PROBABILITY OF A FEASIBLE FLOW IN A STOCHASTIC TRANSPORTATION NETWORK

By

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ABSTRACT OF THE THESIS

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Electricity is one of the main sources of energy relied upon throughout the world today to power our homes, businesses, and other needs. The electric utility industry is the driving force behind the provision of electric service, responsible for generating and delivering electric power to end use customers on a reliable basis. In order to meet this responsibility, consideration needs to be given to both the supply of and demand for electricity, along with the available capacity through which to deliver it. A resulting fundamental objective of the electric utility industry, then, is to balance supply with customer demand by maintaining sufficient generating and delivery capacity. The probability by which this objective can be accomplished lends itself to representation as a stochastic transportation network. Determining this probability is the primary problem addressed in this paper.

The general construct of this paper is largely a continuation of *On The Probability Of A Feasible Flow In A Stochastic Transportation Network* (Prékopa and Boros, 1989).

However, it has been updated to incorporate alternative methods to solve the problem and also include practical examples based on actual data from the industry.

This paper will seek to accomplish the following:

- Section 1 provides a general introduction and overview of the electric utility industry. This section includes statistics and additional details on all aspects of the electric utility industry, a discussion of upcoming challenges currently facing the industry, and a brief overview of a large electric utility company in the United States.
- Section 2 more formally addresses the problem of determining the likelihood that the electric utility has sufficient generating and delivery capacity available to satisfy customer demand by formulating it as a stochastic transportation network. The general formulation is based on the results of a well-established theorem and is improved by incorporating a procedure to increase the efficiency by which the problem can be solved.
- Section 3 introduces several methods that can be used to solve the problem and focuses specifically on three of them that will be incorporated later on in the paper. A description of each applicable method is provided, along with some illustrative examples.
- Sections 4, 5, and 6 are numerical examples of the problem. All three examples are based on the same general formulation and make use of actual data from the electric utility industry. The underlying assumptions, though, are different in each

one, thus providing a range of sensitivity around the results. Each numerical example is solved using all three methods described in Section 3.

– Section 7 provides a summary of the results from the numerical examples and some overall conclusions.

ACKNOWLEDGEMENTS

I express my sincerest thanks and gratitude to Dr. András Prékopa for his mentorship, patience, guidance, and support. I never would have been able to complete this without you and I truly appreciate everything you did for me.

This paper is dedicated to my father.

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1 INTRODUCTION

1a. Electric Utility Industry – General Overview

Within the United States, the electric utility industry employs hundreds of thousands of employees and represents approximately 3% of the gross domestic product. The utility companies responsible for the day-to-day activities within the industry are operated under a few different ownership structures:

- Shareholder-Owned – These utility companies are generally financed through the sale of stocks or bonds to investors.
- Cooperatively-Owned – Each customer of the utility is also a member. These utilities tend to be driven by geographic conditions and are popular in large, rural areas.
- Government-Owned – Utilities may also be owned by federal, state, or local governments.

The majority of electric utilities, particularly in the United States, are shareholder-owned as they serve approximately 70% of all customers in the nation. These utilities are generally regulated by the Federal Energy Regulatory Commission (FERC), and also at the state level by the applicable state regulatory agencies. The Edison Electric Institute (EEI) is the association of U.S. shareholder-owned electric utilities and provides a number of services to its members, including "public policy leadership, critical industry data, market opportunities, and strategic business intelligence."[1]

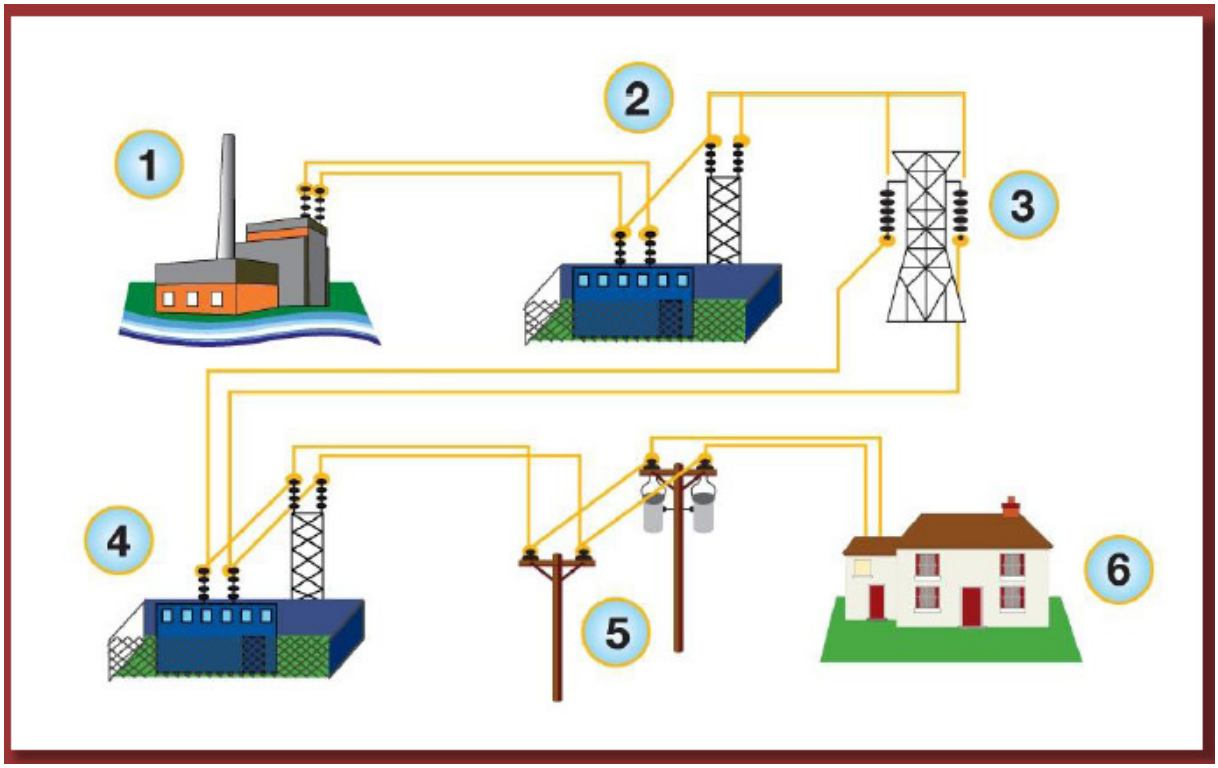
[1] For more information on EEI, the reader is referred to the EEI website: http://www.eei.org/

As with other industries that provide a consumer product, the electric industry functions primarily on the economic concepts of supply and demand. In order to properly balance these concepts, the industry is typically divided into three primary functions:

- Generation – Production of electricity at power plants
- Transmission – Transportation of the power through wires from the power plants to areas closer to the end use customers
- Distribution – Delivery of electricity from the Transmission system to the end use customers via a subsequent system of poles and wires

The image below provides a high level summary of the flow of electricity from the generating power plants to the end use customers.[2]

[2] Picture and corresponding explanation courtesy of EEI.
http://www.eei.org/whoweare/AboutIndustry/Documents/Electricity101.pdf



(1) Electricity is generated at a power plant

(2) Electricity is sent to a substation where the voltage is increased. Typically the power plants are not located near the end use customers, so the increase in the voltage allows the electricity to be transmitted more efficiently to areas closer to the customers.

(3) Power is transmitted across the transmission system and arrives at a substation.

(4) Once the electricity arrives at a substation, the voltage is usually deceased so that it can be delivered the remaining distance to, and ultimately consumed by, the end use customers.

(5) The electricity is then transferred across a local Distribution system.

(6) The electricity arrives at the end use customer.

While the above diagram captures the entire flow of electricity from the power plants to the end use customers, this paper will focus primarily on the Generation and Transmission aspects of the business. As evidenced by the diagram above, the reliability of the Transmission system and the available Generation capacity are critical in ensuring that adequate electricity is available to satisfy customers' demand. The reliability of the Distribution system is relatively insignificant if the combination of available Transmission and Generation capacity is insufficient.

The next section provides some additional details and general statistics on the current Transmission system and available Generation capacity in the U.S.

1b. Electric Utility Industry – Statistics

Before providing statistics, it is important to note how electricity is measured for both supply and demand. The general unit of measure is a watt (W), which represents the amount of power generated or needed at a given point in time. Due to the large number of electric customers, though, a more commonly used measure is a megawatt (MW), which is equivalent to one million watts. The total electricity generated (consumed) over time, then, is a weighted average of individual capacity (demand) values. The applicable time interval typically used to represent this electricity production (consumption) is an hour. So, we say that the amount of electricity produced (consumed) over time is measured in megawatt-hours (MWH), where one MWH is equivalent to a constant demand of one MW for exactly one hour.

We now proceed with some industry statistics.

Generation

There are numerous different fuels that can be used to generate electricity. In fact, according to EEI, no individual fuel type is currently capable of producing enough electricity to satisfy all customer demand in the U.S., which makes fuel diversification a necessity from an economic standpoint. The most common fuel types used today are: coal, natural gas, nuclear, hydropower, oil, and other renewable fuels (solar, wind, etc.).

The table below provides the installed Generation capacity (in MW), by fuel type, in the U.S. in 2009.[3]

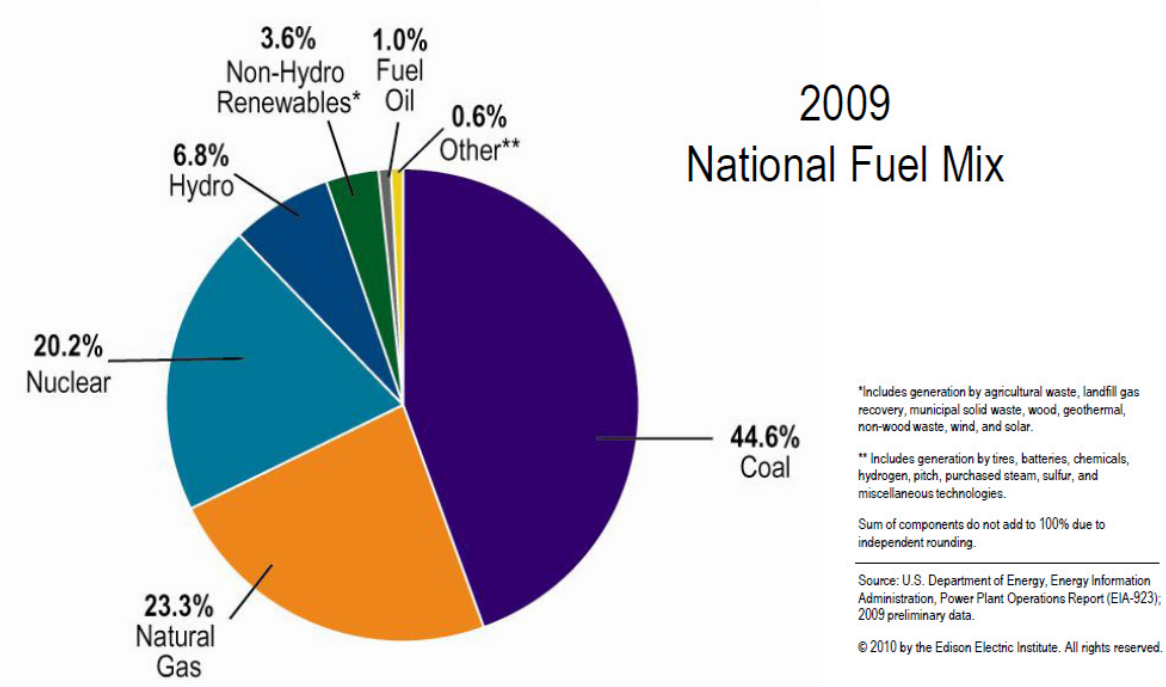
Fuel Type	Number of Generators	Capacity (MW)		
		Installed	Summer	Winter
Coal	1,436	338,723	314,294	316,363
Oil	3,757	63,254	56,781	60,878
Natural Gas	5,568	462,021	403,204	434,208
Nuclear	104	106,618	101,004	102,489
Hydro	4,005	77,910	78,518	78,127
Renewables				
Wind	620	34,683	34,296	34,350
Solar	110	640	619	537
Wood	353	7,829	6,939	6,992
Biomass	1,502	5,007	4,317	4,382
Other				
Geothermal	222	3,421	2,382	2,561
Pumped Storage	151	20,538	22,160	22,063
Other	48	1,042	888	900
	17,876	1,121,686	1,025,402	1,063,850

The second column in the table provides the number of units of each fuel type. The installed capacity in the third column represents the aggregate maximum nameplate capability of all units, while the last two columns show the actual available capacity in the summer and winter months, respectively. Depending on the operational efficiency of the generating units, the available capacity may be slightly higher or lower than the installed nameplate values.

While available capacity is an important measure in determining the adequacy of supply to be able to satisfy customer demand, it is also important to note that individual power plants do not always operate at full capacity. For example, certain units may only be

[3] Data courtesy of the U.S. Energy Information Administration.
http://www.eia.gov/cneaf/electricity/epa/epat1p2.html

called on to run when customer demand is highest. Alternatively, there may be instances when larger units are shut down temporarily for maintenance. As such, we will also consider the overall amount of electricity produced by each fuel type (in MWH). The average fuel mix used to generate electricity in the U.S. in 2009 is provided in the following chart.[4]



We observe that coal produced the most electricity (in MWH) in 2009, even though it is not the highest capacity fuel type. This more than likely is attributable to the high operating efficiency of coal-fired power plants, as compared to other fuel sources. Natural gas, on the other hand, was the highest capacity but produced the second highest

[4] Data courtesy of EEI. http://www.eei.org/ourissues/ElectricityGeneration/FuelDiversity/Documents/pie_fueldiversity.pdf

amount of electricity. Many natural gas-fired units are generally used only during peak demand times on the system, which explains why they may operate less efficiently than coal-fired units.

Transmission

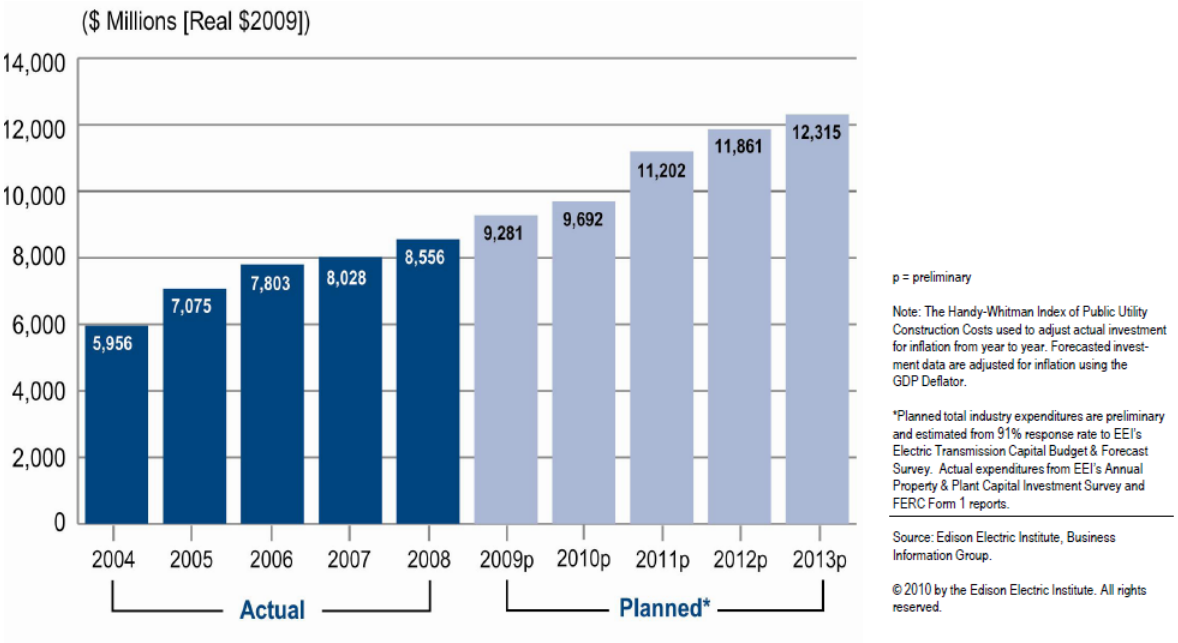
According to EEI, the electric Transmission system in the U.S. consists of approximately 200,000 miles of wires used to carry electricity from power plants to local Distribution systems. New Transmission infrastructure is constructed on an annual basis in order to keep up with changes in Generation capacity and customer demand. The table below shows the estimated Transmission capacity additions for 2011-2016 in miles.[5]

Estimated Transmission Capacity Additions (Miles)						
2011	2012	2013	2014	2015	2016	Average
2,230	3,831	5,503	3,616	3,970	3,451	3,767

We observe that the average annual additions to the Transmission system represent approximately 1.9% of the current total mileage. That is, current estimates indicate that the size of the Transmission system (in miles of wire) will need to increase by approximately 2% every year in order to maintain feasibility of power flow.

[5] Data courtesy of U.S. Energy Information Administration.
http://www.eia.gov/cneaf/electricity/epa/epaxlfile5_5.pdf

To quantify the impact of these infrastructure additions, the chart below shows the amount of actual and estimated annual investment in the Transmission system by U.S. shareholder-owned utility companies from 2004-2013.[6]



From the above chart, it is easy to see an upward trend in the amount of money estimated to be spent on Transmission system enhancements. In fact, according to a report issued by the Brattle Group in 2008, the electric utility industry will need to invest $298 billion in the U.S. Transmission system from 2010 through 2030 in order to maintain reliable service.

[6] Chart courtesy of EEI.
http://www.eei.org/ourissues/ElectricityTransmission/Documents/bar_Transmission_Investment.pdf

1c. Customer Demand for Electricity

As noted in Section 1a, consumers of electricity cover a broad spectrum. However, customers are commonly divided into three main categories:

- Residential – homes, apartment buildings, etc.
- Commercial – stores, restaurants, small businesses, etc.
- Industrial – larger manufacturers, automakers, steel producers, etc.

Obviously, each class of customers has its own unique usage characteristics. Residential customers' demand tends to be relatively weather sensitive due to use of air conditioners in the summer and possibly electric heating options in the winter. Commercial customers may have some sensitivity to weather, but generally not as much as the Residential class. For example, consider a recreation center that is located in a popular vacation destination and uses electricity for lighting, refrigeration, and kitchen appliances. Since use of the center is dependent on the level of vacation activity, it most likely will not have consistent electricity needs throughout the year. A similar example is a ski resort, whose demand will most likely be higher in the winter months than the summer. There are other commercial customers, though, (for example, a shopping mall), that remains open year round and would not be as sensitive to weather.

Industrial customers are the least sensitive to weather because their operations tend to be more round-the-clock in nature. However, there may still be instances when a large industrial customer has a significant change in its demand. Consider a large auto manufacturer that currently is running three shifts of employees. If the customer decides

to temporarily change to two shifts due to economic conditions, this would result in a decrease in the customer's demand until the economy recovers and the plant returns to a round-the-clock operation.

Historically, meters have used to measure customers' demand and electricity consumption for purposes of billing by utility companies. The customers with more sophisticated electric needs tend to also have more sophisticated metering equipment in place. For example, a residential customer's meter may only show a continuous consumption amount so a monthly meter read would entail subtracting the prior month's cumulative consumption total from the current month's. A large steel mill, on the other hand, may have metering in place to measure its demand on an hourly basis and the meter may be read remotely.

The following tables show (i) peak demand across all U.S. electric customers for 2005-2009; (ii) the total number of end use electric customers in the U.S., by customer class, from 2003-2009; and (iii) the total electricity consumed in the U.S., by customer class, from 2003-2010.[7]

U.S. Peak Demand (MW)				
2005	2006	2007	2008	2009
758,876	789,475	782,227	752,470	725,958

[7] Data courtesy of U.S. Energy Information Administration.
http://www.eia.gov/cneaf/electricity/epa/epaxlfile4_1.pdf
http://www.eia.gov/cneaf/electricity/epm/table5_1.html
http://www.eia.gov/cneaf/electricity/epa/epat7p1.html

Number of End Use Customers				
Year	Residential	Commercial	Industrial	Total
2003	117,280,481	16,550,646	713,221	134,544,348
2004	118,763,768	16,607,808	747,600	136,119,176
2005	120,760,839	16,872,458	733,862	138,367,159
2006	122,471,071	17,173,290	759,604	140,403,965
2007	123,949,916	17,377,969	793,767	142,121,652
2008	124,937,469	17,563,453	774,713	143,275,635
2009	125,177,175	17,562,366	757,519	143,497,060

Electricity Consumed (thousand MWH)				
Year	**Residential**	**Commercial**	**Industrial**	**Total**
2003	1,275,824	1,205,538	1,012,373	3,493,735
2004	1,291,982	1,237,649	1,017,850	3,547,481
2005	1,359,227	1,282,585	1,019,156	3,660,968
2006	1,351,520	1,307,102	1,011,298	3,669,920
2007	1,392,241	1,344,488	1,027,832	3,764,561
2008	1,379,981	1,343,681	1,009,300	3,732,962
2009	1,364,474	1,314,949	917,442	3,596,865
2010	1,450,758	1,337,062	962,165	3,749,985

We make a few observations. First, we note that total peak demand in 2009 is less than the available generating capacity provided in Section 1b, which we would expect. It is also important to recall that the table in Section 1b represents the total capacity available, which may or may not be the amount of capacity actually used to serve customers due to various operational reasons.

Second, we notice a decrease in electricity consumption for the Industrial class in recent years, as compared to prior historical levels. This is most likely attributable to the economic downturn as Industrial customers were forced to decreases or eliminate operations. However, it should be noted that the U.S. Energy Information Administration expects electricity consumption to increase 30% by the year 2035.

Third, the differences between customer classes in the amount of electricity consumed are less significant than the differences in the numbers of customers. Most notably, the Residential class has significantly more customers than the Industrial class, but the annual total consumption amounts are relatively much closer. This implies that the Industrial class uses more electricity per customer than the other classes, which makes sense due to the size of an Industrial customer's operation as compared to a home or a small business.

1d. Upcoming Challenges in the Electric Utility Industry

The history of the electric utility industry dates back to Thomas Edison in the late 1800's. While technology enhancements have obviously been made over the years, the general objective of the industry has remained relatively constant – to provide safe and reliable electric service to customers. As discussed above, one of the principal factors in meeting this objective, (and also the primary focus of this paper), continues to be the balancing of supply and demand through adequate Generation and Transmission capacity.

There are several upcoming challenges, though, that could have a significant impact on how the electric industry will continue to achieve its goal of balancing supply with demand. Three of these are described below in more detail.[8]

Environmental Legislation

The second chart in Section 1b under Generation indicates that coal is the largest consumed fuel in the U.S. This is a significant observation in light of the various environmental laws that could go into effect in the near future regarding pollution control and emissions. Below is a version of a popular chart in the industry referred to as the "Environmental Train Wreck", which shows a timeline of the proposed environmental laws that could have an impact on the electric industry.[9]

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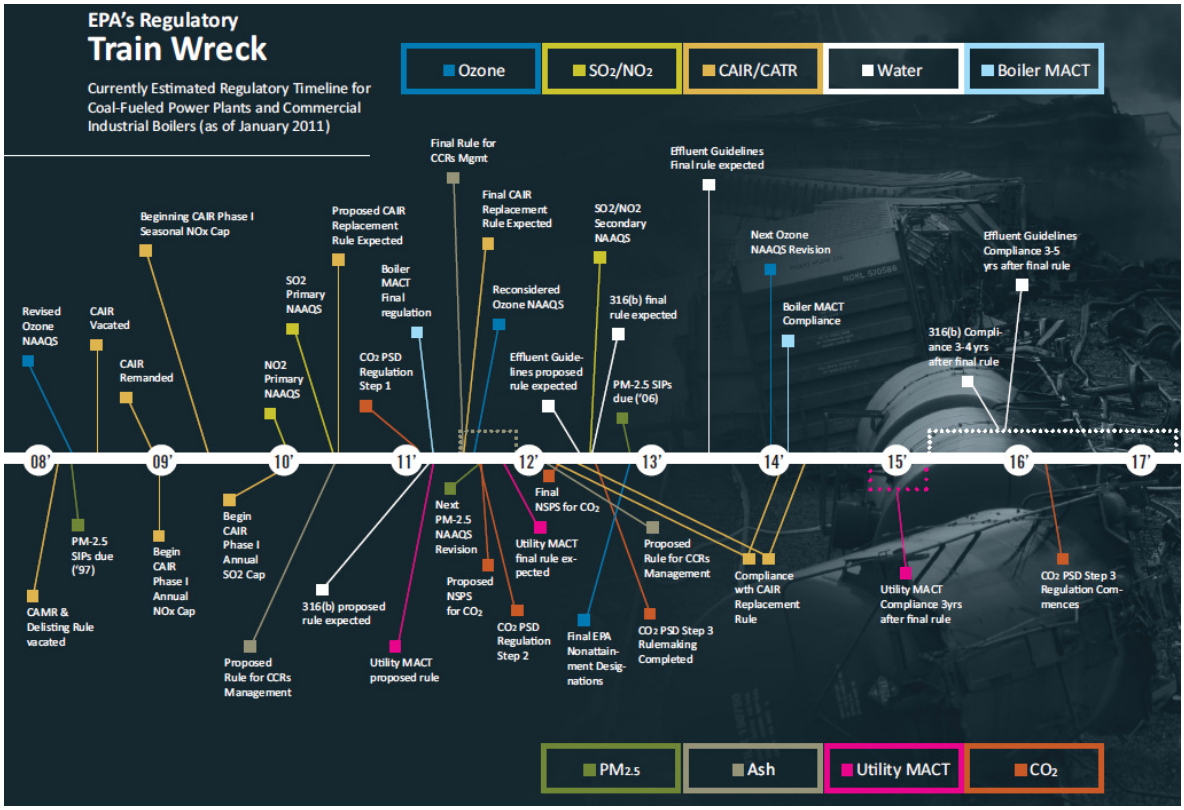
[8] There is significant literature available on various upcoming challenges facing the electric utility industry. For purposes of this paper, a few of these challenges are addressed in general terms to give the reader a better idea of potential impacts to the electric industry.

[9] Slide courtesy of the U.S. Environmental Protection Agency (EPA). http://alec.org/AM/Template.cfm?Section=EPATrainWreck&Template=/CM/ContentDisplay.cfm&ContentID=15348



This chart is not presented for purposes of delving into details of each specific piece of legislation, but rather is intended to simply provide the reader a visual impression of the volume of new laws that could impact the generation of electricity in the U.S. in the next decade. The implementation of further restrictions on emissions or a tax on the amount of coal burned would most likely impact the reliance on coal, which could lead to a shift in focus to alternate fuel sources, such as wind and solar. Thus, the supply side of the industry could be forced to modify its investment and operational strategy.

Energy Efficiency

There are also changes afoot on the demand side of the electricity balancing equation. The recent emphasis on environmental controls has contributed to an increased focus on more efficient use of electricity. Generation and Transmission infrastructure needs to accommodate peak periods, i.e., periods when customer demand is the highest. Accordingly, as customer demand increases, so does the need for more infrastructure to be built, which is expensive In order to combat this need for new capacity, utility companies in several jurisdictions are offering more programs to customers to incent them to decrease usage of electricity during certain peak times and/or to be a more efficient consumer of electricity.

Several popular energy efficiency initiatives currently offered include:[10]

- Lighting Upgrades – For Residential customers, a common approach is to install compact fluorescent light bulbs. Large customers may seek a lighting retrofit of their operation.
- Equipment Replacement – Commercial and Industrial customers that operate machines with large motors may be able to save money on their electric bill in the long run by replacing or improving the efficiency of their machines. Similarly, homeowners can trade in old appliances for more efficient ones.
- Remote Thermostats – Primarily marketed towards Residential customers, these thermostats can monitor home temperature remotely so that the air

[10] For a more comprehensive list of programs, the reader is referred to EEI.

conditioning can be adjusted accordingly, eliminating unnecessary use when no one is home.

In some states, utility companies are even mandated to achieve a certain level of energy efficiency related savings from their customers. For example, Ohio utility companies are order by law to implement energy efficiency programs that achieve a cumulative 22% reduction in electricity consumption reduction by 2025. [11] The table below shows the annual required reductions.

Year(s)	Efficiency-based Energy Reduction
2009	.3%
2010	.5%
2011	.7%
2012	.8%
2013	.9%
2014 – 2018	1% each year
2019 – 2024	2% each year
2025	Cumulatively 22+%

Even though it may result in a reduction of overall customer demand, the implementation of more energy efficiency programs will serve as a significant contributing factor in the balancing of supply and demand in the upcoming years. The impact of these programs will need to be taken into account when making operational and strategic decisions regarding the installation of new Generation or Transmission capacity.

[11] Energy Efficiency targets from Ohio Senate Bill 221, passed in 2008.

Smart Metering

As mentioned above in Section 1c, there are various metering technologies used across customer classes in the electric utility industry. In conjunction with energy efficiency and demand reduction mandates, several jurisdictions are also pursuing more stringent requirements regarding smart metering. Smart meters allow customers' electricity demand and usage to be more transparent, with the intent being that they will be more efficient users of electricity. That is, if a customer's electric demand and corresponding price of electricity are more visible to her, it is believed that she will not knowingly consume wasted energy.[12]

One jurisdiction has already taken legislative action towards promoting the implementation of smart meters. According to Pennsylvania Act 129, which was passed in 2008, the utility companies in the state must submit plans to the Pennsylvania Public Utility Commission to replace all of its existing customer meters with more advanced smart meters over the next 15 years. Some of the provisions that must be included in these plans include:

- Customers must be provided with direct access to hourly pricing information so that they may better manage their electricity consumption
- The utilities must offer different pricing options to customers, e.g., on-peak vs. off-peak pricing, or real time pricing.

[12] While not specifically addressed in this paper, it should be noted that the cost of implementing such smart meter technologies continues to be a debated topic in the industry.

- Customers' electricity consumption must be able to be controlled remotely, either by the customer or another party.[13]

As efforts like those being implemented in Pennsylvania become more popular, and customers use the technology to modify their electricity consumption, smart meters will also play a significant role in the electric industry's ongoing efforts to provide a sufficient supply of electricity.

Environmental legislation, energy efficiency, and smart meters are just a few of the major issues currently facing the electric utility industry, but they should provide the reader with better perspective on where the industry may be headed in the upcoming years.

[13] Source: Pennsylvania Public Utility Commission.

1e. FirstEnergy Corp. – Company Overview

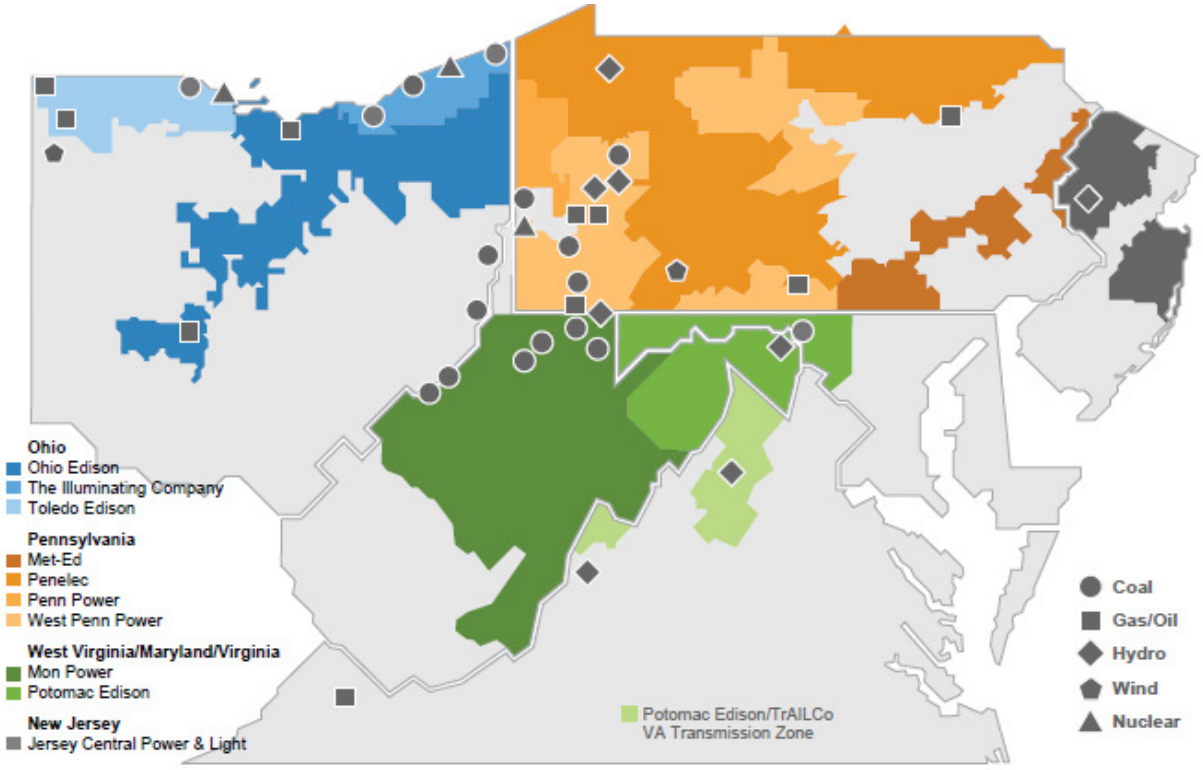
In the final section of the Introduction, we will provide a brief overview of one of the largest investor-owned electric utility companies in the United States, FirstEnergy Corp. (FirstEnergy). Examples provided later in the paper will be based on FirstEnergy's operations.[14]

FirstEnergy is headquartered in Akron, Ohio, but has operations in 6 different states: Ohio, Pennsylvania, West Virginia, New Jersey, Virginia, and Maryland. FirstEnergy serves approximately 6 million customers across its service territory, which is comprised of 10 different operating regions and approximately 67,000 square miles. FirstEnergy does own and operate Distribution assets, but we will continue to focus on its Generation and Transmission infrastructure.

Generation

The map below shows FirstEnergy's service territory along with the location of its Generation assets. Underneath the map is a table showing a breakdown of FirstEnergy's Generation capacity by fuel type.[15]

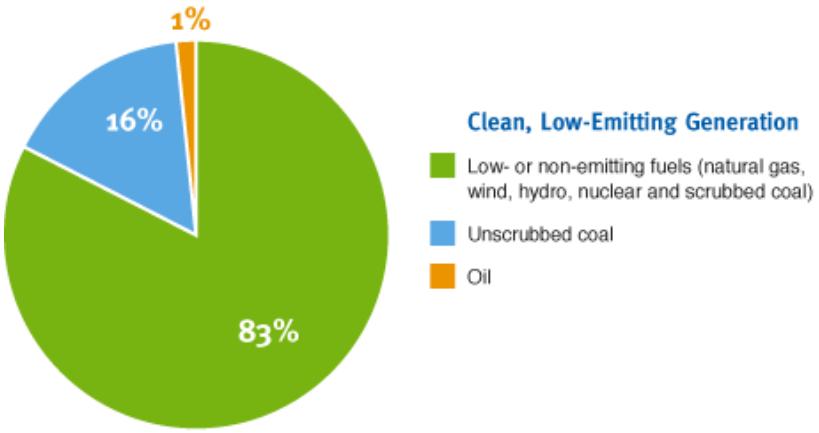
[14] All data and information in this Section is provided by FirstEnergy. (www.firstenergycorp.com)
[15] The colored areas on the map represent the 10 different operating regions, which are also noted in the legend in the bottom left corner. FirstEnergy's power plants are represented by the shapes defined in the bottom right corner of the chart.



Fuel Type	Capacity (MW)	% Total
Coal	14,678	63.1%
Gas/Oil	2,195	9.4%
Hydro	1,832	7.9%
Wind	564	2.4%
Nuclear	3,991	17.2%
	23,260	100.0%

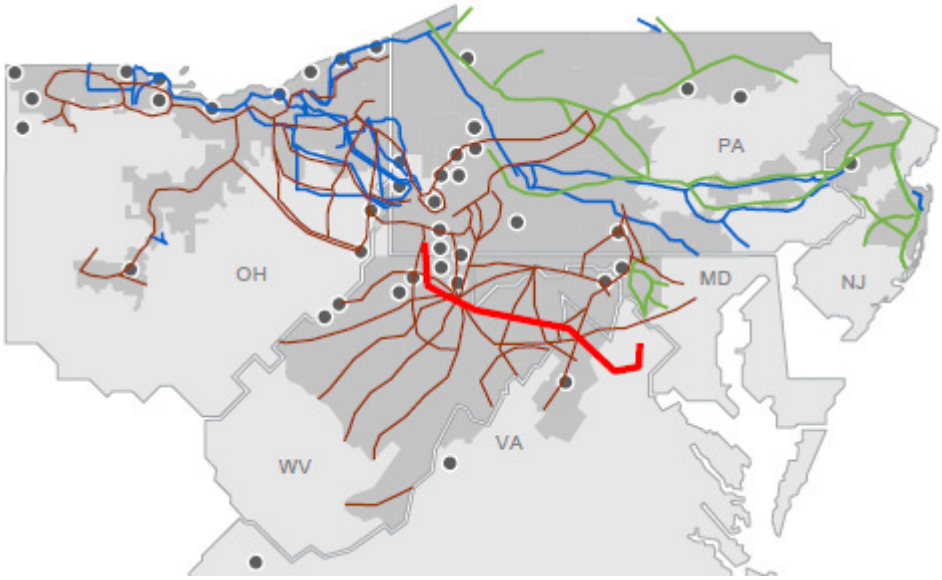
As indicated in the above chart, FirstEnergy has a relatively balanced mix of fuels in its Generation portfolio. In regards to the prior discussion on pending environmental legislation on coal-fired generation, it should be noted that a significant portion of FirstEnergy's coal-fired units have undergone certain operational improvements to mitigate the amount of pollution emitted.

Units that have undergone these improvements are generally referred to as "scrubbed." Taking these scrubbed units into account, the chart below indicates that only 16% of FirstEnergy's Generation capacity is a significant emitter of pollution, which should decrease its risk associated with new environmental laws that may be enacted in the future.



Transmission

FirstEnergy also owns and operates approximately 20,000 miles of Transmission lines across its service territory, which are depicted on the map below.[16]



Planning for the Transmission system is an important part of FirstEnergy's operations, particularly given the breadth of its service territory, which covers a significant portion of the northeastern United States.

Thus, with its large collection of Generation and Transmission assets, FirstEnergy serves as a useful representative of the electric utility industry and provides a practical framework for demonstrating its principal objective, which is to balance supply and demand by maintaining sufficient Generation and Transmission capacity.

[16] The colored lines on the map represent the Transmission system, while the shaded circles represent FirstEnergy's Generation assets.

This Introduction and general overview of FirstEnergy are intended to provide the reader with a basic understanding of the electric utility industry, as well as a practical representation of one its participants. Some numerical examples involving FirstEnergy will be discussed later on in the paper, but first we need to more formally define and formulate our problem from a mathematical perspective.

2 PROBLEM FORMULATION

2a. Definitions and Notations

With this background, we now move to the formulation of the primary problem to be discussed in this paper. As a preliminary matter, we first reiterate that the general construct of this paper is based largely on the work of Prékopa and Boros in their paper, *On the Existence of a Feasible Flow in a Stochastic Transportation Network* (1989). Using this work as a guide, we start with the following notations and definitions, many of which are consistent with common terminologies used in graph theory.

N = Set of all Demand Nodes (locations of customer demand)

y_{ij} = Transmission Capacity Between Node i and Node j

x_i = Random Generating Capacity at Node i

ζ_i = Random Deficiency of Random Generating Capacity at Node i (losses)

$x_i - \zeta_i$ = Available Generating Capacity at Node i (after losses)

η_i = Random Local Demand at Node i

$\xi_i = \eta_i + \zeta_i$ = Random Local Demand (before losses)

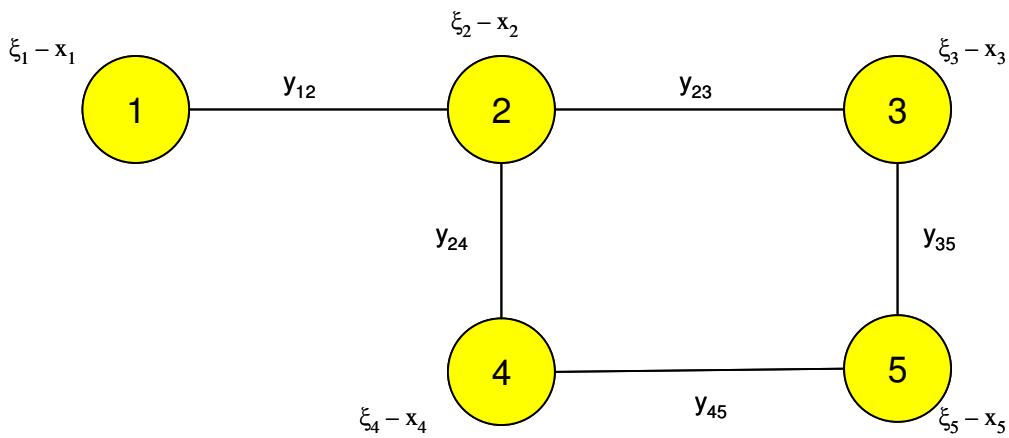
$d(i) = \xi_i - x_i$ = Net Demand (Supply) at Node i (before losses)

$d(i) < 0 \rightarrow$ Surplus of Generation at Node i (have excess power)

$d(i) > 0 \rightarrow$ Shortage of Generation at Node i (need power)

2b. Network Formulation and Problem Statement

We use these definitions to construct a graph representing a stochastic network of (net) demand nodes and available transmission capacities. We will proceed with an example of 5 nodes to demonstrate the formulation.[17] Consider the following network.

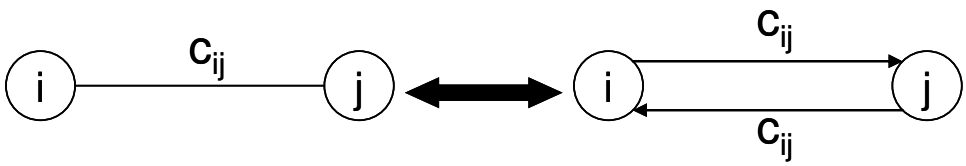


Our primary objective is to determine the probability of a feasible flow in the network. In other words, we want to know how likely it is that all net demand values $d(i) = \xi_i - x_i$ are satisfied simultaneously, based on the given transmission capacities and available generation. It turns out that this objective can be generally represented through a series of inequalities.

[17] This example is based on a subset of FirstEnergy's operations.

2c. Gale-Hoffman Theorem

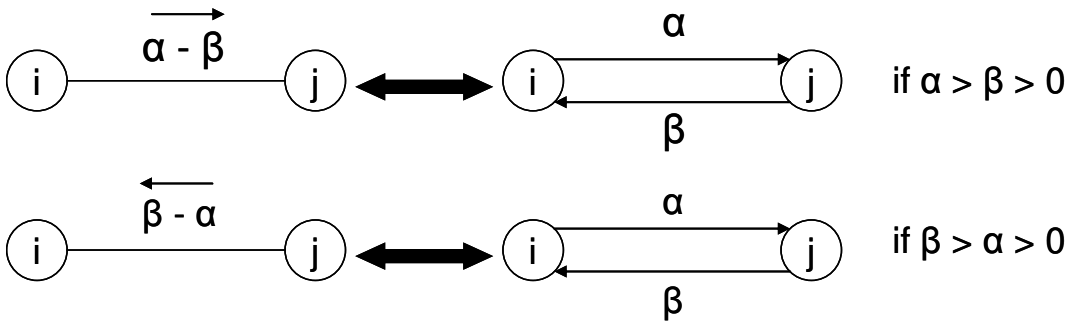
Some additional concepts from graph theory are needed before proceeding with the formulation of the problem. In the network established in the previous section, the edges are undirected, i.e., for each edge (i,j) in the network, power can flow from node i to node j and vice-versa, so long as the sum of the power flowing between the two nodes does not exceed the given capacity c_{ij}. The undirected network, however, can be transformed to a directed network as follows. For each undirected edge (i,j) with capacity c_{ij}, we create directed edges (i,j) and (j,i), each with capacity c_{ij}, as shown in the picture below.[18]



This transformation maintains the same overall capacity as the initial undirected network, but more formally permits the flow of power between any two nodes connected by an edge. We assume that power flow in either direction is non-negative. To demonstrate the equivalence of the undirected and directed edges above, consider the following two examples. If the flow from node i to node j in the undirected network is some positive value $\alpha \leq c_{ij}$, then in the transformed directed network, the flow along edge (i,j) will be α and the flow along (j,i) will be 0. If, on the other hand, there is positive flow along both edges (i,j) and (j,i) in the directed network, then the absolute value of the difference between the two is the flow in the undirected network, where the direction of the flow is

[18] More explicit details behind the transformation from an undirected network to a directed one can be found in *Network Flows* (Ahuja, Magnanti, Orlin).

dependent upon the larger directional flow value from the directed network. These illustrative flow values are depicted below.



Without loss of generality, we will focus on the directed network to more formally define a flow and a feasible demand function d. Let E be the set of directed edges in the transformed directed network defined as G = (N,E), where N is the set of demand nodes. Assume we have a source node s, which is adjacent to only outgoing edges, and a sink node t, which is adjacent to only inward edges.

A common problem in computer science is to calculate the maximum flow between two nodes in the graph, namely s and t, where a flow is a set of numbers $x = \{x_{ij}\}$ assigned to each directed edge (i,j) in the network such that the following conditions hold:

(F1) $\sum_{(j,i)\in E} x_{ji} - \sum_{(i,j)\in E} x_{ij} = 0$ for all nodes $i \in N \setminus \{s,t\}$

(F2) $0 \le x_{ij} \le c_{ij}$ for all directed edges $(i,j) \in E$

Condition (F1) is known as the flow balance constraint because it guarantees that the amount of flow going into a node equals the flow leaving the node, (for all nodes other

than s and t). Condition (F2) is the capacity constraint; i.e., the amount of flow between any two nodes is non-negative and cannot exceed the given capacity c_{ij}. The sum of all flow values leaving node s will be equal to the sum of all flows going into node t, which is the overall value of the s,t-flow in the network.

For purposes of this paper, we focus on the determination of a feasible flow, as opposed to a maximum flow. Returning to the transportation network G = (N,E) with demand function d(i), we say that d(i) is feasible, i.e., the network G admits a feasible flow, if there exists a flow x such that the following similar conditions are met: [19]

(FF1) $\sum_{(j,i)\in E} x_{ji} - \sum_{(i,j)\in E} x_{ij} = d(i)$ for all nodes $i \in N$

(FF2) $0 \leq x_{ij} \leq c_{ij}$ for all directed edges $(i,j) \in E$

Condition (FF1) states that the net power flow at each node is equal to the given demand (supply), while condition (FF2) ensures that the amount of power flowing from node i to node j is non-negative and does not exceed the given capacity c_{ij}.

We note first that feasible flow condition (FF1) can be restated as

$$\sum_{(j,i)\in E} x_{ji} = d(i) + \sum_{(i,j)\in E} x_{ij} .$$

[19] We note here that the network G is generic, i.e., it need not have a source node s or a sink node t.

In the case of a node with a positive demand value d(i) > 0, since all flow values x_{ij} are non-negative by condition (FF2), then we say, equivalently, that the demand function is feasible if

$$\sum_{(j,i)\in E} x_{ji} \geq d(i) \text{ for each node i.}$$

In other words, the flow going into a demand node i from all other nodes in the network that share an edge with node i must be at least as large at the demand at node i.

In the case of a supply node with demand function d(i) < 0, condition (FF1) is also equivalent to the following:

$$d(i) - \sum_{(j,i)\in E} x_{ji} = -\sum_{(i,j)\in E} x_{ij} \rightarrow d(i) \geq -\sum_{(i,j)\in E} x_{ij} \rightarrow -d(i) \leq \sum_{(i,j)\in E} x_{ij}\ .$$

That is, the flow going out of a supply node is at least the net supply at that node. Finally, we note that the demand values at disjoint nodes (or disjoint subsets of nodes) are additive, in which case the above conditions can be extrapolated to the general case defined below.[20]

[20] Similarly, it should further be noted that flow values and capacities are also additive among nodes and edges, respectively.

<u>Definition 1</u> - A demand function d in a network G = (N,E) is said to be feasible if there exists a flow x such that $d(S) \leq \sum_{(i,j)\in(S^C,S)} x_{ij}$, (or equivalently, if $-d(S) \leq \sum_{(i,j)\in(S,S^C)} x_{ij}$), for all $S \subseteq N$.

Therefore, the problem of determining the probability of a feasible flow in a stochastic transportation network can be stated according to the following theorem from Gale and Hoffman.

<u>Theorem</u> – We are given a directed transportation network G = (N,E), a demand function d(i) for each node $i \in N$, and non-negative edge capacities c_{ij} for all $(i,j) \in E$. We also assume that the lower bound on the flow value along any edge in G is zero and that the total demand across all nodes is zero, i.e., $\sum_{i\in N} d(i) = 0$.

The network G admits a feasible flow, or equivalently, the demand function d(i) is feasible, if and only if, for every subset of demand nodes $S \subseteq N$, we have the inequality

$$d(S) \leq \sum_{(i,j)\in(S^C,S)} c_{ij}$$

where $\sum_{(i,j)\in(S^C,S)} c_{ij}$ is the total available capacity into S.

<u>Proof</u>: [21] ($\Rightarrow$) Assume first that the network G admits a feasible flow x and let $S \subseteq N$ be a subset of nodes in the network. Since the network admits a feasible flow, then

$$d(S) \leq \sum_{(i,j)\in(S^C,S)} x_{ij} \text{ by Definition 1.}$$

Further, by condition (FF2) in the definition of a feasible flow, we have

$$d(S) \leq \sum_{(i,j)\in(S^C,S)} x_{ij} \leq \sum_{(i,j)\in(S^C,S)} c_{ij} .$$

Therefore, $d(S) \leq \sum_{(i,j)\in(S^C,S)} x_{ij} \leq \sum_{(i,j)\in(S^C,S)} c_{ij}$.

[21] The structure of this proof is courtesy of *Network Flows* (Ahuja, Magnanti, Orlin). The reader is also referred to *A Theorem on Flows in Networks* (Gale) for an alternative proof.

($\Leftarrow$) Assume that $d(S) \leq \sum_{(i,j)\in(S^C,S)} c_{ij}$ for all $S \subseteq N$. We proceed to construct a modified network:

(i) Create a new node s and let $N^* = N \cup \{s\}$.

(ii) Create an edge (s,i) for every supply node i such that $d(i) < 0$.

(iii) Create an edge (j,s) for every demand node j such that $d(j) > 0$.

(iv) Let $E' = \{(s,i) \cup (j,s) : i,j \in N\}$ and let l_{ij} be the lower bound on the flow along a given edge (i,j). Then define $l_{si} = c_{si} = |d(i)|$ and $l_{js} = c_{js} = |d(j)|$ for all (s,i), $(j,s) \in E'$ and let $E^* = E \cup E'$. (Recall that $l_{ij} = 0$ for all original edges $(i,j) \in E$).

(v) Call this transformed network $G^* = (N^*, E^*)$.

We will refer to node s as the source node. Steps (ii) and (iii) above establish new directed edges from the source node s to all supply nodes and from all demand nodes to the new source node s, respectively. In step (iv) the upper and lower bounds on these new edges are set equal to the absolute value of the demand (supply) at each node in the original network G. That is, the flow along these new edges E' is fixed. Further, we observe that $d(s) = 0$ by construction.

Now that we have some non-zero lower bounds, we slightly modify the feasible flow problem for the case of general lower bounds $l_{ij} \geq 0$. We call a flow in the network $G^* = (N^*,E^*)$ that satisfies the following constraints a feasible circulation. [22]

[22] Unlike condition (F1) the initial definition of a flow, we note that the balance constraint (FC1) in a feasible circulation applies to all nodes in the network.

(FC1) $\sum_{(j,i)\in E^*} x_{ji} - \sum_{(i,j)\in E^*} x_{ij} = 0$ for all nodes $i \in N^*$

(FC2) $l_{ij} \le x_{ij} \le c_{ij}$ for all directed edges $(i,j) \in E^*$

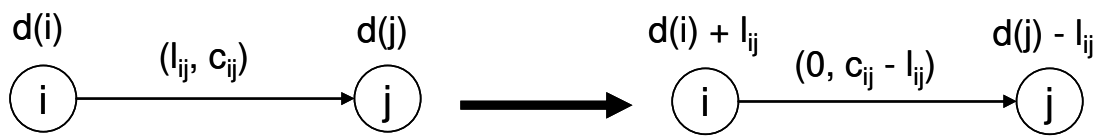
By the construction of G*, we can make the following claim:

<u>Claim 1</u> - The original network G admits a feasible flow if and only if the transformed network G* admits a feasible circulation.

This is because the net demand (supply) values at each node in the network G are forced to zero in the network G* by the creation of the edges E' and associated lower and upper bounds, under the assumption that the net demand (supply) values are met exactly in the network G. That is, the demand (supply) values in G are met exactly for each node if and only if we have a feasible circulation in G*.

In the interest of completeness, we offer an alternative equivalent formation of the feasible circulation problem by transforming the network G* in the above definition to remove the non-zero lower bounds. Introduce the notation x'_{ij} and let $x_{ij} = x'_{ij} + l_{ij}$ for each edge $(i,j) \in E^*$, where x'_{ij} represents the incremental flow above the lower bound on edge (i,j). Since we are sending at least l_{ij} units of flow from node i to node j, we increase d(i) by l_{ij} and decrease d(j) by the same amount because we now need more flow at node i and less at node j. Since we have already accounted for the flow of l_{ij} units, the lower bound and the capacity c_{ij} (upper bound) on the flow along edge (i,j) are both also

decreased by l_{ij} units, with the lower bound going to zero. The transformation is depicted below.



Incorporating this transformation for all nodes in the network G*, we can then restate the definition of a feasible circulation in terms of the incremental flow values x'_{ij}.

(FC1') $\sum_{(j,i)\in E^*} x'_{ji} - \sum_{(i,j)\in E^*} x'_{ij} = d(i)$ for all nodes $i \in N^*$

(FC2') $0 \le x'_{ij} \le c_{ij} - l_{ij}$ for all directed edges $(i,j) \in E^*$

We observe that condition (FC1) in the original network is equivalent to condition (FC1') in the transformed network, since both conditions are simultaneously satisfied if and only if the following equalities hold for each node $i \in N^*$:

$$\sum_{(j,i)\in E^*} x_{ji} - \sum_{(i,j)\in E^*} x_{ij} = 0$$

$$\rightarrow \sum_{(j,i)\in E^*} (x'_{ji} + l_{ji}) - \sum_{(i,j)\in E^*} (x'_{ij} + l_{ij}) = 0$$

$$\rightarrow \sum_{(j,i)\in E^*} x'_{ji} - \sum_{(i,j)\in E^*} x'_{ij} = \sum_{(i,j)\in E^*} l_{ij} - \sum_{(j,i)\in E^*} l_{ji} = d(i),$$

where the last equality follows directly from the transformation to remove non-zero lower bounds.

Similarly, we note that condition (FC2) is equivalent to condition (FC2') as both conditions are satisfied simultaneously if and only the following inequalities hold for all edges (i,j):

$$l_{ij} \leq x_{ij} \leq c_{ij}$$

$$\rightarrow \quad l_{ij} - l_{ij} \leq x_{ij} - l_{ij} \leq c_{ij} - l_{ij}$$

$$\rightarrow \quad 0 \leq x'_{ij} \leq c_{ij} - l_{ij} ,$$

where the last inequality follows directly from the definition of incremental flow values x'_{ij}.

Therefore, we conclude that the original conditions (FC1) and (FC2) are equivalent to the transformed conditions (FC1') and (FC2'), respectively.

Without loss of generality, we return to our original definition of a feasible circulation and restate conditions (FC1) and (FC2) in the general case of subsets of nodes $S \subseteq N^*$ instead of individual nodes:

$$(FC1^*) \qquad \sum_{(S^C,S)\in E^*} x_{ij} - \sum_{(S,S^C)\in E^*} x_{ij} = 0 \qquad \text{for all } S \subseteq N^*$$

$$(FC2^*) \qquad \sum_{(S,S^C)\in E^*} l_{ij} \leq \sum_{(S,S^C)\in E^*} x_{ij} \leq \sum_{(S,S^C)\in E^*} c_{ij} \qquad \text{for all } S \subseteq N^*$$

Combining conditions (FC1*) and (FC2*), we see that

$$\sum_{(S,S^C)\in E^*} l_{ij} \leq \sum_{(S,S^C)\in E^*} x_{ij} = \sum_{(S^C,S)\in E^*} x_{ij} \leq \sum_{(S^C,S)\in E^*} c_{ij} \text{ for all } S \subseteq N^*.$$

We have thus established necessary and sufficient conditions for a feasible circulation.[23]

<u>Claim 2</u> - A feasible circulation exists in the network G* = (N*, E*) if and only if

$$\sum_{(S,S^C)\in E^*} l_{ij} \leq \sum_{(S^C,S)\in E^*} c_{ij} \text{ for all } S \subseteq N^*.$$

At this point in the proof, we have assumed that $d(S) \leq \sum_{(i,j)\in(S^C,S)} c_{ij}$ for all $S \subseteq N$ in the original network G. For a set of nodes $S \subseteq N$, denote by S+ and S– the demand and supply nodes of S, respectively. More formally, let $S+ = \{i \in S : d(i) > 0\}$ and $S- = \{i \in S : d(i) < 0\}$. We will proceed with the remainder of the proof by considering two separate cases.

<u>*Case 1*</u> – Let $S \subseteq N$ be a subset of nodes in the original network G and define $S = S_0$ as the same set of nodes in the transformed network G*.

[23] The reader is also referred to *Network Flows* (Ahuja, Magnanti, Orlin) for more details behind the determination of a feasible circulation and the removal of non-zero lower bounds.

By construction of the transformed network G* at the beginning of the proof, namely step (iv), we observe that

$$\sum_{(S_o, S_o^C) \in E^*} l_{ij} = d(S+).$$

That is, the sum of the lower bounds leaving subset S_0 in G* is equal to the aggregate (positive) demand of all demand nodes in S from the original network. This is because all lower bounds in the original network G were assumed to be zero, and the only outward edges added in the transformed network G* are those associated with demand nodes, as prescribed in steps (iii) and (iv) above.

Similarly, we note that

$$\sum_{(S_o^C, S_o) \in E^*} c_{ij} = \sum_{(S^C, S) \in E} c_{ij} - d(S-).$$

In other words, the total inward capacity to S_0 in the transformed network G* is equal to the inward capacity to S in the original network G plus the absolute value of the aggregate supply of all supply nodes in S.[24] This is because the only inward edges added to the transformed network G* are those associated with supply nodes, as prescribed in steps (ii) and (iv) above.

[24] Since d(S-) < 0, subtracting it is equivalent to adding its absolute value.

Therefore, we have the following:

$$d(S) = d(S+) + d(S-) \qquad \text{because demand values are additive}$$

$$\rightarrow \quad d(S+) + d(S-) \leq \sum_{(i,j)\in(S^C,S)} c_{ij} \qquad \text{by assumption } d(S) \leq \sum_{(i,j)\in(S^C,S)} c_{ij}$$

$$\rightarrow \quad d(S+) \leq \sum_{(i,j)\in(S^C,S)} c_{ij} - d(S-)$$

$$\rightarrow \quad \sum_{(S_o,S_o^C)\in E^*} l_{ij} \leq \sum_{(S_o^C,S_o)\in E^*} c_{ij} \ .$$

<u>*Case 2*</u> - Let $S \subseteq N$ be a subset of nodes in the original network G and define $S \cup \{s\} = S_0$ as the corresponding subset of nodes in the transformed network G*.

We proceed in a similar fashion as in Case 1. First, we note that

$$\sum_{(S_o,S_o^C)\in E^*} l_{ij} = -\, d((S^c)-).$$

Since all edges in the original network are assumed to have zero lower bounds, and node s is included in S_0, then the only outward edges from S_0 in G* (with non-zero lower bounds) are those added in the transformation from G to G* connecting node s to the supply nodes in S^c.

Similarly, we note that

$$\sum_{(S_o^C, S_o) \in E^*} c_{ij} = \sum_{(S^C, S) \in E} c_{ij} + d((S^c)+).$$

This equality is derived from the fact that the only new incoming edges to S_0 in G*, (relative to the incoming edges to S in the original network G), are those coming into node s, namely from the demand nodes in S^c.

It follows that

$$d(S^c) = d((S^c)+) + d((S^c)-)$$

$$\rightarrow \quad -d(S^c) = -d((S^c)+) - d((S^c)-) = d(S) \quad \text{because } d(S) + d(S^c) = 0$$

$$\rightarrow \quad -d((S^c)+) - d((S^c)-) \leq \sum_{(S^C, S) \in E} c_{ij} \qquad \text{by assumption } d(S) \leq \sum_{(i,j) \in (S^C, S)} c_{ij}$$

$$\rightarrow \quad -d((S^c)-) \leq \sum_{(S^C, S) \in E} c_{ij} + d((S^c)+)$$

$$\rightarrow \quad \sum_{(S_o, S_o^C) \in E^*} l_{ij} \leq \sum_{(S_o^C, S_o) \in E^*} c_{ij}.$$

Since node s was the only node added in the formulation of network G*, then the combination of Case 1 and Case 2 covers all possible subsets $S_0 \subseteq N^*$. Thus, we conclude that the transformed network G* admits a feasible circulation by Claim 2 above.

Further, by Claim 1, we know that the original network G admits a feasible flow and the given demand function d(i) is feasible.

Based on the results of this theorem, we conclude that the aggregate capacity going into a subset of nodes S must be at least as large as the net demand of the set S. Otherwise, there is the potential that the net demand of the set S cannot be satisfied because there is insufficient capacity to meet it.

We now return to our 5-node example and set up the Gale-Hoffman inequalities.

(1)	$\xi_1 - x_1$															$\leq$	y_{12}						
(2)			$\xi_2 - x_2$													$\leq$	y_{12}	+	y_{23}	+	y_{24}		
(3)					$\xi_3 - x_3$											$\leq$			y_{23}			+	y_{35}
(4)								$\xi_4 - x_4$								$\leq$					y_{24}		+ y_{45}
(5)											$\xi_5 - x_5$					$\leq$						y_{35}	+ y_{45}
(6)	$\xi_1 - x_1$	+	$\xi_2 - x_2$													$\leq$			y_{23}	+	y_{24}		
(7)	$\xi_1 - x_1$				+ $\xi_3 - x_3$											$\leq$	y_{12}	+	y_{23}			+	y_{35}
(8)	$\xi_1 - x_1$							+ $\xi_4 - x_4$								$\leq$	y_{12}			+	y_{24}		+ y_{45}
(9)	$\xi_1 - x_1$										+ $\xi_5 - x_5$					$\leq$	y_{12}					y_{35}	+ y_{45}
(10)			$\xi_2 - x_2$		+ $\xi_3 - x_3$											$\leq$	y_{12}			+	y_{24}	+ y_{35}	
(11)			$\xi_2 - x_2$					+ $\xi_4 - x_4$								$\leq$	y_{12}	+	y_{23}				+ y_{45}
(12)			$\xi_2 - x_2$								+ $\xi_5 - x_5$					$\leq$	y_{12}	+	y_{23}	+	y_{24}	+ y_{35}	+ y_{45}
(13)					$\xi_3 - x_3$			+ $\xi_4 - x_4$								$\leq$			y_{23}	+	y_{24}	+ y_{35}	+ y_{45}
(14)					$\xi_3 - x_3$						+ $\xi_5 - x_5$					$\leq$			y_{23}				+ y_{45}
(15)								$\xi_4 - x_4$			+ $\xi_5 - x_5$					$\leq$					y_{24}	+ y_{35}	
(16)	$\xi_1 - x_1$	+	$\xi_2 - x_2$		+ $\xi_3 - x_3$											$\leq$					y_{24}	+ y_{35}	
(17)	$\xi_1 - x_1$	+	$\xi_2 - x_2$					+ $\xi_4 - x_4$								$\leq$			y_{23}				+ y_{45}
(18)	$\xi_1 - x_1$	+	$\xi_2 - x_2$								+ $\xi_5 - x_5$					$\leq$			y_{23}	+	y_{24}	+ y_{35}	+ y_{45}
(19)	$\xi_1 - x_1$				+ $\xi_3 - x_3$			+ $\xi_4 - x_4$								$\leq$	y_{12}	+	y_{23}	+	y_{24}	+ y_{35}	+ y_{45}
(20)	$\xi_1 - x_1$				+ $\xi_3 - x_3$						+ $\xi_5 - x_5$					$\leq$	y_{12}	+	y_{23}				+ y_{45}
(21)	$\xi_1 - x_1$							+ $\xi_4 - x_4$			+ $\xi_5 - x_5$					$\leq$	y_{12}			+	y_{24}	+ y_{35}	
(22)			$\xi_2 - x_2$		+ $\xi_3 - x_3$			+ $\xi_4 - x_4$								$\leq$	y_{12}					+ y_{35}	+ y_{45}
(23)			$\xi_2 - x_2$		+ $\xi_3 - x_3$						+ $\xi_5 - x_5$					$\leq$	y_{12}			+	y_{24}		+ y_{45}
(24)			$\xi_2 - x_2$					+ $\xi_4 - x_4$			+ $\xi_5 - x_5$					$\leq$	y_{12}	+	y_{23}			+ y_{35}	
(25)					$\xi_3 - x_3$			+ $\xi_4 - x_4$			+ $\xi_5 - x_5$					$\leq$			y_{23}	+	y_{24}		
(26)	$\xi_1 - x_1$	+	$\xi_2 - x_2$		+ $\xi_3 - x_3$			+ $\xi_4 - x_4$								$\leq$						y_{35}	+ y_{45}
(27)	$\xi_1 - x_1$				+ $\xi_3 - x_3$			+ $\xi_4 - x_4$			+ $\xi_5 - x_5$					$\leq$	y_{12}	+	y_{23}	+	y_{24}		
(28)	$\xi_1 - x_1$	+	$\xi_2 - x_2$					+ $\xi_4 - x_4$			+ $\xi_5 - x_5$					$\leq$			y_{23}			+ y_{35}	
(29)	$\xi_1 - x_1$	+	$\xi_2 - x_2$		+ $\xi_3 - x_3$						+ $\xi_5 - x_5$					$\leq$					y_{24}		+ y_{45}
(30)			$\xi_2 - x_2$		+ $\xi_3 - x_3$			+ $\xi_4 - x_4$			+ $\xi_5 - x_5$					$\leq$	y_{12}						
(31)	$\xi_1 - x_1$	+	$\xi_2 - x_2$		+ $\xi_3 - x_3$			+ $\xi_4 - x_4$			+ $\xi_5 - x_5$					$\leq$	0						

We observe that there are a total of $31 = 2^n - 1$ Gale-Hoffman inequalities, where n = 5. That is, there is an inequality for each combination of the n=5 nodes in the network, with the exception of the case where S is the empty set, which is trivial because d(S) = 0 and all transmission capacities are non-negative.

2d. Elimination of Redundant Inequalities

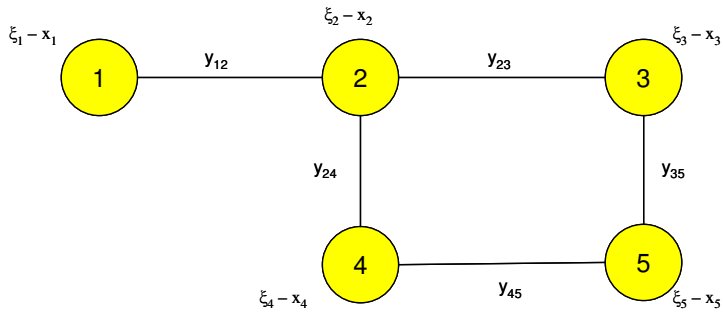
We observe that the number of inequalities ($2^n - 1$) in the network formulation grows exponentially with the number of demand nodes, which can cause run time problems for solving. In an effort to make the problem more manageable, we will discuss and implement in our example an algorithm prescribed by Prékopa and Boros to eliminate any redundant inequalities, i.e., those inequalities that can be written as a sum of others.[25]

The algorithm is described as follows:

0. Let $b(H) = 1$ and $e(H) = 0$ for all $H \subseteq N$, where H is non-empty
1. Choose a non-empty subset $H \subseteq N$ such that $b(H) = 1$ and $e(H) = 0$. (If no such subset exists, then STOP).
2. Let $T \subseteq N \setminus H$ be a maximal subset such that there is no arc between T and H
3. Let $b(V) = 0$ for all $V \subseteq H \cup T$, where $V \cap T$ and $V \cap H$ are both non-empty.
4. Let $e(H) = 1$ and return to Step 1.

Recall the network formulation in our example and note the initial results from Step 0:

[25] See Prékopa and Boros (1989).



Inequality	Set H	b(H)	e(H)
(1)	{1}	1	0
(2)	{2}	1	0
(3)	{3}	1	0
(4)	{4}	1	0
(5)	{5}	1	0
(6)	{1,2}	1	0
(7)	{1,3}	1	0
(8)	{1,4}	1	0
(9)	{1,5}	1	0
(10)	{2,3}	1	0
(11)	{2,4}	1	0
(12)	{2,5}	1	0
(13)	{3,4}	1	0
(14)	{3,5}	1	0
(15)	{4,5}	1	0
(16)	{1,2,3}	1	0
(17)	{1,2,4}	1	0
(18)	{1,2,5}	1	0
(19)	{1,3,4}	1	0
(20)	{1,3,5}	1	0
(21)	{1,4,5}	1	0
(22)	{2,3,4}	1	0
(23)	{2,3,5}	1	0
(24)	{2,4,5}	1	0
(25)	{3,4,5}	1	0
(26)	{1,2,3,4}	1	0
(27)	{1,3,4,5}	1	0
(28)	{1,2,4,5}	1	0
(29)	{1,2,3,5}	1	0
(30)	{2,3,4,5}	1	0
(31)	{1,2,3,4,5}	1	0

We will proceed to process the inequalities in numerical order, starting with H = {1}.

<u>Iteration 1</u>

Step 1: H = {1}

Step 2: T = {3,4,5}

Step 3: Denote by V' all possible subsets of $H \bigcup T$. Then we have the following:

V'	{1}	{3}	{4}	{5}	{1,3}	{1,4}	{1,5}	{3,4}	{3,5}	{4,5}	{1,3,4}	{1,4,5}	{3,4,5}	{1,3,4,5}
$V' \bigcap T$	0	{3}	{4}	{5}	{3}	{4}	{5}	{3,4}	{3,5}	{4,5}	{3,4}	{4,5}	{3,4,5}	{3,4,5}
$V' \bigcap H$	{1}	0	0	0	{1}	{1}	{1}	0	0	0	{1}	{1}	0	{1}
V	NO	NO	NO	NO	YES	YES	YES	NO	NO	NO	YES	YES	NO	YES

It follows that V = {1,3},{1,4},{1,5},{1,3,4},{1,4,5},{1,3,4,5} and b(V) = 0 for all V.

Step 4: e({1}) = 1

We update our initial table of values incorporating the results of Iteration 1:

Inequality	Set H	Iteration 0		Iteration 1	
		b(H)	e(H)	b(H)	e(H)
(1)	{1}	1	0	**1**	**1**
(2)	{2}	1	0	1	0
(3)	{3}	1	0	1	0
(4)	{4}	1	0	1	0
(5)	{5}	1	0	1	0
(6)	{1,2}	1	0	1	0
(7)	{1,3}	1	0	**0**	**0**
(8)	{1,4}	1	0	**0**	**0**
(9)	{1,5}	1	0	**0**	**0**
(10)	{2,3}	1	0	1	0
(11)	{2,4}	1	0	1	0
(12)	{2,5}	1	0	1	0
(13)	{3,4}	1	0	1	0
(14)	{3,5}	1	0	1	0
(15)	{4,5}	1	0	1	0
(16)	{1,2,3}	1	0	1	0
(17)	{1,2,4}	1	0	1	0
(18)	{1,2,5}	1	0	1	0
(19)	{1,3,4}	1	0	**0**	**0**
(20)	{1,3,5}	1	0	1	0
(21)	{1,4,5}	1	0	**0**	**0**
(22)	{2,3,4}	1	0	1	0
(23)	{2,3,5}	1	0	1	0
(24)	{2,4,5}	1	0	1	0
(25)	{3,4,5}	1	0	1	0
(26)	{1,2,3,4}	1	0	1	0
(27)	{1,3,4,5}	1	0	**0**	**0**
(28)	{1,2,4,5}	1	0	1	0
(29)	{1,2,3,5}	1	0	1	0
(30)	{2,3,4,5}	1	0	1	0
(31)	{1,2,3,4,5}	1	0	1	0

Before moving on to the next iteration, we make a few observations. First, we note that each subset $H \subseteq N$ is processed at most once. Obviously, a given subset H cannot be processed more than one time because once its value e(H) changes from 0 to 1, it is no longer a candidate to be processed again in Step 1. Further, there are no other steps in the

algorithm to change the value of e(H) back to 0. Still, not all subsets $H \subseteq N$ need to be processed at all. Consider one of the subsets V from Iteration 1 above. Since b(V) = 0, then V is not an eligible candidate to be processed in Step 1. Similar to the preceding argument regarding a given subset H, once b(V) changes from 1 to 0, there are no other steps in the algorithm to change the value back to 1. Second, we notice that there may be iterations in which there are no subsets T defined in Step 2 of the algorithm, i.e., T can be the empty set. In such instances, there are no subsets V defined by Step 3, in which case the only update made by the algorithm is to change the value of e(H) from 0 to 1. Finally, we comment that a particular subset can be defined as a subset V in Step 3 of multiple iterations of the algorithm. However, since its value b(V) changed from 1 to 0 the first time it was processed as a subset V, the value does not change again and it remains at 0 for the duration of the algorithm.

The binary variable e(H) is simply an indicator of whether a particular subset has been processed through step 1 of the algorithm. The binary variable b(H), though, designates which inequalities can be eliminated due to redundancy. All subsets H with b(H) = 0 at the end of the algorithm correspond to the inequalities that can be eliminated due to redundancy.[26]

We conclude that the inequalities corresponding to the subsets V in Step 3 above can be eliminated because they are the sum of other inequalities. More formally, we have

[26] Please refer to Prékopa and Boros (*On the Existence of a Feasible Flow in a Stochastic Transportation Network*, 1989) for more details behind the development of the algorithm and corresponding supporting theorems.

$V = \{1,3\}$: $(\xi_1 - x_1) + (\xi_3 - x_3) \leq y_{12} + y_{23} + y_{35}$

$= (\xi_1 - x_1 \leq y_{12}) + (\xi_3 - x_3 \leq y_{23} + y_{35}) \Rightarrow (7) = (1) + (3)$

$V = \{1,4\}$: $(\xi_1 - x_1) + (\xi_4 - x_4) \leq y_{12} + y_{24} + y_{45}$

$= (\xi_1 - x_1 \leq y_{12}) + (\xi_4 - x_4 \leq y_{24} + y_{45}) \Rightarrow (8) = (1) + (4)$

$V = \{1,5\}$: $(\xi_1 - x_1) + (\xi_5 - x_5) \leq y_{12} + y_{35} + y_{45}$

$= (\xi_1 - x_1 \leq y_{12}) + (\xi_5 - x_5 \leq y_{35} + y_{45}) \Rightarrow (9) = (1) + (5)$

$V = \{1,3,4\}$: $(\xi_1 - x_1) + (\xi_3 - x_3) + (\xi_4 - x_4) \leq y_{12} + y_{23} + y_{24} + y_{35} + y_{45}$

$= (\xi_1 - x_1 \leq y_{12}) + (\xi_3 - x_3 \leq y_{23} + y_{35}) + (\xi_4 - x_4 \leq y_{24} + y_{45})$

$\Rightarrow (19) = (1) + (3) + (4)$

$V = \{1,4,5\}$: $(\xi_1 - x_1) + (\xi_4 - x_4) + (\xi_5 - x_5) \leq y_{12} + y_{24} + y_{35}$

$= (\xi_1 - x_1 \leq y_{12}) + ((\xi_4 - x_4) + (\xi_5 - x_5) \leq y_{24} + y_{35}) \Rightarrow (21) = (1) + (15)$

$V = \{1,3,4,5\}$: $(\xi_1 - x_1) + (\xi_3 - x_3) + (\xi_4 - x_4) + (\xi_5 - x_5) \leq y_{12} + y_{23} + y_{24}$

$= (\xi_1 - x_1 \leq y_{12}) + ((\xi_3 - x_3) + (\xi_4 - x_4) + (\xi_5 - x_5) \leq y_{23} + y_{24}) \Rightarrow (27) = (1) + (25)$

Based on the above observations and detailed discussion on the results of the first iteration, we can proceed in a somewhat abbreviated fashion to complete the remaining iterations of the algorithm for our particular example.

<u>Iteration 2</u>

Step 1: H = {2}

Step 2: T = {5}

Step 3: V = {2,5} $\Rightarrow$ b({2,5}) = 0

Step 4: e({2}) = 0

V = {2,5}: $(\xi_2 - x_2) + (\xi_5 - x_5) \leq y_{12} + y_{23} + y_{24} + y_{35} + y_{45}$

$= (\xi_2 - x_2 \leq y_{12} + y_{23} + y_{24}) + (\xi_5 - x_5 \leq y_{35} + y_{45}) \Rightarrow (12) = (2) + (5)$

<u>Iteration 3</u>

Step 1: H = {3}

Step 2: T = {1,4}

Step 3: V = {1,3}, {3,4}, {1,3,4} $\Rightarrow$ b(V) = 0 for all V

Step 4: e({3}) = 0

Since b({1,3}) = 0 and b({1,3,4}) = 0 from Iteration 1, we only need to process V = {3,4}.

V = {3,4}: $(\xi_3 - x_3) + (\xi_4 - x_4) \leq y_{23} + y_{24} + y_{35} + y_{45}$

$= (\xi_3 - x_3 \leq y_{23} + y_{35}) + (\xi_4 - x_4 \leq y_{24} + y_{45}) \Rightarrow (13) = (3) + (4)$

<u>Iteration 4</u>

Step 1: H = {4}

Step 2: T = {1,3}

Step 3: V = {1,4}, {3,4}, {1,3,4} $\Rightarrow$ b(V) = 0 for all V

Step 4: e({4}) = 0

All subsets V have already been processed and b(V) = 0, so we move on.

<u>Iteration 5</u>

Step 1: H = {5}

Step 2: T = {1,2}

Step 3: V = {1,5}, {2,5}, {1,2,5} $\Rightarrow$ b(V) = 0 for all V

Step 4: e({5}) = 0

Since b({1,5}) = 0 from Iteration 1 and b({2,5}) = 0 from Iteration 2, we only need to process V = {1,2,5}.

V = {1,2,5}: $(\xi_1 - x_1) + (\xi_2 - x_2) + (\xi_5 - x_5) \leq y_{23} + y_{24} + y_{35} + y_{45}$

$= ((\xi_1 - x_1) + (\xi_2 - x_2) \leq y_{23} + y_{24}) + (\xi_5 - x_5 \leq y_{35} + y_{45}) \Rightarrow$ (18) = (6) + (5)

<u>Iteration 6</u>

Step 1: H = {1,2}

Step 2: T = {5}

Step 3: V = {1,5}, {2,5}, {1,2,5} $\Rightarrow$ b(V) = 0 for all V

Step 4: e({1,2}) = 0

All subsets V have already been processed and b(V) = 0, so we move on.

The next subsets to be processed are {1,3}, {1,4}, and {1,5}, all of which have been eliminated in previous iterations. So, we move on to the next subsets H in the listed order, namely {2,3} and {2,4} in Iterations 7 and 8, respectively. However, since both of these subsets are adjacent to every other node in the network, then set T is the empty set, as defined in Step 2 of the algorithm (and noted in the initial observation following Iteration 1). These two iterations are trivial and the only updates need are to make e({2,3}) = 1 and e({2,4}) = 1.

The next two subsets, {2,5}, and {3,4}, have already been eliminated in Iterations 2 and 3, respectively. Thus, we move on to the next subset in our list.

<u>Iteration 9</u>

Step 1: H = {3,5}

Step 2: T = {1}

Step 3: V = {1,3}, {1,5}, {1,3,5} $\Rightarrow$ b(V) = 0 for all V

Step 4: e({3,5}) = 0

Since b({1,3}) = 0 and b({1,5}) = 0 from Iteration 1, we only need to process V = {1,3,5}.

V = {1,3,5}: $(\xi_1 - x_1) + (\xi_3 - x_3) + (\xi_5 - x_5) \leq y_{12} + y_{23} + y_{45}$

$= (\xi_1 - x_1 \leq y_{12}) + ((\xi_3 - x_3) + (\xi_5 - x_5) \leq y_{23} + y_{45}) \Rightarrow (20) = (1) + (14)$

<u>Iteration 10</u>

Step 1: H = {4,5}

Step 2: T = {1}

Step 3: V = {1,4}, {1,5}, {1,4,5} $\Rightarrow$ b(V) = 0 for all V

Step 4: e({4,5}) = 0

All subsets V have already been processed in Iteration 1 and b(V) = 0, so we move on.

Similar to the arguments made for Iterations 7 and 8 above, we have trivial updates for the following iterations:

Iteration 11: H = {1,2,3} $\Rightarrow$ e({1,2,3}) = 1

Iteration 12: H = {1,2,4} $\Rightarrow$ e({1,2,4}) = 1

[27]Iteration 13: H = {2,3,4} $\Rightarrow$ e({2,3,4}) = 1

Iteration 14: H = {2,3,5} $\Rightarrow$ e({2,3,5}) = 1

Iteration 15: H = {2,4,5} $\Rightarrow$ e({2,4,5}) = 1

<u>Iteration 16</u>

Step 1: H = {3,4,5}

Step 2: T = {1}

Step 3: V = {1,3}, {1,4}, {1,5}, {1,3,4}, {1,4,5}, {1,3,4,5} $\Rightarrow$ b(V) = 0 for all V

Step 4: e({3,4,5}) = 0

All subsets V have already been processed and b(V) = 0, so we move on.

The algorithm terminates following the remaining trivial iterations:

Iteration 17: H = {1,2,3,4} $\Rightarrow$ e({1,2,3,4}) = 1

[28]Iteration 18: H = {1,2,4,5} $\Rightarrow$ e({1,2,4,5}) = 1

Iteration 19: H = {1,2,3,5} $\Rightarrow$ e({1,2,3,5}) = 1

Iteration 20: H = {2,3,4,5} $\Rightarrow$ e({2,3,4,5}) = 1

Iteration 21: H = {1,2,3,4,5} $\Rightarrow$ e({1,2,3,4,5}) = 1

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[27] Subsets {1,2,5}, {1,3,4}, {1,3,5}, and {1,4,5}, while next in sequential order, have already been eliminated in prior iterations and therefore do not need to be processed. Thus, we move on to {2,3,4}.
[28] Subset {1,3,4,5}, while next in sequential order, has already been eliminated in Iteration 1 and therefore does not need to be processed. Thus, we move on to {1,2,4,5}.

The final results of the algorithm are summarized in the table below.

Inequality	Set H	FINAL	
		b(H)	e(H)
(1)	{1}	1	1
(2)	{2}	1	1
(3)	{3}	1	1
(4)	{4}	1	1
(5)	{5}	1	1
(6)	{1,2}	1	1
(7)	{1,3}	0	0
(8)	{1,4}	0	0
(9)	{1,5}	0	0
(10)	{2,3}	1	1
(11)	{2,4}	1	1
(12)	{2,5}	0	0
(13)	{3,4}	0	0
(14)	{3,5}	1	1
(15)	{4,5}	1	1
(16)	{1,2,3}	1	1
(17)	{1,2,4}	1	1
(18)	{1,2,5}	0	0
(19)	{1,3,4}	0	0
(20)	{1,3,5}	0	0
(21)	{1,4,5}	0	0
(22)	{2,3,4}	1	1
(23)	{2,3,5}	1	1
(24)	{2,4,5}	1	1
(25)	{3,4,5}	1	1
(26)	{1,2,3,4}	1	1
(27)	{1,3,4,5}	0	0
(28)	{1,2,4,5}	1	1
(29)	{1,2,3,5}	1	1
(30)	{2,3,4,5}	1	1
(31)	{1,2,3,4,5}	1	1

The algorithm results in the elimination of 10 of the 31 inequalities. The remaining inequalities, renumbered in numerical order, are:

(1)	$\xi_1 - x_1$													$\leq$	y_{12}					
(2)			$\xi_2 - x_2$											$\leq$	y_{12}	+	y_{23}	+	y_{24}	
(3)					$\xi_3 - x_3$									$\leq$			y_{23}			+ y_{35}
(4)								$\xi_4 - x_4$						$\leq$				y_{24}		+ y_{45}
(5)											$\xi_5 - x_5$			$\leq$					y_{35}	+ y_{45}
(6)	$\xi_1 - x_1$	+	$\xi_2 - x_2$											$\leq$			y_{23}	+	y_{24}	
(7)			$\xi_2 - x_2$	+	$\xi_3 - x_3$									$\leq$	y_{12}			+	y_{24}	+ y_{35}
(8)			$\xi_2 - x_2$					+	$\xi_4 - x_4$					$\leq$	y_{12}	+	y_{23}			+ y_{45}
(9)					$\xi_3 - x_3$						+	$\xi_5 - x_5$		$\leq$			y_{23}			+ y_{45}
(10)								$\xi_4 - x_4$			+	$\xi_5 - x_5$		$\leq$				y_{24}	+	y_{35}
(11)	$\xi_1 - x_1$	+	$\xi_2 - x_2$	+	$\xi_3 - x_3$									$\leq$				y_{24}	+	y_{35}
(12)	$\xi_1 - x_1$	+	$\xi_2 - x_2$					+	$\xi_4 - x_4$					$\leq$			y_{23}			+ y_{45}
(13)			$\xi_2 - x_2$	+	$\xi_3 - x_3$			+	$\xi_4 - x_4$					$\leq$	y_{12}				+	y_{35} + y_{45}
(14)			$\xi_2 - x_2$	+	$\xi_3 - x_3$						+	$\xi_5 - x_5$		$\leq$	y_{12}			+	y_{24}	+ y_{45}
(15)			$\xi_2 - x_2$					+	$\xi_4 - x_4$		+	$\xi_5 - x_5$		$\leq$	y_{12}	+	y_{23}		+	y_{35}
(16)					$\xi_3 - x_3$			+	$\xi_4 - x_4$		+	$\xi_5 - x_5$		$\leq$			y_{23}	+	y_{24}	
(17)	$\xi_1 - x_1$	+	$\xi_2 - x_2$	+	$\xi_3 - x_3$			+	$\xi_4 - x_4$					$\leq$					y_{35}	+ y_{45}
(18)	$\xi_1 - x_1$	+	$\xi_2 - x_2$					+	$\xi_4 - x_4$		+	$\xi_5 - x_5$		$\leq$			y_{23}		+	y_{35}
(19)	$\xi_1 - x_1$	+	$\xi_2 - x_2$	+	$\xi_3 - x_3$						+	$\xi_5 - x_5$		$\leq$				y_{24}		+ y_{45}
(20)			$\xi_2 - x_2$	+	$\xi_3 - x_3$			+	$\xi_4 - x_4$		+	$\xi_5 - x_5$		$\leq$	y_{12}					
(21)	$\xi_1 - x_1$	+	$\xi_2 - x_2$	+	$\xi_3 - x_3$			+	$\xi_4 - x_4$		+	$\xi_5 - x_5$		$\leq$	0					

Our problem has been significantly reduced from 31 to 21 inequalities, though our objective remains the same -- find the probability that the remaining inequalities are satisfied simultaneously.[29] Thus, we move on to discuss three alternative approaches for solving this problem.

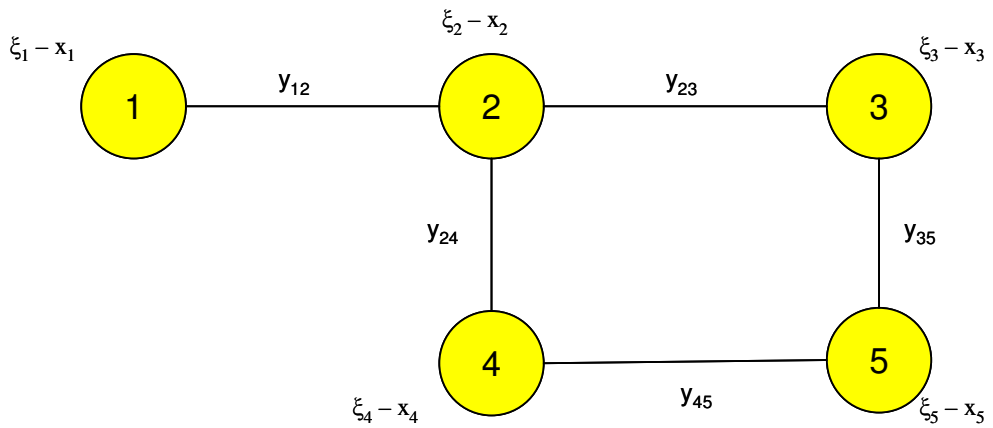
[29] We note that additional inequalities may be eliminated via a subsequent method; however, details around the assumed demand and supply values are needed to do so. We will address a subsequent inequality elimination procedure later on in the paper.

3 METHODS TO SOLVE

3a. Multivariate Normal Probability

Background

Before proceeding with a description of the multivariate normal probability method, we first return to our network and elaborate on its formulation from a stochastic perspective.



We assume that the local generating capacity values x_i are random variables. Consider an individual power plant. Its generating capacity is generally not constant, but rather, fluctuates over time depending on the power plant's efficiency, fuel type, and operational strategy. Individual power plants often have to take outages. These outages can be forced due to operational issues, or planned for maintenance or refueling. Further, some power plants are designed to run continuously, while others may only be used at certain times based on economic or engineering conditions. The overall efficiency of the installed equipment at each power plant also has an impact on its available generating

capacity at any point in time. So the available generating capacity on an individual power plant fluctuates over a range of values from zero to the maximum rated capacity of the unit. Thus, the available generating capacity of an individual power plant is a random variable. It follows that the local generating capacity values x_i , which represent the sum of the capacities of the individual power plants in the area, are also random variables.[30]

Similarly, we note that the local demand values ξ_i are also random variables. Whether in the case of a residential customer who turns her lights and appliances on and off at different times of the day, a commercial customer whose business is not open around the clock, or even a large industrial customer who runs different shifts and may occasionally have to take his own internal maintenance outage, an individual customer's demand for electricity is not constant. Thus, we see that the demand of an individual customer varies over a range of possible values from zero to some finite number based on the quantity and efficiency of the appliances, machines, etc. that are available to be used by the customer at any point in time. Thus, each customer's individual demand can be thought of as a random variable and the local demand values ξ_i , which represent the total demand of all customers in the area, are also random variables. Since x_i and ξ_i are random variables, it follows that the (net) demand values $d(i) = \xi_i - x_i$ are also random variables.[31]

We further assume that the random variables d(i) are identically distributed, i.e., each random variable d(i) has the same probability distribution. The value of each net

[30] This is due to the fundamental property that the sum (difference) of random variables is a random variable. See Springer (1979).
[31] Id. The term "$-x_i$" is simply the scalar -1 multiplied by the random variable x_i, which is also a random variable based on the fundamental properties of random variables.

demand value d(i) is based on the available generating capacity of the local power plants and the behavior of the customers located in the vicinity. Given that the network presented above is based on a contiguous geographic area, we would expect the power plants across the network to have been constructed and operate in a relatively consistent fashion. As noted above, individual generating units can operate at different efficiencies and under different conditions; however, each demand node in the network includes a mix of different power plants that collectively, on average, are not significantly different in the efficiency of their operations as compared to any other node in the network. Further, the overall network represents an area of relatively similar socio-economic conditions. Thus, we would expect the average customer behavior within each area to behave relatively the same. A residential customer's demand profile is going to be different than, say, a large industrial customer's, but each geographic area in the network represents a mix of different customers that collectively, on average, are not significantly different in their demand for electricity as compared to any other node in the network. Thus, the assumption that the random variables d(i) are identically distributed is reasonable.[32]

Finally, we claim that the random variables d(i) are independent. As previously stated, each power plant in the network is operated as a stand-alone entity. Its available capacity generally does not depend on the operation of neighboring generating units or even the

[32] As noted above in Section 1, weather can play a significant role in the supply of and demand for electricity. This assumption that the values d(i) are identically distributed is further supported if the values of the random variables are considered over a time period of similar weather conditions. For example, we would expect customers to behave in a more similar fashion over the summer months than they would over the course of an entire calendar year of different weather conditions.

behavior of end-use customers. [33] We also assume that each end-use customer included in the network consumes electricity independent of other customers and the available generating units. This again is not unreasonable as each customer, regardless of class, has a demand profile that is unique based on his or her individual electricity requirements. Since each power plant and each end-use customer behaves independently, and each net demand value d(i) includes the aggregate net impact of multiple generating units and customers, the assumption that the random variables d(i) are independent is not unreasonable.

Therefore, based on the discussion above, we conclude that the demand values d(i) are independent, identically distributed, random variables. According to the Central Limit Theorem, we know that for large enough samples, the average value of each random variable d(i) can be approximated by the normal distribution.[34] In the context of this paper, we will consider the sample size to be a series of net demand values over a particular time interval. Thus, under these assumptions the net demand values of each random variable d(i), when considered over an appropriate time period, will approximately follow a normal distribution. Further, we recall that the left-hand-side (LHS) values of the remaining inequalities form Section 2 represent the sums of the net demand values of the various combinations of nodes in the network. It follows that these

[33] There may be instances where individual power plants are operated in a manner that follows customer demand in the area, in which case the available generating capacity is somewhat dependent on the customer demand. However, the impact of such instances is not significant for purposes of this analysis, particularly when considering that each node in the network represents a number of different power plants..

[34] For a more detailed description of the Central Limit Theorem, the reader is referred to Feller (1968).

LHS values can also be approximated by the normal distribution.[35] Applying this result to the remaining inequalities from Section 2 above, our problem can be solved using the multivariate normal probability distribution.

Before proceeding with the mathematical formulation, we next note that the assumption of normal distribution, and corresponding use of the multivariate normal probability distribution, are not uncommon. In *Fundamentals of Probability and Statistics for Engineers* (2004), Soong offers an analogous example of gasoline consumption, arguing that the amount of gasoline consumed by all vehicles of a particular brand will tend to behave under a normal distribution. Consider the following excerpt:

> "…(W)hen the randomness in a physical phenomenon is the culmination of many small additive random effects, it tends to a normal distribution irrespective of the distributions of individual effects. For example, the gasoline consumption of all automobiles of a particular brand, supposedly manufactured under identical processes, differs from one automobile to another. This randomness stems from a wide variety of sources, including, among other things: inherent inaccuracies in manufacturing processes, nonuniformities in materials used, differences in weight and other specifications, difference in gasoline quality, and different driver behavior. If one accepts the fact that each of these differences contribute to the randomness in gasoline consumption, the central limit theorem tells us that it tends to a normal distribution."[36]

Obviously, the commodity of gasoline in Soong's example is analogous to electricity. Further, the sources of randomness noted by Soong are comparable to the characteristics of power plants (i.e., "manufacturing processes") and electric consumers (i.e., "drivers") discussed earlier in this section. An analogous problem to the one in this paper, then, would be to determine the probability that sufficient gasoline is manufactured and delivered to gas stations to satisfy drivers' demand. Under Soong's arguments, this

[35] This is because the sum of independent, normally distributed random variables is a normally distributed random variable. See Ross (2000).
[36] See Soong (2004, pg. 200).

problem could be solved using multivariate normal probability distribution, based on the assumption that the relevant data are normally distributed courtesy of the Central Limit Theorem.

Finally, we note that the multivariate normal probability distribution has been studied extensively by others, including Prékopa (1995), Szántai (1985), and Genz and Bretz (2009).[37] We now proceed with the mathematical formulation of this method.

[37] Complete references are provided in the References section at the end of the paper.

Methodology

As discussed above, we assume that the relevant data in our example are normally distributed. In the general case, suppose we have m random variables (demand nodes) and that there are k inequalities remaining after the elimination procedure explained in Section 2d. Based on our assumption, we know that each of the values of each of the m demand nodes is normally distributed and further, that the LHS values of each of the remaining k inequalities, each represented as a linear combination of the m demand nodes, is also a normally distributed random variable. It follows that our problem can be stated as finding the multivariate normal probability distribution of all k random variables (inequalities), i.e., in our example, the probability that all 12 remaining inequalities are satisfied simultaneously. This is further explained in mathematical notation below.

Per our definitions above, recall that d(i) is the net demand (supply) at node i, which we have assumed to be normally distributed for each i = 1,…,m. The LHS of each inequality j can be written as $\eta_j = \sum_{i=1}^{m} a_{ji} d(i)$ for each j=1,…,k, where the coefficient a_{ji} is 1 or 0. By our assumption, η_j is also a normally distributed random variable for each j.

Next, let $\boldsymbol{\eta} = [\eta_1, \eta_2, \ldots, \eta_k]^T$ be the k-dimensional vector representing the LHS values of the remaining k inequalities, each as a linear combination of the m random variables (demand nodes). Then $\boldsymbol{\eta}$ is said to have multivariate normal distribution, denoted by $\boldsymbol{\eta} = N_k(\mu,\Sigma)$, where μ is the k-dimensional vector representing the means (averages) of each random variable η_j and Σ is the $k \times k$ covariance matrix representing the covariance

between each pair of random variables η_i and η_j.[38] Let **b** be the k-dimensional vector representing the right-hand-side (RHS) values of the remaining inequalities, i.e., the sum of the applicable transmission capacities. Our goal, then, is to find the probability P(**η** ≤ **b**).

A common approach to solving for this probability is to first standardize each of the random variables. Consider the jth inequality, where $\eta_j = a_{j1}d(1) + a_{j2}d(2) + \ldots + a_{jm}d(2) \leq b_j$. Then η_j is normally distributed with $E(\eta_j) = \mu_j$ and standard deviation σ_j. We standardize η_j by subtracting μ_j and dividing by σ_j:

$$\eta_j \leq b_j \rightarrow \frac{\eta_j - \mu_j}{\sigma_j} \leq \frac{b_j - \mu_j}{\sigma_j}$$

Let **η'** = $[\eta_1', \eta_2', \ldots, \eta_k']^T$ be the vector of standardized random variables (LHS values of the remaining inequalities) and let **b'** = $[b_1', b_2', \ldots, b_k']^T$ be the comparable vector for the standardized RHS values.

Finding the probability of a feasible flow in the given stochastic network can thus be restated as finding the multivariate normal probability distribution P(**η'** ≤ **b'**). Examples utilizing the multivariate normal distribution method will be provided later on in the paper.

[38] We will use the notation σ_i and σ_{ij} to represent the standard deviation of random variable i and the covariance between random variables η_i and η_j, respectively.

3b. Probability Bounding

The multivariate normal probability method discussed in Section 3a is a direct way to estimate the probability of a feasible flow, based on the assumption that all relevant data are normally distributed courtesy of the Central Limit Theorem. Alternatively, we can also solve the problem by estimating this probability using bounding techniques.

In our problem, we have k events $A_1, A_2, \ldots, A_k$, where each event represents the feasibility of one of the k remaining inequalities from Section 2. As noted in Section 3a immediately above, finding the probability of a feasible flow is equivalent to finding the probability of the intersection of the remaining k inequalities. Conceptually, the intersection operation must include all k events, whereas the union operation need only include at least one event. It follows that the probability of the intersection of k events cannot be greater than the probability of the union of these same k events. That is,

$$P(A_1 \cap A_2 \cap \ldots \cap A_k) \leq P(A_1 \cup A_2 \cup \ldots \cup A_k). \qquad (1)$$

Therefore, finding an appropriate upper bound on the probability of the union of the k events also provides an upper bound on the probability of the intersection of these same k events.

Several bounding techniques are available that could be applied to this example. While this paper will focus on the use of two of these methods to solve the problem, it is worthwhile to make mention of a few of the other methods that were considered.[39]

Recall the general relationship between the probability of the intersection of events and the probability of the union of these same events in formula (1). If all k events were mutually exclusive, i.e., the occurrence of each event was independent of the other k-1 events, then the probability of their collective union would simply be the sum of the individual probabilities. In practice, this rarely occurs as we often have dependence among some of the events. Consider the simple example of two events which are not independent. Then the probability that one occurs is impacted by the probability that the other occurs, in which case adding the individual probabilities would overstate the probability of the union of the events. Thus, we have the following upper bound:

$$P(A_1 \cup A_2 \cup \ldots \cup A_k) \leq \sum_{i=1}^{k} P(A_i). \qquad (2)$$

We note that inequality (2) above, which shows an upper bound on the probability of the union of events, was first obtained by Boole (1854). This upper bound is generally not tight. In fact, it often exceeds one, which is trivial and not very helpful in practice.

[39] This paper only addresses these alternative methods at a very high level to provide the reader with a general understanding of the probability bounding options available to solve the problem. The specific methods chosen to be used for this paper will be discussed in detail later.

Alternatively, consider that for each $j \in \{1,\ldots,k\}$ in inequality (1) above, we have $\binom{k}{j}$ different combinations of size j. We denote by S_j the sum of the probabilities that any j of the remaining k events are satisfied simultaneously, where

$$S_j = \sum_{1 \le i_1 \le i_j} P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{ij}).$$

It follows that $P(A_1 \cup A_2 \cup \ldots \cup A_k) = S_1 - S_2 + \cdots + (-1)^{k-1} S_k$. This is referred to as the inclusion-exclusion formula because of the rotating signs among the terms in the summation between positive and negative.

Bonferroni (1937) utilized the inclusion-exclusion formula to prove the following bounds on the probability of the union of events:

$$P(A_1 \cup A_2 \cup \ldots \cup A_k) \le \sum_{i=1}^{m} (-1)^{i-1} S_i \text{, if m is odd}$$

$$P(A_1 \cup A_2 \cup \ldots \cup A_k) \ge \sum_{i=1}^{m} (-1)^{i-1} S_i \text{, if m is even}$$

We observe that if m = k, then we have the inclusion-exclusion formula and an exact value for the probability of the union of events, This suggests that these Bonferroni inequalities provide tighter bounds as m increases. Unfortunately, in practice we often do not have this much information, which results in the lower and upper bounds not being very close together.

Thus, the inequalities derived by Boole and Bonferroni typically do not provide sufficient approximations for the probability of the union of events so they are not used in this paper.[40]

Other potential bounding techniques that could be applied to this example include Binomial Moment Linear Programming, and Closed Form Two-Sided Bounds. The former utilizes the optimum values of maximization and minimization linear programming problems to provide upper bounds and lower bounds, respectively, on the probability of the union (and probability of the intersection) of events. The constraints in these linear programming problems make use of binomial coefficients as well as the values S_j defined previously. In their work *On the Existence of a Feasible Flow in a Stochastic Transportation Network* (1989), Prékopa and Boros utilize Binomial Moment Linear Programming to solve illustrative examples.

Prékopa (with Boros, 1989; and with M. Subasi and E. Subasi, 2008) also studied Closed Form Two-Sided Bounds on the probability of the union of events based on certain of the values $S_1, S_2, \ldots S_j$.[41]

Others who have contributed significantly to the study of probability bounding methods include Hailperin (1965), Gao (with Prékopa) (2005), Subasi (2008), and Billinton (with Allen, Shahidehpour, and Singh) (1960).[42]

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[40] The reader is also referred to Prékopa (1988) and Prékopa (2005) .for more details on these methods.
[41] See Prékopa, Subasi, Subasi (2008) for further details on these bounding methods.
[42] More complete references are provided in the References section at the end of this paper.

While these aforementioned techniques are valid approaches for bounding the probability of a feasible flow, they are not specifically used in this paper. Rather, in order to distinguish this paper from the work of Prékopa and Boros (1989) and other similar studies, we will instead focus on two other probability bounding methods: Hunter's Upper Bound and Boolean Probability Bounding. Both of these methods are discussed in more detail below.

3c. Hunter's Upper Bound

In Section 3b, we provided some basic inequalities relating to the probability of the union and intersection of events. We continue with some additional relationships to formulate Hunter's Upper Bound.

We denote by A_i^C the complement of event A_i, where $P(A_i^C) = 1 - P(A_i)$. DeMorgan's Law provides a succinct relationship between the probabilities of the intersection of events and the union of their complements. Namely,

$$P(A_1 \cap A_2 \cap \ldots \cap A_k) = 1 - P(A_1^C \cup A_2^C \cup \ldots \cup A_k^C). \qquad (3)$$

In other words, the probability that all k events occur simultaneously is the complement of the instance when at least one of the k events is not satisfied. We will use the results of DeMorgan's Law to derive a lower bound on the probability of the intersection of the k events.

Consider inequality (2) with the complement events $A_1^C, A_2^C, \ldots, A_k^C$. Using the result of (3) we have the following:

$$P(A_1^C \cup A_2^C \cup \ldots \cup A_k^C) \le \sum_{i=1}^{k} P(A_i^c)$$

$$\rightarrow \quad P(A_1^C \cup A_2^C \cup \ldots \cup A_k^C) \le \sum_{i=1}^{k} \left(1 - P(A_i)\right) = k - \sum_{i=1}^{k} P(A_i)$$

$$\rightarrow \quad -P(A_1^C \cup A_2^C \cup \ldots \cup A_k^C) \geq \sum_{i=1}^{k} P(A_i) - k$$

$$\rightarrow \quad 1 - P(A_1^C \cup A_2^C \cup \ldots \cup A_k^C) \geq 1 + \sum_{i=1}^{k} P(A_i) - k$$

$$\rightarrow \quad P(A_1 \cap A_2 \cap \ldots \cap A_k) \geq \sum_{i=1}^{k} P(A_i) - (k-1) \qquad (4)$$

This lower bound on the probability of the intersection of events in (4) was first obtained by Boole (1854).

Combining (1), (2), and (4), we have

$$\begin{aligned} \sum_{i=1}^{k} P(A_i) - (k-1) &\leq P(A_1 \cap A_2 \cap \ldots \cap A_k) \\ &\leq P(A_1 \cup A_2 \cup \ldots \cup A_k) \\ &\leq \sum_{i=1}^{k} P(A_i). \end{aligned}$$

While (2) does provide an upper bound on the probability of the union of events, (and a corresponding upper bound on the probability of their intersection), we strive for a tighter bound to promote more precise results. Hunter's Bound does just that.[43] While the multivariate normal probability distribution considered the joint probability of all k remaining inequalities, for purposes of Hunter's Bound we will utilize only the individual

[43] For more details on Hunter's Bound, the reader is referred to Prékopa (1995).

probabilities p_i and the pairwise joint probabilities p_{ij}. Hunter's Upper Bound on the probability of the union of these events is given by the following:

$$P(A_1 \cup A_2 \cup \ldots \cup A_k) \leq \sum_{i=1}^{k} P(A_i) - \sum_{(i,j)\in T} P(A_i \cap A_j) \tag{5}$$

where each (i,j) is an edge of T, a maximum spanning tree of the complete graph of the k events, and the weight of the edge (i,j) is equal to the joint probability p_{ij} between events i and j. In the general case, the complete graph is depicted by k nodes, (one for each remaining inequality), and an undirected edge connecting each pair of nodes in the graph.

The concept of a maximum spanning tree and a brief description of a commonly used approach to solve it, are provided below.

Kruskal's Algorithm for Finding a Maximum Spanning Tree

In graph theory, we say that a graph is connected if there is a path between every pair of nodes. A tree, then, is an undirected graph in which each pair of nodes is connected by exactly one simple path. From this definition, we note a couple of resulting characteristics of a tree:

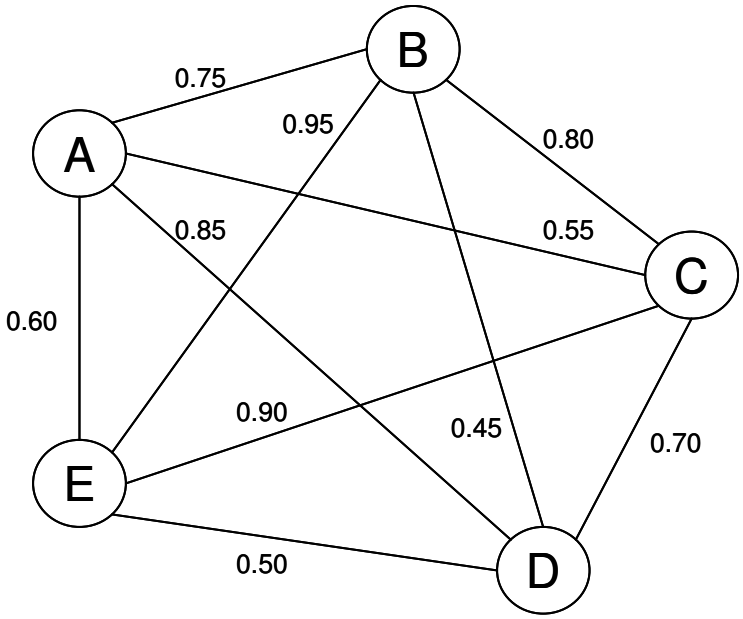
- The number of edges in a tree is one less than the number of nodes.
- There are no cycles in a tree. [44]

A tree can include all nodes in a given graph, or a subset of the nodes. When all nodes in a graph are part of a tree, we say that it is a spanning tree because it spans the entire graph. The edges in a graph are often assigned individual values, typically referred to as weights. Hence, a common task is to find a tree with maximum (or minimum) weight. For purposes of Hunter's Upper Bound, we utilize a maximum spanning tree.

For example, consider the undirected, complete graph of n = 5 nodes below. In the context of Hunter's Upper Bound, this graph can be interpreted as representing 5 possible events, where the edge weights are the pairwise joint probabilities of each pair of nodes (events). The edge weights and corresponding ranking are also provided in the supporting table.

[44] A cycle is a path such that the beginning and ending nodes are the same. For more information on trees and Kruskal's algorithm, the reader is referred to *Algorithms* (Dasgupta, Papadimitriou, Vazirani).

Edge	Weight	Rank
(A,B)	0.75	5
(A,C)	0.55	8
(A,D)	0.85	3
(A,E)	0.60	7
(B,C)	0.80	4
(B,D)	0.45	10
(B,E)	0.95	1
(C,D)	0.70	6
(C,E)	0.90	2
(D,E)	0.50	9

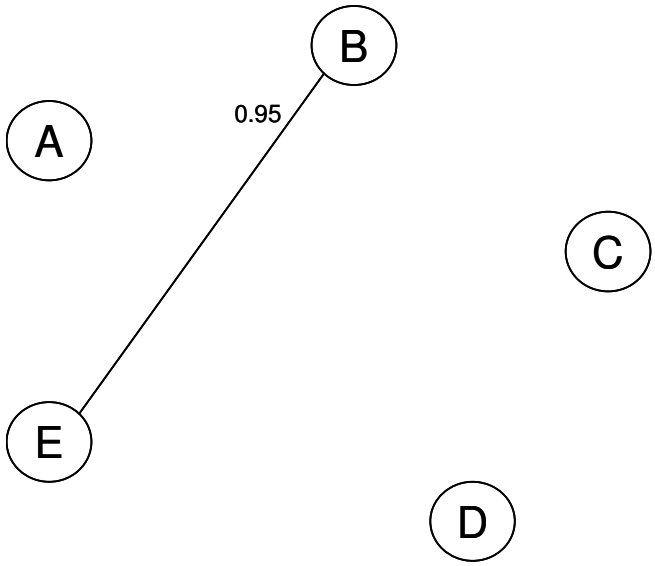


Kruskal's algorithm is prescribed according to the following two basic steps:

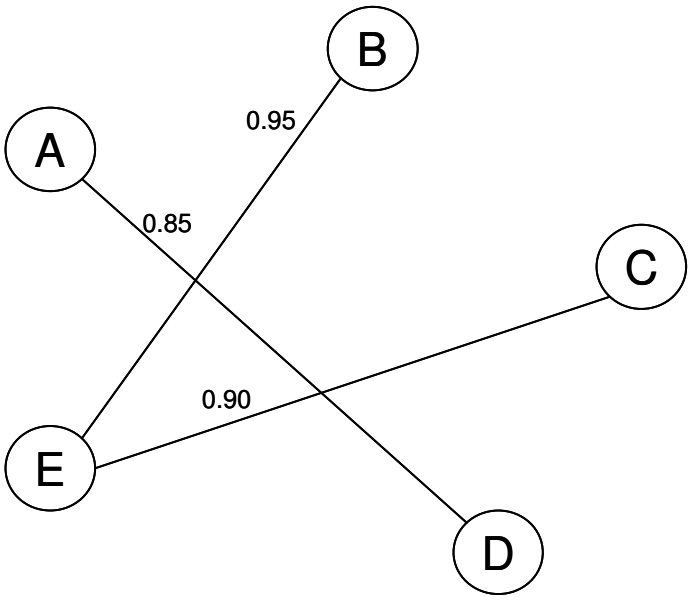
a. Select the heaviest edge in the graph and add it to the tree.

b. Find the next heaviest edge that has not yet been selected and does not create a cycle. Add it to the tree. Repeat step (b) until the tree has n-1 edges.

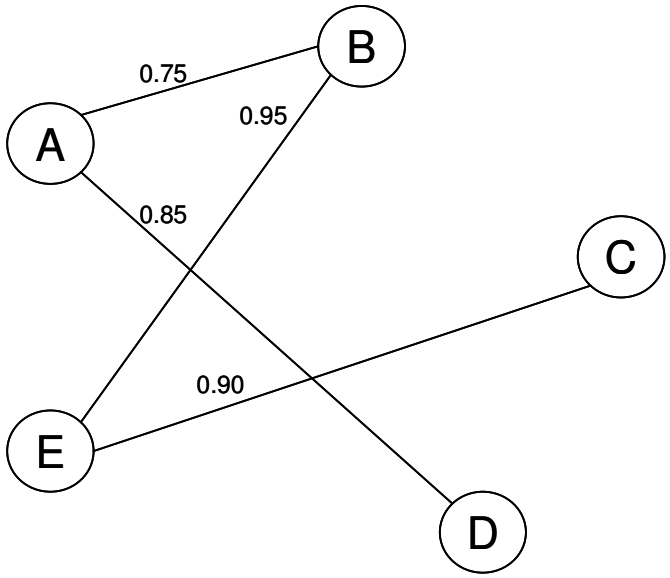
We observe from the table above that the heaviest edge is (B,E). So we add it as the first edge in the tree, per step (a) above.



Continuing with step (b), we proceed in a similar fashion to add the next heaviest edges (C,E) and (A,D) to the tree, respectively.



We note that the next heaviest edge is (B,C); however, if we added this edge to the tree, it would result in a cycle (E,B,C,E), which violates the definition of a tree. Thus, we move on to the next heaviest edge (A,B). Since its addition does not create a cycle, we add it to the tree according to step (b).



Since we have n-1 edges with no cycles, we conclude that the tree depicted above is indeed a spanning tree of total weight 3.45. [45]

We will revisit Hunter's Upper Bound through examples later on in the paper.

[45] The edge weights used in this example are arbitrary and intended only to demonstrate the methodology of Kruskal's algorithm. More practical examples will be provided later in the paper.

3d. Boolean Probability Bounding

In addition to Hunter's Upper Bound, another approach to bounding the union of events is the Boolean Probability Bounding method.[46] This method considers all 2^k possible combinations of the k events and their complements, and focuses on a subset of these combinations for which probabilities are known.

We denote each possible combination I by the set of events which are true as part of the combination, where $I \subseteq \{1,2,3,..., k\}$. [For example, suppose we have k = 6. Then one such combination of events is $(A_1 \cap A_2^C \cap A_3^C \cap A_4 \cap A_5 \cap A_6^C)$, which we would represent as $\{1,4,5\}$]. Further, suppose we know the probabilities of certain of these combinations, specifically the individual stationary probabilities p_i for each event i and the pairwise joint probabilities p_{ij}. This information can be represented as an $m \times n$ matrix A, where m = k + (k choose 2) is the number of combinations with known probabilities (individual and pairs), and $n = 2^k - 1$ is the total number of combinations of events, excluding the combination where none of the k events is true. We will return to the reason for the exclusion of this event in a moment, but first we continue with the formulation of matrix A.

For each combination I with a known probability and all combinations J, we establish the following entries for matrix A:

[46] For additional details on the Boolean Probability Bounding method, the reader is referred to Prékopa (1995).

$$a_{IJ} = \begin{cases} 1 & \text{if } I \subset J \text{ for combinations I and J} \\ 0 & \text{otherwise.} \end{cases}$$

Next, we set up an $m \times 1$ vector **b** representing the known probabilities of each individual event and pair of events and establish an $n \times 1$ vector **x** to represent the probability that each combination of events is true. Based on the known probabilities, we have the system of equations $\mathbf{Ax} = \mathbf{b}$.

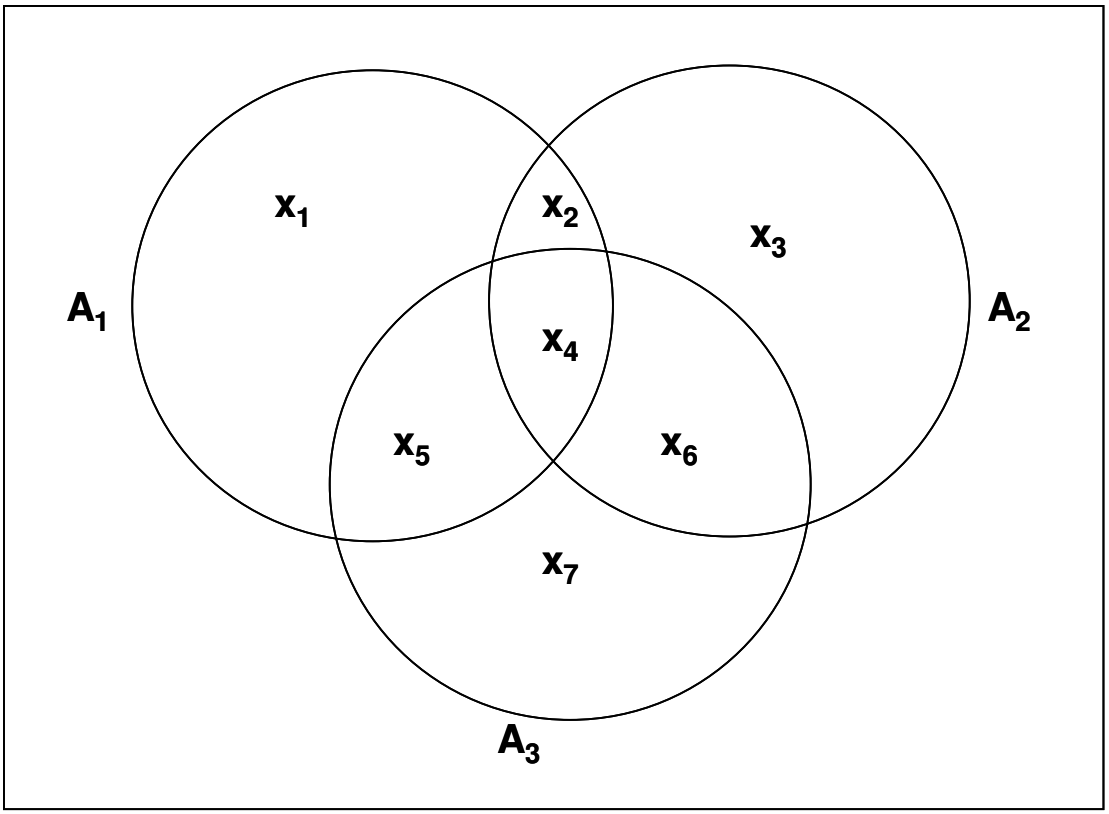
This system of equations lends itself to the following linear programming formulation: [47]

$$\begin{aligned} \text{max (min)} \quad & \sum_{i=1}^{n} x_i \\ \text{s.t.} \quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

If the above LP is solved as a maximization problem, then the optimum value provides an upper bound on the probability of the union of the k events, and vice-versa for a minimization problem.

Consider the following example of 3 events, A_1, A_2, and A_3, depicted by the Venn diagram below, where each combination of events i is denoted by x_i.

[47] The last constraint ensures that the decision variable x takes only non-negative values as it is not possible to have a negative probability.



Suppose that we know the individual stationary probabilities p_i, as well as the pairwise joint probabilities p_{ij}, for all events i and j. These values are shown below.

$P(A_1) = 0.800$	$P(A_1 \cap A_2) = 0.725$
$P(A_2) = 0.850$	$P(A_1 \cap A_3) = 0.685$
$P(A_3) = 0.750$	$P(A_2 \cap A_3) = 0.645$

Each of these known probabilities can be written as a sum of the sections of Venn diagram. For example, $P(A_1) = x_1 + x_2 + x_4 + x_5 = 0.800$. The variables x_i, then, represent the probability that each combination (section) i is satisfied, while the known

probabilities listed above serve as the RHS vector **b** in the linear programming formulation.

Extrapolating this concept to the entire diagram and all known probabilities, we have the following linear program:

$$\text{max (min)} \quad \sum_{i=1}^{7} x_i$$

$$\begin{aligned}
\text{s.t.} \quad & x_1 + x_2 + x_4 + x_5 = 0.800 \\
& x_2 + x_3 + x_4 + x_6 = 0.850 \\
& x_4 + x_5 + x_6 + x_7 = 0.750 \\
& x_2 + x_4 = 0.725 \\
& x_4 + x_5 = 0.685 \\
& x_4 + x_6 = 0.645 \\
& x_i \geq 0 \text{ for all } i
\end{aligned}$$

Before proceeding to solve the LP in this example, we return to the formulation of the matrix A above, where we excluded one possible event. If we had created a variable x_0 representing the event ($A_1^C \cap A_2^C \cap A_3^C$), then we would have had one more equality constraint

$$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 1$$

and this would be the only constraint where x_0 is represented. The LHS of this constraint, though, would be the same as the objective function, in which case the resulting optimum value will always be 1 (assuming the LP is feasible). In terms of the formulation of an upper bound, this would be a trivial result. For this reason, the variable x_0 can be regarded as a slack variable and excluded from the LP formulation. Hence, we use $n = 2^k - 1$ and combination J cannot represent the empty set.

Several solver packages are available to solve the above LP. For purposes of this particular example, we will use AMPL. The AMPL code (shown as a maximization problem) and associated data file are provided below.

```
set Columns;
set Rows;
param A {r in Rows, c in Columns};
param B {r in Rows};
var X {c in Columns};

maximize Bound: sum {c in Columns} X[c];
subject to Constraint {r in Rows}: sum {c in Columns} A[r,c] * X[c] = B[r];
subject to NonNegative {c in Columns}: X[c] >= 0;
```

```
data;

set Columns:= 1 2 3 4 5 6 7;

set Rows:= 1 2 3 4 5 6;

param A:  1  2  3  4  5  6  7  :=
      1   1  1  0  1  1  0  0
      2   0  1  1  1  0  1  0
      3   0  0  0  1  1  1  1
      4   0  1  0  1  0  0  0
      5   0  0  0  1  1  0  0
      6   0  0  0  1  0  1  0  ;

param B:= [1] 0.80  [2] 0.85  [3] 0.75  [4] 0.725  [5] 0.685  [6] 0.645;
```

When the objective function is maximized, the resulting probability is 0.99: [48]

```
ampl: option solver cplex;
ampl: solve;
CPLEX 11.2.0: optimal solution; objective 0.99
0 dual simplex iterations (0 in phase I)
ampl: display X;
X [*] :=
1  0.035
2  0.08
3  0.125
4  0.645
5  0.04
6  0
7  0.065
;
```

We confirm that the optimal solution is feasible:

MAX	x_1	x_2	x_3	x_4	x_5	x_6	x_7	=	Sum
Objective	0.035	0.080	0.125	0.645	0.040	-	0.065	=	0.990
Constraint 1	0.035	0.080	-	0.645	0.040	-	-	=	0.800
Constraint 2	-	0.080	0.125	0.645	-	-	-	=	0.850
Constraint 3	-	-	-	0.645	0.040	-	0.065	=	0.750
Constraint 4	-	0.080	-	0.645	-	-	-	=	0.725
Constraint 5	-	-	-	0.645	0.040	-	-	=	0.685
Constraint 6	-	-	-	0.645	-	-	-	=	0.645

[48] AMPL results from CPLEX solver package.

Alternatively, when the objective function is minimized, the optimum value is slightly less, at 0.955:

```
ampl: option solver cplex;
ampl: solve;
CPLEX 11.2.0: optimal solution; objective 0.955
0 dual simplex iterations (0 in phase I)
ampl: display X;
X [*] :=
1  0
2  0.115
3  0.09
4  0.61
5  0.075
6  0.035
7  0.03
;
```

MIN	x_1	x_2	x_3	x_4	x_5	x_6	x_7	=	Sum
Objective	-	0.115	0.090	0.610	0.075	0.035	0.030	=	0.955
Constraint 1	-	0.115	-	0.610	0.075	-	-	=	0.800
Constraint 2	-	0.115	0.090	0.610	-	0.035	-	=	0.850
Constraint 3	-	-	-	0.610	0.075	0.035	0.030	=	0.750
Constraint 4	-	0.115	-	0.610	-	-	-	=	0.725
Constraint 5	-	-	-	0.610	0.075	-	-	=	0.685
Constraint 6	-	-	-	0.610	-	0.035	-	=	0.645

These results imply the following bounds on the probability of the union of events in this example:

$$0.955 \leq P(A_1 \cup A_2 \cup A_3) \leq 0.990.$$

It follows that the probability of the intersection of events $P(A_1 \cap A_2 \cap A_3)$ is also bounded above by 0.99; however we cannot say solely from these bounds how the probability of the intersection compares to 0.955, the lower bound on the probability of the union.

The Boolean Probability Bounding method will be applied through examples later on in the paper.

4 NUMERICAL EXAMPLE #1
Summer Demand & Average Annual Generating Capacity

4a. Assumptions

For the first example, we consider customer demand in the summer months. Specifically, we will focus on the hourly demand of a subset of FirstEnergy's customers from June 1, 2010 – August 31, 2010, which is comprised of 2,208 hourly data points. As noted in the Introduction section above, customer load is typically highest in the summer months, so evaluating the feasibility of the network in the summer should be a strong indicator of the feasibility of the network year round.

As for the corresponding generating capacity, (which, like the customer demand mentioned above, is also based on a subset of FirstEnergy's total fleet), we assume that the average capacity across all applicable generating units is approximately 78%. This is based on the average annual capacity factor achieved by FirstEnergy's entire generation fleet from 2006-2010. The table below provides historical average annual generating capacity factors for all of FirstEnergy, by fuel type.

Fuel Type	2006	2007	2008	2009	2010	Average
Nuclear	87%	89%	93%	84%	88%	88%
Coal	89%	80%	84%	72%	76%	80%
Other *	69%	71%	64%	31%	52%	57%

* Other consists primarily of natural gas, oil, hydro, and wind

When applying the average factors calculated above to the total available generating capacity applicable to this example (14,975 MW), we see that the weighted average capacity factor is approximately 78%.

Using Average Capacity Factors			
Fuel Type	**Total (MW)**	**Average (MW)**	**Average Factor**
Nuclear	4,200	3,696	88.0%
Coal	8,110	6,488	80.0%
Other	2,665	1,519	57.0%
	14,975	11,703	78.2%

For purposes of this particular example, hourly capacity values for each applicable generating unit were randomly generated via a normal distribution based on maximum available capacities and assumed arithmetic means and standard deviations.[49] These hourly generating capacity values were then aligned with the actual customer demand data to represent hourly net demand (supply) values at each node in the network. The results are presented in the table below.

Example #1 Assumptions			
Fuel Type	**Total (MW)**	**Average (MW)**	**Average Factor**
Nuclear	4,200	3,502	83.4%
Coal	8,110	6,652	82.0%
Other	2,665	1,524	57.2%
	14,975	11,678	78.0%

Thus, we see that while the average factors used in this example are slightly different than the prior table by fuel type, the overall result is consistent at approximately 78%.

[49] The reasonableness of the assumption of normal distribution is supported by the Central Limit Theorem, as discussed above in Section 3. Random number generation was performed in Microsoft Excel.

Using the historical average annual generating capacity factors to derive the hourly generating capacity values in this example is a fairly conservative assumption for a couple of reasons: (i) The summer generating capacity is generally higher in the summer months than the rest of the year to accommodate the summer customer demand. For example, planned power plant outages are generally scheduled more often in the winter months when customer demand is lower, thus driving down the annual average generating capacity as compared to the available capacity in the summer months only. (ii) Using the straight 5-year average of historical capacity factors, as opposed to a normalized approach, may include certain extraordinary occurrences that are not reflective of normal operations and therefore, are driving down the 5-year average relative to expected performance going forward. Nonetheless, this assumption is reasonable for our initial example and we will proceed accordingly.

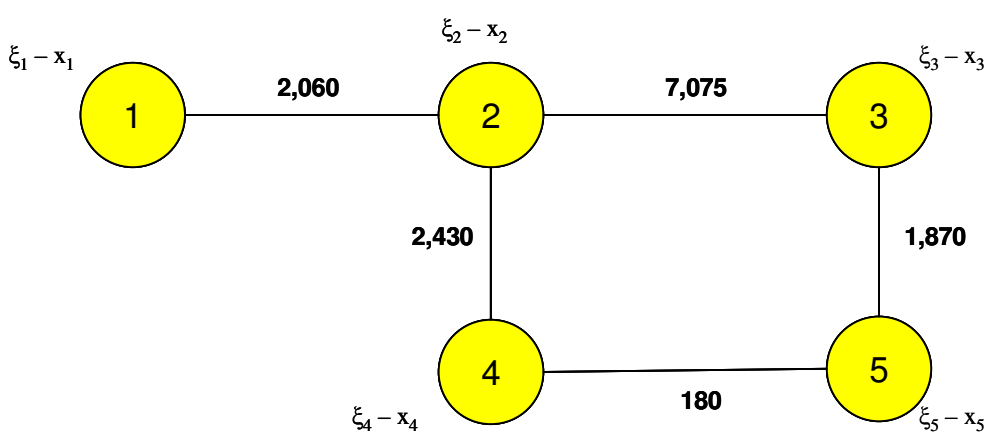
Finally, we assume for purposes of this paper that transmission capacities are constant. [50]

[50] Historical hourly customer demand data, available generating capacity, and historical average generation capacity factors are based on publicly available information from FirstEnergy. Again, the data is based only on a subset of FirstEnergy's assets.

4b. Elimination of Trivial Inequalities by Upper Bounds

Recall the elimination procedure in Section 2d where we eliminated all redundant inequalities. We will consider next any additional remaining trivial inequalities, i.e., those inequalities whose LHS is always less than or equal to its RHS. Since these trivial inequalities are automatically satisfied, they can also be removed from the problem formulation. To put it in terms of electric power flow, we can remove any inequality whose maximum net demand (LHS) is always less than or equal to the available transmission capacity (RHS) because feasibility is guaranteed.

In order to determine which inequalities can be removed in this step, we need to identify (i) transmission capacity values y_{ij}; and (ii) upper bounds on the demands for each set S appearing on the LHS of the remaining inequalities. The picture below provides the installed transmission capacity values for each edge in the network formulation (in MW), based on an applicable subset of FirstEnergy's Transmission assets.



Next, we need to determine upper bounds on the demand values $d(i) = \xi_i - x_i$ for each node i. In this particular example, based on the assumptions described in Section 4a above, we have the following summarized demand and generating capacity data: [51]

Node	Demand ξ_i			Generating Capacity x_i		
	Min	Max	Average	Min	Max	Average
1	849	2,006	1,353	0	1,663	1,348
2	1,827	5,024	3,249	0	7,921	6,157
3	1,503	4,043	2,562	0	3,280	2,744
4	319	964	590	0	895	735
5	1,028	3,298	1,868	0	1,085	694

In terms of feasibility of the network, the extreme case occurs when available generating capacity is zero at a particular demand node i and the corresponding local demand value is at its respective maximum. [52] Accordingly, for purposes of finding an upper bound on the LHS of the remaining inequalities, we consider the following maximum net demand values.

i	ξ_i
1	2,006
2	5,024
3	4,043
4	964
5	3,298

[51] All demand and generating capacity values presented in terms of MW.
[52] Available generating capacity may be zero in instances of power plant outages. These outages can be planned, (e.g., nuclear refueling), or forced (e.g., unexpected breakdown or malfunction of equipment). Either way, the feasibility of the network needs to take these extreme events into consideration.

We return now to the remaining 21 inequalities from Section 2d and plug in the corresponding LHS and RHS values based on the information above to determine which ones are trivial and can be removed.

	2,006	5,024	4,043	964	3,298		LHS		2,060	7,075	2,430	1,870	180		RHS
(1)	$\xi_1 - x_1$					=	2,006	<	y_{12}					=	2,060
(2)		$\xi_2 - x_2$				=	5,024	<	y_{12} +	y_{23} +	y_{24}			=	11,565
(3)			$\xi_3 - x_3$			=	4,043	<		y_{23}		+ y_{35}		=	8,945
(4)				$\xi_4 - x_4$		=	964	<			y_{24}		+ y_{45}	=	2,610
(5)					$\xi_5 - x_5$	=	3,298					y_{35}	+ y_{45}	=	2,050
(6)	$\xi_1 - x_1$ +	$\xi_2 - x_2$				=	7,030	<		y_{23}	+ y_{24}			=	9,505
(7)		$\xi_2 - x_2$ +	$\xi_3 - x_3$			=	9,067		y_{12}		+ y_{24}	+ y_{35}		=	6,360
(8)		$\xi_2 - x_2$		+ $\xi_4 - x_4$		=	5,988	<	y_{12} +	y_{23}			+ y_{45}	=	9,315
(9)			$\xi_3 - x_3$		+ $\xi_5 - x_5$	=	7,341			y_{23}			+ y_{45}	=	7,255
(10)				$\xi_4 - x_4$	+ $\xi_5 - x_5$	=	4,262	<			y_{24}	+ y_{35}		=	4,300
(11)	$\xi_1 - x_1$ +	$\xi_2 - x_2$ +	$\xi_3 - x_3$			=	11,073				y_{24}	+ y_{35}		=	4,300
(12)	$\xi_1 - x_1$ +	$\xi_2 - x_2$		+ $\xi_4 - x_4$		=	7,994			y_{23}			+ y_{45}	=	7,255
(13)		$\xi_2 - x_2$ +	$\xi_3 - x_3$	+ $\xi_4 - x_4$		=	10,030		y_{12}			+ y_{35}	+ y_{45}	=	4,110
(14)		$\xi_2 - x_2$ +	$\xi_3 - x_3$		+ $\xi_5 - x_5$	=	12,365		y_{12}		+ y_{24}		+ y_{45}	=	4,670
(15)		$\xi_2 - x_2$		+ $\xi_4 - x_4$	+ $\xi_5 - x_5$	=	9,286	<	y_{12} +	y_{23}		+ y_{35}		=	11,005
(16)			$\xi_3 - x_3$	+ $\xi_4 - x_4$	+ $\xi_5 - x_5$	=	8,305	<		y_{23} +	y_{24}			=	9,505
(17)	$\xi_1 - x_1$ +	$\xi_2 - x_2$ +	$\xi_3 - x_3$	+ $\xi_4 - x_4$		=	12,037					y_{35}	+ y_{45}	=	2,050
(18)	$\xi_1 - x_1$ +	$\xi_2 - x_2$		+ $\xi_4 - x_4$	+ $\xi_5 - x_5$	=	11,292			y_{23}		+ y_{35}		=	8,945
(19)	$\xi_1 - x_1$ +	$\xi_2 - x_2$ +	$\xi_3 - x_3$		+ $\xi_5 - x_5$	=	14,371				y_{24}		+ y_{45}	=	2,610
(20)		$\xi_2 - x_2$ +	$\xi_3 - x_3$	+ $\xi_4 - x_4$	+ $\xi_5 - x_5$	=	13,329		y_{12}					=	2,060
(21)	$\xi_1 - x_1$ +	$\xi_2 - x_2$ +	$\xi_3 - x_3$	+ $\xi_4 - x_4$	+ $\xi_5 - x_5$	=	15,335		0					=	0

The table above shows that the following 9 inequalities can be removed from the formulation because they are trivial: (1), (2), (3), (4), (6), (8), (10), (15), and (16). These inequalities are trivial because in the extreme case where no generating capacity is available and the local demand is at its relative maximum, there is still sufficient transmission capacity available for a feasible flow in the network. The remaining 12 inequalities are summarized in the next table below, renumbered in numerical order:

$$
\begin{array}{lcccccccccccccccc}
(1) & & & & & & & & & \xi_5 - x_5 & \leq & & & & & y_{35} & + & y_{45} \\
(2) & & & \xi_2 - x_2 & + & \xi_3 - x_3 & & & & & \leq & y_{12} & & + & y_{24} & + & y_{35} & & \\
(3) & & & & & \xi_3 - x_3 & & & + & \xi_5 - x_5 & \leq & & y_{23} & & & & & + & y_{45} \\
(4) & \xi_1 - x_1 & + & \xi_2 - x_2 & + & \xi_3 - x_3 & & & & & \leq & & & & y_{24} & + & y_{35} & & \\
(5) & \xi_1 - x_1 & + & \xi_2 - x_2 & & & + & \xi_4 - x_4 & & & \leq & & y_{23} & & & & & + & y_{45} \\
(6) & & & \xi_2 - x_2 & + & \xi_3 - x_3 & + & \xi_4 - x_4 & & & \leq & y_{12} & & & & + & y_{35} & + & y_{45} \\
(7) & & & \xi_2 - x_2 & + & \xi_3 - x_3 & & & + & \xi_5 - x_5 & \leq & y_{12} & & + & y_{24} & & & + & y_{45} \\
(8) & \xi_1 - x_1 & + & \xi_2 - x_2 & + & \xi_3 - x_3 & + & \xi_4 - x_4 & & & \leq & & & & & & y_{35} & + & y_{45} \\
(9) & \xi_1 - x_1 & + & \xi_2 - x_2 & & & + & \xi_4 - x_4 & + & \xi_5 - x_5 & \leq & & y_{23} & & & + & y_{35} & & \\
(10) & \xi_1 - x_1 & + & \xi_2 - x_2 & + & \xi_3 - x_3 & & & + & \xi_5 - x_5 & \leq & & & & y_{24} & & & + & y_{45} \\
(11) & & & \xi_2 - x_2 & + & \xi_3 - x_3 & + & \xi_4 - x_4 & + & \xi_5 - x_5 & \leq & y_{12} & & & & & & & \\
(12) & \xi_1 - x_1 & + & \xi_2 - x_2 & + & \xi_3 - x_3 & + & \xi_4 - x_4 & + & \xi_5 - x_5 & \leq & 0 & & & & & & &
\end{array}
$$

Decreasing the number of total inequalities from the original 31 down to 12 is a significant enhancement in terms of solving the problem more efficiently.

We can now proceed to solve the problem through the three different methods discussed above in Section 3.

4c. Solve By Method 1 – Multivariate Normal Probability

As discussed in Section 3a, we assume that the net demand (supply) $\xi_i - x_i$ is a normally distributed random variable for each node i, and that the LHS of each of the remaining 12 inequalities is also normally distributed as a linear combination of normally distributed random variables.

Recall the notation used in Section 3a. We denote the LHS of each of the remaining inequalities by η_j, for j=1,…,12. Further, we let $b_j^{'}$ be the normalized RHS of each inequality,

$$b_j^{'} = \frac{b_j - \mu_j}{\sigma_j},$$

where b_j is the initial (non-normalized) RHS value of inequality j, and μ_j and σ_j represent the average and standard deviation of the initial (non-normalized) LHS value of inequality j, respectively. Denote by **b'** the 12×1 vector of all normalized RHS values and by **η'** the corresponding normalized LHS values.

Our objective for solving the problem via this method is to find the multivariate normal probability distribution of the remaining inequalities, P(**η'**≤ **b'**). We will utilize a solver package known as NORTEST to compute this probability.[53] NORTEST requires the following information as input values:

(1) Coefficients of Correlation, denoted ρ_{ij}, for each pair of random variables η_i and η_j

(2) Normalized RHS values $b_j^{'}$ of each remaining inequality j

[53] NORTEST solver developed by Dr. Tamas Szántai (Budapest University of Technology and Economics).

First, we consider the Coefficient of Correlation matrix. Based on the hourly demand and generating capacity data described in Section 4a, we can compute the following averages, standard deviations and variances for the random variables η_j associated with each remaining inequality.

Inequality	$E(\eta_j)$	$\sigma(\eta_j)$	$\sigma^2(\eta_j)$
1	1,174	449	201,212
2	-3,090	1,620	2,623,435
3	993	1,070	1,144,496
4	-3,085	1,815	3,294,744
5	-3,048	1,451	2,104,382
6	-3,234	1,723	2,969,247
7	-1,916	1,915	3,667,697
8	-3,229	1,920	3,686,833
9	-1,874	1,735	3,011,757
10	-1,911	2,121	4,499,319
11	-2,060	2,024	4,097,741
12	-2,055	2,231	4,975,640

Next we need to find the Covariance Matrix representing the covariance values between each pair of remaining inequalities.[54]

Covariance Matrix												
	1	2	3	4	5	6	7	8	9	10	11	12
1	201,212	421,525	391,928	501,681	353,082	463,641	622,736	543,798	554,293	702,893	664,853	745,010
2	421,525	2,623,435	1,312,482	2,887,350	2,131,101	2,758,143	3,044,960	3,022,058	2,552,625	3,308,875	3,179,668	3,443,583
3	391,928	1,312,482	1,144,496	1,509,801	863,381	1,418,631	1,704,410	1,615,949	1,255,309	1,901,729	1,810,558	2,007,877
4	501,681	2,887,350	1,509,801	3,294,744	2,444,471	3,045,197	3,389,031	3,452,590	2,946,152	3,796,425	3,546,878	3,954,272
5	353,082	2,131,101	863,381	2,444,471	2,104,382	2,301,311	2,484,182	2,614,681	2,457,463	2,797,552	2,654,392	2,967,762
6	463,641	2,758,143	1,418,631	3,045,197	2,301,311	2,969,247	3,221,784	3,256,300	2,764,952	3,508,838	3,432,888	3,719,941
7	622,736	3,044,960	1,704,410	3,389,031	2,484,182	3,221,784	3,667,697	3,565,856	3,106,919	4,011,768	3,844,521	4,188,593
8	543,798	3,022,058	1,615,949	3,452,590	2,614,681	3,256,300	3,565,856	3,686,833	3,158,479	3,996,388	3,800,098	4,230,630
9	554,293	2,552,625	1,255,309	2,946,152	2,457,463	2,764,952	3,106,919	3,158,479	3,011,757	3,500,446	3,319,245	3,712,772
10	702,893	3,308,875	1,901,729	3,796,425	2,797,552	3,508,838	4,011,768	3,996,388	3,500,446	4,499,319	4,211,731	4,699,281
11	664,853	3,179,668	1,810,558	3,546,878	2,654,392	3,432,888	3,844,521	3,800,098	3,319,245	4,211,731	4,097,741	4,464,951
12	745,010	3,443,583	2,007,877	3,954,272	2,967,762	3,719,941	4,188,593	4,230,630	3,712,772	4,699,281	4,464,951	4,975,640

[54] Covariance Matrix solved using Microsoft Excel. In order to ensure that the covariance matrix is positive definite, the variance values down the diagonal of the covariance matrix were rounded up slightly.

Using the covariance values for each pair of random variables, along with the individual standard deviations, we can compute the Coefficient of Correlation Matrix. [55]

Correlation Coefficient Matrix												
ρ_{ij}	1	2	3	4	5	6	7	8	9	10	11	12
1	1.000000	0.580177	0.816717	0.616155	0.542607	0.599834	0.724904	0.631369	0.712037	0.738735	0.732194	0.744578
2	0.580177	1.000000	0.757445	0.982095	0.906998	0.988231	0.981634	0.971720	0.908117	0.963101	0.969782	0.953127
3	0.816717	0.757445	1.000000	0.777502	0.556331	0.769554	0.831898	0.786672	0.676135	0.838046	0.836052	0.841406
4	0.616155	0.982095	0.777502	1.000000	0.928350	0.973602	0.974918	0.990621	0.935264	0.986031	0.965302	0.976633
5	0.542607	0.906998	0.556331	0.928350	1.000000	0.920640	0.894179	0.938706	0.976146	0.909165	0.903921	0.917154
6	0.599834	0.988231	0.769554	0.973602	0.920640	1.000000	0.976285	0.984180	0.924601	0.959990	0.984157	0.967806
7	0.724904	0.981634	0.831898	0.974918	0.894179	0.976285	1.000000	0.969707	0.934810	0.987565	0.991684	0.980499
8	0.631369	0.971720	0.786672	0.990621	0.938706	0.984180	0.969707	1.000000	0.947854	0.981223	0.977678	0.987765
9	0.712037	0.908117	0.676135	0.935264	0.976146	0.924601	0.934810	0.947854	1.000000	0.950911	0.944837	0.959100
10	0.738735	0.963101	0.838046	0.986031	0.909165	0.959990	0.987565	0.981223	0.950911	1.000000	0.980878	0.993193
11	0.732194	0.969782	0.836052	0.965302	0.903921	0.984157	0.991684	0.977678	0.944837	0.980878	1.000000	0.988826
12	0.744578	0.953127	0.841406	0.976633	0.917154	0.967806	0.980499	0.987765	0.959100	0.993193	0.988826	1.000000

Next, we need to calculate the normalized RHS values $b_j^{'}$. Again, based on the hourly data described in Section 4a, we can compute the average μ_j and standard deviation σ_j of the random variable η_j associated with each remaining inequality. The initial RHS values b_j are also known based on the assumed Transmission capacities provided in Section 4b. Thus, we have the following values:

[55] Recall that the Coefficient of Correlation ρ_{ij} between random variables η_i and η_j is given by $\rho_{ij} = \dfrac{\sigma_{ij}}{\sigma_i \sigma_j}$, where σ_j is the standard deviation of random variable j and σ_{ij} is the covariance between random variables i and j.

Inequality	RHS	$E(\eta_j)$	$\sigma(\eta_j)$	b'_j
1	2,050	1,174	449	1.952661
2	6,360	-3,090	1,620	5.834415
3	7,255	993	1,070	5.853807
4	4,300	-3,085	1,815	4.068604
5	7,255	-3,048	1,451	7.102224
6	4,110	-3,234	1,723	4.262147
7	4,670	-1,916	1,915	3.438899
8	2,050	-3,229	1,920	2.749528
9	8,945	-1,874	1,735	6.233994
10	2,610	-1,911	2,121	2.131380
11	2,060	-2,060	2,024	2.035392
12	0	-2,055	2,231	0.921406

The Coefficients of Correlation and normalized RHS values are entered into the NORTEST solver, which proceeds to solve the multivariate normal probability. The output from NORTEST is provided in the next table.

Results for the distribution function		

Error code	=	0
Estimated value (crude)	=	0.820500
Std. deviation (crude)	=	0.003838
Time	=	0.007812
SADMVN value	**=**	**0.818375**
SADER value	=	0.000019
Time	=	0.023438
Lower bound (0,9)	=	0.818380
Time	=	0.000000
Error code	=	0
Estimated value	=	0.818380
Std. deviation	=	0.000000
Time	=	0.000000
Efficiency	=	**********
Upper bound (1,8)	=	0.818380
Time	=	0.007812

The above result indicates that the probability of a feasible flow in our network is approximately 0.818375.

4d. Solve By Method 2 – Hunter's Bound

Before proceeding with Hunter's Upper Bound on the probability of the union of the remaining 12 inequalities, we will first derive a lower bound on the intersection using inequality (4) from Section 3b above. All we need for this calculation are the individual probabilities for each of the remaining 12 inequalities. These probabilities can be calculated using NORTEST and the results are provided below.

Inequality	p_i
1	0.974570
2	1.000000
3	1.000000
4	0.999976
5	1.000000
6	0.999990
7	0.999708
8	0.997016
9	1.000000
10	0.983471
11	0.979094
12	0.821581
	11.755406

Thus, we have the following lower bound:

$$P(A_1 \cap A_2 \cap \ldots \cap A_{12}) \geq \sum_{i=1}^{12} P(A_i) - (k-1)$$

$$\rightarrow \quad P(A_1 \cap A_2 \cap \ldots \cap A_{12}) \geq 11.755406 - (12-1)$$

$$\rightarrow \quad P(A_1 \cap A_2 \cap \ldots \cap A_{12}) \geq 0.755406$$

We conclude the probability of a feasible flow in the transportation network in our example is at least 0.755406. We note that the result of the multivariate normal distribution from Section 4c was 0.818375, which exceeds this lower bound.

Now we turn our attention to Hunter's Upper Bound. Recall the definition of Hunter's Upper Bound on the union of k events $A_1, A_2, \ldots, A_k$. As explained in Section 3b,

$$P(A_1 \cup A_2 \cup \ldots \cup A_k) \leq \sum_{i=1}^{k} P(A_i) - \sum_{(i,j)\in T} P(A_i \cap A_j),$$

where (i,j) are the edges of T, a maximum spanning tree of the complete graph of the k events, with the weight of the edge (i,j) equal to the joint probability p_{ij} between events i and j. [56]

The first term on the RHS of the inequality is simply the sum of the individual stationary probabilities, which is equal to 11.755406, as provided above. In order to calculate the second term on the RHS of the inequality above, we need to first identify a maximum spanning tree of the complete graph. The pairwise joint probabilities p_{ij}, which are used as the edge weights in the complete graph, are provided below as calculated by NORTEST.

Pairwise Joint Probabilities												
p_{ij}	1	2	3	4	5	6	7	8	9	10	11	12
1		0.974570	0.974570	0.974566	0.974570	0.974568	0.974525	0.973035	0.974570	0.965204	0.961851	0.818381
2	0.974570		1.000000	0.999976	1.000000	0.999990	0.999708	0.997016	1.000000	0.983471	0.979094	0.821581
3	0.974570	1.000000		0.999976	1.000000	0.999990	0.999708	0.997016	1.000000	0.983471	0.979094	0.821581
4	0.974566	0.999976	0.999976		0.999976	0.999975	0.999708	0.997016	0.999976	0.983471	0.979094	0.821581
5	0.974570	1.000000	1.000000	0.999976		0.999990	0.999708	0.997016	1.000000	0.983471	0.979094	0.821581
6	0.974568	0.999990	0.999990	0.999975	0.999990		0.999708	0.997016	0.999990	0.983471	0.979094	0.821581
7	0.974525	0.999708	0.999708	0.999708	0.999708	0.999708		0.997015	0.999708	0.983471	0.979094	0.821581
8	0.973035	0.997016	0.997016	0.997016	0.997016	0.997016	0.997015		0.997016	0.983470	0.979094	0.821581
9	0.974570	1.000000	1.000000	0.999976	1.000000	0.999990	0.999708	0.997016		0.983471	0.979094	0.821581
10	0.965204	0.983471	0.983471	0.983471	0.983471	0.983471	0.983471	0.983470	0.983471		0.977322	0.821581
11	0.961851	0.979094	0.979094	0.979094	0.979094	0.979094	0.979094	0.979094	0.979094	0.977322		0.821581
12	0.818381	0.821581	0.821581	0.821581	0.821581	0.821581	0.821581	0.821581	0.821581	0.821581	0.821581	

[56] The complete graph in this case is depicted by 12 nodes, (one for each constraint), and an undirected edge connecting each pair of nodes in the graph.

We note that we have $\frac{k(k-1)}{2} = 66$ total edges in the complete graph. We proceed by Kruskal's algorithm to find a maximum spanning tree. The tables below show the ranking of the weights of the edges in descending order and the corresponding decision as to whether to add the edge to the tree (denoted "Tree") or not to add it because it creates a cycle (denoted "Cycle"). Once we have k-1 = 11 edges in the tree, then the algorithm terminates.

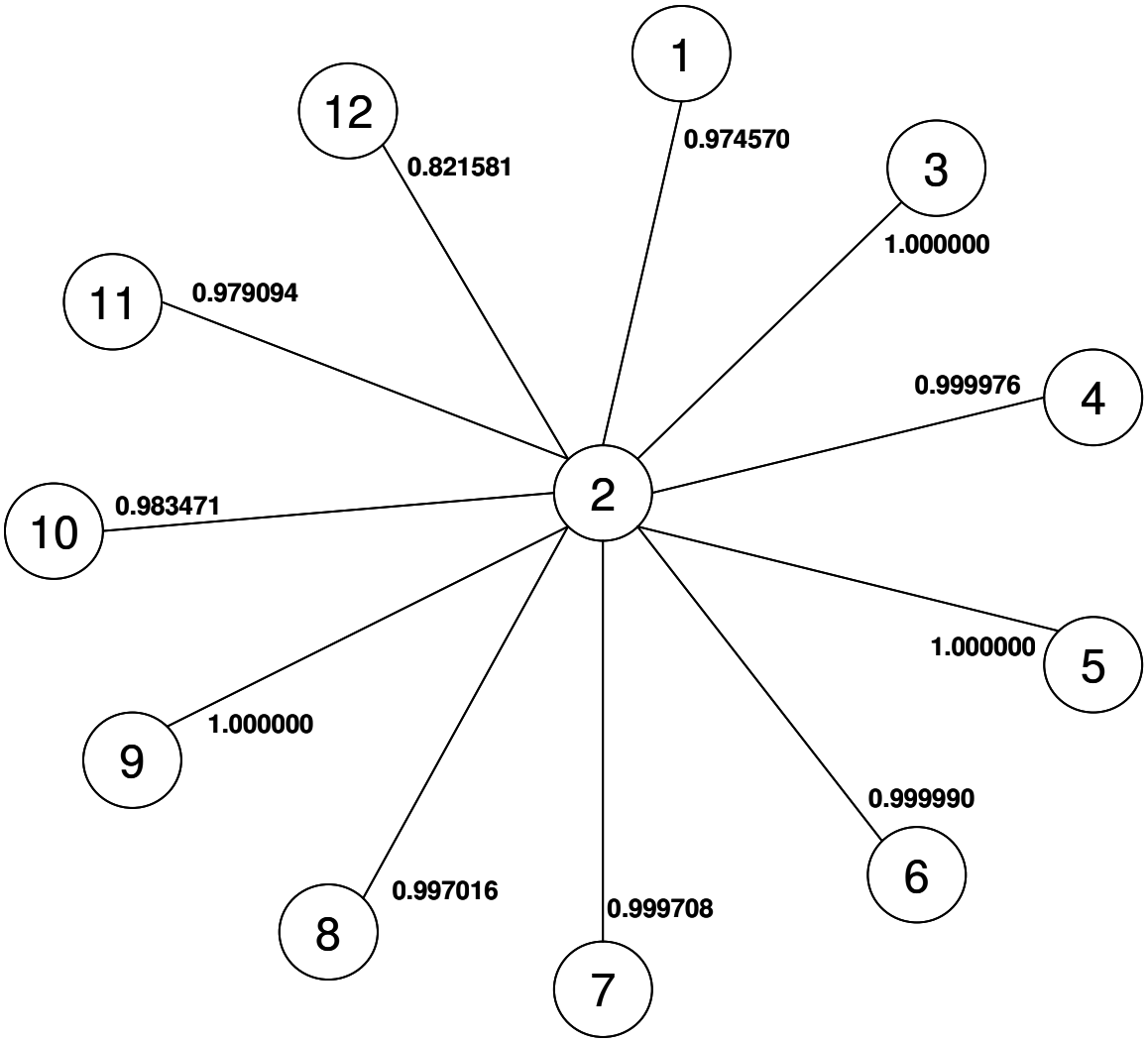
No.	Edge	Weight	Decision
1	(2,3)	1.000000	Tree
2	(2,5)	1.000000	Tree
3	(2,9)	1.000000	Tree
4	(3,5)	1.000000	Cycle
5	(3,9)	1.000000	Cycle
6	(5,9)	1.000000	Cycle
7	(2,6)	0.999990	Tree
8	(3,6)	0.999990	Cycle
9	(5,6)	0.999990	Cycle
10	(6,9)	0.999990	Cycle
11	(2,4)	0.999976	Tree
12	(3,4)	0.999976	Cycle
13	(4,5)	0.999976	Cycle
14	(4,9)	0.999976	Cycle
15	(4,6)	0.999975	Cycle
16	(2,7)	0.999708	Tree
17	(3,7)	0.999708	Cycle
18	(4,7)	0.999708	Cycle
19	(5,7)	0.999708	Cycle
20	(6,7)	0.999708	Cycle
21	(7,9)	0.999708	Cycle
22	(2,8)	0.997016	Tree

No.	Edge	Weight	Decision
23	(3,8)	0.997016	Cycle
24	(4,8)	0.997016	Cycle
25	(5,8)	0.997016	Cycle
26	(6,8)	0.997016	Cycle
27	(8,9)	0.997016	Cycle
28	(7,8)	0.997015	Cycle
29	(2,10)	0.983471	Tree
30	(3,10)	0.983471	Cycle
31	(4,10)	0.983471	Cycle
32	(5,10)	0.983471	Cycle
33	(6,10)	0.983471	Cycle
34	(7,10)	0.983471	Cycle
35	(9,10)	0.983471	Cycle
36	(8,10)	0.983470	Cycle
37	(2,11)	0.979094	Tree
38	(3,11)	0.979094	Cycle
39	(4,11)	0.979094	Cycle
40	(5,11)	0.979094	Cycle
41	(6,11)	0.979094	Cycle
42	(7,11)	0.979094	Cycle
43	(8,11)	0.979094	Cycle
44	(9,11)	0.979094	Cycle

No.	Edge	Weight	Decision
45	(10,11)	0.977322	Cycle
46	(1,2)	0.974570	Tree
47	(1,3)	0.974570	Cycle
48	(1,5)	0.974570	Cycle
49	(1,9)	0.974570	Cycle
50	(1,6)	0.974568	Cycle
51	(1,4)	0.974566	Cycle
52	(1,7)	0.974525	Cycle
53	(1,8)	0.973035	Cycle
54	(1,10)	0.965204	Cycle
55	(1,11)	0.961851	Cycle
56	(2,12)	0.821581	Tree
57	(3,12)	0.821581	N/A
58	(4,12)	0.821581	N/A
59	(5,12)	0.821581	N/A
60	(6,12)	0.821581	N/A
61	(7,12)	0.821581	N/A
62	(8,12)	0.821581	N/A
63	(9,12)	0.821581	N/A
64	(10,12)	0.821581	N/A
65	(11,12)	0.821581	N/A
66	(1,12)	0.818381	N/A

The results of Kruskal's algorithm, including the associated spanning tree, are provided below.

No.	Edge	Weight
1	(2,3)	1.000000
2	(2,5)	1.000000
3	(2,9)	1.000000
4	(2,6)	0.999990
5	(2,4)	0.999976
6	(2,7)	0.999708
7	(2,8)	0.997016
8	(2,10)	0.983471
9	(2,11)	0.979094
10	(1,2)	0.974570
11	(2,12)	0.821581
		10.755406



We note that the edges added to the tree are all connected to node 2 directly, which confirms that we do not have a cycle.

It follows that Hunter's Upper Bound on the probability of the union of the 12 events is

$$P(A_1 \cup A_2 \cup \ldots \cup A_k) \leq 11.755406 - 10.755406 = 1.000000.$$

In terms of probability, this is a trivial result as all probabilities are less than or equal to 1, by definition. Nonetheless, as indicated in Section 3b, this result also serves as an upper bound on the probability of the intersection of the remaining events.

Thus, combining Hunter's Upper Bound with the lower bound results presented earlier in this section, we have the following:

$$0.755406 \leq P(A_1 \cap A_2 \cap \ldots \cap A_{12}) \leq 1.000000.$$

We conclude that the probability of a feasible flow in our example is between these bounds. This range is validated by the results of the multivariate normal distribution from Section 4c, which indicated that the probability of a feasible flow is 0.818375.

4e. Solve By Method 3 – Boolean Probability Bounding

As noted above in Section 3c, the Boolean Probability Bounding method finds upper and lower bounds on the probability of the union of events by utilizing linear programming and known probabilities. Similar to the previous discussion on Hunter's Upper Bound, we will use the individual stationary probabilities p_i and the joint pairwise probabilities p_{ij} for purposes of this example. These values have already been provided in Section 4d.

In our example, we have k=12. Using the notation from Section 3c, we denote each possible combination of events I by the events which are true as part of the combination, where $I \subseteq \{1,2,3,...,12\}$. For each combination I with a known probability and all combinations J, we can establish the following entries for an $m \times n$ matrix A, consistent with the methodology discussed in Section 3c:

$$a_{IJ} = \begin{cases} 1 & \text{if } I \subset J \text{ for combinations I and J} \\ 0 & \text{otherwise.} \end{cases}$$

In this example, m = 78 (the number of known probabilities, 12 individual and 66 pairs) and $n = 2^{12} - 1 = 4{,}095$.[57]

Next, we set up a 78×1 vector **b** representing the known probabilities of each individual event and pair of events. A brief excerpt of vector b is provided below.

[57] Matrix A is too large to present in its entirety, so it is just discussed in general terms. For purposes of this example, matrix A was constructed using Microsoft Excel.

$$\mathbf{b} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{10} \\ p_{11} \\ p_{12} \\ p_{1,2} \\ p_{1,3} \\ p_{1,4} \\ \vdots \\ p_{10,11} \\ p_{10,12} \\ p_{11,12} \end{bmatrix} = \begin{bmatrix} 0.974570 \\ 1.000000 \\ 1.000000 \\ \vdots \\ 0.983471 \\ 0.979094 \\ 0.821581 \\ 0.974570 \\ 0.974570 \\ 0.974566 \\ \vdots \\ 0.977322 \\ 0.821581 \\ 0.821581 \end{bmatrix}$$

Finally, let **x** be an n×1 vector (decision variable). We now have the following LP.

$$\begin{aligned} \text{max (min)} \quad & \sum_{i=1}^{4,095} x_i \\ \text{s.t.} \quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

This LP can be solved using a number of different solver packages; however, for purposes of this example, we will utilize MOSEL. The MOSEL code is provided below.[58]

[58] The associated data file is too large to present in this paper. However, the format is consistent with the AMPL data file shown in the example in Section 3c.

```
model Boolean12                                  ! Start new model

uses "mmxprs"                                    ! Load the optimizer library

declarations
        m = 78
        n = 4095
        Rows= 1..m                               ! Set of rows in A matrix
        Columns= 1..n                            ! Set of columns in A matrix
        A: array(Rows, Columns) of real          ! Binary matrix representing each combination of events
        B: array(Rows) of real                   ! Individual and joint probabilities
        X: array(Columns) of mpvar               ! Decision variable for each combination of events

end - declarations

initializations from "Boolean12.dat"
        A  B
end - initializations

        Bound:= sum(c in Columns) X(c)

        forall(r in Rows) sum(c in Columns) A(r,c) * X(c) = B(r)
        forall(c in Columns) X(c) >= 0

        minimize (Bound)

        writeln("Bound is: ", getobjval)
        forall(c in Columns | getsol(X(c)) > 0)
                writeln("Optimal Solution of combination "+ c + " is: ", getsol(X(c)))
        writeln("All other variables (combinations) are zero in the optimal solution.")

end - model
```

When the objective function is solved as a maximum, we see that the upper bound on the union of the 12 events is 1, as presented in the output below from MOSEL.

```
Bound is: 1
Optimal Solution of combination 475 is: 7e-006
Optimal Solution of combination 805 is: 2e-006
Optimal Solution of combination 1127 is: 1e-006
Optimal Solution of combination 1154 is: 1.3e-005
Optimal Solution of combination 1616 is: 2e-006
Optimal Solution of combination 2050 is: 0.000225
Optimal Solution of combination 2512 is: 4.1e-005
Optimal Solution of combination 2973 is: 0.001203
Optimal Solution of combination 2977 is: 1e-006
Optimal Solution of combination 3303 is: 0.001489
Optimal Solution of combination 3632 is: 0.003941
Optimal Solution of combination 3797 is: 0.007832
Optimal Solution of combination 3801 is: 1e-006
Optimal Solution of combination 3962 is: 0.002796
Optimal Solution of combination 3963 is: 0.001772
Optimal Solution of combination 4017 is: 0.003352
Optimal Solution of combination 4072 is: 0.012271
Optimal Solution of combination 4083 is: 0.14347
Optimal Solution of combination 4094 is: 0.0032
Optimal Solution of combination 4095 is: 0.818381
All other variables (combinations) are zero in the optimal solution.
```

When the LP is instead solved as a minimization problem, we obtain a different optimal solution, but the same optimum value of 1. These results imply the following bounds:

$$1 \leq P(A_1 \cup A_2 \cup \ldots \cup A_{12}) \leq \mathbf{1}$$

$$\rightarrow \quad P(A_1 \cup A_2 \cup \ldots \cup A_{12}) = 1.$$

Given that more than one of the individual probabilities p_i is equal to 1, it makes sense that the probability of the union of all 12 events, i.e., the probability that at least one of the events is satisfied, is equal to 1.

As discussed in Section 3b, we know that an upper bound on the probability of the union of events is also an upper bound on the probability of the intersection of all 12 events. Thus, we have the following:

$$P(A_1 \cap A_2 \cap \ldots \cap A_{12}) \leq 1.$$

This again is a trivial result. However, it is worth noting that this upper bound is consistent with Hunter's Upper Bound and that the result from the multivariate normal distribution method fits within this bound.

4f. Summary of Results

The results of this first example across the three methods are consistent -- the multivariate normal distribution method provided the most precise probability, and this result was also within the bounds generated via Hunter's Upper Bound and the Boolean Probability Bounding methods. Thus, we conclude that the probability of a feasible flow in the network formulation in Example #1 is approximately 82%.

Recall that our generating capacity assumptions for this example were relatively conservative, which suggests that the probability could improve with more aggressive assumptions. Hence, we proceed with another example.

5 NUMERICAL EXAMPLE #2
Summer Demand & Maximum Annual Generating Capacity

5a. Assumptions

Consistent with the first example, Example #2 will consider customer demand in the summer months, i.e., hourly demand data from June 1, 2010 – August 31, 2010, which is comprised of 2,208 hourly data points. As noted in the Introduction section above, customer load (and generating capacity) are typically highest in the summer months, so evaluating the feasibility of the network in the summer should be a strong indicator of the feasibility of the network year round.

Instead of using the historical average annual generating capacity factors as we did in Example #1, this time we will incorporate a more aggressive approach and assume that the average hourly capacity factor across all applicable generating units is approximately 86%. This figure represents the weighted average of the maximum capacity factors for each fuel type within FirstEnergy's overall generating fleet from 2006-2010. Returning to the historical performance of FirstEnergy's entire generating fleet, we note the historical maximum capacity factors by fuel type.

Fuel Type	2006	2007	2008	2009	2010	MAX
Nuclear	87%	89%	**93%**	84%	88%	93%
Coal	**89%**	80%	84%	72%	76%	89%
Other *	69%	**71%**	64%	31%	52%	71%

* Other consists primarily of natural gas, oil, hydro, and wind

If these maximum capacity factors are applied to the total available generating capacity relevant to this example (14,975 MW), we see that the weighted average is approximately 86.9%.[59]

Using MAX Capacity Factors			
Fuel Type	**Total (MW)**	**Average (MW)**	**Average Factor**
Nuclear	4,200	3,906	93.0%
Coal	8,110	7,218	89.0%
Other	2,665	1,892	71.0%
	14,975	13,016	86.9%

Similar to Example #1, the hourly capacity values for each applicable generating unit were randomly generated via a normal distribution based on maximum available capacities and assumed arithmetic means and standard deviations. These hourly generating capacity values were then aligned with the actual customer demand data to represent hourly net demand (supply) values at each node in the network.

Applying this approach with the aforementioned assumptions, the resulting average capacity factors, by fuel type, are summarized in the table below.

Example #2 Assumptions			
Fuel Type	**Total (MW)**	**Average (MW)**	**Average Factor**
Nuclear	4,200	3,932	93.6%
Coal	8,110	7,217	89.0%
Other	2,665	1,752	65.7%
	14,975	12,901	86.2%

[59] The total available generating capacity (in MW) did not change from Example #1 – only the average capacity across the 2,208 hourly data points is impacted by this modified assumption.

We note that the overall weighted average generating capacity factor assumed in Example #2 is approximately 86.2%, which is consistent with the weighted average derived using the maximum historical capacity factors, by fuel type, in the preceding table. This is an increase over the comparable assumption from Example #1, which was a weighted average of approximately 78%.

We also reiterate our previous assumption that transmission capacities are constant.

5b. Elimination of Trivial Inequalities by Upper Bounds

At this point, we recall the 21 remaining inequalities following the elimination procedure described in Section 2d. Since we have new generating capacity values in Example #2, we need to again consider the subsequent elimination procedure, which eliminates trivial inequalities based on upper bounds.

In order to do so, we first identify upper bounds on the demand values $d(i) = \xi_i - x_i$ for each node i. In this particular example, based on the assumptions described in Section 5a above, we have the following summarized demand and generating capacity data: [60]

Node	Demand ξ_i			Generating Capacity x_i		
	Min	Max	Average	Min	Max	Average
1	849	2,006	1,353	188	1,665	1,462
2	1,827	5,024	3,249	1,939	8,050	6,922
3	1,503	4,043	2,562	652	3,280	2,989
4	319	964	590	0	895	795
5	1,028	3,298	1,868	0	1,085	733

We observe that the demand values ξ_i did not change from those presented in Example #1 because we are using the same hourly data. The minimum generating capacity values x_i did increase, though, due to our assumption in Example #2 of higher average generating capacity factors. As noted in Section 4b, the feasibility of the network is dependent on its ability to satisfy customer demand when the net demand values $d(i) = \xi_i - x_i$ are the highest. This occurs when available generating capacity is at its minimum at a particular demand node i and the corresponding local gross demand value is at its respective

[60] All demand and generating capacity values presented in terms of MW.

maximum.[61] Accordingly, for purposes of finding an upper bound on the LHS on the remaining inequalities, we consider the following maximum demand values d(i).

Node	Max ξ_i	Min x_i	Max d(i)
1	2,006	188	1,818
2	5,024	1,939	3,085
3	4,043	652	3,390
4	964	0	964
5	3,298	0	3,298

We return now to the remaining 21 inequalities and plug in the corresponding LHS and RHS values based on the information above to determine which ones are trivial and can be removed.

	1,818	3,085	3,390	964	3,298		LHS		2,060	7,075	2,430	1,870	180		RHS
(1)	$\xi_1 - x_1$					=	1,818	<	y_{12}					=	2,060
(2)		$\xi_2 - x_2$				=	3,085	<	y_{12}	+ y_{23}	+ y_{24}			=	11,565
(3)			$\xi_3 - x_3$			=	3,390	<		y_{23}		+ y_{35}		=	8,945
(4)				$\xi_4 - x_4$		=	964	<			y_{24}		+ y_{45}	=	2,610
(5)					$\xi_5 - x_5$	=	3,298					y_{35}	+ y_{45}	=	2,050
(6)	$\xi_1 - x_1$	+ $\xi_2 - x_2$				=	4,904	<		y_{23}	+ y_{24}			=	9,505
(7)		$\xi_2 - x_2$	+ $\xi_3 - x_3$			=	6,476		y_{12}		+ y_{24}	+ y_{35}		=	6,360
(8)		$\xi_2 - x_2$		+ $\xi_4 - x_4$		=	4,049	<	y_{12}	+ y_{23}			+ y_{45}	=	9,315
(9)			$\xi_3 - x_3$		+ $\xi_5 - x_5$	=	6,689	<		y_{23}			+ y_{45}	=	7,255
(10)				$\xi_4 - x_4$	+ $\xi_5 - x_5$	=	4,262	<			y_{24}	+ y_{35}		=	4,300
(11)	$\xi_1 - x_1$	+ $\xi_2 - x_2$	+ $\xi_3 - x_3$			=	8,294				y_{24}	+ y_{35}		=	4,300
(12)	$\xi_1 - x_1$	+ $\xi_2 - x_2$		+ $\xi_4 - x_4$		=	5,867	<		y_{23}			+ y_{45}	=	7,255
(13)		$\xi_2 - x_2$	+ $\xi_3 - x_3$	+ $\xi_4 - x_4$		=	7,439		y_{12}			+ y_{35}	+ y_{45}	=	4,110
(14)		$\xi_2 - x_2$	+ $\xi_3 - x_3$		+ $\xi_5 - x_5$	=	9,774		y_{12}		+ y_{24}		+ y_{45}	=	4,670
(15)		$\xi_2 - x_2$		+ $\xi_4 - x_4$	+ $\xi_5 - x_5$	=	7,347	<	y_{12}	+ y_{23}		+ y_{35}		=	11,005
(16)			$\xi_3 - x_3$	+ $\xi_4 - x_4$	+ $\xi_5 - x_5$	=	7,652	<		y_{23}	+ y_{24}			=	9,505
(17)	$\xi_1 - x_1$	+ $\xi_2 - x_2$	+ $\xi_3 - x_3$	+ $\xi_4 - x_4$		=	9,258					y_{35}	+ y_{45}	=	2,050
(18)	$\xi_1 - x_1$	+ $\xi_2 - x_2$		+ $\xi_4 - x_4$	+ $\xi_5 - x_5$	=	9,165			y_{23}		+ y_{35}		=	8,945
(19)	$\xi_1 - x_1$	+ $\xi_2 - x_2$	+ $\xi_3 - x_3$		+ $\xi_5 - x_5$	=	11,592				y_{24}		+ y_{45}	=	2,610
(20)		$\xi_2 - x_2$	+ $\xi_3 - x_3$	+ $\xi_4 - x_4$	+ $\xi_5 - x_5$	=	10,738		y_{12}					=	2,060
(21)	$\xi_1 - x_1$	+ $\xi_2 - x_2$	+ $\xi_3 - x_3$	+ $\xi_4 - x_4$	+ $\xi_5 - x_5$	=	12,556		0					=	0

[61] Available generating capacity may be zero in instances of power plant outages. These outages can be planned, (e.g., nuclear refueling), or forced (e.g., unexpected breakdown or malfunction of equipment). Either way, the feasibility of the network needs to take these extreme events into consideration.

The table above shows that 11 inequalities can be removed from the formulation because they are trivial, as compared to the 9 that were eliminated in Section 4b for Example #1. These eliminated inequalities are: (1), (2), (3), (4), (6), (8), (9), (10), (12), (15), and (16). These inequalities are trivial because in the extreme case where generating capacity is at its minimum and the local demand is at its gross maximum, there is still sufficient transmission capacity available for a feasible flow in the network.

The remaining 10 inequalities are summarized in the next table below, renumbered in numerical order:

(1)									$\xi_5 - x_5$	$\leq$						y_{35}	+	y_{45}
(2)			$\xi_2 - x_2$	+	$\xi_3 - x_3$					$\leq$	y_{12}		+	y_{24}	+	y_{35}		
(3)	$\xi_1 - x_1$	+	$\xi_2 - x_2$	+	$\xi_3 - x_3$					$\leq$				y_{24}	+	y_{35}		
(4)			$\xi_2 - x_2$	+	$\xi_3 - x_3$	+	$\xi_4 - x_4$			$\leq$	y_{12}				+	y_{35}	+	y_{45}
(5)			$\xi_2 - x_2$	+	$\xi_3 - x_3$			+	$\xi_5 - x_5$	$\leq$	y_{12}		+	y_{24}			+	y_{45}
(6)	$\xi_1 - x_1$	+	$\xi_2 - x_2$	+	$\xi_3 - x_3$	+	$\xi_4 - x_4$			$\leq$						y_{35}	+	y_{45}
(7)	$\xi_1 - x_1$	+	$\xi_2 - x_2$			+	$\xi_4 - x_4$	+	$\xi_5 - x_5$	$\leq$		y_{23}			+	y_{35}		
(8)	$\xi_1 - x_1$	+	$\xi_2 - x_2$	+	$\xi_3 - x_3$			+	$\xi_5 - x_5$	$\leq$				y_{24}			+	y_{45}
(9)			$\xi_2 - x_2$	+	$\xi_3 - x_3$	+	$\xi_4 - x_4$	+	$\xi_5 - x_5$	$\leq$	y_{12}							
(10)	$\xi_1 - x_1$	+	$\xi_2 - x_2$	+	$\xi_3 - x_3$	+	$\xi_4 - x_4$	+	$\xi_5 - x_5$	$\leq$	0							

We now proceed to solve the problem through the three different methods discussed above in Section 3.

5c. Solve By Method 1 – Multivariate Normal Probability

Recall that our objective for solving the problem via this method is to find the multivariate normal probability distribution of the remaining inequalities, $P(\boldsymbol{\eta}' \leq \mathbf{b}')$, where $\boldsymbol{\eta}'$ and $\mathbf{b}'$ represent the normalized LHS and RHS values of the remaining 10 inequalities, respectively, as described in Section 3a. Similar to Example #1, we will utilize the solver package NORTEST to compute this probability. NORTEST requires the following information as input values:

(1) Coefficients of Correlation, denoted ρ_{ij}, for each pair of random variables η_i and η_j

(2) Normalized RHS values b'_j of each remaining inequality j

First, we consider the Coefficient of Correlation matrix. Based on the hourly demand and generating capacity data described in Section 5a, we can compute the following averages, standard deviations and variances for the random variables η_j associated with the LHS of each remaining inequality.

Inequality	$E(\eta_j)$	$\sigma(\eta_j)$	$\sigma^2(\eta_j)$
1	1,136	439	192,682
2	-4,099	1,525	2,324,292
3	-4,209	1,712	2,930,189
4	-4,304	1,643	2,700,040
5	-2,964	1,837	3,372,762
6	-4,413	1,832	3,357,341
7	-2,852	1,660	2,756,949
8	-3,073	2,033	4,132,067
9	-3,169	1,959	3,839,067
10	-3,278	2,156	4,649,776

Next we need to find the Covariance Matrix representing the covariance values between each pair of remaining inequalities.[62]

Covariance Matrix										
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	192,682	427,894	504,598	473,173	620,576	549,877	556,027	697,280	665,855	742,559
2	427,894	2,324,292	2,576,942	2,479,256	2,752,186	2,731,907	2,346,569	3,004,837	2,907,150	3,159,801
3	504,598	2,576,942	2,930,189	2,757,609	3,081,541	3,110,855	2,690,396	3,434,787	3,262,207	3,615,453
4	473,173	2,479,256	2,757,609	2,700,040	2,952,429	2,978,392	2,572,456	3,230,781	3,173,213	3,451,565
5	620,576	2,752,186	3,081,541	2,952,429	3,372,762	3,281,783	2,902,596	3,702,117	3,573,005	3,902,359
6	549,877	2,731,907	3,110,855	2,978,392	3,281,783	3,357,341	2,916,283	3,660,731	3,528,269	3,907,217
7	556,027	2,346,569	2,690,396	2,572,456	2,902,596	2,916,283	2,756,949	3,246,422	3,128,483	3,472,310
8	697,280	3,004,837	3,434,787	3,230,781	3,702,117	3,660,731	3,246,422	4,132,067	3,928,061	4,358,011
9	665,855	2,907,150	3,262,207	3,173,213	3,573,005	3,528,269	3,128,483	3,928,061	3,839,067	4,194,124
10	742,559	3,159,801	3,615,453	3,451,565	3,902,359	3,907,217	3,472,310	4,358,011	4,194,124	4,649,776

Using the covariance values for each pair of random variables, along with the individual standard deviations, we can compute the Coefficient of Correlation Matrix.

Correlation Coefficient Matrix										
ρ_{ij}	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1.000000	0.639397	0.671548	0.656015	0.769806	0.683671	0.762888	0.781453	0.774187	0.784502
2	0.639397	1.000000	0.987442	0.989671	0.982968	0.977964	0.926987	0.969598	0.973216	0.961166
3	0.671548	0.987442	1.000000	0.980392	0.980228	0.991823	0.946573	0.987116	0.972637	0.979487
4	0.656015	0.989671	0.980392	1.000000	0.978366	0.989234	0.942863	0.967250	0.985601	0.974126
5	0.769806	0.982968	0.980228	0.978366	1.000000	0.975258	0.951874	0.991685	0.992951	0.985413
6	0.683671	0.977964	0.991823	0.989234	0.975258	1.000000	0.958556	0.982848	0.982768	0.988903
7	0.762888	0.926987	0.946573	0.942863	0.951874	0.958556	1.000000	0.961849	0.961627	0.969813
8	0.781453	0.969598	0.987116	0.967250	0.991685	0.982848	0.961849	1.000000	0.986238	0.994234
9	0.774187	0.973216	0.972637	0.985601	0.992951	0.982768	0.961627	0.986238	1.000000	0.992687
10	0.784502	0.961166	0.979487	0.974126	0.985413	0.988903	0.969813	0.994234	0.992687	1.000000

Next, we need to calculate the normalized RHS values $b_j^{'}$. Recall from Section 3a that

$$b_j^{'} = \frac{b_j - \mu_j}{\sigma_j},$$

[62] Covariance Matrix solved using Microsoft Excel. In order to ensure that the covariance matrix is positive definite, the variance values down the diagonal of the covariance matrix were rounded up slightly.

where b_j is the initial (non-normalized) RHS value of inequality j, and μ_j and σ_j represent the average and standard deviation of the initial (non-normalized) LHS value of inequality j, respectively.

Using the hourly data described in Section 5a, we can compute the average μ_j and standard deviation σ_j of the random variable η_j associated with each remaining inequality. The initial RHS values b_j are also known based on the assumed Transmission capacities provided in Section 4b. Thus, we have the following values:

Inequality	RHS	$E(\eta_j)$	$\sigma(\eta_j)$	b'_j
1	2,050	1,136	439	2.083324
2	6,360	-4,099	1,525	6.860423
3	4,300	-4,209	1,712	4.970596
4	4,110	-4,304	1,643	5.120582
5	4,670	-2,964	1,837	4.156602
6	2,050	-4,413	1,832	3.527500
7	8,945	-2,852	1,660	7.104643
8	2,610	-3,073	2,033	2.795747
9	2,060	-3,169	1,959	2.668494
10	0	-3,278	2,156	1.520145

The Coefficients of Correlation and normalized RHS values are entered into the NORTEST solver, which proceeds to solve the multivariate normal probability. The output from NORTEST is provided in the next table.

Results for the distribution function		

Error code	=	0
Estimated value (crude)	=	0.927600
Std. deviation (crude)	=	0.002591
Time	=	0.023438
SADMVN value	**=**	**0.930576**
SADER value	=	0.000000
Time	=	0.000000
Lower bound (0,9)	=	0.930589
Time	=	0.000000
Error code	=	0
Estimated value	=	0.930589
Std. deviation	=	0.000000
Time	=	0.000000
Efficiency	=	**********
Upper bound (1,8)	=	0.930589
Time	=	0.046875

The above result indicates that the probability of a feasible flow in our network in this example is 0.930576. This is a significant improvement over the probability calculated in Example #1, primarily attributable to the assumption of higher average generating capacity.

We now move on to Hunter's Upper Bound method to solve Example #2.

5d. Solve By Method 2 – Hunter's Bound

Similar to our approach in Section 4d, we will first derive a lower bound on the probability of the intersection of the remaining 10 inequalities using inequality (4) from Section 3b above. The individual probabilities for each of the remaining 10 inequalities are shown in the table below, as calculated by NORTEST.

Inequality	p_i
1	0.981389
2	1.000000
3	1.000000
4	1.000000
5	0.999984
6	0.999790
7	1.000000
8	0.997411
9	0.996190
10	0.935763
	9.910527

Thus, we have the following lower bound:

$$P(A_1 \cap A_2 \cap \ldots \cap A_{10}) \geq \sum_{i=1}^{10} P(A_i) - (k-1)$$

$$\rightarrow \quad P(A_1 \cap A_2 \cap \ldots \cap A_{10}) \geq 9.910527 - (10-1)$$

$$\rightarrow \quad P(A_1 \cap A_2 \cap \ldots \cap A_{10}) \geq 0.910527$$

We conclude the probability of a feasible flow in the transportation network in our example is at least 0.910527. We note that the result of the multivariate normal distribution from Section 5c was 0.930576, which exceeds this lower bound.

We now turn our attention to Hunter's Upper Bound on the union of k events $A_1, A_2, \ldots, A_k$. As explained in Section 3b,

$$P(A_1 \cup A_2 \cup \ldots \cup A_k) \leq \sum_{i=1}^{k} P(A_i) - \sum_{(i,j)\in T} P(A_i \cap A_j).$$

The first term on the RHS of the inequality is simply the sum of the individual stationary probabilities, which is equal to 9.910527, as provided above. In order to calculate the second term on the RHS of the inequality above, we need to first identify a maximum spanning tree T of the complete graph. The pairwise joint probabilities p_{ij}, which are used as the edge weights in the complete graph, are provided below as calculated by NORTEST.

Pairwise Joint Probabilities										
p_{ij}	1	2	3	4	5	6	7	8	9	10
1		0.981389	0.981389	0.981389	0.981389	0.981336	0.981389	0.980591	0.979967	0.930590
2	0.981389		1.000000	1.000000	0.999984	0.999790	1.000000	0.997411	0.996190	0.935763
3	0.981389	1.000000		1.000000	0.999984	0.999790	1.000000	0.997411	0.996190	0.935763
4	0.981389	1.000000	1.000000		0.999984	0.999790	1.000000	0.997411	0.996190	0.935763
5	0.981389	0.999984	0.999984	0.999984		0.999790	0.999984	0.997411	0.996190	0.935763
6	0.981336	0.999790	0.999790	0.999790	0.999790		0.999790	0.997411	0.996190	0.935763
7	0.981389	1.000000	1.000000	1.000000	0.999984	0.999790		0.997411	0.996190	0.935763
8	0.980591	0.997411	0.997411	0.997411	0.997411	0.997411	0.997411		0.995991	0.935763
9	0.979967	0.996190	0.996190	0.996190	0.996190	0.996190	0.996190	0.995991		0.935763
10	0.930590	0.935763	0.935763	0.935763	0.935763	0.935763	0.935763	0.935763	0.935763	

We note that we have $\frac{k(k-1)}{2} = 45$ total edges in the complete graph. We proceed by Kruskal's algorithm to find a maximum spanning tree. The tables below show the ranking of the weights of the edges in descending order and the corresponding decision as to whether to add the edge to the tree (denoted "Tree") or not to add it because it creates a

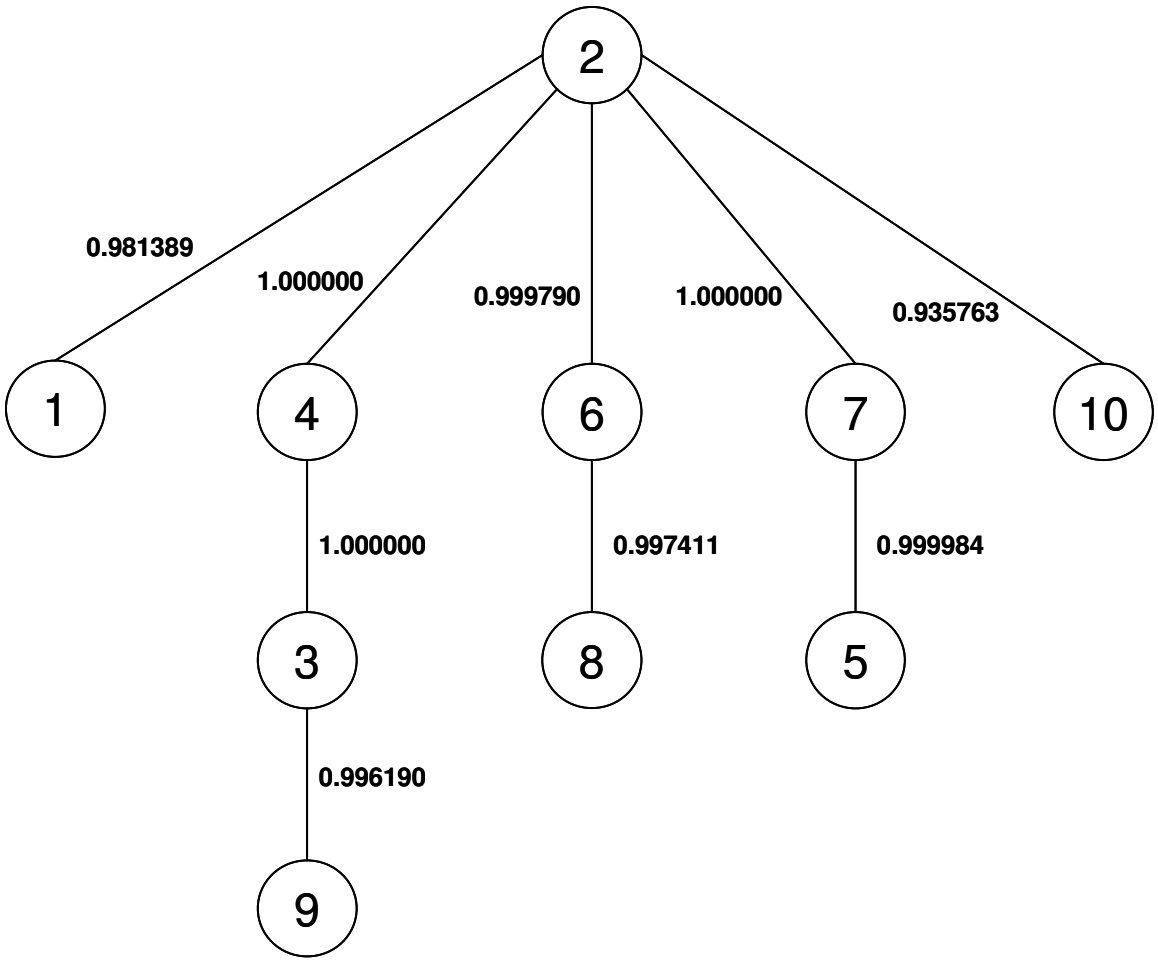
cycle (denoted “Cycle”). Once we have k-1 = 9 edges in the tree, then the algorithm terminates.

No.	Edge	Weight	Decision
1	(2,4)	1.000000	Tree
2	(2,7)	1.000000	Tree
3	(3,4)	1.000000	Tree
4	(3,7)	1.000000	Cycle
5	(4,7)	1.000000	Cycle
6	(2,3)	1.000000	Cycle
7	(5,7)	0.999984	Tree
8	(2,5)	0.999984	Cycle
9	(3,5)	0.999984	Cycle
10	(4,5)	0.999984	Cycle
11	(2,6)	0.999790	Tree
12	(3,6)	0.999790	Cycle
13	(4,6)	0.999790	Cycle
14	(5,6)	0.999790	Cycle
15	(6,7)	0.999790	Cycle
16	(6,8)	0.997411	Tree
17	(2,8)	0.997411	Cycle
18	(3,8)	0.997411	Cycle
19	(4,8)	0.997411	Cycle
20	(5,8)	0.997411	Cycle
21	(7,8)	0.997411	Cycle
22	(3,9)	0.996190	Tree
23	(2,9)	0.996190	Cycle

No.	Edge	Weight	Decision
24	(4,9)	0.996190	Cycle
25	(5,9)	0.996190	Cycle
26	(6,9)	0.996190	Cycle
27	(7,9)	0.996190	Cycle
28	(8,9)	0.995991	Cycle
29	(1,2)	0.981389	Tree
30	(1,3)	0.981389	Cycle
31	(1,4)	0.981389	Cycle
32	(1,5)	0.981389	Cycle
33	(1,7)	0.981389	Cycle
34	(1,6)	0.981336	Cycle
35	(1,8)	0.980591	Cycle
36	(1,9)	0.979967	Cycle
37	(2,10)	0.935763	Tree
38	(3,10)	0.935763	N/A
39	(4,10)	0.935763	N/A
40	(5,10)	0.935763	N/A
41	(6,10)	0.935763	N/A
42	(7,10)	0.935763	N/A
43	(8,10)	0.935763	N/A
44	(9,10)	0.935763	N/A
45	(1,10)	0.930590	N/A

The results of Kruskal’s algorithm, including the associated maximum spanning tree, are provided below.

No.	Edge	Weight
1	(2,4)	1.000000
2	(2,7)	1.000000
3	(3,4)	1.000000
4	(5,7)	0.999984
5	(2,6)	0.999790
6	(6,8)	0.997411
7	(3,9)	0.996190
8	(1,2)	0.981389
9	(2,10)	0.935763
		8.910527



We note from the picture that there are k – 1 = 9 edges and no cycles, which confirms that we have a spanning tree.

It follows that Hunter's Upper Bound on the probability of the union of the 10 events is

$$P(A_1 \cup A_2 \cup \ldots \cup A_{10}) \leq 9.910527 - 8.910527 = 1.000000.$$

This is the same trivial result as achieved in Example #1. This result also serves as an upper bound on the probability of the intersection of all events, and when combined with the lower bound results presented earlier in this section, we have the following:

$$0.910527 \leq P(A_1 \cap A_2 \cap \ldots \cap A_{10}) \leq 1.000000.$$

We conclude that the probability of a feasible flow in our example is between these bounds. While the upper bound is the same as Example #1, the lower bound has increased which suggests more precise results. This range is validated by the results of the multivariate normal distribution from Section 5c, which indicated that the probability of a feasible flow is 0.930576.

5e. Solve By Method 3 – Boolean Probability Bounding

Finally, we proceed using the Boolean Probability Bounding method to find upper and lower bounds on the probability of the union of events. Similar to Example #1, we will utilize the individual stationary probabilities p_i and the joint pairwise probabilities p_{ij} for purposes of this example. These values have already been provided in Section 5d.

We have k=10 so the dimensions of the m×n matrix A used in the LP formulation are m = 55 (10 individual probabilities and 45 joint probabilities) and $n = 2^{10} - 1 = 1{,}023$.[63]

Next, we set up a 55× 1 vector **b** representing the known probabilities of each individual event and pair of events. A brief excerpt of vector **b** is provided below.

$$\mathbf{b} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_8 \\ p_9 \\ p_{10} \\ p_{1,2} \\ p_{1,3} \\ p_{1,4} \\ \vdots \\ p_{8,9} \\ p_{8,10} \\ p_{9,10} \end{bmatrix} = \begin{bmatrix} 0.981389 \\ 1.000000 \\ 1.000000 \\ \vdots \\ 0.997411 \\ 0.996190 \\ 0.935763 \\ 0.981389 \\ 0.981389 \\ 0.981389 \\ \vdots \\ 0.995991 \\ 0.935763 \\ 0.935763 \end{bmatrix}$$

[63] Details behind the formulation of the matrix A are provided in Section 3c.

We now construct the following LP with decision variable x, represented as an n×1 vector.

$$\text{max (min)} \quad \sum_{i=1}^{1,023} x_i$$

$$\text{s.t.} \quad \mathbf{Ax = b}$$

$$\mathbf{x \geq 0}$$

We will utilize the same MOSEL code that we used in Example #1 to solve this LP, updated to reflect the new dimensions of matrix A. [64]

```
model Boolean10                                  ! Start new model

uses "mmxprs"                                    ! Load the optimizer library

declarations
        m = 55
        n = 1023
        Rows= 1..m                               ! Set of rows in A matrix
        Columns= 1..n                            ! Set of columns in A matrix
        A: array(Rows, Columns) of real          ! Binary matrix representing each combination of events
        B: array(Rows) of real                   ! Individual and joint probabilities
        X: array(Columns) of mpvar               ! Decision variable for each combination of events

end - declarations

initializations from "Boolean10.dat"
        A  B
end - initializations

        Bound:= sum(c in Columns) X(c)

        forall(r in Rows) sum(c in Columns) A(r,c) * X(c) = B(r)
        forall(c in Columns) X(c) >= 0

        maximize (Bound)

        writeln("Bound is: ", getobjval)
        forall(c in Columns | getsol(X(c)) > 0)
                writeln("Optimal Solution of combination "+ c + " is: ", getsol(X(c)))
        writeln("All other variables (combinations) are zero in the optimal solution.")

end - model
```

[64] The associated data file is too large to present in this paper. However, the format is consistent with the AMPL data file shown in the example in Section 3c.

When the objective function is solved as a maximization problem, the upper bound on the probability of the union of the 10 events is 1, as shown in the output from MOSEL below.

```
Bound is: 1
Optimal Solution of combination 262 is: 1.6e-005
Optimal Solution of combination 513 is: 0.000141
Optimal Solution of combination 639 is: 5.3e-005
Optimal Solution of combination 764 is: 0.001634
Optimal Solution of combination 848 is: 0.000546
Optimal Solution of combination 932 is: 0.000597
Optimal Solution of combination 968 is: 0.000823
Optimal Solution of combination 969 is: 0.000199
Optimal Solution of combination 1004 is: 0.01105
Optimal Solution of combination 1013 is: 0.049178
Optimal Solution of combination 1022 is: 0.005173
Optimal Solution of combination 1023 is: 0.93059
All other variables (combinations) are zero in the optimal solution.
```

When we instead solve the LP as a minimization problem, we obtain the same optimal solution and optimum value of 1. Thus, we have the following bounds,

$$1 \leq P(A_1 \cup A_2 \cup \ldots \cup A_{10}) \leq 1$$

$$\rightarrow \quad P(A_1 \cup A_2 \cup \ldots \cup A_{10}) = 1.$$

Similar to Example #1, this result is not unexpected as there are several individual probabilities p_i equal to 1.

As discussed in Section 3b, we know that an upper bound on the probability of the union of events is also an upper bound on the probability of the intersection of all 10 events, which implies the following trivial upper bound:

$$P(A_1 \cap A_2 \cap \ldots \cap A_{10}) \leq 1.$$

As we saw in Example #1, this trivial bound is consistent with Hunter's Upper Bound. We also note that the result from the multivariate normal distribution method, which was 0.930576, fits within this upper bound.

5f. Summary of Results

The results for this second example across the three methods are consistent, i.e., the upper bounds obtained via Hunter's Upper Bound and the Boolean Probability Bounding methods are both equal to 1, and the multivariate normal distribution method is within the range resulting from this trivial upper bound and the lower bound derived in Section 5d. We conclude that the probability of a feasible flow in the network formulation in Example #2 is approximately 93%. This is higher than the 82% probability calculated in Example #1 -- the increased probability is primary attributable to the increased assumed average generating capacity. In Example #3, we will consider our most aggressive assumption for average available generating capacity, which we expect to yield even higher results.

6 NUMERICAL EXAMPLE #3
Summer Demand & Peak Annual Generating Capacity

6a. Assumptions

Consistent with the first two examples and for the same reasons noted previously, Example #3 will consider customer demand in the summer months, i.e., hourly demand data from June 1, 2010 – August 31, 2010, which is comprised of 2,208 hourly data points.

With the demand values assumed to be the same as in the prior examples, we turn our attention to the assumed generating capacity values. Recall in Example #1 that this assumption was based on average annual capacity factors across FirstEnergy's entire fleet from 2006-2010, which was approximately 78%. Example #2 more aggressively assumed that the capacity factor for each fuel type of generation was equal to its respective maximum capacity factor achieved from 2006-2010, which was approximately 86% on a weighted average basis. In this final example, we will take this one step further and assume that the average capacity across all fuel types of generation in our model is equal to the maximum capacity factor achieved by any individual fuel type from 2006-2010. In other words, we assume that the average generating capacity for Example #3 is equal to the peak generating capacity factor achieved by any fuel type in FirstEnergy's generating fleet from 2006-2010, which was approximately 93%.[65] The rationale behind this assumption is to model the case of peak historical performance.

[65] As noted previously, the overall available generating capacity did not change from the previous examples – only the weighted average capacity factors are impacted by this modified assumption Also, see the

Similar to the previous 2 examples, the hourly capacity values for each generating unit were randomly generated via a normal distribution based on maximum available capacities and assumed arithmetic means and standard deviations. These hourly generating capacity values were then aligned with the actual customer demand data to represent hourly net demand (supply) values at each node in the network. The resulting average generating capacity values are summarized in the table below.

Example #3 Assumptions			
Fuel Type	**Total (MW)**	**Average (MW)**	**Average Factor**
Nuclear	4,200	3,932	93.6%
Coal	8,110	7,575	93.4%
Other	2,665	2,487	93.3%
	14,975	13,994	93.4%

Finally, we reiterate our previous assumption that transmission capacities remain constant.

tables provided in Sections 4a and 5a for historical capacity factors, by fuel type, achieved by FirstEnergy's entire generating fleet.

6b. Elimination of Trivial Inequalities by Upper Bounds

We first need to analyze the hourly data resulting from the assumptions above to determine which of the initially remaining 21 inequalities from Section 2d can be removed due to triviality.

The following table summarizes the maximum demand values $d(i) = \xi_i - x_i$ for each node i that are produced by the assumptions described above.[66]

Node	Demand ξ_i			Generating Capacity x_i		
	Min	Max	Average	Min	Max	Average
1	849	2,006	1,353	365	1,665	1,561
2	1,827	5,024	3,249	3,475	8,050	7,516
3	1,503	4,043	2,562	650	3,280	3,074
4	319	964	590	0	895	830
5	1,028	3,298	1,868	304	1,085	1,013

We observe that the demand values ξ_i are consistent with those presented in the previous two examples because we are using the same hourly data. The minimum generating capacity values x_i increased again due to our assumed increase in average generating capacity. As noted in Section 4b, the feasibility of the network is dependent on its ability to satisfy customer demand when the net demand values $d(i) = \xi_i - x_i$ are the highest. This occurs when available generating capacity is at its minimum at a particular demand node i and the corresponding local demand value is at its respective maximum. Accordingly, for purposes of finding an upper bound on the LHS on the remaining inequalities, we consider the following maximum demand values d(i):

[66] All demand and generating capacity values presented in terms of MW.

Node	Max ξ_i	Min x_i	Max d(i)
1	2,006	365	1,641
2	5,024	3,475	1,549
3	4,043	650	3,392
4	964	0	964
5	3,298	304	2,994

We now plug these maximum d(i) values into the LHS of the remaining 21 inequalities, along with the assumed constant transmission capacities on the RHS, in order to determine which ones are trivial and can be removed.

	1,641	1,549	3,392	964	2,994		LHS		2,060	7,075	2,430	1,870	180		RHS
(1)	$\xi_1 - x_1$					=	1,641	<	y_{12}					=	2,060
(2)		$\xi_2 - x_2$				=	1,549	<	y_{12} +	y_{23} +	y_{24}			=	11,565
(3)			$\xi_3 - x_3$			=	3,392	<		y_{23}		+ y_{35}		=	8,945
(4)				$\xi_4 - x_4$		=	964	<			y_{24}		+ y_{45}	=	2,610
(5)					$\xi_5 - x_5$	=	2,994					y_{35}	+ y_{45}	=	2,050
(6)	$\xi_1 - x_1$ +	$\xi_2 - x_2$				=	3,191	<		y_{23} +	y_{24}			=	9,505
(7)		$\xi_2 - x_2$ +	$\xi_3 - x_3$			=	4,942	<	y_{12}		+ y_{24} +	y_{35}		=	6,360
(8)		$\xi_2 - x_2$		+ $\xi_4 - x_4$		=	2,513	<	y_{12} +	y_{23}			+ y_{45}	=	9,315
(9)			$\xi_3 - x_3$		+ $\xi_5 - x_5$	=	6,386	<		y_{23}			+ y_{45}	=	7,255
(10)				$\xi_4 - x_4$	+ $\xi_5 - x_5$	=	3,958	<			y_{24} +	y_{35}		=	4,300
(11)	$\xi_1 - x_1$ +	$\xi_2 - x_2$ +	$\xi_3 - x_3$			=	6,583				y_{24} +	y_{35}		=	4,300
(12)	$\xi_1 - x_1$ +	$\xi_2 - x_2$		+ $\xi_4 - x_4$		=	4,154	<		y_{23}			+ y_{45}	=	7,255
(13)		$\xi_2 - x_2$ +	$\xi_3 - x_3$	+ $\xi_4 - x_4$		=	5,905		y_{12}			+ y_{35}	+ y_{45}	=	4,110
(14)		$\xi_2 - x_2$ +	$\xi_3 - x_3$		+ $\xi_5 - x_5$	=	7,936		y_{12}		+ y_{24}		+ y_{45}	=	4,670
(15)		$\xi_2 - x_2$		+ $\xi_4 - x_4$	+ $\xi_5 - x_5$	=	5,507	<	y_{12} +	y_{23}		+ y_{35}		=	11,005
(16)			$\xi_3 - x_3$	+ $\xi_4 - x_4$	+ $\xi_5 - x_5$	=	7,350	<		y_{23} +	y_{24}			=	9,505
(17)	$\xi_1 - x_1$ +	$\xi_2 - x_2$ +	$\xi_3 - x_3$	+ $\xi_4 - x_4$		=	7,546					y_{35}	+ y_{45}	=	2,050
(18)	$\xi_1 - x_1$ +	$\xi_2 - x_2$		+ $\xi_4 - x_4$	+ $\xi_5 - x_5$	=	7,148	<		y_{23}		+ y_{35}		=	8,945
(19)	$\xi_1 - x_1$ +	$\xi_2 - x_2$ +	$\xi_3 - x_3$		+ $\xi_5 - x_5$	=	9,577				y_{24}		+ y_{45}	=	2,610
(20)		$\xi_2 - x_2$ +	$\xi_3 - x_3$	+ $\xi_4 - x_4$	+ $\xi_5 - x_5$	=	8,899		y_{12}					=	2,060
(21)	$\xi_1 - x_1$ +	$\xi_2 - x_2$ +	$\xi_3 - x_3$	+ $\xi_4 - x_4$	+ $\xi_5 - x_5$	=	10,541		0					=	0

The table above shows that 13 inequalities can be removed from the formulation because they are trivial, as compared to the 13 and 11 that were eliminated in Example #1 and

Example #2, respectively. These eliminated inequalities are: (1), (2), (3), (4), (6), (7), (8), (9), (10), (12), (15), (16), and (18).

The remaining 8 inequalities are summarized in the next table below, renumbered in numerical order:

(1)							$\xi_5 - x_5$	$\leq$						y_{35}	+	y_{45}
(2)	$\xi_1 - x_1$	+	$\xi_2 - x_2$	+	$\xi_3 - x_3$			$\leq$				y_{24}	+	y_{35}		
(3)			$\xi_2 - x_2$	+	$\xi_3 - x_3$	+ $\xi_4 - x_4$		$\leq$	y_{12}				+	y_{35}	+	y_{45}
(4)			$\xi_2 - x_2$	+	$\xi_3 - x_3$		+ $\xi_5 - x_5$	$\leq$	y_{12}		+	y_{24}			+	y_{45}
(5)	$\xi_1 - x_1$	+	$\xi_2 - x_2$	+	$\xi_3 - x_3$	+ $\xi_4 - x_4$		$\leq$						y_{35}	+	y_{45}
(6)	$\xi_1 - x_1$	+	$\xi_2 - x_2$	+	$\xi_3 - x_3$		+ $\xi_5 - x_5$	$\leq$				y_{24}			+	y_{45}
(7)			$\xi_2 - x_2$	+	$\xi_3 - x_3$	+ $\xi_4 - x_4$	+ $\xi_5 - x_5$	$\leq$	y_{12}							
(8)	$\xi_1 - x_1$	+	$\xi_2 - x_2$	+	$\xi_3 - x_3$	+ $\xi_4 - x_4$	+ $\xi_5 - x_5$	$\leq$	0							

We now proceed to solve the problem through the three different methods discussed above in Section 3.

6c. Solve By Method 1 – Multivariate Normal Probability

We again use NORTEST to compute the multivariate normal probability distribution $P(\boldsymbol{\eta}' \leq \mathbf{b}')$, where $\boldsymbol{\eta}'$ and $\mathbf{b}'$ represent the normalized LHS and RHS values of the remaining 8 inequalities, respectively, as described in Section 3a. Recall the input data required by NORTEST:

(1) Coefficients of Correlation, denoted ρ_{ij}, for each pair of random variables η_i and η_j

(2) Normalized RHS values $b_j^{'}$ of each remaining inequality j

Based on the hourly demand and generating capacity data described in Section 6a, we have the following averages, standard deviations and variances for the random variables associated with the LHS η_j of each remaining inequality.

Inequality	$E(\eta_j)$	$\sigma(\eta_j)$	$\sigma^2(\eta_j)$
1	855	410	167,868
2	-4,987	1,599	2,555,589
3	-5,018	1,509	2,276,936
4	-3,923	1,721	2,960,483
5	-5,226	1,715	2,940,665
6	-4,132	1,930	3,726,573
7	-4,163	1,838	3,380,062
8	-4,371	2,049	4,199,358

Next we need to find the Covariance Matrix representing the covariance values between each pair of remaining inequalities.[67]

[67] Covariance Matrix solved using Microsoft Excel. In order to ensure that the covariance matrix is positive definite, the variance values down the diagonal of the covariance matrix were rounded up slightly.

Covariance Matrix								
	1	2	3	4	5	6	7	8
1	167,868	501,558	467,629	591,642	545,413	669,426	635,497	713,281
2	501,558	2,555,589	2,375,637	2,706,363	2,726,421	3,057,147	2,877,195	3,227,979
3	467,629	2,375,637	2,276,936	2,556,924	2,563,278	2,843,266	2,744,565	3,030,907
4	591,642	2,706,363	2,556,924	2,960,483	2,894,447	3,298,005	3,148,566	3,486,089
5	545,413	2,726,421	2,563,278	2,894,447	2,940,665	3,271,833	3,108,691	3,486,077
6	669,426	3,057,147	2,843,266	3,298,005	3,271,833	3,726,573	3,512,692	3,941,259
7	635,497	2,877,195	2,744,565	3,148,566	3,108,691	3,512,692	3,380,062	3,744,188
8	713,281	3,227,979	3,030,907	3,486,089	3,486,077	3,941,259	3,744,188	4,199,358

Using the covariance values for each pair of random variables, along with the individual standard deviations, we have the following Coefficient of Correlation Matrix.

Correlation Coefficient Matrix								
ρ_{ij}	1	2	3	4	5	6	7	8
1	1.000000	0.765758	0.756383	0.839255	0.776280	0.846376	0.843659	0.849542
2	0.765758	1.000000	0.984825	0.983919	0.994545	0.990640	0.978952	0.985357
3	0.756383	0.984825	1.000000	0.984830	0.990598	0.976084	0.989318	0.980180
4	0.839255	0.983919	0.984830	1.000000	0.980983	0.992922	0.995335	0.988703
5	0.776280	0.994545	0.990598	0.980983	1.000000	0.988356	0.986035	0.992025
6	0.846376	0.990640	0.976084	0.992922	0.988356	1.000000	0.989744	0.996297
7	0.843659	0.978952	0.989318	0.995335	0.986035	0.989744	1.000000	0.993811
8	0.849542	0.985357	0.980180	0.988703	0.992025	0.996297	0.993811	1.000000

Next, we need to calculate the normalized RHS values $b_j^{'}$. Recall from Section 3a that

$$b_j^{'} = \frac{b_j - \mu_j}{\sigma_j},$$

where b_j is the initial (non-normalized) RHS value of inequality j, and μ_j and σ_j represent the average and standard deviation of the initial (non-normalized) LHS η_j of inequality j, respectively.

Using the hourly data described in Section 6a, we can compute the average μ_j and standard deviation σ_j of the random variable associated with the LHS η_j of each remaining inequality. The initial RHS values b_j are also known based on the assumed Transmission capacities provided in Section 4b. Thus, we have the following values:

Inequality	RHS	$E(\eta_j)$	$\sigma(\eta_j)$	b'_j
1	2,050	855	410	2.915627
2	4,300	-4,987	1,599	5.809409
3	4,110	-5,018	1,509	6.049363
4	4,670	-3,923	1,721	4.994377
5	2,050	-5,226	1,715	4.243246
6	2,610	-4,132	1,930	3.492287
7	2,060	-4,163	1,838	3.384716
8	0	-4,371	2,049	2.133021

The Coefficients of Correlation and normalized RHS values are entered into the NORTEST solver, which proceeds to solve the multivariate normal probability. The output from NORTEST is provided in the next table.

```
Results for the distribution function
-------------------------------------
Error code                    =      0
Estimated value (crude)       =   0.981900
Std. deviation (crude)        =   0.001330
Time                          =   0.015625
SADMVN value                  =   0.983261
SADER value                   =   0.000039
Time                          =   0.000000
Lower bound (0,9)             =   0.000977
Time                          =   0.000000
Error code                    =      0
Estimated value               =   0.000977
Std. deviation                =   0.000000
Time                          =   0.000000
Efficiency                    =   ***********
Upper bound (1,8)             =   0.000977
Time                          =   0.000000
-------------------------------------
```

The above result indicates that the probability of a feasible flow in our network in this example is 0.983261. This is another improvement over the probability calculated in the prior examples, primarily attributable to the assumption of higher average generating capacity.

We now move on to Hunter's Upper Bound method to solve Example #3.

6d. Solve By Method 2 – Hunter's Bound

To derive a lower bound on the probability of the intersection of the remaining 8 inequalities, we first note the individual probabilities, which are shown in the table below, as calculated by NORTEST.

Inequality	p_i
1	0.998225
2	1.000000
3	1.000000
4	1.000000
5	0.999989
6	0.999761
7	0.999644
8	0.983538
	7.981157

Thus, we have the following lower bound:

$$P(A_1 \cap A_2 \cap \ldots \cap A_8) \geq \sum_{i=1}^{8} P(A_i) - (k-1)$$

$$\rightarrow \quad P(A_1 \cap A_2 \cap \ldots \cap A_8) \geq 7.981157 - (8-1)$$

$$\rightarrow \quad P(A_1 \cap A_2 \cap \ldots \cap A_8) \geq 0.981157$$

We conclude the probability of a feasible flow in the transportation network in Example #3 is at least 0.981157. We note that the result of the multivariate normal distribution from Section 6c was 0.983261, which slightly exceeds this lower bound.

We turn our attention to Hunter's Upper Bound on the union of k events $A_1, A_2, \ldots, A_k$. As explained in Section 3b,

$$P(A_1 \cup A_2 \cup \ldots \cup A_k) \leq \sum_{i=1}^{k} P(A_i) - \sum_{(i,j)\in T} P(A_i \cap A_j).$$

The first term on the RHS of the inequality is simply the sum of the individual stationary probabilities, which is equal to 7.981157, as provided above. In order to calculate the second term on the RHS of the inequality above, we need to first identify a maximum spanning tree T of the complete graph. The pairwise joint probabilities p_{ij}, which are used as the edge weights in the complete graph, are provided below as calculated by NORTEST.

Pairwise Joint Probabilities								
p_{ij}	1	2	3	4	5	6	7	8
1		0.998225	0.998225	0.998225	0.998223	0.998146	0.998084	0.983270
2	0.998225		1.000000	1.000000	0.999989	0.999761	0.999644	0.983538
3	0.998225	1.000000		1.000000	0.999989	0.999761	0.999644	0.983538
4	0.998225	1.000000	1.000000		0.999989	0.999761	0.999644	0.983538
5	0.998223	0.999989	0.999989	0.999989		0.999761	0.999644	0.983538
6	0.998146	0.999761	0.999761	0.999761	0.999761		0.999624	0.983538
7	0.998084	0.999644	0.999644	0.999644	0.999644	0.999624		0.983538
8	0.983270	0.983538	0.983538	0.983538	0.983538	0.983538	0.983538	

We note that we have $\frac{k(k-1)}{2} = 28$ total edges in the complete graph. We proceed by Kruskal's algorithm to find a maximum spanning tree. The tables below show the ranking of the weights of the edges in descending order and the corresponding decision as to whether to add the edge to the tree (denoted "Tree") or not to add it because it creates a

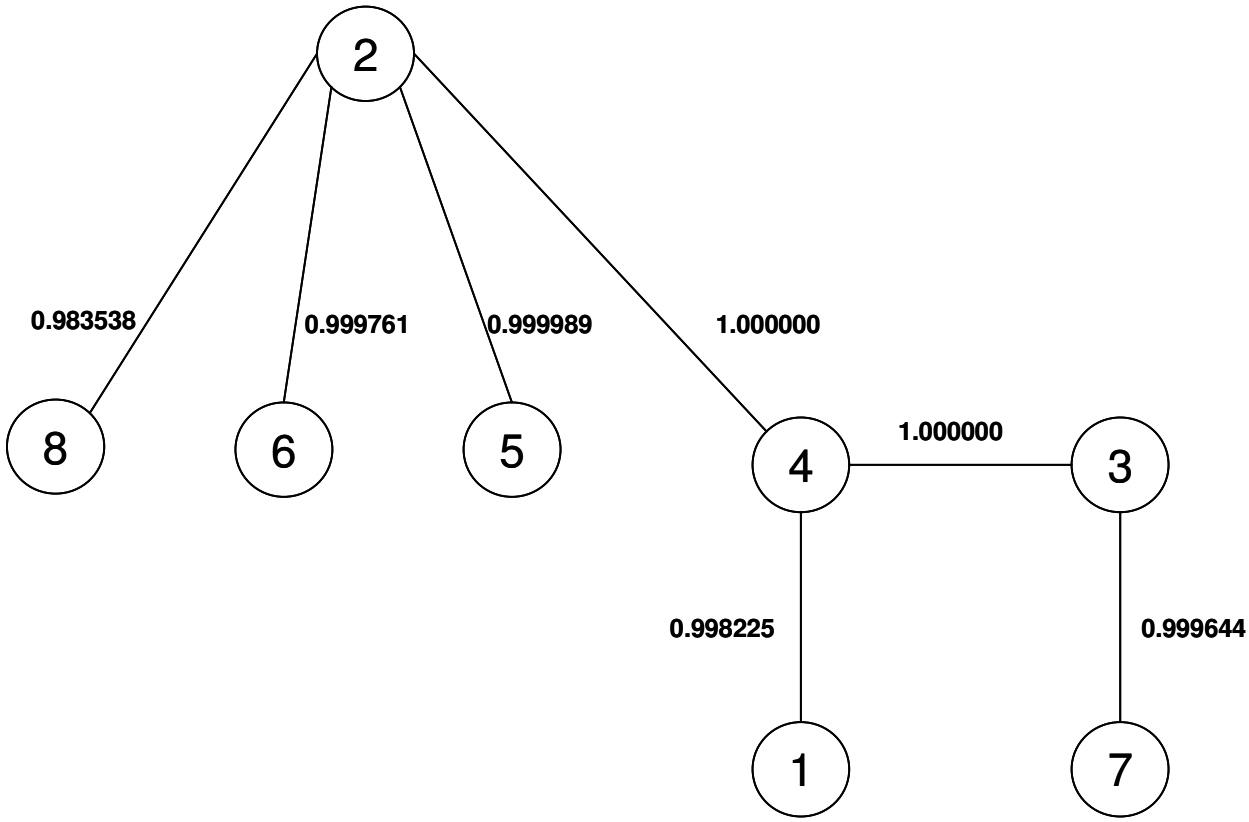
cycle (denoted "Cycle"). The algorithm terminates once we have k – 1 = 7 edges in the tree.

No.	Edge	Weight	Decision
1	(2,4)	1.000000	Tree
2	(3,4)	1.000000	Tree
3	(2,3)	1.000000	Cycle
4	(2,5)	0.999989	Tree
5	(3,5)	0.999989	Cycle
6	(4,5)	0.999989	Cycle
7	(2,6)	0.999761	Tree
8	(3,6)	0.999761	Cycle
9	(4,6)	0.999761	Cycle
10	(5,6)	0.999761	Cycle
11	(3,7)	0.999644	Tree
12	(2,7)	0.999644	Cycle
13	(4,7)	0.999644	Cycle
14	(5,7)	0.999644	Cycle

No.	Edge	Weight	Decision
15	(6,7)	0.999624	Cycle
16	(1,4)	0.998225	Tree
17	(1,2)	0.998225	Cycle
18	(1,3)	0.998225	Cycle
19	(1,5)	0.998223	Cycle
20	(1,6)	0.998146	Cycle
21	(1,7)	0.998084	Cycle
22	(2,8)	0.983538	Tree
23	(3,8)	0.983538	N/A
24	(4,8)	0.983538	N/A
25	(5,8)	0.983538	N/A
26	(6,8)	0.983538	N/A
27	(7,8)	0.983538	N/A
28	(1,8)	0.983270	N/A

The results of Kruskal's algorithm, including the associated spanning tree, are provided below.

No.	Edge	Weight
1	(2,4)	1.000000
2	(3,4)	1.000000
3	(2,5)	0.999989
4	(2,6)	0.999761
5	(3,7)	0.999644
6	(1,4)	0.998225
7	(2,8)	0.983538
		6.981157



We observe from the picture above that we in fact have a spanning tree because there are no cycles and there are k – 1 = 7 edges.

It follows that Hunter's Upper Bound on the probability of the union of the 8 events is

$$P(A_1 \cup A_2 \cup \ldots \cup A_8) \leq 7.981157 - 6.981157 = 1.000000.$$

This is the same trivial result as the previous 2 examples. Since this also serves as an upper bound on the probability of the intersection of all events, we can combine it with

the lower bound results presented earlier in this section and we have the following range of probabilities:

$$0.981157 \leq P(A_1 \cap A_2 \cap \ldots \cap A_8) \leq 1.000000.$$

We conclude that the probability of a feasible flow in our example is between these bounds. Recall that the multivariate normal probability from Section 6c was 0.983261, which is satisfied by these bounds. Finally, we note that while the trivial upper bound is consistent with the prior examples, the range between the lower and upper bounds has tightened significantly, which suggests more precise results.

6e. Solve By Method 3 – Boolean Probability Bounding

We again utilize the individual stationary probabilities p_i and the joint pairwise probabilities p_{ij} in order to set up the LP according to the Boolean Probability Bounding method. These values have already been provided in Section 6d.

We have k=8 so the dimensions of the m×n matrix A used in the LP formulation are m = 36 (8 individual probabilities and 28 joint probabilities) and $n = 2^8 - 1 = 255$.[68]

Next, we set up a 36× 1 vector **b** representing the known probabilities of each individual event and pair of events. A brief excerpt of vector **b** is provided below.

$$\mathbf{b} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_6 \\ p_7 \\ p_8 \\ p_{1,2} \\ p_{1,3} \\ p_{1,4} \\ \vdots \\ p_{6,7} \\ p_{6,8} \\ p_{7,8} \end{bmatrix} = \begin{bmatrix} 0.998225 \\ 1.000000 \\ 1.000000 \\ \vdots \\ 0.999761 \\ 0.999644 \\ 0.983538 \\ 0.998225 \\ 0.998225 \\ 0.998225 \\ \vdots \\ 0.999624 \\ 0.983538 \\ 0.983538 \end{bmatrix}$$

We now construct the following LP with decision variable x, represented as an n×1 vector.

[68] Details behind the formulation of the matrix A are provided in Section 3c.

$$\text{max (min)} \quad \sum_{i=1}^{255} x_i$$

$$\text{s.t.} \quad \mathbf{Ax} = \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

This time, we will return to the AMPL code developed in Section 3c, which is shown as a maximization problem in the picture below.[69]

```
set Columns;
set Rows;
param A {r in Rows, c in Columns};
param B {r in Rows};
var X {c in Columns};

minimize Bound: sum {c in Columns} X[c];
subject to Constraint {r in Rows}: sum {c in Columns} A[r,c] * X[c] = B[r];
subject to NonNegative {c in Columns}: X[c] >= 0;
```

When the objective function is solved as a maximization problem, we see that the upper bound on the probability of the union of the 8 events is **1**. The AMPL output and corresponding optimal solution are provided below. [70]

```
ampl: option solver cplex;
ampl: solve;
CPLEX 11.2.0: optimal solution; objective 1
100 dual simplex iterations (0 in phase I)
```

[69] In Examples #1 and #2, matrix A was too large to process in the student version of AMPL.
[70] Output from CPLEX solver in AMPL.

```
ampl: display X;
X [*] :=
  1      0          65      0          129     0          193     0
  2      0          66      0          130     0          194     0
  3      0          67      0          131     0          195     0
  4      0          68      0          132     0          196     0
  5      0          69      0          133     0          197     0
  6      0          70      0          134     0          198     7.5e-05
  7      0          71      0          135     0          199 2e-05
  8      0          72      0          136     0          200     0
  9      0          73      0          137     0          201     0
 10      0          74      0          138     0          202     0
 11      0          75      0          139     0          203     0
 12      0          76      0          140     0          204     0
 13      0          77      0          141     0          205     0
 14      0          78      0          142     0          206     0
 15      0          79      0          143     0          207     0
 16      0          80      0          144     0          208     0
 17      0          81      0          145     0          209     0
 18      0          82      0          146     0          210     0
 19      0          83      0          147     0          211     0
 20      0          84      0          148     0          212     0
 21      0          85      0          149     0          213     0
 22      0          86      0          150     0          214     0
 23      0          87      0          151     0          215     0
 24      0          88      0          152     0          216     0
 25      0          89      0          153     0          217     0
 26      0          90      0          154     0          218     0
 27      0          91      0          155     0          219     6.2e-05
 28      0          92      0          156     0          220     0
 29      0          93 2e-06           157     0          221     0
 30      0          94      0          158     0          222     0
 31      0          95      0          159     0          223     0
 32      0          96      0          160     0          224     0
 33      0          97      0          161     0          225     0
 34      0          98      0          162     0          226     0
 35      0          99      0          163     7.7e-05    227     0
 36      0         100      0          164     0          228     0
 37      0         101      0          165     0          229     0
 38      0         102      0          166     0          230     0
 39      0         103      0          167     0          231     0
 40      0         104      0          168     0          232     0
 41      0         105      0          169     0          233     0
 42      0         106      0          170     0          234     0
 43      0         107      0          171     0          235     0
 44      0         108      0          172     0          236     0
 45      0         109      0          173     0          237     0
 46      0         110      0          174     0          238     0
 47      0         111      0          175     0          239     0
 48      0         112      0          176     0          240     0.001272
 49      0         113      0          177     0          241     0
 50      0         114      0          178     0          242     0
 51      0         115      0          179     0          243     0
 52      0         116      0          180     0          244     0
 53      0         117      0          181     0          245     0
 54      0         118      0          182     0          246     0
 55      0         119      0          183     0          247     0.014814
 56      0         120      0          184     0          248     0
 57      0         121      0          185     0          249     0
 58 9e-06          122      0          186     0          250     0
 59      0         123      0          187     0          251     0
 60      0         124      0          188     0          252     0
 61      0         125      0          189     0          253     0
 62      0         126      0          190     0          254     0.000268
 63      0         127      0          191     0          255     0.98327
 64      0         128      0.000131   192     0
;
```

The optimal solution is summarized in the following table, where all decision variables x_j not shown have an optimal value of zero:

i	x_i
58	0.000009
93	0.000002
128	0.000131
163	0.000077
198	0.000075
199	0.000020
219	0.000062
240	0.001272
247	0.014814
254	0.000268
255	0.983270
	1.000000

When the LP is solved instead as a minimization problem, the resulting optimal value is still 1 and the optimal solution is the same as the preceding maximization problem. These results imply that the probability of the union of the remaining 8 events is 1,

$$1 \leq P(A_1 \cup A_2 \cup \ldots \cup A_8) \leq 1$$

$$\rightarrow \quad P(A_1 \cup A_2 \cup \ldots \cup A_8 = 1.$$

Again, we note that this result is reasonable based on the fact that several of the individual stationary probabilities p_i are equal to 1. Consequently, as noted in the previous two examples, we know that the probability of the intersection of the remaining 8 events is trivially bounded above by 1, which is the same as Hunter's Upper Bound,

$$P(A_1 \cap A_2 \cap \ldots \cap A_8) \leq 1.$$

6f. Summary of Results

We again see consistency in the trivial upper bounds derived from the bounding techniques offered by Hunter and Boolean Probability. The result of the multivariate normal probability distribution in Section 6c (0.983261) is obviously less than this trivial upper bound, but also slightly greater than the lower bound derived in Section 6d, thereby providing credence to the probability range derived in this section. Thus, we conclude that the probability of a feasible flow in the network formulation in Example #3 is approximately 98%. This is the best performance we have seen in the three examples, primarily due to the more aggressive assumption of peak generating capacity.

7 CONCLUSION

The probabilities of a feasible flow in each of three network examples are summarized in the table below. The second column provides the average generating capacity based on the assumptions discussed in Sections 4a, 5a, and 6a. The third column contains the results of the multivariate normal probability distribution, while the corresponding lower and upper bounds obtained from Hunter's Upper Bound and the Boolean Probability Bounding methods are shown in the last two columns, respectively.

Example	Avg. Gen. Capacity	Estimated Probability	Lower Bound	Upper Bound
1	78.0%	0.818375	0.755406	1.000000
2	86.2%	0.930576	0.910527	1.000000
3	93.4%	0.983261	0.981157	1.000000

It is not a coincidence that the estimated probabilities of a feasible flow in the three examples, as calculated by the multivariate normal distribution, increased with the assumed increase in average generating capacity. Since the same hourly demand values were used in each example and the transmission capacities were assumed to be constant, the only variable impacting the feasibility of the network across the examples was the assumed hourly output of the applicable generating units. When more generating capacity is present in the network, it makes sense that the probability of a feasible flow increases.

In each example, the upper bounds on the network feasibility were consistent between Hunter's Upper Bound and the Boolean Probability Bounding methods; all upper bounds

were 1. This implied that the probability of the intersection of the remaining events was also bounded above by 1, which is a trivial result. The lower bounds on the probability of the intersection, though, increased with the assumed increase in average generating capacity, so the range between the lower and upper bounds tightened each time, which suggests a greater level of accuracy. Finally, we note the result from the multivariate normal probability distribution was within the applicable range obtained by the probability bounding in each example, which supports the reasonableness of our assumption in Section 3a that the relevant date are normally distributed.

While these examples were only based on a subset of the assets of one major electric utility company, they should provide the reader with a better understanding of one of the fundamental issues facing the utility industry: satisfying customer demand with available generating and transmission capacity. While there are changes to the electric utility industry on the horizon, this fundamental problem will remain at the core of its day-to-day operations. Therefore, analyses such as those presented in this paper will continue to be vital to ensuring the ongoing viability of electric utilities to sufficiently satisfy customer demand for electricity.

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