PRODUCTS OF DISTINCT WHITTAKER COEFFICIENTS ON THE METAPLECTIC GROUP AND THE RELATIVE TRACE FORMULA

by Cesar Valverde

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Abstract

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In this thesis we obtain a geometric identity between Relative Trace Formula on the metaplectic group and the general linear group. As a consequence of the spectral analysis, we expect to obtain a relation between products of distinct Whittaker coefficients of a cuspidal automorphic representation on the metaplectic group and a non-split period of a related representation on the general linear group. This would generalize famous work of Kohnen and Waldspurger.

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Contents

1. Introduction

Let F be a number field with ring of adeles A. Let G be a reductive group over F and let H be the subgroup of G fixed by an involutive automorphism of G.

Let π be a cuspidal automorphic representation of $G(A)$ and χ be a character of $H(A)$ trivial on $H(F)$. We consider the period integral on the space of π:

$$P_{(H,\chi)}(\phi) = \int\limits_{H(F)\backslash H(A)} \phi(h)\chi(h)dh. \tag{1.1}$$

If there exists ϕ in the space of π with $P_{(H,\chi)}(\phi) \neq 0$, then π is said to be $(H,\chi)-$distinguished.

Such representations are of interest because they are expected to arise as functorial images of Langlands liftings. Moreover, the value of this period integral is expected to be related to special values of L-functions.

Let us explain one classical case of the relation between a period integral and a special value for an L-function. Let k be an even integer, let N be odd and square-free. Let f be a newform on $\Gamma_0(N)$ of weight $2k$ and let g be the Hecke eigenform on $\Gamma_0^+(4N)$ of weight $k+\frac{1}{2}$ associated to f via Shimura correspondence. Let m and n be fundamental discriminants. Kohnen has proved in [Koh] the following equality:

$$\overline{c(n)}c(m) = \frac{< g,g >}{< f,f >}(-1)^{k/2}2^k r_{k,N}(f;n,m), \tag{1.2}$$

here $c(m)$ is the m-th Fourier coefficient of g. The period integral $r_{k,N}$ is given by

$$\sum_{Q=[a,b,c]} \omega_n(Q) \int\limits_{C_Q} \frac{f(z)}{(az^2+bz+c)^{k-1}}dz.$$

Here $Q = [a, b, c]$ runs over a set of $\Gamma_0(N)$ inequivalent integral binary quadratic forms of discriminant $|Q| = nm$ with $N|a$, $\omega_n(Q) \in \{-1, 0, 1\}$. C_Q is the image of $a|z|^2 + b\mathrm{Re}\ z + c = 0$ in $\Gamma_0(N)\backslash H$.

On the other hand, in [Wa1] and [Wa2], Waldspurger has proved that

$$|c(m)|^2 \sim m^{k-\frac{1}{2}} L_f(k, \chi_m) \tag{1.3}$$

where $L_f(k, \chi_m)$ is the L-series attached to f twisted by the real character $\chi_m(n) = \left(\frac{m}{n}\right)$.

Combining equations (1.2) and (1.3), we obtain that the period integral $r_{k,N}(f; m, m)$ is related to the central value of the L-series for f twisted by $\chi_m(n)$. This also follows from results of Waldspurger and Martin and Whitehouse ([MaWh]).

1.1. **Relative Trace Formula.** In order to study distinction, Jacquet introduced the Relative Trace Formula (see [J-L]). For $i = 1, 2$, let H_i be closed subgroups of G with χ_i global automorphic characters of $H_i(A)$ trivial on $H_i(F)$. For f a Schwartz function on $G(A)$, let $K_f(x, y)$ denote the kernel function for the regular representation $\rho(f)$ acting on $L^2(G(F)\backslash G(A))$.

We consider the following distribution:

$$I_G(f : H_1, \chi_1; H_2, \chi_2) = \int_{H_1(F)\backslash H_1(A)} \int_{H_2(F)\backslash H_2(A)} K_f(h_1, h_2)\chi_1(h_1)\chi_2(h_2)\, dh_2\, dh_1. \tag{1.4}$$

The kernel $K_f(x, y)$ has a 'geometric' expression of the form

$$K_f(x, y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y).$$

On the other hand, formally we have that the kernel for the right regular representation by f admits a decomposition where the cuspidal term is of the form

$$\sum_{\Pi}\sum_{\phi_i}(\rho(f)\phi_i)(x)\overline{\phi_i(y)} \tag{1.5}$$

where the first summation is over all irreducible cuspidal representations of G and $\{\phi_i\}$ is an orthonormal basis for the space of Π.

We obtain that the distribution $I_G(f : H_1, \chi_1; H_2, \chi_2)$ is equal to

$$\sum_{\Pi}\sum_{\phi_i}\int_{H_1(F)\backslash H_1(A)}\int_{H_2(F)\backslash H_2(A)}(\rho(f)\phi_i)(h_1)\chi_1(h_1)\overline{\phi_i(h_2)}\chi_2(h_2)dh_1dh_2 \tag{1.6}$$

or

$$\sum_{\Pi}\sum_{\phi_i}P_{(H_1,\chi_1)}(\rho(f)\phi_i)P_{(H_2,\chi_2)}(\overline{\phi_i}). \tag{1.7}$$

Suppose that $f = \otimes f_v \in \mathcal{S}(G(A))$, then for almost all places v, we have that $f_v \in \mathcal{H}(G_v, K_v)$. Suppose that there exists a morphism between the $L-$groups of G' and G. Then by Satake isomorphism, we have a map $\lambda_v : \mathcal{H}(G_v, K_v) \to \mathcal{H}(G'_v, K'_v)$ between the Hecke algebras. We define a set of maps $\{\epsilon_v : \mathcal{S}(G_v) \to \mathcal{S}(G'_v)\}$ to be admissible if for almost all places v, we have $\epsilon_v = \lambda_v$. We say a relative trace identity

$$I_G(f : H_1, \chi_1, H_2, \chi_2) = I_{G'}(f' : H'_1, \chi'_1, H'_2, \chi'_2)$$

holds if there exists a set of admissible maps such that the above equality holds for $f' = \otimes\epsilon_v(f_v)$.

We see that an equality of geometric sides of the form:

$$I_G(f : H_1, \chi_1; H_2, \chi_2) = I_{G'}(f' : H_1', \chi_1'; H_2', \chi_2') \tag{1.8}$$

should lead to a relation between periods of the form (1.7).

We consider the following case. Let η be an irreducible, automorphic, cuspidal representation of $GL_{2n}(A)$ with η self-dual. In this case, $L(s, \eta \otimes \eta)$ has a simple pole at $s = 1$. We have that

$$L(s, \eta \otimes \eta) = L(s, \eta, sym^2) L(s, \eta, \Lambda^2), \tag{1.9}$$

and assume that $L(s, \eta, \Lambda^2)$ has a simple pole at $s = 1$. In this case, η is a lift from the group SO_{2n+1}.

Assume furthermore that $L(1/2, \eta) \neq 0$ and fix a nontrivial additive character ψ of $F \backslash A$. Then the 'backward lift' from η to a representation on SO_{2n+1} is lifted from the metaplectic group $\widetilde{Sp_{2n}}(A)$ via theta correspondence associated to ψ. This suggests, for appropriate f and $\widetilde{f}$, a relative trace identity of the form

$$I_{\mathrm{GL}_{2n}}(f : \mathrm{GL}_n \times \mathrm{GL}_n, 1; N, \theta) = I_{\widetilde{\mathrm{Sp}}_n}(\widetilde{f} : N', \theta'^{-1}; N', \theta'). \tag{1.10}$$

where θ is the non-degenerate character of the maximal standard unipotent N of GL_{2n} defined by

$$\theta(n) = \psi(n_{1,2} + \ldots + n_{2n-1,2n}) \tag{1.11}$$

and θ' is the degenerate character of the maximal standard unipotent N' of Sp_n given by

$$\theta'(n) = \psi(n_{1,2} + \ldots + n_{n,n+1}). \tag{1.12}$$

The equality of the geometric sides of equation (1.10) was proved by Mao and Rallis in [MR].

Now fix a non-square $\tau \in F^\times$ and let K denote the quadratic extension of F given by τ. The map $a + b\tau \mapsto \begin{pmatrix} a & b\tau \\ b & a \end{pmatrix}$ induces an embedding of $GL_n(K)$ into $GL_{2n}(F)$. We denote the image of this map by $GL_{(n,K)}(F)$. We define a degenerate character θ'_τ on N' by

$$\theta'_\tau(n') = \psi(n'_{1,2} + \ldots + n'_{n-1,n} + \tau n'_{n,n+1}). \tag{1.13}$$

The purpose of the present thesis is to obtain the following equality of geometric sides of relative trace formula:

Theorem 1.1. *We have the relative trace identity*

$$I_{GL_{2n}}(f : GL_{(n,K)}, 1; N, \theta) = I_{\widetilde{Sp_n}}(\tilde{f} : N', \theta'^{-1}_\tau; N', \theta'). \tag{1.14}$$

This suggests, for a representation Π lifted from Π' and for $\{\varphi_\alpha\}$ and $\{\widetilde{\varphi}_\alpha\}$ orthonormal bases of Π and Π' respectively, the following identity:

$$\sum_{\varphi_\alpha} \mathcal{P}_\tau(\Pi(f)(\varphi_\alpha))\overline{\mathcal{W}(\varphi_\alpha)} = \sum_{\widetilde{\varphi_\alpha}} \widetilde{\mathcal{W}_\tau}(\Pi'(\widetilde{f})(\widetilde{\varphi}_\alpha))\overline{\widetilde{\mathcal{W}}(\widetilde{\varphi}_\alpha)} \tag{1.15}$$

where $\mathcal{W}$, $\widetilde{\mathcal{W}}$ and $\widetilde{\mathcal{W}_\tau}$ are Whittaker functionals and the 'non-split' period is given by

$$\mathcal{P}_\tau(\Pi(f)\varphi_\alpha) = \int\limits_{C_{2n}(A)GL_{n,K}(F)\backslash GL_{n,K}(A)} (\Pi(f)\varphi_\alpha)(h)dh \tag{1.16}$$

where C_{2n} denotes the center of GL_{2n}.

Equation 1.16 can be thought of as a generalization of equation (1.2). The work of Friedberg and Jacquet in [FJ] and of Friedberg and Bump in [BF] gives that

$$\mathcal{P}_1(\phi) \sim L(1/2,\pi)\mathrm{Res}_{s=1}L(s,\pi,\Lambda^2)W(\phi) \tag{1.17}$$

where π is an automorphic cuspidal representation of G, ϕ is some cusp form in the space of π and $L(s,\pi,\Lambda^2)$ is the partial exterior square L function. By $\sim$, we mean equality up to local factors.

Combining equation (1.17) and equation (1.15) for $\tau = 1$, we have:

$$|\widetilde{\mathcal{W}}(\widetilde{\varphi}_\alpha)|^2 = L(1/2,\pi)\mathrm{Res}_{s=1}L(s,\pi,\Lambda^2)|\mathcal{W}(\varphi_\alpha)|^2. \tag{1.18}$$

This would generalize equation (1.3). From equations (1.18) and (1.15) one obtains

$$|\mathcal{P}_\tau(\varphi_\alpha)|^2 \sim L(1/2,\pi)L(1/2,\pi\otimes\chi_\tau)\mathrm{Res}_{s=1}L(s,\pi,\Lambda^2)^2|\mathcal{W}(\varphi_\alpha)|^2 \tag{1.19}$$

where χ_τ is a quadratic character of idele class group $A^\times/F^\times$ attached to the quadratic extension K. Thus we would recover an equation from work of Guo in [G] and upcoming work of Feigon, Whitehouse and Martin ([FWM]). In the case $n = 1$, we would obtain results of Waldspurger and Martin and Whitehouse ([MaWh]).

1.2. **Sketch of proof.** Let us sketch the method of descent. Let τ be an irreducible, automorphic, cuspidal, self dual representation of $GL_{2n}(A)$ with $L(s,\tau,\Lambda^2)$ having a pole at $s = 1$ and with $L(1/2,\tau) \neq 0$. We construct the Eisenstein series $E(g, f^{\phi}_{\tau,s})$ (precise definition in equation (3.13)), it has a pole at $s = 1$. We consider the residual Eisenstein series $E_1(g,\phi) = Res_{s=1}E(g, f^{\phi}_{\tau,s})$.

We consider the space $V_{\tau,k}$ spanned by Fourier-Jacobi type coefficients of the residual Eisenstein series, in other words, we consider the space $V_{\tau,k}$ spanned by functions $p_k(h)$ with

$$p_k(h) = p_k(h,\phi) = \int\limits_{N^k(F)\backslash N^k(A)} E_1(vh,\phi)\Theta_{\psi_k^{-1}}(j_k(v)h)\psi_k^{-1}(v)dv.$$

Here $\Theta_{\psi_k^{-1}}$ is a Theta series and N^k is given by

$$N^{(k)} = \left\{ n = \begin{pmatrix} z & u & * & * & * \\ & 1 & x & y & * \\ & & I_{2k} & x' & * \\ & & & 1 & u' \\ & & & & z^* \end{pmatrix} |z \in N_{2n-(k+1)} \right\}. \tag{1.20}$$

The representation $\sigma_k(\tau)$ of $\widetilde{Sp}_k$ is obtained by right traslation on the space $V_{\tau,k}$.

By ([GRS2, Main Theorem (global)]), this space is nonzero when $k = n$. To prove that $V_{\tau,k} = 0$ for $k < n$, the authors make use of two observations. First, the $Sp_n \times Sp_n$ period of the residual Eisenstein series $E_1(g,\phi)$ is related to the $GL_n \times GL_n$ period of ϕ, this is [GRS1, Theorem 2]. In particular, because of our assumptions on τ, this period is nonzero. On the other hand, in [GRS1, Section 3] it is proved that the existence of non-trivial $Sp_n \times Sp_n$ period implies that $V_{\tau,k}$ is zero.

In the present case, we prove in Theorem 3.10 that the residual Eisenstein series $E_1(g,\phi)$ has $Sp_{(n,K)}$ period which is related to the $GL_{(n,K)}$ period of ϕ. Reflecting this relation we prove in Section 3, through a matching of relevant orbits, the equation:

$$I_{GL_{2n}}(f : GL_{n,K}, 1; N_1, \theta) = I_{Sp_{2n}}(f' : Sp_{n,K}, 1; N_3, \theta_3). \tag{1.21}$$

It is possible for the residual Eisenstein series to have nontrivial degenerate Whittaker model and Fourier-Jacobi model. The next identity reflects this fact:

$$I_{Sp_{2n}}(f' : Sp_{n,K}, 1; N_3, \theta_3) = I_{Sp_{2n}}(f'' : Sp_{n,K}, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}}). \tag{1.22}$$

Here $\Theta^{\Phi}_{\psi^{-1}}$ is a Theta series defined below by equation (4.37). This is proved using global methods of Ginzburg, Soudry and Rallis, as applied by Mao and Rallis in [MR]. This is carried out in Section 4. The main obstacle is to prove that $Sp_{n,K}$-invariant functionals and $(N^{(k)}, \chi_{k,\alpha})$-eigenfunctionals are disjoint. For $Sp_n \times Sp_n$ this is in [GRS1, Section 3.2]; it is Theorems 4.2 and 4.4 in the present paper.

Finally, the standard method of comparing orbital integrals proves the identity

$$I_{Sp_{2n}}(f'' : Sp_{n,K}, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}}) = I_{\widetilde{Sp_n}}(\tilde{f} : N', \theta'^{-1}_{\tau}; N', \theta'). \tag{1.23}$$

In Section 5 we compute the orbital integrals arising from equation (1.23); in Section 6, we reduce their comparison to a suitable fundamental lemma.

In Section 7, we prove the fundamental lemma. The unit Hecke element case is done by a calculation, while the general Hecke element case follows from a Plancherel formula, as in [MR1]. This argument is detailed in Section 7. Section 8 proves the main theorem.

The Placherel formula needed in Section 7 follows from an explicit calculation of spherical functions on $Sp_{n,K}\backslash Sp_{2n}$ and on $N_2\backslash\widetilde{Sp_n}$. The first calculation is done in Appendix A, more general results are obtained in [Sak]. The calculation for the second case is done in Appendix B and follows [BFH]. Appendix C defines the orbital integral of spherical functions on $Sp_{n,K}\backslash Sp_{2n}$ and proves that this corresponds, up to constants, to spherical functions on $N_2\backslash\widetilde{Sp_n}$.

2. Preliminaries

• F is a number field with ring of adeles A, the completion of F at a local place v is denoted F_v.

• $\tau \in F \backslash F^2$.

• K is the quadratic extension of F given by τ.

• T is the square block diagonal matrix of size $2n$ with diagonal consisting of the 2×2 matrices $\begin{pmatrix} & \tau \\ 1 & \end{pmatrix}$.

• $\mathbf{T}$ is the square block diagonal matrix of size $4n$ with diagonal consisting of the 2×2 matrices $\begin{pmatrix} & \tau \\ 1 & \end{pmatrix}$.

• τ_n (resp. 1_n) denote the $n \times n$ diagonal matrices consisting of τ (resp. 1) on the diagonal.

• $\mathrm{E} = \begin{pmatrix} & \tau_n & & \\ 1_n & & & \\ & & & \tau_n \\ & & 1_n & \end{pmatrix}$

• $\sigma = \begin{pmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ 1 & & & \end{pmatrix}$

• $\mathrm{J} = \begin{pmatrix} & -\sigma \\ \sigma & \end{pmatrix}$

• $\mathrm{Sp}_n = \{g \in GL_{2n} | {}^t g J g = J\}$

• $\mathrm{GSp}_n = \{g \in GL_{2n} | {}^t g J g = \lambda(g) J; \lambda(g) \in F^\times\}$

- ψ is either a nontrivial additive character of A/F or of F_v.
- $\mathrm{GL}_{n,K} = \{g \in GL_{2n} | g^{-1}Tg = T\}$
- N_1 =standard maximal unipotent for GL_{2n}
- A_1 =set of diagonal matrices in GL_{2n}
- W_1 =Weyl group corresponding to A_1
- $\mathrm{N}_{1,g} = \{n \in N_1 | n^{-1}gn = g\}$
- $\mathrm{N}'_{1,\gamma} = \gamma^{-1}H_1\gamma \cap N_1$
- θ_1 is a character of N_1 with $\theta_1(n) = \psi(n_{1,2} + ... + n_{2n-1,2n})$
- $\mathrm{Sp}_{n,K} = \{g \in Sp_{2n} | g^{-1}Tg = T\}$
- N_3 =standard maximal unipotent for Sp_{2n}
- A_3 =set of diagonal matrices in Sp_{2n}
- W_3 =Weyl group corresponding to A_3
- P_3 =Maximal Siegel parabolic in Sp_{2n}
- V_3 = Siegel unipotent radical for Sp_{2n}
- K_3 =Maximal compact subgroup of Sp_{2n}
- $\mathrm{K}_{Sp_{n,K}} = K_3 \cap Sp_{n,K}$
- $\mathrm{V}_{Sp_{n,K}} = V_3 \cap Sp_{n,K}$
- $\mathrm{N}_{3,g} = \{n' \in N_3 | n'^{-1}gn' = g\}$
- $\mathrm{N}'_{3,\gamma} := \gamma^{-1}H_3\gamma \cap N_3$
- θ_3 is a (degenerate) character of N_3 with $\theta_3(n) = \psi(n_{1,2} + ... + n_{2n-1,2n})$
- $\mathrm{g}^* = \sigma^t g^{-1}\sigma$
- S_n is the set of matrices $g \in \mathrm{GL}_n$ satisfying $\sigma_n g$ is a symmetric matrix.
- For $g \in GL_n$, the map $i_1 : GL_n \to Sp_n$ is given by $i_1(g) = \begin{pmatrix} g & \\ & g* \end{pmatrix}$.

- For $g \in GL_n$, the map $i_\tau : GL_n \to GSp_n$ is given by $i_\tau(g) = \begin{pmatrix} g & \\ & \tau g* \end{pmatrix}$.

Let F be a number field with ring of adeles A and let τ a nonsquare in F. Denote by K the quadratic extension $F[\sqrt{\tau}]$, then an element in K embeds in $GL_2(F)$ via $a + b\sqrt{\tau} \mapsto \begin{pmatrix} a & b\tau \\ b & a \end{pmatrix}$, and this naturally extends to an embedding of $GL_n(K)$ into $GL_{2n}(F)$. We denote the image of $GL_n(K)$ in $GL_{2n}(F)$ under this embedding by $GL_{(n,K)}(F)$ and remark that if we denote by T the nonsplit torus

$\begin{pmatrix} & \tau & & & & \\ 1 & & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & & \tau \\ & & & & 1 & \end{pmatrix}$, then $GL_{(n,K)}(F)$ consists of $g \in GL_{2n}(F)$ with $g^{-1}Tg = T$.

We define an injection $j : Sp_n \to Sp_{2n}$ by $j(g) = \begin{pmatrix} 1_n & & \\ & g & \\ & & 1_n \end{pmatrix}$

$G_2 = \widetilde{Sp_n}$

N_2 =maximal unipotent for Sp_n

$N_{2,w'a'} = (w'a')^{-1}N_2(w'a') \cap N_2$

$N'_{2,w'a'} := (w'a')N_2(w'a')^{-1} \cap N_2$

$I_2 = I_{\widetilde{Sp_{n,K}}}(\widetilde{f} : N_2, \theta_2^{-1}; N_2, \theta_2)$

U^1_{wa} is defined by equation (5.15)

U^2_{wa} is defined by equation (5.17)

$\theta_2(n) = \psi(n_{1,2} + ... + n_{n-1,n} + n_{n,n+1})$ for $n \in N_2$

$\theta_{2,\tau}(n) = \psi(n_{1,2} + ... + n_{n-1,n} + \tau n_{n,n+1})$ for $n \in N_2$

2.0.1. *Characters.* • ψ is either a nontrivial additive character of A/k or of k_v.

• θ_1 is a character of N_1 with

$$\theta_1(n) = \psi(n_{1,2} + \ldots + n_{2n-1,2n}).$$

• θ_2 is a character of N_2 with

$$\theta_2(n) = \psi(n_{1,2} + \ldots + n_{n-1,n} + n_{n,n+1}).$$

• θ_3 is a character of N_3 with

$$\theta_3(n) = \psi(n_{1,2} + \ldots + n_{2n-1,2n}).$$

• θ_4 is a character of N_3 defined by (4.38).

2.0.2. *Weil representation and Theta function.* Recall for a fixed ψ, the Weil representation is defined for the metaplectic group $\widetilde{\mathrm{Sp}}_n$. We use $\gamma(*, \psi)$ to denote the Weil constant, and ω_ψ to denote the Weil representation. We describe explicitly a model of the Weil representation.

Let $\Phi \in \mathcal{S}(A^n)$. Then

$$\omega_\psi(\widetilde{m(g)})\Phi(X) = |\det g|^{1/2} \frac{\gamma(1, \psi)}{\gamma(\det g, \psi)} \Phi(Xg), \ g \in \mathrm{GL}_n . \tag{2.1}$$

$$\omega_\psi(\left(\begin{smallmatrix} 1_n & V \\ & 1_n \end{smallmatrix}\right), 1)\Phi(X) = \psi(\mathrm{tr}(X^t V \sigma_n X))\Phi(X), \ V \in \mathcal{S}_n. \tag{2.2}$$

$$\omega_\psi(\widetilde{J_n})\Phi(X) = \gamma(1, \psi)^{-n} \widehat{\Phi}(X), \tag{2.3}$$

where

$$\widehat{\Phi}(X) = \int_{A^n} \psi(\mathrm{tr}(X^t \sigma_n Y))\Phi(Y)\, dY.$$

The above describes the action of the metaplectic group on $\mathcal{S}(A^n)$ under the Weil representation.

We use $\Theta^{\Phi}_{\psi^{-1}}$ to denote the Theta function defined by (4.37).

• The space $\mathcal{S}(G(F_v))$ of Schwartz functions on a reductive group G over a local field F_v at a non-archimedean place v consists of smooth functions of local support; at an archimedean place, we use the definition of Casselman.

3. The trace identity between GL_{2n} and Sp_{2n} and an identity of periods

We have a local correspondence of Schwartz functions on $Sp_{2n}(F_v)$ and $GL_{2n}(F_v)$ given by $f_v \in \mathcal{S}(Sp_{2n}(F_v)) \mapsto f_v^{'} \in \mathcal{S}(GL_{2n}(F_v))$ where f_v' is given by

$$f_v'(g) = \int\limits_{u \in V_3(F_v)} \int\limits_{k \in K_3 \cap Sp_{n,K}(F_v)} f_v(k i_1(g) u) |\det(g)|_v^{n+1} dk du \tag{3.1}$$

where K_3 and V_3 are the maximal compact subgroup of Sp_{2n} and the unipotent radical for the maximal Siegel parabolic in Sp_{2n} respectively.

From [JR], one expects a relation between the inner period on a Levi subgroup and the outer period on the group. In our case, the embedding of GL_{2n} as the Levi subgroup of the Siegel parabolic of Sp_{2n} suggests the following relative trace formula:

Theorem 3.1. *Let $f = \otimes f_v \in \mathcal{S}(Sp_{2n}(A))$ and $f' = \otimes f_v' \in \mathcal{S}(GL_{2n}(A))$ where for all v, f_v and f_v' are related by correspondence (3.1) then*

$$I_{GL_{2n}}(f : GL_{(n,K)}, 1; N_1, \theta_1) = I_{Sp_{2n}}(f' : Sp_{(n,K)}, 1; N_3, \theta_3). \tag{3.2}$$

Moreover, at a p-adic place v, the map $f_v \mapsto f_v'$ restricts to a Hecke algebra homomorphism.

3.1. **Comparison of orbits.** We introduce a space isomorphic to $GL_{n,K} \backslash GL_{2n}$ (resp. $Sp_{n,K} \backslash Sp_{2n}$), namely, we define an involution θ on GL_{2n} (resp. Sp_{2n}) given by $\theta(g) = TgT^{-1}$, then the space $Y_1 = \{g^{-1}\theta(g)T | g \in GL_{2n}\}$ (resp. $Y_3 = \{g^{-1}\theta(g)T | g \in Sp_{2n}\}$) satisfies $Y_1 \cong GL_{n,K} \backslash GL_{2n}$ (resp. $Y_3 \cong Sp_{n,K} \backslash Sp_{2n}$). We exhibit a Bruhat decomposition for elements in these spaces. We first state a well known lemma:

Lemma 3.2. *Let U be an algebraic connected unipotent group over F. Let ϑ be an automorphism of $U(F)$ with $\vartheta^2 = 1$. If $x \in U(F)$ verifies $x\vartheta(x) = 1$ then there is $u \in U$ with $x = \vartheta(u^{-1})u$.* □

Lemma 3.3. *If $y = g^{-1}Tg \in Y_1$ (resp. Y_3), then y admits the decomposition $y = n^{-1}wan$, with $n \in N_1$ (resp. N_3), $w \in W_1$ (resp. W_3), $a \in A_1$ (resp. A_3)*

Proof. By the Bruhat decomposition we write $y = n_1^{-1}wan_2$. Since $\tau y^{-1} = y$, we have that $\tau n_2^{-1}(wa)^{-1}n_1 = n_1^{-1}(wa)n_2$, this implies that $(wa)^2 = \tau$, so that we get $n_2^{-1}wan_1 = n_1^{-1}wan_2$, i.e. $n_2n_1^{-1}wan_2n_1^{-1} = wa$. Write $n = n_2n_1^{-1}$, then $y = n_2^{-1}nwan_2$ so that we may assume $y = nwa$ with $nwan = wa$.

For any $w \in W$, there exists a subgroup $N_w \subset N$, namely $N_w = wNw^{-1} \cap N$, with the property that $w^{-1}N_ww = N_w$. Thus if $y = nwa$ with $nwan = wa$ then $n \in N_w$ and $n^{-1}(wa)^{-1}n^{-1}wa = 1$. We define an involution ϑ on N_w given by $\vartheta(n) := (wa)^{-1}nwa$. We have that $n^{-1}\vartheta(n^{-1}) = 1$ so by Lemma 3.2, there exists $u \in N_w$ with $n^{-1} = \vartheta(u^{-1})u$, i.e. $n^{-1} = (wa)^{-1}u^{-1}wau$ so that $y = u^{-1}\frac{wa}{\tau}u(wa)(wa) = u^{-1}(wa)u$ as desired. $\square$

The right action of N_1 on GL_{2n} (resp. N_3 on Sp_{2n}), composed with the map $GL_{2n} \to Y_1$ (resp. $Sp_{2n} \to Y_3$) gives rise to an action of N_1 on Y_1 (resp. N_3 on Y_3) by conjugation. Upon observing that if $g \in Sp_{2n}$ then ${}^t(g^{-1}Tg)J(g^{-1}Tg) = \tau J$, we get the following:

Corollary 3.4. *The orbits of N_1 (resp. N_3) in Y_1 (resp. Y_3) admit representatives of the form wa with $(wa)^2 = \tau$ (resp. $(wa)^2 = \tau$ and ${}^t(wa)J(wa) = \tau J$)*

Definition 3.5. *We call wa (resp. $w'a'$) relevant if θ_1 (resp. θ_3) is trivial on $N_{1,wa}$ (resp. $N_{3,w'a'}$).*

We remark that if $g^{-1}Tg = w'a'$ then $w'a'$ is relevant if and only if θ_3 is trivial on the set of $n \in N_3$ with $n^{-1}w'a'n = w'a'$, this condition is equivalent to $n^{-1}g^{-1}Tgn = g^{-1}Tg$ which is equivalent to $gng^{-1} \in Sp_{n,K}$ and equivalent to $n \in g^{-1}Sp_{n,K}g \cap N_3 = N'_{3,g}$. Therefore we say that $g_3 \in Sp_{n,K}\backslash Sp_{2n}/N_3$ is relevant if θ_3 is trivial on N'_{3,g_3}. A similar computation leads us to define $g_1 \in GL_{n,K}\backslash GL_{2n}/N_1$ to be relevant if θ_1 is trivial on N'_{1,g_1}.

Lemma 3.6. *The element $w'a'$ with $w' \in W_3$ and a' a diagonal matrix of size $4n$ satisfying $(w'a')^2 = \tau$ and ${}^t(w'a')J(w'a') = \tau J$ is relevant if and only if $w'a' = i_\tau(wa)$ where $w \in W_1$, a is a diagonal matrix of size $2n$, $(wa)^2 = \tau$ and wa is relevant.*

Proof. It's easy to check that $i_\tau(wa)$ is relevant if and only if wa is; we show that for $w'a'$ not of the desired form, we can find a unipotent n' with the property that $n'^{-1}w'a'n' = w'a'$ and $\theta_3(n') \neq 1$. Let us write $w' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A, B, C, D represent $2n \times 2n$ blocks; clearly, if B is zero, so is C and $w'a'$ is of the desired form, so let us assume that B is nonzero. B is anti-symmetric with respect to the anti-diagonal, so let i be the smallest positive integer such that $w'(i) \geq 4n + 2 - i$, we consider the root $X_{i-1,i}$.

Claim 1. The element $w'a'$ cannot map $X_{i-1,i}$ to a positive non-simple root.

Suppose $w'a'$ maps $X_{i-1,i}$ to a positive non-simple root. We consider the matrix n' with 1 on the diagonal and with entries $n'_{l,k}$ given by x if $(l, k) = (i-1, i)$, x multiplied by the elements in the $\sigma(i-1)$ row and the $\sigma(i)$ column of $w'a'$ divided by τ if $(l, k) = (\sigma(i-1), \sigma(i))$, $-x$ if $(l, k) = (4n-i+1, 4n-i+2)$ and $-x$ multiplied by the elements in the $\sigma(4n-i+1)$ row and the $\sigma(4n-i+2)$ column divided by τ if $(l, k) = (\sigma(4n-i+1), \sigma(4n+2-i))$. One has that $n' \in Sp_{2n}$ with $n'^{-1}w'a'n' = w'a'$, and $\theta_3(n') \neq 1$ contradicting relevancy and proving our claim.

Claim 2. $w'(j) \neq 4n + 1 - j$ $\forall j$.

We let $\alpha = w'a'$ and assume $w'(j) = 4n+1-j$. The $(j, 4n+1-j)$ coordinate of ${}^t\alpha J\alpha = \tau J$ is given by $\tau J_{j,4n+1-j} = -\tau$. On the other hand, we have that $\alpha_{4n+1-j,j}J_{4n+1-j,j}\alpha_{j,4n+1-j} =$

τ, this is absurd and proves our claim.

Claim 3. $w'(i) = 4n + 2 - i, w'(i-1) = 4n + 1 - i$.

By minimality of i, we have that $w'(i-1) \leq 4n+1-(i-1)$. By claim 2, we may assume $w'(i-1) \leq 4n - i + 1$. On the other hand, Claim 1 implies that $w'(i-1) \geq w'(i) - 1 \geq 4n - i + 1$, thus $w'(i-1) = 4n - i + 1$ and using $w'(i-1) = w'(i) - 1$, we have that $w'(i) = 4n - i + 2$ and our claim is proved.

Claim 4. An element $w'a'$ as before with $(w'a')^2 = \tau$, ${}^t(w'a')J(w'a') = \tau J$ and w' satisfying Claim 3 is nonrelevant.

To prove Claim 4 we are reduced to considering the case $\beta = \begin{pmatrix} & & b\tau & \\ & & & -b\tau \\ 1/b & & & \\ & -1/b & & \end{pmatrix}$.

In this case the we consider $n = \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{pmatrix}$. We have that n stabilizes β through conjugation and that n has a nontrivial character action, contradicting the relevancy of β.

Therefore, if $w'a'$ is relevant with $w' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then $B = C = 0$. This proves that $w'a'$ is of the form $w'a' = i_\tau(wa)$. $\square$

3.2. **Computation and comparison of the distributions.**

Lemma 3.7. *The bijection between relevant $w'a'$ and relevant wa induces a bijection between representatives of relevant orbits $g_1 \in GL_{n,K} \backslash GL_{2n}/N_1$ and $g_3 \in Sp_{n,K} \backslash Sp_{2n}/N_3$ where $g_1^{-1}Tg_1 = wa$ and $g_3^{-1}Tg_3 = w'a'$ This bijection is given by $g_1 \mapsto g_3 = i_1(g_1)$.*

Proof. It is easy to see that if $g_1^{-1}Tg_1 = wa$ then $i_1(g_1)^{-1}Ti_1(g_1) = w'a'$. On the other hand, suppose $w'a'$ is relevant, then we know $w'a' = i_\tau(wa)$. We have that if $w'a'$ is of the form $w'a' = g_3^{-1}Tg_3$, then the eigenvalues of $w'a'$ are $\sqrt{\tau}$ and $-\sqrt{\tau}$ with multiplicities $2n$ respectively. We note that the set of eigenvalues of $w'a'$ is the union of the set of eigenvalues for wa and the set of eigenvalues for $\tau(wa)^*$. Using that $(wa)^2 = \tau$, we get that wa has zero trace. Thus the eigenvalues of wa consist of $\sqrt{\tau}$ and $-\sqrt{\tau}$ with multiplicities n respectively. Such wa are seen to be of the form $wa = g_1^{-1}Tg_1$, this proves our lemma. □

Once we have this bijection we proceed with a formal proof of the relative trace identity we have in mind. We have

$$\begin{aligned} I_{Sp_{2n}}(f' : Sp_{(n,K)}, 1; N_3, \theta_3) &= \int\limits_{l_3 \in Sp_{n,K}(F)\backslash Sp_{n,K}(A)} \int\limits_{n_3 \in N_3(F)\backslash N_3(A)} K_f(l_3, n_3)\theta_3(n_3)dl_3dn_3 \\ &= \int\limits_{l_3 \in Sp_{n,K}(F)\backslash Sp_{n,K}(A)} \int\limits_{n_3 \in N_3(F)\backslash N_3(A)} \sum_{g_3 \in Sp_{2n}(F)} f_3(l_3^{-1}g_3n_3)\theta_3(n_3)dl_3dn_3 \\ &= \sum_{g_3 \in Sp_{n,K}\backslash Sp_{2n}/N_3} \int\limits_{l_3 \in Sp_{n,K}(A)} \int\limits_{n_3 \in N'_{3,g_3}(F)\backslash N_3(A)} f_3(l_3^{-1}g_3n_3)\theta_3(n_3)dl_3dn_3 \end{aligned}$$

where $N'_{3,g_3} := g_3^{-1}Sp_{n,K}g_3 \cap N_3$.

We factor this integral as

$$\sum_{g_3 \in Sp_{n,K}\backslash Sp_{2n}/N_3} \int\limits_{l_3 \in Sp_{n,K}(A)} \int\limits_{n_3 \in N'_{3,g_3}(A)\backslash N_3(A)} \int\limits_{m_3 \in N'_{3,g_3}(F)\backslash N'_{3,g_3}(A)} f_3(l_3^{-1}g_3m_3n_3)\theta_3(n_3)\theta_3(m_3)dl_3dn_3dm_3$$

or

$$\sum_{g_3\in Sp_{n,K}\backslash Sp_{2n}/N_3}\int_{l_3\in Sp_{n,K}(A)}\int_{n_3\in N'_{3,g_3}(A)\backslash N_3(A)} f_3(l_3^{-1}g_3m_3n_3)\theta_3(n_3)dl_3dn_3 \int_{m_3\in N'_{3,g_3}(F)\backslash N'_{3,g_3}(A)} \theta_3(m_3)dm_3.$$

The integral over m_3 is zero if g_3 is non-relevant. We get that $I_{Sp_{2n}}(f' : Sp_{(n,K)}, 1; N_3, \theta_3)$ is given by

$$(3.3)\qquad \sum_{g_3\in Sp_{n,K}\backslash Sp_{2n}/N_3 relevant}\int_{l_3\in Sp_{n,K}(A)}\int_{n_3\in N'_{3,g_3}(A)\backslash N_3(A)} f_3(l_3^{-1}g_3n_3)\theta_3(n_3)dl_3dn_3.$$

A similar computation shows that $I_{GL_{2n}}(f : GL_{(n,K)}, 1; N_1, \theta_1)$ is equal to

$$(3.4)\qquad \sum_{g_1\in GL_{n,K}\backslash GL_{2n}/N_1, relevant}\int_{l_1\in GL_{n,K}(A)}\int_{n_1\in N'_{1,g_1}(A)\backslash N_1(A)} f_1(l_1^{-1}g_1n_1)\theta_1(n_1)dl_1dn_1.$$

Lemma 3.8. *(Comparison of the distributions). For any relevant $g_1 \in GL_{2n}$ and $g_3 \in Sp_{2n}$ with $g_3 = i_1(g_1)$, for any place v of F, with f'_v given by correspondence (3.1), we have*

$$|\det(g_1)|_v^{-n-1}\int_{l_1\in GL_{n,K}(F_v)}\int_{n_1\in N'_{1,g_1}(F_v)\backslash N_1(F_v)} f'_v(l_1^{-1}g_1n_1)\theta_1(n_1)dl_1dn_1$$
$$= \int_{l_3\in Sp_{n,K}(F_v)}\int_{n_3\in N'_{3,i_1(g_1)}(F_v)\backslash N_3(F_v)} f_3(l_3^{-1}i_1(g_1)n_3)\theta_3(n_3)dl_3dn_3.$$

Proof. We fix a place v and drop the reference to F_v in the notation. We write $g_3 = i_1(g_1)$, we have to consider

$$\int_{l_3\in Sp_{n,K}}\int_{n_3\in N'_{3,g_3}\backslash N_3} f_3(l_3^{-1}i_1(g_1)n_3)\theta_3(n_3)dl_3dn_3.$$

We write $n \in N_3$ as $n_3 = vi(n_1)$ with $v \in V_3$ and $n_1 \in N_1$, then $\theta_3(n_3) = \theta_1(n_1)$. An explicit computation shows that if $n_3 \in N'_{3,g_3}$ then $n_1 \in N'_{1,g_1}$ and $i_1(g_1)vm(i_1(g_1))^{-1} \in V_3 \cap Sp_{n,K}$.

Our integral is

$$\int\limits_{l_3 \in Sp_{n,K}} \int\limits_{n_1 \in N'_{1,g_1} \backslash N_1} \int\limits_{v \in V_3 \cap Sp_{n,K} \backslash V_3} f_3(l_3^{-1} i_1(g_1) v i(n_1)) \theta_1(n_1) dl_3 dn_1.$$

We make a change of variables $v \mapsto i_1(g_1)^{-1} v i_1(g_1)$ to get

$$\int\limits_{l_3 \in Sp_{n,K}} \int\limits_{n_1 \in N'_{1,g_1} \backslash N_1} \int\limits_{v \in V_3 \cap Sp_{n,K} \backslash V_3} f_3(l_3^{-1} v i_1(g_1 n_1)) \theta_1(n_1) |\det(g_1)|^{-(n+1)} dl_3 dn_1 dv.$$

From the Iwasawa decomposition one has that $Sp_{n,K} = (P_3 \cap Sp_{n,K})(K_3 \cap Sp_{n,K})$, so we write $l_3^{-1} = k i_1(h_1) u$ with $k \in K_3 \cap Sp_{n,K}, u \in V_3 \cap Sp_{n,K}, h_1 \in GL_{n,K}$, we combine the u and v integrals to get

$$\int\limits_{k \in K_3 \cap Sp_{n,K}} \int\limits_{h_1 \in GL_{n,K}} \int\limits_{n_1 \in N'_{1,g_1} \backslash N_1} \int\limits_{u \in V_3} f_3(ki(h_1) u i_1(g_1 n_1)) \theta_1(n_1) |\det(h_1 g_1^{-1})|^{-(n+1)} du dn_1 dh_1 dk.$$

We change variables $u \mapsto i(g_1 n_1) u i(g_1 n_1)^{-1}$, we obtain

$$\int\limits_{k \in K_3 \cap Sp_{n,K}} \int\limits_{h_1 \in GL_{n,K}} \int\limits_{n_1 \in N'_{1,g_1} \backslash N_1} \int\limits_{u \in V_3} f_3(ki(h_1 g_1 n_1) u) \theta_1(n_1) |\det(h_1)|^{n+1} du dn_1 dh_1 dk.$$

We may write this as

$$|\det(g_1)|^{-n-1} \int\limits_{h_1 \in GL_{n,K}} \int\limits_{n_1 \in N'_{1,g_1} \backslash N_1} \int\limits_{k \in K_3 \cap Sp_{n,K}} \int\limits_{u \in V_3} f_3(ki(h_1 g_1 n_1) u) |\det(h_1 g_1)|^{n+1} \theta_1(n_1) du\, dk\, dh_1\, dn_1$$

$$= |\det(g_1)|^{-n-1} \int\limits_{h_1 \in GL_{n,K}} \int\limits_{n_1 \in N'_{1,g_1} \backslash N_1} f'(h_1 g_1 n_1) \theta_1(n_1) dn_1 dh_1$$

which proves the lemma. $\square$

Proof of Theorem 3.1. As the product of $|\det(g_1)|_v$ over all places equals 1, we get the equality of the distributions from equations (3.4), (3.3) and Lemma 3.8.

Now we work over a p-adic field F. For $z \in \mathbb{C}^{2n}$, let χ_z be an unramified character on A_1 given by

$$\chi_z \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{2n} \end{pmatrix} = |a_1|^{z_1} \dots |a_{2n}|^{z_{2n}}. \tag{3.5}$$

We now define, for $f \in \mathcal{H}(Sp_{2n}, K_3)$, $f' \in \mathcal{H}(GL_{2n}, K_1)$:

$$\hat{f}(z) = \int_{a \in A_1(F)} \int_{n \in N_3(F)} f(i_1(a)n)\chi_z(a)\delta_3^{1/2}(i_1(a))dnda, \tag{3.6}$$

$$\widehat{f'}(z) = \int_{a \in A_1(F)} \int_{n \in N_1(F)} f'(an)\chi_z(a)\delta_1^{1/2}(i_1(a))dnda. \tag{3.7}$$

Here δ_3, δ_1 denote the modulus functions of the Borel subgroup of $Sp_{2n}(F)$ and $GL_{2n}(F)$ respectively.

We define a Hecke algebra homomorphism $\lambda_1 : \mathcal{H}(Sp_{2n}, K_3) \to \mathcal{H}(GL_{2n}, K_1)$ so that when $f' = \lambda_1(f)$, we have

$$\widehat{f'}(z - \frac{1}{2}) = \hat{f}(z). \tag{3.8}$$

We let ϵ_1' be the map on $\mathcal{S}(Sp_{2n}(F))$ given by equation (3.1). Then when $f \in \mathcal{H}(Sp_{2n}, K_3)$:

$$\epsilon_1'(f) = \int\limits_{u \in V_3(F)} \int\limits_{k \in K_3 \cap GL_{(2n,K)}(F)} f(i_1(g)u) |\det(g)|^{n+1} dk du.$$

From Iwasawa decomposition we get that $\hat{f}(z) = \widehat{\epsilon_1'(f)}(z - \frac{1}{2})$. Thus $\epsilon_1'(f) = \lambda_1(f)$, i.e. ϵ_1 restricts to a Hecke algebra homomorphism. □

Theorem (3.1) gives a map from $f \in \mathcal{S}(Sp_{2n}(A))$ to $f' \in \mathcal{S}(GL_{2n}(A))$ with

$$I_{GL_{2n}}(f' : GL_{n,K}, 1; N_1, \theta_1) = I_{Sp_{2n}}(f : Sp_{n,K}, 1; N_3, \theta_3).$$

We want to construct a map in the other direction.

Corollary 3.9. *For any place v there exists maps $\varepsilon_{1,v} : \mathcal{S}(GL_{2n}(F_v)) \to \mathcal{S}(Sp_{2n}(F_v))$, such that $I_{GL_{2n}}(f_v' : GL_{(n,K)}, 1; N_1, \theta_1) = I_{Sp_{2n}}(f_v : Sp_{(n,K)}, 1; N_3, \theta_3)$ for $f = \otimes f_v$ and $f' = \otimes f_v'$ where*

1. $f_v' = \lambda_{1,v}(f_v)$ for $v \notin S$ a finite set of places containing bad places.

2. $f_v = \varepsilon_{1,v}(f_v')$ for $v \in S$.

Proof. Given $f_v' \in \mathcal{S}(\mathrm{GL}_{2n}(F_v))$, define $f_{1,v}(p)$ on the Siegel parabolic subgroup $P_3(F_v)$ by setting $f_{1,v}(i_1(m)u) = f_v'(m)\phi(u)$ where $m \in \mathrm{GL}_{2n}(F_v)$, $u \in V_3(F_v)$ and $\phi(u)$ is a Schwartz function on $V_3(F_v)$ such that $\int_{V_3(F_v)} \phi(u)\, du = 1$. Define

$$f_{2,v}(p) = \int_{k \in K_3 \cap Sp_{n,K} \cap P_3(F_v)} f_{1,v}(kp)\, dk.$$

Then $f_{2,v}$ is left $K_3 \cap Sp_{n,K} \cap P_3(F_v)$ invariant. We extend $f_{2,v}$ to a function $f_{3,v}$ on $Sp_{n,K}P_3(F_v)$ as follows: using the Iwasawa decomposition, any element in $Sp_{n,K}P_3(F_v)$ has the form kp with $k \in K_3 \cap Sp_{n,K}(F_v)$ and $p \in P_3(F_v)$; we let $f_{3,v}(kp) = f_{2,v}(p)$.

As $Sp_{n,K}P_3$ is a closed subset of Sp_{2n}, the restriction map from $\mathcal{S}(\mathrm{Sp}_{2n}(F_v))$ to $\mathcal{S}(Sp_{n,K}P_3(F_v))$ is surjective. Thus, there is a function $f_v \in \mathcal{S}(\mathrm{Sp}_{2n}(F_v))$ that restricts to $f_{3,v}$. We will let $f_v = \epsilon_{1,v}(f_v')$.

We now check that the equality

$$I_{GL_{2n}}(f'_v : GL_{(n,K)}, 1; N_1, \theta_1) = I_{Sp_{2n}}(f_v : Sp_{(n,K)}, 1; N_3, \theta_3)$$

holds under the conditions in the corollary.

For the given f'_v, define a function on $\mathrm{GL}_{2n}(F_v)$:

$$f''_v(g) = \int_{K_1 \cap GL_{n,K}(F_v)} f'_v(kg)\, dk.$$

Then $f'' = \otimes f''_v \in \mathcal{S}(\mathrm{GL}_{2n}(A))$ and

$$I_{GL_{2n}}(f'' : GL_{n,K}, 1; N_1, \theta_1) = I_{GL_{2n}}(f' : GL_{n,K}, 1; N_1, \theta_1). \tag{3.9}$$

When $v \notin S$, we have that $f'_v = \lambda_1(f_v)$; since f'_v is in $\mathcal{H}(GL_{2n}, K_1)$, we have $f''_v = f'_v$. Thus from the last statement of Theorem 3.1 (and its proof), we have f_v and f''_v satisfy equation (3.1). When $v \in S$, $f_v = \epsilon_{1,v}(f'_v)$; we can check that f_v and f''_v again satisfy equation (3.1). It follows from Theorem 3.1 that $I_{Sp_{2n}}(f : Sp_{n,K}, 1; N_3, \theta_3)$ equals $I_{GL_{2n}}(f'' : GL_{n,K}, 1; N_1, \theta_1)$. From (3.9) we get the claim of the corollary. □

3.3. **Identity of periods.** The previous section was motivated by the conjectural relation (see [JR] and [JiMR]) between the inner period on the Levi factor and the outer period on the group. In the case when τ is an irreducible cuspidal representation of GL_{2n} with its exterior square L-function having a pole at $s=1$ and with $L(1/2,\tau)\neq 0$, τ has a nontrivial $GL_n\times GL_n$ period. On the other hand, the residual Eisenstein series on Sp_{2n} constructed from τ has a nontrivial period along the subgroup $Sp_n\times Sp_n$. The relation between the periods in this case is given as Theorem D in [GRS1].

In our present case, the residual Eisenstein series on Sp_{2n} should have $Sp_{(n,K)}$ period which is related to the $GL_{(n,K)}$ period of τ. This is the content of Theorem 3.10 below. For the convenience of the reader, we reproduce the material on Eisenstein series from [GRS1].

Let $P=MU$ be the Siegel parabolic subgroup of Sp_{2n}, we have a natural identification $M\cong GL_{2n}$ via $i_1(g)\mapsto g$. Let τ be an irreducible, automorphic, cuspidal, self-dual representation of $GL_{2n}(A)$. Let $\phi\in Ind_{P(A)}^{Sp_{2n}(A)}\tau$, i.e. ϕ is a smooth function on $Sp_{2n}(A)$ with values in the space of τ with

$$\phi(mug;r)=\delta_P^{1/2}(m)\phi(g;rm) \tag{3.10}$$

for $m\in M(A)$, $u\in U(A), g\in Sp_{2n}(A), r\in GL_{2n}(A)$. We realize ϕ as a complex function on $Sp_{2n}(A)\times GL_{2n}(A)$ such that $r\mapsto\phi(g;r)$ is a cusp form in the space of τ; we assume ϕ is right K_3-finite where K_3 is the standard maximal compact subgroup of $Sp_{2n}(A)$. If $g\in Sp_{2n}(A)$ has Iwasawa decomposition $g=auk$ where $a\in GL_{2n}(A), u\in U(A), k\in K_3$, we define for $s\in\mathbb{C}$,

$$\varphi^{\phi}_{\tau,s}(g;m)=H(g)^{s-1/2}\phi(g;m) \tag{3.11}$$

and

$$f^{\phi}_{\tau,s}(g)=\varphi^{\phi}_{\tau,s}(g;1) \tag{3.12}$$

where $H(g) := |\det(a)|$ for g with Iwasawa decomposition as above. We also denote by χ_c the characteristic function of all real numbers larger than c; by χ^c, the characteristic function of the interval $(0, c]$.

Consider the Eisenstein series

$$E(g, f^{\phi}_{\tau,s}) = \sum_{\gamma \in P_{Sp_{2n}}(F) \backslash Sp_{2n}(F)} f^{\phi}_{\tau,s}(\gamma g) \tag{3.13}$$

The constant term along U is given by

$$E^U(g, f^{\phi}_{\tau,s}) = \int_{U(A)/U(F)} E(ug, f^{\phi}_{\tau,s}) du = f^{\phi}_{\tau,s}(g) + M(s) f^{\phi}_{\tau,s}(g) \tag{3.14}$$

where M is the intertwining operator given by

$$M(s) f^{\phi}_{\tau,s}(g) = \int_{U(A)} f^{\phi}_{\tau,s}(w^{-1} ug) du \tag{3.15}$$

with $w = \begin{pmatrix} & I_{2n} \\ -I_{2n} & \end{pmatrix}$; write

$$E_1(g, \phi) = Res_{s=1} E(g, f^{\phi}_{\tau,s}) \tag{3.16}$$

We wish to prove the following theorem

Theorem 3.10. *For a suitable choice of measures we have*

$$\int_{Sp_{(n,K)}(F)/Sp_{(n,K)}(A)} E_1(h, \phi) dh = \int_{K_{Sp_{n,K}}} \int_{C_{2n}(A) GL_{n,K}(F) \backslash GL_{n,K}(A)} \phi(k; a) da dk \tag{3.17}$$

with $K_{Sp_{(n,K)}} = K \cap Sp_{n,K}$ *and* C_{2n} *the center of* GL_{2n}.

As in [GRS1], we apply the truncation operator Λ^c to $E(g, f^{\phi}_{\tau,s})$, we get

$$\Lambda^c E(g, f^{\phi}_{\tau,s}) = E(g, f^{\phi}_{\tau,s}) - \sum_{\gamma \in P_{Sp_{2n}}(F)\backslash Sp_{2n}(F)} E^U(\gamma g, f^{\phi}_{\tau,s})\chi_c(H(\gamma g)). \tag{3.18}$$

By (3.14) and (3.18) we have

$$\Lambda^c E(g, f^{\phi}_{\tau,s}) = E(g, f^{\phi}_{\tau,s}) - \sum_{\gamma \in P_{Sp_{2n}}(F)\backslash Sp_{2n}(F)} (f^{\phi}_{\tau,s}(\gamma g) + M(s) f^{\phi}_{\tau,s}(\gamma g))\chi_c(H(\gamma g)) \tag{3.19}$$

$$= \sum_{\gamma \in P_{Sp_{2n}}(F)\backslash Sp_{2n}(F)} f^{\phi}_{\tau,s}(\gamma g)\chi^c(H(\gamma g)) - \sum_{\gamma \in P_{Sp_{2n}}(F)\backslash Sp_{2n}(F)} M(s) f^{\phi}_{\tau,s}(\gamma g)\chi_c(H(\gamma g)) \tag{3.20}$$

Denote

$$\theta^c_1(g, f^{\phi}_{\tau,s}) := \sum_{\gamma \in P_{Sp_{2n}}(F)\backslash Sp_{2n}(F)} f^{\phi}_{\tau,s}(\gamma g)\chi^c(H(\gamma g)), \tag{3.21}$$

$$\theta^c_2(g, f^{\phi}_{\tau,s}) := \sum_{\gamma \in P_{Sp_{2n}}(F)\backslash Sp_{2n}(F)} M(s) f^{\phi}_{\tau,s}(\gamma g)\chi_c(H(\gamma g)). \tag{3.22}$$

Applying Λ^c to E_1 and noticing that $f^{\phi}_{\tau,s}$ is holomorphic, we get

$$\Lambda^c E_1(g, \phi) = E_1(g, \phi) - \theta^c_3(g, \phi) \tag{3.23}$$

where

$$\theta^c_3(g, \phi) = \sum_{\gamma \in P_{Sp_{2n}}(F)\backslash Sp_{2n}(F)} M_1(f^{\phi}_{\tau,s}(\gamma g)\chi_c(H(\gamma g))) \tag{3.24}$$

and $M_1 := Res_{s=1} M(s)$. It is enough to prove the following proposition:

Proposition 3.11. *The following formulae are valid with certain choice of measure:*

$$(3.25)\qquad \int\limits_{Sp_{n,K}(F)\backslash Sp_{n,K}(A)} \theta_1^c(h, f^{\phi}_{\tau,s})dh = \frac{c^{s-1}}{s-1} \int\limits_{K_{Sp_{n,K}}} \int\limits_{C_{2n}(A)GL_{n,K}(F)\backslash GL_{n,K}(A)} \phi(k,a)\, da\, dk,$$

(3.26)
$$\int\limits_{Sp_{n,K}(F)\backslash Sp_{n,K}(A)} \theta_2^c(h, f^{\phi}_{\tau,s})dh = \frac{c^{-s}}{s} \int\limits_{K_{Sp_{n,K}}} \int\limits_{C_{2n}(A)GL_{n,K}(F)\backslash GL_{n,K}(A)} M(s)(\varphi^{\phi}_{\tau,s})(k;a)\, da\, dk,$$

(3.27)
$$\int\limits_{Sp_{n,K}(F)\backslash Sp_{n,K}(A)} \theta_3^c(h, \phi)dh = c^{-1} \int\limits_{K_{Sp_{n,K}}} \int\limits_{C_{2n}(A)GL_{n,K}(F)\backslash GL_{n,K}(A)} M_1(\varphi_{\tau,1})^{\phi}(k;a)\, da\, dk.$$

Assuming Proposition 3.11, let us show Theorem 3.10. By equation (3.20), we have that $\Lambda^c E(g, f^{\phi}_{\tau,s})$ is equal to

$$\sum_{\gamma\in P_{Sp_{2n}}(F)\backslash Sp_{2n}(F)} f^{\phi}_{\tau,s}(\gamma g)\chi^c(H(\gamma g)) - \sum_{\gamma\in P_{Sp_{2n}}(F)\backslash Sp_{2n}(F)} M(s) f^{\phi}_{\tau,s}(\gamma g)\chi_c(H(\gamma g)).$$

Integrating along $Sp_{n,K}(F)\backslash Sp_{n,K}(A)$ and using equations (3.25) and (3.26), we obtain that $\int\limits_{Sp_{n,k}(F)\backslash Sp_{n,K}(A)} \Lambda^c E(h, f^{\phi}_{\tau,s})\, dh$ is equal to

$$\frac{c^{s-1}}{s-1}\int\int \phi(k,a)dadk - \frac{c^{-s}}{s}\int\int M(s)(\varphi^{\phi}_{\tau,s})(k;a)\, da\, dk,$$

here the integrals are over K_3 and $C_{2n}(A)GL_{n,K}(F)\backslash GL_{n,K}(A)$.

Taking residues at $s = 1$, we get

$$\int\limits_{Sp_{n,K}(F)\backslash Sp_{n,K}(A)} \Lambda^c E_1(h, f^{\phi}_{\tau,s})\, dh = \int\int \phi(k,a)\, da\, dk - c^{-1}\int\int M_1(\varphi^{\phi}_{\tau,1})(k,a)\, da\, dk.$$

Using equation (3.27), this is

$$\int\limits_{Sp_{n,K}(F)\backslash Sp_{n,K}(A)} \Lambda^c E_1(h, f^{\phi}_{\tau,s})\, dh = \int\int \phi(k,a)\, da\, dk - \int\limits_{Sp_{n,K}(F)\backslash Sp_{n,K}(A)} \theta^c_3(h,\phi)\, dh. \tag{3.28}$$

On the other hand, using equation (3.23), we get

$$\int\limits_{Sp_{n,K}(F)\backslash Sp_{n,K}(A)} \Lambda^c E_1(h, f^{\phi}_{\tau,s})\, dh = \int\limits_{Sp_{n,K}(F)\backslash Sp_{n,K}(A)} E_1(h,\phi)\, dh - \int\limits_{Sp_{n,K}(F)\backslash Sp_{n,K}(A)} \theta^c_3(h,\phi)\, dh. \tag{3.29}$$

Comparing equations (3.28) and (3.29) proves Theorem 3.10.

Proof of Proposition 3.11. To prove this proposition we write the integrals $\int\limits_{Sp_{n,K}(F)\backslash Sp_{n,K}(A)} \theta_j(h)dh$ in terms of $I_{j,d}$ (defined in equation (3.33)) and show that $I_{j,d} = 0$ for $d < n$; the case $d = n$ gives the result.

The functions θ^c_j have the form $\theta_j(g) = \sum\limits_{\gamma\in P(F)\backslash G(F)} \xi_j(\gamma g)$ with

$$\xi_j(g) = \begin{cases} f^{\phi}_{\tau,s}(g)\chi^c(H(g)), & \text{j= 1} \\ M(s)(f^{\phi}_{\tau,s})(g)\chi_c(H(g)), & \text{j = 2} \\ M_1(f^{\phi}_{\tau,1})(g)\chi_c(H(g)), & \text{j = 3} \end{cases} \tag{3.30}$$

Proceeding formally,

$$\int\limits_{Sp_{n,K}(F)\backslash Sp_{n,K}(A)} \theta_j(h)dh = \int\limits_{Sp_{n,K}(F)\backslash Sp_{n,K}(A)} \sum_{\gamma\in P(F)\backslash Sp_{2n}(F)} \xi_j(\gamma h)dh, \tag{3.31}$$

and this is

$$\sum_{\gamma\in P(F)\backslash Sp_{2n}(F)/Sp_{n,K}(F)} \int_{\gamma^{-1}P(F)\gamma\cap Sp_{n,K}(F)\backslash Sp_{n,K}(A)} \xi_j(\gamma h)dh. \tag{3.32}$$

Recall the description of the double cosets $N_3(F)\backslash Sp_{2n}(F)/Sp_{(n,K)}(F)$ under the map $g \mapsto gTg^{-1}$: namely, we may take as representatives elements $g \in Sp_{2n}$ with $gTg^{-1} = wa$ with $(wa)^2 = \tau$ and ${}^t(wa)J(wa) = \tau J$. We remark that by considering the effect of the parabolic part and after conjugation by an appropriate element of the form $\begin{pmatrix} W & \\ & W^* \end{pmatrix}$ with $W \in W(GL_n)$, we may take wa to be of the form $wa = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with

$$A = \begin{pmatrix} 0 & \tau & & & & & & \\ 1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & \tau & & & \\ & & & 1 & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}, \quad B = \begin{pmatrix} & & & & & & & & 0 \\ & & & & & & & \cdot & \\ & & & & & & \cdot & & \\ & & & & & 0 & & & \\ & & & \tau & 0 & & & & \\ & & & 0 & -\tau & & & & \\ & & \cdot & & & & & & \\ & \cdot & & & & & & & \\ \tau & 0 & & & & & & & \\ 0 & -\tau & & & & & & & \end{pmatrix},$$

$$C = \begin{pmatrix} & & & & & & 1 & 0 \\ & & & & & & 0 & -1 \\ & & & & & \cdot & & \\ & & & & \cdot & & & \\ & & & 1 & 0 & & & \\ & & & 0 & -1 & & & \\ & & 0 & & & & & \\ & \cdot & & & & & & \\ \cdot & & & & & & & \\ 0 & & & & & & & \end{pmatrix}, \quad D = \begin{pmatrix} 0 & & & & & & & & \\ & \cdot & & & & & & & \\ & & \cdot & & & & & & \\ & & & 0 & & & & & \\ & & & & 0 & \tau & & & \\ & & & & 1 & 0 & & & \\ & & & & & & \cdot & & \\ & & & & & & & \cdot & \\ & & & & & & & 0 & \tau \\ & & & & & & & 1 & 0 \end{pmatrix}.$$

We attach a parameter d to such wa, where $2d$ is the number of nonzero rows on the A block, and we label such wa as wa_d.

We have that $gTg^{-1} = \begin{pmatrix} & & & & \tau & \\ & & & & & -\tau \\ & & & \cdot & & \\ & & \cdot & & & \\ 1 & & & & & \\ & -1 & & & & \end{pmatrix}$ where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}$ with

$A = \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & 1 & \\ & & & & & 0 \end{pmatrix}$, $B = \begin{pmatrix} & & & & & 0 \\ & & & & -1 & \\ & & & \cdot & & \\ & & \cdot & & & \\ & 0 & & & & \\ -1 & & & & & \end{pmatrix}$, $C = \begin{pmatrix} & & & & & 1 \\ & & & & 0 & \\ & & & \cdot & & \\ & & \cdot & & & \\ & 1 & & & & \\ 0 & & & & & \end{pmatrix}$, $D = \begin{pmatrix} 0 & & & & & \\ & 1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & 0 & \\ & & & & & 1 \end{pmatrix}$. If the square matrix $\begin{pmatrix} & & & & \tau & \\ & & & & & -\tau \\ & & & \cdot & & \\ & & \cdot & & & \\ 1 & & & & & \\ & -1 & & & & \end{pmatrix}$ is of size $2r$, then we denote the corresponding g as g_r.

We have that if $\gamma_d = \begin{pmatrix} Id_{2d} & & \\ & g_r & \\ & & Id_{2d} \end{pmatrix}$, then $\gamma_d T \gamma_d^{-1} = wa_d$.

As in [GRS1], we let $Q_d = \gamma_d^{-1} P \gamma_d \cap Sp_{n,K}$ and we need to compute the integrals

$$I_{j,d} = \int\limits_{Q_d(F)\backslash Sp_{n,K}(A)} \xi_j(\gamma_d h) \tag{3.33}$$

for $0 \leq d \leq n$ and $j = 1, 2, 3$. We assume that $\gamma_d T \gamma_d^{-1} = wa_d$ and remark that to find Q_d it is enough to find $p \in P$ with $\gamma^{-1} p \gamma \in Sp_{(n,K)}(F)$, which is equivalent to $wap = pwa$.

Case $d = 0$:

In this case we have $wa = \begin{pmatrix} & \alpha \\ \beta & \end{pmatrix}$ with

$$\alpha = \begin{pmatrix} & & & \tau & \\ & & & & -\tau \\ & & \cdot & & \\ & \cdot & & & \\ \tau & & & & \\ & -\tau & & & \end{pmatrix} \text{ and } \beta = \begin{pmatrix} & & & 1 & \\ & & & & -1 \\ & & \cdot & & \\ & \cdot & & & \\ 1 & & & & \\ & -1 & & & \end{pmatrix} \text{ and if } p = \begin{pmatrix} P_1 & P_2 \\ & P_3 \end{pmatrix} \text{ with}$$

$wap = pwa$ then $\begin{pmatrix} & \alpha P_3 \\ \beta P_1 & \beta P_2 \end{pmatrix} = \begin{pmatrix} P_2\beta & P_1\alpha \\ P_3\beta & \end{pmatrix}$ then $P_2 = 0$ and since $P_3 = \sigma^t P_1^{-1} \sigma$ we have $\beta P_1 = \sigma^t P_1^{-1} \sigma \beta$ which implies ${}^t P_1 \sigma \beta P_1 = \sigma \beta$ but

$$\sigma\beta = \begin{pmatrix} 0 & -1 & & & & \\ 1 & 0 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & 0 & -1 \\ & & & & 1 & 0 \end{pmatrix} \text{ so that } p = i(P_1) = \begin{pmatrix} P_1 & \\ & P_1 * \end{pmatrix} \text{ where } P_1 \in Sp_n. \text{ As } P_1$$

runs over Sp_n, so does $Q_d = \gamma_0^{-1} i(P_1) \gamma_0$, embedded via i_2 as a subgroup of $Sp_{n,K}$.

We have

$$I_{j,0} = \int\limits_{i_2(Sp_n(F))\backslash Sp_{2n}(A_K)} \xi_j(\gamma_0 h) dh. \tag{3.34}$$

We write $h = i_2(a)b$ where $i_2(a) \in i_2(Sp_n(F))\backslash i_2(Sp_n(A_F))$ and $b \in i_2(Sp_n(A_F))\backslash Sp_n(A_K)$.

We factor the integral as

$$I_{j,0} = \int\limits_{i_2(a) \in i_2(Sp_n(F))\backslash i_2(Sp_n(A_F))} \int\limits_{b \in i_2(Sp_n(A_F))\backslash Sp_n(A_K)} \xi_j(\gamma_0 i_2(a) \gamma_0^{-1} \gamma_0 b) di_2(a) db. \tag{3.35}$$

This includes the integration of a cusp form in τ along $i_2(Sp_n(F))\backslash i_2(Sp_n(A_F))$ which equals zero by [Jacquet-Rallis, Prop 1].

Case $0 < d < n$:

In this case we write $wa = \begin{pmatrix} \alpha & & & \\ & & \beta & \\ & \beta/\tau & & \\ & & & \alpha \end{pmatrix}$ with

$$\alpha = \begin{pmatrix} & \tau & & & & \\ 1 & & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & & \tau \\ & & & & 1 & \end{pmatrix} \text{ and } \beta = \begin{pmatrix} & & & & \tau & \\ & & & & & -\tau \\ & & & \cdot & & \\ & & \cdot & & & \\ \tau & & & & & \\ & -\tau & & & & \end{pmatrix} \text{ and } p = \begin{pmatrix} P_1 & P_4^1 & P_4^2 & P_5 \\ & P_2 & & P_6^1 \\ & & & P_6^2 \\ & & & P_3 \end{pmatrix}$$

where P_1, P_2, P_3, P_5 are squares matrices of sizes $2d, 4(n-d), 2d, 2d$ respectively, P_4^1 and P_4^2 are of sizes $2d \times 2(n-d)$, P_6^1 and P_6^2 of sizes $2(n-d) \times 2d$ and p satisfies $wap = pwa$ then

$$\begin{pmatrix} \alpha P_1 & \alpha P_4^1 & \alpha P_4^2 & \alpha P_5 \\ & \delta P_2 & & \beta P_6^2 \\ & & & \frac{\beta}{\tau} P_6^1 \\ & & & \alpha P_3 \end{pmatrix} = \begin{pmatrix} P_1\alpha & P_4^2\frac{\beta}{\tau} & P_4^1\beta & P_5\alpha \\ & P_2\delta & & P_6^1\alpha \\ & & & P_6^2\alpha \\ & & & P_3\alpha \end{pmatrix} \tag{3.36}$$

where $\delta = \begin{pmatrix} & \beta \\ \beta/\tau & \end{pmatrix}$.

Write this as a semidirect product $M_d \times V_d$, write the Iwasawa decomposition in $Sp_{n,K}(A)$, $h = vmk$ where $m \in M_d(A), v \in V_d(A), k \in K_{Sp_{n,K}}, dh = \delta^{-1}(m)dvdmdk$, where δ is the modulus function of the parabolic subgroup Q_d, then

$$I_{j,d} = \int\limits_{K_{Sp_{n,K}}} \int\limits_{M_d(A)/M_d(F)} \int\limits_{V_d(A)/V_d(F)} \xi_j(\gamma_d v \gamma_d^{-1}.\gamma_d m k)\delta^{-1}(m) dv dm dk. \tag{3.37}$$

We let $P_4 = \begin{pmatrix} P_4^1 & P_4^2 \end{pmatrix}$ and $P_6 = \begin{pmatrix} P_6^1 \\ P_6^2 \end{pmatrix}$. One has that $\gamma_d^{-1} P \gamma = \begin{pmatrix} P_1 & P_4 g_r & P_5 \\ & g_r^{-1} P_2 g & g^{-1} P_6 \\ & & P_3 \end{pmatrix}$, from equation (3.36) we have that $P_1, P_5, P_3 \in GL_{n,K}$. The equation $\alpha P_4 = P_4 \delta$ implies $TP_4 = P_4 g_r T g_r^{-1}$ so that $P_4 g_r \in GL_{n,K}$. Similarly, we have that $g_r^{-1} P_2 g_r, g_r^{-1} P_6 \in GL_{n,K}$. Thus, the projection to $Sp_{(n,K)}$ of $\gamma_d v \gamma_d^{-1}$ as v varies in V_d is a unipotent radical in GL_{2n}. We have $H(\gamma_d v \gamma_d^{-1}) = 1$ and $I_{j,d}$ involves an integration of a cusp form in τ along $V_d(F) \backslash V_d(A)$, so $I_{j,d} = 0$ for $0 < d < 2n$, $j = 1, 2, 3$.

Case $d = n$:

We have $wa = \begin{pmatrix} & \tau & & & & \\ 1 & & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & & \tau \\ & & & & 1 & \end{pmatrix}$ and if $p = \begin{pmatrix} P_1 & P_2 \\ & P_3 \end{pmatrix}$ with $pwa = wap$ then

$$P_i \begin{pmatrix} & \tau & & & & \\ 1 & & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & & \tau \\ & & & & 1 & \end{pmatrix} = \begin{pmatrix} & \tau & & & & \\ 1 & & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & & \tau \\ & & & & 1 & \end{pmatrix} P_i,$$ i.e. $P_i \in GL_{n,K}(F)$ for $i = 1, 2, 3$.

Using the Iwasawa decomposition $Sp_{n,K} = K_{Sp_{n,K}} i_1(GL_{n,K}) V_{Sp_{n,K}}$, where $K_H = K_3 \cap Sp_{n,K}$ and $V_{Sp_{n,K}} = V_3 \cap Sp_{n,K}$ we have

$$I_{j,n} = \int_{K_{Sp_{n,K}}} \int_{GL_{n,K}(F)\backslash GL_{n,K}(A)} \xi_j\left(\begin{pmatrix} a & \\ & a^* \end{pmatrix} k\right) |\det(a)|^{-(n+1)} dadk \tag{3.38}$$

$$= \int_{K_{Sp_{n,K}}} \int_{GL_{n,K}(F)\backslash GL_{n,K}(A)^0} \int_{F^\times\backslash A^\times} \xi_j\left(\begin{pmatrix} ta & \\ & t^{-1}a^* \end{pmatrix} k\right) |t|^{-2n(n+1)} d^\times t dadk. \tag{3.39}$$

When $j = 1$ we get

(3.40)

$$I_{1,d} = \int_{K_{Sp_{n,K}}} \int_{GL_{n,K}(F)\backslash GL_{n,K}(A)^0} \int_{F^\times\backslash A^\times} f^\phi_{\tau,s}\left(\begin{pmatrix} ta & \\ & t^{-1}a^* \end{pmatrix} k\right) \chi^c\left(H\begin{pmatrix} ta & \\ & t^{-1}a^* \end{pmatrix}\right) |t|^{-2n(n+1)} d^\times t dadk$$

(3.41)

$$= \int_{K_{Sp_{n,K}}} \int_{GL_{n,K}(F)\backslash GL_{n,K}(A)^0} \int_{F^\times\backslash A^\times} \varphi^\phi_{\tau,s}\left(\begin{pmatrix} ta & \\ & t^{-1}a^* \end{pmatrix} k; 1\right) \chi^c\left(H\begin{pmatrix} ta & \\ & t^{-1}a^* \end{pmatrix}\right) |t|^{-2n(n+1)} d^\times t dadk$$

$$= \int_{K_{Sp_{n,K}}} \int_{GL_{n,K}(F)\backslash GL_{n,K}(A)^0} \int_{F^\times\backslash A^\times} \left(H\begin{pmatrix} ta & \\ & t^{-1}a^* \end{pmatrix}\right)^{s-1/2} \phi\left(\begin{pmatrix} ta & \\ & t^{-1}a^* \end{pmatrix} k; 1\right)$$

$$\times \chi^c(|t|^{2n}) |t|^{-2n(n+1)} d^\times t dadk$$

(3.42)

$$= \int\limits_{K_{Sp_{n,K}}} \int\limits_{GL_{n,K}(F)\backslash GL_{n,K}(A)^0} \int\limits_{F^\times\backslash A^\times, |t|^{2n}\leq c} |t|^{2n(s-1/2)}\phi(k;a)\delta_{P_{Sp_{2n}}}\begin{pmatrix} ta & \\ & t^{-1}a* \end{pmatrix}^{1/2} |t|^{-2n(n+1)} dad^\times tdk.$$

But we have that $\delta_{P_{Sp_{2n}}}\begin{pmatrix} ta & \\ & t^{-1}a^* \end{pmatrix} = |t^2|^{2n(\frac{2n+1}{2})}$ so that we get

$$= \int\limits_{K_{Sp_{n,K}}} \int\limits_{GL_{n,K}(F)\backslash GL_{n,K}(A)^0} \phi(k;a)d(a)dk \int\limits_{t\in F^\times\backslash A^\times, |t|^{2n}\leq c} |t|^{2n(s-1)}d^*t \tag{3.43}$$

We choose d^*t so that the integral over t equals $\int\limits_0^c t^{s-1}\frac{dt}{t} = \frac{c^{s-1}}{s-1}$, for $Re(s) > 1$. $\square$

3.4. **Convergence of the integrals.** We need to prove the absolute convergence of the integrals

$$\int\limits_{\gamma_d^{-1}P\gamma_d \cap Sp_{2n,K}(F)\backslash Sp_{2n,K}(A)} \xi_j(\gamma_d h). \tag{3.44}$$

For this we write the Iwasawa decomposition for elements in $Sp_{4n,K}$ as $M'VK$ where $M' = \begin{pmatrix} P_1 & & \\ & P' & \\ & & P_1^* \end{pmatrix}$ and $V = \begin{pmatrix} 1 & \begin{pmatrix} P_2 & P_3 \end{pmatrix} & P_4 \\ & 1 & \begin{pmatrix} P_2 \\ P_3 \end{pmatrix}^* \\ & & 1 \end{pmatrix}$

where $P_1 \in GL_{2d,K}$, $P' \in Sp_{4(n-d),K}$, $P_2, P_3 \in M_{2d\times 2(n-d),K}$, $P_4 \in GL_{2d,K}$ and K denotes the maximal compact subgroup of $Sp_{4n,K}$.

Now P satisfies $\gamma_d^{-1}P\gamma_d \in Sp_{4n,K}$ if and only if $hP = Ph$, this implies that $\gamma_d^{-1}P\gamma_d$ is of the form $\begin{pmatrix} P_1 & \begin{pmatrix} P_2 & P_3 \end{pmatrix} & P_4 \\ & \begin{pmatrix} P_6 & \\ & P_6^* \end{pmatrix} & \begin{pmatrix} P_2 \\ P_3 \end{pmatrix}^* \\ & & P_1^* \end{pmatrix}$, with ${}^tP_6\sigma\beta^{-1}P_6 = \sigma\beta^{-1}$, $(P_2P_3)\delta = \alpha(P_2P_3)$, $\alpha P_i = P_i\alpha$ for $i = 1, 4$ and $P_6^* \equiv \beta^{-1}P_6\beta$.

The conjugation by γ gives

$$\gamma^{-1}P\gamma \cap Sp_{4n,K} = \begin{pmatrix} P_1 & \begin{pmatrix} P_2 & P_3 \end{pmatrix}\gamma & P_4 \\ & \gamma^{-1}\begin{pmatrix} P_6 & \\ & P_6^* \end{pmatrix}\gamma & \gamma^{-1}\begin{pmatrix} P_2 \\ P_3 \end{pmatrix}^* \\ & & P_1^* \end{pmatrix} \tag{3.45}$$

We define $M := \begin{pmatrix} P_1 & & \\ & \gamma^{-1}\begin{pmatrix} P_6 & \\ & P_6^* \end{pmatrix}\gamma & \\ & & P_1^* \end{pmatrix}$ with $P_1 \in GL_{2d,K}$ and $P_6 \in Sp_{2(n-d)}$.

We thus get that $r \in \mathrm{M}(F)\backslash M(A)$ is of the form $r = \begin{pmatrix} P_1 & & \\ & \gamma^{-1}\begin{pmatrix} P_6 & \\ & P_6^* \end{pmatrix}\gamma & \\ & & P_1^* \end{pmatrix}$, where $P_1 \in GL_{2d,K}(F)\backslash GL_{2d,K}(A)$ and $P_6 \in Sp_{2(n-d)}(F)\backslash Sp_{2(n-d)}(A)$.

For such r, we have that $\gamma r\gamma^{-1} = \begin{pmatrix} P_1 & & & \\ & P_6 & & \\ & & P_6^* & \\ & & & P_1^* \end{pmatrix}$ with the projection to GL_{2n}

of its Levi component being of the form $\begin{pmatrix} a & \\ & b \end{pmatrix}$ with $a \in GL_{2d,K}(F)\backslash GL_{2d,K}(A)$ and $b \in Sp_{2(n-d)}(F)\backslash Sp_{2(n-d)}(A)$.

Similarly, we define $V := \begin{pmatrix} 1 & \begin{pmatrix} P_2 & P_3 \end{pmatrix} & P_4 \\ & 1 & \begin{pmatrix} P_2 \\ P_3 \end{pmatrix}^* \\ & & 1 \end{pmatrix}$ with $P_2, P_3 \in M_{2d\times 2(n-d),K}$ and $P_4 \in GL_{2d,K}$. Thus $v \in V(F)\backslash V(A)$ is of the form $v = \begin{pmatrix} 1 & \begin{pmatrix} P_2 & P_3 \end{pmatrix} & P_4 \\ & 1 & \begin{pmatrix} P_2 \\ P_3 \end{pmatrix}^* \\ & & 1 \end{pmatrix}$ with $P_2', P_3' \in M_{2d\times 2(n-d),K}(F)\backslash M_{2d\times 2(n-d),K}(A)$ and $P_4 \in GL_{2d,K}(F)\backslash GL_{2d,K}(A)$.

For v as above, we have that $\gamma v \gamma^{-1}$ is of the form $\begin{pmatrix} 1 & \begin{pmatrix} P_2 & P_3 \end{pmatrix}\gamma^{-1} & P_4 \\ & 1 & \gamma\begin{pmatrix} P_2 \\ P_3 \end{pmatrix}^* \\ & & 1 \end{pmatrix}$ with P_2, P_3 and P_4 as above. The projection to GL_{2n} of the Levi component of $\gamma v \gamma^{-1}$ is thus of the form $\begin{pmatrix} I_{2d} & y \\ & I_{2(n-d)} \end{pmatrix}$ with $y \in M_{2d\times 2(n-d)}(F)\backslash M_{2d\times 2(n-d)}(A)$.

We remark that the Iwasawa decomposition for $\gamma^{-1}P\gamma \cap Sp_{4n,K}$ is MV.

As P_6 varies in $Sp_{2(n-d)}(F)$, $\gamma^{-1}\begin{pmatrix} P_6 & \\ & P_6^* \end{pmatrix}\gamma$ varies over $Sp_{4(n-d),K}(F)$ where the embedding from F into $GL_{2,K}(F)$ is given by $a \mapsto \begin{pmatrix} a & \\ & a \end{pmatrix}$ with $a \in F$. We take the Iwasawa decomposition of $\gamma^{-1}\begin{pmatrix} P_6 & \\ & P_6^* \end{pmatrix}\gamma$ according to the Borel subgroup. Thus we may take $m' \in M(A)\backslash M'(A)$ to be of the form $\begin{pmatrix} I_{2d} & & \\ & m & \\ & & I_{2d} \end{pmatrix}$ with $m = m(A) = m_{2(n-d)-1} \times m_{2(n-d)-2} \times \ldots \times m_1 \times t$ where $t = \begin{pmatrix} A^* & \\ & A \end{pmatrix}$ with $A = \begin{pmatrix} a_1 & \tau & & & \\ 1 & a_1 & & & \\ & & \ddots & & \\ & & & a_{n-d} & \tau \\ & & & 1 & a_{n-d} \end{pmatrix}$

and $m_i = \begin{pmatrix} I_2 & \ldots & X_\alpha & & \\ & I_2 & & X_{\alpha+1} & \\ & & \ddots & & X_\omega \\ & & & I_2 & \vdots \\ & & & & I_2 \end{pmatrix}$ where the $X's$ lie on the ith superdiagonal with $X_j = \begin{pmatrix} & x_j\tau \\ x_j & \end{pmatrix}$, $\alpha = 2(i-1)(n-d) - \frac{i(i-1)}{2} + 1$ and $\omega = \alpha + 2(n-d) - i - 1$. We also remark that the $X's$ are related by the symplectic structure on m_i.

For γm with m as above, we write the Iwasawa decomposition according to the Borel subgroup as $\gamma m = l_m u_m k_m$ and $dm := \prod\limits_i \frac{dx_i}{|x_i^2 - \tau|}$. We also write $|g|$ for $|\det(g)|$.

Lemma 3.12. *For $m(A)$ as above, we have:*

1) $|l_m| \leq 1$.

2) The integral $\int\limits_m |l_m|^t dm$ converges, provided t is large enough.

Proof. We denote by r the matrix obtained by taking the even rows of m. We have that $|l_m|^{-1}$ is greater than or equal to the maximum of the determinant of the $2(n-d)\times 2(n-d)$ minors of the matrix r, note in particular that these minors are upper triangular.
For $1 \leq j \leq n-d$ we have that the last entries of the columns $2(n-d)+2j-1$ and $2(n-d)+2j$ are up to signs, 1 and a_j respectively, both located at the row $n-d+j$.
Also, the last entry of the columns $2(n-d)-2j+1$ and $2(n-d)-2j+2$ are up to signs, $\frac{1}{a_j^2-\tau}$ and $\frac{a_j}{a_j^2-\tau}$ respectively, both located at the row $n-d-j+1$.

For $a_j^2-\tau$ small, we choose the columns $2(n-d)+2j-1)$ and $2(n-d)-2j+1$, we remark that the product of their last entries is $1\times\frac{1}{a_j^2-\tau}$.
For $a_j^2-\tau$ large, we choose the columns $2(n-d)+2j$ and $2(n-d)-2j+2$, we remark that the product of their last entries is $a_j\times\frac{a_j}{a_j^2-\tau}$.
In either case, we have that $\max\left\{\frac{1}{a_j^2-\tau}, \frac{a_j^2}{a_j^2-\tau}\right\}\geq 1$. Thus after choosing columns as described, we get that $|l_m|\leq 1$.
We remark that by the symplectic structure of m, it is enough to consider the convergence over the variables x_i where x_i lies on a column k with $k\geq 2(n-d)+1$.

For such a variable x, we let j be such that (after a change of variables) the elements a_jx, τx appear on the columns $2(n-d)+2j+1$ and $2(n-d)+2j+2$ respectively with $1\leq j\leq n-d$ and j minimal with respect to this property. We also let i be the row for such a_jx and τx.

We choose the columns $2(n-d)+2j+1$ and $2(n-d)+2j+2$ with j as above and for the remaining columns we make a choice similar to the one used in the proof of 1). The determinant of the minor obtained is $|a_j^2-\tau|\times|x|$.

We choose the columns $2(n-d)-2j+1$ and $2(n-d)+2j-1$ with j as above and for the remaining columns we follow the proof of 1) as before, we get that the determinant of the minor thus obtained is $|\frac{1}{a_j^2-\tau}|$.

Combining the last two observations with the fact that $|l_m| \leq 1$, we see that $|l_m| \leq min\,\{1, |a_i^2-\tau|, |a_i^2-\tau|^{-1}|x_j|^{-1}\}$. When $a_i^2-\tau$ is large, we have $|l_m| \leq |a_i^2-\tau|^{-1}|x_j|^{-1}$. On the other hand, when $|a_i^2-\tau|$ is small, we have $|l_m| \leq |a_i^2-\tau|^{\alpha}|x_i^2-\tau|^{\alpha-1}|x_j|^{\alpha-1}$ where $1/2 < \alpha < 1$. We get that $|l_m| \leq |a_i^2-\tau|^{2\alpha-1}|x_j|^{\alpha-1}$ which gives a convergent integral in x. Repeating this over $x's$, we obtain the second assertion in the Lemma. $\square$

Our integral becomes

(3.46)
$$I_{j,d} = \int\limits_{K_H}\int\limits_{m\in M(A)\backslash M'(A)}\int\limits_{r\in M(F)\backslash M(A)}\int\limits_{v\in V_d(F)\backslash V_d(A)} \xi_j(\gamma v\gamma^{-1}.\gamma r\gamma^{-1}.\gamma_d mk)\delta^{-1}(r)dvdmdk.$$

By the definition of ξ_j, this is

$$\int \phi_j(k;\begin{pmatrix} I_{2d} & y \\ & I_{2(n-d)}\end{pmatrix}\begin{pmatrix} a & \\ & b\end{pmatrix}\begin{pmatrix} I_{2d} & \\ & l_m\end{pmatrix})|a|^{s_j}|l_m|^{s'_j}\chi_{j,c}(|a||l_m|)dadbdydm \tag{3.47}$$

where $y \in M_{2d\times 2(n-d)}(F)\backslash M_{2d\times 2(n-d)}(A)$, $a \in GL_{2d,K}(F)\backslash GL_{2d,K}(A)$, $b \in Sp_{2(n-d)}(F)\backslash Sp_{2(n-d)}(A)$ and l_m as above.

Case j =1, d = 0 Our integral becomes

$$\int\limits_{m\in M(A)\backslash M'(A),|l_m|<c}\ \int\limits_{b\in Sp_{2(n-d)}(F)\backslash Sp_{2(n-d)}(A)} \varphi(bl_m)|l_m|^{s+n}dbdm. \tag{3.48}$$

Since $|l_m| \leq 1$ by Lemma 3.12 and $c > 1$, we have that $|l_m| \leq c$ always holds. Since φ is bounded and $Sp_{2n}(F)\backslash Sp_{2n}(A)$ has finite volume, it is enough to consider $\int |l_m|^{s+n} dm$. By Lemma 3.12, this converges for $Re(s)$ large enough.

Case $j = 2,3,\ d = 0$ Our integral becomes

$$\int\limits_{m \in M(A)\backslash M'(A), |l_m| > c} \int\limits_{b \in Sp_{2(n-d)}(F)\backslash Sp_{2(n-d)}(A)} \varphi(bl_m)|l_m|^{1-s+n} dbdm. \tag{3.49}$$

Here the domain of integration is empty by Lemma 3.12 and the fact that $c > 1$.

The proof for the cases where $0 < d < n$ is similar to [GRS1], with Lemma 3.12 in place of Lemmas 6 and 7.

4. Some global identities on Sp_{2n}

We want to relate the distributions $I_{Sp_{2n}}(f : Sp_{n,K}, 1; N_3, \theta_3)$ and $I_{Sp_{2n}}(f : Sp_{n,K}, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$. We use (modified) results from [GRS2] and [GRS1]. We recall the setup in [MR].

For $f \in \mathcal{S}(Sp_{2n}(A))$, define

$$\Psi_f(g) = \int\limits_{l \in Sp_{n,K}(F)\backslash Sp_{n,K}(A)} K_f(l, g)dl.$$

Then $\Psi_f(g)$ is a left $Sp_{2n}(F)$ invariant form on $Sp_{2n}(A)$ satisfying the moderate growth condition: $|\Psi_f(g)|$ is bounded by a polynomial in $||g||$ where

$$||g|| = \prod_v ||g_v||_v = \prod_v (\max_{i,j}\{|g_{i,j,v}|_v, |g^{-1}_{i,j,v}|_v\}). \tag{4.1}$$

Clearly $f \mapsto \Psi_f(g)$ is a linear map. When $f_{g'}(g) = f(gg')$, we have $\Psi_{f_{g'}}(g) = \Psi_f(gg')$.

4.1. Definition of $I_1(f)$.

We recall the definition of sets $\mathcal{X}_0$ and $Y^*_{n-1,n}$, elements ν_0 and ω in [GRS2, §4].

Let $\widetilde{\omega}$ be a permutation matrix in GL_{2n} such that

$$\widetilde{\omega}_{2i,i} = 1, \;\; \widetilde{\omega}_{2i-1,n+i} = 1, \;\; i = 1, \dots, n.$$

Recall i_1 is map from GL_{2n} to Sp_{2n}: $i_1(g) = \left(\begin{smallmatrix} g & \\ & g^* \end{smallmatrix}\right)$. Let $\omega = i_1(\widetilde{\omega})$. Let

$$a = \mathrm{diag}[b, \dots, b, b^*, \dots, b^*] \in \mathrm{Sp}_{2n}, \;\; b = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right),$$

and ν be the Weyl element in Sp_{2n} such that

$$\nu_{i,2i-1} = \nu_{n+i,2n+2i-1} = \nu_{3n+i,2n+2i} = 1, \;\; \nu_{2n+i,2i} = -1, \;\; i = 1, \dots, n.$$

Let $\nu_0 = \nu a$. We only need to note here that ν_0 and ω are elements in $\mathrm{Sp}_{2n}(F)$; and over a $p-$adic place v where p is odd, ν_0 and ω lie in the maximal compact subgroup of $\mathrm{Sp}_{2n}(F_v)$.

Recall that $\sigma \in \mathrm{GL}_n$ denotes the longest Weyl element, and the set $\mathcal{S}_n$ is the set of matrices $g \in \mathrm{GL}_n$ satisfying σg is a symmetric matrix. Let

$$\mathcal{X}_0 = \{x \in \mathcal{S}_{2n} \,|\, x \text{ is nilpotent and upper triangular}\}. \tag{4.2}$$

For $x \in \mathcal{X}_0$, let

$$\bar{l}(x) = \begin{pmatrix} 1_{2n} & \\ x & 1_{2n} \end{pmatrix}. \tag{4.3}$$

Let $T(n) \subset \mathrm{GL}_{2n}$ be defined as in [GRS2, (4.34),(4.35)], then $T(n) = \widetilde{\omega} N_{\setminus n} \widetilde{\omega}^{-1}$ where $N_{\setminus n}$ denotes the subgroup of N_1 consisting of matrices whose $n-$th row has only one nonzero entry. Let $Y^*_{n-1,n}$ be the set

$$\{i_1(T) \,|\, T \in T(n),\, T \text{ is lower triangular}\} \subset \mathrm{Sp}_{2n}\,. \tag{4.4}$$

Define:

$$I_1(f) = \int\limits_{y^* \in Y^*_{n-1,n}(A)} \int\limits_{x \in \mathcal{X}_0(A)} \int\limits_{N_3(F)\backslash N_3(A)} \Psi_f(n\bar{l}(x)\nu_0 y^* \omega)\theta_3(n)\, dn\, dx\, dy^*. \tag{4.5}$$

This definition is motivated by the Corollary on p.895 of [GRS2]. The integral over $\mathcal{X}_0$ and $Y^*_{n-1,n}$ are absolutely convergent, which is clear from equation (4.26) after applying the Dixmier-Malliavin Theorem.

4.2. **Definition of $I_2(f)$.** Let j be the injection from Sp_n to Sp_{2n}:

$$j:\ g \mapsto j(g) = \begin{pmatrix} 1_n & & \\ & g & \\ & & 1_n \end{pmatrix}. \tag{4.6}$$

We define some subgroups of N_3. Let Z_i be the maximal unipotent subgroup of GL_i consisting of upper triangular matrices with unit diagonal. Let

$$\hat{N}^k = \{v = \begin{pmatrix} z & * & * \\ & 1_{4n-2k+2} & * \\ & & z^* \end{pmatrix} \in N_3 |\, z \in Z_{k-1}\}. \tag{4.7}$$

Then $\hat{N}^k$ is a normal subgroup of $\hat{N}^j$ whenever $k < j \leq 2n+1$.

Define a subgroup U^n of N_3:

$$U^n = \{\eta(\mathbf{x}, \mathbf{y}, t) = \begin{pmatrix} 1_{n-1} & & & & & \\ & 1 & \mathbf{x} & \mathbf{y} & t & \\ & & 1_n & 0 & * & \\ & & & 1_n & * & \\ & & & & 1 & \\ & & & & & 1_{n-1} \end{pmatrix}\}. \tag{4.8}$$

Then U^n is a Heisenberg group and is isomorphic to $\hat{N}^n \backslash \hat{N}^{n+1}$. Let U_0^n be the normal subgroup of U^n consisting of $\eta(\mathbf{0}, \mathbf{y}, t)$.

Define $\tilde{N}^n$ to be $U_0^n \hat{N}^n$. Define a character $\tilde{\chi}_n$ on $\tilde{N}^n(A)$, such that for $n = \eta(\mathbf{0}, \mathbf{y}, t)n'$ with $n' \in \hat{N}^n$:

$$\tilde{\chi}_n(\eta(\mathbf{0}, \mathbf{y}, t)n') = \psi\left(\sum_{i=1}^{n-1} n'_{i,i+1} + t\right). \tag{4.9}$$

Note that $j(N_2)\tilde{N}^n$ is a group with $\tilde{N}^n$ being a normal subgroup. Define

$$I_2(f) = \int\limits_{n_2 \in N_2(F)\backslash N_2(A)} \int\limits_{v \in \tilde{N}^n(F)\backslash \tilde{N}^n(A)} \Psi_f(vj(n_2))\theta_2(n_2)\tilde{\chi}_n^{-1}(v)\, dv\, dn_2. \tag{4.10}$$

This expression can be rewritten as $I_{Sp_{2n}}(f : Sp_{n,K}, 1; j(N_2)\tilde{N}^n, \theta_2\tilde{\chi}^{-1})$, which is absolutely convergent.

4.3. **Global identity 1: between $I_1(f)$ and $I_2(f)$.**

Proposition 4.1. *The equation $I_1(f) = I_2(f)$ holds for any $f \in \mathcal{S}(\mathrm{Sp}_{2n}(A))$.*

Proof. This is Proposition 3.1 in [MR] and we follow its proof, namely, we consider $\Psi_f(g)$ in place of $\mathcal{E}(g,\phi) = Res_{s=1}E(g,\phi_{\eta,s})$ in the setting of [GRS2, Theorem 2] to obtain

$$I_2(f) = \int\limits_{Y^*_{n-1,n}(A)} \int\limits_{u \in E_{2n}(F)\backslash E_{2n}(A)} \Psi_f(uy^*\omega)\psi^{2n}(u)\, du\, dy^*. \tag{4.11}$$

We remark that [GRS2, Theorem 5.2] is a general statement about automorphic forms on $Sp_{2n}(A)$, hence our present use of it is justified.

Let us assume equation [GRS2, (5.16)] with $\Psi_f(g)$ in place of $Res_{s=1}E(g,\phi_{\eta,s})$. We obtain:

$$\int\limits_{E_{2n}(F)\backslash E_{2n}(A)} \Psi_f(u)\psi^{2n}(u)\, du = \int\limits_{x \in \mathcal{X}_0(A)} \int\limits_{N_3(F)\backslash N_3(A)} \Psi_f(n_3\bar{l}(x)\nu_0)\theta_3(n_3)\, dn_3\, dx. \tag{4.12}$$

From equations (4.11), (4.12) and the definition of $I_1(f)$ in (4.5), we obtain the proof of the Proposition. $\square$

We must now justify our use of [GRS2, (5.16)] in the present case. The argument leading up to [GRS2, (5.16)] applies in our case up to equation [GRS2, (5.2)], namely

$$(4.13)\qquad \int\limits_{N^{(k)}(F)\backslash N^{(k)}(A)} \Psi_f(n)\chi_{k,\alpha}^{-1}(n)dn = 0$$

whenever $1 \leq k < n$. The groups $N^{(k)}$ are defined below in equation (5).

Equation (4.13) follows from a result on disjointness of $Sp_{(n,K)}$-invariant functionals and $(N^{(k)}, \chi_{k,\alpha})-$eigenfunctionals. The local results which give disjointness of $Sp_n \times Sp_n-$invariant functionals and $(N^{(k)}, \chi_{k,\alpha})-$eigenfunctionals are given by [GRS1, Theorem 16,17]. In our case, the following Theorem is analogous to [GRS1, Theorem 16], it implies the analogous statement to [GRS1, Theorem 17] and justifies equation (4.13) and our use of [GRS2, (5.16)].

For $0 \leq k < n$, the subgroup $N^{(k)}$ is defined by

$$(4.14)\qquad N^{(k)} = \left\{ n = \begin{pmatrix} z & u & * & * & * \\ & 1 & 0 & y & * \\ & & I_{2k} & 0 & * \\ & & & 1 & u' \\ & & & & z^* \end{pmatrix} |z \in Z_{2n-(k+1)} \right\}$$

and its character $\chi_{k,\alpha}$ is defined by $\chi_{k,\alpha}(n) = \psi(z_{1,2} + z_{2,3} + \ldots + z_{2n-k-2,2n-k-1} + u_{2n-k-1})\psi(y)$. Note that on $N^{(k)}$ the characters $\chi_{n,1}$ and $\widetilde{\chi_n}$ are equal.

Theorem 4.2. *For $0 \leq k < n$, the Jacquet module $J_{N^{(k)},\chi_{k,\alpha}}({}^cInd_{Sp_{n,K}}^{Sp_{2n}}1)$ is zero.*

Proof. We prove this by standard Bruhat theory, we consider the double cosets $Sp_{n,K}\backslash Sp_{2n}/N^{(k)}$ and show that for all $g \in Sp_{2n}(F_v)$ one has $\chi_{k,\alpha}|g^{-1}Sp_{n,K}g \cap N^{(k)} \neq 1$. As before, we consider a symmetric space isomorphic to $Sp_{n,K}\backslash Sp_{2n}$, namely given $g \in Sp_{2n}$ we define the involution $\theta(g) := \mathbf{T}g\mathbf{T}^{-1}$. The centralizer of $\mathbf{T}$ in Sp_{2n} is $Sp_{n,K}$. The symmetric space Y is defined by $Y = \{g^{-1}\theta(g)\mathbf{T} | g \in Sp_{2n}\} = \{g^{-1}\mathbf{T}g | g \in Sp_{2n}\}$ and $Y \cong Sp_{n,K}\backslash Sp_{2n}$. We recall that $y \in Y$ admits a decomposition

$$y = n^{-1}wan \text{ where } n \in N_3, (wa)^2 = \tau, {}^t(wa)J(wa) = \tau J. \tag{4.15}$$

Now an element $v \in N^{(k)}$ is in the stabilizer of $n^{-1}wan$ if and only if $v^{-1}n^{-1}wanv = n^{-1}wan$, that is, $(nvn^{-1})^{-1}wa(nvn^{-1}) = wa$.

Let $U^{(k)} = \left\{ \begin{pmatrix} I_{2n-k-1} & & \\ & u & \\ & & I_{2n-k-1} \end{pmatrix} | u \in N_{Sp_{k+1}(F)} \right\}$.

Clearly, $N_3 = N^{(0)} = U^{(k)}N^{(k)}$, $U^{(k)}$ normalizes $N^{(k)}$ and for $u \in U^{(k)}, v \in N^{(k)}$, we have

$$\chi_{k,\alpha}(uvu^{-1}) = \chi_{k,\alpha}(v). \tag{4.16}$$

Note that $U^{(k)} \cap N^{(k)}$ is the center of $U^{(k)}$. From the decomposition in equation (4.15), we have that the $N^{(k)}$ orbits on Y admit representatives of the form

$$u^{-1}wau$$

with $u \in U^{(k)}$ and wa as in (4.15). Now an element $v \in N^{(k)}$ is in the stabilizer of $u^{-1}wau$ if and only if $(uvu^{-1})^{-1}wa(uvu^{-1}) = wa$ and by equation (4.16), it suffices to assume $u = 1$. Thus we are reduced to solving $n^{-1}wan = wa, n \in N^{(k)}$.

We call wa satisfying $(wa)^2 = \tau, {}^t(wa)Jwa = \tau J$ nonrelevant, if the above equation admits solutions n in $N^{(k)}$ such that $\chi_{k,\alpha}(n) \neq 1$. Otherwise, we call wa relevant. In order to prove our theorem, we have to show all wa with $(wa)^2 = \tau, {}^t(wa)Jwa = \tau J$ are nonrelevant if $k < n$. We first need a Lemma.

Lemma 4.3. *With definitions as in the previous theorem and for a fixed k, $1 \leq k \leq n$, if wa is relevant then wa is of the form*

$$wa = \begin{pmatrix} 0_{2n-k} & * & 0_{2n-k} \\ * & * & * \\ 0_{2n-k} & * & 0_{2n-k} \end{pmatrix} \tag{4.17}$$

where 0_{2n-k} is the zero square matrix of size $2n-k$.

Proof. We note that solving the equation $n^{-1}wan = wa, n \in N^{(k)}$ is equivalent to solving $(wa)n(wa) = \tau n$. We denote the element $n_{i,j}$ by (i,j), so that the elements of $n \in N^{(k)}$ on which $\chi_{k,\alpha}$ acts are $(1,2),(3,4),\ldots,(2n-k-1,2n-k),(2n-k,2n+k+1)$. We remark that under the action of left and right multiplication by wa, the element (i,j) maps to $(\sigma(i),\sigma(j))$ where σ denotes the natural action of wa.
As a first step, we note that it is enough by the symplectic structure of w to show that $(i,j) = 0$ for $1 \leq i \leq 2n-k, 4n-i+1 \leq j \leq 4n$ and for $1 \leq i \leq 2n-k, 1 \leq j \leq i$.

We prove the first assertion by induction on the rows. We remark that the entries of w satisfy $(i, 4n-i+1) = 0$, which in particular gives the initial case in our induction. Now let us assume that $(i,j) = 0$ for $1 \leq i \leq l-1, 4n-i+1 \leq j \leq 4n$. We want to show that $\sigma(l) \notin \{4n-l+2, 4n-l+3, \ldots, 4n\}$.

We do this by contradiction, we assume $\sigma(l) = 4n-\gamma$ with $\gamma \in \{0,1,\ldots,l-3\}$. We consider the root $X = (l-1,l)$ and remark that w maps X to $(\sigma(l-1), 4n-\gamma)$. This allows us to construct $n \in N^{(k)}$ with non trivial character action (contradicting the relevancy of w) unless $\sigma(l-1) \in \{4n-\gamma-1, 4n-\gamma, \ldots 4n\}$, but this is impossible by the induction hypothesis. This proves that $\sigma(l) \notin \{4n-l+3,\ldots,4n\}$.
We now assume that $\sigma(l) = 4n-l+2$ and again consider the root $X = (l-1,l)$. As before, w carries X to $(\sigma(l-1), 4n-l+2)$ and this gives rise to $n \in N^{(k)}$ with nontrivial character action unless $\sigma(l-1) \in \{4n-l+1, 4n-l+2, 4n-l+3, \ldots, 4n\}$.

The induction hypothesis gives that $\sigma(l-1) \notin \{4n-l+2,\ldots,4n\}$ so that we are left to consider the case $\sigma(l-1) = 4n-l+1$. In this case the root X is mapped to the element $(4n-l+1, 4n-l+2)$ which is related by the symplectic structure of n to X, this allows us once more to define $n \in N^{(k)}$ with nontrivial character action, contradicting the relevancy of w. This shows that $\sigma(l) \notin \{4n-l+2, 4n-l+3,\ldots,4n\}$ and concludes the induction.

We similarly prove the second claim by induction on the rows. By hypothesis we have that $(i,j) = 0$ for $i \in \{2n-k, 2n-k-1,\ldots,2n-k-(l-1)\}$ with $1 \leq j \leq i$. We assume that $\sigma(2n-k-l+1) \in \{1,2,\ldots,2n-k-l\}$, and we consider the root $(2n-k-(l-1), 2n-k-l+2)$. This root gets mapped by w to $(\sigma(2n-k-l+1), \sigma(2n-k-l+1))$. This gives rise to $n \in N^{(k)}$ with nontrivial character action contradicting the relevancy of w unless $\sigma(2n-k-l+2) \in \{1,2,\ldots,\sigma(2n-k-l+1)+1\}$. This last condition is impossible by the induction hypothesis. This proves the induction step.

For the initial case, we have to prove $(i,j) = 0$ whenever $i = 2n-k$ and $1 \leq j \leq i$. We assume that $\sigma(2n-k) \in \{1,2,\ldots,2n-k-1\}$. We consider the root $(2n-k, 2n+k+1)$ and remark that this root gets mapped by w to the element $(\sigma(2n-k), \sigma(2n+k+1))$. This implies the existence of $n \in N^{(k)}$ with nontrivial character action contradicting the relevancy of w unless $\sigma(2n+k+1) \in \{1,2,\ldots\sigma(2n-k)-1\}$. This is impossible as a consequence of the first assertion in the proof. This proves the initial case of our induction. $\square$

We remark that if $k < n$ the matrix wa of type (4.17) is not invertible, since w is in the Weyl group of $Sp_{2n}(F_v)$. This proves Theorem 4.2. $\square$

The following is the equivalent of [GRS1, Theorem 17] our case, the proof is the same as in [GRS1], we present it here for completeness.

Theorem 4.4. *Let π be an irreducible, admissible representation of $Sp_{2n}(F_v)$. Assume that the space of π admits a nontrivial $Sp_{(n,K)}$-invariant functional. Then for $0 \le k < n$, $J_{N^{(k)},\chi_k}(\pi) = 0$, i.e. π has no nontrivial $(N^{(k)}, \chi_k)$-eigenfunctionals.*

Proof. By [MVW, P. 91], we have that $\tilde{\pi} \cong \pi^\delta$ where π^δ stands for the composition of π with conjugation by $\check{\delta}$ where $\check{\delta} = \begin{pmatrix} I_{2n} & \\ & \delta I_{2n} \end{pmatrix} \in GSp_{2n}(F_v)$ and δ is a non square in F_v. We remark that conjugation by $\check{\delta}$ preserves $Sp_{(n,K)}$. Then if π acts on the space V_π and l is a $Sp_{(n,K)}$ invariant functional on V_π, then $l(\pi^\delta(h)\xi) = l(\pi(h^\delta)\xi) = l(\xi)$ for $h \in Sp_{(n,K)}$ and $\xi \in V_{\tilde{\pi}} = V_{\pi^\delta}$ so that l serves as an $Sp_{(n,K)}$ invariant functional for π^δ as well, hence π admits $Sp_{(n,K)}$ invariant functionals if and only if $\tilde{\pi}$ does. This happens if and only if $\tilde{\pi}$ embeds in $\mathrm{Ind}_{Sp_{n,K}}^{Sp_{2n}} 1 = C^\infty(Sp_{n,K} \backslash Sp_{2n})$, which is the same as the existence of a surjection ${}^c\mathrm{Ind}_{Sp_{n,K}}^{Sp_{2n}} 1 \to \tilde{\tilde{\pi}} \cong \pi$ and by the exactness of Jacquet functors and Theorem 4.2, we get $J_{N^{(k)},\chi_k}(\pi) = 0$ □

To relate the distributions $I_{Sp_{2n}}(f : Sp_{n,K}, 1; N_3, \theta_3)$ and $I_{Sp_{2n}}(f : Sp_{n,K}, 1; N_3, \theta_4\Theta^\Phi_{\psi^{-1}})$, we require two other trace identities. The following trace identities are in [MR], we state them here for completeness.

4.4. **Global identity 2: between $I_1(f)$ and $I_{Sp_{2n}}(f : Sp_{n,K}, 1; N_3, \theta_3)$.** Recall:

$$I_{Sp_{2n}}(f : Sp_{n,K}, 1; N_3, \theta_3) = \int\limits_{N_3(F)\backslash N_3(A)} \Psi_f(n)\theta_3(n)\, dn. \tag{4.18}$$

Theorem 4.5. *There exist maps $\epsilon_{2,v}$ from $\mathcal{S}(\mathrm{Sp}_{2n}(F_v))$ to itself, such that*

(1) the equation

$$I_{Sp_{2n}}(f : Sp_{n,K}, 1; N_3, \theta_3) = I_1(f') \tag{4.19}$$

holds for $f = \otimes f_v$, $f' = \otimes f'_v$ when $f'_v = \epsilon_{2,v}(f_v)$.

(2) for v a good place, $\epsilon_{2,v}$ restricts to identity map on Hecke algebra $\mathcal{H}(\widetilde{Sp_n}, K_2)_v$.

Similarly there exist maps $\epsilon'_{2,v}$ satisfying condition (2) such that (4.19) holds when $f_v = \epsilon'_{2,v}(f'_v)$.

Proof. We set $f^1_v(g) = f'_v(g\omega)$ for all places v. Then we have that

$$I_1(f') = \int_{y^* \in Y^*_{n-1,n}(A)} \int_{x \in \mathcal{X}_0(A)} \int_{N_3(F)\backslash N_3(A)} \Psi_{f^1}(n\bar{l}(x)\nu_0 y^*)\theta_3(n)dndxdy^*. \tag{4.20}$$

Recall that ω is in the maximal compact subgroup of $Sp_{2n}(F_v)$ for v an odd p-adic place. Thus we have that if f'_v is a Hecke function at a good place v, then $f^1_v(g) = f'_v(g)$.

Next we note that $Y^*_{n-1,n}$ is an abelian group which can be written as $Y^*_{n-1,n} = \prod_{i=1}^{n-1} K_i$ where

$$K_i = \left\{ i_1(1_{2n} + \sum_{j=1}^{i} t_j e_{2i,2j-1}) \right\}. \tag{4.21}$$

Let $K^i = \prod_{l=1}^{i} K_l$, note that $K^0 = \{1_{4n}\}$ and $K^{n-1} = Y^*_{n-1,n}$.

Let

$$h_f(g) = \int_{x \in \mathcal{X}_0(A)} \int_{N_3(F)\backslash N_3(A)} \Psi_f(n\bar{l}(x)\nu_o g)\theta_3(n)dndx. \tag{4.22}$$

Let L be a space of smooth functions on $Sp_{2n}(A)$ such that $f(g) \in L$ implies $f(ug) = \psi^{2n}(u^{-1})f(g)$ for $u \in E_{2n}$. From equation (4.12), we have that $h_f(g) \in L$. The righthand side of equation (4.20) is

$$\int_{K^{n-1}(A)} h_{f^1}(y)dy. \tag{4.23}$$

From the Theorem of Dixmier-Mallavin [DMa], we have that f_v^1 can be written as

$$f_v^1(g) = \sum_{\alpha_v} \int\limits_{F_v^i} \phi_{\alpha_v}(x_1, \ldots, x_i) f_{\alpha_v}(gr_i(x_1, \ldots, x_i)) d(x_1, \ldots, x_i) \tag{4.24}$$

for some $f_{\alpha_v} \in \mathcal{S}(Sp_{2n}(F_v))$ and $\phi_{\alpha_v} \in \mathcal{S}(F_v^i)$; here r_i is the homomorphism from A^{i-1} to $Sp_{2n}(A)$ given by

$$r_i(t_1, \ldots, t_{i-1}) = i_1(1_{2n} + \sum_{j=1}^{i-1} t_j e_{2j-1,2i}). \tag{4.25}$$

Moreover, at good places f_v^1 is a Hecke function and can be expressed as above with a single α_v with ϕ_{α_v} being the characteristic function of the integer lattice and $f_{\alpha_v} = f_v^1$.

We have that

$$h_{f^1}(g) = \sum_{\alpha} \int\limits_{A^i} \phi_\alpha(x_1, \ldots, x_i) h_{f\alpha}(gr_i(x_1, \ldots, x_i)) d(x_1, \ldots, x_i) \tag{4.26}$$

for some $f_\alpha \in \mathcal{S}(\mathrm{Sp}_{2n}(A))$ and $\phi_\alpha \in \mathcal{S}(A^i)$.

From the proof of [GRS2, Lemma 5.1], we get the following:

Lemma 4.6. *For fixed i and a function $h_i(g) \in L$, such that $h_i(g)$ equals*

$$\sum_{\alpha} \int\limits_{A^i} \phi_\alpha(x_1, \ldots, x_i) h_\alpha(gr_i(x_1, \ldots, x_i))\, d(x_1, \ldots, x_i)$$

for some $h_\alpha \in L$ and $\phi_\alpha \in \mathcal{S}(A^i)$, we have

$$\int\limits_{K^i(A)} h_i(y)\, dy = \int\limits_{K^{i-1}(A)} h_{i-1}(y)\, dy, \tag{4.27}$$

where

$$h_{i-1}(g) = \sum_{\alpha} \int\limits_{A^i} \widehat{\phi_\alpha}(x_1, \ldots, x_i) h_\alpha(gr_i(x_1, \ldots, x_i))\, d(x_1, \ldots, x_i),$$

$\widehat{\phi_\alpha}$ *is the Fourier transform of* ϕ_α*:*

$$\widehat{\phi_\alpha}(x_1, \ldots, x_i) = \int \phi_\alpha(t_1, \ldots, t_i)\psi(\sum_{j=1}^{i} x_i t_i)\, d(t_1, \ldots, t_i).$$

From equation(4.27), we get a $f^1_{n-2} = \otimes f^1_{n-2,v} \in \mathcal{S}(\mathrm{Sp}_{2n}(A))$ defined by

$$f^1_{n-2,v}(g) = \sum_{\alpha_v} \int\limits_{F_v^i} \widehat{\phi_{\alpha_v}}(x_1, \ldots, x_i) f_{\alpha_v}(gr_i(x_1, \ldots, x_i))\, d(x_1, \ldots, x_i), \tag{4.28}$$

satisfying

$$\int\limits_{K^{n-1}(A)} h_{f^1}(y)dy = \int\limits_{K^{n-2}(A)} h_{f^1_{n-2}}(y)\, dy. \tag{4.29}$$

Note that from (4.28), $f^1_{n-2,v} = f^1_v$ when f^1_v is a Hecke function at a good place v. Continuing the procedure we get eventually $f^2 = f^1_0 \in \mathcal{S}(\mathrm{Sp}_{2n}(A))$, with $f^2_v = f^1_v$ when f^1_v is a Hecke function at a good place v, and

$$\int\limits_{K^{n-1}(A)} h_{f^1}(y)dy = \int\limits_{K^0(A)} h_{f^1_0}(y)\, dy = h_{f^2}(1_{4n}).$$

We get that the righthand side of (4.20) equals:

$$\int\limits_{x \in \mathcal{X}_0(A)} \int\limits_{N_3(F)\backslash N_3(A)} \Psi_{f^2}(n\bar{l}(x)\nu_0)\theta_3(n)\, dn\, dx. \tag{4.30}$$

Moreover $f^2_v = f^1_v$ when f^1_v is a Hecke function at a good place v.

By letting $f^3(g) = f^2(g\nu_0)$, we get (4.30) equals

$$\int_{x\in\mathcal{X}_0(A)} \int_{N_3(F)\backslash N_3(A)} \Psi_{f^3}(n\bar{l}(x))\theta_3(n)\,dn\,dx. \tag{4.31}$$

Clearly $f_v^3 = f_v^2$ when f_v^2 is a Hecke function at a good place v.

We now consider

$$\bar{h}_f(g) = \int_{N_3(F)\backslash N_3(A)} \Psi_f(ng)\theta_3(n)\,dn.$$

We have that $\bar{h}_f(g) \in \bar{L}$, where $\bar{L}$ consists of functions satisfying $\phi(ng) = \theta_3(n^{-1})\phi(g)$. As in Lemma 4.6, we have that the equation

$$\int_{\bar{K}^i(A)} \bar{h}_i(y)\,dy = \int_{\bar{K}^{i-1}(A)} \bar{h}_{i-1}(y)\,dy \tag{4.32}$$

holds for $\bar{h}_i, \bar{h}_{i-1} \in \bar{L}$ related as in Lemma 4.6. Here the subgroups $\bar{K}^i$ are defined as on [GRS2, p.897]; we note that $\bar{K}^{2n} = \bar{l}(\mathcal{X}_0)$ and $\bar{K}^1 = \{1_{4n}\}$. Similar to the above argument, using (4.32) we get a function $f^4 \in \mathcal{S}(\mathrm{Sp}_{2n}(A))$, with $f_v^4 = f_v^3$ when f_v^3 is a Hecke function at a good place v, such that (4.31) equals $\bar{h}_{f^4}(1_{4n})$. Since $\bar{h}_{f^4}(1_{4n})$ is just $I_{Sp_{2n}}(f^4 : Sp_{n,K}, 1; N_3, \theta_3)$, we can set $f = \epsilon_2'(f') = f^4$. Then $f_v = f_v'$ when f_v^1 is a Hecke function at a good place v, and the equality (4.19) holds.

As each of the steps above can be reversed, given f, we can find $f' = \epsilon_2(f)$ to make the equality (4.19) hold. $\square$

4.5. **Heisenberg representation and definition of $I_{Sp_{2n}}(f : Sp_{n,K}, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$.** Recall the definition of the map η in (4.8). The character ψ determines an irreducible unitary representation of Heisenberg group U^n acting on $\mathcal{S}(F^n)$, denoted again by ω_ψ; then:

$$\omega_\psi(\eta(\mathbf{x},0,0))\Phi(X) = \Phi(X+\mathbf{x}). \tag{4.33}$$

$$\omega_\psi(\eta(0,\mathbf{y},t))\Phi(X) = \psi(t+2\operatorname{tr}(\mathbf{y}\sigma_n X))\Phi(X). \tag{4.34}$$

$$\omega_\psi(\widetilde{g})\omega_\psi(j(g)^{-1}uj(g))\Phi(X) = \omega_\psi(u)\omega_\psi(\widetilde{g})\Phi(X),\ g\in \mathrm{Sp}_n,\ u\in U^n. \tag{4.35}$$

For $n\in N_3$, we will use $\mathrm{pr}(n)$ to denote the middle $2n+2\times 2n+2$ block of n. Then $\mathrm{pr}(n)=j(n_2)\eta(\mathbf{x},\mathbf{y},t)$ for some $\eta(\mathbf{x},\mathbf{y},t)$ in the Heisenberg group and $n_2\in N_2$. We define:

$$\omega_{\psi^{-1}}(n)\Phi(X) = \omega_{\psi^{-1}}(\widetilde{n_2})\omega_{\psi^{-1}}(\eta(\mathbf{x},\mathbf{y},t))\Phi(X),\ \mathrm{pr}(n)=j(n_2)\eta(\mathbf{x},\mathbf{y},t). \tag{4.36}$$

Clearly the above defines an action of N_3 on the space $\mathcal{S}(A^n)$. Define the Theta function

$$\Theta^\Phi_{\psi^{-1}}(n) = \sum_{X\in F^n}\omega_{\psi^{-1}}(n)\Phi(X). \tag{4.37}$$

Define a character θ_4 on N_3 by setting

$$\theta_4(n) = \psi\left(\sum_{i=1}^{n-1} -n_{i,i+1}\right)\theta_2(n_2),\ \text{ if }\ \mathrm{pr}(n)=j(n_2)\eta(\mathbf{x},\mathbf{y},t). \tag{4.38}$$

Define $I_{Sp_{2n}}(f: Sp_{n,K},1;N_3,\theta_4\Theta^\Phi_{\psi^{-1}})$ to be:

$$\int\limits_{l\in Sp_{n,K}(F)\backslash Sp_{n,K}(A)}\ \int\limits_{n\in N_3(F)\backslash N_3(A)} K_f(l,n)\theta_4(n)\Theta^\Phi_{\psi^{-1}}(n)\,dn\,dl. \tag{4.39}$$

4.6. **Global identity 3: between $I_2(f)$ and $I_{Sp_{2n}}(f: Sp_{n,K},1;N_3,\theta_4\Theta^\Phi_{\psi^{-1}})$.**

Theorem 4.7. *We have $I_{Sp_{2n}}(f: Sp_{n,K},1;N_3,\theta_4\Theta^\Phi_{\psi^{-1}}) = I_2(f')$ when*

$$f'(g) = \int\limits_{X\in A^n}\Phi(X)f(g\eta(X,\mathbf{0},0))\,dX. \tag{4.40}$$

Proof. We have that $I_2(f')$ is equal to

$$\int_{n_2\in N_2(F)\backslash N_2(A)}\int_{v\in \tilde{N}^n(F)\backslash \tilde{N}^n(A)} \Psi_{f'}(vj(n_2))\theta_2(n_2)\tilde{\chi}_n^{-1}(v)\, dv\, dn_2$$

which is

$$\int_{n_2\in N_2(F)\backslash N_2(A)}\int_{v\in \tilde{N}^n(F)\backslash \tilde{N}^n(A)}\int_{X\in A^n} \Psi_f(vj(n_2)\eta(X,0,0))\Phi(X)\theta_2(n_2)\tilde{\chi}_n^{-1}(v)\, dX\, dv\, dn_2. \tag{4.41}$$

From equations (2.1), (2.2) and (4.33), we get

$$\Phi(X) = \omega_{\psi^{-1}}(\widetilde{n_2})\omega_{\psi^{-1}}(\eta(X,0,0))\Phi(0). \tag{4.42}$$

Also, recall that $U^n = \{\eta(\mathbf{x},\mathbf{y},t)\}$, while $U_0^n = \{\eta(\mathbf{0},\mathbf{y},t)\}$ and U_0^n is normal in U^n. Thus, we may identify $\eta(\mathbf{X},\mathbf{0},0)$ with $U_0^n\backslash U^n(A)$. Then $I_2(f')$ is equal to

$$\int_{n_2\in N_2(F)\backslash N_2(A)}\int_{v\in \tilde{N}^n(F)\backslash \tilde{N}^n(A)}\int_{u\in U_0^n\backslash U^n(A)} \Psi_f(vj(n_2)u)\omega_{\psi^{-1}}(\widetilde{n_2})\omega_{\psi^{-1}}(u)\Phi(0)\theta_2(n_2)\tilde{\chi}_n^{-1}(v)\, du\, dv\, dn_2. \tag{4.43}$$

Note that $j(N_2)$ acts on U^n by conjugation and it stabilizes U_0^n; we change variables $u \mapsto j(n_2)^{-1}uj(n_2)$, we obtain

$$\int_{N_2(F)\backslash N_2(A)}\int_{v\in \tilde{N}^n(F)\backslash \tilde{N}^n(A)}\int_{u\in U_0^n\backslash U^n(A)} \Psi_f(vuj(n_2))\omega_{\psi^{-1}}(\widetilde{n_2})\omega_{\psi^{-1}}(j(n_2)^{-1}uj(n_2))\Phi(0)\theta_2(n_2)\tilde{\chi}_n^{-1}(v)\, du\, dv\, dn_2 \tag{4.44}$$

By equation (4.35), this is

(4.45)

$$\int_{n_2\in N_2(F)\backslash N_2(A)}\int_{v\in\tilde{N}^n(F)\backslash\tilde{N}^n(A)}\int_{u\in U_0^n\backslash U^n(A)}\Psi_f(vuj(n_2))\omega_{\psi^{-1}}(u)\omega_{\psi^{-1}}(\widetilde{n_2})\Phi(0)\theta_2(n_2)\tilde{\chi}_n^{-1}(v)\,du\,dv\,dn_2.$$

Since by definition, $\tilde{N}^n = U_0^n\hat{N}^n$, we get the above expression is

$$\int_{n_2\in N_2(F)\backslash N_2(A)}\int_{u\in U_0^n(F)\backslash U^n(A)}\int_{v\in\hat{N}^n(F)\backslash\hat{N}^n(A)}\Psi_f(vuj(n_2))\omega_{\psi^{-1}}(u)\omega_{\psi^{-1}}(\widetilde{n_2})\Phi(0)\theta_2(n_2)\tilde{\chi}_n^{-1}(v)\,du\,dv\,dn_2,$$

which is

$$\int_{N_2(F)\backslash N_2(A)}\int_{u\in U^n(F)\backslash U^n(A)}\sum_{u'\in U_0^n(F)\backslash U^n(F)}\int_{v\in\hat{N}^n(F)\backslash\hat{N}^n(A)}\Psi_f(vu'uj(n_2))\omega_{\psi^{-1}}(u'u)\omega_{\psi^{-1}}(\widetilde{n_2})\Phi(0)\theta_2(n_2)\tilde{\chi}_n^{-1}(v).$$

We change variables $v\mapsto u'v(u')^{-1}$, we get

$$\int_{N_2(F)\backslash N_2(A)}\int_{u\in U^n(F)\backslash U^n(A)}\int_{v\in\hat{N}^n(F)\backslash\hat{N}^n(A)}\sum_{u'\in U_0^n(F)\backslash U^n(F)}\Psi_f(u'vuj(n_2))\omega_{\psi^{-1}}(u'u)\omega_{\psi^{-1}}(\widetilde{n_2})\Phi(0)\theta_2(n_2)\tilde{\chi}_n^{-1}(v).$$

Since Ψ_f is $Sp_{2n}(F)-$left invariant and $\tilde{\chi}_n$ is stabilized under conjugation by U^n, we obtain

$$\int_{N_2(F)\backslash N_2(A)}\int_{u\in U^n(F)\backslash U^n(A)}\int_{v\in\hat{N}^n(F)\backslash\hat{N}^n(A)}\Psi_f(vuj(n_2))\sum_{u'\in U_0^n(F)\backslash U^n(F)}\omega_{\psi^{-1}}(u'u)\omega_{\psi^{-1}}(\widetilde{n_2})\Phi(0)\theta_2(n_2)\tilde{\chi}_n^{-1}(v).$$

Identifying $U_0^n\backslash U^n$ with $\eta(\mathbf{X},\mathbf{0},0)$ and using equation (4.33), we obtain

$$\sum_{u'\in U_0^n(F)\backslash U^n(F)}\omega_{\psi^{-1}}(u'u)\omega_{\psi^{-1}}(\widetilde{n_2})\Phi(0)=\sum_{X\in F^n}\omega_{\psi^{-1}}(u)\omega_{\psi^{-1}}(\widetilde{n_2})\Phi(X).$$

So we have that $I_2(f')$ is

(4.46)
$$\int_{N_2(F)\backslash N_2(A)} \int_{u\in U^n(F)\backslash U^n(A)} \int_{v\in \hat{N}^n(F)\backslash \hat{N}^n(A)} \Psi_f(vuj(n_2)) \sum_{X\in F^n} \omega_{\psi^{-1}}(u)\omega_{\psi^{-1}}(\widetilde{n_2})\Phi(X)\theta_2(n_2)\tilde{\chi}_n^{-1}(v).$$

On the other hand, we have that $I_{Sp_{2n}}(f : Sp_{n,K}, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$ is equal to

$$\int_{n\in N_3(F)\backslash N_3(A)} \Psi_f(n)\theta_4(n)\Theta^{\Phi}_{\psi^{-1}}(n)\, dn.$$

Since $\hat{N}^{n+1}\backslash N_3 \cong j(N_2)$ and $\hat{N}^n\backslash \hat{N}^{n+1} \cong U^n$, we obtain

$$\int_{N_2(F)\backslash N_2(A)} \int_{U^n(F)\backslash U^n(A)} \int_{\hat{N}^n(F)\backslash \hat{N}^n(A)} \Psi_f(vuj(n_2))\theta_4(vuj(n_2))\Theta^{\Phi}_{\psi^{-1}}(vuj(n_2))\, dv\, du\, dn_2.$$

We have that

$$\Theta^{\Phi}_{\psi^{-1}}(vuj(n_2)) = \sum_{X\in F^n} \omega_{\psi^{-1}}(u)\omega_{\psi^{-1}}(\widetilde{n_2})\Phi(X)$$

and $\theta_4(vuj(n_2)) = \theta_4(v)\theta_2(n_2)$. Furthermore, θ_4 agrees with $\tilde{\chi}_n^{-1}$ on $\hat{N}^n$. Thus we obtain $I_{Sp_{2n}}(f : Sp_{n,K}, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$ is equal to equation (4.46). $\square$

Corollary 4.8. *There exist maps $\epsilon_{3,v}$ from $\mathcal{S}(\mathrm{Sp}_{2n}(F_v))$ to $\mathcal{S}(\mathrm{Sp}_{2n}(F_v))\otimes\mathcal{S}(F_v^n)$, such that:*

(1) at a good place v, $\epsilon_{3,v}(f_v) = f_v \otimes \Phi_{0,v}$ when f_v is a Hecke function and $\Phi_{0,v}$ is the characteristic function of $\mathcal{O}_v^n$.

(2) when $\epsilon_3 = \otimes\epsilon_{3,v}$ and $f\otimes\Phi = \epsilon_3(f')$:

(4.47)
$$I_2(f') = I_{Sp_{2n}}(f : Sp_{n,K}, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}}).$$

Proof. From Theorem 4.7, to define $\epsilon_{3,v}(f'_v)$ so that (4.47) holds, we only need to find f and Φ so that (4.40) holds. The map $(f,\Phi) \mapsto f'$ defined by (4.40) is a convolution, it clearly factors into local maps. The existence of f_v and Φ_v follows from the result of Dixmier-Malliavin [DMa]. For v a good place, it is clear that when f_v is a Hecke function and $\Phi_v = \Phi_{0,v}$,

$$\int_{X\in F_v^n} f_v(g\eta(X,0,0))\Phi_{0,v}(X)\, dX = f_v(g). \tag{4.48}$$

Thus at good place v, we can choose $f_v = f'_v$ and Φ_v to be $\Phi_{0,v}$. □

We remark that equation (4.40) defines the map $\epsilon'_{3,v}$ from $\mathcal{S}(\mathrm{Sp}_{2n}(F_v)) \otimes \mathcal{S}(F_v^n)$ to $\mathcal{S}(\mathrm{Sp}_{2n}(F_v))$, with the property that at good places $\epsilon'_{3,v}(f_v \otimes \Phi_{0,v}) = f_v$ when f_v is a Hecke function, and equation (4.47) holds when $f' = \epsilon'_3(f\otimes\Phi)$.

4.7. **Conclusion.** Combining the three global identities on Sp_{2n}, we get:

Corollary 4.9. *There exist maps $\epsilon_{4,v}$ from $\mathcal{S}(Sp_{2n}(F_v))$ to $\mathcal{S}(Sp_{2n}(F_v))\otimes\mathcal{S}(F_v^n)$ such that:*

(1) at a good place v, $\epsilon_{4,v}(f_v) = f_v \otimes \Phi_{0,v}$ when f_v is a Hecke function and $\Phi_{0,v}$ is the characteristic function of $\mathcal{O}_v^n$.

(2) when $\epsilon_4 = \otimes\epsilon_{4,v}$ and $f\otimes\Phi = \epsilon_4(f')$:

$$I_{Sp_{2n}}(f' : Sp_{n,K}, 1; N_3, \theta_3) = I_{Sp_{2n}}(f : Sp_{n,K}, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}}). \tag{4.49}$$

Proof. Define $\epsilon_{4,v} = \epsilon_{3,v}\epsilon_{2,v}$. The claim follows from Proposition 4.1, Theorem 4.5 and Corollary 4.8. □

We remark that one can also define the maps $\epsilon'_{4,v} = \epsilon'_{2,v}\epsilon'_{3,v}$ from $\mathcal{S}(Sp_{2n}(F_v)) \otimes \mathcal{S}(F_v^n)$ to $\mathcal{S}(Sp_{2n}(F_v))$, such that at a good place $\epsilon'_{4,v}(f_v\otimes\Phi_{0,v}) = f_v$ when f_v is a Hecke function, and equation (4.49) holds when $f' = \epsilon'_4(f\otimes\Phi)$.

5. Orbital integral decompositions

In this section we wish to relate the distributions $I_{Sp_{2n}}(f : Sp_{(n,K)}, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$ and $I_{\widetilde{Sp_n}}(\widetilde{f} : N_2, \theta_{2,\tau}^{-1}; N_2, \theta_2)$.

As before, we consider a space isomorphic to $Sp_{n,K}\backslash Sp_{2n}$, namely given $g \in Sp_{2n}$ we define the involution $\theta(g) := \mathbf{T}g\mathbf{T}^{-1}$. The centralizer of $\mathbf{T}$ in Sp_{2n} is $Sp_{n,K}$. The space Y is defined by $Y = \{g^{-1}\theta(g)\mathbf{T} | g \in Sp_{2n}\} = \{g^{-1}\mathbf{T}g | g \in Sp_{2n}\}$ and we have that $Y \cong Sp_{n,K}\backslash Sp_{2n}$.

We unwind the distribution $I_{Sp_{2n}}(f : Sp_{(n,K)}, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$, we have that it equals

$$\int\limits_{l_3 \in Sp_{n,K}(F)\backslash Sp_{n,K}(A)} \int\limits_{n_3 \in N_3(F)\backslash N_3(A)} K_f(l_3, n_3)\theta_4(n_3)\Theta^{\Phi}_{\psi^{-1}}(n_3) dl_3 dn_3 \tag{5.1}$$

$$= \int\limits_{l_3 \in Sp_{n,K}(F)\backslash Sp_{n,K}(A)} \int\limits_{n_3 \in N_3(F)\backslash N_3(A)} \sum_{g_3 \in Sp_{2n}(F)} f_3(l_3^{-1} g_3 n_3)\theta_4(n_3)\Theta^{\Phi}_{\psi^{-1}}(n_3) dl_3 dn_3 \tag{5.2}$$

$$= \sum_{g_3 \in Sp_{n,K}(F)\backslash Sp_{2n}(F)/N_3(F)} \int\limits_{l_3 \in H_3(A)} \int\limits_{n_3 \in N'_{3,g_3}(F)\backslash N_3(A)} f_3(l_3^{-1} g_3 n_3)\theta_4(n_3)\Theta^{\Phi}_{\psi^{-1}}(n_3) dl_3 dn_3 \tag{5.3}$$

where $N'_{3,g_3} := g_3^{-1} Sp_{n,K} g_3 \cap N_3$. We now define

$$F(g^{-1}\mathbf{T}g) := \int\limits_{l_3 \in Sp_{n,K}(A)} f(l_3^{-1} g) dl_3 \tag{5.4}$$

then the distribution $I_{Sp_{2n}}(f : Sp_{(n,K)}, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$ is

$$(5.5) \qquad \sum_{g_3 \in Sp_{n,K}(F)\backslash Sp_{2n}(F)/N_3(F)} \int_{n_3 \in N'_{3,g_3}(F)\backslash N_3(A)} F(n_3^{-1} g_3^{-1} \mathbf{T} g_3 n_3) \theta_4(n_3) \Theta^{\Phi}_{\psi^{-1}}(n_3) dn_3.$$

We recall from Lemma (3.3) and Corollary (3.4) that elements in Y admit a decomposition $y = n^{-1}wan$ where $w \in W_3$ and a is a diagonal matrix of size $4n$ satisfying $(wa)^2 = \tau$ and ${}^t(wa)J(wa) = \tau J$. Moreover, with the action of N_3 by conjugation, to each representative $g_3 \in Sp_{n,K}\backslash Sp_{2n}/N_3$ there corresponds a wa as above such that $g^{-1}\mathbf{T}g$ and wa are in the same N_3 orbits of Y.

Lemma 5.1. *If $g \in Sp_{2n}$ satisfies $g^{-1}\mathbf{T}g = wa$, then $N'_{3,g} = N_{3,wa}$*

Proof. The condition $n \in N_{3,wa}$ is equivalent to $n \in N_3$ with $n^{-1}wan = wa$, or $n^{-1}g^{-1}\mathbf{T}gn = g^{-1}\mathbf{T}g$. This is equivalent to $gn^{-1}g^{-1}\mathbf{T}gng^{-1} = \mathbf{T}$, or $n \in g^{-1}Sp_{n,K}g \cap N_3$. □

We get that $I_{Sp_{2n}}(f : Sp_{(n,K)}, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$ equals

$$(5.6) \qquad \sum_{wa, (wa)^2 = \tau, {}^t waJ = Jwa} \int_{n_3 \in N_{3,wa}(F)\backslash N_3(A)} F(n_3^{-1} wan_3) \theta_4(n_3) \Theta^{\Phi}_{\psi^{-1}}(n_3) dn_3.$$

We factor the integral as

$$(5.7) \qquad \int_{n_3 \in N_{3,wa}(A)\backslash N_3(A)} F(n^{-1}wan) \int_{n' \in N_{3,wa}(F)\backslash N_{3,wa}(A)} \theta_4(n'n_3) \Theta^{\Phi}_{\psi^{-1}}(n'n_3) dn' dn_3.$$

Recall that the subgroup $N^{(k)}$ is defined by

$$N^{(k)} = \left\{ n = \begin{pmatrix} z & u & * & * & * \\ & 1 & 0 & y & * \\ & & I_{2k} & 0 & * \\ & & & 1 & u' \\ & & & & z^* \end{pmatrix} |z \in Z_{2n-(k+1)} \right\}.$$

The subgroup $N^{(n)}$ can also be described as follows. We have that an element of $\widetilde{N}^n$ can be written as $\eta(\mathbf{0}, \mathbf{y}, t)n$ with $n \in \widehat{N}^n$; $N^{(n)}$ consists of elements with $\mathbf{y} = 0$. Then for $n' \in N^{(n)}$ we have $\theta_4(n'n_3)\Theta^{\Phi}_{\psi^{-1}}(n'n_3) = \widetilde{\chi}_n^{-1}(n')\theta_4(n_3)\Theta^{\Phi}_{\psi^{-1}}(n_3)$. For the inner integral in equation (5.7) to be nonzero, $\widetilde{\chi}_n$ must be trivial on $N^{(n)} \cap N_{3,wa}$.

Lemma 5.2. *If $\widetilde{\chi}_n$ is trivial on $N^{(n)} \cap N_{3,wa}$, then w has the form $w = \begin{pmatrix} 0 & * & * & 0 \\ * & 0 & 0 & * \\ * & 0 & 0 & * \\ 0 & * & * & 0 \end{pmatrix}$, where each entry represents an $n \times n$ block.*

Proof. Using Lemma (4.3) in the case $k = n$, we get that w is of the form:

$$w = \begin{pmatrix} 0 & A & B & 0 \\ C & D & E & F \\ G & H & I & J \\ 0 & K & L & 0 \end{pmatrix}$$

where each entry represents a $n \times n$ block. Since w^2 is a diagonal matrix, we get that

$$\begin{pmatrix} A & B \\ K & L \end{pmatrix} \begin{pmatrix} D & E \\ H & I \end{pmatrix} = 0, \quad \begin{pmatrix} A & B \\ K & L \end{pmatrix} \begin{pmatrix} C & F \\ G & J \end{pmatrix} \text{ is invertible.}$$

Thus $D = E = H = I = 0$ and w is of the form in the Lemma. $\square$

We wish to define a bijection between $w'a' \in Sp_n$ and $wa \in Sp_{2n}$ with $(wa)^2 = \tau, {}^t(wa)J = J(wa)$; for this purpose we introduce the matrix $E := \begin{pmatrix} & \tau_n & & \\ 1_n & & & \\ & & & \tau_n \\ & & 1_n & \end{pmatrix}$ and consider the map on $g \in Sp_n$ given by

$$P(g) = j(g)^{-1} E j(g) \tag{5.8}$$

Lemma 5.3. *The map P defines a bijection from the set of $w'a' \in Sp_n$ to the set wa where $w \in W_3$ and a is a diagonal matrix of size $4n$ with $(wa)^2 = \tau, {}^t(wa)J(wa) = \tau J$, with w as in Lemma 5.2.*

Proof. We note that $E^2 = \tau$, hence $P(w'a')^2 = \tau$. We also have that ${}^tP(w'a')JP(w'a') = \tau J$ using that ${}^tEJE = \tau E$. Also, $P(w'a')$ is clearly of the form in Lemma 5.2.

For g a square matrix of size $4n$, we denote by $\rho'(g)$ its middle $2n \times 2n$ block. It's clear that $\rho'(EP(w'a')/\tau) = w'a'$, so P is an injection.

On the other hand, given wa as in Lemma 5.2, we have that $E(wa)/\tau$ verifies

$${}^t(E(wa)/\tau)J(E(wa)/\tau) = {}^t(wa)J(wa)/\tau = J,$$

thus $E(wa)/\tau \in Sp_{2n}$. The element $Ewa/\tau \in Sp_{2n}$ is of the form $\begin{pmatrix} * & 0 & 0 & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & 0 & 0 & * \end{pmatrix}$ and any $g \in Sp_{2n}$ of this form satisfies $\rho'(g) \in Sp_n$. Hence $\rho'(Ewa/\tau)$ has the form $w'a'$ where $w' \in W_2$ and a' is diagonal in Sp_n.

We prove surjectivity by showing the identity

$$P(\rho'(E(wa)/\tau)) = wa. \tag{5.9}$$

Write $wa = \begin{pmatrix} 0 & a & b & 0 \\ c & 0 & 0 & d \\ e & 0 & 0 & f \\ 0 & g & h & 0 \end{pmatrix}$ so that $\rho'(Ewa/\tau) = \begin{pmatrix} a/\tau & b/\tau \\ g & h \end{pmatrix}$. Denote this matrix $w'a'$.

It suffices to show

$$\tau = Ej(w'a')(wa)j(w'a')^{-1}$$

and this is easily checked.

∎

From the previous lemma, we see that the distribution $I_{Sp_{2n}}(f : Sp_{(n,K)}, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$ may be written as

$$\sum_{wa,(wa)^2=\tau,{}^t(wa)J(wa)=\tau J} \int_{n_3\in N_{3,wa}(F)\backslash N_3(A)} F(n_3^{-1}wan_3)\theta_4(n_3)\Theta^{\Phi}_{\psi^{-1}}(n_3)dn_3 \tag{5.10}$$

(5.11)

$$= \sum_{wa,(wa)^2=\tau,{}^t(wa)J(wa)=\tau J} \int_{n_3\in N_{3,wa}(A)\backslash N_3(A)} F(n^{-1}wan) \int_{n'\in N_{3,wa}(F)\backslash N_{3,wa}(A)} \theta_4(n'n_3)\Theta^{\Phi}_{\psi^{-1}}(n'n_3)dn'dn_3.$$

Given $w'a'$ with w' in the Weyl group of Sp_n and a' a diagonal matrix in Sp_n, we define $N'_{2,w'a'} = (w'a')N_2(w'a')^{-1} \cap N_2$. We wish to prove the following:

Proposition 5.4. *We have*

$$\int_{n'\in N_{3,wa}(F)\backslash N_{3,wa}(A)} \theta_4(n')\Theta^{\Phi}_{\psi^{-1}}(n')dn' = c(w'a')\omega_{\psi^{-1}}(\widetilde{w'a'})\Phi(0) \tag{5.12}$$

where $wa = P(w'a')$ and $c(w'a')$ is defined by

$$c(w'a') = \int_{n' \in N'_{2,w'a'}(F)\backslash N'_{2,w'a'}(A)} \theta_2((w'a')^{-1}n'(w'a'))\theta_{2,\tau}^{-1}(n')dn'. \tag{5.13}$$

We first describe the sets $N_{3,wa}$. When $wa = E$, we get from the definition that

$$N_{3,E} = \left\{ u(n,B,T) := \begin{pmatrix} n & & nB & nT\tau \\ & n & nT & nB \\ & & n^* & \\ & & & n^* \end{pmatrix} \in N_3 | n \in Z_n, B, T \in \mathcal{S}_n \right\} \tag{5.14}$$

We use V_E to denote the intersection of $N_{3,E}$ with the Siegel unipotent. It consists of $u(1_n, B, T)$. Define U_E^1 to be the subgroup consisting of $u(1_n, B, 0)$.

Lemma 5.5. *If $wa = P(w'a')$, then $N_{3,wa} = j(w'a')^{-1}N_{3,E}j(w'a') \cap N_3$*

Proof. If $n \in j(w'a')^{-1}N_{3,E}j(w'a') \cap N_3$ then $j(w'a')n^{-1}j(w'a')^{-1}Ej(w'a')nj(w'a')^{-1} = E$ which implies that $n \in N_{3,wa}$. On the other hand, given $n \in N_{3,wa}$ we have that $j(w'a')nj(w'a')^{-1}$ fixes E through conjugation and has the form $\begin{pmatrix} n & * & * \\ & * & * \\ & & n^* \end{pmatrix}$ where $n \in Z_n$; any element of the above form fixing E through conjugation must lie in N_3, so $j(w'a')nj(w'a')^{-1} \in N_{3,E}$. □

Define the group

$$U_{wa}^1 := j(w'a')^{-1}U_E^1 j(w'a'). \tag{5.15}$$

We have that U_{wa}^1 is a normal subgroup of $N_{3,wa}$ as U_E^1 is a normal subgroup of $N_{3,E}$.

Lemma 5.6. *We have* $\int_{u \in U_{wa}^1(F)\backslash U_{wa}^1(A)} \theta_4(u)\Theta_{\psi^{-1}}^{\Phi}(u)du = \omega_{\psi^{-1}}(\widetilde{w'a'})\Phi(0).$

Proof. Since $U^1_{wa} := j(w'a')^{-1}\{u(1_n, B, 0)\}\, j(w'a')$, we have that $\theta_4(u) = 1$ for $u \in U^1_{wa}$. Using Poisson summation, we get that

$$\begin{aligned}\Theta^{\Phi}_{\psi^{-1}}(u) &= \sum_{X\in F^n} \omega_{\psi^{-1}}(u)\Phi(X)\\ &= \sum_{X\in F^n} \omega_{\psi^{-1}}(\widetilde{w'a'})\omega_{\psi^{-1}}(u)\Phi(X).\end{aligned}$$

So that the integral in the Lemma is given by

$$\int\limits_{u\in U^1_{wa}(F)\backslash U^1_{wa}(A)} \sum_{X\in F^n} \omega_{\psi^{-1}}(\widetilde{w'a'})\omega_{\psi^{-1}}(u)\Phi(X)\, du.$$

We write $u = j(w'a')^{-1}u(1_n, B, 0)j(w'a')$ with $B \in \mathcal{S}_n$. The above integral is

$$\int\limits_{\mathcal{S}_n(F)\backslash\mathcal{S}_n(A)} \sum_{X\in F^n} \omega_{\psi^{-1}}(\widetilde{w'a'})\omega_{\psi^{-1}}(j(w'a')^{-1}u(1_n, B, 0)j(w'a'))\Phi(X)\, dB.$$

By equation (4.35), this is

$$\int\limits_{\mathcal{S}_n(F)\backslash\mathcal{S}_n(A)} \sum_{X\in F^n} \omega_{\psi^{-1}}(u(1_n, B, 0))\omega_{\psi^{-1}}(\widetilde{w'a'})\Phi(X)\, dB.$$

By equation (4.34), this is

$$\int\limits_{\mathcal{S}_n(F)\backslash\mathcal{S}_n(A)} \sum_{X\in F^n} \omega_{\psi^{-1}}(\widetilde{w'a'})\Phi(X)\psi^{-1}(2\langle B, X\rangle)\, dB.$$

where $\langle B, X\rangle$ denotes the inner product of the last row of B with X. The integral over B is zero unless $X = 0$. In this case we obtain

$$\int\limits_{\mathcal{S}_n(F)\backslash\mathcal{S}_n(A)} \omega_{\psi^{-1}}(\widetilde{w'a'})\Phi(0)\, dB = \omega_{\psi^{-1}}(\widetilde{w'a'})\Phi(0)$$

as desired. $\square$

Note that $U_E^2 = u(n, 0, T)$ is isomorphic to N_2 through the embedding $i_2 : N_2 \to Sp_{2n}$ given by

$$i_2 \begin{pmatrix} n & nT \\ & n^* \end{pmatrix} = u(n, 0, T). \tag{5.16}$$

Let

$$U_{wa}^2 = j(w'a')^{-1} U_E^2 j(w'a') \cap N_3. \tag{5.17}$$

From Lemma 5.5, we have that $N_{3,wa} = j(w'a')^{-1} N_{3,E} j(w'a') \cap N_3$, so that $U_{wa}^2 \cong U_{wa}^1 \backslash N_{3,wa}$. Recall that $N'_{2,w'a'} := (w'a')N_2(w'a')^{-1} \cap N_2$.

Lemma 5.7. *We have that* $U_{wa}^2 = j(w'a')^{-1} i_2(N'_{2,w'a'}) j(w'a')$.

Proof. Since $U_{wa}^2 = j(w'a')^{-1} U_E^2 j(w'a') \cap N_3$, we have that the group $j(w'a')U_{wa}^2 j(w'a')^{-1}$ consists of $i_2(n), n \in N_2$ with $j(w'a')^{-1} i_2(n) j(w'a') \in N_3$. Explicitly, we have that

$$\begin{pmatrix} 1 & & \\ & (w'a')^{-1} & \\ & & 1 \end{pmatrix} \begin{pmatrix} n & & & nT\tau \\ & n & nT & \\ & & n^* & \\ & & & n^* \end{pmatrix} \begin{pmatrix} 1 & & \\ & (w'a') & \\ & & 1 \end{pmatrix} = \begin{pmatrix} n & & nT\tau \\ & (w'a')^{-1} \begin{pmatrix} n & nT \\ & n^* \end{pmatrix} (w'a') & \\ & & n^* \end{pmatrix} \tag{5.18}$$

is an element in N_3. This is equivalent to $(w'a')^{-1} n_2 (w'a') \in N_2$ or $n_2 \in N'_{2,w'a'}$. $\square$

Proof of Proposition 5.4. From Lemma (5.6) we get that

$$\int\limits_{n' \in N_{3,wa}(F) \backslash N_{3,wa}(A)} \theta_4(n') \Theta_{\psi^{-1}}^{\Phi}(n') dn' = \int\limits_{n' \in U_{wa}^2(F) \backslash U_{wa}^2(A)} \theta_4(n') \omega_{\psi^{-1}}(\widetilde{w'a'}\, n') \Phi(0) dn' \tag{5.19}$$

Since $U_{wa}^2 = j(w'a')^{-1} i_2(N'_{2,w'a'}) j(w'a')$ by Lemma (5.7), our integral becomes

$$\int\limits_{n' \in N'_{2,w'a'}(F) \backslash N'_{2,w'a'}(A)} \theta_4(j(w'a')^{-1} i_2(n') j(w'a')) \omega_{\psi^{-1}}(i_2(n')) \omega_{\psi^{-1}}(\widetilde{w'a'}) \Phi(0) dn', \tag{5.20}$$

here we used equations (4.34) and (4.36).

Write $n' \in N_2$ as $n' = \begin{pmatrix} n'' & n''T \\ & n''^{*} \end{pmatrix}$, from the formulas (2.1), (2.2), (4.34) and (4.36), we have

$$\omega_{\psi^{-1}}(i_2(n')) \omega_{\psi^{-1}}(\widetilde{w'a'}) \Phi(0) = \omega_{\psi^{-1}}(\widetilde{w'a'}) \Phi(0) \psi^{-1}(T_{n,1}.\tau) \tag{5.21}$$

where $T_{n,1}$ stands for the lower left entry of T.

From equation (4.38) we obtain

$$\theta_4(j(w'a')^{-1} i_2(n') j(w'a')) = \psi^{-1}\left(\sum_{i=1}^{n-1} n''_{i,i+1}\right) \theta_2((w'a')^{-1} n'(w'a')) \tag{5.22}$$

and

$$\theta_{2,\tau}(n') = \psi\left(\sum_{i=1}^{n-1} n''_{i,i+1} + T_{n,1}.\tau\right). \tag{5.23}$$

Thus our integral (5.20) becomes

$$\int\limits_{n' \in N'_{2,w'a'}(F) \backslash N'_{2,w'a'}(A)} \theta_2((w'a')^{-1} n'(w'a')) \theta_{2,\tau}^{-1}(n') \omega_{\psi^{-1}}(\widetilde{w'a'}) \Phi(0) dn'. \tag{5.24}$$

We recall we defined

$$c(w'a') = \int\limits_{n' \in N'_{2,w'a'}(F) \backslash N'_{2,w'a'}(A)} \theta_2((w'a')^{-1} n'(w'a')) \theta_{2,\tau}^{-1}(n') dn',$$

so our integral is

$$c(w'a')\omega_{\psi^{-1}}(\widetilde{w'a'})\Phi(0) \tag{5.25}$$

as desired. $\square$

We recall from equations (5.6) and (5.7) that our distribution $I_{Sp_{2n}}(f : Sp_{(n,K)}, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$ is equal to

(5.26)
$$\sum_{w'a' \in Sp_n(F), wa = P(w'a')} \int_{n_3 \in N_{3,wa}(A)\backslash N_3(A)} F(n^{-1}wan) \int_{n' \in N_{3,wa}(F)\backslash N_{3,wa}(A)} \theta_4(n'n_3)\Theta^{\Phi}_{\psi^{-1}}(n'n_3)dn'dn_3.$$

Using proposition 5.4 this is equal to

$$\sum_{w'a' \in Sp_n(F), wa = P(w'a')} c(w'a') \int_{n_3 \in N_{3,wa}(A)\backslash N_3(A)} F(n^{-1}wan)\theta_4(n)\omega_{\psi^{-1}}(\widetilde{w'a'}.n)\Phi(0)dn \tag{5.27}$$

which is

(5.28)
$$\sum_{w'a' \in Sp_n(F), wa = P(w'a')} c(w'a') \prod_v \int_{n_3 \in N_{3,wa}(F_v)\backslash N_3(F_v)} F(n^{-1}wan)\theta_4(n)\omega_{\psi^{-1}}(\widetilde{w'a'}.n)\Phi(0)dn.$$

We have proved the following proposition:

Proposition 5.8. *When $f = \otimes f_v$, $\Phi = \otimes \Phi_v$, $F = \otimes F_v$, we have that the distribution $I_{Sp_{2n}}(f : Sp_{(n,K)}, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$ is equal to*

$$\sum_{w'a' \in Sp_n(F)} c(w'a') \prod_v I_{w'a'}(F_v, \Phi_v) \tag{5.29}$$

where

$$I_{w'a'}(F_v, \Phi_v) = \int_{n \in N_{3,\mathrm{P}(w'a')}(F_v)\backslash N_3(F_v)} F_v(n^{-1}\,\mathrm{P}(w'a')n)\theta_4(n)\omega_{\psi^{-1}}(\widetilde{w'a'} \cdot n)\Phi_v(0)\,dn \tag{5.30}$$

We have a similar decomposition for $I_{\widetilde{Sp_n}}(\widetilde{f} : N_2, \theta_{2,\tau}^{-1}; N_2, \theta_2)$:

Proposition 5.9. *When $f = \otimes f_v$, we have*

$$I_{\widetilde{Sp_n}}(\widetilde{f} : N_2, \theta_{2,\tau}^{-1}; N_2, \theta_2) = \sum_{w'a' \in Sp_n(F)} c(w'a') \prod_v J_{w'a'}(\widetilde{f_v}) \tag{5.31}$$

where

$$J_{w'a'}(\widetilde{f_v}) = \int\limits_{n_2 \in N_{2,w'a'}(F_v) \backslash N_2(F_v)} \int\limits_{n_1 \in N_2(F_v)} \widetilde{f_v}(\widetilde{n_1}^{-1}.\widetilde{w'a'}.\widetilde{n_2}) \theta_{2,\tau}(n_1^{-1}) \theta_2(n_2) dn_1 dn_2 \tag{5.32}$$

Proof. Using the Bruhat decomposition we have $I_{\widetilde{Sp_n}}(\widetilde{f} : N_2, \theta_{2,\tau}^{-1}; N_2, \theta_2)$ equals

$$\int\limits_{N_2(F) \backslash N_2(A)} \int\limits_{N_2(F) \backslash N_2(A)} \sum_{\gamma \in N_2(F) \backslash Sp_n(F) / N_2(F)} \widetilde{f}(\widetilde{n_1}^{-1} \widetilde{\gamma} \widetilde{n_2}) \theta_{2,\tau}^{-1}(n_1) \theta_2(n_2) dn_1 dn_2 \tag{5.33}$$

$$= \sum_{w'a' \in Sp_n(F)} \int\limits_{n_2 \in N_{2,w'a'}(F) \backslash N_2(A)} \int\limits_{n_1 \in N_2(A)} \widetilde{f}(\widetilde{n_1}^{-1} \widetilde{w'a'} \widetilde{n_2}) \theta_{2,\tau}^{-1}(n_1) \theta_2(n_2) dn_1 dn_2 \tag{5.34}$$

where $N_{2,w'a'} = (w'a')^{-1} N_2 (w'a') \cap N_2$.

We may write this integral as

$$\int\limits_{n_2 \in N_{2,w'a'}(A) \backslash N_2(A)} \int\limits_{n' \in N_{2,w'a'}(F) \backslash N_{2,w'a'}(A)} \int\limits_{n_1 \in N_2(A)} \widetilde{f}(\widetilde{n_1}^{-1} \widetilde{w'a'} \widetilde{n'} \widetilde{n_2}) \theta_{2,\tau}^{-1}(n_1) \theta_2(n' n_2) \tag{5.35}$$

Since for $n' \in N_{2,w'a'}$ we have that $(w'a') n' (w'a')^{-1} \in N_2$, we can make a change of variable $n_1 \mapsto (w'a') n' (w'a')^{-1} n_1$. Using the fact that when $n \in N_2$ we have $\widetilde{n} \cdot \widetilde{g} = \widetilde{ng}$ and $\widetilde{g} \cdot \widetilde{n} = \widetilde{gn}$, we get:

$$\widetilde{(w'a'n'(w'a')^{-1}n_1)}^{-1} \widetilde{w'a'} \widetilde{n'} = \widetilde{n_1}^{-1} \widetilde{w'a'} \widetilde{n'}^{-1} \widetilde{w'a'}^{-1} \widetilde{w'a'} \widetilde{n'} = \widetilde{n_1}^{-1} \widetilde{w'a'}. \tag{5.36}$$

Our integral becomes

(5.37)
$$\int\limits_{n_2\in N_{2,w'a'}(A)\backslash N_2(A)}\ \int\limits_{n'\in N_{2,w'a'}(F)\backslash N_{2,w'a'}(A)}\ \int\limits_{n_1\in N_2(A)} \widetilde{f}({\widetilde{n_1}}^{-1}\widetilde{w'a'}\widetilde{n_2})\theta_{2,\tau}^{-1}(w'a'n'(w'a')^{-1}\cdot n_1)\theta_2(n'n_2)dn_1dn_2dn'.$$

Clearly $n' \in N_{2,w'a'}$ iff $(w'a')n'(w'a')^{-1} \in N'_{2,w'a'}$, so our integral equals

$$\begin{aligned}(5.38)\qquad &\int\limits_{n_2\in N_{2,w'a'}(A)\backslash N_2(A)}\ \int\limits_{n_1\in N_2(A)} \widetilde{f}({\widetilde{n_1}}^{-1}\cdot\widetilde{w'a'}\widetilde{n_2})\theta_{2,\tau}^{-1}(n_1)\theta_2(n_2)dn_1\,dn_2\\ &\qquad\times\int\limits_{n\in N'_{2,w'a'}(F)\backslash N'_{2,w'a'}(A)} \theta_{2,\tau}^{-1}(n)\theta_2((w'a')^{-1}nw'a')dn_1dn_2dn\\ &= c(w'a')\int\limits_{n_2\in N_{2,w'a'}(A)\backslash N_2(A)}\ \int\limits_{n_1\in N_2(A)} \widetilde{f}({\widetilde{n_1}}^{-1}\cdot\widetilde{w'a'}\widetilde{n_2})\theta_{2,\tau}^{-1}(n_1)\theta_2(n_2)dn\end{aligned}$$

Factoring this integral over places v completes the proof. □

6. Comparison of orbital integrals

To compare the distributions $I_{\widetilde{Sp_n}}(\widetilde{f}: N_2, \theta_{2,\tau}^{-1}; N_2, \theta_2)$ and $I_{Sp_{2n}}(f: Sp_{n,K}, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$, we only need to compare the local orbital integrals $I_{w'a'}(F_v, \Phi_v)$ and $J_{w'a'}(\widetilde{f_v})$. In this section, we fix a place v and omit it in our notation.

Recall that $U^n = \left\{ \eta(\mathbf{x}, \mathbf{y}, t) := \begin{pmatrix} 1_{n-1} & & & & & \\ & 1 & \mathbf{x} & \mathbf{y} & t & \\ & & 1_n & 0 & * & \\ & & & 1_n & * & \\ & & & & 1 & \\ & & & & & 1_{n-1} \end{pmatrix} \right\}$ and

$$\widehat{N}^k := \left\{ \begin{pmatrix} z & * & * \\ & 1_{4n-2k+2} & * \\ & & z^* \end{pmatrix} \in N_3 | z \in Z_{k-1} \right\}.$$

$\hat{N}^{n+1} = U^n \hat{N}^n$ is a normal subgroup of N_3 with $N_3/\hat{N}^{n+1} \cong N_2$. Note also $N_{3,E} \cap \hat{N}^{n+1}$ is just the group $U_E^1 = u(1_n, B, 0)$. Recall that $Y \cong Sp_{n,K} \backslash Sp_{2n}$. Given a function $F \in \mathcal{S}(Y(F))$, and $\Phi \in \mathcal{S}(F^n)$, we define a genuine function on $\widetilde{\mathrm{Sp}}_n(F)$:

$$\Psi_{F,\Phi}(\widetilde{g}) = \int\limits_{u \in U_E^1 \backslash \hat{N}^{n+1}} F(j(g)^{-1}u^{-1}Euj(g))\theta_4(u)\omega_{\psi^{-1}}(u)\omega_{\psi^{-1}}(\widetilde{g})\Phi(0)\, du. \tag{6.1}$$

6.1. **Comparison of $I_{w'\mathbf{a}'}(F, \Phi)$ and $J_{w'\mathbf{a}'}(\tilde{f})$.** For a compatible choice of measures, we have:

Lemma 6.1.

$$I_{w'a'}(F, \Phi) = \int\limits_{n \in N_{2,w'a'} \backslash N_2} \Psi_{F,\Phi}(\widetilde{w'a'} \cdot \widetilde{n})\theta_2(n)\, dn. \tag{6.2}$$

Proof. For $g \in Sp_n$, we let $U^1_{\mathrm{P}(g)} = j(g)^{-1} U^1_E j(g) \subset N_3$. Then from Lemma (5.5) we have that $N_{3,\mathrm{P}(w'a')} \cap \hat{N}^{n+1} = U^1_{\mathrm{P}(w'a')}$. From (5.30), we get that $I_{w'a'}(F, \Phi)$ equals

$$\int_{n \in N_{3,\mathrm{P}(w'a')}\hat{N}^{n+1} \backslash N_3} \int_{u \in U^1_{\mathrm{P}(w'a')} \backslash \hat{N}^{n+1}} F(n^{-1}u^{-1}\,\mathrm{P}(w'a')un)$$

$$\theta_4(un)\omega_{\psi^{-1}}(\widetilde{w'a'})\omega_{\psi^{-1}}(un)\Phi(0)\,du\,dn.$$

As $N_3 = j(N_2)\hat{N}^{n+1}$ and $j(N_2) \cap \hat{N}^{n+1}$ consists just of identity element, we can write an element in N_3 in a unique way as $j(n)u$ such that $n \in N_2$ and $u \in \hat{N}^{n+1}$.

Since $N_{3,\mathrm{P}(w'a')} = U^1_{\mathrm{P}(w'a')} U^2_{\mathrm{P}(w'a')}$, with $U^1_{\mathrm{P}(w'a')} \subset \hat{N}^{n+1}$, we have

$$N_{3,\mathrm{P}(w'a')}\hat{N}^{n+1} = U^2_{\mathrm{P}(w'a')}\hat{N}^{n+1}.$$

As $U^2_{\mathrm{P}(w'a')} = j(w'a')^{-1} i_2(N'_{2,w'a'}) j(w'a')$ from Lemma 5.7,

$$U^2_{\mathrm{P}(w'a')}\hat{N}^{n+1} = j((w'a')^{-1}N'_{2,w'a'}(w'a'))\hat{N}^{n+1} = j(N_{2,w'a'})\hat{N}^{n+1}.$$

Thus we can choose the representatives in $N_{3,\mathrm{P}(w'a')}\hat{N}^{n+1} \backslash N_3$ as $j(n)$ with $n \in N_{2,w'a'} \backslash N_2$. The above integral equals:

$$\int_{n \in N_{2,w'a'} \backslash N_2} \int_{u \in U^1_{\mathrm{P}(w'a')} \backslash \hat{N}^{n+1}} F(j(n)^{-1}u^{-1}\,\mathrm{P}(w'a')uj(n))$$

$$\theta_4(uj(n))\omega_{\psi^{-1}}(\widetilde{w'a'})\omega_{\psi^{-1}}(uj(n))\Phi(0)\,du\,dn.$$

As $j(w'a')$ stabilizes $\hat{N}^{n+1}$ through conjugation, we can make a change of variable $u \mapsto j(w'a')^{-1}uj(w'a')$. Notice that $\theta_4(j(w'a')^{-1}uj(w'a')) = \theta_4(u)$, and from (4.35), (4.36) and

(5.15), the above integral is the same as:

$$\int_{n\in N_{2,w'a'}\backslash N_2}\int_{u\in U_E^1\backslash \hat{N}^{n+1}} F(j(w'a'n)^{-1}u^{-1}Euj(w'a'n))$$

$$\theta_4(uj(n))\omega_{\psi^{-1}}(u)\omega_{\psi^{-1}}(\widetilde{w'\mathbf{a}'n})\Phi(0)\,du\,dn.$$

From the definition of $\Psi_{F,\Phi}$ in equation (6.1), the above integral equals:

$$\int_{n\in N_{2,w'a'}\backslash N_2} \Psi_{F,\Phi}(\widetilde{w'a'n})\theta_4(j(n))dn.$$

Since $\theta_4(j(n))=\theta_2(n)$ for $n\in N_2$, we get the Lemma. $\square$

We have the following

Corollary 6.2. *If $\tilde{f}\in\mathcal{S}(\widetilde{\mathrm{Sp}}_n(F))$, $f\in\mathcal{S}(\mathrm{Sp}_{2n}(F))$ and $\Phi\in\mathcal{S}(F^n)$ satisfying (for F defined by (5.4))*

$$\Psi_{F,\Phi}(\widetilde{g})=\int_{N_2}\tilde{f}(\widetilde{n}^{-1}\cdot\widetilde{g})\theta_{2,\tau}^{-1}(n)dn, \tag{6.3}$$

then $I_{w'a'}(F,\Phi)=J_{w'a'}(\tilde{f})$.

Proof. From Lemma 6.1, we have

$$I_{w'a'}(F,\Phi)=\int_{n\in N_{2,w'a'}\backslash N_2} \Psi_{F,\Phi}(\widetilde{w'a'}\cdot\widetilde{n})\theta_2(n)\,dn.$$

By our assumption on $\Psi_{F,\Phi}$, this is

$$\int_{n_2\in N_{2,w'a'}\backslash N_2}\int_{n_1\in N_2}\tilde{f}(\widetilde{n_1}^{-1}\cdot\widetilde{w'a'}\widetilde{n_2})\theta_{2,\tau}^{-1}(n_1)\theta_2(n_2)\,dn_1\,dn_2,$$

which is $J_{w'a'}(\widetilde{f})$. $\square$

6.2. **Properties of $\Psi_{F,\Phi}$ and matching.** The function $\Psi_{F,\Phi}(\widetilde{g})$ defined by (6.1) has the following equivariance property:

Lemma 6.3. *The function $\Psi_{F,\Phi}(\widetilde{g})$ satisfies for all $n \in N_2$:*

$$\Psi_{F,\Phi}(\widetilde{n} \cdot \widetilde{g}) = \theta_{2,\tau}^{-1}(n)\Psi_{F,\Phi}(\widetilde{g}). \tag{6.4}$$

Proof. We only need to establish the identity in the case g is identity, which we now assume. Recall the definition of $u(n,B,T) \in N_{3,E}$ in (5.14):

$$u(n,B,T) := \left\{ \begin{pmatrix} n & & nB & nT\tau \\ & n & nT & nB \\ & & n^* & \\ & & & n^* \end{pmatrix} \in N_3 | n \in Z_n, B, T \in \mathcal{S}_n \right\}.$$

From (6.1) $\Psi_{F,\Phi}(\widetilde{n})$ equals:

$$\int\limits_{u \in U_E^1 \backslash \hat{N}^{n+1}} F(j(n)^{-1}u^{-1}Euj(n))\theta_4(u)\omega_{\psi^{-1}}(uj(n))\Phi(0)\, du. \tag{6.5}$$

Write n as $\left(\begin{smallmatrix} n' & n'T \\ & (n')^* \end{smallmatrix}\right)$.

Case (1): when n' is identity. We observe $u(1_n, 0, T) \in N_{3,E}$ and $j(n)u(1_n,0,T)^{-1} \in \hat{N}^{n+1}$. Since $j(N_2)$ and $u(1_n,0,T)$ fix the group $U_E^1 = u(1_n, B, 0)$ by conjugation, we can make a change of variable $u \mapsto u(1_n,0,T)uj(n)^{-1}$ in $\hat{N}^{n+1}$. Notice that $u(1_n,0,T)$ fixes E through conjugation; the above integral becomes:

$$\int\limits_{u \in U_E^1 \backslash \hat{N}^{n+1}} F(u^{-1}Eu)\theta_4(u(1_n,0,T)uj(n)^{-1})\omega_{\psi^{-1}}(u(1_n,0,T)u)\Phi(0)\, du.$$

Clearly $\theta_4(u(1_n,0,T)uj(n)^{-1}) = \theta_4(u)$. From (4.34) and (4.36) we have

$$\omega_{\psi^{-1}}(u(1_n,0,T)u)\Phi(0) = \psi^{-1}(T_{n,1} \cdot \tau)\omega_{\psi^{-1}}(u)\Phi(0)$$

where $T_{n,1}$ is the lower left entry of T. The above integral becomes:

$$\psi^{-1}(T_{n,1}\cdot\tau)\int\limits_{u\in U_E^1\backslash\hat{N}^{n+1}} F(u^{-1}Eu)\theta_4(u)\omega_{\psi^{-1}}(u)\Phi(0)\,du$$

which is $\theta_{2,\tau}^{-1}(n)\Psi_{F,\Phi}(\widetilde{1_{2n}})$ in the case n' is identity.

Case (2): when $T'=0$. Now $u(n',0,0)\in N_{3,E}$ and $u(n',0,0)j(n)^{-1}\in\hat{N}^{n+1}$. Since $u(n',0,0)$ and $j(n)$ fixes the group U_E^1 by conjugation, we can change u to $u(n',0,0)uj(n)^{-1}$. Using that $u(n',0,0)^{-1}Eu(n',0,0)=E$, the integration (6.5) becomes:

$$\int\limits_{u\in U_E^1\backslash\hat{N}^{n+1}} F(u^{-1}Eu)\theta_4(u(n',0,0)uj(n)^{-1})\omega_{\psi^{-1}}(u(n',0,0)u)\Phi(0)\,du.$$

Clearly $\theta_4(u(n',0,0)uj(n)^{-1})=\theta_4(u)\theta_2^{-1}(n)$ (in our case $T=0$). From (2.1) and (4.36) we get

$$\omega_{\psi^{-1}}(u(n',0,0)u)\Phi(0)=\omega_{\psi^{-1}}(u)\Phi(0).$$

Thus the above integral is just

$$\theta_2^{-1}(n)\int\limits_{u\in U_E^1\backslash\hat{N}^{n+1}} F(u^{-1}Eu)\theta_4(u)\omega_{\psi^{-1}}(u)\Phi(0)\,du$$

which is $\theta_{2,\tau}(n)^{-1}\Psi_{F,\Phi}(\widetilde{1_{2n}})$. Here we used that for $n=\begin{pmatrix} n' & \\ & (n')^* \end{pmatrix}$ we have that $\theta_{2,\tau}(n)=\theta_2(n)$.

From the above two cases, identity (6.4) holds for any $n\in N_2$ when g is identity, thus holds in general. $\square$

We also need to consider the behavior of the function $\Psi_{F,\Phi}(\widetilde{g})$ when $g=\mathrm{diag}[\mathbf{a},\mathbf{a}^*]$ where $\mathbf{a}=\mathrm{diag}[a_1,\ldots,a_n]$ is a diagonal matrix. The proof of the following Lemma is similar to that of [MR, Lemma 5.4] and we omit it.

Lemma 6.4. *When $F \in \mathcal{S}(Y)$ and $\Phi \in \mathcal{S}(F^n)$, as function of* $\mathbf{a}$, $\Psi_{F,\Phi}(\widetilde{\mathrm{diag}[\mathbf{a},\mathbf{a}^*]})$ *is a Schwartz function on* $(F^\times)^n$.

The following is [MR2, Lemma 5.6].

Lemma 6.5. *Let T be a Schwartz function on $N_2\backslash\widetilde{Sp_n}$ with $T(\tilde{n}g) = \theta(n)T(g)$. Then there exists a Schwartz function $\widetilde{f}$ on $\widetilde{Sp_n}$ with*

$$\int\limits_{N_2} \widetilde{f}(\tilde{n}^{-1}g)\theta(n)\,dn = T(g).$$

Moreover, the function $\tilde{f}$ may be defined by

$$\widetilde{f}(nak) = \lambda(n)T(ak),$$

where λ is any Schwartz function on N_2 with

$$\int\limits_{N_2} \lambda(n^{-1})\theta(n)\,dn = 1.$$

Corollary 6.6. *Given any $f \in \mathcal{S}(\mathrm{Sp}_{2n})$ and $\Phi \in \mathcal{S}(F^n)$, there is $\tilde{f} \in \mathcal{S}(\widetilde{\mathrm{Sp}_n})$ such that equation (6.3) holds.*

Proof. From Lemmas 6.4 and 6.3, $\Psi_{F,\Phi}$ is Schwartz function on $N_2\backslash\widetilde{Sp_n}$ satisfying the equivariance property

$$\Psi_{F,\Phi}(\widetilde{n}\cdot\widetilde{g}) = \theta_{2,\tau}(n)^{-1}\Psi_{F,\Phi}(\widetilde{g}).$$

By Lemma 6.5, we obtain the desired $\tilde{f}$ satisfying

$$\Psi_{F,\Phi}(\widetilde{g}) = \int\limits_{N_2} \tilde{f}(\widetilde{n}^{-1}\cdot\widetilde{g})\theta_{2,\tau}^{-1}(n)dn.$$

$\square$

From Corollary (6.2), we get:

Corollary 6.7. *There is a map* ϵ_5 *from* $\mathcal{S}(\mathrm{Sp}_{2n}) \otimes \mathcal{S}(F^n)$ *to* $\mathcal{S}(\widetilde{\mathrm{Sp}}_n)$ *such that when* $\tilde{f} = \epsilon_5(F \otimes \Phi)$, $I_{w'\mathbf{a}'}(F, \Phi) = J_{w'\mathbf{a}'}(\tilde{f})$.

7. Fundamental Lemma

7.1. **Statement of the result.** Let v be a nonarchimedean place with odd residue characteristic, and where ψ is unramified. We will omit v in the notations.

Let $\mathcal{O}$ be the ring of integers in F. Recall $K_3 = Sp_{2n}(\mathcal{O})$ and K_2 is the image of an embedding of $Sp_n(\mathcal{O})$ in $\widetilde{Sp_n}$, (the covering splits over $Sp_n(\mathcal{O})$). Let $\mathcal{H}(Sp_{2n}, K_3)$ (and $\mathcal{H}(\widetilde{Sp_n}, K_2)$) be the algebra of Hecke functions on Sp_{2n} (and $\widetilde{Sp_n}$ respectively). For $\tilde{f} \in \mathcal{H}(\widetilde{Sp_n}, K_2)$, define

$$\widehat{\tilde{f}}(z) = \int\limits_{a \in T_n} \int\limits_{n \in N_2} \tilde{f}(\widetilde{i_1(a)} \cdot \widetilde{n}) \gamma(\det a, \psi^{-1})^{-1} \chi_z(a) \delta_2^{\frac{1}{2}}(i_1(a))\, dn\, da$$

where δ_2 is the modulus functions of the Borel subgroup of Sp_n and χ_z is the unramified character defined on the subgroup of diagonal matrices in GL_n by

$$\chi_z \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = |a_1|_v^{z_1} \dots |a_n|_v^{z_n}.$$

We can define a homomorphism $f \mapsto \tilde{f}$ between $\mathcal{H}(Sp_{2n}, K_3)$ and $\mathcal{H}(\widetilde{Sp_n}, K_2)$ so that:

$$\widehat{f}(z_1 - \frac{1}{2}, z_1 + \frac{1}{2}, \dots, z_n - \frac{1}{2}, z_n + \frac{1}{2}) = \widehat{\tilde{f}}(z_1, \dots, z_n). \tag{7.1}$$

We prove

Proposition 7.1. *If $f \in \mathcal{H}(Sp_{2n}, K_3)$ and $\tilde{f} \in \mathcal{H}(\widetilde{Sp_n}, K_2)$ are related by (7.1), then when Φ_0 is the characteristic function of $\mathcal{O}^n$, we have*

$$\Psi_{F,\Phi}(\widetilde{g}) = \int\limits_{N_2} \tilde{f}(n^{-1} \cdot \widetilde{g}) \theta_{2,\tau}^{-1}(n)\, dn, \tag{7.2}$$

where F is defined by (5.4).

From Corollary 6.2, we get:

Corollary 7.2. *When $f \in \mathcal{H}(Sp_{2n}, K_3)$ and $\tilde{f} \in \mathcal{H}(\widetilde{Sp_n}, K_2)$ are such that (7.1) holds, then when Φ_0 is the characteristic function of $\mathcal{O}^n$, we have $I_{w'a'}(F, \Phi_0) = J_{w'a'}(\tilde{f})$ where F is defined by (5.4).*

This identity of orbital integrals is the fundamental lemma for the case at hand. The rest of the section gives the proof of Proposition 7.1.

7.2. **Unit element case.** We prove the Proposition first in the case when both f and $\tilde{f}$ are unit elements. In this case we denote the functions by f_0 and $\tilde{f}_0$ respectively. Then f_0 is the characteristic functions of K_3, while $\tilde{f}_0$ takes value 1 over K_2, and vanishes outside the inverse image of $\mathrm{Sp}_n(\mathcal{O})$ in $\widetilde{Sp_n}$. Let F_0 be the function associated to f_0 by (5.4), namely

$$F_0(g^{-1}Tg) = \int\limits_{l \in Sp_{n,K}} f_0(l^{-1}g)dl.$$

Lemma 7.3. *The function F_0 is the characteristic function of $Y \cap K_3$.*

Proof. Clearly F_0 is a K_3-invariant function on Y. As $lg \in K_3$ for $l \in Sp_{n,K}$ implies $g^{-1}Tg \in K_3$, we get F_0 vanishes outside $Y \cap K_3$. By Lemma (A.1) below, we see $Y \cap K_3$ is a single K_3-orbit of T. Hence we get F_0 is constant on $Y \cap K_3$. Putting $g = 1_{4n}$ in the definition of F_0 shows that F_0 is the characteristic function of $Y \cap K_3$. □

Denote the right hand side of (7.2) by $\Psi_{\tilde{f}}(\widetilde{g})$, i.e.

$$\Psi_{\tilde{f}}(\widetilde{g}) = \int\limits_{N_2} \tilde{f}(n^{-1} \cdot \widetilde{g})\theta_{2,\tau}^{-1}(n)\, dn.$$

Then

$$\Psi_{\tilde{f}_0}(\widetilde{n}\cdot\widetilde{g}) = \theta_{2,\tau}^{-1}(n)\Psi_{\tilde{f}_0}(\widetilde{g}),\ n\in N_2.$$

Lemma 6.3 shows that Ψ_{F_0,Φ_0} satisfies the same left N_2-equivariance condition. Both functions $\Psi_{\tilde{f}_0}$ and Ψ_{F_0,Φ_0} are clearly right K_2-invariant. Thus from the Iwasawa decomposition to show the identity (7.2), we only need to show it holds when $\widetilde{g}=\widetilde{a}$ where

$$a = \mathrm{diag}[a_1,\dots,a_n,a_n^{-1},\dots,a_1^{-1}]$$

is a diagonal matrix.

It is easy to see that $\Psi_{\tilde{f}_0}(\widetilde{a}) = 1$ when $|a_i| = 1$ for $i = 1,\dots,n$, and $\Psi_{\tilde{f}_0}(\widetilde{a}) = 0$ otherwise. Thus the Proposition in this case follows from

Lemma 7.4. *When $|a_i| = 1$ for $i = 1,\dots,n$, $\Psi_{F_0,\Phi_0}(\widetilde{a}) = 1$. Otherwise $\Psi_{F_0,\Phi_0}(\widetilde{a}) = 0$.*

Proof. Recall that

$$\Psi_{F_0,\Phi_0}(\widetilde{g}) = \int_{u\in U_E^1\backslash\hat{N}^{n+1}} F_0(j(g)^{-1}u^{-1}Euj(g))\theta_4(u)\omega_{\psi^{-1}}(u)\omega_{\psi^{-1}}(\widetilde{g})\Phi_0(0)\,du.$$

For $u\in\hat{N}^{n+1}$, it has the form $\begin{pmatrix} n_1 & \begin{pmatrix} * & * \\ & * \end{pmatrix} \\ & n_1^* \end{pmatrix}$ with n_1 having the form $\left(\begin{smallmatrix} n_2 & v \\ & 1_n \end{smallmatrix}\right)$, where $n_2\in Z_n$ the maximal unipotent subgroup of GL_n. The matrix $j(a)^{-1}u^{-1}Euj(a)$ lies in the Siegel parabolic subgroup; it has the form $\left(\begin{smallmatrix} A & * \\ & A^* \end{smallmatrix}\right)$ where

$$A = \begin{pmatrix} 1_n & \\ & b^{-1} \end{pmatrix}\begin{pmatrix} n_2^{-1} & v' \\ & 1_n \end{pmatrix}\begin{pmatrix} & \tau_n \\ 1_n & \end{pmatrix}\begin{pmatrix} n_2 & v \\ & 1_n \end{pmatrix}\begin{pmatrix} 1_n & \\ & b \end{pmatrix}$$

with $b = \mathrm{diag}[a_1, \ldots, a_n]$, and $v' = -n_2^{-1}v$. A computation shows:

$$A = \begin{pmatrix} v'n_2 & \tau_n n_2^{-1}b + v'vb \\ b^{-1}n_2 & b^{-1}vb \end{pmatrix}.$$

If $|a_i| < 1$ for some i, we get from looking at the lower left block of A that

$$F_0(j(a)^{-1}u^{-1}Euj(a)) = 0.$$

Thus in this case $\Psi_{F_0,\Phi_0}(\widetilde{a}) = 0$.

On the other hand, since

$$\Psi_{F_0,\Phi_0}(\widetilde{n} \cdot \widetilde{g} \cdot \bar{k}) = \theta_{2,\tau}^{-1}(n)\Psi_{F_0,\Phi_0}(\widetilde{g})$$

when $n \in N_2$ and $\bar{k} \in K_2$, we get $\Psi_{F_0,\Phi_0}(\widetilde{a}) = 0$ whenever $|a_i| > |a_{i+1}|$ for some i or $|a_n| > |a_n|^{-1}$.

Therefore, $\Psi_{F_0,\Phi_0}(\widetilde{a})$ is nonzero only when $|a_n|^{-1} \geq |a_n| \geq \ldots \geq |a_1|$ and $|a_1| \geq 1$. This condition is only satisfied when $|a_i| = 1$ for all i, in which case using the K_3-invariance of F_0 and K_2-invariance of Φ_0, we get

$$\Psi_{F_0,\Phi_0}(\widetilde{a}) = \Psi_{F_0,\Phi_0}(\widetilde{1_{2n}}).$$

Consider now the case $a = 1_{2n}$. Use the computation of the matrix A again. From the lower left block and lower right block of A, we see in this case if $u^{-1}Eu \in K_3 \cap Y$, n_2 is in $\mathrm{GL}_n(\mathcal{O})$ and v has integral entries. Thus $u = \begin{pmatrix} n_1 & v_1 \\ & n_1^* \end{pmatrix}$ with $n_1 \in \mathrm{GL}_{2n}(\mathcal{O})$.

Write u as $i_1(n_1)v$ with v in the Siegel unipotent subgroup of Sp_{2n}. Since $n_1 \in \mathrm{GL}_{2n}(\mathcal{O})$, we get $v^{-1}Ev \in K_3 \cap Y$. Write $v = \begin{pmatrix} 1_{2n} & \mathbf{v} \\ & 1_{2n} \end{pmatrix}$, and $\mathbf{v} = \begin{pmatrix} B_1 & T \\ & B_2 \end{pmatrix}$, then the condition is equivalent to all entries in T and $B_1 - B_2$ are integers. Since $U_E^1 = u(1, B, 0)$ consists of v of the above form with $T = 0$ and $B_1 = B_2$, we see over the subdomain where

$F_0(u^{-1}Eu) \neq 0$, the representatives of $U_E^1 \backslash \hat{N}^{n+1}$ can be chosen to be in K_3. Thus:

$$\Psi_{F_0,\Phi_0}(\widetilde{1_{2n}}) = \int_{u \in U_E^1(\mathcal{O}) \backslash \hat{N}^{n+1}(\mathcal{O})} \theta_4(u)\omega_{\psi^{-1}}(u)\Phi(0)\, du.$$

Over the domain $\theta_4(u) = 1$ and $\omega_{\psi^{-1}}(u)\Phi(0) = \Phi(0) = 1$. Thus we get $\Psi_{F_0,\Phi_0}(\widetilde{1_{2n}}) = 1$. We get the claim in the Lemma. □

7.3. **General Hecke element case.** The result we need follows from a Plancherel formula and the fact that the orbital integrals of spherical functions on $Sp_{n,K} \backslash Sp_{2n}$ are related to spherical functions on $N_2 \backslash \widetilde{Sp_n}$. The argument is essentially that of Mao and Rallis in [MR1], where the split case is treated. Here we carry out the main argument and relegate some details to the Appendix.

Let F be a $p-$adic field with p odd. Let $\mathcal{O}$ be the ring of integers in F. Let π be a prime in $\mathcal{O}$ and let $q = |\pi|^{-1}$. Let G be a reductive group over F with maximal compact subgroup K. Let H be a closed unimodular subgroup of G with the property that there exists a Borel $B \subset G$ with BH open in G. Let χ be a unitary character on H, trivial on $H \cap K$. Denote by $C_K^\infty(H \backslash G, \chi)$ the space of complex functions on G with $f(hgk) = \chi(h)f(g)$. Denote by $\mathcal{S}_K(H \backslash G, \chi)$ the subspace consisting of functions of compact support modulo H in $C_K^\infty(H \backslash G, \chi)$.

We consider $G_1 = Sp_{2n}$, $H_1 = Sp_{n,K}$, $K_1 = Sp_{2n}(O)$ and χ_1 the trivial character, $G_2 = \widetilde{Sp_n}$, $H_2 =$ maximal unipotent of G_2 and $K_2 = Sp_n(\mathcal{O})$. We take the character χ_2 to be $\theta_{2,\tau}^{-1}$.

Denote by $\mathcal{H}(G, K)$ the Hecke algebra of G with respect to K. It consists of compactly supported functions on G satisfying $f(k_1gk_2) = f(g)$ for all $g \in G$, $k_1, k_2 \in K$. The multiplication is given by the convolution product.

The Hecke algebra $\mathcal{H}(G,K)$ acts on $C_K^\infty(H\backslash G,\chi)$ by

$$(f*\phi)(g)=\int_G f(h)\phi(gh)dh, \qquad f\in\mathcal{H}(G,K), \phi\in C_K^\infty(H\backslash G,\chi). \tag{7.3}$$

A spherical function in $C_K^\infty(H\backslash G,\chi)$ is an eigenfunction $\Psi(g)$ in $C_K^\infty(H\backslash G,\chi)$ under the action of $\mathcal{H}(G,K)$ normalized so that $\Psi(1)=1$.

Call an element $g\in G$ relevant if χ is trivial on $gKg^{-1}\cap H$. The subset of relevant elements in G is denoted G^{rel}. We have the following characterization of relevant elements:

Lemma 7.5. *Let*

$$\Lambda_n^+=\{\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)\in\mathbf{Z}^n|\lambda_1\ge\lambda_2\ge\ldots\ge\lambda_n\ge 0\}. \tag{7.4}$$

There are injective maps $\Pi_1:\Lambda_n^+\to Sp_{2n}$, $\Pi_2:\Lambda_n^+\to\widetilde{Sp_n}$ such that $\widetilde{Sp_n}^{rel}=\cup_{\lambda\in\Lambda_n^+}N_2\Pi_2(\lambda)K_2$ and $Sp_{2n}^{rel}=\cup_{\lambda\in\Lambda_n^+}Sp_{n,K}\Pi_2(\lambda)K_3$ as a disjoint union. Moreover, $\Pi_1(0,\ldots,0)$ and $\Pi_2(0,\ldots,0)$ are elements in K_3 and K_2 respectively.

Proof. In the second case, this follows if we let Π_2 be given by $\Pi_2(\lambda)=(\tilde{\Pi}^\lambda,1)$ where

$$\tilde{\Pi}^\lambda=\mathrm{diag}[\pi^{\lambda_1},\ldots,\pi^{\lambda_n},\pi^{-\lambda_n},\ldots,\pi^{-\lambda_1}].$$

In the first case, it follows from Lemma A.1 below. $\square$

We recall an explicit linear map from $\mathcal{S}_{K_3}(Sp_{n,K}\backslash Sp_{2n},1)$ to $\mathcal{S}_{K_2}(N_2\backslash\widetilde{Sp_n},\theta_{2,\tau}^{-1})$.

Recall $E = \begin{pmatrix} & \tau_n & & \\ 1_n & & & \\ & & & \tau_n \\ & & 1_n & \end{pmatrix}$ and ϵ_0 is chosen so that $E = \epsilon_0^{-1}\mathbf{T}\epsilon_0$. The subgroup $\hat{N}^{n+1}$ of N_3 is given by $\hat{N}^{n+1} = \left\{ \begin{pmatrix} u & * & * \\ & 1_{2n} & * \\ & & u^* \end{pmatrix} | u \in Z_n \right\}$.

Recall the Weil representation $\omega_{\psi^{-1}}$ of $\widetilde{Sp}_n$ acts on the space $\mathcal{S}(F^n)$ of Schwartz functions on F^n. For $\Phi \in \mathcal{S}(F^n)$, in equation (6.1) we defined, for $g \in Sp_n$ and $\zeta \in \{\pm 1\}$:

$$T_\Phi(F)(g,\zeta) = \int_{u \in U_E^1 \backslash \hat{N}^{n+1}} F(j(g)^{-1}u^{-1}Euj(g))\theta_4(u)\omega_{\psi^{-1}}(u)\omega_{\psi^{-1}}(\widetilde{g})\Phi(0)\,du. \tag{7.5}$$

Then T_Φ is a linear map from $\mathcal{S}_{K_3}(Sp_{n,K}\backslash Sp_{2n})$ to the set of genuine functions on $\widetilde{Sp}_n$.

For $\lambda \in \Lambda_n^+$, define ch_λ to be a function in $\mathcal{S}_K(H\backslash G, \chi)$ such that $ch_\lambda(\Pi(\lambda')) = 0$ unless $\lambda = \lambda'$, in which case $ch_\lambda(\Pi(\lambda')) = 1$. We will use $ch_{\mathbf{0}}^1$ to denote the function on $Sp_{n,K}\backslash Sp_{2n}$ corresponding to the function $ch_{\mathbf{0}}$ on X (through the identification in section 3.1), and let $ch_{\mathbf{0}}^2$ be the function $ch_{\mathbf{0}}$ on $\widetilde{Sp_n}$. Let Φ_0 be the characteristic function of $\mathcal{O}^n$.

We remark that for $f_1 \in \mathcal{H}(Sp_{2n}, K_3)$, we have that $f_1 * ch_{\mathbf{0}}^1 = F$ where F is related to f_1 by equation (5.4). Similarly, for $f_2 \in \mathcal{H}(\widetilde{Sp_n}, K_2)$, we have $f_2 * ch_{\mathbf{0}}^2$ is equal to the right hand side of (7.2). Thus to prove Proposition 7.1 it is enough to prove:

Proposition 7.6. *For $f_1, f_2 \in \mathcal{H}(Sp_{2n}, K_3), \mathcal{H}(\widetilde{Sp_n}, K_2)$ respectively, such that $\tilde{f}_1(z) \equiv \tilde{f}_2(z)$, we have $T_{\Phi_0}(f_1 * ch_{\mathbf{0}}^1) = f_2 * ch_{\mathbf{0}}^2$.*

7.4. **Preliminary results.** Let S be the reduced root system of type C_n, let R be the root system of type BC_n. The root systems R and S are inside the same vector space identified with $\mathbf{C}^n$. Let ϵ_i, $i = 1, \dots, n$ be the standard basis of $\mathbf{C}^n$, then

$$S = \{\pm\epsilon_i \pm \epsilon_j, \pm\epsilon_i, 1 \le i \le n, i < j \le n.\};$$

$$R = \{\pm\epsilon_i \pm \epsilon_j, \pm\epsilon_i, \pm 2\epsilon_i, 1 \le i \le n, i < j \le n.\}.$$

The root systems R and S have the same Weyl group W which is the Weyl group of Sp_n. There is a natural action of W on $\mathbf{C}^n$.

A Macdonald polynomial has the form ([Mc1], equation (10.1))

$$Q_\lambda^t(z) = P_\lambda^t(e^{\epsilon_i}) = V_\lambda(t)^{-1} \sum_{w \in W} w \left(e^\lambda \prod_{\alpha \in R^+} \frac{1 - t_\alpha t_{2\alpha}^{\frac{1}{2}} e^{-\alpha}}{1 - t_{2\alpha}^{\frac{1}{2}} e^{-\alpha}} \right). \tag{7.6}$$

Here $\lambda \in \Lambda_n^+$ is identified with dominant weights of R, R^+ is the set of positive roots, and e^{ϵ_i} are the independent variables of the polynomial P_λ^t; Q_λ^t and P_λ^t are related through the equation $e^{\epsilon_i} = q^{-z_i}$. The data t_α are parameters such that when α is not a root in R, $t_\alpha = t_\alpha^{\frac{1}{2}} = 1$. Thus the parameters t are determined by values of t_α when α is a long root in S, and t_α with $t_{2\alpha}^{\frac{1}{2}}$ when α is a short root in S. $V_\lambda(t)$ are nonzero constants independent of variables e^{ϵ_i}; they are defined in [Mc1] (denoted $W_\lambda(t)$ there).

Let $\mathbf{C}[q^z, q^{-z}]^W$ be the space of functions on $(z_1, \dots, z_n) \in \mathbf{C}^n$ that are polynomials in q^{z_i} and q^{-z_i} and that are invariant under the action of W. We have that $\{Q_\lambda^t(z) | \lambda \in \Lambda_n^+\}$ forms a basis of $\mathbf{C}[q^z, q^{-z}]^W$.

Theorems A.2 and B.1 below give:

Theorem 7.7. *For $i = 1, 2$, there are choices of real numbers parameters t_α^i for α roots in R and nonzero values $a_i(\lambda)$ for $\lambda \in \Lambda_n^+$, such that for all $z = (z_1, \dots, z_n) \in \mathbf{C}^n$,*

$$\Psi_z^i(\Pi_i(\lambda)) = a_i(\lambda) Q_\lambda^{t^i}(z), \qquad \lambda \in \Lambda_n^+, \tag{7.7}$$

determines a spherical function Ψ_z^1 in $C_{K_3}^\infty(Sp_{n,K} \backslash Sp_{2n}, 1)$, Ψ_z^2 in $C_{K_2}^\infty(N_2 \backslash \widetilde{Sp_n}, \theta_{2,\tau}^{-1})$.

For $f \in \mathcal{H}(G,K)$, recall that $\tilde{f}(z)$ is the eigenvalue of $f^*(g) = f(g^{-1})$ acting on $\overline{\Psi_z}$ through equation (7.3), that is:

$$\tilde{f}(z) = \int f(g^{-1})\overline{\Psi_z(g)}dg. \tag{7.8}$$

Then $f \mapsto \tilde{f}(z)$ is an algebra homomorphism from $f \in \mathcal{H}(G,K)$ to $\mathbf{C}[q^{\bar{z}}, q^{-\bar{z}}]^W$. We can consider $\mathbf{C}[q^{\bar{z}}, q^{-\bar{z}}]^W$ as a $\mathcal{H}(G,K)$ module through the multiplication by $\tilde{f}(z)$.

When $\phi \in \mathcal{S}_K(H\backslash G, \chi)$, we define its Fourier transform $\hat{\phi}(z)$ by

$$\hat{\phi}(z) = \langle \phi, \Psi_z \rangle = \int_{H\backslash G} \phi(g)\overline{\Psi_z(g)}dg \tag{7.9}$$

Clearly $\hat{\phi}(z) \in \mathbf{C}[q^{\bar{z}}, q^{-\bar{z}}]^W$. Recall $\mathcal{S}_K(H\backslash G, \chi)$ is a $\mathcal{H}(G,K)$ module through (7.3).

Theorem 7.8. *The map $\phi \mapsto \hat{\phi}$ gives an isomorphism of $\mathcal{H}(G,K)-$modules between $\mathcal{S}_K(H\backslash G, \chi)$ and $\mathbf{C}[q^{\bar{z}}, q^{-\bar{z}}]^W$.*

Proof. A basis of $S_K(H\backslash G, \chi)$ is given by functions ch_λ, $\lambda \in \Lambda_n^+$ where $ch_\lambda(\Pi(\lambda'))$ is given by $\delta_\lambda^{\lambda'}$ where δ is the Kronecker delta. On the other hand, the set of $Q_\lambda^t(z)$ for $\lambda \in \Lambda_n^+$ gives a basis for $\mathbf{C}[q^z, q^{-z}]^W$.

We have that

$$\widehat{ch_\lambda}(z) = \int\limits_{H\backslash G} ch_\lambda(g)\overline{\Psi_z(g)}dg$$

and this is, up to a volume factor, equal to $\overline{a(\lambda)Q_\lambda^t(z)}$. Thus the map $\phi \mapsto \widehat{\phi}$ establishes a bijection between bases of $S_K(H\backslash G, \chi)$ and of $\mathbf{C}[q^{\bar{z}}, q^{-\bar{z}}]^W$.

We now show it is a $\mathcal{S}_K(H\backslash G, \chi)-$ homomorphism. Take $f \in \mathcal{H}(G,K)$, $\phi \in \mathcal{S}_K(H\backslash G, \chi)$, then

$$\widehat{f * \phi}(z) = \int_G \int_{H\backslash G} f(g)\phi(hg)\overline{\Psi_z(h)}dhdg.$$

Changing $h \mapsto hg^{-1}$, we get

$$\begin{aligned}
\widehat{f * \phi}(z) &= \int_G \int_{H\backslash G} f(g)\phi(h)\overline{\Psi_z(hg^{-1})}dhdg \\
&= \int_{H\backslash G} \int_G f(g^{-1})\phi(h)\overline{\Psi_z(hg)}dhdg \\
&= \tilde{f}(z)\hat{\phi}(z)
\end{aligned}$$

as desired. $\square$

Theorem 7.9. *Let* $\Delta^t(z) = \prod_{\alpha \in R} \frac{1-t_{2\alpha}^{\frac{1}{2}} e^{\alpha}}{1-t_{2\alpha}^{\frac{1}{2}} e^{\alpha}}$. *Denote by* D_n *the direct product of* n *copies of* $\sqrt{-1}\mathbf{R}/((2\pi/\log q)\mathbf{Z})$. *For* $\phi \in \mathcal{S}_K(H\backslash G, \chi)$, *we have*

$$\phi(g) = \int_{D_n} \hat{\phi}(z)\Psi_z(g)d_\mu(z) \tag{7.10}$$

where the measure $d_\mu(z) = \frac{1}{|W|}V_0(t)\Delta^t(z)dz$.

Proof. In the case of $\widetilde{Sp_n}$, this holds with $d_\mu^2(z) = \frac{1}{|W|}\Delta^{t^{II}}(z)$. It follows from the fact that the volume of $N_2\backslash N_2(\tilde{\Pi}^\lambda, 1)K_2$ equals $\delta^{-1}(\tilde{\Pi}^\lambda)$ using the argument in [Mc2].

In the case of Sp_{2n}, it is a consequence (c.f. [MR1], Section 2.3.) of Lemma 7.5, Theorem 7.7, Corollary A.4 and the fact that

$$f * \Phi = f^* * \Phi$$

for all $\Phi \in C^\infty_{K_3}(Sp_{n,K}\backslash Sp_{2n}, 1)$ and $f \in \mathcal{H}(Sp_{2n}, K_3)$, where $f^*(g) = f(g^{-1})$.

This last property holds since $f \in \mathcal{H}(Sp_{2n}, K_3)$ implies, by Cartan decomposition, that $f(g) = f(g^{-1})$. $\square$

Suppose T is a map from $\mathcal{S}_{K_1}(H_1\backslash G_1,\chi_1)$ to $\mathcal{S}_{K_2}(H_2\backslash G_2,\chi_2)$ satisfying $T(\Psi_z^1)=c(z)\Psi_z^2$. Then from Theorem 7.9, formally we have:

$$T(\phi_1)=\int\limits_{D_n}\hat{\phi}_1(z)\Psi_z^2c(z)d_\mu^1(z),\forall\phi_1\in\mathcal{S}_{K_1}(H_1\backslash G_1,\chi_1). \tag{7.11}$$

Lemma 7.10. *Let T be a linear map satisfying*

(1) equation (7.11) holds for some function $c(z)$,

(2) $T(S_1^{-1}(1))=S_2^{-1}(1)$,

then $S_1=S_2T$ on $\mathcal{S}_{K_1}(H_1\backslash G_1,\chi_1)$.

Proof. Let $\phi_i\in\mathcal{S}_{K_i}(H_i\backslash G_i,\chi_i)$ such that $S_1(\phi_1)=S_2(\phi_2)$, we show $\phi_2=T(\phi_1)$.

By Theorem 7.9,

$$\phi_2=\int\limits_{D_n}\hat{\phi}_2(z)\Psi_z^2d_\mu^2(z).$$

Since $\hat{\phi}_1(z)=\hat{\phi}_2(z)$, we get there is a function $c'(z)$ on D_n with:

$$T(\phi_1)-\phi_2=\int\limits_{D_n}\hat{\phi}_2(z)\Psi_z^2c'(z)dz. \tag{7.12}$$

When $\hat{\phi}_2(z)\equiv 1$, condition (2) implies:

$$\int\limits_{D_n}\Psi_z^2(\Pi_2(\lambda))c'(z)dz\equiv 0,\ \forall\lambda\in\Lambda_n^+. \tag{7.13}$$

When $z\in D_n$, $\bar{z}=-z$; thus we can consider $\hat{\phi}_2(z)$ as a polynomial in $\mathbf{C}[q^z,q^{-z}]^W$ For any $\lambda\in\Lambda_n^+$ and any ϕ_2, $\hat{\phi}_2(z)\Psi_z^2(\Pi_2(\lambda))\in\mathbf{C}[q^z,q^{-z}]^W$, thus it is a linear combination $\sum c_i\Psi_z^2(\Pi_2(\lambda_i))$. From (7.12) and (7.13), we get $(T(\phi_1)-\phi_2)(\Pi_2(\lambda))=0$; thus $T(\phi_1)=\phi_2$. □

7.5. **Proof of Proposition 7.6.**

Lemma 7.11. *For $F \in \mathcal{S}_{K_3}(Sp_{n,K}\backslash Sp_{2n})$, and Φ_0 the characteristic function of $\mathcal{O}^n$,*

$$T_{\Phi_0}(F) = \int\limits_{D_n} \hat{F}(z)c(z)\Psi_z^2 d_\mu^1 z.$$

Proof. From Theorem 7.9,

$$T_{\Phi_0}(F) = T_{\Phi_0}\left(\int\limits_{D_n} \hat{F}(z)\Psi_z^1 d_\mu^1 z\right).$$

As T_{Φ_0} is an iterated integral over a fixed compact set, we can interchange the integral and operator T_{Φ_0} and use Proposition C.2 to get:

$$\begin{aligned} T_{\Phi_0}(F) &= \int\limits_{D_n} \hat{F}(z)T_{\Phi_0}(\Psi_z^1)d_\mu^1 z \\ &= \int\limits_{D_n} \hat{F}(z)c(z)\Psi_z^2 d_\mu^1 z. \end{aligned}$$

□

We have already proved that

Lemma 7.12. $T_{\Phi_0}(ch_{\mathbf{0}}^1) = ch_{\mathbf{0}}^2$.

Since clearly $S_1(ch_{\mathbf{0}}^1) = S_2(ch_{\mathbf{0}}^2) = 1$, we checked the two conditions in Lemma 7.10 are satisfied for the map T_{Φ_0}. From Theorem 7.8, for $f_1, f_2 \in \mathcal{H}(Sp_{2n}, K_3), \mathcal{H}(\widetilde{Sp_n}, K_2)$, $S_1(f_1 * ch_{\mathbf{0}}^1) = S_2(f_2 * ch_{\mathbf{0}}^2)$ whenever $\tilde{f}_1(z) \equiv \tilde{f}_2(z)$. Lemma 7.10 gives that $S_2T(f_1 * ch_{\mathbf{0}}^1) = S_2(f_2 * ch_{\mathbf{0}}^2)$ and using that S_2 is an isomorphism, we obtain Proposition 7.6.

8. Proof of Theorem 1.1

From the previous sections, we get the following trace identity:

Theorem 8.1. *There exists maps $\epsilon_{5,v}$ from $\mathcal{S}(Sp_{2n}(F_v))\otimes\mathcal{S}(F_v^n)$ to $\mathcal{S}(\widetilde{\mathrm{Sp}}_n(F_v))$, such that*

$$I_{Sp_{2n}}(f : Sp_{n,K}, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}}) = I_{\widetilde{Sp_n}}(\tilde{f} : N_2, \theta^{-1}_{2,\tau}; N_2, \theta_2) \tag{8.1}$$

when

(1) at $v \notin S$ where S is a finite set of places containing all bad places, f_v is a Hecke function and Φ_v is the characteristic function of the lattice $\mathcal{O}_v^n$, $\tilde{f}_v$ is the Hecke function associated to f_v by (7.1).

(2) at $v \in S$, $\tilde{f}_v = \epsilon_{5,v}(f_v \otimes \Phi_v)$.

Proof. : Given f_v, Φ_v, we find $\tilde{f}_v$ through Corollary 6.7. The identity follows from Propositions 5.8, 5.9, and Corollary 7.2. □

Proof of Theorem 1.1:

Given $f_v \in \mathcal{S}(\mathrm{GL}_{2n}(F_v))$, we find $f'_v \in \mathcal{S}(\mathrm{Sp}_{2n}(F_v))$ through Corollary 3.9, then we find a pair $f''_v \in \mathcal{S}(\mathrm{Sp}_{2n}(F_v))$ and $\Phi_v \in \mathcal{S}(F_v^n)$ through Corollary 4.9, then we find $\tilde{f}_v \in \mathcal{S}(\widetilde{\mathrm{Sp}}_n(F_v))$ through Theorem 8.1. This gives the map ϵ_v which is $\epsilon_{5,v}\epsilon_{4,v}\epsilon_{1,v}$.

At a good place v, the Hecke algebra homomorphism $\lambda_v : f_v \mapsto \tilde{f}_v$ from $\mathcal{H}(GL_{2n}, K_1)$ to $\mathcal{H}(\widetilde{Sp_n}, K_2)$ is defined so that

$$\widehat{f_v}(z_1, -z_1, z_2, -z_2, \ldots, z_n, -z_n) = \widehat{\tilde{f}_v}(z_1, \ldots, z_n).$$

Let S be a finite set of places containing archimedean places and even places and places where ψ is not unramified. Assume $f = \otimes f_v$ and $\tilde{f} = \otimes_{v\in S}\epsilon_v(f_v) \otimes_{v\notin S} \lambda_v(f_v)$. We need to show the equality

$$I_{GL_{2n}}(f : GL_{n,K}, 1; N_1, \theta_1) = I_{\widetilde{Sp_n}}(\widetilde{f} : N_2, \theta_{2,\tau}^{-1}; N_2, \theta_2). \tag{8.2}$$

For $v \notin S$, there is $f'_v \in \mathcal{H}(Sp_{2n}, K_3)$ such that equation (7.1) holds. From Theorem 8.1, equation (8.1) holds when we replace $f \otimes \Phi$ by

$$\otimes_{v \in S} \epsilon_{4,v} \epsilon_{1,v}(f_v) \otimes_{v \notin S} (f'_v \otimes \Phi_{0,v}).$$

From Corollary 4.9, we get $I_{\widetilde{Sp_n}}(\tilde{f} : N_2, \theta_{2,\tau}^{-1}; N_2, \theta_2)$ equals $I_{Sp_{2n}}(f' : Sp_{n,K}, 1; N_3, \theta_3)$ when $f' = \otimes_{v \in S} \epsilon_{1,v}(f_v) \otimes_{v \notin S} f'_v$.

From Corollary 3.9, we get $I_{\widetilde{Sp_n}}(\tilde{f} : N_2, \theta_{2,\tau}^{-1}; N_2, \theta_2)$ equals $I_{GL_{2n}}(f_1 : GL_{n,K}, 1; N_1, \theta_1)$ where

$$f_1 = \otimes_{v \in S} f_v \otimes_{v \notin S} f_{1,v}, \;\; f_{1,v} = \lambda_{1,v}(f'_v).$$

We have the following relationship between f_v and $f_{1,v}$:

Lemma 8.2. *For all $z \in \mathbf{C}^n$:*

$$\widehat{f_v}(z_1, -z_1, z_2, -z_2, \ldots, z_n, -z_n) = \widehat{f_{1,v}}(z_1, -z_1, z_2, -z_2, \ldots, z_n, -z_n). \tag{8.3}$$

Proof. The left hand side of the equation is $\widehat{\tilde{f}}(z_1, \ldots, z_n)$, which equals

$$\widehat{f'_v}(z_1 + \frac{1}{2}, z_1 - \frac{1}{2}, \ldots, z_n + \frac{1}{2}, z_n - \frac{1}{2}).$$

Using the invariance under the Weyl group of Sp_{2n}, the above equals:

$$\widehat{f'_v}(z_1 + \frac{1}{2}, -z_1 + \frac{1}{2}, \ldots, z_n + \frac{1}{2}, -z_n + \frac{1}{2}).$$

Using the relation (3.8) we get the above equals the right hand side of the equation. □

To complete the proof of identity (8.2), we only need to show

$$I_{GL_{2n}}(f_1 : GL_{n,K}, 1; N_1, \theta_1) = I_{GL_{2n}}(f : GL_{n,K}, 1; N_1, \theta_1).$$

From the orbital integral decomposition in equation (3.4), this will follow from

Lemma 8.3. *When equation (8.3) holds for all $z \in C^n$,*

$$\int_{GL_{n,K}(F_v)} f_v(hg)\, dh = \int_{GL_{n,K}(F_v)} f_{1,v}(hg) dh.$$

Proof. We work over a place v, which we omit it in the notation. Let $C^C_{K_1}(GL_{n,K}\backslash GL_{2n})$ be the space of right K_1 and left $GL_{n,K}$ invariant functions compactly supported on $GL_{2n}(F_v)$. Then the Hecke algebra $\mathcal{H}(GL_{2n}, K_1)$ acts on this space by:

$$f * \phi(g) = \int \phi(gh) f(h^{-1})\, dh,\ \phi \in C^C_{K_1}(GL_{n,K}\backslash GL_{2n}),\ f \in \mathcal{H}(GL_{2n}, K_1).$$

Let $f_{0,v}$ be the unit Hecke function on GL_{2n}, and define

$$\Xi_{0,v}(g) = \int\limits_{GL_{n,K}(F_v)} f_{0,v}(hg) dh.$$

Then $\Xi_{0,v} \in C^C_{K_1}(GL_{n,K}\backslash GL_{2n})$. It is clear the two sides of the equation in Lemma are $f_v * \Xi_{0,v}$ and $f_{1,v} * \Xi_{0,v}$.

By [O, Proposition 4.9], we have that $C^C_{K_1}(GL_{n,K}\backslash GL_{2n})$ is isomorphic to $\mathbf{C}[q^z, q^{-z}]^{W_1}$ as $\mathcal{H}(GL_{2n}, K_1)-$modules. In particular, the action of f is determined by the values of $\widehat{f}(z_1, -z_1, \ldots, z_n, -z_n)$ and thus we have $f_v * \phi = f_{1,v} * \phi$ for all $\phi \in C^C_{K_1}(GL_{n,K}\backslash GL_{2n})$, when f_v and $f_{1,v}$ satisfy (8.3).

□

This completes the proof of Theorem 1.1. □

Appendix A. Spherical functions on $Sp_{n,K}\backslash Sp_{2n}$

Recall that $\mathbf{T}$ is the square matrix of size $4n$ consisting of $\begin{pmatrix} & \tau \\ 1 & \end{pmatrix}$ on the diagonal and zero elsewhere; the group $Sp_{n,K}$ consists of g in Sp_{2n} with $g^{-1}\mathbf{T}g = \mathbf{T}$. Let X' be the space of antisymmetric matrices in Sp_{2n}. We have a map $Sp_{n,K}\backslash Sp_{2n} \to X'$ given by $g \mapsto g^{-1}\mathbf{T}gJ$. We denote the image of this map by $X \subset X'$. The group Sp_{2n} acts on X by $g \cdot x = gx^tg$.

Recall that

$$\Lambda_n^+ = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbf{Z}^n | \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0\}.$$

We denote by Π_a the matrix $\begin{pmatrix} & \pi^a \\ -\pi^a & \end{pmatrix}$; for $\lambda \in \Lambda_n^+$, we denote by $\Pi^\lambda \in X$ the matrix with the diagonal being $[\tau\Pi_{\lambda_1}, \tau\Pi_{\lambda_2}, \dots, \tau\Pi_{\lambda_n}, -\Pi_{-\lambda_n}, \dots, -\Pi_{-\lambda_1}]$.

Lemma A.1. *As a disjoint union,*

$$X = \cup K \cdot \Pi^\lambda, \qquad \lambda \in \Lambda_n^+. \tag{A.1}$$

Proof. Let $x \in X$. Let x_{i_0,j_0} be the entry with the largest norm in x. Since x is antisymmetric, we have that $i_0 \neq j_0$ and so through the action of a suitable Weyl element $w \in K_3$ and a diagonal matrix $a \in K_3$, we have that $x' = wa \cdot x$ has one of the following two properties: either $x'_{1,2} = -x'_{2,1} = \pi^{-\lambda_1}$ is the element with the largest norm or $x'_{1,4n}$ is the element with the largest norm in x'.

We now consider the second case, we assume that for elements in $K \cdot x$ the entry with the largest norm lies on the antidiagonal. In this case, there is an element y in the orbit $K \cdot x$ which is antidiagonal. But then y must satisfy $yJ^2 = \tau$, this is a contradiction.

In the first case, there exists a lower triangular matrix $n \in K_3$ with $x'' = n \cdot x'$ has the property that $x''_{1,i} = 0$ when $i \neq 2$ and $x''_{2,j} = 0$ when $j \neq 1$. Since $x \in X$ satisfies

$x^tJx = \tau J$, we have that $x''_{1,i} = x''_{2,j} = x''_{i,1} = x''_{j,2} = x''_{4n+1-j,4n-1} = x''_{4n+1-i,4n} = 0$ unless $i = 2$ or $j = 1$; furthermore, $x''_{4n,4n-1} = -x''_{4n-1,4n} = \pi^{\lambda_1}$. Thus x'' is a diagonal matrix with elements in the diagonal being Π_{λ_1}, y and $-\Pi_{\lambda_1}$ where y is an antisymmetric square matrix of size $4(n-1)$ with $(yJ)^2 = \tau$ and ${}^tyJy = \tau J$. Moreover, the maximum norm of the entries of y is q^{λ_2} with $\lambda_2 \leq \lambda_1$. Continuing this process, we see that $x \in K \cdot \Pi^{-\lambda}$ with $\lambda \in \Lambda_n^+$. Moreover, the sets $K \cdot \Pi^{-\lambda}$ are clearly disjoint for distinct $\lambda \in \Lambda_n^+$. $\square$

Let $C^\infty_{K_3}(X)$ be the space of all K_3-invariant complex functions on X and $\mathcal{S}_{K_3}(X)$ the subspace consisting of all compactly supported functions in $C^\infty_{K_3}(X)$. With the identification between $Sp_{n,K}\backslash Sp_{2n}$ and X, the action of $\mathcal{H}(Sp_{2n}, K_3)$ on $C^\infty_{K_3}(X)$ becomes the convolution product:

$$(f * \Phi)(x) = \int_{Sp_{2n}} f(g)\Phi(g^{-1} \cdot x)dg \;\; f \in \mathcal{H}(Sp_{2n}, K_3), \Phi \in C^\infty_{K_3}(X). \tag{A.2}$$

A spherical function on X is by definition an eigenfunction $\Psi(x)$ in $C^\infty_{K_3}(X)$ of all the convolutions defined by elements in $\mathcal{H}(Sp_{2n}, K_3)$ normalized such that $\Psi(\Pi^{\mathbf{0}}) = 1$. Here $\mathbf{0}$ is the vector in Λ_n^+ where all entries are 0. The spherical functions on X are clearly in one-one correspondence with spherical functions on $C^\infty_{K_3}(Sp_{n,K}\backslash Sp_{2n}, 1)$.

Recall by equation (10.1) of [Mc1] that a Macdonald polynomial has the form

$$Q^t_\lambda(z) = P^t_\lambda(e^{\epsilon_i}) = V_\lambda(t)^{-1} \sum_{w \in W} w \left(e^\lambda \prod_{\alpha \in R^+} \frac{1 - t_\alpha t_{2\alpha}^{\frac{1}{2}} e^{-\alpha}}{1 - t_{2\alpha}^{\frac{1}{2}} e^{-\alpha}} \right),$$

where the parameters t are determined by the values of t_α when α is a long root in S, and t_α with $t_{2\alpha}^{\frac{1}{2}}$ when α is a short root in S.

We will prove the following result:

Theorem A.2. *For $z = (z_1, \dots, z_n) \in C$, $\Psi_z(x)$ defined by the following equation is a spherical function on X:*

$$\Psi_z(\Pi^\lambda) = q^{b(\lambda)} \frac{V_\lambda(t^I)}{V_0(t^I)} Q_\lambda^{t^I}, \qquad \lambda \in \Lambda_n^+ \tag{A.3}$$

where $b(\lambda) = -\sum_{i=1}^{n}(2(n-i+1)-1/2)\lambda_i$. The parameter t_α^I in the definition of $Q_\lambda^{t^I}$ is given by: when α is a long root in S, $t_\alpha^I = q^{-2}$; when α is a short root in S, we set $t_\alpha^I = q^{-1}$, ${t_{2\alpha}^I}^{1/2} = q^{-1/2}$.

We will construct a function $\Psi_z(x)$ on X that is an eigenfunction under the Hecke algebra action, thus a spherical function, then we establish formula (A.3) for $\Psi_z(x)$.

For $x \in X$, denote by $\mathcal{P}f_i(x)$ $(1 \leq i \leq n)$ the Pfaffian of the lower right $2i \times 2i$ block of x. Define the integral

$$\zeta(x; s) = \zeta(x; s_1, \dots, s_n) = \int_{K_3} \prod_{i=1}^{n} |\mathcal{P}f_i(k \cdot x)|^{s_i} dk \tag{A.4}$$

where $x \in X$ and $s \in \mathbf{C}^n$. The integral is taken over the open subset

$$\{k \in K_3 | \prod_{i=1}^{n} |\mathcal{P}f_i(k \cdot x)| \neq 0\}.$$

Set

$$\Psi_z(x) = \zeta(x; s)/\zeta(\Pi^{\mathbf{0}}, s), \; x \in X, \tag{A.5}$$

where $z = (z_1, \dots, z_n)$ satisfies the relation:

$$\sum_{i=j}^{n} s_i = -z_j - 2(n-j) - \frac{3}{2}, \; j = 1, \dots, n. \tag{A.6}$$

A.0.1. *Hecke algebra action.* We first prove $\Psi_z(x)$ is an eigenfunction of the Hecke algebra $\mathcal{H}(Sp_{2n}, K_3)$.

Recall that A_3 is the set of diagonal matrices in Sp_{2n} and N_3 is the standard maximal unipotent subgroup of Sp_{2n}. Let $B = A_3N_3$ be the standard Borel subgroup. Given $\nu = (\nu_1, \ldots, \nu_{2n}) \in \mathbf{C}^{2n}$, let $\Phi_\nu(g)$ be the K_3-invariant vector in the induced representation $I(\chi_\nu) = \mathrm{Ind}_B^{Sp_{2n}} \chi_\nu$, where

$$\chi_\nu(au) = \prod_{i=1}^{2n} |a_i|^{\nu_i},\ a = diag[a_1, \ldots, a_{2n}, a_{2n}^{-1}, \ldots, a_1^{-1}], u \in N_3.$$

We normalize Φ_ν so that

$$\Phi_\nu(auk) = \prod_{i=1}^{2n} |a_i|^{\nu_i+(2n-i+1)}, k \in K_3. \tag{A.7}$$

The Satake transform $\hat{f}(\nu)$ of $f \in \mathcal{H}(Sp_{2n}, K_3)$ is defined by

$$\hat{f}(\nu) = \int_{Sp_{2n}} f(g)\Phi_\nu(g)dg.$$

By the Iwasawa decomposition, this agrees with the definition in equation (3.6).

We will let

$$\omega_f(z) = \hat{f}(-z_1 + \frac{1}{2}, -z_1 - \frac{1}{2}, \ldots, -z_n + \frac{1}{2}, -z_n - \frac{1}{2}). \tag{A.8}$$

The following proposition shows Ψ_z is a spherical function.

Proposition A.3. *When* $f \in \mathcal{H}(Sp_{2n}, K_3)$,

$$(f * \Psi_z)(x) = \omega_f(z)\Psi_z(x),\ x \in X. \tag{A.9}$$

Proof. Let $f \in \mathcal{H}(Sp_{2n}, K_3)$. Recall from equation (A.5) that

$$\zeta(\Pi^0; s)\Psi_z(x) = \zeta(x; s).$$

We compute

$$\begin{aligned}
\zeta(\Pi^{\mathbf{0}};s)(f*\Psi_z)(x) &= \zeta(\Pi^{\mathbf{0}};s)\int\limits_{Sp_{2n}} f(g)\Psi_z(g^{-1}\cdot x)dg \\
&= \int\limits_{Sp_{2n}} f(g)\zeta(g^{-1}\cdot x;s)dg \\
&= \int\limits_{Sp_{2n}} f(g)\left\{\int_{K_3}\prod_{i=1}^{n}|\mathcal{P}f_i(kg^{-1}\cdot x)|^{s_i}dk\right\}dg.
\end{aligned}$$

Changing variables on g and using that $f \in \mathcal{H}(Sp_{2n}, K_3)$, we get that this is

$$\int\limits_{Sp_{2n}} f(g)\prod_{i=1}^{n}|\mathcal{P}f_i(g^{-1}\cdot x)|^{s_i}\,dg.$$

By the Iwasawa decomposition, we write g as $g = k^{-1}b$ where $k \in K_3$, $b \in B$. We get that

$$\zeta(\Pi^{\mathbf{0}};s)(f*\Psi_z)(x) = \int\limits_{K_3}\int\limits_{B} f(k^{-1}b)\prod_{i=1}^{n}|\mathcal{P}f_i(b^{-1}k\cdot x)|^{s_i}\,dk\,db,$$

here d_*b is a right invariant measure on B.

Note that for $b \in B$,

$$|\mathcal{P}f_i(b\cdot x)|^{s_i} = |d_i(b)|^{s_i}|\mathcal{P}f_i(x)|^{s_i}, \tag{A.10}$$

where $d_i(b)$ is the determinant of the lower right $2i \times 2i$ block of b.

Thus

$$\zeta(\Pi^{\mathbf{0}};s)(f*\Psi_z)(x) = \int\limits_{K_3}\prod_{i=1}^{n}|\mathcal{P}f_i(k\cdot x)|^{s_i}dk\int\limits_{B} f(b)\prod_{i=1}^{n}|d_i(b^{-1})|^{s_i}d_*b.$$

The relation of z and s gives that

$$\omega_f(z) = \hat{f}(-z_1 + \frac{1}{2}, -z_1 - \frac{1}{2}, \ldots, -z_n + \frac{1}{2}, -z_n - \frac{1}{2}) = \int\limits_B f(b) \prod_{i=1}^{n} |d_i(b^{-1})|^{s_i} d_*b.$$

Therefore we obtain

$$\zeta(\Pi^{\mathbf{0}}; s)(f * \Psi_z(x)) = \zeta(x; s)\omega_f(z).$$

This equation implies Proposition A.3. □

Corollary A.4. *The map from $\mathcal{H}(Sp_{2n}, K_3)$ to $\mathbf{C}[q^{\bar{z}}, q^{-\bar{z}}]^{W_3}$ given by $f \mapsto \omega_f(z)$ is onto.*

A.0.2. *Another definition of $\Psi_z(x)$.* Given $x \in X$, define

$$F_z^x(g) = \prod_{i=1}^{n} |\mathcal{P}f_i(g \cdot x)|^{s_i}, \; g \in Sp_{2n}. \tag{A.11}$$

where z and s are related by

$$\sum_{i=j}^{n} s_i = -z_j - 2(n-j) - \frac{3}{2}, \; j = 1, \ldots, n.$$

Then $\zeta(x; s) = \int\limits_{K_3} F_z^x(k)dk$. By equation (A.10) we have

$$F_z^x(bg) = \prod_{i=1}^{n} |d_i(b)|^{s_i} F_z^x(g). \tag{A.12}$$

Therefore F_z^x defines a distribution on the space of $I(\chi_{-\nu(z)})$, where

$$\nu(z) = (z_1 - \frac{1}{2}, z_1 + \frac{1}{2}, \ldots, z_n - \frac{1}{2}, z_n + \frac{1}{2}).$$

The distribution is given by

$$F_z^x(\Phi) = \int\limits_{B\backslash Sp_{2n}} F_z^x(g)\Phi(g)dg. \tag{A.13}$$

for $\Phi \in I(\chi_{-\nu(z)})$.

When $x = \mathbf{T}J$, since $h \cdot \mathbf{T}J = \mathbf{T}J$ for $h \in Sp_{n,K}$, we get $F_z^{\mathbf{T}J}(gh) = F_z^{\mathbf{T}J}(g)$ for $h \in Sp_{n,K} \subset Sp_{2n}$. Therefore if $L(\Phi) = F_z^{\mathbf{T}J}(\Phi)$, then L is a $Sp_{n,K}-$invariant linear form on $I(\chi_{-\nu(z)})$.

Proposition A.5. *Let L be the linear form above, and $\Phi_{-\nu(z)}$ be the vector in $I(\chi_{-\nu(z)})$ defined by (A.7). Denote the action of Sp_{2n} on $I(\chi_{-\nu(z)})$ by ρ. Then when $x = g \cdot \mathbf{T}J$, $g \in Sp_{2n}$,*

$$\zeta(x; s) = L(\rho(g^{-1})\Phi_{-\nu(z)}). \tag{A.14}$$

Proof. The right hand side of the equation is:

$$\int_{B\backslash Sp_{2n}} F_z^{\mathbf{T}J}(h)\rho(g^{-1})\Phi_{-\nu(z)}(h)dh = \int_{B\backslash Sp_{2n}} \prod_{i=1}^{n} |\mathcal{P}f_i(h \cdot \mathbf{T}J)|^{s_i}\Phi_{-\nu(z)}(hg^{-1})dh.$$

Making a change of variable $h \mapsto hg$, this becomes

$$\int_{B\backslash Sp_{2n}} \prod_{i=1}^{n} |\mathcal{P}f_i(h \cdot x)|^{s_i}\Phi_{-\nu(z)}(h)dh.$$

By the Iwasawa decomposition, this is

$$\int_{K_3} \prod_{i=1}^{n} |\mathcal{P}f_i(k \cdot x)|^{s_i}\Phi_{-\nu(z)}(k)dk.$$

Since $\Phi_{-\nu(z)}(k) = 1$, we get the equation above. $\square$

A.1. **Computation of $\Psi_z(\Pi^\lambda)$.** The calculation of $\zeta(x; s)$, as in [MR], follows Casselman's method ([C]). We assume $\nu(z)$ is such that the numbers $\{z_i \pm \frac{1}{2} | i = 1, \dots n\}$ are all distinct. The analytic continuation would give the formula for all cases of z.

A.1.1. *Expansion in the basis* $\{f_w^z\}$. Let B_0 be the Iwahori subgroup of Sp_{2n}, define

$$\xi_z^x(g) = \int\limits_{B_0} F_z^x(gb)db, \tag{A.15}$$

here the measure is normalized so B_0 has volume 1. We have that $\xi_z^x(g)$ is right B_0-invariant. From (A.12), we see that $\xi_z^x(g)$ is a B_0 fixed vector in $I(\chi_{\nu(z)})$, where

$$\nu(z) = (z_1 - \frac{1}{2}, z_1 + \frac{1}{2}, \dots, z_n - \frac{1}{2}, z_n + \frac{1}{2}).$$

In [C], Casselman defined a basis $\{f_w^z | w \in W_3\}$ of the space of B_0 fixed vectors in $I(\chi_{\nu(z)})$. We have that there exist functions $\{a_w(x;z) | w \in W_3\}$, so that

$$\xi_z^x(g) = \sum_{w \in W_3} a_w(x;z) f_w^z(g). \tag{A.16}$$

Here by definition:

$$a_w(x;z) = T_w^z(\xi_z^x)(1) \tag{A.17}$$

where T_w^z is the intertwining operator of w from the space $I(\chi_{\nu(z)})$ to $I(w\chi_{\nu(z)})$, defined by analytic continuation of the following integration:

$$T_w^z(\varphi)(g) = \int_{N_3 \cap w N_3 w^{-1} \backslash N_3} \varphi(w^{-1}ug)du.$$

As $\zeta(x;s) = \int\limits_{K_3} F_z^x(k)dk = \int\limits_{K_3} \xi_z^x(k)dk$, we get

$$\zeta(x;s) = \sum_{w \in W_3} a_w(x;z) \int\limits_{K_3} f_w^z(k)dk. \tag{A.18}$$

The integral $\int\limits_{K_3} f_w^z(k)dk$ is computed in [C]. Recall that the root system of Sp_n is given by S, we will denote by S_{2n} the root system of Sp_{2n}. Let S_{2n}^+ and S_{2n}^- be the set of positive and negative roots respectively. We define e^α for a given $z' = (z_1', z_2', \dots, z_{2n}')$ by $e^{\epsilon_i} = q^{-z_i}$.

Let

$$c_w(z') = \prod_{\alpha\in S_{2n}^+, w\alpha\in S_{2n}^-} \frac{1-q^{-1}e^{\alpha}}{1-e^{\alpha}}.$$

Let w_l be the longest Weyl element in W_3. Then we have

$$\int_{K_3} f_w^z(k)dk = Q^{-1}c_{w_l}(w_l w\nu(z))/c_w(\nu(z)). \tag{A.19}$$

Here Q is some constant independent of z defined in [C].

A.1.2. *Vanishing of* $a_w(x;z)$. We now turn attention to the computation of $a_w(x;z)$. As in [Hi], define an intertwining operator $\tilde{T}_w^z$ from $I(\chi_{-\nu(z)})^*$ to $I(w\chi_{-\nu(z)})^*$, where $I(\chi)^*$ is the dual of $I(\chi)$, (see Proposition 1.6 in [Hi]). By equation (A.13), we have that $F_z^x \in I(\chi_{-\nu(z)})^*$. Then as in [Hi], $\tilde{T}_w^z$ extends the intertwining operator T_w^z.

Similar to Proposition 1.7 in [Hi], we have

$$a_w(x;z) = T_w^z(\xi_z^x)(1) = \int_{B_0} \tilde{T}_w^z(F_z^x)(b)db. \tag{A.20}$$

From now on we assume $x = \Pi^{\lambda}$ for some $\lambda \in \Lambda_n^+$. We show then $a_w(x;z) = 0$ for most $w \in W_3$. Define $Y \subset Sp_{2n}$:

$$Y = \{g \in Sp_{2n} | F_z^{\mathbf{T}J}(g) \neq 0\}. \tag{A.21}$$

Clearly Y is an open subset of Sp_{2n}.

Lemma A.6. *Let $b \in B$, then $g \in Sp_{2n}$ is in Y if and only if $bg \in Y$.*

Proof. This follows from equation (A.12). ☐

Lemma A.7. *When $g \in Sp_{2n}$ is such that $g \cdot \mathbf{T}J = x = \Pi^{\lambda}$ for some $\lambda \in \Lambda_n^+$, $F_z^{\mathbf{T}J}(bg) = F_z^x(b) = F_z^x(1)$ when $b \in B_0$.*

Proof. It is clear from (A.12) that when $b \in B_0 \cap B$,

$$F_z^x(b) = \prod_{i=1}^{n} |d_i(b)|^{s_i} F_z^x(1) = F_z^x(1).$$

We now assume $b \in B_0 \cap \bar{B}$ where $\bar{B}$ is the set of lower triangular matrices in Sp_{2n}. We show in this case $|\mathcal{P}f_i(b \cdot x)| = |\mathcal{P}f_i(x)|$ for $i = 1, \ldots, n$.

Write x as $\begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix}$ where A_2 is the lower right $2i \times 2i$ block of x. Write b as $\begin{pmatrix} b_1 & \\ S & b_2 \end{pmatrix}$, where b_2 is the lower right $2i \times 2i$ block of b. Thus b_2 is a lower-triangular matrix, in the Iwahori subgroup of GL_{2i}. With these notations, we see

$$|\mathcal{P}f_i(b \cdot x)| = |\mathcal{P}f(bx\,{}^t b)| = |\mathcal{P}f(b_2 A_2\,{}^t b_2 + S A_1\,{}^t S)|.$$

Since the entries of S are in the prime ideal $\mathcal{P} \subset \mathcal{O}$, with our choice of x, we see

$$b_2 A_2\,{}^t b_2 + S A_1\,{}^t S = b_2 A_2\,{}^t b_2 \bmod P^{1-\lambda_i}.$$

Thus

$$|\mathcal{P}f(b_2 A_2\,{}^t b_2 + S A_1\,{}^t S)| = |\mathcal{P}f(b_2 A_2\,{}^t b_2)| = |\mathcal{P}f(A_2)| = |\mathcal{P}f_i(x)|.$$

We have proved $|\mathcal{P}f_i(b \cdot x)| = |\mathcal{P}f_i(x)|$, thus the identity $F_z^x(b) = F_z^x(1)$. $\square$

Proposition A.8. *Let $x = \Pi^\lambda$ for some $\lambda \in \Lambda_n^+$. If the distribution $\tilde{T}_w^z(F_z^{\mathbf{T}J}) \in I(w\chi_{-\nu(z)})^*$ is supported away from Y, then $a_w(x; z) = 0$.*

Proof. Let Φ_0^w be the vector in $I(w\chi_{-\nu(z)})$, such that Φ_0^w is supported on BB_0, and $\Phi_0^w(b) = 1$ when $b \in B_0$. Then we have that $\tilde{T}_w^z(F_z^x)(\Phi_0^w)$ is equal to

$$\int_{B\backslash Sp_{2n}} \tilde{T}_w^z(F_z^x)(g)\Phi_0^w(g)dg = \int_{B_0} \tilde{T}_w^z(F_z^x)(b)db. \tag{A.22}$$

Thus from (A.20), we have that

$$a_w(x;z) = \tilde{T}_w^z(F_z^x)(\Phi_0^w).$$

Assume $x = g \cdot \mathbf{T}J$, let ρ_w denote the representation on the space $I(w\chi_{-\nu(z)})^*$ and let ρ'_w denote the representation on the space $I(w\chi_{-\nu(z)})$. We calculate that for $\phi \in I(w\chi_{-\nu(z)})$ we have that

$$\begin{aligned}
\rho_w(g)\tilde{T}_w^z(F_z^{\mathbf{T}J})(\phi) &= \tilde{T}_w^z(F_z^{\mathbf{T}J})(\rho'_w(g^{-1})\phi) \\
&= \int_{B\backslash Sp_{2n}} \int_{N_3\cap wN_3w^{-1}\backslash N_3} F_z^{\mathbf{T}J}(w^{-1}uh)\phi(hg^{-1})du\,dh \\
&= \int_{B\backslash Sp_{2n}} \int_{N_3\cap wN_3w^{-1}\backslash N_3} \prod_{i=1}^n |\mathcal{P}f_i(w^{-1}uh\cdot\mathbf{T}J)|^{s_i}\phi(hg^{-1})du\,dh \\
&= \int_{B\backslash Sp_{2n}} \int_{N_3\cap wN_3w^{-1}\backslash N_3} \prod_{i=1}^n |\mathcal{P}f_i(w^{-1}uh\cdot x)|^{s_i}\phi(h)du\,dh \\
&= \int_{B\backslash Sp_{2n}} \int_{N_3\cap wN_3w^{-1}\backslash N_3} F_z^x(w^{-1}uh)\phi(h)du\,dh.
\end{aligned}$$

Thus, $\tilde{T}_w^z(F_z^x) = \rho_w(g)\tilde{T}_w^z(F_z^{\mathbf{T}J})$ and

$$a_w(x;z) = \tilde{T}_w^z(F_z^{\mathbf{T}J})(\rho'_w(g^{-1})\Phi_0^w).$$

We show $\rho'_w(g^{-1})\Phi_0^w$ is supported on Y; then if the assumption of the Proposition holds, $a_w(x;z) = 0$.

The support of $\rho'_w(g^{-1})\Phi_0^w$ is BB_0g. Let $b_1 \in B$, $b_2 \in B_0$. From Lemma A.7, $F_z^{\mathbf{T}J}(b_2g) = F_z^x(b_2) = F_z^x(1) \neq 0$, thus from Lemma A.6 we have that $F_z^{\mathbf{T}J}(b_1b_2g) \neq 0$. We have shown that $\rho'_w(g^{-1})\Phi_0^w$ is supported on Y, this gives the Proposition. □

Proposition A.9. *The distribution $\tilde{T}_w^z(F_z^{\mathbf{T}J}) \in I(w\chi_{-\nu(z)})^*$ is supported away from Y unless $w\chi_{\nu(z)} = \chi_{\nu(w'z)}$ for some w' in W_2.*

Proof. Assume $\tilde{T}_w^z(F_z^{\mathbf{T}J})$ is not supported away from Y. Let $I_Y(w\chi_{-\nu(z)})$ be the subspace of $I(w\chi_{-\nu(z)})$ consisting of sections supported on Y. Then $\tilde{T}_w^z(F_z^{\mathbf{T}J})$ defines a nontrivial $Sp_{n,K}-$invariant linear form on $I_Y(w\chi_{-\nu(z)})$.

Note that $Y = P_2g_0Sp_{n,K}$ for some g_0 in Sp_{2n}, where P_2 is the parabolic group containing B and with 2×2 blocks on the diagonal. Let $Sp_{n,K\,g_0} = g_0^{-1}P_2g_0 \cap Sp_{n,K}$ and $\sigma = \mathrm{Ind}_B^{P_2} w\chi_{-\nu(z)}$. Consider the $Sp_{n,K}-$module $\mathrm{Ind}_{Sp_{n,K\,g_0}}^{Sp_{n,K}}(\sigma^{g_0}\delta_{P_0}^{g_0})$ consisting of modulo $Sp_{n,K\,g_0}$ compactly supported functions ϕ on $Sp_{n,K}$ with values in the space of σ, satisfying:

$$\phi(h_0h) = \delta_{P_0}(g_0h_0g_0^{-1})\sigma(g_0h_0g_0^{-1})\phi(h),\ h_0 \in Sp_{n,K\,g_0}.$$

As $Sp_{n,K}-$modules, we have $I_Y(w\chi_{-\nu(z)}) \cong \mathrm{Ind}_{Sp_{n,K\,g_0}}^{Sp_{n,K}}(\sigma^{g_0}\delta_{P_0}^{g_0})$.

From Frobenius reciprocity, the existence of nontrivial $Sp_{n,K}-$invariant linear form on $I_Y(w\chi_{-\nu(z)})$ implies that as a representation of $Sp_{n,K\,g_0}$, $\sigma^{g_0}\delta_{P_0}^{g_0}$ contains a trivial representation. Equivalently, as the representation of $g_0Sp_{n,K\,g_0}g_0^{-1}$, $\sigma\delta_{P_0}$ contains a trivial representation. Notice that $g_0Sp_{n,K\,g_0}g_0^{-1}$ equals n copies of SL_2 sitting in the diagonal 2×2 blocks of P_2, over which δ_{P_0} is trivial; we see that σ contains the trivial representation of $SL_2 \times \ldots \times SL_2$. With our assumption that z is in the general position, it is only possible when w is as described in the Proposition. □

For each $w' \in W_2$, there is a unique $w \in W_3$ such that

$$w\chi_{\nu(z)} = \chi_{\nu(w'z)}.$$

We will write $w = \sigma(w')$ if this is the case. Furthermore, if w, $\alpha \in W_3$ and w', $\alpha' \in W_2$ satisfy $\sigma(w') = w$ and $\sigma(\alpha') = \alpha$, then

$$w\alpha\chi_{\nu(z)} = w\chi_{\nu(\alpha' z)} = \chi_{\nu(w'\alpha' z)},$$

so that σ is a group homomorphism.

Corollary A.10. *$a_w(x; z) = 0$ unless $w = \sigma(w')$ for some w' in the Weyl group of Sp_n.*

Given $w = \sigma(w')$ with w' in the Weyl group of Sp_n, then $F^x_{w'z} \in I(w\chi_{-\nu(z)})^*$. Write $x = g \cdot \mathbf{T}J = \Pi^\lambda$. As $T^z_w(F^{\mathbf{T}J}_z)$ and $F^{\mathbf{T}J}_{w'z}$ satisfy the same left equivariance condition over P_2 and right $Sp_{n,K}-$invariance condition, we get

$$T^z_w(F^{\mathbf{T}J}_z)(hg) = \delta(w', z)F^{\mathbf{T}J}_{w'z}(hg),$$

or

$$T^z_w(F^x_z)(h) = \delta(w', z)F^x_{w'z}(h)$$

for some number $\delta(w', z)$ independent of $h \in Sp_{2n}$.

From (A.20), we get

$$a_w(x; z) = \delta(w', z)\int_{B_0} F^x_{w'z}(b)db. \tag{A.23}$$

From Lemma A.7, we see that $a_w(x; z) = \delta(w', z)F^x_{w'z}(1)$. From (A.11),

$$F^x_{w'z}(1) = \prod_{i=1}^{n} |\mathcal{P}f_i(\Pi^\lambda)|^{s_i} = e^{w'\lambda}q^{b(\lambda)} \tag{A.24}$$

when $x = \Pi^\lambda$. Here $b(\lambda)$ is as defined in Theorem A.2, by

$$b(\lambda) = -\sum_{i=1}^{n}(2(n - i + 1) - 1/2)\lambda_i$$

and e^λ is defined with $e^{\epsilon_i} = q^{-z_i}$. Note that considered as function of z, $F^x_{w'z}(1)$ is an additive character of z.

Summarizing the results so far, we get

Lemma A.11. *When $x = \Pi^\lambda$ for some $\lambda \in \Lambda_n^+$, we have:*

$$\zeta(\Pi^\lambda; z) = \sum_{w \in W_2} Q^{-1} q^{b(\lambda)} c(w, z) e^{w\lambda}. \tag{A.25}$$

where $c(w, z) = c_{w_l}(w_l \sigma(w) \nu(z)) \delta(w, z)/c_{\sigma(w)}(\nu(z))$.

A.1.3. *Functional equations.* We see when $w = e$ is the identity, $c(e, z)$ in the lemma equals $c_{w_l}(w_l \nu(z))$ which is:

$$(1 + q^{-1})^n \prod_{\alpha \in S^{+L}} \left(\frac{1 - q^{-2} e^{-\alpha}}{1 - e^{-\alpha}} \frac{1 - q^{-1} e^{-\alpha}}{1 - q e^{-\alpha}} \right) \prod_{\alpha \in S^{+S}} \left(\frac{1 - q^{-1} e^{-2\alpha}}{1 - e^{-2\alpha}} \frac{1 - q^{-\frac{3}{2}} e^{-\alpha}}{1 - q^{\frac{1}{2}} e^{-\alpha}} \right) \tag{A.26}$$

where S^{+L} and S^{+S} are the set of long and short positive roots. We will use the functional equations of $\zeta(x; s)$ to determine $c(w, z)$ for other $w \in W_2$.

Proposition A.12. *The function $\Psi_z(x) = \zeta(x; s)/\zeta(\Pi^{\mathbf{0}}; s)$ satisfies the functional equation $\Psi_{wz}(x) = \Psi_z(x)$ for all $w \in W_2$.*

Proof. Let $w_0 \in W_2$. Then from (A.25):

$$\Psi_{w_0 z}(\Pi^\lambda) = \frac{\sum_{w \in W_2} q^{b(\lambda)} c(w, w_0 z) e^{w w_0 \lambda}}{\sum_{w \in W_2} c(w, w_0 z)}.$$

Let $w_1 \in W_2$. We compare the coefficient of $e^{w_1 w_0 \lambda}$ for $\Psi_z(\Pi^\lambda)$ and $\Psi_{w_0 z}(\Pi^\lambda)$. They are

$$q^{b(\lambda)} c(w_1 w_0, z) / \sum_{w \in W_2} c(w, z) \tag{A.27}$$

and

$$q^{b(\lambda)}c(w_1, w_0 z)/\sum_{w\in W_2} c(w, w_0 z) \tag{A.28}$$

From the definition of $c(w, z)$, we see that the quotient of $c(ww', z)$ by $c(w, w'z)$ is given by

$$\left\{c_{w_l}(w_l\sigma(ww')\nu(z))\delta(ww', z)/c_{\sigma(ww')}(\nu(z))\right\}\times c_{\sigma(w)}(\nu(w'z))/\left\{c_{w_l}(w_l\sigma(w)\nu(w'z))\delta(w, w'z)\right\}.$$

Using that σ is a homomorphism and that $\sigma(w')\nu(z) = \nu(w'z)$, this is equal to

$$\frac{\delta(ww', z)}{\delta(w, w'z)}\frac{c_{\sigma(w)}(\nu(w'z))}{c_{\sigma(ww')}(\nu(z))}.$$

It is well known that

$$c_{\sigma(ww')}(\nu(z)) = c_{\sigma(w)}(\nu(w'z))c_{\sigma(w')}(\nu(z)),$$

so our expression is

$$\frac{\delta(ww', z)}{\delta(w, w'z)c_{\sigma(w')}(\nu(z))}.$$

Since the intertwining operator $T^z_{ww'}$ is given by the composition of $T^{w'z}_w$ and $T^z_{w'}$, we see that

$$\delta(ww', z) = \delta(w, w'z)\delta(w', z).$$

Thus we obtain that

$$\frac{c(ww', z)}{c(w, w'z)} = \frac{\delta(w', z)}{c_{\sigma(w')}(\nu(z))}. \tag{A.29}$$

From equation (A.29), the quotient of (A.27) by (A.28) is given by

$$\frac{\sum_{w\in W_2} c(w, w_0 z)\delta(w_0, z)/c_{\sigma(w_0)}(\nu(z))}{\sum_{w\in W_2} c(w, z)}.$$

Using equation (A.29), we see that

$$c(w, w_0 z)\delta(w_0, z)/c_{\sigma(w_0)}(\nu(z)) = c(ww_0, z),$$

so that the quotient of $c(ww', z)$ by $c(w, w'z)$ is equal to

$$\frac{\sum_{w\in W_2} c(ww_0, z)}{\sum_{w\in W_2} c(w, z)} = 1.$$

We have shown the coefficients of $e^{w_1 w_0 \lambda}$ for $\Psi_z(\Pi^\lambda)$ and $\Psi_{w_0 z}(\Pi^\lambda)$ equal. Thus $\Psi_z(x) = \Psi_{w_0 z}(x)$. $\square$

To introduce a more precise functional equation, let

$$\Gamma_1(z) = \prod_{\alpha\in S^{+L}} \frac{1 - qe^{-\alpha}}{1 - q^{-1}e^{-\alpha}}$$

and

$$\Gamma_2(z) = \prod_{\alpha\in S^{+S}} \frac{1 - q^{\frac{1}{2}}e^{-\alpha}}{1 - q^{-\frac{1}{2}}e^{-\alpha}}.$$

Proposition A.13. *Let* $\tilde{\zeta}(x; z) = \Gamma_1(z)\Gamma_2(z)\zeta(x; s)$. *Then*

(A.30) $$\tilde{\zeta}(x; wz) = \tilde{\zeta}(x, z)$$

for all $w \in W_2$.

We will give the proof in the next section. From Proposition A.13, equations (A.25) and (A.26), and the linear independence of the characters $F^x_{wz}(1)$ (as characters of z), we get for $\lambda \in \Lambda_n^+$, $\tilde{\zeta}(\Pi^\lambda; z)$ equals

$$Q^{-1}(1+q^{-1})^n q^{b(\lambda)} \sum_{w\in W_2} w(e^\lambda \Gamma_1(z)\Gamma_2(z)c(e,z)),$$

which is

(A.31)
$$Q^{-1}(1+q^{-1})^n q^{b(\lambda)} \sum_{w\in W_2} w\left(e^\lambda \prod_{\alpha\in S^{+L}} \frac{1-q^{-2}e^{-\alpha}}{1-e^{-\alpha}} \prod_{\alpha\in S^{+S}} \frac{(1+q^{-\frac{1}{2}}e^{-\alpha})(1-q^{-\frac{3}{2}}e^{-\alpha})}{1-e^{-2\alpha}}\right).$$

Comparing this with the definition of the Macdonald polynomial $Q^t_\lambda(z)$, we see that

$$\tilde{\zeta}(\Pi^\lambda; z) = Q^{-1}(1+q^{-1})^n q^{b(\lambda)} V_\lambda(t^I) Q^{t^I}_\lambda(z) \tag{A.32}$$

and

$$\tilde{\zeta}(\Pi^0; z) = Q^{-1}(1+q^{-1})^n V_{\mathbf{0}}(t^I) Q^{t^I}_{\mathbf{0}}(z)$$

when $\lambda \in \Lambda_n^+$ and t^I is the parameter defined in Theorem A.2 by $t^I_\alpha = q^{-2}$ when α is a long root in S and $t^I_\alpha = q^{-1}$, ${t^I_{2\alpha}}^{1/2} = q^{-1/2}$ for α a short root in S.

Since

$$\tilde{\zeta}(x;z)/\tilde{\zeta}(\Pi^{\mathbf{0}};z) = \Psi_z(x),$$

we get

$$\Psi_z(\Pi^\lambda) = q^{b(\lambda)} V_\lambda(t^I) Q^{t^I}_\lambda(z)/(Q^{t^I}_{\mathbf{0}}(z) V_{\mathbf{0}}(t^I)) \tag{A.33}$$

when $\lambda \in \Lambda_n^+$. From [Mc1], we see $Q^{t^I}_{\mathbf{0}}(z) = 1$. Thus the function Ψ_z defined in (A.5) by

$$\Psi_z(x) = \zeta(x;s)/\zeta(\Pi^{\mathbf{0}}, s),\ x \in X.$$

is given by

$$q^{b(\lambda)}\frac{V_\lambda(t^I)}{V_0(t^I)}Q_\lambda^{t^I}, \qquad \lambda \in \Lambda_n^+.$$

We have proved Theorem A.2.

A.2. **Proof of Proposition A.13.** We prove (A.30) here. The Weyl group W_2 is generated by elements σ_i ($1 \le i \le n-1$) and r_n, where σ_i fixes ϵ_j if $j \ne i, i+1$, and switches ϵ_i with ϵ_{i+1}; r_n fixes ϵ_j if $j < n$ and maps ϵ_n to $-\epsilon_n$.

A.2.1. *Functional equation for σ_i.* Assume $n > 1$. We fix an $i \le n-1$. Recall that s and z are related in equation (A.6) by

$$\sum_{i=j}^{n} s_i = -z_j - 2(n-j) - \frac{3}{2},\ j = 1, \ldots, n.$$

From this relation, the following lemma is clear.

Lemma A.14. *The ordered set of complex numbers* $(s_1, \ldots, s_{i-2}, s_{i-1}+\frac{s_i}{2}, \frac{s_i}{2}+s_{i+1}, s_{i+2}, \ldots, s_n)$ *is invariant under the map $z \mapsto \sigma_i z$.*

Let $Y' = Y \cdot \mathbf{T}J$, then by definition of Y in (A.21), $Y = \{g \in Sp_{2n} | F_z^{\mathbf{T}J}(g) \ne 0\}$, we see $Y' = \{x \in X |\, F_s^x(1) \ne 0\}$. Recall that P_2 is the parabolic subgroup of Sp_{2n} whose Levi subgroup is a product of GL_2's and whose unipotent subgroup consists of upper triangular matrices. We have the following lemma.

Lemma A.15. *The set Y' is transitive under P_2. Any $x \in Y'$ has a decomposition $x = p \cdot \Pi^{\lambda(x)}$ where $\lambda(x) \in \mathbf{Z}^n$ and the Levi part of $p \in P_2$ lies in products of $GL_2(\mathcal{O})$.*

We will embed $K_4 = GL_4(\mathcal{O})$ in Sp_{2n} as follows: If $\{e_j | j = 1, \ldots, 4n\}$ is the standard basis of the vector space Sp_{2n} acting on, then $k \in K_4$ acts trivially on the space generated by $\{e_1, \ldots, e_{2i-2}, e_{2i+3}, \ldots, e_{4n-2i-2}, e_{4n-2i+3}, \ldots, e_{4n}\}$; acts by multiplication of k on the space generated by $\{e_{2i+j}|j=-1,0,1,2\}$ and by multiplication of k^* on the space generated by $\{e_{4n-2i+j}|j=-1,0,1,2\}$. Then

$$\zeta(x;s)=\int_{k\in K_3}\int_{k'\in K_4}\prod_{j=1}^{n}|\mathcal{P}f_j(k'k\cdot x)|^{s_j}dk'dk. \tag{A.34}$$

The proof of the following lemma is as in [MR1].

Lemma A.16. *Given any $x\in X$, the expression*

$$\frac{1-q^{z_i-z_{i+1}+1}}{1-q^{z_i-z_{i+1}-1}}\int_{k\in K_4}\prod_{j=1}^{n}|\mathcal{P}f_j(k\cdot x)|^{s_j}dk \tag{A.35}$$

is invariant under the action $z\mapsto\sigma_i z$.

From the Lemma, after multiplying by $\frac{1-q^{z_i-z_{i+1}+1}}{1-q^{z_i-z_{i+1}-1}}$ the inner integral in (A.34) is invariant under $z\mapsto\sigma_i z$. Thus we get

Lemma A.17. *The expression $\frac{1-q^{z_i-z_{i+1}+1}}{1-q^{z_i-z_{i+1}-1}}\zeta(x;s)$ is invariant under $z\mapsto\sigma_i z$.*

A.2.2. *Functional equation for r_n.* We first consider the case $n=1$. Here $z\in\mathbf{C}$ and $r_n z=-z$. From Proposition A.12, to get an explicit function equation, we only need to compute $\zeta(\Pi^0;s)$.

Lemma A.18. *When $n=1$,*

$$\zeta(\Pi^0;s)=\frac{1-q^{z-\frac{1}{2}}}{(1+q^{-1})(1-q^{z+\frac{1}{2}})}. \tag{A.36}$$

Proof. Let K^m be the set of $k\in K_3$ such that $|\mathcal{P}f_1(k\cdot\Pi^0)|=q^{-m}$. Then

$$\zeta(\Pi^0;s)=\sum_{m=0}^{\infty}vol(K^m)q^{-sm}. \tag{A.37}$$

The set $K \cdot \Pi^0$ is given by the elements in X whose entries are all in $\mathcal{O}$. This set can be described as:

$$\left\{ \begin{pmatrix} & a & b_1 & b_2 \\ -a & & -b_2 & b_3 \\ -b_1 & b_2 & & d \\ -b_2 & -b_3 & -d & \end{pmatrix} \mid b_2^2 + b_1 b_3 = \tau + ad,\ a, d, b_1, b_2, b_3 \in \mathcal{O} \right\}.$$

Let X_m be the subset with $|d| = q^{-m}$. Choose a $g_0 \in K_3$ such that $g_0 \cdot \mathbf{T}J = \Pi^0$, and let $Sp'_{n,K} = g_0 Sp_{n,K} g_0^{-1}$. Then $Sp'_{n,K} \cdot \Pi^0 = \Pi^0$.

Lemma A.19. *Let $m > 1$, let K_m be the set of $k \in K_3$ with $k = 1$ mod P^m. Then $\rho : k \mapsto k \cdot \Pi^0$ induces a bijection between $K_m \backslash K_3 / K_3 \cap Sp'_{n,K}$ and $K_3 \cdot \Pi^0$ mod P^m.*

Proof. Surjectivity follows as in [MR1]. To show injectivity, we do a counting of the number of the double cosets $K_m \backslash K_3 / K_3 \cap Sp'_{n,K}$ and the number of the cosets $K_3 \cdot \Pi^0$ mod P^m.

Notice that $|K_m \backslash K_3 / K_3 \cap Sp'_{n,K}|$ is equal to the number of K_3 mod P^m divided by the number of $K_3 \cap Sp'_{n,K}$ mod P^m.

We have that

$$|K_3 \text{ mod } P^m| = q^{3m}(1 - q^{-2}) q^{7m}(1 - q^{-4}) \tag{A.38}$$

and that

$$|K_3 \cap Sp'_{n,K} \text{ mod } P^m| = q^{6m}(1 - q^{-4}). \tag{A.39}$$

Thus we obtain that

$$|K_m \backslash K_3 / K_3 \cap Sp'_{n,K}| = \frac{q^{10m}(1 - q^{-2})(1 - q^{-4})}{q^{6m}(1 - q^{-4})} = q^{4m}(1 - q^{-2}). \tag{A.40}$$

On the other hand, the number of $K_3 \cdot \Pi^0$ mod P^m is equal to the number of solutions of $b_2^2 + b_1 b_3 = \tau + ad$ in $(O/P)^5$. In the case when b_1 is a unit, we obtain $q^{4m}(1 - q^{-1})$ solutions. When b_1 is not a unit, but d is a unit, we have $q^{4m}(q^{-1} - q^{-2})$ solutions. The case when b_1, $d \in P$ contributes no solutions. Thus the number of cosets $K_3 \cdot \Pi^0$ mod P^m is equal to $q^{4m}(1 - q^{-2})$. This proves that the map $k \mapsto k \cdot \Pi^0$ induces a bijection between $K_m \backslash K_3 / K_3 \cap Sp'_{n,K}$ and $K_3 \cdot \Pi^0$ mod P^m. □

We continue with the proof of Lemma A.18. We have that

$$\mathrm{vol}(K^m) = \mathrm{vol}(K_{m+1}) |X_m \bmod P^m| |K_3 \cap Sp'_{n,K} \bmod P^{m+1}|. \tag{A.41}$$

We already know that

$$|K_3 \cap Sp'_{n,K} \bmod P^{m+1}| = q^{6(m+1)}(1 - q^{-4}). \tag{A.42}$$

To count the cosets of X_m, we remark that since $|d| = q^{-m} < 1$, we have the cases $|b_1| = 1$ and $|b_1| < 1$. The first case contributes $q^{3(m+1)}(q-1)(1-q^{-1})$ elements, while the second case contributes no solutions. Thus,

$$|X_m \bmod P^m| = q^{3(m+1)}(q-1)(1-q^{-1}) \tag{A.43}$$

With our assumption that $\mathrm{vol}(K_3) = 1$, we obtain that

$$\mathrm{vol}(K_{m+1}) = q^{-3(m+1)}(1-q^{-2})^{-1} q^{-7(m+1)}(1-q^{-4})^{-1}. \tag{A.44}$$

Thus, for $m > 0$, we obtain that

$$\mathrm{vol}(K^m) = q^{-(m+1)}(q-1)(1+q^{-1})^{-1} = q^{-m}\frac{1-q^{-1}}{1+q^{-1}}. \tag{A.45}$$

Thus,

$$\text{vol}(K^0) = 1 - \sum_{m=1}^{\infty} \text{vol}(K^m) = 1 - \frac{1-q^{-1}}{1+q^{-1}}.\frac{q^{-1}}{1-q^{-1}} = \frac{1}{1+q^{-1}}. \tag{A.46}$$

Using $\zeta(\Pi^0; s) = \sum_{m=0}^{\infty} \text{vol}(K^m) q^{-sm}$, we obtain that

$$\zeta(\Pi^0; s) = \frac{1}{1+q^{-1}} + \frac{1-q^{-1}}{1+q^{-1}} \sum_{m=1}^{\infty} q^{m(-1-s)} = \frac{1-q^{-2-s}}{(1+q^{-1})(1-q^{-1-s})}. \tag{A.47}$$

Using the relation between s and z in equation (A.6), we get that $s = -z - \frac{3}{2}$, so that

$$\zeta(\Pi^0; s) = \frac{1-q^{z-\frac{1}{2}}}{(1+q^{-1})(1-q^{z+\frac{1}{2}})} \tag{A.48}$$

as desired. ☐

We have the following corollary:

Corollary A.20. *When $n = 1$, $s = -z - \frac{3}{2}$, then $\frac{1-q^{z+\frac{1}{2}}}{1-q^{z-\frac{1}{2}}}\zeta(x; s)$ is invariant under $z \mapsto -z$.*

Proof. We have that $\frac{1-q^{z+\frac{1}{2}}}{1-q^{z-\frac{1}{2}}}\zeta(x; s)$ is equal to

$$\frac{1-q^{z+\frac{1}{2}}}{1-q^{z-\frac{1}{2}}}\zeta(\Pi^0; s)\Psi_z(x) = (1+q^{-1})^{-1}\Psi_z(x)$$

and this is invariant under $z \mapsto -z$ by Proposition A.12. ☐

Now assume $n > 1$. Let K_2' be $Sp_2(\mathcal{O})$, and embed it into K_3 with the embedding of the F^4 into the subspace of F^{4n} generated by part of the standard basis e_j with $j = 2n-1, 2n, 2n+1, 2n+2$. Then

$$\zeta(x; s) = \int\limits_{k \in K_3} \int\limits_{k' \in K_2'} \prod_{j=1}^{n} |\mathcal{P} f_j(k'k \cdot x)|^{s_j} dk' dk. \tag{A.49}$$

Lemma A.21. *Given any $x \in X$, the expression*

$$\frac{1-q^{z_n+\frac{1}{2}}}{1-q^{z_n-\frac{1}{2}}} \int\limits_{k\in K_2'} \prod_{j=1}^{n} |\mathcal{P}f_j(k\cdot x)|^{s_j} dk \tag{A.50}$$

is invariant under the action $z \mapsto r_n z$.

Proof. Without loss of generality, we can assume $x \in Y'$ and has the form $x = p \cdot \Pi^{\lambda(x)}$ with $p \in P_2$ as in Lemma A.15. Let P_2' be the subgroup of P_2 where the middle 4×4 block is the identity, a change of variable shows that the integral in (A.50) remains the same with x replaced by $p\cdot x$ with $p \in P_2'$. Since $P_2 \subset P_2' K_2'$, we only need to consider the case $x = \Pi^{\lambda(x)}$, which we now assume.

Let $\tilde{x}$ be the middle 4×4 block of x. Then clearly $|\mathcal{P}f_j(k\cdot x)| = |\mathcal{P}f_j(x)|$ when $j \neq n$, and $|\mathcal{P}f_n(k\cdot x)| = |\mathcal{P}f_{n-1}(x)||\mathcal{P}f(k\cdot\tilde{x})|$. Thus (A.50) equals:

$$\begin{aligned}
&\frac{1-q^{z_n+\frac{1}{2}}}{1-q^{z_n-\frac{1}{2}}}\left(\prod_{j=1}^{n-2}|\mathcal{P}f_j(x)|^{s_j}\right)|\mathcal{P}f_{n-1}(x)|^{s_{n-1}+s_n}\int\limits_{K_2'}|\mathcal{P}f(k\cdot\tilde{x})|^{s_n}dk\\
=\;&\frac{1-q^{z_n+\frac{1}{2}}}{1-q^{z_n-\frac{1}{2}}}\left(\prod_{j=1}^{n-2}|\mathcal{P}f_j(x)|^{s_j}\right)|\mathcal{P}f_{n-1}(x)|^{s_{n-1}+s_n}\zeta(\tilde{x};s_n).
\end{aligned}$$

From (A.6), the ordered set $(s_1,\ldots,s_{n-2},s_{n-1}+s_n)$ is invariant under $z_n \mapsto -z_n$. Also from (A.6), $s_n = -z_n - \frac{3}{2}$, thus our Lemma follows from Corollary A.20. $\square$

From the Lemma, after multiplying by $\frac{1-q^{z_n+\frac{1}{2}}}{1-q^{z_n-\frac{1}{2}}}$ the inner integral in (A.49) is invariant under $z \mapsto r_n z$. Thus we get

Lemma A.22. *The expression $\frac{1-q^{z_n+\frac{1}{2}}}{1-q^{z_n-\frac{1}{2}}}\zeta(x;s)$ is invariant under $z \mapsto r_n z$.*

A.2.3. *Proof of Proposition A.13.*

Proof. Since r_n reflects the long positive roots to long positive roots, it leaves $\Gamma_1(z)$ invariant. The reflection r_n also fixes all positive short roots except when $\alpha = \epsilon_n$, in which case:

$$\frac{1-q^{\frac{1}{2}}e^{-\alpha}}{1-q^{-\frac{1}{2}}e^{-\alpha}} = \frac{1-q^{z_n+\frac{1}{2}}}{1-q^{z_n-\frac{1}{2}}}.$$

Thus from Lemma A.22, $\Gamma_2(z)\zeta(x;s)$ is invariant under r_n. Thus $\tilde{\zeta}(x;z)$ is invariant under r_n.

Since σ_i maps the short positive roots to short positive roots, it leaves $\Gamma_2(z)$ invariant. It also acts as a permutation of $S^{+L}\backslash\alpha_i$ where $\alpha_i = \epsilon_i - \epsilon_{i+1}$. Thus

$$\Gamma_1(z)\left(\frac{1-qe^{-\alpha_i}}{1-q^{-1}e^{-\alpha_i}}\right)^{-1}$$

is invariant under σ_i. Since

$$\frac{1-qe^{-\alpha_i}}{1-q^{-1}e^{-\alpha_i}} = \frac{1-q^{z_i-z_{i+1}+1}}{1-q^{z_i-z_{i+1}-1}},$$

from Lemma A.17, we see $\Gamma_1(z)\zeta(x;s)$ is invariant under σ_i. Thus $\tilde{\zeta}(x;z)$ is invariant under σ_i.

As the Weyl group W_2 is generated by σ_i and r_n, we get $\tilde{\zeta}(x;z)$ is invariant under W_2. $\square$

Appendix B. Unramified Whittaker function on metaplectic group

Denote an element in $\widetilde{Sp_n}$ by (g,ζ) with $g \in Sp_n$ and $\zeta = \pm 1$. Recall the functions in $C^\infty_{K_2}(N_2\backslash\widetilde{Sp_n}, \theta^{-1}_{2,\tau})$ are *genuine*; namely for $f(g,\zeta)$ in this space, we have $f(g,\zeta) = \zeta f(g,1)$

Recall that the character $\theta_{2,\tau}$ is defined on N_2 as follows:

(B.1) $$\theta_{2,\tau}(u,\zeta) = \zeta\psi(u_{12} + u_{2,3} + \ldots + \tau u_{n,n+1})$$

where ψ is an additive character trivial on $\mathcal{O}$ but nontrivial on $\pi^{-1}\mathcal{O}$.

We are interested in computing the spherical functions in $C^\infty_{K_2}(N_2\backslash\widetilde{Sp_n}, \theta^{-1}_{2,\tau})$. Let $\mathcal{T}$ denote the diagonal matrix $\mathrm{diag}[\tau^n, \tau^{n-1}, \ldots, 1, 1, \tau^{-1} \ldots \tau^{-n}] \in K_2$. Let ${}_\tau\theta_2$ denote the character on N_2 given by

$$_{\tau}\theta_2(u,\zeta) = \zeta\psi(\tau u_{12} + \tau u_{2,3} + \ldots + \tau u_{n,n+1}).$$

Then we have an isomorphism of $\mathcal{H}(\widetilde{Sp_n}, K_2)-$modules:

$$\mathcal{S}_{K_2}(N_2\backslash\widetilde{Sp_n}, \theta_{2,\tau}^{-1}) \to \mathcal{S}_{K_2}(N_2\backslash\widetilde{Sp_n}, \,{}_{\tau}\theta_2^{-1}) \tag{B.2}$$

given by $f \mapsto f^{\mathcal{T}}$ where $f^{\mathcal{T}} \in \mathcal{S}_{K_2}(N_2\backslash\widetilde{Sp_n}, \,{}_{\tau}\theta_2^{-1})$ is given by $f^{\mathcal{T}}(g) = f(\mathcal{T}g\mathcal{T}^{-1})$.

We first exhibit spherical functions in $\mathcal{S}_{K_2}(N_2\backslash\widetilde{Sp_n}, \,{}_{\tau}\theta_2^{-1})$.

Let $\lambda \in \Lambda_n^+$, and let $\tilde{\Pi}^\lambda$ be the diagonal matrix

$$\mathrm{diag}[\pi^{\lambda_1}, \ldots, \pi^{\lambda_n}, \pi^{-\lambda_n}, \ldots, \pi^{-\lambda_1}].$$

It is well known that $\widetilde{Sp_n}^{rel} = \cup_{\lambda\in\Lambda_n^+} N_2(\tilde{\Pi}^\lambda, 1)K_2$ as a disjoint union.

Let $\chi = \chi_z$ be an unramified character on the group A_2 of diagonal matrices given by

$$\chi(\tilde{\Pi}^\lambda) = \prod_{i=1}^{n} \alpha_i^{\lambda_i} = \prod_{i=1}^{n} q^{z_i\lambda_i}. \tag{B.3}$$

This character extends to a genuine character $\tilde{\chi}$ of $\tilde{A}_2$ the double cover of A_2:

$$\tilde{\chi}(\mathbf{a},\zeta) = \chi(\mathbf{a})\zeta\gamma_{\psi_\tau}(\mathbf{a})^{-1}, \ \mathbf{a} \in A_2. \tag{B.4}$$

Here ψ_τ is the character ψ composed with multiplication by τ and γ_{ψ_τ} is a fourth root of unity defined by the equation following (1.5) in [BFH]. The unramified Whittaker function $W_{\tilde{\chi}}(g)$ in the principal series representation $I(\tilde{\chi})$ will be normalized so that $W_{\tilde{\chi}}(1_{2n}, 1) = 1$ and will satisfy $W_{\tilde{\chi}}(g) = W_{w\tilde{\chi}}(g)$ for all $w \in W_2$, where $w\tilde{\chi}(\mathbf{a}) = \tilde{\chi}(\tilde{w}^{-1}\mathbf{a}\tilde{w})$, with $\tilde{w}$ being the inverse image of w in $\widetilde{Sp}_n$.

The proof in [BFH] shows $W_{\tilde{\chi}}(\tilde{\Pi}^\lambda, 1)$ equals

$$\gamma_{\psi_\tau}(\tilde{\Pi}^\lambda)^{-1}\delta_2^{\frac{1}{2}}(\tilde{\Pi}^\lambda)\sum_{w\in W_2} w\left(\prod_{i=1}^{n}\alpha_i^{-\lambda_i}\frac{1-q^{-1}\alpha_i^2}{(1+(p,p)q^{-\frac{1}{2}}\alpha_i)(1-\alpha_i^2)}\prod_{i>j}\frac{1}{(1-\alpha_i\alpha_j)(1-\alpha_j\alpha_i^{-1})}\right).$$

Here (p,p) is a Hilbert symbol taking value ± 1 and δ_2 is the modulus function of the Borel subgroup of N_2. We rewrite above formula in terms of Macdonald polynomial:

Theorem B.1. *For $z = (z_1, \ldots, z_n)$ in $\mathbf{C}^n$, let χ and $\tilde{\chi}$ be defined as in (B.3) and (B.4), then*

$$\Psi_z^2(\tilde{\Pi}^\lambda, 1) = W_{\tilde{\chi}}(\tilde{\Pi}^\lambda, 1) = V_\lambda(t^{II})\gamma_{\psi_\tau}(\tilde{\Pi}^\lambda)^{-1}\delta_2^{\frac{1}{2}}(\tilde{\Pi}^\lambda)Q_\lambda^{t^{II}}(z) \tag{B.5}$$

is a spherical function in $C^\infty_{K_2}(N_2\backslash\widetilde{Sp_n},\ {}_\tau\theta_2^{-1})$. Here when $\beta \in S$, $t^{II}_\beta = 0$; when β is a short root in S, $(t^{II}_{2\beta})^{\frac{1}{2}} = -(p,p)q^{-\frac{1}{2}}$.

As a consequence of the isomorphism (B.2) and the fact that $\tilde{\Pi}^\lambda$ is stabilized under conjugation by $\mathcal{T}$ we get that $\{\Psi_z^2\}$ also serve as spherical functions in $\mathcal{S}_{K_2}(N_2\backslash\widetilde{Sp_n}, \theta_{2,\tau}^{-1})$.

It is well known in this case that the map $f \mapsto \tilde{f}(z)$ from $\mathcal{H}(\widetilde{Sp_n}, K_2)$ to $\mathbf{C}[q^{\bar{z}}, q^{-\bar{z}}]^{W_2}$ is onto. Since $t^{II}_\beta = 0$ for all $\beta \in S$, from the definition of $V_\lambda(t^{II})$ in equations (3.8) and (10.1) of [Mc1] we have

$$V_\lambda(t^{II}) = 1\ \forall \lambda \in \Lambda_n^+.$$

Note that it is clear that the volume of $N_2\backslash N_2(\tilde{\Pi}^\lambda, 1)K_2$ equals $\delta_2^{-1}(\tilde{\Pi}^\lambda)$.

Appendix C. Proof of $T(\Psi_z^1) = c(z)T(\Psi_z^2)$

C.1. **Definition of the integral in (6.1) when $F = \Psi_z^1$.** The integral in (6.1) is clearly well defined if F is compactly supported. For Ψ_z the spherical function described in Theorem A.2, through the identification $Sp_{n,K}\backslash Sp_{2n} \cong X$, $\Psi_z^1(g) = \Psi_z(g^{-1}TgJ)$ defines a spherical function on $Sp_{n,K}\backslash Sp_{2n}$. The integral (6.1) still can be defined for $F = \Psi_z^1$, though the definition is more subtle.

Denote by π_z the induced representation $I(\chi_{-\nu(z)})$. Then π_z is induced from an unramified representation τ_z of GL_{2n}. Thus a model of π_z is given by a space of functions of two variables $\phi(g,h)$ with $g \in Sp_{2n}, h \in GL_{2n}$ satisfying:

1. $\phi(\begin{pmatrix} h_1 & \\ & h_1^* \end{pmatrix} g, h) = \phi(g, hh_1)$,
2. $\phi(g,h)$ as a function of g is compactly supported over $P\backslash Sp_{2n}$.
3. For fixed g, $\phi(g,h)$ is a vector in τ_z with model in $Ind_{GL_{n,K}}^{GL_{2n}} 1$. Here $GL_{n,K}$ is thought of as a subgroup of GL_{2n} and GL_{2n} embeds in Sp_{2n} as the Levi factor of the Siegel parabolic.

Let V_E be the subgroup of elements $u \in V$ such that $Eu = uE$, concretely

$$V_E = \left\{ \begin{pmatrix} 1_{2n} & v \\ & 1_{2n} \end{pmatrix}, v = \begin{pmatrix} X & \tau Y \\ Y & X \end{pmatrix} | X, Y \in \mathcal{S}_n \right\}.$$

Lemma C.1. *There is an unramified vector $\phi_z(g,h)$ in the space of π_z such that*

$$\Psi_z^1(\epsilon_0 g) = \Psi_z(g^{-1}EgJ) = \int_{v \in V_E} \phi_z(J_{2n}\epsilon_0 vg, 1_{2n})dv, \tag{C.1}$$

and the above integral converges absolutely.

Proof. Use $L_g(\phi)$ to denote the above integral with ϕ_z replaced by ϕ a vector in space of π_z. We let $\phi'_g(g') = \phi(g'g, 1_{2n})$ where $g' \in Sp_{n,K}$. Then ϕ'_g is a left-invariant under the Levi subgroup $GL_{n,K}$ of $Sp_{n,K}$.

We remark that if $u \in V_E$, then $\epsilon_0 u^{-1}\epsilon_0^{-1} T \epsilon_0 u \epsilon_0^{-1} = T$. Thus we have that $\epsilon_0 V_E \epsilon_0^{-1}$ is the unipotent subgroup for the parabolic subgroup in $Sp_{n,K}$ with Levi subgroup $GL_{n,K}$.

Thus $L_{\epsilon_0^{-1}}$ is an intertwining operator on π_z considered as a representation of $Sp_{n,K}$; it satisfies $L_{\epsilon_0^{-1}}(\pi_z(h)\phi) = L_{\epsilon_0^{-1}}(\phi)$. Thus $L_{\epsilon_0^{-1}}$ is a $Sp_{n,K}$ invariant linear form on π_z. By work of Zhang in [Zha], such a form is unique up to multiple. Proposition A.5 implies the Lemma. □

With the previous Lemma, the precise definition of integral (6.1) in the case $F = \Psi_z^1$ is through a series of compactly supported integrals, as in Lemmas 8.3 and 8.4 of [MR1].

C.2. **The image of $T_{\Phi_0}(\Psi_z^1)$.**

Proposition C.2. *There is a function $c(z)$ on $\mathbf{C}^n$ such that $T_{\Phi_0}(\Psi_z^1) = c(z)\Psi_z^2$ for $z \in \mathbf{C}^n$.*

Proof. In the definition of $T_\Phi(\Psi_z^1)$ in [MR1] we can replace ϕ_z by any vector ϕ in π_z. Let F_ϕ be the function on $Sp_{n,K}\backslash Sp_{2n}$ such that $F_\phi(\epsilon_0 g)$ is given by the integral in (C.1) with ϕ_z replaced by ϕ. Through the process of iterated integration in [MR1], we can define a linear map T on the space of $\pi_z \otimes \omega_{\psi^{-1}}$ to functions on $\widetilde{Sp}_n$.

It is shown in Lemma 6.3 that T is a map from $\pi_z \otimes \omega_{\psi^{-1}}$ to $Ind_{N_2}^{\widetilde{Sp}_n}\theta_{2,\tau}^{-1}$ where θ_2 is defined in (B.1). Observe that $\widetilde{Sp}_n$ acts on $\pi_z \otimes \omega_{\psi^{-1}}$ through the embedding j of Sp_n in Sp_{2n}. Since $\pi_z(j(h))\phi_z(j(g)) = \phi_z(j(gh))$ for $g, h \in Sp_n$, we get:

Lemma C.3. *The map T is a $\widetilde{Sp}_n$-module homomorphism from $\pi_z \otimes \omega_{\psi^{-1}}$ to $Ind_{N_2}^{\widetilde{Sp}_n}\theta_{2,\tau}^{-1}$.*

Recall that $\hat{N}^{n+1}$ is defined by

$$\hat{N}^{n+1} = \left\{ \begin{pmatrix} u & * & * \\ & 1_{2n} & * \\ & & u^* \end{pmatrix} \in Sp_{2n} | u \in Z_n \right\}.$$

Lemma C.4. *Let $u \in \hat{N}^{n+1}$, then $T(\pi_z(u)\phi \otimes \omega_{\psi^{-1}}(u,1)\Phi) = T(\phi \otimes \Phi)\theta'(u)$, where for $u = (u_{i,j}) \in \hat{N}^{n+1}$*

$$\theta'(u) = \psi(-u_{1,2} - u_{2,3} - \ldots - u_{n-1,n}).$$

Proof. We have that $T(\pi_z(u)\phi \otimes \omega_{\psi^{-1}}(u,1)\Phi)(g,1)$ is equal to

$$\int_{w\in U_E^1\backslash \hat{N}^{n+1}} \phi(\epsilon_0 wj(g)u)\theta_4(w)\omega_{\psi^{-1}}(w)\omega_{\psi^{-1}}(g)\omega_{\psi^{-1}}(u)\Phi(0).$$

Since $j(g)$ normalizes $\hat{N}^{n+1}$, we write $j(g)u = u'j(g)$ and change variables $w \mapsto wu'^{-1}$. We obtain the above integral is

$$\theta_4(u'^{-1}) \int_{w\in U_E^1\backslash \hat{N}^{n+1}} \phi(\epsilon_0 wj(g))\theta_4(w)\omega_{\psi^{-1}}(w)\omega_{\psi^{-1}}(g)\Phi(0).$$

Since $\theta_4(u'^{-1}) = \theta'(u)$, we obtain the Lemma. □

The Jacquet module $J_{\hat{N}^{n+1},\theta'}(\pi_z \otimes \omega_{\psi^{-1}})$ is considered in [GRS4]. It is a $\widetilde{Sp}_n$ module defined in (1.6) of [GRS4]. The above lemma shows that T factors through to a map $\overline{T}$ from $J_{\hat{N}^{n+1},\theta'}(\pi_z \otimes \omega_{\psi^{-1}})$ to $Ind_{N_2}^{\widetilde{Sp_n}}\theta_{2,\tau}^{-1}$. It follows from Theorem B of [GRS4] that

$$J_{\hat{N}^{n+1},\theta'}(\pi_z \otimes \omega_{\psi^{-1}}) \cong \bar{\pi}_z,$$

where $\bar{\pi}_z = I(\tilde{\chi})$ when $\alpha_i = q^{z_i}$ in the definition (B.3) of $\tilde{\chi}$.

Since there is a unique Whittaker model for $\bar{\pi}_z$, the map $\overline{T}$ is the unique (up to scalar multiple) map of $\bar{\pi}_z$ into $Ind_{N_2}^{\widetilde{Sp_n}}\theta_{2,\tau}^{-1}$. The function $T_{\Phi_0}(\Psi_z^1)$ is the image of the unramified vector in $\pi_z \otimes \omega_{\psi^{-1}}$. The image is clearly an unramified vector in $Ind_{N_2}^{\widetilde{Sp_n}}\theta_{2,\tau}^{-1}$, thus $T_{\Phi_0}(\Psi_z^1)$ corresponds to the image of the unramified vector of $\bar{\pi}_z$ under the Whittaker map to $Ind_{N_2}^{\widetilde{Sp_n}}\theta_{2,\tau}^{-1}$. This image is just the unramified Whittaker function of $\bar{\pi}_z$. Thus we have proved that $T_{\Phi_0}(\Psi_z^1)(g)$ is a multiple of $\Psi_z^2(g)$. □

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CESAR VALVERDE

Rutgers University - Newark, Department of Mathematics and Computer Science,
101 Warren St. (Smith 216), Newark, New Jersey 07102
cvalverd@rutgers.edu

CITIZENSHIP:

- Born October 1, 1984 in Guayaquil, Ecuador. Citizen of the United States and of Ecuador.

EDUCATION

2012 Ph.D. in Mathematics (expected May 2012)

- **Thesis:** *Products of distinct Whittaker coefficients on the metaplectic group and the Relative Trace Formula*, Rutgers University - Newark.
- **Advisor:** Zhengyu Mao

2006 B.S. in Mathematics from Rutgers University - Newark.

Awards and Scholarships

2011 Rutgers - Newark Graduate Dissertation Fellowship.

2006 Samuel Greitzer Award - awarded to the top undergraduate student in Mathematics. Rutgers - Newark.

Talks given

- *A relative trace formula between the metaplectic and general linear groups* (CUNY Collaborative Number Theory Seminar, Feb. 2012.)
- *A relative trace formula and a generalization of work of Kohnen* (University of Michigan Group, Lie and Number Theory Seminar, Jan. 2012.)
- *A generalization of a result of Kohnen* (2011 Fall AMS Eastern Sectional Meeting, Cornell University, Ithaca, NY, Sept. 2011.)

PAPERS

- *Products of distinct Whittaker coefficients on the metaplectic group and the Relative Trace Formula*, Thesis.
- *On induced representations distinguished by orthogonal groups*, to appear in *Canadian Mathematical Bulletin*, 14 pages.
- *Periods of Eisenstein Series on* Sp_{2n} , in preparation.