

**OBSTACLES, SLOPES AND TIC-TAC-TOE: AN
EXCURSION IN DISCRETE GEOMETRY AND
COMBINATORIAL GAME THEORY**

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A dissertation submitted to the
Graduate School—New Brunswick
Rutgers, The State University of New Jersey
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy
Graduate Program in Mathematics

Written under the direction of
János Pach and Mario Szegedy
and approved by

New Brunswick, New Jersey

October, 2011

ABSTRACT OF THE DISSERTATION

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The minimum number of slopes used in a straight-line drawing of G is called the slope number of G . We show that every cubic graph can be drawn in the plane with straight-line edges using only the four basic slopes $\{0, \pi/4, \pi/2, -\pi/4\}$. We also prove that four slopes have this property if and only if we can draw K_4 with them.

Given a graph G , an *obstacle representation* of G is a set of points in the plane representing the vertices of G , together with a set of obstacles (connected polygons) such that two vertices of G are joined by an edge if and only if the corresponding points can be connected by a segment which avoids all obstacles. The *obstacle number* of G is the minimum number of obstacles in an obstacle representation of G . We show that there are graphs on n vertices with obstacle number $\Omega(n/\log n)$.

We show that there is an $m = 2n + o(n)$, such that, in the Maker-Breaker game played on \mathbb{Z}^d where Maker needs to put at least m of his marks consecutively in one of n given winning directions, Breaker can force a draw using a pairing strategy. This improves the result of Kruczek and Sundberg who showed that such a pairing strategy exists if $m \geq 3n$. A simple argument shows that m has to be at least $2n + 1$ if Breaker is only allowed to use a pairing strategy, thus the main term of our bound is optimal.

Acknowledgements

I would like to thank my parents for always being a pillar of strength for me and supporting me through everything in life and being patient with me despite all my pitfalls.

I would also like to thank my brother for guiding me through my most vital decisions in life and for always supporting me.

I would like to thank Radoš Radoičić for initiating me into discrete geometry, Mario Szegedy for his valuable guidance as an advisor, and, Jeff Kahn, József Beck, Michel Saks, William Steiger, Doron Zeilberger, Van Vu for teaching me combinatorics and discrete geometry.

I would also like to thank János Pach for his guidance as a mentor, for encouraging me into mathematical research with new problems, for trying to teach me to write papers, and above all, for bringing me to Switzerland where I met my husband.

I would also like to thank all my friends at Rutgers, CUNY and EPFL for making my stay at all places most enjoyable.

I would like to thank Linda Asaro at the International Student Center, Rutgers, for being the most helpful advisor and for her punctuality with every request.

I would also like to thank Pat Barr, Maureen Clausen, Lynn Braun, Demetria Carpenter at Administration, Mathematics Department, for always being helpful.

Lastly, I would like to thank my husband, without whom this PhD would not have been possible and meeting whom would not have been possible without the PhD. His support, guidance and encouragement have shaped not just this PhD, but more aspects of me than I can describe.

Various sections of the dissertation have appeared in independent published articles. The interested reader is referred to [53, 54, 55, 56, 57].

Dedication

This thesis is dedicated to my mother, who taught me that above all, it is important to be a good human being. This thesis is also dedicated to my father, who taught me to love math.

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Chapter 1

Introduction

The field of Graph Theory is said to have first come to light with Euler's Königsberg bridge problem in 1736. Since then, it has seen much development and also boasts of being a subfield of combinatorics that sees intense application. In the beginning, graph theory solely comprised of treating a graph as an abstract combinatorial object. It could even suffice to call it a set system with some predefined properties. This outlook was in itself sufficient to devise and capture some remarkably elegant problems (e.g. traveling salesman, vertex and edge coloring, extremal graph theory). Besides the large number of areas it can be applied to (Computer Science, Operations Research, Game Theory, Decision Theory), some independent and naturally intriguing problems of combinatorial nature were studied in graph theory. Around the same time however, a new field in graph theory arose. It can be said that, because of the simplicity of representing so many things as graphs, the natural question of the simplicity of representing graphs themselves, on paper or otherwise, came up. Thus started the yet nascent field of graph drawing, where now the concern was mostly of representing a graph in the plane and in particular, how simply can it be represented.

This idea in itself had many far reaching applications. Among the first of it was the four color theorem, where the simplicity of drawing maps was the concern. Since then, many more questions have arisen. An important one of these was drawing graphs in the plane without crossings, or in other words, estimating the crossing number of graphs. Although initially the idea of edges was restricted to being topological curves, before long, a natural further restriction was introduced. To discretize the problem further and to add to the aspect of naturally representing graphs, the branch of straight-line drawings of graphs started. With straight-line drawings, besides the old questions like

crossing number etc., some new, purely geometrical concerns arose. For example, one way of simplifying a drawing of a graph could be to try to reduce the number of slopes used in the drawing. This led to the general notion (introduced by Wade and Chu) of the slope number, which for a graph is the minimum number of slopes required to draw it.

The slope number of graphs is at least half their maximum degree. This led to the intuitive belief that bounded degree graphs might allow for small slope numbers. This was shown to the contrary, with a counting argument, that even graphs with maximum degree at most five need not have a bounded slope number. Graphs with maximum degree two can trivially be shown to require at most three slopes. This restricts our attention to graphs with maximum degree three and four. Maximum degree four, still, to the best of our knowledge, remains an exciting open problem. For maximum degree three although, using some previous results, we can provide an exact answer. We show that four slopes, even the four fixed slopes of North, East, Northeast, Northwest are sufficient to draw all graphs with maximum degree three. Since K_4 requires at least four slopes in the plane, this indeed is an exact answer.

Another interesting notion about straight-line graphs, that arises in many natural contexts is that of representing it as a visibility graph. Given a set of points and a set of polygons (obstacles) in the plane, a visibility graph's edges comprise of exactly all mutually visible vertex pairs. Visibility graphs have numerous applications in Computer Science (Vision, Graphics, Robot motion planning). A natural question that arises from considering the simplicity of such a representation is to find the smallest number of obstacles one has to use in the plane to represent a graph. This is defined as the obstacle number of the graph. It was shown that there are graphs on n vertices that require $\Omega(\sqrt{\log n})$ obstacles. We improve this to show that there are graphs which require $\Omega(\frac{n}{\log n})$ obstacles, which can be further improved for nicer obstacles. In particular, we show that there are graphs which require $\Omega(\frac{n^2}{\log n})$ segment obstacles.

The final chapter of the thesis deals with positional games. Many combinatorial games (Tic-Tac-Toe, hex, Shannon switching game) can be thought of as played on a

hypergraph in which a point is claimed by one of the two players at every turn. Winning in such a game is characterized by the capture of a “winning set” by a player. All the winning sets form the edges in our hypergraph. Such games are called positional games. If the second player wins if there is a draw, then the game is called a *Maker-Breaker* game, and the players are called respectively, *Maker* and *Breaker*. We may also note that if the Breaker can find a pairing of the vertices such that every winning set contains a pair, then he can achieve a draw, called a *pairing strategy draw*.

The classical Tic-Tac-Toe game can be generalized to the hypergraph \mathbb{Z}^d with winning sets as consecutive m points in n given directions. For example, in the Five-in-a-Row game $d = 2$, $m = 5$ and $n = 4$, the winning directions are the vertical, the horizontal and the two diagonals with slope 1 and -1 . It was shown by Hales and Jewett, that for the four above given directions of the two dimensional grid and $m = 9$ the second player can achieve a pairing strategy draw. In the general version, it was shown by Kruczek and Sundberg that the second player has a pairing strategy if $m \geq 3n$ for any d . They conjectured that there is always a pairing strategy for $m \geq 2n + 1$, generalizing the result of Hales and Jewett. We show that their conjecture is asymptotically true, i.e. for $m = 2n + o(n)$. In fact we prove the stronger result where $m - 1 = p \geq 2n + 1$, p a prime. This is indeed stronger because there is a prime between n and $n + o(n)$.

Chapter 2

Slope number

2.1 Introduction

A *straight-line drawing* of a graph, G , in the plane is obtained if the vertices of G are represented by distinct points in the plane and every edge is represented by a straight-line segment connecting the corresponding pair of vertices and not passing through any other vertex of G . If it leads to no confusion, in notation and terminology we make no distinction between a vertex and the corresponding point, and between an edge and the corresponding segment. The *slope* of an edge in a straight-line drawing is the slope of the corresponding segment. Wade and Chu [69] defined the *slope number*, $sl(G)$, of a graph G as the smallest number s with the property that G has a straight-line drawing with edges of at most s distinct slopes.

Our terminology is somewhat unorthodox: by the *slope* of a line ℓ , we mean the angle α modulo π such that a counterclockwise rotation through α takes the x -axis to a position parallel to ℓ . The slope of an edge (segment) is the slope of the line containing it. In particular, the slopes of the lines $y = x$ and $y = -x$ are $\pi/4$ and $-\pi/4$, and they are called *Northeast* (or *Southwest*) and *Northwest* (or *Southeast*) lines, respectively. Directions are often abbreviated by their first letters: N, NE, E, SE, etc. These four directions are referred to as *basic*. That is, a line ℓ is said to be of one of the four basic directions if ℓ is parallel to one of the axes or to one of the NE and NW lines $y = x$ and $y = -x$.

Obviously, if G has a vertex of degree d , then its slope number is at least $\lceil d/2 \rceil$. Dujmović et al. [25] asked if the slope number of a graph with bounded maximum degree d could be arbitrarily large. Pach and Pálvölgyi [63] and Barát, Matoušek, Wood [15] (independently) showed with a counting argument that the answer is yes for $d \geq 5$.

In [44], it was shown that cubic (3-regular) graphs could be drawn with five slopes. The major result from which this was concluded was that subcubic graphs¹ can be drawn with the four basic slopes. We note here that the proof of this was slightly incorrect. We give below a stronger version of that theorem, in which the shortcomings of the incorrect proof can be overcome. Before the statement of the theorem, we clarify some terminology used in it.

For any two points $p_1 = (x_1, y_1), p_2 = (x_2, y_2) \in \mathbf{R}^2$, we say that p_2 is *to the North* (or *to the South* of p_1 if $x_2 = x_1$ and $y_2 > y_1$ (or $y_2 < y_1$)). Analogously, we say that p_2 is to the Northeast (to the Northwest) of p_1 if $y_2 > y_1$ and $p_1 p_2$ is a Northeast (Northwest) line.

Theorem 2.1.1 *Let G be a graph without components that are cycles and whose every vertex has degree at most three. Suppose that G has at least one vertex of degree less than three, and denote by v_1, \dots, v_m the vertices of degree at most two ($m \geq 1$).*

Then, for any sequence x_1, x_2, \dots, x_n of real numbers, linearly independent over the rationals, G has a straight-line drawing with the following properties:

- (1) Vertex v_i is mapped into a point with x -coordinate $x(v_i) = x_i$ ($1 \leq i \leq m$);
- (2) The slope of every edge is $0, \pi/2, \pi/4$, or $-\pi/4$.
- (3) No vertex is to the North of any vertex of degree *two*.
- (4) No vertex is to the North or to the Northwest of any vertex of degree *one*.
- (5) The x -coordinates of all the vertices are a linear combination with rational coefficients of x_1, \dots, x_n .

Therefore, cubic graphs require one additional slope and hence, five slopes. We improve this as following.

Theorem 2.1.2 *Every connected cubic graph has a straight-line drawing with only four slopes.*

The above theorem gives a drawing of connected cubic graphs with four slopes, one

¹A graph is subcubic if it is a proper subgraph of a cubic graph, i.e. the degree of every vertex is at most three and it is not cubic (not 3-regular).

of which is not a basic slope. Further, for disconnected cubic graphs, we require 5 slopes. We show a reduction of cubic graphs with triangles (Lemma 2.3.4) because of which, instead of the above theorem, our focus will be to prove the following.

Theorem 2.1.3 *Every triangle-free connected cubic graph has a straight-line drawing with only four slopes.*

We note that the four slopes used above are not the four basic slopes. Towards this, it was shown by Max Engelstein [29] that 3-connected cubic graphs with a Hamiltonian cycle can be drawn with the four basic slopes.

We later improve all these results by the following

Theorem 2.1.4 *Every cubic graph has a straight-line drawing with only the four basic slopes.*

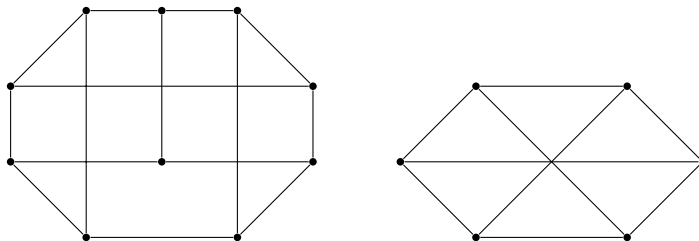


Figure 2.1: The Petersen graph and $K_{3,3}$ drawn with the four basic slopes.

This is the first result about cubic graphs that uses a nice, fixed set of slopes instead of an unpredictable set, possibly containing slopes that are not rational multiples of π . Also, since K_4 requires at least 4 slopes, this settles the question of determining the minimum number of slopes required for cubic graphs.

We also prove

Theorem 2.1.5 *Call a set of slopes good if every cubic graph has a straight-line drawing with them. Then the following statements are equivalent for a set S of four slopes.*

1. S is good.

2. S is an affine image of the four basic slopes.
3. We can draw K_4 with S .

The problem whether the slope number of graphs with maximum degree four is unbounded or not remains an interesting open problem.

There are many other related graph parameters. The *thickness* of a graph G is defined as the smallest number of planar subgraphs it can be decomposed into [58]. It is one of the several widely known graph parameters that measures how far G is from being planar. The *geometric thickness* of G , defined as the smallest number of *crossing-free* subgraphs of a straight-line drawing of G whose union is G , is another similar notion [41]. It follows directly from the definitions that the thickness of any graph is at most as large as its geometric thickness, which, in turn, cannot exceed its slope number. For many interesting results about these parameters, consult [23, 28, 25, 26, 30, 39].

A variation of the problem arises if (a) two vertices in a drawing have an edge between them if and only if the slope between them belongs to a certain set S and, (b) collinearity of points is allowed. This violates the condition stated before that an edge cannot pass through vertices other than its end points. For instance, K_n can be drawn with one slope. The smallest number of slopes that can be used to represent a graph in such a way is called the *slope parameter* of the graph. Under these set of conditions, [10] proves that the slope parameter of subcubic outerplanar graphs is at most 3. It was shown in [45] that the slope parameter of every cubic graph is at most seven. If only the four basic slopes are used, then the graphs drawn with the above conditions are called Queen's graphs and [9] characterizes certain graphs as Queen's graphs. Graph theoretic properties of some specific Queen's graphs can be found in [18].

Another variation for planar graphs is to demand a planar drawing. The *planar slope number* of a planar graph is the smallest number of distinct slopes with the property that the graph has a straight-line drawing with non-crossing edges using only these slopes. Dujmović, Eppstein, Suderman, and Wood [24] raised the question whether there exists a function f with the property that the planar slope number of every planar graph with maximum degree d can be bounded from above by $f(d)$. Jelinek et

al. [40] have shown that the answer is yes for *outerplanar* graphs, that is, for planar graphs that can be drawn so that all of their vertices lie on the outer face. Eventually the question was answered in [43] where it was proved that any bounded degree planar graph has a bounded planar slope number.

Finally we would mention a slightly related problem. Didimo et al. [21] studied drawings of graphs where edges can only cross each other in a right angle. Such a drawing is called an RAC (right angle crossing) drawing. They showed that every graph has an RAC drawing if every edge is a polygonal line with at most three bends (i.e. it consists of at most four segments). They also gave upper bounds for the maximum number of edges if less bends are allowed. Later Arikushi et al. [12] showed that such graphs can have at most $O(n)$ edges. Angelini et al. [11] proved that every cubic graph admits an RAC drawing with at most one bend. It remained an open problem whether every cubic graph has an RAC drawing with straight-line segments. If besides orthogonal crossings, we also allow two edges to cross at 45° , then it is a straightforward corollary of Theorem 2.1.4 that every cubic graph admits such a drawing with straight-line segments.

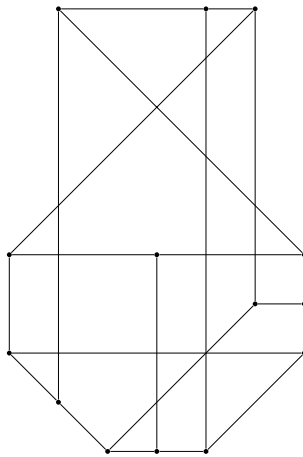


Figure 2.2: The Heawood graph drawn with the four basic slopes.

2.2 Correct Proof of the Subcubic Theorem

We would like the reader to note that this is a modification of the proof as it appears in [44].

Note that it is enough to establish the theorem for connected graphs, because if the different components of G are drawn separately and placed far above each other, then none of the properties will be violated.

2.2.1 Embedding Cycles

Let C be a straight-line drawing of a cycle in the plane. A vertex v of C is said to be a *turning point* if the slopes of the two edges meeting at v are not the same.

We start with two simple auxiliary statements.

Lemma 2.2.1 *Let C be a straight-line drawing of a cycle such that the slope of every edge is 0 , $\pi/4$, or $-\pi/4$. Then the x -coordinates of the vertices of C are not independent over the rational numbers.*

Moreover, there is a vanishing linear combination of the x -coordinates of the vertices, with as many nonzero (rational) coefficients as many turning points C has.

Proof. Let v_1, v_2, \dots, v_n denote the vertices of C in cyclic order ($v_{n+1} = v_1$). Let $x(v_i)$ and $y(v_i)$ be the coordinates of v_i . For any i ($1 \leq i \leq n$), we have $y(v_{i+1}) - y(v_i) = \lambda_i (x(v_{i+1}) - x(v_i))$, where $\lambda_i = 0, 1$, or -1 , depending on the slope of the edge $v_i v_{i+1}$. Adding up these equations for all i , the left-hand sides add up to zero, while the sum of the right-hand sides is a linear combination of the numbers $x(v_1), x(v_2), \dots, x(v_n)$ with integer coefficients of absolute value at most *two*.

Thus, we are done with the first statement of the lemma, unless all of these coefficients are zero. Obviously, this could happen if and only if $\lambda_1 = \lambda_2 = \dots = \lambda_n$, which is impossible, because then all points of C would be collinear, contradicting our assumption that in a proper *straight-line drawing* no edge is allowed to pass through any vertex other than its endpoints.

To prove the second statement, it is sufficient to notice that the coefficient of $x(v_i)$

vanishes if and only if v_i is not a turning point. \square

Lemma 2.2.1 shows that Theorem 2.1.1 does not hold if G is a cycle. Nevertheless, according to the next claim, cycles satisfy a very similar condition. Observe, that the main difference is that here we have an exceptional vertex, denoted by v_0 .

Lemma 2.2.2 *Let C be a cycle with vertices v_0, v_1, \dots, v_m , in this cyclic order.*

Then, for any real numbers x_1, x_2, \dots, x_m , linearly independent over the rationals, C has a straight-line drawing with the following properties:

- (1) Vertex v_i is mapped into a point with x -coordinate $x(v_i) = x_i$ ($1 \leq i \leq m$);
- (2) The slope of every edge is $0, \pi/4$, or $-\pi/4$.
- (3) No vertex is to the North of any other vertex.
- (4) No vertex has a larger y -coordinate than $y(v_0)$.
- (5) The x -coordinate of v_0 is a linear combination with rational coefficients of x_1, \dots, x_m .

Proof. We can assume without loss of generality that $x_2 > x_1$. Place v_1 at the point $(x_1, 0)$ of the x -axis. Assume that for some $i < m$, we have already determined the positions of v_1, v_2, \dots, v_i , satisfying conditions (1)–(3). If $x_{i+1} > x_i$, then place v_{i+1} at the (unique) point Southeast of v_i , whose x -coordinate is x_{i+1} . If $x_{i+1} < x_i$, then put v_{i+1} at the point West of v_i , whose x -coordinate is x_{i+1} . Clearly, this placement of v_{i+1} satisfies (1)–(3), and the segment $v_i v_{i+1}$ does not pass through any point v_j with $j < i$.

After m steps, we obtain a noncrossing straight-line drawing of the path $v_1 v_2 \dots v_m$, satisfying conditions (1)–(3). We still have to find a right location for v_0 . Let R_W and R_{SE} denote the rays (half-lines) starting at v_1 and pointing to the West and to the Southeast. Further, let R be the ray starting at v_m and pointing to the Northeast. It follows from the construction that all points v_2, \dots, v_m lie in the convex cone below the x -axis, enclosed by the rays R_W and R_{SE} .

Place v_0 at the intersection point of R and the x -axis. Obviously, the segment $v_m v_0$ does not pass through any other vertex v_j ($0 < j < m$). Otherwise, we could find a drawing of the cycle $v_j v_{j+1} \dots v_m$ with slopes $0, \pi/4$, and $-\pi/4$. By Lemma 2.2.1, this would imply that the numbers x_j, x_{j+1}, \dots, x_m are *not* independent over the rationals,

contradicting our assumption. It is also clear that the horizontal segment v_0v_1 does not pass through any vertex different from its endpoints because all other vertices are below the horizontal line determined by v_0v_1 . Hence, we obtain a proper straight-line drawing of C satisfying conditions (1),(2), and (4). Note that (5) automatically follows from Lemma 2.2.1.

It remains to verify (3). The only thing we have to check is that $x(v_0)$ does not coincide with any other $x(v_i)$. Suppose it does, that is, $x(v_0) = x(v_i) = x_i$ for some $i > 0$. By the second statement of Lemma 2.2.1, there is a vanishing linear combination

$$\lambda_0x(v_0) + \lambda_1x_1 + \lambda_2x_2 + \dots + \lambda_mx_m = 0$$

with rational coefficients λ_i , where the number of nonzero coefficients is at least the number of turning points, which cannot be smaller than *three*. Therefore, if in this linear combination we replace $x(v_0)$ by x_i , we still obtain a nontrivial rational combination of the numbers x_1, x_2, \dots, x_m . This contradicts our assumption that these numbers are independent over the rationals. \square

2.2.2 Subcubic Graphs - Proof of Theorem 2.1.1

First we settle Theorem 2.1.1 in a special case.

Lemma 2.2.3 *Let $m, k \geq 2$ and let G be a graph consisting of two disjoint cycles, $C = \{v_0, v_1, \dots, v_m\}$ and $C' = \{v'_0, v'_1, \dots, v'_m\}$, connected by a single edge $v_0v'_0$.*

Then, for any sequence $x_1, x_2, \dots, x_m, x'_1, x'_2, \dots, x'_k$ of real numbers, linearly independent over the rationals, G has a straight-line drawing satisfying the following conditions:

- (1) The vertices v_i and v'_j are mapped into points with x -coordinates $x(v_i) = x_i$ ($1 \leq i \leq m$) and $x(v'_j) = x'_j$ ($1 \leq j \leq k$).
- (2) The slope of every edge is $0, \pi/2, \pi/4$, or $-\pi/4$.
- (3) No vertex is to the North of any vertex of degree *two*.

(4) The x -coordinates of all the vertices are a linear combination with rational coefficients of $x_1, x_2, \dots, x_m, x'_1, x'_2, \dots, x'_k$.

Proof. Apply Lemma 2.2.2 to cycle C with vertices v_0, v_1, \dots, v_m and with assigned x -coordinates x_1, x_2, \dots, x_m , and analogously, to the cycle C' , with vertices v'_0, v'_1, \dots, v'_k and assigned x -coordinates x'_1, x'_2, \dots, x'_k . For simplicity, the resulting drawings are also denoted by C and C' .

Let x_0 and x'_0 denote the x -coordinates of $v_0 \in C$ and $v'_0 \in C'$. It follows from (5) of Lemma 2.2.2 that x_0 is a linear combination of x_1, x_2, \dots, x_m , and x'_0 is a linear combination of x'_1, x'_2, \dots, x'_k with rational coefficients. Therefore, if $x_0 = x'_0$, then there is a nontrivial linear combination of $x_1, x_2, \dots, x_m, x'_1, x'_2, \dots, x'_k$ that gives 0, contradicting the assumption that these numbers are independent over the rationals. Thus, we can conclude that $x_0 \neq x'_0$. Assume without loss of generality that $x_0 < x'_0$. Reflect C' about the x -axis, and shift it in the vertical direction so that v'_0 ends up to the Northeast from v_0 . Clearly, we can add the missing edge $v_0v'_0$. Let D denote the resulting drawing of G . We claim that D meets all the requirements. Conditions (1), (2), (3) and (4) are obviously satisfied, we only have to check that no vertex lies in the interior of an edge. It follows from Lemma 2.2.2 that the y -coordinates of v_1, \dots, v_m are all smaller than or equal to the y -coordinate of v_0 and the y -coordinates of v'_1, \dots, v'_k are all greater than or equal to the y -coordinate of v'_0 . We also have $y(v_0) < y(v'_0)$. Therefore, there is no vertex in the interior of $v_0v'_0$. Moreover, no edge of C (resp. C') can contain any vertex of v'_0, v'_1, \dots, v'_k (resp. v_0, v_1, \dots, v_m) in its interior. \square

The rest of the proof is by induction on the number of vertices of G . The statement is trivial if the number of vertices is at most *two*. Suppose that we have already established Theorem 2.1.1 for all graphs with fewer than n vertices.

Suppose that G has n vertices, it is not a cycle and not the union of two cycles connected by one edge.

Unfortunately we have to distinguish several cases. Most of these are very special and less interesting instances that prevent us from using our main argument, which is considered last after clearing all obstacles, as Case 9.

Case 1: G has a vertex of degree one.

Assume, without loss of generality, that v_1 is such a vertex. If G has no vertex of degree *three*, then it consists of a simple path $P = v_1v_2 \dots v_m$, say. Place v_m at the point $(x_m, 0)$. In general, assuming that v_{i+1} has already been embedded for some $i < m$, and $x_i < x_{i+1}$, place v_i at the point West of v_{i+1} , whose x -coordinate is x_i . If $x_i > x_{i+1}$, then put v_i at the point Northeast of v_{i+1} , whose x -coordinate is x_i . The resulting drawing of $G = P$ meets all the requirements of the theorem. To see this, it is sufficient to notice that if v_j would be Northwest of v_m for some $j < m$, then we could apply Lemma 2.2.1 to the cycle $v_jv_{j+1} \dots v_m$, and conclude that the numbers x_j, x_{j+1}, \dots, x_m are dependent over the rationals. This contradicts our assumption.

Assume next that v_1 is of degree *one*, and that G has at least one vertex of degree *three*. Suppose without loss of generality that $v_1v_2 \dots v_kv$ is a path in G , whose internal vertices are of degree *two*, but the degree of w is *three*. Let G' denote the graph obtained from G by removing the vertices v_1, v_2, \dots, v_k . Obviously, G' is a connected graph, in which the degree of w is *two*.

If G' is a *cycle*, then apply Lemma 2.2.2 to $C = G'$ with w playing the role of the vertex v_0 which has no preassigned x -coordinate. We obtain an embedding of G' with edges of slopes $0, \pi/4$, and $-\pi/4$ such that $x(v_i) = x_i$ for all $i > k$ and there is no vertex to the North, to the Northeast, or to the Northwest of w . By (5) of Lemma 2.2.2, $x(w)$ is a linear combination of x_{k+1}, \dots, x_m with rational coefficients. Therefore, $x(w) \neq x_k$, so we can place v_k at the point to the Northwest or to the Northeast of w , whose x -coordinate is x_k , depending on whether $x(w) > x_k$ or $x(w) < x_k$. After this, embed v_{k-1}, \dots, v_1 , in this order, so that v_i is either to the Northeast or to the West of v_{i+1} and $x(v_i) = x_i$. According to property (4) in Lemma 2.2.1, the path $v_1v_2 \dots v_k$ lies entirely above G' , so no point of G can lie to the North or to the Northwest of v_1 .

If G' is *not a cycle*, then use the induction hypothesis to find an embedding of G' that satisfies all conditions of Theorem 2.1.1, with $x(w) = x_k$ and $x(v_i) = x_i$ for every $i > k$. Now place v_k very far from w , to the North of it, and draw v_{k-1}, \dots, v_1 , in this order, in precisely the same way as in the previous case. Now if v_k is far enough, then none of the points v_k, v_{k-1}, \dots, v_1 is to the Northwest or to the Northeast of any vertex

of G' . It remains to check that condition (4) is true for v_1 , but this follows from the fact that there is no point of G whose y -coordinate is larger than that of v_1 .

From now on, we can and will assume that G has *no vertex of degree one*.

A graph with *four* vertices and *five* edges between them is said to be a Θ -graph.

Case 2: G contains a Θ -subgraph.

Suppose that G has a Θ -subgraph with vertices a, b, c, d , and edges ab, bc, ac, ad, bd . If neither c nor d has a third neighbor, then G is identical to this graph, which can easily be drawn in the plane with all conditions of the theorem satisfied.

If c and d are connected by an edge, then all four points of the Θ -subgraph have degree *three*, so that G has no other vertices. So G is a complete graph of four vertices, and it has a drawing that meets the requirements.

Suppose that c and d have a common neighbor $e \neq a, b$. If e has no further neighbor, then a, b, c, d, e are the only vertices of G , and again we can easily find a proper drawing. Thus, we can assume that e has a third neighbor f . By the induction hypothesis, $G' = G \setminus \{a, b, c, d, e\}$ has a drawing satisfying the conditions of Theorem 2.1.1. In particular, no vertex of G' is to the North of f (and to the Northwest of f , provided that the degree of f in G' is *one*). Further, consider a drawing H of the subgraph of G induced by the vertices a, b, c, d, e , which satisfies the requirements. We distinguish two subcases.

If the degree of f in G' is *one*, then take a very small *homothetic* copy of H (i.e., similar copy in parallel position), and rotate it about e in the clockwise direction through $3\pi/4$. There is no point of this drawing, denoted by H' , to the Southeast of e , so that we can translate it into a position in which e is to the Northwest of $f \in V(G')$ and very close to it, to a sufficient distance so that (5) is satisfied. Connecting now e to f , we obtain a drawing of G satisfying the conditions. Note that it was important to make H' very small and to place it very close to f , to make sure that none of its vertices is to the North of any vertex of G' whose degree is at most *two*, or to the Northwest of any vertex of degree *one* (other than f).

If the degree of f in G' is *two*, then we follow the same procedure, except that now H' is a small copy of H , rotated by π . We translate H' into a position in which e is to the North of f , and connect e to f by a vertical segment. It is again clear that the resulting drawing of G meets the requirements in Theorem 2.1.1. Thus, we are done if c and d have a common neighbor e .

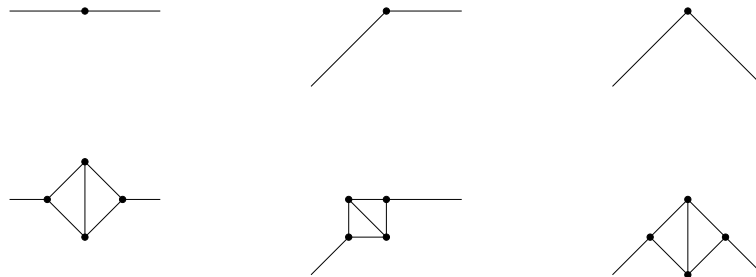
Suppose now that only one of c and d has a third neighbor, different from a and b . Suppose, without loss of generality, that this vertex is c , so that the degree of d is *two*. Then in $G' = G \setminus \{a, b, d\}$, the degree of c is *one*. Apply the induction hypothesis to G' so that the x -coordinate originally assigned to d is now assigned to c (which had no preassigned x -coordinate in G). In the resulting drawing, we can easily reinsert the remaining vertices, a, b, d , by adding a very small square whose lowest vertex is at c and whose diagonals are parallel to the coordinate axes. The highest vertex of this square will represent d , and the other two vertices will represent a and b .

We are left with the case when both c and d have a third neighbor, other than a and b , but these neighbors are different. Denote them by c' and d' , respectively. Create a new graph G' from G , by removing a, b, c, d and adding a new vertex v , which is connected to c' and d' . Draw G' using the induction hypothesis, and reinsert a, b, c, d in a small neighborhood of v so that they form the vertex set of a very small square with diagonal ab . (See Figure 2.3.) As before, we have to choose this square sufficiently small to make sure that a, b, c, d are not to the North of any vertex $w \neq c', d', v$ of G' , whose degree is at most *two*, or to the Northwest of any vertex of degree *one* and pick an appropriate scaling to make sure that (5) is satisfied. Thus, we are done if G has a Θ -subgraph.

So, from now on we assume that G has *no Θ -subgraph*.

Case 3: G has *no cycle that passes through a vertex of degree two*.

Since G is not three-regular, it contains at least one vertex of degree *two*. Consider a decomposition of G into 2-connected blocks and edges. If a block contains a vertex of degree *two*, then it consists of a single edge. The block decomposition has a treelike structure, so that there is a vertex w of degree *two*, such that G can be obtained as the

Figure 2.3: Replacing v by Θ .

union of two graphs, G_1 and G_2 , having only the vertex w in common, and there is no vertex of degree *two* in G_1 .

By the induction hypothesis, for any assignment of rationally independent x -coordinates to all vertices of degree less than *three*, G_1 and G_2 have proper straight-line embeddings (drawings) satisfying conditions (1)–(5) of the theorem. The only vertex of G_1 with a preassigned x -coordinate is w . Applying a vertical translation, if necessary, we can achieve that in both drawings w is mapped into the same point. Using the induction hypothesis, we obtain that in the union of these two drawings, there is no vertex in G_1 or G_2 to the North or to the Northwest of w , because the degree of w in G_1 and G_2 is *one* (property (4)). This is stronger than what we need: indeed, in G the degree of w is *two*, so that we require only that there is no point of G to the North of w (property (3)).

The superposition of the drawings of G_1 and G_2 satisfies all conditions of the theorem. Only two problems may occur:

1. A vertex of G_1 may end up at a point to the North of a vertex of G_2 with degree *two*.
2. The (unique) edges in G_1 and G_2 , incident to w , may partially overlap.

Notice that both of these events can be avoided by enlarging the drawing of G_1 , if necessary, from the point w , and rotating it about w by $\pi/4$ in the clockwise direction. The latter operation is needed only if problem 2 occurs. This completes the induction step in the case when G has no cycle passing through a vertex of degree *two*.

Case 4: G has two adjacent vertices of degree two.

Take a longest path that contains only degree two vertices. Without loss of generality, assume that this path is $v_1v_2 \dots v_k$. Denote the degree three neighbor of v_1 by u and the degree three neighbor of v_k by w . Let $G' = G \setminus \{v_1 \dots v_k\}$. Now we distinguish two subcases depending on whether these two vertices are the same or not.

Case 4/a: $u \neq w$.

First suppose that G' is connected.

If G' is not a cycle, embed it using induction with x_1 being the prescribed x -coordinate of u and x_k being the prescribed x -coordinate of w . Now place the v_i vertices one by one high above this drawing, starting with v_1 , using NW and NE directions. Finally we embed v_k above w and we are done.

If G' is a cycle, then embed it using Lemma 2.2.2 with $v_0 = u$ and prescribed x -coordinate x_k for w . Remember that there are no vertices above u . So first, we can place v_1 to the NW or NE from u . Then we place the v_i vertices one by one using NW and NE directions. Finally we embed v_k above w and we are done.

Now suppose G' has two components. If none of them is a cycle, embed both of them using induction, high above each other, with x_1 being the prescribed x -coordinate of u and x_k being the prescribed x -coordinate of w . Now place the v_i vertices one by one high above the so far drawn components, starting with v_1 , using NW and NE directions. Finally we embed v_k above w and we are done.

Finally, if $G' = G \setminus \{v_1 \dots v_k\}$ has two components one of which, say the one containing w , is a cycle, then embed the component of u using induction with prescribed x -coordinate x_1 for u or, if it is a cycle, Lemma 2.2.2 with $v_0 = u$. It is easy to see that we can embed the v_i vertices one by one, starting with v_1 , just like in the previous cases, and then the rest of the v_i vertices one by one using NW and NE directions. Finally we embed the cycle containing w using Lemma 2.2.2 with $v_0 = w$, but upside down, so that w has (one of) the smallest y -coordinate(s). Shift this cycle vertically such that the edge $v_k w$ has NW or NE direction and we are done.

Note that this last case even works for $k = 1$.

Case 4/b: $u = w$.

Denote the third neighbor of u by t . If the degree of t is two, then deleting the longest path containing t that contains only degree two vertices, the remaining graph will have two components, one of which is a cycle. Thus we end up exactly in the last subcase of Case 4/a, thus we are done.

If the degree of t is three, apply Lemma 2.2.2 with $v_0 = u$ to the cycle $C = uv_1 \dots v_k$. Denote the x -coordinate of u by x_0 . If $G \setminus C$ is a cycle, we can use Lemma 2.2.3. Otherwise, embed $G \setminus C$ using induction with x_0 being the prescribed x -coordinate of t . Now place C sufficiently high above this drawing.

Case 5 (Main case): *G has a cycle passing through a vertex of degree two.*

By assumption, G itself is not a cycle. Therefore, we can also find a *shortest* cycle C whose vertices are denoted by v, u_1, \dots, u_k , in this order, where the degree of v is *two* and the degree of u_1 is *three*. The length of C is $k + 1$.

It follows from the minimality of C that u_i and u_j are not connected by an edge of G , for any $|i - j| > 1$. Moreover, if $|i - j| > 2$, then u_i and u_j do not even have a common neighbor ($1 \leq i \neq j \leq k$). This implies that any vertex $v \in V(G \setminus C)$ has at most *three* neighbors on C , and these neighbors must be consecutive on C . However, *three* consecutive vertices of C , together with their common neighbor, would form a Θ -subgraph in G (see Case 2). Hence, we can assume that every vertex belonging to $G \setminus C$ is joined to at most *two* vertices on C .

Consider the list v_1, v_2, \dots, v_m of all vertices of G with degree *two*. (Recall that we have already settled the case when G has a vertex of degree *one*.) Assume without loss of generality that $v_1 = v$ and that v_i belongs to C if and only if $1 \leq i \leq j$ for some $j \leq m$.

Let \mathbf{x} denote the *assignment* of x -coordinates to the vertices of G with degree *two*, that is, $\mathbf{x} = (x(v_1), x(v_2), \dots, x(v_m)) = (x_1, x_2, \dots, x_m)$. Given G , C , \mathbf{x} , and a real parameter L , we define the following so-called EMBEDDING PROCEDURE(G, C, \mathbf{x}, L) to construct a drawing of G that meets all requirements of the theorem, and satisfies the additional condition that the y -coordinate of every vertex of C is at least L higher than

the y -coordinates of all other vertices of G .

Let u'_1 be the neighbor of u_1 in $G \setminus C$. We mark two different cases here and all the steps in the Embedding Procedure will be defined for both the cases. If u'_1 is a vertex of degree three in G , we will call it Subcase 5(a), and we define $G' = G \setminus C$. On the other hand, if u'_1 is a vertex of degree two, then by Case 4, its other neighbor (besides v), say u''_1 , is a degree three vertex. We call this Subcase 5(b) and define $G' = G \setminus (C \cup \{u'_1\})$. The main idea of the Embedding procedure is to inductively embed G' and place the rest of the graph in a convenient way.

Let B_i denote the set of all vertices of G' that have precisely i neighbors on C ($i = 0, 1, 2$). Thus, we have $V(G') = B_0 \cup B_1 \cup B_2$. Further, $B_1 = B_1^2 \cup B_1^3$, where an element of B_1 belongs to B_1^2 or B_1^3 , according to whether its degree in G is *two* or *three*.

STEP 1: If G' is *not* a cycle, then construct recursively a drawing of G' satisfying the conditions of Theorem 2.1.1 with the assignment \mathbf{x}' of x -coordinates $x(v_i) = x_i$ for $j < i \leq m$, and $x(u'_1) = x_1$ in Subcase 5(a), and, $x(u''_1) = x(u'_1)$ in Subcase 5(b).

If G' is a cycle, then, by assumption, there are at least two edges between C and G' . One of them connects u_1 to u'_1 . Let $u_\alpha u'_\alpha$ be another such edge, where $u_\alpha \in C$ and $u'_\alpha \in G'$. Since the maximum degree is three, $u'_1 \neq u'_\alpha$. Now construct recursively a drawing of G' satisfying the conditions of Lemma 2.2.2, with the exceptional vertex as u'_α .

We note here that if G' is disconnected, but the components are not cycles, then we just place them vertically far apart and we still have a good recursive drawing of G' . Suppose that it is disconnected and some components are cycles. If the component connected to u_1 or u'_1 (based on Subcase 5(a) or 5(b)) is a cycle, we draw the cycle exactly as in the preceding paragraph. For all other components that are cycles, we note that since G is connected, there must be at least one vertex of the cycle connected to $G \setminus G'$ (in fact at least two because of Case 4). This is a degree three vertex in G and we will call this the exceptional vertex and draw the cycle using Lemma 2.2.2. At the end, we shift all components vertically to place them sufficiently far apart. We note that this drawing of G' will satisfy all the conditions in Theorem 2.1.1.

STEP 2: For each element of $B_1^2 \cup B_2$, take two rays starting at this vertex, pointing to the Northwest and to the North. Further, take a vertical ray pointing to the North from each element of B_1^3 and each element of the set $B_x := \{(x_2, 0), (x_3, 0), \dots, (x_j, 0)\}$. Let \mathcal{R} denote the set of all of these rays. Choose the x -axis above all points of G' and all intersection points between the rays in \mathcal{R} .

For any u_h ($1 \leq h \leq k$) whose degree in G is *three*, define $N(u_h)$ as the unique neighbor of u_h in G' . If u_h has degree *two* in G , then $u_h = v_i$ for some $1 \leq i \leq j$, and let $N(u_h)$ be the point $(x_i, 0)$.

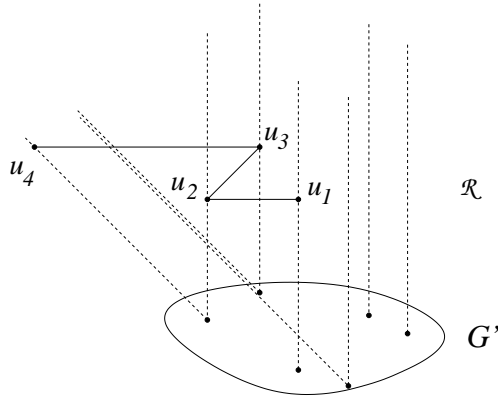


Figure 2.4: Recursively place u_1, u_2, \dots, u_k on the rays belonging to \mathcal{R} .

STEP 3: Recursively place u_1, u_2, \dots, u_k on the rays belonging to \mathcal{R} , as follows. In Subcase 5(a), place u_1 on the vertical ray starting at $N(u_1) = u'_1$ such that $y(u_1) = L$. In Subcase 5(b), place u'_1 on the vertical ray starting at $N(u'_1) = u''_1$ such that $y(u'_1) = L$. If $x_1 < x(u'_1)$ then place u_1 to the West of u'_1 on the line $x = x_1$, otherwise place u_1 to the Northeast of u_1 , again on $x = x_1$. Suppose that for some $i < k$ we have already placed u_1, u_2, \dots, u_i , so that $L \leq y(u_1) \leq y(u_2) \leq \dots \leq y(u_i)$ and there is no vertex to the West of u_i . Next we determine the place of u_{i+1} .

If $N(u_{i+1}) \in B_1^2$, then let $r \in \mathcal{R}$ be the ray starting at $N(u_{i+1})$ and pointing to the Northwest. If $N(u_{i+1}) \in B_1^3 \cup B_x$, let $r \in \mathcal{R}$ be the ray starting at $N(u_{i+1})$ and pointing to the North. In both cases, place u_{i+1} on r : if u_i lies on the left-hand side of r , then put u_{i+1} to the Northeast of u_i ; otherwise, put u_{i+1} to the West of u_i .

If $N(u_{i+1}) \in B_2$, then let $r \in \mathcal{R}$ be the ray starting at $N(u_{i+1})$ and pointing to the North, or, if we have already placed a point on this ray, let r be the other ray from $N(u_{i+1})$, pointing to the Northwest, and proceed as before.

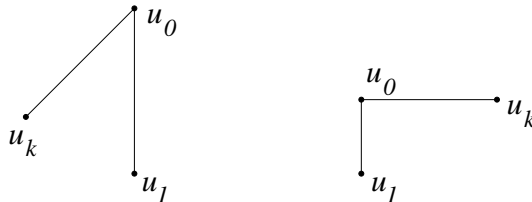


Figure 2.5: Finding the right position for u_0 .

STEP 4: Suppose we have already placed u_k . It remains to find the right position for $u_0 := v$, which has only two neighbors, u_1 and u_k . Let r be the ray at u_1 , pointing to the North. If u_k lies on the left-hand side of r , then put u_0 on r to the Northeast of u_k ; otherwise, put u_0 on r , to the West of u_k .

During the whole procedure, we have never placed a vertex on any edge, and all other conditions of Theorem 2.1.1 are satisfied \square .

Remark that the y -coordinates of the vertices $u_0 = v, u_1, \dots, u_k$ are at least L higher than the y -coordinates of all vertices in $G \setminus C$. If we fix G, C , and \mathbf{x} , and let L tend to infinity, the coordinates of the vertices given by the above EMBEDDING PROCEDURE(G, C, \mathbf{x}, L) change continuously.

2.3 Proof of Theorem 2.1.3

2.3.1 Assumptions

This subsection is dedicated to showing that assuming that the cubic graph is bridgeless and triangle free does not restrict generality.

We would use theorem 2.1.1 to patch together different components of a cubic graph obtained after removal of some edges. For this we would want to note that we could rotate the components by any multiple of $\pi/4$ and still have a graph with the four basic slopes.

Claim 2.3.1 *A cubic graph with a bridge or a minimal two-edge disconnecting set can be drawn with the four basic slopes.*

Proof. We note that the above method cannot be extended to a minimal disconnecting set with more edges, as then, one of the components might be a cycle and then the above theorem cannot be invoked.

Both components obtained by removing the bridge can be drawn with four slopes using Theorem 2.1.1. Both have the north direction free for the vertex of degree two. To put these together, rotate the second one by π and place the degree two vertices above each other. Move the components far enough so that none of the other vertices or edges overlap.

For a two-edge disconnecting set, we may note that these edges must be vertex-disjoint or the graph would contain a bridge. Then, the same procedure as above can be used, now keeping the distance between the two vertices of degree two the same in both components. \square

Claim 2.3.2 *A cubic graph with a cut-vertex or a two-vertex disconnecting set can be drawn with the four basic slopes.*

Proof. If the graph has a cut-vertex, then it has a bridge. If it has a two-vertex disconnecting set, then it has a two-edge disconnecting set. In both cases we can then invoke Claim 2.3.1 to draw the graph with four slopes. \square

Remark 2.3.3 *A consequence of the above discussion is that any cubic graph that cannot be drawn with the four basic slopes (N, E, NE, NW) must be three vertex and edge connected.*

Claim 2.3.4 *Any cubic graph with a triangle can be drawn with four slopes*

Proof. First we note that by using the above claims, we may assume that we only consider cubic graphs in which all triangles are connected to the rest of the graph by

vertex disjoint edges. If not, then the graph is either K_4 or has a two-vertex disconnecting set. A K_4 can be drawn using the vertices of a square. In the later case, we can draw the graph with four slopes using Claim 2.3.2.

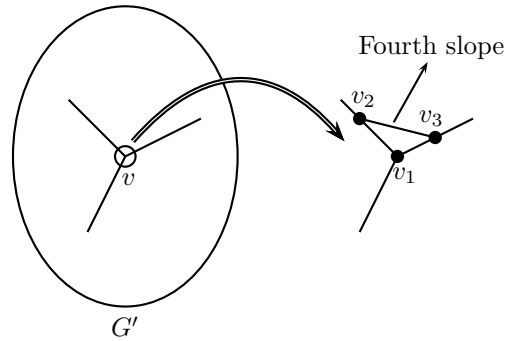


Figure 2.6: Adding the triangle to the drawing of G' with four slopes.

We now prove the claim by contradiction. Suppose there exist cubic graphs with triangles that cannot be drawn with four slopes. By the preceding discussion all triangles in these graphs are necessarily connected to the graphs with vertex-disjoint edges. Of all such graphs consider the one with minimum number of vertices, say G . The graph G' obtained by contracting the edges of the triangle $\{v_1, v_2, v_3\}$ is also cubic and has fewer vertices. Either all triangles in G' are connected to the rest of G' with vertex-disjoint edges, in which case we invoke the minimality of G to conclude that G' can be drawn with 4 slopes (note: here the method of drawing the graph is unknown. We just know there exists a drawing of G' with four slopes). Or, some triangles in G' could be connected to the rest of G' with edges that are not vertex-disjoint. Here we can use Theorem 2.1.1 and the argument of the preceding paragraph to draw G' using four slopes. And lastly, G' could be a triangle-free graph. In this case we use Theorem 2.1.3 to draw G' . Hence, G' can always be drawn with four slopes. In G' , we call the vertex formed by contracting the edges of the triangle as v . Since there is one slope that is not used by the edges incident on v , we draw a segment with this slope in a very small neighborhood of v as shown in the figure, to obtaining a drawing of G with four slopes. This contradicts the existence of a minimal counterexample and hence all

graphs with triangles can be drawn with four slopes.

□

Remark 2.3.5 *We note here that the preceding Lemma also holds in stricter conditions. To be precise, if the set of basic slopes are sufficient to draw all triangle-free cubic graphs, then they are sufficient to draw all cubic graphs.*

Remark 2.3.6 *It must be noted that this also gives an algorithm for drawing cubic graphs with triangles, namely, we contract triangles until we get a graph that can be drawn with either the Claims 2.3.1, 2.3.2, 2.3.4 or Theorem 2.1.1 or with our drawing strategy for triangle-free bridgeless graphs. Then we can backtrack with placing a series of edges which give us back all the contracted triangles.*

2.3.2 Drawing strategy

Because of the above claims, we would now only focus on graphs that are bridgeless and triangle-free. Since the graph is bridgeless, Petersen's theorem implies that it has a matching. We fix the slope of all the edges in the matching to be $\pi/2$ so that they all lie on (distinct) vertical lines (Figure 2.3.2). If this matching is removed, then the graph consists of disjoint cycles. Next we isolate one special edge from each cycle. Our method of drawing the graph with four slopes then is as follows: For each cycle, remove the selected edge and draw the remaining path by going between corresponding vertical lines of the cycle alternating with slopes $\pi/4, 3\pi/4$ depending on whether we draw the edges with increasing/decreasing x -coordinate. This ensures that the cycles all grow upwards. Since we have the freedom to place the cycles where we want, we place them vertically on the matching so that they are very far apart (non-intersecting). Also, if the special edge of each cycle was between adjacent vertical lines then this edge would not pass through any other vertex of the graph either. Then, the only thing we would need is that the final edge in each cycle is drawn with the same slope. Figure 2.3.2 illustrates this and the next remark is followed by a formal description of the problem.

Remark 2.3.7 *In [29] a similar strategy of drawing the matching on vertical lines*

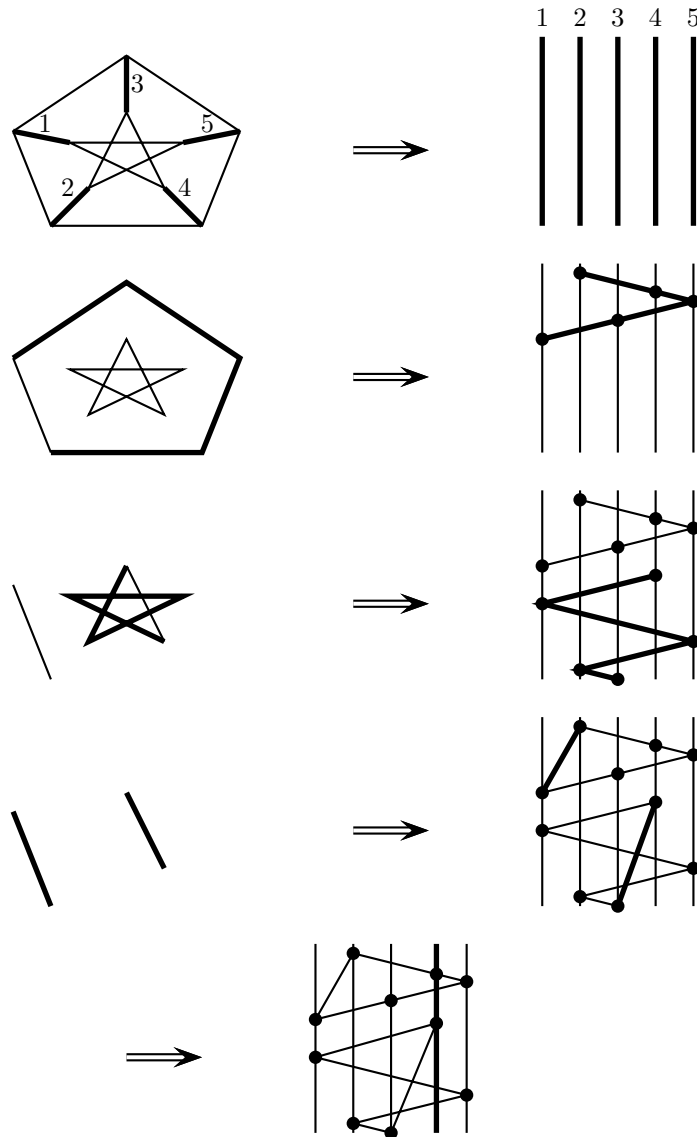


Figure 2.7: Process of drawing the cycles.

was employed. However, the cycles were drawn with alternating $\pi/4, 3\pi/4$ slopes for adjacent edges, so that the cycles were not “growing upwards” as in our construction. It leads to a different algebraic formulation of the problem giving tight bounds for the case when the cubic graph contains a Hamiltonian cycle.

Let M be a matching in G . Each cycle C in $E(G) \setminus M$ can be represented as a cyclic sequence $C = (v_1, \dots, v_k)$, where each v_i is an element of M . The sequence represents the elements of M as we go around the cycle. We can assume (by Claim 2.3.4) that $k \geq 4$. An *edge* of C by definition is (v_i, v_{i+1}) (all indices are understood mod k), which is although formally a pair formed by two distinct elements of M , also corresponds to an actual edge of the cycle. Notice that each element of M is either shared by two cycles or occurs twice in a single cycle.

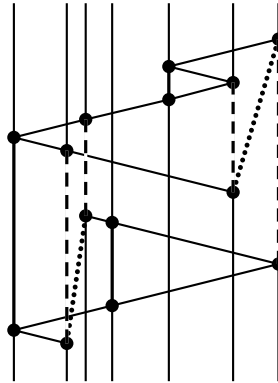


Figure 2.8: Distinguished “matching-edges” of Figure 1 are represented by dashed lines while distinguished cycle-edges are represented by dotted lines.

We now want to pick a distinguished edge (as in Figure 2.3.2) (v_i, v_{i+1}) in C (and in other cycles) such that the set of distinguished cycle-edges will satisfy certain properties.

Notation: Each distinguished cycle-edge is adjacent with two edges from the matching. These would be called the distinguished matching-edges of the cycle. In particular, the collection of distinguished edges from all cycles form the set of *distinguished matching-edges*. We would hope that distinguished matching-edges corresponding to a distinguished cycle-edge can be drawn as adjacent vertical lines for all cycles so that

this would naturally enforce that the distinguished cycle-edge would not go through any other vertex of the graph.

Definition 2.3.8 *Two cycles are connected if they share a distinguished matching-edge, and two cycles belong to the same component if they can be reached one from another by going through connected cycles. (An alternate way of looking at this would be that two cycles are adjacent iff the sets of distinguished matching-edges corresponding to the two cycles have a non-empty intersection). In other words, we define a graph on the cycles that we call the **cycle-connectivity graph**. Notice that in this graph each cycle can have at most two neighbors, thus the graph is a union of paths and cycles. The set of distinguished matching-edges associated with the component where cycle C belongs is denoted by $D(C)$. (Clearly, if C_1 and C_2 belong to the same component, then $D(C_1) = D(C_2)$).*

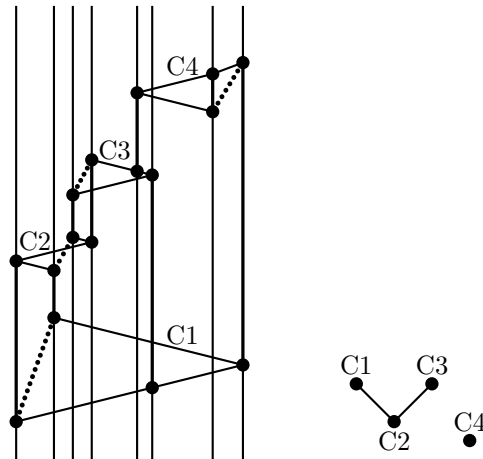


Figure 2.9: Graph and its connectivity graph.

Remark 2.3.9 *We note that in the cycle-connectivity graph two cycles are not necessarily connected if they share a matching-edge but only if they share a distinguished matching-edge. We can define another graph, where two cycles are connected if they share any matching-edge. It is easy to see that G is connected iff the latter graph is connected.*

Remark 2.3.10 We also note that we may get a multigraph for the cycle-connectivity graph in the event that two cycles pick distinguished cycle-edges between the same set of matching-edges. Condition I below avoids that scenario also.

Condition I: The cycle-connectivity graph does not contain cycles (only paths). Equivalently, we can enumerate the distinguished matching-edges associated with the cycles of a component in some linear order y_1, \dots, y_l in such a way that the pairs of consecutive matching-edges of this order are exactly the distinguished cycle-edges associated with the cycles in the component.

Condition II: In each component there is at most one cycle C such that $C \subseteq D(C)$.

Assume that the lines of the matching are ordered v_1, \dots, v_n . From Condition I, we can ensure that every distinguished cycle-edge takes up two adjacent lines in this ordering. A drawing of these lines would be completely determined by the distance between consecutive lines. If v_i, v_{i+1} form a distinguished cycle-edge of the k^{th} cycle, then call the distance between these lines x_k . Otherwise fix this distance to be some arbitrary positive constant c_i . This is illustrated in Figure 2.3.2.

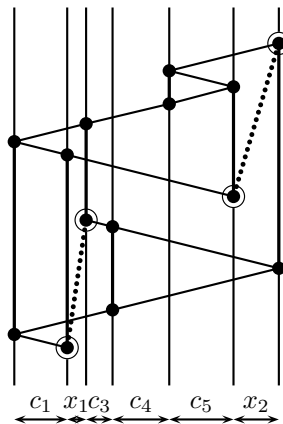


Figure 2.10: Definition of variables x_i and c_i .

Now draw a cycle by starting at one of the distinguished matching-edges and first drawing the path obtained by removing the distinguished cycle-edge. If an edge of the cycle is v_k, v_l where $k < l$ then use a slope of $\pi/4$ and $3\pi/4$ otherwise. Notice that

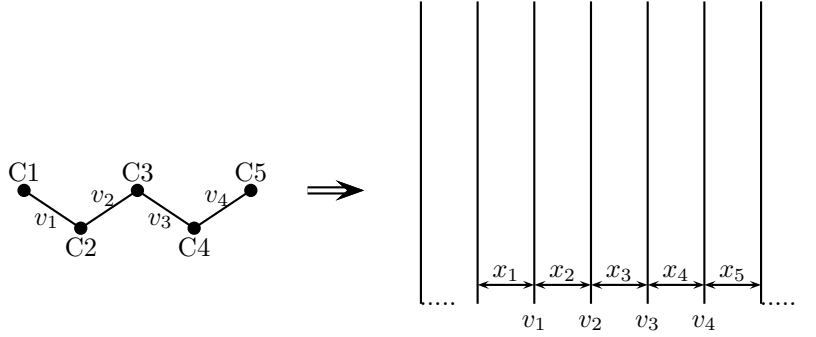


Figure 2.11: Paths of cycles will have adjacent distinguished cycle-edges in the drawing (because of the distinguished matching-edge they share). Hence it is necessary to not have cycles in the connectivity graph.

the vertical distance traveled across this edge is equal to the distance between the lines v_k and v_l . Hence the slope of the distinguished cycle-edge would look like $g_i = \frac{\mathcal{L}_i(\mathbf{x})}{x_i}$ where $\mathcal{L}_i(\mathbf{x}) = a_{i,0} + \sum_{j=1}^n a_{i,j}x_j$ for $1 \leq i \leq m$ (m being the number of cycles) is a linear equation on \mathbf{x} with non-negative coefficients. We will use the following Solvability Theorem to ensure that these slopes can always be matched. This will be proved in the next subsection.

Theorem 2.3.11 *Let $\mathcal{L}_i(\mathbf{x}) = a_{i,0} + \sum_{j=1}^n a_{i,j}x_j$ for $1 \leq i \leq n$ be linear forms, such that all coefficients are non-negative. Define a directed graph, $\mathcal{G} = \mathcal{G}(\overline{\mathcal{L}})$ with vertex set $V(\mathcal{G}) = \{0, 1, \dots, n\}$ and edge set $E(\mathcal{G}) = \{(j, i) \mid a_{i,j} \neq 0\}$. Let $g_i = \frac{\mathcal{L}_i(\mathbf{x})}{x_i}$ for $1 \leq i \leq n$. Assume that in $\mathcal{G}(\overline{\mathcal{L}})$ every node can be reached from 0. Then*

$$g_1(\mathbf{x}) = g_2(\mathbf{x}) = \dots = g_n(\mathbf{x}) \quad (2.1)$$

has an all-positive solution.

Definition 2.3.12 *We define $r(i) = \text{dist}(0, i)$ in the above graph $\mathcal{G}(\overline{\mathcal{L}})$ and for a cycle C if the variable was x_i for its distinguished cycle-edge, we would denote $r(C)$ to mean $r(i)$.*

Theorem 2.3.13 *If Conditions I and II hold then we can use Theorem 2.3.11 to prove*

that every connected graph G is implementable with four directions.

Proof. Condition I ensures that the slope associated with the distinguished cycle-edge of each cycle i can be expressed as $g_i(\mathbf{x})$ (as we have seen). Condition II is sufficient for the reachability condition (for \mathcal{G}) of Theorem 2.3.11. We will in fact show that $r(C) \leq 2$ for every cycle C . The linear expression for cycle C has a non-zero constant term iff $C \setminus D(C) \neq \emptyset$. Consider a fixed component. By Condition II all cycles, except perhaps one, have associated linear expressions with non-zero constant terms, therefore they have $r = 1$.

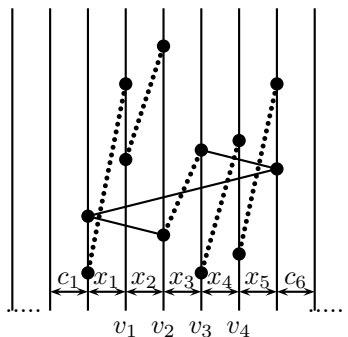


Figure 2.12: Here the dotted edges represent a set of adjacent distinguished cycle-edges. $r(C) \neq 1$ if all edges of the cycle span over these adjacent distinguished cycle-edges. But all v_i 's in the figure have both vertices of the matching-edges used up by cycles. So C could at best be a 4 cycle (since the graph is triangle-free) using up the first and the last vertical lines of this contiguous block and one distinguished cycle-edge.

It is sufficient to show that the single cycle C for which $C \subseteq D(C)$, if exists, has $r(C) = 2$. Indeed, let y_1, \dots, y_l be the distinguished matching-edges belonging to this component in this linear order, and let y_p and y_{p+1} be the distinguished matching-edges that belong to cycle C . Since C is at least a four cycle, it either contains some other $y_{p'} \notin \{y_p, y_{p+1}\}$, in which case indeed, it is geometrically easy to see that one of the other variables from the component has to occur in \mathcal{L}_C or C is a four cycle and both y_p and y_{p+1} occur with multiplicity two in it. In the latter case C would form a separate K_4 component, thus $G = K_4$. In the former case the variable has $r = 1$, so $r(C) = 2$.

□

We are left with proving that we can pick distinguished cycle-edges from the cycles such that Conditions I and II are satisfied. Indeed, start from any cycle, and pick an edge for a distinguished cycle-edge, which has at least one adjacent matching-edge y that is common with a different cycle. If there is none, the cycle is the single (Hamiltonian) cycle, and if we distinguish any edge, Conditions I and II are clearly satisfied. Otherwise, in the cycle that contains y , pick one of the two edges adjacent to y , look at the other adjacent matching-edge, y' , of this edge, look for another cycle that is adjacent with y' , etc. The process ends when we get back to any cycle (including the current one) that has already been visited. There is one reason for back-track and this is when we return to the other adjacent matching-edge, z , of the starting edge. In this case we choose the other edge (recall we always have two choices). It would be fatal to get back to z , since then Condition I would not hold.

Assume that the above procedure has gone through. Then we have distinguished at most three matching-edges adjacent to any cycle. But this is not all. We have to do the same procedure from z as well. The procedure terminates when we encounter a cycle that has already been encountered. Thus in the final step we might create a fourth distinguished matching-edge adjacent to one of the cycles, but only in one of them. This can be the single cycle C in the component for which $C \subseteq D(C)$. And because the graph is triangle-free, all the other components would have $C \setminus D(C) \neq \emptyset$.

Once we are done with creating the first component, we select a cycle not involved in it, and start the same procedure as before with the only difference that in subsequent rounds we also stop if we encounter a cycle visited in one of the previous rounds. It is easy to see, that now for the distinguished cycle-edges that we have selected Conditions I and II hold.

2.3.3 Solvability

Before we prove Theorem 2.3.11, we will look at the following special case when all the constant terms in \mathcal{L}_i are positive.

Theorem 2.3.14 *Let $B_1, \dots, B_n > 0$ be positive constants, $\mathcal{L}_i(\mathbf{x}) = \sum_{j=1}^n a_{i,j} x_j$ for*

$1 \leq i \leq n$ be linear forms. Let $g_i = \frac{B_i + \mathcal{L}_i(\mathbf{x})}{x_i}$ for $1 \leq i \leq n$. Then

$$g_1(\mathbf{x}) = g_2(\mathbf{x}) = \cdots = g_n(\mathbf{x}) \quad (2.2)$$

has an all-positive solution.

Proof. The intuition behind the proof is this: Let ϵ be very small and $\alpha_1, \dots, \alpha_n > 0$ be fixed. If we set $x_i = \epsilon B_i \alpha_i^{-1}$ then $g_i(\mathbf{x}) \approx \epsilon^{-1} \alpha_i$. In particular, let $\boldsymbol{\alpha}$ range in the $[1, 2]^n$ solid cube. Then, if ϵ is small enough, the vector $(g_1(\mathbf{x}), \dots, g_n(\mathbf{x}))$ will range roughly in the $[\epsilon^{-1}, 2\epsilon^{-1}]^n$ cube, thus $\epsilon^{-1}(1.5, \dots, 1.5)$, which is the center of this cube, has to be in the image.

To make this proof idea precise we will use the following version of Brouwer's well known fix point theorem:

Theorem 2.3.15 (Brouwer) *Let $f : [1, 2]^n \rightarrow [1, 2]^n$ be a continuous function. Then f has a fix point, i.e. an $\mathbf{x}_0 \in [1, 2]^n$ for which $f(\mathbf{x}_0) = \mathbf{x}_0$.*

We will use the fix point theorem as below. We first define

$$h(\alpha_1, \dots, \alpha_n) = (\epsilon g_1(\mathbf{x}), \dots, \epsilon g_n(\mathbf{x})),$$

where $\mathbf{x} = \epsilon(\alpha_1^{-1} B_1, \dots, \alpha_n^{-1} B_n) = \epsilon \mathbf{x}'$, and we think of ϵ as some fixed positive number. Notice that \mathbf{x}' is just a function of $\boldsymbol{\alpha}$, independent of ϵ . It is sufficient to show that if ϵ is small enough, there are $\alpha_1, \dots, \alpha_n$ such that $h(\boldsymbol{\alpha}) = (1.5, \dots, 1.5)$, since then \mathbf{x} satisfies (2.2) with common value $1.5\epsilon^{-1}$. We have:

$$\epsilon g_i(\mathbf{x}) = \epsilon \frac{B_i + \mathcal{L}_i(\mathbf{x})}{\epsilon \alpha_i^{-1} B_i} = \alpha_i (1 + \epsilon B_i^{-1} \mathcal{L}_i(\mathbf{x}')).$$

Here we used that $\mathcal{L}_i(\epsilon \mathbf{x}') = \epsilon \mathcal{L}_i(\mathbf{x}')$. We would like to have

$$\alpha_i (1 + \epsilon B_i^{-1} \mathcal{L}_i(\mathbf{x}')) = 1.5 \quad \text{for } 1 \leq i \leq n. \quad (2.3)$$

Define

$$\begin{aligned} K &= \max_i \sup_{\alpha \in [1,2]^n} B_i^{-1} \mathcal{L}_i(\mathbf{x}'); \\ \epsilon &= 1/(10K). \end{aligned}$$

To use the fix point theorem we consider the map

$$f : (\alpha_1, \dots, \alpha_n) \rightarrow \left(\frac{1.5}{1 + \epsilon B_1^{-1} \mathcal{L}_1(\mathbf{x}')} , \dots , \frac{1.5}{1 + \epsilon B_n^{-1} \mathcal{L}_n(\mathbf{x}')} \right)$$

on the cube $[1, 2]^n$. The image is contained in $[1, 2]^n$, since if $\alpha \in [1, 2]^n$ then for $1 \leq i \leq n$ we have

$$1 < \frac{1.5}{1 + 0.1} = \frac{1.5}{1 + \epsilon K} \leq \frac{1.5}{1 + \epsilon B_i^{-1} \mathcal{L}_i(\mathbf{x}')} \leq \frac{1.5}{1 - \epsilon K} = \frac{1.5}{1 - 0.1} < 2.$$

Therefore, by Theorem 2.3.15 there is an $\alpha \in [1, 2]^n$ such that $\alpha_i = \frac{1.5}{1 + \epsilon B_i^{-1} \mathcal{L}_i(\mathbf{x}')}$ for $1 \leq i \leq n$, which is equivalent to (2.3). \square

In Theorem 2.3.14 all linear forms have non-zero constant terms. We can, however generalize this to Theorem 2.3.11. We discuss its proof below.

Remark 2.3.16 *The non-negativity of the coefficients can be relaxed such that the theorem becomes a true generalization of Theorem 2.3.14. Since the more general condition is slightly technical, we will stay with the simpler non-negativity condition, which is sufficient for us.*

Proof. For $1 \leq i \leq n$ let $r(i) = \text{dist}(0, i)$ in $\mathcal{G}(\overline{\mathcal{L}})$. (In Theorem 2.3.14 each $r(i)$ was 1.) Define

$$x_i = \epsilon^{r(i)} x'_i,$$

where $\epsilon > 0$ will be a small enough number that we will appropriately fix later, but as of now we think about it as a quantity tending to zero. We can rewrite (2.2) as:

$$\epsilon g_1(\mathbf{x}) = \epsilon g_2(\mathbf{x}) = \dots = \epsilon g_n(\mathbf{x}).$$

If we fix \mathbf{x}' and take epsilon tending to zero, then,

$$\epsilon g_i(\mathbf{x}) \rightarrow \frac{\beta_i(\mathbf{x}')}{x'_i},$$

where $\beta_i(\mathbf{x}') = a_{i,0}/x'_i$ if $r(i) = 1$, otherwise

$$\beta_i(\mathbf{x}') = \sum_{j: r(j)=r(i)-1} a_{i,j} x'_j.$$

We can now solve the system

$$\frac{\beta_i(\mathbf{x}')}{x'_i} = 1.5$$

and even the system

$$\frac{\beta_i(\mathbf{x}')}{x'_i} = \alpha_i, \tag{2.4}$$

where $1 \leq \alpha_i \leq 2$ for $1 \leq i \leq n$. Indeed, the solution can be obtained iteratively, by first computing the values of the variables x_i with $r(i) = 0$, then with $r(i) = 1$, etc. We can again use the fix point theorem of Brouwer to show that if ϵ is sufficiently small, the system

$$\epsilon g_i(\mathbf{x}) = 1.5 \quad \text{for } 1 \leq i \leq n$$

has a solution. For this we again parameterize \mathbf{x}' with $\boldsymbol{\alpha}$. When $\boldsymbol{\alpha}$ ranges in the solid cube $[1, 2]^n$ then \mathbf{x}' will range in some domain D , where we obtain D by solving the system (2.4) for all $\alpha_i \in [1, 2]^n$. Now we have to set ϵ small enough such that everywhere in D it should hold that

$$0.9 \leq \frac{\beta_i(\mathbf{x}')/x'_i}{\epsilon g_i(\mathbf{x})} = \frac{\alpha_i}{\epsilon g_i(\mathbf{x})} \leq 1.1 \quad \text{for } 1 \leq i \leq n. \tag{2.5}$$

This is easily seen to be possible, since D is contained in a closed cube in the strictly positive orthant. We then apply the fix point theorem to

$$f : \boldsymbol{\alpha} \rightarrow \boldsymbol{\gamma},$$

where

$$\gamma_i = \frac{1.5\alpha_i}{\epsilon g_i(\mathbf{x})}.$$

The fix point theorem applies, since the range of f remains in the $[0.9 \cdot 1.5, 1.1 \cdot 1.5]^n \subset [1, 2]^n$ cube by Equation (2.5). For the fixed point $\alpha_i = \frac{1.5\alpha_i}{\epsilon g_i(\mathbf{x})}$ for $1 \leq i \leq n$, which implies $\epsilon g_i(\mathbf{x}) = 1.5$ for $1 \leq i \leq n$. \square

2.4 Proof of Theorem 2.1.4

We start with some definitions we will use throughout this section.

2.4.1 Definitions

Throughout this section \log always denotes \log_2 , the logarithm in base 2.

We recall that the girth of a graph is the length of its shortest cycle.

Definition 2.4.1 *Define a supercycle as a connected graph where every degree is at least two and not all are two. Note that a minimal supercycle will look like a “ θ ” or like a “dumbbell”.*

We recall that a *cut* is a partition of the vertices into two sets. We say that an edge is in the cut if its ends are in different subsets of the partition. We also call the edges in the cut the *cut-edges*. The *size* of a cut is the number of cut-edges in it.

Definition 2.4.2 *We say that a cut is an M -cut if the cut-edges form a matching, in other words, if their ends are pairwise different vertices. We also say that an M -cut is suitable if after deleting the cut-edges, the graph has two components, both of which are supercycles.*

We refer the reader to Section 2.1 for the exact statement of Theorem 2.1.1 [44] about subcubic graphs.

Note that Theorem 2.1.1 proves the result of Theorem 2.1.4 for subcubic graphs. Another minor observation is that we may assume that the graph is connected. Since

we use the basic four slopes, if we can draw the components of a disconnected graph, then we just place them far apart in the plane so that no two drawings intersect. So we will assume for the rest of the section that the graph is cubic and connected.

2.4.2 Preliminaries

The results in this subsection are also interesting independent of the current problem we deal with. The following is also called the Moore bound.

Lemma 2.4.3 *Every connected cubic graph on n vertices contains a cycle of length at most $2\lceil\log(\frac{n}{3} + 1)\rceil$.*

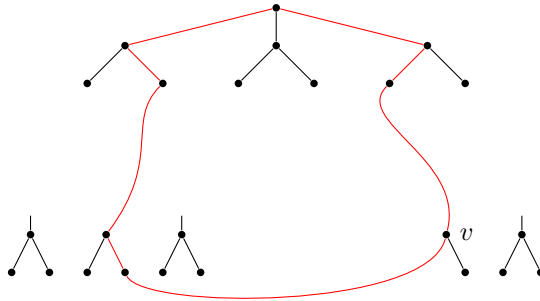


Figure 2.13: Finding a cycle in the BFS tree using that the left child of v already occurred.

Proof. Start at any vertex of G and conduct a breadth first search (BFS) of G until a vertex repeats in the BFS tree. We note here that by iterations we will (for the rest of the subsection) mean the number of levels of the BFS tree. Since G is cubic, after k iterations, the number of vertices visited will be $1+3+6+12+\dots+3\cdot 2^{k-2} = 1+3(2^{k-1}-1)$. And since G has n vertices, some vertex must repeat after $k = \lceil\log(\frac{n}{3} + 1)\rceil + 1$ iterations. Tracing back along the two paths obtained for the vertex that reoccurs, we find a cycle of length at most $2\lceil\log(\frac{n}{3} + 1)\rceil$. \square

Lemma 2.4.4 *Every connected cubic graph on n vertices with girth g contains a supercycle with at most $2\lceil\log(\frac{n+1}{g})\rceil + g - 1$ vertices.*

Proof. Contract the vertices of a length g cycle, obtaining a multigraph G' with $n - g + 1$ vertices, that is almost 3-regular, except for one vertex of degree g , from which we start a BFS. It is easy to see that the number of vertices visited after k iterations is at most $1 + g + 2g + 4g + \dots + g \cdot 2^{k-2} = g(2^{k-1} - 1) + 1$. And since G' has $n - g + 1$ vertices, some vertex must repeat after $k = \lceil \log(\frac{n-g+1}{g} + 1) \rceil + 1 = \lceil \log(\frac{n+1}{g}) \rceil + 1$ iterations. Tracing back along the two paths obtained for the vertex that reoccurs, we find a cycle (or two vertices connected by two edges) of length at most $2\lceil \log(\frac{n+1}{g}) \rceil$ in G' . This implies that in G we have a supercycle with at most $2\lceil \log(\frac{n+1}{g}) \rceil + g - 1$ vertices. \square

Lemma 2.4.5 *Every connected cubic graph on $n > 2s - 2$ vertices with a supercycle with s vertices contains a suitable M -cut of size at most $s - 2$.*

Proof. The supercycle with s vertices, A , has at least two vertices of degree 3. The size of the $(A, G - A)$ cut is thus at most $s - 2$. This cut need not be an M -cut because the edges may have a common neighbor in $G - A$. To repair this, we will now add, iteratively, the common neighbors of edges in the cut to A , until no edges have a common neighbor in $G - A$. Note that in any iteration, if a vertex, v , adjacent to exactly two cut-edges was chosen, then the size of A increases by 1 and the size of the cut decreases by 1 (since, these two cut-edges will get added to A along with v , but since the graph is cubic, the third edge from v will become a part of the cut-edges). If a vertex adjacent to three cut-edges was chosen, then the size of A increases by 1 while the number of cut-edges decreases by 3. From this we can see that the maximum number of vertices that could have been added to A during this process is $s - 3$. Now there are three conditions to check.

The first condition is that this process returns a non-empty second component. This would occur if

$$(n - s) - (s - 3) > 0$$

or,

$$n > 2s - 3.$$

The second condition is that the second component should not be a collection of disjoint cycles. For this we note that it is enough to check that at every stage, the number of cut-edges is strictly smaller than the number of vertices in $G - A$. But since in the above iterations, the number of cut-edges decreases by a number greater than or equal to the decrease in the size of $G - A$, it is enough to check that before the iterations, the number of cut-edges is strictly smaller than the number of vertices in $G - A$. This is the condition

$$n - s > s - 2$$

or,

$$n > 2s - 2.$$

Note that if this inequality holds then the non-emptiness condition will also hold.

Finally, we need to check that both components are connected. A is connected but $G - A$ need not be. We pick a component in $G - A$ that has more vertices than the number of cut-edges adjacent to it. Since the number of cut-edges is strictly smaller than the number of vertices in $G - A$, there must be one such component, say B , in $G - A$. We add every other component of $G - A$ to A . Note that the size of the cut only decreases with this step. Since B is connected and has more vertices than the number of cut-edges, B cannot be a cycle. \square

Corollary 2.4.6 *Every connected cubic graph on $n \geq 18$ vertices contains a suitable M -cut.*

Proof. Using the first two lemmas, we have a supercycle with $s \leq 2\lceil \log(\frac{n+1}{g}) \rceil + g - 1$ vertices where $3 \leq g \leq 2\lceil \log(\frac{n}{3} + 1) \rceil$. Then using the last lemma, we have an M -cut with both partitions being a supercycle if $n > 2s - 2$. So all we need to check is that n is indeed big enough. Note that

$$\begin{aligned} s &\leq 2\log\left(\frac{n+1}{g}\right) + g + 1 = 2\log(n+1) + g + 1 - 2\log g \leq \\ &\leq 2\log(n+1) + 2\log\left(\frac{n}{3} + 1\right) - 2\log(2\log\left(\frac{n}{3} + 1\right)) + 1 \end{aligned}$$

where the last inequality follows from the fact that $x - 2 \log x$ is increasing for $x \geq 2/\log_e 2 \approx 2.88$. So we can bound the right hand side from above by $4 \log(n+1)+1$. Now we need that

$$n > 2(4 \log(n+1) + 1) - 2 = 8 \log(n+1)$$

which holds if $n \geq 44$.

The statement can be checked for $18 \leq n \leq 42$ with code that can be found in the Appendix. It outputs for a given value of n , the g for which $2s - 2$ is maximum and this maximum value. Based on the output we can see that for $n \geq 18$, this value is smaller. \square

2.4.3 Proof

Lemma 2.4.7 *Let G be a connected cubic graph with a suitable M -cut. Then, G can be drawn with the four basic slopes.*

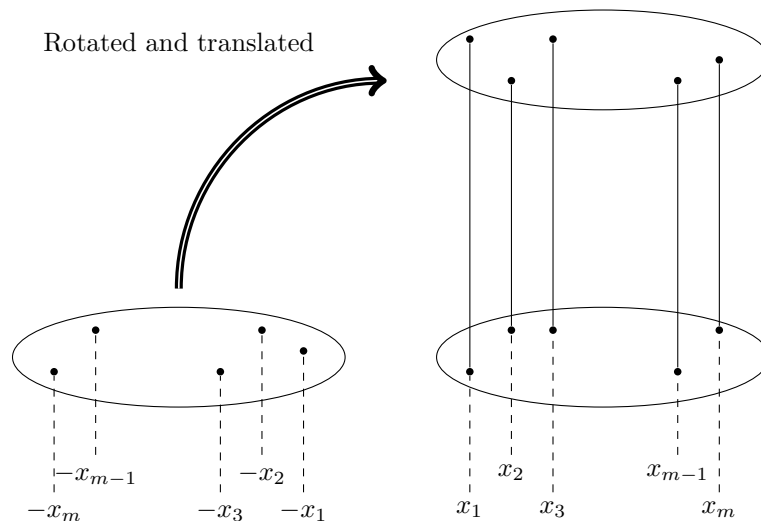


Figure 2.14: The x -coordinates of the degree 2 vertices is suitably chosen and one component is rotated and translated to make the M -cut vertical.

Proof. The proof follows rather straightforwardly from Theorem 2.1.1. Note that the two components are subcubic graphs and we can choose the x -coordinates of the vertices of the M -cut (since they are the vertices with degree two in the components). If we picked coordinates x_1, x_2, \dots, x_m in one component, then for the neighbors of these vertices in the other component we pick the x -coordinates $-x_1, -x_2, \dots, -x_m$. We now rotate the second component by π and place it very high above the other component so that the drawings of the components do not intersect and align them so that the edges of the M -cut will be vertical (slope $\pi/2$). Also, since Theorem 2.1.1 guarantees that degree two vertices have no other vertices on the vertical line above them, hence the drawing we obtain above is a valid representation of G with the basic slopes. \square

By combining Lemma 2.4.6 and Lemma 2.4.7, we can see that Theorem 2.1.4 is true for all cubic graphs with $n \geq 18$. For smaller graphs, we reduce the number of graphs we have to check with the help of Lemma 2.3.2 and Remark 2.3.3 as a consequence of which, a graph that cannot be drawn with the four basic slopes must be three vertex and edge connected.

We also employ the following theorem by Max Engelstein [29].

Lemma 2.4.8 *Every 3-connected cubic graph with a Hamiltonian cycle can be drawn in the plane with the four basic slopes.*

Note that combining this with Lemma 2.3.2 we even get

Corollary 2.4.9 *Every cubic graph with a Hamiltonian cycle can be drawn in the plane with the four basic slopes.*

The graphs which now need to be checked satisfy the following conditions:

1. the number of vertices is at most 16
2. the graph is 3-connected
3. the graph does not have a Hamiltonian cycle.

Remark 2.4.10 *We now bring the attention of our reader to Remark 2.3.5 to add that we may also add to the above list that the graph does not contain a triangle. However,*

we use our girth lemmas to have an easy way to analyze the graphs excluded by only the above three assertions.

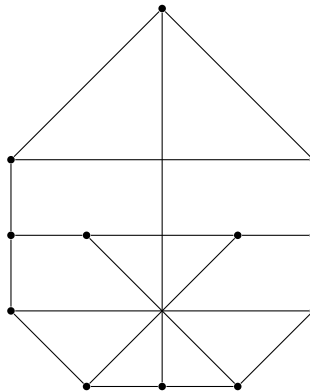


Figure 2.15: The Tietze's graph drawn with the four basic slopes.

Note that if the number of vertices is at most 16, then it follows from Lemma 2.4.3 that the girth is at most 6. Luckily there are several lists available of cubic graphs with a given number of vertices, n and a given girth, g .

If $g = 6$, then there are only two graphs with at most 16 vertices (see [1, 52]), both containing a Hamiltonian cycle.

If $g = 5$ and $n = 16$, then Lemma 2.4.4 gives a supercycle with at most 8 vertices, so using Lemma 2.4.5 we are done.

If $g = 5$ and $n = 14$, then there are only nine graphs (see [1, 52]), all containing a Hamiltonian cycle.

If $g \leq 4$ and $n = 16$, then Lemma 2.4.4 gives a supercycle with at most 8 vertices, so using Lemma 2.4.5 we are done.

If $g \leq 4$ and $n = 14$, then Lemma 2.4.4 gives a supercycle with at most 7 vertices, so using Lemma 2.4.5 we are done.

Finally, all graphs with at most 12 vertices are either not 3-connected or contain a Hamiltonian cycle, except for the Petersen graph and Tietze's Graph (see [2]). For the drawing of these two graphs, see the respective Figures.

2.5 Which four slopes? and other concluding questions

After establishing Theorem 2.1.4 the question arises whether we could have used any other four slopes. Call a set of slopes *good* if every cubic graph has a straight-line drawing with them. In this section we prove Theorem 2.1.5 that claims that the following statements are equivalent for a set S of four slopes.

1. S is good.
2. S is an affine image of the four basic slopes.
3. We can draw K_4 with S .

Proof. Since affine transformation keeps incidences, any set that is the affine image of the four basic slopes is good.

On the other hand, if a set $S = \{s_1, s_2, s_3, s_4\}$ is good, then K_4 has a straight-line drawing with S . Since we do not allow a vertex to be in the interior of an edge, the four vertices must be in general position. This implies that two incident edges cannot have the same slope. Therefore there are two slopes, without loss of generality s_1 and s_2 , such that we have two edges of each slope. These four edges must form a cycle of length four, which means that the vertices are the vertices of a parallelogram. But in this case there is an affine transformation that takes the parallelogram to a square. This transformation also takes S into the four basic slopes. \square

Note that a similar reasoning shows that no matter how many slopes we take, their set need not be good, because we cannot even draw K_4 with them unless they satisfy some correlation. The above proofs use the four basic slopes only in a few places (for rotation invariance and to start induction). Thus, we make the following conjecture.

Conjecture 2.5.1 *There is a (not necessarily connected, finite) graph such that a set of slopes is good if and only if this graph has a straight-line drawing with them.*

This finite graph would be the disjoint union of K_4 , maybe the Petersen graph and other small graphs. We could not even rule out the possibility that K_4 (or maybe

another, connected graph) is alone sufficient. Note that we can define a partial order on the graphs this way. Let $G < H$ if any set of slopes that can be used to draw H can also be used to draw G . This way of course $G \subset H \Rightarrow G < H$ but what else can we say about this poset?

Is it possible to use this new method to prove that the slope parameter of cubic graphs is also four?

The main question remains to prove or disprove whether the slope number of graphs with maximum degree four is unbounded.

Chapter 3

Obstacle number

3.1 Introduction

Consider a set P of points in the plane and a set of closed polygonal obstacles whose vertices together with the points in P are in *general position*, that is, no *three* of them are on a line. The corresponding *visibility graph* has P as its vertex set, two points $p, q \in P$ being connected by an edge if and only if the segment pq does not meet any of the obstacles. Visibility graphs are extensively studied and used in computational geometry, robot motion planning, computer vision, sensor networks, etc.; see [20], [35], [59], [60], [68].

Recently, Alpert, Koch, and Laison [8] introduced an interesting new parameter of graphs, closely related to visibility graphs. Given a graph G , we say that a set of points and a set of polygonal obstacles as above constitute an *obstacle representation* of G , if the corresponding visibility graph is isomorphic to G . A representation with h obstacles is also called an h -obstacle representation. The smallest number of obstacles in an obstacle representation of G is called the *obstacle number* of G and is denoted by $\text{obs}(G)$. If we are allowed to use only *convex* obstacles, then the corresponding parameter $\text{obs}_c(G)$ is called the *convex obstacle number* of G . Of course, we have $\text{obs}(G) \leq \text{obs}_c(G)$ for every G , but the two parameters can be very far apart.

A special instance of the obstacle problem has received a lot of attention, due to its connection to the Szemerédi-Trotter theorem on incidences between points and lines [66], [65], and other classical problems in incidence geometry [62]. It is an exciting open problem to decide whether the obstacle number of \overline{K}_n , the empty graph on n vertices, is $O(n)$ if the obstacles must be *points*. The best known upper bound is $n2^{O(\sqrt{\log n})}$; see Pach [61], Dumitrescu et al. [27], Matoušek [50], and Aloupis et al. [7].

Alpert et al. [8] constructed a bipartite graph and a split graph (a graph whose vertex set is the union of a complete graph and an independent set), both with a fixed number of vertices, with obstacle number at least *two*. In [54] another graph, whose vertex set is the union of two complete subgraphs, was shown to have obstacle number at least two. Consequently, no graph of obstacle number *one* can contain a subgraph isomorphic to these graphs. Using this and some extremal graph theoretic tools developed by Erdős, Kleitman, Rothschild, Frankl, Rödl, Prömel, Steger, Bollobás, Thomason and others, the following was proved.

Theorem 3.1.1 ([54]) *For any fixed positive integer h , the number of graphs on n (labeled) vertices with obstacle number at most h is at most $2^{o(n^2)}$.*

Since the number of bipartite graphs with n labeled vertices is $\Omega(2^{n^2/4})$, this also implies that there exist bipartite graphs with arbitrarily large obstacle number.

For every sufficiently large n , Alpert et al. constructed a graph with n vertices with obstacle number at least $\Omega(\sqrt{\log n})$. By using the existence of graphs with obstacle number at least 2 and a result by Erdős and Hajnal [31], we show the existence of graphs with much larger obstacle numbers.

Theorem 3.1.2 *For every $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon)$ such that for all $n \geq n_0$, there are graphs G on n vertices such that their obstacle numbers satisfy*

$$\text{obs}(G) \geq \Omega(n^{1-\varepsilon}).$$

In Section 3.3, we improve on the last two corollaries, using some estimates on the number of different *order types* of n points in the Euclidean plane, discovered by Goodman and Pollack [36], [37] (see also Alon [5]). We establish the following results.

Theorem 3.1.3 *For any fixed positive integer h , the number of graphs on n (labeled) vertices with obstacle number at most h is at most*

$$2^{O(hn \log^2 n)}.$$

Theorem 3.1.4 *For every n , there exist graphs G on n vertices with obstacle numbers*

$$\text{obs}(G) \geq \Omega(n/\log^2 n).$$

Note that the last statement directly follows from Theorem 3.1.3. Indeed, since the total number of (labeled) graphs with n vertices is $2^{\Omega(n^2)}$, as long as $2^{O(hn \log^2 n)}$ is smaller than this quantity, there is a graph with obstacle number larger than h .

We prove a slightly better bound for convex obstacle numbers.

Theorem 3.1.5 *For every n , there exist graphs G on n vertices with convex obstacle numbers*

$$\text{obs}_c(G) \geq \Omega(n/\log n).$$

If we only allow segment obstacles, we get an even better bound. Following Alpert et al., we define the *segment obstacle number* $\text{obs}_s(G)$ of a graph G as the minimal number of obstacles in an obstacle representation of G , in which each obstacle is a straight-line segment.

Theorem 3.1.6 *For every n , there exist graphs G on n vertices with segment obstacle numbers*

$$\text{obs}_s(G) \geq \Omega(n^2/\log n).$$

We then improve the bound for the general obstacle number as follows.

Theorem 3.1.7 *For every n , there exists a graph G on n vertices with obstacle number*

$$\text{obs}(G) \geq \Omega(n/\log n).$$

This comes close to answering the question in [8] whether there exist graphs with n vertices and obstacle number at least n . However, we have no upper bound on the maximum obstacle number of n -vertex graphs, better than $O(n^2)$.

Given any placement (embedding) of the vertices of G in general position in the plane, a *drawing* of G consists of the image of the embedding and the set of *open*

segments connecting all pairs of points that correspond to the edges of G . If there is no danger of confusion, we make no notational difference between the vertices of G and the corresponding points, and between the pairs uv and the corresponding open segments. The complement of the set of all points that correspond to a vertex or belong to at least one edge of G falls into connected components. These components are called the *faces* of the drawing. Notice that if G has an obstacle representation with a particular placement of its vertex set, then

- (1) each obstacle must lie entirely in one face of the drawing, and
- (2) each non-edge of G must be blocked by at least one of the obstacles.

3.2 Extremal methods and proof of Theorem 3.1.2

In order to prove Theorem 3.1.2, we need the following result, which shows that if G avoids at least one induced subgraph with k vertices, for some $k \ll \log n$, then the Erdős-Szekeres bound on $\text{hom}(G)$ can be substantially improved. We note that $\text{hom}(G)$ for a graph G is defined as the size of the largest clique or independent set in the graph. Also, a graph is k -universal if it contains every graph on k vertices as induced subgraph.

Theorem 3.2.1 (Erdős, Hajnal [31]) *For any fixed positive integer t , there is an $n_0 = n_0(t)$ with the following property. Given any graph G on $n > n_0$ vertices and any integer $k < 2^{c\sqrt{\log n}/t}$, either G is t -universal or we have $\text{hom}(G) \geq k$. (Here $c > 0$ is a suitable constant.)*

We now prove Theorem 3.1.2.

Proof. For the sake of clarity of the presentation, we systematically omit all floor and ceiling functions wherever they are not essential. Let H be a graph of t vertices that does not admit a 1-obstacle representation. Fix any $0 < \varepsilon < 1$, and choose an integer $N \geq n_0$, that satisfies the inequality

$$2^{c\sqrt{\varepsilon \log N}/t} > 2 \log N, \tag{3.1}$$

where c, n_0 are constants that appear in the previous theorem.

For any $n \geq N$, we set $m = n^{1-\varepsilon}$. According to a theorem of Erdős [32], there exists a graph G with n vertices such that

$$\text{hom}(G) < 2 \log n < 2^c \sqrt{\log(n/m)}/t.$$

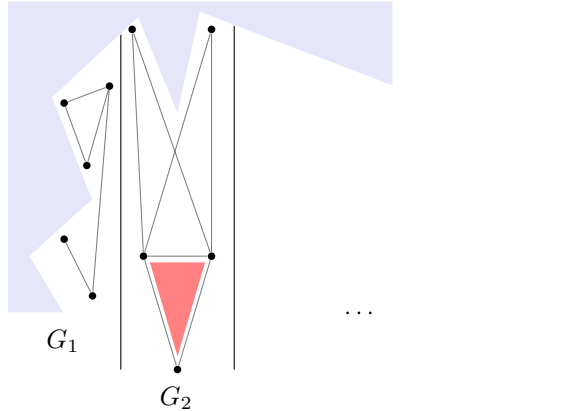


Figure 3.1: Division of the graph into m parts each with n/m points. The light grey obstacle is a common exterior obstacle, while the darker one is an internal obstacle of G_2 .

Consider an obstacle representation of G with the smallest number h of obstacles. Suppose without loss of generality that in our coordinate system all points of G have different x -coordinates. By vertical lines, partition the plane into m strips, each containing n/m points. Let G_i denote the subgraph of G induced by the vertices lying in the i -th strip ($1 \leq i \leq m$).

Obviously, we have

$$\text{hom}(G_i) \leq \text{hom}(G) < 2^c \sqrt{\log(n/m)}/t,$$

for every i . Hence, applying Theorem 3.2.1 to each G_i separately, we conclude that each must be t -universal. In particular, each G_i contains an induced subgraph isomorphic to H . That is, we have $\text{obs}(G_i) > 1$ for every i , which means that each G_i requires at least *two* obstacles.

As was explained at the end of the Introduction, each obstacle must be contained in an interior or in the exterior face of the graph. Therefore, in an h -obstacle representation of G , each G_i must have at least one internal face that contains an obstacle, and there must be at least one additional obstacle (which may possibly be contained in the interior face of every G_i). At any rate, we have $h > m = n^{1-\varepsilon}$, as required.

□

3.3 Encoding graphs of low obstacle number

The aim of this section is to prove Theorems 3.1.3–3.1.6. The idea is to find a short encoding of the obstacle representations of graphs, and to use this to give an upper bound on the number of graphs with low obstacle number.

We need to review some simple facts from combinatorial geometry. Two sets of points, P_1 and P_2 , in general position in the plane are said to have the same *order type* if there is a one to one correspondence between them with the property that the orientation of any triple in P_1 is the same as the orientation of the corresponding triple in P_2 . Counting the number of different order types is a classical task, see e.g.

Theorem 3.3.1 (Goodman, Pollack [36]) *The number of different order types of n points in general position in the plane is $2^{O(n \log n)}$.*

Observe that the same upper bound holds for the number of different order types of n labeled points, because the number of different permutations of n points is $n! = 2^{O(n \log n)}$.

In a graph drawing, the *complexity* of a face is the number of line segment sides bordering it. The following result was proved by Arkin, Halperin, Kedem, Mitchell, and Naor (see Matoušek, Valtr [51] for its sharpness).

Theorem 3.3.2 (Arkin et al. [13]) *The complexity of a single face in a drawing of a graph with n vertices is at most $O(n \log n)$.*

Note that this bound does not depend of the number of edges of the graph. We are now ready to prove Theorem 3.1.3.

Proof. For any graph G with n vertices that admits an h -obstacle representation, fix such a representation. Consider the visibility graph G of the vertices in this representation. As explained at the end of the Introduction, any obstacle belongs to a single face in this drawing. In view of Theorem 3.3.2, the complexity of every face is $O(n \log n)$. Replacing each obstacle by a slightly shrunken copy of the face containing it, we can achieve that every obstacle *is* a polygonal region with $O(n \log n)$ sides.

Notice that the order type of the sequence S starting with the vertices of G , followed by the vertices of the obstacles (listed one by one, in cyclic order, and properly separated from one another) completely determines G . That is, we have a sequence of length N with $N \leq n + c_1 h n \log n$. According to Theorem 3.3.1 (and the following comment), the number of different order types with this many points is at most

$$2^{O(N \log N)} < 2^{c h n \log^2 n},$$

for a suitable constant $c > 0$. This is a very generous upper bound: most of the above sequences do not correspond to any visibility graph G . \square

If in the above proof the average number of sides an obstacle can have is small, then we obtain

Theorem 3.3.3 *The number of graphs admitting an obstacle representation with at most h obstacles, having a total of at most hs sides, is at most*

$$2^{O(n \log n + hs \log(hs))}.$$

In particular, for segment obstacles ($s = 2$), Theorem 3.3.3 immediately implies Theorem 3.1.6. Indeed, as long as the bound in Theorem 3.3.3 is smaller than $2^{\binom{n}{2}}$, the total number of graphs on n labeled vertices, we can argue that there is a graph with segment obstacle number larger than h .

We now show how to prove Theorem 3.1.5 with an easier way to encode the drawing of a graph and its convex obstacles.

Proof. As before, it is enough to bound the number of graphs that admit an obstacle

representation with at most h convex obstacles. Let us fix such a graph G , together with a representation. Let V be the set of points representing the vertices, and let O_1, \dots, O_h be the convex obstacles. For any obstacle O_i , rotate an oriented tangent line ℓ along its boundary in the clockwise direction. We can assume without loss of generality that ℓ never passes through two points of V . Let us record the sequence of points met by ℓ . If $v \in V$ is met at the right side of ℓ , we add the symbol v_+ to the sequence, otherwise we add v_- (Figure 2.2). When ℓ returns to its initial position, we stop. The resulting sequence consists of $2n$ characters. From this sequence, it is easy to reconstruct which pairs of vertices are visible in the presence of the single obstacle O_i . Hence, knowing these sequences for every obstacle O_i , completely determines the visibility graph G . The number of distinct sequences assigned to a single obstacle is at most $(2n)!$, so that the number of graphs with convex obstacle number at most h cannot exceed $((2n)!)^h/h! < (2n)^{2hn}$. As long as this number is smaller than $2^{\binom{n}{2}}$, there is a graph with convex obstacle number larger than h . \square

3.4 Proof of Theorem 3.1.7

Here we prove Theorem 3.1.7 that claims that for every n , there exists a graph G on n vertices with obstacle number $\text{obs}(G) \geq \Omega(n/\log n)$.

Proof. The proof will be a counting argument. From Theorem 3.1.3 we know that the number of graphs on k (labeled) vertices with obstacle number at most one is at most $2^{o(k^2)}$. Now we will count the graphs with obstacle number less than $n/2k$. Suppose G has a representation with less than $n/2k$ obstacles. Fix one such representation. There are $n!$ possibilities for the order of the vertices of G from left to right (we can suppose that no two are below each other). We divide the vertices into n/k groups of size k , from left to right. Denote the respective induced graphs by G_i .

Claim 3.4.1 *At least half of the G_i 's require at most one obstacle.*

Proof. By contradiction, suppose that at least half of the G_i 's require at least two obstacles. One of each of these obstacles must be in an interior face of the respective

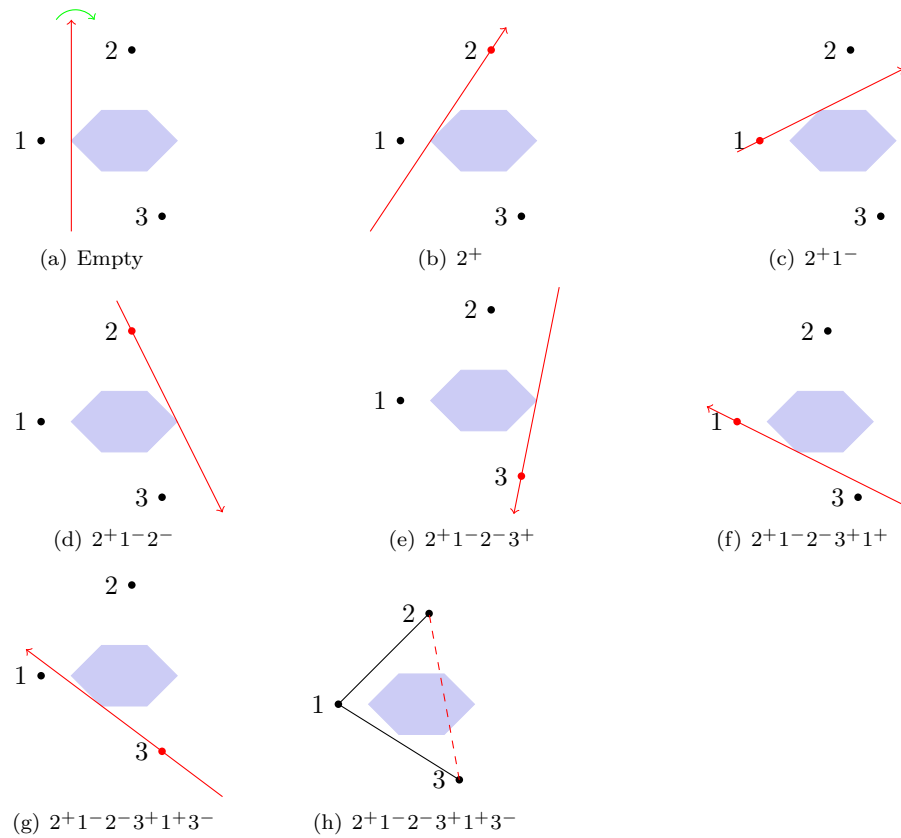


Figure 3.2: Parts (a) to (g) show the construction of the sequence and (h) shows the visibilities. The arrow on the tangent line indicates the direction from the point of tangency in which we assign $+$ as a label to the vertex. The additional arrow in (a) indicates that the tangent line is rotated clockwise around the obstacle.

G_i 's. Thus these obstacles are pair-wise separated by edges and must be different. This together would be at least $n/2k$ obstacles which contradicts the choice of G . \square

For the subset of G_i 's that require at most one obstacle there are less than $2^{n/k}$ possibilities. Since the number of graphs on k vertices whose obstacle number is at most one is $2^{o(k^2)}$, the probability that a G_i has a representation with at most one obstacle is $2^{o(k^2) - \binom{k}{2}}$. Therefore, the probability that a random graph G has obstacle number at most $n/2k$ is at most

$$n! \cdot 2^{n/k} \cdot (2^{o(k^2) - \binom{k}{2}})^{n/2k} = 2^{n \log n - \frac{kn}{4} + o(kn)}.$$

If $k = \Omega(5 \log n)$, this number tends to zero. Therefore some graphs need at least $\Omega(n/\log n)$ obstacles. \square

3.5 Further properties

In this section, we describe further properties of obstacle numbers. We start with another question from [8].

Theorem 3.5.1 *For every h , there exists a graph with obstacle number exactly h .*

Proof. Pick a graph G with obstacle number $h' > h$. (The existence of such a graph follows, e.g., from Corollary 3.1.1.) Let n denote the number of vertices of G . Consider a complete graph K_n on $V(G)$. Its obstacle number is *zero*, and G can be obtained from K_n by successively deleting edges. Observe that as we delete an edge from a graph G' , its obstacle number cannot increase by more than *one*. This follows from the fact that by blocking the deleted edge with an additional small obstacle that does not intersect any other edge of G' , we obtain a valid obstacle representation of the new graph. (Of course, the obstacle number of a graph can also *decrease* by the removal of an edge.) Since at the beginning of the process, K_n has obstacle number *zero*, at the end G has obstacle number $h' > h$, and whenever it increases, the increase is *one*, we can conclude that at some stage we obtain a graph with obstacle number precisely h . \square

The same argument applies to the convex obstacle number, to the segment obstacle number, and many similar parameters.

Let H be a fixed graph. According to a classical conjecture of Erdős and Hajnal [31], any graph with n vertices that does not have an induced subgraph isomorphic to H contains an independent set or a complete subgraph of size at least $n^{\varepsilon(H)}$, for some positive constant $\varepsilon(H)$. It follows that for any hereditary graph property there exists a constant $\varepsilon > 0$ such that every graph G on n vertices with this property satisfies $\text{hom}(G) \geq n^\varepsilon$.

Here we show that the last statement holds for the property that the graph has bounded obstacle number.

Theorem 3.5.2 *For any fixed integer $h > 0$, every graph on n vertices with $\text{obs}_c(G) \leq h$ satisfies $\text{hom}(G) \geq \frac{1}{2}n^{\frac{1}{h+1}}$.*

Proof. We proceed by induction on h . For $h = 1$, Alpert et al. [8] showed that all graphs with convex obstacle number *one* are so-called "circular interval graphs" (intersection graphs of a collection of arcs along the circle). It is known that all such graphs G whose maximum complete subgraph is of size x has an independent set of size at least $\frac{n}{2x}$; see [67]. Setting $x = \sqrt{n/2}$, it follows that $\text{hom}(G) \geq \frac{1}{2}\sqrt{n}$.

Let $h > 1$, and assume that the statement has already been verified for all graphs with convex obstacle number smaller than h . Let G be a graph that requires h convex obstacles, and consider one of its representations. Then we have $G = \cap_i G_i$, where G_i denotes the visibility graph of the same set of points after the removal of all but the i -th obstacle.

If the size of the largest independent set in G_1 is at least $\frac{1}{2}n^{\frac{1}{h+1}}$, then the statement holds, because this set is also an independent set in G . If this is not the case, then, by the above property of circular arc graphs, G must have a complete subgraph K of size at least $n^{\frac{h}{h+1}}$. Consider now the subgraph of $\cap_{i=2}^h G_i$ induced by the vertices of K . This graph requires only $h - 1$ obstacles. Thus, we can apply the induction hypothesis to obtain that it has a complete subgraph or an independent set of size at least $\frac{1}{2}(n^{\frac{h}{h+1}})^{\frac{1}{h}} = \frac{1}{2}n^{\frac{1}{h+1}}$. \square

It is easy to see that every graph G on n vertices with convex obstacle number at most h has the following stronger property, which implies that they satisfy the Erdős-Hajnal conjecture: There exists a constant $\varepsilon = \varepsilon(h)$ such that G contains a complete subgraph of size at least εn or two sets of size at least εn such that no edges between them belongs to G (cf. [33]).

Finally, we make a comment on higher dimensional representations.

Proposition 3.5.3 In dimensions $d = 4$ and higher, every graph can be represented with one convex obstacle.

Proof. Let G be a graph with n vertices. Consider the moment curve

$$\{(t, t^2, t^3, t^4) : t \in \mathbf{R}\}.$$

Pick n points $v_i = (t_i, t_i^2, t_i^3, t_i^4)$ on this curve, $i = 1, \dots, n$. The convex hull of these points is a *cyclic polytope* P_n . The vertex set of P_n is $\{v_1, \dots, v_n\}$, and any segment connecting a pair of vertices of P_n is an edge of P_n (lying on its boundary). Denote the midpoint of the edge $v_i v_j$ by v_{ij} , and let O be the convex hull of the set of all midpoint v_{ij} , for which v_i and v_j are not connected by an edge in G . Obviously, the points v_i and the obstacle O (or its small perturbation, if we wish to attain general position) show that G admits a representation with a single convex obstacle. \square

3.6 Open Problems

The problems we have considered in the last few sections were to ascertain the obstacle number of graphs when we restrict the kind of obstacles we use, namely, general polygons, convex polygons and segments. Two other ways to consider the problem would be, firstly, to consider restrictions on the placement of the obstacles, and secondly, to consider restrictions on the kind of graphs we consider. For the first question, an interesting problem raised in [8] was to determine graphs which require only one obstacle in their outer face.

For the second problem, we realize from Theorem 3.1.1 that the problem is more interesting if we consider sparse graphs. In [8], it was shown that outerplanar graphs can be drawn with exactly one obstacle in the outer face that was not necessarily convex. Hence, they raised the question whether outerplanar graphs can be drawn with a finite number of convex obstacles. To this, in [34] it was shown that outerplanar graphs can be drawn with only five convex obstacles. Since every tree is an outerplanar graph, this also settles the question for trees. It is an interesting open problem if planar graphs can be drawn with a finite number of (convex) obstacles.

Any graph with e edges can be drawn with $2e$ segment obstacles, by placing a segment very close to every vertex between any two adjacent edges in the drawing. Hence, a sublinear bound on obstacle (or convex obstacle) number of planar graphs would also be interesting.

In three dimensions, it is easy to see that every graph can be represented with one obstacle. It is interesting, however, to find a bound when we restrict ourselves to convex obstacles.

Finally, the upper bound of obstacle numbers is wide open and nothing better is known than $2n^2/3$ (this can be achieved since a graph with e edges needs at most $n(n-1)/2 - e$ obstacles, or $2e$ obstacles from the above observation). Hence, even a subquadratic bound would be interesting.

Chapter 4

Tic-Tac-Toe

4.1 Introduction

A central topic of combinatorial game theory is the study of positional games. The interested reader can find the state of the art methods in Beck's Tic-Tac-Toe book [16]. In general, positional games are played between two players on a *board*, the points of which they alternately occupy with their marks and whoever first fills a *winning set* completely with her/his marks wins the game. Thus a positional game can be played on any hypergraph, but in this chapter, we only consider *semi-infinite* games where all winning sets are finite. If after countably many steps none of them occupied a winning set, we say that the game ended in a draw. It is easy to see that we can suppose that the next move of the players depends only on the actual position of the board and is deterministic.¹ We say that a player has a *winning strategy* if no matter how the other player plays, she/he always wins. We also say that a player has a *drawing strategy* if no matter how the other player plays, she/he can always achieve a draw (or win). A folklore strategy stealing argument shows that the second player (who puts *his* first mark after the first player puts *her* first mark, as ladies go first) cannot have a winning strategy, so the best that he can hope for is a draw. Given any semi-infinite game, either the first player has a winning strategy, or the second player has a drawing strategy. We say that the second player can achieve a *pairing strategy draw* if there is a matching among the points of the board such that every winning set contains at least one pair. It is easy to see that the second player can now force a draw by putting his

¹This is not the case for infinite games and even in semi-infinite games it can happen that the first player can always win the game but there is no N such that the game could be won in N moves. For interesting examples, we refer the reader to the antique papers [4, 17, 19].

mark always on the point which is matched to the point occupied by the first player in the previous step (or anywhere, if the point is unmatched). Note that in a relaxation of the game for the first player, by allowing her to win if she occupies a winning set (not necessarily first), the pairing strategy still lets the second player to force a draw. Such drawing strategies are called *strong draws*. Since in these games only the first player is trying to complete a winning set and the second is only trying to prevent her from doing so, the first player is called *Maker*, the second *Breaker*, and the game is called a *Maker-Breaker* game.

This chapter is about a generalization of the Five-in-a-Row game² which is the more serious version of the classic Tic-Tac-Toe game. This generalized game is played on the d -dimensional integer grid, \mathbb{Z}^d , and the winning sets consist of m consecutive gridpoints in n previously given directions. For example, in the Five-in-a-Row game $d = 2$, $m = 5$ and $n = 4$, the winning directions are the vertical, the horizontal and the two diagonals with slope 1 and -1 . Note that we only assume that the greatest common divisor of the coordinates of each direction is 1, so a direction can be arbitrarily long, e.g. $(5, 0, 24601)$. The question is, for what values of m can we guarantee that the second player has a drawing strategy? It was shown by Hales and Jewett [16], that for the four above given directions of the two dimensional grid and $m = 9$ the second player can achieve a pairing strategy draw. In the general version, a somewhat weaker result was shown by Kruczek and Sundberg [48], who showed that the second player has a pairing strategy if $m \geq 3n$ for any d . They conjectured that there is always a pairing strategy for $m \geq 2n + 1$, generalizing the result of Hales and Jewett.³

Conjecture 4.1.1 (Kruczek and Sundberg) *If $m = 2n + 1$, then in the Maker-Breaker game played on \mathbb{Z}^d , where Maker needs to put at least m of his marks consecutively in one of n given winning directions, Breaker can force a draw using a pairing strategy.*

²Aka Go-Muku and Amóba.

³It is not hard to show that if $m = 2n$, then such a strategy might not exist, we show why in Section 3.

Our main result asymptotically solves their conjecture.

Theorem 4.1.2 *There is an $m = 2n + o(n)$ such that in the Maker-Breaker game played on \mathbb{Z}^d , where Maker needs to put at least m of his marks consecutively in one of n given winning directions, Breaker can force a draw using a pairing strategy.*

In fact we prove the following theorem, which is clearly stronger because of the classical result [38] showing that there is a prime between n and $n + o(n)$.

Theorem 4.1.3 *If $p = m - 1 \geq 2n + 1$ is a prime, then in the Maker-Breaker game played on \mathbb{Z}^d , where Maker needs to put at least m of his marks consecutively in one of n given winning directions, Breaker can force a draw using a pairing strategy.*

The proof of the theorem is by reduction to a game played on \mathbb{Z} and then using the following recent number theoretic result of Preissmann and Mischler. Later this result was independently rediscovered by Kohen and Sadofski [46] and by Karasev and Petrov [42], they both gave a short proof using the Combinatorial Nullstellansatz [6]. The latter paper also gives an even shorter topological proof and generalizations.

Lemma 4.1.4 [64] *Given d_1, \dots, d_n and $p \geq 2n + 1$ prime, we can select $2n$ numbers, $x_1, \dots, x_n, y_1, \dots, y_n$ all different modulo p such that $x_i + d_i \equiv y_i \pmod{p}$.*

We prove our theorem in the next section and end the chapter with some additional remarks.

4.2 Proof of Theorem 4.1.3

We consider the winning directions to be the primitive vectors⁴ $\vec{v}_1, \dots, \vec{v}_n$. Using a standard compactness argument it is enough to show that there is a pairing strategy if the board is $[N]^d$, where $[N]$ stands for $\{1, \dots, N\}$. For interested readers, the compactness argument is discussed in detail at the end of this section.

First we reduce the problem to one dimension. Take a vector $\vec{r} = (r_1, r_2, \dots, r_d)$ and transform each grid point \vec{v} to $\vec{v} \cdot \vec{r}$. If \vec{r} is such that $r_j > 0$ and $r_{j+1} > N(r_1 + \dots + r_j)$

⁴A vector $(v_1, \dots, v_d) \in \mathbb{Z}^d$ is primitive if $\gcd(v_1, \dots, v_d) = 1$.

for all j , then this transformation is injective from $[N]^d$ to \mathbb{Z} and each winning direction is transformed to some number, $d_i = |\vec{r} \cdot \vec{v}_i|$.⁵ So we have these n differences, d_1, \dots, d_n , and the problem reduces to avoiding arithmetic progressions of length m with these differences. From the reduction it follows that if we have a pairing strategy for this game, we also have one for the original.

Let p be a prime such that $2n + 1 \leq p \leq 2n + 1 + o(n)$. (In [38] it was shown that we can always find such a p). If we pick a vector \vec{u} uniformly at random from $[p]^d$, then for any primitive vector \vec{v} , $\vec{u} \cdot \vec{v}$ will be divisible by p with probability $1/p$. Since each winning direction was a primitive vector, using the union bound, the probability that at least one of the $\vec{u} \cdot \vec{v}_i$ is divisible by p is at most $n/p < 1/2$. So, there is a $\vec{u}' = (u'_1, u'_2, \dots, u'_d) \in [p]^d$ such that none of $\vec{u}' \cdot \vec{v}_i$ is divisible by p . If we now take $\vec{r} = (r_1, r_2, \dots, r_d)$ such that $r_j = u'_j + (pN)^{j-1}$, then the dot product with \vec{r} is injective from $[N]^d$ to \mathbb{Z} and none of the $d_i = \vec{r} \cdot \vec{v}_i$ are divisible by p , since $\forall j \ r_j \equiv u'_j \pmod{p}$.

We now apply Lemma 4.1.4 for d_1, \dots, d_n to get $2n$ distinct numbers $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$ such that $0 \leq x_i, y_i < p$ and $x_i + d_i \equiv y_i \pmod{p}$. Our pairing strategy is, for every $x \equiv x_i \pmod{p}$, x is paired to $x + d_i$ and if $x \equiv y_i \pmod{p}$, then x is paired to $x - d_i$.

To see that this is a good pairing strategy, consider an arithmetic progression a_1, \dots, a_m of $m = p + 1$ numbers with difference, say, d_i . Since p and d_i are coprimes, one of the numbers a_1, \dots, a_{m-1} , say a_j , must be such that $a_j \equiv x_i \pmod{p}$. Hence a_j, a_{j+1} must be paired in our pairing strategy, showing both cannot be occupied by Maker. \square

For completeness here we sketch how the compactness argument goes. We show that it is sufficient to show that a pairing strategy exists for every finite $[N]^d$ board. For this we use the following lemma.⁶

Lemma 4.2.1 [47] (*König's Infinity Lemma*) *Let V_0, V_1, \dots be an infinite sequence of disjoint non-empty finite sets, and let G be a graph on their union. Assume that every vertex v in a set V_N with $N \geq 1$ has a neighbor $f(v)$ in V_{N-1} . Then G contains an*

⁵It is even possible that some of these numbers are zero, we will take care of this later.

⁶We use the version stated in [22].

infinite path, $v_0v_1\dots$ with $v_N \in V_N$ for all N .

Given a pairing strategy for $[N_0]^d$, consider a smaller board $[N]^d$ where $N < N_0$. We can think of a pairing strategy as, essentially, a partition of $[N_0]^d$ into pairs and unpaired elements.⁷ We can construct a good pairing strategy for the smaller board by taking the restriction of these set of pairs to $[N]^d$ and leave the elements paired outside $[N]^d$ as unpaired elements. We call this as a restriction of the pairing strategy to the new board. As long as we do not change the length of the winning sets and the prescribed directions, any winning set in the $[N]^d$ board is also a winning set in the $[N_0]^d$ board and hence must have a pair from the restriction. Hence, the Breaker can block all winning pairs and the restriction of the pairing strategy is a valid strategy for Breaker for the smaller board.

We can now prove the following theorem,

Theorem 4.2.2 *Given a fixed set S , $|S| = n$, of winning directions, and positive integer m , if Breaker has a pairing strategy for all boards $[N]^d$ and length of winning sets equal to m , then Breaker also has a pairing strategy for the \mathbb{Z}^d board.*

We will apply König's Infinity Lemma to prove the theorem. Let V_N be the set of all pairing strategies on the $\{-N, \dots, N\}^d$ board with winning sets as defined in the theorem. We say a strategy in V_{N-1} and a strategy in V_N have an edge between them if the former is a restriction of the latter. It is easy to see that every vertex in V_N does have an edge to its restriction in V_{N-1} . Hence, by the lemma, we must have an infinite path $v_0v_1\dots$. The union of all these pairing strategies gives a valid pairing strategy for the infinite game.

4.3 Possible further improvements and remarks

As we said before, if $m \leq 2n$, then the second player cannot have a pairing strategy draw. This can be seen as follows. On one hand, in any pairing strategy, from any m

⁷Note that a pairing strategy does not guarantee that every element is paired. It only states that every winning set has a pair. Hence there might be many unpaired elements in a pairing strategy.

consecutive points in a winning direction, there must be at least two points paired to each other in this direction. On the other hand, there must be a winning direction in which at most $1/n$ of all points are matched to another in this direction. If we pick a set of size $m - 1$ uniformly randomly in this direction, then the expected number of points matched in this direction will be at most $(m - 1)/n < 2$. Thus, there is a set of size $m - 1$ that contains only one such point. Its matching point can now be avoided by extending this set to one way or the other, thereby giving us a winning set with no matched pair.

If $n = 1$ or 2 , then a not too deep case analysis shows that the first player has a winning strategy if $m = 2n$, even in the *strong game*, where the second player also wins if he occupies a winning set. Moreover, the second player has a pairing strategy for $m = 2n + 1$ if $n = 1$ or 2 , thus, in this case, the conjecture is tight. However, for higher values, it seems that Breaker can always do better than just playing a pairing strategy, so we should not expect this strategy the best to achieve a draw. Quite tight bounds have been proved for Maker-Breaker games with *potential* based arguments, for the latest in generalization of Tic-Tac-Toe games, see [49]. Despite this, from a combinatorial point of view, it still remains an interesting question to determine the best pairing strategy. Unfortunately our proof can only give $2n + 2$ (if $2n + 1$ is a prime) which is still one bigger than the conjecture.

One could hope that maybe we could achieve a better bound using a stronger result than Lemma 4.1.4 (see for example the conjecture of Roland Bacher in [64], whom we would like to thank for directing us to it [14]), however, already for $n = 3$, our method cannot work. Consider the three directions $(1, 0), (0, 1), (1, 1)$. Optimally, we would hope to map them to three numbers, d_1, d_2, d_3 , all coprime to 6, such that we can find $x_1, x_2, x_3, y_1, y_2, y_3$ all different modulo 6 such that $x_i + d_i \equiv y_i \pmod{6}$. But this is impossible since $d_3 = d_1 + d_2$, so we cannot even fulfill the condition that the differences have to be coprimes to 6. But even if we forget about that condition, it would still be impossible to find a triple satisfying $d_3 = d_1 + d_2$. If we consider a pairing strategy where the pair of any grid point \vec{v} , depends only on $v \cdot r$, then the above argument shows that such a pairing strategy does not exist for the three vectors $(1, 0), (0, 1), (1, 1)$. However,

it is not hard to find a suitable periodic pairing strategy for these three vectors. We would like to end with an equivalent formulation of Conjecture 1.

Conjecture 4.3.1 (Kruczek and Sundberg, reformulated) *Suppose we are given n primitive vectors, \vec{v}_i of \mathbb{Z}_{2n}^d for $i \in [n]$. Is it always possible to find a partition of \mathbb{Z}_{2n}^d into \vec{x}_i^j, \vec{y}_i^j for $i \in [n], j \in [2n]$ such that $\vec{x}_i^j + \vec{v}_i = \vec{y}_i^j$ and $\vec{x}_i^j - \vec{x}_i^{j'}$ is not a multiple of \vec{v}_i for $j \neq j'$?*

Chapter 5

Program code

The following code is in Maple.

```

#For accessing log, ceil functions.
with(MTM);

#fmax is a procedure that computes the girth for which a graph on N
#vertices will have the largest supercycle.
#Here, mg denotes the maximum possible girth, max and g will have the
#values of the maximum size of the supercycle and the girth at which
#it occurs respectively. The procedure returns 2s-2, if this value is
#less than N, we can apply Lemma 2.6 and 2.8 to draw the graphs on N
#vertices.
fmax := proc (N) local g, mg, max, i, exp;

#Initializations
max := -1;
g := 0;
mg := 2*ceil(evalf(log2((1/3)*N+1)));

if mg < 3 then RETURN([N, 2*max-2, mg, g]) fi;

#Main search cycle.
for i from 3 while i <= mg do
    exp := 2*ceil(evalf(log2((N+1)/i)))+i-1;

```

```
        if max < exp then max := exp; g := i fi
    end do;

RETURN([N, 2*max-2, mg, g])
end proc;

seq(fmax(i), i = 6 .. 42, 2);
[6,10,4,3], [8,12,4,4], [10,14,6,5], [12,16,6,6], [14,16,6,6],
[16,16,6,4], [18,16,6,4], [20,18,6,5], [22,20,8,8], [24,20,8,6],
[26,20,8,6], [28,22,8,7], [30,22,8,7], [32,24,8,8], [34,24,8,8],
[36,24,8,8], [38,24,8,8], [40,24,8,8], [42,24,8,8]
```


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