denote the collection of nonempty, closed, bounded subsets of $S_v$ with the Hausdorff metric $D_v$. Then since $S_v$ is a complete metric space, we know that $\mathcal{B}(S_v)$ is a complete metric space. So the finite Cartesian product $\prod_{v \in V} \mathcal{B}(S_v)$ with the sup metric is also a complete metric space. Define the map $\Theta : \prod_{v \in V} \mathcal{B}(S_v) \to \prod_{v \in V} \mathcal{B}(S_v)$ by

$$
\Theta((A_v)_{v \in V}) = \left( \bigcup_{(v,e) \in \Gamma_u} \theta_{(v,e)}(A_v) \right)_{u \in V}.
$$

Note that for any $u \in V$, $\bigcup_{(v,e) \in \Gamma_u} \theta_{(v,e)}(A_v)$ is nonempty because $\Gamma_u$ is nonempty by assumption, and $A_v$ is nonempty for each $v \in V$. Also it is bounded because $\theta_{(v,e)}(A_v)$, being the image of a bounded set $A_v$ under a Lipschitz map $\theta_{(v,e)}$, is bounded, and a finite union of bounded sets is bounded. Thus the map $\Theta$ is well-defined.

We claim that $\Theta$ is a contraction map. Let $A = (A_v)_{v \in V}$ and $B = (B_v)_{v \in V}$ be in $\prod_{v \in V} \mathcal{B}(S_v)$. Then $D(A, B) = \max_{v \in V} D_v(A_v, B_v)$ and

$$
D(\Theta(A), \Theta(B)) = \max_{u \in V} D_u \left( \bigcup_{(v,e) \in \Gamma_u} \theta_{(v,e)}(A_v), \bigcup_{(v,e) \in \Gamma_u} \theta_{(v,e)}(B_v) \right).
$$

Let $\delta > D(A, B)$, and take any $(v, e) \in \Gamma_u$, $a_v \in A_v$. Since $D_v(A_v, B_v) \leq D(A, B) < \delta$, there exists $b_v \in B_v$ such that $d_v(a_v, b_v) < \delta$. Then

$$
d_u(\theta_{(v,e)}(a_v), \theta_{(v,e)}(b_v)) \leq c d_v(a_v, b_v) < c \delta.
$$

This shows that $\bigcup_{(v,e) \in \Gamma_u} \theta_{(v,e)}(A_v)$ is contained in a $c \delta$-neighborhood of

$$
\bigcup_{(v,e) \in \Gamma_u} \theta_{(v,e)}(B_v).
$$

Interchanging the roles of $A_v$ and $B_v$ yields a corresponding inclusion, so

$$
D_u \left( \bigcup_{(v,e) \in \Gamma_u} \theta_{(v,e)}(A_v), \bigcup_{(v,e) \in \Gamma_u} \theta_{(v,e)}(B_v) \right) \leq c \delta
$$

for all $u \in V$. Hence, $D(\Theta(A), \Theta(B)) \leq c \delta$. Since $\delta > D(A, B)$ was arbitrary, we have proved that $D(\Theta(A), \Theta(B)) \leq c D(A, B)$.

Therefore we have a contraction map $\Theta$ on a complete metric space. By the contraction mapping theorem, $\Theta$ has a unique fixed point, say $(C_v)_{v \in V}$. Thus we have $C_u = \bigcup_{(v,e) \in \Gamma_u} \theta_{(v,e)}(C_v)$. 

To see that $C_v$ is compact for all $v \in V$, let us restrict the map $\Theta$ to $\prod_{v \in V} K(S_v)$, where $K(S_v)$ denotes the collection of nonempty, compact subsets of $S_v$ with the Hausdorff metric $D_v$. We have seen in the proof of Theorem 2.3.1 that for any $v \in V$, $K(S_v)$ is a complete metric space. Therefore $\prod_{v \in V} K(S_v)$ is a complete metric space. Then $\Theta$ maps the complete metric space $\prod_{v \in V} K(S_v)$ into itself, and is a contraction map as seen above. Thus $\Theta$ has a unique fixed point in $\prod_{v \in V} K(S_v)$. The fixed point must be the same as $(C_v)_{v \in V}$, since otherwise the original map $\Theta$ would have two fixed points. Therefore $C_v$ is compact for all $v \in V$, and the theorem is proved.

3.3 A Few Remarks

Remark 3.3.1. The assumption $\Gamma_u \neq \emptyset$ for all $u \in V$ in the previous theorem may be too strong for some examples. A weaker assumption under which we can prove the existence of an invariant list is $V_{\infty} \neq \emptyset$. Note that $\Gamma_u \neq \emptyset$ for all $u \in V$ implies $V_{\infty} = V$, so $V_{\infty} \neq \emptyset$.

First we claim that $u \in V_{\infty}$ implies that there exists $v \in V_{\infty}$ with $(v, e) \in \Gamma_u$ for some $e \in E$. Suppose not. Then for all $(v, e) \in \Gamma_u$, $v \notin V_{\infty}$. This implies, since $V$ is a finite set, that there exists $n \geq 1$ such that $\Gamma^{(n)}_n = \emptyset$ for all $(v, e) \in \Gamma_u$. But since $u \in V_{\infty}$, there exists $[(v_1, e_1), (v_2, e_2), \ldots, (v_{n+1}, e_{n+1})] \in \Gamma^{(n+1)}_u$, which implies $[(v_2, e_2), \ldots, (v_{n+1}, e_{n+1})] \in \Gamma^{(n)}_u$ and since $\alpha(v_1, e_1) = u, (v_1, e_1) \in \Gamma_u$. This contradicts $\Gamma^{(n)}_u = \emptyset$ for all $(v, e) \in \Gamma_u$, hence the claim.

Now consider the map $\Theta : \prod_{v \in V_{\infty}} B(S_v) \to \prod_{v \in V_{\infty}} B(S_v)$ by

$$\Theta((A_v)_{v \in V_{\infty}}) = \left( \bigcup_{(v, e) \in \Gamma_u \atop v \in V_{\infty}} \theta_{(v, e)}(A_v) \right)_{u \in V_{\infty}}.$$

Note that $\Theta$ is well defined because of the above claim. Again by the contraction mapping theorem, we have $(C_v)_{v \in V_{\infty}}, C_v \subset S_v$ is compact such that

$$C_u = \bigcup_{(v, e) \in \Gamma_u \atop v \in V_{\infty}} \theta_{(v, e)}(C_v).$$

Equivalently, under the assumption that $V_{\infty} \neq \emptyset$, all we have done is replaced $V$ by $\hat{V} := V_{\infty}$, $\Gamma$ by $\hat{\Gamma} := \{(v, e) \in \Gamma : v \in \hat{V}\}$ and $\alpha$ by $\hat{\alpha} := \alpha|_{\hat{\Gamma}}$, and applied Theorem 3.2.1.
Remark 3.3.2. Applying the previous result to Example 3.1.2, we get nonempty compact sets $C_k \subset T_k$ for $k \in V_\infty$ such that

$$C_k = \bigcup_{\nu(i,l)=k, l \in V_\infty} \theta_i(C_l) \text{ for } k \in V_\infty.$$  

If we let $C = \bigcup_{l \in V_\infty} C_l$, then $C$ is a nonempty compact set and it satisfies

$$C = \bigcup_{i=1}^m \theta_i(C).$$

Thus we have a compact invariant set for the family of maps $\theta_i$, $1 \leq i \leq m$.

Remark 3.3.3. We can relax the condition Lip($\theta_{(v,e)}$) $\leq c < 1$ for all $(v,e) \in \Gamma$ in Theorem 3.2.1 to the following weaker condition. Suppose that for some fixed $n \geq 1$, the composition of any $n$ of the maps $\theta_{(v,e)}$, whenever the composition is defined, is Lipschitz with Lipschitz constant $\leq c < 1$.

Then we see that $\Theta^n$ is a contraction map on a complete metric space, where $\Theta$ is the map defined in the proof of Theorem 3.2.1.

It is then well known that the map $\Theta$ has a unique fixed point. Thus the conclusion of Theorem 3.2.1 holds under this weaker assumption.
Chapter 4

Perron-Frobenius Operators

The classical Krein-Rutman theorem (see [12]) considers a positive (in the sense of mapping a suitable cone to itself), compact, linear map $T : X \rightarrow X$ which has positive spectral radius $r$ and asserts the existence of a positive eigenvector $v$ with $T(v) = rv$. Generalizations, particularly allowing noncompact $T$, can be found in [1, 13, 14, 22, 23, 26]. In this chapter we study a positive linear operator $L$ on a Banach space of continuous functions, and use generalizations of the Krein-Rutman theorem to prove the existence of a positive eigenvector with eigenvalue equal to the spectral radius of the operator. Analogues of the operator $L$ we consider are sometimes called “Perron-Frobenius operators” or “Frobenius-Ruelle operators”, although the theory originally developed by Perron and Frobenius is restricted to matrices with nonnegative entries, and generalizations to infinite dimensions pose substantial difficulties.

4.1 The Positive Operator

From now on, let $V = \{1, 2, \ldots, p\}$ with $S_1, S_2, \ldots, S_p$ the corresponding complete metric spaces. We do not necessarily assume that $S_j$, $1 \leq j \leq p$, is compact. Let

$$X_i = C_b(S_i) = \{f : S_i \rightarrow \mathbb{R} : f \text{ is continuous and bounded}\}$$

for $1 \leq i \leq p$ with $\|f\| = \sup_{s \in S_i} |f(s)|$.

Define a linear map $A : X_1 \times X_2 \times \cdots \times X_p \rightarrow X_1 \times X_2 \times \cdots \times X_p$ by

$$(Af)_j(s) = \sum_{e \in E_j} b(j,e)(s)f_{a(j,e)}(\theta(j,e)(s)) \quad \text{for } s \in S_j, 1 \leq j \leq p,$$

(4.1)

where $f = (f_1, f_2, \ldots, f_p)$ and the functions $b(j,e) \in X_j$ are given. We assume throughout this section that $E_j = \{e \in \mathcal{E} : (j, e) \in \Gamma\}$ is nonempty for all $j \in V$. 

Define for $M > 0$, $\lambda \geq 0$ and $1 \leq j \leq p$,

$$K_j(M, \lambda) = \{ f \in X_j : 0 \leq f(s) \leq f(t) \exp(M(d_j(s,t))^\lambda) \text{ for all } s, t \in S_j \}. \quad (4.2)$$

**Remark 4.1.1.** From the definition, it follows that if $f \in K_j(M, \lambda)$ and $f(t) = 0$ for some $t \in S_j$, then $f(s) = 0$ for all $s \in S_j$. Thus $f \in K_j(M, \lambda)$ implies that either $f$ is identically zero on $S_j$ or $f$ is strictly positive on $S_j$.

If $Y$ is a real Banach space, a closed set $K \subset Y$ is called a closed *cone* if $\lambda K + \mu K \subset K$ for all $\lambda \geq 0$, $\mu \geq 0$, and $K \cap (-K) = \{0\}$.

The following lemma follows by the same argument used in Lemma 5.4 in [23]. We give the proof for the reader’s convenience.

**Lemma 4.1.2.** Let $K_j := K_j(M, \lambda)$ be as defined by (4.2) with $\lambda > 0$. Then $K_j$ is a closed cone in $(X_j, \|\|)$, and $\{ f \in K_j : \|f\| \leq 1 \}$ is equicontinuous.

**Proof.** It follows immediately from the definition of $K_j$ that $K_j$ is a closed cone. To prove the equicontinuity of $\{ f \in K_j : \|f\| \leq 1 \}$ let $f \in K_j$ with $\|f\| \leq 1$. We claim that for any $s, t \in S_j$ we have

$$|f(s) - f(t)| \leq M(d_j(s,t))^\lambda.$$

According to the previous remark, either $f$ is identically zero on $S_j$ or $f$ is strictly positive on $S_j$. The inequality is obvious in the first case. In the latter case, we may assume that $0 < f(s) \leq f(t) \leq 1$. The definition of $K_j$ implies that

$$|\ln(f(s)) - \ln(f(t))| \leq M(d_j(s,t))^\lambda.$$

The mean value theorem implies that for some $\xi$ with $\ln(f(s)) \leq \xi \leq \ln(f(t)) \leq 0$ we have

$$|f(s) - f(t)| = \exp(\ln(f(t))) - \exp(\ln(f(s)))
= \exp(\xi)|\ln(f(t)) - \ln(f(s))| \leq M(d_j(s,t))^\lambda.$$

Since this is true for any $f \in K_j$ with $\|f\| \leq 1$, equicontinuity follows. □
Lemma 4.1.3. Assume that for some $M_0 > 0$ and $\lambda > 0$, $b(j,e) \in K_j(M_0, \lambda)$ for all $(j, e) \in \Gamma$. Then there exists $M > 0$ so that the map $A$ defined by (4.1) maps $\prod_{i=1}^p K_i(M, \lambda)$ into itself.

Proof. Let $f_i \in K_i(M, \lambda)$ for $1 \leq i \leq p$ and $s, t \in S_j$. Then

$$(Af)_j(s) = \sum_{e \in E_j} b(j,e)(s) f_{\alpha(j,e)}(s).$$

Since $b(j,e) \in K_j(M_0, \lambda)$, $b(j,e)(s) \leq b(j,e)(t) \exp(M_0(d_j(s,t))^\lambda)$. Also

$$f_{\alpha(j,e)}(s) \leq f_{\alpha(j,e)}(t) \exp(M(d_{\alpha(j,e)}(s,t))^\lambda) \leq f_{\alpha(j,e)}(t) \exp(Mc^\lambda(d_j(s,t))^\lambda).$$

Thus

$$(Af)_j(s) = \sum_{e \in E_j} b(j,e)(s) f_{\alpha(j,e)}(s) \leq \sum_{e \in E_j} b(j,e)(t) f_{\alpha(j,e)}(t) \exp((M_0 + Mc^\lambda)(d_j(s,t))^\lambda) = (Af)_j(t) \exp((M_0 + Mc^\lambda)(d_j(s,t))^\lambda).$$

So, if we choose $M$ such that $M_0 + Mc^\lambda \leq M$, which can be done because $c < 1$ and $\lambda > 0$, then

$$(Af)_j(s) \leq (Af)_j(t) \exp(M(d_j(s,t))^\lambda),$$

so $(Af)_j \in K_j(M, \lambda)$ for $1 \leq j \leq p$. \hfill \Box

We should note that observations similar to Lemma 4.1.3 have been made earlier by other authors. See the proof of Theorem 5.4 in [23] and [2], for example.

We shall use the following notation:

$$\Gamma^{(n)} := \{([j_1, e_1], \ldots, [j_n, e_n]) : (j_i, e_i) \in \Gamma, 1 \leq i \leq n, \alpha(j_i, e_i) = j_{i+1}, 1 \leq i < n\}.$$

$$\bar{\Gamma}^{(n)} := \{([j_1, e_1], \ldots, [j_n, e_n]) \in \Gamma^{(n)} : j_1 = j\}.$$

We shall also use $(J, E)$, where $J = (j_1, \ldots, j_n)$, $E = (e_1, \ldots, e_n)$ as a shorthand notation for $([j_1, e_1], \ldots, [j_n, e_n]) \in \bar{\Gamma}^{(n)}$. 


Let Lemma 4.2.1. This computation suggests the formula for $A$

We show that (4.3) is correct by using an induction on $n$

**Proof.**

Also the operator norm of $A$

and

$$
\theta_{(J,E)}(s) := \theta_{(j_n,e_n)} \circ \cdots \circ \theta_{(j_1,e_1)}(s).
$$

### 4.2 Iterates of the Operator $A$

Let us compute $A^2$:

$$(A^2 f)_{j_1}(s) = (A(Af))_{j_1}(s) = \sum_{\epsilon_1:(j_1,e_1)\in \Gamma} b_{(j_1,e_1)}(s)(Af)_{\alpha(j_1,e_1)}(\theta_{(j_1,e_1)}(s)).$$

Using

$$(Af)_{\alpha(j_1,e_1)}(\theta_{(j_1,e_1)}(s)) = \sum_{\epsilon_2:(j_2,e_2)\in \Gamma_{j_2=\alpha(j_1,e_1)}} b_{(j_2,e_2)}(\theta_{(j_1,e_1)}(s))f_{\alpha(j_2,e_2)}(\theta_{(j_2,e_2)}(\theta_{(j_1,e_1)}(s)))$$

we get

$$(A^2 f)_{j_1}(s) = \sum_{(J,E)\in \Gamma^{(2)}_{j_1}} b_{(J,E)}(s)f_{\alpha(j_2,e_2)}(\theta_{(J,E)}(s)).$$

This computation suggests the formula for $A^n$ given in the following lemma.

**Lemma 4.2.1.** Let $n \geq 1$. Then for $f = (f_1, f_2, \ldots, f_p) \in \prod_{i=1}^p X_i$ and $1 \leq j_1 \leq p$,

$$(A^n f)_{j_1}(s) = \sum_{(J,E)\in \Gamma^{(n)}_{j_1}} b_{(J,E)}(s)f_{\alpha(j_n,e_n)}(\theta_{(J,E)}(s)), \quad s \in S_{j_1}. \quad (4.3)$$

Also the operator norm of $A^n$ is given by

$$
\|A^n\| = \max_{1 \leq j \leq p} \sup_{s \in S_j} \sum_{(J,E)\in \Gamma^{(n)}_j} b_{(J,E)}(s).
$$

**Proof.** We show that (4.3) is correct by using an induction on $n$. For $n = 1$, (4.3) is the same as (4.1). Suppose that $n \geq 2$ and assume that (4.3) holds for $n - 1$. Let $1 \leq j_1 \leq p$ and $s \in S_{j_1}$ be fixed. Then

$$(A^n f)_{j_1}(s) = (A(A^{n-1} f))_{j_1}(s) = \sum_{\epsilon_1:(j_1,e_1)\in \Gamma} b_{(j_1,e_1)}(s)(A^{n-1} f)_{\alpha(j_1,e_1)}(\theta_{(j_1,e_1)}(s)).$$
Assuming that (4.3) holds for $n - 1$, we have

$$(A^{n-1}f)_{\alpha(j_1,e_1)}(\theta_{(j_1,e_1)}(s)) = \sum_{(J,E)\in \Gamma^{(n-1)}_{\alpha(j_1,e_1)}} b_{(J,E)}(\theta_{(j_1,e_1)}(s))f_{\alpha(j_n,e_n)}(\theta_{(j,E)}(\theta_{(j_1,e_1)}(s))),$$

where $(J,E) = [(j_2,e_2), \ldots, (j_n,e_n)] \in \Gamma^{(n-1)}_{\alpha(j_1,e_1)}$ in the above sum. Using this formula for $A^{n-1}$ in the previous equation we obtain (4.3) for $n$. Thus by induction (4.3) holds for all $n \geq 1$.

If $f \in \prod_{i=1}^p X_i$ with $\|f\| \leq 1$, i.e., $|f_j(s)| \leq 1$ for all $s \in S_j, 1 \leq j \leq p$, then (4.3) gives

$$|(A^n f)_j(s)| \leq \sum_{(J,E)\in \Gamma^{(n)}_j} b_{(J,E)}(s).$$

Taking the supremum over $s \in S_j$ and then the maximum over $1 \leq j \leq p$ gives

$$\|A^n\| \leq \max \sup_{1 \leq j \leq p} \sum_{s \in S_j} \sum_{(J,E)\in \Gamma^{(n)}_j} b_{(J,E)}(s).$$

If we take $f = (f_1, f_2, \ldots, f_p)$, where $f_j$ is identically equal to one on $S_j$, then

$$(A^n f)_j(s) = \sum_{(J,E)\in \Gamma^{(n)}_j} b_{(J,E)}(s).$$

Therefore we get the equation for $\|A^n\|$. \qed

**Lemma 4.2.2.** Let $(S_j)_{j=1}^p$ be bounded, complete metric spaces. Assume that $\Gamma_j := \{ (k, e) \in \Gamma : \alpha(k, e) = j \}$ is nonempty for $1 \leq j \leq p$, and let $(C_j)_{j=1}^p$ be the unique invariant list of compact sets given by Theorem 3.2.1. Let $(J,E) = [(j_1,e_1), \ldots, (j_n,e_n)] \in \Gamma^{(n)}$ and $\theta_{(J,E)}(s) = \theta_{(j_n,e_n)} \circ \cdots \circ \theta_{(j_1,e_1)}(s), s \in S_{j_1}$. Then there exists $M_1 > 0$ such that for all $n \geq 1$,

$$d_{\alpha(j_n,e_n)}(\theta_{(J,E)}(s), C_{\alpha(j_n,e_n)}) \leq M_1 c^n \quad \forall s \in S_{j_1},$$

where $c < 1$ is the constant such that $\text{Lip}(\theta_{(j,e)}) \leq c$ for all $(j, e) \in \Gamma$.

**Proof.** Since the metric spaces $(S_j), 1 \leq j \leq p$ are bounded, we can find $M_1$ so that $d_j(s, C_j) \leq M_1$ for all $s \in S_j, 1 \leq j \leq p$. Let $(j, e) \in \Gamma$ and $s \in S_j$. Then we can find $t \in C_j$ such that $d_j(s, t) \leq M_1$. Since $\theta_{(j,e)}(t) \in C_{\alpha(j,e)}$,

$$d_{\alpha(j,e)}(\theta_{(j,e)}(s), C_{\alpha(j,e)}) \leq d_{\alpha(j,e)}(\theta_{(j,e)}(s), \theta_{(j,e)}(t)) \leq c d_j(s, t) \leq c M_1.$$
So the result is true for $n = 1$. The result now follows by an induction on $n$. To see this, let us assume that the result is true for $n - 1$. If $(J, E) = [(j_1, e_1), \ldots, (j_n, e_n)] \in \bar{\Gamma}(n)$, we let $(J', E') = [(j_1, e_1), \ldots, (j_{n-1}, e_{n-1})] \in \bar{\Gamma}(n-1)$. Let $s \in S_{j_1}$. Since $C_{j_n}$ is compact, we can find $t \in C_{j_n}$ such that $d_{j_n}(\theta_{(J', E')}(s), C_{j_n}) = d_{j_n}(\theta_{(J', E')}(s), t)$. Then we have

$$d_{\alpha(j_n, e_n)}(\theta_{(J, E)}(s), C_{\alpha(j_n, e_n)}) = d_{\alpha(j_n, e_n)}(\theta_{(J, E)}(s), C_{\alpha(j_n, e_n)})$$

$$\leq d_{\alpha(j_n, e_n)}(\theta_{(J, E)}(s), \theta_{(j_n, e_n)}(t))$$

$$\leq c d_{j_n}(\theta_{(J', E')}(s), t),$$

where the first inequality follows by using that $\theta_{(j_n, e_n)}(t) \in \theta_{(j_n, e_n)}(C_{j_n}) \subset C_{\alpha(j_n, e_n)}$, and the second inequality follows by using that $\text{Lip}(\theta_{(j, e)}) \leq c$ for all $(j, e) \in \Gamma$. By our choice of $t$, we get

$$d_{\alpha(j_n, e_n)}(\theta_{(J, E)}(s), C_{\alpha(j_n, e_n)}) \leq c d_{j_n}(\theta_{(J', E')}(s), C_{j_n}).$$

By the induction hypothesis $d_{j_n}(\theta_{(J', E')}(s), C_{j_n}) \leq c^{n-1}M_1$, so

$$d_{\alpha(j_n, e_n)}(\theta_{(J, E)}(s), C_{\alpha(j_n, e_n)}) \leq c^n M_1.$$

Thus we are done by induction. \(\Box\)

### 4.3 Kuratowski’s Measure of Noncompactness and the Cone Spectral Radius

Let us recall the definition of Kuratowski’s measure of noncompactness $\beta$. If $(S, d)$ is a metric space and $B \subset S$ is a bounded set, then $\beta(B)$ is defined by

$$\beta(B) = \inf \{\delta > 0 : B = \bigcup_{j=1}^{k} B_j, k < \infty \text{ and } \text{diam}(B_j) \leq \delta \text{ for } 1 \leq j \leq k\}. \quad (4.5)$$

Suppose that $K$ is a closed cone in a Banach space $Y$ and $L : Y \to Y$ is a bounded linear map with $L(K) \subset K$. Define

$$\|L\|_K = \sup \{\|L(y)\| : y \in K, \|y\| \leq 1\}.$$ 

Define $r_K(L)$, the cone spectral radius of $L$, and $\sigma_K(L)$, by

$$r_K(L) := \lim_{n \to \infty} \|L^n\|_K^{1/n} \text{ and}$$
\[ \sigma_K(L) := \limsup_{n \to \infty} (\beta(L^n(U)))^{1/n}, \]

where \( U = \{ y \in K : \| y \| \leq 1 \} \).

It is a special case of Theorem 4.10 in [14] that if \( \sigma_K(L) < r_K(L) \), then there exists \( y \in K \setminus \{ 0 \} \) with \( L(y) = ry, \ r = r_K(L) \). Note that the definition in [14] of \( \rho_K(L) \), the cone essential spectral radius of \( L \), satisfies \( \rho_K(L) \leq \sigma_K(L) \). The definition of \( \rho_K(L) \) in [14] differs from that in [13] and [22]. It is shown in [14] that the earlier definition has some serious deficiencies. We shall use this result to prove the existence of a nonzero eigenvector for the map \( A \) given by (4.1). Alternatively, if all the sets \( S_j \) were compact, an argument similar to the one used to prove Theorem 5.4 in [23] would give the following theorem.

**Theorem 4.3.1.** Consider the map \( A \) defined on \( \prod_{j=1}^p C_b(S_j) \) by

\[
(Af)_j(s) = \sum_{e \in E_j} b_{(j,e)}(s) f_{\alpha_{(j,e)}(\theta_{(j,e)}(s))} \quad \text{for } s \in S_j,
\]

where \( f = (f_1, f_2, \ldots, f_p) \). Assume that \( S_j \) is a bounded, complete metric space for \( 1 \leq j \leq p \). Also assume that \( \Gamma_j \neq \emptyset \) and \( E_j \neq \emptyset \) for \( 1 \leq j \leq p \) and that for some \( M_0 > 0 \) and \( \lambda > 0, b_{(j,e)} \in K_j(M_0, \lambda) \) for all \( (j,e) \in \Gamma \). Let \( K \) be the cone \( \prod_{j=1}^p K_j(M, \lambda) \), where \( M_0 + \lambda M \leq M \). Then \( \| A^n \|_K = \| A^n \| \) for all \( n \geq 1 \), where \( \| A^n \| \) is given by (4.4), and \( r_K(A) = \lim_{n \to \infty} \| A^n \|^\frac{1}{n} \). If \( r_K(A) > 0 \), there exists \( u = (u_1, u_2, \ldots, u_p) \in K \setminus \{ 0 \} \) with \( Au = ru \), where \( r = r_K(A) \). If \( b_{(j,e)}(s) > 0 \) for all \( (j,e) \in \Gamma \) and all \( s \in S_j \), then \( r_K(A) > 0 \).

**Proof.** It is enough to show that \( \sigma_K(A) < r_K(A) \). Let \( (C_j)_{j=1}^p \) be the unique invariant list of nonempty compact sets given by Theorem 3.2.1 and let \( C = \prod_{j=1}^p C_j \).

Let \( \mathcal{U} = \{ f \in K : \| f \| \leq 1 \} \). From Lemma 4.1.2, we know that \( \mathcal{U} \) is equicontinuous. Let us write \( \mathcal{U}|_C = \{ f|_C : f \in \mathcal{U} \} \). Then \( \mathcal{U}|_C \) is a bounded equicontinuous family of functions from the compact set \( C \) into \( \mathbb{R}^p \). So, by Ascoli’s theorem, it is totally bounded. Therefore, given \( \epsilon > 0 \), we can write \( \mathcal{U} = \bigcup_{i=1}^q \mathcal{U}_i, \ q < \infty, \) such that \( \| f|_C - g|_C \| \leq \epsilon \) provided \( f \) and \( g \) are in the same \( \mathcal{U}_i \). Let \( f, g \in \mathcal{U} \) and \( 1 \leq j \leq p \). Then using...
Lemma 4.2.1, we have for \( s \in S_j \),

\[
|\langle A^n f \rangle_j(s) - \langle A^n g \rangle_j(s)\rangle \leq \sum_{(J,E) \in \Gamma_j^{(n)}} b_{(J,E)}(s)|f_{\alpha(j_n, e_n)}(\theta_{(J,E)}(s)) - g_{\alpha(j_n, e_n)}(\theta_{(J,E)}(s))|.
\]

Using Lemma 4.2.2, there exists \( \tau \in C_{\alpha(j_n, e_n)} \) with \( d(\theta_{(J,E)}(s), \tau) \leq M_1 c^n \).

Since \( f_{\alpha(j_n, e_n)} \in K_{\alpha(j_n, e_n)}(M, \lambda) \) and \( \|f\| \leq 1 \), we have as in the proof of Lemma 4.1.2,

\[
|f_{\alpha(j_n, e_n)}(\theta_{(J,E)}(s)) - f_{\alpha(j_n, e_n)}(\tau)| \leq M d(\theta_{(J,E)}(s), \tau) \leq M(M_1 c^n)\lambda.
\]

The same is true for \( g \). Also, if we assume that \( f, g \in \mathcal{U}_l \) for some \( l, 1 \leq l \leq q \), then

\[
|f_{\alpha(j_n, e_n)}(\tau) - g_{\alpha(j_n, e_n)}(\tau)| \leq \epsilon. \quad \text{Therefore, by the triangle inequality,}
\]

\[
|f_{\alpha(j_n, e_n)}(\theta_{(J,E)}(s)) - g_{\alpha(j_n, e_n)}(\theta_{(J,E)}(s))| \leq \epsilon + 2M(M_1 c^n)\lambda \quad \text{if } f, g \in \mathcal{U}_l.
\]

So, if \( f, g \in \mathcal{U}_l \) and \( 1 \leq j \leq p \),

\[
|\langle A^n f \rangle_j(s) - \langle A^n g \rangle_j(s)\rangle \leq (\epsilon + 2M(M_1 c^n)\lambda) \sum_{(J,E) \in \Gamma_j^{(n)}} b_{(J,E)}(s).
\]

Taking the supremum over \( s \in S_j \) and the maximum over \( 1 \leq j \leq p \), and using (4.4), we get

\[
\|A^n f - A^n g\| \leq (\epsilon + 2M(M_1 c^n)\lambda)\|A^n\|
\]

for \( f, g \in \mathcal{U}_l \), \( 1 \leq l \leq q \).

Thus \( A^n(\mathcal{U}) = \bigcup_{l=1}^q A^n(\mathcal{U}_l) \) with

\[
\text{diam}(A^n(\mathcal{U}_l)) \leq (\epsilon + 2M(M_1 c^n)\lambda)\|A^n\|,
\]

so \( \beta(A^n(\mathcal{U})) \leq (\epsilon + 2M(M_1 c^n)\lambda)\|A^n\| \).

Since \( \epsilon > 0 \) was arbitrary, \( \beta(A^n(\mathcal{U})) \leq 2M(M_1 c^n)\lambda\|A^n\| \) which implies that

\[
(\beta(A^n(\mathcal{U})))^{\frac{1}{n}} \leq (2M(M_1 \lambda)\lambda^\frac{1}{n})^{\frac{1}{n}} c^\lambda\|A^n\|^{\frac{1}{n}}.
\]

In general, it is obviously true that \( \|A^n\|_K \leq \|A^n\| \). On the other hand, if \( f = (f_1, f_2, \ldots, f_p) \) and \( f_j(s) = 1 \) for all \( s \in S_j \), then \( f \in K \), and we have seen in the proof of Lemma 4.2.1 that \( \|A^n(f)\| = \|A^n\| \). It follows that \( \|A^n\|_K = \|A^n\| \) for all \( n \geq 1 \) and
that \( r_K(A) = \lim_{n \to \infty} \|A^n\|^{\frac{1}{n}} = r(A) \), where \( \|A^n\| \) is given by (4.4) and \( r(A) \) denotes the spectral radius of \( A \). Taking the limit in our estimate for \((\beta(A^n(U)))^{\frac{1}{n}}\) gives

\[
\left( \frac{\sigma_K(A)}{\lambda} \right) = \lim_{n \to \infty} \left( \frac{\beta(A^n(U))}{\lambda_n} \right) = \frac{1}{\lambda} \left( \frac{\sigma_K(A)}{\lambda} \right).
\]

If \( r_K(A) > 0 \), it follows (because \( 0 < c < 1 \)) that \( \sigma_K(A) < r_K(A) \), and we are done. If we assume that \( b_{j,e}(s) > 0 \) for all \((j,e) \in \Gamma \) and \( s \in S_j \), then because \( b_{j,e} \in K_j(M_0, \lambda) \), there exists \( \delta > 0 \) such that \( b_{j,e}(s) \geq \delta \) for all \((j,e) \in \Gamma \) and \( s \in S_j \), so \( b_{j,e}(s) \geq \delta^n \) for all \((J,E) \in \bar{\Gamma}^{(n)}\). By using (4.4) of Lemma 4.2.1 it follows that \( \|A^n\| \geq \delta^n \). Since we have seen in the previous paragraph that \( r_K(A) = \lim_{n \to \infty} \|A^n\|^{\frac{1}{n}} \), it follows that \( r_K(A) \geq \delta > 0 \).

\[
\square
\]

### 4.4 A Few Remarks

**Remark 4.4.1.** Suppose that \( u = (u_1, u_2, \ldots, u_p) \) is a nonzero eigenvector of the linear map \( A \) with eigenvalue \( r(A) \) given by Theorem 4.3.1. Then for any \( 1 \leq j \leq p \), either \( u_j \) is identically zero or \( u_j \) is strictly positive on \( S_j \). To see this note that \( u_j \in K_j(M, \lambda) \), so

\[
0 \leq u_j(s) \leq u_j(t) \exp(M(d_j(s,t))^{\lambda}) \quad \text{for all } s, t \in S_j.
\]

Thus \( u_j(t) = 0 \) for some \( t \in S_j \) will imply that \( u_j(s) = 0 \) for all \( s \in S_j \). Also, since \( u \) is nonzero, at least one of the coordinate functions \( u_j \) is strictly positive.

**Remark 4.4.2.** In general (in the context of the Krein-Rutman Theorem), if \( r_K(L) = 0 \), it need not be true that there exist \( v \in K \setminus \{0\} \) with \( L(v) = 0 \). Suppose, however, that \( K \) and \( A \) are as in Theorem 4.3.1, that \( b_{j,e} \in K_j(M_0, \lambda) \) for all \((j,e) \in \Gamma \) and that \( r_K(A) = 0 \). We claim that there exists an integer \( N \) such that \( A^N = 0 \) and that there exists \( v \in K \setminus \{0\} \) with \( A(v) = 0 \). Because \( S_j \) is bounded and \( b_{j,e} \in K_j(M_0, \lambda) \), we have already seen that either \( b_{j,e}(s) = 0 \) for all \( s \in S_j \) or there exists \( \delta_{j,e} > 0 \) with \( b_{j,e}(s) \geq \delta_{j,e} \) for all \( s \in S_j \). Let \( \mathcal{P} = \{(j,e) \in \Gamma : b_{j,e}(s) > 0 \text{ for all } s \in S_j\} \.

Because \( \mathcal{P} \) is a finite set (since \( \Gamma \) is finite), there exists \( \delta > 0 \) with \( b_{j,e}(s) \geq \delta \) for all \( s \in S_j \) and for all \((j,e) \in \mathcal{P} \). For \( n \geq 1 \), define \( \bar{\mathcal{P}}^{(n)} \subset \bar{\Gamma}^{(n)} \) by

\[
\bar{\mathcal{P}}^{(n)} = \{(J,E) \in \bar{\Gamma}^{(n)} : (j_k, e_k) \in \mathcal{P} \text{ for } 1 \leq k \leq n\}.
\]
If \( \mathcal{P}^{(n)} \) is nonempty, it easily follows from (4.4) that \( \| A^n \| \geq \delta^n \), so if \( \mathcal{P}^{(n)} \) is nonempty for all \( n \geq 1 \), \( r_K(A) \geq \delta \), contrary to our assumption. Thus there must exist an integer \( N \) such that \( \mathcal{P}^{(n)} \) is empty for all \( n \geq N \). However, if \( (J,E) \in \Gamma^{(n)} \) and \( (J,E) \notin \mathcal{P}^{(n)} \), \( b(J,E)(s) = 0 \) for all \( s \in S_j \), so we find that \( A^n = 0 \) for all \( n \geq N \). If \( w \in K \setminus \{0\} \), let \( p \leq N \) be the least positive integer such that \( A^n(w) = 0 \). If we define \( v = A^{p-1}(w) \in K \setminus \{0\} \), \( A(v) = 0 \).

**Remark 4.4.3.** If \( \beta \) denotes the Kuratowski’s measure of noncompactness on a Banach space \( X \), \( K \) denotes a closed cone in \( X \) and \( \Lambda : X \to X \) is a bounded linear map such that \( \Lambda(K) \subset K \), define \( \beta_K(\Lambda) \) by

\[
\beta_K(\Lambda) := \inf \{ \lambda > 0 : \beta(\Lambda(T)) \leq \lambda \beta(T) \text{ for all bounded sets } T \subset K \}.
\]

If \( L : X \to X \) is a bounded linear map such that \( L(K) \subset K \), one can prove that \( \lim_{n \to \infty} (\beta_K(L^n))^{\frac{1}{n}} \) exists. If \( \lim_{n \to \infty} (\beta_K(L^n))^{\frac{1}{n}} < r_K(L) \), Theorem 2.2 in [22] implies that there exists \( u \in K \setminus \{0\} \) with \( L(u) = ru \) and \( r := r_K(L) \).

One might hope that Theorem 2.2 in [22] could be used to prove Theorem 4.3.1. However, if the metric spaces \( S_j \), \( 1 \leq j \leq p \), in Theorem 4.3.1 are not compact, Theorem 2.2 in [22] is, in general, not applicable, even in very simple special cases. To illustrate this, we work in the Hilbert space \( l^2(\mathbb{N}) \), and we let \( \{e_j : j \geq 1\} \) denote the usual orthonormal basis of \( l^2(\mathbb{N}) \). In the notation of Theorem 4.3.1, we take \( S_1 = \{x \in l^2(\mathbb{N}) : \|x\| \leq 1\} \), \( V = \{1\} \) and \( \mathcal{E} \) to be a set with exactly one point. Let \( X_1 = C_b(S_1) \) and, in the notation of Lemma 4.1.2, let \( K = K(M,1) \), where \( M \) is a fixed constant with \( M \geq 1 \). Define a bounded linear operator \( A : X_1 \to X_1 \) by

\[
(Af)(x) = f \left( \frac{1}{2} x \right)
\]

for \( x \in S_1 \). Obviously, \( A \) is a trivial example of the operators considered in Theorem 4.3.1, and \( A(K(M,1)) \subset K(M,1) \) and \( Au = u \), where \( u \in K(M,1) \) denotes the function identically equal to 1. However, we claim that for each integer \( n \geq 1 \), there is a bounded set \( T_n \subset K(M,1) \) such that \( \beta(A^n(T_n)) = \beta(T_n) = (\frac{1}{2})^n \); this implies that \( \lim_{n \to \infty} (\beta_K(A^n))^{\frac{1}{n}} = 1 = r_K(A) \), so Theorem 2.2 of [22] cannot be used (at least with the cones \( K(M,1) \)) even in this trivial situation.
To construct the sets $T_n$, for each integer $n \geq 1$ define $\phi_n : [0, 1] \to [1, 1 + 2^{-n}]$ by
\[
\phi_n(s) = \begin{cases} 
1 + s & \text{if } 0 \leq s \leq 2^{-n} \\
1 + 2^{-n} & \text{if } 2^{-n} \leq s \leq 1
\end{cases}
\]
Then $\phi_n$ is Lipschitz with Lipschitz constant 1. Let $\langle \cdot, \cdot \rangle$ denote the inner product on $l^2(\mathbb{N})$, and define $\psi_{j,n}(x)$ and $T_n$ by
\[
\psi_{j,n}(x) = \phi_n(|x_j|) \quad \text{and} \quad T_n = \{\psi_{j,n} : j \geq 1\},
\]
where $x_j = \langle x, e_j \rangle$ for $j \geq 1$. Let us first show that $T_n \subset K(M, 1)$ with $M = 1$ for all $n \geq 1$. Let $n \geq 1$, $j \geq 1$ and $x, y \in S_1$. Since $\phi_n$ is Lipschitz with Lipschitz constant 1, we have $|\phi_n(|x_j|) - \phi_n(|y_j|)| \leq ||x_j| - |y_j|| \leq |x_j - y_j| \leq \|x - y\|$. Since $\phi_n(|x_j|) \geq 1$ and $\phi_n(|y_j|) \geq 1$, we have $\ln(\phi_n(|x_j|)) - \ln(\phi_n(|y_j|)) \leq |\phi_n(|x_j|) - \phi_n(|y_j|)|$. So $\ln(\psi_{j,n}(x)) - \ln(\psi_{j,n}(y)) \leq \|x - y\|$ which implies that $\psi_{j,n}(x) \leq \psi_{j,n}(y) \exp(\|x - y\|)$. This shows that $\psi_{j,n} \in K(1, 1)$ for all $j \geq 1$ and $n \geq 1$.

Also for a fixed $n \geq 1$, and for $1 \leq j < k < \infty$,
\[
\|\psi_{j,n} - \psi_{k,n}\| = \sup\{|\psi_{j,n}(x) - \psi_{k,n}(x)| : x \in S_1\} = |\psi_{j,n}(e_j) - \psi_{k,n}(e_j)| = 2^{-n}.
\]
Here we have used the fact that for any $x \in S_1$, $\phi_n(|x_j|)$ and $\phi_n(|x_k|)$ are in the interval $[1, 1 + 2^{-n}]$ which is of length $2^{-n}$. The above equation shows that any set with diameter less than $2^{-n}$ can contain at most one element of $T_n$. So by the definition of Kuratowski’s measure of noncompactness (see (4.5)), we get that $\beta(T_n) \geq 2^{-n}$. Also by the above equation $\text{diam}(T_n) = 2^{-n}$, so $\beta(T_n) \leq 2^{-n}$. Thus we have shown that $\beta(T_n) = 2^{-n}$.

Since $(A^n\psi_{j,n})(x) = \phi_n(2^{-n}|x_j|)$, the same argument shows that for a fixed $n \geq 1$ and for $1 \leq j < k < \infty$,
\[
\|A^n(\psi_{j,n}) - A^n(\psi_{k,n})\| = |\psi_{j,n}(2^{-n}e_j) - \psi_{k,n}(2^{-n}e_j)| = 2^{-n},
\]
and therefore $\beta(A^n(T_n)) = 2^{-n}$.  

Chapter 5

Hausdorff Dimension

5.1 Infinitesimal Similitudes

5.1.1 The Definition

Recall that a metric space \((S, d)\) is called perfect if every point of \(S\) is a limit point of \(S\), i.e., for each \(s \in S\), there exists a sequence \((s_k)_k\) in \(S\) such that \(s_k \neq s\) for all \(k\) and \(s_k \to s\) as \(k \to \infty\).

Let \((S_1, d_1)\) be a perfect metric space and \((S_2, d_2)\) a metric space. A map \(\theta : S_1 \to S_2\) is said to be an infinitesimal similitude at \(s \in S_1\) if for any sequences \((s_k)_k\) and \((t_k)_k\) in \(S_1\) with \(s_k \neq t_k\) for \(k \geq 1\) and \(s_k \to s, t_k \to s\), the limit

\[
\lim_{k \to \infty} \frac{d_2(\theta(s_k), \theta(t_k))}{d_1(s_k, t_k)} =: (D\theta)(s) \tag{5.1}
\]

exists and is independent of the particular sequences \((s_k)_k\) and \((t_k)_k\). We shall say that \(\theta\) is an infinitesimal similitude on \(S_1\) if \(\theta\) is an infinitesimal similitude at \(s\) for all \(s \in S_1\). Notice that the assumption that \(S_1\) is perfect implies that for every \(s \in S_1\), there exist sequences \((s_k)_k\) and \((t_k)_k\) as above.

We should note that the concept of infinitesimal similitude depends on the particular metrics \(d_1\) and \(d_2\). Also note that we shall usually take \(d_1 = d_2\) if \(S_1 = S_2\).

5.1.2 Basic Properties

We list some basic properties of infinitesimal similitudes that we shall need.

**Lemma 5.1.1.** Let \((S_1, d_1)\) be a perfect metric space and \((S_2, d_2)\) a metric space. If \(\theta : S_1 \to S_2\) is an infinitesimal similitude, then \(s \mapsto (D\theta)(s)\) is continuous.
Proof. We argue by contradiction and assume that \( s \mapsto (D\theta)(s) \) is not continuous. Then there exist \( \epsilon > 0 \) and \( s \in S_1 \) and a sequence \((s_k)_k\) in \( S_1 \) with \( d_1(s_k, s) \to 0 \) as \( k \to \infty \) such that

\[
| (D\theta)(s_k) - (D\theta)(s) | > \epsilon > 0.
\]

Since \( S_1 \) is perfect and \( \theta \) is an infinitesimal similitude, for each \( k \geq 1 \), there exist \( t_k \) and \( w_k \) in \( S_1 \) with \( t_k \neq w_k \), \( 0 < d_1(s_k,t_k) < \frac{1}{k} \), \( 0 < d_1(s_k,w_k) < \frac{1}{k} \) and

\[
\left| \frac{d_2(\theta(t_k),\theta(w_k))}{d_1(t_k, w_k)} - (D\theta)(s_k) \right| < \frac{\epsilon}{4}.
\]

Since \( s_k \to s \) as \( k \to \infty \), it follows that \( t_k \to s \) and \( w_k \to s \) as \( k \to \infty \) as well. So, by definition,

\[
\frac{d_2(\theta(t_k),\theta(w_k))}{d_1(t_k, w_k)} \to (D\theta)(s) \text{ as } k \to \infty.
\]

So, for \( k \) large enough, \(|(D\theta)(s_k) - (D\theta)(s)| < \frac{\epsilon}{4}\), a contradiction. Hence, \( s \mapsto (D\theta)(s) \) is continuous.

The following lemma states an analogue of the “chain rule” for infinitesimal similitudes.

**Lemma 5.1.2.** Let \((S_j,d_j), j = 1,2\), be perfect metric spaces and let \((S_3,d_3)\) be a metric space. Let \( \theta : S_1 \to S_2 \) and \( \psi : S_2 \to S_3 \) be given. If \( \theta \) is an infinitesimal similitude at \( s \in S_1 \) and \( \psi \) is an infinitesimal similitude at \( \theta(s) \in S_2 \), then \( \psi \circ \theta \) is an infinitesimal similitude at \( s \) and

\[
(D(\psi \circ \theta))(s) = (D\psi)(\theta(s))(D\theta)(s). \tag{5.2}
\]

Proof. Let \((s_k)_k\) and \((t_k)_k\) be sequences in \( S_1 \) with \( s_k \neq t_k \), \( k \geq 1 \), \( s_k \to s \), \( t_k \to s \) as \( k \to \infty \). Then \( \theta(s_k) \to \theta(s), \theta(t_k) \to \theta(s) \) as \( k \to \infty \). We consider two cases.

**Case I.** Assume that \((D\theta)(s) \neq 0\). We claim that there exists a positive integer \( k_0 \) with \( \theta(s_k) \neq \theta(t_k) \) for all \( k \geq k_0 \). If not, there exists a subsequence \( k_i \to \infty \) such that \( \theta(s_{k_i}) = \theta(t_{k_i}) \) for \( i \geq 1 \). Writing \( \sigma_i = s_{k_i} \) and \( \tau_i = t_{k_i} \), we have that \( \sigma_i \to s, \tau_i \to s, \sigma_i \neq \tau_i \) and \((D\theta)(s) = \lim_{i \to \infty} \frac{d_2(\theta(\sigma_i),\theta(\tau_i))}{d_1(\sigma_i,\tau_i)} = 0\), which contradicts our assumption. It follows that, for \( k \geq k_0 \), we can write

\[
\frac{d_2(\psi(\theta(s_k)),\psi(\theta(t_k)))}{d_1(s_k,t_k)} = \frac{d_2(\psi(\theta(s_k)),\psi(\theta(t_k)))}{d_2(\theta(s_k),\theta(t_k))} \frac{d_2(\theta(s_k),\theta(t_k))}{d_1(s_k,t_k)}.
\]
As $k \to \infty$, the limit of the right-hand side exists and equals $(D\psi)(\theta(s))(D\theta)(s)$, so the limit of the left-hand side exists and eq. (5.2) is satisfied.

Notice that if $s_k$ and $t_k$ are sequences with $s_k \to s$, $t_k \to s$, $s_k \neq t_k$ for all $k$ and $\theta(s_k) \neq \theta(t_k)$ for all $k \geq k_0$, then the argument above proves that
\[
\lim_{k \to \infty} \frac{d_3(\psi(\theta(s_k)), \psi(\theta(t_k)))}{d_1(s_k, t_k)} = (D\psi)(\theta(s))(D\theta)(s),
\]
even if $(D\theta)(s) = 0$.

**Case II.** Assume that $(D\theta)(s) = 0$. Let $s_k$ and $t_k$ be sequences in $S_1$ with $s_k \to s$, $t_k \to s$ and $s_k \neq t_k$ for all $k \geq 1$. If there exists $k_0 \geq 1$ such that $\theta(s_k) \neq \theta(t_k)$ for all $k \geq k_0$, the argument above shows that
\[
\lim_{k \to \infty} \frac{d_3(\psi(\theta(s_k)), \psi(\theta(t_k)))}{d_1(s_k, t_k)} = (D\psi)(\theta(s))(D\theta)(s) = 0.
\]
If there exists $k_1$ such that $\theta(s_k) = \theta(t_k)$ for all $k \geq k_1$, we certainly have that
\[
\lim_{k \to \infty} \frac{d_3(\psi(\theta(s_k)), \psi(\theta(t_k)))}{d_1(s_k, t_k)} = 0.
\]
Thus we can assume that $K_1 := \{k \geq 1 : \theta(s_k) \neq \theta(t_k)\}$ and $K_2 := \{k \geq 1 : \theta(s_k) = \theta(t_k)\}$ are infinite sets. However, our previous argument (Case I) shows that
\[
\lim_{k \to \infty, k \in K_1} \frac{d_3(\psi(\theta(s_k)), \psi(\theta(t_k)))}{d_1(s_k, t_k)} = (D\psi)(\theta(s))(D\theta)(s) = 0,
\]
and it is clear that
\[
\lim_{k \to \infty, k \in K_2} \frac{d_3(\psi(\theta(s_k)), \psi(\theta(t_k)))}{d_1(s_k, t_k)} = 0,
\]
so we conclude that
\[
\lim_{k \to \infty} \frac{d_3(\psi(\theta(s_k)), \psi(\theta(t_k)))}{d_1(s_k, t_k)} = 0.
\]

The following lemma gives a “mean value theorem” and will be crucial in the proof of the main theorem.

**Lemma 5.1.3.** Suppose that $(S_1, d_1)$ and $(S_2, d_2)$ are bounded, complete metric spaces, that $(S_1, d_1)$ is perfect and that $\theta : S_1 \to S_2$ is an infinitesimal similitude. Also, assume that $\theta$ is Lipschitz and that $(D\theta)(s) > 0$ for all $s \in S_1$. Let $K \subset S_1$ be a compact,
nonempty set. Then for each $\mu > 1$, there exists an open neighborhood $U_\mu$ of $K$ and a positive number $\epsilon = \epsilon(\mu)$ such that for every $s, t \in U_\mu$ with $0 < d_1(s, t) < \epsilon(\mu)$,

$$
\mu^{-1}(D\theta)(s) \leq \frac{d_2(\theta(s), \theta(t))}{d_1(s, t)} \leq \mu(D\theta)(s).
$$

Proof. For $(s, t) \in S_1 \times S_1$ with $s \neq t$, define $F(s, t) = \frac{d_2(\theta(s), \theta(t))}{d_1(s, t)}$. If $(s, s) \in S_1 \times S_1$, define $F(s, s) = (D\theta)(s)$. Because we assume that $\theta$ is Lipschitz on $S_1$, there is a constant $M_1$ with $F(s, t) \leq M_1$ for all $(s, t) \in S_1 \times S_1$.

We claim that $F$ is continuous on $S_1 \times S_1$. It suffices to prove that if $(s_k, t_k) \to (s, s)$, then $F(s_k, t_k) \to F(s, s) = (D\theta)(s)$. If $s_k \neq t_k$ for all $k \geq k_0$, we know that $F(s_k, t_k) \to (D\theta)(s)$ by the definition of $(D\theta)(s)$. If $s_k = t_k$ for all $k \geq k_1$, then $F(s_k, t_k) = (D\theta)(s_k)$ for $k \geq k_1$, and Lemma 5.1.1 implies that $(D\theta)(s_k) \to (D\theta)(s)$. Thus we can assume that $J_1 := \{k : s_k \neq t_k\}$ and $J_2 := \{k : s_k = t_k\}$ are infinite sets. But in this case, the same reasoning implies that $\lim_{k \to \infty, k \in J_1} F(s_k, t_k) = (D\theta)(s)$ and $\lim_{k \to \infty, k \in J_2} F(s_k, t_k) = (D\theta)(s)$.

Lemma 5.1.1 implies that $s \mapsto (D\theta)(s)$ is continuous on $S_1$, and $(s, t) \mapsto F(s, t)$ is continuous on $S_1 \times S_1$. Thus, since $(D\theta)(s) > 0$, if we define $G(s, t)$ by $G(s, t) = \frac{F(s, t)}{(D\theta)(s)}$, $(s, t) \mapsto G(s, t)$ is continuous on $S_1 \times S_1$ and $G(s, s) = 1$. Since $K \times K$ is compact, $G|_{K \times K}$ is uniformly continuous, so given $\mu > 1$, there exists $\epsilon(\mu) > 0$ with

$$
\mu^{-1} < G(s, t) < \mu
$$

for all $(s, t) \in K \times K$ with $d_1(s, t) \leq \epsilon(\mu)$. We claim that there exists an open neighborhood $U_\mu$ of $K$ such that for all $s, t \in U_\mu$ with $d_1(s, t) \leq \epsilon(\mu), \mu^{-1} < G(s, t) < \mu$.

We argue by contradiction and suppose that the claim is false. For a positive integer $m$, let $V_m = \{s \in S_1 : d_1(s, K) < \frac{1}{m}\}$. By assumption, there exist $s_m, t_m \in V_m$ with $d_1(s_m, t_m) \leq \epsilon(\mu)$ and $G(s_m, t_m) \leq \mu^{-1}$ or $G(s_m, t_m) > \mu$. Because $d_1(s_m, K) \to 0$ and $d_1(t_m, K) \to 0$, we can, by taking a subsequence, assume that $s_m \to s \in K$ and $t_m \to t \in K$ and $d_1(s, t) \leq \epsilon(\mu)$. By continuity of $G$, we either have $G(s, t) \leq \mu^{-1}$ or $G(s, t) \geq \mu$. However, because $s, t \in K$ and $d_1(s, t) \leq \epsilon(\mu), \mu^{-1} < G(s, t) < \mu$, a contradiction. Thus an open set $U_\mu$ exists and, in fact, we can take $U_\mu = V_m$ for some $m \geq 1$. \qed
Remark 5.1.4. For each \( \epsilon > 0 \), define \( \mu(\epsilon) \geq 1 \) to be the infimum of numbers \( \mu > 1 \) such that \( \mu^{-1}(D\theta)(s) \leq \frac{d_2(\theta(s),\theta(t))}{d_1(s,t)} \leq \mu(D\theta)(s) \) for \( s, t \in K \) with \( 0 < d_1(s,t) \leq \epsilon \). Lemma 5.1.3 implies that \( \lim_{\epsilon \to 0^+} \mu(\epsilon) = 1 \), and clearly \( \mu(\epsilon) \) is an increasing function of \( \epsilon \) for \( \epsilon > 0 \).

5.2 The Main Assumptions

Throughout this section we shall make the following assumption.

H5.1 Let \( V = \{1, 2, \ldots, p\} \) and \( S_1, S_2, \ldots, S_p \) be bounded, complete, perfect metric spaces. Let \( \mathcal{E} \) be a finite set, \( \Gamma \subset V \times \mathcal{E} \) and \( \alpha : \Gamma \to V \). For each \( (j,e) \in \Gamma \), \( \theta_{(j,e)} : S_j \to S_{\alpha(j,e)} \) is a Lipschitz map with \( \text{Lip}(\theta_{(j,e)}) \leq c < 1 \). Also, assume that \( \Gamma_i = \{(j,e) \in \Gamma : \alpha(j,e) = i\} \neq \emptyset \) for \( 1 \leq i \leq p \) and \( E_j = \{e \in \mathcal{E} : (j,e) \in \Gamma\} \neq \emptyset \) for \( 1 \leq j \leq p \).

If H5.1 is satisfied then Theorem 3.2.1 implies that there exists a unique list of nonempty, compact sets \( C_j \subset S_j, 1 \leq j \leq p \), with

\[
C_i = \bigcup_{(j,e) \in \Gamma_i} \theta_{(j,e)}(C_j) \quad \text{for } 1 \leq i \leq p. \tag{5.3}
\]

We shall further assume the following.

H5.2 For each \( (j,e) \in \Gamma \), the map \( \theta_{(j,e)} : S_j \to S_{\alpha(j,e)} \), given in H5.1, is an infinitesimal similitude and there exist an \( m > 0 \) with \( (D\theta_{(j,e)})(s) > m \) for all \( s \in S_j \).

Notice that since \( \theta_{(j,e)} \) is Lipschitz with \( \text{Lip}(\theta_{(j,e)}) \leq c \), if \( \theta_{(j,e)} \) is an infinitesimal similitude, we have \( (D\theta_{(j,e)})(s) \leq c \) for all \( s \in S_j \).

Assume that H5.1 and H5.2 are satisfied. For \( \sigma \geq 0 \), define

\[
L_\sigma : \prod_{j=1}^{p} C_b(S_j) \to \prod_{j=1}^{p} C_b(S_j)
\]

by

\[
(L_\sigma f)_j(s) = \sum_{e \in E_j} ((D\theta_{(j,e)})(s))^\sigma f_{\alpha(j,e)}(\theta_{(j,e)}(s)) \quad \text{for } s \in S_j, 1 \leq j \leq p. \tag{5.4}
\]

Recall that a map \( f : (S,d) \to \mathbb{R} \) is said to be Hölder continuous with Hölder exponent \( \lambda > 0 \) if there exists a constant \( C \in (0, \infty) \) such that

\[
|f(s) - f(t)| \leq C(d(s,t))^\lambda \quad \text{for all } s, t \in S.
\]
Let us assume the following hypothesis.

**H5.3** For each \((j,e) \in \Gamma\), the map \(s \mapsto (D\theta_{j,e})(s)\) is Hölder continuous with Hölder exponent \(\lambda > 0\), where \(\lambda\) is independent of \((j,e) \in \Gamma\).

**Lemma 5.2.1.** If H5.1, H5.2 and H5.3 hold, and \(\sigma \geq 0\), then the map

\[
s \mapsto ((D\theta_{j,e})(s))^\sigma
\]

is in the cone \(K_j(M_0, \lambda)\) defined by (4.2) for some \(M_0 > 0\) (depending on \(\sigma\)).

**Proof.** Fix \((j,e) \in \Gamma\) and \(\sigma \geq 0\). Let \(f(s) = (D\theta_{j,e})(s)\). The hypotheses H5.1 and H5.2 imply that \(0 < m < f(s) \leq c\). By H5.3,

\[
|f(s) - f(t)| \leq C(d_j(s,t))^{\lambda}\text{ for all } s, t \in S_j.
\]

Let \(s, t \in S_j\). By the mean value theorem, there exists \(\xi\) between \(f(s)\) and \(f(t)\) such that

\[
|\ln(f(s)) - \ln(f(t))| = \frac{1}{\xi} |f(s) - f(t)| \leq \frac{1}{\xi} C(d_j(s,t))^{\lambda} \leq \frac{1}{m} C(d_j(s,t))^{\lambda}.
\]

So, \(f(s) \leq f(t) \exp(\frac{1}{m} C(d_j(s,t))^{\lambda})\), which implies that

\[
f(s)^\sigma \leq f(t)^\sigma \exp(M_0(d_j(s,t))^{\lambda}), \text{ where } M_0 = \frac{\sigma C}{m}.
\]

This completes the proof. \(\square\)

Now applying Theorem 4.3.1 to the linear map \(L_\sigma\) defined in (5.4), we get an eigenvector \(u_\sigma \in K \setminus \{0\}\) with \(L_\sigma u_\sigma = r(L_\sigma)u_\sigma\) and \(r(L_\sigma) > 0\).

**Lemma 5.2.2.** The map \(\sigma \mapsto r(L_\sigma)\) is continuous and strictly decreasing. Furthermore, there is a unique \(\sigma_0 \geq 0\) such that \(r(L_{\sigma_0}) = 1\).

**Proof.** Let \(u_\sigma\) be the positive eigenvector of \(L_\sigma\) with eigenvalue \(r(L_\sigma)\). Let us write \(b_{j,e}(t) = (D\theta_{j,e})(t)\) for \((j,e) \in \Gamma\). We know that \(0 < m \leq b_{j,e}(t) \leq c < 1\) for all \(t \in S_j\). Let \(0 \leq \sigma < \sigma'\). Then

\[
(b_{j,e}(t))^{\sigma'} = (b_{j,e}(t))^{\sigma' - \sigma}(b_{j,e}(t))^\sigma \leq c^{\sigma' - \sigma}(b_{j,e}(t))^\sigma.
\]
Therefore, \((b_{(j,e)}(t))^\sigma \geq \mu (b_{(j,e)}(t))^\sigma'\), where \(\mu = (\frac{1}{\nu})^{\sigma' - \sigma} > 1\). It follows that for all \(t \in S_j\),

\[
(L_\sigma u_{\sigma'})_j(t) = \sum_{e \in E_j} (b_{(j,e)}(t))^\sigma (u_{\sigma'})_{\alpha(j,e)}(\theta_{(j,e)}(t)) \\
\geq \mu \sum_{e \in E_j} (b_{(j,e)}(t))^\sigma' (u_{\sigma'})_{\alpha(j,e)}(\theta_{(j,e)}(t)) \\
= \mu (L'_\sigma u_{\sigma'})_j(t) = \mu r(L_{\sigma'})(u_{\sigma'})_j(t),
\]

so \(L_\sigma u_{\sigma'} \geq \mu r(L_{\sigma'})u_{\sigma'}\), where the inequality has the natural coordinate-wise interpretation. Iterating this inequality \(k\) times, we obtain

\[
L^k_\sigma u_{\sigma'} \geq (\mu r(L_{\sigma'}))^k u_{\sigma'}.
\]

If \(e\) denotes the function identically equal to one in each component, we have \(u_{\sigma'} \leq \|u_{\sigma'}\|e\). Thus

\[
L^k_\sigma u_{\sigma'} \leq L^k_\sigma (\|u_{\sigma'}\|e) \leq \|u_{\sigma'}\|L^k_\sigma (e).
\]

Taking norms, we get

\[
\|u_{\sigma'}\||L^k_\sigma (e)\| \geq \|L^k_\sigma u_{\sigma'}\| \geq (\mu r(L_{\sigma'}))^k \|u_{\sigma'}\|.
\]

So, \(\|L^k_\sigma\| \geq \|L^k_\sigma (e)\| \geq (\mu r(L_{\sigma'}))^k\), from which it follows that

\[
r(L_{\sigma}) = \lim_{k \to \infty} \|L^k_\sigma\|^{1/k} \geq \mu r(L_{\sigma'}).
\]

Since \(\mu > 1\), we have proved that \(r(L_{\sigma}) > r(L_{\sigma'})\).

Next we prove the continuity of \(\sigma \mapsto r(L_{\sigma})\). Let \(\sigma > 0\) be fixed. Given \(\nu < 1\), select \(\delta > 0\) such that

\[
\nu (b_{(j,e)}(t))^\sigma \leq (b_{(j,e)}(t))^\sigma' \leq \nu^{-1} (b_{(j,e)}(t))^\sigma \quad \text{for} \quad t \in S_j, \quad |\sigma - \sigma'| \leq \delta.
\]

Then, using the argument as above, we have \(\nu r(L_{\sigma}) \leq r(L_{\sigma'}) \leq \nu^{-1} r(L_{\sigma})\) whenever \(|\sigma - \sigma'| \leq \delta\). Since \(\nu < 1\) was arbitrary, this proves that \(\sigma \mapsto r(L_{\sigma})\) is continuous.

Since \(\|L^0_\sigma\| \geq \|L^0_\sigma (e)\| \geq 1\), we see that \(r(L_0) \geq 1\). Also if \(|E|\) denotes the cardinality of \(E\), then from the definition of \(L_{\sigma}\) and using the fact that \(D\theta_{(j,e)}(t) \leq c\) for all \(t \in S_j\) and \((j,e) \in \Gamma\), it is clear that \(\|L_{\sigma}\| \leq |E|c^\sigma \to 0\) as \(\sigma \to \infty\), so \(r(L_{\sigma}) \leq \|L_{\sigma}\| \to 0\) as
\( \sigma \to \infty \). It follows, by the continuity and strict monotonicity of \( \sigma \to r(L_{\sigma}) \), that there exists a unique \( \sigma_0 \geq 0 \) such that \( r(L_{\sigma_0}) = 1 \). \( \qed \)

**Definition 5.2.3.** We define **strong connectedness** to be the property that for each pair \( j \) and \( k \) in \( V \), there exists for some \( n \geq 1 \), \( (J, E) = [(j_1, e_1), \ldots, (j_n, e_n)] \) such that \( (j_i, e_i) \in \Gamma_i \), for \( 1 \leq i \leq n \), \( j_1 = j \), \( \alpha(j_i, e_i) = j_{i+1}, 1 \leq i < n \) and \( \alpha(j_n, e_n) = k \). Note that in this case we have a map \( \theta(j, e) = \theta(j, e_n) \circ \cdots \circ \theta(j_1, e_1) \) which maps \( S_j \) into \( S_k \).

**Lemma 5.2.4.** Assume that the hypotheses \( H5.1 \), \( H5.2 \), \( H5.3 \) and \( H5.4 \) are satisfied and let \( u_{\sigma} \in K \setminus \{0\} \) be an eigenvector of \( L_{\sigma} \) with eigenvalue \( r(L_{\sigma}) \). Then each component \( (u_{\sigma})_j \) is a strictly positive function on \( S_j \) for \( 1 \leq j \leq p \). Furthermore, there are constants \( l \) and \( L \) with \( 0 < l \leq L < \infty \) such that for every \( j \), \( 1 \leq j \leq p \),

\[
l \leq (u_{\sigma})_j(t) \leq L \quad \text{for all } t \in S_j,
\]

**(5.5)**

**Proof.** Suppose for some \( j \), \( 1 \leq j \leq p \), that \( (u_{\sigma})_j \) equals zero at some point in \( S_j \). Then, since \( (u_{\sigma})_j \in K_j(M, \lambda) \), it follows, as shown in Remark 4.1.1, that \( (u_{\sigma})_j \) is identically equal to zero on \( S_j \). Fix a \( k, 1 \leq k \leq p \). By strong connectedness, there exist \( n \geq 1 \) and \((J', E') = [(j_1, e_1), \ldots, (j_n, e_n)] \in \bar{\Gamma}^{(n)}_j \) with \( j_1 = j \) and \( \alpha(j_n, e_n) = k \).

Since \( L_{\sigma}u_{\sigma} = r(L_{\sigma})u_{\sigma} \), it follows that \( L^n_{\sigma}u_{\sigma} = (r(L_{\sigma}))^nu_{\sigma} \). So, using the formula for \( L^n_{\sigma} \) given by Lemma 4.2.1 with \( A \) replaced by \( L_{\sigma} \), we get

\[
(r(L_{\sigma}))^n(u_{\sigma})_j(s) = \sum_{(J, E) \in \bar{\Gamma}^{(n)}_j} b_{(J, E)}(s)(u_{\sigma})_{\alpha(j_n, e_n)}(\theta(J, E)(s)),
\]

where \( b_{(J, E)}(s) = ((D\theta_{(j, e)})(s))^n \). The left-hand side in the above equation is zero because \( (u_{\sigma})_j(s) = 0 \). Thus, since each term in the sum on the right-hand side is nonnegative, it follows that each term equals zero. In particular, \( (u_{\sigma})_k(\theta(J, E)(s)) = 0 \)
since $b_{(J,E)}(s)$ is strictly positive by H5.2. This implies that $(u_\sigma)_k$ is identically equal to zero on $S_k$. Since this is true for any $k$, $1 \leq k \leq p$, we arrive at a contradiction that $u_\sigma$ is identically zero. Thus $(u_\sigma)_j$ is a strictly positive function on $S_j$ for $1 \leq j \leq p$.

Since each $S_j$ is bounded, there is a $D < \infty$ such that $\text{diam}(S_j) \leq D$ for $1 \leq j \leq p$. Then, since $(u_\sigma)_j \in K_j(M, \lambda)$, it follows that

\[ 0 < (u_\sigma)_j(s) \leq (u_\sigma)_j(t) \exp(MD\lambda) \quad \text{for all } s, t \in S_j. \]

From this it follows that there are constants $0 < l \leq L < \infty$ such that

\[ l \leq (u_\sigma)_j(t) \leq L \quad \text{for all } t \in S_j, 1 \leq j \leq p. \]

\[ \square \]

5.3 The Hausdorff Dimension

Let $C_j \subset S_j$, $1 \leq j \leq p$, be the invariant list of nonempty, compact sets such that

\[ C_i = \bigcup_{(j,e) \in \Gamma_i} \theta_{(j,e)}(C_j) \quad \text{for } 1 \leq i \leq p. \]

Our goal is to determine the Hausdorff dimension of the sets $C_i$, $1 \leq i \leq p$.

First we shall prove that the Hausdorff dimension of $C_i$ is independent of $i$, $1 \leq i \leq p$, under the assumption of strong connectedness.

**Lemma 5.3.1.** Assume that the hypotheses H5.1, H5.2, H5.3 and H5.4 of the previous section are satisfied and let $C_j \subset S_j$, $1 \leq j \leq p$, be the unique invariant list of compact, nonempty sets such that

\[ C_i = \bigcup_{(j,e) \in \Gamma_i} \theta_{(j,e)}(C_j) \quad \text{for } 1 \leq i \leq p. \]

Then $\dim(C_j)$, the Hausdorff dimension of $C_j$, is independent of $j$ for $1 \leq j \leq p$.

**Proof.** First we claim that $\dim(\theta_{(j,e)}(C_j)) = \dim(C_j)$ for any $(j,e) \in \Gamma$. Since $\theta_{(j,e)}$ is a Lipschitz map with Lipschitz constant $c$, $\mathcal{H}^\sigma(\theta_{(j,e)}(C_j)) \leq c^\sigma \mathcal{H}^\sigma(C_j)$ for any $\sigma \geq 0$. This implies that $\dim(\theta_{(j,e)}(C_j)) \leq \dim(C_j)$. To prove the other inequality, we first claim that there exist $m_0 > 0$ and $\delta > 0$ such that $d(\theta_{(j,e)}(s), \theta_{(j,e)}(t)) \geq m_0 d(s, t)$ for
all $s,t \in C_j$ with $d(s,t) \leq \delta$. We abuse notation here by letting $d$ denote $d_j$ and $d_{\alpha(j,e)}$.

We argue by contradiction. If the claim is false, then for each positive integer $k$ there exist $s_k, t_k \in C_j$ with $0 < d(s_k, t_k) \leq k^{-1}$ and $d(\theta_{(j,e)}(s_k), \theta_{(j,e)}(t_k)) < k^{-1}d(s_k, t_k)$.

Since $C_j$ is compact, by taking a subsequence, we can assume that $s_k \to s$ and $t_k \to t$. But this implies that $D\theta_{(j,e)}(s) = 0$, which contradicts H 5.2. Thus $m_0 > 0$ and $\delta > 0$ exist, and since $C_j$ is compact, we can write $C_j = \bigcup_{i=1}^p C_{j,i}$, where diam$(C_{j,i}) \leq \delta$ and $p < \infty$. It follows easily from the definition of Hausdorff dimension that there exists $l$ such that the Hausdorff dimension of $C_{j,i}$ equals the Hausdorff dimension of $C_j$. Also, by our construction, $\theta_{(j,e)}|_{C_{j,i}}$ is one-to-one and $(\theta_{(j,e)}|_{C_{j,i}})^{-1}$ is Lipschitz. This implies that

$$\dim(C_j) = \dim(C_{j,i}) \leq \dim(\theta_{(j,e)}(C_{j,i})) \leq \dim(\theta_{(j,e)}(C_j)),$$

and we have shown that $\dim(C_j) = \dim(\theta_{(j,e)}(C_j))$.

Now, since $\theta_{(j,e)}(C_j) \subset C_{\alpha(j,e)}$, $\dim(C_{\alpha(j,e)}) \geq \dim(\theta_{(j,e)}(C_j)) = \dim(C_j)$ for all $(j,e) \in \Gamma$. Let $1 \leq j \leq p$ and $1 \leq k \leq p$. By strong connectedness, there exists $[(j_1, e_1), \ldots, (j_n, e_n)]$ such that $j_1 = j$, $\alpha(j_i, e_i) = j_{i+1}, 1 \leq i < n$ and $\alpha(j_n, e_n) = k$. So,

$$\dim(C_k) = \dim(C_{\alpha(j_n, e_n)}) \geq \dim(C_{j_n}) \geq \dim(C_{j_{n-1}}) \geq \cdots \geq \dim(C_{j_1}) = \dim(C_j).$$

Since $j$ and $k$ were arbitrary, it follows that $\dim(C_j) = \dim(C_k)$ for all $1 \leq j, k \leq p$. \hfill \Box

### 5.3.1 The Upper Bound

We introduce a ‘weighted’ Hausdorff measure using the strictly positive eigenvector $u_\sigma$ of $L_\sigma$ with eigenvalue $r(L_\sigma)$. Let $1 \leq j \leq p$. Define for $A_j \subset S_j$ and $\epsilon > 0$,

$$\hat{\mathcal{H}}^\sigma_\epsilon(A_j) = \inf \left\{ \sum_{k=1}^\infty (u_\sigma)_j(\xi_{jk})(\text{diam}(A_{jk}))^\sigma : A_j \subset \bigcup_{k=1}^\infty A_{jk}, \right. \left. \xi_{jk} \in A_{jk}, \text{diam}(A_{jk}) < \epsilon \right\}. \tag{5.6}$$

From Lemma 5.2.4, we know that there exist constants $0 < l \leq L < \infty$ such that for $1 \leq j \leq p$, $l \leq (u_\sigma)_j(t) \leq L$ for all $t \in S_j$. This implies that, for $A_j \subset S_j$, $\hat{\mathcal{H}}^\sigma_\epsilon(A_j)$ and $\mathcal{H}^\sigma_\epsilon(A_j)$ are equivalent:

$$l\mathcal{H}^\sigma_\epsilon(A_j) \leq \hat{\mathcal{H}}^\sigma_\epsilon(A_j) \leq L\mathcal{H}^\sigma_\epsilon(A_j). \tag{5.7}$$
Theorem 5.3.2. Assume that the hypotheses H5.1, H5.2, H5.3 and H5.4 are satisfied and let $C_j \subset S_j$, $1 \leq j \leq p$, be the unique invariant list of compact, nonempty sets such that

$$C_i = \bigcup_{(j,e) \in \Gamma_i} \theta_{(j,e)}(C_j) \text{ for } 1 \leq i \leq p.$$ 

If $\dim(C_i)$ denotes the Hausdorff dimension of $(C_i)$ and $\sigma_0$ denotes the unique nonnegative real number such that $r(L_{\sigma_0}) = 1$, then $\dim(C_i) \leq \sigma_0$ for $1 \leq i \leq p$.

Proof. Fix $\epsilon > 0$. Take $\delta > 0$ and $\sigma > 0$. We can choose a covering $\{A_{jk}\}_{k=1}^\infty$ of $C_j$ and points $\xi_{jk} \in A_{jk}$ such that $\text{diam}(A_{jk}) < \epsilon$ and

$$\sum_{k=1}^\infty (u_{\sigma})_j(\xi_{jk})(\text{diam}(A_{jk}))^\sigma \leq \tilde{H}_\epsilon^\sigma(C_j) + \delta. \quad (5.8)$$

Since $C_i = \bigcup_{(j,e) \in \Gamma_i} \theta_{(j,e)}(C_j)$, we have that $\{\theta_{(j,e)}(A_{jk}) : 1 \leq k < \infty, (j,e) \in \Gamma_i\}$ is a covering of $C_i$ with

$$\text{diam}(\theta_{(j,e)}(A_{jk})) \leq c \text{ diam}(A_{jk}) < c\epsilon.$$ 

Furthermore, using Lemma 5.1.3 and Remark 5.1.4, it is easy to see that there exists $\mu_{(j,e)}(\epsilon) > 1$ with $\mu_{(j,e)}(\epsilon) \to 1$ as $\epsilon \to 0^+$ such that

$$\text{diam}(\theta_{(j,e)}(A_{jk})) \leq \mu_{(j,e)}(\epsilon)(D\theta_{(j,e)})(\xi_{jk})\text{diam}(A_{jk}). \quad (5.9)$$

Let $\mu(\epsilon) = \max_{(j,e) \in \Gamma} \mu_{(j,e)}(\epsilon)$. Then we have

$$\tilde{H}_\epsilon^\sigma(C_i) \leq \sum_{k=1}^\infty \sum_{(j,e) \in \Gamma_i} (u_{\sigma})_i(\theta_{(j,e)}(\xi_{jk}))(\text{diam}(\theta_{(j,e)}(A_{jk})))^\sigma \leq (\mu(\epsilon))^\sigma \sum_{k=1}^\infty \sum_{(j,e) \in \Gamma_i} (u_{\sigma})_i(\theta_{(j,e)}(\xi_{jk}))(D\theta_{(j,e)})(\xi_{jk})^\sigma(\text{diam}(A_{jk}))^\sigma.$$

Summing over $i$, $1 \leq i \leq p$, we have

$$\sum_{i=1}^p \tilde{H}_\epsilon^\sigma(C_i) \leq (\mu(\epsilon))^\sigma \sum_{i=1}^p \sum_{k=1}^\infty \sum_{(j,e) \in \Gamma_i} (u_{\sigma})_i(\theta_{(j,e)}(\xi_{jk}))(D\theta_{(j,e)})(\xi_{jk})^\sigma(\text{diam}(A_{jk}))^\sigma.$$
Rearranging the sum, we get
\[ \sum_{i=1}^{p} \tilde{H}_{\sigma}^\sigma(C_i) \leq (\mu(\epsilon))^{\sigma} \sum_{j=1}^{p} \sum_{k=1}^{\infty} (\text{diam}(A_{jk}))^{\sigma} \sum_{i:(j,e) \in \Gamma_i} (u_\sigma)_{j}(\theta_{(j,e)}(\xi_{jk}))((D\theta_{(j,e)})(\xi_{jk}))^{\sigma} \]
\[ = (\mu(\epsilon))^{\sigma} \sum_{j=1}^{p} \sum_{k=1}^{\infty} (L_{\sigma} u_\sigma)_{j}(\xi_{jk})(\text{diam}(A_{jk}))^{\sigma} \]
\[ = (\mu(\epsilon))^{\sigma} r(L_{\sigma}) \sum_{j=1}^{p} \sum_{k=1}^{\infty} (u_\sigma)_{j}(\xi_{jk})(\text{diam}(A_{jk}))^{\sigma}. \]

Thus, using (5.8), we get
\[ \sum_{i=1}^{p} \tilde{H}_{\sigma}^\sigma(C_i) \leq (\mu(\epsilon))^{\sigma} r(L_{\sigma}) \sum_{j=1}^{p} \tilde{H}_{\sigma}^\sigma(C_j) + \delta. \]

Since \( c < 1 \), \( \tilde{H}_{\epsilon}^\sigma(C_i) \leq \tilde{H}_{\epsilon c}^\sigma(C_i) \). Also \( \delta > 0 \) was arbitrary. Therefore,
\[ \sum_{i=1}^{p} \tilde{H}_{\epsilon}^\sigma(C_i) \leq (\mu(\epsilon))^{\sigma} r(L_{\sigma}) \sum_{i=1}^{p} \tilde{H}_{\epsilon}^\sigma(C_i). \quad (5.10) \]

Using Lemma 5.2.2, \( r(L_{\sigma}) < 1 \) for all \( \sigma > \sigma_0 \). Since \( \mu(\epsilon) \to 1 \) as \( \epsilon \to 0 \), given \( \sigma > \sigma_0 \), we can choose \( \epsilon > 0 \) small so that \( (\mu(\epsilon))^{\sigma} r(L_{\sigma}) < 1 \). By the definition, \( \tilde{H}_{\epsilon}^\sigma(C_i) < \infty \) because we can take a finite \( \epsilon \)-cover of the compact set \( C_i \). Thus, if \( \sigma > \sigma_0 \), (5.10) can hold only if
\[ \lim_{\epsilon \to 0^+} \sum_{i=1}^{p} \tilde{H}_{\epsilon}^\sigma(C_i) = 0. \]

This implies that for each \( i, 1 \leq i \leq p \) and \( \sigma > \sigma_0 \), \( \lim_{\epsilon \to 0^+} \tilde{H}_{\epsilon}^\sigma(C_i) = 0 \), and hence using (5.7), \( \lim_{\epsilon \to 0^+} \tilde{H}_{\epsilon}^\sigma(C_i) = 0 \), i.e., \( \tilde{H}_{\sigma}^\sigma(C_i) = 0 \) for all \( \sigma > \sigma_0 \). Thus, by the definition of Hausdorff dimension, \( \dim(C_i) \leq \sigma_0 \).

5.3.2 The Lower Bound

We define for \( 0 < \eta < \epsilon \), and \( A_j \subset S_j \),
\[ \tilde{H}_{\sigma,\eta}(A_j) := \inf \left\{ \sum_{k=1}^{\infty} (u_\sigma)_{j}(\xi_{jk})(\text{diam}(A_{jk}))^{\sigma} : A_j \subset \bigcup_{k=1}^{\infty} A_{jk}, \xi_{jk} \in A_{jk}, \eta < \text{diam}(A_{jk}) < \epsilon \right\}. \quad (5.11) \]

The quantity \( \tilde{H}_{\sigma,\eta}(A_j) \) will be technically useful later, primarily because it is strictly positive whenever it is defined. However, caution is necessary in using this tool. It is
easy to see that $\tilde{H}_c^\sigma(A_j)$ and $\tilde{H}_c^\sigma(A_j)$ depend only on the metric space $(A_j, d_j)$. In contrast, $\tilde{H}_c^\sigma(A_j)$ depends also on $S_j$. If $A_j \subset T_j \subset S_j$ one could give an analogous definition in which one only allows sets $A_{jk} \subset T_j$:

\[
\tilde{H}_{c,\eta}^\sigma(A_j; T_j) := \inf \left\{ \sum_{k=1}^{\infty} (u_n)_{j}(\xi_{jk})(\text{diam}(A_{jk}))^\sigma : A_j \subset \bigcup_{k=1}^{\infty} A_{jk} \subset T_j, \right. \\
\left. \xi_{jk} \in A_{jk}, \eta < \text{diam}(A_{jk}) < \epsilon \right\}.
\] (5.12)

In this cumbersome notation, $\tilde{H}_c^\sigma(A_j) = \tilde{H}_{c,\eta}^\sigma(A_j; S_j)$ for $A_j \subset S_j$. If $A_j$ is compact and $A_j \subset T_j \subset S_j$, it may happen that $\tilde{H}_{c,\eta}^\sigma(A_j; T_j)$ and $\tilde{H}_{c,\eta}^\sigma(A_j; S_j)$ are both defined but are unequal. For the unique list of nonempty compact sets $C_j \subset S_j$, $1 \leq j \leq p$, ensured by Theorem 3.2.1 and for $A_j \subset C_j$, we shall, in our later work, sometimes consider $\tilde{H}_{c,\eta}^\sigma(A_j; C_j)$ rather than $\tilde{H}_{c,\eta}^\sigma(A_j; S_j)$.

In general, if $A_j \subset T_j \subset S_j$ and $A_j$ is compact and $0 < \eta < \epsilon$, in order that $\tilde{H}_{c,\eta}^\sigma(A_j; T_j)$ be defined and finite, it is necessary and sufficient that there exist sets $A_{jk} \subset T_j, 1 \leq k \leq m < \infty$, with $\eta < \text{diam}(A_{jk}) < \epsilon$ for $1 \leq k \leq m$ and $A_j \subset \bigcup_{k=1}^{m} A_{jk}$.

The existence or nonexistence of such sets may be a delicate question. If $A_j$ contains an isolated point of $T_j$, such sets $A_{jk}$ do not exist for all small $\epsilon$ and all $\eta$ with $0 < \eta < \epsilon$.

If $T_j$ is a complete perfect metric space, $A_j \subset T_j$ is compact and $\epsilon > 0$, $\tilde{H}_{c,\eta}^\sigma(A_j; T_j)$ will be defined for all sufficiently small $\eta > 0$. To see this, use compactness of $A_j$ to find finitely many open balls $B_k \subset T_j, 1 \leq k \leq n$, with radius $r < \epsilon/2$ and centers in $A_j$, such that $A_j \subset \bigcup_{k=1}^{n} B_k$. Since $T_j$ is perfect, each $B_k$ contains an accumulation point, so $\text{diam}(B_k) > \eta_k > 0$ and $\text{diam}(B_k) > \eta := \min\{\eta_k : 1 \leq k \leq n\}$ and $\text{diam}(B_k) \leq 2r < \epsilon$.

**Lemma 5.3.3.** Assume that H5.1, H5.2, H5.3 and H5.4 are satisfied. Let $1 \leq j \leq p$ and let $A_j$ be a compact subset of $S_j$. If $\sigma \geq 0$ and $\epsilon > 0$, then

\[
\lim_{\eta \to 0+} \tilde{H}_{c,\eta}^\sigma(A_j) = \tilde{H}_c^\sigma(A_j).
\]

**Proof.** For $0 < \eta < \epsilon$, we have $\tilde{H}_{c,\eta}^\sigma(A_j) \geq \tilde{H}_c^\sigma(A_j)$ because the infimum is taken over a smaller set. So,

\[
\lim_{\eta \to 0+} \tilde{H}_{c,\eta}^\sigma(A_j) \geq \tilde{H}_c^\sigma(A_j).
\]
To prove the reverse inequality, take $\delta > 0$ and choose a covering $\{A_{jk} : 1 \leq k < \infty\}$ of $A_j$ by sets $A_{jk}$ with $\text{diam}(A_{jk}) < \epsilon$, $1 \leq k < \infty$ such that

$$\inf\left\{ \sum_{k=1}^{\infty} (u_\sigma)_j(\xi_{jk})(\text{diam}(A_{jk}))^\sigma : \xi_{jk} \in A_{jk} \right\} \leq \tilde{\mathcal{H}}^\sigma_\epsilon(A_j) + \delta.$$

Without loss of generality, we can assume that the sets $A_{jk}$, $k \geq 1$, are open. Since $A_j$ is compact, there exists a finite open subcover of $A_j$, so there exists an integer $l < \infty$ such that

$$A_j \subset \bigcup_{k=1}^{l} A_{jk}.$$

Let $0 < \eta_0 < \epsilon$ be such that $\eta_0 < \min_{1 \leq k \leq l} \text{diam}(A_{jk})$. Then, for $0 < \eta < \eta_0$, we have

$$\tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(A_j) \leq \inf\left\{ \sum_{k=1}^{l} (u_\sigma)_j(\xi_{jk})(\text{diam}(A_{jk}))^\sigma : \xi_{jk} \in A_{jk} \right\}.$$

So, for every $\delta > 0$, there exists $\eta_0 > 0$ such that for $0 < \eta < \eta_0$,

$$\tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(A_j) \leq \tilde{\mathcal{H}}^\sigma_\epsilon(A_j) + \delta.$$

This shows

$$\lim_{\eta \to 0^+} \tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(A_j) \leq \tilde{\mathcal{H}}^\sigma_\epsilon(A_j)$$

and completes the proof of the lemma.

\[ \square \]

**Lemma 5.3.4.** Assume that H5.1, H5.2, H5.3 and H5.4 are satisfied. Let $1 \leq j \leq p$ and let $A_j$ be a compact subset of $S_j$. Let $\sigma > 0$ be such that $\mathcal{H}^\sigma(A_j) = 0$. Then for every $\epsilon_1$ and $\epsilon_2$ with $0 < \epsilon_1 < \epsilon_2$, there exists an $\eta_0 > 0$ such that for any $B_j \subset A_j$,

$$\tilde{\mathcal{H}}^\sigma_{\epsilon_1,\eta}(B_j) = \tilde{\mathcal{H}}^\sigma_{\epsilon_2,\eta}(B_j) \quad \text{for} \quad 0 < \eta < \eta_0.$$

**Proof.** Since $\mathcal{H}^\sigma(A_j) = 0$, it follows that $\tilde{\mathcal{H}}^\sigma_\epsilon(A_j) = 0$ for every $\epsilon > 0$. By using (5.7), it also follows that $\tilde{\mathcal{H}}^\sigma_\epsilon(A_j) = 0$ for every $\epsilon > 0$. So, by Lemma 5.3.3,

$$\lim_{\eta \to 0^+} \tilde{\mathcal{H}}^\sigma_{\epsilon_2,\eta}(A_j) = 0.$$

This implies that there exists $\eta_0 > 0$ such that for $0 < \eta < \eta_0$,

$$\tilde{\mathcal{H}}^\sigma_{\epsilon_2,\eta}(A_j) < l \epsilon_1^\sigma,$$
where, as before, \( l > 0 \) is such that \((u_\sigma)_j(t) \geq l\) for all \( t \in S_j\).

If \( B_j \subset A_j \), then \( \tilde{H}^\sigma_{\epsilon_2, \eta}(B_j) \leq \tilde{H}^\sigma_{\epsilon_2, \eta}(A_j) < l\epsilon_1\). Therefore, given \( \delta > 0 \), there exists a covering \( \{B_{jk} : k \geq 1\} \) of \( B_j \) such that \( \eta < \text{diam}(B_{jk}) < \epsilon_2 \) for \( k \geq 1 \) and

\[
\inf \left\{ \sum_{k=1}^{\infty} (u_\sigma)_j(\xi_{jk})(\text{diam}(B_{jk}))^\sigma : \xi_{jk} \in B_{jk} \right\} \leq \tilde{H}^\sigma_{\epsilon_2, \eta}(B_j) + \delta < l\epsilon_1\sigma.
\]

Next we claim that actually \( \text{diam}(B_{jk}) < \epsilon_1 \) for all \( k \geq 1 \). Suppose not. Then there exists an index \( k_1 \) such that \( \text{diam}(B_{j_{k_1}}) \geq \epsilon_1 \). By considering the term corresponding to index \( k_1 \) in the sum and using \((u_\sigma)_j(\xi_{jk}) \geq l\), we get

\[
\inf \left\{ \sum_{k=1}^{\infty} (u_\sigma)_j(\xi_{jk})(\text{diam}(B_{jk}))^\sigma : \xi_{jk} \in B_{jk} \right\} \geq l\epsilon_1\sigma,
\]

which gives a contradiction. Thus \( \text{diam}(B_{jk}) < \epsilon_1 \) for all \( k \geq 1 \) and we conclude that

\[
\tilde{H}^\sigma_{\epsilon_1, \eta}(B_j) \leq \tilde{H}^\sigma_{\epsilon_2, \eta}(B_j) + \delta \quad \text{for } 0 < \eta < \eta_0.
\]

Since \( \delta > 0 \) was arbitrary, we conclude that

\[
\tilde{H}^\sigma_{\epsilon_1, \eta}(B_j) \leq \tilde{H}^\sigma_{\epsilon_2, \eta}(B_j) \quad \text{for } 0 < \eta < \eta_0.
\]

Since \( \tilde{H}^\sigma_{\epsilon_2, \eta}(B_j) \) is a decreasing function of \( \epsilon \), the reverse inequality is obvious. Thus, we obtain

\[
\tilde{H}^\sigma_{\epsilon_1, \eta}(B_j) = \tilde{H}^\sigma_{\epsilon_2, \eta}(B_j) \quad \text{for } 0 < \eta < \eta_0.
\]

If \( C_j \subset S_j \), \( 1 \leq j \leq p \), is the unique invariant list guaranteed by Theorem 3.2.1, it is convenient in the arguments below to work in the compact sets \( C_j \) rather than \( S_j \). For this to be permissible, we must first show that each set \( C_j \) is a perfect metric space. Our first lemma in this direction has essentially been established in the proof of Lemma 5.3.1 but we give the proof for completeness.

**Lemma 5.3.5.** Assume that the hypotheses H5.1, H5.2, H5.3 and H5.4 are satisfied and let \( C_j \subset S_j \), \( 1 \leq j \leq p \), be the invariant sets guaranteed by Theorem 3.2.1. Then there exist \( m_0 > 0 \) and \( \delta_0 > 0 \) such that for all \((j,e) \in \Gamma \) and for all \( s,t \in S_j \) with \( s \in C_j \) and \( d_j(s,t) < \delta_0 \),

\[
d_{\alpha(j,e)}(\theta(j,e)(s),\theta(j,e)(t)) \geq m_0d_j(s,t).
\]
Proof. We argue by contradiction. If the lemma is not true, then for each positive integer \( k \), there exist \( s_k, t_k \in S_j \) with \( s_k \in C_j, 0 < d_j(s_k, t_k) < k^{-1} \) and \( d_{n(j,e)}(\theta_{j,e}(s_k), \theta_{j,e}(t_k)) < k^{-1}d_j(s_k, t_k) \). Since \( C_j \) is compact, by taking a subsequence, we can assume that \( s_k \to s \). Since \( 0 < d_j(s_k, t_k) < k^{-1} \), it follows, by taking a corresponding subsequence, that \( t_k \to s \). But this implies that \( D\theta_{j,e}(s) = 0 \), which contradicts H5.2. Thus \( m_0 > 0 \) and \( \delta > 0 \) satisfying the lemma exist.

Lemma 5.3.6. Assume that the hypotheses H5.1, H5.2, H5.3 and H5.4 are satisfied and let \( C_j \subset S_j, 1 \leq j \leq p, \) be the invariant list guaranteed by Theorem 3.2.1. Then either (a) each \( C_j, 1 \leq j \leq p, \) is a perfect metric space or (b) each \( C_j, 1 \leq j \leq p, \) is a finite set. If \( \theta_{j,e}|C_j \) is one-to-one for all \((j, e) \in \Gamma \) and each \( C_j \) is a finite set, then each \( C_j \) contains exactly one element for \( 1 \leq j \leq p \).

Proof. Assume that \( C_i \) is not a finite set for some \( i, 1 \leq i \leq p \). It follows that there exists \( \tau_i \in C_i \) such that \( \tau_i \) is an accumulation point of \( C_i \). If \( 1 \leq k \leq p \), then H5.4 implies that there exist \((j_l, e_l) \in \Gamma, 1 \leq l \leq n, \) with \( \alpha(j_l, e_l) = j_{l+1} \) for \( 1 \leq l < n, j_1 = i \) and \( \alpha(j_n, e_n) = k \). Writing \((J, E) = [(j_1, e_1), (j_2, e_2), \ldots, (j_n, e_n)], \theta_{(j,e)}(\tau_i) \in C_k. \) Also, because each \( \theta_{(j,e)} \) is a contraction, Lemma 5.3.5 implies that \( \theta_{(j,e)} \) is one-to-one on \( B_{r_0}(\tau_i) := \{ t \in C_i : d_i(t, \tau_i) < r_0 \} \), where \( r_0 = \delta_0/2 \) and \( \delta_0 \) is as in Lemma 5.3.5. Using this fact, we see that \( \theta_{(j,e)}(\tau_i) := \tau_k \) is an accumulation point of \( C_k \).

If \( V = \{1, 2, \ldots, p\}, \Theta \) is as in the proof of Theorem 3.2.1 and \( A_j \) is any closed, bounded, nonempty subset of \( S_j \) for \( 1 \leq j \leq p \), then, because \( \Theta \) is a contraction map in the Hausdorff metric, \( \Theta^n((A_1, A_2, \ldots, A_p)) \) converges in the Hausdorff metric to \((C_1, C_2, \ldots, C_p)\) as \( n \to \infty \). We apply this result to \( A_j = \{ \tau_j \}, \) where \( \tau_j \) is an accumulation point of \( C_j \) for \( 1 \leq j \leq p \). If \( k, 1 \leq k \leq p, \) and \( \sigma \in C_k \) are fixed, it follows that for each \( n \geq 1 \), there exists \((J^n, E^n) = [(j_1^n, e_1^n), (j_2^n, e_2^n), \ldots, (j_n^n, e_n^n)] \) with \( (j^n_i, e^n_i) \in \Gamma \) for \( 1 \leq i \leq n, \alpha(j^n_i, e^n_i) = (j_{i+1}^n, e_{i+1}^n) \) for \( 1 \leq i < n, \alpha(j^n_n, e^n_n) = k \) and \( \theta_{(J^n, E^n)}(\tau^n_{j_i}) \to \sigma \) as \( n \to \infty \). For notational convenience, write \( \tau^n_{j_i} := j_i^n \). If \( r_0 \) and \( \delta_0 \) are as above and \( V_n = \{ t \in C_{\tau^n} : d(t, \tau^n_{j_i}) < r_0 \}, \) then \( \theta_{(J^n, E^n)}|V_n \) is one-to-one and \( \text{diam}(\theta_{(J^n, E^n)}(V_n)) \leq \delta_0 c^n \), where \( \text{Lip}(\theta_{(j,e)}) \leq c < 1 \) for all \((j, e) \in \Gamma \). Because we also know that \( \tau^n_{j_i} \) is an accumulation point of \( C_{\tau^n_{j_i}}, \) it follows that there exist points...
$s_n, t_n \in \theta_{(j^n,k^n)}(V_n)$, $s_n \neq t_n$ and necessarily $\lim_{n \to \infty} s_n = \lim_{n \to \infty} t_n = \sigma$. This, in turn, implies that $\sigma$ is an accumulation point of $C_k$ and that $C_k$ is a perfect metric space for $1 \leq k \leq p$.

If $\theta_{(j,e)}|C_j$ is one-to-one for each $(j, e) \in \Gamma$ and if each $C_j$ is a finite set, we claim that each $C_j$ is a one-point set. If not, then there exists $j$, $1 \leq j \leq p$, such that $C_j$ contains at least two points. Using $H5.4$ and the fact that each $\theta_{(i,e)}|C_i$ is one-to-one, it follows that $C_k$ contains at least two points for $1 \leq k \leq p$. Define $\rho = \inf\{d_j(s, t) : s, t \in C_j, s \neq t, 1 \leq j \leq p\}$, so $\rho > 0$, and select $j$ and $s,t \in C_j$ with $d_j(s, t) = \rho$. Because $\theta_{(j,e)}$ is a contraction and is one-to-one on $C_j$, $\theta_{(j,e)}(s) \neq \theta_{(j,e)}(t)$ and $d_{\alpha(j,e)}(\theta_{(j,e)}(s), \theta_{(j,e)}(t)) < \rho$, a contradiction.

\[\square\]

Assume $H5.1$, $H5.2$, $H5.3$ and $H5.4$. If $f = (f_1, f_2, \ldots, f_p) \in \prod_{j=1}^{p} C(C_j)$, where $(C_1, C_2, \ldots, C_p)$ is as in Theorem 3.2.1, we can define $\Lambda_\sigma : \prod_{j=1}^{p} C(C_j) \rightarrow \prod_{j=1}^{p} C(C_j)$ by (5.4). Furthermore, if each $C_j$ is a perfect metric space (compare Lemma 5.3.6) and $\psi_{(j,e)} : C_j \rightarrow C_{\alpha(j,e)}$ is defined by $\psi_{(j,e)}(s) = \theta_{(j,e)}(s)$ for $s \in C_j$, then $D\psi_{(j,e)}(s)$ is defined for all $s \in C_j$ and $D\psi_{(j,e)}(s) = D\theta_{(j,e)}(s)$ for $s \in C_j$. In any event, by using Lemma 5.2.4, it follows easily that the spectral radius of $L_\sigma$ equals the spectral radius of $\Lambda_\sigma$ and is the eigenvalue of the strictly positive eigenvector $u_\sigma$ of Lemma 5.2.4. The corresponding positive eigenvector $v_\sigma$ for $\Lambda_\sigma$ is given by $(v_\sigma)_j(s) = (u_\sigma)_j(s)$ for $s \in C_j$. If each $C_j$ is a finite set, $\prod_{j=1}^{p} C(C_j)$ is finite-dimensional and determining $r(\Lambda_\sigma) = r(L_\sigma)$ is a finite-dimensional problem.

**Lemma 5.3.7.** Assume that the hypotheses $H5.1$, $H5.2$, $H5.3$ and $H5.4$ are satisfied and let $C_j \subset S_j$, $1 \leq j \leq p$, be the unique invariant list such that

$$C_i = \bigcup_{(j,e) \in \Gamma_i} \theta_{(j,e)}(C_j)$$

for $1 \leq i \leq p$.

Also assume that the map $\theta_{(j,e)}|C_j$ is one-to-one for all $(j, e) \in \Gamma$ and that, for some $j$, $C_j$ has more than one element. Suppose that for all $(j, e) \in \Gamma$ and $(j', e') \in \Gamma$ such that $(j, e) \neq (j', e')$ and $\alpha(j, e) = \alpha(j', e')$, we have that $\theta_{(j,e)}(C_j) \cap \theta_{(j',e')}(C_{j'}) = \emptyset$. Finally, suppose, for $\nu > 0$, that $\{\Lambda_\beta^i : \beta \in A_i(\nu)\}$ is a partition of $C_i$ consisting of compact subsets $\Lambda^i_\beta$ of $C_i$ with $\text{diam}(\Lambda^i_\beta) < \nu$. Then there exist $\epsilon_0 > 0$ and $\nu_0 > 0$ such that for
0 < \epsilon < \epsilon_0, 0 < \nu < \nu_0, 0 < \eta < m\epsilon and 1 \leq i \leq p,

\mu(\nu) r(L_\sigma) \tilde{\mathcal{H}}_{\mu_1(\nu),\epsilon,\eta}(\Lambda^j_\beta; C_i) \leq \sum_{e \in E_i} \tilde{\mathcal{H}}_{\mu_1(\nu),\epsilon,\eta}(\theta(t,i,e)(\Lambda^j_\beta); C_{\alpha(i,e)}), \quad (5.13)

where \mu(\nu) \to 1 and \mu_1(\nu) \to 1 as \nu \to 0+ and (D\theta(t,i,e))(t) > m for all t \in C_i and all \((i,e) \in \Gamma\).

Remark. For compact sets \(B \subset C_k\), we shall always use \(\tilde{\mathcal{H}}_{\epsilon_1,\eta}(B; C_k)\) in the following proof, so for notational simplicity we shall write \(\tilde{\mathcal{H}}_{\epsilon_1,\eta}(B)\) instead of \(\tilde{\mathcal{H}}_{\epsilon_1,\eta}(B; C_k)\).

We interpret (5.13) as meaning that \(\eta\) with \(0 < \eta < m\epsilon\) is such that \(\mathcal{H}_{\epsilon_1,\eta}(\theta(t,i,e)(\Lambda^j_\beta))\) is defined and finite for all \(e \in E_i\) and that this implies that for \(e^* := \mu_1(\nu)e, \tilde{\mathcal{H}}^\sigma_{\epsilon_1,\eta}(\Lambda^j_\beta)\) is defined and finite and (5.13) is satisfied.

Proof. Lemma 5.3.6 implies that each \(C_j, 1 \leq j \leq p\), is a compact, perfect metric space. Since we assume that \(\theta(t,i,e)(C_j) \cap \theta(t',i,e')(C_{j'}) = \emptyset\) whenever \((j,e) \neq (j',e')\) and \(\alpha(j,e) = \alpha(j',e')\), we can select \(\epsilon_0\) so that \(d_k(s,t) > \epsilon_0\) whenever \((j,e) \neq (j',e')\) \(\in \Gamma\), \(\alpha(j,e) = \alpha(j',e') = k\), \(s \in \theta(t,i,e)(C_j)\) and \(t \in \theta(t',i,e')(C_{j'})\). Since, for \(1 \leq k \leq p\), \(C_k = \bigcup_{(j,e) \in \Gamma_k} \theta(t,i,e)(C_j)\), it follows that, as a subset of the metric space \((C_k, d_k)\), \(\theta(t,i,e)(C_j)\) is both compact and open for all \((j,e) \in \Gamma\) with \(\alpha(j,e) = k\). Furthermore, denoting by \(N_\delta(B)\) the closed \(\delta\)-neighborhood in \(C_k\) of a compact set \(B \subset C_k\), \(N_\delta(\theta(t,i,e)(C_j)) = \theta(t,i,e)(C_j)\) for all \((j,e) \in \Gamma\) with \(\alpha(j,e) = k\) and all \(\delta\) with \(0 < \delta \leq \epsilon_0\).

Fix \(\nu > 0\), \(1 \leq i \leq p\), \(\Lambda^i_\beta\) with \(\beta \in \mathcal{A}_i(\nu)\). Suppose that \(\epsilon_0\) is as above. By decreasing \(\epsilon_0\) further we can also assume that \(\epsilon_0 \leq (1 - \epsilon)\nu/(2m)\). Suppose that

\[0 < \epsilon \leq \epsilon_0, 0 < \eta < m\epsilon\]

and that \(\mathcal{H}_{\epsilon_1,\eta}(\Lambda^j_\beta)\) is defined and finite for all \(e \in E_i\).

For any \(\delta > 0\), there exists an open (in the relative topology of \(C_{\alpha(i,e)}\), necessarily finite covering of \(\theta(t,i,e)(\Lambda^j_\beta)\), \(\{A_j : 1 \leq j \leq k\}\), in \(C_{\alpha(i,e)}\), such that \(\eta < \text{diam}(A_j) < m\epsilon\) for \(1 \leq j \leq k\) and

\[\inf \left\{ \sum_{j=1}^{k} (u_{\sigma})_{\alpha(i,e)}(\xi_j)(\text{diam}(A_j))\right\} \leq \mathcal{H}_{\epsilon_1,\eta}(\theta(t,i,e)(\Lambda^j_\beta)) + \delta. \quad (5.14)\]

We denote by \(N_{m\epsilon}(\theta(t,i,e)(\Lambda^j_\beta))\) the closed \(m\epsilon\)-neighborhood of \(\theta(t,i,e)(\Lambda^j_\beta)\) in \(C_{\alpha(i,e)}\), and because \(m\epsilon \leq \epsilon_0\), we observe that \(N_{m\epsilon}(\theta(t,i,e)(\Lambda^j_\beta)) \subset \theta(t,i,e)(C_i)\). We have that \(\text{diam}(A_j) < \]

\[\cdots \]
\[ \theta_{(i,e)}(A^i_{\beta}) \subset \bigcup_{j=1}^{k} A_j \subset N_{m\epsilon}(\theta_{(i,e)}(A^i_{\beta})) \subset \theta_{(i,e)}(C_i). \quad (5.15) \]

Since \( \text{diam}(\theta_{(i,e)}(A^i_{\beta})) \leq c \ \text{diam}(A^i_{\beta}) < cv, \)

\[ \text{diam}(N_{m\epsilon}(\theta_{(i,e)}(A^i_{\beta}))) \leq cv + 2m\epsilon < v \quad (5.16) \]

since we assumed that \( \epsilon < \epsilon_0 \leq (1 - c)\nu/(2m) \). Since \( \theta_{(i,e)}|_{C_i} \) is one-to-one, we derive from (5.15) that we have \( \theta^{-1}_{(i,e)} : \theta_{(i,e)}(C_i) \to C_i \) and

\[ A^i_{\beta} \subset \bigcup_{j=1}^{k} \theta^{-1}_{(i,e)}(A_j) \subset C_i. \quad (5.17) \]

Using Lemma 5.1.3 and Remark 5.1.4, there exists \( \mu_1(\nu) \) with \( \mu_1(\nu) \to 1+ \) as \( \nu \to 0+ \) and

\[ \mu_1(\nu)^{-1}(D\theta^{-1}_{(i,e)})(y) \leq \frac{d(\theta^{-1}_{(i,e)}(x), \theta^{-1}_{(i,e)}(y))}{d(x, y)} \leq \mu_1(\nu)(D\theta^{-1}_{(i,e)})(y) \quad (5.18) \]

for all \( x, y \in \theta_{(i,e)}(C_i) \) with \( 0 < d(x, y) < \nu \) and, in particular, for all \( x \) and \( y \) in \( N_{m\epsilon}(\theta_{(i,e)}(A^i_{\beta})) \). Note that \( \mu_1(\nu) \) can be chosen to be independent of \( (i, e) \in \Gamma. \) We write \( d \) instead of \( d_i \) or \( d_{\alpha(i,e)} \) here and below. In particular, for any \( x, y \in A_j, 1 \leq j \leq k, \)

\[ d(\theta^{-1}_{(i,e)}(x), \theta^{-1}_{(i,e)}(y)) \leq \mu_1(\nu)(D\theta^{-1}_{(i,e)})(y)d(x, y) \leq \mu_1(\nu) \left( \frac{1}{(D\theta_{(i,e)})(\theta^{-1}_{(i,e)}y)} \right) \text{diam}(A_j). \]

Using the compactness of \( \theta^{-1}_{(i,e)}(N_{m\epsilon}(\theta_{(i,e)}(A^i_{\beta}))) \), there exists \( \tau_1 \in \theta^{-1}_{(i,e)}(N_{m\epsilon}(\theta_{(i,e)}(A^i_{\beta}))) \) with

\[ \text{diam}(\theta^{-1}_{(i,e)}(A_j)) \leq \mu_1(\nu) \frac{1}{(D\theta_{(i,e)})(\tau_1)} \text{diam}(A_j). \quad (5.19) \]

Since \( \text{diam}(A_j) < m\epsilon, (5.19) \) implies

\[ \text{diam}(\theta^{-1}_{(i,e)}(A_j)) < \mu_1(\nu) \left( \frac{1}{m} \right) m\epsilon = \mu_1(\nu)\epsilon. \]

By choosing \( x_0, y_0 \in N_{m\epsilon}(\theta_{(i,e)}(A^i_{\beta})) \) with \( d(x_0, y_0) = \text{diam}(A_j) \), we also obtain

\[ \text{diam}(\theta^{-1}_{(i,e)}(A_j)) \geq d(\theta^{-1}_{(i,e)}(x_0), \theta^{-1}_{(i,e)}(y_0)) \geq \mu_1(\nu)^{-1} \left( \frac{1}{(D\theta_{(i,e)})(\theta^{-1}_{(i,e)}y_0)} \right) \text{diam}(A_j). \]
Since $\text{diam}(A_j) > \eta$ and $(D\theta_{(i,e)}(\theta_{(i,e)}^{-1}(y_0))) \leq c < 1$, we find that $\text{diam}(\theta_{(i,e)}^{-1}(A_j)) \geq \mu_1(\nu)^{-1}c^{-1}\eta$. It follows that, assuming we originally chose $\nu_0$ such that $\mu_1(\nu)c < 1$ for $0 < \nu < \nu_0$, $\text{diam}(\theta_{(i,e)}^{-1}(A_j)) > \eta$. Since $\Lambda^i_j \subset \bigcup_{j=1}^k \theta_{(i,e)}^{-1}(A_j)$, we conclude that

$$\mathcal{H}_{\mu_1(\nu)c,\eta}(\Lambda^i_j) \leq \inf \left\{ \sum_{j=1}^k (u_\sigma)_i(\zeta_j)(\text{diam}(\theta_{(i,e)}^{-1}(A_j)))^\sigma : \zeta_j \in \theta_{(i,e)}^{-1}(A_j) \right\}.$$  

Using (5.19) and writing $e^* = \mu_1(\nu)c$, we have

$$\mathcal{H}_{e^*,\eta}(\Lambda^i_j) \leq \frac{\mu_1(\nu)^\sigma}{(\text{diam}(\theta_{(i,e)}^{-1}(\eta)))^\sigma} \inf \left\{ \sum_{j=1}^k \frac{(u_\sigma)_i(\zeta_j)(\text{diam}(A_j))^{\sigma}(u_\sigma)_i(\zeta_j)}{(u_\sigma)_i(\theta_{(i,e)}(\zeta_j))} : \zeta_j \in \theta_{(i,e)}^{-1}(A_j) \right\}.$$  

Choose $\tau_2 \in \theta_{(i,e)}^{-1}(N_{me}(\theta_{(i,e)}(\Lambda^i_j)))$ such that

$$\frac{(u_\sigma)_i(\tau_2)}{(u_\sigma)_i(\theta_{(i,e)}(\tau_2))} \geq \frac{(u_\sigma)_i(\zeta_j)}{(u_\sigma)_i(\theta_{(i,e)}(\zeta_j))}$$

for all $\zeta_j \in \theta_{(i,e)}^{-1}(A_j)$, $1 \leq j \leq k$. Using this together with (5.14), we get

$$\mathcal{H}_{e^*,\eta}(\Lambda^i_j) \leq \frac{\mu_1(\nu)^\sigma}{(\text{diam}(\theta_{(i,e)}^{-1}(\eta)))^\sigma} \frac{(u_\sigma)_i(\tau_2)}{(u_\sigma)_i(\theta_{(i,e)}(\tau_2))} \mathcal{H}_{me,\eta}(\theta_{(i,e)}(\Lambda^i_j)) + \delta.$$  

Since $\delta > 0$ was arbitrary, we get

$$\mathcal{H}_{e^*,\eta}(\Lambda^i_j) \leq \frac{\mu_1(\nu)^\sigma}{(\text{diam}(\theta_{(i,e)}^{-1}(\eta)))^\sigma} \frac{(u_\sigma)_i(\tau_2)}{(u_\sigma)_i(\theta_{(i,e)}(\tau_2))} \mathcal{H}_{me,\eta}(\theta_{(i,e)}(\Lambda^i_j)).$$  

(5.20)

The final step consists of replacing $\tau_1, \tau_2$ by $\xi \in \Lambda^i_j$. From (5.18), it follows that

$$\text{diam}(\theta_{(i,e)}^{-1}(N_{me}(\theta_{(i,e)}(\Lambda^i_j)))) \leq \mu_1(\nu)\frac{1}{m}c \leq \kappa\nu,$$

where $\kappa$ is independent of $\nu$. In particular, we have that $d(\tau_1, \tau_2) \leq \kappa\nu$, so by continuity, there exists a function $\mu_2(\nu)$ such that $\mu_2(\nu) \to 1$ as $\nu \to 0$ and

$$\mu_2(\nu)((D\theta_{(i,e)}(\xi)))^\sigma\frac{(u_\sigma)_i(\xi)}{(u_\sigma)_i(\zeta_j)} \leq ((D\theta_{(i,e)}(\xi)))^\sigma\frac{(u_\sigma)_i(\theta_{(i,e)}(\xi))}{(u_\sigma)_i(\theta_{(i,e)}(\tau_2))},$$

$\xi \in \Lambda^i_j$. Using this, (5.20) implies that

$$\mathcal{H}_{me,\eta}(\theta_{(i,e)}(\Lambda^i_j)) \geq \mu(\nu)\frac{(u_\sigma)_i(\theta_{(i,e)}(\xi))}{(u_\sigma)_i(\xi)}((D\theta_{(i,e)}(\xi)))^\sigma\mathcal{H}_{e^*,\eta}(\Lambda^i_j),$$

where $\mu(\nu) = \mu_2(\nu)(\mu_1(\nu))^{-\sigma}$. Now, we sum over $e \in E_i$, and use the fact that

$$\sum_{e \in E_i} ((D\theta_{(i,e)}(\xi)))^\sigma(u_\sigma)_{a(i,e)}(\theta_{(i,e)}(\xi)) = (L_\sigma u_\sigma)_i(\xi) = r(L_\sigma)(u_\sigma)_i(\xi)$$

to obtain (5.13). This completes the proof. □
Now we are ready to prove the remaining inequality.

**Theorem 5.3.8.** Assume that the hypotheses H5.1, H5.2, H5.3 and H5.4 are satisfied and let \( C_j \subset S_j, 1 \leq j \leq p \), be the unique invariant list such that

\[
C_i = \bigcup_{(j,e) \in \Gamma_i} \theta_{(j,e)}(C_j) \quad \text{for } 1 \leq i \leq p.
\]

Also assume that \( \theta_{(j,e)}|_{C_j} \) is one-to-one for all \( (j,e) \in \Gamma \) and that \( \theta_{(j,e)}(C_j) \cap \theta_{(j',e')}((C_{j'})) \) is empty whenever \( \alpha(j,e) = \alpha(j',e') \) and \( (j,e) \neq (j',e') \). Let \( \beta_0 \) be the unique non-negative real number such that \( r(L_{\sigma_0}) = 1 \) and let \( \beta_0 \) denote the common Hausdorff dimension of \( C_i \) for \( 1 \leq i \leq p \). Then \( \beta_0 \geq \sigma_0 \).

**Proof.** We make the same notational conventions as in Lemma 5.3.7. If \( B \) is any compact subset of \( C_j, 1 \leq j \leq p \), and \( \epsilon > 0 \), it is convenient to note that there is a positive, decreasing function \( \phi(\epsilon) \), independent of \( j \) and \( B \), such that \( \tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(B;C_j) := \mathcal{H}^\sigma_{\epsilon,\eta}(B) \) is defined whenever \( 0 < \eta < \phi(\epsilon) \).

By Lemma 5.3.6 either (a) each \( C_j \) is a compact, perfect metric space or (b) each \( C_j \) is a single point. In case (b), our assumptions imply that for each \( i, 1 \leq i \leq p \), there is a unique \( (j,e) \in \Gamma \) with \( \alpha(j,e) = i \). Using strong connectedness it follows that for each \( 1 \leq j \leq p \), \( E_j = \{e_j\} \), a singleton. The linear map \( L_{\sigma} \) then takes a simple form

\[
(L_{\sigma}f)(t_j) = (D\theta_{(j,e_j)})^\sigma f_{\alpha(j,e_j)}(\theta_{(j,e_j)}(t_j)) \quad \text{for } t_j \in S_j, 1 \leq j \leq p.
\]

It follows that \( L_{\sigma}g = g \), where \( g = (g_1, g_2, \ldots, g_p) \) is the function which is identically equal to 1 in each component. This shows that \( r(L_{\sigma}) = 1 \) and hence \( \sigma_0 = 0 \). Clearly \( \beta_0 = 0 \) in case (b) because the Hausdorff dimension of any finite set is zero. So \( \sigma_0 = \beta_0 = 0 \) in case (b).

Thus we shall assume that we are in case (a).

Suppose \( \beta_0 < \sigma_0 \). Then there exists a \( \sigma < \sigma_0 \) such that \( \mathcal{H}^\sigma(C_i) = 0 \) for \( 1 \leq i \leq p \). This implies that for every \( \epsilon > 0 \), \( \mathcal{H}^\sigma_\epsilon(C_i) = 0 \) and using (5.7), we have

\[
\tilde{\mathcal{H}}^\sigma_\epsilon(C_i) = 0 \quad \text{for } \epsilon > 0. \quad (5.21)
\]

Let \( (J,E) = [(j_1, e_1), \ldots, (j_n, e_n)] \in \Gamma(n) \) and \( \theta_{(j,E)} = \theta_{(j_1, e_1)} \circ \cdots \circ \theta_{(j_n, e_n)} \). Then \( \text{diam}(\theta_{(j,E)}(C_{j_n})) \leq c^n \text{diam}(C_{j_n}) \). So, given \( \nu > 0 \), we can choose \( n = n(\nu) \) large enough such that

\[
\text{diam}(\theta_{(J,E)}(C_{j_n})) < \nu \quad \text{for all } (J,E) \in \Gamma(n).
\]
For $1 \leq i \leq p$, we have

$$C_i = \bigcup_{(J,E) \in \Gamma_i^{(n)}} \theta_{(J,E)}(C_{j_n}) \quad (5.22)$$

with the union being pairwise disjoint because of the disjointness assumption and the assumption that $\theta_{(J,E)}|C_j$ is one-to-one for $(j,e) \in \Gamma$. By Lemma 5.3.7, writing $\epsilon^* = \mu_1(\nu)\epsilon$, we have

$$\mu(\nu) r(L_\sigma) \tilde{H}^\sigma_{\epsilon^*,\eta}(\theta_{(J,E)}(C_{j_n})) \leq \sum_{e \in E_i} \tilde{H}^\sigma_{\epsilon,\eta}(\theta(\theta_{(J,E)}(C_{j_n}))) \quad (5.23)$$

where $\mu(\nu) \to 1$ as $\nu \to 0$. Since $H^\sigma(C_j) = 0$ for $1 \leq j \leq p$, using Lemma 5.3.4, we get $\eta_0 > 0$ such that for $0 < \eta < \eta_0$,

$$\tilde{H}^\sigma_{\epsilon,\eta}(\theta_{(J,E)}(C_{j_n})) = \tilde{H}^\sigma_{\epsilon,\eta}(\theta_{(J,E)}(C_{j_n}))$$

and

$$\tilde{H}^\sigma_{\epsilon^*,\eta}(\theta_{(J,E)}(C_{j_n})) = \tilde{H}^\sigma_{\epsilon,\eta}(\theta_{(J,E)}(C_{j_n})).$$

Therefore, the previous inequality becomes

$$\mu(\nu) r(L_\sigma) \tilde{H}^\sigma_{\epsilon,\eta}(C_i) \leq \sum_{e \in E_i} \tilde{H}^\sigma_{\epsilon,\eta}(\theta(\theta_{(J,E)}(C_{j_n}))). \quad (5.23)$$

Now from (5.22), since the union is disjoint, we can choose $\epsilon > 0$ small enough so that

$$N_\epsilon(\theta_{(J,E)}(C_{j_n})) \cap N_\epsilon(\theta_{(J',E')}(C_{j_n})) = \emptyset$$

for all $(J,E), (J',E') \in \Gamma_i^{(n)}$, $(J,E) \neq (J',E')$. This implies that

$$\tilde{H}^\sigma_{\epsilon,\eta}(C_i) = \sum_{(J,E) \in \Gamma_i^{(n)}} \tilde{H}^\sigma_{\epsilon,\eta}(\theta_{(J,E)}(C_{j_n})).$$

Therefore, we can sum (5.23) over all $(J,E) \in \Gamma_i^{(n)}$ to obtain

$$\mu(\nu) r(L_\sigma) \tilde{H}^\sigma_{\epsilon,\eta}(C_i) \leq \sum_{(J,E) \in \Gamma_i^{(n)}} \sum_{e \in E_i} \tilde{H}^\sigma_{\epsilon,\eta}(\theta(\theta_{(J,E)}(C_{j_n}))).$$

Now we sum over $i = 1, 2, \ldots, p$ to get

$$\mu(\nu) r(L_\sigma) \sum_{i=1}^p \tilde{H}^\sigma_{\epsilon,\eta}(C_i) \leq \sum_{i=1}^p \sum_{(J,E) \in \Gamma_i^{(n)}} \sum_{e \in E_i} \tilde{H}^\sigma_{\epsilon,\eta}(\theta(\theta_{(J,E)}(C_{j_n}))).$$
Note that \( \theta_{(i,e)}(\theta_{(J,E)}(C_{j_n})) \subset C_{\alpha(i,e)} \), so collecting the terms with \( \alpha(i, e) = j, 1 \leq j \leq p \), we get

\[
\mu(\nu)r(L_\sigma) \sum_{i=1}^{p} \hat{H}_{\epsilon,\eta}^\sigma(C_i) \leq \sum_{j=1}^{p} \sum_{(i,e) \in \Gamma_j(J,E) \in \Gamma_i^{(n)}} \hat{H}_{\epsilon,\eta}^\sigma(\theta_{(i,e)}(\theta_{(J,E)}(C_{j_n}))).
\]

Since \( C_j = \bigcup_{(i,e) \in \Gamma_j \cup (J,E) \in \Gamma_i^{(n)}} \theta_{(i,e)}(\theta_{(J,E)}(C_{j_n})) \) with disjoint union, we get

\[
\mu(\nu)r(L_\sigma) \sum_{i=1}^{p} \hat{H}_{\epsilon,\eta}^\sigma(C_i) \leq \sum_{j=1}^{p} \hat{H}_{\epsilon,\eta}^\sigma(C_j).
\] (5.24)

Since \( \sigma < \sigma_0 \), Lemma 5.2.2 implies that \( r(L_\sigma) > 1 \), so we can choose \( \nu > 0 \) small enough so that \( \mu(\nu)r(L_\sigma) > 1 \). But, for \( 1 \leq i \leq p \), we have \( \hat{H}_{\epsilon,\eta}^\sigma(C_i) > 0 \) by the definition of \( \hat{H}_{\epsilon,\eta}^\sigma \). Also, we know that \( \hat{H}_{\epsilon,\eta}^\sigma(C_i) \) is defined and finite for \( \eta \) small enough. So, (5.24) cannot be true. Therefore, our initial assumption must be wrong. Thus, \( \beta_0 \geq \sigma_0 \). \[ \square \]

Combining Theorem 5.3.2 and Theorem 5.3.8, we have proved the following theorem.

**Theorem 5.3.9.** Assume that the hypotheses H5.1, H5.2, H5.3 and H5.4 are satisfied and let \( C_j \subset S_j, 1 \leq j \leq p \), be the unique invariant list such that

\[
C_i = \bigcup_{(j,e) \in \Gamma_i} \theta_{(j,e)}(C_j) \quad \text{for } 1 \leq i \leq p.
\]

Also assume that \( \theta_{(j,e)}|C_j \) is one-to-one for all \( (j,e) \in \Gamma \) and that \( \theta_{(j,e)}(C_j) \cap \theta_{(j',e')}(C_{j'}) \) is empty whenever \( \alpha(j, e) = \alpha(j', e') \) and \( (j, e) \neq (j', e') \). Let \( \sigma_0 \) be the unique non-negative real number such that \( r(L_{\sigma_0}) = 1 \). Then the Hausdorff dimension of each \( C_i \) for \( 1 \leq i \leq p \) is the same, and if \( \beta_0 \) denotes the common Hausdorff dimension of \( C_i \), \( 1 \leq i \leq p \), then \( \beta_0 = \sigma_0 \).

**5.4 Choice of an Appropriate Metric**

In this section we shall see that by choosing an appropriate metric the theory can be applied to examples where we cannot work directly with the Euclidean metric. For instance, we discuss the “Carathéodory-Reiffen-Finsler (CRF) metric” on bounded open subsets of \( \mathbb{C} \).

We need to recall the definition of the Carathéodory-Reiffen-Finsler (CRF) metric on bounded, open, connected sets in Banach spaces. Let \( G \) be a bounded, open, connected
subset of a complex Banach space \((X, \|\cdot\|)\) and let \(U\) denote the open unit disc in \(\mathbb{C}\). Let \(Hol(G, U)\) be the family of all holomorphic functions \(f : G \to U\). Define \(\alpha : G \times X \to \mathbb{R}\) by

\[
\alpha(x, v) = \sup\{|Dg(x)v| : g \in Hol(G, U)\}
\]

where \(Dg(x)\) denotes the Fréchet derivative of \(g\) at \(x\). We should note that \(\alpha(x, v)\) is finite because \(|Dg(x)v| \leq \frac{\|v\|}{\text{dist}(x, \partial G)}\) for all \(g \in Hol(G, U)\) by the Cauchy estimates (see Theorem 9.3 in [8]). Given any two points \(x\) and \(y\) in \(G\), consider the (nonempty) family of curves \(\gamma : [0, 1] \to G\) that have piecewise continuous derivatives and \(\gamma(0) = x, \gamma(1) = y\). Call such a curve admissible and define its length by

\[
L(\gamma) = \int_0^1 \alpha(\gamma(t), \gamma'(t)) \, dt.
\]

We now define the distance between \(x\) and \(y\) by

\[
\rho(x, y) = \inf\{L(\gamma) : \gamma\text{ is admissible with }\gamma(0) = x\text{ and }\gamma(1) = y\}.
\]

\(\rho\) is called the CRF metric on \(G\). For a detailed discussion of the CRF metric we refer the reader to [8].

Let \(G\) be a bounded, open, connected set in \(\mathbb{C}\) and let \(\theta : G \to G\) be a holomorphic map such that \(\overline{\theta(G)}\) is a compact subset of \(G\). If \(\rho\) denotes the CRF metric on \(G\) then it is known (see Theorem 13.1 in [8]) that \(\theta\) is a strict contraction on \(G\) with respect to the CRF metric \(\rho\). Also, on a compact subset \(C\) of \(G\), \(\rho\) is a complete metric and is equivalent to the standard Euclidean metric, i.e., there exist positive constants \(m\) and \(M\) such that

\[
m|z - w| \leq \rho(z, w) \leq M|z - w|\text{ for all }z, w \in C.
\]

Let \(G\) be a bounded, open set in \(\mathbb{C}\) and assume, for \(1 \leq i \leq N\), that \(\theta_i : G \to G\) is a holomorphic map such that \(C_i = \overline{\theta_i(G)}\) is a compact subset of \(G\) and \(\theta'_i(z) \neq 0\) for all \(z \in G\). Define \(C := \bigcup_{i=1}^N C_i\). Then, by Theorem 2.3.1 and the above remarks, there exists a unique nonempty compact set \(K\) with \(K = \bigcup_{i=1}^N \theta_i(K)\). For \(k \geq 1\), define \(I_k = \{I = (i_1, i_2, \ldots, i_k) : 1 \leq i_j \leq N \text{ for } 1 \leq j \leq k\}\). For \(I = (i_1, i_2, \ldots, i_k) \in I_k\), define \(\theta_I = \theta_{i_k} \circ \cdots \circ \theta_{i_2} \circ \theta_{i_1}\). It is easy to see that \(K = \bigcup_{I \in I_k} \theta_I(K)\). We claim that, for
large $k$, $\theta_I$ is a strict contraction map with respect to the Euclidean metric. Suppose that $z, w \in C$, $z \neq w$. Then
\[
\frac{|\theta_I(z) - \theta_I(w)|}{|z - w|} \leq \frac{1}{M} \rho(\theta_I(z), \theta_I(w)) \leq \frac{M}{m} c^k,
\]
where $c < 1$ is the maximum of the contraction ratios of the maps $\theta_i$, $1 \leq i \leq N$, with respect to the metric $\rho$. If we choose $k$ large enough so that $\frac{M}{m} c^k < 1$, then it follows that $\theta_I$ is a contraction map for all $I \in \mathcal{I}_k$ with respect to the Euclidean metric. Thus, if $\theta_I(K) \cap \theta_J(K) = \emptyset$ for $I, J \in \mathcal{I}_k$, $I \neq J$ (which is certainly true if $\{\theta_i(K)\}_{i=1}^N$ are pairwise disjoint and $\theta_i|_K$ is one-to-one for $1 \leq i \leq N$), the Hausdorff dimension of the invariant set $K$ is given by Theorem 5.3.9 by considering the iterated function system given by the maps $\{\theta_I\}_{I \in \mathcal{I}_k}$ and the standard Euclidean metric. Note that in this case $(D\theta_I)(z)$ is nothing but $|\theta'_I(z)|$. If we write, for $\sigma \geq 0$, $(L_\sigma f)(z) = \sum_{i=1}^N |\theta'_i(z)|^\sigma f(\theta_i(z))$ and $(\tilde{L}_\sigma f)(z) = \sum_{I \in \mathcal{I}_k} |\theta'_I(z)|^\sigma f(\theta_I(z))$, where $k$ is as chosen above, then it is easy to see that $\tilde{L}_\sigma = L_\sigma^k$. It follows from Lemma 5.4.1 below that $r(\tilde{L}_{\sigma_0}) = 1$ if and only if $r(L_{\sigma_0}) = 1$. Thus, to find the Hausdorff dimension of the invariant set $K$, it is enough to find $\sigma_0$ such that $r(L_{\sigma_0}) = 1$.

The following lemma is well known, and the proof is given only for completeness.

**Lemma 5.4.1.** Let $X$ be a Banach space, $L : X \to X$ be a bounded linear map and $k \geq 1$ be a positive integer. Then $r(L^k) = (r(L))^k$, where $r(L)$ denotes the spectral radius of $L$.

**Proof.** We have that
\[
\begin{align*}
    r(L^k) &= \lim_{n \to \infty} \| (L^k)^n \|^{1/n} \\
    &= \lim_{n \to \infty} \left( \| L^{kn} \|^{1/kn} \right)^k \\
    &= \left( \lim_{n \to \infty} \| L^{kn} \|^{1/kn} \right)^k \\
    &= (r(L))^k.
\end{align*}
\]
Chapter 6

Infinite Graph-Directed Systems

In this chapter we shall obtain results for the case of graph-directed systems where we allow the edge-set \( E \) to be countably infinite but the vertex-set \( V \) is assumed to be a finite set. This, clearly, generalizes the infinite iterated system case which is obtained by taking the vertex set to be a singleton. The simpler infinite iterated function system case is discussed in Section 5 of [24].

6.1 Introduction

Throughout this section we shall make the following basic assumption.

**H6.1** Let \( V = \{1, 2, \ldots, p\} \) be a finite vertex-set and \( S_1, S_2, \ldots, S_p \) be compact, perfect metric spaces. Let \( E \) be a countably infinite set. Let \( \Gamma \) be an infinite subset of \( V \times E \) and \( \alpha : \Gamma \to V \) be a Lipschitz map with \( \text{Lip}(\theta_{(j,e)}) \leq c < 1 \). Also assume that \( \Gamma_i := \{(j, e) \in \Gamma : \alpha(j, e) = i\} \) is nonempty and \( E_i := \{e \in E : (i, e) \in \Gamma\} \) is nonempty for \( 1 \leq i \leq p \).

Let \( \Gamma^\infty := \{\omega = ((j_1, e_1), (j_2, e_2), \ldots) : (j_i, e_i) \in \Gamma, \alpha(j_{i+1}, e_{i+1}) = j_i, i \geq 1\} \). For \( \omega \in \Gamma^\infty \) and \( n \geq 1 \), let \( \omega|_n = ((j_1, e_1), (j_2, e_2), \ldots, (j_n, e_n)) \). Consider the map \( \theta_{\omega|_n} := \theta_{(j_1, e_1)} \circ \cdots \circ \theta_{(j_n, e_n)} : S_{j_n} \to S_{\alpha(j_1, e_1)} \). For \( \omega \in \Gamma^\infty \), \( \{\theta_{\omega|_n}(S_{j_n})\}_{n \geq 1} \) is a decreasing sequence of compact sets in \( S_{\alpha(j_1, e_1)} \). Furthermore, \( \text{diam}(\theta_{\omega|_n}(S_{j_n})) \leq c^n \text{diam}(S_{\alpha(j_1, e_1)}) \leq c^n \max\{\text{diam}(S_i) : 1 \leq i \leq p\} \), which goes to zero as \( n \to \infty \). It follows that the intersection \( \bigcap_{n=1}^{\infty} \theta_{\omega|_n}(S_{j_n}) \) is a singleton in \( S_{\alpha(j_1, e_1)} \) and we denote it by \( \pi(\omega) \). Thus we have defined a map

\[
\pi : \Gamma^\infty \to \bigcup_{j=1}^{p} S_j \tag{6.1}
\]

from \( \Gamma^\infty \) to the disjoint union of the compact spaces \( S_j, 1 \leq j \leq p \).
For $\omega = ((j_1, e_1), (j_2, e_2), \ldots) \in \Gamma^\infty$, define $\alpha(\omega) = \alpha(j_1, e_1)$. For $1 \leq j \leq p$, define $C_j := \{\pi(\omega) : \omega \in \Gamma^\infty, \alpha(\omega) = j\}$. We shall call $(C_1, C_2, \ldots, C_p)$ the limit set list of the graph-directed system.

**Theorem 6.1.1.** Assume that the hypothesis H6.1 is satisfied and let $(C_1, C_2, \ldots, C_p)$ be the limit set list defined above. Then

$$C_i = \bigcup_{(j,e) \in \Gamma_i} \theta_{(j,e)}(C_j), \quad 1 \leq i \leq p.$$  \hfill (6.2)

**Proof.** Fix $1 \leq i \leq p$. Let $(j, e) \in \Gamma_i$ and $\pi(\omega) \in C_j$. Then $\theta_{(j,e)}(\pi(\omega)) = \pi((j, e), \omega)$ which is in $C_i$ because $\alpha(j, e) = i$. Thus $\theta_{(j,e)}(C_j) \subset C_i$ for all $(j, e) \in \Gamma_i$. Conversely, if $\pi(\omega) \in C_i$, then $\omega = ((j_1, e_1), (j_2, e_2), \ldots) \in \Gamma^\infty$ with $\alpha(\omega) = i$. If we denote by $\sigma(\omega)$ the left shift $((j_2, e_2), (j_3, e_3), \ldots)$ of $\omega$, then $\pi(\omega) = \theta_{(j_1, e_1)}(\pi(\sigma(\omega)))$. Since $\alpha(\sigma(\omega)) = \alpha(j_2, e_2) = j_1$, $\pi(\omega) \in \theta_{(j_1, e_1)}(C_{j_1})$. Since $(j_1, e_1) \in \Gamma_i$, we have proved the opposite containment $C_i \subset \bigcup_{(j,e) \in \Gamma_i} \theta_{(j,e)}(C_j)$. This completes the proof of the theorem. \hfill $\square$

**Remark 6.1.2.** Notice that the sets $C_i$, $1 \leq i \leq p$, need not be compact. But if the set $\Gamma$ is finite, i.e., if we have a finite graph-directed system, then the sets $C_i$, $1 \leq i \leq p$, defined as above using the map $\pi$, are compact. Therefore, by the uniqueness in Theorem 3.2.1, for a finite graph-directed system, $\{C_i\}_{1 \leq i \leq p}$ defined in this section is the same as the unique list of nonempty, compact sets obtained in Theorem 3.2.1.

As in the case of finite graph-directed system, we define, for $M > 0$, $\lambda \geq 0$ and $1 \leq j \leq p$,

$$K_j(M, \lambda) = \{f \in C(S_j) : 0 \leq f(s) \leq f(t) \exp(M(d_j(s, t))^{\lambda}) \text{ for all } s, t \in S_j\}.$$

In addition to H6.1, we shall assume the following.

**H6.2** For each $(j, e) \in \Gamma$, the map $\theta_{(j,e)} : S_j \to S_{\alpha(j,e)}$ is an infinitesimal similitude, and there exist $M_0 > 0$ and $\lambda > 0$ such that for all $(j, e) \in \Gamma$, $D\theta_{(j,e)} \in K_j(M_0, \lambda)$ and $D\theta_{(j,e)}(t) > 0$ for all $t \in S_j$. Also assume that there exist $\sigma > 0$ and $t_j \in S_j$ with $\sum_{e \in E_j}(D\theta_{(j,e)}(t_j))^\sigma < \infty$ for $1 \leq j \leq p$. 


Lemma 6.1.3. Assume that H6.1 and H6.2 are satisfied and let $1 \leq j \leq p$ be fixed. If $\sum_{e \in E_j} (D\theta_{(j,e)}(t_s))^s < \infty$ for some $t_s \in S_j$ and $s > 0$, then for any $\sigma \geq s$, $\sum_{e \in E_j} (D\theta_{(j,e)}(t))^s < \infty$ for all $t \in S_j$.

Proof. Since $D\theta_{(j,e)}(t) \leq c < 1$ for all $t \in S_j$, $(D\theta_{(j,e)}(t))^s \leq (D\theta_{(j,e)}(t))^s$ for $\sigma \geq s$. So $\sum_{e \in E_j} (D\theta_{(j,e)}(t))^s < \infty$ implies that $\sum_{e \in E_j} (D\theta_{(j,e)}(t))^s < \infty$ for $\sigma \geq s$. Thus we only have to prove that for any $t \in S_j$, $\sum_{e \in E_j} (D\theta_{(j,e)}(t))^s < \infty$. Let $t \in S_j$. Since $D\theta_{(j,e)} \in K(M_0,\lambda)$, $D\theta_{(j,e)}(t) \leq D\theta_{(j,e)}(t_s)\exp(M_0d_j(t,t_s)^\lambda)$. Therefore,

$$(D\theta_{(j,e)}(t))^s \leq (D\theta_{(j,e)}(t_s))^s \exp(sM_0(d(t,t_s))^\lambda),$$

from which the result follows. \qed

Lemma 6.1.4. Assume that H6.1 and H6.2 are satisfied. Let $1 \leq j \leq p$ be fixed and let $\sigma > 0$ be such that $\sum_{e \in E_j} (D\theta_{(j,e)}(t_s))^\sigma < \infty$ for some $t_s \in S_j$. If we define $b(t) = \sum_{e \in E_j} (D\theta_{(j,e)}(t))^\sigma$ for $t \in S_j$, then $b : S_j \rightarrow (0,\infty)$ is continuous.

Proof. The previous lemma implies that $b(t) < \infty$ for all $t \in S_j$. Also H6.2 implies that $b(t) > 0$ for all $t \in S_j$. Since $D\theta_{(j,e)} \in K(M_0,\lambda)$, it follows that $b(t) \leq b(t_s)\exp(\sigma M_0(d(t,t_s))^\lambda)$. Since $S_j$ is compact, it is bounded and hence there is a constant $R < \infty$ such that $b(t) \leq R$ for all $t \in S_j$. Now to prove that $b$ is continuous, let $s,t \in S_j$. Then we have $b(s) \leq b(t)\exp(\sigma M_0(d(s,t))^\lambda)$, which gives $|b(s) - b(t)| \leq R[\exp(\sigma M_0(d(s,t))^\lambda) - 1]$. The right-hand side approaches 0 as $d(s,t) \rightarrow 0$, so $|b(s) - b(t)| \rightarrow 0$ as $d(s,t) \rightarrow 0$. This proves the continuity of $b$. \qed

Assume that H6.1 and H6.2 are satisfied. For $\sigma > 0$ with $\sum_{e \in E_j} (D\theta_{(j,e)}(t_j))^\sigma < \infty$, $1 \leq j \leq p$, we define $L_\sigma : \prod_{j=1}^p C(S_j) \rightarrow \prod_{j=1}^p C(S_j)$ by

$$(L_\sigma f)_j(t) = \sum_{e \in E_j} (D\theta_{(j,e)}(t))^\sigma f_{(j,e)}(\theta_{(j,e)}(t))$$

for $t \in S_j, 1 \leq j \leq p$. (6.3)

Note that from the previous lemma $t \mapsto \sum_{e \in E_j} (D\theta_{(j,e)}(t))^\sigma$ is continuous for all $1 \leq j \leq p$. It follows that if $f \in \prod_{j=1}^p C(S_j)$, then $(L_\sigma f)_j$ is continuous on $S_j$. Thus $L_\sigma$ is well-defined and is a bounded linear operator on $\prod_{j=1}^p C(S_j)$.

Let $\sigma_0 = \inf\{\sigma > 0 : \sum_{j=1}^p \sum_{e \in E_j} (D\theta_{(j,e)}(t_j))^\sigma < \infty \text{ for all } t_j \in S_j\}$. Then for all $\sigma > \sigma_0$, $L_\sigma$ defined above is a well-defined bounded linear operator on $\prod_{j=1}^p C(S_j)$.
Lemma 6.1.5. Assume that H6.1 and H6.2 are satisfied and let \( \sigma > 0 \) is such that the linear operator \( L_\sigma \) is defined. Then there exists \( M > 0 \) (depending on \( \sigma \)) such that \( L_\sigma \) maps the cone \( \prod_{j=1}^{p} K_j(M, \lambda) \) into itself. Furthermore, \( L_\sigma \) has a non-zero eigenvector \( u_\sigma \in \prod_{j=1}^{p} K_j(M, \lambda) \) with eigenvalue \( r_\sigma = r(L_\sigma) > 0 \).

Proof. Let \( \sigma > 0 \) be fixed. Since \( c < 1 \) and \( \lambda > 0 \), we can choose \( M > 0 \) so that \( \sigma M_0 + Mc^\lambda \leq M \). We claim that \( L_\sigma(K) \subset K \), where \( K := \prod_{j=1}^{p} K_j(M, \lambda) \). For \( 1 \leq j \leq p \), let \( f_j \in K_j(M, \lambda) \), \( e \in E_j \) and \( s, t \in S \). Then we have

\[
\begin{align*}
    f_{\alpha(j,e)}(\theta_{(j,e)}(s)) &\leq f_{\alpha(j,e)}(\theta_{(j,e)}(t)) \exp(M(d_{\alpha(j,e)}(\theta_{(j,e)}(s), \theta_{(j,e)}(t))^{\lambda})) \\
    &\leq f_{\alpha(j,e)}(\theta_{(j,e)}(t)) \exp(Mc^{\lambda}(d_j(s,t)))
\end{align*}
\]

Also, since \( D\theta_{(j,e)} \in K_j(M_0, \lambda) \), we have

\[
(D\theta_{(j,e)}(s))^{\sigma} \leq (D\theta_{(j,e)}(t))^{\sigma} \exp(\sigma M_0(d_j(s,t)))
\]

Thus

\[
(L_\sigma f)_j(s) = \sum_{e \in E_j} (D\theta_{(j,e)}(s))^{\sigma} f_{\alpha(j,e)}(\theta_{(j,e)}(s)) \\
\leq \sum_{e \in E_j} (D\theta_{(j,e)}(t))^{\sigma} f_{\alpha(j,e)}(\theta_{(j,e)}(t)) \exp((\sigma M_0 + Mc^{\lambda})d_j(s,t)) \\
\leq (L_\sigma f)_j(s) \exp(Md_j(s,t))
\]

This proves that \( (L_\sigma f)_j \in K_j(M, \lambda) \) whenever \( f \in K \), i.e., \( L_\sigma(K) \subset K \).

By Lemma 4.1.2, for each \( 1 \leq j \leq p \), \( \{f_j \in K_j : \|f_j\| \leq 1\} \) is an equicontinuous family of functions on compact metric space \( S_j \), and hence it is compact by Ascoli’s theorem. It follows that \( L|_K \) is a compact map. If \( g = (g_1, g_2, \ldots, g_p) \), where \( g_j \) is identically 1 on \( S_j \), then \( g \in K \), so

\[
r_K(L) \geq \lim_{n \to \infty} \|L^n(g)\|^{\frac{1}{n}} = r(L).
\]

The opposite inequality is obviously true, so \( r_K(L) = r(L) \). Since we know, from Lemma 5.1.1, that \( D\theta_{(j,e)} \) is continuous on compact \( S_j \) and \( D\theta_{(j,e)}(t) > 0 \) for all \( t \in S_j \), there exists \( \delta > 0 \) such that \( \sum_{e \in E_j} (D\theta_{(j,e)}(t))^{\sigma} \geq \delta \) for all \( t \in S_j \) and \( 1 \leq j \leq p \). So \( L_\sigma g \geq \delta g \), where \( g \) is the constant function defined above. It follows that \( r(L_\sigma) \geq \delta > 0 \).
Since \( L_\sigma|_K \) is a compact map and \( r_K(L) = r(L) > 0 \), a theorem of Bonsall [1] implies that there exists \( u_\sigma \in K, \|u_\sigma\| = 1 \), with \( L_\sigma(u_\sigma) = r_\sigma u_\sigma, r_\sigma = r(L_\sigma) \).

**Lemma 6.1.6.** The map \( \sigma \mapsto r(L_\sigma) \) is strictly decreasing and continuous for \( \sigma > \sigma_0 \). Also, \( r(L_\sigma) \to 0 \) as \( \sigma \to \infty \).

**Proof.** The proof that \( \sigma \mapsto r(L_\sigma) \) is strictly decreasing and continuous for \( \sigma > \sigma_0 \) is exactly the same as in the proof of Lemma 5.2.2, and we omit the details. To prove the last part of the lemma, fix \( s > 0 \) with \( \max\{1 \leq j \leq p \mid \sup_{t \in S_j} \sum_{e \in E_j} (D\theta_{(j,e)}(t))^s \leq K < \infty \} \). Let \( \sigma > s \). Then, since \( D\theta_{(j,e)}(t) \leq c \),

\[
\sum_{e \in E_j} (D\theta_{(j,e)}(t))^\sigma \leq c^{\sigma - s} \sum_{e \in E_j} (D\theta_{(j,e)}(t))^s.
\]

Therefore, \( \|L_\sigma\| \leq \max_{1 \leq j \leq p} \left( \sup_{t \in S_j} \sum_{e \in E_j} (D\theta_{(j,e)}(t))^\sigma \right) \leq c^{\sigma - s} K \), which implies \( \|L_\sigma\| \to 0 \) as \( \sigma \to \infty \) because \( c < 1 \). Since \( r(L_\sigma) \leq \|L_\sigma\| \), the result follows.

We should note that in the case of infinite graph-directed system there need not be a value of \( \sigma \) for which \( r(L_\sigma) = 1 \) because we cannot guarantee a \( \sigma \) for which \( r(L_\sigma) \geq 1 \). It is possible that \( r(L_\sigma) < 1 \) for all the values of \( \sigma \) for which \( L_\sigma \) is defined. Let

\[
\sigma_\infty = \inf\{\sigma > \sigma_0 : r(L_\sigma) < 1\}.
\]

We claim that, under natural further assumptions, the Hausdorff dimension of the invariant set list is equal to \( \sigma_\infty \).

By Lemma 5.1.3 and Remark 5.1.4, we know that given \( \epsilon > 0 \), there exists a \( \mu_{(j,e)}(\epsilon) \) such that for every \( s, t \in S_j \) with \( 0 < d_j(s, t) < \epsilon \),

\[
\mu_{(j,e)}(\epsilon)^{-1}(D\theta_{(j,e)}(t))(t) \leq \frac{d(\theta_{(j,e)}(s), \theta_{(j,e)}(t))}{d_j(s, t)} \leq \mu_{(j,e)}(\epsilon)(D\theta_{(j,e)}(t))(t)
\]

and \( \lim_{\epsilon \to 0^+} \mu_{(j,e)}(\epsilon) = 1 \). In the case of finitely many \( \theta_{(j,e)} \)'s, a uniform \( \mu(\epsilon) \) satisfying the above property could be chosen by taking the maximum over \( (j,e) \in \Gamma \). But for the infinite case, we cannot guarantee a uniform \( \mu(\epsilon) \) which would work for each \( \theta_{(j,e)} \). So instead we shall make the assumption that a uniform \( \mu(\epsilon) \) can be chosen. For a specific problem we would have to check that this condition is indeed satisfied.
For some important examples such as complex continued fractions, which have been studied by other authors (see Section 6 of [16]), we will show that this condition can be easily verified.

**H6.3** Given \( \epsilon > 0 \), there exists a \( \mu(\epsilon) \geq 1 \) such that for each \((j, e) \in \Gamma \) and for every \( s, t \in S_j \) with \( 0 < d_j(s, t) < \epsilon \),

\[
\mu(\epsilon)^{-1}(D\theta_{(j, e)})(t) \leq \frac{d(\theta_{(j, e)}(s), \theta_{(j, e)}(t))}{d_j(s, t)} \leq \mu(\epsilon)(D\theta_{(j, e)})(t)
\]

and \( \lim_{\epsilon \to 0^+} \mu(\epsilon) = 1 \).

Recall the definition of strong connectedness. The definition for infinite graph-directed case is exactly the same as Definition 5.2.3 which was given for the finite graph-directed case. For the rest of this section we shall assume that strong connectedness is satisfied.

First we shall prove that Hausdorff dimension of \( C_i \) is independent of \( i \), \( 1 \leq i \leq p \), under the assumption of strong connectedness. The proof is similar to the proof of Lemma 5.3.1.

**Lemma 6.1.7.** Assume that the hypotheses H6.1 and H6.2 are satisfied. Also assume that strong connectedness is satisfied. Let \( \dim(C_j) \) denote the Hausdorff dimension of \( C_j \), where \((C_1, C_2, \ldots, C_p)\) is the limit set list satisfying

\[
C_i = \bigcup_{(j, e) \in \Gamma_i} \theta_{(j, e)}(C_j), \quad 1 \leq i \leq p.
\]

Then \( \dim(C_j) \) is independent of \( j \) for \( 1 \leq j \leq p \).

**Proof.** First we claim that \( \dim(\theta_{(j, e)}(C_j)) = \dim(C_j) \) for any \((j, e) \in \Gamma \). Since \( \theta_{(j, e)} \) is a Lipschitz map with Lipschitz constant \( c \), \( \mathcal{H}^{\sigma}(\theta_{(j, e)}(C_j)) \leq c^\sigma \mathcal{H}^{\sigma}(C_j) \) for any \( \sigma \geq 0 \). This implies that \( \dim(\theta_{(j, e)}(C_j)) \leq \dim(C_j) \). To prove the other inequality, we first claim that there exist \( m_0 > 0 \) and \( \delta > 0 \) such that \( d(\theta_{(j, e)}(s), \theta_{(j, e)}(t)) \geq m_0 d(s, t) \) for all \( s, t \in S_j \) with \( d(s, t) \leq \delta \). We abuse notation here by letting \( d \) denote \( d_j \) and \( d_\alpha(j, e) \).

We argue by contradiction. If the claim is false, then for each positive integer \( k \), there exist \( s_k, t_k \in S_j \) with \( 0 < d(s_k, t_k) \leq k^{-1} \) and \( d(\theta_{(j, e)}(s_k), \theta_{(j, e)}(t_k)) < k^{-1} d(s_k, t_k) \). Since \( S_j \) is compact, by taking a subsequence we can assume that \( s_k \to s \) and \( t_k \to s \).
But this implies that $D\theta_{(j,e)}(s) = 0$, which contradicts \text{H6.2}. Thus $m_0 > 0$ and $\delta > 0$ exist. Since $C_j$ is contained in the compact set $S_j$, it is totally bounded, so we can write $C_j = \bigcup_{l=1}^p C_{j,l}$, where $\text{diam}(C_{j,l}) \leq \delta$ and $p < \infty$. It follows easily from the definition of Hausdorff dimension that there exists $l$ such that the Hausdorff dimension of $C_{j,l}$ equals the Hausdorff dimension of $C_j$. Also, by our construction, $\theta_{(j,e)}|C_{j,l}$ is one-to-one and $(\theta_{(j,e)}|C_{j,l})^{-1}$ is Lipschitz. This implies that

$$\dim(C_j) = \dim(C_{j,l}) \leq \dim(\theta_{(j,e)}(C_{j,l})) \leq \dim(\theta_{(j,e)}(C_j)),$$

and we have shown that $\dim(C_j) = \dim(\theta_{(j,e)}(C_j))$.

Now, since $\theta_{(j,e)}(C_j) \subset C_{\alpha(j,e)}$, we get $\dim(C_{\alpha(j,e)}) \geq \dim(\theta_{(j,e)}(C_j)) = \dim(C_j)$ for all $(j,e) \in \Gamma$. Let $1 \leq j \leq p$ and $1 \leq k \leq p$. By strong connectedness, there exists $[(j_1,e_1), \ldots, (j_n,e_n)]$ such that $j_1 = j$, $\alpha(j_i,e_i) = j_{i+1}$, $1 \leq i < n$ and $\alpha(j_n,e_n) = k$. So,

$$\dim(C_k) = \dim(C_{\alpha(j_n,e_n)}) \geq \dim(C_{j_n}) \geq \dim(C_{j_{n-1}}) \geq \cdots \geq \dim(C_{j_1}) = \dim(C_j).$$

Since $j$ and $k$ were arbitrary, it follows that $\dim(C_j) = \dim(C_k)$ for all $1 \leq j, k \leq p$.  

Now we are ready to prove the upper bound for the Hausdorff dimension of $C_i$, $1 \leq i \leq p$. The proof is very similar to the proof of Theorem 5.3.2 but we provide the proof for the sake of completeness.

\textbf{Theorem 6.1.8.} Assume that H6.1, H6.2 and H6.3 are satisfied. Let $\dim(C_j)$ denote the Hausdorff dimension of $C_j$, where $(C_1,C_2,\ldots,C_p)$ is the limit set list satisfying

$$C_i = \bigcup_{(j,e) \in \Gamma_i} \theta_{(j,e)}(C_j), \quad 1 \leq i \leq p.$$

Let $\sigma_\infty$ be as defined in (6.4). Then $\dim(C_i) \leq \sigma_\infty$ for $1 \leq i \leq p$.

\textit{Proof.} Fix $\epsilon > 0$. Take $\delta > 0$ and $\sigma > 0$. We can choose a covering $\{A_{jk}\}_{k=1}^\infty$ of $C_j$ and points $\xi_{jk} \in A_{jk}$ such that $\text{diam}(A_{jk}) < \epsilon$ and

$$\sum_{k=1}^\infty (u_\sigma)_j(\xi_{jk})(\text{diam}(A_{jk}))^\sigma \leq \tilde{H}_\epsilon^\sigma(C_j) + \delta. \quad (6.5)$$

Since $C_i = \bigcup_{(j,e) \in \Gamma_i} \theta_{(j,e)}(C_j)$, we have that $\{\theta_{(j,e)}(A_{jk}) : 1 \leq k < \infty, (j,e) \in \Gamma_i\}$ is a covering of $C_i$ with

$$\text{diam}(\theta_{(j,e)}(A_{jk})) \leq c \text{diam}(A_{jk}) < c\epsilon.$$
Furthermore, by H6.3, there exists $\mu(\epsilon) > 1$ with $\mu(\epsilon) \to 1$ as $\epsilon \to 0+$ such that
\[
\operatorname{diam}(\theta_{j,e}(A_{jk})) \leq \mu_{j,e}(\epsilon)(D\theta_{j,e})(\xi_{jk}) \operatorname{diam}(A_{jk})
\]  
(6.6)
for each $(j, e) \in \Gamma_1$ and $k \geq 1$. Thus we have
\[
\hat{H}^\sigma_{ce}(C_i) \leq \sum_{k=1}^{\infty} \sum_{(j,e) \in \Gamma_1} (u_{\sigma})_{i}(\theta_{j,e}(\xi_{jk})) (\operatorname{diam}(\theta_{j,e}(A_{jk})))^\sigma
\]
\[
\leq (\mu(\epsilon))^\sigma \sum_{k=1}^{\infty} \sum_{(j,e) \in \Gamma_1} (u_{\sigma})_{i}(\theta_{j,e}(\xi_{jk})) ((D\theta_{j,e})(\xi_{jk}))^\sigma (\operatorname{diam}(A_{jk}))^\sigma.
\]
Summing over $i$, $1 \leq i \leq p$, we have
\[
\sum_{i=1}^{p} \hat{H}^\sigma_{ce}(C_i) \leq (\mu(\epsilon))^\sigma \sum_{j=1}^{p} \sum_{k=1}^{\infty} (\operatorname{diam}(A_{jk}))^\sigma \sum_{i:(j,e) \in \Gamma_1} (u_{\sigma})_{i}(\theta_{j,e}(\xi_{jk})) ((D\theta_{j,e})(\xi_{jk}))^\sigma
\]
\[
= (\mu(\epsilon))^\sigma r(L_{\sigma}) \sum_{j=1}^{p} \sum_{k=1}^{\infty} (u_{\sigma})_{j}(\xi_{jk}) (\operatorname{diam}(A_{jk}))^\sigma.
\]
Rearranging the sum, we get
\[
\sum_{i=1}^{p} \hat{H}^\sigma_{ce}(C_i) \leq (\mu(\epsilon))^\sigma r(L_{\sigma}) \sum_{j=1}^{p} \left( \hat{H}^\sigma_{e}(C_j) + \delta \right).
\]
Thus, using (6.5), we get
\[
\sum_{i=1}^{p} \hat{H}^\sigma_{e}(C_i) \leq (\mu(\epsilon))^\sigma r(L_{\sigma}) \sum_{j=1}^{p} \hat{H}^\sigma_{e}(C_j).
\]
Since $c < 1$, $\hat{H}^\sigma_{e}(C_i) \leq \hat{H}^\sigma_{ce}(C_i)$. Also $\delta > 0$ was arbitrary. Therefore,
\[
\sum_{i=1}^{p} \hat{H}^\sigma_{e}(C_i) \leq (\mu(\epsilon))^\sigma r(L_{\sigma}) \sum_{i=1}^{p} \hat{H}^\sigma_{e}(C_i).
\]
(6.7)
By the definition of $\sigma_{\infty}$, $r(L_{\sigma}) < 1$ for all $\sigma > \sigma_{\infty}$. Since $\mu(\epsilon) \to 1$ as $\epsilon \to 0$, given $\sigma > \sigma_{\infty}$, we can choose $\epsilon > 0$ small so that $(\mu(\epsilon))^\sigma r(L_{\sigma}) < 1$. By the definition, for $1 \leq i \leq p$, $\hat{H}^\sigma_{e}(C_i) < \infty$ because we can take a finite $\epsilon$-cover of the totally bounded set $C_i$ contained in the compact set $S_i$. Thus, if $\sigma > \sigma_{\infty}$, (5.10) can hold only if
\[
\lim_{\epsilon \to 0+} \sum_{i=1}^{p} \hat{H}^\sigma_{e}(C_i) = 0.
\]
This implies that for each $1 \leq i \leq p$ and $\sigma > \sigma_{0}$, $\lim_{\epsilon \to 0+} \hat{H}^\sigma_{e}(C_i) = 0$, and hence using (5.7), $\lim_{\epsilon \to 0+} \hat{H}^\sigma_{e}(C_i) = 0$, i.e., $\hat{H}^\sigma(C_i) = 0$ for all $\sigma > \sigma_{\infty}$. Thus, by the definition of Hausdorff dimension, $\dim(C_i) \leq \sigma_{\infty}$. 

\qed
To prove the other half, \( \dim(C_j) \geq \sigma_\infty \), we shall consider the infinite graph-directed system as a limit of finite graph-directed systems and use the result that we have for the finite case.

We first enumerate the countable set \( \Gamma \), and for \( N \geq 1 \), let \( \Gamma_{(N)} \) be the set containing the first \( N \) elements of \( \Gamma \). Then clearly we have \( \Gamma_{(N)} \subseteq \Gamma_{(N+1)} \) and \( \Gamma = \bigcup_{N=1}^{\infty} \Gamma_{(N)} \).

Let \( V_{(N)} = \{ j \in V : (j,e) \in \Gamma_{(N)} \text{ for some } e \in \mathcal{E} \} \) and \( \mathcal{E}_{(N)} = \{ e \in \mathcal{E} : (j,e) \in \Gamma_{(N)} \text{ for some } j \in V \} \). In other words, \( V_{(N)} \) and \( \mathcal{E}_{(N)} \) are the projection onto the first and second components of \( \Gamma_{(N)} \subseteq V \times \mathcal{E} \), so \( \Gamma_{(N)} \subseteq V_{(N)} \times \mathcal{E}_{(N)} \). For each \( N \geq 1 \), \( (V_{(N)}, \mathcal{E}_{(N)}, \Gamma_{(N)}, \alpha|_{\Gamma_{(N)}}) \) gives a finite graph-directed system as discussed in Chapter 3.

For \( 1 \leq i \leq p \), define \( \Gamma_{(N),i} = \{ (j,e) \in \Gamma_{(N)} : \alpha(j,e) = i \} \) and \( E_{(N),i} = \{ e \in \mathcal{E}_{(N)} : (i,e) \in \Gamma_{(N)} \} \). By the assumptions that \( \Gamma_i \neq \emptyset \) and \( E_i \neq \emptyset \), it follows that \( \Gamma_{(N),i} \neq \emptyset \), \( E_{(N),i} \neq \emptyset \) and \( V_{(N)} = V \) for all but finitely many \( N \)'s. Ignoring those finitely many \( N \)'s, there exists \( N_0 \) such that for \( N \geq N_0 \), Theorem 3.2.1 is applicable and we get a unique list \( \{ C_{(N),i} : 1 \leq i \leq p \} \) of nonempty compact sets \( C_{(N),i} \subseteq S_i \) satisfying

\[
C_{(N),i} = \bigcup_{(j,e) \in \Gamma_{(N),i}} \theta_{(j,e)}(C_{(N),j})
\]

for \( 1 \leq i \leq p \). If we assume that our infinite graph-directed system is strongly connected, it follows that the corresponding finite graph-directed systems are also strongly connected for all but finitely many \( N \)'s. We shall assume the following hypothesis.

**H6.4** For each \( N \geq N_0 \), \( \theta_{(j,e)}(C_{(N),j}) \cap \theta_{(j',e')}(C_{(N),j'}) = \emptyset \) for \( (j,e) \neq (j',e') \), and \( \theta_{(j,e)}|_{C_{(N),j}} \) is one-to-one for all \( (j,e) \in \Gamma_{(N)} \).

Define for \( \sigma \geq 0 \), \( N \geq N_0 \) and \( f = (f_1, f_2, \ldots, f_p) \in \prod_{j=1}^{p} C(S_j) \),

\[
(L_{N,\sigma} f)_j(t) = \sum_{e \in E_{(N),j}} (D_{\theta_{(j,e)}(t)})^\sigma f_{\alpha(j,e)}(\theta_{(j,e)}(t)) \text{ for } t \in S_j, 1 \leq j \leq p. \quad (6.8)
\]

Let \( \sigma_N \) be the unique nonnegative real number such that \( r(L_{N,\sigma_N}) = 1 \). By Theorem 5.3.9, we know, assuming H6.1, H6.2 and H6.4, that \( \dim(C_{(N),i}) = \sigma_N \) for \( 1 \leq i \leq p \).

**Lemma 6.1.9.** For each \( N \geq N_0 \) and \( 1 \leq i \leq p \), \( C_{(N),i} \subseteq C_{(N+1),i} \) and \( C_{(N),i} \subseteq C_i \).

Hence, \( \dim(C_i) \geq \dim(C_{(N),i}) = \sigma_N \).

**Proof.** Let \( N \geq N_0 \) and \( 1 \leq i \leq p \). By Remark 6.1.2, we know that \( C_{(N),i} = \{ \pi(\omega) :
\[ \omega \in \Gamma_{\infty}^{(N)}, \alpha(\omega) = i \}. \] Using this, since \( \Gamma_{(N)} \subset \Gamma_{(N+1)} \) and \( \Gamma_{(N)} \subset \Gamma \) for all \( N \geq 1 \), it follows that \( C_{(N),i} \subset C_{(N+1),i} \) and \( C_{(N),i} \subset C_i \). \( \square \)

Using H6.2, we see that \( \|L_{\sigma} - L_{N,\sigma}\| \to 0 \) as \( N \to \infty \) for \( \sigma > \sigma_0 \).

**Remark 6.1.10.** Let \( X \) be a real or complex Banach space and \( L : X \to X, L_k : X \to X, k \geq 1 \) be bounded linear operators. Assume that \( \lim_{k \to \infty} \|L_k - L\| = 0 \).

Then we have that \( \limsup_{k \to \infty} r(L_k) \leq r(L) \). But, in general, it is not true that \( \lim_{k \to \infty} r(L_k) \to r(L) \). In fact, Kakutani has given an example of a sequence of bounded linear operators \( L_k \) on a Hilbert space which converges in the operator norm to an operator \( L \) and satisfies \( r(L_k) = 0 \) for all \( k \geq 1 \) and \( r(L) > 0 \). The example can be found on pages 282-283 of [25]. If, in addition, we know that \( \rho(L) < r(L) \), where \( \rho(L) \) is the essential spectral radius of \( L \), then it is true that \( r(L_k) \to r(L) \). To see this, note that by using the natural extension of \( L \) to the complexification of \( X \), we can assume that \( X \) is a complex Banach space. If \( \sigma(L) \) denotes the spectrum of \( L \), recall that \( \sigma(L) \cap \{ z \in \mathbb{C} : |z| > \rho(L) \} \) consists of isolated points each of which is an eigenvalue of \( L \) of finite algebraic multiplicity. Then exactly the argument on pages 227-228 of [21] proves that \( r(L_k) \to r(L) \).

The following lemma is known. The proof is included for the reader’s convenience.

**Lemma 6.1.11.** Let \((S_j, d_j), 1 \leq j \leq p\), be compact metric spaces and suppose that \( L : X = \prod_{j=1}^{p} C(S_j) \to X \) is a positive bounded linear map, i.e., \( f_j(t_j) \geq 0 \) for all \( t_j \in S_j, 1 \leq j \leq p \) implies that \( (L f_j)(t_j) \geq 0 \) for all \( t_j \in S_j, 1 \leq j \leq p \). Let \( e \) denote the function identically equal to 1 in each component. If \( r(L) \) denotes the spectral radius of \( L \), we have \( r(L) = \lim_{n \to \infty} \|L^n e\|^{\frac{1}{n}} \). Furthermore, if \( u \in X \) is such that \( u_j > 0 \) on \( S_j \) for \( 1 \leq j \leq p \), then \( r(L) = \lim_{n \to \infty} \|L^n u\|^{\frac{1}{n}} \). Finally, if \( Lu = ru \) with \( u(t) > 0 \) for all \( t \in S \), then \( r(L) = r \).

**Proof.** We shall write, for \( f, g \in X, f \leq g \) to mean \( f_j(t) \leq g_j(t) \) for all \( t \in S_j, 1 \leq j \leq p \).

Since \( L \) is linear and maps nonnegative functions to nonnegative functions, it follows that \( Lf \leq Lg \) whenever \( f \leq g \). If \( f \in X \) with \( \|f\| \leq 1 \), we have \(-e \leq f \leq e \). So, \(-L^n e \leq L^n f \leq L^n e \) which implies that \( |(L^n f)(t_j)| \leq |(L^n e)(t_j)| \) for all \( t \in S_j \) and
1 \leq j \leq p. Thus \|L^nf\| \leq \|L^ne\| whenever \|f\| \leq 1 which gives \|L^n\| = \|L^ne\|. Taking the nth root and taking the limit as n goes to \infty, we get \( r(L) = \lim_{n \to \infty} \|L^ne\|^{\frac{1}{n}} \).

Now let \( u \in X \) such that \( u_j(t) > 0 \) for all \( t \in S_j \) and \( 1 \leq j \leq p \). Since \( S_j \) is compact for \( 1 \leq j \leq p \), there exist \( 0 < m \leq M < \infty \) such that \( me \leq u \leq Me \). This implies \( mL^ne \leq L^nu \leq ML^ne \), so \( m\|L^ne\| \leq \|L^nu\| \leq M\|L^ne\| \). Taking the nth root and taking the limit, we get \( \lim_{n \to \infty} \|L^nu\|^{\frac{1}{n}} = \lim_{n \to \infty} \|L^ne\|^{\frac{1}{n}} = r(L) \). To see the last part, note that \( Lu = ru \) implies \( L^n u = r^n u \). So, \( \|L^nu\|^{\frac{1}{n}} = r\|u\|^{\frac{1}{n}} \). Since \( \|u\| > 0 \), \( \|u\|^{\frac{1}{n}} \to 1 \). Thus we get \( r(L) = r \).

**Corollary 6.1.12.** Let \((S_j, d_j)\), \(1 \leq j \leq p\), be compact metric spaces and suppose that \( L : X = \prod_{j=1}^p C(S_j) \to X \) and \( L_k : X \to X, k \geq 1 \), are positive bounded linear maps. Assume that \( \|L_k - L\| \to 0 \) as \( k \to \infty \). Suppose \( Lu = ru \) with \( u_j > 0 \) on \( S_j \) for \( 1 \leq j \leq p \). Then \( r(L_k) \to r(L) \).

**Proof.** First we know that \( r = r(L) \) by Lemma 6.1.11. Now we have that \( \|L_ku - Lu\| \to 0 \) as \( k \to \infty \). Because each \( u_j \) is strictly positive, given \( \delta > 0 \), there exists \( k_0 \) such that \( (L_ku)_j \geq (1 - \delta)ru_j \) for all \( k \geq k_0 \) and \( 1 \leq j \leq p \). This implies for any \( n \geq 1 \), \( \|L_k^nu\| \geq (1 - \delta)^n\|ru\| \) for \( k \geq k_0 \). By Lemma 6.1.11, \( r(L_k) = \lim_{n \to \infty} \|L_k^nu\|^{\frac{1}{n}} \geq (1 - \delta)r \) for \( k \geq k_0 \). Since \( \delta > 0 \) was arbitrary, \( \liminf_{k \to \infty} r(L_k) \geq r \). Thus we are done because we always have \( \limsup_{k \to \infty} r(L_k) \leq r(L) \).

**Lemma 6.1.13.** Assume that H6.1 and H6.2 are satisfied. Also assume that strong connectedness is satisfied and let \( \sigma > \sigma_0 \) be fixed. Then we have that \( r(L_{N,\sigma}) \uparrow r(L_{\sigma}) \) as \( N \to \infty \).

**Proof.** Let \( \sigma > \sigma_0 \). Clearly \( L_{\sigma,N}e \leq L_{\sigma,N+1}e \), where the inequality is component-wise and \( e \) denotes the function in \( X = \prod_{j=1}^p C(S_j) \) which is identically equal to 1 in each component. So \( \|L_{\sigma,N}e\| \leq \|L_{\sigma,N+1}e\| \). By Lemma 6.1.11, we get that \( r(L_{\sigma,N}) \leq r(L_{\sigma,N+1}) \). Using H6.2, we see that \( \|L_{\sigma} - L_{N,\sigma}\| \to 0 \) as \( N \to \infty \). From Lemma 6.1.5 we know that \( L_{\sigma} \) has a nonnegative eigenvector \( u_{\sigma} \) with eigenvalue \( r(L_{\sigma}) \), and by strong connectedness \( (u_{\sigma})_j \) is strictly positive on \( S_j \) for \( 1 \leq j \leq p \) (compare Lemma 5.2.4). Thus by Corollary 6.1.12, we must have \( r(L_{\sigma,N}) \to r(L_{\sigma}) \).
Now we can prove the lower bound for the Hausdorff dimension of \( C_i, \ 1 \leq i \leq p \), and obtain the following theorem.

**Theorem 6.1.14.** Assume that H6.1, H6.2, H6.3 and H6.4 are satisfied. Also assume strong connectedness. Let \( \dim(C_i) \) denote the Hausdorff dimension of \( C_i \) for \( 1 \leq i \leq p \) and \( \sigma_\infty \) be as defined in eq. (6.4). Then \( \dim(C_i) = \sigma_\infty \) for \( 1 \leq i \leq p \).

**Proof.** Fix \( i \) with \( 1 \leq i \leq p \). By Theorem 6.1.8, it suffices to prove that \( \dim(C_i) \geq \sigma_\infty \).

If \( \sigma_0 < \sigma_\infty \), then for \( \sigma_0 < \sigma < \sigma_\infty \), \( L_\sigma \) is defined and \( r(L_\sigma) > 1 \) using the definition of \( \sigma_\infty \) and the fact that \( \sigma \mapsto r(L_\sigma) \) is strictly decreasing. So, by Lemma 6.1.13, there exists \( N_\sigma \) such that \( r(L_\sigma, N) > 1 \) for all \( N \geq N_\sigma \). Since \( r(L_{N, \sigma}) = 1 \), \( \sigma_N > \sigma \) for all \( N \geq N_\sigma \). Therefore, by Lemma 6.1.9, \( \dim(C_i) \geq \dim(C_{(N), i}) = \sigma_N \) for all \( \sigma < \sigma_\infty \), and hence \( \dim(C_i) \geq \sigma_\infty \). This completes the proof of the theorem.

6.2 An Example - Complex Continued Fractions

To illustrate Theorem 6.1.14, we discuss a special infinite iterated function system that is generated by complex continued fractions. This was introduced in [7] and has further been studied by Mauldin and Urbański in Section 6 of [16]. We show that our theory is applicable to this particular example. In the next chapter we shall obtain some numerical results for this example.

Let \( I \) be an infinite subset of \( \{ m + ni : m \in \mathbb{N}, n \in \mathbb{Z} \} \), where \( \mathbb{Z} \) is the set of integers and \( \mathbb{N} \) is the set of positive integers. Let \( X \subset \mathbb{C} \) be the closed disc centered at the point \( \frac{1}{2} \) with radius \( \frac{1}{2} \). For \( b \in I \) one can easily verify that \( \theta_b(X) \subset X \), where

\[
\theta_b(z) = \frac{1}{b + z}.
\]

The mappings \( \theta_b, \ b \in I, \) may not all be strict contractions in the Euclidean metric; \( \theta_1 \) is not a strict contraction because \( |\theta'_1(0)| = 1 \). Therefore we consider the system
\{\theta_b \circ \theta_c : b, c \in I\}. It is easy to verify that \(\theta_b \circ \theta_c\) is a strict contraction for each \(b, c \in I\) with a uniform Lipschitz constant \(\kappa < 1\). Let \(J = \{\pi(\omega) : \omega \in \Omega^\infty\}\) be the limit set for this system. Then we know that \(J = \bigcup_{b \in I} \theta_b(J)\). First note that \(\theta_b(z) = \theta_c(w)\) implies that \(|z - w| = |b - c|\). So, if \(|b - c| > 1\) then \(\theta_b(X) \cap \theta_c(X) = \emptyset\). Furthermore, if \(|b - c| = 1\) then \(\theta_b(z) = \theta_c(w)\) implies that \(z\) and \(w\) belong to the boundary of \(X\) and \(|z - w| = 1\).

**Lemma 6.2.1.** \(\theta_b \circ \theta_c(X)\) is contained in the interior of \(X\).

**Proof.** First we claim that \(\theta_b(z) \in \partial X\) implies \(z = 0\). Let \(b = m + ni, m \in \mathbb{N}, n \in \mathbb{Z}\) and \(z = x + yi \in X\). Then \(\theta_b(z) \in \partial X\) implies \(|\frac{1}{b+z} - \frac{1}{2}| = \frac{1}{2}\) which implies \(2 - b - z = |b+z|\). Therefore, \((2 - m - x)^2 + (n + y)^2 = (m + x)^2 + (n + y)^2\) which implies \(m + x = 1\), i.e., \(x = 1 - m\). Since \(m \geq 1\) and \(x \geq 0\) for \(z \in X\), it follows that \(m = 1\) and \(x = 0\). But \(x = 0\) implies that \(z = 0\).

Now suppose that \(\theta_b \circ \theta_c(z) \in \partial X\) for some \(z \in X\). Then, by the above claim, \(\theta_c(z) = 0\), which is impossible by the definition of \(\theta_c(z)\). \(\square\)

Let us verify H6.4 first. By the previous lemma, we know that for any \(b, c \in I\), \(\theta_b \circ \theta_c(X)\) is a compact set contained in the interior of \(X\). So if we take finitely many \(b_i, c_i\), the union of the images would still be a compact subset of the interior of \(X\). So, for any \(N\), \(J_N\) is a compact subset of the interior of \(X\) which means \(\text{diam}(J_N) < 1\). We claim that if \(\hat{X} \subset \text{int}(X)\) with \(\text{diam}(\hat{X}) < 1\), then \(\theta_{b_1} \circ \theta_{c_1}(\hat{X}) \cap \theta_{b_2} \circ \theta_{c_2}(\hat{X}) = \emptyset\) for any \((b_1, c_1) \neq (b_2, c_2)\). Suppose instead that \(\theta_{b_1}(\theta_{c_1}(z)) = \theta_{b_2}(\theta_{c_2}(w))\) with \(z, w \in \hat{X}\). This implies that \(b_1 + \theta_{c_1}(z) = b_2 + \theta_{c_2}(w)\). If \(b_1 = b_2\), this would imply that \(\theta_{c_1}(z) = \theta_{c_2}(w)\), i.e., \(c_1 + z = c_2 + w\), which is impossible because \(|z - w| < 1\) and \(|c_1 - c_2| \geq 1\). If \(b_1 \neq b_2\), we must have \(|\theta_{c_1}(z) - \theta_{c_2}(w)| = 1\), which is possible only if both \(\theta_{c_1}(z)\) and \(\theta_{c_2}(w)\) belong to the boundary of \(X\), which is possible only if \(z = w = 0\). This is a contradiction to the fact that \(\hat{X}\) is in the interior of \(X\). Thus the disjointness condition in H6.4 is satisfied. Also for any \(b \in I\), the map \(\theta_b\) is clearly one-to-one.

For \(b \in I\), we have \(D\theta_b(z) = |\theta'_b(z)| = \frac{1}{|z + b|^2}\). So, \(D\theta_b(z) > 0\) for all \(z \in X\). We claim that there exists \(0 < M_0 < \infty\) such that \(D\theta_b \in K(M_0, \lambda)\) with \(\lambda = 1\). Let \(z, w \in X\).
We have
\[(D\theta_b)(z) \leq (D\theta_b)(w) \exp(M_0|z - w|)\]
\[
\iff \frac{1}{|z + b|^2} \leq \frac{1}{|w + b|^2} \exp(M_0|z - w|)
\]
\[
\iff M_0 \geq 2\frac{\ln \left| \frac{w + b}{z + b} \right|}{|w - z|}
\]
But \(\ln \left| \frac{w + b}{z + b} \right| = \ln \left| 1 + \frac{w - z}{z + b} \right| \leq \frac{|w - z|}{|z + b|} \). Therefore, \(2\ln \frac{|w + b|}{|w - z|} \leq \frac{2}{|z + b|} \leq 2\). So, we can choose \(M_0\) independent of \(b \in I\).

To complete the verification of H6.2 note that
\[
\sum_{b \in I} (D\theta_b)^\sigma(0) = \sum_{b \in I} \frac{1}{|b|^{2\sigma}} \leq \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \frac{1}{(m^2 + n^2)^{2\sigma}}
\]
which converges for \(\sigma > 1\).

To verify H6.3 it is enough to show that given \(\epsilon > 0\) there exist \(\mu_1(\epsilon)\) and \(\mu_2(\epsilon)\) such that \(\lim_{\epsilon \to 0^+} \mu_1(\epsilon) = \lim_{\epsilon \to 0^+} \mu_2(\epsilon) = 1\) and \(\mu_1(\epsilon) \leq \frac{1}{(D\theta_b)(z)} \frac{|\theta_b(z) - \theta_b(w)|}{|z - w|} \leq \mu_2(\epsilon)\) for all \(b \in I\) whenever \(0 < |z - w| < \epsilon\). But using \(\theta_b(z) = \frac{1}{z + b}\), we get
\[
\frac{1}{(D\theta_b)(z)} \frac{|\theta_b(z) - \theta_b(w)|}{|z - w|} = \left| \frac{z + b}{w + b} \right| = \left| 1 + \frac{z - w}{w + b} \right|, \text{ which is bounded between } 1 - \frac{|z - w|}{|w + b|} \text{ and } 1 + \frac{|z - w|}{|w + b|}. \text{ Since for any } w \in X |w + b|^{-1} = |\theta_b(w)| \leq 1, \text{ taking } \mu_1(\epsilon) = 1 - \epsilon \text{ and } \mu_2(\epsilon) = 1 + \epsilon \text{ does the job.}
\]

Thus we see that all the hypotheses of the theorem are satisfied for this particular example and hence the Hausdorff dimension of the limit set \(J\) is given by the value of \(\sigma_\infty\).
Chapter 7

Numerical Results

In this chapter we focus our attention on the complex continued fractions that we talked about in Section 6.2. This particular example was first introduced and studied in [7]. Let $J$ be the limit set for the countably infinite iterated function system. It was shown in [7] that the Hausdorff dimension of $J$ is strictly between 1 and 2. These estimates were improved by Mauldin and Urbański in [16], where they showed that the Hausdorff dimension of $J$ is between 1.2484 and 1.9. We will show that the lower estimate can be improved substantially by using our methods involving spectral radius.

Let $X = \{ z \in \mathbb{C} : |z - \frac{1}{2}| \leq \frac{1}{2} \}$ and $I = \{ m + ni : m \in \mathbb{N}, n \in \mathbb{Z} \}$. For any $b \in I$, $\theta_b(z) := \frac{1}{z+b}$ maps $X$ into itself. For $\sigma > 1$, let $L_\sigma : C(X) \to C(X)$ be the Perron-Frobenius operator

$$(L_\sigma f)(z) = \sum_{b \in I} |\theta'_b(z)|^\sigma f(\theta_b(z)).$$

We saw in Section 6.2 that the hypotheses for our theorem are satisfied and hence the Hausdorff dimension of $J$ is given by the unique $\sigma_0$ such that $r(L_{\sigma_0}) = 1$. Since for any positive integer $k$, $r(L_k) = r(L_{\sigma_0})^k$ (by Lemma 5.4.1), we may use the operator $L_k$ instead of $L_\sigma$. Notice that

$$(L_k^\sigma f)(z) = \sum_{\omega=(b_1,\ldots,b_k) \in I^k} |\theta'_\omega(z)|^\sigma f(\theta_\omega(z)), \quad (7.1)$$

where $\theta_\omega(z) = \theta_{b_1} \circ \theta_{b_2} \circ \cdots \circ \theta_{b_k}(z)$ if $\omega = (b_1, b_2, \ldots, b_k)$.

$\theta_\omega$ is a Möbius transformation because it is a composition of finitely many Möbius transformations. In fact, if $\omega = (b_1, b_2, \ldots, b_k) \in I^k$, then

$$\theta_\omega(z) = \frac{p_k(z)z + p_k(\omega)}{q_k(z)z + q_k(\omega)}, \quad (7.2)$$
Lemma 7.0.2. Let \( p_0(\omega) = 0, p_1(\omega) = q_0(\omega) = 1, q_1(\omega) = b_1, q_l(\omega) = b_l q_{l-1}(\omega) + q_{l-2}(\omega) \) and \( p_l(\omega) = b_l p_{l-1}(\omega) + p_{l-2}(\omega) \), for \( 2 \leq l \leq k \). For simplicity, we will write \( q_k \) and \( p_k \) instead of \( q_k(\omega) \) and \( p_k(\omega) \). So, \( \theta_\omega(z) = \frac{p_{k-1} z + p_k}{q_{k-1} z + q_k} \) which gives
\[
\theta'_\omega(z) = \frac{p_{k-1} q_k - p_k q_{k-1}}{(q_{k-1} z + q_k)^2} = \frac{(-1)^k}{(q_{k-1} z + q_k)^2},
\]
and thus
\[
|\theta'_\omega(z)| = \frac{1}{|q_{k-1}|^2 |z + \frac{q_k}{q_{k-1}}|^2}.
\]

The following lemma gives formulas for the center and radius of the disk which is the image under a Möbius transformation of a closed disk in the complex plane. The proof is an easy exercise but we give the proof for completeness.

Lemma 7.0.2. Let \( \phi(z) = \frac{az + b}{cz + d} \), where \( ad - bc \neq 0 \) and \( c \neq 0 \). Let \( D \) be the closed disk centered at \( z_0 \) and with radius \( r_0 \). If \( \frac{z_0 d}{c} \notin D \), then \( \phi(D) \) is a closed disk with center
\[
\tilde{z} = \frac{a}{c} + \left( \frac{|z_0 + \frac{d}{c}|^2}{|z_0 + \frac{d}{c}|^2 - r_0^2} \right) \left( \frac{bc - ad}{c^2} \right) \frac{1}{|z_0 + \frac{d}{c}|}
\]
and with radius
\[
\tilde{r} = \frac{|bc - ad|}{|c|^2} \frac{r_0}{|z_0 + \frac{d}{c}|^2 - r_0^2}.
\]

Proof. First note that since \( c \neq 0 \), \( \phi(z) \) can be written as
\[
\phi(z) = \frac{a}{c} + \frac{1}{c^2} \left( \frac{bc - ad}{z + \frac{d}{c}} \right).
\]
If we let \( w = z + d/c \), then \( w \neq 0 \) for all \( z \in D \) by the assumption that \( -d/c \notin D \).

Since \( z \mapsto w \) is a translation by \( d/c, D_0 := \{w : z \in D\} \) is the closed disk centered at \( \zeta_0 = z_0 + d/c \) and radius \( r_0 \), and \( 0 \notin \tilde{D} \). Therefore, \( w \mapsto 1/w \) maps the closed disk \( D_0 \) to a closed disk \( \tilde{D} \) whose center \( \tilde{\zeta}_0 \) and radius \( \tilde{r}_0 \) can be calculated as follows. Consider the line through the origin and passing through the center \( \zeta_0 \) of the disk \( D_0 \). This line intersects the circle \( \{w : |w - \zeta_0| = r_0\} \) at the points \( \zeta_1 = \zeta_0 - r_0 \frac{\zeta_0}{|\zeta_0|} \) and \( \zeta_2 = \zeta_0 + r_0 \frac{\zeta_0}{|\zeta_0|} \).

It follows that \( 1/\zeta_1 \) and \( 1/\zeta_2 \) are points on the circle which is the boundary of \( \tilde{D} \). This implies that the center \( \tilde{\zeta}_0 = \frac{1}{2} \left( \frac{1}{\zeta_1} + \frac{1}{\zeta_2} \right) \) and the radius \( \tilde{r}_0 = \frac{1}{2} \left| \frac{1}{\zeta_2} - \frac{1}{\zeta_1} \right| \). A simple algebra now gives that \( \tilde{\zeta}_0 = \frac{|\zeta_0|^2}{|\zeta_0|^2 - r_0^2} \zeta_0 \) and \( \tilde{r}_0 = \frac{r_0}{|\zeta_0|^2 - r_0^2} \). Thus we have found the
center $\zeta_0$ and radius $r_0$ of the disk $\{1/(z + d/c) : z \in D\}$. From (7.3) we see that $\phi(z)$ is the composition of a multiplication of $z \mapsto \frac{1}{z + d/c}$ by $\frac{bc-ad}{c^2}$ followed by a translation by $\frac{a}{c}$, which immediately gives the formulas for the center $\tilde{z}$ and radius $\tilde{r}$ given in the lemma.

From the above lemma, it follows that if $\omega = (b_1, b_2, \ldots, b_k) \in I^k$ and $X$ is the closed disk of radius $1/2$ centered at $1/2$, then $\theta_\omega(X)$ is the closed disc with center

$$\tilde{z} = \frac{pk_{k-1}}{q_{k-1}} - \frac{1}{2} + \frac{q_k}{q_{k-1}} \frac{1}{2}^2 - \frac{(-1)^k}{q_{k-1}} \frac{1}{2} + \frac{q_k}{q_{k-1}}$$

and with radius

$$\tilde{r} = \frac{2}{|q_{k-1} + 2q_k|^2 - |q_{k-1}|^2}.$$  

In what follows we shall give an algorithm to compute a lower bound for the Hausdorff dimension of $J$.

Fix a $k \geq 1$, and let $\tilde{I}$ be a finite subset of $I$. Let $\tilde{X} = \bigcup_{\omega \in \tilde{I}^k} \theta_\omega(X)$. Then it is clear that $\theta_\omega(\tilde{X}) \subset \tilde{X}$ for all $\omega \in \tilde{I}^k$. We have the map $L^k_\sigma$, given by

$$(L^k_\sigma f)(z) = \sum_{\omega \in \tilde{I}^k} |\theta_\omega'(z)|^\sigma f(\theta_\omega(z)).$$

Note that $L^k_\sigma$ is a bounded linear operator on $C(\tilde{X})$, and for $z \in \theta_\omega'(X)$,

$$(L^k_\sigma f)(z) \geq \sum_{\omega \in \tilde{I}^k} |\theta_\omega'(z)|^\sigma f(\theta_\omega(z)) \geq \sum_{\omega \in \tilde{I}^k} (a_{\omega', \omega})^\sigma f(\theta_\omega(z)), \quad (7.6)$$

where $a_{\omega', \omega} = \min_{z \in \theta_\omega'(X)} |\theta_\omega'(z)|$. Consider the $|\tilde{I}|^k \times |\tilde{I}|^k$ matrix $A_\sigma = (a_{\omega', \omega})_{\omega \in \tilde{I}^k}$. If we denote by $e$ the function identically equal to 1 on $C(\tilde{X})$ and by $u$ the column vector of size $|\tilde{I}|^k$ whose each component is 1, then it follows that $(L^k_\sigma)^n e \geq (A_\sigma)^n u$ for all $n \geq 1$. Since $r(L^k_\sigma) = \lim_{n \to \infty} \|(L^k_\sigma)^n u\|^{1/n}$ and $r(A_\sigma) = \lim_{n \to \infty} \|A^n u\|^{1/n}$, it follows that the spectral radius $r(L^k_\sigma)$ of $L^k_\sigma$ is greater than or equal to the spectral radius $r(A_\sigma)$ of the positive matrix $A_\sigma$. Thus if $r(A_{\sigma_0}) > 1$ for some $\sigma_0$, then $r(L^k_{\sigma_0}) > 1$, which implies that $\sigma_0$ is a lower bound for the Hausdorff dimension of $J$.

Since, given $\omega' \in \tilde{I}^k$, Lemma 7.0.2 gives us explicit formulas for computing the center and the radius of the disk $\theta_\omega'(X)$, $a_{\omega', \omega}$ can be computed as follows. We know
that $|\theta_\omega'(z)| = \frac{1}{|q_{k-1}|^2|z + \frac{q_k}{q_{k-1}}|^2}$, so for $z \in \theta_\omega'(X)$, the minimum of $|\theta_\omega'(z)|$ is attained when $|z + \frac{q_k}{q_{k-1}}|$ is maximum. If $\theta_\omega'(X)$ is the disk with center $\tilde{z}$ and radius $\tilde{r}$, then $\left\{ z + \frac{q_k}{q_{k-1}} : z \in \theta_\omega'(X) \right\}$ is the disk with center $\tilde{z} + \frac{q_k}{q_{k-1}}$ and radius $\tilde{r}$. Therefore, 

$$\max_{z \in \theta_\omega'(X)} \left| z + \frac{q_k}{q_{k-1}} \right| = \left| \tilde{z} + \frac{q_k}{q_{k-1}} \right| + \tilde{r}$$

and

$$a_{\omega', \omega} = \frac{1}{|q_{k-1}|^2 \left( \left| \tilde{z} + \frac{q_k}{q_{k-1}} \right| + \tilde{r} \right)^2}. \quad (7.7)$$

So, the matrix $A_\sigma$ is computable. Furthermore, note that we don’t really need to compute the spectral radius of $A_\sigma$. If we can get a lower estimate for the spectral radius which is greater than 1 for some value of $\sigma_0$, then the Hausdorff dimension of $J$ is greater than $\sigma_0$. If $A$ is an $N \times N$ positive matrix and if for some positive integer $n$, and $\delta > 0$, $A^n u \geq (1 + \delta)u$, where $u^T = (1, 1, \ldots, 1) \in \mathbb{R}^N$, then we have $r(A) \geq 1 + \delta$.

For the computational purpose, we take $k = 1$ and $\bar{J} = \left\{ m + ni : 1 \leq m \leq 43, -43 \leq n \leq 43 \right\}$, and use the Maple software to do the computation. We are able to show that the Hausdorff dimension of $K$ is greater than 1.787, which improves the previous lower bound substantially. The Maple code written for this purpose has been provided in Appendix A.
Chapter 8
Continuity of the Hausdorff Dimension

In this chapter we shall show that the Hausdorff dimension of the invariant set list varies continuously with the functions $\theta_{(j,e)}$, $(j,e) \in \Gamma$ under the assumptions of Theorem 5.3.9. Let $V = \{1, 2, \ldots, p\}$ be the vertex-set, $\mathcal{E}$ be a finite edge-set and $S_1, S_2, \ldots, S_p$ be bounded, complete metric spaces. Let $\Gamma$ be a subset of $V \times \mathcal{E}$ and $\alpha : \Gamma \rightarrow V$. For each integer $m \geq 1$ and $(j,e) \in \Gamma$, suppose that $\theta_{(j,e),m} : S_j \rightarrow S_{\alpha(j,e)}$ is a Lipschitz map with $\text{Lip}(\theta_{(j,e),m}) \leq c < 1$. Assume that, for each $m \geq 1$, $\{\theta_{(j,e),m} : (j,e) \in \Gamma\}$ satisfies the assumptions of Theorem 5.3.9 and let $\sigma_m$ be the Hausdorff dimension of each $C_{j,m}$, $1 \leq j \leq p$, where $\{C_{j,m} : 1 \leq j \leq p\}$ denotes the unique invariant set list for the system $\{\theta_{(j,e),m} : (j,e) \in \Gamma\}$. For $(j,e) \in \Gamma$ and $x \in S_j$, assume that $\lim_{m \rightarrow \infty} \theta_{(j,e),m}(x) = \theta_{(j,e)}(x)$ and $\lim_{m \rightarrow \infty} D\theta_{(j,e),m}(x) = D\theta_{(j,e)}(x)$, where these limits define $\theta_{(j,e)}(x)$ and we assume that the limits are uniform in $x \in S_j$. Assume that for $(j,e) \in \Gamma$, $\theta_{(j,e)}$ satisfies H5.1-H5.4 and that $\theta_{(j,e)}|_{C_j}$ is one-to-one, where $\{C_j : 1 \leq j \leq p\}$ denotes the unique invariant set list for the system $\{\theta_{(j,e)} : (j,e) \in \Gamma\}$.

For $\sigma \geq 0$, we have, in the obvious notation, linear operators $L_{\sigma,m}$ corresponding to $\{\theta_{(j,e),m} : (j,e) \in \Gamma\}$ and $L_{\sigma}$ corresponding to $\{\theta_{(j,e)} : (j,e) \in \Gamma\}$. By Theorem 5.3.9, we know that $r(L_{\sigma,m}) = 1$. Let $\sigma_0$ denote the unique value of $\sigma$ for which $r(L_{\sigma}) = 1$. We shall show that $\lim_{m \rightarrow \infty} \sigma_m = \sigma_0$.

**Lemma 8.0.3.** The notations and assumptions are as in the previous paragraph. Let $\sigma \geq 0$ be fixed. Assume that there exist $M_0 > 0$ and $\lambda > 0$ such that $D\theta_{(j,e),m} \in K_j(M_0, \lambda)$ and $D\theta_{(j,e)} \in K_j(M_0, \lambda)$ for all $(j,e) \in \Gamma$ and for all $m \geq 1$. Choose $M > 0$ satisfying $\sigma M_0 + c^j M \leq M$ and let $K(M, \lambda) = \prod_{j=1}^p K_j(M, \lambda)$. Then $\|L_{\sigma,m} - L_{\sigma}\|_{K(M, \lambda)} \rightarrow 0$ as $m \rightarrow \infty$.

**Proof.** Let $\epsilon > 0$ be given and $1 \leq j \leq p$. By Lemma 4.1.2, we know that $\{f_j \in$
$K_j(M, \lambda) : \|f_j\| \leq 1$ is equicontinuous. Therefore, we can find a $\delta > 0$, independent of $j$, such that $d_j(s, t) < \delta$ implies $|f_j(s) - f_j(t)| < \epsilon$ for all $f_j \in K_j(M, \lambda)$ with $\|f_j\| \leq 1$.

Let $f = (f_1, f_2, \ldots, f_p) \in K(M, \lambda)$, $\|f\| \leq 1$ and let $(j, e) \in \Gamma$. Since as $m \to \infty$, $\theta_{j(e), m} \to \theta_{j(e)}$ and $D\theta_{j(e), m} \to D\theta_{j(e)}$ uniformly on $S_j$, there exists an integer $m_0$ such that $d_{\alpha(j, e)}(\theta_{j(e), m_0}(t), \theta_{j(e)}(t)) < \delta$ and $\|(D\theta_{j(e), m_0}(t))^{\sigma} - (D\theta_{j(e)}(t))^{\sigma}\| < \epsilon$ for all $t \in S_j$ and $m \geq m_0$. This implies that $|f_{\alpha(j, e)}(\theta_{j(e), m_0}(t)) - f_{\alpha(j, e)}(\theta_{j(e)}(t))| < \epsilon$ for any $t \in S_j$ and $m \geq m_0$. So we have for $1 \leq j \leq p$ and $t \in S_j$,

\[
\begin{align*}
|\langle (L_{\sigma, m} f)_{j}(t) - (L_{\sigma} f)_{j}(t) \rangle | & \leq \sum_{e \in E_j} \| (D\theta_{j(e), m}(t))^{\sigma} f_{\alpha(j, e)}(\theta_{j(e), m}(t)) - (D\theta_{j(e)}(t))^{\sigma} f_{\alpha(j, e)}(\theta_{j(e)}(t)) \| \| \leq \sum_{e \in E_j} \| (D\theta_{j(e), m}(t))^{\sigma} - (D\theta_{j(e)}(t))^{\sigma} \| \| f_{\alpha(j, e)}(\theta_{j(e), m}(t)) \| \\
& \quad + \sum_{e \in E_j} \| (D\theta_{j(e)}(t))^{\sigma} \| \| f_{\alpha(j, e)}(\theta_{j(e), m}(t)) - f_{\alpha(j, e)}(\theta_{j(e)}(t)) \| \\
& \leq 2|E_j|\epsilon
\end{align*}
\]

for all $m \geq m_0$, where $|E_j|$ is the cardinality of the set $E_j$. Since $\mathcal{E}$ is a finite set and $E_j \subset \mathcal{E}$, we have proved that $\|L_{\sigma, m} - L_{\sigma}\|_{K(M, \lambda)} \to 0$ as $m \to \infty$.

\textbf{Lemma 8.0.4.} The assumptions are as in the previous lemma. Then $r(L_{\sigma, m}) \to r(L_{\sigma})$ as $m \to \infty$.

\textit{Proof.} Let $u_{\sigma} \in K(M, \lambda) \setminus \{0\}$ be an eigenvector of $L_{\sigma}$ with eigenvalue $r_{\sigma} := r(L_{\sigma})$.

By Lemma 8.0.3, it follows that $\|L_{\sigma, m} u_{\sigma} - L_{\sigma} u_{\sigma}\| \to 0$ as $m \to \infty$. Since $L_{\sigma} u_{\sigma} = r_{\sigma} u_{\sigma}$, $\|L_{\sigma, m} u_{\sigma} - r_{\sigma} u_{\sigma}\| \to 0$ as $m \to \infty$. By Lemma 5.2.4, we know that there exist $l > 0$ such that $(u_{\sigma})_j > l$ on $S_j$ for $1 \leq j \leq p$. So given $\delta > 0$, there exists $m_0$ such that $(1 - \delta)r_{\sigma}(u_{\sigma})_j \leq (L_{\sigma, m_0} u_{\sigma})_j \leq (1 + \delta)r_{\sigma}(u_{\sigma})_j$ on $S_j$ for all $m \geq m_0$ and $1 \leq j \leq p$. This implies that for any $n \geq 1$, $(1 - \delta)^n r_{\sigma}^n (u_{\sigma}) \leq \|L_{\sigma, m} u_{\sigma}\| \leq (1 + \delta)^n r_{\sigma}^n (u_{\sigma})$ for all $m \geq m_0$.

Taking the $n$th root and taking the limit as $n \to \infty$, we get $(1 - \delta)r_{\sigma} \leq r(L_{\sigma, m}) \leq (1 + \delta)r_{\sigma}$ for $m \geq m_0$. Since $\delta > 0$ was arbitrary, we get $r_{\sigma} \leq \lim \inf_{m \to \infty} r(L_{\sigma, m}) \leq \lim \sup_{m \to \infty} r(L_{\sigma, m}) \leq r_{\sigma}$. Thus we get $\lim_{m \to \infty} r(L_{\sigma, m}) = r(L_{\sigma})$. 

$\Box$
**Theorem 8.0.5.** Suppose that \( r(L_{\sigma_m,m}) = 1 \) for \( m \geq 1 \) and \( r(L_{\sigma_0}) = 1 \). Then \( \lim_{m \to \infty} \sigma_m = \sigma_0 \).

**Proof.** We argue by contradiction. Suppose that \( \lim_{m \to \infty} \sigma_m \neq \sigma_0 \). Then there exist \( \delta > 0 \) and a subsequence \( \{m_i\}_{i \geq 1} \) such that either \( \sigma_{m_i} > \sigma_0 + \delta \) for all \( i \geq 1 \) or \( \sigma_{m_i} < \sigma_0 - \delta \) for all \( i \geq 1 \). Assume \( \sigma_{m_i} > \sigma_0 + \delta \) for all \( i \geq 1 \). Then by the strictly decreasing property, \( r(L_{\sigma_{m_i},m_i}) < r(L_{\sigma_0+\delta,m_i}) \) for all \( i \geq 1 \). By Lemma 8.0.4, \( \lim_{i \to \infty} r(L_{\sigma_{0+\delta,m_i}}) = r(L_{\sigma_0+\delta}) \), which is strictly less than \( r(L_{\sigma_0}) = 1 \). On the other hand, \( r(L_{\sigma_{m_i},m_i}) = 1 \) for all \( i \geq 1 \), which gives \( r(L_{\sigma_0+\delta}) \geq 1 \). Thus we arrive at a contradiction. Similarly, \( \sigma_{m_i} < \sigma_0 - \delta \) for all \( i \geq 1 \) leads to a contradiction. Hence we have proved the theorem. \( \square \)

**Remark 8.0.6.** If we assume, in addition to the assumptions made in this section, that \( \theta_{(j,e)}(C_j) \cap \theta_{(j',e')}(C_{j'}) = \emptyset \) whenever \( (j,e) \neq (j',e') \) and \( \alpha_{(j,e)} = \alpha_{(j',e')} \), then Theorem 5.3.9 implies that \( \sigma_0 \) is the Hausdorff dimension of \( C_j \) for \( 1 \leq j \leq p \). Thus the previous Theorem implies that, for \( 1 \leq j \leq p \), the Hausdorff dimension of \( C_j \) is the limit of the Hausdorff dimension of \( C_{j,m} \) as \( m \to \infty \). Also note that if we could allow some overlap in Theorem 5.3.9, for instance, if Theorem 5.3.9 is true under the strong open set condition, then our formula still gives the continuity of the Hausdorff dimension by using elementary facts about positive linear operators as shown above.
Appendix A

Maple code for the example in Chapter 7

> N:=0: > k:=43:
> for m from 1 to k do
> for n from -k to k do
>     N:=N+1;
>     g(N):=m+n*I;
> end do:
> end do:
> for i from 1 to N do
>     for j from 1 to N do
>         b:=g(i);
>         c:=g(j);
>         z0:=evalf((2/(2*c+1))*((abs(2*c+1))^2/((abs(2*c+1))^2-1)));  
>         r:=evalf(2/((abs(2*c+1))^2-1));
>         f(i,j):=evalf((abs(b+z0)+r)^(-2*1.787));
>     end do:
> end do:
M:=Matrix(N,f):
> L:=M^100:
> u:=Vector(1..N,1):
> v:=(L.u)-u;
> for i from 1 to N do
>     if v(i)<0 then print("FAIL"); break end if
> end do:
References


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