Description: PUP Math – Gang of Four Location: David Brearley High School – Kenilworth, NJ Researcher: Professor Carolyn Maher	Transcriber(s): Private Universe Project Verifier(s): Sigley, Robert, Sran, Kiranjeet Date Transcribed: Spring 2000 Page: 1 of 17

NARRATOR: About one month later, the researchers held a group interview with fourth graders: Jeff, Michelle, Milin, and Stephanie. In this session, they were especially interested in what made the students' reasoning convincing?

CAROLYN MAHER: The last one you did in class - remember what that was about?

JEFF: Robin Hood? That was the last one we did.

STEPHANIE: Towers of five.

CAROLYN MAHER: Towers of five. Do you remember what you did with those towers of five?

JEFF: Yeah.

STEPHANIE: Mm-hmm.

CAROLYN MAHER: And tell me about it. What was the problem?

MICHELLE: You had to figure out how many towers you could make, different towers you could make from five blocks up.

CAROLYN MAHER: Any five blocks?

STUDENTS (ALL): They had to be two colors.

CAROLYN MAHER: Two colors, okay. And did you figure out?

STEPHANIE: Yes.

CAROLYN MAHER: And what is it? Do you remember?

STEPHANIE: 32.

CAROLYN MAHER: You're sure of that?

STEPHANIE: Yes.

CAROLYN MAHER: How can you be so sure?

MILIN: 'Cause we checked!

CAROLYN MAHER: How can you be so sure?

JEFF: Remember we did all those - the chart, the thingies, for like all the different patterns? Remember I convinced you up in the whatch-a-ma-callit?

CAROLYN MAHER: Yes, in the room.

JEFF: So, I don't feel like convincing you again.

CAROLYN MAHER: You don't feel like convincing me again. [LAUGHTER]

NARRATOR: This session was a performance assessment to learn more about the students' reasoning.

CAROLYN MAHER: And Stephanie did try to work on towers of six and I asked all of you if you -

MILIN: So did I.

CAROLYN MAHER: You did too? If you were building towers of six, how many would there be?

MILIN: Probably 64.

CAROLYN MAHER: Why do you think 64?

MILIN: Well, because there was a pattern.

CAROLYN MAHER: What's that?

MILIN: You just times them by two.

CAROLYN MAHER: Times what by two?

MILIN: The towers by two, because one is two, and then we figured out two is two, I mean four, and then -

JEFF: You're not making much sense!

MICHELLE: See, if you had only one block up and two colors, then you would have two towers, and we figured out that the other day -

JEFF: Everything is opposite!

MICHELLE: - that you keep on doing, like two times two would be four and then...

CAROLYN MAHER: So four would be for what?

STEPHANIE: All you have to do -

MICHELLE: - four for, there would be four towers for two high.

CAROLYN MAHER: Okay.

JEFF: They're always opposite, though.

CAROLYN MAHER: Wait, let me hear what Michelle is saying.

MICHELLE: And then for this three high, you would eight towers.

CAROLYN MAHER: Do you agree with that?

JEFF: I don't know what you're talking about.

CAROLYN MAHER: Okay, let's get a piece of a paper and write down what you're saying and see if you all agree. I think Jeff hasn't been with us for a while and he doesn't know what we're talking about. But let's take one at a time.

CAROLYN MAHER: There are a couple of ways of approaching the problem. There's the notion of you can start with towers one tall, each color. And once you have the tower - let's say red and blue - you start with a red tower and now you're making a two tall tower. Well, you can either put a red on the top or a blue on the top. So, from that one, you get two. Likewise for the tower with the blue on the bottom. You could either put a red on the top or a blue on the top. From that one, two.

So, from the two towers - one tall - you generate four towers two tall. Each of those towers you can choose to either put a red on the top or a blue on the top, red on the top or blue on the top. From the eight, two more, two more, two more, two more. You could begin to generalize this idea and, later on, you can come up with a nice way of expressing your justification, which leads you to a kind of way of reasoning that we call inductive reasoning.

NARRATOR: For about half an hour, the students shared their different approaches. When it was Stephanie's turn, she presented a version of Proof by Cases to justify her solution for the number of towers three high.

CAROLYN MAHER: How about you, Stephanie?

STEPHANIE: I found it like this. I drew my lines and then I went red/red/red, blue/blue/blue, blue/red/blue, red/blue/blue, blue/blue/red, red/red/blue, red/blue/red, blue/red/red.

CAROLYN MAHER: So, what I'm hearing you say is that you're just -

MILIN: Guessing!

CAROLYN MAHER: - you believe they are eight, but you say guessing. Now, why does that sound like guessing?

MILIN: Because what if you can make more?

STEPHANIE: Okay, this is the three high, right? And you're convinced you can make eight?

MILIN: Yep!

STEPHANIE: I'm convinced I can make eight.

MILIN: But could you convince her?

STEPHANIE: Convince who? Michelle? Him?

MILIN: No. Her.

STEPHANIE: Yeah. All right. I've done this before. OK.

CAROLYN MAHER: Take another piece of paper if you want to. You've got to convince me there are eight and only eight, and no more or fewer.

STEPHANIE: All right, first you have without any blues, with just red/red/red.

CAROLYN MAHER: Okay, no blues.

STEPHANIE: Then you have with one blue - blue/red/red or red/blue/red or red/red/blue.

CAROLYN MAHER: Anything else?

MICHELLE: And you would do the same pattern for -

STEPHANIE: No, not with the blue, not with one blue -

MICHELLE: You would do it, you would do it with one red and two blues?

JEFF: You would alternate-

MICHELLE: You would do it the other way around.

CAROLYN MAHER: Let her finish. That's what you would do. You would alternate. Let's see what Stephanie does. Maybe she's-

STEPHANIE: Well, there's no more of these because if you had to go down another one you'd have to have another block on the bottom. Then you have with three blues - well, not with three blues. I'll go like this.

CAROLYN MAHER: You have no blues and now you have exactly one blue.

STEPHANIE: Now you have exactly two blue. You could put blue/blue/red; you could put red, blue and blue.

MILIN: You could put blue, red and blue.

STEPHANIE: Yeah, but that's not what I'm doing. I'm doing it so that they're stuck together.

CAROLYN MAHER: Okay.

JEFF: There should be one - there could be one with one red and then you could break it up and there's one with two reds and there's one with three reds.

MILIN: Ah, but see - you did the same thing, but there's the blue.

JEFF: There's all reds and then there's three reds, two reds. There should be one with one red. And then you change it to blue.

STEPHANIE: Well, that's not how I do it.

CAROLYN MAHER: Let's hear how Steph - We'll hear that other way; that's interesting. Okay, now, so what you've done so far is-

STEPHANIE: One blue, two blue.

CAROLYN MAHER: Okay, no blues-

STEPHANIE: One blue, two blue.

CAROLYN MAHER: One blue, and two blues, but Milan just said you don't have all two blues, and you said - Why is that?

STEPHANIE: All right, so show me another two blue. With them stuck together, because that's what I'm doing.

MILIN: In that case, here.

CAROLYN MAHER: Okay, so now what are you doing, Stephanie?

MICHELLE: What if you just had two blues and they weren't stuck together:

STEPHANIE: But that's what I'm doing. I'm doing the blues stuck together. Then we have three blues, which you can only make one of. Then you want two blues stuck apart - not stuck apart - took apart.

CAROLYN MAHER: Separated?

STEPHANIE: Yeah, separated. And you can go blue/red/blue -

CAROLYN MAHER: The pattern that Stephanie chooses in justifying that she found all possible of three tall towers selecting from two colors is an organization by cases. Now, her organization by cases does not map into the organization that some of her classmates want. She is adamant about her organization, and it works. It's not "the most elegant" organization by cases, but it's an organization by cases.

JEFF: I have a question. Do you have to make a pattern?

MICHELLE: No.

JEFF: So, then why is everybody going by a pattern?

MILIN: Because we liked it.

STEPHANIE: It's easier.

STEPHANIE: It's easier to find than just going, Ooh, there's a pattern!

MICHELLE: 'Cause if you, 'cause if you just keep on guessing like that, you're not sure if there's going to be more.

STEPHANIE: It's easier, maybe, like Shelly and Milan's pattern was to go put this in a different category.

JEFF: We know their pattern.

STEPHANIE: Okay, but what I'm saying is it's easier, it's just easier to work with a pattern since it's like -

MILIN: Oh, here's another one. Let's see -

MICHELLE: Because you might have a duplicate, and then you may not know.

STEPHANIE: It's harder to check. It's harder to check just having them like come up from out of the blue.

MILIN: Then just going like this and getting two -

JEFF: How do you know there's different things in the pattern?

CAROLYN MAHER: We have naturally, in the solution of these problems, thinking of these ways of organizing that account for all possible towers that can be built.

I'm building towers of a particular height, let's say four tall. In building these four tall towers, I could say, "How many towers can I build that have no red? Only one. How many can I build that have exactly one red?" Well, students will argue, eventually, that there are four of those, if they're four tall, one in each position. "How many can I build that have two red?" That's a little tougher. That kind of reasoning pushes students to begin to have to control for variables.

Is anything else possible? Can I build five? The students will say, "No, I can't do that because you told me the towers had to be four tall." In that kind of reasoning, they're beginning to do proof by contradiction. We see that kind of reasoning also, which is an indirect method of proof.

MILIN: And you can't make any more in this one, so you go on to the next one.

JEFF: How do you know you can't make any more?

MILIN: Because there's not any more colors.

STEPHANIE: Look, okay, start here. Start here - okay, you have the three together. The one, one blue, right? You have the one blue. How could I build another one blue?

JEFF: You can't.

STEPHANIE: All right. So, I've convinced you that there's no more one blue. All right -

MICHELLE: But if you didn't have that pattern, it would be harder to convince you.

STEPHANIE: Okay, so I've convinced you that there's no more one blue?

MICHELLE: Then you have to go to two blue.

STEPHANIE: Two blue. Here's one - all right? Two blue - you have one, blue/blue/red, then we have red/blue/blue. How am I supposed to make another one?

JEFF: Blue/red/blue.

STEPHANIE: No, this is the other. Milin gave me that same argument.

JEFF: But the thing is does it matter that they're together?

MICHELLE: She means stuck together.

STEPHANIE: Stuck together, that means like - Okay, so can I make any more of that kind?

JEFF: No.

MICHELLE: Then you have to move the three, which you're trying to make one.

STEPHANIE: Yeah, you can only make one and then you can make without blue with the three red.

MICHELLE: And then you can make two split apart.

STEPHANIE: Two split apart, which you can only make one of, and then you can find that you can - you can find the opposites right in this. All right, so I've convinced you that there's only eight?

JEFF: Yes.

CAROLYN MAHER: When Jeff says, "Why do you have to have a pattern?" that's really a very important question. My theory is that in Jeff's own exploration with the towers and his finding of many patterns and the opposites of those patterns, they got him in trouble when he was eliminating duplicates. And so all of his pattern finding became really an enormous task of trying to keep track. And I have always wondered if when he says, "Why do you have to have a pattern?" if he doesn't mean that in frustration. But there are other interpretations of what Jeff means. I wonder what other people think. Why do you have to have a pattern?

CAROLYN MAHER: How many if you're making towers of four?

MICHELLE, MILIN: 16.

CAROLYN MAHER: Do you agree, Jeff?

JEFF: Yeah. [laughter]

CAROLYN MAHER: Jeff, why do you agree? Don't let them get by so easily. This could be pressure here.

MICHELLE: Just because say you add a red or a blue, you can add a red or a blue here -

CAROLYN MAHER: Make a "Y" or something to show me...

JEFF: I understand because you can only -

MICHELLE: You can put two colors here, two colors there, two colors - and keep on going.

JEFF: Yeah, you can keep on doing two colors for each one. And that's two, four, six, eight, ten, twelve, fourteen, sixteen.

CAROLYN MAHER: And so that's the towers of-?

JEFF: Four.

CAROLYN MAHER: You made towers of five in class and what did you get?

STEPHANIE: 32.

CAROLYN MAHER: Does that work the same way?

STEPHANIE: Yeah.

MICHELLE: If you get towers of four -

STEPHANIE: The hard part is to make the pattern. Like, from now, we know how to just - Oh, you could give us a problem like how many in 10 and we could just go -

CAROLYN MAHER: How many in 10, and you'd know the answer?

STEPHANIE: Yeah, I know the answer. I figured it out. It's 1,024!

CAROLYN MAHER: 1,024.

ALICE ALSTON: Are you sure?

STEPHANIE: Uh-huh.

JEFF: Don't try to convince her.

CAROLYN MAHER: Try to convince me. [laughter]

JEFF: No!

STEPHANIE: Just give me a couple thousand Unifix® cubes!

CAROLYN MAHER: You can do that later.

NARRATOR: We have seen fourth graders naturally develop two mathematically sound methods of proof-making:

One, Inductive --- building taller towers from shorter towers, one at a time, and using known results about the shorter towers to derive results about the taller ones;

and Two, organizing towers of a given size into different cases which could be studied separately.

What are some of the similarities and/or differences in the mathematical reasoning by the teachers and students that you have observed in this program?