TOPOLOGICAL STRING, SUPERSYMMETRIC GAUGE THEORY AND BPS COUNTING

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ABSTRACT OF THE DISSERTATION

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In this thesis we study the Donaldson-Thomas theory on the local curve geometry, which arises in the context of geometric engineering of supersymmetric gauge theory from type IIA string compactification. The topological A-model amplitude gives the F-term interaction of the compactified theory. In particular, it is related to the instanton partition function via Nekrasov conjecture.

We will introduce ADHM sheaves on curve, as an alternative description of local Donaldson-Thomas theory. We derive the wallcrossing of ADHM invariants and their refinements. We show that it is equivalent to the semi-primitive wallcrossing from supergravity, and the Kontsevich-Soibelman wallcrossing formula.

As an application, we discuss the connection between ADHM moduli space with Hitchin system. In particular we give a recursive formula for the Poincare polynomial of Hitchin system in terms of instanton partition function, from refined wallcrossing.

We also introduce higher rank generalization of Donaldson-Thomas invariant in the context of ADHM sheaves. We study their wallcrossing and discuss their physical interpretation via string duality.

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Dedication

For My Parents.

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Chapter 1

Introduction

Despite the fact that many fundamental issues have not been resolved, for example, statements about string dualities which involve non-perturbative behaviors, string landscape and its connection to physics beyond Standard Model and LHC physics, string theory is still the most promising framework to study quantum field theory and gravity. Many non-perturbative objects like solitons or black holes arise naturally in string theory, and arguments of supersymmetry and string duality make many computations possible, which are otherwise hopeless from a pure field theoretic point of view. They also provide insight such as gauge/gravity duality [1, 2, 10] which can be stated even without the existence of string theory.

One key object of interest is topological string amplitude. For a review see [3, 4]. From the worldsheet point of view, it computes certain correlation functions in the topologically twisted sector coupled with worldsheet gravity. In the case of topological A-model, it can be shown using supersymmetric localization that it only receives contribution from worldsheet instantons, which are holomorphic maps from Riemann surface to some compact two-cycles in spacetime. Physically they correspond to fundamental strings wrapping compact cycles which give non-perturbative corrections to the string scattering amplitudes.

From the target space point of view, topological string amplitude encodes certain interactions between moduli fields and gravity multiplets, and quite remarkably, these terms in the spacetime Lagrangian are protected by supersymmetry and do not receive quantum corrections from certain deformations. Thus these quantities give exact informations about low energy supergravity, as well as their compactification down to lower dimensions.

Because of string duality, the topological string amplitude can sometimes be interpreted as certain supersymmetric index, which counts weighted degeneracies of the Hilbert space and is description invariant. Here by description I mean either UV/IR, strong/weak coupling or other dualities. These dualities are non-trivial and usually involve non-perturbative effects on either side. Topological string amplitude provides a necessary condition to test such dualities.

String theory provides a possible UV description of supergravity and super Yang-Mills theory in four dimensions, via compactifying along six-dimensional manifolds. Different geometries give rise to different theories, or the same theory in different vacua. Therefore there is a close relation between geometry and field theories. Interestingly enough, they live in completely transverse directions and talk to each other only through string compactification. As an example the Nekrasov Conjecture [80] which relates six-manifolds with four-manifolds are thus physically natural but totally mysterious from mathematical point of view.

The key objects which play an important role in string compactification are D-branes [5], which are sources of Ramond-Ramond fields. They are higher dimensional BPS solitons of string theory, i.e. they form short representations of the supersymmetry algebra. In particular, their transformations under string dualities have simple descriptions. In some regimes they become light and are the perturbative degrees of freedom of the theory. Under compactification, depending on what cycles of internal manifolds they wrap, they become either solitons, Dirac strings or higher dimensional defects in the compactified theory. In particular, consider $\mathcal{N} = 2$ supergravity in 4 dimensions resulting from type IIA compactification on Calabi-Yau 3-fold. When the mass of the D-brane is large, it becomes charged black hole. Therefore it provides a microscopic description of black holes, via studying the worldvolume theory on D-branes, which is usually the dimension reduction of super Yang-Mills theory with matters. Along this direction, progress has been made to reproduce the famous Bekenstein-Hawking black hole thermodynamics [6, 7, 31].

As another example [8], type IIA string theory compactified on ADE singularities will have enhanced gauge symmetry at the point in the moduli space when certain 2-cycles shrink to zero volume. D2-branes wrapping the 2-cycles form vector multiplets which are charged under $U(1)$'s and become massless, which give rise to enhanced gauge symmetry. This allows one to geometrically engineer various super Yang-Mills theories with matter, and study their non-perturbative behaviors using string theory. It will be explained in more detail in Chapter 2.

Another way of studying supersymmetric gauge theories from string theory is via brane construction. In flat space, open string dynamics on the brane in IR has simple description of supersymmetric gauge theories. The Lagrangian can be written explicitly and one can study the moduli space of supersymmetric vacua[1]. This is particularly useful due to the following reason. First the moduli space is identified with the classical configuration space of D-branes. This usually gives an alternative description of the moduli space in question, in terms of solutions of F and D-term equations modulo gauge transformations, which can be further related to an

[1]For D-branes wrapping curved space, see for example [17]

algebraic space via Donaldson-Uhlenbeck-Yau type correspondence. For example the ADHM construction of instanton moduli space [14, 11, 12], and Nahm construction of monopole moduli space[13]. Secondly, in the compactified theory one wants to study the single-particle Hilbert space of these D-particles, the usual collective coordinate quantization says that the quantum mechanics in question is a supersymmetric sigma model into the same moduli space. For a review of quantization of solitons see [15]. Supersymmetric ground states correspond to different cohomology theories, depending on what supersymmetry the theory has. The Witten index is usually the Euler character of certain bundles over the moduli space. There could be extra symmetries which act on the field theory, they induce actions on the moduli space and lead to more general equivariant indices, which physically correspond to turning on relevant chemical potential of the symmetry.

The thesis is organized as follows: In chapter 2 we give an overview of topological string amplitude on Calabi-Yau manifolds. We will show that it admits several different descriptions coming from string dualities. It also makes connection with low energy effective prepotential and instanton counting. In the rest of the thesis we will study a particular class of Calabi-Yau geometry, namely the total space of rank 2 bundle over Riemann surface of genus g. It arises as the large fiber volume limit of local ruled surface geometry, which engineers $\mathcal{N} = 2$ $SU(2)$ gauge theories with g massless adjoint hypermultiplets in four dimensions from type IIA string compactification. In particular, we study the local Donaldson-Thomas theory, which physically is the twisted $U(1)$ super Yang-Mills theory on Calabi-Yau, which counts the BPS degeneracy of D6-D2-D0 bound states.

In chapter 3 we study the generalized Donaldson-Thomas theory on local curve geometry via studying ADHM sheaves using the fomalism of [65] in [33, 28, 27]. Moduli spaces of ADHM sheaves are constructed using a natural stability condition depending on a real parameter $\delta \in \mathbb{R}$. In particular for fixed numerical invariants $\gamma = (r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ there is a finite set of critical stability parameters dividing the real axis into stability chambers. Note that $\delta = 0$ is critical for all $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. Residual ADHM invariants $A_\delta(r, e)$ are defined in each chamber by equivariant virtual integration [33]. The asymptotic invariants $A_{+\infty}(r, e)$ corresponding to $\delta >> 0$ are identified with local stable pair invariants in [34]. Wallcrossing formulas for ADHM invariants are derived in [28] using the formalism of Joyce [61, 62, 63, 64] and Joyce and Song [65]. It provides in particular a mathematical framework for the local wallcrossing picture studied by Jafferis and Moore [60]. The resulting wallcrossing formulas are also shown to be in agreement with the Kontsevich-Soibelman formula [69]. Note that the theory of Joyce and Song also yields residual generalized Donaldson-Thomas invariants $H(r, e)$ counting semistable

Higgs sheaves on X with numerical invariants $(r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. These invariants enter the wallcrossing formulas for $A_\delta(r,e)$ derived in [28].

Section (3.2.3) summarizes the main results of motivic wallcrossing in Kontsevich-Soibelman theory. Since the virtual enumerative theory of ADHM sheaves has been studied in [33, 28] employing Joyce-Song theory, these conjectures can be also viewed as a refinement of their generalized Donaldson-Thomas formalism. In particular, the conjectural invariants take in general values in a field of rational functions in one or two formal variables and are conjecturally related to the quantum Donaldson-Thomas invariants of Kontsevich and Soibelman by a refined multicover formula.

In order to make contact with previous results, note that refined wallcrossing formulas have been derived in physical theories defined by a Seiberg-Witten curve in [36, 38, 26, 39, 44], and conjectured to hold in more general situations. Moreover, motivic wallcrossing formulas for Donaldson-Thomas invariants of quivers with potential have been also announced in [68]. The wallcrossing formulas conjectured in this paper for refined generalized Donaldson-Thomas invariants, are related to those of [38, 26, 39, 44] by a refined multicover formula, as explained in more detail below. In addition, it is worth noting that the invariants conjectured here are also equivariant residual invariants with respect to a torus action. Therefore a rigorous construction would require an equivariant localization theorem for motivic Donaldson-Thomas invariants. Although the conjectures below are specifically formulated for ADHM sheaves, analogous conjectures can be formulated in more general situations including abelian categories of coherent sheaves or coherent perverse sheaves on Calabi-Yau threefolds. Previous results and conjectures in the mathematics literature are presented in [24, 76].

We will then present an application of ADHM sheaves to computations of Betti and Hodge numbers of moduli spaces of stable Hitchin pairs on the curve X. As a brief history of the subject, note that the Poincaré polynomial of the moduli space of stable bundles on a curve has been recursively computed in [32], [50] using number theoretic methods, respectively [23] using gauge theoretic methods. The Hodge polynomial of the same moduli spaces, has been recursively computed in [40], and also in [30, 74, 75, 73] for bundles of rank two and three. The Poincaré polynomial of the moduli space of stable Hitchin pairs with coprime rank and degree has been computed by Hithchin in [54] for rank two, and Gothen, [47], for rank three. Using number theoretic techniques, a conjectural formula for any rank has been derived by Hausel and Rodriguez-Villegas in [53] and generalized to Hodge polynomials by Hausel in [52]. Similar results for parabolic rank three Higgs bundles have been obtained in [45]. Finally, the motive of the moduli space of rank four Hitchin pairs in the Grothedieck ring of algebraic varieties is computed in the upcomig work [22].

We present a string theoretic perspective on this subject based on wallcrossing and refined generalized Donaldson-Thomas invariants. There are currently two theories of Donaldson-Thomas invariants, the Kontsevich-Soibelman theory [69] and the Joyce-Song theory [65]. The former is based on a construction of motivic Donaldson-Thomas invariants which specialize to integer valued invariants in a semiclassical limit. The later constructs $\mathbb{Q}$-valued generalized Donaldson-Thomas invariants which are conjecturally related to these integral invariants by a multicover formula [65, Sect. 6.2]. The application presented below relies on the motivic Donaldson-Thomas theory of Kontsevich and Soibelman applied to ADHM sheaves, or, equivalently, on a conjectural refinement of Joyce-Song theory.

The main application of the conjectures in section (3.2.3) is a recursive formula presented in section (3.3.2). This formula determines the Poincaré and Hodge polynomial of moduli spaces of Hitchin pairs with coprime rank and degree in terms of asymptotic motivic ADHM invariants. The later are in turn determined by string theoretic techniques, the results being summarized in section (3.4). In section (3.5) it is checked by direct computation that the resulting expressions are in agreement with the results of [54, 47, 53, 52] in many concrete examples. This provides strong evidence for the validity of the conjectural formalism proposed here. It has also been proved to agree with Hausel-Rodriguez-Villegas formula for the Hodge polynomial of Hitchin system [53, 72]. Note that Higgs sheaves on curves are also employed in [46] as a computational device for local BPS invariants of toric surfaces.

In chapter 4 we will study a further generalization of ADHM invariants allowing higher rank framing sheaves. In contrast to [93, 85], the invariants constructed here count local objects with nontrivial D2-rank, in physics terminology. Similar rank two Donaldson-Thomas invariants of Calabi-Yau threefolds are defined and computed in [86] using both wallcrossing and direct virtual localization methods.

Local invariants with higher D6-rank are also interesting on physical grounds. Explicit results for such invariants are required in order to test the OSV conjecture [7] for magnetically charged black holes. In particular, such results would be needed in order to extend the work of [37] to local D-brane configuration with nonzero D6-rank. According to [31], counting invariants with higher D6-rank are also expected to determine certain subleading corrections to the OSV formula [7]. Moreover, walls of marginal stability for BPS states with nontrivial D6-charge in a local conifold model have been studied from a supergravity point of view in [60]. The construction presented below should be viewed as a rigorous mathematical framework for the microscopic theory of such BPS states. A detailed comparison will appear elsewhere.

From the point of view of six dimensional gauge theory dynamics, the invariants constructed in this paper can be thought of as a higher rank generalization of local Donaldson-Thomas invariants of curves. It should be noted however that they are not the same as the higher rank local DT invariants defined in [34], which, from a gauge theoretic point of view, are Coulomb branch invariants (see also [60, 51] for a noncommutative gauge theory approach.) Instead, employing a different treatment of boundary conditions in the six dimensional gauge theory, the approach presented below yields Higgs branch invariants.

Section (4.1) consists of a step-by-step construction of counting invariants for objects of $\mathcal{C}_{\mathcal{X}}$ following [65]. The required stability conditions, chamber structure and moduli stacks are presented in sections (4.1.1), (4.1.2), (4.1.4) respectively. Some basic homological algebra results are provided in section (4.1.3). The construction is concluded in section (4.1.5). Given a stability parameter $\delta \in \mathbb{R}$ the geometric data $\mathcal{X}$ determines a function $A_\delta : \mathbb{Z}^{\times 3} \to \mathbb{Q}$, which assigns to any triple $\gamma = (r, e, v)$ the virtual number of δ-semistable ADHM sheaves on X of type γ. This function is supported on $\mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. In physics terms, the integers (r, e, v) correspond to D2, D0 and D6-brane charges respectively. In the derivation of wallcrossing formulas, it is more convenient to use the alternative notation $\gamma = (\alpha, v)$, $\alpha = (r, e) \in \mathbb{Z} \times \mathbb{Z}$. Moreover, the invariants $A_\delta(\alpha, 0)$ are manifestly independent on δ, and will be denoted by $H(\alpha)$ since they are counting invariants for Higgs sheaves on X.

Note that for a fixed type γ there is a finite set $\Delta(\gamma) \subset \mathbb{R}$ of critical stability parameters dividing the real axis in stability chambers (see lemma (4.1.9)). The invariants $A_\delta(\gamma)$ are constant when δ varies within a stability chamber. The chamber $\delta > \max \Delta(\gamma)$ will be referred to as the asymptotic chamber, and the corresponding invariants will be also denoted by $A_\infty(\gamma)$. The main result of this paper is a wallcrossing formula for $v = 2$ ADHM invariants at a critical stability parameter $\delta_c > 0$ of type $(\alpha, 2)$, for arbitrary $\alpha = (r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. Certain preliminary definitions will be needed in the formulation of this result, as follows.

For any integer $l \in \mathbb{Z}_{\geq 1}$, and any $v \in \{1, 2\}$ let $\mathcal{HN}_-(\alpha, v, \delta_c, l, l-1)$ denote the set of ordered sequences $((\alpha_i))_{1 \leq i \leq l}$, $\alpha_i \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, $1 \leq i \leq l$ satisfying the following conditions

$$\alpha_1 + \cdots + \alpha_l = \alpha \tag{1.1}$$

and

$$\frac{e_1}{r_1} = \cdots = \frac{e_{l-1}}{r_{l-1}} = \frac{e_l + v\delta_c}{r_l} = \frac{e + v\delta_c}{r} \tag{1.2}$$

For any integer $l \in \mathbb{Z}_{\geq 2}$, let $\mathcal{HN}_-(\alpha, 2, \delta_c, l, l-2)$ denote the set of ordered sequences $((\alpha_i))_{1 \leq i \leq l}$, $\alpha_i \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, $1 \leq i \leq l$ satisfying condition (1.1),

$$\frac{e_1}{r_1} = \cdots = \frac{e_{l-2}}{r_{l-2}} = \frac{e_{l-1} + \delta_c}{r_{l-1}} = \frac{e_l + \delta_c}{r_l} = \frac{e + 2\delta_c}{r}, \tag{1.3}$$

and

$$1/r_{l-1} < 1/r_l. \tag{1.4}$$

Let $0 < \delta_- < \delta_c < \delta_+$ be stability parameters so that there are no critical stability parameters of type $(\alpha, 2)$ in the intervals $[\delta_-,\ \delta_c)$, $(\delta_c,\ \delta_+]$. For any triple (β, v), $\beta \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 1}$, $v \in \{1, 2\}$, the invariants $A_{\delta_\pm}(\beta, v)$ will be denoted by $A_\pm(\beta, v)$. Then the following result holds for δ_-, δ_+ sufficiently close to δ_c.

Theorem 1.0.1. *The $v = 2$ ADHM invariants satisfy the following wallcrossing formula*

$$\begin{aligned}
&A_-(\alpha,2) - A_+(\alpha,2) = \\
&\sum_{l\geq 2} \frac{1}{(l-1)!} \sum_{(\alpha_i)\in \mathcal{HN}_-(\alpha,2,\delta_c,l,l-1)} A_+(\alpha_l,2) \prod_{i=1}^{l-1} f_2(\alpha_i) H(\alpha_i) \\
&- \frac{1}{2}\sum_{l\geq 1} \frac{1}{(l-1)!} \sum_{(\alpha_i)\in \mathcal{HN}_-(\alpha,2,\delta_c,l+1,l-1)} g(\alpha_{l+1},\alpha_l) A_+(\alpha_l,1) A_+(\alpha_{l+1},1) \prod_{i=1}^{l-1} f_2(\alpha_i) H(\alpha_i) \\
&+ \frac{1}{2} \sum_{(\alpha_1,\alpha_2)\in \mathcal{HN}_-(\alpha,2,\delta_c,2,0)} \sum_{l_1\geq 1}\sum_{l_2\geq 1} \frac{1}{(l_1-1)!}\frac{1}{(l_2-1)!} \sum_{(\alpha_{1,i})\in \mathcal{HN}_-(\alpha_1,1,\delta_c,l_1,l_1-1)} \\
&\sum_{(\alpha_{2,i})\in \mathcal{HN}_-(\alpha_2,1,\delta_c,l_2,l_2-1)} g(\alpha_1,\alpha_2) A_+(\alpha_{1,l_1},1) A_+(\alpha_{2,l_2},1) \prod_{i=1}^{l_1-1} f_1(\alpha_{1,i}) H(\alpha_{1,i}) \prod_{i=1}^{l_2-1} f_1(\alpha_{2,i}) H(\alpha_{2,i})
\end{aligned} \tag{1.5}$$

where

$$f_v(\alpha) = (-1)^{v(e-r(g-1))} v(e - r(g-1)), \qquad v = 1, 2$$
$$g(\alpha_1, \alpha_2) = (-1)^{e_1-e_2-(r_1-r_2)(g-1)} (e_1 - e_2 - (r_1 - r_2)(g-1))$$

for any $\alpha = (e, r)$ respectively $\alpha_i = (r_i, e_i)$, $i = 1, 2$, and the sum in the right hand side of equation (1.5) *is finite.*

Theorem (1.0.1) is proven in section (4.2.2) using certain stack function identities established in section (4.2.1). Formula (1.5) is shown to agree with the wallcrossing formula of Kontsevich and Soibelman in section (4.3).

An application of theorem (1.0.1) to genus zero invariants is presented in section (4.4). Consider the following generating functions

$$Z_{\mathcal{X},v}(u, q) = \sum_{r\geq 1} \sum_{n\in\mathbb{Z}} u^r q^n A_\infty(r, n - r, v) \tag{1.6}$$

where $v = 1, 2$. Using the wallcrossing formula (1.5) and the comparison result of section (4.3), the following closed formulas are proven in section (4.4).

Corollary 1.0.2. *Suppose X is a genus 0 curve and $M_1 \simeq \mathcal{O}_X(d_1)$, $M_2 \simeq \mathcal{O}_X(d_2)$ where $(d_1, d_2) = (1,1)$ or $(0,2)$. Then*

$$Z_{\mathcal{X},1}(u,q) = \prod_{n=1}^{\infty}(1-u(-q)^n)^{(-1)^{d_1-1}n}$$

$$Z_{\mathcal{X},2}(u,q) = \frac{1}{4}\prod_{n=1}^{\infty}(1-uq^n)^{2(-1)^{d_1-1}n} - \frac{1}{2}\sum_{\substack{r_1>r_2\geq 1,\ n_1,n_2\in\mathbb{Z} \\ \text{or } r_1=r_2\geq 1,\ n_2>n_1 \\ \text{or } r_1\geq 1,\ n_1\in\mathbb{Z},\ r_2=n_2=0}} (n_1-n_2)(-1)^{(n_1-n_2)} \qquad (1.7)$$

$$A_\infty(r_1, n_1-r_1, 1)A_\infty(r_2, n_2-r_2, 1)u^{r_1+r_2}q^{n_1+n_2}.$$

Chapter 2

Topological String Amplitude

Topological A-model amplitude, from the worldsheet point of view, is the path integral over holomorphic maps to the target Calabi-Yau, also known as worldsheet instantons, weighted by e^{-A}, where A is the area of the image, as a function of complexified Kähler moduli. It has been a subject of much interest because of its close relations with string compactification and supersymmetric gauge theory. More specifically, consider type IIA string theory compactified on a Calabi-Yau 3-fold, the resulting $\mathcal{N} = 2$ supergravity will have $U(1)$ vector multiplets, whose scalars correspond to the complexified Kähler moduli t_i. There will be F-term coupling: $\int d^4x F_g(t_i) R_+^2 F_+^{2g-2}$, where $F_g(t_i)$ is the genus g topological string amplitude, R_+ is the self-dual part of the Riemann tensor, F_+ is the self-dual part of the graviphoton field strength. It can be shown that these vector multiplets do not have interactions with dilaton, which is in a hypermultiplet. Thus the above F-terms do not receive stringy corrections and can be computed in both weak and strong string coupling. One can further decouple gravity by sending $M_p \to \infty$ and engineer pure $\mathcal{N} = 2$ gauge theory.

More specifically, consider type IIA string compactification along a Calabi-Yau 3-fold X which is a ADE type singularities fibration over a base $\mathbb{P}^1$. Let $\tilde{X}$ denote the blowup of X. Exceptional curves $\{\Gamma_i\}$ in $\tilde{X}$ will have the same intersection matrix as the corresponding Cartan matrix. Denote the corresponding complexified Kähler moduli as t_i.

D2-branes wrapping Γ_i are BPS particles, they are charged under corresponding $U(1)$ gauge fields and have a Higgs-like coupling with t_i. On a generic point in the Kähler moduli space, they have mass proportional to $g_s^{-1} t_i$. As the exceptional curve shrinks to zero, they become massless and give rise to enhanced gauge symmetry. One can further decouple gravity as follows:

The bare gauge coupling $1/g_{YM}^2$ at string scale is proportional to the area of the base t_b. So we want $t_b \to \infty$ due to asymptotic freedom. Also, in order to have a finite W-boson mass, we want $t_f \sim M_W/M_p \to 0$. These two limits are related via renormalization group flow as:

$$\frac{1}{g_{YM}^2} \sim \log \frac{M_W}{\Lambda}$$

Thus we have:

$$t_b \sim -\log t_f$$

It is particularly useful to see the relations between worldsheet instantons and gauge theory instantons. Recall that type IIA compactified on K3 surface has a field equation in 6-d:

$$d * (e^{-2\phi} H) = trR \wedge R - trF \wedge F \tag{2.1}$$

Now imagine a worldsheet instanton wrapping the base $\mathbb{P}^1$ n times at a point x in the non-compact 4 dimension. Integrate on a S^3 surrounding x gives:

$$\int_{S^3} e^{-2\phi} H = n$$

which according to Eq.(2.1) identifies gauge theory instantons with worldsheet instanton wrapping the base $\mathbb{P}^1$. Here the term $\int trR \wedge R$ is ignored since we are taking the limit $M_p \to \infty$.

In fact, quite remarkably the genus 0 topological A-model amplitude gives the full perturbative and non-perturbative corrections to the low-energy effective prepotential. This provides a powerful tool to study non-perturbative effects of a wide class of $\mathcal{N} = 2$ theories in four dimensions.

Mathematically, for a given Calabi-Yau 3-fold X, let $\overline{M}_g(X, \beta)$ be the moduli space of stable maps from genus g curve to X, with image class $\beta \in H_2(X, \mathbb{Z})$.

$$N_{g,\beta} = \int_{[\overline{M}_g(X,\beta)]^{vir}} 1$$

is the topological A-model invariants which count the number of such holomorphic maps. Let

$$F_{GW}(X, u, v) = \sum_{\beta \neq 0} \sum_{g \geq 0} N_{g,\beta} u^{2g-2} v^{\beta}$$

denote the reduced Gromov-Witten free energy(excluding constant maps), where v is the Kähler moduli, and $v^{\beta} = e^{-\int_{\beta} K}$ is the volume of the homology class β. The reduced Gromov-Witten partition function is:

$$Z_{GW}(X, u, v) = \exp\left(F_{GW}(X, u, v)\right)$$

2.1 BPS Degeneracy of D6-D2-D0 System

The target space field theory corresponding to topological A-model involves summing over all Kähler metrics on Calabi-Yau, for a fixed complex structure. It turned out that the path integral can be rewritten as an integration over (possibly singular) $U(1)$ gauge fields [90]. Singular $U(1)$ gauge fields are best described by ideal sheaves and they correspond to D2-D0 systems bound to a single D6-brane wrapping the entire Calabi-Yau. This suggests the connection between the topological A-model amplitude with D6-D2-D0 bound state degeneracies.

Mathematically this is the conjectural relation between Gromov-Witten invariant with Donaldson-Thomas invariant, and has been proved in special cases such as toric Calabi-Yau. Let $[I_n(X,\beta)]$ be the moduli space of ideal sheaves $\mathcal{I}$ with support $Y \subset X$ satisfying:

$$\chi(\mathcal{O}_Y) = n$$

and

$$[Y] = \beta \in H_2(X, \mathbb{Z})$$

The Donaldson-Thomas invariant is defined via:

$$\tilde{N}_{n,\beta} = \int_{[I_n(X,\beta)]^{vir}} 1$$

and the partition function:

$$Z_{DT}(X, q, v) = \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{n \in \mathbb{Z}} \tilde{N}_{n,\beta} q^n v^\beta$$

The conjectural GW/DT correspondence [81, 82] states that:

$$Z_{GW}(X, u, v) = Z_{DT}(X, -e^{iu}, v)$$

The connection can also be seen from string compactification. Again consider type IIA string theory compactified on a Calabi-Yau 3-fold X. There are charged BPS solitons which correspond to D6-D2-D0 systems wrapping the Calabi-Yau. They form short representations of the $\mathcal{N} = 2$ SUSY algebra, and are characterized by the fact that their mass is equal to the modulus of the central charge Z. Their spectra are protected by supersymmetry, i.e. the spectrum is invariant with respect to small deformation of the theory. They are minimally coupled to the gauge fields and have one-loop contributions to the low energy effective action:

$$S = \ln det(\Delta + m^2 + \sigma_L F) = \int_\epsilon^\infty \frac{ds}{s} Tr e^{-s(\Delta + m^2 + \sigma_L F)} = \int_\epsilon^\infty \frac{ds}{s} \frac{Tr(-1)^F e^{-sm^2 - 2se\sigma_L F}}{(2\sin(seF/2))^2}$$

for each particle, where $F_{12} = F_{34} = F$ is the self-dual graviphoton field strength, and the result depends on the charge vector γ and the spin of the BPS particle. Once we sum it over all BPS particles, the result will depend on the BPS index, which is defined as:

$$\Omega(\gamma, t_i) = -\frac{1}{2} Tr_{\mathcal{H}_\gamma^{BPS}} (-1)^{2J_3} (2J_3)^2$$

Therefore if we know the full BPS spectrum, the low energy theory can be solved exactly.

To compute such degeneracies, one can write down the supersymmetric quantum mechanics corresponding the D6-D2-D0 system, using the effective action on the branes and collective coordinates, which leads to a sigma-model with target space being the moduli space of branes, each ground state corresponds to a cohomology element of the moduli space and the vacuum degeneracy is the Euler character.

Another explanation is via typeIIA/M-theory duality. At strong coupling, one can lift to M-theory and the worldsheet instanton becomes M2-branes wrapping the 2-cycle, with genus expansion weighted by $g_s^{(2g-2)}$. Viewing the non-compact $\mathbb{R}^4$ as a circle fibration over $\mathbb{R}^3$, which I will explain in more detail in next section, one can compactify along the fiber S^1 and have a dual type IIA picture. The M2-branes become D2-D0 branes bound to a single D6. In fact these two type IIA pictures are related by a TST duality. The TST duality will be important in the context of Nekrasov conjecture, which we will discuss later in this chapter.

In a generic Calabi-Yau compactification without decoupling of gravity, the BPS states of the resulting $\mathcal{N}=2$ 4-d SUGRA are represented as supersymmetric black hole solutions and their multi-center generalizations [31]. This provides a microscopic description of black holes, which in many cases reproduces Bekenstein-Hawking entropy and other thermodynamic properties of black holes. This leads to the Ooguri-Strominger-Vafa conjecture which relates the black hole partition function with topological string amplitude.

Another interesting property of BPS states is that their spectrum is piecewise constant as a function of the Kähler moduli. The degeneracy can jump across some codimension one walls in the Kähler moduli space. Physically what is happening is that some bound state of BPS particles (which is also BPS) may become unstable and decays into multiple BPS particles, as one varies the Kähler moduli, which can be thought of as the vacuum expectation of the scalar fields t_i. For example, consider a BPS particle of charge γ with mass:

$$M = |Z(\gamma, t_i)|$$

If there exists a point t_i^* in Kähler moduli space, and a pair of charge vectors $\gamma_1 + \gamma_2 = \gamma$ such that:

$$|Z(\gamma, t_i^*)| = |Z(\gamma_1, t_i^*)| + |Z(\gamma_2, t_i^*)|$$

On one side of t_i^*, the LHS will be bigger and the BPS particle of charge γ becomes unstable. Thus the BPS index will jump across these so called walls of marginal stability. The wallcrossing of BPS spectrum has been studied extensively in the context of multi-centered black hole solutions in $\mathcal{N}=2$ SUGRA.

Mathematically the moduli space of branes undergo a change of topology known as birational transformation when crossing the wall. Some cycles shrink to zero while new ones emerge via blowup, which gives rise to the jump in Poincare polynomial and Euler characters.

2.2 M-Theory Interpretation

As explained in the last section, the partition function of BPS states does not depend on the string coupling, so one can go to strong coupling and the light states are D2-D0 branes. At strong coupling, they are nothing but M2-branes with Kaluza-Klein momentum along the extra circle. More precisely, consider M-theory on $CY \times \mathbb{R}^4 \times S^1_\beta$, where we are taking a thermal partition function of M2-branes wrapping 2-cycles in CY, S^1_β is the thermal circle with radius β and it is also the M-theory circle. Thus the corresponding type IIA theory will have string coupling $g_s \sim \beta$.

After compactification on CY, those M2-branes become BPS particles, which form representation of the spatial rotation group $SO(4) \simeq SU(2)_L \times SU(2)_R$. One can study quantum mechanics of these M2-branes [18, 19], namely the 8 transverse scalars split into 4 scalars which describe the 4-momentum in the non-compact direction, the other 4 are paired up with the fermions and their collective coordinate quantization gives a supersymmetric sigma model into the moduli space of holomorphic 2-cycles in CY. The $SU(2)$ R-symmetry turns out to be the same as $SU(2)_R \subset SO(4)$, the spatial rotation group, because of the twisting. Thus the $sl(2)$ Lefschetz multiplet is naturally identified with the spin-j_R multiplet.

Let $N_Q^{(m_L,m_R)}$ denote the degeneracy of M2-branes with charge vector $Q \in H_2(CY,\mathbb{Z})$, and spin quantum number (m_L, m_R). They might belong to multiplets of different (j_L, j_R). We can turn on the chemical potential $\lambda_{L,R}$ for these spin quantum numbers, physically they correspond to the graviphoton field strength. These charged particles in the field strength $\lambda_{L,R}$ will also have a Landau level (n_L, n_R) which are the orbital angular momentum. In sum, the contribution of one such particle with quantum number (m_L, m_R, n_L, n_R) to the partition function is:

$$\left(1 - e^{t\cdot Q+(m_L+n_L+1)\beta\lambda_L+(m_R+n_R)\beta\lambda_R}\right)^{\pm 1}$$

The exponent ± 1 depends on if it is a boson or fermion. Thus the full partition function under a generic graviphoton background is:

$$Z(\beta, \lambda_L, \lambda_R, t) = \prod_{Q,m_L,m_R} \prod_{n_L,n_R\geq 0} \left(1 - e^{t\cdot Q+(m_L+n_L+1)\beta\lambda_L+(m_R+n_R)\beta\lambda_R}\right)^{\pm 1}$$

In the situation when $\lambda_R = 0$ (self-dual graviphoton strength), let

$$\lambda = \beta\lambda_L$$

and

$$N_Q^m = \sum_{m_R} (-1)^{m_R} N_Q^{(m,m_R)}$$

, the above expression simplifies to:

$$Z(\lambda, t) = \prod_{Q,m} \left[\prod_{n\geq 0} (1 - e^{t\cdot Q + (m+n)\lambda})^n \right]^{N_Q^m}$$

There is a more unified picture as follows [91]: Consider M-theory on $CY \times TN \times S^1_\beta$, where TN is a Taub-NUT space with the following metric:

$$ds^2 = R^2 \left(\frac{1}{(1 + |\vec{x}|^{-1})}(d\theta + \vec{A} \cdot d\vec{x})^2 + (1 + |\vec{x}|^{-1}) d\vec{x}^2\right)$$

where $\nabla \times \vec{A} = \nabla(1/|\vec{x}|)$. It can be viewed as a S^1 fibration over $\mathbb{R}^3$, where the fiber shrinks to zero at origin, and has radius R at infinity. Thus we have two circles which we can compactify on. If one compactifies along the thermal circle, one gets the previous picture relating topological string amplitude with M2 degeneracies. On the other hand, if one compactifies along the fiber S^1, because it degenerates at origin, it corresponds to a D6-brane wrapping the entire CY and localized at the origin of $\mathbb{R}^3$. This is exactly the picture with Donaldson-Thomas theory on CY.

2.3 Instanton Partition Function

When compactifying from 5-d to 4-d and going to IR [70], the low energy effective action can be derived via integrating out massive particles. There are two types of particles one needs to consider, namely those perturbative degrees of freedom and BPS solitons. The solitons in 5-d come from lifting 4-d instanton solutions. The collective coordinate quantization gives a supersymmetric sigma model into the moduli space of instanton solutions in $\mathbb{R}^4$.

The instanton moduli space $\mathcal{M}(N, k)$ with gauge group $SU(N)$ and instanton number k can be described using ADHM data:

$$\mathcal{M}(N,k) = \left\{ (B_1, B_2, I, J) \;\middle|\; \begin{array}{c} \text{(i)}[B_1, B_2] + IJ = 0 \\ \text{(ii)(stability) there exists no proper subspace S} \subsetneq \mathbb{C}^k \\ \text{such that } B_\alpha(S) \subsetneq S (\alpha = 1,2) \text{and im} I \subset S. \end{array} \right\} \Big/ GL_k(\mathbb{C})$$

where $B_{1,2} \in \mathrm{End}(\mathbb{C}^k)$, $I \in \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^k)$ and $J \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^N)$, and $GL_k(\mathbb{C})$ acts as:

$$g \cdot (B_1, B_2, I, J) = (gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1})$$

for $g \in GL_k(\mathbb{C})$.

This can be seen either via explicit construction of the one-to-one map from instanton solution to ADHM data [14], or the brane construction of instantons using $D(p-4)$-Dp system, where $D(p-4)$-brane in the Higgs branch corresponds to small instanton living on Dp-branes. It can be seen from the Ramond-Ramond coupling $C_{p-3} \wedge Tr(F \wedge F)$ on Dp worldvolume. The D-term equations for $D(p-4)$-Dp and $D(p-4)$-$D(p-4)$ fields coincide with those in ADHM description.

It can be shown [16] that the supersymmetric quantum mechanics in question has mass deformations, the Witten index gives the equivariant Hirzbruch χ_y-genus of the moduli space [56], where the group action is $SU(N) \times SO(4)$, where $SU(N)$ is the gauge transformation on gauge fields at infinity, and $SO(4)$ action is the rotation in $\mathbb{R}^4$. Physically, these equivariant parameters correspond to the vacuum expectation of adjoint scalars, and the background graviphoton field strength.

The Nekrasov conjecture thus provides a connection between instanton moduli space and the moduli space of ideal sheaves on Calabi-Yau. This will be explained in more detail in 3.4. It can also be seen directly from string duality, see for example [48]. However, a direct geometric connection is still unknown.

Chapter 3

ADHM Theory of Curves, Instanton Counting and Hitchin System

In this chapter we will study the theory of ADHM sheaves on curves. It was first studied in [34], as an alternative description of the local Donaldson-Thomas theory on curves. Physically it counts the D6-D2-D0 bound state degeneracy in the large radius limit, where D2-branes wrap the base curve. It employs a real stability parameter which corresponds to the central charge of D6-brane. In section 3.1 we will review the definition of ADHM invariants, the construction of stability and moduli space of stable objects. In section 3.2 we will study the wallcrossing and its refinement using Joyce-Song formalism [61, 62, 63, 64, 65] and show that they agree with Kontsevich-Soibelman motivic wallcrossing formula [68]. In section 3.3 we study the connection with Hitchin moduli space, and derive a recursive formula for the Poincare polynomial of Hitchin moduli space, in terms of refined local Donaldson-Thomas invariants. In section 3.4 we compute the refined local Donaldson-Thomas invariants via studying the instanton partition function of $\mathcal{N} = 2$ $SU(2)$ gauge theory with g adjoint hypermultiplets. In section 3.5 we show some lower rank examples and compare with results from literatures. This chapter is based on [28, 29].

3.1 Review of ADHM Sheaves, Stability and Chamber Structure

Let X be a smooth projective curve over $\mathbb{C}$ of genus $g \geq 2$. Let M_1, M_2 be line bundles on X so that $M_1 \otimes_X M_2 \simeq K_X^{-1}$, and fix such an isomorphism in the following. Let $\deg(M_1) = p$, $\deg(M_2) = -2g - 2 - p$, $p \in \mathbb{Z}$ and $\mathcal{X} = (X, M_1, M_2)$.

The abelian category $\mathcal{C}_\mathcal{X}$ of ADHM sheaves is defined as follows. The objects of $\mathcal{C}_\mathcal{X}$ are collections $\mathcal{E} = (E, V, \Phi_1, \Phi_2, \phi, \psi)$ on X where E is a coherent sheaf on X, V is a finite dimensional complex vector space, and $\Phi_i : E \otimes_X M_i \to E$, $i = 1, 2$, $\phi : E \otimes_X M_1 \otimes_X M_2 \to V \otimes \mathcal{O}_X$, $\psi : V \otimes \mathcal{O}_X \to E$ are morphisms of $\mathcal{O}_X$-modules satisfying the ADHM relation

$$\Phi_1 \circ (\Phi_2 \otimes 1_{M_1}) - \Phi_2 \circ (\Phi_1 \otimes 1_{M_2}) + \psi \circ \phi = 0. \tag{3.1}$$

The morphisms of $\mathcal{C}_\mathcal{X}$ are natural morphisms of quiver sheaves.

An object $\mathcal{E}$ of $\mathcal{C}_{\mathcal{X}}$ will be called locally free if E is a coherent locally free $\mathcal{O}_X$-module. Given a coherent $\mathcal{O}_X$-module E we will denote by $r(E)$, $d(E)$, $\mu(E)$ the rank, degree, respectively slope of E if $r(E) \neq 0$. The type of an object $\mathcal{E}$ of $\mathcal{C}_{\mathcal{X}}$ is the collection $(r(\mathcal{E}), d(\mathcal{E}), v(\mathcal{E})) = (r(E), d(E), \dim(V))) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$.

The dual of a locally free ADHM sheaf $\mathcal{E} = (E, V, \Phi_1, \Phi_2, \phi, \psi)$ is defined by

$$\begin{aligned} \widetilde{E} &= E^\vee \otimes_X M^{-1} \\ \widetilde{\Phi}_i &= (\Phi_i^\vee \otimes 1_{M_i}) \otimes 1_{M^{-1}} : \widetilde{E} \otimes M_i \to \widetilde{E} \\ \widetilde{\phi} &= \psi^\vee \otimes 1_{M^{-1}} : \widetilde{E} \otimes_X M \to V^\vee \otimes \mathcal{O}_X \\ \widetilde{\psi} &= \phi^\vee : V^\vee \otimes \mathcal{O}_X \to \widetilde{E} \end{aligned} \tag{3.2}$$

where $i = 1, 2$. Obviously, if $\mathcal{E}$ is of type (r, e, v), $\widetilde{\mathcal{E}}$ is of type $(r, -e + 2r(g-1), v)$.

Note that the objects of $\mathcal{C}_{\mathcal{X}}$ with $v(\mathcal{E}) = 0$ are triples $\mathcal{E} = (E, \Phi_1, \Phi_2)$ so that

$$\Phi_1 \circ (\Phi_2 \otimes 1_{M_1}) - \Phi_2 \circ (\Phi_1 \otimes 1_{M_2}) = 0. \tag{3.3}$$

and they form a full abelian subcategory of $\mathcal{C}_{\mathcal{X}}$. These are known as Higgs sheaves on X with coefficient bundle $M_1 \oplus M_2$ (see [33, App. A] for a brief summary of definitions and properties.)

Any real parameter $\delta \in \mathbb{R}$ determines a stability condition on $\mathcal{C}_{\mathcal{X}}$ [33, 93]. An object $\mathcal{E}$ of $\mathcal{C}_{\mathcal{X}}$ is δ-(semi)stable if any proper nontrivial subobject $0 \subset \mathcal{E}' \subset \mathcal{E}$ satisfies the inequality

$$r(\mathcal{E})(d(\mathcal{E}') + \delta v(\mathcal{E}')) \ (\leq) \ r(\mathcal{E}')(d(\mathcal{E}) + \delta v(\mathcal{E})). \tag{3.4}$$

Standard arguments show that the δ-stability condition satisfies the Harder-Narasimhan as well as Jordan-Hölder property for any $\delta \in \mathbb{R}$. Moreover the following properties hold for any object $\mathcal{E} = (E, V, \Phi_1, \Phi_2 \phi, \psi)$ of $\mathcal{C}_{\mathcal{X}}$ with $r(\mathcal{E}) \geq 1$ and $v(\mathcal{E}) = 1$ [33, Sect 3]

($S.1$) If $\mathcal{E}$ is δ-semistable for some $\delta \in \mathbb{R}$, then $\mathcal{E}$ is locally free. In addition, if $\delta > 0$ then ψ is not identically zero; if $\delta < 0$, ϕ is not identically zero.

($S.2$) If $\mathcal{E}$ is δ-stable for some $\delta \in \mathbb{R}$, the endomorphism ring of $\mathcal{E}$ in $\mathcal{C}_{\mathcal{X}}$ is canonically isomorphic to $\mathbb{C}$.

($S.3$) $\mathcal{E}$ is δ-(semi)stable if and only if the dual $\widetilde{\mathcal{E}}$ is $(-\delta)$-(semi)stable.

One also has the following boundedness results [33, Lemm. 2.6, Lemm. 2.7, Cor. 2.8]

($B.1$) The set of isomorphism classes of locally free ADHM sheaves of fixed type $(r, e, 1)$ which are δ-semistable for some $\delta \in \mathbb{R}$ is bounded.

$(B.2)$ For any $r \geq 1$ there exists an integer $c(r) \in \mathbb{Z}$ so that any δ-semistable ADHM sheaf of type $(r, e, 1)$ for some $\delta > 0$ satisfies $e \geq c(r)$. Note that the integer $c(r)$ is not unique unless required to be optimal with this property. In fact the proof of [33, Lemm. 2.6] implies that any integer

$$c(r) \leq -2(r-1)^2 \max\{|\deg(M_1)|, |\deg(M_2)|\}$$

satisfies this condition.

Note that for $v = 0$ objects, δ-stability is independent of δ and reduces to standard slope stability for Higgs sheaves on X.

A straightforward corollary of the above results is the existence of an algebraic moduli stack of finite type $\mathfrak{M}^{ss}_\delta(\mathcal{X}, r, e)$ of δ-semistable ADHM sheaves on X of type $(r, e, 1)$ for any $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ and any $\delta \in \mathbb{R}$. The substack $\mathfrak{M}^{s}_\delta(\mathcal{X}, r, e)$ of δ-stable objects is separated and has the structure of a $\mathbb{C}^\times$-gerbe over an algebraic moduli space $M^{ss}_\delta(\mathcal{X}, r, e)$. Property $(S.3)$ also yields a canonical isomorphism

$$\mathfrak{M}^{ss}_\delta(\mathcal{X}, r, e) \simeq \mathfrak{M}^{ss}_\delta(\mathcal{X}, r, -e + 2r(g-1)) \tag{3.5}$$

for any $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ and any $\delta \in \mathbb{R}$.

Moreover there is a stability chamber structure on $\mathbb{R}_{>0}$ as follows [33, Sect. 4]. For a fixed type $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, three exists a finite set $\Delta(r, e) \subset \mathbb{R}_{>0}$ of critical stability parameters so that

$(C.1)$ For any $\delta \in \mathbb{R}_{>0} \setminus \Delta(r, e)$, δ-semistability is equivalent to δ-stability i.e. $\mathfrak{M}^{ss}_\delta(\mathcal{X}, r, e) = \mathfrak{M}^{s}_\delta(\mathcal{X}, r, e)$.

$(C.2)$ For any $\delta > \max \Delta(\mathrm{r}, \mathrm{e})$ δ-stability is equivalent with the following asymptotic stability condition. An object $\mathcal{E} = (E, V, \Phi_i, \phi, \psi)$ with $v = 1$ is asymptotically stable if E is locally free, ψ nontrivial, and there is no proper saturated subsheaf $0 \subset E' \subset E$ preserved by Φ_i, $i = 1, 2$ so that $Im(\psi) \subseteq E'$.

Finally note that there is a torus $\mathbf{S} = \mathbb{C}^\times$ action on the moduli stacks $\mathfrak{M}^{ss}_\delta(\mathcal{X}, r, e)$ so that

$$t \times (E, V, \Phi_1, \Phi_2, \phi, \psi) \to (E, V, t^{-1}\Phi_1, t\Phi_2, \phi, \psi) \tag{3.6}$$

on closed points. According to [33, Thm. 1.5], for noncritical stability parameter $\delta \in \mathbb{R}_{>0} \setminus \Delta(r, e)$, the stack theoretic fixed locus $\mathfrak{M}^{ss}_\delta(\mathcal{X}, r, e)^{\mathbf{S}}$ is universally closed over $\mathbb{C}$. Moreover, the algebraic moduli space $M^{ss}_\delta(\mathcal{X}, r, e)$ has a perfect obstruction theory. Therefore residual

δ-ADHM invariants $A_\delta(r,e) \in \mathbb{Z}$ can be defined in each chamber by equivariant virtual localization. We will discuss their wallcrossing and its refinement in next section.

In order to conclude this section, note that the stacks $\mathfrak{Higgs}^{ss}(\mathcal{X}, r, e)$ have the following simple properties. By analogy with (3.5), there is a canonical torus equivariant isomorphism

$$\mathfrak{Higgs}^{ss}(\mathcal{X}, r, e) \simeq \mathfrak{Higgs}^{ss}(\mathcal{X}, r, -e + 2r(g-1)) \tag{3.7}$$

In addition, taking tensor product by a fixed degree one line bundle on X yields an equivariant isomorphism

$$\mathfrak{Higgs}^{ss}(\mathcal{X}, r, e) \simeq \mathfrak{Higgs}^{ss}(\mathcal{X}, r, e + r) \tag{3.8}$$

for any $(r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. Finally note that for (r,e) coprime slope semistability is equivalent to slope stability, and the stack $\mathfrak{Higgs}^{ss}(\mathcal{X}, r, e)$ has a $\mathbb{C}^\times$-gerbe structure over a quasi-projective scheme $Higgs^{ss}(\mathcal{X}, r, e)$.

3.2 Wallcrossing

Let $\delta_c \in \mathbb{R}_{\geq 0}$ be a critical stability parameter of type $(r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, possibly zero, and $\delta_+ > \delta_c$, $\delta_- < \delta_c$ be stability parameters so that there are no critical stability parameters of type (r,e) in the interval $[\delta_-, \delta_+]$. In order to simplify the formulas, we will denote the numerical invariants by $\alpha = (r,e)$, and use the notation

$$\mu_\delta(\alpha) = \frac{e+\delta}{r}, \qquad \mu(\alpha) = \frac{e}{r}$$

for any $\alpha = (r,e)$ with $r \geq 1$, and any $\delta \in \mathbb{R}$.

For fixed $\alpha = (r,e)$, $\delta_c \geq 0$ and $l \in \mathbb{Z}_{\geq 2}$ let $\mathsf{S}^{(l)}_{\delta_c}(\alpha)$ be the set of all ordered decompositions

$$\alpha = \alpha_1 + \cdots + \alpha_l, \qquad \alpha_i = (r_i, e_i) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}, \qquad i = 1, \ldots, l \tag{3.9}$$

satisfying

$$\mu(\alpha_1) = \cdots = \mu(\alpha_{l-1}) = \mu_{\delta_c}(\alpha_l) = \mu_{\delta_c}(\alpha). \tag{3.10}$$

Note that the union $\mathsf{S}_{\delta_c}(\alpha) = \bigcup_{l\geq 2} \mathsf{S}^{(l)}_{\delta_c}(\alpha)$ is a finite set for fixed $\delta_c \geq 0$. Then the following theorem is proven in [33]:

Theorem 3.2.1. *(i) The following wallcrossing formula holds for $\delta_c > 0$*

$$\begin{aligned} &A^{\mathbf{S}}_{\delta_+}(\alpha) - A^{\mathbf{S}}_{\delta_-}(\alpha) = \\ &\sum_{l\geq 2} \frac{(-1)^{l-1}}{(l-1)!} \sum_{(\alpha_1,\ldots,\alpha_l)\in \mathsf{S}^{(l)}_{\delta_c}(\alpha)} A^{\mathbf{S}}_{\delta_-}(\alpha_l) \prod_{j=1}^{l-1} [(-1)^{e_j - r_j(g-1)}(e_j - r_j(g-1)) H(\alpha_j)]. \end{aligned} \tag{3.11}$$

(ii) The following wallcrossing formula holds for $\delta_c = 0$.

$$
\begin{aligned}
&A^{\mathbf{S}}_{\delta_+}(\alpha) - A^{\mathbf{S}}_{\delta_-}(\alpha) = \\
&\sum_{l\geq 2} \frac{(-1)^{l-1}}{(l-1)!} \sum_{(\alpha_1,\ldots,\alpha_l)\in \mathsf{S}_0^{(l)}(\alpha)} A^{\mathbf{S}}_{\delta_-}(\alpha_l) \prod_{j=1}^{l-1} [(-1)^{e_j - r_j(g-1)}(e_j - r_j(g-1))H(\alpha_j)] \\
&+ \sum_{l\geq 1} \frac{(-1)^l}{l!} \sum_{(\alpha_1,\ldots,\alpha_l)\in \mathsf{S}_0^{(l)}(\alpha)} \prod_{j=1}^{l} [(-1)^{e_j - r_j(g-1)}(e_j - r_j(g-1))H(\alpha_j)]
\end{aligned}
\tag{3.12}
$$

Moreover, if $g \geq 1$*, the right hand sides of equations* (3.11), (3.12) *vanish.*

Here $H(\alpha)$ are generalized Donaldson-Thomas type invariants for Higgs sheaves with numerical invariants $\alpha = (r, e)$ on X defined in the previous section.

Some applications of Theorem (3.2.1) are presented below. For any $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ let $A^{\mathbf{S}}_{\pm\infty}(r, e)$ denote the ADHM invariants in the asymptotic chambers $\delta >> 0$, $\delta << 0$ respectively. Then let

$$
Z_{\pm\infty}(q)_r = \sum_{e\in\mathbb{Z}} q^{e-r(g-1)} A^{\mathbf{S}}_{\pm\infty}(r, e). \tag{3.13}
$$

be the formal generating function of such invariants for fixed rank $r \geq 1$. According to [34, Cor. 1.12] $Z_{+\infty}(q)_r$ is the generating function of degree r local stable pair invariants of the data $\mathcal{X} = (X, M_1, M_2)$. Note that [33, Lemm. 2.3] implies that $A^{\mathbf{S}}_{+\infty}(r, e) = A^{\mathbf{S}}_{-\infty}(r, -e + 2r(g-1))$ for all $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. On the other hand, for curves X of genus $g \geq 1$, Theorem (3.2.1) implies that $A^{\mathbf{S}}_{+\infty}(r, e) = A^{\mathbf{S}}_{-\infty}(r, e)$ for all $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ since in this case the invariants $H(r, e)$ are zero. Therefore the following holds.

Corollary 3.2.2. *If* $g \geq 1$*,* $Z_{+\infty}(q)_r$ *is a polynomial in* q, q^{-1} *invariant under* $q \leftrightarrow q^{-1}$.

This implies that the strong rationality conjecture formulated in [96] holds for the local stable pair theory of curves of genus $g \geq 1$.

Further applications of Theorem (3.2.1) are presented in section (3.2.1) where it is shown that the wallcrossing formula (3.11) is in agreement with the wallcrossing formula of Kontsevich and Soibelman [69]. Moreover, it is also shown that formula (3.11) is in agreement with the halo wallcrossing formula for D6-D2-D0 bound states derived by Denef and Moore [31, Sect. 6.1.2] using supergravity arguments.

3.2.1 Comparison with Kontsevich-Soibelman formula

In this section we specialize the wallcrossing formula of Kontsevich and Soibelman [69] to ADHM invariants, and prove that it implies equation (3.11). Recall that locally free ADHM quiver sheaves on X have a numerical invariants of the form $(r, e, v) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. The pair (r, e) is denoted by α in theorem (3.2.1). Let $\mathsf{e}_\alpha = \lambda^\alpha$, $\mathsf{f}_\alpha = \lambda^{(\alpha,1)}$, $\alpha \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ be alternative notation for the generators of the Lie algebra $\mathsf{L}(\mathcal{X})_{\geq 1}$. Therefore

$$\begin{aligned} &[\mathsf{e}_{\alpha_1}, \mathsf{e}_{\alpha_2}]_{\leq 1} = 0 \\ &[\mathsf{f}_{\alpha_1}, \mathsf{f}_{\alpha_2}]_{\leq 1} = 0 \\ &[\mathsf{f}_{\alpha_1}, \mathsf{e}_{\alpha_2}]_{\leq 1} = \chi(\alpha_1, \alpha_2)\mathsf{f}_{\alpha_1+\alpha_2} \end{aligned} \tag{3.14}$$

where $\chi(\alpha_1, \alpha_2) = (-1)^{e_2 - r_2(g-1)}(e_2 - r_2(g-1))$.

Let $\delta_c \in \mathbb{R}_{>0}$ be a critical stability parameter of type $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ as in theorem (3.2.1). Then there exist $\alpha, \beta \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, with

$$\mu_c(\alpha) = \mu(\beta) = \mu_c(\alpha) \tag{3.15}$$

so that any $\eta \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ with

$$\mu_c(\eta) = \mu_c(\alpha)$$

is uniquely written as

$$\eta = \alpha + q\beta, \qquad q \in \mathbb{Z}_{\geq 0}$$

and any $\rho \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ with

$$\mu(\rho) = \mu_c(\alpha)$$

is uniquely written as

$$\rho = q\beta, \qquad q \in \mathbb{Z}_{\geq 0}.$$

Therefore α and β generate a subcone of $\mathbb{Z}_{\geq 1} \times \mathbb{Z}$ consisting of elements of δ_c-slope equal to $\mu_c(\alpha)$.

For any $q \in \mathbb{Z}_{\geq 0}$ define to be the following formal expressions

$$U_{\alpha+q\beta} = \exp(f_{\alpha+q\beta}) \qquad U_{q\beta} = \exp(\sum_{m\geq 1} \frac{e_{mq\beta}}{m^2}) \tag{3.16}$$

In this context, the wallcrossing formula of Kontsevich and Soibelman [69] reads

$$\prod_{q\geq 0,\ q\uparrow} U_{q\beta}^{\overline{H}(q\beta)} \prod_{q\geq 0,\ q\uparrow} U_{\alpha+q\beta}^{A_+^{\mathbf{S}}(\alpha+q\beta)} = \prod_{q\geq 0,\ q\uparrow} U_{\alpha+q\beta}^{A_-^{\mathbf{S}}(\alpha+q\beta)} \prod_{q\geq 0,\ q\downarrow} U_{q\beta}^{\overline{H}(q\beta)}, \tag{3.17}$$

where an up, respectively down arrow means that the factors in the corresponding product are taken in increasing, respectively decreasing order of q. Integer invariants $\overline{H}(r, e)$ are defined using the multicover formula

$$H(r, e) = \sum_{\substack{m\geq 1 \\ m|r,\ m|e}} \frac{1}{m^2}\overline{H}(r/m, e/m). \tag{3.18}$$

In the following we will prove that equation (4.41) implies the wallcrossing formula (3.11). First note that given equation (4.39), the formal operators U commute within each product over q in equation (4.41). Therefore (4.41) can be rewritten as

$$\exp\big(\sum_{m\geq 1}\sum_{q\geq 0}\overline{H}(mq\beta)\frac{\mathsf{e}_{mq\beta}}{m^2}\big)\prod_{q\geq 0}U_{\alpha+q\beta}^{A_+^{\mathbf{S}}(\alpha+q\beta)} = \prod_{q\geq 0}U_{\alpha+q\beta}^{A_-^{\mathbf{S}}(\alpha+q\beta)}\exp\big(\sum_{m\geq 1}\sum_{q\geq 0}\overline{H}(mq\beta)\frac{\mathsf{e}_{mq\beta}}{m^2}\big)$$

This formula can be rewritten in terms of the rational invariants $H(\alpha)$. We obtain

$$\exp\big(\sum_{q\geq 0}H(q\beta)\mathsf{e}_{q\beta}\big)\prod_{q\geq 0}U_{\alpha+q\beta}^{A_+^{\mathbf{S}}(\alpha+q\beta)} = \prod_{q\geq 0}U_{\alpha+q\beta}^{A_-^{\mathbf{S}}(\alpha+q\beta)}\exp\big(\sum_{q\geq 0}H(q\beta)\mathsf{e}_{q\beta}\big) \tag{3.19}$$

Let us denote by

$$\mathbb{H} = \sum_{q\geq 0}H(q\beta)\mathsf{e}_{q\beta}.$$

Therefore we obtain

$$\prod_{q\geq 0}U_{\alpha+q\beta}^{A_+^{\mathbf{S}}(\alpha+q\beta)} = \exp(-\mathbb{H})\prod_{q\geq 0}U_{\alpha+q\beta}^{A_-^{\mathbf{S}}(\alpha+q\beta)}\exp(\mathbb{H}). \tag{3.20}$$

Using again the Lie algebra structure (4.39), note that

$$\prod_{q\geq 0}U_{\alpha+q\beta}^{A_\pm^{\mathbf{S}}(\alpha+q\beta)} = \exp\big(\sum_{q\geq 0}A_\pm^{\mathbf{S}}(\alpha+q\beta)\mathsf{f}_{\alpha+q\beta}\big)$$

Therefore equation (4.46)) simplifies to

$$\exp\big(\sum_{q\geq 0}A_+^{\mathbf{S}}(\alpha+q\beta)\mathsf{f}_{\alpha+q\beta}\big) = \exp(-\mathbb{H})\exp\big(\sum_{q\geq 0}A_-^{\mathbf{S}}(\alpha+q\beta)\mathsf{f}_{\alpha+q\beta}\big)\exp(\mathbb{H}). \tag{3.21}$$

Now let us recall the following form of the BCH formula:

$$\begin{aligned}\exp(A)\exp(B)\exp(-A) &= \exp(\sum_{n=0}\frac{1}{n!}(Ad(A))^n B)\\ &= \exp(B+[A,B]+\frac{1}{2}[A,[A,B]]+\cdots)\end{aligned} \tag{3.22}$$

Using this formula in (4.48), we obtain

$$\begin{aligned}&\exp\big(\sum_{q\geq 0}A_+^{\mathbf{S}}(\alpha+q\beta)\mathsf{f}_{\alpha+q\beta}\big) =\\ &\exp\big(\sum_{q\geq 0}A_-^{\mathbf{S}}(\alpha+q\beta)\sum_{l\geq 1}\sum_{q_1,\ldots,q_l\geq 1}\frac{(-1)^l}{l!}\prod_{i=1}^{l}(-1)^{\chi(\alpha,q_i\beta)}\chi(\alpha,q_i\beta)H(q_i\beta)\mathsf{f}_{\alpha+(q+q_1+\cdots+q_l)\beta}\big)\end{aligned} \tag{3.23}$$

Finally, identifying the coefficients of a given Lie algebra generator $\mathsf{f}_{\alpha+p\beta}$ we obtain the wallcrossing formula (3.11).

3.2.2 Comparison with Denef-Moore halo formula

Suppose X is a genus zero curve such that the Higgs sheaf invariants $H(r,e)$ may be nontrivial. Employing the notation introduced in the previous subsection, consider the following generating functions

$$Z_{\pm}(q,v) = \sum_{p\geq 0} A^{\mathbf{S}}_{\delta_\pm}(\alpha+p\beta)q^{pn(\beta)}v^{pr(\beta)},$$

where $\alpha = (r(\alpha), e(\alpha))$, $\beta = (r(\beta), e(\beta))$ and $n(\beta) = e(\beta) - r(\beta)(g-1)$. Then the wallcrossing formula (3.11) yields

$$Z_{+}(q,v) = \sum_{l\geq 0} \frac{(-1)^l}{l!}\Big(\sum_{p\geq 0} A^{\mathbf{S}}_{\delta_-}(\alpha+p\beta)q^{pn(\beta)}v^{pr(\beta)}\Big) \Big(\sum_{p\geq 0}(-1)^{\chi(\alpha,p\beta)}\chi(\alpha,p\beta)H(p\beta)q^{pn(\beta)}v^{pr(\beta)}\Big)^l.$$

Using the multicover formula (3.18), this expression becomes

$$\begin{aligned} Z_{+}(q,v) &= Z_{-}(q,v)\mathrm{exp}\Big[-\sum_{k,p\geq 0}\frac{1}{k}(-1)^{kp\chi(\alpha,\beta)}\chi(\alpha,p\beta)\overline{H}(p\beta)q^{kpn(\beta)}v^{kpr(\beta)}\Big] \\ &= Z_{-}(q,v)\mathrm{exp}\Big[\sum_{p\geq 0}\chi(\alpha,p\beta)\overline{H}(p\beta)\mathrm{ln}\big(1-(-1)^{p\chi(\alpha,\beta)}q^{pn(\beta)}\big)\Big] \\ &= Z_{-}(q,v)\prod_{p\geq 0}\big(1-(-1)^{p\chi(\alpha,\beta)}q^{pn(\beta)}v^{pr(\beta)}\big)^{\chi(\alpha,p\beta)\overline{H}(p\beta)} \end{aligned}$$

This formula in agreement with the halo formula [31, Eqn. 6.17].

3.2.3 Refined Wallcrossing Formulas

In order to fix the notation, let $\Delta(r,e) \subset \mathbb{R}_{>0}$ be the (finite) set of positive critical stability parameters of type $(r,e) \in \mathbb{Z}_{\geq 1}\times\mathbb{Z}$. For any $n\in\mathbb{Z}$, and any formal variable y let

$$[n]_y = \frac{y^n - y^{-n}}{y-y^{-1}} \in \mathbb{Q}(y)$$

Conjecture 3.2.3. *Let $\gamma = (r,e) \in \mathbb{Z}_{\geq 1}\times\mathbb{Z}$. Then there exist refined equivariant residual ADHM invariants $A_\delta(r,e)(y) \in \mathbb{Q}(y)$, for any $\delta\in\mathbb{R}$, and refined equivariant residual Higgs sheaf invariants $H(r,e)(y) \in \mathbb{Q}(y)$ so that $A_\delta(r,e)(1) = A_\delta(r,e)$, $H(r,e)(1) = H(r,e)$ and the following wallcrossing formulas hold.*

(i) Let $\delta_c \in \Delta(r,e)$ be critical stability parameter and $\delta_{c-} < \delta_c$, $\delta_{c+} > \delta_c$ be noncritical stability parameters so that $[\delta_{c-},\delta_c)\cap\Delta(r,e) = \emptyset$, $(\delta_c,\delta_{c+}]\cap\Delta(r,e) = \emptyset$. The following wallcrossing

formula holds for $\delta_{c\pm}$ sufficiently close to δ_c

$$A_{\delta_c+}(\gamma)(y) - A_{\delta_c-}(\gamma)(y) =$$

$$\sum_{l\geq 2} \frac{1}{(l-1)!} \sum_{\substack{\gamma_1+\cdots+\gamma_l=\gamma \\ \mu_{\delta_c}(\gamma_1)=\mu(\gamma_2)=\cdots=\mu(\gamma_l)}} A_{\delta_{c-}}(\gamma_1) \prod_{i=2}^{l} (-1)^{e_i - r(g-1)} [e_i - r_i(g-1)]_y H(\gamma_i)(y) \tag{3.24}$$

where the sum in the right hand side of (3.24) is finite. Moreover $[\delta_{c-}, \delta_c) \cap \Delta(r_1, e_1) = \emptyset$, $(\delta_c, \delta_{c+}] \cap \Delta(r_1, e_1) = \emptyset$ for all $\gamma_1 = (r_1, e_1)$ in the right hand side of (3.24).

(ii) Let $\delta_- < 0$, $\delta_+ > 0$ be noncritical stability parameters so that $[\delta_-, 0) \cap \Delta(r, e) = \emptyset$, $(0, \delta_+] \cap \Delta(r, e) = \emptyset$. The following wallcrossing formula holds for $\delta_\pm$ sufficiently close to 0

$$\begin{aligned} &A_{\delta_+}(\gamma)(y) - A_{\delta_-}(\gamma)(y) = \\ &\sum_{l\geq 1} \frac{1}{l!} \sum_{\substack{\gamma_1+\cdots+\gamma_l=\gamma \\ \mu(\gamma_i)=\mu(\gamma),\ 1\leq i\leq l}} \prod_{i=1}^{l} (-1)^{e_i - r(g-1)} [e_i - r_i(g-1)]_y H(\gamma_i)(y) \\ &+ \sum_{l\geq 2} \frac{1}{(l-1)!} \sum_{\substack{\gamma_1+\cdots+\gamma_l=\gamma \\ \mu(\gamma_i)=\mu(\gamma),\ 1\leq i\leq l}} A_{\delta_-}(\gamma_1)(y) \prod_{i=2}^{l} (-1)^{e_i - r(g-1)} [e_i - r_i(g-1)]_y H(\gamma_i)(y) \end{aligned} \tag{3.25}$$

where the sum in the right hand side of (3.25) is finite. Moreover, $[\delta_-, 0) \cap \Delta(r_1, e_1) = \emptyset$, $(0, \delta_+] \cap \Delta(r_1, e_1) = \emptyset$ for all $\gamma_1 = (r_1, e_1)$ in the second line of the right hand side of equation (3.25).

Moreover $A_\delta(r, e) \in \mathbb{Z}[y, y^{-1}]$ if $\delta \in \mathbb{R}$ is noncritical, and $H(r, e)(y) \in \mathbb{Z}[y, y^{-1}]$ if (r, e) are coprime.

As mentioned above the invariants $A_\delta(r, e) \in \mathbb{Z}[y, y^{-1}]$, $H(r, e)(y)$ are conjecturally related to residual equivariant Kontsevich-Soibelman invariants $\overline{A}_\delta(r, e)(y) \in \mathbb{Z}[y, y^{-1}]$, $\overline{H}(r, e)(y) \in \mathbb{Z}[y, y^{-1}]$ by a refined multicover formula. For $v = 1$ invariants this formula states simply that $A_\delta(r, e)(y) = \overline{A}_\delta(r, e)(y)$, while the explicit formula for $v = 0$ is given below.

Conjecture 3.2.4. *Under the same hypothesis as in conjecture (3.2.3), the following relation holds for any $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$*

$$H(r, e)(y) = \sum_{\substack{k\in\mathbb{Z},\ k\geq 1 \\ k|r,\ k|e}} \frac{1}{k\,[k]_y} \overline{H}\left(\frac{r}{k}, \frac{e}{k}\right)(y^k). \tag{3.26}$$

The refined wallcrossing formulas (3.24), (3.25) are formal quantum generalizations of the wallcrossing formulas derived in [28]. Refined, or quantum, wallcrossing formulas have been physically derived in [38, 26, 39] using arguments analogous to [31]. In particular a refinement of the semiprimitive wallcrossing formula of [31] has been formulated in [39]. A motivic wallcrossing formula has been also announced in [68]. By analogy with [28, Sect. 4], [27, Sect. 4], the wallcrossing formulas conjectured in (3.2.3) can be shown to agree with the refined semiprimitive wallcrossing formulas of [26, 39, 44], once the multicover formula (3.26) is properly taken into account. In particular the above refined multicover formula can be easily inferred from [26, Sect 4.]. The details are similar to those in [28, Sect. 4], [27, Sect. 4], hence will be omitted.

Finally note that a refined formula has been also derived in [36] for primitive wallcrossing using arguments analogous to [31], and shown to be in a agreement with wallcrossing formulas for Poincaré and Hodge polynomials of moduli spaces of stable sheaves on surfaces [49, 94, 95]. The formula derived in [36] is in fact doubly refined, the BPS states being simultaneously graded by spin and $U(1)_R$-charge quantum numbers. This motivates the following further refinement of conjecture (3.2.3), which can be physically justified using arguments analogous to [31, 36, 39].

Let (u,v) be formal variables, and $(u^{1/2},v^{1/2})$ be formal square roots. For any $n\in\mathbb{Z}$ set

$$[n]_{(u,v)} = \frac{(uv)^{n/2}-(uv)^{-n/2}}{(uv)^{1/2}-(uv)^{-1/2}} \in \mathbb{Q}(u^{1/2},v^{1/2}).$$

Conjecture 3.2.5. *Under the same conditions as in conjecture (3.2.3) there exist doubly refined equivariant residual ADHM invariants* $A_\delta(r,e)(u,v)\in\mathbb{Q}(u^{1/2},v^{1/2})$*, and doubly refined Higgs sheaf invariants* $H(r,e)(u,v)\in\mathbb{Q}(u^{1/2},v^{1/2})$ *so that*

(i) $A_\delta(r,e)(u,u)=A_\delta(r,e)(u)$, $H(r,e)(u,u)=H(r,e)(u)$,
$A_\delta(r,e)(u,v)\in\mathbb{Z}[u^{1/2},u^{-1/2},v^{1/2},v^{-1/2}]$ *if* δ *is noncritical and*
$H(r,e)(u,v)\in\mathbb{Z}[u^{1/2},u^{-1/2},v^{1/2},v^{-1/2}]$ *if* (r,e) *are coprime.*

(ii) $A_\delta(r,e)(u,v)$ *satisfy wallcrossing formulas obtained by substituting* $A_\delta(\gamma_i)(u,v), H(\gamma_i)(u,v), [e_i-r_i(g-1)]_{(u,v)}$ *for* $A_\delta(\gamma_i)(y), H(\gamma_i)(y), [e_i-r_i(g-1)]_y$ *in* (3.24), (3.25).

(iii) There exist alternative Higgs sheaf invariants $\overline{H}(r,e)(u,v)\in\mathbb{Z}[u^{1/2},u^{-1/2},v^{1/2},v^{-1/2}]$, $(r,e)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}$ *so that* $H(r,e)(u,v)$, $\overline{H}(r,e)(u,v)$ *satisfy a multicover formula obtained by making the same substitutions in* (3.26).

Note that the same notation $A_\delta(r,e)$, $H(r,e)$; $A_\delta(r,e)(y)$, $H(r,e)(y)$; $A_\delta(r,e)(u,v)$ and $H(r,e)(u,v)$ is (abusively) employed for rational, respectively motivic and refined motivic invariants. By convention, the distinction will reside only in the number of arguments of these rational functions. Therefore if no arguments are present, $A_\delta(r,e)$, $H(r,e)$ are rational numbers, if one argument is present they are rational functions of one variable etc. Moreover, the invariants $H(r,e)(y)$ will be called refined Higgs invariants in the following. The invariants $A_{\delta_\pm}(r,e)(y)$ with $\delta_\pm$ close to 0 as in (3.2.3.*ii*) will be denoted by $A_{0\pm}(r,e)(y)$. Similarly the invariants $A_\delta(r,e)(y)$, with $\delta > \max\Delta(r,e)$ respectively $\delta < \min\Delta(r,e)$ will be denoted by $A_{\pm\infty}(r,e)(y)$ and referred to as asymptotic invariants.

Finally note that the duality isomorphisms (3.5), (3.7) yield relations of the form

$$A_\delta(r,e)(y) = A_{-\delta}(r,-e+2r(g-1))(y) \qquad H(r,e)(y) = H(r,-e+2r(g-1))(y) \qquad (3.27)$$

for all $(r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. Moreover, the isomorphisms (3.8) imply that

$$H(r,e)(y) = H(r,e+r)(y). \qquad (3.28)$$

for any $(r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. Therefore for fixed r there are only r a priori distinct invariants $H(r,e)(y)$. Obviously entirely analogous formulas hold for the refined motivic invariants $A_\delta(r,e)(u,v)$, $H(r,e)(u,v)$.

3.2.4 Remarks on refined wallcrossing conjectures

This subsection consists of several remarks on conjectures (3.2.3) (3.2.5). It can be skipped with no loss of essential information.

(*i*) First note that given any two objects $\mathcal{E}_1, \mathcal{E}_2$ of $\mathcal{C}_\mathcal{X}$ with $v(\mathcal{E}_1) + v(\mathcal{E}_2) \leq 1$, it has been proven in [33, Lemm. 7.4] that the expression

$$\dim \mathrm{Ext}^0_{\mathcal{C}_\mathcal{X}}(\mathcal{E}_1,\mathcal{E}_2) - \dim \mathrm{Ext}^1_{\mathcal{C}_\mathcal{X}}(\mathcal{E}_1,\mathcal{E}_2) - \dim \mathrm{Ext}^0_{\mathcal{C}_\mathcal{X}}(\mathcal{E}_2,\mathcal{E}_1) + \dim \mathrm{Ext}^1_{\mathcal{C}_\mathcal{X}}(\mathcal{E}_2,\mathcal{E}_1) \qquad (3.29)$$

depends only on the numerical types of the two objects. Moreover, if $\mathcal{E}_1, \mathcal{E}_2$ determine closed points in the stack theoretic fixed locus $\mathfrak{Ob}(\mathcal{C}_\mathcal{X})^{\mathbf{S}}$, there is an induced torus action on all the extension groups in (3.29) and the same statement holds for the alternating sum of dimensions of fixed, respectively moving parts. This technical condition makes both Joyce-Song and Kontsevich-Soibelman theories applicable to non-Calabi-Yau categories, which is the present case.

(*iii*) As pointed out in [38], the quantum Donaldson-Thomas invariants of Kontsevich and Soibelman can be naturally identified with the refined topological string invariants constructed in [59] via the refined topological vertex formalism. The asymptotic invariants $A_{\pm\infty}(r,e)(y)$ are refinements of the integral invariants $A_{\pm\infty}(r,e)$, which are in turn identical to local stable pair invariants according to [34]. Therefore it entirely natural to expect these invariants to be determined by the refined BPS counting invariants of a local curve. The later can be inferred from the Nekrasov partition function of a five dimensional gauge theory as explained in section (3.4).

(v) Finally note that assuming an equivariant localization result for motivic invariants one can conjecture more refined wallcrossing formulas for the residual contributions of individual components of the fixed loci. This follows from the stack function relations derived in [28, Sect. 3].

3.3 Hitchin System and Its Cohomology

3.3.1 Connection with Hitchin pairs

Let L be a fixed line bundle on X. Recall that a Hitchin pair [54, 83] on X with coefficient bundle L is defined is a pair (E, Φ) where E is a coherent sheaf on X and $\Phi : E \to E \otimes_X L$ a morphism of coherent sheaves. Such a pair is called (semi)stable if any proper nontrivial subsheaf $0 \subset E' \subset E$ so that $\Phi(E') \subset E' \otimes_X L$ satisfies the inequality

$$r(E)d(E') \ (\leq) \ r(E')d(E). \tag{3.30}$$

Note that if $r(E) > 0$, semistability implies that E is locally free. In the following L be either K_X or a line bundle on X of degree $d(L) > 2g - 2$. This will be implicitly assumed in all statements below.

Well-known results in the literature [54, 83, 25, 84, 87, 88] establish the existence of an algebraic stack of finite type $\mathfrak{H}(X, L, r, e)$ of semistable Hitchin pairs of fixed type $(r(E), d(E)) = (r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. Moreover, if (r, e) are coprime, this stack is a $\mathbb{C}^\times$-gerbe over a smooth quasi-projective variety $H(X, L, r, e)$. For $L = K_X$, $H(X, L, r, e)$ is commonly referred to as the Hitchin integrable system.

Note that there is a torus $\mathbb{C}^\times$ action on the stack $\mathfrak{H}(X, L, r, e)$ given by $t \times (E, \Phi) \to (E, t^{-1}\Phi)$ on closed points. The stack theoretic fixed locus is universally closed. In particular, for (r, e) coprime, there is an induced torus action on the moduli scheme $H(X, L, r, e)$, and the fixed locus is a smooth projective scheme over $\mathbb{C}$.

The relation between ADHM sheaves and Hitchin pairs is summarized in the following simple observations.

($AH.1$) Suppose $M_1 = \mathcal{O}_X$, $M_2 = K_X^{-1}$ and let $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ be coprime. Then there is an isomorphism

$$\mathfrak{Higgs}(\mathcal{X}, r, e) \simeq \mathbb{C} \times \mathfrak{H}(X, K_X, r, e). \tag{3.31}$$

($AH.2$) Suppose M_2 is a line bundle of degree $2 - 2g - p$, where $p \in \mathbb{Z}_{>0}$. Then there is an isomorphism

$$\mathfrak{Higgs}(\mathcal{X}, r, e) \simeq \mathfrak{H}(X, M_2^{-1}, r, e). \tag{3.32}$$

Both statements rely on the fact that for coprime (r,e) slope semistability is equivalent to slope stability. Therefore the endomorphism ring of any semistable object $\mathcal{E}$ is canonically isomorphic to $\mathbb{C}$.

Then note that in the first case, given any semistable object $\mathcal{E} = (E, \Phi_1, \Phi_2)$ the relation (3.3) implies that $\Phi_1 : E \to E$ is an endomorphism of $\mathcal{E}$ since it obviously commutes with itself. Therefore it must be of the form $\Phi_1 = \lambda 1_E$ for some $\lambda \in \mathbb{C}$. In particular, it preserves any subsheaf $E' \subset E$. Generalizing this observation to flat families it follows that there is an forgetful morphism

$$\mathfrak{Higgs}(\mathcal{X}, r, e) \to \mathfrak{H}(X, K_X, r, e)$$

projecting (E, Φ_1, Φ_2) to $(E, \Phi_2 \otimes 1_{K_X})$. The isomorphism (3.31) then follows easily.

In the second case, note that given a semistable Higgs sheaf (E, Φ_1, Φ_2), of type (r,e), the data

$$\mathcal{E}' = \left(E \otimes_X M_1^{-1}, \Phi_1 \otimes 1_{M_1^{-1}}, \Phi_2 \otimes 1_{M_1^{-1}} \right)$$

determines a semistable Higgs sheaf of type $(r, e - r\deg(M_1)) = (r, e - rp)$. Relation (3.3) implies that $\Phi_1 \otimes 1_{M_1^{-1}}$ is a morphism of (semistable) Higgs sheaves. However $\mu(\mathcal{E}) > \mu(\mathcal{E}')$ since $p > 0$, therefore any such morphism must vanish. This completes the proof.

3.3.2 Recursion formula for refined Higgs invariants

For the purpose of the present paper, the main application of conjectures (3.2.3), (3.2.5) is a recursion formula for the invariants $H(r,e)(y)$, $H(r,e)(u,v)$ which determines inductively all invariants $H(r,e)(y)$, $H(r,e)(u,v)$, $(r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ in terms of the asymptotic invariants $A_{+\infty}(r,e)(y)$, $A_{+\infty}(r,e)(u,v)$.

In the following X is assumed to be a smooth projective curve of genus $g \geq 2$ and $p = \deg(M_1) \geq 0$. For any $\gamma = (r,e)$, let $\widetilde{\gamma} = (r, -e + 2r(g-1))$, $\widetilde{e} = -e + 2r(g-1)$. For a stability parameter δ let $\mu_\delta(\gamma) = (e+\delta)/r$, $\mu(\gamma) = e/r$. Given $\gamma = \mathbb{Z} \times \mathbb{Z}$, the notation $\gamma = (r(\gamma), e(\gamma))$ will also be used on occasion.

The recursion formula will be written in detail only for the refined invariants $H(r,e)(y)$ since the analogous formula for the doubly refined invariants $H(r,e)(u,v)$ follows by obvious substitutions, as explained in conjecture (3.2.5). Let $\gamma = (r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ be an arbitrary numerical type. Then the following wallcrossing formula holds.

$$\begin{aligned}
&(-1)^{e-r(g-1)}[e-r(g-1)]_y H(\gamma)(y) = A_{+\infty}(\gamma)(y) - A_{+\infty}(\widetilde{\gamma})(y)\\
&+\sum_{l\geq 2}\frac{(-1)^{l-1}}{(l-1)!}\sum_{\substack{\gamma_1,\ldots,\gamma_l\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}\\ \gamma_1+\cdots+\gamma_l=\gamma\\ \mu(\gamma)<\mu(\gamma_i),\ 2\leq i\leq l}} A_{+\infty}(\gamma_1)(y)\prod_{i=2}^{l}(-1)^{e_i-r_i(g-1)}[e_i-r_i(g-1)]_y H(\gamma_i)(y)\\
&-\sum_{l\geq 2}\frac{(-1)^{l-1}}{(l-1)!}\sum_{\substack{\gamma_1,\ldots,\gamma_l\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}\\ \gamma_1+\cdots+\gamma_l=\widetilde{\gamma}\\ \mu(\widetilde{\gamma})\leq\mu(\gamma_i),\ 2\leq i\leq l}} A_{+\infty}(\gamma_1)(y)\prod_{i=2}^{l}(-1)^{e_i-r_i(g-1)}[e_i-r_i(g-1)]_y H(\gamma_i)(y)\\
&-\sum_{l\geq 2}\frac{1}{l!}\sum_{\substack{\gamma_1,\ldots,\gamma_l\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}\\ \gamma_1+\cdots+\gamma_l=\gamma\\ \mu(\gamma)=\mu(\gamma_i),\ 1\leq i\leq l}}\prod_{i=1}^{l}(-1)^{e_i-r_i(g-1)}[e_i-r_i(g-1)]_y H(\gamma_i)(y)
\end{aligned}\tag{3.33}$$

where the sum in the right hand side of equation (3.33) contains only finitely many nontrivial terms. The derivation of the recursion formula (3.33) from the wallcrossing formulas (3.24), (3.25) is presented in section (3.3.3).

Remark 3.3.1. (*i*) *Note that only invariants $H(r_i, e_i)(y)$ with $r_i < r$ enter the sum in right hand side of* (3.33). *Therefore this relation completely determines all invariants $H(r,e)$, $(r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ if all invariants $A_{+\infty}(r,e)(y)$ are known. A conjectural formula for the asymptotic refined ADHM invariants $A_{+\infty}(r,e)(y)$ will be derived in the next section using string duality.*

(*ii*) *Given relations* (3.27), (3.28), *equations* (3.33) *are in fact an overdetermined set of recursion relations for refined Higgs invariants. If conjecture (3.2.3) holds, all these equations are compatible, and one can choose the most economical one for concrete computations. In fact, one can obtain a simpler relation by taking $e > 2r(g-1) - c(r)$ in* (3.33). *This results in $A_{+\infty}(\widetilde{\gamma}) = 0$ and the second line in the right hand side is zero as well. However, the simpler relation obtained this way is not necessarily the most efficient as far as computer time is concerned. Concrete examples and computations will be presented in section (3.5).*

3.3.3 Derivation of recursion formula

The purpose of this section is to prove the recursion relation (3.33), given the wallcrossing formulas (3.24), (3.25). The proof is analogous to the proof of [28, Lemm. 3.8]. The main steps will be outlined below for completeness.

According to property $(B.2)$ in section (3.1) for any fixed $r \geq 1$ there exists an integer $c(r) \in \mathbb{Z}$ so that all invariants $A_\delta(r,e)(y)$, for any $\delta > 0$, are identically zero if $e < c(r)$. Moreover, this integer is not unique unless required to be optimal with this property; any integer $c(r) \leq -(r-1)^2(2g-2+p)$ satisfies this condition. In the following set

$$c(r) = -r(r-1)(2g-2+p) \qquad c(r') = -r'(r-1)(2g-2+p) \tag{3.34}$$

for any $r \in \mathbb{Z}_{\geq q1}$, $1 \leq r' \leq r$. This is not an optimal choice, but it will facilitate the derivation of formula (3.33), as shown below.

Next note that the wallcrossing formula (3.24) is equivalent to

$$\begin{gathered} A_{\delta_c-}(\gamma)(y) - A_{\delta_c+}(\gamma)(y) = \\ \sum_{l\geq 2} \frac{(-1)^{l-1}}{(l-1)!} \sum_{\substack{\gamma_1+\cdots+\gamma_l=\gamma \\ \mu_{\delta_c}(\gamma_1)=\mu(\gamma_2)=\cdots=\mu(\gamma_l)}} A_{\delta_{c+}}(\gamma_1) \prod_{i=2}^{l} (-1)^{e_i - r(g-1)} [e_i - r_i(g-1)]_y H(\gamma_i)(y) \end{gathered} \tag{3.35}$$

For any $n \in \mathbb{Z}_{\geq 1}$ and any collection of n positive integers $(l_1, \ldots, l_n) \in \mathbb{Z}^n_{\geq 1}$, define

$$\begin{gathered} \mathsf{S}^{(l_1,\ldots,l_n)}_{0,+\infty}(\gamma) = \Big\{ (\gamma_1, \eta_{1,1}, \ldots, \eta_{1,l_1}, \ldots, \eta_{n,1}, \ldots, \eta_{n,l_n}) \in (\mathbb{Z}_{\geq 1} \times \mathbb{Z})^{\times(l_1+\ldots+l_n+1)} \Big| \\ \gamma_1 + \sum_{i=1}^{n}\sum_{j=1}^{l_i} \eta_{i,j} = \gamma, \quad \mu_0(r) \leq \mu(\gamma) < \mu(\eta_{1,1}) = \cdots = \mu(\eta_{1,l_1}) < \\ \mu(\eta_{2,1}) = \cdots = \mu(\eta_{2,l_2}) < \cdots < \mu(\eta_{n,1}) = \cdots = \mu(\eta_{n,l_n}) < \mu_\delta(\gamma), \ \mu_0(r) \leq \mu(\gamma_1) \Big\} \end{gathered} \tag{3.36}$$

where $\mu_0(r) = c(r)/r$. Then it straightforward to check that the union

$$\bigcup_{n\geq 1} \bigcup_{l_1,\ldots,l_n \geq 1} \mathsf{S}^{(l_1,\ldots,l_n)}_{0,+\infty}(\gamma) \tag{3.37}$$

is a finite set.

Let $(\gamma_1, \eta_{1,1}, \ldots, \eta_{1,l_1}, \ldots, \eta_{n,1}, \ldots, \eta_{n,l_n}) \in \mathsf{S}^{(l_1,\ldots,l_n)}_{+,\delta}(\gamma)$ be an arbitrary element, for some $n \geq 1$ and $l_1, \ldots, l_n \geq 1$. Let μ_i, $1 \leq i \leq n$ denote the common value of the slopes $\mu(\eta_{i,j})$, $1 \leq j \leq l_i$. If $n \geq 2$, let also

$$\gamma_{n-i+2} = \gamma_1 + \eta_{i,1} + \cdots + \eta_{n,l_n}$$

for $2 \leq i \leq n$. Define the stability parameters δ_i, $1 \leq i \leq n$ by

$$\begin{aligned} \mu_{\delta_1}(\gamma_1) &= \mu_n \\ \mu_{\delta_i}(\gamma_i) &= \mu_{n+1-i}, \qquad 2 \leq i \leq n \qquad (\text{if } n \geq 2). \end{aligned} \tag{3.38}$$

By construction, δ_i is a critical stability parameter of type γ_i for all $1 \leq i \leq n$. Given the slope inequalities in (3.36), it is straightforward to check that

$$0 < \delta_n < \delta_{n-1} < \cdots < \delta_1. \tag{3.39}$$

Moreover, $\mu(\gamma_i) \geq \mu_0(r)$ for all $1 \leq i \leq n$ since the integers $c(r')$, $1 \leq r' \leq r$ defined in (3.34) satisfiy

$$\frac{c(r')}{r'} = -(r-1)(2g-2+p) = \mu_0(r). \tag{3.40}$$

Next note that the set Δ_γ of all stability parameters constructed this way, for all $n \geq 1$ and any possible values of $l_1, \ldots, l_n$ is finite, since the set (3.37) is finite. Therefore one can choose stability parameters $0 < \delta_{0+} < \min \Delta_\gamma$, $\delta_{+\infty} > \max \Delta_\gamma$. By construction Δ_γ contains all possible decreasing finite sequences of stability parameters of the form (3.39) with the property that there exists

$$(\gamma_1, \eta_{1,1}, \ldots, \eta_{1,l_1}, \ldots, \eta_{n,1}, \ldots, \eta_{n,l_n}) \in (\mathbb{Z}_{\geq} \times \mathbb{Z})^{\times(l_1+\cdots+l_n+1)}$$

for some $l_1, \ldots, l_n \geq 1$ so that

(a) $\gamma_1 + \eta_{1,1} + \cdots + \eta_{n,l_n} = \gamma$

(b) Conditions (3.38) hold.

In conclusion, successive applications of the wallcrossing formula (3.35) yield

$$\begin{aligned} &A_{0+}(\gamma) - A_{+\infty}(\gamma) = \\ &\sum_{n=1}^{\infty} \sum_{l_1,\ldots,l_n \geq 1} \prod_{i=1}^{n} \frac{(-1)^{l_i}}{l_i!} \sum_{\substack{\gamma_1+\eta_{1,1}+\cdots+\eta_{1,l_1}+\cdots+\eta_{n,1}+\cdots+\eta_{n,l_n}=\gamma, \\ \mu_0(r)\leq\mu(\gamma)<\mu(\eta_{1,1})=\cdots\mu(\eta_{1,l_1})<\cdots<\mu(\eta_{n,1})=\cdots=\mu(\eta_{n,l_n}) \\ \mu_0(r)\leq\mu(\gamma_1)}} \\ &\qquad A_{+\infty}(\gamma_1)(y) \prod_{i=1}^{n}\prod_{j=1}^{l_i} (-1)^{e_{i,j}-r_{i,j}(g-1)}[e_{i,j} - r_{i,j}(g-1)]_y H(\eta_{i,j})(y) \end{aligned} \tag{3.41}$$

where $\gamma = (r_1, e_1) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, $\eta_{i,j} = (e_{i,j}, r_{i,j}) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, $1 \leq i \leq n$, $1 \leq j \leq l_i$. T Moreover, the sum in the right hand side of equation (3.41) is finite for any fixed $\gamma = (r, e)$.

Then in equation (3.25) $A_{\delta_-}(\gamma) = A_{\delta_+}(\widetilde{\gamma})$ and

$$\begin{aligned} &\sum_{l\geq 2} \frac{1}{(l-1)!} \sum_{\substack{\gamma_1+\cdots+\gamma_l=\gamma \\ \mu(\gamma_i)=\mu(\gamma), 1\leq i\leq l}} A_{\delta_-}(\gamma_1)(y) \prod_{i=2}^{l-1} e^{e_i - r_i(g-1)}[e_i - r_i(g-1)]_y H(\gamma_i)(y) = \\ &\sum_{l\geq 2} \frac{1}{(l-1)!} \sum_{\substack{\gamma_1+\cdots+\gamma_l=\gamma \\ \mu(\gamma_i)=\mu(\gamma), 1\leq i\leq l}} A_{\delta_+}(\widetilde{\gamma}_1)(y) \prod_{i=2}^{l-1} e^{e_i - r_i(g-1)}[e_i - r_i(g-1)]_y H(\gamma_i)(y) = \\ &\sum_{l\geq 2} \frac{(-1)^{l-1}}{(l-1)!} \sum_{\substack{\gamma_1+\cdots+\gamma_l=\widetilde{\gamma} \\ \mu(\gamma_i)=\mu(\widetilde{\gamma}), 1\leq i\leq l}} A_{\delta_+}(\gamma_1)(y) \prod_{i=2}^{l-1} e^{e_i - r_i(g-1)}[e_i - r_i(g-1)]_y H(\gamma_i)(y) = \end{aligned} \tag{3.42}$$

by a redefinition of variables. Substituting (3.41) and (3.42) in equation (3.25), equation (3.33) follows by simple combinatorics.

3.3.4 Higgs invariants and cohomology of moduli spaces of Hitchin pairs

The goal of this subsection is to formulate one more conjecture relating refined Higgs invariants to the cohomology of moduli spaces of stable Hitchin pairs on X, for coprime numerical invariants $(r,e) \in \mathbb{Z}_{\geq} \times \mathbb{Z}$. In the following it is still assumed that the genus of X is $g \geq 2$, and $p = \deg(M_1) \geq 0$. Moreover, $M_1 \simeq \mathcal{O}_X$ if $p = 0$.

First recall that a Hitchin pair on X with coefficient line bundle L is a coherent sheaf E equipped with a morphism $\Phi : E \to E \otimes_X L$. The moduli theory of such objects has been extensively and intensively studied in the mathematics literature [54, 83, 25, 84, 87, 88]. In particular, as recalled in section (3.3.1), there is a natural stability condition which yields an algebraic moduli stack $\mathfrak{H}(X, L, r, e)$ of finite type. Moreover, suppose $\deg(L) \geq 2g-2$ and $L \simeq K_X$ if $\deg(L) = 2g-2$. There also exists a coarse moduli scheme $H^s(X, L, r, e)$ parameterizing isomorphism classes of stable objects. If $(r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ are coprime, any semistable Hitchin pair is stable, and $H^s(X, L, r, e)$ will be denoted by $H(X, L, r, e)$.

The connection between Higgs sheaves and Hitchin pairs is based on the observation that there is a natural forgetful morphism of moduli stacks

$$\mathfrak{Higgs}(X, M_1, M_2, r, e) \to \mathfrak{H}(X, M_2^{-1}, r, e)$$

which simply forgets $\Phi_1 : E \otimes_X M_1 \to M_1$. Moreover, under the current assumptions, this morphism is compatible with stability for (r,e) coprime, and has a very simple structure as explained in section (3.3.1). This leads to the conjecture formulated below.

First note that for $(r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ coprime, the degree of the Poincaré polynomial $P_y(H(X, L, r, e))$ of the smooth moduli space $H(X, L, r, e)$ is an even integer $2m(r,e)$, $m(r,e) \in \mathbb{Z}_{\geq 0}$. Under the same conditions, let $H_{(u,v)}(H(X, L, r, e))$ denote the Hodge polynomial of $H(X, L, r, e)$ (see [53, Sect. 2.1], [52, Sect. 2] for definition and properties.)

Conjecture 3.3.2. *Under the above assumptions, let $L \simeq M_2^{-1}$. Then*

$$\begin{aligned} H(r,e)(y) &= (-1)^{e-r(g-1-p)} y^{-n(r,e)} P_{(-y)}(H(X, L, r, e)) \\ H(r,e)(u,v) &= (-1)^{e-r(g-1-p)} (uv)^{-n(r,e)/2} H_{(-u,-v)}(H(X, L, r, e)) \end{aligned} \tag{3.43}$$

where

$$n(r,e) = r^2(g-1) + r(r-1)p + m(r,e).$$

Remark 3.3.3. (i) *The recursion relation* (3.33) *and conjecture (3.3.2) determine all Hodge polynomials $H_{(u,v)}(H(X, L, r, e))$ with $(r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ coprime if the asymptotic refined ADHM* *invariants are known for all $(r,e)\in \mathbb{Z}_{\geq 1}\times\mathbb{Z}$. Conjectural formulas for these asymptotic invariants are presented in the next subsection.*

(ii) Note that the recursion formula (3.33) determines in fact all invariants $H(r,e)(y)$, $H(r,e)(u,v)$, including non-coprime pairs. A priori, the Higgs invariants $H(r,e)(y)$ with (r,e) not coprime are not related in any direct way to the cohomology of moduli spaces of semistable Hitchin pairs with the same numerical invariants. However, a conjectural relation based on the multicover formula (3.26) will be formulated in the next subsection.

3.4 Asymptotic refined ADHM invariants from gauge theory

As explained above, the invariants $H(r,e)(y)$, $H(r,e)(u,v)$ are completely determined by the recursion relation (3.33) if all asymptotic refined ADHM invariants are known. In this section we present a conjectural formula for the generating function of asymptotic refined ADHM invariants from string duality. Basically, this generating function is determined by the Nekrasov partition function [80] of a five dimensional supersymmetric gauge theory.

As shown below, the resulting formula involves an infinite formal sum over Young tableaus Y. In order to fix conventions, note that a nonempty Young tableau Y is identified with a partition

$$|Y| = Y_1 + \cdots + Y_{l(Y)}$$

where Y denotes the total number of boxes of Y and $l(Y)$ denotes the number of rows. For any $1\leq i\leq l(Y)$, Y_i denotes the length of the i-th row, and $Y_1\geq Y_2\geq\cdots\geq Y_{l(Y)}$. Boxes of Y will be labeled by $(i,j)\in\mathbb{Z}\times\mathbb{Z}$, $1\leq i\leq l(Y)$, $1\leq j\leq Y_i$.

Conjecture 3.4.1. *Let $\mathcal{X}=(X,M_1,M_2)$ be a triple as above and let $p=\deg(M_1)$. Let*

$$\mathcal{Z}_{+\infty}(\mathcal{X},r;\lambda,y)=\sum_{e\in\mathbb{Z}}\lambda^e A_{+\infty}(r,e)(y) \tag{3.44}$$

be the generating function for the rank $r\in\mathbb{Z}_{\geq 1}$ asymptotic refined ADHM invariants conjectured in (3.2.3). Then

$$Z_{+\infty}(\mathcal{X},r;\lambda,y)=\sum_{|Y|=r}\Omega_Y^{(g,p)}(\lambda,y) \tag{3.45}$$

where

$$\begin{aligned}\Omega_Y^{(g,p)}(\lambda,y) =&(-1)^{p|Y|}y^{-p\sum_{(i,j)\in Y}(i+j-2)+(g-1)\sum_{(i,j)\in Y}(-2i+2j+1-2Y_i+Y_j^t)}\\ &\lambda^{-p\sum_{(i,j)\in Y}(-i+j)+(g-1)\sum_{(i,j)\in Y}(2i+2j-1-2Y_i-Y_j^t)}\\ &\prod_{(i,j)\in Y}F(\lambda^{-i-j+Y_i+Y_j^t+1}y^{i-j+Y_i-Y_j^t},y)\end{aligned} \tag{3.46}$$

and

$$F(q,z) = z^{1-g}\frac{(1-q)^{2g}}{(1-qz)(1-qz^{-1})}.$$

By convention $\Omega^{(p)}_{\emptyset}(\lambda,y)=1$.

The generating function of asymptotic doubly refined ADHM invariants

$$\mathcal{Z}_{+\infty}(\mathcal{X},r;\lambda,u,v) = \sum_{e\in\mathbb{Z}} \lambda^e A_{+\infty}(r,e)(u,v) \tag{3.47}$$

is conjecturally determined as follows.

Conjecture 3.4.2. *Under the same conditions as in conjecture (3.4.1),*

$$Z_{+\infty}(\mathcal{X},r;\lambda,u,v) = \sum_{|Y|=r} \Omega^{(g,p)}_{Y}(\lambda,u,v) \tag{3.48}$$

where

$$\begin{aligned}\Omega^{(g,p)}_{Y}(\lambda,u,v) =& (-1)^{p|Y|}(uv)^{-p\sum_{(i,j)\in Y}(i+j-2)/2+(g-1)\sum_{(i,j)\in Y}(-2i+2j+1-2Y_i+Y^t_j)/2}\\ &\lambda^{-p\sum_{(i,j)\in Y}(-i+j)+(g-1)\sum_{(i,j)\in Y}(2i+2j-1-2Y_i-Y^t_j)}\\ &\prod_{(i,j)\in Y} G(\lambda^{-i-j+Y_i+Y^t_j+1}(uv)^{(i-j+Y_i-Y^t_j)/2},(uv)^{1/2},(uv^{-1})^{1/2})\end{aligned} \tag{3.49}$$

and

$$G(q,z,w) = z^{(1-g)}\frac{(1-qw)^g(1-qw^{-1})^g}{(1-qz)(1-qz^{-1})}.$$

By convention $\Omega^{(p)}_{\emptyset}(\lambda,u,v)=1$.

Using the recursion relation (3.33) and conjectures (3.3.2), (3.4.1), (3.4.2), one can derive explicit formulas for the Hodge polynomials of the moduli spaces $H(X,L,r,e)$ with (r,e)-coprime. Note in particular that formulas (3.45), (3.48) imply that all invariants $A_{+\infty}(\gamma_1)(y)$ in the right hand side of equation (3.33) are trivial if $\mu(\gamma_1) < -(r-1)(2g-2+p)$. Concrete computations are presented in section (3.5) for $r=1,2,3$ and various values of $g\geq 2$, $p\geq 0$. In all cases, the resulting formulas are in agreement with the direct localization computations of Hitchin [54], Gothen [47] as well as the Hausel-Rodriguez-Villegas formula [53, 52]. Moreover, direct computations in all examples considered in section (3.5) support the following intriguing conjecture.

Conjecture 3.4.3. *Under the same conditions as in conjecture (3.4.1), for fixed* $r\geq 1$, *the refined invariants* $\overline{H}(r,e)(y)$, $\overline{H}(r,e)(u,v)$ *are independent of* $e\in\mathbb{Z}$. *In particular, they take the same value for all pairs* (r,e), *coprime or not.*

The recursion relation (3.33) has been beautifully solved by Mozgovoy in [72], and the solution has been proven to be in agreement with the Hausel-Rodriguez-Villegas invariants. Furthermore, Mozgovoy's solution also satisfies the multicover formula (3.26) and has the property stated in conjecture (3.4.3).

Conjecture (3.4.1) will be shown to follow from type IIA/M-theory duality using arguments analogous to [70, 41, 80, 57, 58, 42, 55, 67, 71, 59]. Summarizing these results, the topological string amplitudes of certain toric Calabi-Yau threefolds (as well as some nontoric configurations of local rational curves) were identified with the instanton partition function of five dimensional gauge theory compactified on a circle of finite radius. The later has been identified in [80] with the generating function for the equivariant Hirzebruch genus of the moduli space of torsion free framed sheaves on the projective plane. A mathematical exposition can be found for example in [78, 79]. The relation between topological strings and five dimensional gauge theory has been subsequently refined in [59]. Moreover, the refined topological string partition function constructed in [59] has been conjecturally identified in [38] with the generation function of refined Donaldson-Thomas invariants. The present problem requires a version of this identification for higher genus local curves.

3.4.1 Geometric engineering via local ruled surfaces

Working under the same assumptions as in section (3.3.1), M_1, M_2 are line bundles on the curve X so that $M_1 \otimes_X M_2 \simeq K_X^{-1}$, $p = d(M_1) \geq 0$ and $M_1 \simeq \mathcal{O}_X$ if $p = 0$. Let Y be the total space of the rank two vector bundle $M_1^{-1} \oplus M_2^{-1}$ on X, which is a noncompact Calabi-Yau threefold under the current assumptions. There is a torus action $\mathbf{S} \times Y \to Y$ scaling M_1^{-1}, M_2^{-1} with characters t, t^{-1}, so that Y is equivariantly K-trivial. In principle the relevant five dimensional gauge theory should be constructed by geometric engineering, that is identifying the low energy effective action of an M-theory supersymmetric background defined by $S^1 \times Y$. This direct approach is somewhat problematic in the present case. A much clearer picture emerges considering a different local Calabi-Yau threefold constructed as follows.

Let S be the total space of the projective bundle $\mathbb{P}(\mathcal{O}_X \oplus M_1)$. S is a smooth geometrically ruled surface over X and it has two canonical sections X_1, X_2 with normal bundles

$$N_{X_1/S} \simeq M_1^{-1}, \qquad N_{X_2/S} \simeq M_1$$

respectively. Note that the cone of effective curve classes on S is generated by the section class $[X_2]$ and the fiber class.

Let Z be the total space of the canonical bundle K_S, which is again a noncompact Calabi-Yau threefold. The normal bundle to X_1 in Z is

$$N_{X_1/Z} \simeq M_1^{-1} \oplus K_X \otimes_X M_1 \simeq M_1^{-1} \otimes M_2^{-1},$$

therefore the total space of $N_{X_1/Z}$ is isomorphic to Y. Moreover, there is a torus action $\mathbf{S} \times Z \to Z$ so that Z is equivariantly Calabi-Yau and the induced torus action on $N_{X_1/Z}$ is compatible with the torus action on Y.

Now the main observation is that the local threefold Z engineers a supersymmetric five dimensional $SU(2)$ gauge theory with g adjoint hypermultiplets on $\mathbb{C}^2 \times S^1$, where g is the genus of X [66]. The integer $p = \deg(M_1)$ corresponds to the level of the five dimensional Chern-Simons term [89]. Therefore by analogy with [70, 41, 80, 57, 58, 42, 55, 67, 71, 59], the refined topological string partition function of Z should be related with the equivariant instanton partition function $\mathcal{Z}^{(p)}_{inst}(Q, \epsilon_1, \epsilon_2, a_1, a_2, y)$, which has been constructed in [80]. As explained in detail in the next subsection, $\mathcal{Z}^{(p)}_{inst}(Q, \epsilon_1, \epsilon_2, a_1, a_2, y)$ is the generating function for the χ_y-genus of a certain holomorphic bundle on a partial compactification of the instanton moduli space. In particular ϵ_1, ϵ_2, a_1, a_2 are equivariant parameters for a natural torus action, Q is a formal variable counting instanton charge, and y is another formal variable.

In order to make string duality predictions more precise, let Q_f, Q_b be formal symbols associated to the fiber class, respectively section class $[X_1]$ on Z. Then string duality predicts that there is a factorization

$$\mathcal{Z}_{ref}(Z; Q_f, Q_b, q, y) = \mathcal{Z}^{pert}_{ref}(Z; Q_f, q, y)\mathcal{Z}^{nonpert}_{ref}(Z; Q_f, Q_b, q, y) \tag{3.50}$$

into a perturbative, respectively nonperturbative parts. Moreover, and there is an identification

$$\mathcal{Z}^{nonpert}_{ref}(Z; Q_f, Q_b, q, y) = \mathcal{Z}^{(p)}_{inst}(Q, \epsilon_1, \epsilon_2, a_1, a_2, y)$$

subject to certain duality relations between the formal parameters in the two partition functions.

Next note that only non-negative powers of Q_b, Q_f can appear in $\mathcal{Z}_{ref}(Z; Q_f, Q_b, q, y)$ since the section class $[X_1]$ and the fiber class generate the Mori cone of S. Similarly, only non-negative powers of Q_f can appear in $\mathcal{Z}^{pert}_{ref}(Z; Q_f, q, y)$, which represents the contribution of pure fiber classes to $\mathcal{Z}_{ref}(Z; Q_f, Q_b, q, y)$. Therefore $\mathcal{Z}_{ref}(Z; Q_f, Q_b, q, y)$, $\mathcal{Z}^{pert}_{ref}(Z; Q_f, q, y)$ have well defined specialization at $Q_f = 0$. Moreover, by construction $\mathcal{Z}^{pert}_{ref}(Z; Q_f, q, y)\big|_{Q_f=0} = 1$. Therefore $\mathcal{Z}^{nonpert}_{ref}(Z; Q_f, Q_b, q, y)$ has well defined specialization at $Q_f = 0$ as well, which is determined by the instanton expansion $\mathcal{Z}^{(p)}_{inst}(Q, \epsilon_1, \epsilon_2, a_1, a_2, y)$. The refined theory of the local threefold Y is then determined by identifying the contributions of curves supported on the section X_1 to $\mathcal{Z}^{nonpert}_{ref}(Z; Q_f, Q_b, q, y)\big|_{Q_f=0}$. Computations will be carried out in detail in the next subsections, resulting in explicit formulas for the instanton partition function and duality relations among formal variables.

3.4.2 Hirzebruch genus

Let $M(r,k)$ denote the moduli space of rank r framed torsion-free sheaves (F,f) on $\mathbb{P}^2$ with second Chern class $k \in \mathbb{Z}_{\geq 0}$. The framing data is an isomorphism

$$f : F|_{\mathbb{P}^1_\infty} \to \mathcal{O}^{\oplus r}_{\mathbb{P}^1_\infty}. \tag{3.51}$$

$M(r,k)$ is a smooth quasi-projective fine moduli space i.e. there is an universal framed sheaf (F,f) on $M(r,k)\times\mathbb{P}^2$. Let $\mathsf{V} = R^1 p_{1*}\mathsf{F}\otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(-1)$ where $p_1, p_2 : M(r,k)\times\mathbb{P}^2 \to M(r,k), \mathbb{P}^2$ denote the canonical projections. It follows from [77] that V is a locally free sheaf of rank k on $M(r,k)$.

There is a torus $\mathbf{T} = \mathbb{C}^\times \times \mathbb{C}^\times \times (\mathbb{C}^\times)^{\times r}$ action on acting on $M(r,k)$, where the action of the first two factors is induced by the canonical action on $\mathbb{C}^\times\times\mathbb{C}^\times$ on $\mathbb{P}^2$, and the last r factors act linearly on the framing. According to [78] the fixed points of the $\mathbf{T}$-action on $M(r,k)$ are isolated and classified by collections of Young diagrams $\underline{Y} = (Y_1,\ldots,Y_r)$ so that the total number of boxes in all diagrams is $|\underline{Y}| = |Y_1| + \cdots |Y_r| = k$. Let $\mathcal{Y}_{r,k}$ denote the set of all such r-uples of Young diagrams. Note also that both the holomorphic cotangent bundle $T^\vee_{M(r,k)}$ and the bundle V constructed in the previous paragraph carry canonical equivariant structures.

The K-theoretic instanton partition function of an $SU(2)$ theory with g adjoint hypermultiplets and a level p Chern-Simons term is given by the equivariant residual Hirzebruch genus of the holomorphic $\mathbf{T}$-equivariant bundle

$$(T^\vee_{M(2,k)})^{\oplus g}\otimes(\det\mathsf{V})^{-p}.$$

This is defined by equivariant localization as follows [79, 71]. Let $(\epsilon_1,\epsilon_2,a_1,a_2)$ be equivariant parameters associated to the torus $\mathbf{T}$. Then the localization formula yields [79, 71]

$$\mathcal{Z}^{(g,p)}_{inst}(Q,\epsilon_1,\epsilon_2,a_1,a_2,y) = \sum_{k=0}^{\infty} Q^k \mathcal{Z}^{(g,p)}_k(\epsilon_1,\epsilon_2,a_1,a_2;y) \tag{3.52}$$

where $\mathcal{Z}_0^{(g,p)}(\epsilon_1,\epsilon_2,a_1,a_2;y)=1$ and

$$\begin{aligned}\mathcal{Z}_k^{(g,p)}(\epsilon_1,\epsilon_2,a_1,a_2;y) = \sum_{\underline{Y}\in\mathcal{Y}_{2,k}} \prod_{\alpha=1}^{2}\Big(e^{-|Y_\alpha|a_\alpha}\prod_{(i,j)\in Y_\alpha} e^{(i-1)\epsilon_1+(j-1)\epsilon_2}\Big)^p \\ \prod_{\alpha,\beta=1}^{2}\prod_{(i,j)\in Y_\alpha}\frac{\Big(1-ye^{(Y^t_{\beta,j}-i)\epsilon_1-(Y_{\alpha,i}-j+1)\epsilon_2+a_{\alpha\beta}}\Big)^g}{\Big(1-e^{(Y^t_{\beta,j}-i)\epsilon_1-(Y_{\alpha,i}-j+1)\epsilon_2+a_{\alpha\beta}}\Big)} \\ \prod_{(i,j)\in Y_\beta}\frac{\Big(1-ye^{-(Y^t_{\alpha,j}-i+1)\epsilon_1+(Y_{\beta,i}-j)\epsilon_2+a_{\alpha\beta}}\Big)^g}{\Big(1-e^{-(Y^t_{\alpha,j}-i+1)\epsilon_1+(Y_{\beta,i}-j)\epsilon_2+a_{\alpha\beta}}\Big)}\end{aligned} \tag{3.53}$$

where for any Young tableau Y, Y_i, $i\in\mathbb{Z}_{\geq 1}$ denotes the length of the i-th column and Y^t denotes the transpose of Y. If i is greater than the number of columns of Y, $Y_i=0$. Moreover $a_{\alpha\beta}=a_\alpha-a_\beta$ for any $\alpha,\beta=1,2$.

3.4.3 Comparison with the ruled vertex

A conjectural formula for the unrefined topological string partition function $\mathcal{Z}_{top}(Z;Q_f,Q_b,q)$ of the threefold Z has been derived from large N duality in [35]. The purpose of this subsection, is to show that $\mathcal{Z}_{top}(Z;Q_f,Q_b,q)$ has a factorization of the form (3.50) and there is an identification

$$\mathcal{Z}_{top}^{nonpert}(Z;Q_f,Q_b,q)=\mathcal{Z}_{inst}^{(g,p)}(Q,\epsilon_1,\epsilon_2,a_1,a_2,y)$$

subject to certain duality relations between the formal parameters. This will be a confirmation of duality predictions for local ruled surfaces in the unrefined case. Moreover, it will provide a starting point for understanding this correspondence in the refined case.

By analogy with [57, 71], first set

$$-\epsilon_1=\epsilon_2=\hbar, \qquad y=1. \tag{3.54}$$

Then a straightforward computation yields

$$\begin{aligned}&\mathcal{Z}_{2,k}^{(g,p)}(-\hbar,\hbar,a_1,a_2,1)= \\ &\sum_{\substack{Y_1,Y_2\\|Y_1|+|Y_2|=k}} e^{-p(|Y_1|a_1+|Y_2|a_2)}\prod_{\alpha=1}^{2}\prod_{(i,j)\in Y_\alpha}e^{p(j-i)\hbar}\left(2\sinh\frac{\hbar}{2}(Y_{\alpha,i}+Y^t_{\alpha,j}-i-j+1)\right)^{2(g-1)} \\ &\prod_{(i,j)\in Y_1}\left(2\sinh\frac{1}{2}(a_{1,2}+(Y^t_{2,j}+Y_{1,i}-i-j+1)\hbar)\right)^{2(g-1)} \\ &\prod_{(i,j)\in Y_2}\left(2\sinh\frac{1}{2}(a_{1,2}-(Y^t_{1,j}+Y_{2,i}-i-j+1)\hbar)\right)^{2(g-1)}\end{aligned} \tag{3.55}$$

Using identity [71, Lemm. 4.4], which was conjectured in [57] and proven in [42], it follows that

$$\mathcal{Z}^{(g,p)}_{2,k}(-\hbar,\hbar,a_1,a_2,1) = \sum_{\substack{Y_1,Y_2\\ |Y_1|+|Y_2|=k}} 2^{8(g-1)(|Y_1|+|Y_2|)} e^{-p(|Y_1|a_1+|Y_2|a_2)} e^{p(\kappa(Y_1)+\kappa(Y_2))\hbar/2} \prod_{\alpha,\beta=1}^{2}\prod_{i,j=1}^{\infty}\left(\frac{\sinh\frac{1}{2}(a_{\alpha,\beta}+(Y_{\alpha,i}-Y_{\beta,j}+j-i)\hbar)}{\sinh\frac{1}{2}(a_{\alpha,\beta}+(j-i)\hbar)}\right)^{2(1-g)} \tag{3.56}$$

where for any Young diagram Y

$$\kappa(Y) = 2\sum_{(i,j)\in Y}(j-i) = |Y| + \sum_{i=1}^{l(Y)}(Y_i^2 - 2iY_i),$$

$l(Y)$ being the number of rows of Y. Note that $\kappa(Y) = -\kappa(Y^t)$.

The topological string partition function on Z computed by the ruled vertex formalism [35] is

$$\mathcal{Z}_{top}(Z;q,Q_f,Q_b) = \sum_{Y_1,Y_2}(K_{Y_1,Y_2}(q,Q_f))^{2(1-g)} Q_b^{|Y_1|+|Y_2|} Q_f^{p|Y_2|}(-1)^{p(|Y_1|+|Y_2|)} q^{p(\kappa(Y_2)-\kappa(Y_1))/2} \tag{3.57}$$

where

$$K_{Y_1,Y_2}(q,Q_f) = \sum_{Y} Q_f^{|Y|} W_{Y_2 Y}(q) W_{Y Y_1}(q)$$

and

$$W_{R_1,R_2}(q) = s_{R_2}(q^{-i+1/2}))s_{R_1}(q^{R_{2,i}-i+1/2})$$

for any two Young tableaus R_1, R_2. Here $s_R(x^i)$ denotes the Schur function associated to the Young tableau R.

According to [57, 42], [71, Thm. 7.1], $K_{Y_1,Y_2}(q,Q_f) = K_{Y_2,Y_1}(q,Q_f)$ and

$$\frac{K_{Y_1,Y_2^t}(e^{-z},e^{-b})}{K_{\emptyset,\emptyset}(e^{-z},e^{-b})} = (2^{-4}Q_f^{-1/2})^{|Y_1|+|Y_2|}\prod_{\alpha,\beta=1}^{2}\prod_{i,j=1}^{\infty}\frac{\sinh\frac{1}{2}(b_{\alpha,\beta}+(Y_{\alpha,i}-Y_{\beta,j}+j-i)z)}{\sinh\frac{1}{2}(b_{\alpha,\beta}+(j-i)z)} \tag{3.58}$$

where $b_{1,2} = -b_{2,1} = b$. Therefore (3.57) is equivalent to

$$\mathcal{Z}_{top}(Z;q,Q_f,Q_b) = \sum_{Y_1,Y_2}(K_{Y_2,Y_1^t}(q,Q_f))^{2(1-g)} Q_b^{|Y_1|+|Y_2|} Q_f^{p|Y_2|}(-1)^{p(|Y_1|+|Y_2|)} q^{p(\kappa(Y_1)+\kappa(Y_2))/2} \tag{3.59}$$

Setting

$$\mathcal{Z}^{pert}_{top}(Z;q,Q_f,Q_b) = K_{\emptyset,\emptyset}(q,Q_f)^{2(1-g)}, \qquad \mathcal{Z}^{nonpert}_{top}(Z;q,Q_f,Q_b) = \frac{\mathcal{Z}_{top}(q,Q_f,Q_b)}{K_{\emptyset,\emptyset}(q,Q_f)^{2(1-g)}}.$$

identity (3.58) yields

$$\mathcal{Z}^{nonpert}_{top}(Z;q,Q_f,Q_b) = \sum_{k=0}^{\infty} Q^k \mathcal{Z}^{(g,p)}_{2,k}(-\hbar,\hbar,a_1,a_2;1) \tag{3.60}$$

for the following change of variables

$$Q_f = e^{a_{12}}, \qquad q = e^{\hbar}, \qquad Q = Q_b Q_f^{g-1}, \qquad e^{a_1} = -1. \tag{3.61}$$

This is a concrete confirmation of duality predictions in the unrefined case. The refined case is the subject of the next subsection.

3.4.4 Refinement

As explained at the end of subsection (3.4.1), string duality predicts that the nonperturbative part of the refined topological partition function of Z is determined by instanton partition function $\mathcal{Z}^{(p)}_{inst}(Q, \epsilon_1, \epsilon_2, a_1, a_2, y)$ provided one finds the correct identification of formal parameters as in [55, 59]. Although local ruled surfaces are not discussed in [55, 59], a careful inspection of the cases discussed there leads to the following construction.

Recall that the contribution of a fixed point $(Y_1, Y_2) \in \mathcal{Y}_{2,k}$ for some arbitrary $k \geq 1$ to the right hand side of the localization formula (3.53) is

$$\begin{gathered}\prod_{\alpha=1}^{2}\left(e^{-|Y_\alpha|a_\alpha}\prod_{(i,j)\in Y_\alpha} e^{(i-1)\epsilon_1+(j-1)\epsilon_2}\right)^p \\ \prod_{\alpha,\beta=1}^{2}\prod_{(i,j)\in Y_\alpha}\frac{\left(1-ye^{(Y^t_{\beta,j}-i)\epsilon_1-(Y_{\alpha,i}-j+1)\epsilon_2+a_\alpha-a_\beta}\right)^g}{\left(1-e^{(Y^t_{\beta,j}-i)\epsilon_1-(Y_{\alpha,i}-j+1)\epsilon_2+a_\alpha-a_\beta}\right)} \\ \prod_{(i,j)\in Y_\beta}\frac{\left(1-ye^{-(Y^t_{\alpha,j}-i+1)\epsilon_1+(Y_{\beta,i}-j)\epsilon_2+a_\alpha-a_\beta}\right)^g}{\left(1-e^{-(Y^t_{\alpha,j}-i+1)\epsilon_1+(Y_{\beta,i}-j)\epsilon_2+a_\alpha-a_\beta}\right)}\end{gathered} \tag{3.62}$$

Let $\mathcal{Z}^{(g,p)}_{(\emptyset,Y)}(q_1, q_2, Q_f, y)$ be the expression obtained by setting $q_1 = e^{-\epsilon_1}$, $q_2 = e^{-\epsilon_2}$ and

$$Q_f = e^{a_{12}}, \qquad e^{a_1} = -1$$

in (3.62). Note that a simple power counting argument shows that the expression

$$Q_f^{(g-1)|Y|}\mathcal{Z}^{(g,p)}_{(Y,\emptyset)}(q_1, q_2, Q_f, y)$$

has well defined specialization $\mathcal{Z}^{(g,p)}_{(Y,\emptyset)}(q_1, q_2, y)^{(0)}$ at $Q_f = 0$, for any Y. Then, for any $r \in \mathbb{Z}_{\geq 1}$, any Young diagram Y with $|Y| = r$, and any $p \in \mathbb{Z}$ let

$$\Omega^{(g,p)}_Y(\lambda, y) = y^{2|Y|}\lambda^{(g-1)|Y|}\mathcal{Z}^{(g,p)}_{(Y,\emptyset)}(\lambda^{-1}y, \lambda y, y^{-1})^{(0)}. \tag{3.63}$$

Then string duality predicts that the generating function of asymptotic singly refined ADHM invariants is given by

$$\mathcal{Z}_{+\infty}(\mathcal{X}, r; \lambda, y) = \sum_{|Y|=r}\Omega^{(g,p)}_Y(\lambda, y).$$

Formula (3.46) follows by a straightforward computation.

3.4.5 Double Refinement

Physical arguments [36] present compelling evidence for the existence of a doubly refined BPS counting function, which is graded by $U(1)_R$ charge in addition to spin quantum number. In this section it is conjectured that the doubly refined partition function of asymptotic ADHM invariants is obtained again from the equivariant instanton sum (3.53) by a different specialization of the equivariant parameters. Namely, for $r \in \mathbb{Z}_{\geq 1}$, any Young diagram Y with $|Y| = r$, and any $p \in \mathbb{Z}$ let

$$\Omega^{(g,p)}_{(Y,\emptyset)}(\lambda, u^{1/2}, v^{1/2}) = u^{(g+1)|\mu|}v^{(g-1)|\mu|}\mathcal{Z}^{(g,p)}_{(Y,\emptyset)}(\lambda^{-1}(uv)^{1/2}, \lambda(uv)^{1/2}, u^{-1})^{(0)}. \tag{3.64}$$

The generating function of doubly refined asymptotic ADHM invariants is then conjectured to be

$$\mathcal{Z}_{+\infty}(\mathcal{X}, r; \lambda, u, v) = \sum_{|Y|=r} \Omega^{(g,p)}_{Y}(\lambda, u^{1/2}, v^{1/2}).$$

A straightforward computation yields formula (3.49). In conjunction with the doubly refined wallcrossing conjecture (3.2.5), the above formula has been shown to yield correct results for the Hodge polynomial of the Hitchin moduli space in many examples, see appendix B of [29].

3.4.6 Localization interpretation for $r = 2$

Suppose the conditions of section (3.3.1) are satisfied, that is $p \geq 0$, and $M_1 = \mathcal{O}_X$, $M_2 = K_X^{-1}$ if $p = 0$. The goal of this section is to discuss the geometric interpretation of conjecture (3.4.1) for $r = 1, 2$. The main observation is that in these cases, equation (3.45) can be interpreted as a sum of contributions of torus fixed loci in the moduli space $\mathfrak{M}^{ss}_{+\infty}(\mathcal{X}, r, e)$. However, a rigorous geometric computation would require a localization theorem for the refined Donaldson-Thomas invariants defined in [69], which has not been been formulated and proven so far.

First let $r = 1$. The moduli stack of δ-semistable ADHM sheaves of type $(1, e)$ on X with $\delta > 0$ and $e \geq 0$ is a $\mathbb{C}^\times$-gerbe over the smooth variety

$$S^e(X) \times H^0(X, M_1^{-1}) \times H^0(X, M_2^{-1}). \tag{3.65}$$

A $\mathbb{C}$-valued point of $\mathfrak{M}^{ss}_{\delta}(\mathcal{X}, 1, e)$ is an ADHM sheaf of the form $(E, \Phi_1, \Phi_2, 0, \psi)$ where E is a degree e line bundle on X, $\Phi_1 \in \mathrm{Hom}_X(E \otimes_X M_1, E) \simeq H^0(X, M_1^{-1})$, $\Phi_2 \in \mathrm{Hom}_X(E \otimes_X M_2, E) \simeq H^0(X, M_2^{-1})$ and $\psi \in H^0(X, E)$. The δ-stability condition, $\delta > 0$ is equivalent to ψ not identically zero. Obviously, the moduli stack is empty if $e < 0$.

The fixed point conditions require $\Phi_1 = 0$, $\Phi_2 = 0$. Therefore the torus fixed locus is a $\mathbb{C}^\times$-gerbe over the symmetric product $S^e(X)$.

Conjecture (3.4.1) and equation (3.46) yield

$$\mathcal{Z}_{+\infty}(\mathcal{X},1;\lambda,y) = (-1)^p y^{1-g} \frac{(1-\lambda)^{2g}}{(1-\lambda y)(1-\lambda y^{-1})}. \tag{3.66}$$

Now recall Macdonald's formula

$$\sum_{n\geq 0} P_z(S^n(X))x^n = \frac{(1-xz)^{2g}}{(1-x)(1-xz^2)}. \tag{3.67}$$

for the generating function of Poincaré polynomials of symmetric products of X. Then equations (3.66) and (3.67) imply

$$\mathcal{Z}_{+\infty}(\mathcal{X},1;\lambda,y) = (-1)^p \sum_{e\geq 0} \lambda^e y^{1-g-e} P_y(S^e(X)) \tag{3.68}$$

for all $e \in \mathbb{Z}_{\geq 0}$.

Next let $r = 2$. Property $(B.2)$ implies that the moduli space $\mathfrak{M}^{ss}_{+\infty}(\mathcal{X}, r, e)$ is empty unless $e \geq 2-2g$. Assuming this to be the case, a straightforward analysis shows that the components of the torus fixed locus are of two types. The ADHM sheaves corresponding to the $\mathbb{C}$-valued fixed points are presented as follows.

(i) $E \simeq E_{-1} \oplus E_0$, $\Phi_2 = 0$, $\mathrm{Im}(\psi) \subseteq E_0$ and

$$\Phi_1 = \begin{bmatrix} 0 & \varphi \\ 0 & 0 \end{bmatrix}$$

with $\varphi : E_0 \otimes_X M_1 \to E_{-1}$ a nontrivial morphism of line bundles. Components of this type are isomorphic to $\mathbb{C}^\times$-gerbes over the smooth varieties

$$S^{e_0}(X) \times S^{e_{-1}-e_0-p}(X)$$

where $0 \leq e_0 \leq e_{-1} - p$ and $e_0 + e_{-1} = e$.

(ii) $E \simeq E_0 \oplus E_1$, $\Phi_1 = 0$, $\mathrm{Im}(\psi) \subseteq E_0$ and

$$\Phi_2 = \begin{bmatrix} 0 & 0 \\ \varphi & 0 \end{bmatrix}$$

with $\varphi : E_0 \otimes_X M_2 \to E_1$ a nontrivial morphism of line bundles. Components of this type are isomorphic to $\mathbb{C}^\times$-gerbes over the smooth varieties

$$S^{e_0}(X) \times S^{e_1-e_0+2g-2+p}(X)$$

where $0 \leq e_0 \leq e_1 + 2g - 2 + p$ and $e_0 + e_1 = e$.

Note that in both cases, the moduli stack of asymptotically stable ADHM sheaves is not smooth along the fixed loci, although the fixed loci are smooth.

Conjecture (3.4.1) and equation (3.46) yield

$$\begin{aligned}
&\mathcal{Z}_{+\infty}(\mathcal{X},2;\lambda,y) = \Omega^{(g,p)}_{\square\square}(\lambda,y) + \Omega^{(g,p)}_{\genfrac{}{}{0pt}{}{\square}{\square}}(\lambda,y)\\
&\quad\Omega^{(g,p)}_{\genfrac{}{}{0pt}{}{\square}{\square}}(\lambda,y) = (\lambda^{-1}y)^{-p}y^{2-2g}\frac{(1-\lambda^2y^{-1})^{2g}(1-\lambda)^{2g}}{(1-\lambda^2)(1-\lambda^2y^{-2})(1-\lambda y)(1-\lambda y^{-1})}\\
&\quad\Omega^{(g,p)}_{\square\square}(\lambda,y) = (\lambda y)^{-p}y^{4-4g}\lambda^{2-2g}\frac{(1-\lambda^2y)^{2g}(1-\lambda)^{2g}}{(1-\lambda^2)(1-\lambda^2y^2)(1-\lambda y)(1-\lambda y^{-1})}
\end{aligned}\tag{3.69}$$

A straightforward computation using equation (3.67) yields

$$\begin{aligned}
\Omega^{(g,p)}_{\genfrac{}{}{0pt}{}{\square}{\square}}(\lambda,y) &= \sum_{e\geq p}\lambda^e \sum_{\substack{e_0+e_{-1}=e\\ 0\leq e_0\leq e_{-1}-p}} y^{2-2g-p-e_0}\ y^{-e_0}P_y(S^{e_0}(X))\\
&\qquad\qquad y^{-e_{-1}+e_0+p}P_y(S^{e_{-1}-e_0-p}(X))\\
\Omega^{(g,p)}_{\square\square}(\lambda,y) &= \sum_{e\geq 2-2g-p}\lambda^e \sum_{\substack{e_0+e_1=e\\ 0\leq e_0\leq e_1+2g-2+p}} y^{e_0-p}\ y^{-e_0}P_y(S^{e_0}(X))\\
&\qquad\qquad y^{-e_1+e_0-2g+2-p}P_y(S^{e_1-e_0+2g-2+p}(X))
\end{aligned}\tag{3.70}$$

Given the explicit description of the fixed loci, equations (3.66), (3.68), (3.69), (3.70) clearly suggest an equivariant localization theorem for refined ADHM invariants. Such a formula would presumably allow a rigorous computation of the polynomial weights assigned to each component of the fixed locus.

For future reference, let us record the expressions $\Omega^{(p)}_Y(\lambda,y)$ for $|Y|=3$.

$$\begin{aligned}
&\Omega^{(g,p)}_{\genfrac{}{}{0pt}{}{\square}{\genfrac{}{}{0pt}{}{\square}{\square}}}(\lambda,y) = (-1)^p(\lambda^3y^{-3})^py^{3-3g}\frac{(1-\lambda)^{2g}(1-\lambda^2y^{-1})^{2g}(1-\lambda^3y^{-2})^{2g}}{(1-\lambda y)(1-\lambda y^{-1})(1-\lambda^2y^{-2})(1-\lambda^2)(1-\lambda^3y^{-3})(1-\lambda^3y^{-1})}\\
&\Omega^{(g,p)}_{\genfrac{}{}{0pt}{}{\square}{\square\square}}(\lambda,y) = (-1)^py^{2p}y^{5-5g}\lambda^{2-2g}\frac{(1-\lambda)^{4g}(1-\lambda^3)^{2g}}{(1-\lambda y)^2(1-\lambda y^{-1})^2(1-\lambda^3y)(1-\lambda^3y^{-1})}\\
&\Omega^{(g,p)}_{\square\square\square}(\lambda,y) = (-1)^p(\lambda^{-3}y^{-3})^py^{9-9g}\lambda^{6-6g}\frac{(1-\lambda)^{2g}(1-\lambda^2y)^{2g}(1-\lambda^3y^2)^{2g}}{(1-\lambda y)(1-\lambda y^{-1})(1-\lambda^2y^2)(1-\lambda^2)(1-\lambda^3y^3)(1-\lambda^3y)}
\end{aligned}\tag{3.71}$$

3.5 Examples, comparison with existing results

This section will present several concrete results for Poincaré polynomials of moduli spaces of Hitchin pairs obtained from the recursion relation (3.33). It will be checked that all these results are in agreement with the computations of Hitchin [54] and Gothen [47], as well as the conjecture of Hausel and Rodriguez-Villegas [53]. In order to simplify the notation set

$\widetilde{H}(r,e)(y) = (-1)^{e-r(g-1)}H(r,e)(y)$ for all $(r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. Then equation (3.33) becomes

$$\begin{aligned}
&[e - r(g-1)]_y \widetilde{H}(\gamma)(y) = A_{+\infty}(\gamma)(y) - A_{+\infty}(\widetilde{\gamma})(y) \\
&+ \sum_{l\geq 2} \frac{(-1)^{l-1}}{(l-1)!} \sum_{\substack{\gamma_1,\ldots,\gamma_l \in \mathbb{Z}_{\geq 1}\times\mathbb{Z} \\ \gamma_1+\cdots+\gamma_l=\gamma \\ \mu_0(r)\leq\mu(\gamma)<\mu(\gamma_i),\ 2\leq i\leq l, \\ \mu_0(r)\leq\mu(\gamma_1)}} A_{+\infty}(\gamma_1)(y) \prod_{i=2}^{l} [e_i - r_i(g-1)]_y \widetilde{H}(r_i,e_i)(y) \\
&- \sum_{l\geq 2} \frac{(-1)^{l-1}}{(l-1)!} \sum_{\substack{\gamma_1,\ldots,\gamma_l \in \mathbb{Z}_{\geq 1}\times\mathbb{Z} \\ \gamma_1+\cdots+\gamma_l=\widetilde{\gamma} \\ \mu_0(r)\leq\mu(\widetilde{\gamma})\leq\mu(\gamma_i),\ 2\leq i\leq l \\ \mu_0(r)\leq\mu(\gamma_1)}} A_{+\infty}(\gamma_1)(y) \prod_{i=2}^{l} [e_i - r_i(g-1)]_y \widetilde{H}(r_i,e_i)(y) \qquad (3.72) \\
&- \sum_{l\geq 2} \frac{1}{l!} \sum_{\substack{\gamma_1,\ldots,\gamma_l \in \mathbb{Z}_{\geq 1}\times\mathbb{Z} \\ \gamma_1+\cdots+\gamma_l=\gamma \\ \mu(\gamma)=\mu(\gamma_i),\ 1\leq i\leq l}} \prod_{i=1}^{l} [e_i - r_i(g-1)]_y \widetilde{H}(r_i,e_i)(y)
\end{aligned}$$

where $\mu_0(r) = -(r-1)(2g-2+p)$, and the sum in the right hand side of equation (3.33) is finite.

3.5.1 Rank $r = 1$

There are no positive critical parameters of type $(1,e)$ for any $e \in \mathbb{Z}_{\geq 0}$ The wallcrossing formula (3.25) at $\delta_c = 0$ reads

$$A_{+\infty}(1,e) - A_{+\infty}(1,-e+2(g-1)) = [e-g+1]_y \widetilde{H}(1,e). \qquad (3.73)$$

Expanding the right hand side of equation (3.68) in powers of λ yields

$$A_{+\infty}(1,e) = y^{1-g} \sum_{\substack{0\leq k\leq 2g \\ m,l\geq 0,\ k+l+m=e}} (2g,k)(-1)^k y^{l-m}$$

for any $e \geq 0$, where $(2g,k) = \frac{(2g)!}{k!(2g-k)!}$ are binomial coefficients. A series of elementary manipulations further yield

$$\begin{aligned}
A_{+\infty}(1,e) &= y^{1-g} \sum_{\substack{0\leq k\leq 2g \\ l\geq 0,\ l+k\leq e}} (2g,k)(-1)^k y^{2l+k-e} \\
&= y^{1-g} \sum_{\substack{0\leq k\leq 2g \\ l\geq 0,\ l+k\leq e}} (2g,k)(-1)^k y^{k-e} \frac{1-y^{2e-2k+2}}{1-y^2} \\
&= \frac{y^{1-g}}{1-y^2} \sum_{\substack{0\leq k\leq 2g \\ k\leq e}} (2g,k)(-1)^k \left(y^{k-e} - y^{e-k+2}\right)
\end{aligned}$$

for any $e \geq 0$. In order to compute the left hand side of equation (3.73), it is convenient to consider three cases.

a) $0 \le e \le 2g-2$. Then

$$\begin{aligned}
&A_{+\infty}(1,e) - A_{+\infty}(1,-e+2(g-1))A_{+\infty}(1,e) = \\
&\frac{y^{1-g}}{1-y^2}\left[\sum_{k=0}^{e}(2g,k)(-1)^k y^{k-e} + \sum_{k=0}^{2g-2-e}(2g,k)(-1)^k y^{2g-2-e-k}\right] \\
&-\frac{y^{1-g}}{1-y^2}\left[\sum_{k=0}^{e}(2g,k)(-1)^k y^{e-k+2} + \sum_{k=0}^{2g-2-e}(2g,k)(-1)^k y^{k+e-2g+2}\right] \\
&= \frac{y^{1-g}}{1-y^2}y^{-e}\left[\sum_{k=0}^{e}(2g,k)(-1)^k y^{k} + \sum_{k=e+2}^{2g-2}(2g,k)(-1)^k y^{k}\right] \\
&\quad -\frac{y^{1-g}}{1-y^2}y^{e+2}\left[\sum_{k=0}^{e}(2g,k)(-1)^k y^{-k} + \sum_{k=e+2}^{2g-2}(2g,k)(-1)^k y^{-k}\right] \\
&= -\frac{y^{1-g}}{1-y^2}\left[y^{e+2}(1-y^{-1})^{2g} - y^{-e}(1-y)^{2g}\right] = \frac{y^{e-g+1}-y^{-e+g-1}}{y-y^{-1}}\frac{(1-y)^{2g}}{y^{2g-1}}
\end{aligned}$$

b) $e = 2g-1$. Then $A_{+\infty}(-e+2g-2) = 0$ and

$$\begin{aligned}
A_{+\infty}(1,2g-1) &= \frac{y^{1-g}}{1-y^2}\sum_{k=0}^{2g-1}(2g,k)(-1)^k\left(y^{k-2g+1} - y^{2g-k+1}\right) \\
&= \frac{y^{1-g}}{1-y^2}\left[y^{1-2g}(1-y)^{2g} - y^{2g+1}(1-y^{-1})^{2g}\right] = \frac{y^{g}-y^{-g}}{y-y^{-1}}\frac{(1-y)^{2g}}{y^{2g-1}}
\end{aligned}$$

c) $e \ge 2g$. Then $A_{+\infty}(-e+2g-2) = 0$ and a similar computation yields

$$A_{+\infty}(1,e) = \frac{y^{e-g+1}-y^{-e+g-1}}{y-y^{-1}}\frac{(1-y)^{2g}}{y^{2g-1}}.$$

In conclusion,

$$\widetilde{H}(1,e)(y) = \frac{(1-y)^{2g}}{y^{2g-1}} \tag{3.74}$$

for all $e \ge 0$, hence also for all $e \in \mathbb{Z}$.

The moduli space of rank one semistable Hitchin pairs of any degree $e \in \mathbb{Z}$ is isomorphic to

$$H^0(X, M_2^{-1}) \times J_e(X)$$

where $J_e(X)$ is the degree e Jacobian of X. Obviously the formula (3.74) can be rewritten as

$$\widetilde{H}(1,e)(y) = y^{1-2g}P_y(J_e(X))$$

for any $e \in \mathbb{Z}$.

3.5.2 Rank $r = 2$

According to property (B.2) in section (3.1), all invariants $A_\delta(1,e)(y)$ are zero for $e < 0$. It will be convenient to distinguish two cases, depending on the parity of e. By convention, any sum in the following formulas is zero if the lower summation bound exceeds the upper summation bound.

a) $e = 2n,\ n \in \mathbb{Z}$. Then equation (3.72) reduces to

$$\begin{aligned}
&[2n-2g+2]_y \widetilde{H}(2,2n)(y) = A_{+\infty}(2,2n) - A_{+\infty}(2,-2n+4g-4) \\
&- \sum_{e_1=0}^{n-1} A_{+\infty}(1,e_1)(y)[2n-e_1-g+1]_y \widetilde{H}(1,2n-e_1)(y) \\
&+ \sum_{e_1=0}^{2g-2-n} A_{+\infty}(1,e_1)(y)[3g-3-2n-e_1]_y \widetilde{H}(1,4g-4-2n)(y) \\
&- \frac{1}{2}[n-g+1]_y^2 \widetilde{H}(1,n)(y)^2.
\end{aligned} \tag{3.75}$$

b) $e = 2n+1,\ n \in \mathbb{Z}$. Then equation (3.33) reduces to

$$\begin{aligned}
&[2n-2g+3]_y \widetilde{H}(2,2n+1)(y) = A_{+\infty}(2,2n+1) - A_{+\infty}(2,4g-5-2n) \\
&- \sum_{e_1=0}^{n} A_{+\infty}(1,e_1)(y)[2n-e_1-g+2]_y \widetilde{H}(1,2n+1-e_1)(y) \\
&+ \sum_{e_1=0}^{2g-3-n} A_{+\infty}(1,e_1)(y)[3g-4-2n-e_1]_y H(1,4g-4-2n-e_1)(y)
\end{aligned} \tag{3.76}$$

Some concrete results are recorded below. $\widetilde{H}^{(p)}(r,e)$ denotes the motivic Higgs invariant of type (r,e) with coefficient bundles (M_1, M_2) of degrees $(p, 2-2g-p)$, $p \geq 0$. Under the current assumptions, $M_1 \simeq \mathcal{O}_X$ if $p = 0$.

$\boxed{g=2}$

$$\widetilde{H}^{(0)}(2,1) = \frac{(1-y)^4(1+y^2)(1-4y^3+2y^4)}{y^9}$$

$$\widetilde{H}^{(0)}(2,0) = \frac{(1-y)^4(2+4y^2-8y^3+7y^4-12y^5+14y^6-4y^7+5y^8)}{2y^9(1+y^2)}$$

$\boxed{g=3}$

$$\widetilde{H}^{(0)}(2,1) = \frac{(1-y)^6}{y^{17}}(1+y^2-6y^3+2y^4-6y^5+17y^6-12y^7+18y^8-32y^9+18y^{10}-12y^{11}+3y^{12})$$

$$\begin{aligned}
\widetilde{H}^{(0)}(2,0) = \frac{(1-y)^6}{2y^{17}(1+y^2)}&(2+4y^2-12y^3+6y^4-24y^5+38y^6-36y^7+71y^8-82y^9+87y^{10} \\
&-68y^{11}+57y^{12}-18y^{13}+7y^{14})
\end{aligned}$$

$\boxed{g=4}$

$$\widetilde{H}^{(0)}(2,1) = \frac{(1-y)^8}{y^{25}}(1+y^2)(1-8y^3+2y^4+28y^6-16y^7+3y^8-56y^9+56y^{10}-24y^{11} + 74y^{12}-112y^{13}+56y^{14}-24y^{15}+4y^{16})$$

$$\widetilde{H}^{(0)}(2,0) = \frac{(1-y)^8}{2y^{25}(1+y^2)}(2+4y^2-16y^3+6y^4-32y^5+64y^6-48y^7+122y^8-176y^9 + 180y^{10}-304y^{11}+379y^{12}-424y^{13}+548y^{14}-488y^{15}+450y^{16} - 264y^{17}+156y^{18}-40y^{19}+9y^{20})$$

$\boxed{g=5}$

$$\widetilde{H}^{(0)}(2,1) = \frac{(1-y)^{10}}{y^{33}}(1+y^2-10y^3+2y^4-10y^5+47y^6-20y^7+48y^8-140y^9+93y^{10} - 150y^{11}+304y^{12}-270y^{13}+349y^{14}-532y^{15}+560y^{16}-652y^{17} + 770y^{18}-784y^{19}+560y^{20}-400y^{21}+140y^{22}-40y^{23}+5y^{24})$$

In all the above cases, similar computations also show that the invariants $\widetilde{H}(2,e)$ depend only on the parity of $e \in \mathbb{Z}$. Note also that for even e the rank two motivic Higgs invariants are rational functions of y rather than polynomials in y^{-1}, y. By analogy with the theory of generalized Donaldson-Thomas invariants [65], this reflects the fact that in this case the moduli stack $\mathfrak{Higgs}^{ss}(\mathcal{X}, 2, e)$ contains strictly semistable $\mathbb{C}$-valued points.

3.5.3 Rank $r = 3$.

According to property (B.2) in section (3.1), all invariants $A_\delta(2,e)(y)$ are zero for $e < 2-2g-p$. Suppose $e = 3n+1$, $n \in \mathbb{Z}$. Then equation (3.72) reduces to

$$\begin{aligned}
&[3n-3g+4]_y \widetilde{H}(3,3n+1) = A_{+\infty}(3,3n+1)(y) - A_{+\infty}(3,-3n+6g-7)(y) \\
&- \sum_{e_1=2-2g-p}^{2n} A_{+\infty}(2,e_1)[3n+2-g-e_1]_y \widetilde{H}(1,3n+1-e_1)(y) \\
&- \sum_{e_1=0}^{n} A_{+\infty}(1,e_1)[3n+3-2g-e_1]_y \widetilde{H}(2,3n+1-e_1)(y) \\
&+ \frac{1}{2} \sum_{e_1=0}^{n-1} \sum_{e_2=n+1}^{2n-e_1} A_{+\infty}(1,e_1)[e_2-g+1]_y [3n+2-g-e_1-e_2]_y \widetilde{H}(1,3n+1-e_1-e_2)(y)^2 \\
&+ \sum_{e_1=2-2g-p}^{4g-2n-5} A_{+\infty}(2,e_1)[5g-6-3n-e_1]_y \widetilde{H}(1,6g-7-3n-e_1)(y) \\
&+ \sum_{e_1=0}^{2g-n-3} A_{+\infty}(1,e_1)[4g-5-3n-e_1]_y \widetilde{H}(2,6g-7-3n-e_1)(y) \\
&- \frac{1}{2} \sum_{e_1=0}^{2g-3-n} \sum_{e_2=2g-2-n}^{4g-2n-5-e_1} A_{+\infty}(1,e_1)[e_2-g+1]_y [5g-6-3n-e_1-e_2]_y \\
&\widetilde{H}(1,6g-7-3n-e_1-e_2)(y)^2
\end{aligned} \tag{3.77}$$

Again, some concrete results are recorded below.

$\boxed{g = 2}$

$$\begin{aligned}
\widetilde{H}^{(0)}(3,1) = H_{g=2}(3,2) = &\frac{(1-y)^4}{y^{19}} \\
&(1+y^2-4y^3+3y^4-8y^5+10y^6-16y^7+29y^8-32y^9+48y^{10} \\
&-64y^{11}+67y^{12}-68y^{13}+48y^{14}-24y^{15}+6y^{16})
\end{aligned}$$

$$\begin{aligned}
\widetilde{H}^{(1)}(3,1) = \frac{(1-y)^4}{y^{25}} &(6y^{22}-36y^{21}+96y^{20}-168y^{19}+207y^{18}-216y^{17}+210y^{16}-184y^{15} \\
&+149y^{14}-120y^{13}+92y^{12}-72y^{11}+49y^{10}-32y^9 \\
&+29y^8-16y^7+10y^6-8y^5+3y^4-4y^3+y^2+1)
\end{aligned}$$

$$\widetilde{H}^{(2)}(3,1)=\frac{(1-y)^4}{y^{31}}(10y^{28}-64y^{27}+184y^{26}-344y^{25}+477y^{24}-560y^{23}+583y^{22}$$
$$-560y^{21}+522y^{20}-464y^{19}+386y^{18}-320y^{17}+267y^{16}-208y^{15}$$
$$+158y^{14}-124y^{13}+93y^{12}-72y^{11}+49y^{10}-32y^{9}+29y^{8}-16y^{7}$$
$$+10y^{6}-8y^{5}+3y^{4}-4y^{3}+y^{2}+1)$$

$g=3$

$$\widetilde{H}^{(0)}(3,1)=\frac{(1-y)^6}{y^{37}}(15y^{32}-120y^{31}+480y^{30}-1260y^{29}+2355y^{28}-3486y^{27}$$
$$+4189y^{26}-4416y^{25}+4315y^{24}-3922y^{23}+3399y^{22}-2860y^{21}$$
$$+2309y^{20}-1872y^{19}+1433y^{18}-1072y^{17}+861y^{16}-604y^{15}$$
$$+446y^{14}-336y^{13}+212y^{12}-176y^{11}+105y^{10}-62y^{9}$$
$$+58y^{8}-24y^{7}+19y^{6}-12y^{5}+3y^{4}-6y^{3}+y^{2}+1)$$

$$\widetilde{H}^{(1)}(3,1)=\frac{(1-y)^6}{y^{43}}(15y^{38}-150y^{37}+690y^{36}-2010y^{35}+4110y^{34}-6542y^{33}$$
$$+8598y^{32}-9930y^{31}+10427y^{30}-10254y^{29}+9672y^{28}-8800y^{27}$$
$$+7705y^{26}-6600y^{25}+5598y^{24}-4600y^{23}+3723y^{22}-3006y^{21}$$
$$+2363y^{20}-1884y^{19}+1434y^{18}-1072y^{17}+861y^{16}-604y^{15}+446y^{14}$$
$$-336y^{13}+212y^{12}-176y^{11}+105y^{10}-62y^{9}+58y^{8}-24y^{7}$$
$$+19y^{6}-12y^{5}+3y^{4}-6y^{3}+y^{2}+1)$$

$$\widetilde{H}^{(2)}(3,1)=\frac{(1-y)^6}{y^{49}}(21y^{44}-216y^{43}+1026y^{42}-3090y^{41}+6621y^{40}-11094y^{39}$$
$$+15375y^{38}-18672y^{37}+20712y^{36}-21584y^{35}+21450y^{34}-20552y^{33}$$
$$+19178y^{32}-17460y^{31}+15503y^{30}-13546y^{29}+11706y^{28}-9952y^{27}$$
$$+8316y^{26}-6912y^{25}+5736y^{24}-4650y^{23}+3741y^{22}-3012y^{21}$$
$$+2364y^{20}-1884y^{19}+1434y^{18}-1072y^{17}+861y^{16}-604y^{15}$$
$$+446y^{14}-336y^{13}+212y^{12}-176y^{11}+105y^{10}-62y^{9}+58y^{8}$$
$$-24y^{7}+19y^{6}-12y^{5}+3y^{4}-6y^{3}+y^{2}+1)$$

$\boxed{g=4}$

$$\widetilde{H}^{(0)}(3,1) = \frac{(1-y)^8}{y^{55}}(28y^{48} - 336y^{47} + 2016y^{46} - 7896y^{45} + 22218y^{44} - 48328y^{43}$$
$$+ 84084y^{42} - 122616y^{41} + 155235y^{40} - 176912y^{39} + 186320y^{38}$$
$$- 185408y^{37} + 176976y^{36} - 163656y^{35} + 146930y^{34} - 128936y^{33}$$
$$+ 111544y^{32} - 94416y^{31} + 78918y^{30} - 65392y^{29} + 53178y^{28} - 43392y^{27}$$
$$+ 34620y^{26} - 27288y^{25} + 21936y^{24} - 16728y^{23} + 13005y^{22} - 10064y^{21}$$
$$+ 7290y^{20} - 5760y^{19} + 4077y^{18} - 2880y^{17} + 2278y^{16} - 1416y^{15} + 1071y^{14}$$
$$- 744y^{13} + 416y^{12} - 368y^{11} + 185y^{10} - 112y^{9} + 99y^{8} - 32y^{7} + 32y^{6}$$
$$- 16y^{5} + 3y^{4} - 8y^{3} + y^{2} + 1)$$

In addition similar computations show that $H^{(p)}(3,2) = H^{(p)}(3,1)$ in all above examples.

3.5.4 Hausel-Rodriguez-Villegas Formula

This subsection is a brief summary of the formulas of Hausel and Rodriguez-Villegas [53], [52] for the Poincaré, respectively Hodge polynomial of the moduli space $H(X, K_X, r, e)$ with $(r,e) \in \mathbb{Z}_{\geq} \times \mathbb{Z}$ coprime. Construct the following formal series

$$\mathcal{Z}(q,x,y,T) = 1 + \sum_{k\geq 1} T^k A_k(q,x,y) = 1 + \sum_{k\geq 1} T^k \left(\sum_{|Y|=k} A_Y(q,x,y) \right)$$

where:

$$A_Y(q,x,y) = \prod_{z\in Y} \frac{(qxy)^{l(z)(2-2g)}(1+q^{h(z)}y^{l(z)}x^{l(z)+1})^g(1+q^{h(z)}x^{l(z)}y^{l(z)+1})^g}{(1-q^{h(z)}(xy)^{l(z)+1})(1-q^{h(z)}(xy)^{l(z)})}$$

where for $z=(i,j)\in Y$:

$$a(z) = Y_i - j,\ l(z) = Y_j^t - i,\ h(z) = a(z) + l(z) + 1$$

Define $H_r(q,x,y)$ in terms of the following recursive formula:

$$\sum_{r\geq 1}\sum_{k\geq 1} H_r(q^k, -(-x)^k, -(-y)^k) B_r(q^k, -(-x)^k, -(-y)^k) \frac{T^{kr}}{k} = \log \mathcal{Z}(q,x,y,T)$$

by comparing the coefficient of T^{nk}, where:

$$B_r(q,x,y) = \frac{(qxy)^{(1-g)r(r-1)}(1+qx)^g(1+qy)^g}{(1-qxy)(1-q)}$$

Then

$$E_r(u, v) = H_r(1, u, v) \tag{3.78}$$

is conjectured in [52] to be Hodge polynomial of the moduli space $H(X, K_X, r, e)$.

Chapter 4
Higher Rank D6-D2-D0 Invariants

4.1 Higher rank ADHM invariants

4.1.1 Definitions and basic properties

Let X be a smooth projective curve of genus $g \in \mathbb{Z}_{\geq 0}$ over an infinite field K of characteristic 0 equipped with a very ample line bundle $\mathcal{O}_X(1)$. Let M_1, M_2 be fixed line bundles on X equipped with a fixed isomorphism $M_1 \otimes_X M_2 \simeq K_X^{-1}$. Set $M = M_1 \otimes_X M_2$. For fixed data $\mathcal{X} = (X, M_1, M_2)$, let $\mathcal{Q}_{\mathcal{X},s}$ denote the abelian category of (M_1, M_2)-twisted coherent ADHM quiver sheaves. An object of $\mathcal{Q}_{\mathcal{X}}$ is given by a collection $\mathcal{E} = (E, E_\infty, \Phi_1, \Phi_2, \phi, \psi)$ where

- E, E_∞ are coherent $\mathcal{O}_X$-modules
- $\Phi_i : E \otimes_X M_i \to E$, $i = 1, 2$, $\phi : E \otimes_X M_1 \otimes_X M_2 \to E_\infty$, $\psi : E_\infty \to E$ are morphisms of $\mathcal{O}_X$-modules satisfying the ADHM relation

$$\Phi_1 \circ (\Phi_2 \otimes 1_{M_1}) - \Phi_2 \circ (\Phi_1 \otimes 1_{M_2}) + \psi \circ \phi = 0. \tag{4.1}$$

The morphisms are natural morphisms of quiver sheaves i.e. collections $(\xi, \xi_\infty) : (E, E_\infty) \to (E', E'_\infty)$ of morphisms of $\mathcal{O}_X$-modules satisfying the obvious compatibility conditions with the ADHM data.

Let $\mathcal{C}_{\mathcal{X}}$ be the full abelian subcategory of $\mathcal{Q}_{\mathcal{X}}$ consisting of objects with $E_\infty = V \otimes \mathcal{O}_X$, where V is a finite dimensional vector spaces over K (possibly trivial.) Note that given any two objects $\mathcal{E}, \mathcal{E}'$ of $\mathcal{C}_{\mathcal{X}}$, the morphisms $\xi_\infty : V \otimes \mathcal{O}_X \to V' \otimes \mathcal{O}_X$ must be of the form $\xi_\infty = f \otimes 1_{\mathcal{O}_X}$, where $f : V \to V'$ is a linear map.

An object $\mathcal{E}$ of $\mathcal{C}_{\mathcal{X}}$ will be called locally free if E is a coherent locally free $\mathcal{O}_X$-module. Given a coherent $\mathcal{O}_X$-module E we will denote by $r(E)$, $d(E)$, $\mu(E)$ the rank, degree, respectively slope of E if $r(E) \neq 0$. The type of an object $\mathcal{E}$ of $\mathcal{C}_{\mathcal{X}}$ is the collection $(r(\mathcal{E}), d(\mathcal{E}), v(\mathcal{E})) = (r(E), d(E), \dim(V))) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. An object of $\mathcal{O}_X$ will be called an ADHM sheaf in the following. Throughout this paper, the integer $v(\mathcal{E})$ will be called the rank of $\mathcal{E}$, as opposed to the terminology used in [34, 33, 28], where the rank of $\mathcal{E}$ was defined to be $r(\mathcal{E})$. Note that the objects of $\mathcal{C}_{\mathcal{X}}$ with $v(\mathcal{E}) = 0$ form a full abelian category which is naturally equivalent to the abelian category of Higgs sheaves on X with coefficient bundles (M_1, M_2) (see for example [33, App. A] for brief summary of the relevant definitions.)

Let $\delta \in \mathbb{R}$ be a stability parameter. The δ-degree of an object $\mathcal{E}$ of $\mathcal{C}_{\mathcal{X}}$ is defined by

$$\deg_{\underline{\delta}}(\mathcal{E}) = d(\mathcal{E}) + \delta v(\mathcal{E}). \tag{4.2}$$

If $r(\mathcal{E}) \neq 0$, the δ-slope of $\mathcal{E}$ is defined by

$$\mu_\delta(\mathcal{E}) = \frac{\deg_\delta(\mathcal{E})}{r(\mathcal{E})}. \tag{4.3}$$

Definition 4.1.1. *Let $\delta \in \mathbb{R}$ be a stability parameter. A nontrivial object $\mathcal{E}$ of $\mathcal{C}_{\mathcal{X}}$ is δ-(semi)stable if*

$$r(E)\deg_\delta(\mathcal{E}')\ (\leq)\ r(E')\deg_\delta(\mathcal{E}) \tag{4.4}$$

for any proper nontrivial subobject $0 \subset \mathcal{E}' \subset \mathcal{E}$.

The following lemmas summarize some basic properties of δ-semistable ADHM sheaves. The proofs are either standard or very similar to those of [33, Lemm. 2.4], [33, Lemm 3.7] and will be omitted.

Lemma 4.1.2. *Suppose $\mathcal{E}$ is a δ-semistable framed ADHM sheaf with $r(\mathcal{E}) > 0$ for some $\delta \in \mathbb{R}$. Then*

(*i*) *E is locally free.*

(*ii*) *If $\delta > 0$, there is no nontrivial linear subspace $0 \subset V' \subseteq V$ so that $\psi|_{V'\otimes\mathcal{O}_X}$ is identically zero. Similarly, if $\delta < 0$, there is no proper linear subspace $0 \subseteq V' \subset V$ so that $\mathrm{Im}(\phi) \subseteq V' \otimes \mathcal{O}_X$.*

(*iii*) *If $\mathcal{E}$ is δ-stable any endomorphism of $\mathcal{E}$ in $\mathcal{C}_{\mathcal{X}}$ is either trivial or an isomorphism. If the ground field K is algebraically closed, the endomorphism ring of $\mathcal{E}$ is canonically isomorphic to K.*

Lemma 4.1.3. *For fixed $(r, e, v) \in \mathbb{Z}_{>0} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ there is a constant $c \in \mathbb{R}$ (depending only on $\mathcal{X}$ and (r,e,v)) so that for any $\delta \in \mathbb{R}$, any δ-semistable framed ADHM sheaf of type (r, e, v) satisfies*

$$\mu_{\max}(\mathrm{E}) < c.$$

In particular, the set of isomorphism classes of framed ADHM sheaves of fixed type (r, e, v) which are δ-semistable for some $\delta \in \mathbb{R}$ is bounded.

Given a locally free ADHM sheaf $\mathcal{E} = (E, \Phi_1, \Phi_2, \phi, \psi)$ on X of type $(r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$, the data

$$\begin{aligned} \widetilde{E} &= E^\vee \otimes_X M^{-1} \\ \widetilde{\Phi}_i &= (\Phi_i^\vee \otimes 1_{M_i}) \otimes 1_{M^{-1}} : \widetilde{E} \otimes M_i \to \widetilde{E} \\ \widetilde{\phi} &= \psi^\vee \otimes 1_{M^{-1}} : \widetilde{E} \otimes_X M \to V^\vee \otimes \mathcal{O}_X \\ \widetilde{\psi} &= \phi^\vee : V^\vee \otimes \mathcal{O}_X \to \widetilde{E} \end{aligned} \tag{4.5}$$

with $i = 1, 2$, determines a locally free ADHM sheaf $\widetilde{\mathcal{E}}$ of type $(r, -e + 2r(g-1), v)$ where g is the genus of X. $\widetilde{\mathcal{E}}$ will be called the dual of $\mathcal{E}$ in the following. Then the following lemma is straightforward.

Lemma 4.1.4. *Let $\delta \in \mathbb{R}$ be a stability parameter and let $\mathcal{E}$ be a locally free ADHM sheaf on X. Then $\mathcal{E}$ is δ-(semi)stable if and only if $\widetilde{\mathcal{E}}$ is $(-\delta)$-(semi)stable.*

4.1.2 Chamber structure

This subsection summarizes the main properties of δ-stability chambers.

Definition 4.1.5. *An ADHM sheaf $\mathcal{E}$ of type $(r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ is asymptotically (semi)stable if the following conditions hold*

(i) E is locally free, $\psi : V \otimes \mathcal{O}_X \to E$ is not identically zero, and there is no saturated proper nontrivial subobject $0 \subset \mathcal{E}' \subset \mathcal{E}$ in $\mathcal{C}_\mathcal{X}$ so that $v(\mathcal{E}')/r(\mathcal{E}') > v/r$.

(ii) Any proper nontrivial subobject $0 \subset \mathcal{E}' \subset \mathcal{E}$ with $v(\mathcal{E}')/r(\mathcal{E}') = v/r$ satisfies the slope inequality $\mu(E')$ $(\leq)$ $\mu(E)$.

Here a subobject $\mathcal{E}' \subset \mathcal{E}$ is called saturated in the underlying coherent sheaf E' is saturated in E. Note that according to [33, Lemm. 3.10], any proper subobject $0 \subset \mathcal{E}' \subset \mathcal{E}$ admits a canonical saturation $\overline{\mathcal{E}'} \subset \mathcal{E}$.

Lemma 4.1.6. *The set of isomorphism classes of asymptotically semistable ADHM sheaves of fixed type $(r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 1}$ is bounded.*

Proof. The proof is based on Maruyama's boundedness theorem. Suppose $\mathcal{E}$ is asymptotically semistable of type (r, e, v), and the underlying coherent sheaf E is not semistable. Then there is a nontrivial Harder-Narasimhan filtration

$$0 \subset E_1 \subset \cdots \subset E_h = E$$

with $h \geq 2$ so that $\mu(E_j) > \mu(E)$ and $r(E_j) < r$ for all $1 \leq j \leq h-1$. Suppose E_j is Φ_i-invariant, $i = 1, 2$, and $\mathrm{Im}(\psi) \subseteq E_j$ for some $1 \leq j \leq h-1$. Then the data $\mathcal{E}_j = (E_j, \Phi_i|_{E_j \otimes_X M_i}, \phi|_{E_j \otimes_X M}, \psi)$ is subobject of $\mathcal{E}$ with

$$v(\mathcal{E}_j)/r(\mathcal{E}_j) = \frac{v}{r(\mathcal{E}_j)} > \frac{v}{r}.$$

Since $E_j \subset E$ is saturated, it follows that $\mathcal{E}_j$ violates condition (i) in definition (4.1.5). Therefore for any $1 \leq j \leq h$, E_j is either not preserved by some Φ_i, $i = 1, 2$, or it does not contain the image of ψ. From this point on the proof is identical to the proof of [34, Prop. 2.7].

□

Definition 4.1.7. *Let $\delta \in \mathbb{R}_{>0}$. A δ-semistable ADHM sheaf $\mathcal{E}$ of type $(r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ is generic if it is either δ-stable or any proper nontrivial subobject $0 \subset \mathcal{E}' \subset \mathcal{E}$ of type $(r', e', v') \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ satisfies*

$$\frac{e'}{r'} = \frac{e}{r} \qquad \frac{v'}{r'} = \frac{v}{r}. \tag{4.6}$$

The stability parameter $\delta \in \mathbb{R}_{>0}$ is called generic of type (r, e, v) if any δ-semistable ADHM sheaf of type (r, e, v) is generic. The stability parameter $\delta \in \mathbb{R}_{>0}$ is called critical of type (r, e, v) if there exists a nongeneric δ-semistable ADHM sheaf of type (r, e, v).

Lemma (4.1.3) implies the following.

Lemma 4.1.8. *For fixed $(r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 1}$ there exists $\delta_\infty \in \mathbb{R}_{>0}$ so that for all $\delta \geq \delta_\infty$ an ADHM sheaf $\mathcal{E}$ of type (r, e, v) is δ-(semi)stable if and only if it is asymptotically (semi)stable.*

Proof. The proof if similar to the proof of lemma [33, Lemm. 4.7]. Some details will be provided for convenience. It is straightforward to prove that asymptotic stability implies δ-stability for sufficiently large δ using lemma (4.1.3). The converse is slightly more involved. First note that given any nontrivial locally free ADHM sheaf $\mathcal{E}$, any linear subspace $V' \subset V$, determines a canonical subobject $\mathcal{E}_{V'} \subset \mathcal{E}$. $\mathcal{E}_{V'}$ is the saturation of the subobject of $\mathcal{E}$ generated by $V' \otimes \mathcal{O}_X$ by successive applications of the ADHM morphisms ψ, Φ_i, ϕ. Since $\mathcal{E}_{V'}$ is canonically determined by V' and $\mathcal{E}$, lemma (4.1.3) implies that the set of isomorphism classes of subobjects $\mathcal{E}_{V'}$, where $\mathcal{E}$ is a δ-semistable ADHM sheaf of type (r, e, v) for some $\delta > 0$ is bounded. Moreover, by construction, any subobject $0 \subset \mathcal{E}' \subset \mathcal{E}$ contains the canonical subobject $\mathcal{E}_{V'}$.

Now suppose that for any $\delta > 0$ there exists a δ-semistable ADHM sheaf $\mathcal{E}$ of type (r, e, v) which is not asymptotically stable. Let $0 \subset \mathcal{E}' \subset \mathcal{E}$ be a saturated nontrivial proper saturated subobject violating the asymptotic stability conditions. Note that $\mathcal{E}'$ cannot violate condition

(ii) in definition (4.1.5) since $\mathcal{E}$ is δ-semistable. Therefore it must violate condition (i) i.e. $v'/r' > v/r$ where $r' = r(\mathcal{E}')$. In particular $v' = v(\mathcal{E}') > 0$. Then the subobject $\mathcal{E}_{V'}$ also violates condition (i) since

$$\frac{v(\mathcal{E}_{V'})}{r(\mathcal{E}_{V'})} = \frac{v'}{r(\mathcal{E}_{V'})} \geq \frac{v'}{r'} > v/r.$$

Since $\mathcal{E}$ is δ-semistable $\mu_\delta(\mathcal{E}_{V'}) \leq \mu_\delta(\mathcal{E})$. However, as noted above, the set of isomorphism classes of all $\mathcal{E}_{V'}$ is bounded, therefore the set of all types $(r(\mathcal{E}_{V'}), d(\mathcal{E}_{V'}), v(\mathcal{E}_{V'}))$ is finite. Taking δ sufficiently large, this leads to a contradiction.

□

By analogy with [33, Lemm. 4.4], [33, Lemm. 4.6], lemmas (4.1.8) and (4.1.4) imply the following.

Lemma 4.1.9. *Let $(r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 1}$ be a fixed type. Then there is a finite set $\Delta(r, e, v) \subset \mathbb{R}$ of critical stability parameters of type (r, e, v). Given any two stability parameters $\delta, \delta' \in \mathbb{R}$, $\delta < \delta'$ so that $[\delta,\ \delta'] \cap \Delta(r, e, v) = \emptyset$, the set of δ-semistable ADHM sheaves of type (r, e, v) is identical to the set of δ'-semistable ADHM sheaves of type (r, e, v).*

Remark 4.1.10. *It is straightforward to check that $\Delta(1, e, v) = \{0\}$ for any $v \geq 1$.*

Lemma 4.1.11. *Let $(r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 1}$ and let $\delta_c > 0$ be a critical stability parameter of type (r, e, v). Let $\delta_\pm > 0$ be stability parameters so that $\delta_- < \delta_c < \delta_+$ and $[\delta_-,\ \delta_c) \cap \Delta(r, e, v) = \emptyset$, $(\delta_c,\ \delta_+] \cap \Delta(r, e, v) = \emptyset$. If $\mathcal{E}$ is a $\delta_\pm$-semistable ADHM sheaf of type (r, e, v), then $\mathcal{E}$ is also δ_c-semistable.*

Definition 4.1.12. *Let $(r, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$.*

(a) A positive admissible configuration of type (r, v) is an ordered sequence of integral points $(\rho_i = (r_i, v_i) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0})_{1 \leq i \leq h,\, h \geq 1}$ satisfying the following conditions

- $\rho_1 + \cdots + \rho_h = (r, v)$.
- $(v_1 + \cdots + v_i)/(r_1 + \cdots + r_i) > v/r$ *and* $v_i/r_i > v_{i+1}/r_{i+1}$ *for all* $i = 1, \ldots, h-1$.

(b) A negative admissible configuration of type (r, v) is an ordered sequence of integral points $(\rho_i = (r_i, v_i) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0})_{1 \leq i \leq h,\, h \geq 1}$ satisfying the following conditions

- $\rho_1 + \cdots + \rho_h = (r, v)$.
- $(v_1 + \cdots + v_i)/(r_1 + \cdots + r_i) < v/r$ *and* $v_i/r_i < v_{i+1}/r_{i+1}$ *for all* $i = 1, \ldots, h-1$.

Remark 4.1.13. *(i) It is straightforward to prove that for fixed $(r, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ the set of positive, respectively negative, admissible configurations is finite. These sets will be denoted by $\mathcal{HN}_\pm(r, v)$.*

(ii) The only positive, respectively negative admissible configuration of type (r,v) with $h=1$ is $(\rho=(r,v))$.

Lemma 4.1.14. *Let $\delta_c \in \mathbb{R}_{>0}$ be a critical stability parameter of type $(r,e,v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 1}$. Then the following hold.*

(i) There exists $\epsilon_+ > 0$, so that $(\delta_c,\ \delta_c + \epsilon_+] \cap \Delta(r,e,v) = \emptyset$ and the following holds for any $\delta_+ \in (\delta_c,\ \delta_c + \epsilon_+)$. A locally free ADHM sheaf $\mathcal{E}$ of type (r,e,v) on X is δ_c-semistable if and only if it is either δ_+-semistable or there exists a unique filtration of the form

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_h = \mathcal{E} \tag{4.7}$$

with $h \geq 2$ satisfying the following conditions

- *The successive quotients $\mathcal{F}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$, $i = 1,\ldots,h$ of the filtration (4.7) are locally free ADHM sheaves with numerical types $(r_i,e_i,v_i) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. δ_+ is noncritical of type (r_i,e_i,v_i), $\mathcal{F}_i$ is δ_+-semistable and $\mu_{\delta_c}(\mathcal{F}_i) = \mu_{\delta_c}(\mathcal{E})$ for all $i = 1,\ldots,h$.*
- *The sequence $\rho_i = (r_i,v_i)$, $i = 1,\ldots,h$ is a positive admissible configuration of type (r,e,v).*

(ii) There exists $\epsilon_- > 0$, so that $[\delta_c - \epsilon_-, \delta_c) \cap \Delta(r,e,v) = \emptyset$ and the following holds for any $\delta_- \in (\delta_c - \epsilon_-,\ \delta_c)$. A locally free ADHM sheaf $\mathcal{E}$ of type (r,e,v) on X is δ_c-semistable if and only if it is either δ_--semistable or there exists a unique filtration of the form

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_h = \mathcal{E} \tag{4.8}$$

with $h \geq 2$ satisfying the following conditions

- *The successive quotients $\mathcal{F}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$, $i = 1,\ldots,h$ of the filtration (4.8) are locally free ADHM sheaves with numerical types $(r_i,e_i,v_i) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. δ_- is noncritical of type (r_i,e_i,v_i), $\mathcal{F}_i$ is δ_--semistable and $\mu_{\delta_c}(\mathcal{F}_i) = \mu_{\delta_c}(\mathcal{E})$ for all $i = 1,\ldots,h$.*
- *The sequence $\rho_i = (r_i,v_i)$, $i = 1,\ldots,h$ is a negative admissible configuration of type (r,e,v).*

Proof. The proof is similar to the proof of [33, Lemm. 4.13]. Details are included below for completeness. Note that it suffices to prove statement (i) since the proof of (ii) is analogous.

Let $\delta_+ > \delta_c$ be an arbitrary noncritical stability parameter of type (r,e,v) so that $(\delta_c, \delta_+] \cap \Delta(r,e,v) = \emptyset$. Suppose $\mathcal{E}$ is a δ_c-semistable ADHM sheaf on X. Then $\mathcal{E}$ is either δ_+-stable or there is a Harder-Narasimhan filtration of $\mathcal{E}$ with respect to δ_+-semistability

$$0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_h = \mathcal{E} \tag{4.9}$$

where $h \geq 2$. It is straightforward to check that $\mathcal{E}_l$, $1 \leq l \leq h$ must have $r(\mathcal{E}_l) \geq 1$ and the successive quotients $\mathcal{F}_l$, $0 \leq l \leq h-1$ must also have $r_l \geq 1$. Then by the general properties of Harder-Narasimhan filtrations

$$\mu_{\delta_+}(\mathcal{E}_1) > \mu_{\delta_+}(\mathcal{E}_2/\mathcal{E}_1) > \cdots > \mu_{\delta_+}(\mathcal{E}_h/\mathcal{E}_{h-1}) \tag{4.10}$$

and

$$\mu_{\delta_+}(\mathcal{E}_l) > \mu_{\delta_+}(\mathcal{E}) \tag{4.11}$$

for all $1 \leq l \leq h-1$. Since $\mathcal{E}$ is δ_c-semistable by assumption, inequalities (4.11) imply that

$$v(\mathcal{E}_l)/r(\mathcal{E}_l) > v/r \tag{4.12}$$

for all $l = 1, \ldots, h$. Note that $v(\mathcal{E}_l) = v_1 + \cdots + v_l$, $r(\mathcal{E}_l) = r_1 + \cdots + r_l$ for any $l = 1, \ldots, h$.

Moreover, using the δ_c-semistability condition and inequalities (4.11) we have

$$\delta_+ \left(\frac{v}{r} - \frac{v(\mathcal{E}_l)}{r(E_l)} \right) < \mu(E_l) - \mu(E) \leq \delta_c \left(\frac{v}{r} - \frac{v(\mathcal{E}_l)}{r(E_l)} \right) \tag{4.13}$$

for all $l = 1, \ldots, h$.

Now let $\gamma > \delta_c$ be a fixed stability parameter so that $(\delta_c, \gamma] \cap \Delta(r, e, v) = \emptyset$. Using Grothendieck's lemma and lemma (4.1.3), inequalities (4.13) imply that the set of isomorphism classes of locally free ADHM sheaves $\mathcal{E}'$ on X satisfying condition $(\star)$ below is bounded.

($\star$) There exists a δ_c-semistable ADHM sheaf $\mathcal{E}$ of type (r, e, v) and a stability parameter $\delta_+ \in (\delta_c\ \gamma]$ so that $\mathcal{E}' \simeq \mathcal{E}_l$ for some $l \in \{1, \ldots, h\}$, where $0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_h = \mathcal{E}$, $h \geq 1$, is the Harder-Narasimhan filtration of $\mathcal{E}$ with respect to δ_+-semistability.

Then it follows that the set of numerical types (r', e', v') of locally free ADHM sheaves $\mathcal{E}'$ satisfying property $(\star)$ is finite. This implies that there exists $0 < \epsilon_+ < \gamma - \delta_c$ so that for any $\delta_+ \in (\delta_c,\ \delta_c + \epsilon_+)$, and any δ_c-semistable ADHM sheaf $\mathcal{E}$ of type (r, e, v) inequalities (4.13) can be satisfied only if

$$\mu_{\delta_c}(\mathcal{E}_l) = \mu_{\delta_c}(\mathcal{E}) \tag{4.14}$$

for all $l = 1, \ldots, h$. Hence also

$$\mu_{\delta_c}(\mathcal{E}_l/\mathcal{E}_{l-1}) = \mu_{\delta_c}(\mathcal{E})$$

for all $l = 2, \ldots, h$. Then inequalities (4.10), (4.12) imply that the sequence $\rho_l = (r_l, v_l)$, $l = 1, \ldots, h$ is a positive admissible configuration. Therefore for all $\delta_+ \in (\delta_c,\ \delta_c + \epsilon_+)$, any locally free δ_c-semistable ADHM sheaf $\mathcal{E}$ of type (r, e, v) is either δ_+-stable or has a Harder-Narasimhan filtration with respect to δ_+-semistability as in lemma (4.1.14.i).

Next note that the set of numerical types

$$\mathsf{S}_{\delta_c}(r,e,v) = \{(r',e',v') \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0} \,|\, 0 < r' \leq r,\ 0 \leq v' \leq v,\ r(e'+\delta_c v') = r'(e+\delta_c v)\} \tag{4.15}$$

is finite. Therefore $0 < \epsilon_+ < \gamma - \delta_i$ above may be chosen so that there are no critical stability parameters of type (r', e', v') in the interval $(\delta_c,\ \delta_c + \epsilon_+)$ for any $(r', e', v') \in \mathsf{S}_{\delta_c}(r, e, v)$. In particular, δ_+ is noncritical of type (r_i, e_i, v_i), $i = 1, \ldots, h$ for any Harder-Narasihan filtration as above.

Conversely, suppose $\mathcal{E}$ is a locally free ADHM sheaf of type (r, e, v) on X which has a filtration of the form (4.7) with $\mathcal{E}'$ δ_+-stable and satisfying the conditions of lemma (4.1.14.i) for some $\delta_+ \in (\delta_c,\ \delta_c + \epsilon_+)$. By the above choice of ϵ_+, there are no critical stability parameters of type (r_i, e_i, v_i) in the interval $(\delta_c,\ \delta_c + \epsilon_+)$, for any $i = 1, \ldots, h$. Since $\mathcal{F}_i$ are δ_+-semistable, lemma (4.1.11) implies that $\mathcal{F}_i$ is also δ_c-semistable, for any $i = 1, \ldots, h$. Hence $\mathcal{E}$ is also δ_c-semistable since the $\mathcal{F}_i$ have equal δ_c-slopes.

□

4.1.3 Extension groups

Let $\mathcal{E}', \mathcal{E}''$ be nontrivial locally free objects in $\mathcal{C}_{\mathcal{X}}$ of types $(r', e', v'), (r'', e'', v'') \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. Let $\mathcal{C}(\mathcal{E}'', \mathcal{E}')$ be the three term complex

$$0 \to \mathcal{H}om_X(E'', E') \xrightarrow{d_1} \begin{array}{c} \mathcal{H}om_X(E'' \otimes_X M_1, E') \\ \oplus \\ \mathcal{H}om_X(E'' \otimes_X M_2, E') \\ \oplus \\ \mathcal{H}om_X(E'' \otimes_X M, V' \otimes \mathcal{O}_X) \\ \oplus \\ \mathcal{H}om_X(V'' \otimes \mathcal{O}_X, E') \end{array} \xrightarrow{d_2} \mathcal{H}om_X(E'' \otimes_X M, E') \to 0 \tag{4.16}$$

where

$$\begin{aligned} d_1(\alpha) = (&-\alpha \circ \Phi_1'' + \Phi_1' \circ (\alpha \otimes 1_{M_1}), -\alpha \circ \Phi_2'' + \Phi_2' \circ (\alpha \otimes 1_{M_2}), \\ &\phi' \circ (\alpha \otimes 1_M), -\alpha \circ \psi'') \end{aligned}$$

for any local sections (α, α_∞) of the first term and

$$\begin{aligned} d_2(\beta_1, \beta_2, \gamma, \delta) =& \beta_1 \circ (\Phi_2'' \otimes 1_{M_1}) - \Phi_2' \circ (\beta_1 \otimes 1_{M_2}) - \beta_2 \circ (\Phi_1'' \otimes 1_{M_2}) \\ &+ \Phi_1' \circ (\beta_2 \otimes 1_{M_1}) + \psi' \circ \gamma + \delta \circ \phi'' \end{aligned}$$

for any local sections $(\beta_1, \beta_2, \gamma, \delta)$ of the middle term. The degrees of the three terms in (4.16) are $0, 1, 2$ respectively.

Let $C(\mathcal{C}(\mathcal{E}'', \mathcal{E}'))$ be the double complex obtained from $\mathcal{C}(\mathcal{E}'', \mathcal{E}')$ by taking Čech resolutions and let $D(\mathcal{E}', \mathcal{E}'')$ be the diagonal complex of $C(\mathcal{C}(\mathcal{E}'', \mathcal{E}'))$. Note that there is a canonical linear map

$$\mathrm{Hom}(V'', V') \to D^1(\mathcal{E}', \mathcal{E}'') = C^0(\mathcal{C}^1(\mathcal{E}'', \mathcal{E}')) \oplus C^1(\mathcal{C}^0(\mathcal{E}'', \mathcal{E}'))$$
$$f \to \begin{bmatrix} {}^t(0, 0, -(f \otimes 1_{\mathcal{O}_X}) \circ \phi'', \psi' \circ (f \otimes 1_{\mathcal{O}_X})) \\ 0 \end{bmatrix}$$

Given the above expressions for the differentials d_1, d_2 it is straightforward to check that this map yields a morphism of complexes

$$\varrho : \mathrm{Hom}(V'', V')[-1] \to D(\mathcal{E}'', \mathcal{E}')$$

Let $\widetilde{D}(\mathcal{E}'', \mathcal{E}')$ denote the cone of ϱ. Then the lemma below follows either by explicit Čech cochain computations as in [34, Sect. 4] or using the methods of [43].

Lemma 4.1.15. *The extension groups* $\mathrm{Ext}^k_{\mathcal{C}_\mathcal{X}}(\mathcal{E}'', \mathcal{E}')$, $k = 0, 1$ *are isomorphic to the cohomology groups* $H^k(\widetilde{D}(\mathcal{E}'', \mathcal{E}'))$, $k = 0, 1$. *Moreover there is an exact sequence*

$$0 \longrightarrow \mathbb{H}^0(\mathcal{C}(\mathcal{E}'', \mathcal{E}')) \longrightarrow \mathrm{Ext}^0_{\mathcal{C}_\mathcal{X}}(\mathcal{E}'', \mathcal{E}') \longrightarrow Hom(V'', V') \tag{4.17}$$

$$\longrightarrow \mathrm{Ext}^1_{\mathcal{C}_\mathcal{X}}(\mathcal{E}'', \mathcal{E}') \longrightarrow \mathbb{H}^1(\mathcal{C}(\mathcal{E}'', \mathcal{E}')) \longrightarrow 0$$

where $\mathbb{H}^k(\mathcal{C}(\mathcal{E}'', \mathcal{E}'))$, $k = 0, 1$ *are hypercohomology groups of the complex* $\mathcal{C}(\mathcal{E}'', \mathcal{E}')$.

Corollary 4.1.16. *Given any two locally free objects* $\mathcal{E}', \mathcal{E}''$

$$\begin{aligned} &\dim(\mathrm{Ext}^0_{\mathcal{C}_\mathcal{X}}(\mathcal{E}'', \mathcal{E}')) - \dim(\mathrm{Ext}^1_{\mathcal{C}_\mathcal{X}}(\mathcal{E}'', \mathcal{E}')) - \dim(\mathrm{Ext}^0_{\mathcal{C}_\mathcal{X}}(\mathcal{E}', \mathcal{E}'')) \\ &+ \dim(\mathrm{Ext}^1_{\mathcal{C}_\mathcal{X}}(\mathcal{E}', \mathcal{E}'')) = v'e'' - v''e' - (v'r'' - v''r')(g-1) \end{aligned} \tag{4.18}$$

Proof. Follows from the exact sequence (4.17) and the fact that the hypercohomology groups of the complex $\mathcal{C}(\mathcal{E}'', \mathcal{E}')$ satisfy the duality relation

$$\mathbb{H}^k(\mathcal{C}(\mathcal{E}'', \mathcal{E}')) \simeq \mathbb{H}^{3-k}(\mathcal{C}(\mathcal{E}', \mathcal{E}''))^\vee$$

for $k = 0, \ldots, 3$.

□

4.1.4 Moduli stacks

In the following let the ground field K be $\mathbb{C}$. Let $\mathfrak{Ob}(\mathcal{X})$ denote the moduli stack of all objects of the abelian category $\mathcal{C}_\mathcal{X}$ and let $\mathfrak{Ob}(\mathcal{X}, r, e, v)$ denote the open and closed component of type

$(r,e,v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. Standard arguments analogous to [61, Sect. 9], [61, Sect. 10] prove that $\mathfrak{Ob}(\mathcal{X})$ is an algebraic stack locally of finite type and it satisfies conditions [61, Assumption 7.1], [61, Assumption 8.1]. Given the boundedness result (4.1.3), the following is also standard.

Proposition 4.1.17. *For fixed type $(r,e,v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ and fixed $\delta \in \mathbb{R}_{>0}$ there is an algebraic moduli stack of finite type $\mathfrak{M}^{ss}_{\delta}(\mathcal{X}, r, e, v)$ of δ-semistable objects of type (r,e,v) of $\mathcal{C}_{\mathcal{X}}$. If $\delta < \delta'$ are two stability parameters so that $[\delta,\ \delta'] \cap \Delta(r,e,v) = \emptyset$, the corresponding moduli stacks are canonically isomorphic. Moreover, for any $\delta \in \mathbb{R}$ there are canonical open embeddings*

$$\mathfrak{M}^{ss}_{\delta}(\mathcal{X}, r, e, v) \hookrightarrow \mathfrak{Ob}(\mathcal{X}, r, e, v) \hookrightarrow \mathfrak{Ob}(\mathcal{X}). \tag{4.19}$$

4.1.5 ADHM invariants

ADHM invariants will be defined applying the formalism of Joyce and Song [65] to δ-semistable ADHM sheaves on X. Given corollary (4.1.16), the required results on Behrend constructible functions are a straightforward generalization of the analogous statements proven in [33, Sect. 7] for ADHM sheaves with $v = 1$. Therefore the construction of generalized Donaldson-Thomas invariants via Behrend's constructible functions [65] applies to the present case.

Let $\mathsf{L}(\mathcal{X})$ be the Lie algebra over $\mathbb{Q}$ spanned by $\{\lambda(\gamma) \,|\, \gamma \in \mathbb{Z}^3\}$ with Lie bracket

$$[\lambda(\gamma'), \lambda(\gamma'')] = (-1)^{\chi(\gamma',\gamma'')} \chi(\gamma', \gamma'') \lambda(\gamma' + \gamma'')$$

where

$$\chi(\gamma', \gamma'') = v''e' - v'e'' - (v''r' - v'r'')(g-1)$$

for any $\gamma' = (r', e', v')$, $\gamma'' = (r'', e'', v'')$. Then there is a Lie algebra morphism

$$\Psi : \mathsf{SF}^{\mathsf{ind}}_{\mathsf{al}}(\mathfrak{Ob}(\mathcal{X})) \to \mathsf{L}(\mathcal{X}) \tag{4.20}$$

so that for any stack function of the form $[(\mathfrak{X}, \rho)]$, whith $\rho : \mathfrak{X} \hookrightarrow \mathfrak{Ob}(\mathcal{X}, \gamma) \hookrightarrow \mathfrak{Ob}(\mathcal{X})$ an open embedding, and $\mathfrak{X}$ a $\mathbb{C}^{\times}$-gerbe over an algebraic space X,

$$\Psi([(\mathfrak{X}, \rho)]) = -\chi^B(\mathsf{X}, \rho^*\nu)\lambda(\gamma)$$

where ν is Behrend's constructible function of the stack $\mathfrak{Ob}(\mathcal{X})$.

In order to define ADHM invariants note that for any $\delta \in \mathbb{R}$, the canonical open embedding stack $\mathfrak{M}^{ss}_{\delta}(\mathcal{X}, \gamma) \hookrightarrow \mathfrak{Ob}(\mathcal{X})$ determines a stack function $\mathfrak{d}_{\delta}(\gamma) \in \underline{\mathsf{SF}}(\mathfrak{Ob}(\mathcal{X}))$. For $v = 0$, the resulting stack functions are independent of stability parameters and will be denoted by $\mathfrak{h}(\gamma)$.

According to [63, Thm. 8.7] the associated log stack function

$$\mathfrak{e}_{\delta}(\gamma) = \sum_{l \geq 1} \frac{(-1)^{l-1}}{l} \sum_{\substack{\gamma_1 + \cdots + \gamma_l = \gamma \\ \mu_{\delta}(\gamma_i) = \mu_{\delta}(\gamma),\ 1 \leq i \leq l}} \mathfrak{d}_{\delta}(\gamma_1) * \cdots * \mathfrak{d}_{\delta}(\gamma_l) \tag{4.21}$$

belongs to $\mathsf{SF}_{\mathsf{al}}^{\mathsf{ind}}(\mathfrak{Ob}(\mathcal{X}))$, and is supported in $\mathfrak{Ob}(\mathcal{X},\gamma)$. Note that for fixed γ and δ the sum in the right hand side is finite, therefore there are no convergence issues in the present case.

Then, for $\gamma \in \mathbb{Z}_{\geq 1}\times\mathbb{Z}\times\mathbb{Z}_{\geq 0}$, the δ-ADHM invariant $A_\delta(\gamma)$ is defined by

$$\Psi(\mathfrak{e}_\delta(\gamma)) = -A_\delta(\gamma)\lambda(\gamma). \tag{4.22}$$

Note that $\mathfrak{e}_\delta(\gamma)$ is independent of δ if $v=0$. Then the corresponding invariants will be denoted by $H(\gamma)$.

By analogy with [65], define the invariants $\overline{A}_\delta(r,e,v)$ by the multicover formula

$$\overline{A}_\delta(r,e,v) = \sum_{\substack{m\geq 1\\ m|r,\ m|e,\ m|v}} \frac{1}{m^2}\overline{A}_\delta(r/m,e/m,v/m). \tag{4.23}$$

Conjecturally, $\overline{A}_\delta(r/m,e/m,v/m)$ are integral. Obviously, for $v=0$ the alternative notation $\overline{H}(r,e)$ will be used.

4.2 Wallcrossing formulas

4.2.1 Stack function identities

Let $\gamma=(r,e,v)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}\times\mathbb{Z}_{\geq 1}$ and let $\delta_c>0$ be a critical stability parameter of type γ. Let $\delta_-<\delta_c$, $\delta_+>\delta_c$ be stability parameters as in lemma (4.1.14). Recall that $\mathcal{HN}_\pm(r,v)$ denote the set of positive, respectively negative admissible configurations of type (r,v) introduced in definition (4.1.12). For any $h\in\mathbb{Z}_{\geq 2}$ let $\mathcal{HN}_\pm(\gamma,\delta_c,h)$ denote the set of ordered sequences of triples $(\gamma_i=(r_i,e_i,v_i)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}\times\mathbb{Z}_{\geq 0})_{1\leq i\leq h}$ so that $(\rho_i=(r_i,v_i))_{1\leq i\leq h}\in\mathcal{HN}_\pm(r,v)$,

$$e_1+\cdots+e_h=e \qquad \text{and} \qquad \frac{e_i+v_i\delta_c}{r_i}=\frac{e+v\delta_c}{r} \qquad \text{for all } 1\leq i\leq h.$$

More generally, given $h\in\mathbb{Z}_{\geq 2}$, for any $0\leq k\leq h-1$ let $\mathcal{HN}_+(\gamma,\delta_c,h,k)$ denote the set of ordered sequences $(\gamma_i=(r_i,e_i,v_i)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}\times\mathbb{Z}_{\geq 0})_{1\leq i\leq h}$ so that

- $\gamma_1+\cdots+\gamma_h=\gamma$, $v_{h-k+1}=\cdots=v_h=0$, $v_i>0$ for $1\leq i\leq h-k$, and

$$\frac{e_1+v_1\delta_c}{r_1}=\cdots=\frac{e_{h-k}+v_{h-k}\delta_c}{r_{h-k}}=\frac{e_{h-k+1}}{r_{h-k+1}}=\cdots=\frac{e_h}{r_h}=\frac{e+v\delta_c}{r}$$

- The sequence $(\rho_j=(r_j,v_j))_{1\leq j\leq h-k}$ belongs to $\mathcal{HN}_+\left(r-\sum_{i=1}^k r_i, v\right)$.

Similarly, for any $0\leq k\leq h-1$ let $\mathcal{HN}_-(\gamma,\delta_c,h,k)$ denote the set of ordered sequences $(\gamma_i=(r_i,e_i,v_i)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}\times\mathbb{Z}_{\geq 0})_{1\leq i\leq h}$ so that

- $\gamma_1 + \cdots + \gamma_h = \gamma$, $v_1 = \cdots = v_k = 0$, $v_i > 0$ for $k+1 \leq i \leq h$, and

$$\frac{e_1}{r_1} = \cdots = \frac{e_k}{r_k} = \frac{e_{k+1} + v_{k+1}\delta_c}{r_{k+1}} = \cdots = \frac{e_h + v_h\delta_c}{r_h} = \frac{e + v\delta_c}{r}$$

- The sequence $(\rho_j = (r_{k+j}, v_{k+j}))_{1\leq j\leq h-k}$ belongs to $\mathcal{HN}_-\left(r - \sum_{i=1}^k r_i, v\right)$.

Remark 4.2.1. *(i) Obviously, in both cases $v_i > 0$ for all $1 \leq i \leq h$ if $k = 0$. Moreover,*

$$\mathcal{HN}_\pm(\gamma, \delta_c, h) = \mathcal{HN}_\pm(\gamma, \delta_c, h, 0) \cup \mathcal{HN}_\pm(\gamma, \delta_c, h, 1).$$

If $k = h-1$ the condition that the sequence $(\rho_j)_{1\leq j\leq h-k}$ belong to $\mathcal{HN}_\pm\left(r - \sum_{i=1}^k r_i, v\right)$ is empty.

(ii) For fixed γ and $\delta_c > 0$ it straightforward to check that the following set is finite

$$\bigcup_{h\geq 2} \bigcup_{0\leq k\leq h-1} \mathcal{HN}_\pm(\gamma, \delta_c, h, k),$$

i.e. the set $\mathcal{HN}_\pm(\gamma, \delta_c, h, k)$ is nonempty only for a finite set of pairs (h, k).

For any triple $\gamma' = (r', e', v') \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 1}$ let $\mathfrak{d}_\pm(\gamma'), \mathfrak{d}_c(\gamma')$ be the stack functions determined by the open embeddings $\mathfrak{M}^{ss}_{\delta_\pm}(\mathcal{X}, r', e', v') \hookrightarrow \mathfrak{Ob}(\mathcal{X})$, respectively $\mathfrak{M}^{ss}_{\delta_c}(\mathcal{X}, r', e', v') \hookrightarrow \mathfrak{Ob}(\mathcal{X})$. The alternative notation $\mathfrak{h}(\gamma')$ will be used if $v' = 0$.

Lemma 4.2.2. *The following relations hold in the stack function algebra $\underline{\mathsf{SF}}(\mathfrak{Ob}(\mathcal{X}))$*

$$\mathfrak{d}_c(\gamma) = \mathfrak{d}_\pm(\gamma) + \sum_{h\geq 2} \sum_{(\gamma_i)\in\mathcal{HN}_\pm(\gamma,\delta_c,h)} \mathfrak{d}_\pm(\gamma_1) * \cdots * \mathfrak{d}_\pm(\gamma_h) \tag{4.24}$$

$$\begin{aligned} &\mathfrak{d}_-(\gamma) + \sum_{h\geq 2} \sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,h,0)} \mathfrak{d}_-(\gamma_1) * \cdots * \mathfrak{d}_-(\gamma_h) = \\ &\mathfrak{d}_c(\gamma) + \sum_{h\geq 2} (-1)^{h-1} \sum_{(\gamma_i)\in\mathcal{HN}(\gamma,\delta_c,h,h-1)} \mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_{h-1}) * \mathfrak{d}_c(\gamma_h) \end{aligned} \tag{4.25}$$

Proof. Equation (4.24) follows directly from lemma (4.1.14). In order to prove formula (4.25) it will be first proven by induction that the following formula holds for any $l \in \mathbb{Z}_{\geq 1}$.

$$\begin{aligned} &\mathfrak{d}_-(\gamma) + \sum_{h\geq 2} \sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,h,0)} \mathfrak{d}_-(\gamma_1) * \cdots * \mathfrak{d}_-(\gamma_h) = \\ &\mathfrak{d}_c(\gamma) + \sum_{k=2}^{l} (-1)^{k-1} \sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,k,k-1)} \mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_{k-1}) * \mathfrak{d}_c(\gamma_k) \\ &+ (-1)^l \sum_{h\geq l+1} \sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,h,l)} \mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_l) * \mathfrak{d}_-(\gamma_{l+1}) * \cdots * \mathfrak{d}_-(\gamma_h) \end{aligned} \tag{4.26}$$

First note that remark (4.2.1.ii) implies that all sums in equation (4.26) are finite for any $l \geq 1$.

Next, if $l = 1$, equation (4.26) is equivalent to (4.24). Suppose it holds for some $l \geq 1$. Then note that equation (4.24) is valid for any triple $\gamma = (r, e, v)$ and any stability parameter δ_c. If δ_c is not critical of type γ as assumed above, it reduces to a trivial identity. In particular setting $\gamma = \gamma_{l+1}$ in equation (4.24) yields

$$\mathfrak{d}_-(\gamma_{l+1}) = \mathfrak{d}_c(\gamma_{l+1}) - \sum_{m\geq 2} \sum_{(\eta_i)\in\mathcal{HN}_-(\gamma_{l+1},\delta_c,m,1)} \mathfrak{h}(\eta_1) * \mathfrak{d}_-(\eta_2) * \cdots * \mathfrak{d}_-(\eta_m)$$
$$- \sum_{m\geq 2} \sum_{(\eta_i)\in\mathcal{HN}_-(\gamma_{l+1},\delta_c,m,0)} \mathfrak{d}_-(\eta_1) * \mathfrak{d}_-(\eta_2) * \cdots * \mathfrak{d}_-(\eta_m)$$

Using this expression, the third term in the right hand side of equation (4.26) can be rewritten as follows.

$$(-1)^l \sum_{h\geq l+1} \sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,h,l)} \mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_l) * \mathfrak{d}_-(\gamma_{l+1}) * \cdots * \mathfrak{d}_-(\gamma_h) = \tag{4.27}$$
$$(-1)^l \sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,l+1,l)} \Bigg[\mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_l) * \mathfrak{d}_c(\gamma_{l+1}) -$$
$$\sum_{m\geq 2} \sum_{(\eta_i)\in\mathcal{HN}_-(\gamma_{l+1},\delta_c,m,1)} \mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_l) * \mathfrak{h}(\eta_1) * \mathfrak{d}_-(\eta_2) * \cdots * \mathfrak{d}_-(\eta_m)$$
$$- \sum_{m\geq 2} \sum_{(\eta_i)\in\mathcal{HN}_-(\gamma_{l+1},\delta_c,m,0)} \mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_l) * \mathfrak{d}_-(\eta_1) * \mathfrak{d}_-(\eta_2) * \cdots * \mathfrak{d}_-(\eta_m)\Bigg]$$
$$+(-1)^l \sum_{h\geq l+2} \sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,h,l)} \mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_l) * \mathfrak{d}_-(\gamma_{l+1}) * \cdots * \mathfrak{d}_-(\gamma_h)$$

By construction

$$\bigcup\nolimits_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,l+1,l)} \mathcal{HN}_-(\gamma_{l+1}, \delta_c, m, j) = \mathcal{HN}_-(\gamma, \delta_c, l+m, l+j)$$

for any $m \in \mathbb{Z}_{\geq 2}$, $j \in \{0, 1\}$. Therefore the last two terms in the right hand side of equation (4.27) cancel, and formula (4.27) reduces to

$$(-1)^l \sum_{h\geq l+1} \sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,h,l)} \mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_l) * \mathfrak{d}_-(\gamma_{l+1}) * \cdots * \mathfrak{d}_-(\gamma_h) =$$
$$(-1)^l \sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,l+1,l)} \mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_l) * \mathfrak{d}_c(\gamma_{l+1}) - \tag{4.28}$$
$$+ (-1)^{l+1} \sum_{h\geq l+2} \sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,h,l+1)} \mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_{l+1}) * \mathfrak{d}_-(\gamma_{l+2}) * \cdots * \mathfrak{d}_-(\gamma_h)$$

Substituting (4.28) in (4.26) it follows that formula (4.26) also holds if l is replaced by $(l+1)$. This concludes the inductive proof of formula (4.26).

In order to conclude the proof of equation (4.25), it suffices to observe that for sufficiently large l, equation (4.26) stabilizes to equation (4.25) using remark (4.2.1.ii).

□

Now note that equations (4.24), (4.25) yield a recursive algorithm expressing $\mathfrak{d}_-(\gamma)$ in terms of $\mathfrak{d}_+(\gamma_i)$, $1 \leq i \leq h$, $h \geq 1$. This follows observing that in the left hand side of (4.25) $0 < v_i < v$ for all stack functions $\mathfrak{d}_-(\gamma_i)$ occuring in the sum

$$\sum_{h\geq 2} \sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,h,0)} \mathfrak{d}_-(\gamma_1) * \cdots * \mathfrak{d}_-(\gamma_h).$$

Therefore, once a formula for the difference $\mathfrak{d}_-(\gamma) - \mathfrak{d}_+(\gamma)$, has been derived for triples of the form $\gamma = (r, e, v)$, one can recursively derive an analogous formula for triples of the form $\gamma = (r, e, v+1)$. For $v = 1$, equations (4.24), (4.25) easily imply

$$\mathfrak{d}_-(\gamma) = \mathfrak{d}_+(\gamma) + \sum_{l\geq 2}(-1)^l \sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,l,l-1)} \mathfrak{h}(\gamma_1) * \cdots * [\mathfrak{d}_+(\gamma_l), \mathfrak{h}(\gamma_{l-1})] \tag{4.29}$$

Employing the above recursive algorithm one can determine in principle analogous formulas for $v \geq 2$. Since the resulting expressions quickly become cumbersome, explicit formulas will be given below only for $v = 2$.

Corollary 4.2.3. *Suppose $\gamma = (r, e, 2)$ with $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. The following relations hold in the stack function algebra $\underline{\mathsf{SF}}(\mathfrak{Ob}(\mathcal{X}))$*

$$\begin{aligned}\mathfrak{d}_-(\gamma) = \mathfrak{d}_+(\gamma) &+ \sum_{l\geq 2}(-1)^l \sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,l,l-1)} \mathfrak{h}(\gamma_1) * \cdots * [\mathfrak{d}_+(\gamma_l), \mathfrak{h}(\gamma_{l-1})] \\ &+ \sum_{(\gamma_1,\gamma_2)\in\mathcal{HN}_+(\gamma,\delta_c,2,0)} \mathfrak{d}_+(\gamma_1) * \mathfrak{d}_+(\gamma_2) - \sum_{(\gamma_1,\gamma_2)\in\mathcal{HN}_-(\gamma,\delta_c,2,0)} \mathfrak{d}_-(\gamma_1) * \mathfrak{d}_-(\gamma_2) \\ &+ \sum_{l\geq 2}(-1)^l \sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,l+1,l-1)} \mathfrak{h}(\gamma_1) * \cdots * [\mathfrak{d}_+(\gamma_{l+1}) * \mathfrak{d}_+(\gamma_l), \mathfrak{h}(\gamma_{l-1})]\end{aligned} \tag{4.30}$$

where $\mathfrak{d}_-(\gamma_1), \mathfrak{d}_-(\gamma_2)$ are given by equation (4.29).

4.2.2 Wallcrossing for $v = 2$ invariants

Let $\gamma = (r, e, 2)$, $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, $\delta_c > 0$ a critical stability parameter of type γ, and $\delta_\pm$ two noncritical stability parameters as in lemma (4.1.14). The main goal of this section is to convert the stack function relation (4.30) to a wallcrossing formula for generalized Donaldson-Thomas invariants of ADHM sheaves.

As mentioned in the introduction the alternative notation $\alpha = (r, e)$ will be used for pairs $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. Using this notation, the sets $\mathcal{HN}_-(\alpha, v, \delta_c, h, k)$, $v \in \{1, 2\}$, $k \in \{0, h-2, h-1\}$, can be identified with sets of ordered sequences $(\alpha_i)_{1\leq i\leq h}$ satisfying the conditions listed above theorem (1.0.1). For convenience, recall that $\mathcal{HN}_-(\alpha, v, \delta_c, l, l-1)$, $l \in \mathbb{Z}_{\geq 1}$, $v \in \{1, 2\}$, denotes the set of ordered sequences $((\alpha_i))_{1\leq i\leq l}$, $\alpha_i \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, $1 \leq i \leq l$ so that

$$\alpha_1 + \cdots + \alpha_l = \alpha \tag{4.31}$$

and

$$\frac{e_1}{r_1} = \cdots = \frac{e_{l-1}}{r_{l-1}} = \frac{e_l + v\delta_c}{r_l} = \frac{e + v\delta_c}{r} \tag{4.32}$$

Similarly, $\mathcal{HN}_-(\alpha, v, \delta_c, l, l-2)$, $l \in \mathbb{Z}_{\geq 2}$, denotes the set of ordered sequences $((\alpha_i))_{1\leq i\leq l}$, $\alpha_i \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, $1 \leq i \leq l$ satisfying condition (1.1),

$$\frac{e_1}{r_1} = \cdots = \frac{e_{l-2}}{r_{l-2}} = \frac{e_{l-1} + \delta_c}{r_{l-1}} = \frac{e_l + \delta_c}{r_l} = \frac{e + 2\delta_c}{r} \tag{4.33}$$

and $1/r_{l-1} < 1/r_l$.

Note that the sets $\mathcal{HN}_-(\alpha, 2, \delta_c, h, 0)$ are nonempty if and only if $h = 2$, in which case they consist of ordered pairs (α_1, α_2) so that $\alpha_1 + \alpha_2 = \alpha$, $1/r_1 < 1/r_2$, and

$$\frac{e_1 + \delta_c}{r_1} = \frac{e_2 + \delta_c}{r_2} = \frac{e + 2\delta_c}{r}$$

Moreover the set $\mathcal{HN}_-(\alpha, 2, \delta_c, 1, 0)$ consists of only of the element (α).

It is straightforward to check that for fixed $\alpha = (r, e)$ and δ_c, the union

$$\begin{gathered}\bigcup_{l\geq 1} [\mathcal{HN}_-(\alpha, 2, \delta_c, l, l-1) \cup \mathcal{HN}_-(\alpha, 2, \delta_c, l+1, l-1)] \\ \bigcup_{(\alpha_1,\alpha_2)\in\mathcal{HN}_-(\alpha,2,\delta_c,2,0)} \bigcup_{l_1\geq 1} \bigcup_{l_2\geq 1} [\mathcal{HN}_-(\alpha_1, 1, \delta_c, l_1, l_1-1) \times \mathcal{HN}_-(\alpha_2, 1, \delta_c, l_2, l_2-1)]\end{gathered} \tag{4.34}$$

is a finite set.

Now let $0 < \delta_- < \delta_c < \delta_+$ be stability parameters so that there are no critical stability parameters of type $(\alpha, 2)$ in the intervals $[\delta_-,\ \delta_c)$, $(\delta_c,\ \delta_+]$. Since the set (4.34) is finite δ_-, δ_+ can be chosen so that the same holds for all numerical types (α_i, v_i) in all ordered sequences in (4.34). Then the following lemma holds.

Lemma 4.2.4. *The following relations hold in the stack function algebra* $\underline{\mathsf{SF}}(\mathfrak{Ob}(\mathcal{X}))$

$$\mathfrak{d}_-(\alpha, 1) = \sum_{l\geq 1} \frac{(-1)^{l-1}}{(l-1)!} \sum_{(\alpha_i)\in\mathcal{HN}_-(\alpha,1,\delta_c,l,l-1)} [\mathfrak{g}(\alpha_1), [\cdots[\mathfrak{g}(\alpha_{l-1}), \mathfrak{d}_+(\alpha_l, 1)]\cdots] \tag{4.35}$$

$$\begin{aligned}
\mathfrak{d}_-(\alpha,2) = & \sum_{l\geq 1} \frac{(-1)^{l-1}}{(l-1)!} \sum_{(\alpha_i)\in \mathcal{HN}_-(\alpha,2,\delta_c,l,l-1)} [\mathfrak{g}(\alpha_1),[\cdots[\mathfrak{g}(\alpha_{l-1}),\mathfrak{d}_+(\alpha_l,2)]\cdots] \\
& + \sum_{l\geq 1} \frac{(-1)^{l-1}}{(l-1)!} \sum_{(\alpha_i)\in \mathcal{HN}_-(\alpha,2,\delta_c,l+1,l-1)} [\mathfrak{g}(\alpha_1),[\cdots[\mathfrak{g}(\alpha_{l-1}),\mathfrak{d}_+(\alpha_{l+1},1)*\mathfrak{d}_+(\alpha_l,1)]\cdots] \\
& - \sum_{(\alpha_1,\alpha_2)\in \mathcal{HN}_-(\alpha,2,\delta_c,2,0)} \sum_{l_1\geq 1}\sum_{l_2\geq 1} \frac{(-1)^{l_1-1}}{(l_1-1)!}\frac{(-1)^{l_2-1}}{(l_2-1)!} \sum_{(\alpha_{1,i})\in \mathcal{HN}_-(\alpha_1,1,\delta_c,l_1,l_1-1)} \\
& \sum_{(\alpha_{2,i})\in \mathcal{HN}_-(\alpha_2,1,\delta_c,l_2,l_2-1)} \big([\mathfrak{g}(\alpha_{1,1}),[\cdots[\mathfrak{g}(\alpha_{1,l_1-1}),\mathfrak{d}_+(\alpha_{1,l_1},1)]\cdots] \\
& \qquad * [\mathfrak{g}(\alpha_{2,1}),[\cdots[\mathfrak{g}(\alpha_{2,l_2-1}),\mathfrak{d}_+(\alpha_{2,l_2},1)]\cdots]\big).
\end{aligned} \tag{4.36}$$

Proof. Formulas (4.35), (4.36) follow from equations (4.30), (4.29) by repeating the computations in the proof of [33, Lemm. 2.6] in the present context.

□

Proof of Theorem (1.0.1.) The proof consists of two steps. First the stack function identities (4.35), (4.36) must be converted into similar identities for the log stack functions (4.21). As explained in [64, Sect. 6.5], [65, Sect. 3.5], applying the morphism (4.20) to the log stack function identities (4.35), (4.36) yields certain relations in the universal enveloping algebra $U(\mathsf{L}(\mathcal{X}))$ of the Lie algebra $\mathsf{L}(\mathcal{X})$. These relations imply in turn a wallcrossing formula for generalized Donaldson-Thomas invariants by identifying the coefficients of generators of the generators of $\mathsf{L}(\mathcal{X}) \subset U(\mathsf{L}(\mathcal{X}))$.

Given the above choice of $\delta_\pm$, for $v=1$, equation (4.21) reduces to $\mathfrak{e}_\pm(\gamma)=\mathfrak{d}_\pm(\gamma)$, while for $v=2$

$$\mathfrak{e}_\pm(\gamma) = \mathfrak{d}_\pm(\gamma) - \frac{1}{2}\mathfrak{d}_\pm(\gamma/2)*\mathfrak{d}_\pm(\gamma/2). \tag{4.37}$$

The second term in the right hand side of (4.37) is by convention trivial unless (r,e) are even.

Equations (4.37), (4.36), (1.5) yield the following identity in the universal enveloping algebra of the Lie algebra $\mathsf{L}(\mathcal{X})$

$$\begin{aligned}
& \sum_\alpha (A_-(\alpha,2)-A_+(\alpha,2))\lambda(\alpha,2) = \\
& \sum_\alpha \sum_{l\geq 2} \frac{1}{(l-1)!} \sum_{(\alpha_i)\in \mathcal{HN}_-(\alpha,2,\delta_c,l,l-1)} \left(A_+(\alpha_l,2)\prod_{i=1}^{l-1} f_2(\alpha_i)H(\alpha_i)\right)\lambda(\alpha,2)
\end{aligned} \tag{4.38}$$

$$-\sum_{\alpha}\sum_{l\geq 1}\frac{1}{(l-1)!}\sum_{(\alpha_i)\in\mathcal{HN}_-(\alpha,2,\delta_c,l+1,l-1)}\left(A_+(\alpha_l,1)A_+(\alpha_{l+1},1)\prod_{i=1}^{l-1}H(\alpha_i)\right)$$

$$[\lambda(\alpha_1),[\cdots[\lambda(\alpha_{l-1}),\lambda(\alpha_{l+1},1)\star\lambda(\alpha_l,1)]\cdots]$$

$$+\sum_{\alpha}\sum_{(\alpha_1,\alpha_2)\in\mathcal{HN}_-(\alpha,2,\delta_c,2,0)}\sum_{l_1\geq 1}\sum_{l_2\geq 1}\frac{1}{(l_1-1)!}\frac{1}{(l_2-1)!}\sum_{(\alpha_{1,i})\in\mathcal{HN}_-(\alpha_1,1,\delta_c,l_1,l_1-1)}$$

$$\sum_{(\alpha_{2,i})\in\mathcal{HN}_-(\alpha_2,1,\delta_c,l_2,l_2-1)}A_+(\alpha_{1,l_1})A_+(\alpha_{2,l_2})\prod_{i=1}^{l_1-1}f_1(\alpha_{1,i})H(\alpha_{1,i})\prod_{i=1}^{l_2-1}f_1(\alpha_{2,i})H(\alpha_{2,i})$$

$$\lambda(\alpha_1,1)\star\lambda(\alpha_2,1)$$

$$+\frac{1}{2}\sum_{\alpha}(A_-(\alpha/2,1)^2-A_+(\alpha/2,1)^2)\lambda(\alpha/2,1)\star\lambda(\alpha/2,1)$$

$$-\sum_{\alpha}\sum_{l\geq 1}\frac{1}{(l-1)!}\sum_{(\alpha_i)\in\mathcal{HN}_-(\alpha,2,\delta_c,l,l-1)}\left(A_+(\alpha_l/2,1)^2\prod_{i=1}^{l-1}H(\alpha_i)\right)$$

$$[\lambda(\alpha_1),[\cdots,[\lambda(\alpha_{l-1},\lambda(\alpha_l/2,1)\star\lambda(\alpha_l/2,1)]\cdots]$$

where $\star$ denotes the associative product in the universal enveloping algebra. By conventions the invariants of the form $A_+(\alpha/2,1)$ are trivial unless $\alpha=2\alpha'$ for some $\alpha'=(r',e')=\mathbb{Z}_{\geq 1}\times\mathbb{Z}$.

Next, the identity [64, Eqn. 127] or [65, Eqn. 45] yields the following relations in the universal enveloping algebra

$$\begin{aligned}\lambda(\alpha_{l+1},1)\star\lambda(\alpha_l,1)&=\frac{1}{2}g(\alpha_{l+1},\alpha_l)\lambda(\alpha_l+\alpha_{l+1},2)+\cdots\\ \lambda(\alpha_1,1)\star\lambda(\alpha_2,1)&=\frac{1}{2}g(\alpha_1,\alpha_2)\lambda(\alpha_1+\alpha_2,2)+\cdots\\ \lambda(\alpha/2,1)\star\lambda(\alpha/2,1)&=\cdots\\ \lambda(\alpha_l/2,1)\star\lambda(\alpha_l/2,1)&=\cdots\end{aligned}$$

where $\cdots$ stands for linear combinations of generators of $U(\mathsf{L}(\mathcal{X}))$ not in $\mathsf{L}(\mathcal{X})$. Since the left hand side of equation (4.38) must belong to the Lie algebra $\mathsf{L}(\mathcal{X})$ according to [63, Thm. 8.7], it follows that all higher order terms must cancel. Then equation (1.5) follows by straightforward computations.

□

4.3 Comparison with Kontsevich-Soibelman Formula

The goal of this section is to prove that formula (1.5) is in agreement with the wallcrossing formula of Kontsevich and Soibelman [69], which will be referred to as the KS formula in the following.

As in section (4.2.2), numerical types of ADHM sheaves will be denoted by $\gamma = (\alpha, v)$, $\alpha = (r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, $v \in \mathbb{Z}_{\geq 0}$. In order to streamline the computations, let $\mathsf{L}(\mathcal{X})_{\leq 2}$ denote the truncation of the Lie algebra $\mathsf{L}(\mathcal{X})$ defined by

$$[\lambda(\alpha_1, v_1), \lambda(\alpha_2, v_2)]_{\leq 2} = \begin{cases} [\lambda(\alpha_1, v_1), \lambda(\alpha_2, v_2)] & \text{if } v_1 + v_2 \leq 2 \\ 0 & \text{otherwise.} \end{cases} \tag{4.39}$$

Furthermore, it will be more convenient to use the alternative notation $\mathsf{e}_\alpha = \lambda(\alpha, 0)$, $\mathsf{f}_\alpha = \lambda(\alpha, 1)$, and $\mathsf{g}_\alpha = \lambda(\alpha, 2)$.

Given a critical stability parameter δ_c of type $(r, e, 2)$, $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, there exist two pairs $\alpha = (r_\alpha, e_\alpha)$ and $\beta = (r_\beta, e_\beta)$ with

$$\frac{e_\alpha + \delta_c}{r_\alpha} = \frac{e_\beta}{r_\beta} = \mu_{\delta_c}(\gamma)$$

so that any $\eta \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ with $\mu_{\delta_c}(\eta) = \mu_{\delta_c}(\gamma)$ can be uniquely written as $\eta = (q\beta, 0), (\alpha + q\beta, 1)$, or $(2\alpha + q\beta, 2)$, with $q \in \mathbb{Z}_{\geq 0}$.

For any $q \in \mathbb{Z}_{\geq 0}$ the following formal expressions will be needed in the KS formula,

$$U_{\alpha+q\beta} = \exp(\mathsf{f}_{\alpha+q\beta} + \frac{1}{4}\mathsf{g}_{2\alpha+2q\beta}) \quad , \quad U_{2\alpha+q\beta} = \exp(\mathsf{g}_{2\alpha+q\beta}) \quad , \quad U_{q\beta} = \exp(\sum_{m\geq 1} \frac{\mathsf{e}_{mq\beta}}{m^2}) \ . \tag{4.40}$$

Moreover, let

$$\mathbb{H} = \sum_{q \geq 0} H(q\beta)\mathsf{e}_{q\beta},$$

where the invariants $H(\alpha)$ are defined in (4.22). Then the wallcrossing formula of Kontsevich and Soibelman reads

$$\begin{aligned} &\exp(\mathbb{H}) \prod_{q\geq 0,\ q\downarrow} U_{2\alpha+q\beta}^{\overline{A}_+(2\alpha+q\beta,2)} \prod_{q\geq 0,\ q\downarrow} U_{\alpha+q\beta}^{A_+(\alpha+q\beta,1)} \\ &= \prod_{q\geq 0,\ q\uparrow} U_{\alpha+q\beta}^{A_-(\alpha+q\beta,1)} \prod_{q\geq 0,\ q\uparrow} U_{2\alpha+q\beta}^{\overline{A}_-(2\alpha+q\beta,2)} \exp(\mathbb{H}) \end{aligned} \tag{4.41}$$

where an up, respectively down arrow means that the factors in the corresponding product are taken in increasing, respectively decreasing order of q. Note that $\overline{A}_\pm(2\alpha + q\beta, 2)$ are the invariants defined in section (4.1.5) by the multicover formula (4.23). In this case equation (4.23) reduces to

$$A_\pm(2\alpha + q\beta, 2) = \overline{A}_\pm(2\alpha + q\beta, 2) + \frac{1}{4}A_\pm(\alpha + q\beta/2, 1).$$

Expanding the right hand side, equation (4.41) yields

$$\begin{aligned}
&\exp(\sum_{q\geq 0} A_-(2\alpha+q\beta,2)\mathsf{g}_{2\alpha+q\beta}+\\
&\sum_{q_2>q_1\geq 0}\frac{1}{2}g(q_1\beta,q_2\beta)A_-(\alpha+q_1\beta,1)A_-(\alpha+q_2\beta,1)\mathsf{g}_{2\alpha+(q_1+q_2)\beta})=\\
&\exp(\mathbb{H})\exp(\sum_{q\geq 0} A_+(2\alpha+q\beta,2)\mathsf{g}_{2\alpha+q\beta}\\
&+\sum_{q_1>q_2\geq 0}\frac{1}{2}g(q_1\beta,q_2\beta)A_+(\alpha+q_1\beta,1)A_+(\alpha+q_2\beta,1)\mathsf{g}_{2\alpha+(q_1+q_2)\beta})\exp(-\mathbb{H}),
\end{aligned} \tag{4.42}$$

modulo terms involving f_γ. These terms are omitted since they enter $v=1$ wallcrossing formula derived in [28]. The BCH formula

$$\begin{aligned}
\exp(A)\exp(B)\exp(-A) &= \exp(\sum_{n=0}\frac{1}{n!}(Ad(A))^n B)\\
&= \exp(B+[A,B]+\frac{1}{2}[A,[A,B]]+\cdots),
\end{aligned} \tag{4.43}$$

yields

$$\begin{aligned}
&\exp(\mathbb{H})\exp(\mathsf{g}_{2\alpha+q\beta})\exp(-\mathbb{H}) = \exp(\mathsf{g}_{2\alpha+q\beta}+\sum_{q_1>0} f_2(q_1\beta)H(q_1\beta)\mathsf{g}_{2\alpha+(q+q_1)\beta}\\
&+\frac{1}{2!}\sum_{q_1>0,q_2>0} f_2(q_1\beta)H(q_1\beta)f_2(q_2\beta)H(q_2\beta)\mathsf{g}_{2\alpha+(q+q_1+q_2)\beta}+\cdots)\\
&=\exp\Big(\sum_{l\geq 0,q_i>0}\frac{1}{l!}(\prod_{i=1}^{l} f_2(q_i\beta)H(q_i\beta))\mathsf{g}_{2\alpha+(q+q_1+\cdots+q_l)\beta}\Big)
\end{aligned} \tag{4.44}$$

Substituting (4.44) in (4.42) results in

$$\begin{aligned}
&\exp\big(\sum_{q\geq 0} A_-(2\alpha+q\beta,2)\mathsf{g}_{2\alpha+q\beta}+\sum_{q_2>q_1\geq 0}\frac{1}{2}g(q_1\beta,q_2\beta)A_-(\alpha+q_1\beta,1)A_-(\alpha+q_2\beta,1)\mathsf{g}_{2\alpha+(q_1+q_2)\beta}\big)\\
&=\exp\big(\sum_{\substack{q\geq 0,l\geq 0\\ q_i>0}} A_+(2\alpha+q\beta,2)\frac{1}{l!}(\prod_{i=1}^{l} f_2(q_i\beta)H(q_i\beta))\mathsf{g}_{2\alpha+(q+q_1+\cdots+q_l)\beta}\\
&+\sum_{\substack{q_1'>q_2'\geq 0\\ l\geq 0,q_i>0}}\frac{1}{2}g(q_1'\beta,q_2'\beta)A_+(\alpha+q_1'\beta,1)A_+(\alpha+q_2'\beta,1)\frac{1}{l!}(\prod_{i=1}^{l} f_2(q_i\beta)H(q_i\beta))\mathsf{g}_{2\alpha+(q_1'+q_2'+q_1+\cdots+q_l)\beta}\big)
\end{aligned} \tag{4.45}$$

In order to further simplify the notation, let

$$A_\pm(v\alpha+q\beta,v)\equiv A_\pm(q,v),\qquad \mathsf{g}_{2\alpha+q\beta}\equiv\mathsf{g}_q.$$

Comparing the coefficients of g_Q in (4.42), yields

$$\begin{aligned} A_-(Q,2) = & \sum_{\substack{q'\geq 0,\ l\geq 0,\ q_i>0 \\ q'+q_1+\cdots+q_l=Q}} A_+(q',2)\frac{1}{l!}(\prod_{i=1}^{l} f_2(q_i\beta)H(q_i\beta)) \\ & + \frac{1}{2} \sum_{\substack{q_1'>q_2'\geq 0 \\ l\geq 0,\ q_i>0 \\ q_1'+q_2'+q_1+\cdots+q_l=Q}} g(q_1'\beta,q_2'\beta)A_+(q_1',1)A_+(q_2',1)\frac{1}{l!}(\prod_{i=1}^{l} f_2(q_i\beta)H(q_i\beta)) \\ & - \frac{1}{2} \sum_{q_2'>q_1'\geq 0,\ q_1'+q_2'=Q} g(q_1'\beta,q_2'\beta)A_-(q_1',1)A_-(q_2',1)\ . \end{aligned} \tag{4.46}$$

Using the $v=1$ wallcrossing formula [28, Thm. 1.1] the last term in (4.46) becomes

$$\begin{aligned} & -\frac{1}{2} \sum_{q_2>q_1\geq 0,\ q_1+q_2=Q} g(q_1\beta,q_2\beta)A_-(q_1,1)A_-(q_2,1) \\ & = -\frac{1}{2} \sum_{\substack{q_2>q_1\geq 0 \\ q_1+q_2=Q \\ l\geq 0,\ \tilde{l}\geq 0 \\ q_1'\geq 0,\ q_2'\geq 0 \\ n_i>0,\ \tilde{n}_i>0 \\ q_1'+n_1+\cdots+n_l=q_1 \\ q_2'+\tilde{n}_1+\cdots+\tilde{n}_{\tilde{l}}=q_2}} g(q_1\beta,q_2\beta)A_+(q_1',1)A_+(q_2',1)\frac{1}{l!}(\prod_{i=1}^{l} f_1(n_i\beta)H(n_i\beta))\frac{1}{\tilde{l}!}(\prod_{i=1}^{\tilde{l}} f_1(\tilde{n}_i\beta)H(\tilde{n}_i\beta))\ . \end{aligned} \tag{4.47}$$

Therefore the final wallcrossing formula for $v=2$ invariants is

$$\begin{aligned} A_-(Q,2) = & \sum_{\substack{q'\geq 0,\ l\geq 0,\ q_i>0 \\ q'+q_1+\cdots+q_l=Q}} A_+(q',2)\frac{1}{l!}(\prod_{i=1}^{l} f_2(q_i\beta)H(q_i\beta)) \\ & + \frac{1}{2} \sum_{\substack{q_1'>q_2'\geq 0 \\ l\geq 0,\ q_i>0 \\ q_1'+q_2'+q_1+\cdots+q_l=Q}} \frac{1}{2}g(q_1'\beta,q_2'\beta)A_+(q_1',1)A_+(q_2',1)\frac{1}{l!}(\prod_{i=1}^{l} f_2(q_i\beta)H(q_i\beta)) \\ & - \frac{1}{2} \sum_{\substack{q_2>q_1\geq 0 \\ q_1+q_2=Q \\ l\geq 0,\ \tilde{l}\geq 0 \\ q_1'\geq 0,\ q_2'\geq 0 \\ n_i>0,\tilde{n}_i>0 \\ q_1'+n_1+\cdots+n_l=q_1 \\ q_2'+\tilde{n}_1+\cdots+\tilde{n}_{\tilde{l}}=q_2}} g(q_1\beta,q_2\beta)A_+(q_1',1)A_+(q_2',1)\frac{1}{l!}(\prod_{i=1}^{l} f_1(n_i\beta)H(n_i\beta))\frac{1}{\tilde{l}!}(\prod_{i=1}^{\tilde{l}} f_1(\tilde{n}_i\beta)H(\tilde{n}_i\beta))\ . \end{aligned} \tag{4.48}$$

This formula agrees with (1.5) since the bilinear function $g(\quad,\quad)$ is antisymmetric.

4.4 Asymptotic invariants in the $g=0$ theory

In this subsection X will be a smooth genus 0 curve over a $\mathbb{C}$-field K, and $M_1 \simeq \mathcal{O}_X(d_1)$, $M_2 \simeq \mathcal{O}_X(d_2)$, with $(d_1,d_2)=(1,1)$ or $(d_1,d_2)=(0,2)$. In this case any coherent locally free sheaf E on X is isomorphic to a direct sum of line bundles. Let $E_{\geq 0}$ denote the direct sum of all summands of non-negative degree, and $E_{<0}$ denote the direct sum of all summands of negative degree.

Lemma 4.4.1. *Let $\mathcal{E} = (E, V, \Phi_1, \Phi_2, \phi, \psi)$ be a nontrivial δ-semistable ADHM sheaf of type $(r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 1}$, for some $\delta > 0$. Then $E_{<0} = 0$ and ϕ is identically zero.*

Proof. Since $\delta > 0$, lemma (4.1.2.*ii*) implies that ψ is not identically zero. Then obviously $E_{\geq 0}$ must be nontrivial and $\mathrm{Im}(\psi) \subseteq E_{\geq 0}$. Since $M \simeq K_X^{-1} \simeq \mathcal{O}_X(2)$, $E_{\geq 0} \otimes_X M \subseteq \mathrm{Ker}(\phi)$. Moreover, since $\deg(M_1) \geq 0$, $\deg(M_2) \geq 0$, $\Phi_i(E_{\geq 0} \otimes_X M_i) \subseteq E_{\geq 0}$. It follows that the data

$$\mathcal{E}_{\geq 0} = (E_{\geq 0}, V \otimes \mathcal{O}_X, \Phi_i|_{E_{\geq 0} \otimes_X M_i}, 0, \psi)$$

is a nontrivial subobject of $\mathcal{E}$. If $E_{<0}$ is not the zero sheaf, $\mathcal{E}_{\geq 0}$ is a proper subobject of $\mathcal{E}$. Then δ-semistability condition implies $r(\mathcal{E}_{\geq 0}) < r(\mathcal{E})$, hence

$$\frac{d(\mathcal{E}_{\geq 0}) + v(\mathcal{E}_{\geq 0})\,\delta}{r(\mathcal{E}_{\geq 0})} \leq \frac{e + v\,\delta}{r}. \tag{4.49}$$

However $e < d(\mathcal{E}_{\geq 0})$ and $0 < r(\mathcal{E}_{\geq 0}) < r$ under the current assumptions. Since also $v(\mathcal{E}_{\geq 0}) = v$ and $\delta, d(\mathcal{E}_{\geq 0}) > 0$, inequality (4.49) leads to a contradiction. Therefore $E_{<0} = 0$ and ϕ must be identically zero.

□

Let $\mathcal{C}^0_{\mathcal{X}}$ be the full abelian subcategory of $\mathcal{C}_{\mathcal{X}}$ consisting of ADHM sheaves $\mathcal{E}$ with $\phi = 0$. For any $\delta \in \mathbb{R}$, an object $\mathcal{E}$ of $\mathcal{C}^0_{\mathcal{X}}$ will be called δ-semistable if it is δ-semistable as an object of $\mathcal{C}_{\mathcal{X}}$. Note that given an object $\mathcal{E}$ of $\mathcal{C}^0_{\mathcal{X}}$, any subobject $\mathcal{E}' \subset \mathcal{E}$ must also belong to $\mathcal{C}^0_{\mathcal{X}}$. In particular all test subobjects in definition (4.1.1) also belong to $\mathcal{C}^0_{\mathcal{X}}$, and one obtains a stability condition on the abelian category $\mathcal{C}^0_{\mathcal{X}}$. Then the properties of δ-stability and moduli stacks of semistable objects in $\mathcal{C}^0_{\mathcal{X}}$ are analogous to those of $\mathcal{C}_{\mathcal{X}}$. In particular for fixed $(r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 1}$ there are finitely many critical stability parameters of type (r, e, v) dividing the real axis into stability chambers. The main difference between $\mathcal{C}^0_{\mathcal{X}}$ and $\mathcal{C}_{\mathcal{X}}$ is the presence of an empty chamber, as follows.

Lemma 4.4.2. *For any $(r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 1}$ the moduli stack of δ-semistable objects of $\mathcal{C}^0_{\mathcal{X}}$ of type (r, e, v) is empty if $\delta < 0$.*

Proof. Given an ADHM sheaf $\mathcal{E} = (E, V, \Phi_i, \psi)$ of type (r, e, v), it is straightforward to check that for $\delta < 0$ the proper nontrivial object $(E, 0, \Phi_i, 0)$ is always destabilizing if $\delta < 0$.

□

Lemma 4.4.3. *Let $\mathcal{E}$ be a δ-semistable object of $\mathcal{C}^0_{\mathcal{X}}$ of type $(r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ for some $\delta \geq 0$. If $e \geq 0$, then $E_{<0} = 0$ and ϕ is identically zero.*

Proof. For $\delta > 0$ and $v > 0$, this obviously follows from lemma (4.4.1). If $\delta = 0$ or $v = 0$ note that $E_{\geq 0}$ cannot be the zero sheaf since $e \geq 0$. Then the proof of lemma (4.4.1) also applies to this case as well.

□

Lemma 4.4.4. *Let $\mathcal{E} = (E, 0, \Phi_i, 0, 0)$ be a semistable object of $\mathcal{C}^0_{\mathcal{X}}$ of type $(r, e, 0)$, $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. If $(d_1, d_2) = (1, 1)$, E must be isomorphic to $\mathcal{O}_X(n)^{\oplus r}$ for some $n \in \mathbb{Z}$, and $\Phi_i = 0$ for $i = 1, 2$. If $(d_1, d_2) = (0, 2)$, E must be isomorphic to $\mathcal{O}_X(n)^{\oplus r}$ for some $n \in \mathbb{Z}$, and $\Phi_2 = 0$.*

Proof. In both cases, let $E \simeq \oplus_{s=1}^{r} \mathcal{O}_X(n_s)$ for some $n_s \in \mathbb{Z}$ so that $n_1 \leq n_2 \leq \cdots \leq n_r$. Since $d_1, d_2 \geq 0$, any subsheaf of the form

$$\oplus_{s=s_0}^{r} \mathcal{O}_X(n_s)$$

for some $1 \leq s_0 \leq r$ must be Φ_i-invariant, $i = 1, 2$. Therefore the semistability condition implies

$$\frac{n_{s_0} + \cdots + n_r}{r - s_0 + 1} \leq \frac{n_1 + \ldots + n_r}{r}$$

for any $1 \leq s_0 \leq r$. Then it is straightforward to check that $n_1 = \cdots = n_r = n$. The rest is obvious.

□.

Corollary 4.4.5. *Under the same conditions as in lemma (4.4.4),*

$$H(r, e) = \begin{cases} \frac{(-1)^{d_1 - 1}}{r^2} & \text{if } e = rn, \ n \in \mathbb{Z} \\ \\ 0 & \text{otherwise.} \end{cases} \tag{4.50}$$

Proof. If $(d_1, d_2) = (1, 1)$, lemma (4.4.4) implies that the moduli stack $\mathfrak{M}^{ss}(\mathcal{X}, r, e, 0)$ is isomorphic to the quotient stack $[*/GL(r)]$ if $e = rn$ for some $n \in \mathbb{Z}$, and empty otherwise. Alternatively, if $e = rn$, the moduli stack $\mathfrak{M}^{ss}(\mathcal{X}, r, e, 0)$ can be identified with the moduli stack of trivially semistable representations of dimension r of a quiver consisting of only one vertex and no arrows. Recall that the trivial semistability condition for quiver representations is King stability with all stability parameters associated to the vertices set to zero [65, Ex. 7.3].

If $(d_1, d_2) = (0, 2)$, lemma (4.4.4) implies that the moduli stack $\mathfrak{M}^{ss}(\mathcal{X}, r, rn, 0)$, $n \in \mathbb{Z}$, is isomorphic to the moduli stack of trivially semistable representations of dimension r of a quiver consisting of one vertex and one arrow joining the unique vertex with itself. If e is not a multiple of r, the moduli stack $\mathfrak{M}^{ss}(\mathcal{X}, r, e, 0)$ is empty.

Then corollary (4.4.5) follows by a computation very similar to [65, Sect. 7.5.1].

□

Remark 4.4.6. *The same arguments as in the proof of corollary (4.4.5) imply that for any $\delta > 0$,*

$$A_\delta(0,0,1) = 1 \qquad A_\delta(0,0,2) = \frac{1}{4}. \tag{4.51}$$

Extension groups in $\mathcal{C}^0_{\mathcal{X}}$ can be determined by analogy with those of $\mathcal{C}_{\mathcal{X}}$. Given two locally free objects $\mathcal{E}'', \mathcal{E}'$ of $\mathcal{C}^0_{\mathcal{X}}$, let $\widetilde{\mathcal{C}}(\mathcal{E}'', \mathcal{E}')$ be the three term complex of locally free $\mathcal{O}_X$-modules

$$0 \to \begin{array}{c} \mathcal{H}om_X(E'', E') \\ \oplus \\ \mathcal{H}om_X(V''\otimes \mathcal{O}_X, V'\otimes \mathcal{O}_X) \end{array} \xrightarrow{d_1} \begin{array}{c} \mathcal{H}om_X(E''\otimes_X M_1, E') \\ \oplus \\ \mathcal{H}om_X(E''\otimes_X M_2, E') \\ \oplus \\ \mathcal{H}om_X(V''\otimes \mathcal{O}_X, E') \end{array} \xrightarrow{d_2} \mathcal{H}om_X(E''\otimes_X M, E') \to 0 \tag{4.52}$$

where

$$\begin{aligned} d_1(\alpha, f) = (& -\alpha\circ\Phi_1'' + \Phi_1'\circ(\alpha\otimes 1_{M_1}), -\alpha\circ\Phi_2'' + \Phi_2'\circ(\alpha\otimes 1_{M_2}), \\ & -\alpha\circ\psi'' + \psi'\circ f) \end{aligned}$$

for any local sections (α, f) of the first term and

$$\begin{aligned} d_2(\beta_1,\beta_2,\gamma) =& \beta_1\circ(\Phi_2''\otimes 1_{M_1}) - \Phi_2'\circ(\beta_1\otimes 1_{M_2}) - \beta_2\circ(\Phi_1''\otimes 1_{M_2}) \\ & + \Phi_1'\circ(\beta_2\otimes 1_{M_1}) \end{aligned}$$

for any local sections $(\beta_1, \beta_2, \gamma)$ of the middle term. The degrees of the three terms in (4.16) are $0, 1, 2$ respectively. By analogy with lemma(4.1.15), the following holds.

Lemma 4.4.7. *Under the current assumptions,* $\mathrm{Ext}^k_{\mathcal{C}^0_{\mathcal{X}}}(\mathcal{E}'', \mathcal{E}') \simeq \mathbb{H}^k(\widetilde{\mathcal{C}}(\mathcal{E}'', \mathcal{E}'))$ *for* $k = 0, 1$.

Lemma 4.4.8. *Let $\mathcal{E}', \mathcal{E}''$ be two nontrivial locally free objects of $\mathcal{C}^0_{\mathcal{X}}$ of types $(r', e', v'), (r'', e'', v'') \in \mathbb{Z}_{\geq 1}\times\mathbb{Z}\times\mathbb{Z}_{\geq 0}$. Suppose that $E'_{<0} = 0$, $E''_{<0} = 0$ for both underlying locally free sheaves E', E''. Then*

$$\begin{aligned} & \dim(\mathrm{Ext}^0_{\mathcal{C}_{\mathcal{X}}}(\mathcal{E}'', \mathcal{E}')) - \dim(\mathrm{Ext}^1_{\mathcal{C}_{\mathcal{X}}}(\mathcal{E}'', \mathcal{E}')) - \dim(\mathrm{Ext}^0_{\mathcal{C}_{\mathcal{X}}}(\mathcal{E}', \mathcal{E}'')) \\ & + \dim(\mathrm{Ext}^1_{\mathcal{C}_{\mathcal{X}}}(\mathcal{E}', \mathcal{E}'')) = v'(e''+r'') - v''(e'+r'). \end{aligned} \tag{4.53}$$

Proof. Note that the complex (4.52) can be written as the cone of a morphism of locally free complexes on X

$$\varrho : \mathcal{H}[-1] \to \mathcal{V}$$

where $\mathcal{H}$ is the complex obtained from $\widetilde{\mathcal{C}}(\mathcal{E}'', \mathcal{E}')$ by omitting all direct summands depending on V', V'' (as well as making some obvious changes of signs), and $\mathcal{V}$ is the two term complex

$$\begin{aligned} \mathcal{H}om_X(V''\otimes\mathcal{O}_X, V'\otimes\mathcal{O}_X) \longrightarrow & \mathcal{H}om_X(V''\otimes\mathcal{O}_X, E') \\ f \longrightarrow & \psi'\circ f \end{aligned}$$

with degrees $0, 1$. The morphism ϱ is determined by the map

$$\begin{gathered}\mathcal{H}om_X(E'', E') \longrightarrow \mathcal{H}om_X(V'' \otimes \mathcal{O}_X, E') \\ \alpha \longrightarrow -\alpha \circ \psi''\end{gathered}$$

Therefore there is a long exact sequence of hypercohomology groups

$$\begin{array}{l} 0 \longrightarrow \mathbb{H}^0(\mathcal{V}) \longrightarrow \mathrm{Ext}^0_{\mathcal{C}^0_{\mathcal{X}}}(\mathcal{E}'', \mathcal{E}') \longrightarrow \mathbb{H}^0(\mathcal{H}(\mathcal{E}'', \mathcal{E}')) \\ \longrightarrow \mathbb{H}^1(\mathcal{V}) \longrightarrow \mathrm{Ext}^1_{\mathcal{C}^0_{\mathcal{X}}}(\mathcal{E}'', \mathcal{E}') \longrightarrow \mathbb{H}^1(\mathcal{H}(\mathcal{E}'', \mathcal{E}')) \\ \longrightarrow \mathbb{H}^2(\mathcal{V}) \longrightarrow \cdots \end{array} \tag{4.54}$$

Since $E'_{<0} = 0$ and X is rational, $\mathbb{H}^2(\mathcal{V}) = 0$. Obviously, there is a similar exact sequence with $\mathcal{E}', \mathcal{E}''$ interchanged. Then equation (4.53) easily follows observing that

$$\mathbb{H}^k(\mathcal{H}(\mathcal{E}'', \mathcal{E}')) \simeq \mathbb{H}^{3-k}(\mathbb{H}(\mathcal{E}', \mathcal{E}''))^\vee$$

for all $0 \leq k \leq 3$.

□

Proof of Corollary (1.0.2). Let $\gamma = (r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ be an arbitrary numerical type, and $\delta \in \mathbb{R}_{\geq 0}$. Given any decomposition $\gamma = \gamma_1 + \cdots + \gamma_l$, $l \geq 1$ so that

$$\frac{e_1 + v_1\delta}{r_1} = \cdots = \frac{e_l + v_l\delta}{r_l} = \frac{e + v\delta}{r}$$

it is obvious that if $v_i = 0$ for some $1 \leq i \leq l$ then $e_i \geq 0$. Moreover, if $\delta = 0$, then $e_i \geq 0$ for all $1 \leq i \leq l$. In particular this holds for all terms in the right hand side of the defining equation of log stack functions (4.21). It also holds for all possible numerical types of Harder-Narasimhan filtrations associated to a critical stability parameter $\delta_c \geq 0$ as in lemma (4.1.14). Note that if $\delta_c = 0$, the last quotient $\mathcal{F}_h$ in the Harder-Narasimhan filtration with respect to δ_+-stability, respectively the first quotient $\mathcal{F}_1$ in the Harder-Narasimhan filtration with respect to δ_--stability is allowed to be isomorphic to the object $O_v = (0, \mathbb{C}^v, 0, 0, 0)$, $v \geq 1$. In conclusion, the definition of generalized Donaldson-Thomas invariants, and derivation of wallcrossing formulas carry over to the present set-up for semistable objects of positive degree and stability parameters $\delta \geq 0$. In this case the resulting invariants will be denoted by $A^0_\delta(\gamma)$, or $A^0_\delta(\alpha, v)$ by analogy with section (4.2.2). Lemmas (4.4.1) and (4.4.3) imply that the invariants $A^0_\delta(\alpha, 2)$ satisfy the wallcrossing formula (1.5) at a positive critical stability parameter δ_c of type $(\alpha, 2)$. If $\delta_c = 0$, a modification of formula (1.5) is required, reflecting the presence of objects isomorphic to $\mathcal{O}_v$, $v = 1, 2$ in the Harder-Narasimhan filtrations. Basically one has to set $\delta_c = 0$ in conditions (1.2)-(1.4), and allow elements $(\alpha_i)_{1\leq i\leq l}$ so that α_i, $1 \leq i \leq l-1$ satisfy conditions (1.2)-(1.4), and $\alpha_l = (0, 0)$. This will result in extra terms in the right hand side of (1.5) which can be easily written down using (4.51). Since this is an easy exercise, explicit formulas will be omitted (see [28, Thm. 1.*ii.*] for the $v = 1$ case). Finally, note that one can also check compatibility with the Kontsevich-Soibelman formula at $\delta_c = 0$ repeating the calculations in section (4.3).

Then the proof of corollary (1.0.2) will be based on the KS wallcrossing formula relating δ-invariants for $\delta < 0$ to δ-invariants with $\delta >> 0$. Let $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}$ and let $\delta_+ \in \mathbb{R}_{>0} \setminus \mathbb{Q}$ an irrational stability parameter so that δ_+ is asymptotic of type (r', e') for all $1 \leq r' \leq r$, $0 \leq e' \leq e$, $1 \leq v \leq 2$. Moreover, assume that $re < \delta_+$. Then the KS formula reads

$$\prod_{(r,n,v)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 0}\times\{0,1,2\}\cup\{0,0,1\}} U_{\lambda(r,n,v)}^{\overline{A^0}_-(r,e,v)} = \prod_{(r,n,v)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 0}\times\{0,1,2\}\cup\{0,0,1\}} U_{\lambda(r,n,v)}^{\overline{A^0}_+(r,n,v)} \tag{4.55}$$

where in each term the factors are ordered in increasing order of $\delta_\pm$-slopes from left to right. The alternative notation introduced in section (4.3) will be used in the following. Then corollary (4.4.5) and equation (4.51) imply that the left hand side of (4.55) reads

$$\exp(\mathsf{f}_{00} + \frac{1}{4}\mathsf{g}_{00}) \prod_{n=0}^{\infty} U_{\mathsf{e}_{1n}}, \tag{4.56}$$

where

$$U_{\mathsf{e}_{rn}} = \exp\left((-1)^{d_1-1}\sum_{k=1}^{\infty}\frac{\mathsf{e}_{kr,kn}}{k^2}\right).$$

Moreover, given the above choice of δ_+,

$$e < \frac{\delta_+}{r} < \cdots \frac{e+\delta_+}{r} < \frac{\delta_+}{r-1} < \cdots < \frac{e+\delta_+}{r-1} < \cdots < \delta_+ + e < \frac{2\delta_+}{r} < \cdots < 2\delta_+ + e.$$

Therefore, in the right hand side of equation (4.55), the factors of the form $U_{\lambda(r',e',v)}^{\overline{A}_+(r',e',v)}$, with $v \in \{0, 1, 2\}$, and $1 \leq r' \leq r$, $1 \leq e' \leq e$ occur in the following order

$$\begin{gathered}\prod_{n=0}^{e} U_{\mathsf{e}_{1n}} \prod_{n=0}^{e} U_{\mathsf{f}_{r,n}}^{\overline{A}_+(r,n,1)} \prod_{n=0}^{e} U_{\mathsf{f}_{r-1,n}}^{\overline{A}_+(r-1,n,1)} \cdots \prod_{n=0}^{e} U_{\mathsf{f}_{1,n}}^{\overline{A}_+(1,n,1)} U_{\mathsf{f}_{0,0}}^{\overline{A}_+(0,0,1)} \\ \prod_{n=0}^{e} U_{\mathsf{g}_{r,n}}^{\overline{A}_+(r,n,2)} \cdots \prod_{n=0}^{e} U_{\mathsf{g}_{r-1,n}}^{\overline{A}_+(r-1,n,2)} \cdots \prod_{n=0}^{e} U_{\mathsf{g}_{1,n}}^{\overline{A}_+(1,n,2)},\end{gathered} \tag{4.57}$$

where

$$U_{\mathsf{f}_{rn}} = \exp(\mathsf{f}_{rn} + \frac{1}{4}\mathsf{g}_{2r,2n}), \qquad U_{\mathsf{g}_{rn}} = \exp(\mathsf{g}_{rn}).$$

In addition, the right hand side of (4.55) contains of course extra factors of the form $U_{\lambda(r',e',v)}^{\overline{A}_+(r',e',v)}$, with $v \in \{0, 1, 2\}$, and either $r' > r$ or $e' > e$. Some of these extra factors may in fact occur between the factors listed in (4.57). However, they can be ignored for the purpose of this computation since commutators involving such factors are again expressed in terms of generators $\lambda(r',e',v)$ with either $r'>r$ or $e'>e$. Therefore, using the BCH formula, (4.55) yields

$$\begin{aligned}
&(\prod_{n=0}^{e} U_{\mathsf{e}_{1n}})^{-1}\exp(\mathsf{f}_{00}+\frac{1}{4}\mathsf{g}_{00})\prod_{n=0}^{\infty} U_{\mathsf{e}_{1n}} = \\
&\exp\Big(\sum_{1\leq s\leq r,\ 0\leq n\leq e} A_+(s,n,1)\,\mathsf{f}_{sn} + \sum_{1\leq s\leq r,\ 0\leq n\leq e} A_+(s,n,2)\,\mathsf{g}_{sn} + \\
&\sum_{\substack{r_1>r_2\geq 1,\ r_1+r_2\leq r,\ n_1,\ n_2\geq 0, n_1+n_2\leq e\\ \text{or}\ 1\leq r_1=r_2\leq r/2,\ 0\leq n_1<n_2,\ n_1+n_2\leq e\\ \text{or}\ 1\leq r_1\leq r,\ 0\leq n_1\leq e,\ r_2=n_2=0}} \frac{1}{2}(n_1-n_2+r_1-r_2)(-1)^{(n_1-n_2+r_1-r_2)} \\
&A_+(r_1,n_1,1)A_+(r_2,n_2,1)\,\mathsf{g}_{r_1+r_2,n_1+n_2}+\cdots\Big)
\end{aligned} \tag{4.58}$$

where $\cdots$ are terms involving generators $\lambda(r',e',v)$ with either $r'>r$ or $e'>e$. For fixed $e\geq 1$, let $\mathcal{H}_e$ be defined by

$$\exp(\mathcal{H}_e)\equiv\prod_{n=0}^{e} U_{\mathsf{e}_{1,n}} = \exp\Big((-1)^{d_1-1}\sum_{0\leq n\leq e,\ k\geq 1}\frac{\mathsf{e}_{k,kn}}{k^2}\Big). \tag{4.59}$$

Using the BCH formula, the left hand side of equation (4.58) becomes

$$\exp\Big(\mathsf{f}_{00}+\frac{1}{4}\mathsf{g}_{00}+\sum_{j=1}^{\infty}\frac{1}{j!}[\underbrace{-\mathcal{H}_e,\cdots[-\mathcal{H}_e}_{\text{j times}},\mathsf{f}_{00}+\frac{1}{4}\mathsf{g}_{00}]\cdots]\Big) \tag{4.60}$$

modulo terms involving generators $\lambda(r',e',v)$ with either $r'>r$ or $e'>e$.

Next, the Lie algebra commutators

$$[\mathsf{e}_{r_1,n_1},\mathsf{f}_{r_2,n_2}] = (-1)^{n_1+r_1}(n_1+r_1)\,\mathsf{f}_{r_1+r_2,n_1+n_2}$$

$$[\mathsf{e}_{r_1,n_1},\mathsf{g}_{r_2,n_2}] = 2(n_1+r_1)\,\mathsf{g}_{r_1+r_2,n_1+n_2},$$

yield

$$[\underbrace{-\mathcal{H}_e,\cdots[-\mathcal{H}_e}_{\text{j times}},\mathsf{f}_{00}]\cdots] = \sum_{n_1,\ldots,n_j=0}^{e}\ \sum_{k_1,\ldots,k_j\geq 1}(-1)^{j(d_1-1)}\prod_{i=1}^{j}\frac{n_i+1}{k_i}(-1)^{(n_i+1)k_i-1}\,\mathsf{f}_{k_1+\cdots+k_j,k_1n_1+\cdots+k_jn_j}$$

and

$$[\underbrace{-\mathcal{H}_e,\cdots[-\mathcal{H}_e}_{\text{j times}},\mathsf{g}_{00}]\cdots] = \sum_{n_1,\ldots,n_j=0}^{e}\ \sum_{k_1,\ldots,k_j\geq 1}(-1)^{j(d_1-1)}\prod_{i=1}^{j}(-2)\frac{n_i+1}{k_i}\,\mathsf{g}_{k_1+\cdots+k_j,k_1n_1+\cdots+k_jn_j}$$

Therefore, identifying the coefficients of the generators f_{rn} in (4.58) it follows that the invariant $A_+(r',e',1)$ with $1\leq r'\leq r$ and $0\leq e'\leq e$ equals the coefficient of the monomial $u^{r'}q^{e'+r'}$ in the expression

$$\sum_{j=0}^{\infty}\frac{1}{j!}\left(\ln\left(\prod_{n=0}^{e}(1-u(-q)^{n+1})^{(-1)^{d_1-1}(n+1)}\right)\right)^j = \prod_{n=1}^{e+1}(1-u(-q)^n)^{(-1)^{d_1-1}n}.$$

Similarly, identifying the coefficients of the generators g_{rn} in (4.58) proves that the invariant $A_+(r', e', 2)$ with $1 \leq r' \leq r$ and $0 \leq e' \leq e$ equals the coefficient of the monomial $u^{r'} q^{e'+r'}$ in the expression

$$\frac{1}{4}\prod_{n=1}^{e+1}(1-uq^n)^{2(-1)^{d_1-1}n} - \sum_{\substack{r_1>r_2\geq 1,\ r_1+r_2\leq r,\ n_1,\ n_2\geq 0, n_1+n_2\leq e \\ \text{or } 1\leq r_1=r_2\leq r/2,\ 0\leq n_1<n_2,\ n_1+n_2\leq e \\ \text{or } 1\leq r_1\leq r,\ 0\leq n_1\leq e,\ r_2=n_2=0}}$$

$$\frac{1}{2}(n_1+r_1-n_2-r_2)(-1)^{(n_1+r_1-n_2-r_2)}A_+(r_1,n_1,1)A_+(r_2,n_2,1)q^{r_1+r_2}u^{n_1+n_2}.$$

Since this holds for any $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}$ (with a suitable choice of δ_+), corollary (1.0.2) follows.

$\square$

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