MODELING FINANCIAL MARKETS USING MIXED MINORITY/MAJORITY GAMES

by

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ABSTRACT OF THE THESIS

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Thesis Director:

Professor Ivan Marsic

Financial markets are considered to be a system formed due to the interaction between heterogeneous individuals. Many models have tried emulating it and have tried to uncover the working behind it. Minority Game Model is one such model which has tried to emulate it. It is a game consisting of heterogeneous agents who believe that to gain profit one needs to be in the minority. However, it has been proved that the financial market consists of both fundamentalists (i.e. individuals gaining profit by being in the minority) as well as noise traders (individuals gaining profit by following the herd). So, we have used the Mixed Game Model to emulate financial markets which consists of Minority and Majority game players. Although it has been proved that the mixed game model is a suitable model to imitate financial world, we have observed that it still has many limitations like the two groups of agents have same properties and thus they lack in heterogeneity and also that the life of each agent is constant. But in real world, every individual has a unique memory and learning ability and will join and leave the markets as well. To improve on these limitations we have created the model, "Highly

Heterogeneous Model” which removes both of these limitations. Also, we show that the new improved game improves the performance of the majority game players by 2.35 % and minority game players by 4.45 %. Apart from this we observed that all the models which have emulated financial market by using Minority Game have concentrated on the combined effect of agents of financial factors like prices, returns and volatilities i.e. they are synchronous. With the availability of high frequency data, its analysis has been continuously gaining importance in recent years. We have thus also studied this behavior of market using the asynchronous form of the game known as the “Asynchronous Mixed Game Model”. We finally also prove that the Highly Heterogeneous Game represents the daily time series and the Asynchronous Mixed Game represents the high frequency time series of real financial world.

Dedication

To my Parents

Acknowledgement

I would like to thank Professor Ivan Marsic for allowing me to work on this highly engaging topic. I would like to thank him for all the guidance and constant feedback and also for inspiring me throughout. I would like to thank my parents for supporting and encouraging me throughout this process. Also, I would like to thank my friends without whose help and motivation all this would not have been possible.

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Chapter 1: Introduction

1.1 Motivation

The study of financial markets has been the topic of interest from quite some time. The modeling of financial markets by means of individual based simulations has attracted a significant amount of attention in recent years [1, 4, 7, 29, 23]. Many tools and applications have been developed that can emulate financial world [1, 21, 25, 31]. However, the **Efficient Market Hypothesis(EMH)** states that that current stock prices fully reflect available information about the value of the firm, and there is no way to earn excess profits, (more than the market overall), by using this information[8]. There has always been a clash between believers of EMH and individuals who believe that financial market can be imitated by making use of tools and methods. Out of the various tools and methods examined to emulate financial markets, **Agent Based Models** [22] have gained great significance and a lot of work is being done towards it. Agent Based Models deal with simulating the actions and interactions of heterogeneous agents with a view to evaluating their effects on the system [22]. The reason behind the rise and popularity of Agent Based Models as models for impersonating financial markets is that financial markets deal with the flow of information between the heterogeneous individuals [31]. Agent Based Models like financial markets also deal with the interactions between heterogeneous agents [22] and thus they are used in re-creating and predicting the behavior of the real world financial markets. Agent based models pertaining to financial markets consist of heterogeneous individuals whose interactions lead to the formation of virtual market. These interactions among agents lead to the price formation and the formation of various distinct statistical patterns know as stylized facts [9, 10]. These stylized facts are financial patterns such as volatility [47], excess kurtosis [43, 47], and non Gaussian behavior [47] and are not easily replicated by any single representative agent model. Only an agent based model which creates a virtual market that suffices all these properties is considered an authentic agent based model for financial markets. Thus to create an agent based model, first the market needs to set up followed by setting up the agents. Further all agents based on their perceptions and goals interact which leads to the formation of the time series and the virtual market. This time series is finally verified by means of statistical patterns – stylized facts [9, 10, 47]. The stylized facts have already been established by studying the previous time series of the real world financial markets and they are considered to be the most authentic tool for validating financial time series.

The Minority Game Model [7] proposed by Challet and Zhang is one such agent based model which has become a very important tool in emulating financial market [22, 23]. It is basically an extended implementation of El Farol Bar problem proposed by Brian Arthur [2]. El Farol Bar problem deals with making the right decision by learning through past experiences. It also deals with choosing the right decision without any knowledge of the decisions made by others. It is basically a problem related to inductive reasoning and bounded rationality [2]. Economists suggest that beyond certain complicatedness, our logical apparatus ceases to cope i.e. our rationality is bounded [2]. Also, when there is no awareness about the decisions made by others, the reasoning has to be inductive. The Minority Game Model [7] deals with these situations. The Minority

Game is a model of interaction between heterogeneous odd numbers of agents. In the Minority Game Model, each of the agents has to decide independently whether to go to a bar or not. The decision is based on the fact that the bar is enjoyable only if the bar is not crowded. Each of the agents takes a decision; +1 indicating to go to the bar and -1 indicating to stay at home without any knowledge about the actions of the other agents. As the number of agents is odd, there will be some agents whose decision will be the minority out of the total number of agents and these agents are considered winners of the game.

The Minority Game [7] has gained great popularity in terms of research because of its simplicity as well as its property of being molded into the most versatile game structure. The Minority Game Model has been used as a model for representing the financial market from quite some time [1, 21, 25, 31]. The original model in its most simple setup is not able to produce the stylized facts due to the lack of heterogeneity among agents [14]. Various agent based models have tried to emulate financial markets by making changes to the basic minority game model [1, 21, 25, 31]. Each model has tried to remove the limitations in Minority Game in different ways, for instance the Grand Canonical Minority Game tries to do this by giving the agents an option of abstaining at any trading period [21, 28]. One of the other models is the $ Game [1] which deals with the speculative nature of the market. It is based on the fact that a price at time t say p(t) is only profitable if one is able to sell at a higher price say p(t1) where t1>t [1]. In case of these various models, apart from the random assignments of strategies, there is no heterogeneity in the different types of agents, whereas real markets are composed of different groups of traders, with different objectives, impact on market, and trading behavior [14]. Moreover, these games do not exactly represent the actual financial market. The actual financial market does not just deal with gaining profit by being in the minority. It has the “noise traders” [1, 31, 30, 6, 37, 15] who effectively play a majority game; while others are “foundation traders” [15] who follow the learning process and play the minority game [17]. So we have used the **Mixed Game** model [16] proposed by Chengling Gou for emulating financial markets. It is referred as Mixed Game or Synchronous Mixed Game. It consists of both the majority and minority game players. This also solves the issue of less heterogeneity up till some extent because the presence of majority and minority player groups provides the heterogeneity aspect to the game. However, we have also observed several disadvantages in the Mixed Game as a system for modeling financial markets. For instance, all the agents in the Mixed Game start playing the game and play till the very end which is not the case in the real financial world. In real financial world, we have individuals coming and going and dealing with the financial assets. Also, we have new individuals entering at various instances in the market. We aim at including this factor also in our research and analyzing the stylized facts under this condition. In addition, the Mixed Game has each group of agents having the same lengths of memories for making decisions. However, in real time markets, different agents have different memories. Due to this, they have different ways of reacting in different situations. We have even included this factor in our new model known as **“Highly Heterogeneous Mixed Game”.**

Apart from these limitations observed in the Minority Game, we have observed that all models[1,31,21,25] related to the Minority Game which have worked in the direction of making use of it as a financial model are synchronous. By synchronous, we mean that these models deal with joint behavior of all the agents in determining financial factors like price, returns etc. This can be regarded as modeling the financial market based on the daily prices of the assets. With the availability of high frequency data [42], its analysis has been continuously gaining importance in recent years. Traders and researchers are not contented with low frequency financial market data like monthly and weekly data anymore. The demand for high-quality high frequency data: intra-day data, intra-hour data and intra-minute data is elevated. Traders study the high frequency data to make decision and trade strategy. These data have their own distinctiveness and need corresponding analyzing methods. According to our research, none of the minority game models [1,31,21,25], till date has attempted to generate stylized facts such that the individual effect of each agent is taken into account. In this condition, the financial factors like price, return and others are calculated asynchronously i.e. all agents do not synchronize to calculate them. We have attempted this model known as the **"Asynchronous Mixed Game"** and then compared it to the real financial world and stated where exactly does this model fit in as compared to the real financial world.

1.1 Contribution of the Thesis

Our contribution in this thesis firstly is choosing the most relevant available game to model financial markets which is the Mixed Game. By choosing this game we provide the necessary heterogeneity aspect in terms of minority and majority agents which is not present in games like $Game, Grand Canonical game and others [1, 31, 21, 25]. Also we recognize the limitations in the Mixed Game like the heterogeneity limited by the **use of same length of memories for minority and majority group agents** and also that all agents play the game from start till end. We have removed these limitations by providing each agent with unique memories (unique memories in terms of length as well as content) and by including the life-death scenario which lets agents die in between and also new agents are born in place of them. By removing these limitations we make the model more relative to the real world financial market as in real world, each of the individuals investing in stocks have different memories in terms of length as well as content and this leads to them reacting differently each time new information arrives. Also, this increases heterogeneity content in our game as observed in real world. Moreover, individuals who trade assets in real world enter and leave the markets whenever required and new agents too come in the market to start trading. We remove this limitation by causing the death of agents and also by introducing new agents into the game. We further validate our model as a model for representing financial markets by producing stylized facts [9, 10] using these models. Stylized facts are the patterns observed in the seemingly random variations of asset prices [9, 10]. They act as a validation tool to check whether a model represents financial world or not. We generate stylized facts for the Synchronous Mixed Game and the Highly Heterogeneous Mixed Game and show how they relate to the daily prices of the real financial world. Apart from this we also delve into the unnoticed area of asynchronous markets [2] and generate stylized facts for the high frequency data [42] and relate them to the high frequency prices observed in real financial world. For better understanding, we describe all the games with the brief objective of each game in the table 1.1.

Name of the Game	**Objective**	**Contribution**
Minority Game	To prove that original Minority Game **cannot represent financial market in its most basic form**. Other better versions of the game are required.	Already established.
Synchronous Mixed Game	• To relate to the real world financial market by having minority as well as majority agents • To provide heterogeneity by having two types of agent groups; majority and minority group.	Already established.
Highly Heterogeneous Mixed Game	To include: • Agents with unique memories • Life-death scenario	Our contribution
Asynchronous Mixed Game	To study the **individual effect of each agent on the financial market** which is equivalent to analysis of high frequency data in real world.	Our contribution

Table 1.1 Summary of the games

1.2 Outline

In this chapter (chapter 1) we have provided the backdrop of our work, the various models and what really inspired us to conduct this research. The rest of the thesis is organized in the remaining four chapters. In chapter 2, we have given the overview of the financial market and have described the common financial terms. We have even described the stylized facts which form a very important aspect of our research. Further, we have described the origin of our research which is the El Farol Bar Problem followed by the Minority Game adaptation to it. We have described the limitations of Minority Game as a model for depicting financial markets. We have further described the previous works related to our research and how the Minority Game fits in as a financial model and also the reasons behind choosing the Mixed Game. In chapter 3, we start by describing the Synchronous Mixed Game Model. We have described the various factors of the model like agent's decision making, rewarding the agents and others. We have even described the algorithm and the simulator design for it. We have then described our new game models; Heterogeneous Mixed Game Model and Asynchronous Mixed Game Model and their overall working. Next in chapter 4, we describe the implementations and results for Basic Minority Game model, Mixed Game model, the improvement made to it which is the Highly Heterogeneous Mixed Game model and the Asynchronous Mixed Game model. Finally in the last chapter 5, we discuss the conclusion of our thesis and future possible work.

Chapter 2: Stylized Facts and the emergence of Minority Game as a Market Model

2.1 Financial Time Series and Stylized Facts

In this section, we describe some basic financial terms followed by the stylized facts.

2.1.1 Some Basic Financial Terms

The financial market generates tons of data every single day. It shows some unique properties and traits and it is believed that that we can get some very important information from it. Here, we describe few important terms with respect to financial time series.

The change of price of an asset over a period of time is known as **return**. It can be represented as:

$$R(t) = P(t+\Delta t) - P(t) \tag{2.1}$$

The net return of an asset is given by:

$$R(t) = [P(t+\Delta t) - P(t)] / P(t) \tag{2.2}$$

Where, R(t) = change in price over the time periods, t and $t+\Delta t$

Demand in a financial market is the amount of shares that an investor desires to buy at a particular time. It is correlated with the supply which is the amount of shares present in a market. The demand of an asset is calculated by:

$$D(t) = \text{no of buying orders}(t-1) - \text{no of selling orders}(t-1) \quad (2.4)$$

Where D(t) = demand at time t

The price of an asset at a particular time is driven the demand of the stock at that time [13]. The **price** of an asset is given by:

$$P(t) = D(t) + P(t-1) \quad (2.5)$$

Where P(t) = Price of an asset at time t

2.1.2 Stylized facts

Stylized Facts are the empirical facts emerging from the statistical analysis of price variations in various types of financial markets [9]. The seemingly random variations of asset prices share some quite important statistical properties. The large amount of financial data is being studied from many years and these properties are so evident in some form, that they have become a tool for establishing financial factors. They are highly important in the studies of models trying to emulate financial market as they provide evidence in establishing whether the data actually resembles financial data or not [9, 10]. Basically these stylized facts are observed on day to day financial data and we try to attain them on our models, thus proving that these models emulate the real financial world [9, 10]. Some of the widely known stylized facts are absence of autocorrelation, heavy tails, intermittency, volatility clustering etc. In this research, we focus on the stylized patterns like absence of autocorrelation in returns, slowly decaying autocorrelation in absolute returns, fat tail distribution of returns and volatility clustering.

The **reason behind choosing these stylized facts** is that it has been established that most notably all real world financial markets generate these stylized facts as they are independent of the specific trading rules or external circumstances like crashes at the given market place [33]. The two mentioned phenomena, i.e. volatility clustering and fat tails, have been detected in almost every financial return series that was subject to statistical analysis and they are of paramount importance for any individual or institution engaging in the financial markets, as well as for financial economists trying to understand their mode of operation [18]. Also, these stylized facts cover the main phenomenon's like power-law statistics and non-Gaussian behavior that are observed in real world financial market [33]. The fat tail distributions show the non-Gaussian behavior and the autocorrelation in returns and volatility clustering show the power law statistics [9, 10].

2.1.2.1 Absence of Autocorrelation in Returns

It has been established by several studies [9, 10] that the daily returns themselves show no correlation but the autocorrelation of absolute returns is always positive and significant, and decays slowly. This is a signature of the well-known phenomenon of volatility clustering [32]: large price variations are more likely to be followed by large price variations [9,10]. Volatility clustering shows that volatilities are correlated as small changes tend to be followed by small changes and large changes by large. The autocorrelations of absolute returns for tick-tick data (high frequency data like intra minute data or change in a stock's price from one trade to the next) should be negative [47, 42]. Similarly, absolute returns show similar property as it shows significant autocorrelation which decays slowly [9, 10]. This stylized fact states that as no correlation should exist between the returns at different time intervals i.e. one cannot predict the behavior of returns at a particular instance with the help of the prior information regarding returns. This again manifests the fact that the returns need not be uniform. This is also related to the fact that returns observe **pareto law** [48]. The pareto law states that 20% of the input creates 80% of the result. It is further established that the numbers don't have to be "20%" and "80%" exactly. The key point is that **most things in life are not distributed evenly – some contribute more than others** [48]. For our case, it just signifies the fact that returns are not uniform and are not correlated.

The autocorrelation function can be defined as:

$$C(\tau) = E[(R_t - \mu)\,(R_{t+\tau} - \mu)]/\sigma^2 \qquad (2.5)$$

Where, τ = *lag*

R_t = *return at time t*

μ = *mean of return*

σ^2 = *variance of return*

2.1.2.2 Volatility Clustering

The rate of increase and decrease of price of an asset is known as volatility. Volatility clustering means that large changes tend to be followed by large changes, of either sign and small changes tend to be followed by small changes [32]. The reason behind this is that unlike returns volatilities are correlated with time which means that information about volatility can be predicted from over the previous results of volatilities. Volatility basically represents swings in supply and demand of asset. Volatility clustering in real financial market is a result of information asymmetry. The information asymmetry is because of the heterogeneous individuals whose invest in the assets and their different way of reacting to new information as and when it arrives in the market [37, 10]. For our games, the information asymmetry is provided by the heterogeneity among agents which is explained in later chapters 3 and 4.

The volatility is given by the standard deviation of the prices over a period of time [16].

$$V[t] = \sigma p' * \sigma p' \tag{2.6}$$

$$\sigma p' = \sqrt{i / t \sum_{i=1}^{t} (p' - \overline{p'})}$$

Where $p' = p(t) - p(t-1)$

$V(t)$ = *Volatility at time t*

2.1.2.3 Fat Tail Distribution of Returns

Many studies[9,10] concerning the analysis of properties of returns have shown that returns in financial market show the property of fat tailed distribution i.e. the probability distribution function for returns of financial assets have sharper peak around zero when compared to the Gaussian distribution [18]. There are two main factors to be observed for the fat tail distribution of returns: 1) fat tails and 2) high kurtosis [43]. Kurtosis means that the distribution should have a distinct peak near the mean, declines rather rapidly around the mean, and then decays slowly leading to heavy tails. The kurtosis of a normal distribution is 3[44]. Kurtosis less than 3 indicates a distribution that is flat at the mean and decays fast, while kurtosis and larger than 3 indicates a distribution with a sharp peak and heavy tails [44]. Fat tail distribution of returns should have kurtosis larger than 3. The fat tails in the distribution can be characterized by the fat tail index also known as the fat tail exponent. The fat tail index provides measure of fatness of tails. It has been established the fat tail in index for the fat tail distribution should have a value greater than 2 [38].

Also, the curve remains well above the horizontal axis for large changes whereas Gaussian distribution has almost attained zero [26]. This suggests that, compared to the normal distribution, the distribution of the returns is fat tailed, i.e., the probability of large losses and gains is much higher than would be implied by a time–invariant unconditional

Gaussian distribution [18]. Also, it has been established that the distribution of returns seems to display a power-law or Pareto-like tail [9].

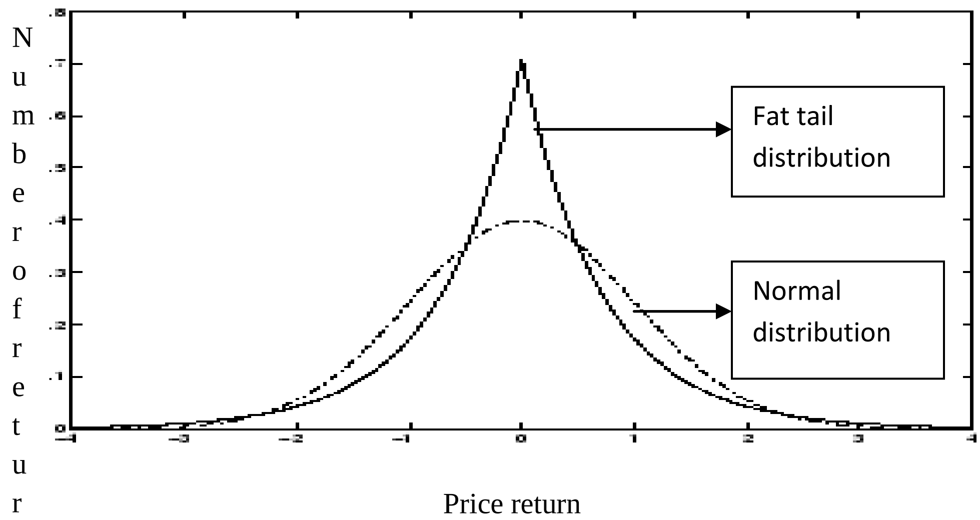


Figure 2.1 Sample Fat Distributions of Returns [39]

The above diagram shows the comparison between the sample fat tail distribution curve for returns and the normal distribution. The longer curve with high peaks represents the fat tailed distribution while the other curve represents the normal distribution [46]. As observed from the diagram the fat tailed distribution has high kurtosis [43] and fatter tails as compared to the normal distribution.

2.1.2.4 Stylized Facts Observed in Real World Financial Market

Figures 2.2 to 2.8 show samples of the stylized facts observed in real world financial market. These graphs represent stylized facts examined using DJIA data ranging starting October 1 1928 up till next September 10 1940. For plotting these graphs, we obtained **daily price data of DJIA from yahoo finance** [49]. Figures 2.2 shows the daily price series of DJIA for the above mentioned time period. The price series show different patterns for different stocks and indexes like Dow Jones, Apple etc at different time intervals. So the price series is not a stylized fact. It just shows the relative growth of the price with respect to time. But there are several factors like returns and volatility which are derived from the price series and are important elements in modeling stylized facts like return series, autocorrelation in returns, autocorrelation in absolute returns and others. Figure 2.3 represents the return series i.e. the returns vs. time graph corresponding to daily DJIA data. The returns series shows the herding behavior i.e. small changes in price are followed by small and similarly large are followed by large. This is in agreement with the principle of volatility clustering (Section 2.1.2.2). In case of volatility clustering also, small fluctuations tend to be followed by small fluctuations and similarly vice versa [32]. The volatility clustering chart is shown in figure 2.4. The daily returns should exhibit very less or no correlation and the autocorrelation function of absolute returns decays slowly as a function of the time lag [9]. This is observed in Figure 2.6 which represents the autocorrelation in returns which exhibits correlation in the initial lags but dies eventually. Also, the autocorrelation in absolute returns represented through figure 2.5 decays slowly as a function of time lag. Finally the fat tailed distribution of returns [18] is evidently observed in figure 2.7. In case of fat-tailed distribution [18] of returns, there are two main factors to be observed: fat tails and high kurtosis [43]. Figure 2.7 shows long peaks which rise at the center. This is high kurtosis. We have further shown a section of the enlarged version of figure 2.7 in figure 2.8 which represents the fat tails observed at the bottom for distribution of returns. Figure 2.9 shows the fat tail distribution in log version which again clearly showcases the fat tails and high peaks (kurtosis).

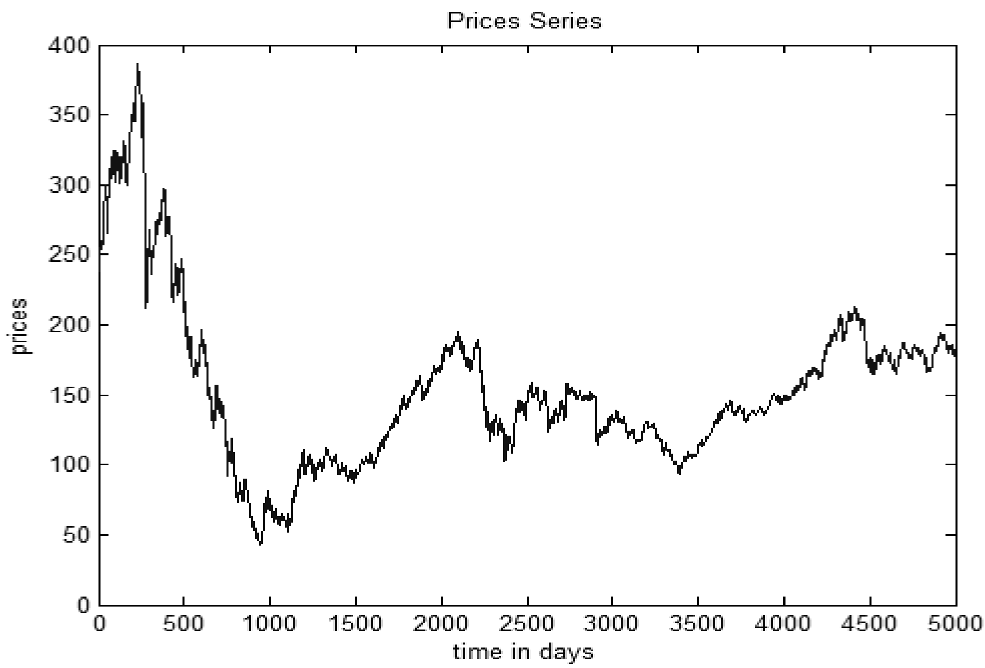


Figure 2.2 Price Series chart of DJIA

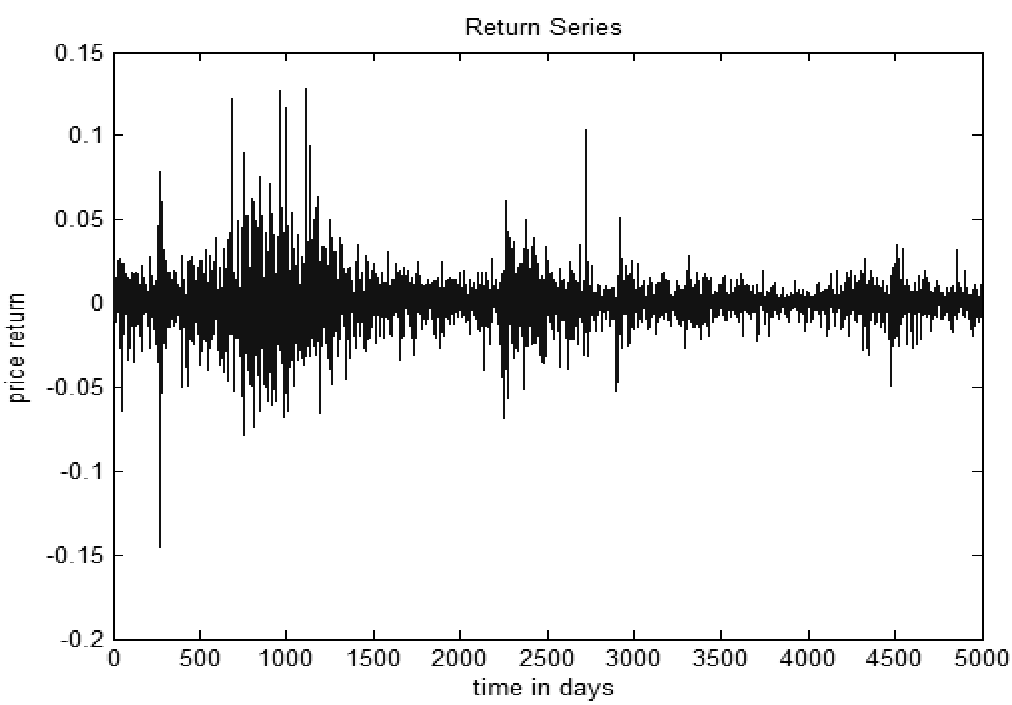


Figure 2.3 Return Series chart of DJIA

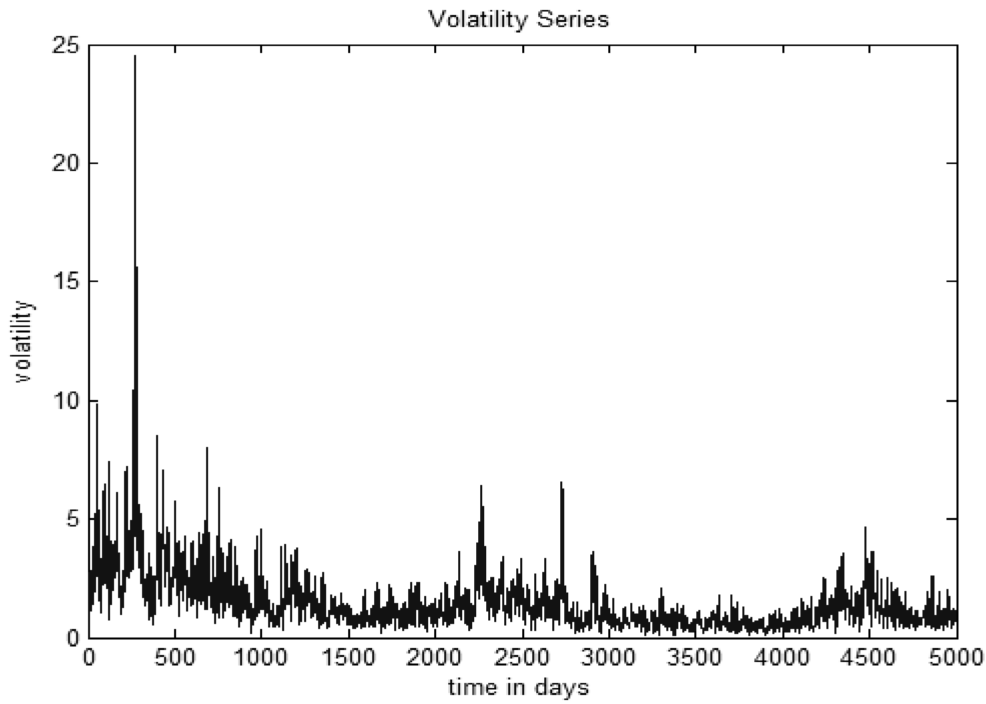


Figure 2.4 Volatility Clustering observed in DJIA

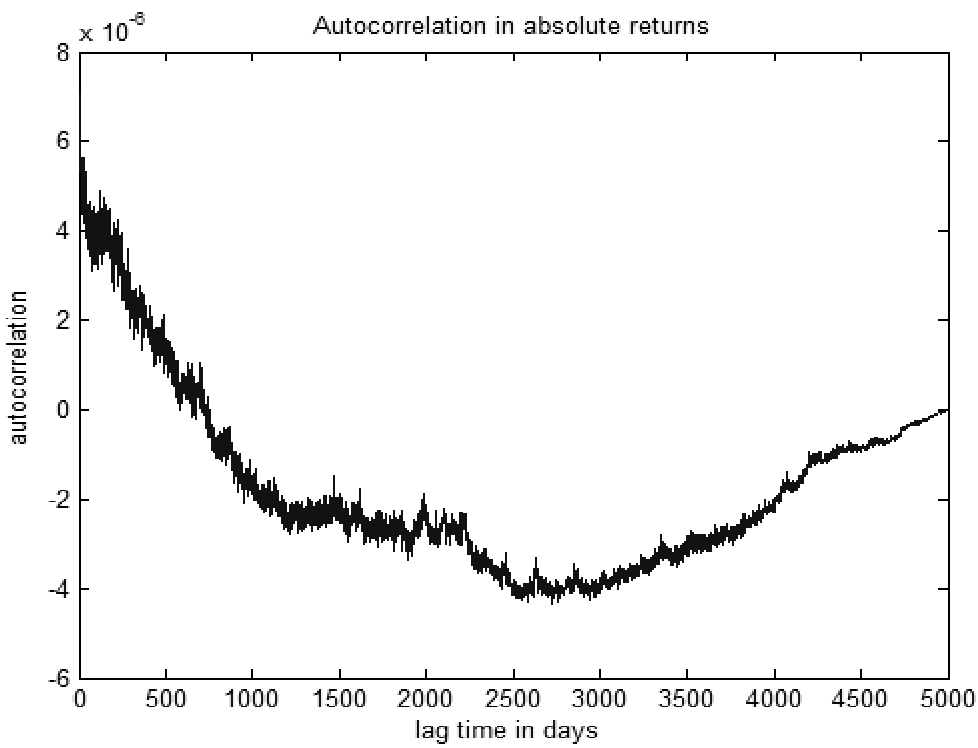


Figure 2.5 Slowly Decaying Autocorrelation of absolute returns observed in DJIA

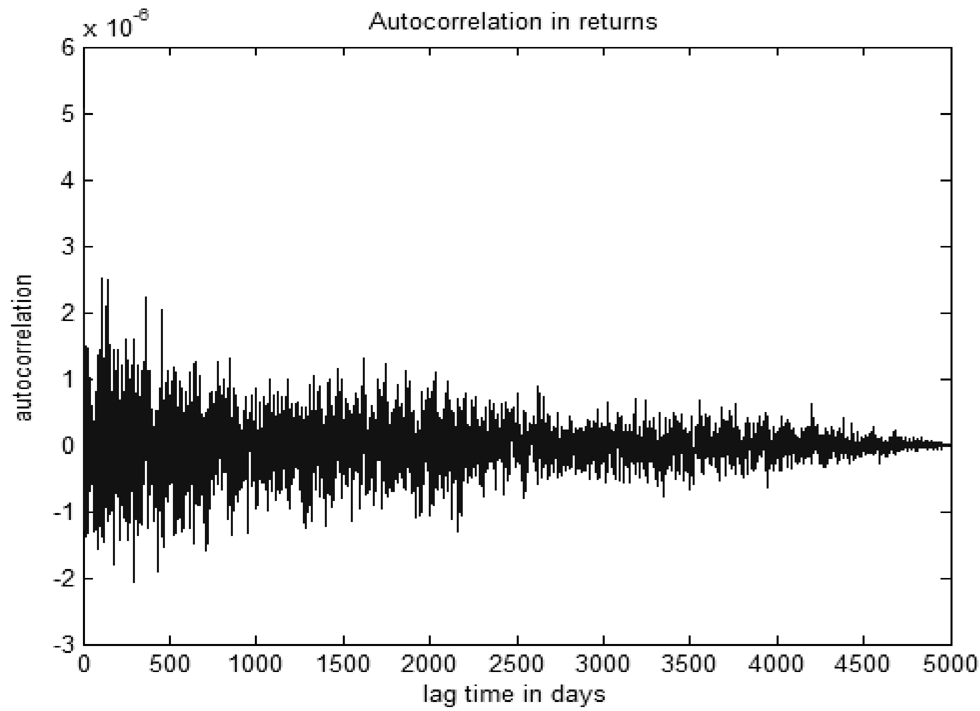


Figure 2.6 Autocorrelation of returns observed in DJIA

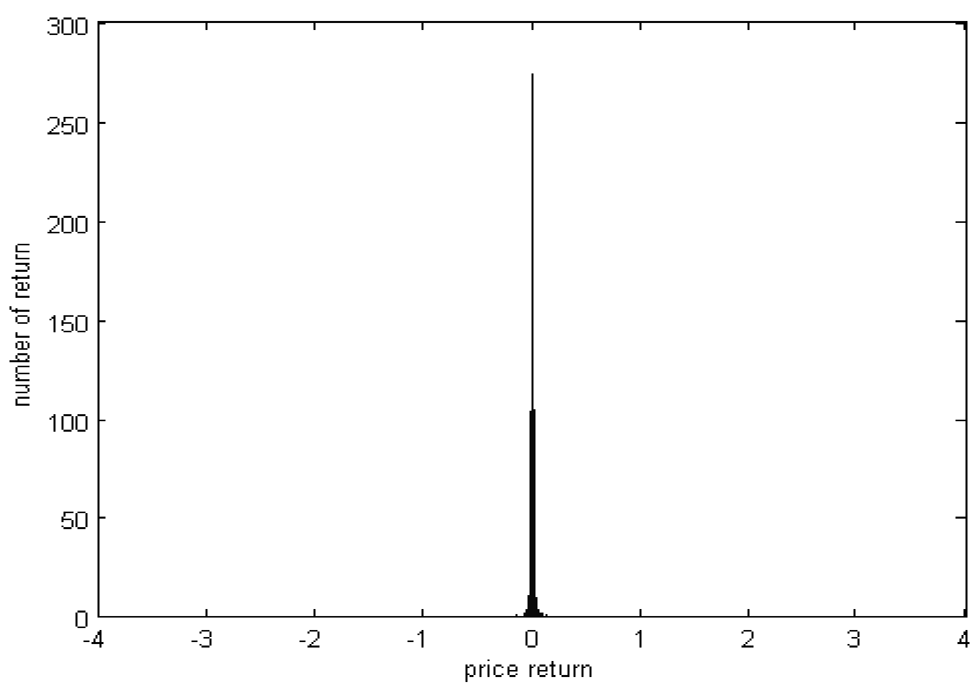


Figure 2.7 Distribution of returns observed in DJIA

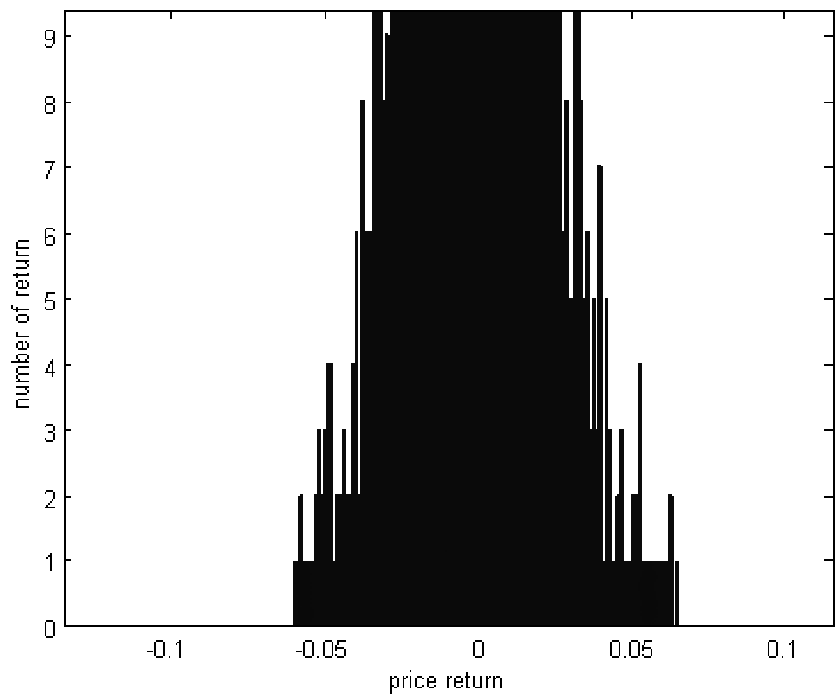


Figure 2.8 Fat tails (Enlarged version of figure 2.7)

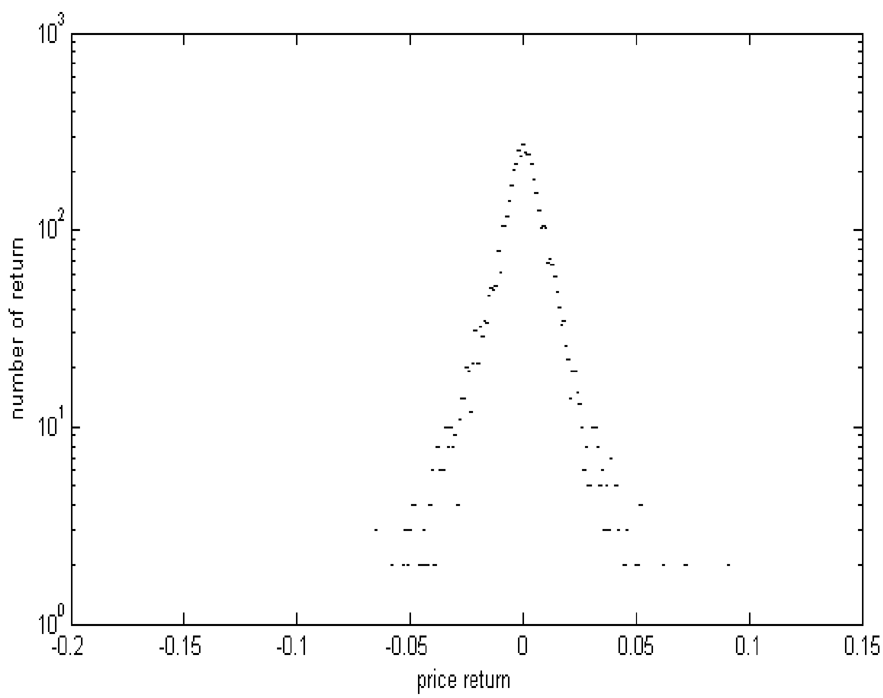


Figure 2.9 Fat Tails Log Version

2.2 El Farol Bar Problem

The El Farol Bar problem created by W. Brian Arthur, in 1994 is a problem of game theory. It is a problem related to inductive reasoning and bounded rationality. Economists suggest beyond certain complicatedness, our logical apparatus ceases to cope i.e our rationality is bounded [2]. In this case, humans use characteristic and predictable methods of reasoning known as inductive reasoning [2]. One of the reasons behind using inductive reasoning is that the individuals have bounded rationality [2] which means that the information provided is not sufficient to derive a logical solution. In such cases the deductive i.e. rational thinking is replaced by inductive reasoning [2]. The El Farol Bar problem addresses this concept of inductive reasoning. It is described as follows:

A particular number of people (say N agents) have to decide whether to go to bar on a particular day based on the assumption that the bar will be crowded or not. Every agent has to take decision at the same time without knowing what others have decided. Because, the agents do not have any information about other agents, there is no perfect deductive solution to this problem. It's possible that all agents assume that every other agent has decided to visit the bar and so each agent ends up staying at home or vice versa. The only information that the agents have at the time of making their decisions is the history of the past decisions. For example, considering the total number of agents as 100, the no of agents who attended the bar is given by:

63 42 72 53 49 36 70 39 51 40 44 84 35 19 47 54 41

Here, one of the possible solutions of this problem is that agents can foretell the solution based on some predicators. Some of the possible predicators are:

- Same as 3 weeks ago: 47
- Mirror image around 50 of last week's attendance: 59
- Minimum of last 5 weeks: 19
- Rounded average of last 3 weeks: 48

Each of the agents chooses one of the predicators and at the end, rewards the predicator which predicted the correct outcome. As the time proceeds, agents start choosing the predicators with the highest points to get the correct outcome. Computer simulation demonstrated that attendance fluctuated around 60% [2]. The reason behind this somewhat surprising feature is that agents adapt to the aggregate environment that they jointly create. The aggregate behavior of the agents adapt to what is known as a form of Nash equilibrium [45]. Nash equilibrium is an action profile with the property that no single player can obtain a higher pay off by deviating unilaterally from this profile [24, 45].The Nash Equilibrium is a concept mostly found to explain agent based games and it shows that through simulations, the game finally comes to a point that any change made to it won't improve the results of the game. In the El Farol Bar Problem, the game runs for various rounds and then finally when there comes a point which is approximately that 40% people take a particular action and the remaining 60% take the other action. After this equilibrium point, all agents have attained their best aggregate state and no change in their action will help them to improve their performance or the current equilibrium state.

2.3 Minority Game Adaptation to El Farol Bar Problem

The Minority game [7] is proposed by Yi-Cheng Zhang and Damien Challet from the University of Fribourg. It is an adaptation to the El Farol Bar problem. Like El Farol Bar Problem it is also based on the fundamental of inductive reasoning and bounded rationality [7]. The game consists of N odd numbers of agents. Every agent of the game is to choose one of two choices independently at each turn. They have no information regarding the decisions of others. So as information provided is not sufficient to derive a logical solution, logical reasoning is replaced by inductive reasoning. The winning criterion of the game is to come in the minority after each round of the game. All the agents chose actions; +1 representing to go to the bar and -1 representing staying at home. After all the agents have provided with their actions, the agents who chose the minority action out of the total number of actions win the game.

All the agents decide whether to go to the bar or not based on their previous memories. Each agent is attributed with a short term memory and long term memory. Each long term consists of various strategies. For size of memory say M, there are 2^M possible inputs and the total number of possible strategy sets are $2^{2^\wedge M}$. Table 2.1 shows a sample strategy with memory M=3.

History	Action
000	+1
001	+1
010	-1
011	+1
100	-1
101	-1
110	+1
111	-1

Table 2.1 Sample strategy with memory M=3

Out of the $2^{2^{\wedge}M}$ strategies, each agent chooses a particular amount of strategies say S. Each strategy chosen has a strategy score associated with it. This is incremented when the agent wins a round (when the agent is in minority) on choosing that particular strategy and decremented vice versa. In each round, each agent takes an action either to go to the bar (+1) or to stay at home (-1). This action is chosen on comparing the strategy with the highest score with the short term memory. There is an action (+1 or -1) associated with each option in each strategy as shown in the table 2.1. The bits stored in short term memory are matched with the bits of the strategy having the highest score and according the action associated with that strategy is selected.

At the end of each round, the **differential head count which is the sum of actions** of all agents for a round is calculated. The differential headcount can never be 0 as the total no of agents is always odd and the actions are either +1s or -1s. The differential headcount is given by:

$$A(t) = \sum_{j=1}^{N} a_j(t) \tag{2.7}$$

Where A(t) = Differential Headcount of round t

j= agent number (1≤ j≤ N, no of agents)

- If A(t) < 0, then this indicates majority of agents stayed at home (taking action -1) and thus the bar at time t was enjoyable.
- If A(t) > 0, then this indicates majority of agents went to the bar (taking action +1) and thus the bar at time t was crowded.

The payoff of an agent i.e. whether an agent won or lost a round is given by the function.

$$g_i(t) = (-1) * a_i(t) * A(t) \tag{2.8}$$

Where g(t) =payoff of agent of round t

i= agent number (1≤ i≤ N, no of agents)

The absolute value of g_i represents the margin by which agent won or lost the round. The main use of payoff is to figure out whether an agent ends up in the minority or majority at the end of the round.

If $g_i(t) > 0$, then agent won for round t i.e. the agent was in the minority.

If $g_i(t) < 0$, then agent lost for round t i.e the agent was in the majority.

The agents can retain the last bar outcomes equivalent to the size of the short term memory. The short term memory acts as a shift register. After each round, a new bit will push the oldest bit out. Each agent makes their next decision based only on these *M* bits of historical data.

Initially at the start of the game, all strategies have score 0. So initially, all agents choose strategy randomly out of the S chosen strategies. Also, at an instance of time, if there are two or more strategies having the same highest score, then at that time the strategy is chosen randomly out of those strategies. The winning strategy is awarded a virtual point and thereafter the strategy with maximum points is chosen.

$$\sigma_{ij}(t) = \sigma_{ij}(t-1) + 1, \text{ if } (a_{ij}.A(t)) > 0 \qquad (2.9)$$

$$\sigma_{ij}(t-1) - 1, \text{ if } (a_{ij}.A(t)) < 0$$

$\sigma_{ij}(t)$ = *Strategy Score of round t*

i= *agent number* ($1 \leq i \leq N$, *no of agents*)

$1 \leq j \leq S$, *where S represents strategy*

Since agents keep track of how their strategies are performing, update their points, and pick the strategy that is performing best, the agents are constantly adapting.

2.4 Minority Game as a Financial Game Model

The minority game is a game involving interactions between heterogeneous agents. Studies [17, 30] reveal that the interaction between heterogeneous agents can represent financial markets [37]. Also in the minority game, the binary decisions of the agents are interpreted as 'buying' versus 'selling', and hence the aggregate action corresponds to an excess demand in this model market, and leads to movements of the price [14]. Traders in the Minority Game take their decisions based on the strategies having high scores. This is related to the past price history of the market or to information provided externally. For simplicity, it is assumed that there is just one type of stock. At the end of each round, the new price of the stock is calculated based on the demand. Based on the new price, agents then buy or sell the stock in the next round.

Various models of minority games have been applied to markets like Grand Canonical Minority game [21, 25], $ game [1], Lux Marchesi [31], Mixed Game [16, 17] and others. To achieve each of these models, some changes have been made to original minority game to make them fit as market game models.

2.5 Limitations of Minority Game as a model for depicting Financial Markets

Though minority game is considered to be a good model for depicting financial markets, it has some limitations as well which are as follows:

1. In original minority game, the diversity of the agents is limited, since all the agents have the same lengths of memories.
2. The minority game concentrates just on gaining profit by being in the minority i.e. they act like tendency followers (who gain profit by studying history of price variations). In real financial world, some agents are tendency-followers who try to gain profit by being in the minority as well as noise traders [1, 31, 30, 6, 37, 15] who try to gain profit by being in the majority.
3. The minority game is synchronous. All agents co-ordinate their decisions at all time intervals. Trading decisions on most financial markets however, are taken asynchronously [3].
4. All agents have to either buy or sell at each time step. There is no condition for holding the stock.
5. All agents begin playing at the start of the game and play till the end. There are no new agents born or none of the agents die during the game.
6. Also, all agents buy and sell equal quantity amount of stock every time.

We discuss the limitations that have been removed and also the limitations we have removed in the section 2.6.

2.6 Related Work

The Minority Game has been the subject of interest for a long time. Many attempts have been made to relate the minority game as a financial model. The original minority game in its most basic form is unable to represent the financial model [14]. As a result, various changes have been made to the original minority game to make it suitable for representing financial markets. The main limitation behind the original minority game as a financial model is the lack of heterogeneity which is a very important element of the real financial world [37]. Apart from this limitation, there are various other limitations which are described in section 2.5. Each of the previous works have tried to remove these limitations in some or the other manner as in the Grand Canonical Minority Game introduced the option of holding the stock if the agent is not doing well (Section 2.5, Limitation 4). For this, some of the agents in Grand Canonical are provided with a zero strategy which can be used to hold the decision if required [21, 25]. Next the $ Game proposed by Anderon and Sornette works on the limitation that profit in financial market is not just gained by being in the minority (Section 2.5, Limitation 2). To remove this limitation, it offers a different payoff function where the gain at time t depends on the trading action of agents at time t-1, thus making the game a two step game [1]. The Lux Marchesi Model [31] works on the removing the limitation of less diversity among agents. To address this issue, the original minority game has been modified such that it has pools of agents: fundamentalists and minority games with different historical memories. It works on providing diversity through the movements of individuals from one group to another together with the (exogenous) changes of the fundamental value and the (endogenous) price changes resulting from the agents' market operations [31]. **In all of the above mentioned models, apart from the random assignment of strategies there is no heterogeneity in the different types of agents, whereas real markets are composed of different groups of traders, with different objectives, impact on the market, and trading behavior[14].** Also one interesting thing to be noticed regarding all these variations is that all these models concentrate on the combined effect of agents on financial factors like prices, returns and others i.e. all these models are synchronous. **To our best knowledge, none of the models concentrate on the individual effect of price variations i.e. asynchronous models.** Our paper focuses on both all these limitations and removing limitations 1, 2, 3 and 5 (Section 2.5) by our new game models namely Highly Heterogeneous Game model and Asynchronous Mixed Game model.

For removing the limitation 2, we have used Mixed Game Model [16, 17]. As mentioned in limitation 2(Section 2.5), in real markets, some agents try to gain profit by doing analysis of the previous histories of prices know as tendency followers [17]. For our game these are the agents who try to gain profit by being in the minority. Also, in real markets there exist individuals, who try to gain by following the herd known as “noise traders [1, 31, 30, 6, 37, 15]. In our game, these are the agents who try to gain by being in the majority. The main goal behind the market is to make money whether it is made by being in minority or majority does not matter. Thus the Mixed Game Model removes limitation 2(Section 2.5) and also limitation 1(Section 2.5) as two different types of agents who try to gain profit by using opposite ideologies (being in minority and majority to gain profit) accentuates the diversity feature of the game. But in real markets, individuals not only have different trading strategies but have unique memories (both in length and content) which lead to them reacting differently to different situations. The Highly Heterogeneous Mixed Game (Section 3.2) addresses this situation by providing different unique short term memories and long term memories to each of the agents. This further adds to the diversity aspect and eliminates limitation 1(Section 2.5). We have further included the scenario where agents die in between the game and also new agents are born in Highly Heterogeneous Game to remove limitation 5 (Section 2.5). Finally, to eliminate limitation 3 (Section 2.5) we have introduced the Asynchronous Mixed Game (Section 3.3) which calculates price after each agent takes its action and thus the effect of each individual is taken into account. We have further related this model to real world financial market which is discussed in further chapters 3 and 4.

Chapter 3: Synchronous and Asynchronous Mixed Game

This chapter we describe the general structure of the Mixed Game. We first describe each individual feature as in the types of agents in the game, how agents tend to make their decisions, how they are rewarded and calculating the various factors like price, returns etc for the Mixed Game. After describing each of this individually, we have described the block diagrams followed by step by step flow chart of the game. We further discuss the improvement to the Mixed Game which is the Highly Heterogeneous Mixed Game and discuss how it differs from the original Mixed Game. Finally we describe the Asynchronous Mixed Game, its use and how it differs from the Synchronous game.

3.1 Synchronous Mixed Game

Chengling Gou introduced the Mixed Game model and used it to predict the Shanghai markets [16, 17]. The mixed game is so called because it consists of two groups of agents; one group playing the majority game and one group playing the minority game. The minority and majority groups have different set of parameters i.e. they have different number of agents, values of memory lengths, long term memories, number of strategies selected and time horizons. Table 3.1 represents the different components of the mixed game. We describe all of these one by one in the sections 3.1.1 to 3.1.3

Group 1	**Group 2**
Majority Game	Minority Game

No of agents N1	No of agents N2
Noise Traders (agents who vie to be in the majority to win the game)	Fundamentalists (agents who vie to be in the minority to win the game)
Memory m1	Memory m2
Strategies $2^{2^{\wedge}m1}$	Strategies $2^{2^{\wedge}m2}$
Time Horizon T1	Time Horizon T2

Table 3.1 Components of Mixed Game Model

3.1.1 Types of Agents

In the mixed games, there are basically two types of agents: N1 Noise Traders and N2 Fundamentalists. The total number of agents is N=N1+N2. The noise traders are the agents which play the majority game [16, 17].They are named so because they just follow the noise i.e. they follow the majority decision. They believe in gaining profit by following others. In real markets, these are the people who follow the buzz and act irrationally. They do not make use of any fundamental information available. They imitate the other traders and as soon as they notice even little rise or fall, they jump to irrational conclusions.

The other type of agents is the fundamentalists [16, 17] who play the minority game. They try gaining profit by being in the minority. They strongly follow the concept of price being driven by demand in the market [13]. If the demand is more, the price will increase and selling is a better option in this case. Similarly, if the demand is less then profit can be gained by buying the stock. Thus, being in minority can help one gain considerable profit. The minority game players work according to this criterion. These different types of agents provide heterogeneity to the game. In real world, the investors are completely heterogeneous in nature. They have different memories and tendencies to invest in the stock market. By having noise traders and fundamentalists with different memory lengths and number of strategies, we increase the heterogeneity nature of the game. This leads to various interesting observations which will be discussed in the later chapter 4.

3.1.2 Agent's Decision Making

The agents make their decisions to buy or sell the stock based on their respective strategies. The noise traders and fundamentalists are provided with different memory sizes say m1 and m2 respectively. Based on the memory sizes, they have long term memories and short term memories. The majority game players have short term memory with length m1 and long term memories with length $2^{2^\wedge m1}$. Similarly, the minority game players have short term memory with length m2 and long term memories with length $2^{2^\wedge m2}$. Out of each of these strategies of long term memory, the noise traders as well as the fundamentalists chose a particular amount of strategies say s1 and s2. The short term memory is like a m1 bit register for majority game players and m2 bit register for minority game players. It stores the last m1 and m2 actions of the majority and minority agents respectively. The m1 and m2 bits of the short term memory of the majority and minority agents are compared to the strategy having the maximum score out of s1 and s2 selected strategies and accordingly the action associated with it is taken into consideration.

At the end of each round the **differential head count which is the sum of actions** of all agents for a round is calculated. The differential headcount can never be 0 as the total no of agents is always odd and the actions are either +1s or -1s. The differential headcount is given by equation 3.1 which is as follows:

$$A(t) = \sum_{j=1}^{N} a_j(t) \qquad (3.1)$$

Where A(t) = Differential Headcount of round t

- If A(t) < 0, then this indicates majority of N took action -1, minority took action +1
- If A(t) > 0, then this indicates majority of N took action +1, minority took action -1

This action further is stored in the last bit of m1 and m2 bit register for majority and minority agents respectively. This leads to each of the bits being shifted forward further resulting in the discarding of the first bit of the register.

At the beginning of the game, out of the $2^{2^{\wedge}m1}$ sized long term memory, the majority game players randomly chose s1 strategies and similarly out of the $2^{2^{\wedge}m2}$ sized long term memory the minority game players randomly chose s2 strategies. Then onwards, for each round, each agent chooses the strategy having the highest score. The majority agents has **time horizon counter T1** and the minority agents have **time horizon counter T2**. The agents collect their points over the time horizons and then from there on select the strategy with the highest score. The greater the time horizons; the more past information deposits and influences decision-making of agents. Therefore, these parameters represent the learning rates of agents who learn from their past performances. A fast learning rate means that agents can adapt themselves to their environment fast. **The essence of the time horizon counters is to enhance the concept of bounded rationality. Agents when need to select their best strategies from the limited time horizon, their rationality gets bounded. Also, for the first time horizon, agents choose their strategies randomly, so instead of sticking to just one strategy, agents get to choose and experience the game with different combinations of the strategies. Moreover, the time horizons help to give importance to the most recent decisions as the agents would be selecting strategies from the recent time horizons.**

The bits in the short term memory are compared with the strategy having the highest score and the corresponding action is taken. The strategies of the minority game players are awarded a point if they are in the minority at the end of the round. But in case of the majority game players, their strategies are awarded if they tend to appear in the majority at the end of the round. This is given by the table 3.2. Also, each agent is allocated with an agent score which is incremented by 1 if the agent wins the round and is decremented by 1 if the agent loses the round. This is given by equations 3.5 and 3.6. The majority agents win the round if they end up in majority and the minority agents win the round if they end up in minority. The agent scores are used at the end of the game to calculate the mean average score of majority and minority group and basically measure the performance of the game (Table 3.2).

The **payoff of an agent** i.e. whether an agent won or lost a round is given by the equation 3.2.

$$g_i(t) = (-1) * a_i(t) * A(t) \qquad (3.2)$$

Where g(t) =payoff of agent of round t

The main use of payoff is to figure out whether an agent ends up in the minority or majority at the end of the round which is mentioned in table 3.2.

Majority agents	***Minority agents***
If $g_i(t) < 0$, then agent lost for round t i.e the agent was in the majority. ***If agent ends in majority at the end of round*** $\sigma_{ij}(t) = \sigma_{ij}(t-1) + 1$ ***or else*** $\sigma_{ij}(t-1) - 1$ (3.3) $\sigma_{ij}(t)$ = *Strategy Score of round t* $\pi_{ij}(t) = \pi_{ij}(t-1) + 1$ ***or else*** $\pi_{ij}(t-1) - 1$ (3.5)	**If $g_i(t) > 0$**, then agent won for round t i.e. the agent was in the minority ***If agent ends in minority at the end of round*** $\sigma_{ij}(t) = \sigma_{ij}(t-1) + 1$ ***or else*** $\sigma_{ij}(t-1) - 1$ (3.4) $\sigma_{ij}(t)$ = *Strategy Score of round t* $\pi_{ij}(t) = \pi_{ij}(t-1) + 1$ ***or else*** $\pi_{ij}(t-1) - 1$ (3.6)

π_{ij} ***(t) =*** *Agent score of round*	π_{ij} ***(t) =*** *Agent score of round*

Table 3.2 Technique for updating strategy scores and agent scores for majority and minority agents

This compared to real world suggests that some traders gain profit by being in the majority i.e. by following the trend while some others gain profit by taking an individual path i.e. by doing fundamental analysis and basically taking an independent decision.

Initially, when the game has just begun the strategies do not have any scores allocated to them. So, at that time any random strategy is selected. Also, if at an instance of time, there are two or more strategies having the highest score, any of those strategies is selected randomly. This entire process is followed in each round and thus the agents try to choose the winning strategies. This when compared to the real world financial market is equivalent to observing the patterns in the past prices and acting accordingly. The agents in mixed game are following the patterns based on the history of wins. In real world, people observe financial patterns [28] generated earlier and try to gain profit by acting accordingly.

Chengling Gou , the founder of the Mixed Game Model has already proved and established that the learning power of the majority agents is faster than that of the minority agents [17]. So, the memory lengths, number of strategies selected and the time horizons of the majority agents should be less than that of minority agents. Also, he has established that an active financial market must be dominated by agents who play a minority game, otherwise, the market would die [17]. This means the number of minority game players must be more than the majority game players. Thus N1 < N*0.5 [17]. We follow the same fundamental established by him throughout our game.

3.1.3 Calculating Price, Volatility and Returns

At the end of each of the round the price, volatility and returns are calculated. These are directly influenced by the interactions between the heterogeneous agents. Price is said to be a measure of the demand. As a result, in order to calculate price, we first need to calculate demand.

The demand[16,17] of a given round is given by the difference between the number of buying orders and the number of selling orders i.e. the number of 1's and -1's of previous round respectively.

$$D(t) = n_{buying\text{-}orders}[t-1] - n_{selling\text{-}orders}[t-1] \qquad (3.7)$$

Where D(t) = demand at time t

The price of the stock is said to be driven by the demand at that time [13]. So, the price is of the stock is given by:

$$P(t) = D(t) + P(t-1) \qquad (3.8)$$

Where P(t) = Price at time t

P(t-1) = Price at time t-1

$$D(t) = Demand\ at\ time\ t$$

The volatility is given by the standard deviation of the prices over a period of time [16]. The main motive of calculating the volatility is to measure the swings in demand and supply over a period of time. The volatility is given by equation 3.9.

$$V(t) = \sigma p' * \sigma p' \qquad (3.9)$$

$$\sigma p' = \sqrt{i / t \sum_{i=1}^{t} (p' - \overline{p'})}$$

$$Where\ p' = p(t) - p(t-1)$$

$$V(t) = Volatility\ at\ time\ t$$

Return is given by the change of price of an asset over a period of time. The return is given by the equation 3.10.

$$R(t) = p(t) - p(t-1) \qquad (3.10)$$

$$Where\ P(t) = Price\ at\ time\ t$$

$$P(t-1) = Price\ at\ time\ t-1$$

3.1.4 Algorithm

This section describes the generic algorithm that we have developed for simulation of our Mixed Game model. This algorithm is used for the analysis of price series and to obtain the stylized facts (Section 2.1.2). We made subtle modifications to this Mixed Game Algorithm to perform other simulation tests which are discussed in later chapters

(Chapter 3 and 4). We have used the modular approach and have divided the algorithm into four modules

1. Market Setup
2. Agents Setup
3. Agents' Trading and Market Operation
4. Agent Adaptation and Interaction

3.1.4.1 Market Setup

1. Initialize the round counter (i.e. counter representing the number of rounds) to 0.
2. Initialize the differential headcount (Equation 3.1) to 0.
3. Initialize arrays to hold price series (Equation 3.8), volatility (Equation 3.9) and returns (Equation 3.10).

3.1.4.2 Agents Setup

1. Set the number of majority and minority game players to N1 and N2 respectively such that N1<N/0.5. The reason behind this is the fundamental established by Chengling Gou, the founder of Mixed Game[16] that in order for the market to run, minority game players should dominate the market or else the market will be unstable [16].

2. Set the size of memories for majority and minority game players to m1 and m2 respectively such that m1<m2<6. Set the number of strategies selected for majority and minority game players to s1 and s2 respectively such that s1<s2. Set the time horizons for majority and minority agents to T1 and T2 respectively such that T1≤T2. The reason behind the values of parameters of majority game players being less than minority game players is that Chengling Gou established that agents who play a majority game should have a faster learning rate than those who play a minority game for the stability of the Mixed Game Model [16]. This means that the majority game players should be able to learn the game faster with the available lesser memories and time horizons [16].
3. Generate randomly $2^{2^\wedge m1}$ and $2^{2^\wedge m2}$ strategies which form the long term memories for majority and minority game players respectively.
4. Also generate the initial short term memories which are again randomly generated strings with length equivalent to memory length which is m1 and m2 respectively for majority and minority game players.
5. Setup counters for counting the success rate of each strategy. Initially the counters for all the strategies are assigned to 0.
6. Setup counters for counting the success rate of each agent. Initially the counters for all the agents are assigned to 0.

3.1.4.3 Agents' Trading & Market Operation

1. At the start of the game, when the scores allocated to each of the strategies are 0, agents pick the strategy randomly from the s1 and s2 strategies for majority and minority game agents respectively. Further as the game advances and scores start getting allocated to strategies, the agents choose the strategy with the highest score. The selection of the strategies with highest scores is bounded by the time horizons T1 and T2.
2. Once all the trading decisions are made, the number of buyers (number of 1s) and the number of sellers (number of -1s) are calculated. The demand is influenced by the amount of sellers and buyers and thus the current demand is calculated as per equation 3.7.
3. The price is driven by the value of demand [13]. The current price is calculated using the price of the previous round and the previous value of the demand. It is calculated as per the equation 3.8 and stored in the price series array.
4. The volatility representing the swings in the prices can be calculated by making use of the prices. It is calculated as per the equation 3.9. The returns are calculated based on the equation 3.10.

3.1.4.4 Agent Adaptation and Interaction

1. The counters that represent the success rates of strategies are updated as per table 3.2.
    - In case of group playing the majority game, the agents win if they are in the majority. So, if the strategy they chose places them in the

majority, then the strategy scores are incremented by one or else they are decremented by one. Also, the agent's score is incremented by one if the agent wins the round.

- Similarly for the group of agents playing the minority game, the strategy scores are incremented by one if they are in the minority or decremented otherwise. Again, the agent's score is decremented by one if the agent loses the round.

As the agents use the strategies based on the constantly updated scores of the strategies, the agents are constantly adapting. This when compared to the real world financial market is equivalent to observing the patterns [28] in the past prices and acting accordingly. The agents in mixed game are following the patterns based on the history of wins. In real world, people observe financial patterns [28] generated earlier and try to gain profit by acting accordingly.

2. The T1 and T1 counters are also incremented by 1.
3. The round counter is incremented by one (if it has not reached the total number of rounds).

3.1.5 Simulator Design for Mixed Game Model

In this section, we describe the design of the simulator for the mixed game. Figure 3.1 shows the general design of the simulator for the mixed game. Module 1(figure 3.2) represents the module for market setup where the overall market for the mixed game is set. Module 1 is described in section 3.1.5.1. Module 2 (figure 3.3) is the module for setting up the agents and basically initializing agents with long term memory and short term memory. Module 2 is described in section 3.1.5.2. In module 3(figure 3.4), the actual trading between the heterogeneous agents take place (Section 3.1.5.3) and in module 4(figure 3.5), the interactions between the agents leading to adaptations take place.

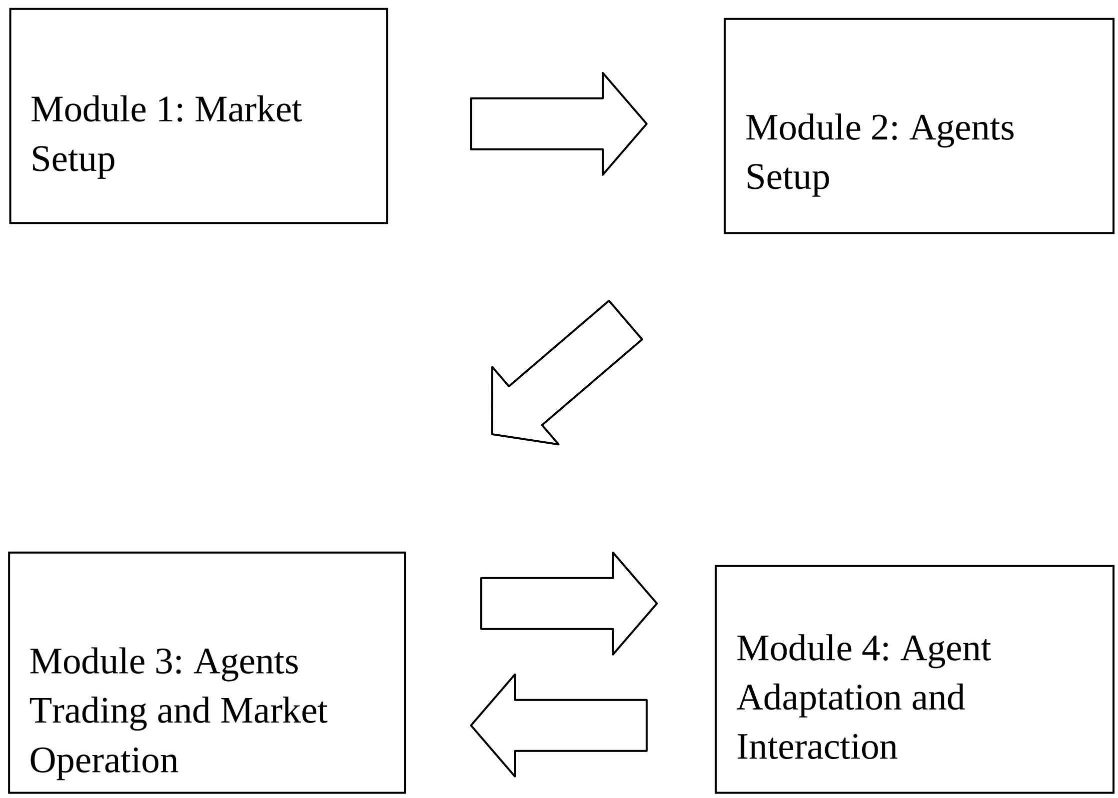


Figure 3.1 Overall module Structure

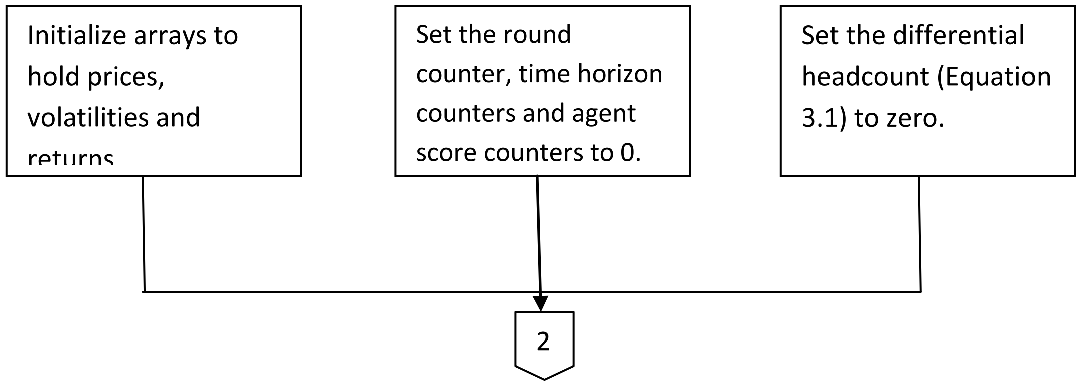


Figure 3. 2 Module 1: Market Setup

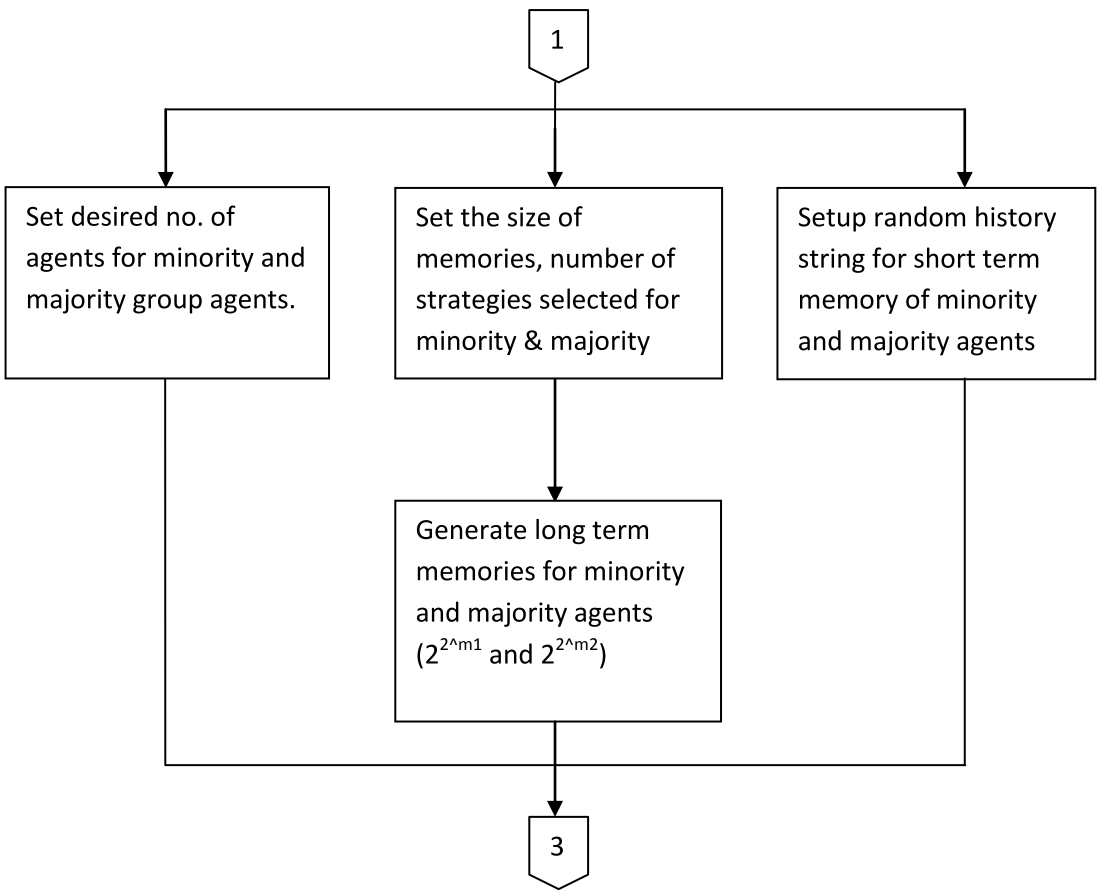


Figure 3. 3 Module 2: Agents Setup

In the **Market Setup module** (figure 3.2), the entire market required for the processing of the mixed game is set up. Arrays are initialized to hold prices, volatilities and returns. Also the counter indicating the number of rounds is set to 0 (i.e. start of the game) and the differential headcount is also set to 0. Also, the time horizon counters (T1 and T2) are set to 0.

In the **Agents Setup module** (figure 3.3), firstly the number of majority and minority agents is set up. Then for all of the majority and minority agent groups, the random memory string of size m1 and m2 for short term memories is set up. Further, the $2^{2^{\wedge}m1}$ and $2^{2^{\wedge}m2}$ strategies are set up randomly which constitute the long term memories for majority and minority game agents respectively. Also, out of these long term memories, a particular amount of strategies, s1 and s2 is picked and allocated to the minority and majority game agents respectively.

In the module 3(figure 3.4), **Agents Trading and Market Operation**, each agent picks its strategy with the highest score. In the beginning when no scores have been allocated to the strategies or in the case where there are two or more strategies having the same maximum scores, then the strategy is selected randomly. After each agent selects an action based on the best strategy, the demand is calculated based on the actions of all the agents (Equation 3.7). Then the price of the asset is calculated based on the value of the demand (Equation 3.8). Further the volatility and return is calculated based on equations 3.9 and 3.10 respectively.

In module 4(figure 3.5), **Agent Adaptation and Interaction**, the basic adaptation of the agents takes place based on the selection of strategies and the wins and losses earned by them. After each round, the strategy scores and the agent scores are incremented by 1 if an agent wins using that strategy and are decremented by 1 if the agent loses by using that strategy. For majority agents, they are in minority and the minority agents win a round if they are in the minority at the end of the round.

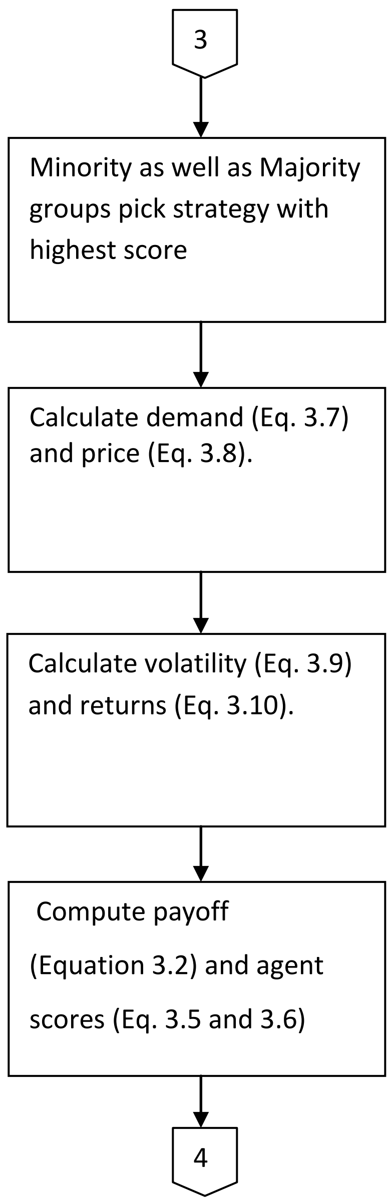


Figure 3. 4 Module 3: Agents Trading and Market Operation

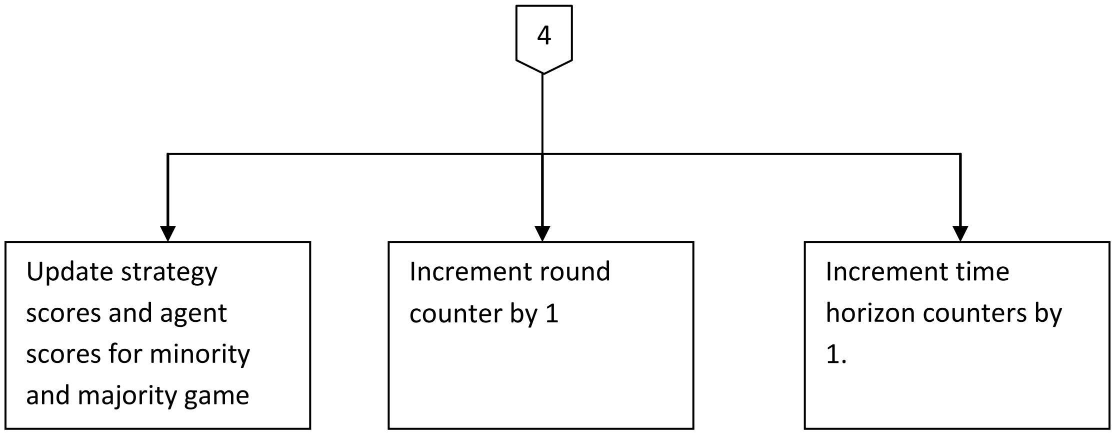
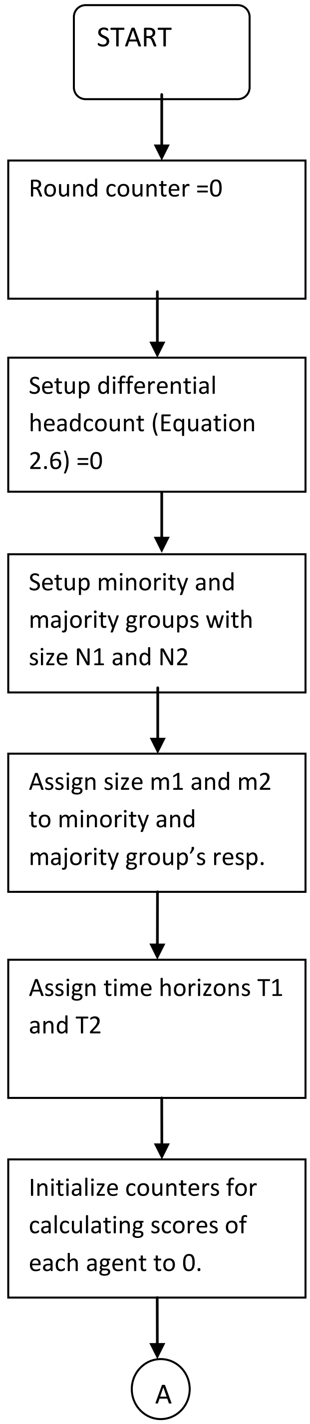
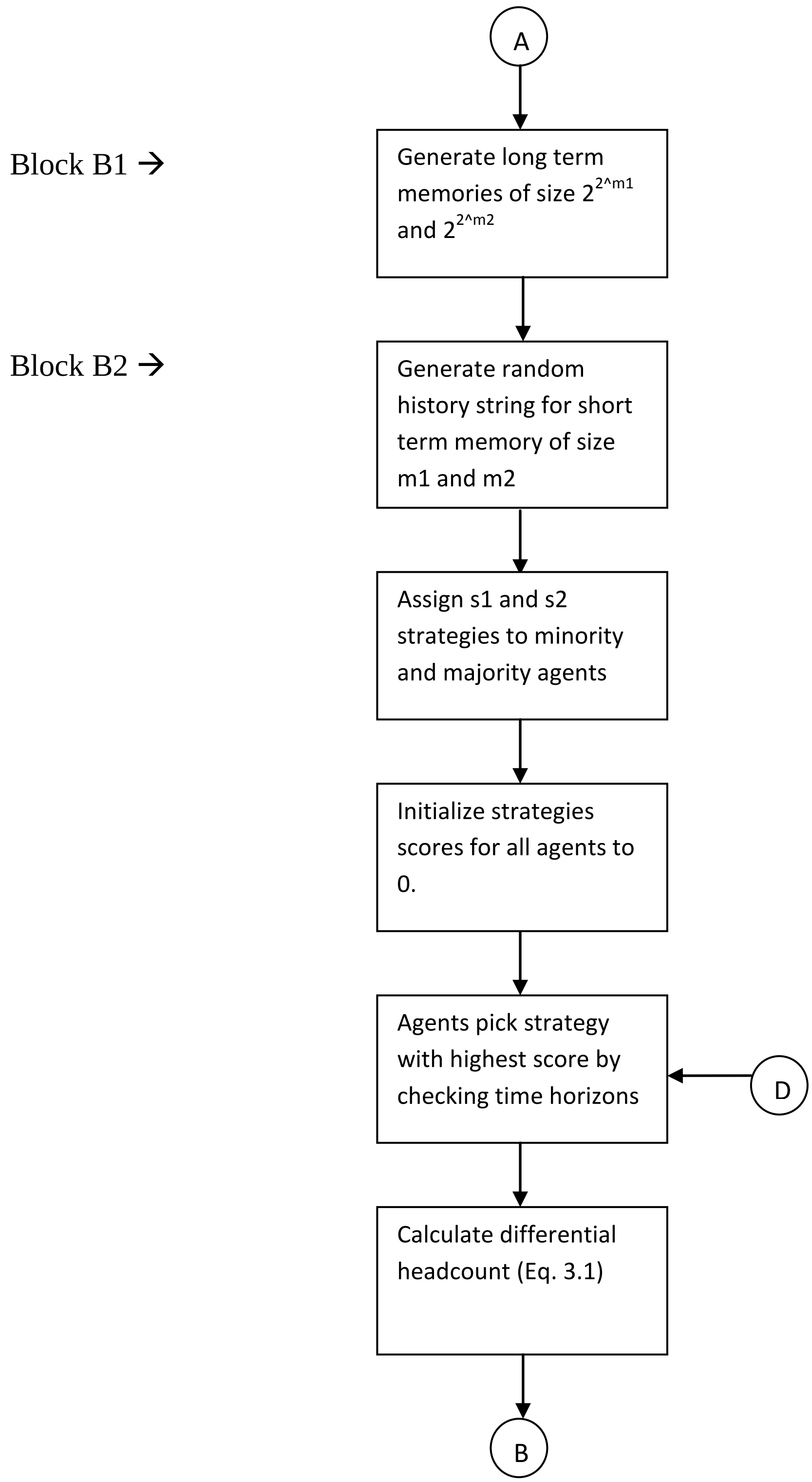


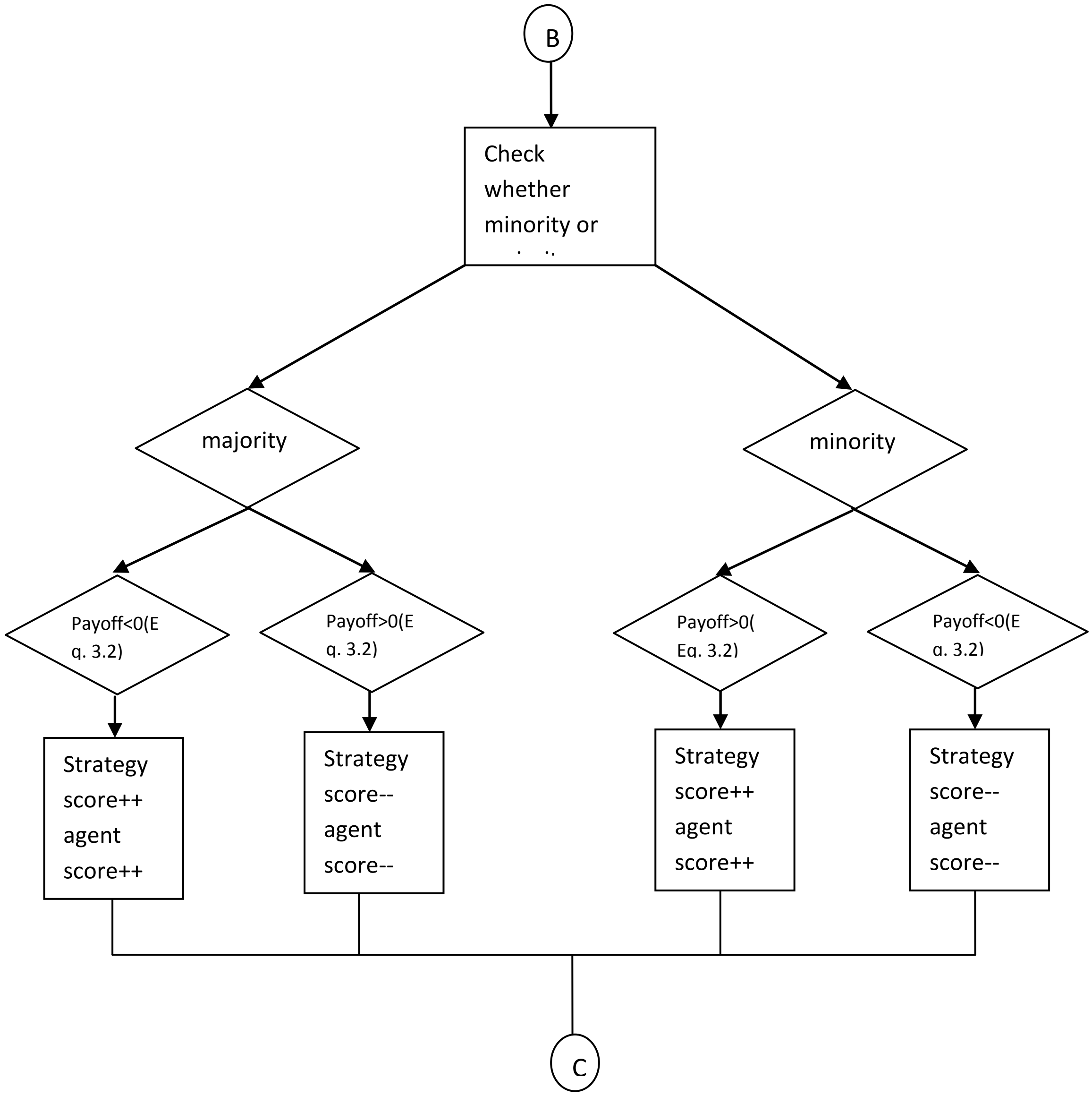
Figure 3. 5 Module 4: Agents' Adaptation & Interaction

3.1.6 Flowchart

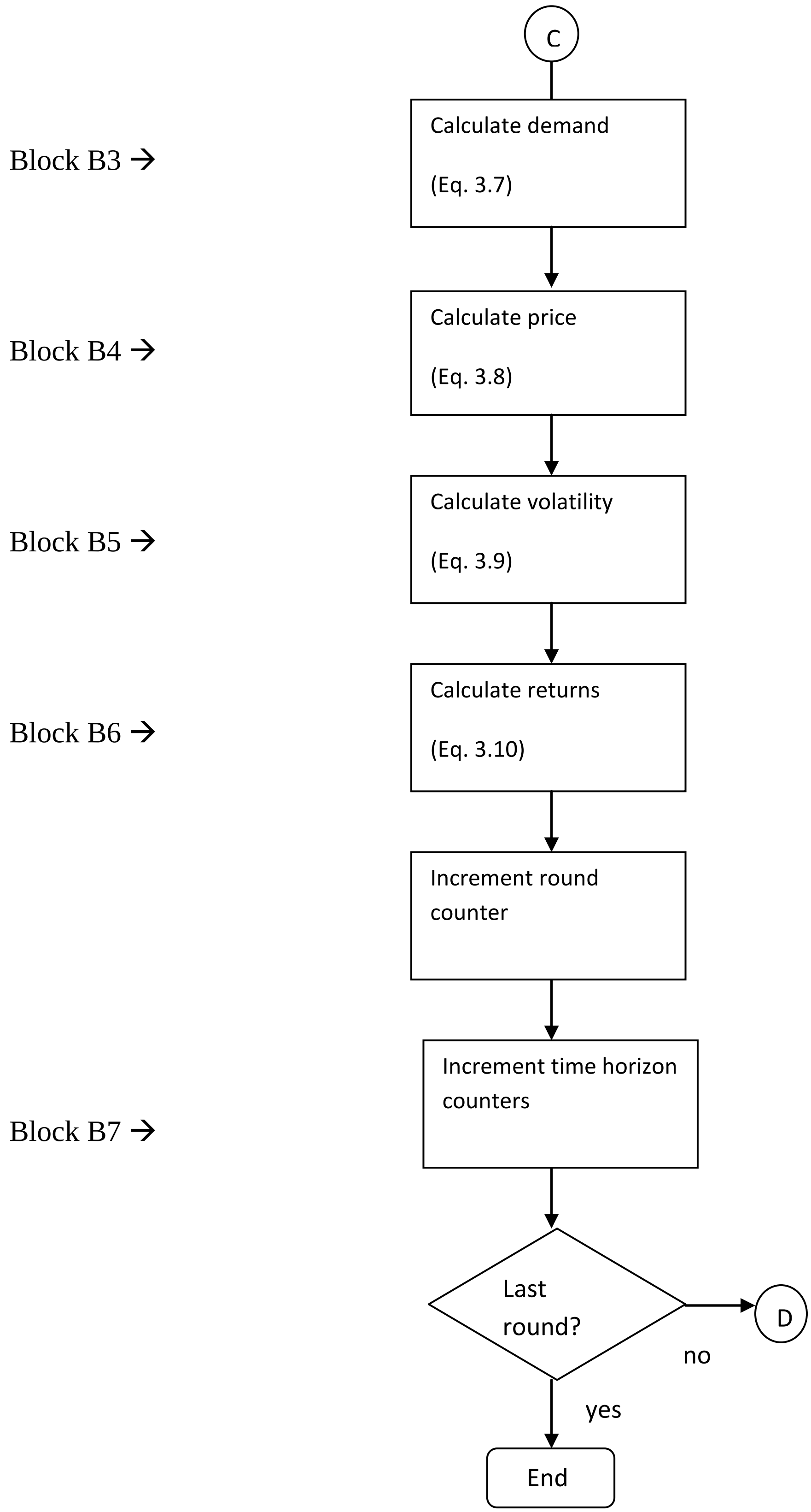
Section 3.1.6 represented the modular structure of our game by dividing the game into four modules. In this section, we show the flowchart of the game which consists of combination of all four modules. This flowchart gives a pictorial description of all the steps required to model the Mixed Game. It basically gives a step by step analysis of each round in the Mixed Game.


START
Round counter =0
Setup differential headcount (Equation 2.6) =0
Setup minority and majority groups with size N1 and N2
Assign size m1 and m2 to minority and majority group's resp.
Assign time horizons T1 and T2
Initialize counters for calculating scores of each agent to 0.
A


A
Block B1 →
Generate long term memories of size $2^{2^{m1}}$ and $2^{2^{m2}}$
Block B2 →
Generate random history string for short term memory of size m1 and m2
Assign s1 and s2 strategies to minority and majority agents
Initialize strategies scores for all agents to 0.
Agents pick strategy with highest score by checking time horizons
D
Calculate differential headcount (Eq. 3.1)
B



B
Check whether minority or
majority
minority
Payoff<0(E q. 3.2)
Payoff>0(E q. 3.2)
Payoff>0(Eq. 3.2)
Payoff<0(E q. 3.2)
Strategy score++ agent score++
Strategy score-- agent score--
Strategy score++ agent score++
Strategy score-- agent score--
C


C
Block B3 →
Calculate demand
(Eq. 3.7)
Block B4 →
Calculate price
(Eq. 3.8)
Block B5 →
Calculate volatility
(Eq. 3.9)
Block B6 →
Calculate returns
(Eq. 3.10)
Increment round counter
Block B7 →
Increment time horizon counters
Last round?
no
D
yes
End

3.2 Highly Heterogeneous Mixed Game

We chose Mixed Game to emulate the financial world as it provides with diversity in form of majority and minority game players. When compared to the real financial world, majority game players represent the noise traders [1, 31, 30, 6, 37, 15] who try to gain profit by following the herd(being in the majority) and minority game players represent fundamentalists who try to gain profit by doing the analysis of the prices and patterns [28] . Though the game provided heterogeneity in form of two groups; minority and majority, the minority group agents have same length of memories and the majority group agents have same length of memories (Section 2.5 Limitation 1). In real world, each of the individual possesses distinct memories which lead to them reacting in a distinct manner in every situation. To improve upon this we have randomly generated memories (following guidelines of Section 4.4) for all the agents to make each of them distinct. Because of this, each agent has a distinct learning tendency towards the game. **The reason behind random generation of memories is that in real life, memory is a function of brain and each human brain is as unique as a human face [12]. By providing memories randomly to agents, we try to achieve this feature.** Also in the Mixed Game, all agents start playing at the start of the game and play till the end (Section 2.5 Limitation 5). But in real world, we have traders coming and going in the financial market. So, we have made half of the worst performing agents to die and new agents are born (with distinct memories) for every alternate time horizon. The reason for causing the death of agents for every alternate time horizon is that for the first time horizon (i.e. number of rounds equivalent to the time horizon), the agents pick the strategies randomly. When the next time horizon appears the agents start picking the strategy with the highest score from the previous time horizon. For example, if the time horizon is 12 then for the first 12 rounds, new agents will pick strategies randomly but from the 13th round, the agents will select the best strategy from the last 12 rounds and continue accordingly. Thus, when the T1 (time horizon for majority game players) time horizon ends every alternate time, half of the majority agents die and new majority agents are born and similarly when the T2 (time horizon for majority game players) time horizon ends, half of the minority agents die and new minority agents are born. The reason for the death of the worst performing agents is that in real financial markets, the investors who lose money tend to leave the market and also new investors enter the markets to invest.

For this we have maintained exactly the same structure of the as that of mixed game (flowchart in Section 3.1.6) except for the two modifications:

- The memory lengths are generated randomly for all the agents (following the guidelines in Section 4.4). For this, block B1 and B2 of flowchart (Section 3.1.6) are modified. Instead of m1 (memory length for majority agents) and m2 (memory length for minority agents), each majority group agent is given a distinct m1 and each minority group agent is given a distinct m2 (following the guidelines in Section 4.4).
- At every second instance of time horizon T1, half of worst performing majority agents die and new agents are born in place of them and at every second instance of time horizon T2, half of minority agents die and new agents are born in place of them. This is reflected in the flowchart (Section 3.1.6) at time step of block B7 where N1/2(number

of majority agents/2) and N2/2 (number of minority agents/2) agents die and new agents are born with distinct memories.

3.3 Asynchronous Mixed Game Model

All the models related to minority game model like the $Game Model [1], Lux Marchesi Model [31] and even Mixed Game [16] are synchronous i.e. the agents coordinate their actions that are taken at discrete moments in time. In synchronous Mixed Game, in each round, all agents take an action and then based on this combined effect of actions of all the agents, the price, return and volatility is calculated. We develop a new game - Asynchronous Mixed Game in which the effect of each action is taken into account at that moment. In each round, as and when each agent takes an action, the price, return and volatility is calculated. Thus for each round, we have price series equivalent to the number of agents i.e. number of elements in price series = N (number of majority agents+ number of minority agents). In case of synchronous game just one price is calculated for each round.

The basis for modeling Asynchronous Mixed Game is that trading decisions on most financial markets are taken asynchronously [3]. With the availability of high frequency data, its analysis has been continuously gaining importance in recent years. Traders and researchers are not contented with low frequency financial market data like monthly and weekly data anymore. The demand for high-quality high frequency data: intra minute or tick-tick data (change in a stock's price from one trade to the next) is elevated. These data have their own distinctiveness and need corresponding analyzing methods. Traders nowadays use this data to make decisions like whether to trade or not [36].This very important feature of financial markets has hardly been taken into account till date [3]. As mentioned in the previous works[1,31,21,25], research of minority game as a tool for modeling financial market is going on from quite some time but according to our review of the recent literature[1,31,21,25], the asynchronous aspect of the game has not been captured. We are making an attempt in this direction and we gain some very useful information and findings through this which are mentioned in section 4.9.

When compared with the real financial world, the asynchronous game seems to resemble the tick by tick market also known as the high frequency data market. The term "tick" refers to a change in a stock's price from one trade to the next. It is basically the smallest possible change in the value of an asset. Traders nowadays aim to study the changes in tick-tick data to decide their trading decisions based on it. The tick-tick data provided the traders with patterns [28] based on each and every transaction which shows them exact picture of the financial market. We expect that by modeling the stylized facts based on the asynchronous market, we should achieve patterns in them similar to the patterns observed in high frequency data. The reason behind this is high frequency data or the tick-tick data provides us with the information about the smallest changes in the price from one trade to another and asynchronous mixed game works on the similar pattern.

3.3.1 Algorithm and structure of the game

The structure of the asynchronous mixed game is exactly same as that of the synchronous game described in section 3.1 except for the fact that every factor is looked upon after action (+1 for buying and -1 for selling) of each individual agent instead of combined actions of all agents in a round. Just as in the synchronous mixed game, there are two type of agents; N1 majority and N2 minority. The majority game players compete to me in the majority and the minority game players compete to be in the minority. The majority and minority game players are provided with long term memories of size $2^{2^{\wedge}m1}$ and $2^{2^{\wedge}m2}$ and short term memories of size m1 and m2 respectively. They choose strategies s1 and s2 respectively. They have time horizons of T1 and T2 respectively. The agents collect their points over the time horizons and then from there on select the strategy with the highest score. The greater the time horizons; the more past information deposits and influences decision-making of agents.

The first agent will first chose the strategy having the highest score based on its time horizon and compare it with the bits of short term memory to get the action (similar to the Synchronous Mixed Game model, section 3.1.2). Based on this one action, the demand is calculated. The demand is given by equation 3.11.

$$D[(t)] = n_{buying\text{-}orders}^{(t)} - n_{selling\text{-}orders}^{(t)} \qquad (3.11)$$

where, $n_{buying\text{-}orders}^{(t)}$ *= no of buying orders till that agent has taken action*

$n_{selling\text{-}orders}^{(t)}$ *= no of selling orders till that agent has taken action*

Once the demand has been calculated, the price is calculated based on it. The price is calculated based on previous price (i.e. price calculated by previous agent) and the demand. The price is given by the equation 3.12.

$$P\,[(t)] = D(t) + P(t-1) \qquad (3.12)$$

Where D(t) = demand

P(t-1) = price calculated by the previous agent

Once all calculations are done for an agent, the next agent comes into picture and he plays the game and calculates price on the basis of the price calculated by the previous agent. After all the agents are done in this way, one round ends. So, at the end of 1 round, we have price series equivalent to the total number of agents. In case of synchronous minority game, we have a single price at the end of 1 round. This price is calculated by taking into account the combined actions of all the agents. The volatility and the returns are calculated in the same manner as that of Mixed Game (Equations 3.9 and 3.10 respectively). The only difference is that here time t does not represent the round but it represents the time an agent takes an action. As and when the agents take an action, the return and volatility are calculated. Hence, time t-1 here represents time at which the precious agent took action.

The volatility is calculated by the equation 3.13.

$$V[t] = \sigma p' * \sigma p' \qquad (3.13)$$

$$\sigma p' = \sqrt{i/t\sum_{i=1}^{t}(p' - \overline{p'})}$$

Where p' = p(t) – p(t-1)

p(t) = price calculated at time t(time when current agent takes action)

p(t-1)=price calculated at time t-1(time when previous agent takes action)

V(t) = Volatility at time t (time when current agent takes action)

The return is calculated by the equation 3.14.

$$R(t) = P(t) – P(t-1) \qquad (3.14)$$

Where p(t) = price calculated at time t(time when current agent takes action)

p(t-1)=price calculated at time t-1(time when previous agent takes action

For Synchronous Mixed Game, the flowchart in section 3.1.6 is run once for each round i.e. after all agents take their action (+1 for buying, -1 for selling) for that round. In case of Asynchronous Mixed Game, the same flow chart of section 3.1.6 is run for each agent i.e. after each agent takes its action. Thus for one round, the flowchart is run N (total number of agents) times. The equations in blocks B3, B4, B5 and B6 are replaced by equations 3.11, 3.12, 3.13 and 3.14 respectively.

Chapter 4: Implementations and Results

4.1 Implementation Overview

In this section, we discuss the implementation and results for various versions of the Minority Game which are as follows:

- Basic Minority Game (discussed in section 2.3)
- Synchronous Mixed Game (discussed in section 3.1)
- Highly Heterogeneous Mixed Game (discussed in section 3.2)
- Asynchronous Mixed Game (discussed in section 3.3)

Also, we have mentioned the parameters that we used for running each of these games. The **first version is the Basic Minority Game'**(Section 2.3) in which there are odd number of agents having same length of long term memories, short term memories and number of strategies selected. All the agents have the same tendency of playing which is rewarding the strategies if by using those; they land up in the minority. **The results of Basic Minority Game are discussed in Section 4.5.**

Next we have modeled the **Synchronous Mixed Game** (Section 3.1) version of the minority game in which there are two groups; majority and minority playing the game. Each of the groups has same length of long term memories, short term memories and strategies selected. The tendency towards the game is different for the majority and minority groups. The agents belonging to the majority game aim at coming in the majority while the agents belonging to the minority game aim at coming in the minority at the end of the game. **The results of Synchronous Mixed Game are discussed in Section 4.6.**

We further observed several disadvantages in the Mixed Game as a system for modeling financial markets. For instance, each group – majority and minority have same lengths of memories. Also, all the agents in the Mixed Game start playing the game and play till the very end which is not the case in the real financial world. In real financial world, we have individuals coming and going and dealing with the financial assets. Also, we have new individuals entering at various instances in the market. Thus we have simulated the **Heterogeneous Mixed Game** (Section 3.2) which has randomly generated memories for all agents and agents dying and coming at regular time intervals. **The results of Highly Heterogeneous Mixed Game are discussed in Section 4.7.**

Finally we delve into a completely different model which is the **Asynchronous Mixed Game** (Section 3.3) model as discussed in section 3.2. In this we have modeled the mixed game but in an asynchronous fashion i.e. the price, returns and volatilities are examined after each of the agent acts. The remaining conditions are same as that of the mixed game model. **The results of Asynchronous Mixed Game are discussed in Section 4.9.**

4.2 Platform and Tools

The entire simulator has been developed using Java. The reasons behind selecting Java for the development are modularity in structure, portability and the fact that it is architecture neutral. Also, it has less overhead as it is interpreted, threaded and dynamic. It has inbuilt class facilitating random number generation which helps our application greatly in generating the initial long term memories and short term memories. Matlab is considered to be one of the best tools for plotting financial charts. As java can be easily interfaced with Matlab, we have plotted our graphs in Matlab. So, the basic code of the various games is written in Java and at the end of the game, the graphs are plotted in Matlab.

4.3 Validation Benchmarks

4.3.1 Benchmarks for stylized facts produced through daily price series

1. The probability distribution of return should demonstrate a fat tail distribution with fat tail index greater than 2[38]. The distribution should demonstrate high kurtosis with value greater than 3[44].The autocorrelation function of absolute returns decays slowly as a function of the time lag [9, 10].
2. The autocorrelations of asset returns should be insignificant (less than 0.01) for lag of more than 30 trading cycles [9].
3. Different measures of volatility display a positive autocorrelation over several days, which quantify the fact that high-volatility events tend to cluster in time [9, 10].

4.3.2 Benchmarks for stylized facts produced through tick-tick (high frequency) data

1. The high frequency data returns should display a fat tail distribution with fat tail index greater than 2 and kurtosis value greater than 3[38][44][47].
2. The autocorrelations of absolute returns for tick-tick data should be negative [47, 42].
3. The volatility clustering for tick-tick data has a “U” shape i.e. it should be high at the start of the and then again it should become high at the end. [11, 35].

4.4 Selection of Parameters

There are two things that are required for an agent based model based on finance to work which is **1) positive correlation among winning rate of the agents and the volatilities and 2) the volatilities should not approach to 0** as a market with 0 volatility is considered to be a dead market[16]. For Mixed Game, to achieve both of these factors, the learning power of the majority agents should be faster than that of the minority agents [17] which mean that the memory lengths, number of strategies selected and the time horizons of the majority agents should be less than that of minority agents. This has been established and proved by Chengling Gou, the author of Mixed Game [16, 17]. We follow the same fundamental established by him throughout our game [16, 17]. The reason behind this is that when the conditions m1<m2, T1<T2 and s1<s1 are not satisfied, then the correlations among majority agents, minority agents and the volatilities become negative and the volatilities become 0. Also, the memory length of minority agents should be less than 6 (m1<m2<6) [16]. Time horizon should typically be in the range of 8 to 100. This range guarantees a reasonable learning rate of agents. It's absolutely fine to choose time horizon outside this range as well, however very high value of it results in extremely slow learning rate for agents [16, 17]. Moreover, maximum number of strategies selected should be 10. The reason behind this is that the average performance of agents tends to degrade significantly [16, 17] if the number of assigned strategies is more than 10 and also if the memory length is greater than 6. The reason behind this is that the game mainly depends on the learning capability of the agents which directly is related to learning to select the right strategy. If the number of strategies or length of memory is too high, then it becomes tough for the agents to get to the topmost strategies within the provided time horizons. Also, the number of majority agents should be less than half of the total agents. This is to make sure that the majority agents don't dominate the game and the minority nature of the game is maintained [16, 17]. If the number of majority agents is more than half of the total agents then, the **"majority game effect"**[17] arises which means that the interaction among majority game agents leads to the volatilities to fall and this leads to a dead market (market with volatility approaching to 0). These fundamentals have been researched and established by Chengling Gou, the founder of Mixed Game.

4.5 Obtaining Stylized Facts with Minority Game Original

The following are the results of the original minority game. It has been established that in its original formulation the statistics of the MG are mostly Gaussian (if large populations of agents are considered), and there are no signs of volatility clustering and other stylized facts in the time-series generated by the most basic MGs [14]. Figures

4.1 to 4.6 show the results of the game in terms of various graphs. When compared with the sample DJIA stylized facts, we observed that these results are not able to produce the authentic stylized facts observed in the real world. This complies with the fact that the original minority game without any change is unable to produce stylized facts. The reason behind this is that the original minority game is a negative sum game [39]. As the total number of agents N in the game is an odd integer, the minority side can always be determined and the number of winners is always less than the number of losers, implying the Minority Game to be a negative sum game [40]. In contrast to this, the real financial market is a zero sum game [19]. These results are obtained on the basis of the original Minority Game played with the below listed parameters. It has been established that beyond size of memory>6 and s>5, the performance of the game starts to deteriorate [7]. So we have chosen the parameters as mentioned in table 4.1.

Parameter	Value
Size of memory	3
No of strategies selected	5
No of agents	201
No of rounds	1000

Table 4.1 Simulation Parameters for Basic Minority Game

Figure 4.1 represents the price series of the game. It is clear that it does not comply with the real world scenario (DJIA price series, figure 2.2). It shows some of the most unrealistic values and fluctuations. It is also not able to produce the remaining stylized facts. For instance figure 4.2 which intends to produce the volatility clustering does not show properties related to it. In case of volatility clustering, the small fluctuations should cluster together and also the large fluctuations should cluster together. In figure 4.2, there is an initial large peak of changes followed by small peaks representing minor changes. This large peak can be attributed to the learning time required by the agents to adapt to the game. Also, there is absence of any further clusters in the diagram. Figure 4.3 represents the return series which again shows some of the unreasonable fluctuations. The returns should follow fat tail distribution with longer peaks and fatter tails. The distribution plotted using the minority game is shown in figure 4.4. We observe that it does not follow fat tail distribution and shows some irreparable traits. Also, the autocorrelation of returns showed in figure 4.5 in not acceptable. Moreover the autocorrelation of absolute returns shown in figure 4.6 should be constantly decaying [8] but it shows a lot of spikes at various places. The reason behind this inability to produce stylized facts can be attributed to the fact that all the agents in the original minority game have the same length of the long term and short term memories. They even follow the same strategy of playing that is rewarding the strategies that lead to the minority result. Thus each of the agents goes through the same learning process with the same parameters. So, there is hardly any heterogeneity except having different strategies.

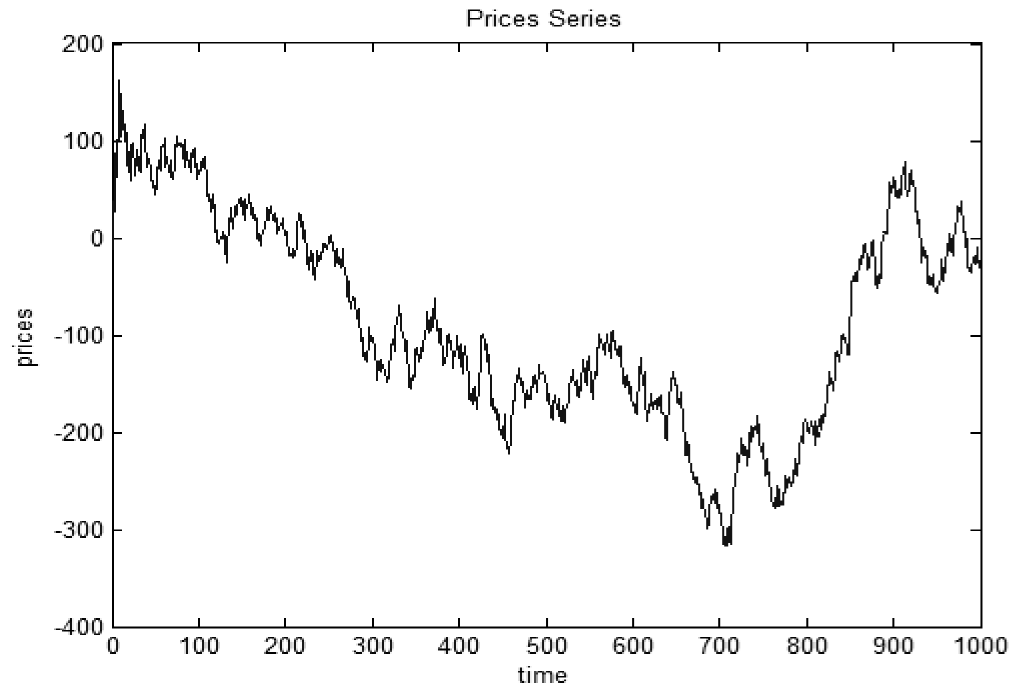


Figure 4.1 Price Series (Price vs. Time)

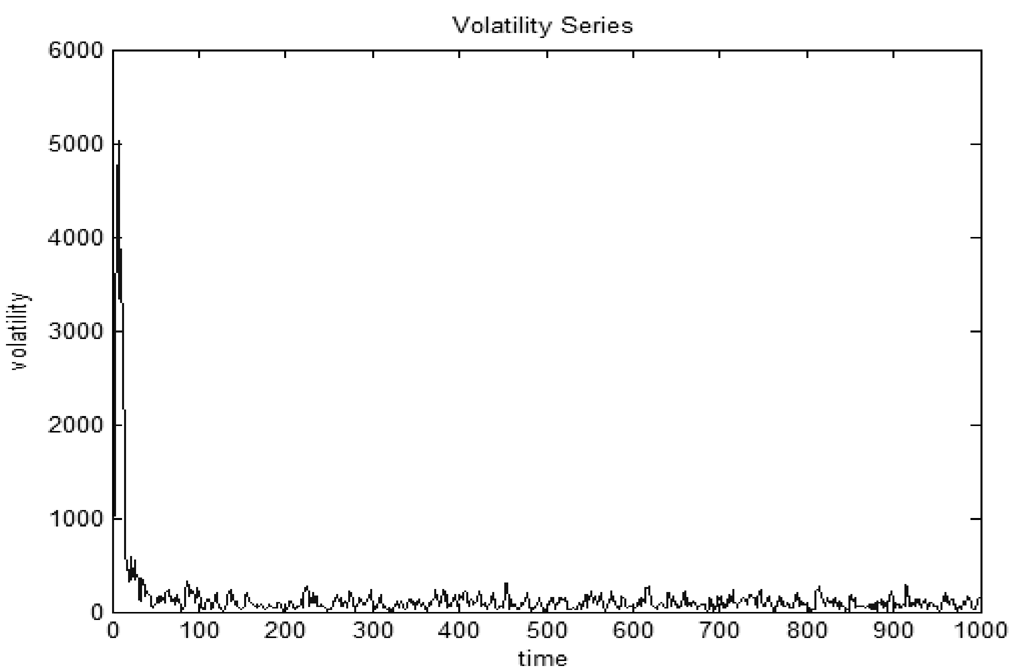


Figure 4.2 Volatility Clustering

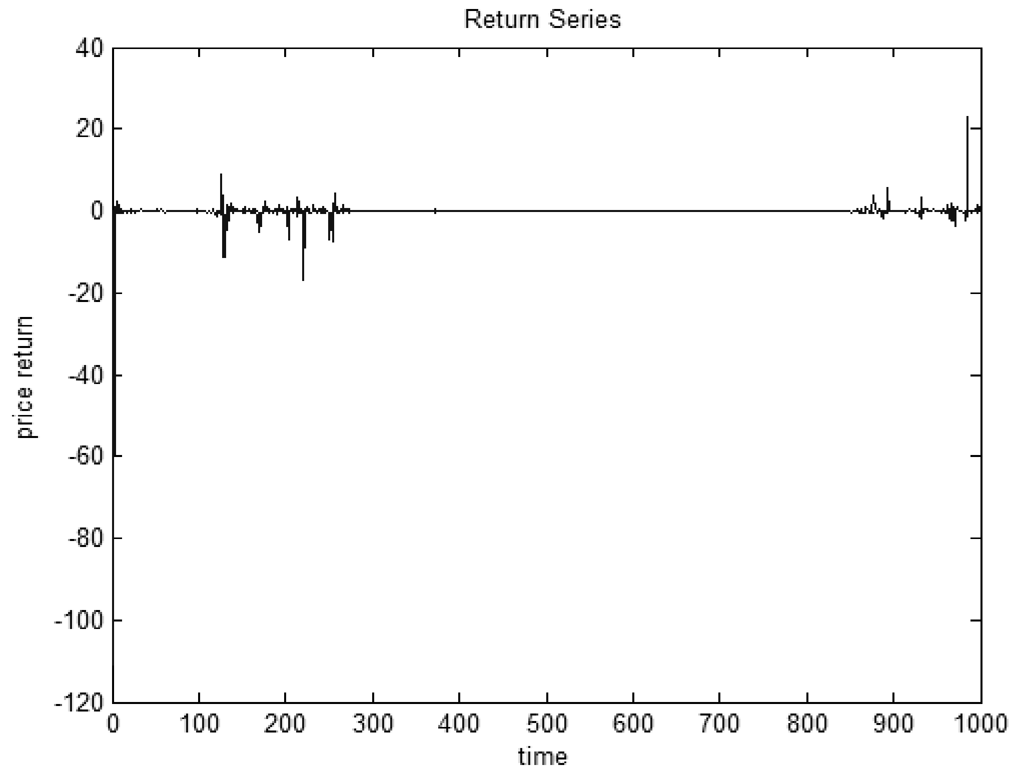


Figure 4.3 Return Series (Return vs. Time)

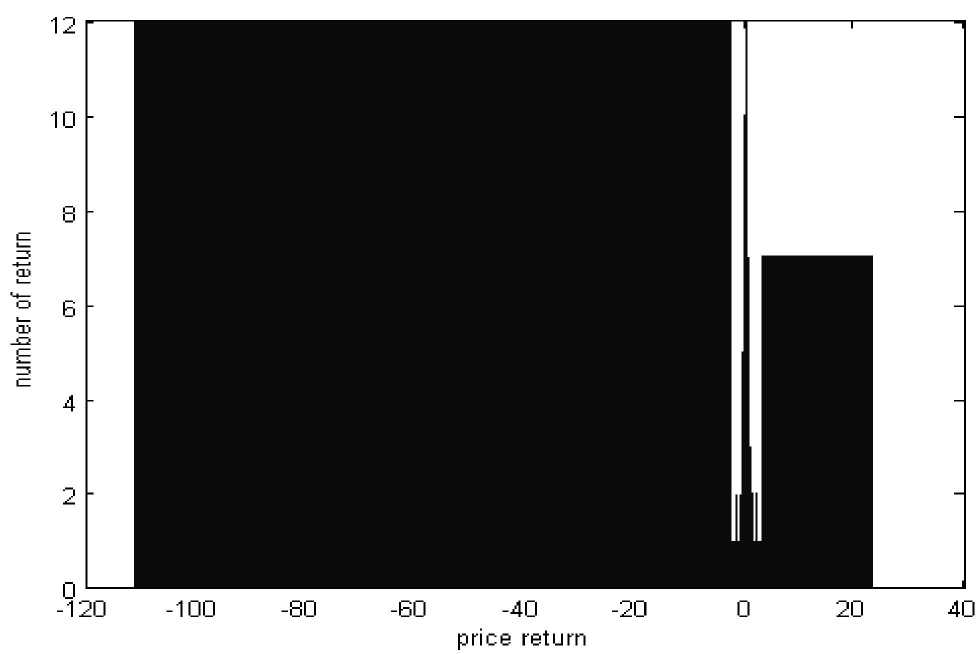


Figure 4.4 Distribution of returns

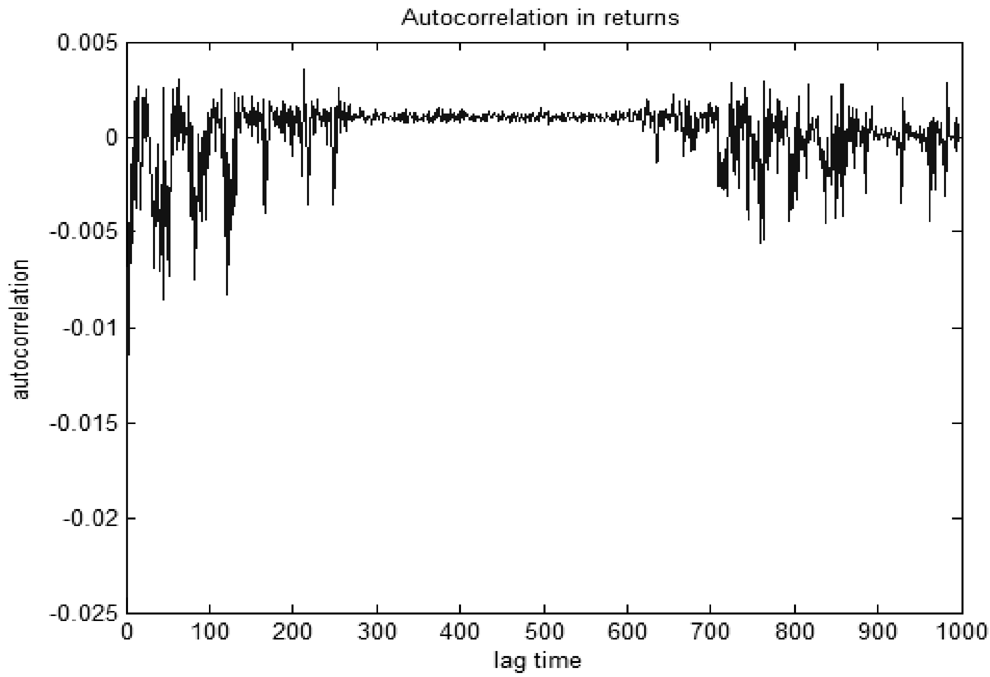


Figure 4.5 Autocorrelation in returns

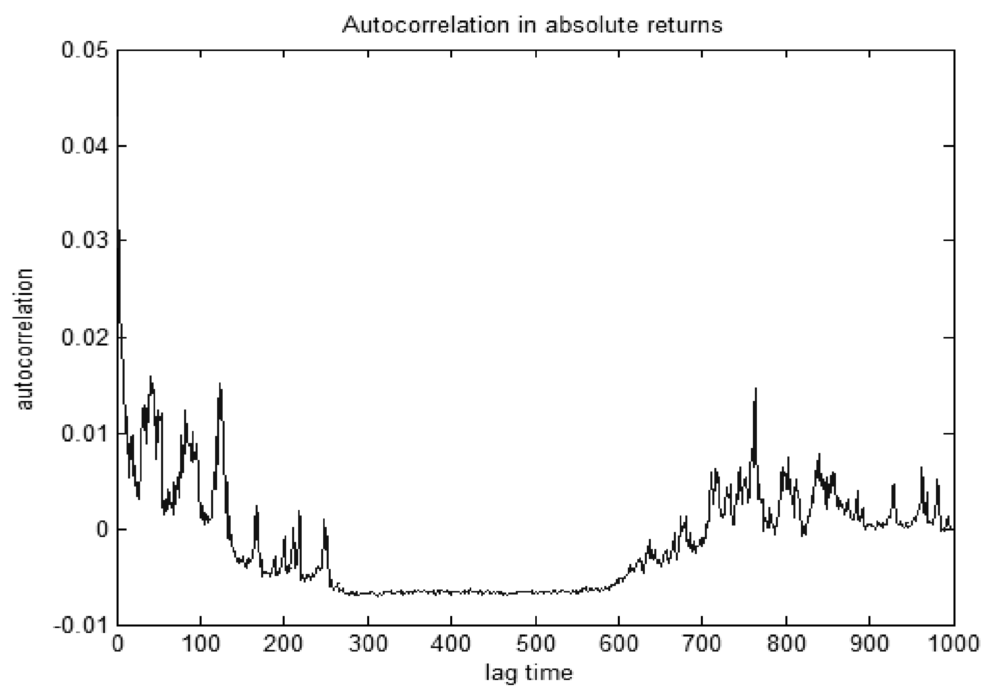


Figure 4.6 Autocorrelation in absolute returns

4.6 Obtaining Stylized Facts with Mixed Game

The following are the results of the Mixed Game. Figures 4.7 to 4.12 show the results of the game in terms of various graphs. All the graphs are plotted against time also know as lag in case of autocorrelation. **Here time represents round of the game when all the agents take their actions** .When compared with the sample DJIA stylized facts, we observed that these results look pretty similar to them and the price series of it is able to produce the authentic stylized facts observed in the real world. These results are obtained on the basis of the game played with the parameters (following the conditions of selection of parameters in 4.4) listed in table 4.2

Parameter	Value
Total no of agents, N	201
No of majority game players	40
No of minority game players	161
Size of memory for majority game players	3
Size of memory for minority game players	6
Time horizon for majority game players	12
Time horizon for majority game players	60
No of strategies selected for majority game players	2
No of strategies selected for minority game	2

players	

Table 4.2 Simulation Parameters for Synchronous Mixed Game

As seen from the simulation results, the mixed game satisfactorily produces all the stylized facts. Figure 4.7 is the chart for the price series for the game. The price series demonstrate good resemblance to real world market data, for instance Dow Jones Industrial average price series that we discussed in chapter 2, figure 2.2 As we discussed in chapter 2, price series tends to exhibit different patterns across different markets and assets. However these price series are conventional in the sense that over the long period the asset value is always rising. Also, markets would demonstrate intermittent non-periodic crashes and recovery from it. As price series is not a stylized fact, the comparison is not in terms of absolute values of asset but in terms of its overall trait. Figure 4.8 shows the volatility clustering for the mixed game. As observed in the figure there is a clustering of similar length spikes. This is just in agreement with the fact that extreme events tend to be followed by extreme events (following Section 4.3.1, Benchmark 3). The reason behind this can be attributed to the higher range of heterogeneity provided by the different groups of agents. Both groups; majority and minority play completely opposite games and vie to be in different states at all times. Also, the fact that the two groups of agents have different time horizon (window over which the strategy scores are collected) adds to the heterogeneity aspect of the game. This leads to various intermittent non-periodic crashes and also recovery from it which is a basic factor in real world market. Also, this reinforces the belief in financial market that volatilities are not independent. Various studies of financial markets have shown that volatility displays significant autocorrelation [10, 21, 34]. Figure 4.9 shows the return series obtained from this game. As observed from the diagram it fluctuates in the beginning and then it stabilizes later. The reason behind the early fluctuations is that the price starts from 0 and also the fact it takes some time to adapt to the new information which arrives in the market. Figure 4.10 shows the distribution of returns. As described in the benchmarks for stylized facts in section 4.3.1, the returns should demonstrate a fat tail demonstration with fat tail index greater than 2 and kurtosis greater than 3. We observe the exact same phenomenon of fat tails and high kurtosis in the distribution of returns followed in the figure 4.10. The fat tail distribution has the fat tail index value of 2.67 and a kurtosis of 11.67 which is in accordance with the benchmark (Section 4.3.1). Figure 4.11 shows the chart for the autocorrelation in returns. It has been established by various papers that the autocorrelation in returns should be negligible or 0[10, 43]. As seen in the figure, the autocorrelation shows some fluctuations in the beginning but then finally converges to 0. Even the autocorrelations shown in the beginning are in between -0.0002 and 0.0035 which is negligible adhering to section 4.3.1, benchmark 2. Figure 4.12 shows that the autocorrelation in absolute returns decays slowly as a function of time lag and thus follows section 4.3.1, benchmark 1. Thus the Mixed Game is able to follow all the benchmarks for establishing that it can represent **daily time series** of financial markets.

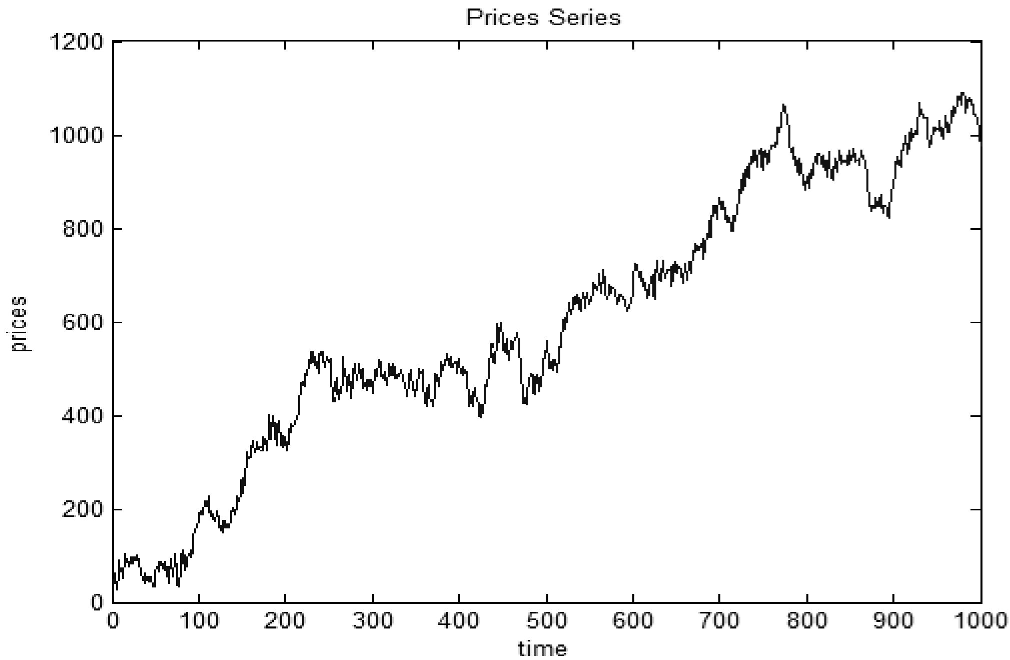


Figure 4.7 Price Series

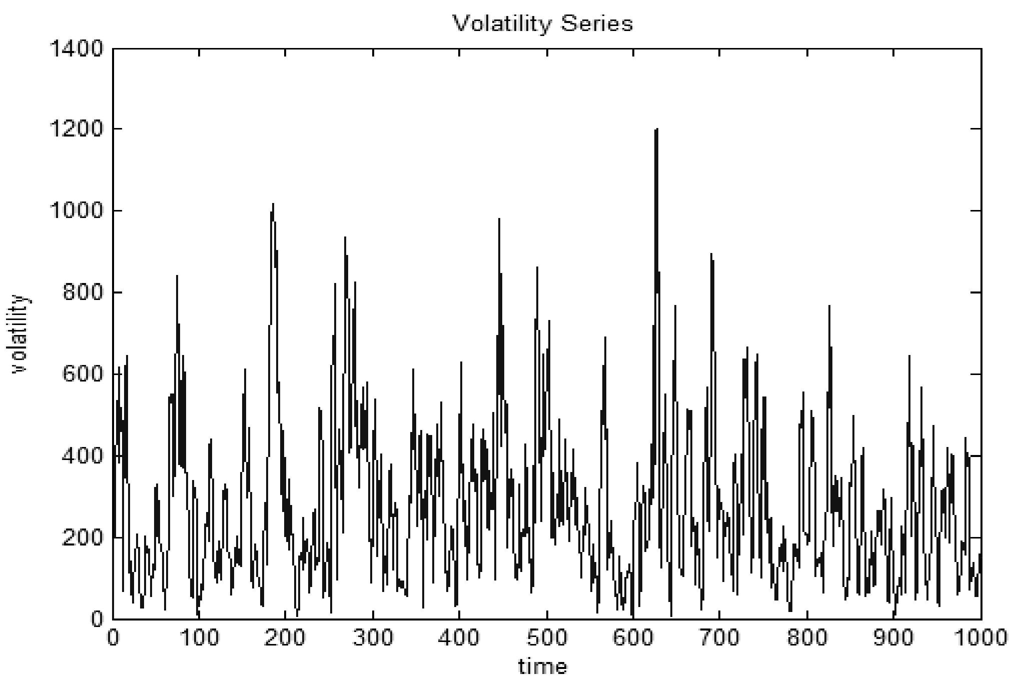


Figure 4.8 Volatility Clustering

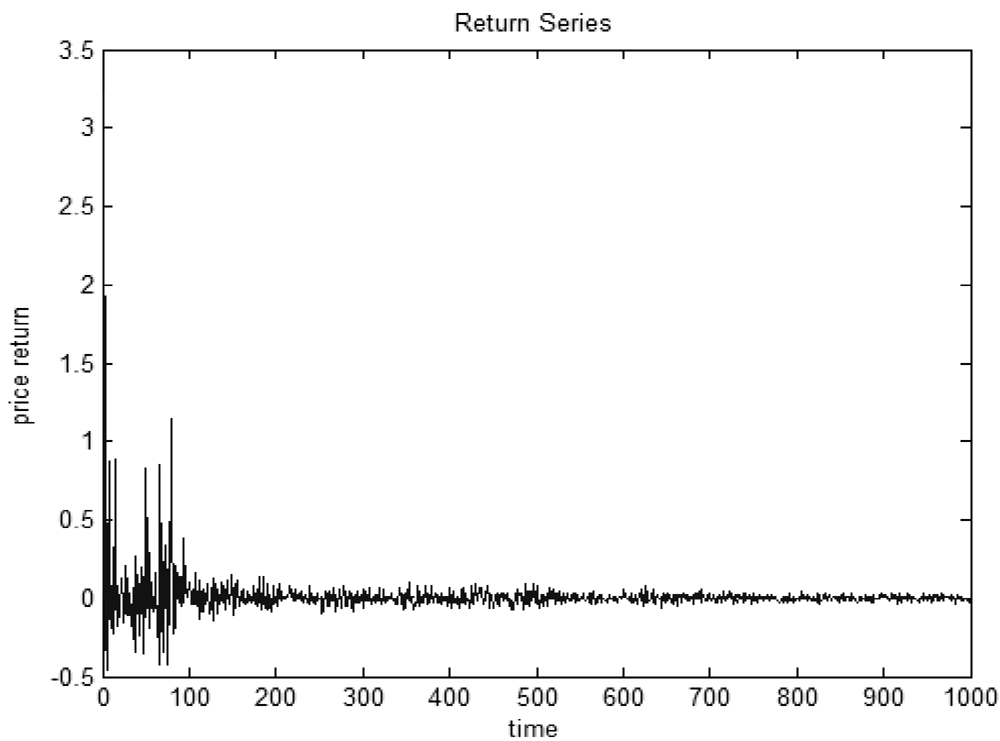


Figure 4.9 Return Series

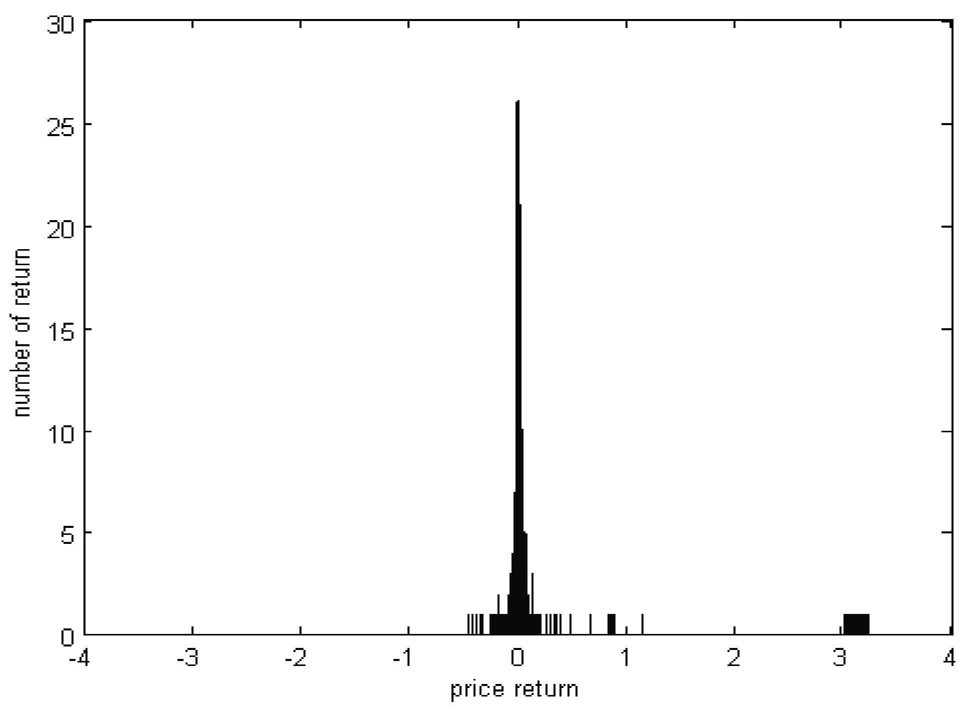


Figure 4.10 Distribution of returns

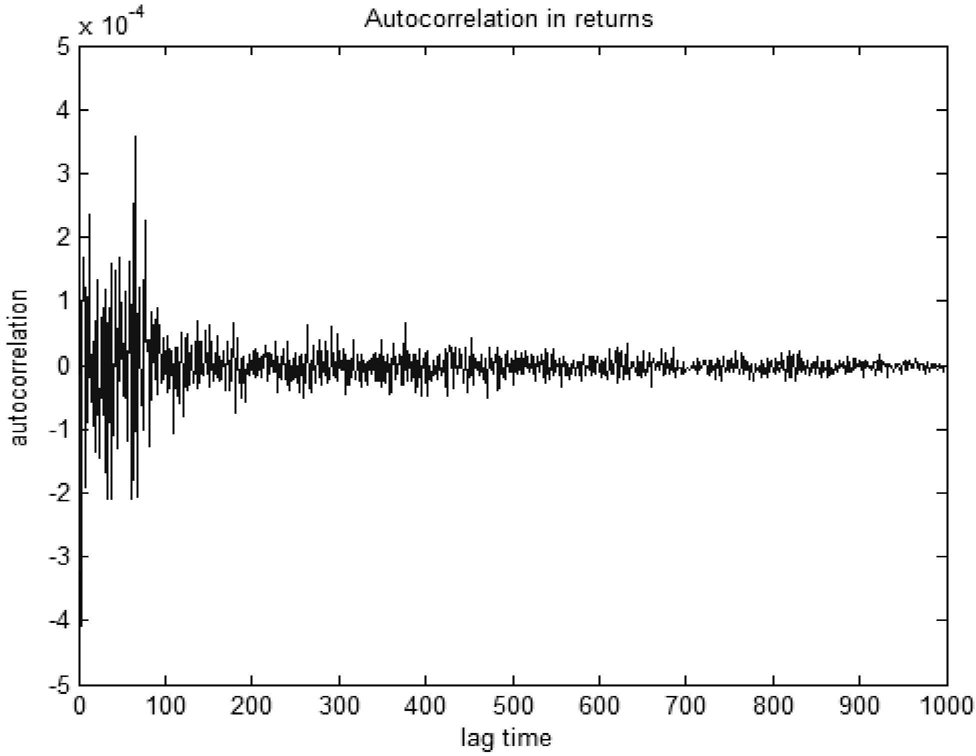


Figure 4.11 Autocorrelation in returns

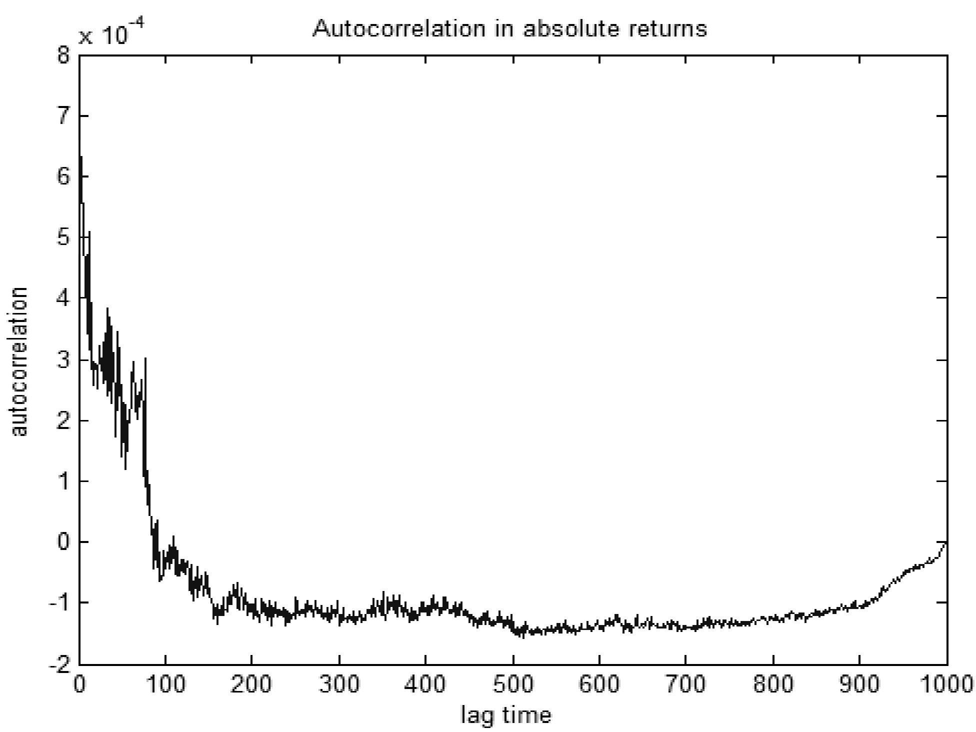


Figure 4.12 Autocorrelation in absolute returns

4.7 Obtaining Stylized Facts with Highly Heterogeneous Mixed Game

As mentioned in section 2.6, the Mixed Game has several limitations as a financial market model. In this section, we focus on removing two of the major limitations (Section 2.5, Limitations 1 and 5) to form a better model to imitate financial markets through our Highly Heterogeneous Mixed Game model. In Mixed Game, the two groups; the majority and minority have fixed length of long term and short term memories. This limits them in terms of heterogeneity and complexity. But in real life scenario, each individual is unique and he has different memories. Each individual according to his ability and thinking capability reacts to different situations in a completely different manner. We are capturing this very basic concept in our game. To remove this limitation, we have randomly provided all agents with different memory lengths at the start of the game. We have followed the property established by the mixed game i.e. the learning ability of the majority game players should be more than the minority game players. So we have provided the majority game players with randomly generated memories between 1 and 3 and the minority game players with randomly generated memories between 3 and 6. Following the same ideology, we have provided majority game agents with randomly generated strategies between 1 and 5 and minority agents with randomly generated strategies between 1 and 10.

Apart from this limitation, we are also focusing on the limitation 5 of the life-death scenario mentioned in section 2.6. This limitation states that in Mixed Game, the agents playing the game are active throughout the game. In actual markets, there are individuals coming and going. We have decided to solve this limitation by causing the death of half of the agents at every second instance of time horizon. We do this by depleting the memories of half of the agents and providing them with new memories and strategies. These new memories and strategies are also generated randomly for all the new agents. This definitely makes our Mixed Game much more realistic and removes the above mentioned limitations. By providing different memories and including the life death scenario, we increase the learning phase of the agents. Every time a new agent is born, he has to apply the new strategies and learn the game all over again. This adds up to the heterogeneity aspect already endowed by providing different lengths of long term memories and short term memories. When compared with the real financial world market, this is like absorbing new information as and when it arrives. We have named our new game as Highly Heterogeneous Mixed Game because removing both of these limitations leads to increase in the heterogeneity aspect of the game.

Figures 4.13 - 4.18 represent the diagrams showing results produced by Highly Heterogeneous Mixed Game. All the graphs are plotted against time also know as lag in case of autocorrelation. **Here time represents round of the game when all the agents take their actions.** The graphs are produced by using the parameters (following the conditions of selection of parameters in 4.4) in table 4.3.

Parameter	**Value**
Total no of agents, N	201
No of majority game players	40

No of minority game players	161
Size of memory for majority game players	Randomly generated (1-3)
Size of memory for minority game players	Randomly generated(1-6)
Time horizon for majority game players	12
Time horizon for majority game players	60
No of strategies selected for majority game players	Randomly generated (2-5)
No of strategies selected for minority game players	Randomly generated (2-10)

Table 4.3 Simulation Parameters for Highly Heterogeneous Mixed Game

Figures 4.13 represent the graph for the price series generated for the Highly Heterogeneous Mixed Game. It shows a normal price series curve which has the property of rising with the time. When compared with the DJIA price series of figure 2.1 and figure 4.8 of Mixed Game, it shows good resemblance. As price series is not a stylized fact, the comparison is not in terms of absolute values of asset but of the series in its rather overall trait entirety. Figure 4.14 represents the volatility clustering diagram for this game. As observed from the diagram the same peak volatilities are clustered together (thus following Section 4.3.1; Benchmark 3). Also, when compared with the Synchronous Mixed Game, it is pretty evident that there are higher peaks of clustering's in the Heterogeneous Mixed Game though both of these games use the same set of parameters. Basically, the level of volatility is higher as compared to the normal Synchronous Mixed Game. The reason can be attributed to the high level of heterogeneity of agents provided by instantiating the agents with different memories and strategies. Also, including the life-death scenario adds to the heterogeneity aspect of the game. This is because each time new agents join the game; they have to learn the game all over again. Wang found a positive relationship between heterogeneity of information and price volatility. Information asymmetry among investors can cause price volatility to increase [37]. This is exactly the situation in our game. The presence of different memories (in terms of length and content) for each of the agents is like the information asymmetry for our game which leads to high volatility. Figure 4.15 shows the return series for the game. When compared with the DJIA returns of figure 2.3 and the returns of Mixed Game they show great resemblance. It shows some fluctuation in the beginning and then becomes stable. The reason behind the early fluctuations is that the price starts from 0 and moreover it takes some time to learn and adapt to the new information which arrives in the market. Also, figure 4.16 representing the distribution of returns evidently shows distribution with fat tails and high peak adhering to 4.3.1, benchmark 1. Moreover the fat tail index for the distribution is 2.77 and has a kurtosis of 12.11 which again satisfies benchmark 1 of section 4.3.1. Figure 4.17 shows the autocorrelation in returns. Various studies have established that the autocorrelation in returns should be negligible or 0[10, 47]. Like autocorrelation of DJIA returns (figure 2.5), it shows some fluctuations in the beginning but then finally congregates to 0 adhering to section 4.3.1, benchmark 2. The few fluctuations in the beginning are attributed to the learning and adapting to the new information that arrives at the start of the game. Figure 4.18 represents the autocorrelation in absolute returns. Figure 4.18 shows that the autocorrelation is decaying gradually but while decaying it shows some spikes. This again is due to the presence of high volatility in the game which is due to distinct memories of all agents and introduction of the life-death scenario. Thus our Highly Heterogeneous game which is a more relevant game model to the real financial market is able to produce all the stylized facts (adhering to the benchmarks in section 4.3.1) and thus can be used as modeling financial markets pertaining to **daily time series**

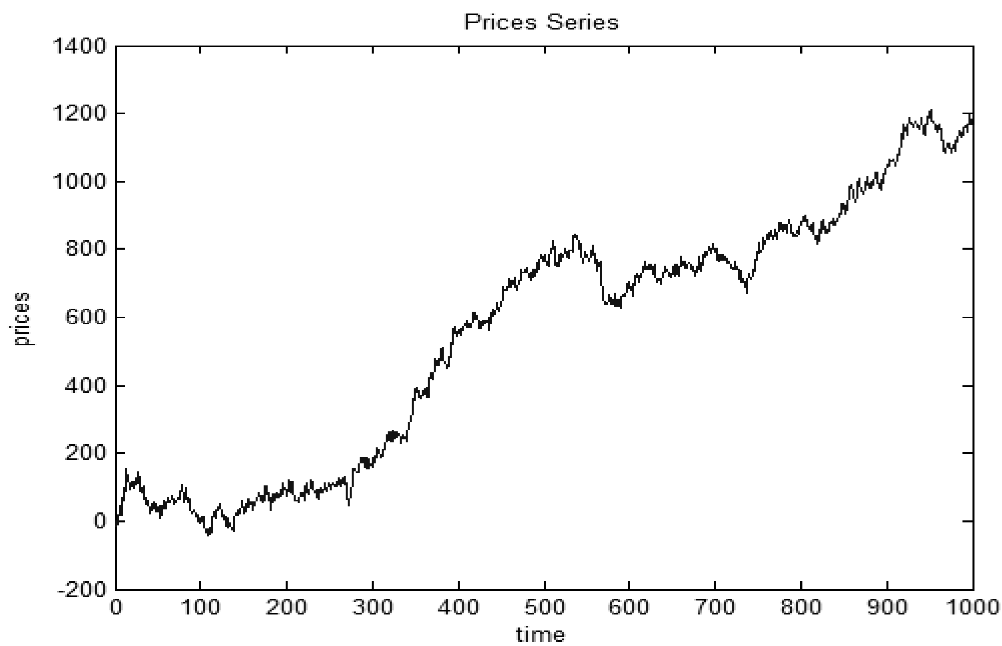


Figure 4.13 Price Returns

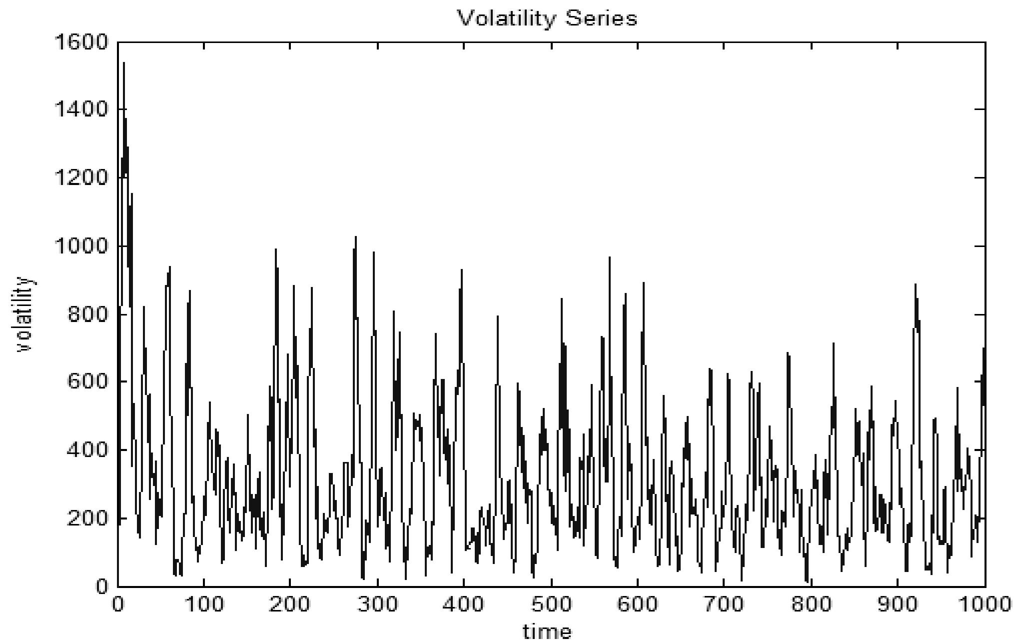


Figure 4.14 Volatility Clustering

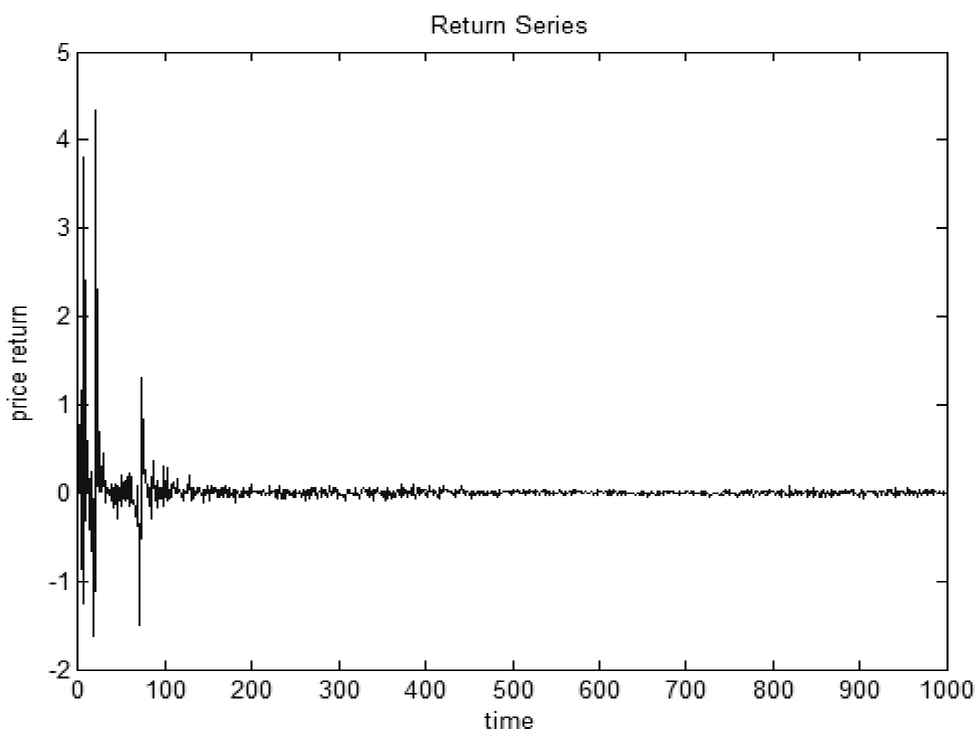


Figure 4.15 Return Series

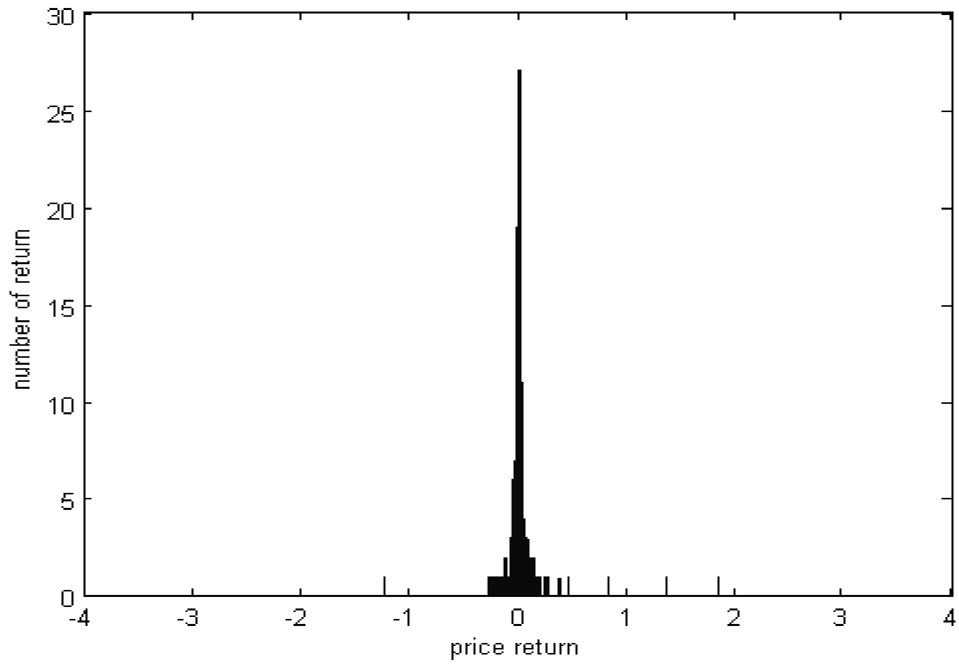


Figure 4.16 Distribution of returns

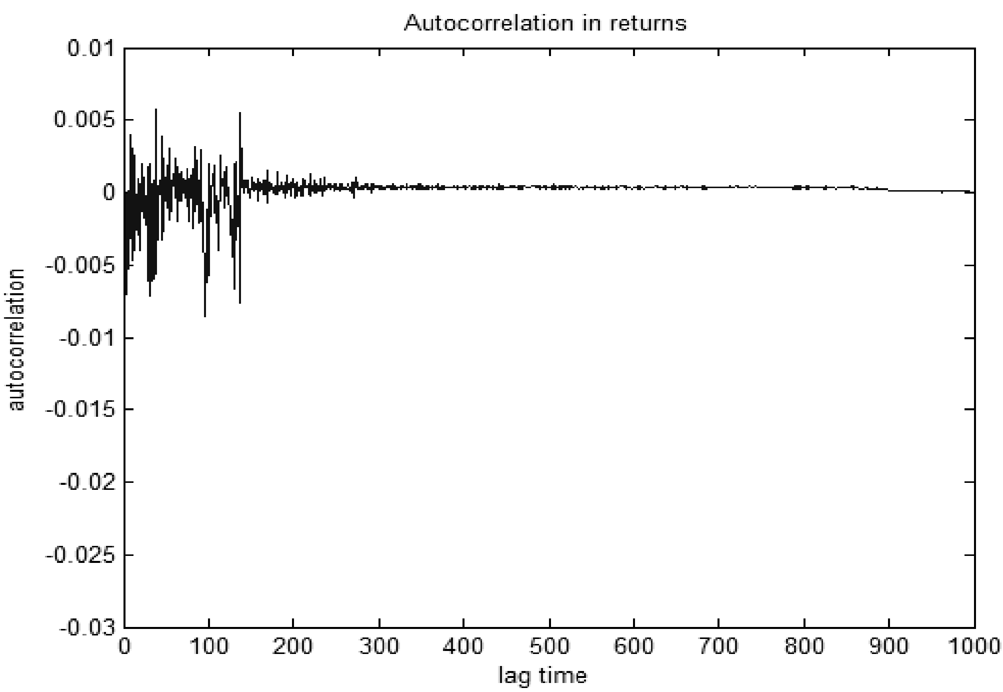


Figure 4.17 Autocorrelation in returns

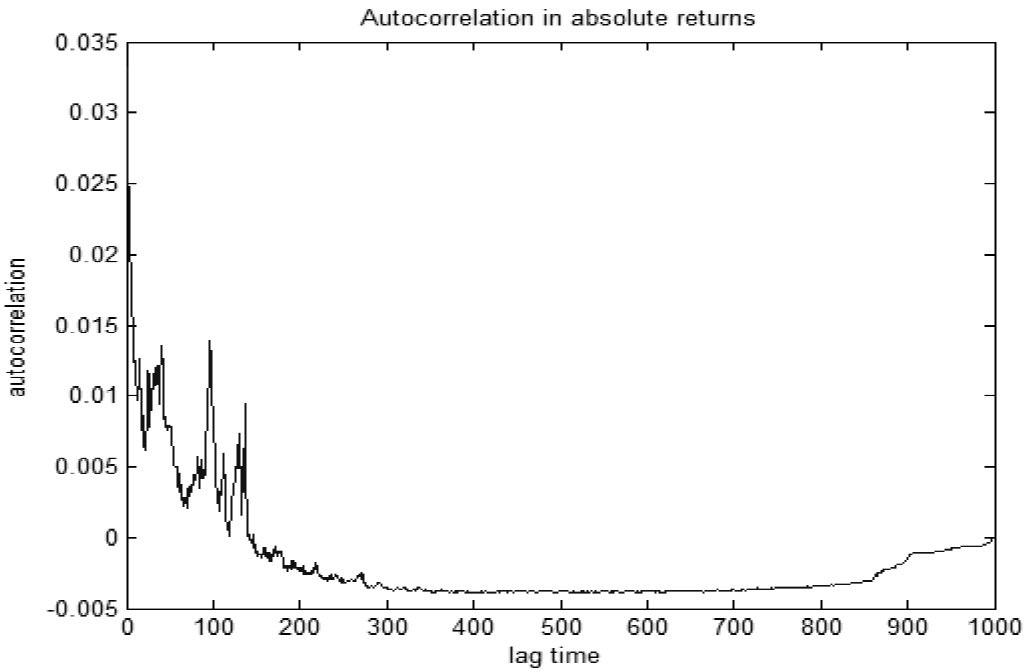


Figure 4.18 Autocorrelation in absolute returns

4.8 Comparison between the Synchronous Mixed Game and the Highly Heterogeneous Mixed Game

The Mixed Game has been created by Chengling Gou and though it has been able to generate the stylized facts satisfactorily, it has several limitations in terms of being considered as an authentic model for financial markets. Each of the groups; majority and minority has agents with same length of memories which limits the diversity in the game which is present in real financial world. Secondly it does not take care of the real life scenario which is the life-death scenario i.e. agents coming in and leaving the financial markets. The Highly Heterogeneous game removes these limitations and thus is considered a better authentic model. Most agent based models including the Mixed Game quantify their results in terms of the efficiency of the agents in the game. The efficiency of the agents; majority group and the minority group is quantified by their average winning score. Each agent has an agent score which is incremented if the agent wins the game (Table 3.2). The majority agents win the game if at the end of the round they appear in the majority and similarly the minority agents win the game if they appear in the minority. Further the scores of the majority groups and minority groups are averaged out and thus we have the Majority Game players score and the Minority Game Players score. We have compared these scores of both of the groups to review the Mixed Game and the improvement of the Mixed Game which is the Highly Heterogeneous Mixed Game.

	Mixed Game	**Highly Heterogeneous Game**	**% Improvement**
Majority Game Players Score (R1)	0.5443	0.5571	2.35%
Minority Game Players Score (R2)	0.4713	0.4923	4.45%

Table 4.4 Comparison between Mixed and Highly Heterogeneous Game

Table 4.4 shows the mean scores of the majority game players and the minority game players for the Mixed Game and the Highly Heterogeneous Mixed Game. As shown in the table, **the mean scores of the Highly Heterogeneous Mixed Game are better than that of the original Mixed Game with an increase of 2.35% for Majority Game players and an improvement of 4.45% for the Minority Game players**. These scores are averaged over 50 runs of the games which run with different acceptable parameters based on the conditions mentioned in section 4.4.

In Agent Based Models based on finance, there are two factors which are of prime importance for the performance of the game (Section 4.4):

1. Positive correlation between average winnings of agents and volatilities
2. Volatilities not converging to 0 (zero volatility indicates dead markets)

We have studied and compared both of these factors in the Mixed Game and the Highly Heterogeneous Game as the % improvement is characterized by these factors [16].

The correlation among two factors means that when the value of one factor increases, the value of the other factor also increases. As mentioned in the selection of parameters (Section 4.4), the correlations among the average winnings of agents and the volatility is positive when all conditions mentioned in the section 4.4 are observed. The more the positive correlation among the average winnings of the agents and the volatilities, the better is the performance for the agents.

Firstly, we observe the correlation between average winnings of agents and volatilities for different conditions of memories m1 (memory length of majority game players) and m2 (memory length of minority game players) i.e. when m1<m2, m1=m2 and m1>m2. All other parameters apart from memory lengths (m1 and m2) are kept constant in this case. As observed from figures 4.19 to 4.24, the correlation among average winnings of agents and volatilities is positive when m1<m2. Also, from figures 4.19 to 4.21, it is observed that the correlation among average winnings of the minority and majority agents is more for Highly Heterogeneous Game than for Mixed Game. This directly affects the winning scores of the majority and minority groups (Table 4.4). Next we observe the correlations among average winnings of agents and volatilities during

various conditions for time horizons (T1<T2, T1=T2, T1>T2). All other parameters are kept constant and they follow the conditions mentioned in the selection of parameters (Section 4.4). It is observed from figures 4.22 to 4.24, that the correlations among average winnings of agents and volatilities is higher when the value of T1 (time horizon for majority game players) is less than or equal to T2 (time horizon for minority game players). Also, figures 4.22 to 4.24 signify the fact that the correlations in case of Highly Heterogeneous Game are higher than that of Mixed Game. Figure 4.25 shows the correlation among the average winnings of the minority and majority agents. As observed from the figure 4.25, the correlation for Heterogeneous game is more than that of mixed game and thus acts as a factor in increasing the performance of the agents and thus the performance of the overall game (Table 4.4). Similarly, figure 4.26 shows the correlation among average winnings of majority game players and volatilities and even in this case, the Highly Heterogeneous Game (Section 3.2) fares better than the Mixed Game (Section 3.1). Finally, we observed the correlations among average winnings of agents and volatilities when the game is run using different combinations of acceptable parameters (following the conditions in section 4.4). From figure 4.27, we can clearly observe that the correlation among average winnings of minority game players and volatilities is better for Highly Heterogeneous Game than Mixed Game. All these observations (figures 4.19 to 4.27) show that the correlations in case of Highly Heterogeneous Game are more than that of Mixed Game. This directly influences the performance of the agents and as seen in table 4.4, the performance of agents for Highly Heterogeneous Game is better than that of Mixed Game players. All of these readings are averaged over 50 runs of the games which run with different acceptable parameters based on the conditions mentioned in section

4.4. **The reason that the correlations being better and basically the overall game being better for Highly Heterogenous game is thatit focusses on two very important aspscts of a financial model which are heterogeneity and also the death of the worst performing agents is caused and the new agents are born in place of them. This helps to improve the correlation among average winnings of agents and volatilities and thus the overall performance of the agents and the game.**

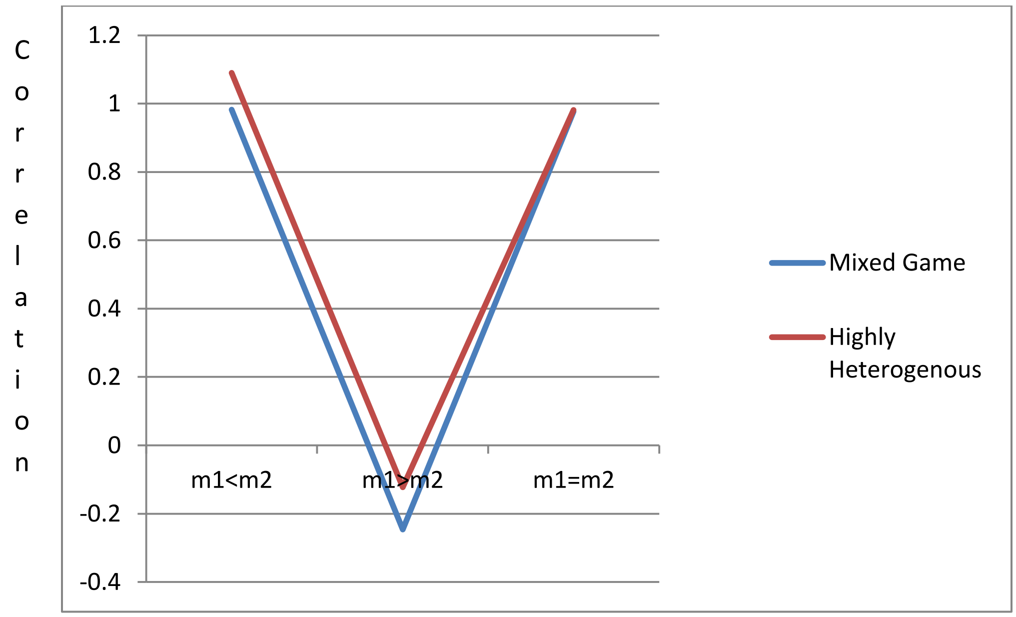


Figure 4.19 Correlation among average winnings of the minority and majority agents

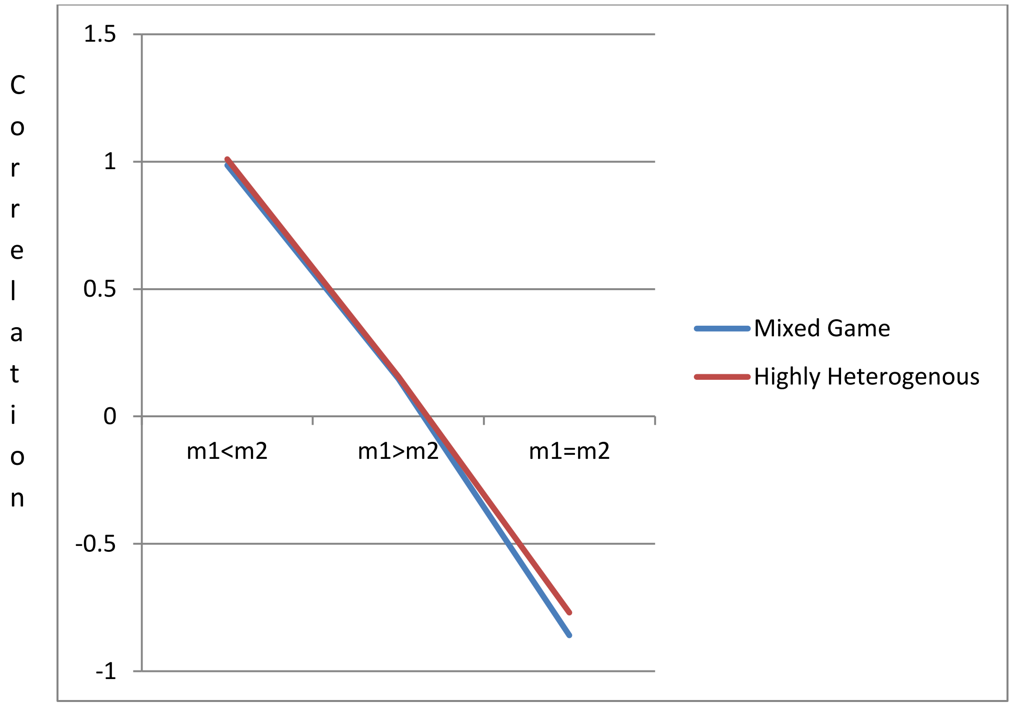


Figure 4.20 Correlation among average winnings of majority game players and volatilities

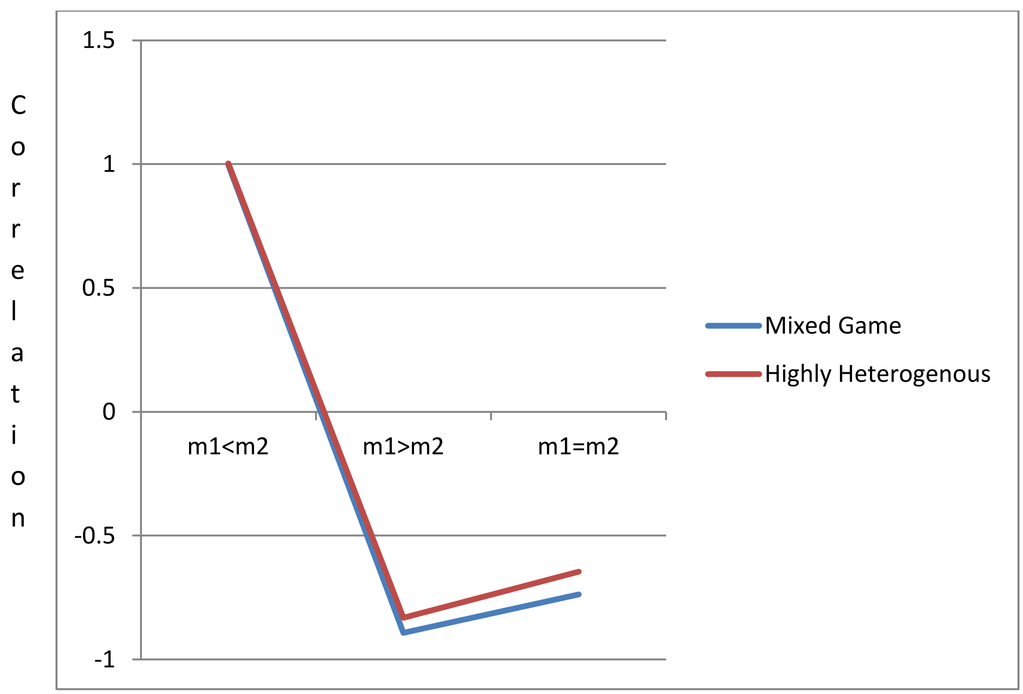


Figure 4.21 Correlation among average winnings of minority game players and volatilities

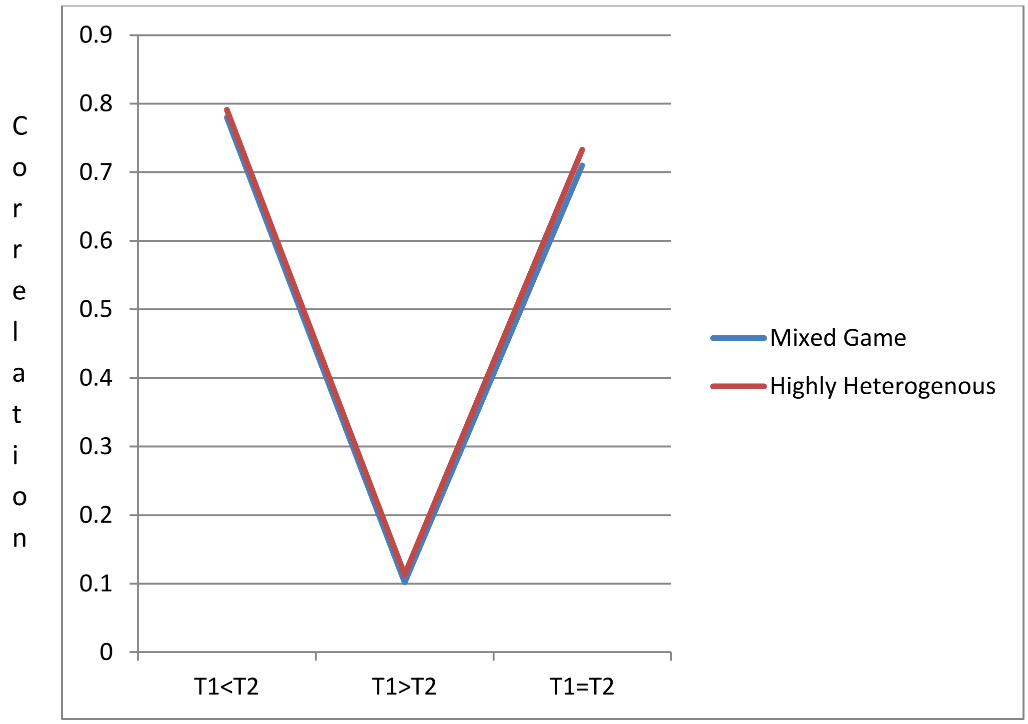


Figure 4.22 Correlation among average winnings of the minority and majority agents

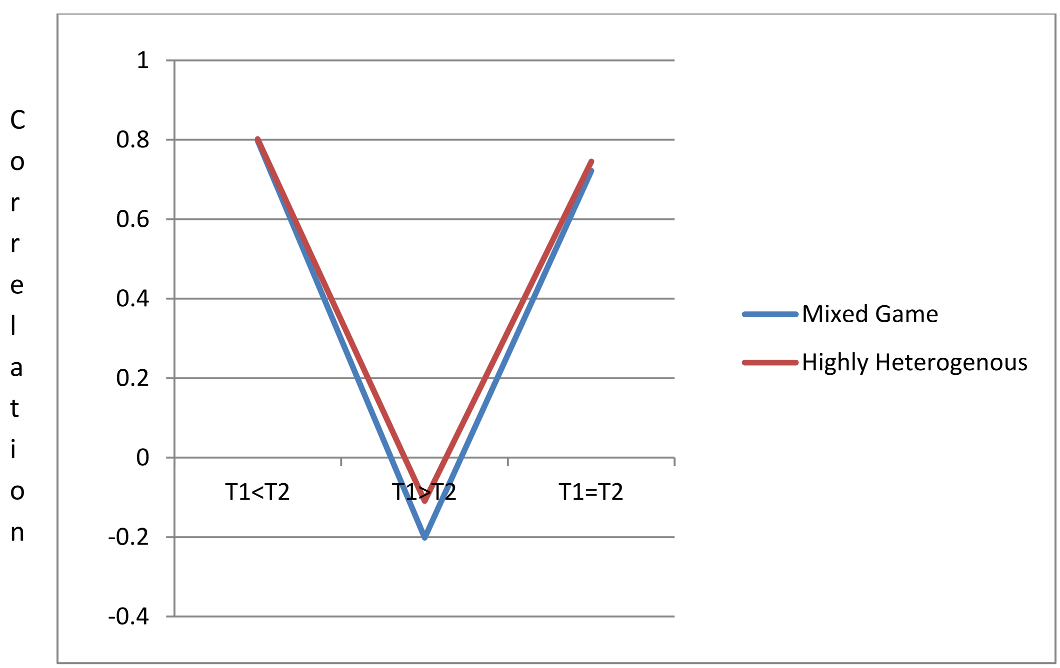


Figure 4.23 Correlation among average winnings of majority game players and volatilities

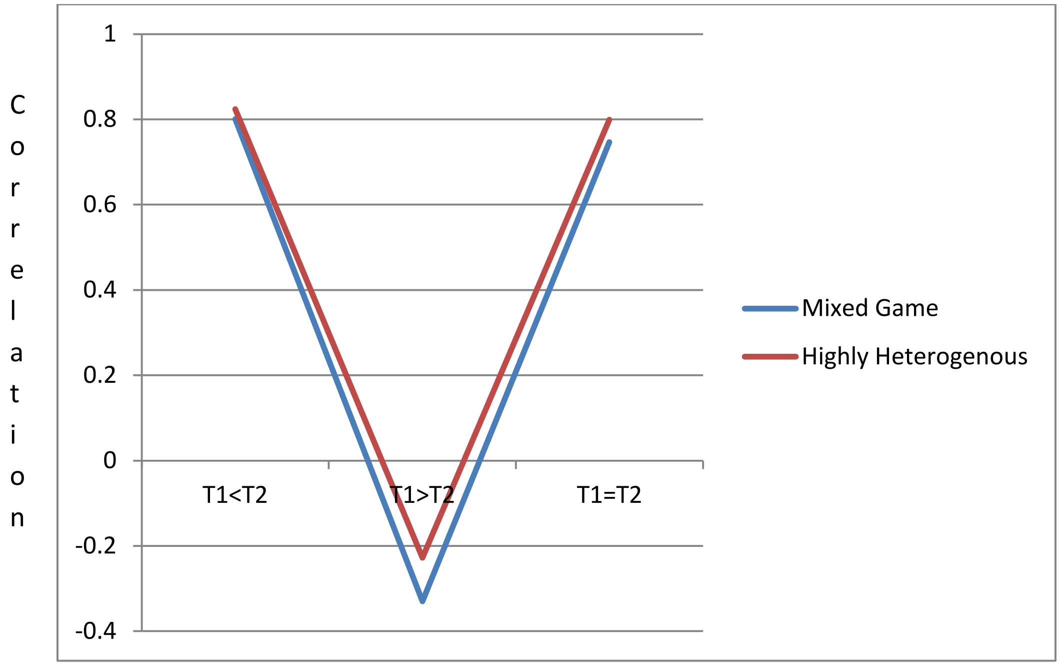


Figure 4.24 Correlation among average winnings of minority game players and volatilities

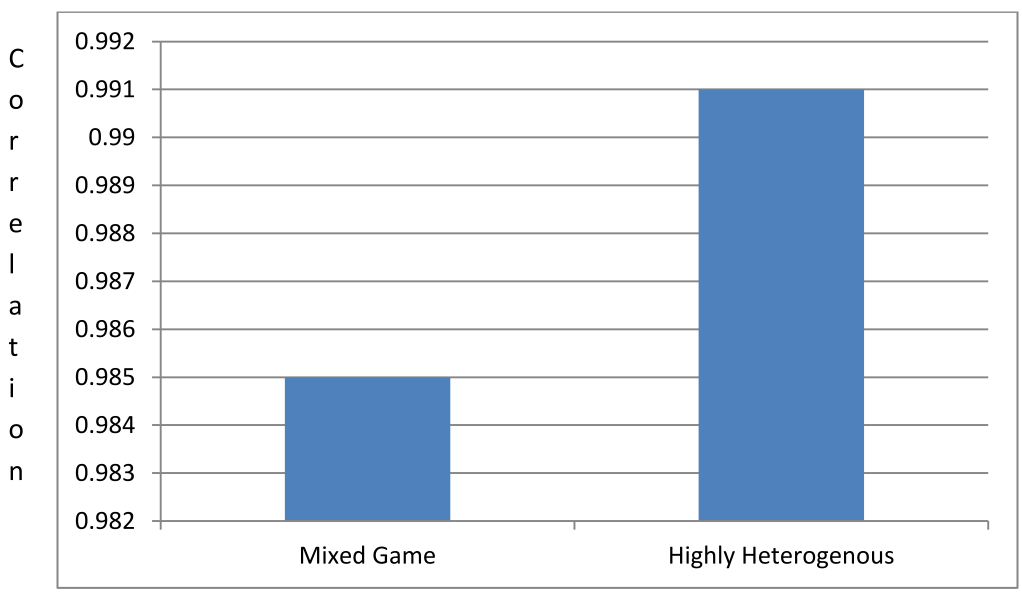


Figure 4.25 Correlation among average winnings of the minority and majority game players

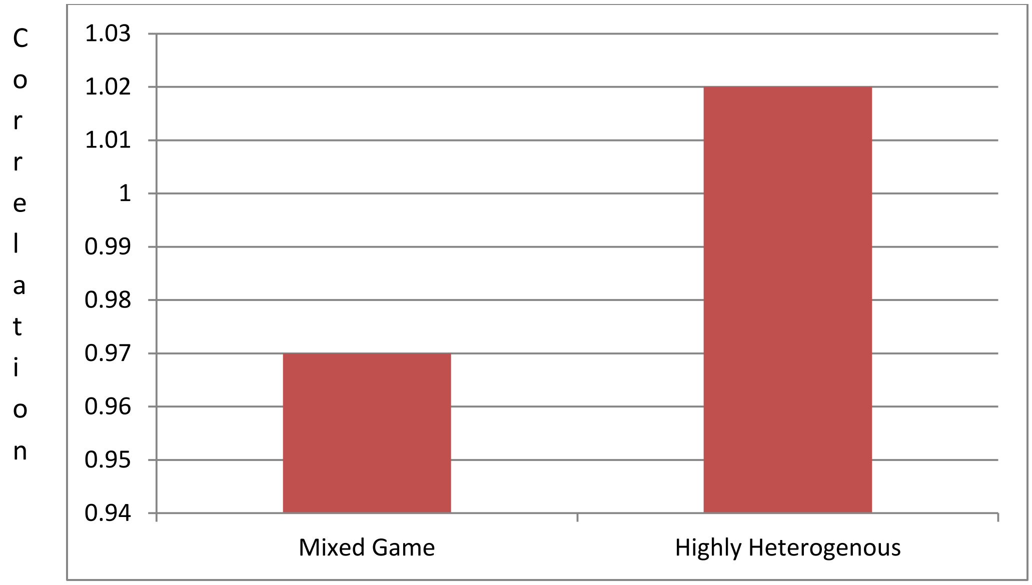


Figure 4.26 Correlation among average winnings of majority game players and volatilities

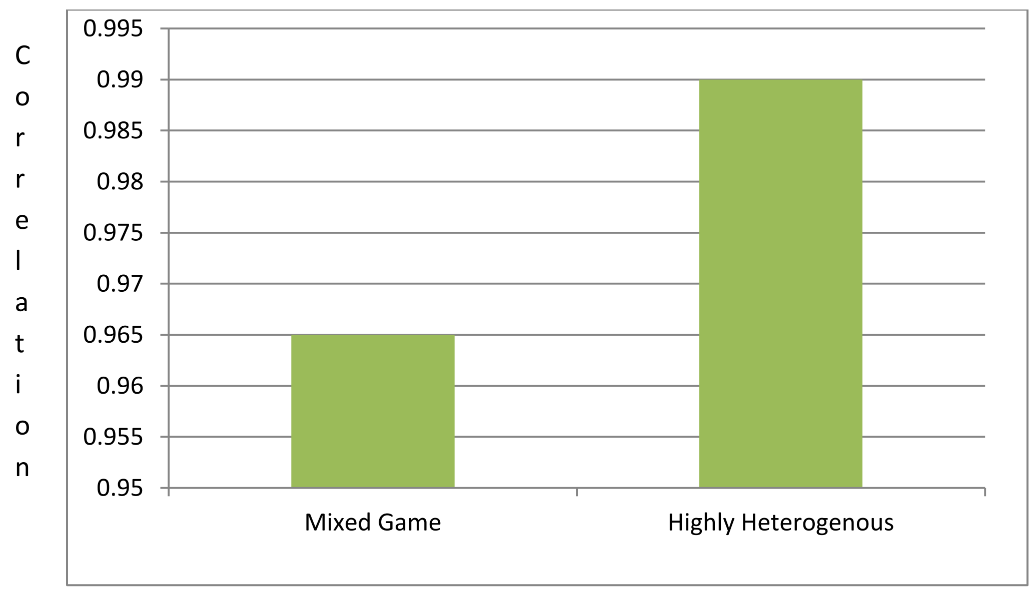


Figure 4.27 Correlation among average winnings of minority game players and volatilities

4.9 Obtaining Stylized Facts with Asynchronous Mixed Game

Here we describe the stylized facts generated on simulating the Asynchronous Mixed game. The parameters used in generating the stylized facts are similar to that of the synchronous mixed game but because these results show the observations of one round, the time horizon factors does not come into play. The reason behind running this game for just one round is that we are aiming at recording just one action of each agent and calculating their effects on prices one by one. We are trying to prove that actions of one round should produce facts of high frequency data. Figures 4.28 to 4.33 show the results obtained by simulating the Asynchronous Mixed Game. All the graphs are plotted against time also know as lag in case of autocorrelation. **Here time represents the time at which each individual agent takes an action**. The results are obtained when the asynchronous game is played using the parameters (following the conditions of selection of parameters in 4.4) listed in table 4.5.

Parameter	**Value**
Total no of agents, N	201
No of majority game players	40
No of minority game players	161
Size of memory for majority game players	3
Size of memory for minority game players	6
No of strategies selected for majority game players	2

No of strategies selected for minority game players	2

Table 4.5 Simulation Parameters for Asynchronous Mixed Game

Figure 4.28 shows the price series generated through the asynchronous mixed game. As mentioned earlier this is no stylized fact. Because this price series is equivalent to just one round we don't observe any major fluctuations in it. If compared with the synchronous game it is equivalent to just the one first price calculated in the first round in that game. So, we don't see any kind of major variation in the price series. Figure 4.29 represents the volatility clustering observed in the game. According to the benchmarks mentioned in section 4.3.2, it should be a "U" curve. The 'U" curve suggests that the trading frequency is higher at the beginning and towards the end of each day. Here the volatility clustering has 2 peaks in the beginning and then it lowers and again at the end of the round it has a high peak. This is not exactly like a "U" but it reinforces the basic idea that the trading frequency is high at the beginning and at the end. Figure 4.30 shows the return series of the game. It is almost similar to that of the synchronous mixed game and (figure 4.9). It shows some fluctuations in the beginning and then converges to 0. The fluctuations in returns are less as compared to the Mixed Game and DJIA returns as this game just represents the affect of each single trade unlike the combined effect of various trades observed in the Synchronous Mixed Game (figure 4.9) or daily DJIA returns (figure 2.3). Figure 4.31 shows the distribution of the returns. Adhering to the asynchronous benchmarks (Section 4.3.2, Benchmark 1), it has high peak and fat tails. Figure 4.32 represents the autocorrelations in returns observed through the asynchronous mixed game. As mentioned in the benchmarks (Section 4.3.2, Benchmark 2), it shows negative correlation and thus again follows the property of high frequency data. Figure 4.33 which represents the autocorrelation in absolute returns also shows negative correlation. This again is in agreement with the property of high frequency data (Section 4.3.2, Benchmark 2) the autocorrelation in returns should be negative. Thus all the results agree with the benchmarks of high frequency data and thus we establish that Asynchronous Mixed Game model can be used as a model for representing **high frequency data** for real financial markets.

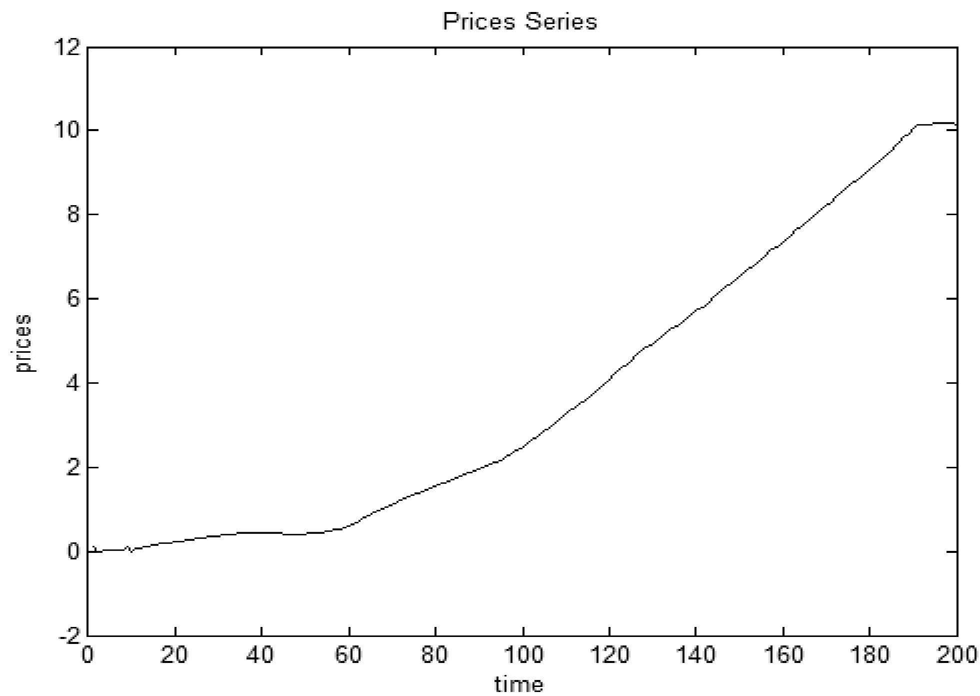


Figure 4.28 Price Series

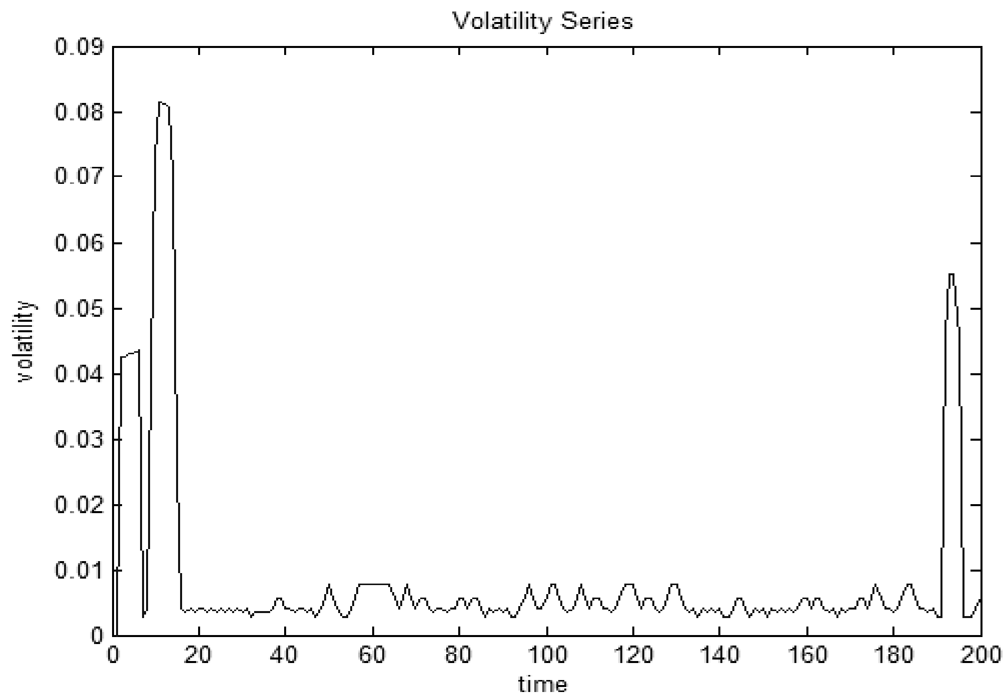


Figure 4.29 Volatility Clustering

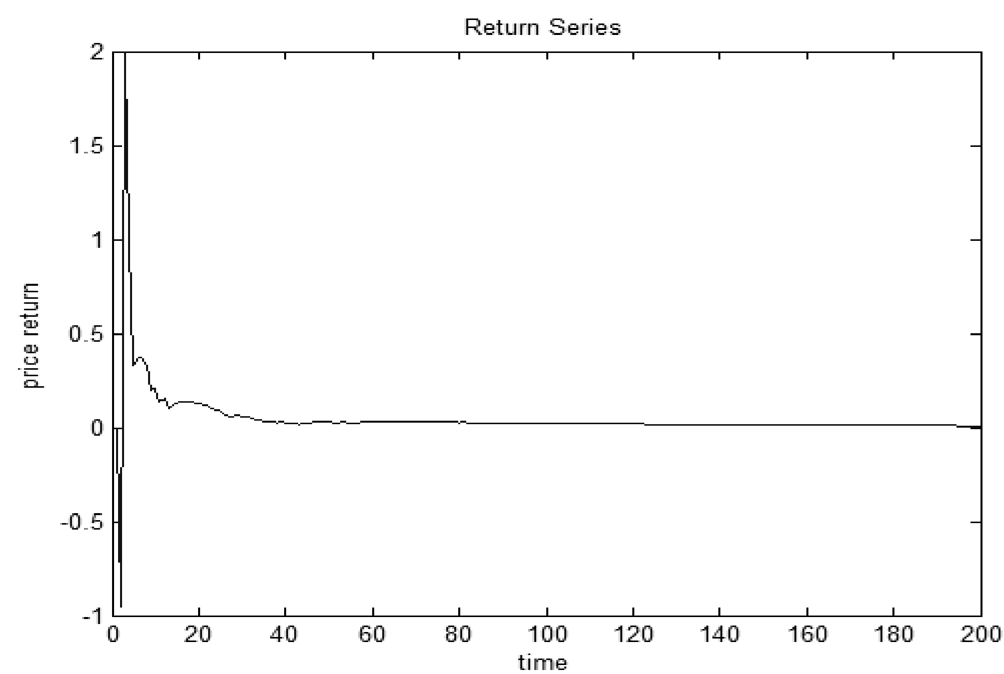


Figure 4.30 Return Series

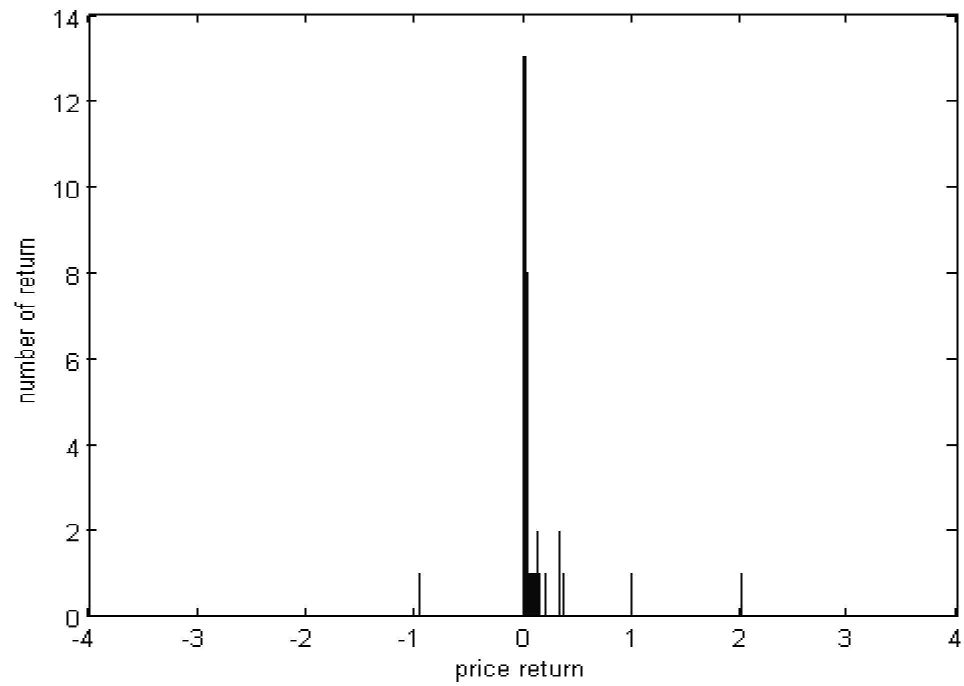


Figure 4.31 Distribution of Returns

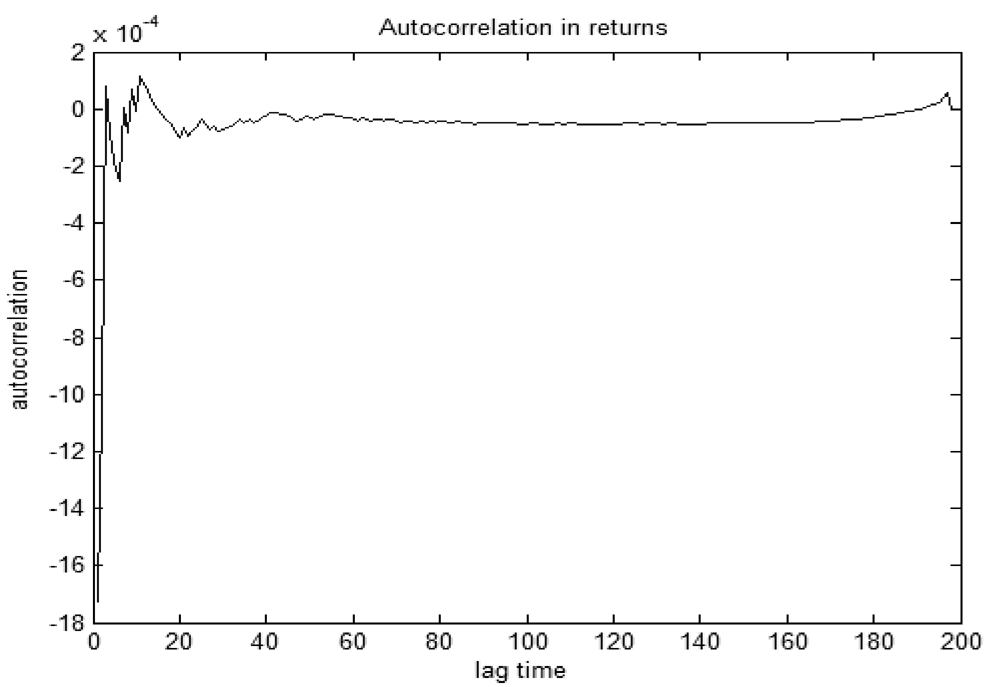


Figure 4.32 Autocorrelation in returns

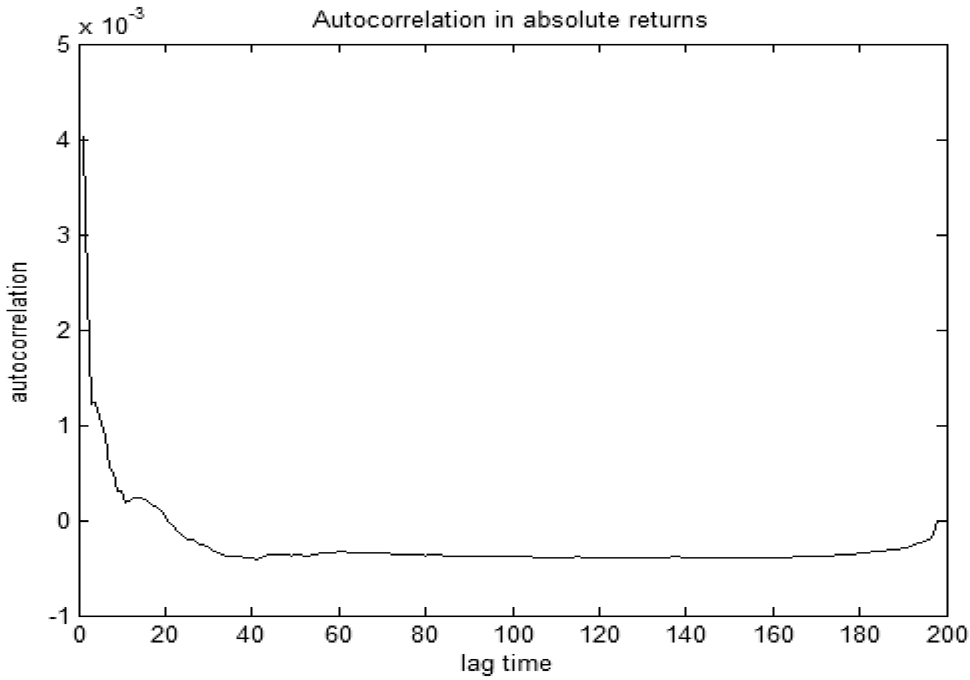


Figure 4.33 Autocorrelation in absolute returns

Chapter 5 Conclusion

5.1 Conclusion and Future Work

The core objective of this thesis was to remove limitations in minority game as a financial model and produce stylized facts under those conditions. By choosing Mixed Game as the system for modeling financial market, we firstly removed the limitation that in financial market one can gain profit by being in the minority as well as the majority. The Mixed Game model consists of both the players; noise traders who gain profit by being in the majority and fundamentalists who gain profit by being in the minority. Further we developed our own model which is the Highly Heterogeneous game model which removed the limitation of diversity in agents by providing each of the agents with different lengths as well as content for long term and short term memories. By introducing the life death scenario for agents, we removed the limitation of each agent playing the game from start until the end. We have been able to successfully produce stylized facts like autocorrelation in returns, volatility clustering and decay in autocorrelations to validate our models and relate them to the daily price series of financial market.

We have also studied the less researched topic of Asynchronous Markets in context with the Mixed Game. We have obtained stylized facts for the Asynchronous Mixed Game also and we have related them to high frequency financial markets. According to the best of our knowledge, there has been hardly any work done on any kind of asynchronous models with respect to minority game and according to our awareness, this is a first of a kind. Table 5.1 briefly describes the game models used by us and the objectives and results we achieved through them.

Objective	**Status**	**Game Model**	**Financial Model**
The game model should consist of both minority and majority game agents.	Achieved	Mixed Game Model	Can represent **daily time series** of financial market.
Each agent should have distinct memories and agents should enter and leave markets.	Achieved	Highly Heterogeneous Mixed Game Model	Can represent **daily time series** of financial market.
The trading should be done asynchronously.	Achieved	Asynchronous Mixed Game Model	Can represent **high frequency** time series of financial market.

Table 5.1 Conclusion

Thus we can conclude that the Synchronous Mixed Game and Highly Heterogeneous Mixed Game produce stylized facts equivalent to the daily time series of real financial world and the Asynchronous Mixed Game Model is able to produce the stylized facts equivalent to high frequency time series of real financial world.

Further studies in this area could be:

- Exploring the asynchronous aspect of the game much more and trying it using different models.

- Constructing a model of Mixed Game with all limitations removed which can exactly be related to financial world like introducing the hold option and all the limitations listed in section 2.6.
- Providing random assets to each agent and then playing accordingly.

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