ON SOME NONLOCAL ELLIPTIC AND PARABOLIC EQUATIONS

BY TIANLING JIN

A dissertation submitted to the

Graduate School—New Brunswick

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

YanYan Li

and approved by

New Brunswick, New Jersey

May, 2012

ABSTRACT OF THE DISSERTATION

On some nonlocal elliptic and parabolic equations

by Tianling Jin

Dissertation Director: YanYan Li

We prove some results on the existence and compactness of solutions of a fractional Nirenberg problem involving nonlocal conformally invariant operators. Regularity properties for solutions of some degenerate elliptic equations as well as a Liouville type theorem are established, and used in our blow up analysis. We also introduce a fractional Yamabe flow and show that on the conformal spheres $(\mathbb{S}^n, [g_{\mathbb{S}^n}])$ it converges to the standard sphere up to a Möbius diffeomorphism. These arguments can be applied to obtain extinction profiles of solutions of some fractional porous medium equations, which are further used to improve a Sobolev inequality via a quantitative estimate of the remainder term.

Acknowledgements

My advisor, Professor YanYan Li, has influenced my life more than anyone else outside my family. His insights and inspirations taught me more about doing mathematics than I could ever have learned without him. I am very grateful for his invaluable guidance, encouragement, and unfailing patience throughout my Ph.D study. Working with him is my most valuable experience and priceless treasures to my future career. I own him a debt of gratitude that can never be repaid.

I would like to thank my thesis committee members Professor Zheng-Chao Han, Professor Natasa Sesum and Professor Yannick Sire for their help during my writing of this dissertation and comments during my graduate career. My appreciation also goes to Professor Haïm Brezis for his interests in our work and his strong support.

I wish to thank Professors Sagun Chanillo, Xiaojun Huang, Michael Kiessling, Feng Luo, Roger Nussbaum, Xiaochun Rong, Jian Song, and other professors at Rutgers for their help. I also thank Jingang Xiong for many discussions on mathematics. I appreciate Professor YanYan Li, Professor Vladimir Maz'ya, Professor Jean Van Schaftingen, and Jingang Xiong allowing me to include our joint work in my thesis.

I want to acknowledge to my fellows and friends Ellen Shiting Bao, Biao Yin, Luc Nguyen, Po Lam Yung, Longzhi Lin, Yuan Zhang, Seng Kee Chua, Ali Maalaoui, Yuan Yuan, Yu Wang, Haigang Li, Yunpeng Wang, Hui Wang, Tian Yang, Jingwei Yang, Wanke Yin, Min Xiao, Xukai Yan, Bin Guo, Jianguo Xiao and many other friends at Rutgers, for their help. Thanks go to my friends outside of Rutgers for their constant support and belief in me.

I thank the Department of Mathematics of Rutgers University for the support and the Graduate School for awarding me fellowships for the academic years 2010-2011 and 2011-2012.

I thank my classmates in my grade for helping me get used to the life in USA.

Finally, I would like to express my appreciation to my family for supporting me everyday, and special thanks to my wife, Ke Wang, for her love. She has walked with me along this path, making it bright. I consider myself extremely fortunate to have her as my wife.

Dedication

To my parents Ruigen Jin, and Zhoufen Ma.

To my wife Ke Wang.

Table of Contents

Chapter 1

Introduction

The Nirenberg problem concerns the following: For which positive function K on the standard sphere $(\mathbb{S}^n, g_{\mathbb{S}^n}), n \geq 2$, there exists a function w on $\mathbb{S}^n$ such that the scalar curvature (Gauss curvature in dimension $n = 2$) R_g of the conformal metric $g = e^w g_{\mathbb{S}^n}$ is equal to K on $\mathbb{S}^n$? The problem is equivalent to solving

$$-\Delta_{g_{\mathbb{S}^n}} w + 1 = K e^{2w}, \quad \text{on } \mathbb{S}^2,$$

and

$$-\Delta_{g_{\mathbb{S}^n}} v + c(n) R_0 v = c(n) K v^{\frac{n+2}{n-2}}, \quad \text{on } \mathbb{S}^n \text{ for } n \geq 3,$$

where $c(n) = (n-2)/(4(n-1)), R_0 = n(n-1)$ is the scalar curvature of $(\mathbb{S}^n, g_{\mathbb{S}^n})$ and $v = e^{\frac{n-2}{4} w}$.

The first work on this problem is by Koutroufiotis [91], where the solvability on $\mathbb{S}^2$ is established when K is assumed to be an antipodally symmetric function which is close to 1. Moser [112] established the solvability on $\mathbb{S}^2$ for all antipodally symmetric functions K which are positive somewhere. Without any symmetry assumption on K, sufficient conditions were given in dimension $n = 2$ by Chang and Yang [38] and [39], and in dimension $n = 3$ by Bahri and Coron [8]. Compactness of all solutions in dimensions $n = 2, 3$ can be found in work of Chang, Gursky and Yang [37], Han [73] and Schoen and Zhang [119]. In these dimensions, a sequence of solutions can not blow up at more than one point. Compactness and existence of solutions in higher dimensions were studied by Li in [95] and [96]. The situation is very different, as far as the compactness issues are concerned: In dimension $n \geq 4$, a sequence of solutions can blow up at more than one point, as shown in [96].

Attentions have been attracted to study similar questions for higher order curvatures such as the Q-curvatures, even on general Riemannian manifolds M^n. These involve the Paneitz operator which is a fourth order differential operator, and the GJMS operators (see [68]) P_k^g for all positive integers k if n is odd, and for $k \in \{1, \cdots, n/2\}$ if n is even. Moreover, P_1^g is the conformal Laplacian $L_g := -\Delta_g + c(n)R_g$ and P_2^g is the Paneitz operator. The construction in [68] is based on the ambient metric construction of [61]. Up to positive constants, $P_1^g(1)$ is the scalar curvature of g and $P_2^g(1)$ is the Q-curvature. One feature of these operators is that they are conformally invariant. In these directions of prescribing higher order curvatures, we refer to [52, 53, 54, 62, 131, 132] and references therein.

Recently, there is work by Qing and Raske [116], González, Mazzeo and Sire [66], González and Qing [67] about finding a conformal metric on certain given Riemannian manifolds (M, g) whose so-called σ-curvature (or fractional Q-curvature in some context) is constant, where σ is a fractional number and $\sigma \in (0, \frac{n}{2})$ except at most finite values. These σ-curvatures R_σ^g can be defined by $R_\sigma^g := P_\sigma^g(1)$, where P_σ^g is a normalized scattering operator (see Graham and Zworski [69], Chang and González [36]) on the conformal infinity of asymptotically hyperbolic manifolds (see Mazzeo [105], Mazzeo and Melrose [106]). Moreover, P_σ^g is conformally invariant. Chang and González in [36] reconciled the way of Graham and Zworski to define P_σ^g and the localization method of Caffarelli and Silvestre [29] for factional Laplacian $(-\Delta)^\sigma$ on the Euclidean space $\mathbb{R}^n$.

We focus on the typical case, that is the standard conformal spheres $(\mathbb{S}^n, [g_{\mathbb{S}^n}])$ which are the conformal infinity of Poincaré disks $(\mathbb{B}^{n+1}, g_{\mathbb{B}^{n+1}})$. In this case, σ-curvature can be expressed in the following explicit way. Let g be a representative in the conformal class $[g_{\mathbb{S}^n}]$ and write $g = v^{\frac{4}{n-2\sigma}} g_{\mathbb{S}^n}$, where v is positive and smooth on $\mathbb{S}^n$. Then we have

$$P_\sigma^g(\phi) = v^{-\frac{n+2\sigma}{n-2\sigma}} P_\sigma^{g_{\mathbb{S}^n}}(\phi v) \quad \text{for any } \phi \in C^\infty(\mathbb{S}^n), \tag{1.1}$$

and hence the σ-curvature for $(\mathbb{S}^n, g)$ can be computed as

$$R_\sigma^g = v^{-\frac{n+2\sigma}{n-2\sigma}} P_\sigma^{g_{\mathbb{S}^n}}(v). \tag{1.2}$$

$P_\sigma^{g_{\mathbb{S}^n}}$, which is simply written as P_σ, is an *intertwining operator* and has the formula (see, e.g., [17])

$$P_\sigma = \frac{\Gamma(B+\frac{1}{2}+\sigma)}{\Gamma(B+\frac{1}{2}-\sigma)}, \quad B = \sqrt{-\Delta_{g_{\mathbb{S}^n}} + \left(\frac{n-1}{2}\right)^2}, \tag{1.3}$$

where Γ is the Gamma function and $\Delta_{g_{\mathbb{S}^n}}$ is the Laplace-Beltrami operator on $(\mathbb{S}^n, g_{\mathbb{S}^n})$. Let $Y^{(k)}$ be a spherical harmonic of degree $k \geq 0$. Since $-\Delta_{g_{\mathbb{S}^n}} Y^{(k)} = k(k+n-1)Y^{(k)}$,

$$B\Big(Y^{(k)}\Big) = \left(k + \frac{n-1}{2}\right) Y^{(k)} \quad \text{and} \quad P_\sigma\Big(Y^{(k)}\Big) = \frac{\Gamma(k+\frac{n}{2}+\sigma)}{\Gamma(k+\frac{n}{2}-\sigma)} Y^{(k)}. \tag{1.4}$$

The operator P_σ can be seen more concretely on $\mathbb{R}^n$ using stereographic projection. The stereographic projection from $\mathbb{S}^n \backslash \{\mathcal{N}\}$ to $\mathbb{R}^n$ is the inverse of

$$F : \mathbb{R}^n \to \mathbb{S}^n \setminus \{\mathcal{N}\}, \quad x \mapsto \left(\frac{2x}{1+|x|^2}, \frac{|x|^2-1}{|x|^2+1}\right),$$

where $\mathcal{N}$ is the north pole of $\mathbb{S}^n$. Then it follows from the conformal invariance of P_σ that

$$(P_\sigma(\phi)) \circ F = |J_F|^{-\frac{n+2\sigma}{2n}} (-\Delta)^\sigma (|J_F|^{\frac{n-2\sigma}{2n}} (\phi \circ F)), \quad \text{for } \phi \in C^\infty(\mathbb{S}^n) \tag{1.5}$$

where

$$|J_F| = \left(\frac{2}{1+|x|^2}\right)^n,$$

and $(-\Delta)^\sigma$ is the fractional Laplacian operator (see, e.g., page 117 of [125]). It is also well-known (see, e.g., [111]) that P_σ is the inverse of the spherical Riesz potential

$$K^\sigma(f)(\xi) = c_{n,\sigma} \int_{\mathbb{S}^n} \frac{f(\zeta)}{|\xi-\zeta|^{n-2\sigma}} \, \mathrm{d}vol_{g_{\mathbb{S}^n}}(\zeta), \quad f \in L^p(\mathbb{S}^n) \tag{1.6}$$

where $c_{n,\sigma} = \frac{\Gamma(\frac{n-2\sigma}{2})}{2^{2\sigma}\pi^{n/2}\Gamma(\sigma)}$, $1 \leq p < \infty$ and $|\cdot|$ is the Euclidean distance in $\mathbb{R}^{n+1}$. On the other hand, the inverses of spherical Riesz potentials have been constructed in terms of singular integrals in [115]. When $\sigma \in (0,1)$, Pavlov and Samko [115] showed that if $v = K^\sigma(f)$ for some $f \in L^p(\mathbb{S}^n)$, then

$$P_\sigma(v)(\xi) = P_\sigma(1)v(\xi) + c_{n,-\sigma} \int_{\mathbb{S}^n} \frac{v(\xi)-v(\zeta)}{|\xi-\zeta|^{n+2\sigma}} \, \mathrm{d}vol_{g_{\mathbb{S}^n}}(\zeta), \tag{1.7}$$

where $c_{n,-\sigma} = \frac{2^{2\sigma}\sigma\Gamma(\frac{n+2\sigma}{2})}{\pi^{\frac{n}{2}}\Gamma(1-\sigma)}$ and $\int_{\mathbb{S}^n}$ is understood as $\lim\limits_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}$ in $L^p(\mathbb{S}^n)$ sense.

From (1.2), we consider

$$P_\sigma(v) = c_{n,\sigma} K v^{\frac{n+2\sigma}{n-2\sigma}} \quad \text{on } \mathbb{S}^n, \tag{1.8}$$

where $c_{n,\sigma} = P_\sigma(1) = \frac{\Gamma(\frac{n}{2}+\sigma)}{\Gamma(\frac{n}{2}-\sigma)}$, and $K > 0$ is a continuous function on $\mathbb{S}^n$.

When $K = 1$, (1.8) is the Euler-Lagrange equation for a functional associated to the following sharp Sobolev inequality (see [10])

$$\left(\fint_{\mathbb{S}^n} |v|^{\frac{2n}{n-2\sigma}} \,\mathrm{d}vol_{g_{\mathbb{S}^n}}\right)^{\frac{n-2\sigma}{n}} \leq \frac{\Gamma(\frac{n}{2}-\sigma)}{\Gamma(\frac{n}{2}+\sigma)} \fint_{\mathbb{S}^n} vP_\sigma(v)\,\mathrm{d}vol_{g_{\mathbb{S}^n}}, \quad v \in H^\sigma(\mathbb{S}^n) \tag{1.9}$$

where $\fint_{\mathbb{S}^n} = \frac{1}{|\mathbb{S}^n|}\int_{\mathbb{S}^n}$ and $H^\sigma(\mathbb{S}^n)$ is the closure of $C^\infty(\mathbb{S}^n)$ under the norm

$$\int_{\mathbb{S}^n} vP_\sigma(v)\,\mathrm{d}vol_{g_{\mathbb{S}^n}}.$$

The extremal functions of (1.9) follows from [101] and some classifications of solutions of (1.8) with $K \equiv 1$ can be found in [41] and [97]. All positive solutions must be of the form

$$v_{\xi_0,\lambda}(\xi) = \left(\frac{2\lambda}{2+(\lambda^2-1)(1-\cos dist_{g_{\mathbb{S}^n}}(\xi,\xi_0))}\right)^{\frac{n-2\sigma}{2}}, \quad \xi \in \mathbb{S}^n \tag{1.10}$$

for some $\xi_0 \in \mathbb{S}^n$ and positive constant λ.

In general, (1.8) may have no positive solution, since if v is a positive solution of (1.8) with $K \in C^1(\mathbb{S}^n)$ then it has to satisfy the Kazdan-Warner type condition

$$\int_{\mathbb{S}^n} \langle \nabla_{g_{\mathbb{S}^n}} K, \nabla_{g_{\mathbb{S}^n}} \xi\rangle v^{\frac{2n}{n-2\sigma}}\,\mathrm{d}\xi = 0. \tag{1.11}$$

Consequently, if $K(\xi) = \xi_{n+1}+2$, (1.8) has no solutions. The proof of (1.11) is provided in Appendix 4.5.

We study (1.8) with $\sigma \in (0,1)$, a *fractional Nirenberg problem.* Throughout the thesis, we assume that $\sigma \in (0,1)$ without otherwise stated. The following is one of our main existence results, which will be proved in Section 4.2.

Theorem 1.1. *Let $K \in C^{1,1}(\mathbb{S}^n)$ be an antipodally symmetric function, and be positive somewhere on $\mathbb{S}^n$. If there exists a maximum point x_0 of K near which $K(x) = K(x_0) + o(|x-x_0|^d)$ for some $d \geq n-2\sigma$, then* (1.8) *has at least one positive C^2 solution.*

When $\sigma = 1$, the above theorem was proved by Escobar and Schoen [59] for $n \geq 3$. On $\mathbb{S}^2$, the existence of solutions of $-\Delta_{g_{\mathbb{S}^n}} v + 1 = Ke^{2v}$ for such K was proved by Moser [112].

Our local analysis of solutions of (1.8) relies on a localization method introduced by Caffarelli and Silvestre in [29] for the factional Laplacian $(-\Delta)^\sigma$ on the Euclidean space $\mathbb{R}^n$, through which (1.8) is connected to a degenerate elliptic differential equation in one dimension higher

$$\begin{cases} \operatorname{div}\big(t^{1-2\sigma}\nabla U(x,t)\big) = 0, & \forall\ (x,t) \in \mathbb{R}^{n+1}_+, \\ -\lim\limits_{t\to 0} t^{1-2\sigma}\partial_t U(x,t) = K(x)U(x,0)^p, & \forall\ x \in \partial\mathbb{R}^{n+1}_+, \end{cases} \tag{1.12}$$

with $p \le \frac{n+2\sigma}{n-2\sigma}$. This leads us to establish regularity and isolated singularity properties for solutions of such degenerate equations in Chapter 3 and a Liouville theorem in Section 4.1, which are used to prove existence and compactness of solutions of (1.8) in Chapter 4. These results are in the joint work [82, 83] with Yanyan Li and Jingang Xiong.

We also study P_σ and σ-curvatures in the evolution equation point of view. Let us recall the Yamabe flow first.

Let (M, g_0) be a compact Riemannian manifold of dimension $n \ge 2$. The following evolution equation for the metric g

$$\frac{\partial}{\partial t} g(t) = -(R_{g(t)} - r_{g(t)})g(t), \quad g(0) = g_0 \tag{1.13}$$

was introduced by Hamilton in [72], and is known as the Yamabe flow. Here, $R_{g(t)}$ is the scalar curvature of $g(t)$ and $r_{g(t)} = vol_{g(t)}(M)^{-1}\int_M R_{g(t)}\mathrm{d}vol_{g(t)}$ is the average of $R_{g(t)}$. The existence and convergence of solutions of (1.13) were established through [72], [43], [134], [120], [20] and [22]. Some higher integer order curvature flows involving the Paneitz operator or the GJMS operators P^g_k, such as Q-curvature flow, have been studied in [19, 104, 9, 21, 76] and so on. We study some flow of this fractional order curvature R^g_σ associated with P^g_σ on the standard conformal sphere $(\mathbb{S}^n, [g_{\mathbb{S}^n}])$, which is the conformal infinity of the Poincaré disk.

Consider the normalized total σ-curvature functional

$$S(g) = vol_g(\mathbb{S}^n)^{\frac{2\sigma-n}{n}} \int_{\mathbb{S}^n} R^g_\sigma \,\mathrm{d}vol_g, \quad g \in [g_{\mathbb{S}^n}].$$

The negative gradient flow of S takes the form

$$\frac{\partial g}{\partial t} = -\frac{n-2\sigma}{2n}(vol_g(\mathbb{S}^n))^{\frac{2\sigma-n}{n}}(R^g_\sigma - r^g_\sigma)g,$$

where r_σ^g is the average of R_σ^g. It is easy to verify that this flow preserves the conformal class and the volume of $\mathbb{S}^n$. By a rescaling of the time variable, we obtain the following evolution equation

$$\frac{\partial g}{\partial t} = -(R_\sigma^g - r_\sigma^g)g. \tag{1.14}$$

If we write $g(t) = v^{\frac{4}{n-2\sigma}}(\cdot,t)g_{\mathbb{S}^n}$, then after rescaling the time variable, (1.14) can be written in an equivalent form

$$\frac{\partial v^N}{\partial t} = -P_\sigma(v) + r_\sigma^g v^N, \quad \text{on } \mathbb{S}^n, \tag{1.15}$$

where $N = (n+2\sigma)/(n-2\sigma)$.

Let $\mathcal{N}$ be the north pole of $\mathbb{S}^n$ and F be the inverse of stereographic projection from $\mathbb{S}^n \backslash \{\mathcal{N}\}$ to $\mathbb{R}^n$. Then $u(x,t) := |J_F|^{\frac{n-2\sigma}{2n}} v(F(x),t)$ satisfies

$$\frac{\partial u^N}{\partial t} = -(-\Delta)^\sigma u + r_\sigma^g u^N, \quad \text{in } \mathbb{R}^n. \tag{1.16}$$

We will call (1.14), (1.15) or (1.16) as a (normalized) *fractional Yamabe flow* when $\sigma \in (0,1)$. The following is a long time existence and convergence result which will be proved in Chapter 5.

Theorem 1.2. *Let $g(0) \in [g_{\mathbb{S}^n}]$ be a smooth metric on $\mathbb{S}^n$ for $n \geq 2$. Then the fractional Yamabe flow* (1.14) *with initial metric $g(0)$ exists for all time $0 < t < \infty$. Furthermore, there exists a smooth metric $g_\infty \in [g_{\mathbb{S}^n}]$ such that*

$$R_\sigma^{g_\infty} = r_\sigma^{g_\infty} \quad \text{and} \quad \lim_{t\to\infty} \|g(t) - g_\infty\|_{C^l(\mathbb{S}^n)} = 0$$

for all positive integers l.

As observed in [36] that the operator $P_{1/2}^g$ is related to the Yamabe problem on manifolds with boundary (see, e.g., [42, 56, 57, 74]), this fractional Yamabe flow (1.14) with $\sigma = 1/2$ is related to some generalization of Yamabe flow for manifolds with boundary studied in [18].

We also consider the unnormalized fractional Yamabe flow

$$\frac{\partial v^N}{\partial t} = -P_\sigma(v) \quad \text{on } \mathbb{S}^n \times (0,\infty), \quad \text{or} \quad \frac{\partial u^N}{\partial t} = -(-\Delta)^\sigma u \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

The second one is a fractional porous medium equation studied, e.g., in [3, 48, 33, 49, 89], where it is taken the form

$$\begin{cases} u_t = -(-\Delta)^\sigma(|u|^{m-1}u) \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^n, \end{cases} \tag{1.17}$$

with $m = \frac{n-2\sigma}{n+2\sigma}$, $\sigma \in (0,1)$. Models of this kind of fractional diffusion equations arise, e.g., in statistical mechanics [79, 80, 81] and heat control [3].

We are interested in analyzing the exact behavior of solutions of (1.17) near the extinction time for fast decaying initial data. In the classical case $\sigma = 1$, the extinction profiles of solutions of porous medium equations have been described in the results of [63, 50, 46, 13, 14] and so on. We obtain the asymptotic behaviors of solutions of (1.17) in Theorem 5.3 when t approaches the extinction time T.

An application of Theorem 5.3 is an improvement of some Sobolev inequality. A sharp form of the standard Sobolev inequality in $\mathbb{R}^n$ $(n \geq 3)$ asserts that

$$S_n\|\nabla u\|_{L^2(\mathbb{R}^n)} - \|u\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \geq 0 \tag{1.18}$$

for all $u \in \dot{H}^1(\mathbb{R}^n) = \{u \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) : \nabla u \in L^2(\mathbb{R}^n)\}$, where S_n is the sharp constant obtained in [4] and [127].

There have been many results on remainder terms of Sobolev inequalities (see, e.g., [26, 25, 12, 45, 35, 55]), which give various lower bounds of the left-handed side of (1.18).

For any $\sigma \in (0,1)$, the Sobolev inequality (see, e.g., [125] or [51]) asserts that

$$\|u\|^2_{L^{2^*(\sigma)}} \leq S_{n,\sigma}\|u\|^2_{\dot{H}^\sigma}, \quad \forall\ u \in \dot{H}^\sigma(\mathbb{R}^n) \tag{1.19}$$

where $2^*(\sigma) = \frac{2n}{n-2\sigma}$, $S_{n,\sigma}$ is the optimal constant and $\dot{H}^\sigma(\mathbb{R}^n)$ is the closure of $C_c^\infty(\mathbb{R}^n)$ under the norm

$$\|u\|_{\dot{H}^\sigma} = \|(-\Delta)^{\sigma/2}u\|_{L^2(\mathbb{R}^n)}. \tag{1.20}$$

The optimal constant $S_{n,\sigma}$ in the Sobolev inequality (1.19) is obtained by Lieb [101] and is achieved by $u(x) = \left(1+|x|^2\right)^{-\frac{n-2\sigma}{2}}$. The Hardy-Littlewood-Sobolev inequality

$$S_{n,\sigma}\|u\|^2_{L^{\frac{2n}{n+2\sigma}}} \geq \int_{\mathbb{R}^n} u(-\Delta)^{-\sigma}u\,\mathrm{d}x, \quad \forall\ u \in L^{\frac{2n}{n+2\sigma}}(\mathbb{R}^n) \tag{1.21}$$

involves the same optimal constant $S_{n,\sigma}$, where $(-\Delta)^{-\sigma}$ is a Riesz potential defined by

$$(-\Delta)^{-\sigma}u(x) = c_{n,\sigma}\int_{\mathbb{R}^n}\frac{u(y)}{|x-y|^{n-2\sigma}}\mathrm{d}y. \tag{1.22}$$

We will improve (1.19) via a quantitative estimate of the remainder term, which is stated in Theorem 5.6.

The operators P_σ and $(-\Delta)^\sigma$ are nonlocal, pseudo-differential operators. Generally speaking, strong maximum principles and Harnack inequalities might fail for nonlocal operators, see, e.g., a counterexample in [87]. The counterexample in [87] shows that the local non-negativity of solutions of certain nonlocal equations is not enough to guarantee local strong maximum principles and Harnack inequalities. However, if solutions are assumed to be globally nonnegative, then various strong maximum principles and Harnack inequalities have been obtained in, e.g., [27], [128] and [82].

We establish a strong maximum principle and a Hopf lemma for odd solutions of some linear nonlocal parabolic equations, which should be of independent interest. Our proofs make use of the expression (5.1) of $(-\Delta)^\sigma$. The odd function in Lemma 5.4 will serve as a barrier function, which allows us to obtain a Hopf lemma.

These results on fractional Yamabe flows are in the joint work [85] with Jingang Xiong.

My thesis also contains the next chapter on solutions of elliptic equations in divergence form with continuous coefficients, the joint work [84] with Maz'ya and Van Schaftingen, in which we solved some open problems posed by H. Brezis [1, 23].

Chapter 2

Elliptic problems in divergence form with continuous coefficients

2.1 Regularity

Let $\Omega \subset \mathbb{R}^n, n \geq 2$, be a domain, i.e., a bounded connected open set in $\mathbb{R}^n$. Consider the equation

$$-\operatorname{div} A\nabla u = 0 \quad \text{in } \Omega, \tag{2.1}$$

where $A : \Omega \to \mathbb{R}^{n\times n}$ is bounded, measurable and uniformly elliptic, i.e.,

$$\lambda|\xi|^2 \leq (A(x)\xi)\cdot\xi \leq \Lambda|\xi|^2, \quad \xi \in \mathbb{R}^n,$$

with $0 < \lambda < \Lambda < \infty$ for every $x \in \Omega$. One can define a weak solution $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ of (2.1) by requiring that for every $\varphi \in C^1_c(\Omega)$,

$$\int_\Omega (A\nabla u)\cdot\nabla\varphi = 0.$$

We are interested in the regularity properties of u. A fundamental result of De Giorgi [47] states that if u is a weak solution of (2.1) and moreover $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, then u is locally Hölder continuous. In particular, u is then locally bounded. In the same direction, Meyers [107] also proved that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ for some $p > 2$.

Serrin [121] showed that the assumption $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is essential in De Giorgi's result by constructing for every $p \in (1,2)$ a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ that solves such an elliptic equation but which is not locally bounded. In these counterexamples, A is not continuous. Serrin [121] conjectured that if A was Hölder continuous, then any weak solution $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is in $W^{1,2}_{\mathrm{loc}}(\Omega)$, and one can then apply De Giorgi's theory.

This conjecture was confirmed for $u \in W^{1,p}(\Omega)$ with $p > 1$ by Hager and Ross [71], and recently, solved completely by Brezis [1, 23] for $u \in W^{1,1}(\Omega)$. The proof of Brezis extends to the case where A is Dini continuous, i.e., the modulus of continuity of A

$$\omega_A(t,\Omega) = \sup_{\substack{x,y\in\Omega \\ |x-y|\le t}} |A(x)-A(y)|, \tag{2.2}$$

satisfies the Dini condition

$$\int_0^1 \frac{\omega_A(s,\Omega)}{s}\, ds < \infty. \tag{2.3}$$

In the case where $A \in C(\Omega;\mathbb{R}^{n\times n})$, which is merely continuous, Brezis obtained the following result.

Theorem 2.1 (Brezis [1, 23])**.** *Assume that $A \in C(\Omega;\mathbb{R}^{n\times n})$ is uniformly elliptic. If $u \in W^{1,p}_{\text{loc}}(\Omega)$ for some $p > 1$ is a weak solution of* (2.1)*, then $u \in W^{1,q}_{\text{loc}}(\Omega)$ for every $q \in [p, +\infty)$.*

Brezis asked two questions: Does Theorem 2.1 hold in the two limiting cases: $p = 1$ and/or $q = \infty$? The answer to both questions was known to be positive if A is Dini continuous. We answer both questions in the next section.

2.2 Some counterexamples

In the joint work [84] with Maz'ya and Van Schaftingen, we answered the above two questions raised by Brezis. We denote B_τ as the ball in $\mathbb{R}^n$ of radius τ centered at the origin. First we have

Proposition 2.1. *There exists $u \in W^{1,1}_{\text{loc}}(B_1)$ and a uniformly elliptic $A \in C(B_1;\mathbb{R}^{n\times n})$ such that u is a weak solution of* (2.1)*, but $u \notin W^{1,p}_{\text{loc}}(B_1)$ for every $p > 1$.*

Proposition 2.1 shows that Theorem 2.1 does not hold in the limiting case $p = 1$. As a byproduct, we obtain an answer to a further question (Open problem 3 in [1]) raised by Brezis.

Proposition 2.2. *There exists $A \in C(B_1;\mathbb{R}^{n\times n})$ such that the problem*

$$\begin{cases} -\operatorname{div}(A\nabla u) = 0 & \text{in } B_1, \\ \qquad\qquad\quad u = 0 & \text{on } \partial B_1 \end{cases} \tag{2.4}$$

has a nontrivial solution.

Brezis asked us the question whether those counterexamples could be improved by taking Du that belongs to $L\log L$ or to the Hardy space $\mathcal{H}^1$. Our construction in Proposition 2.1 answers the question.

Proposition 2.3. *There exists* $u \in W^{1,1}_{\mathrm{loc}}(B_1)$ *and a uniformly elliptic* $A \in C(B_1;\mathbb{R}^{n\times n})$ *such that* u *is a weak solution of* (2.1), $Du \in (L\log L)_{\mathrm{loc}}(B_1)$ *but* $u \notin W^{1,p}_{\mathrm{loc}}(B_1)$ *for every* $p > 1$*.*

In particular, in this case, Du belongs locally to the Hardy space $\mathcal{H}^1$ (see [124]).

Concerning the possibility of Lipschitz estimates, we have

Proposition 2.4. *There exists* $u \in W^{1,1}_{\mathrm{loc}}(B_1)$ *and a uniformly elliptic* $A \in C(B_1;\mathbb{R}^{n\times n})$ *such that* u *is a weak solution of* (2.1), $u \in W^{1,p}_{\mathrm{loc}}(B_1)$ *for every* $p > 1$, $Du \in \mathrm{BMO}_{\mathrm{loc}}(B_1)$ *but* $u \notin W^{1,\infty}_{\mathrm{loc}}(B_1)$*.*

Proposition 2.4 shows that Theorem 2.1 does not hold in the limiting case $q = \infty$. Brezis asked whether $Du \in BMO_{\mathrm{loc}}(B_1)$ for any weak solution u of (2.1) as in Theorem 2.1. The answer is still negative.

Proposition 2.5. *There exists* $u \in W^{1,1}_{\mathrm{loc}}(B_1)$ *and a uniformly elliptic* $A \in C(B_1;\mathbb{R}^{n\times n})$ *such that* u *is a weak solution of* (2.1), $u \in W^{1,p}_{\mathrm{loc}}(B_1)$ *for every* $p \in (1,\infty)$ *but* $Du \notin \mathrm{BMO}_{\mathrm{loc}}(B_1)$*.*

The construction of the counterexamples are made by explicit formulas, inspired by the construction of Serrin [121]. They can also be obtained from asymptotic formulas of Kozlov and Maz'ya [92, 93]. Our counterexamples rely on the following computational lemma.

Lemma 2.1. *Let* $v \in C^2((0,\tau))$ *and* $\alpha \in C^1((0,\tau))$*. Define* $A(x) = (a_{ij}(x))_{\substack{1\le i\le n\\ 1\le j\le n}}$ *by*

$$a_{ij}(x) = \delta_{ij} + \alpha(|x|)\Big(\delta_{ij} - \frac{x_i x_j}{|x|^2}\Big).$$

Then for every $x \in B_\tau \setminus \{0\}$,

$$\operatorname{div}\big(A(x)\nabla(x_1 v(|x|))\big) = x_1\Big(v''(|x|) + \frac{n+1}{|x|}v'(|x|) - \frac{n-1}{|x|^2}\alpha(|x|)v(|x|)\Big). \tag{2.5}$$

Proof. One has,

$$
\begin{aligned}
&\operatorname{div}\big(A(x)\nabla(x_1 v(|x|))\big)\\
&= x_1 \operatorname{div}\big(A(x)\nabla v(|x|)\big) + 2\nabla x_1 \cdot \big(A(x)\nabla v(|x|)\big) + v(|x|)\operatorname{div}\big(A(x)\nabla x_1\big).
\end{aligned}
$$

For the first two terms, one notices that $A(x)\nabla v(|x|) = \nabla v(|x|)$, and hence

$$
x_1 \operatorname{div}\big(A(x)\nabla v(|x|)\big) + 2\nabla x_1 \cdot \big(A(x)\nabla v(|x|)\big) = x_1\big(v''(|x|) + \frac{n+1}{|x|}v'(|x|)\big).
$$

For the last term, one has

$$
\operatorname{div}\big(A(x)\nabla x_1\big) = \operatorname{div}\big(\alpha(|x|)(e_1 - \frac{x_1 x}{|x|^2})\big) = -\frac{(n-1)x_1}{|x|^2}\alpha(|x|).
$$

The desired formula follows by adding these two together. □

Remark 2.1. *If P is a homogeneous harmonic polynomial of degree k, the formula generalizes to*

$$
\begin{aligned}
&\operatorname{div}\big(A(x)\nabla(P(x)v(|x|))\big)\\
&= P(x)\Big(v''(|x|) + \frac{n+2k-1}{|x|}v'(|x|) - \frac{k(n+k-2)}{|x|^2}\alpha(|x|)v(|x|)\Big).
\end{aligned}
$$

Proof of Proposition 2.1. Choose $\beta > 1$, and define for some $r_0 > 1$, for $r \in (0,1)$,

$$
v(r) = \frac{1}{r^n(\log\frac{r_0}{r})^\beta}. \tag{2.6}
$$

One takes then

$$
\alpha(r) = \frac{r^2 v''(r) + (n+1)rv'(r)}{(n-1)v(r)} = \frac{-\beta n}{(n-1)\log\frac{r_0}{r}} + \frac{\beta(\beta+1)}{(n-1)\big(\log\frac{r_0}{r}\big)^2}. \tag{2.7}
$$

One can take r_0 large enough so that $\alpha \geq -\frac{1}{2}$ on $(0,1)$; the coefficient matrix A is then uniformly elliptic. Define now $u(x) = x_1 v(|x|)$. One checks that $u \in W^{1,1}(B_1)$ and that u is a weak solution of (2.1). Indeed, it is a classical solution on $B_1 \setminus \{0\}$ by Lemma 2.1. Taking, $\varphi \in C_c^1(B_1)$ and $\rho \in (0,1)$, and integrating by parts we obtain

$$
\begin{aligned}
\int_{B_1\setminus B_\rho} \nabla\varphi \cdot (A\nabla u) &= -\int_{\partial B_\rho} \varphi\nabla u \cdot (A\frac{x}{\rho})\\
&= -\int_{\partial B_\rho} \varphi\nabla u \cdot \frac{x}{\rho}\\
&= -\int_{\partial B_\rho} \varphi x_1\Big(\frac{v(\rho)}{\rho} + v'(\rho)\Big)\\
&= -\int_{\partial B_\rho} (\varphi(x) - \varphi(0))x_1\Big(\frac{v(\rho)}{\rho} + v'(\rho)\Big).
\end{aligned}
$$

Since $\varphi \in C^1_c(B_1)$, one has

$$\Big|\int_{B_1\setminus B_\rho} \nabla\varphi\cdot(A\nabla v)\Big| \le C\rho^n(|v(\rho)| + \rho|v'(\rho)|).$$

Since the right-hand side goes to 0 as $\rho \to 0$, u is a weak solution. □

Remark 2.2. *The examples constructed in the case of merely measurable coefficients by Serrin [121] to show that a solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ need not be in $W^{1,2}_{\mathrm{loc}}(\Omega)$ and by Meyers [107] to show that for every $p > 2$, that a solution in $W^{1,2}_{\mathrm{loc}}(\Omega)$ need not be in $W^{1,p}_{\mathrm{loc}}(\Omega)$ can be recovered with the same construction, by taking $v(r) = r^\alpha$. The ellipticity condition requires $\alpha < n-1$ or $\alpha > 1$. This covers all the cases when $n = 2$; a descent argument finishes the construction in higher dimension.*

Proof of Proposition 2.3. One checks that the counterexample constructed in the proof of Proposition 2.1 satisfies $Du \in (L\log L)_{\mathrm{loc}}(B_1)$ when $\beta > 2$. □

Similar examples can be obtained following the results of Kozlov and Maz'ya [93]. By (4) therein, if $A \in C(B_1;\mathbb{R}^{n\times n})$, $A(Rx) = RA(x)R$ where R is the reflection with respect to the x_1 variable and A satisfies some regularity assumptions, then the equation $-\operatorname{div}(A\nabla u) = 0$ has a solution that is odd with respect to the x_1 variable and that behaves like

$$\frac{x_1}{|x|^n}\exp\Big(\int_{B_1\setminus B_{|x|}} \mathcal{R}(y)\,dy\Big)$$

around 0, where $\mathcal{R}$ is defined following [93, (3)] (The reader should correct the misprint in [93, (3)] and read $|S^{n-1}_+|$ instead of $|S^{n-1}|$.)

$$\begin{aligned}\mathcal{R}(x)&\\ &= \frac{(e_1\cdot(A(x)-A(0))e_1)(x\cdot A(0)^{-1}x) - n(e_1\cdot(A(x)-A(0))A(0)^{-1}x)(e_1\cdot x)}{|\partial B(0,1)||\det A(0)|^{\frac12}(x\cdot A(0)^{-1}x)^{\frac n2+1}}.\end{aligned} \tag{2.8}$$

Taking A as in Lemma 2.1 with $\lim_{r\to 0}\alpha(r) = 0$, one has $\mathcal{R}(x) = \alpha(|x|)(|x|^2 - {x_1}^2)/(|\partial B_1||x|^{n+2})$. Therefore, there is a solution that behaves like

$$\frac{x_1}{|x|^n}\exp\Big(\frac{n-1}{n}\int_{|x|}^1 \alpha(r)\,\frac{dr}{r}\Big).$$

In particular, if one takes $\alpha(r) = -\beta n/((n-1)\log\frac{r}{r_0})$, one obtains a solution that behaves like $\frac{x_1}{|x|^n}(\log\frac{r_0}{r})^{-\beta}$. One could also take $a_{ij}(x) = \delta_{ij} + \kappa(|x|)(\delta_{ij} - n\delta_{i1}\delta_{j1}\frac{{x_1}^2}{|x|^2})$ and continue the computations with now $\mathcal{R}(x) = \kappa(|x|)(|x|^2 - n{x_1}^2)^2/(|\partial B_1||x|^{n+2})$.

Proof of Proposition 2.2. Let u be given by the proof of Proposition 2.1. Notice that u is smooth on ∂B_1. Since A is bounded and elliptic, the problem

$$\begin{cases} -\operatorname{div}(A\nabla w) = 0 & \text{in } B_1, \\ \qquad\qquad w = u & \text{on } \partial B_1. \end{cases}$$

has a unique solution in $w \in W^{1,2}(B_1)$. Since $u \notin W^{1,2}(B_1)$, $u \neq w$. Hence, $u - w \in W^{1,1}(B_1)$ is a nontrivial solution of (2.4). $\square$

Proof of Proposition 2.4. Take for $r \in (0,1)$,

$$v(r) = \log\frac{r_0}{r} \tag{2.9}$$

and

$$\alpha(r) = \frac{1-(n+1)}{(n-1)\log\frac{r_0}{r}} = \frac{-n}{(n-1)\log\frac{r_0}{r}}, \tag{2.10}$$

where r_0 is chosen so that $\alpha(r) > -\frac{1}{2}$ on $(0,1)$. Defining $u(x) = x_1 v(|x|)$, one checks that $Du \in W^{1,p}_{\text{loc}}(B_1)$, $Du \in \text{BMO}(B_1)$, $u \notin W^{1,\infty}(B_1)$ and that u is a weak solution of (2.1). $\square$

As for the previous singular pathological solutions, similar examples can be obtained from results of Kozlov and Maz'ya for solutions [92]. By (4) therein if $A \in C(B_1; \mathbb{R}^{n\times n})$, $A(Rx) = RA(x)R$ where R is the reflection with respect to the x_1 variable and A satisfies some regularity assumptions, then the equation $-\operatorname{div}(A\nabla u) = 0$ has a solution in $W^{1,2}(B_1)$ that is odd with respect to to the x_1 variable and that behaves like

$$x_1 \exp\Big(-\int_{B_1\setminus B_{|x|}} \mathcal{R}(y)\,dy\Big)$$

around 0, where $\mathcal{R}$ is given by (2.8). Taking A as in Lemma 2.1 with $\alpha(r)$ as in (2.10), one recovers the counterexample presented above.

Proof of Proposition 2.5. Define for $r \in (0,1)$,

$$v(r) = \big(\log \frac{r_0}{r}\big)^2.$$

and

$$\alpha(r) = \frac{-2n}{(n-1)\log \frac{r_0}{r}} + \frac{2}{(n-1)(\log \frac{r_0}{r})^2}.$$

Defining $u(x) = x_1 v(|x|)$, one checks that $u \in W^{1,p}(B_1)$ for every $p > 1$ and that u is a weak solution of (2.1). One checks that for every $c > 0$, $\exp(c|Du|) \notin L^1(B_{1/2})$; hence by the John–Nirenberg embedding theorem [86] (see also e.g. [126, Chapter 4, §1.3]), $Du \notin \mathrm{BMO}(B_{1/2})$. $\square$

Chapter 3

Degenerate elliptic equations in divergence form

3.1 A weighted Sobolev space

Let $\sigma \in (0,1)$, $X = (x,t) \in \mathbb{R}^{n+1}$ where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Then $|t|^{1-2\sigma}$ belongs to the Muckenhoupt A_2 class in $\mathbb{R}^{n+1}$, namely, there exists a positive constant C, such that for any ball $\mathcal{B} \subset \mathbb{R}^{n+1}$

$$\left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} |t|^{1-2\sigma}\,\mathrm{d}X\right)\left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} |t|^{2\sigma-1}\,\mathrm{d}X\right) \leq C.$$

Let D be an open set in $\mathbb{R}^{n+1}$. Denote $L^2(|t|^{1-2\sigma}, D)$ as the Banach space of all measurable functions U, defined on D, for which

$$\|U\|_{L^2(|t|^{1-2\sigma},D)} := \left(\int_D |t|^{1-2\sigma} U^2\,\mathrm{d}X\right)^{\frac{1}{2}} < \infty.$$

We say that $U \in H(|t|^{1-2\sigma}, D)$ if $U \in L^2(|t|^{1-2\sigma}, D)$, and its weak derivatives ∇U exist and belong to $L^2(|t|^{1-2\sigma}, D)$. The norm of U in $H(|t|^{1-2\sigma}, D)$ is given by

$$\|U\|_{H(|t|^{1-2\sigma},D)} := \left(\int_D |t|^{1-2\sigma} U^2(X)\,\mathrm{d}X + \int_D |t|^{1-2\sigma}|\nabla U(X)|^2\,\mathrm{d}X\right)^{\frac{1}{2}}.$$

It is clear that $H(|t|^{1-2\sigma}, D)$ is a Hilbert space with the inner product

$$\langle U, V\rangle := \int_D |t|^{1-2\sigma}(UV + \nabla U \nabla V)\,\mathrm{d}X.$$

Note that the set of smooth functions $C^\infty(D)$ is dense in $H(|t|^{1-2\sigma}, D)$. Moreover, if D is a domain, i.e. a bounded connected open set, with Lipschitz boundary ∂D, then there exists a linear, bounded extension operator from $H(|t|^{1-2\sigma}, D)$ to $H(|t|^{1-2\sigma}, \mathbb{R}^{n+1})$ (see, e.g., [44]).

Let Ω be an open set in $\mathbb{R}^n$. Recall that $H^\sigma(\Omega)$ is the fractional Sobolev space defined as

$$H^\sigma(\Omega) := \left\{u \in L^2(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{2}+\sigma}} \in L^2(\Omega \times \Omega)\right\}$$

with the norm

$$\|u\|_{H^\sigma(\Omega)} := \left(\int_\Omega u^2 \,\mathrm{d}x + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\sigma}} \,\mathrm{d}x \,\mathrm{d}y \right)^{1/2}.$$

The set of smooth functions $C^\infty(\Omega)$ is dense in $H^\sigma(\Omega)$. If Ω is a domain with Lipschitz boundary, then there exists a linear, bounded extension operator from $H^\sigma(\Omega)$ to $H^\sigma(\mathbb{R}^n)$. Note that $H^\sigma(\mathbb{R}^n)$ with the norm $\|\cdot\|_{H^\sigma(\mathbb{R}^n)}$ is equivalent to the following space

$$\left\{ u \in L^2(\mathbb{R}^n) : |\xi|^\sigma \mathscr{F}(u)(\xi) \in L^2(\mathbb{R}^n) \right\}$$

with the norm

$$\|\cdot\|_{L^2(\mathbb{R}^n)} + \||\xi|^\sigma \mathscr{F}(\cdot)(\xi)\|_{L^2(\mathbb{R}^n)},$$

where $\mathscr{F}$ denotes the Fourier transform operator. It is known that (see, e.g., [102]) there exists $C > 0$ depending only on n and σ such that for $U \in H(t^{1-2\sigma}, \mathbb{R}^{n+1}_+) \cap C(\overline{\mathbb{R}^{n+1}_+})$, $\|U(\cdot, 0)\|_{H^\sigma(\mathbb{R}^n)} \leq C\|U\|_{H(t^{1-2\sigma}, \mathbb{R}^{n+1}_+)}$. Hence by a standard density argument, every $U \in H(t^{1-2\sigma}, \mathbb{R}^{n+1}_+)$ has a well-defined trace $u := U(\cdot, 0) \in H^\sigma(\mathbb{R}^n)$.

We define $\dot{H}^\sigma(\mathbb{R}^n)$ as the closure of the set $C_c^\infty(\mathbb{R}^n)$ of compact supported smooth functions under the norm

$$\|u\|_{\dot{H}^\sigma(\mathbb{R}^n)} = \||\xi|^\sigma \mathscr{F}(u)(\xi)\|_{L^2(\mathbb{R}^n)}.$$

Then there exists a constant C depending only on n and σ such that

$$\|u\|_{L^{\frac{2n}{n-2\sigma}}(\mathbb{R}^n)} \leq C\|u\|_{\dot{H}^\sigma(\mathbb{R}^n)} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n). \tag{3.1}$$

For any $u \in \dot{H}^\sigma(\mathbb{R}^n)$, set

$$U(x, t) = \mathcal{P}_\sigma[u] := \int_{\mathbb{R}^n} \mathcal{P}_\sigma(x - \xi, t) u(\xi) \,\mathrm{d}\xi, \quad (x, t) \in \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, +\infty), \tag{3.2}$$

where

$$\mathcal{P}_\sigma(x, t) = \beta(n, \sigma) \frac{t^{2\sigma}}{(|x|^2 + t^2)^{\frac{n+2\sigma}{2}}}$$

with constant $\beta(n, \sigma)$ such that $\int_{\mathbb{R}^n} \mathcal{P}_\sigma(x, 1) \,\mathrm{d}x = 1$. Then $U \in L^2(t^{1-2\sigma}, K)$ for any compact set K in $\overline{\mathbb{R}^{n+1}_+}$, $\nabla U \in L^2(t^{1-2\sigma}, \mathbb{R}^{n+1}_+)$ and $U \in C^\infty(\mathbb{R}^{n+1}_+)$. Moreover, U satisfies (see [29])

$$\mathrm{div}(t^{1-2\sigma} \nabla U) = 0 \quad \text{in } \mathbb{R}^{n+1}_+, \tag{3.3}$$

$$\|\nabla U\|_{L^2(t^{1-2\sigma},\mathbb{R}^{n+1}_+)} = N_\sigma \|u\|_{\dot{H}^\sigma(\mathbb{R}^n)}, \tag{3.4}$$

and

$$-\lim_{t\to 0} t^{1-2\sigma}\partial_t U(x,t) = N_\sigma(-\Delta)^\sigma u(x), \quad \text{in } \mathbb{R}^n \tag{3.5}$$

in distribution sense, where $N_\sigma = 2^{1-2\sigma}\Gamma(1-\sigma)/\Gamma(\sigma)$. We refer $U = \mathcal{P}_\sigma[u]$ in (3.2) to be the *extension* of u for any $u \in \dot{H}^\sigma(\mathbb{R}^n)$.

For a domain $D \subset \mathbb{R}^{n+1}$ with boundary ∂D, we denote $\partial' D$ as the interior of $\overline{D} \cap \partial\mathbb{R}^{n+1}_+$ in $\mathbb{R}^n = \partial\mathbb{R}^{n+1}_+$ and $\partial'' D = \partial D \setminus \partial' D$. We denote $Q_R = B_R \times (0,R)$ where $B_R \subset \mathbb{R}^n$ is the ball with radius R and centered at 0, $\mathcal{B}_R(X)$ as the ball in $\mathbb{R}^{n+1}$ with radius R and center X, $\mathcal{B}^+_R(X)$ as $\mathcal{B}_R(X) \cap \mathbb{R}^{n+1}_+$. We also write $\mathcal{B}_R(0), \mathcal{B}^+_R(0), B_R(0)$ as $\mathcal{B}_R, \mathcal{B}^+_R, B_R$ for short, respectively.

Lemma 3.1. *Let $u(x) \in C^\infty_c(\mathbb{R}^n)$ and $V(\cdot,t) = \mathcal{P}_\sigma(\cdot,t) * u(\cdot)$. For any $U \in C^\infty_c(\mathbb{R}^{n+1}_+ \cup \partial\mathbb{R}^{n+1}_+)$ with $U(x,0) = u(x)$,*

$$\int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma}|\nabla V|^2 \le \int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma}|\nabla U|^2.$$

Proof. Let $0 \le \eta(x,t) \le 1$, $Supp(\eta) \subset \mathcal{B}^+_{2R}$, $\eta = 1$ in $\mathcal{B}^+_R$ and $|\nabla\eta| \le 2/R$. In the end we will let $R \to \infty$ and hence we may assume that U is supported in $\overline{\mathcal{B}^+_{R/2}}$. Since $\mathrm{div}(t^{1-2\sigma}\nabla V) = 0$, then

$$\begin{aligned} 0 &= \int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma}\nabla V \nabla(\eta(U-V)) \\ &= \int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma}\eta\nabla U\nabla V - \int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma}\eta|\nabla V|^2 - \int_{\mathcal{B}^+_{2R}\setminus\mathcal{B}^+_R} t^{1-2\sigma} V\nabla\eta\nabla V, \end{aligned}$$

where we used $\eta(U-V) = 0$ on the boundary of $\mathcal{B}^+_{2R}$ in the first equality.

Note that for $(x,t) \in \mathcal{B}^+_{2R}\setminus\mathcal{B}^+_R$

$$\begin{aligned} |V(x,t)| &= \beta(n,\sigma)\left|\int_{\mathbb{R}^n} \frac{t^{2\sigma}}{(|x-\xi|^2+t^2)^{\frac{n+2\sigma}{2}}} u(\xi)\,d\xi\right| \\ &\le \beta(n,\sigma)\int_{\mathbb{R}^n} \frac{(|x|^2+t^2)^\sigma}{(|x|^2/4+t^2)^{\frac{n+2\sigma}{2}}} |u(\xi)|\,d\xi \\ &\le C(n,\sigma)(|x|^2+t^2)^{-\frac{n}{2}}\|u\|_{L^1}, \end{aligned}$$

where in the first inequality we have used that U is supported in $\overline{\mathcal{B}^+_{R/2}}$.

Direct computations yield

$$
\begin{aligned}
&\left|\int_{\mathcal{B}_{2R}^+\setminus\mathcal{B}_R^+} t^{1-2\sigma} V\nabla\eta\nabla V\right| \\
&\le \left(\int_{\mathcal{B}_{2R}^+\setminus\mathcal{B}_R^+} t^{1-2\sigma}|\nabla V|^2\right)^{1/2}\left(\int_{\mathcal{B}_{2R}^+\setminus\mathcal{B}_R^+} t^{1-2\sigma}V^2|\nabla\eta|^2\right)^{1/2} \\
&\le \left(\int_{\mathcal{B}_{2R}^+\setminus\mathcal{B}_R^+} t^{1-2\sigma}|\nabla V|^2\right)^{1/2} \\
&\quad \cdot C(n,\sigma)|u|_{L^1(\mathbb{R}^n)}(R^{n+2-2\sigma-2-2n})^{1/2} \to 0 \text{ as } R\to\infty,
\end{aligned}
$$

where we used (3.4) that $\int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma}|\nabla V|^2 < \infty$. Therefore, we have

$$
\int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma}|\nabla V|^2 \le \left|\int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma}\nabla U\nabla V\right|.
$$

Finally, by Hölder inequality,

$$
\int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma}|\nabla V|^2 \le \int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma}|\nabla U|^2.
$$

□

Proposition 3.1. *Let $D=\Omega\times(0,R)\subset\mathbb{R}^n\times\mathbb{R}_+$, $R>0$ and $\partial\Omega$ be Lipschitz.*

(i) If $U\in H(t^{1-2\sigma},D)\cap C(D\cup\partial' D)$, then $u:=U(\cdot,0)\in H^\sigma(\Omega)$, and

$$
\|u\|_{H^\sigma(\Omega)} \le C\|U\|_{H(t^{1-2\sigma},D)},
$$

where C is a positive constant depending only on n,σ,R and Ω. Hence every $U\in H(t^{1-2\sigma},D)$ has a well-defined trace $U(\cdot,0)\in H^\sigma(\Omega)$ on $\partial' D$. Furthermore, there exists $C>0$ depending only on n and σ such that

$$
\|U(\cdot,0)\|_{L^{\frac{2n}{n-2\sigma}}(\Omega)} \le C\|\nabla U\|_{L^2(t^{1-2\sigma},D)} \quad \textit{for all } U\in C_c^\infty(D\cup\partial' D). \tag{3.6}
$$

(ii) If $u\in H^\sigma(\Omega)$, then there exists $U\in H(t^{1-2\sigma},D)$ such that the trace of U on Ω equals to u and

$$
\|U\|_{H(t^{1-2\sigma},D)} \le C\|u\|_{H^\sigma(\Omega)},
$$

where C is a positive constant depending only on n,σ,R and Ω.

Proof. The above results are well-known and here we just sketch the proofs. For (i), by the previously mentioned result on the extension operator, there exists $\tilde{U} \in H(t^{1-2\sigma}, \mathbb{R}^{n+1})$ such that $\tilde{U} = U$ in D and

$$\|\tilde{U}\|_{H(t^{1-2\sigma},\mathbb{R}^{n+1})} \le C\|U\|_{H(t^{1-2\sigma},D)}.$$

Hence by the previously mentioned result on the trace from $H(t^{1-2\sigma}, \mathbb{R}^{n+1}_+)$ to $H^\sigma(\mathbb{R}^n)$, we have

$$\|u\|_{H^\sigma(\Omega)} \le \|\tilde{U}(\cdot,0)\|_{H^\sigma(\mathbb{R}^n)} \le C\|\tilde{U}\|_{H(t^{1-2\sigma},\mathbb{R}^{n+1}_+)} \le C\|U\|_{H(t^{1-2\sigma},D)}.$$

For (3.6), we extend U to be zero in the outside of $\overline{D}$ and let V be the extension of $U(\cdot,0)$ as in (3.2). The inequality (3.6) follows from (3.1), (3.4) and

$$\|\nabla V\|_{L^2(t^{1-2\sigma},\mathbb{R}^{n+1}_+)} \le \|\nabla U\|_{L^2(t^{1-2\sigma},\mathbb{R}^{n+1}_+)},$$

where Lemma 3.1 is used in the above inequality.

For (ii), since $\partial\Omega$ is Lipschitz, there exists $\tilde{u} \in H^\sigma(\mathbb{R}^n)$ such that $\tilde{u} = u$ in Ω and $\|\tilde{u}\|_{H^\sigma(\mathbb{R}^n)} \le C\|u\|_{H^\sigma(\Omega)}$. Then $U = \mathcal{P}_\sigma[u]$, the extension of $\tilde{u}$, satisfies (ii). □

3.2 Weak solutions of degenerate elliptic equations

3.2.1 Existence

Let D be a domain in $\mathbb{R}^{n+1}_+$ with $\partial' D \neq \emptyset$. Let $a \in L_{loc}^{\frac{2n}{n+2\sigma}}(\partial' D)$ and $b \in L^1_{loc}(\partial' D)$. Consider

$$\begin{cases} \mathrm{div}(t^{1-2\sigma}\nabla U(X)) = 0 & \text{in } D, \\ -\lim\limits_{t\to 0^+} t^{1-2\sigma}\partial_t U(x,t) = a(x)U(x,0) + b(x) & \text{on } \partial' D. \end{cases} \tag{3.7}$$

Definition 3.1. *We say that $U \in H(t^{1-2\sigma}, D)$ is a weak solution (resp. supersolution, subsolution) of* (3.7) *in D, if for every nonnegative $\Phi \in C_c^\infty(D \cup \partial' D)$*

$$\int_D t^{1-2\sigma}\nabla U \nabla \Phi = (resp.\ \ge, \le) \int_{\partial' D} aU\Phi + b\Phi. \tag{3.8}$$

Proposition 3.2. *Suppose $a(x) \in L^{\frac{n}{2\sigma}}(B_1)$, $b(x) \in L^{\frac{2n}{n+2\sigma}}(B_1)$. Let $U \in H(t^{1-2\sigma}, Q_1)$ be a weak solution of* (3.7) *in Q_1. There exists $\delta > 0$ depending only on n and σ such* *that if* $\|a^+\|_{L^{\frac{n}{2\sigma}}(B_1)} < \delta$*, then there exists a constant* C *depending only on* n, σ *and* δ *such that*

$$\|U\|_{H(t^{1-2\sigma},Q_{1/2})} \leq C(\|U\|_{L^2(t^{1-2\sigma},Q_1)} + \|b\|_{L^{\frac{2n}{n+2\sigma}}(B_1)}).$$

Consequently, if $a \in L^p(B_1)$ *for* $p > \frac{n}{2\sigma}$*, then* C *depends only on* $n, \sigma, \|a\|_{L^p(B_1)}$.

Proof. Let $\eta \in C_c^\infty(Q_1 \cup \partial' Q_1)$ be a cut-off function which equals to 1 in $Q_{1/2}$ and supported in $Q_{3/4}$. By a density argument, we can choose $\eta^2 U$ as a test function in (3.8). Then we have, by Cauchy-Schwarz inequality,

$$\int_{Q_1} t^{1-2\sigma}\eta^2|\nabla U|^2 \,\mathrm{d}X \leq 4\int_{Q_1} t^{1-2\sigma}|\nabla\eta|^2 U^2 \,\mathrm{d}X + 2\int_{\partial' Q_1} a^+(\eta U)^2 + b\eta^2 U \,\mathrm{d}x.$$

By Hölder inequality and Proposition 3.1,

$$\int_{\partial' Q_1} a^+(\eta U)^2 \,\mathrm{d}x \leq \delta\|\eta U\|^2_{L^{\frac{2n}{n-2\sigma}}(\partial' Q_1)} \leq \delta C(n,\sigma)\|\nabla(\eta U)\|^2_{L^2(t^{1-2\sigma},Q_1)}.$$

By Young's inequality $\forall\, \varepsilon > 0$,

$$\begin{aligned}\int_{\partial' Q_1} b\eta^2 U(\cdot,0) \,\mathrm{d}x &\leq \varepsilon\|\eta U\|^2_{L^{\frac{2n}{n-2\sigma}}(\partial' Q_1)} + C(\varepsilon)\|b\|^2_{L^{\frac{2n}{n+2\sigma}}(\partial' Q_1)} \\ &\leq \varepsilon C(n,\sigma)\|\nabla(\eta U)\|^2_{L^2(t^{1-2\sigma},Q_1)} + C(\varepsilon)\|b\|^2_{L^{\frac{2n}{n+2\sigma}}(\partial' Q_1)}.\end{aligned}$$

The first conclusion follows immediately if δ is sufficiently small.

If $a \in L^p(B_1)$, we can choose r small such that $\|a\|_{L^{\frac{n}{2\sigma}}(B_r(x_0))} < \delta$ for any ball $B_r(x_0) \subset B_1$. Then $\hat{U}(x,t) = r^{\frac{n-2\sigma}{2}} U(rx + x_0, rt)$ satisfies (3.7) with $\hat{a}(x) = r^{2\sigma} a(rx + x_0)$ and $\hat{b}(x,t) = r^{\frac{n+2\sigma}{2}} b(rx + x_0)$ in Q_1. Since $\|\hat{a}\|_{L^{\frac{n}{2\sigma}}(B_1)} < \delta$, applying the above result to $\hat{U}$, we have

$$\|U\|_{H(t^{1-2\sigma},B_{1/2}\times(0,r/2))} \leq C(\|U\|_{L^2(t^{1-2\sigma},Q_1)} + \|b\|_{L^{\frac{2n}{n+2\sigma}}(B_1)}),$$

where C depends only on $n, \sigma, \|a\|_{L^\infty(B_1)}$. This, together with the fact that (3.7) is uniformly elliptic in $B_1 \times (r/4, 1)$, finishes the proof. $\square$

Proposition 3.3. *Suppose that* $a(x) \in L^{\frac{n}{2\sigma}}(B_1)$*. There exists* $\delta > 0$ *which depends only on* n *and* σ *such that if* $\|a^+\|_{L^{\frac{n}{2\sigma}}(B_1)} < \delta$*, then for any* $b(x) \in L^{\frac{2n}{n+2\sigma}}(B_1)$*, there exists a unique solution in* $H(t^{1-2\sigma}, Q_1)$ *to* (3.7) *with* $U|_{\partial'' Q_1} = 0$.

Proof. We consider the bilinear form

$$B[U,V] := \int_{Q_1} t^{1-2\sigma}\nabla U \nabla V \,\mathrm{d}X - \int_{\partial' Q_1} aUV \,\mathrm{d}x, \quad U,V \in \mathcal{A}$$

where $\mathcal{A} := \{U \in H(t^{1-2\sigma}, Q_1) : U|_{\partial'' Q_1} = 0$ in trace sense$\}$. By Proposition 3.1, it is easy to verify that $B[\cdot,\cdot]$ is bounded and coercive provided δ is sufficiently small. Therefore, the proposition follows from the Riesz representation theorem. □

3.2.2 Maximum principles

Lemma 3.2. *Suppose $U \in H(t^{1-2\sigma}, D)$ is a weak supersolution of (3.7) in D with $a \equiv b \equiv 0$. If $U(X) \geq 0$ on $\partial'' D$ in trace sense, then $U \geq 0$ in D.*

Proof. Use U^- as a test function to conclude that $U^- \equiv 0$. □

Lemma 3.3. *There exists $\varepsilon = \varepsilon(n,\sigma)$ such that for all $|a(x)| \leq \varepsilon |x|^{-2\sigma}$, if $U \in H(t^{1-2\sigma}, Q_1)$, $U \geq 0$ on $\partial'' Q_1$, and is a supersolution of (3.7) in Q_1 with $b \equiv 0$, then*

$$U \geq 0 \quad \text{in } Q_1.$$

Proof. By a density argument, we can use U^- as a test function. Hence we have

$$\int_{Q_1} t^{1-2\sigma} |\nabla U^-|^2 \leq \int_{B_1} |a| (U^-(\cdot,0))^2. \tag{3.9}$$

We extend U^- to be zero outside of Q_1 in $\mathbb{R}^{n+1}_+$ and still denote it as U^-. Then the trace

$$U^-(\cdot,0) \in \dot{H}^\sigma(\mathbb{R}^n).$$

Since

$$N_\sigma^2 \|U^-(\cdot,0)\|^2_{\dot{H}^\sigma(\mathbb{R}^n)} = \int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma} |\nabla \mathcal{P}_\sigma * U^-(\cdot,0)|^2 \leq \int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma} |\nabla U^-|^2,$$

we have

$$N_\sigma^2 \|U^-(\cdot,0)\|^2_{\dot{H}^\sigma(\mathbb{R}^n)} \leq \int_{B_1} |a| (U^-(\cdot,0))^2.$$

By Hardy's inequality (see, e.g., [133])

$$C(n,\sigma) \int_{\mathbb{R}^n} |x|^{-2\sigma} (U^-(\cdot,0))^2 \leq \|U^-(\cdot,0)\|^2_{\dot{H}^\sigma(\mathbb{R}^n)},$$

where $C(n,\sigma) = 2^{2\sigma}\frac{\Gamma((n+2\sigma)/4)}{\Gamma((n-2\sigma)/4)}$ is the best constant. Hence if $\varepsilon < N_\sigma^2 C(n,\sigma)$, $U^-(\cdot,0) \equiv 0$ and hence by (3.9), $U^- \equiv 0$ in Q_1. $\square$

Lemma 3.4. *Let $a(x) \in L^\infty(B_1)$. Let $W \in C(\overline{Q_1}) \cap C^2(Q_1)$ satisfying $\nabla_x W \in C(\overline{Q_1})$, $t^{1-2\sigma}\partial_t W \in C(\overline{Q_1})$, and*

$$\begin{cases} -\mathrm{div}(t^{1-2\sigma}\nabla W) & \geq 0 \quad \text{in } Q_1, \\ -\lim\limits_{t\to 0} t^{1-2\sigma}\partial_t W(x,t) & \geq a(x)W(x,0) \quad \text{on } \partial' Q_1, \\ \qquad\qquad W & > 0 \quad \text{in } \overline{Q_1}. \end{cases} \tag{3.10}$$

If $U \in C(\overline{Q_1}) \cap C^2(Q_1)$ satisfying $\nabla_x U \in C(\overline{Q_1})$, $t^{1-2\sigma}\partial_t U \in C(\overline{Q_1})$, and

$$\begin{cases} -\mathrm{div}(t^{1-2\sigma}\nabla U) & \geq 0 \quad \text{in } Q_1, \\ -\lim\limits_{t\to 0} t^{1-2\sigma}\partial_t U(x,t) & \geq a(x)U(x,0) \quad \text{on } \partial' Q_1, \\ \qquad\qquad U & \geq 0 \quad \text{in } \partial'' Q_1, \end{cases} \tag{3.11}$$

then $U \geq 0$ in Q_1.

Proof. Let $V = U/W$. Then

$$\begin{cases} -\mathrm{div}(t^{1-2\sigma}\nabla V) - 2t^{1-2\sigma}\frac{\nabla V \nabla W}{W} - \frac{\mathrm{div}(t^{1-2\sigma}\nabla W)V}{W} & \geq 0 \quad \text{in } Q_1 \\ -\lim\limits_{t\to 0} t^{1-2\sigma}\partial_t V + \frac{V}{W}\big(-\lim\limits_{t\to 0} t^{1-2\sigma}\partial_t W(x,t) - a(x)W(x,0)\big) & \geq 0 \quad \text{on } \partial' Q_1 \\ \qquad\qquad V & \geq 0 \quad \text{in } \partial'' Q_1. \end{cases} \tag{3.12}$$

We are going to show that $V \geq 0$ in Q_1. If not, then we choose k such that $\inf_{Q_1} v < k \leq 0$. Let

$$V_k = V - k \quad \text{and } V_k^- = \max(-V_k, 0).$$

Multiplying V_k^- to (3.12), we have

$$\int_{Q_1} t^{1-2\sigma}|\nabla V_k^-|^2 \leq 2\int_{Q_1} t^{1-2\sigma}W^{-1}V_k^-\nabla V_k^-\nabla W. \tag{3.13}$$

Case 1: Suppose $1-2\sigma \leq 0$. Denote $\Gamma_k = Supp(\nabla V_k^-)$. Then by the Hölder inequality and the bounds of $\nabla_x W$, $t^{1-2\sigma}\partial_t W$,

$$2\int_{Q_1} t^{1-2\sigma}W^{-1}V_k^-\nabla V_k^-\nabla W \leq C\left(\int_{Q_1} t^{1-2\sigma}|\nabla V_k^-|^2\right)^{\frac{1}{2}}\left(\int_{\Gamma_k} t^{1-2\sigma}|V_k^-|^2\right)^{\frac{1}{2}}.$$

Hence it follows from (3.13) that

$$\int_{Q_1} t^{1-2\sigma}|\nabla V_k^-|^2 \le C \int_{\Gamma_k} t^{1-2\sigma}|V_k^-|^2. \tag{3.14}$$

Since $V_k^- = 0$ on $\partial'' Q_1$, by Lemma 2.1 in [128],

$$\left(\int_{Q_1} t^{1-2\sigma}|V_k^-|^{2(n+1)/n}\right)^{\frac{n}{n+1}} \le C \int_{Q_1} t^{1-2\sigma}|\nabla V_k^-|^2. \tag{3.15}$$

By (3.14), (3.15) and Hölder inequality,

$$\int_{\Gamma_k} t^{1-2\sigma} \ge C.$$

This yields a contradiction when $k \to \inf_{Q_1} v$, since $\nabla V = 0$ on the set of $V \equiv \inf_{Q_1} V$.

Case 2: Suppose $1 - 2\sigma > 0$. Denote $\Gamma_k = Supp(V_k^-)$. Then by Hölder inequality and the bounds of $\nabla_x W$, $t^{1-2\sigma}\partial_t W$,

$$\begin{aligned}
\int_{Q_1} t^{1-2\sigma}|\nabla V_k^-|^2 &\le 2\int_{Q_1} t^{1-2\sigma} W^{-1} V_k^- \nabla V_k^- \nabla W \\
&\le C\int_{Q_1} V_k^- \nabla V_k^- \\
&\le C\Big(\int_{Q_1} t^{1-2\sigma}|\nabla V_k^-|^2\Big)^{1/2}\Big(\int_{Q_1} t^{2\sigma-1}|V_k^-|^2\Big)^{1/2}.
\end{aligned}$$

Hence

$$\int_{Q_1} t^{1-2\sigma}|\nabla V_k^-|^2 \int_{Q_1} t^{1-2\sigma}|\nabla V_k^-|^2 \le C\int_{Q_1} t^{1-2\sigma}|\nabla V_k^-|^2 \int_{Q_1} t^{2\sigma-1}|V_k^-|^2.$$

Since $V_k^- = 0$ on $\partial'' Q_1$, by the proof of Lemma 2.3 in [128], for any $\beta > -1$,

$$\int_{Q_1} t^{\beta}|V_k^-|^2 \le C(\beta)\int_{Q_1} t^{1-2\sigma}|\nabla V_k^-|^2.$$

In the following we choose $\beta = \sigma - 1$. Hence,

$$\int_{Q_1} t^{1-2\sigma}|\nabla V_k^-|^2 \int_{Q_1} t^{\sigma-1}|V_k^-|^2 \le C\int_{Q_1} t^{1-2\sigma}|\nabla V_k^-|^2 \int_{Q_1} t^{2\sigma-1}|V_k^-|^2,$$

i.e.

$$\int_{\Gamma_k} t^{1-2\sigma}|\nabla V_k^-|^2 \int_{\Gamma_k} t^{\sigma-1}|V_k^-|^2 \le C\int_{\Gamma_k} t^{1-2\sigma}|\nabla V_k^-|^2 \int_{\Gamma_k} t^{2\sigma-1}|V_k^-|^2.$$

Fixed $\varepsilon > 0$ sufficiently small which will be chosen later. By the strong maximum principle $\inf_{Q_1} V$ has to be attained only on $\partial' Q_1$, then we can choose k sufficiently closed to $\inf_{Q_1} V$ such that $\Gamma_k \subset B_1 \times [0, \varepsilon]$. Then

$$\varepsilon^{-\sigma}\int_{\Gamma_k} t^{2\sigma-1}|V_k^-|^2 \le C\int_{\Gamma_k} t^{\sigma-1}|V_k^-|^2.$$

Choose ε small enough such that $\varepsilon^{-\sigma} > C+1$. It follows that

$$\int_{\Gamma_k} t^{1-2\sigma}|\nabla V_k^-|^2 \int_{\Gamma_k} t^{2\sigma-1}|V_k^-|^2 = 0.$$

Hence one of them has to be zero, which reaches a contradiction immediately. $\square$

3.3 Regularity

3.3.1 Harnack inequalities and Hölder estimates

The following result is a refined version of that in [128]. Such De Giorgi-Nash-Moser type theorems for degenerated equations with Dirichlet boundary conditions have been established in [60].

Proposition 3.4. *Suppose $a, b \in L^p(B_1)$ for some $p > \frac{n}{2\sigma}$.*

(i) Let $U \in H(t^{1-2\sigma}, Q_1)$ be a weak subsolution of (3.7) in Q_1. Then $\forall\ \nu > 0$

$$\sup_{Q_{1/2}} U^+ \leq C(\|U^+\|_{L^\nu(t^{1-2\sigma}, Q_1)} + \|b^+\|_{L^p(B_1)}),$$

where $U^+ = \max(0, U)$, and $C > 0$ depends only on n, σ, p, ν and $\|a^+\|_{L^p(B_1)}$.

(ii) Let $U \in H(t^{1-2\sigma}, Q_1)$ be a nonnegative weak supersolution of (3.7) in Q_1. Then for any $0 < \mu < \tau < 1$, $0 < \nu \leq \frac{n+1}{n}$ we have

$$\inf_{Q_\mu} U + \|b^-\|_{L^p(B_1)} \geq C\|U\|_{L^\nu(t^{1-2\sigma}, Q_\tau)},$$

where $C > 0$ depends only on $n, \sigma, p, \nu, \mu, \tau$ and $\|a^-\|_{L^p(B_1)}$.

(iii) Let $U \in H(t^{1-2\sigma}, Q_1)$ be a nonnegative weak solution of (3.7) in Q_1. Then we have the following Harnack inequality

$$\sup_{Q_{1/2}} U \leq C(\inf_{Q_{1/2}} U + \|b\|_{L^p(B_1)}), \tag{3.16}$$

where $C > 0$ depends only on $n, \sigma, p, \|a\|_{L^p(B_1)}$. Consequently, there exists $\alpha \in (0,1)$ depending only on $n, \sigma, p, \|a\|_{L^p(B_1)}$ such that any weak solution $U(X)$ of (3.7) is of $C^\alpha(\overline{Q_{1/2}})$. Moreover,

$$\|U\|_{C^\alpha(\overline{Q_{1/2}})} \leq C(\|U\|_{L^\infty(Q_1)} + \|b\|_{L^p(B_1)}),$$

where $C > 0$ depends only on $n, \sigma, p, \|a\|_{L^p(B_1)}$.

Proof. The proofs are modifications of those in [128], where the method of Moser iteration is used. Here we only point out the changes. Let $k = \|b^+\|_{L^p(B_1)}$ if $b^+ \not\equiv 0$, otherwise let $k > 0$ be any number which is eventually sent to 0. Define $\overline{U} = U^+ + k$ and, for $m > 0$, let

$$\overline{U}_m = \begin{cases} \overline{U} & \text{if } U < m, \\ k + m & \text{if } U \geq m. \end{cases}$$

Consider the test function

$$\phi = \eta^2(\overline{U}_m^\beta \overline{U} - k^{\beta+1}) \in H(t^{1-2\sigma}, Q_1),$$

for some $\beta \geq 0$ and some nonnegative function $\eta \in C_c^1(Q_1 \cup \partial' Q_1)$. Direction calculations yield that, with setting $W = \overline{U}_m^{\frac{\beta}{2}} \overline{U}$,

$$\frac{1}{1+\beta}\int_{Q_1} t^{1-2\sigma}|\nabla(\eta W)|^2 \leq 16\int_{Q_1} t^{1-2\sigma}|\nabla \eta|^2 W^2 + 4\int_{\partial' Q_1}(a^+ + \frac{b^+}{k})\eta^2 W^2. \quad (3.17)$$

By Hölder's inequality and the choice of k, we have

$$\int_{\partial' Q_1}(a^+ + \frac{b^+}{k})\eta^2 W^2 \leq (\|a^+\|_{L^p(B_1)} + 1)\|\eta^2 W^2\|_{L^{p'}(B_1)},$$

where $p' = \frac{p}{p-1} < \frac{n}{n-2\sigma}$. Choose $0 < \theta < 1$ such that $\frac{1}{p'} = \theta + \frac{(1-\theta)(n-2\sigma)}{n}$. The interpolation inequality gives that, for any $\varepsilon > 0$,

$$\|\eta^2 W^2\|_{L^{p'}(B_1)} \leq \varepsilon \|\eta W\|^2_{L^{\frac{2n}{n-2\sigma}}(B_1)} + \varepsilon^{-\frac{1-\theta}{\theta}}\|\eta^2 W^2\|_{L^1(B_1)}.$$

By the trace embedding inequality in Proposition 3.1, there exists $C > 0$ depending only on n, σ such that

$$\|\eta W\|^2_{L^{\frac{2n}{n-2\sigma}}(B_1)} \leq C\int_{Q_1} t^{1-2\sigma}|\nabla(\eta W)|^2.$$

By Lemma 2.3 in [128], there exist $\delta > 0$ and $C > 0$ both of which depend only on n, σ such that

$$\|\eta^2 W^2\|_{L^1(B_1)} \leq \varepsilon^{\frac{1}{\theta}}\int_{Q_1} t^{1-2\sigma}|\nabla(\eta W)|^2 + \varepsilon^{-\frac{\delta}{\theta}}\int_{Q_1} t^{1-2\sigma}\eta^2 W^2.$$

By choosing ε small, the above inequalities give that

$$\int_{Q_1} t^{1-2\sigma}|\nabla(\eta W)|^2 \leq C(1+\beta)^{\delta/\theta}\int_{Q_1} t^{1-2\sigma}(\eta^2 + |\nabla \eta|^2)W^2,$$

where C depends only on n, σ and $\|a^+\|_{L^p(B_1)}$. Then the proof of Proposition 3.1 in [128] goes through without any change. This finishes the proof of (i) for $\nu = 2$. Then (i) also holds for any $\nu > 0$ which follows from standard arguments. For part (ii) we choose $k = \|b^-\|_{L^p(B_1)}$ if $b^- \not\equiv 0$, otherwise let $k > 0$ be any number which is eventually sent to 0. Then we can show that there exists some $\nu_0 > 0$ for which (ii) holds, by exactly the same proof of Proposition 3.2 in [128]. Finally use the test function $\phi = \overline{U}^{-\beta}\eta^2$ with $\beta \in (0,1)$ to repeat the proof in (i) to conclude (ii) for $0 < \nu \le \frac{n+1}{n}$. Part (iii) follows from (i), (ii) and standard elliptic equation theory. $\square$

Remark 3.1. *Harnack inequality* (3.16)*, without lower order term b, has been obtained earlier in [27] using a different method.*

The above proofs can be improved to yield the following result.

Lemma 3.5. *Suppose* $a \in L^{\frac{n}{2\sigma}}(B_1), b \in L^p(B_1)$ *with* $p > \frac{n}{2\sigma}$ *and* $U \in H(t^{1-2\sigma}, Q_1)$ *is a weak subsolution of* (3.7) *in* Q_1*. There exists* $\delta > 0$ *which depends only on* n *and* σ *such that if* $\|a^+\|_{L^{\frac{n}{2\sigma}}(B_1)} < \delta$*, then*

$$\|U^+(\cdot,0)\|_{L^q(\partial' Q_{1/2})} \le C(\|U^+\|_{H(t^{1-2\sigma},Q_1)} + \|b^+\|_{L^p(B_1)}),$$

where $C > 0$ *depends only on* n, p, σ, δ*, and* $q = \min\left(\frac{2(n+1)}{n-2\sigma}, \ \frac{n(p-1)}{(n-2\sigma)p} \cdot \frac{2n}{n-2\sigma}\right)$.

Remark 3.2. *Analog estimates were established for* $-\Delta u = a(x)u$ *in [24] (see Theorem 2.3 there) and for* $-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = a(x)|u|^{p-2}u$ *in [6] (see Lemma 3.1 there).*

Proof of Lemma 3.5. We start from (3.17), where we choose $\beta = \min\left(\frac{2}{n}, \frac{2(2\sigma p - n)}{(n-2\sigma)p}\right)$. By Hölder inequality and Proposition 3.1,

$$\begin{aligned}\int_{\partial' Q_1} (a^+ + \frac{b^+}{k})\eta^2 W^2 &\le \delta \|\eta^2 W^2\|_{L^{\frac{n}{n-2\sigma}}(B_1)} + \|\eta^2 W^2\|_{L^{p'}(B_1)} \\ &\le C(n,\sigma)\delta \int_{Q_1} t^{1-2\sigma}|\nabla(\eta W)|^2 + C_{n,\sigma,p}\|\overline{U}\|_{H(t^{1-2\sigma},Q_1)}.\end{aligned}$$

By Poincare's inequality in [60], we have

$$\int_{Q_1} t^{1-2\sigma}|\nabla \eta|^2 W^2 \le C_{n,\sigma,p}\|\overline{U}\|_{H(t^{1-2\sigma},Q_1)}.$$

If δ is sufficiently small, the the above together with (3.17) imply that

$$\int_{Q_1} t^{1-2\sigma}|\nabla(\eta W)|^2 \leq C_{n,\sigma,p}\|\overline{U}\|_{H(t^{1-2\sigma},Q_1)}.$$

Hence it follows from Hölder inequality and Proposition 3.1 that, by sending $m \to \infty$,

$$\|\overline{U}(\cdot,0)\|_{L^q(\partial' Q_{1/2})} \leq C_{n,\sigma,p}\int_{Q_1} t^{1-2\sigma}|\nabla(\eta W)|^2 \leq C_{n,\sigma,p}\|\overline{U}\|_{H(t^{1-2\sigma},Q_1)}.$$

This finishes the proof. □

Corollary 3.1. *Suppose that* $K \in L^\infty(B_1)$*,* $U \in H(t^{1-2\sigma},Q_1)$ *and* $U \geq 0$ *in* Q_1 *satisfies, for some* $1 \leq p \leq (n+2\sigma)(n-2\sigma)$*,*

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U(X)) = 0 & \text{in } Q_1 \\ -\lim\limits_{t\to 0^+} t^{1-2\sigma}\partial_t U(x,t) = K(x)U(x,0)^p & \text{on } \partial' Q_1. \end{cases}$$

Then (i) $U \in L^\infty_{loc}(Q_1 \cup \partial' Q_1)$*, and hence* $U(\cdot,0) \in L^\infty_{loc}(B_1)$*.*

(ii) There exist $C > 0$ *and* $\alpha \in (0,1)$ *depending only on* $n,\sigma,p,\|u\|_{L^\infty(B_{3/4})}$ *and* $\|K\|_{L^\infty(B_{3/4})}$ *such that* $U \in C^\alpha(\overline{Q_{1/2}})$ *and*

$$\|U\|_{H(t^{1-2\sigma},Q_{1/2})} + \|U\|_{C^\alpha(\overline{Q_{1/2}})} \leq C.$$

Note that the regularity of solution of $-\Delta u = u^{\frac{n+2}{n-2}}$ was proved by Trudinger in [129].

Proof of Corollary 3.1. By Proposition 3.1, $U(\cdot,0) \in H^\sigma(B_1) \subset L^{\frac{2n}{n-2\sigma}}(B_1)$. Thus $U(\cdot,0)^{p-1} \in L^{\frac{n}{2\sigma}}(B_1)$. Then part (i) follows from Lemma 3.5 and Proposition 3.4. Part (ii) follows from Proposition 3.2 and Proposition 3.4. □

3.3.2 Schauder estimates

Let Ω be a domain in $\mathbb{R}^n$, $a \in L_{loc}^{\frac{2n}{n+2\sigma}}(\Omega)$ and $b \in L^1_{loc}(\Omega)$. We say $u \in \dot{H}^\sigma(\mathbb{R}^n)$ is a weak solution of

$$(-\Delta)^\sigma u = a(x)u + b(x) \quad \text{in } \Omega$$

if for any $\phi \in C^\infty(\mathbb{R}^n)$ supported in Ω,

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{\sigma}{2}} u (-\Delta)^{\frac{\sigma}{2}}\phi = \int_\Omega a(x)u\phi + b(x)\phi.$$

Then by (3.5), $u \in \dot{H}^\sigma(\mathbb{R}^n)$ is a weak solution of

$$(-\Delta)^\sigma u = \frac{1}{N_\sigma}\Big(a(x)u + b(x)\Big) \quad \text{in } B_1$$

if and only if $U = \mathcal{P}_\sigma[u]$, the extension of u defined in (3.2), is a weak solution of (3.7) in Q_1.

For $\alpha \in (0,1)$, $C^\alpha(\Omega)$ denotes the standard Hölder space over domain Ω. For simplicity, we use $C^\alpha(\Omega)$ to denote $C^{[\alpha],\alpha-[\alpha]}(\Omega)$ when $1 < \alpha \notin \mathbb{N}$ (the set of positive integers).

In this part, we shall prove the following local Schauder estimates for nonnegative solutions of fractional Laplace equation.

Theorem 3.1. *Suppose $a(x), b(x) \in C^\alpha(B_1)$ with $0 < \alpha \notin \mathbb{N}$. Let $u \in \dot{H}^\sigma(\mathbb{R}^n)$ and $u \geq 0$ in $\mathbb{R}^n$ be a weak solution of*

$$(-\Delta)^\sigma u = a(x)u + b(x), \quad \textit{in } B_1.$$

Suppose that $2\sigma + \alpha$ is not *an integer. Then $u \in C^{2\sigma+\alpha}(B_{1/2})$. Moreover,*

$$\|u\|_{C^{2\sigma+\alpha}(B_{1/2})} \leq C(\inf_{B_{3/4}} u + \|b\|_{C^\alpha(B_{3/4})}) \tag{3.18}$$

where $C > 0$ depends only on $n, \sigma, \alpha, \|a\|_{C^\alpha(B_{3/4})}$.

Remark 3.3. *Replacing the assumption $u \geq 0$ in $\mathbb{R}^n$ by $u \geq 0$ in B_1, estimate* (3.18) *may fail (see [87]). Without the sign assumption of u,* (3.18) *with $\inf_{B_{3/4}} u$ substituted by $\|u\|_{L^\infty(\mathbb{R}^n)}$ holds, which is proved in [30], [31] and [32] in a much more general setting of fully nonlinear nonlocal equations.*

The following proposition will be used in the proof of Theorem 3.1.

Proposition 3.5. *Let $a(x), b(x) \in C^k(B_1)$, $U(X) \in H(t^{1-2\sigma}, Q_1)$ be a weak solution of* (3.7) *in Q_1, where k is a positive integer. Then we have*

$$\sum_{i=0}^{k} \|\nabla_x^i U\|_{L^\infty(Q_{1/2})} \leq C(\|U\|_{L^2(t^{1-2\sigma}, Q_1)} + \|b\|_{C^k(B_1)}),$$

where $C > 0$ depends only on $n, \sigma, k, \|a\|_{C^k(B_1)}$.

Proof. We know from Proposition 3.4 that U is Hölder continuous in $\overline{Q_{8/9}}$. Let $h \in \mathbb{R}^n$ with $|h|$ sufficiently small. Denote $U^h(x,t) = \frac{U(x+h,t)-U(x,t)}{|h|}$. Then U^h is a weak solution of

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U^h(X)) = 0 & \text{in } Q_{8/9}, \\ -\lim\limits_{t\to 0^+} t^{1-2\sigma}\partial_t U^h(x,t) = a(x+h)U^h + a^h U + b^h & \text{on } \partial' Q_{8/9}. \end{cases} \tag{3.19}$$

By Proposition 3.2 and Proposition 3.4,

$$\begin{aligned} \|U^h\|_{H(t^{1-2\sigma},Q_{2/3})} + \|U^h\|_{C^\alpha(\overline{Q_{2/3}})} &\le C(\|U^h\|_{L^2(t^{1-2\sigma},Q_{3/4})} + \|b\|_{C^1(B_1)}) \\ &\le C(\|\nabla U\|_{L^2(t^{1-2\sigma},Q_{4/5})} + \|b\|_{C^1(B_1)}) \\ &\le C(\|U\|_{L^2(t^{1-2\sigma},Q_1)} + \|b\|_{C^1(B_1)}) \end{aligned}$$

for some $\alpha \in (0,1)$ and positive constant $C > 0$ depending only on $n, \sigma, \|a\|_{C^1(B_1)}$. Hence $\nabla_x U \in H(t^{1-2\sigma}, Q_{2/3}) \cap C^\alpha(\overline{Q_{2/3}})$, and it is a weak solution of

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla(\nabla_x U) = 0 & \text{in } Q_{2/3}, \\ -\lim\limits_{t\to 0^+} t^{1-2\sigma}\partial_t(\nabla_x U) = a\nabla_x U + U\nabla_x a + \nabla_x b & \text{on } \partial' Q_{2/3}. \end{cases}$$

Then this Proposition follows immediately from Proposition 3.2 and Proposition 3.4 for $k = 1$. We can continue this procedure for $k = 2, 3, \cdots$ (by induction). □

To prove Theorem 3.1 we first obtain Schauder estimates for solutions of the equation

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U(X)) = 0 & \text{in } Q, \\ -\lim\limits_{t\to 0^+} t^{1-2\sigma}\partial_t U(x,t) = g(x) & \text{on } \partial' Q, \end{cases} \tag{3.20}$$

where $Q = B_2 \times (0,2)$.

Theorem 3.2. *Let $U(X) \in H(t^{1-2\sigma}, Q)$ be a weak solution of (3.20) and $g(x) \in C^\alpha(B_2)$ for some $0 < \alpha \notin \mathbb{N}$. If $2\sigma + \alpha$ is not an integer, then $U(\cdot,0)$ is of $C^{2\sigma+\alpha}(B_{1/2})$. Moreover, we have*

$$\|U(\cdot,0)\|_{C^{2\sigma+\alpha}(B_{1/2})} \le C(\|U\|_{L^\infty(Q_2)} + \|g\|_{C^\alpha(B_2)}),$$

where $C > 0$ depends only on n, σ, α.

Theorem 3.2 together with Proposition 3.4 implies the following

Theorem 3.3. *Let $U(X) \in H(t^{1-2\sigma}, Q_1)$ be a weak solution of (3.7) with $D = Q_1$ and $a(x), b(x) \in C^{\alpha}(B_1)$ for some $0 < \alpha \notin \mathbb{N}$. If $2\sigma + \alpha$ is not an integer, then $U(\cdot, 0)$ is of $C^{2\sigma+\alpha}(B_{1/2})$. Moreover, we have*

$$\|U(\cdot,0)\|_{C^{2\sigma+\alpha}(B_{1/2})} \leq C(\|U\|_{L^\infty(Q_1)} + \|b\|_{C^\alpha(B_1)}),$$

where $C > 0$ depends only on $n, \sigma, \alpha, \|a\|_{C^\alpha(B_1)}$.

Proof. From Proposition 3.4, U is Hölder continuous in $\overline{Q_{3/4}}$. Theorem 3.3 follows from bootstrap arguments by applying Theorem 3.2 with $g(x) := a(x)U(x,0) + b(x)$. □

Furthermore, we have the following regularity result for solutions of (3.20) in $\overline{Q}$.

Theorem 3.4. *Let $g(x) \in L^\infty(\partial' Q)$, and $U(X) \in H(t^{1-2\sigma}, Q)$ be a weak solution of (3.20). Then $U(\cdot, 0) \in C^{2\sigma}(\partial' Q_1)$ if $\sigma \neq 1/2$. Furthermore, if we assume that $g(x)$ is Dini continuous in $\partial' Q$ then $U \in C^{2\sigma}(\overline{Q_1})$.*

Remark 3.4. *$C^{2\sigma}$ regularity is optimal. For example, $U = t^{2\sigma}$ solves (3.20) with $g(x) \equiv -2\sigma$.*

Remark 3.5. *If we only assume $g(x) \in L^\infty(\partial' Q)$ in the second part of Theorem 3.4, then the same proof implies that $U \in C^{2\sigma'}(\overline{Q_1})$ for any $\sigma' < \sigma$.*

For brevity, we denote $\omega(r)$ as $\omega_g(r, \Omega)$ if there is no ambiguity, where $\omega_g(r, \Omega)$ is as in (2.2).

Theorem 3.5. *Let $g(x) \in L^\infty(\partial' Q)$, and $U(X) \in H(t^{1-2\sigma}, Q)$ be a weak solution of (3.20). Suppose that $U(x, 0) \in C^2(B)$. There exists a constant C which depends only on n and σ such that for any $y_1, y_2 \in B_{1/2}$ with $d = |y_1 - y_2|$,*

$$\begin{aligned} &|\nabla_x^i U(y_1, 0) - \nabla_x^i U(y_2, 0)| \\ &\leq C\left(d|U|_{L^\infty(Q)} + d\int_d^1 r^{2\sigma-2-i}\omega(r)\mathrm{d}r + \int_0^d r^{2\sigma-1-i}\omega(r)\mathrm{d}r\right), \end{aligned} \tag{3.21}$$

where $i = 0, 1, 2$.

Proof. Our arguments are in the spirit of those in [28] and [98]. We denote C as various constants that depend only on n and σ. Let $\rho = \frac{1}{2}$, $Q_k = Q_{\rho^k}(0), \partial' Q_k =$

$B_k, k = 0, 1, 2, \cdots$. (Note that we have abused notations a little. Only in this proof we refer Q_k, B_k as Q_{ρ^k}, B_{ρ^k}.)

Let W_k be the unique weak solution of

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla W_k(X)) = 0 & \text{in } Q_k \\ -\lim_{t\to 0^+} t^{1-2\sigma}\partial_t W_k(x,t) = g(0) - g(x) & \text{on } \partial' Q_k \\ W_k(X) = 0 & \text{on } \partial'' Q_k, \end{cases} \tag{3.22}$$

which is guaranteed by Proposition 3.3. Let $U_k = W_k + U$ in Q_k and $h_{k+1} = U_{k+1} - U_k$ in Q_{k+1}, then

$$\|W_k\|_{L^\infty(Q_k)} \le C\rho^{2\sigma k}\omega(\rho^k). \tag{3.23}$$

Indeed, we can obtain the above estimate by applying the maximum principle to the equation of $\rho^{-2\sigma k}W_k(\rho^k x) \pm (t^{2\sigma} - 3)\omega(\rho^k)$ in Q_0. Hence by the maximum principle again we have

$$\|h_{k+1}\|_{L^\infty(Q_k)} \le C\rho^{2\sigma k}\omega(\rho^k).$$

By Proposition 3.5, we have, for $i = 0, 1, 2, 3$,

$$\|\nabla_x^i h_{k+1}\|_{L^\infty(Q_{k+2})} \le C\rho^{(2\sigma - i)k}\omega(\rho^k). \tag{3.24}$$

Similarly, by applying Proposition 3.5 to U_0, we have

$$\begin{aligned} \|\nabla_x^i U_0\|_{L^\infty(Q_2)} &\le C(\|U_0\|_{L^\infty(Q_1)} + |g(0)|) \\ &\le C(\|U\|_{L^\infty(Q_0)} + \|W_0\|_{L^\infty(Q_0)} + |g(0)|). \end{aligned} \tag{3.25}$$

We decompose U_0 as $U_0 = U_{01} + g(0)U_{02}$, where

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U_{01}(X)) = 0 & \text{in } Q_0 \\ -\lim_{t\to 0^+} t^{1-2\sigma}\partial_t U_{01}(x,t) = 0 & \text{on } \partial' Q_0 \\ U_{01}(X) = U(X) & \text{on } \partial'' Q_0, \end{cases} \tag{3.26}$$

and

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U_{02}(X)) = 0 & \text{in } Q_0 \\ -\lim_{t\to 0^+} t^{1-2\sigma}\partial_t U_{02}(x,t) = 1 & \text{on } \partial' Q_0 \\ U_{02}(X) = 0 & \text{on } \partial'' Q_0. \end{cases} \tag{3.27}$$

By the maximum principles and a weak Harnack inequality in Proposition 3.4,

$$\|U_{02}\|_{L^\infty(Q_0)} = 1/C > 0,$$

which implies that

$$|g(0)| \le C(\|U_0\|_{L^\infty(Q_0)} + \|U_{01}\|_{L^\infty(Q_0)}) \le C(\|U\|_{L^\infty(Q_0)} + \|W_0\|_{L^\infty(Q_0)}).$$

Hence from (3.25),

$$\|\nabla_x^i U_0\|_{L^\infty(Q_2)} \le C(\|U\|_{L^\infty(Q_0)} + \|W_0\|_{L^\infty(Q_0)}) \le C(\|U\|_{L^\infty(Q_0)} + \omega(1)). \tag{3.28}$$

For any given point z near 0, we have

$$\begin{aligned}
&|U(z,0) - U(0,0)| \\
&\le |U_k(0,0) - U(0,0)| + |U(z,0) - U_k(z,0)| + |U_k(z,0) - U_k(0,0)| \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Let k be such that $\rho^{k+4} \le |z| \le \rho^{k+3}$. By (3.23),

$$I_1 + I_2 \le C\rho^{2\sigma k}\omega(\rho^k) \le C\int_0^{|z|} r^{2\sigma-1}\omega(r)\mathrm{d}r.$$

For I_3, by (3.24) and (3.28),

$$\begin{aligned}
I_3 &\le |U_0(z,0) - U_0(0,0)| + \sum_{j=1}^k |h_j(z,0) - h_j(0,0)| \\
&\le C|z|\Big(\|\nabla_x U_0\|_{L^\infty(Q_{k+3})} + \sum_{j=1}^k \|\nabla_x h_j\|_{L^\infty(Q_{k+3})}\Big) \\
&\le C|z|\Big(\|U\|_{L^\infty(Q_0)} + \omega(1) + \sum_{j=1}^k \rho^{(2\sigma-1)j}\omega(\rho^j)\Big) \\
&\le C|z|\Big(\|U\|_{L^\infty(Q_0)} + \int_{|z|}^1 r^{2\sigma-2}\omega(r)\mathrm{d}r\Big).
\end{aligned}$$

Thus

$$|U(z,0) - U(0,0)| \le C|z|\Big(\|U\|_{L^\infty(Q_0)} + \int_{|z|}^1 r^{2\sigma-2}\omega(r)\mathrm{d}r\Big) + C\int_0^{|z|} r^{2\sigma-1}\omega(r)\mathrm{d}r,$$

which shows (3.21) for $i = 0$. Moreover, the above estimate implies that

$$\|\nabla_x U(\cdot,0)\|_{L^\infty(B_1)} \le C\Big(\|U\|_{L^\infty(Q_0)} + \int_0^1 r^{2\sigma-2}\omega(r)\mathrm{d}r\Big). \tag{3.29}$$

Next we will show that for $i = 1$. Applying (3.29) to the equation of W_k, we have, together with (3.23),

$$\begin{aligned}\|\nabla_x W_k(\cdot,0)\|_{L^\infty(B_{k+1})} &\leq C\Big(\rho^{(2\sigma-1)k}\omega(\rho^k) + \int_0^{\rho^k} r^{2\sigma-2}\omega(r)\mathrm{d}r\Big)\\ &\leq C\int_0^{\rho^k} r^{2\sigma-2}\omega(r)\mathrm{d}r.\end{aligned}$$

By (3.24) and (3.28),

$$\begin{aligned}&|\nabla_x U_k(z,0) - \nabla_x U_k(0,0)|\\ &\leq |\nabla_x U_0(z,0) - \nabla_x U_0(0,0)| + \sum_{j=1}^{k} |\nabla_x h_j(z,0) - \nabla_x h_j(0,0)|\\ &\leq C|z|\Big(\|\nabla_x^2 U_0\|_{L^\infty(Q_{k+3})} + \sum_{j=1}^{k} \|\nabla_x^2 h_j\|_{L^\infty(Q_{k+3})}\Big)\\ &\leq C|z|\Big(\|U\|_{L^\infty(Q_0)} + \omega(1) + \sum_{j=1}^{k} \rho^{(2\sigma-2)j}\omega(\rho^j)\Big)\\ &\leq C|z|\Big(\|U\|_{L^\infty(Q_0)} + \int_{|z|}^1 r^{2\sigma-3}\omega(r)\mathrm{d}r\Big).\end{aligned}$$

Hence

$$\begin{aligned}&|\nabla_x U(z,0) - \nabla_x U(0,0)|\\ &\leq |\nabla_x W_k(0,0)| + |\nabla_x W_k(z,0)| + |\nabla_x U_k(z,0) - \nabla_x U_k(0,0)|\\ &\leq C\int_0^{\rho^k} r^{2\sigma-2}\omega(r)\mathrm{d}r + C|z|\Big(\|U\|_{L^\infty(Q_0)} + \int_{|z|}^1 r^{2\sigma-3}\omega(r)\mathrm{d}r\Big).\end{aligned}$$

which shows (3.21) for $i = 1$. Moreover, the above estimate implies that

$$\|\nabla_x^2 U(\cdot,0)\|_{L^\infty(B_1)} \leq C\Big(\|U\|_{L^\infty(Q_0)} + \int_0^1 r^{2\sigma-3}\omega(r)\mathrm{d}r\Big). \tag{3.30}$$

Next we will show that for $i = 2$. Applying (3.30) to the equation of W_k, we have, together with (3.23),

$$\begin{aligned}\|\nabla_x^2 W_k(\cdot,0)\|_{L^\infty(B_{k+1})} &\leq C\Big(\rho^{(2\sigma-2)k}\omega(\rho^k) + \int_0^{\rho^k} r^{2\sigma-3}\omega(r)\mathrm{d}r\Big)\\ &\leq C\int_0^{\rho^k} r^{2\sigma-3}\omega(r)\mathrm{d}r.\end{aligned}$$

By (3.24) and (3.28),

$$
\begin{aligned}
&|\nabla_x^2 U_k(z,0) - \nabla_x^2 U_k(0,0)| \\
&\leq |\nabla_x^2 U_0(z,0) - \nabla_x^2 U_0(0,0)| + \sum_{j=1}^{k} |\nabla_x^2 h_j(z,0) - \nabla_x^2 h_j(0,0)| \\
&\leq C|z|\Big(\|\nabla_x^3 U_0\|_{L^\infty(Q_{k+3})} + \sum_{j=1}^{k} \|\nabla_x^3 h_j\|_{L^\infty(Q_{k+3})}\Big) \\
&\leq C|z|\Big(\|U\|_{L^\infty(Q_0)} + \omega(1) + \sum_{j=1}^{k} \rho^{(2\sigma-3)j}\omega(\rho^j)\Big) \\
&\leq C|z|\Big(\|U\|_{L^\infty(Q_0)} + \int_{|z|}^{1} r^{2\sigma-4}\omega(r)\mathrm{d}r\Big).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&|\nabla_x^2 U(z,0) - \nabla_x^2 U(0,0)| \\
&\leq |\nabla_x^2 W_k(0,0)| + |\nabla_x^2 W_k(z,0)| + |\nabla_x^2 U_k(z,0) - \nabla_x^2 U_k(0,0)| \\
&\leq C\int_0^{\rho^k} r^{2\sigma-3}\omega(r)\mathrm{d}r + C|z|\Big(\|U\|_{L^\infty(Q_0)} + \int_{|z|}^{1} r^{2\sigma-4}\omega(r)\mathrm{d}r\Big),
\end{aligned}
$$

which shows (3.21) for $i = 2$. $\square$

Theorem 3.6. *Let $g(x) \in L^\infty(\partial' Q)$, and $U(X) \in H(t^{1-2\sigma}, Q)$ be a weak solution of (3.20). There exists a constant C which depends only on n and σ such that for any $(z,t) \in Q_{1/2}, d = |(z,t)|$,*

$$
\begin{aligned}
&|U(z,t) - U(z,0)| \\
&\leq C\left(d^{2\sigma}|U|_{L^\infty(Q)} + d^{2\sigma}\int_d^1 r^{-1}\omega(r)\mathrm{d}r + d^{2\sigma-1}\int_0^d \omega(r)\mathrm{d}r\right).
\end{aligned}
\tag{3.31}
$$

Proof. Let ρ, W_k, Q_k, h_k as that in the proof of Theorem 3.5. Let $(z,t) \in Q_{k+3}$ but $(z,t) \notin Q_{k+4}$. By Proposition 3.5, Newton-Leibniz formula and (3.24),

$$
|h_{k+1}(z,t) - k_{k+1}(z,0)| \leq Ct^{2\sigma}\omega(\rho^k).
$$

Hence

$$
\begin{aligned}
&|U(z,t)-U(z,0)| \\
&\leq |U_k(z,0)-U(z,0)|+|U(z,t)-U_k(z,t)|+|U_k(z,t)-U_k(z,0)| \\
&\leq 2\|W_k\|_{L^\infty(Q_k)}+|U_0(z,t)-U_0(z,0)|+\sum_{j=1}^{k}|h_j(z,t)-h_k(z,0)| \\
&\leq C\rho^{2\sigma k}\omega(\rho^k)+Ct^{2\sigma}(\|U_0\|_{L^\infty(Q_0)}+|g(0)|)+Ct^{2\sigma}\sum_{j=1}^{k}\omega(\rho^j) \\
&\leq C\left(d^{2\sigma}|U|_{L^\infty(Q)}+d^{2\sigma}\int_d^1 r^{-1}\omega(r)\mathrm{d}r+d^{2\sigma-1}\int_0^d\omega(r)\mathrm{d}r\right).
\end{aligned}
$$

□

Theorem 3.7. *Let $g(x)\in L^\infty(\partial' Q)$, and $U(X)\in H(t^{1-2\sigma},Q)$ be a weak solution of* (3.20)*. Suppose that $2\sigma\leq 1$. There exists a constant C which depends only on n and σ such that for any $(z_1,t_1)\in Q_{1/4}, z_2\in\partial' Q_{1/4}, d=|(z_1-z_2,t_1)|$,*

$$
|U(z_1,t_1)-U(z_2,0)|\leq Cd^{2\sigma}\left(|U|_{L^\infty(Q)}+\int_0^1 r^{-1}\omega(r)\mathrm{d}r\right). \tag{3.32}
$$

Proof. It follows from Theorem 3.5 and Theorem 3.6. □

The following is Lemma 4.5 in [27], which will be used in the proof of Theorem 3.4.

Lemma 3.6. *(Lemma 4.5 in [27]) Let $g\in C^\alpha(B_1)$ for some $\alpha\in(0,1)$ and $U\in L^\infty(Q_1)\cap H(t^{1-2\sigma},Q_1)$ be a weak solution of* (3.20)*. Then there exists $\beta\in(0,1)$ depending only on n,σ,α such that $t^{1-2\sigma}\partial_t U\in C^\beta(\overline{Q_{1/2}})$. Moreover, there exists a positive constant $C>0$ depending only on n,σ and β such that*

$$
\|t^{1-2\sigma}\partial_t U\|_{C^\beta(\overline{Q_{1/2}})}\leq C(\|U\|_{L^\infty(Q_1)}+\|g\|_{C^\alpha(B_1)}).
$$

Proof of Theorem 3.4. For the first part that if $\sigma\neq 1/2$, then it follows from Theorem 3.5 that $U(\cdot,0)\in C^{2\sigma}(\partial' Q_1)$. To prove the second part, we adapt a method in [130].

Step 1: We first consider the case of $2\sigma<1$. For any fixed $X_1=(x_1,t_1)$, $X_2=(x_2,t_2)\in Q_1$, with $0<t_1\leq t_2<1, t_1<1/2$. Denote $d=|(x_1-x_2,t_1-t_2)|$.

(i) If $t_1>4d$, then $4d<t_1\leq t_2\leq t_1+d\leq 5t_1/4$. Let $\varepsilon=t_1^2$ and $\tilde U(x,t)=U(\varepsilon x,\varepsilon t)$. Hence

$$
\Delta\tilde U(x,t)+(1-2\sigma)t^{-1}\tilde U_t(x,t)=0,\quad x\in B_\varepsilon(0),\quad 1<t<1/\varepsilon.
$$

By the gradient estimates for uniformly elliptic equations (see [65]),

$$\max_{\mathcal{B}_R(X_1/\varepsilon)} |\nabla \tilde{U}| \leq C \frac{1}{R_0} \max_{\mathcal{B}_{R_0}(X_1/\varepsilon)} (\tilde{U} - \inf_{\mathcal{B}_{R_0}(X_1/\varepsilon)} \tilde{U})$$

where $R = \frac{d}{\varepsilon}, R_0 = \frac{t_1}{2\varepsilon}$ and C is a constant depending only on n and σ. Hence, together with Theorem 3.7,

$$\begin{aligned}
|U(x_1,t_1) - U(x_2,t_2)| &\leq C\frac{d}{t_1} osc_{\mathcal{B}_{\frac{t_1}{2}}(X_1)} U \\
&\leq C\frac{d}{t_1} t_1^{2\sigma} \left(|U|_{L^\infty(Q)} + \int_0^1 r^{-1}\omega(r)\mathrm{d}r \right) \\
&\leq C d^{2\sigma} \left(|U|_{L^\infty(Q)} + \int_0^1 r^{-1}\omega(r)\mathrm{d}r \right).
\end{aligned}$$

(ii) If $t_1 \leq 4d$, then $t_2 \leq t_1 + d \leq 5d$. Hence (3.32) leads to

$$\begin{aligned}
|U(x_1,t_1) - U(x_2,t_2)| &\leq |U(x_1,t_1) - U(x_1,0)| + |U(x_1,0) - U(x_2,t_2)| \\
&\leq C \left(|U|_{L^\infty(Q)} + \int_0^1 r^{-1}\omega(r)\mathrm{d}r \right) (t_1^{2\sigma} + (6d)^{2\sigma}) \\
&\leq C d^{2\sigma} \left(|U|_{L^\infty(Q)} + \int_0^1 r^{-1}\omega(r)\mathrm{d}r \right).
\end{aligned}$$

From (i) and (ii), together with uniformly elliptic theory, we see that for any (x_1,t_1), $(x_2,t_2) \in \overline{Q_{1/4}}$ with $d = |(x_1 - x_2, t_1 - t_2)|$,

$$|U(x_1,t_1) - U(x_2,t_2)| \leq C d^{2\sigma} \left(|U|_{L^\infty(Q)} + \int_0^1 r^{-1}\omega(r)\mathrm{d}r \right). \tag{3.33}$$

Step 2: We consider that $2\sigma = 1$. This case is uniformly elliptic and hence the result should be well-known. We include it here for completeness. Let $(z,t) \in Q_{k+3}$ but $(z,t) \notin Q_{k+4}$. Denote $d = (z,t)$. Let W_k, h_k, etc be those as in the proof of Theorem 3.5. Applying (3.33) to W_k, we have, for $\nabla = \nabla_{x,t}$,

$$\|\nabla W_k\|_{L^\infty(Q_{k+1})} \leq C\rho^{(2\sigma-1)k} \int_0^{\rho^k} \frac{\omega(r)}{r} \mathrm{d}r.$$

By Lemma 3.6(Lemma 4.5 in [27]), h_j is Hölder continuous, i.e. there exists $\beta \in (0,1)$ which depends only on n and σ such that

$$|\partial_t h_j(z,t) - \partial_t h_j(0,0)| = C\omega(\rho^k) d^\beta,$$

and

$$|\partial_t U_0(z,t) - \partial_t U_0(0,0)| \leq C d^\beta (\|U_0\|_{L^\infty(Q_0)} + |g(0)|).$$

Hence, together with Proposition 3.5,

$$|\nabla h_j(z,t) - \nabla h_j(0,0)| = C\omega(\rho^k) d^\beta,$$

and

$$|\nabla U_0(z,t) - \nabla U_0(0,0)| \le Cd^\beta(\|U_0\|_{L^\infty(Q_0)} + |g(0)|).$$

Hence

$$\begin{aligned}
&|\nabla U(z,t) - \nabla U(0,0)| \\
&\le |\nabla U_k(z,t) - \nabla U(z,t)| + |\nabla U(0,0) - \nabla U_k(0,0)| + |\nabla U_k(z,t) - \nabla U_k(0,0)| \\
&\le 2\|\nabla W_k\|_{L^\infty(Q_k)} + |\nabla U_0(z,t) - \nabla U_0(0,0)| + \sum_{j=1}^{k} |\nabla h_j(z,t) - \nabla h_j(0,0)| \\
&\le C\int_0^{\rho^k} \frac{\omega(r)}{r}\mathrm{d}r + Cd^\beta(\|U_0\|_{L^\infty(Q_0)} + |g(0)|) + Cd^\beta \sum_{j=1}^{k} \omega(\rho^j) \\
&\le C\left(\int_0^d \frac{\omega(r)}{r}\mathrm{d}r + d^\beta |U|_{L^\infty(Q)} + d^\beta \int_d^1 r^{-1}\omega(r)\mathrm{d}r\right).
\end{aligned}$$

The above estimate implies that for any $(z_1,t_1),(z_2,0) \in Q_{1/4}, d = |(z_1 - z_2, t_1)|$,

$$|\nabla U(z_1,t_1) - \nabla U(z_2,0)| \le C\left(\int_0^d \frac{\omega(r)}{r}\mathrm{d}r + d^\beta |U|_{L^\infty(Q)} + d^\beta \int_d^1 r^{-1}\omega(r)\mathrm{d}r\right). \quad (3.34)$$

For any fixed $X_1 = (x_1,t_1), X_2 = (x_2,t_2) \in Q_1$, with $0 < t_1 \le t_2 < 1, t_1 < 1/2$. Denote $d = |(x_1 - x_2, t_1 - t_2)|$.

(i) If $t_1 > 4d$, then $4d < t_1 \le t_2 \le t_1 + d \le 5t_1/4$. Let $\varepsilon = t_1^2$ and $\tilde{U}(x,t) = U(\varepsilon x, \varepsilon t)$. Hence

$$\Delta \nabla \tilde{U}(x,t) = 0, \quad x \in B_\varepsilon(0), \quad 1 < t < 1/\varepsilon.$$

By the gradient estimates for uniformly elliptic equations (see [65]),

$$\max_{\mathcal{B}_R(X_1/\varepsilon)} |\nabla(\nabla \tilde{U})| \le C\frac{1}{R_0} \max_{\mathcal{B}_{R_0}(X_1/\varepsilon)} (\nabla \tilde{U} - \inf_{\mathcal{B}_{R_0}(X_1/\varepsilon)} \nabla \tilde{U})$$

where $R = \frac{d}{\varepsilon}, R_0 = \frac{t_1}{2\varepsilon}$ and C is a constant depending only on n and σ. Hence, together with (3.34)

$$\begin{aligned}
|\nabla U(x_1,t_1) - \nabla U(x_2,t_2)| &\le C\frac{d}{t_1} osc_{\mathcal{B}_{\frac{t_1}{2}}(X_1)} \nabla U \\
&\le C\frac{d}{t_1}\left(\int_0^{t_1} \frac{\omega(r)}{r}\mathrm{d}r + t_1^\beta |U|_{L^\infty(Q)} + t_1^\beta \int_{t_1}^1 r^{-1}\omega(r)\mathrm{d}r\right) \\
&\le C\left(\int_0^d \frac{\omega(r)}{r}\mathrm{d}r + d^\beta |U|_{L^\infty(Q)} + d^\beta \int_d^1 r^{-1}\omega(r)\mathrm{d}r\right).
\end{aligned}$$

(ii) If $t_1 \leq 4d$, then $t_2 \leq t_1 + d \leq 5d$. Hence (3.34) leads to

$$\begin{aligned}|\nabla U(x_1,t_1) - \nabla U(x_2,t_2)| &\leq |\nabla U(x_1,t_1) - \nabla U(x_1,0)| + |\nabla U(x_1,0) - \nabla U(x_2,t_2)| \\ &\leq C\left(\int_0^d \frac{\omega(r)}{r}\mathrm{d}r + d^\beta |U|_{L^\infty(Q)} + d^\beta \int_d^1 r^{-1}\omega(r)\mathrm{d}r\right).\end{aligned}$$

The (i) and (ii) implies that $\nabla U \in C^0(\overline{Q_1})$.

Step 3: We consider $2\sigma > 1$. By Lemma 4.5 in [27], h_j is Hölder continuous, i.e. there exists $\beta \in (0,1)$ which depends only on n and σ such that

$$|\partial_t h_j(z,t)| = C\omega(\rho^k)t^{2\sigma-1}.$$

and

$$|\partial_t U_0(z,t)| \leq Ct^{2\sigma-1}(\|U_0\|_{L^\infty(Q_0)} + |g(0)|).$$

Hence together with Proposition 3.5,

$$\begin{aligned}&|\nabla_t U(z,t) - \nabla_t U(0,0)| \\ &\leq |\nabla_t U_k(z,t) - \nabla_t U(z,t)| + |\nabla_t U(0,0) - \nabla_t U_k(0,0)| + |\nabla_t U_k(z,t) - \nabla_t U_k(0,0)| \\ &\leq 2\|\nabla_t W_k\|_{L^\infty(Q_k)} + |\nabla_t U_0(z,t)| + \sum_{j=1}^k |\nabla_t h_j(z,t)| \\ &\leq C\rho^{(2\sigma-1)k}\int_0^{\rho^k} \frac{\omega(r)}{r}\mathrm{d}r + Cd^{2\sigma-1}(\|U_0\|_{L^\infty(Q_0)} + |g(0)|) + Cd^{2\sigma-1}\sum_{j=1}^k \omega(\rho^j) \\ &\leq Cd^{2\sigma-1}\left(\int_0^d \frac{\omega(r)}{r}\mathrm{d}r + |U|_{L^\infty(Q)} + \int_d^1 r^{-1}\omega(r)\mathrm{d}r\right) \\ &\leq Cd^{2\sigma-1}\left(\int_0^1 \frac{\omega(r)}{r}\mathrm{d}r + |U|_{L^\infty(Q)}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}&|\nabla_x U(z,t) - \nabla_x U(0,0)| \\ &\leq |\nabla_x U_k(z,t) - \nabla_x U(z,t)| + |\nabla_x U(0,0) - \nabla_x U_k(0,0)| + |\nabla_x U_k(z,t) - \nabla_x U_k(0,0)| \\ &\leq 2\|\nabla_x W_k\|_{L^\infty(Q_k)} + |\nabla_x U_0(z,t) - \nabla_x U_0(0,0)| + \sum_{j=1}^k |\nabla_x h_j(z,t) - \nabla_x h_j(0,0)| \\ &\leq C\rho^{(2\sigma-1)k}\int_0^{\rho^k} \frac{\omega(r)}{r}\mathrm{d}r + Cd^{2\sigma-1}(\|U_0\|_{L^\infty(Q_0)} + |g(0)|) + Cd^{2\sigma-1}\sum_{j=1}^k \omega(\rho^j) \\ &\leq Cd^{2\sigma-1}\left(\int_0^1 \frac{\omega(r)}{r}\mathrm{d}r + |U|_{L^\infty(Q)}\right).\end{aligned}$$

For any fixed $X_1=(x_1,t_1), X_2=(x_2,t_2)\in Q_1$, with $0<t_1\le t_2<1, t_1<1/2$. Denote $d=|(x_1-x_2,t_1-t_2)|$. Using exactly the same proof as that in *step 1* (since $\nabla_x U$ satisfies the same equation as of U), we can show that

$$|\nabla_x U(x_1,t_1)-\nabla_x U(x_2,t_2)|\le Cd^{2\sigma-1}\left(|U|_{L^\infty(Q)}+\int_0^1 r^{-1}\omega(r)\mathrm{d}r\right). \tag{3.35}$$

As to U_t, we only need to consider the case of $t_1>4d$. Then $4d<t_1\le t_2\le t_1+d\le 5t_1/4$. Let $\varepsilon=t_1^2$. Denote $V=U_t$ and $\tilde V(x,t)=V(\varepsilon x,\varepsilon t)$. Differentiating (3.20),

$$\Delta\tilde V(x,t)+(1-2\sigma)t^{-1}\tilde V_t(x,t)-(1-2\sigma)t^{-2}\tilde V=0,\quad x\in B_\varepsilon(0),\quad 1<t<1/\varepsilon.$$

By the gradient estimates for uniformly elliptic equations (see [65]),

$$\max_{\mathcal{B}_R(X_1/\varepsilon)}|\nabla\tilde V|\le C\frac{1}{R_0}\left(\max_{\mathcal{B}_{R_0}(X_1/\varepsilon)}(\tilde V-\inf_{\mathcal{B}_{R_0}(X_1/\varepsilon)}\tilde V)+|\inf_{\mathcal{B}_{R_0}(X_1/\varepsilon)}\tilde V|\right)$$

where $R=\frac{d}{\varepsilon}, R_0=\frac{t_1}{2\varepsilon}$ and C is a constant depending only on n and σ. Hence, together with Theorem 3.7,

$$|V(x_1,t_1)-V(x_2,t_2)|\le C\frac{d}{t_1}\left(osc_{\mathcal{B}_{\frac{t_1}{2}}(x_1,t_1)}V+\|V\|_{L^\infty(\mathcal{B}_{\frac{t_1}{2}}(x_1,t_1))}\right). \tag{3.36}$$

On the other hand, $H=t^{1-2\sigma}U_t$ satisfies (see [29] or [27])

$$\begin{cases}\operatorname{div}(t^{2\sigma-1}\nabla H(X))=0 & \text{in } Q\\ H=g(x) & \text{on } \partial' Q.\end{cases} \tag{3.37}$$

Choose a cut-off function η which is supported in Q and equals to 1 in Q_1. Let H_1 be the solution of

$$\begin{cases}\operatorname{div}(t^{2\sigma-1}\nabla H_1(X))=0 & \text{in } \mathbb{R}^{n+1}_+\\ H_1=\eta g(x) & \text{on } \partial'\mathbb{R}^{n+1}_+.\end{cases} \tag{3.38}$$

Hence by Proposition 3.5 ,

$$\begin{aligned}\|H\|_{L^\infty(Q_{1/4})}&\le\|H_1\|_{L^\infty(Q_{1/4})}+\|H-H_1\|_{L^\infty(Q_{1/4})}\\ &\le\|H_1\|_{L^\infty(Q_{1/4})}+C\|H-H_1\|_{L^2(t^{2\sigma-1},Q_1)}\\ &\le C(\|g(x)\|_{L^\infty(Q)}+\|U\|_{L^\infty(Q)})\\ &\le C(\|g(0)\|+\omega(1)+\|U\|_{L^\infty(Q)})\\ &\le C(\omega(1)+\|U\|_{L^\infty(Q)}),\end{aligned}$$

which leads to

$$|V(x,t)| \le Ct^{2\sigma-1}(\omega(1) + \|U\|_{L^\infty(Q)}), \quad \forall\ (x,t) \in Q_{1/4}.$$

Together (3.36), we have

$$\begin{aligned} |V(x_1,t_1) - V(x_2,t_2)| &\le C\frac{d}{t_1}t_1^{2\sigma-1}\left(|U|_{L^\infty(Q)} + \int_0^1 r^{-1}\omega(r)\mathrm{d}r\right) \\ &\quad + C\frac{d}{t_1}t_1^{2\sigma-1}(\omega(1) + \|U\|_{L^\infty(Q)}) \\ &\le Cd^{2\sigma-1}\left(|U|_{L^\infty(Q)} + \int_0^1 r^{-1}\omega(r)\mathrm{d}r\right). \end{aligned}$$

This and (3.35) result in

$$|\nabla U(x_1,t_1) - \nabla U(x_2,t_2)| \le Cd^{2\sigma-1}\left(|U|_{L^\infty(Q)} + \int_0^1 r^{-1}\omega(r)\mathrm{d}r\right). \tag{3.39}$$

By the uniformly elliptic equation theory we see that for any $(x_1,t_1),(x_2,t_2) \in \overline{Q_{1/4}}$ with $d = |(x_1-x_2,t_1-t_2)|$,

$$|\nabla U(x_1,t_1) - \nabla U(x_2,t_2)| \le Cd^{2\sigma-1}\left(|U|_{L^\infty(Q)} + \int_0^1 r^{-1}\omega(r)\mathrm{d}r\right). \tag{3.40}$$

This finishes the proof of Theorem 3.4. □

Proof of Theorem 3.2. From Proposition 3.4 we have already known that U is Hölder continuous in $\overline{Q_0}$. The case that $\alpha < 1$ then follows from Theorem 3.5. For the case that $\alpha \ge 1$, we may apply ∇_x to (3.20) $[\alpha]$ times, as in the proof of Proposition 3.5, and repeat the three steps. Theorem 3.2 is proved. □

Proof of Theorem 3.1. Since $u \in \dot{H}^\sigma(\mathbb{R}^n)$ is nonnegative, its extension $U \ge 0$ in $\mathbb{R}^{n+1}_+$ and $U \in H(t^{1-2\sigma}, Q_1)$ is a weak solution of (3.7) in Q_1. The theorem follows immediately from Theorem 3.3 and Proposition 3.4. □

Remark 3.6. *Another way to show Theorem 3.1 is the following. Let* $u \in \dot{H}^\sigma(\mathbb{R}^n)$ *and* $u \ge 0$ *in* $\mathbb{R}^n$ *be a solution of*

$$(-\Delta)^\sigma u = g(x), \quad \text{in } B_1,$$

where $g \in C^\alpha(B_1)$. *Let* η *be a nonnegative smooth cut-off function supported in* B_1 *and equal to* 1 *in* $B_{7/8}$. *Let* $v \in \dot{H}^\sigma(\mathbb{R}^n)$ *be the solution of*

$$(-\Delta)^\sigma v = \eta(x)g(x), \quad \text{in } \mathbb{R}^n,$$

where ηg is considered as a function defined in $\mathbb{R}^n$ and supported in B_1, i.e., v is a Riesz potential of ηg

$$v(x) = \frac{\Gamma(\frac{n-2\sigma}{2})}{2^{2\sigma}\pi^{n/2}\Gamma(\sigma)} \int_{\mathbb{R}^n} \frac{\eta(y)g(y)}{|x-y|^{n-2\sigma}} \mathrm{d}y.$$

Then if $2\sigma+\alpha$ and α are not integers, we have (see, e.g., [125])

$$\|v\|_{C^{2\sigma+\alpha}(B_{1/2})} \leq C(\|v\|_{L^\infty(\mathbb{R}^n)} + \|\eta g\|_{C^\alpha(\mathbb{R}^n)}) \leq C\|g\|_{C^\alpha(B_1)}.$$

Let $w = u - v$ which belongs to $\dot{H}^\sigma(\mathbb{R}^n)$ and satisfies

$$(-\Delta)^\sigma w = 0, \quad \text{in } B_{7/8}.$$

Let $W = \mathcal{P}_\sigma[w]$ be the extension of w, and $\tilde{W} = W + \|v\|_{L^\infty(\mathbb{R}^n)} \geq 0$ in $\mathbb{R}^{n+1}_+$. Notice that $\tilde{W}$ is a nonnegative weak solution of (3.7) with $a \equiv b \equiv 0$ and $D = Q_1$. By Proposition 3.5 and Proposition 3.4, we have

$$\begin{aligned} &\|w + \|v\|_{L^\infty(\mathbb{R}^n)}\|_{C^{2\sigma+\alpha}(B_{1/2})} \\ &\leq C\|\tilde{W}\|_{L^2(t^{1-2\sigma}, Q_{7/8})} \leq C \inf_{Q_{3/4}} \tilde{W} \leq C(\inf_{Q_{3/4}} u + \|v\|_{L^\infty(\mathbb{R}^n)}). \end{aligned}$$

Hence

$$\begin{aligned} \|u\|_{C^{2\sigma+\alpha}(B_{1/2})} &\leq \|v\|_{C^{2\sigma+\alpha}(B_{1/2})} + \|w\|_{C^{2\sigma+\alpha}(B_{1/2})} \\ &\leq C(\inf_{B_{3/4}} u + \|g\|_{C^\alpha(B_1)}). \end{aligned}$$

Using bootstrap arguments as that in the proof of Theorem 3.3, we conclude Theorem 3.1.

Remark 3.7. *Indeed, our proofs also lead to the following. If we only assume that $\sigma = \frac{1}{2}$, $a(x), b(x), g(x) \in L^\infty(B_1)$, and let U, u be those in Theorem 3.4 and in Theorem 3.1 respectively, then we have the following log-Lipschitz property: for any $y_1, y_2 \in B_{1/4}, y_1 \neq y_2$,*

$$\frac{|U(y_1,0) - U(y_2,0)|}{|y_1 - y_2|} \leq C_1(\|U\|_{L^\infty(Q_1)} - \|g\|_{L^\infty(B_1)} \log|y_1 - y_2|),$$

$$\frac{|u(y_1) - u(y_2)|}{|y_1 - y_2|} \leq -C_2 \log|y_1 - y_2|(\inf_{B_{3/4}} u + \|b\|_{L^\infty(B_{3/4})}),$$

where $C_1 > 0$ depends only on n, σ and $C_2 > 0$ depends only on $n, \sigma, \|a\|_{L^\infty(B_{3/4})}$.

Next we have

Proposition 3.6. *Suppose that $K \in C^1(B_1)$, $U \in H(t^{1-2\sigma}, Q_1)$ and $U \geq 0$ in Q_1 is a weak solution of*

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U) = 0, & \text{in } Q_1 \\ -\lim\limits_{t\to 0} t^{1-2\sigma}\partial_t U(x,t) = K(x)U^p(x,0), & \text{on } \partial' Q_1, \end{cases} \tag{3.41}$$

where $1 \leq p \leq \frac{n+2\sigma}{n-2\sigma}$. Then there exist $C > 0$ and $\alpha \in (0,1)$ both of which depend only on $n, \sigma, p, \|U\|_{L^\infty(Q_1)}, \|K\|_{C^1(Q_1)}$ such that

$$\nabla_x U \quad \text{and} \quad t^{1-2\sigma}\partial_t U \quad \text{are of} \quad C^\alpha(\overline{Q_{1/2}}),$$

and

$$\|\nabla_x U\|_{C^\alpha(\overline{Q_{1/2}})} + \|t^{1-2\sigma}\partial_t U\|_{C^\alpha(\overline{Q_{1/2}})} \leq C.$$

Proof. We use C and α to denote various positive constants with dependence specified as in the proposition, which may vary from line to line. By Corollary 3.1, $U \in L^\infty_{loc}(Q_1 \cup \partial' Q_1)$ and

$$\|U\|_{C^\alpha(\overline{Q_{8/9}})} \leq C.$$

With the above, we may apply Theorem 3.3 to obtain $U(\cdot, 0) \in C^{1,\sigma}(\overline{B_{7/8}})$ and

$$\|U(\cdot,0)\|_{C^{1,\sigma}(\overline{B_{7/8}})} \leq C.$$

Hence we may differentiate (3.41) with respect to x (which can be justified from the proof of Proposition 3.5) and apply Proposition 3.4 to $\nabla_x U$ to obtain

$$\|\nabla_x U\|_{C^\alpha(\overline{Q_{1/2}})} \leq C.$$

Finally, we can apply Lemma 3.6 to obtain

$$\|t^{1-2\sigma}\partial_t U\|_{C^\alpha(\overline{Q_{1/2}})} \leq C.$$

□

3.4 Isolated singularities: a Bôcher type theorem

The classical Bôcher theorem in harmonic function theory states that a positive harmonic function u in the punctured ball $B_1 \setminus \{0\}$ must be of the form

$$u(x) = \begin{cases} -a \log |x| + h(x), & n = 2, \\ a|x|^{2-n} + h(x), & n \geq 3, \end{cases}$$

where a is a nonnegative constant and h is a harmonic function in B_1.

We are going to establish a similar result, Proposition 3.7, in our setting.

Proposition 3.7. *Let $n \geq 2$. Suppose that for all $\varepsilon \in (0,1)$, $U \in H(t^{1-2\sigma}, \mathcal{B}_1^+ \setminus \overline{\mathcal{B}_\varepsilon^+})$ and $U > 0$ in $\mathcal{B}_1^+ \setminus \overline{\mathcal{B}_\varepsilon^+}$ be a weak solution of*

$$\begin{cases} \operatorname{div}(t^{1-2\sigma} \nabla U) = 0 & \text{in } \mathcal{B}_1^+ \setminus \overline{\mathcal{B}_\varepsilon^+}, \\ -\lim\limits_{t \to 0} t^{1-2\sigma} \partial_t U(x,t) = 0 & \text{on } B_1 \setminus \overline{B_\varepsilon^+}. \end{cases} \tag{3.42}$$

Then

$$U(X) = A|X|^{2\sigma - n} + W(X),$$

where A is a nonnegative constant and $W(X) \in H(t^{1-2\sigma}, \mathcal{B}_1^+)$ satisfies

$$\begin{cases} \operatorname{div}(t^{1-2\sigma} \nabla W) = 0 & \text{in } \mathcal{B}_1^+, \\ -\lim\limits_{t \to 0} t^{1-2\sigma} \partial_t W(x,t) = 0 & \text{on } B_1. \end{cases} \tag{3.43}$$

Proof. We adapt the proof of the Bôcher theorem given in [7].

Define

$$A[U](r) = \frac{\int_{\partial'' \mathcal{B}_r^+} t^{1-2\sigma} U(x,t) dS_r}{\int_{\partial'' \mathcal{B}_r^+} t^{1-2\sigma} dS_r}$$

where $r = |(x,t)| > 0$ and dS_r is the volume element of $\partial'' \mathcal{B}_r$.

By direct computations we have

$$\frac{d}{dr} A[U](r) = \frac{\int_{\partial'' \mathcal{B}_r^+} t^{1-2\sigma} \nabla U(x,t) \cdot \frac{(x,t)}{r} dS_r}{\int_{\partial'' \mathcal{B}_r^+} t^{1-2\sigma} dS_r}.$$

Let

$$f(r) = \int_{\partial'' \mathcal{B}_r^+} t^{1-2\sigma} \nabla U(x,t) \cdot \frac{(x,t)}{r} dS_r.$$

Since U satisfies (3.42), by integration by parts we have

$$f(r_1) = f(r_2), \ \forall \ 0 < r_1, r_2 < 1.$$

Notice that

$$\int_{\partial'' \mathcal{B}_r^+} t^{1-2\sigma} dS_r = r^{n+1-2\sigma} \int_{\partial'' \mathcal{B}_1^+} t^{1-2\sigma} dS_1.$$

Thus there exists a constant b such that

$$\frac{d}{dr} A[U](r) = b r^{-n-1+2\sigma}.$$

So there exist constants a and b such that

$$A[U](r) = a + b r^{2\sigma - n}.$$

Given Lemma 3.7 in the following, the rest of the arguments are rather similar to those in [7] and are omitted here. We refer to [7] for details. □

Lemma 3.7. *There exists a constant $C > 0$ depending only on n and σ such that for every positive function U which satisfies* (3.42),

$$CU(x,t) < U(\tilde{x}, \tilde{t})$$

whenever $0 < |(x,t)| = |(\tilde{x}, \tilde{t})| < 1/2$.

Proof. It follows from Proposition 3.4 and standard Harnack inequalities for uniformly elliptic equations.

□

Chapter 4

A fractional Nirenberg problem

4.1 A Liouville type theorem

We say that $U \in L^\infty_{loc}(\overline{\mathbb{R}^{n+1}_+})$ if $U \in L^\infty(\overline{Q_R})$ for any $R > 0$. Similarly, we say $U \in H_{loc}(t^{1-2\sigma}, \overline{\mathbb{R}^{n+1}_+})$ if $U \in H(t^{1-2\sigma}, \overline{Q_R})$ for any $R > 0$. We start with a Lemma, which is a version of the strong maximum principle.

Proposition 4.1. *Suppose $U(X) \in H(t^{1-2\sigma}, D_\varepsilon) \cap C(\mathcal{B}_1^+ \cup B_1 \setminus \{0\})$ and $U > 0$ in $\mathcal{B}_1^+ \cup B_1 \setminus \{0\}$ is a weak supersolution of (3.7) with $a \equiv b \equiv 0$ and $D = D_\varepsilon := \mathcal{B}_1^+ \setminus \overline{\mathcal{B}_\varepsilon^+}$ for any $0 < \varepsilon < 1$, then*

$$\liminf_{(x,t)\to 0} U(x,t) > 0.$$

Proof. For any $\delta > 0$, let

$$V_\delta = U + \frac{\delta}{|(x,t)|^{n-2\sigma}} - \min_{\partial''\mathcal{B}_{0.8}^+} U.$$

Then V is also a weak supersolution in $D_{\delta^{\frac{2}{n-2\sigma}}}$. Applying Lemma 3.2 to V_δ in $D_{\delta^{\frac{2}{n-2\sigma}}}$ for sufficiently small δ, we have $V_\delta \geq 0$ in $D_{\delta^{\frac{2}{n-2\sigma}}}$. For any $(x,t) \in \mathcal{B}_{0.8}^+ \setminus \{0\}$, we have $\lim_{\delta\to 0} V_\delta(x,t) \geq 0$, i.e., $U(x,t) \geq \min_{\partial''\mathcal{B}_{0.8}^+} U$. □

Theorem 4.1. *Let $U \in H_{loc}(t^{1-2\sigma}, \overline{\mathbb{R}^{n+1}_+})$, $U(X) \geq 0$ in $\mathbb{R}^{n+1}_+$ and $U \not\equiv 0$, be a weak solution of*

$$\begin{cases} \mathrm{div}(t^{1-2\sigma}\nabla U(x,t)) &= 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\ -\lim\limits_{t\to 0} t^{1-2\sigma}\partial_t U(x,t) &= U^{\frac{n+2\sigma}{n-2\sigma}}(x,0) \quad x \in \mathbb{R}^n. \end{cases} \tag{4.1}$$

Then $U(x,0)$ takes the form

$$\left(N_\sigma c_{n,\sigma} 2^{2\sigma}\right)^{\frac{n-2\sigma}{4\sigma}} \left(\frac{\lambda}{1+\lambda^2|x-x_0|^2}\right)^{\frac{n-2\sigma}{2}}$$

where $\lambda > 0$, $x_0 \in \mathbb{R}^n$, $c_{n,\sigma}$ *is the constant in* (1.8) *and* N_σ *is the constant in* (3.4). *Moreover,*

$$U(x,t) = \int_{\mathbb{R}^n} \mathcal{P}_\sigma(x-y,t)U(y,0)\,\mathrm{d}y$$

for $(x,t) \in \mathbb{R}^{n+1}_+$, *where* $\mathcal{P}_\sigma(x)$ *is the kernel given in* (3.2).

Remark 4.1. *If we replace* $U^{\frac{n+2\sigma}{n-2\sigma}}(x,0)$ *by* $U^p(x,0)$ *for* $0 \le p < \frac{n+2\sigma}{n-2\sigma}$ *in* (4.1), *then the only nonnegative solution of* (4.1) *is* $U \equiv 0$. *Moreover, for* $p < 0$, (4.1) *has no positive solution. These can be seen from the proof of Theorem 4.1 with a standard modification (see, e.g., the proof of Theorem 1.2 in [34]). For* $\sigma \in (1/2, 1)$ *and* $1 < p < \frac{n+2\sigma}{n-2\sigma}$, *this nonexistence result has been proved in [16] using a different method.*

Remark 4.2. *We do not make any assumption on the behavior of* U *near* ∞. *If we assume that* $U \in H(t^{1-2\sigma}, \mathbb{R}^{n+1}_+)$, *the theorem in the case of* $p = \frac{n+2\sigma}{n-2\sigma}$ *follows from [41] and [97]. When* $\sigma = \frac{1}{2}$, *the above theorem can be found in [77], [78], [100], [114] and [99].*

The proof of Theorem 4.1 uses the method of moving spheres and is inspired by [100], [99] and [34]. For each $x \in \mathbb{R}^n$ and $\lambda > 0$, we define, $\overline{X} = (x, 0)$, and

$$U_{\overline{X},\lambda}(\xi) := \left(\frac{\lambda}{|\xi - \overline{X}|}\right)^{n-2\sigma} U\left(\overline{X} + \frac{\lambda^2(\xi - \overline{X})}{|\xi - \overline{X}|^2}\right), \quad \xi \in \overline{\mathbb{R}^{n+1}_+} \backslash \{\overline{X}\}, \tag{4.2}$$

the Kelvin transformation of U with respect to the ball $\mathcal{B}_\lambda(\overline{X})$. We point out that if U is a solution of (4.1), then $U_{\bar{x},\lambda}$ is a solution of (4.1) in $\mathbb{R}^{n+1}_+ \setminus \overline{\mathcal{B}^+_\varepsilon}$, for every $\bar{x} \in \partial\mathbb{R}^{n+1}_+$, $\lambda > 0$, and $\varepsilon > 0$.

By Corollary 3.1 any nonnegative weak solution U of (4.1) belongs to $L^\infty_{loc}(\overline{\mathbb{R}^{n+1}_+})$, and hence by Proposition 3.4, U is Hölder continuous and positive in $\overline{\mathbb{R}^{n+1}_+}$. By Theorem 3.2, $U(\cdot, 0)$ is smooth in $\mathbb{R}^n$. From classical elliptic equation theory, U is smooth in $\mathbb{R}^{n+1}_+$.

Lemma 4.1. *For any* $x \in \mathbb{R}^n$, *there exists a positive constant* $\lambda_0(x)$ *such that for any* $0 < \lambda < \lambda_0(x)$,

$$U_{\overline{X},\lambda}(\xi) \le U(\xi), \quad \text{in } \mathbb{R}^{n+1}_+ \backslash \mathcal{B}^+_\lambda(\overline{X}). \tag{4.3}$$

Proof. Without loss of generality we may assume that $x = 0$ and write $U_\lambda = U_{0,\lambda}$.

Step 1. We show that there exist $0 < \lambda_1 < \lambda_2$ which may depend on x, such that

$$U_\lambda(\xi) \le U(\xi),\ \forall\ 0 < \lambda < \lambda_1,\ \lambda < |\xi| < \lambda_2.$$

For every $0 < \lambda < \lambda_1 < \lambda_2$, $\xi \in \partial'' \mathcal{B}_{\lambda_2}$, we have $\frac{\lambda^2 \xi}{|\xi|^2} \in \mathcal{B}^+_{\lambda_2}$. Thus we can choose $\lambda_1 = \lambda_1(\lambda_2)$ small such that

$$\begin{aligned} U_\lambda(\xi) &= \left(\frac{\lambda}{|\xi|}\right)^{n-2\sigma} U\left(\frac{\lambda^2 \xi}{|\xi|^2}\right) \\ &\le \left(\frac{\lambda_1}{\lambda_2}\right)^{n-2\sigma} \sup_{\overline{\mathcal{B}^+_{\lambda_2}}} U \le \inf_{\partial'' \mathcal{B}^+_{\lambda_2}} U \le U(\xi). \end{aligned}$$

Hence

$$U_\lambda \le U \quad \text{on } \partial''(\mathcal{B}^+_{\lambda_2} \backslash \mathcal{B}^+_\lambda)$$

for all $\lambda_2 > 0$ and $0 < \lambda < \lambda_1(\lambda_2)$.

We will show that $U_\lambda \le U$ on $(\mathcal{B}^+_{\lambda_2} \backslash \mathcal{B}^+_\lambda)$ if λ_2 is small and $0 < \lambda < \lambda_1(\lambda_2)$. Since U_λ satisfies (4.1) in $\mathcal{B}^+_{\lambda_2} \setminus \overline{\mathcal{B}^+_{\lambda_1}}$, we have

$$\begin{cases} \operatorname{div}(t^{1-2\sigma} \nabla (U_\lambda - U)) &= 0 \quad \text{in} \quad \mathcal{B}^+_{\lambda_2} \backslash \overline{\mathcal{B}^+_\lambda}, \\ \lim\limits_{t \to 0} t^{1-2\sigma} \partial_t (U_\lambda - U) &= U^{\frac{n+2\sigma}{n-2\sigma}}(x,0) - U_\lambda^{\frac{n+2\sigma}{n-2\sigma}}(x,0) \quad \text{on} \quad \partial'(\mathcal{B}^+_{\lambda_2} \backslash \overline{\mathcal{B}^+_\lambda}). \end{cases} \tag{4.4}$$

Let $(U_\lambda - U)^+ := \max(0, U_\lambda - U)$ which equals to 0 on $\partial''(\mathcal{B}^+_{\lambda_2} \backslash \mathcal{B}^+_\lambda)$. Hence, by a density argument, we can use $(U_\lambda - U)^+$ as a test function in the definition of weak solution of (4.4). We will make use of the narrow domain technique from [11]. With the help of the mean value theorem, we have

$$\begin{aligned} &\int_{\mathcal{B}^+_{\lambda_2} \backslash \mathcal{B}^+_\lambda} t^{1-2\sigma} |\nabla (U_\lambda - U)^+|^2 \\ &= \int_{B_{\lambda_2} \backslash B_\lambda} (U_\lambda^{\frac{n+2\sigma}{n-2\sigma}}(x,0) - U^{\frac{n+2\sigma}{n-2\sigma}}(x,0))(U_\lambda - U)^+ \\ &\le C \int_{B_{\lambda_2} \backslash B_\lambda} ((U_\lambda - U)^+)^2 U_\lambda^{\frac{4\sigma}{n-2\sigma}} \\ &\le C \left(\int_{B_{\lambda_2} \backslash B_\lambda} ((U_\lambda - U)^+)^{\frac{2n}{n-2\sigma}} \right)^{\frac{n-2\sigma}{n}} \left(\int_{B_{\lambda_2} \backslash B_\lambda} U_\lambda^{\frac{2n}{n-2\sigma}} \right)^{\frac{2\sigma}{n}} \\ &\le C \left(\int_{\mathcal{B}^+_{\lambda_2} \backslash \mathcal{B}^+_\lambda} t^{1-2\sigma} |\nabla (U_\lambda - U)^+|^2 \right) \left(\int_{B_{\lambda_2}} U^{\frac{2n}{n-2\sigma}} \right)^{\frac{2\sigma}{n}}, \end{aligned}$$

where Proposition 3.1 is used in the last inequality and C is a positive constant depending only on n and σ. We fix λ_2 small such that

$$C\left(\int_{B_{\lambda_2}} U^{\frac{2n}{n-2\sigma}}\right)^{\frac{2\sigma}{n}} < 1/2.$$

Then $\nabla(U_\lambda - U)^+ = 0$ in $\mathcal{B}^+_{\lambda_2}\backslash\mathcal{B}^+_\lambda$. Since $(U_\lambda - U)^+ = 0$ on $\partial''(\mathcal{B}^+_{\lambda_2}\backslash\mathcal{B}^+_\lambda)$, $(U_\lambda - U)^+ = 0$ in $\mathcal{B}^+_{\lambda_2}\backslash\mathcal{B}^+_\lambda$. We conclude that $U_\lambda \leq U$ on $(\mathcal{B}^+_{\lambda_2}\backslash\mathcal{B}^+_\lambda)$ for $0 < \lambda < \lambda_1 := \lambda_1(\lambda_2)$.

Step 2. We show that there exists $\lambda_0 \in (0, \lambda_1)$ such that $\forall\ 0 < \lambda < \lambda_0$

$$U_\lambda(\xi) \leq U(\xi),\ |\xi| > \lambda_2,\ \xi \in \mathbb{R}^{n+1}_+.$$

Let $\phi(\xi) = \left(\frac{\lambda_2}{|\xi|}\right)^{n-2\sigma} \inf\limits_{\partial''\mathcal{B}_{\lambda_2}} U$, which satisfies

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla\phi) = 0 & \text{in } \mathbb{R}^{n+1}_+ \setminus \mathcal{B}^+_{\lambda_2} \\ -\lim_{t\to 0} t^{1-2\sigma}\partial_t\phi(x,t) = 0 & x \in \mathbb{R}^n \setminus \overline{B_{\lambda_2}}, \end{cases}$$

and $\phi(\xi) \leq U(\xi)$ on $\partial''\mathcal{B}_{\lambda_2}$. By the weak maximum principle Lemma 3.2,

$$U(\xi) \geq \left(\frac{\lambda_2}{|\xi|}\right)^{n-2\sigma} \inf_{\partial''\mathcal{B}_{\lambda_2}} U,\ \forall\ |\xi| > \lambda_2,\ \xi \in \mathbb{R}^{n+1}_+.$$

Let $\lambda_0 = \min(\lambda_1, \lambda_2(\inf\limits_{\partial''\mathcal{B}_{\lambda_2}} U/\sup\limits_{\mathcal{B}_{\lambda_2}} U)^{\frac{1}{n-2\sigma}})$. Then for any $0 < \lambda < \lambda_0,\ |\xi| \geq \lambda_2$, we have

$$U_\lambda(\xi) \leq (\frac{\lambda}{|\xi|})^{n-2\sigma} U(\frac{\lambda^2\xi}{|\xi|^2}) \leq (\frac{\lambda_0}{|\xi|})^{n-2\sigma} \sup_{\mathcal{B}_{\lambda_2}} U \leq (\frac{\lambda_2}{|\xi|})^{n-2\sigma} \inf_{\partial''\mathcal{B}_{\lambda_2}} U \leq U(\xi).$$

Lemma 4.1 is proved. □

With Lemma 4.1, we can define for all $x \in R^n$,

$$\bar{\lambda}(x) = \sup\{\mu > 0 : U_{\overline{X},\lambda} \leq U \text{ in } \mathbb{R}^{n+1}_+\backslash\mathcal{B}^+_\lambda,\ \forall\ 0 < \lambda < \mu\}.$$

By Lemma 4.1, $\bar{\lambda}(x) \geq \lambda_0(x)$.

Lemma 4.2. *If $\bar{\lambda}(x) < \infty$ for some $x \in \mathbb{R}^n$, then*

$$U_{\overline{X},\bar{\lambda}(x)} \equiv U.$$

Proof. Without loss of generality we assume that $x = 0$ and write $U_\lambda = U_{0,\lambda}$ and $\bar\lambda = \bar\lambda(0)$. By the definition of $\bar\lambda$,

$$U_{\bar\lambda} \geq U \text{ in } \overline{\mathcal{B}^+_{\bar\lambda}}\backslash\{0\},$$

and therefore, for all $0 < \varepsilon < \bar\lambda$,

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla(U_\lambda - U)) & = 0 \quad \text{in} \quad \mathcal{B}^+_\lambda\backslash\overline{\mathcal{B}^+_\varepsilon}, \\ -\lim\limits_{t\to 0} t^{1-2\sigma}\partial_t(U_\lambda - U) & \geq 0 \quad \text{on} \quad \partial'(\mathcal{B}^+_\lambda\backslash\overline{\mathcal{B}^+_\varepsilon}). \end{cases} \tag{4.5}$$

We argue by contradiction. If $U_{\bar\lambda}$ is not identically equal to U, applying the Harnack inequality Proposition 3.4 to (4.5), we have

$$U_{\bar\lambda} > U \text{ in } \overline{\mathcal{B}_{\bar\lambda}}\backslash\{\{0\}\cup\partial''\mathcal{B}_{\bar\lambda}\},$$

and in view of Proposition 4.1,

$$\liminf_{\xi\to 0}(U_{\bar\lambda}(\xi) - U(\xi)) > 0.$$

So there exist $\varepsilon_1 > 0$ and $c > 0$ such that $U_{\bar\lambda}(\xi) > U(0) + c$, $\forall\ 0 < |\xi| < \varepsilon_1$. Choose ε_2 small such that

$$\left(\frac{\bar\lambda}{\bar\lambda+\varepsilon_2}\right)^{n-2\sigma}(U(0)+c) > U(0) + \frac{c}{2}.$$

Thus for all $0 < |\xi| < \varepsilon_1$ and $\bar\lambda < \lambda < \bar\lambda + \varepsilon_2$,

$$U_\lambda(\xi) = \left(\frac{\bar\lambda}{\lambda}\right)^{n-2\sigma} U_{\bar\lambda}\left(\frac{\bar\lambda^2\xi}{\lambda^2}\right) \geq \left(\frac{\bar\lambda}{\bar\lambda+\varepsilon_2}\right)^{n-2\sigma}(U(0)+c) \geq U(0) + c/2.$$

Choose ε_3 small such that for all $0 < |\xi| < \varepsilon_3$, $U(0) > U(\xi) - c/4$. Hence for all $0 < |\xi| < \varepsilon_3$ and $\bar\lambda < \lambda < \bar\lambda + \varepsilon_2$,

$$U_\lambda(\xi) > U(\xi) + c/4.$$

For δ small, which will be fixed later, denote $K_\delta = \{\xi \in \mathbb{R}^{n+1}_+ : \varepsilon_3 \leq |\xi| \leq \bar\lambda - \delta\}$. Then there exists $c_2 = c_2(\delta)$ such that

$$U_{\bar\lambda}(X) - U(X) > c_2 \quad \text{in} \quad K_\delta.$$

By the uniform continuous of U on compact sets, there exists $\varepsilon_4 \leq \varepsilon_2$ such that for all $\bar\lambda < \lambda < \bar\lambda + \varepsilon_4$

$$U_\lambda - U_{\bar\lambda} > -c_2/2 \quad \text{in} \quad K_\delta.$$

Hence

$$U_\lambda - U > c_2/2 \quad \text{in} \quad K_\delta.$$

Now let us focus on the region $\{\xi \in \mathbb{R}^{n+1}_+ : \bar{\lambda} - \delta \le |\xi| \le \lambda\}$. Using the narrow domain technique as that in Lemma 4.1, we can choose δ small (notice that we can choose ε_4 as small as we want) such that

$$U_\lambda \ge U \quad \text{in} \quad \{\xi \in \mathbb{R}^{n+1}_+ : \bar{\lambda} - \delta \le |\xi| \le \lambda\}.$$

In conclusion, there exists ε_4 such that for all $\bar{\lambda} < \lambda < \bar{\lambda} + \varepsilon_4$

$$U_\lambda \ge U \quad \text{in} \quad \{\xi \in \mathbb{R}^{n+1}_+ : 0 < |\xi| \le \lambda\}$$

which contradicts with the definition of $\bar{\lambda}$. □

Proof of Theorem 4.1. It follows from Lemma 4.2 and similar arguments in [99] that:

(i) Either $\bar{\lambda}(x) = \infty$ for all $x \in \mathbb{R}^n$ or $\bar{\lambda}(x) < \infty$ for all $x \in \mathbb{R}^n$; (Lemma 2.3 in [99]);

(ii) If $\bar{\lambda}(x) = \infty$ for all $x \in \mathbb{R}^n$, then $U(x,t) = U(0,t)$, $\forall\ (x,t) \in \mathbb{R}^{n+1}_+$; (Lemma 11.3 in [99]);

(iii) If $\bar{\lambda}(x) < \infty$ for all $x \in \mathbb{R}^n$, then by Lemma 11.1 in [99]

$$u(x) := U(x,0) = a\left(\frac{\lambda}{1 + \lambda^2|x - x_0|^2}\right)^{\frac{n-2\sigma}{2}}, \tag{4.6}$$

where $\lambda > 0$, $a > 0$ and $x_0 \in \mathbb{R}^n$.

We claim that (ii) never happens, since this would imply, using (4.1), that

$$U(x,t) = U(0) - U(0)^{\frac{n+2\sigma}{n-2\sigma}}\frac{t^{2\sigma}}{2\sigma},$$

which contradicts to the positivity of U. Then (iii) holds.

We are only left to show that $V := U - \mathcal{P}_\sigma[u] \equiv 0$ where $u(x)$ is given in (4.6) and belongs to $\dot{H}^\sigma(\mathbb{R}^n)$. Hence, V satisfies

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla V) & = 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\ \qquad\qquad V & = 0 \quad \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases}$$

By Lemma 4.2, we know that $V_{\bar{\lambda}}$ can be extended to a Hölder continuous function near 0. Multiplying the above equation by V and integrating by parts, it leads to $\int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma}|\nabla V|^2 = 0$. Hence we have $V \equiv 0$.

Finally, $a = \left(N_\sigma c_{n,\sigma} 2^{2\sigma}\right)^{\frac{n-2\sigma}{4\sigma}}$ follows from (1.5) with $\phi = 1$ and (3.5). □

4.2 An existence result

We have established local Schauder estimates for nonnegative solutions of fractional Laplacian equations in Chapter 3 and a Liouville type theorem in Section 4.1. Now we are ready to prove the following existence result, which has been stated in Theorem 1.1 in the introduction.

Definition 4.1. *For $d > 0$, we say that $K \in C(\mathbb{S}^n)$ has flatness order greater than d at ξ if, in some local coordinate system $\{y_1, \cdots, y_n\}$ centered at ξ, there exists a neighborhood $\mathscr{O}$ of 0 such that $K(y) = K(0) + o(|y|^d)$ in $\mathscr{O}$.*

Theorem 4.2. *Let $n \geq 2$, and $K \in C^{1,1}(\mathbb{S}^n)$ be an antipodally symmetric function, i.e., $K(\xi) = K(-\xi)\ \forall\ \xi \in \mathbb{S}^n$, and be positive somewhere on $\mathbb{S}^n$. If there exists a maximum point of K at which K has flatness order greater than $n - 2\sigma$, then (1.8) has at least one positive C^2 solution.*

For $2 \leq n < 2 + 2\sigma$, $K \in C^{1,1}(\mathbb{S}^n)$ has flatness order greater than $n - 2\sigma$ at every maximum point. As mentioned in the introduction, when $\sigma = 1$, the above theorem was proved by Escobar and Schoen [59] for $n \geq 3$. On $\mathbb{S}^2$, the existence of solutions of $-\Delta_{g_{\mathbb{S}^n}} v + 1 = Ke^{2v}$ for such K was proved by Moser [112].

Denote H^σ_{as} be the set consisting of antipodally symmetric functions in $H^\sigma(\mathbb{S}^n)$. Let

$$\lambda_{as}(K) = \inf_{v \in H^\sigma_{as}} \left\{ \int_{\mathbb{S}^n} v P_\sigma(v) : \int_{\mathbb{S}^n} K|v|^{\frac{2n}{n-2\sigma}} = 1 \right\}.$$

We also denote ω_n as the volume of $\mathbb{S}^n$. The proof of Theorem 4.2 is divided into two steps.

Proposition 4.2. *Let $K \in C^{1,1}(\mathbb{S}^n)$ be antipodally symmetric and positive somewhere. If*

$$\lambda_{as}(K) < \frac{P_\sigma(1)\omega_n^{\frac{2\sigma}{n}} 2^{\frac{2\sigma}{n}}}{(\max_{\mathbb{S}^n} K)^{\frac{n-2\sigma}{n}}}, \tag{4.7}$$

then there exists a positive and antipodally symmetric $C^2(\mathbb{S}^n)$ solution of (1.8).

Proposition 4.3. *Let $K \in C^{1,1}(\mathbb{S}^n)$ be antipodally symmetric and positive somewhere. If there exists a maximum point of K at which K has flatness order greater than $n-2\sigma$,* *then*

$$\lambda_{as}(K) < \frac{P_\sigma(1)\omega_n^{\frac{2\sigma}{n}} 2^{\frac{2\sigma}{n}}}{(\max_{\mathbb{S}^n} K)^{\frac{n-2\sigma}{n}}}. \tag{4.8}$$

Proof of Theorem 4.2. It follows from Proposition 4.2 and Proposition 4.3. □

The proof of Proposition 4.2 uses subcritical approximations. For $1 < p < \frac{n+2\sigma}{n-2\sigma}$, we define

$$\lambda_{as,p}(K) = \inf_{v \in H^\sigma_{as}} \left\{ \int_{\mathbb{S}^n} v P_\sigma(v) : \int_{\mathbb{S}^n} K|v|^{p+1} = 1 \right\}.$$

We begin with a lemma

Lemma 4.3. *Let $K \in C^{1,1}(\mathbb{S}^n)$ be antipodally symmetric and positive somewhere. Then $\lambda_{as,p}(K)$ is achieved by a positive and antipodally symmetric $C^2(\mathbb{S}^n)$ function v_p, which satisfies*

$$P_\sigma(v_p) = \lambda_{as,p}(K) K v_p^p \quad \textit{and} \quad \int_{\mathbb{S}^n} K v_p^{p+1} = 1. \tag{4.9}$$

Proof. The existence of nonnegative minimizer v_p follows from standard variation methods and the inequality $\int_{\mathbb{S}^n} |v| P_\sigma(|v|) \leq \int_{\mathbb{S}^n} v P_\sigma(v)$ for all $v \in H^\sigma(\mathbb{S}^n)$. Then v_p is positive everywhere by the Harnack inequality (see [27], [128] or Proposition 3.4 in the extension point of view). The regularity of v_p follows from Proposition 3.4 and Theorem 3.1. □

Proof of Proposition 4.2. First of all, it is easy to see that

$$\limsup_{p \to \frac{n+2\sigma}{n-2\sigma}} \lambda_{as,p}(K) \leq \lambda_{as}(K).$$

Indeed, for any $\varepsilon > 0$, there exists a nonnegative function $v \in H^\sigma_{as}$ such that

$$\int_{\mathbb{S}^n} v P_\sigma(v) < \lambda_{as} + \varepsilon \quad \text{and} \quad \int_{\mathbb{S}^n} K v^{\frac{2n}{n-2\sigma}} = 1.$$

Let $V_p := \int_{\mathbb{S}^n} K v^{p+1}$. Since $\lim_{p \to \frac{n+2\sigma}{n-2\sigma}} V_p = \int_{\mathbb{S}^n} K v^{\frac{2n}{n-2\sigma}} = 1$, we have, for p closed to $\frac{n+2\sigma}{n-2\sigma}$,

$$\lambda_{as,p}(K) \leq \int_{\mathbb{S}^n} \frac{v}{V_p^{1/(p+1)}} P_\sigma \left(\frac{v}{V_p^{1/(p+1)}} \right) \leq \lambda_{as}(K) + 2\varepsilon.$$

Hence, we may assume that there exists a sequence $\{p_i\} \to \frac{n+2\sigma}{n-2\sigma}$ such that $\lambda_{as,p_i}(K) \to \lambda$ for some $\lambda \leq \lambda_{as}(K)$. Since $\{v_i\}$, which is a sequence of minimizers in Lemma 4.3 for $p = p_i$, is bounded in $H^\sigma(\mathbb{S}^n)$, then there exists $v \in H^\sigma(\mathbb{S}^n)$ such that $v_i \rightharpoonup v$ weakly in

$H^\sigma(\mathbb{S}^n)$ and v is nonnegative. If $v \not\equiv 0$, it follows from (1.7) that $v > 0$ on $\mathbb{S}^n$, and we are done. Now we suppose that $v \equiv 0$. If $\{\|v_i\|_{L^\infty(\mathbb{S}^n)}\}$ is bounded, by the local estimates established in Section 3.3 we have $\{\|v_i\|_{C^2(\mathbb{S}^n)}\}$ is bounded, too. Therefore, $v_i \to 0$ in $C^1(\mathbb{S}^n)$ which leads to $1 = \int_{\mathbb{S}^n} K|v_i|^{p_i+1} \to 0$. This is a contradiction. Thus we may assume that $v_i(x_i) := \max_{\mathbb{S}^n} v_i \to \infty$. Since $\mathbb{S}^n$ is compact, there exists a subsequence of $\{x_i\}$, which will be still denoted as $\{x_i\}$, and $\bar{x}$ such that $x_i \to \bar{x}$. Without loss of generality we assume that $\bar{x}$ is the south pole. Via the stereographic projection F^{-1}, (4.9) becomes

$$(-\Delta)^\sigma u_i(y) = \lambda_{as,p_i}(K) K \circ F(y) \left(\frac{2}{1+|y|^2}\right)^{\varepsilon_i} u_i^{p_i}(y), \quad y \in \mathbb{R}^n \tag{4.10}$$

where $v_i \circ F(y) = (\frac{1+|y|^2}{2})^{\frac{n-2\sigma}{2}} u_i(y)$ and $\varepsilon_i = \frac{n+2\sigma-p_i(n-2\sigma)}{2}$. Thus for any $y \in \mathbb{R}^n$, $u_i(y) \le 2^{\frac{n-2\sigma}{2}} u_i(y_i)$ where $y_i := F^{-1}(x_i) \to 0$. For simplicity, we denote $m_i := u_i(y_i)$. By our assumption on v_i we have $m_i \to \infty$. Define

$$\tilde{u}_i(y) = (m_i)^{-1} u_i\big((m_i)^{\frac{1-p_i}{2\sigma}} y + y_i\big).$$

From (4.10) we see that $\tilde{u}_i(y)$ satisfies, for any $y \in \mathbb{R}^n$,

$$\begin{aligned}(-\Delta)^\sigma \tilde{u}_i(y) =& \lambda_{as,p_i}(K) K \circ F(m_i^{\frac{1-p_i}{2\sigma}} y + y_i) \\ &\cdot \left(\frac{2}{1+|(m_i)^{\frac{1-p_i}{2\sigma}} y + y_i|^2}\right)^{\varepsilon_i} \tilde{u}_i^i(y).\end{aligned} \tag{4.11}$$

Since $0 < \tilde{u}_i \le 2^{\frac{n-2\sigma}{2}}$, by the local estimates in Section 3.3 $\{\tilde{u}_i\}$ is bounded in $C^2_{loc}(\mathbb{R}^n)$. Note that since $\{v_i\}$ is bounded in $H^\sigma(\mathbb{S}^n)$, $\{\tilde{u}_i\}$ is bounded in $\dot{H}^\sigma(\mathbb{R}^n)$. Then there exists $u \in C^2(\mathbb{R}^n) \cap \dot{H}^\sigma(\mathbb{R}^n)$ such that, by passing to a subsequence, $\tilde{u}_i \to u$ in $C^2_{loc}(\mathbb{R}^n)$, $u(0) = 1$, $\tilde{u}_i \rightharpoonup u$ weakly in $\dot{H}^\sigma(\mathbb{R}^n)$ and u weakly satisfies

$$(-\Delta)^\sigma u = \lambda K(\bar{x}) u^{\frac{n+2\sigma}{n-2\sigma}}. \tag{4.12}$$

Hence $\lambda > 0$, $K(\bar{x}) > 0$, and the solutions of (4.12) are classified in [41] and [97] (see also Theorem 4.1).

For $x \in \mathbb{S}^n$ and $r > 0$, we denote $\mathcal{B}(x, r)$ be the geodesic ball centered at x with radius r on $\mathbb{S}^n$, and for $y \in \mathbb{R}^n$ and $R > 0$, we denote $B(y, R)$ be the Euclidean ball in

$\mathbb{R}^n$ of center y and radius R. For any $R>0$, let $\Omega_i := F(B(y_i, m_i^{\frac{1-p_i}{2\sigma}} R))$, we have

$$\begin{aligned}
&\int_{\Omega_i} K v_i^{p_i+1} = \int_{B(y_i, m_i^{\frac{1-p_i}{2\sigma}} R)} K\circ F(y) \left(\frac{2}{1+|y|^2}\right)^{\varepsilon_i} u_i^{p_i+1} \\
&= \int_{B(0,R)} m_i^{\frac{\varepsilon_i}{2}} K\circ F((m_i)^{\frac{1-p_i}{2\sigma}} y + y_i) \left(\frac{2}{1+|(m_i)^{\frac{1-p_i}{2\sigma}} y + y_i|^2}\right)^{\varepsilon_i} \tilde{u}_i^{p_i}(y) \\
&\geq K(\bar{x}) \int_{B(0,R)} u^{\frac{2n}{n-2\sigma}} + o(1)
\end{aligned}$$

as $p_i \to \frac{n+2\sigma}{n-2\sigma}$, where we used that K is positive near $\bar{x}$, $\varepsilon_i \to 0$ and $\tilde{u}_i \to u$ in $C^2_{loc}(\mathbb{R}^n)$. Since K and v_i are antipodally symmetric, we have, by taking δ small and R sufficiently large,

$$\begin{aligned}
1 = \int_{\mathbb{S}^n} K v_i^{p_i+1} &\geq 2\int_{\mathcal{B}(x_1,\delta)} K v_i^{p_i+1} + \int_{\{K<0\}} K v_i^{p_i+1} \\
&= 2K(\bar{x}) \int_{\mathbb{R}^n} u^{\frac{2n}{n-2\sigma}} + \int_{\{K<0\}} K v_i^{p_i+1} + o(1).
\end{aligned} \tag{4.13}$$

We claim that

$$\int_{\{K<0\}} K v_i^{p_i+1} \to 0$$

as $p_i \to \frac{n+2\sigma}{n-2\sigma}$. Indeed, for any $\varepsilon>0$, it is not difficult to show, by blow up analysis, that $\|v_i\|_{L^\infty(\Omega_{\varepsilon/4})} \leq C(\varepsilon)$ where $\Omega_\varepsilon := \{x\in\mathbb{S}^n : K(x) < -\varepsilon\}$ and $C(\varepsilon)$ is independent of p_i. By the local estimates established in Section 3.3, we have $\|v_i\|_{C^2(\Omega_{\varepsilon/2})} \leq C(\varepsilon)$ and hence $v_i \to 0$ in $C^1(\overline{\Omega}_\varepsilon)$ (recall that we assumed that $v_i \rightharpoonup 0$ weakly in $H^\sigma(\mathbb{S}^n)$). Thus when p_i is sufficiently close to $\frac{n+2\sigma}{n-2\sigma}$,

$$\int_{\Omega_\varepsilon} |K| v_i^{p_i+1} < \varepsilon.$$

On the other hand, by Hölder inequality and Sobolev inequality,

$$\int_{-\varepsilon\leq K<0} |K| v_i^{p_i+1} < C(n,\sigma)\varepsilon \|v_i\|^{p_i+1}_{L^{\frac{2n}{n-2\sigma}}} \leq C(n,\sigma,\lambda_{as})\varepsilon,$$

which finishes the proof of our claim. Thus, (4.13) leads to

$$1 \geq 2K(\bar{x}) \int_{\mathbb{R}^n} u^{\frac{2n}{n-2\sigma}} + o(1). \tag{4.14}$$

By the sharp Sobolev inequality (1.9), (4.12) and (4.14), we have

$$
\begin{aligned}
P_\sigma(1)\omega_n^{\frac{2\sigma}{n}} \le \frac{\int_{\mathbb{R}^n} u(-\Delta)^\sigma u}{\left(\int_{\mathbb{R}^n} u^{\frac{2n}{n-2\sigma}}\right)^{\frac{n-2\sigma}{n}}} &= \lambda K(\bar{x}) \left(\int_{\mathbb{R}^n} u^{\frac{2n}{n-2\sigma}}\right)^{\frac{2\sigma}{n}} \\
&\le \lambda_{as}(K) K(\bar{x})(2K(\bar{x}))^{-2\sigma/n} \\
&\le \lambda_{as}(K) 2^{-2\sigma/n} (\max_{\mathbb{S}^n} K)^{1-2\sigma/n},
\end{aligned}
$$

which contradicts with (4.7). □

Next we shall prove Proposition 4.3. The test function we are going to construct is inspired by [75, 117].

Proof of Proposition 4.3. Let ξ_1 be a maximum point of K at which K has flatness order greater than $n-2\sigma$. Suppose ξ_2 is the antipodal point of ξ_1. For $\beta > 1$ and $i = 1, 2$ we define

$$
v_{i,\beta}(x) = \left(\frac{\sqrt{\beta^2-1}}{\beta - \cos r_i}\right)^{\frac{n-2\sigma}{2}} \tag{4.15}
$$

where $r_i = d(x, \xi_i)$ is the geodesic distance between x and ξ_i on the sphere. It is clear that

$$
P_\sigma(v_{i,\beta}) = P_\sigma(1) v_{i,\beta}^{\frac{n+2\sigma}{n-2\sigma}}, \quad \int_{\mathbb{S}^n} v_{i,\beta}^{\frac{2n}{n-2\beta}} = \omega_n.
$$

Let

$$
v_\beta = v_{1,\beta} + v_{2,\beta},
$$

which is antipodally symmetric. Then

$$
\begin{aligned}
\int_{\mathbb{S}^n} v_\beta P_\sigma(v_\beta) &= P_\sigma(1) \int_{\mathbb{S}^n} \sum_{i=1}^2 v_{i,\beta}^{\frac{n+2\sigma}{n-2\sigma}} \sum_{j=1}^2 v_{j,\beta} \\
&= P_\sigma(1) \int_{\mathbb{S}^n} \sum_{i=1}^2 v_{i,\beta}^{\frac{2n}{n-2\sigma}} + 2 v_{1,\beta}^{\frac{n+2\sigma}{n-2\sigma}} v_{2,\beta} \\
&= P_\sigma(1) 2\omega_n \left(1 + \omega_n^{-1} \int_{\mathbb{S}^n} v_{1,\beta}^{\frac{n+2\sigma}{n-2\sigma}} v_{2,\beta}\right).
\end{aligned}
$$

By direct computations with change of variables, we have

$$
\int_{\mathbb{S}^n} v_{1,\beta}^{\frac{n+2\sigma}{n-2\sigma}} v_{2,\beta} = A(\beta-1)^{\frac{n-2\sigma}{2}} + o\big((\beta-1)^{\frac{n-2\sigma}{2}}\big)
$$

for β close to 1, where

$$A = 2^{-\frac{n-2\sigma}{2}}\omega_{n-1}\int_0^{+\infty}\frac{2^n r^{n-1}}{(1+r^2)^{\frac{n+2\sigma}{2}}}dr > 0.$$

Choose a sufficiently small neighborhood V_1 of ξ_1 and let $V_2 = \{x \in \mathbb{S}^n : -x \in V_1\}$. Then K is positive in $V_1 \cup V_2$ and

$$\begin{aligned}
\int_{\mathbb{S}^n} K v_\beta^{\frac{2n}{n-2\sigma}} &= \int_{\cup V_i} K\left(v_{1,\beta}+v_{2,\beta}\right)^{\frac{2n}{n-2\sigma}} + \int_{\mathbb{S}^n\setminus\cup V_i} K v_\beta^{\frac{2n}{n-2\sigma}} \\
&= 2\int_{V_1} K\left(v_{1,\beta}+v_{2,\beta}\right)^{\frac{2n}{n-2\sigma}} + \int_{\mathbb{S}^n\setminus\cup V_i} K v_\beta^{\frac{2n}{n-2\sigma}} \\
&\geq 2\int_{V_1} K\left(v_{1,\beta}^{\frac{2n}{n-2\sigma}} + \frac{2n}{n-2\sigma}v_{1,\beta}^{\frac{n+2\sigma}{n-2\sigma}}v_{2,\beta}\right) + \int_{\mathbb{S}^n\setminus\cup V_i} K v_\beta^{\frac{2n}{n-2\sigma}}.
\end{aligned}$$

Since $K(x)$ is flat of order $n-2\sigma$ at ξ_1, we have in V_1 that,

$$K(x) = K(\xi_1) + o(1)|x-\xi_1|^{n-2\sigma}.$$

Thus

$$\begin{aligned}
\int_{\mathbb{S}^n} K v_\beta^{\frac{2n}{n-2\sigma}} &\geq 2K(\xi_1)\int_{\mathbb{S}^n} v_{1,\beta}^{\frac{2n}{n-2\sigma}} + \frac{4nA}{n-2\sigma}K(\xi_1)(\beta-1)^{\frac{n-2\sigma}{2}} + o\big((\beta-1)^{\frac{n-2\sigma}{2}}\big) \\
&= 2K(\xi_1)\omega_n\left(1 + \frac{2nA}{n-2\sigma}\omega_n^{-1}(\beta-1)^{\frac{n-2\sigma}{2}} + o\big((\beta-1)^{\frac{n-2\sigma}{2}}\big)\right)
\end{aligned}$$

for β close to 1. Hence

$$\frac{\int_{\mathbb{S}^n} v_\beta P_\sigma(v_\beta)}{\left(\int_{\mathbb{S}^n} K v_\beta^{\frac{2n}{n-2\sigma}}\right)^{\frac{n-2\sigma}{n}}} \leq \frac{P_\sigma(1)\omega_n^{\frac{2\sigma}{n}}2^{\frac{2\sigma}{n}}}{K(\xi_1)^{\frac{n-2\sigma}{n}}}\left(1 - \frac{A}{\omega_n}(\beta-1)^{\frac{n-2\sigma}{2}} + o\big((\beta-1)^{\frac{n-2\sigma}{2}}\big)\right),$$

which implies (4.8) holds. $\square$

Theorem 4.2 can be extended to positive functions K which are invariant under some isometry group acting without fixed points (see [75, 117]). Denote $Isom(\mathbb{S}^n)$ as the isometry group of the standard sphere $(\mathbb{S}^n, g_{\mathbb{S}^n})$. Let G be a subgroup of $Isom(\mathbb{S}^n)$. We say that G acts without fixed points if for each $x \in \mathbb{S}^n$, the orbit $O_G(x) := \{g(x)|g \in G\}$ has at least two elements. We denote $|O_G(x)|$ be the number of elements in $O_G(x)$. A function K is called G-invariant if $K \circ g \equiv K$ for all $g \in G$.

Theorem 4.3. *Let G be a finite subgroup of $Isom(\mathbb{S}^n)$ and act without fixed points. Let $K \in C^{1,1}(\mathbb{S}^n)$ be a positive and G-invariant function. If there exists $\xi_0 \in \mathbb{S}^n$ such* *that K has flatness order greater than $n-2\sigma$ at ξ_0, and for any $x \in \mathbb{S}^n$*

$$\frac{K(\xi_0)}{|O_G(\xi_0)|^{\frac{2\sigma}{n-2\sigma}}} \geq \frac{K(x)}{|O_G(x)|^{\frac{2\sigma}{n-2\sigma}}}, \tag{4.16}$$

then (1.8) *possesses a positive and G-invariant $C^2(\mathbb{S}^n)$ solution.*

Denote H_G^σ be the set consisting of G-invariant functions in $H^\sigma(\mathbb{S}^n)$. Let

$$\lambda_G(K) = \inf_{v \in H_G^\sigma} \left\{ \int_{\mathbb{S}^n} v P_\sigma(v) : \int_{\mathbb{S}^n} K|v|^{\frac{2n}{n-2\sigma}} = 1 \right\}.$$

Similar to Theorem 4.2, the proof of Theorem 4.3 is again divided into two steps.

Proposition 4.4. *Let G be a finite subgroup of $Isom(\mathbb{S}^n)$. Let $K \in C^{1,1}(\mathbb{S}^n)$ be a positive and G-invariant function. If for all $x \in \mathbb{S}^n$,*

$$\lambda_G(K) < \frac{P_\sigma(1)\omega_n^{\frac{2\sigma}{n}} |O_G(x)|^{\frac{2\sigma}{n}}}{K(x)^{\frac{n-2\sigma}{n}}}, \tag{4.17}$$

then there exists a positive G-invariant $C^2(\mathbb{S}^n)$ solution of (1.8).

Proposition 4.5. *Let G be a finite subgroup of $Isom(\mathbb{S}^n)$ and act without fixed points. Let $K \in C^{1,1}(\mathbb{S}^n)$ be a positive and G-invariant function. If K has flatness order greater than $n-2\sigma$ at ξ_1 for some $\xi_1 \in \mathbb{S}^n$, then*

$$\lambda_G(K) < \frac{P_\sigma(1)\omega_n^{\frac{2\sigma}{n}} |O_G(\xi_1)|^{\frac{2\sigma}{n}}}{K(\xi_1)^{\frac{n-2\sigma}{n}}}. \tag{4.18}$$

Theorem 4.3 follows from Proposition 4.4 and Proposition 4.5 immediately. The proof of Proposition 4.4 uses subcritical approximations and blow up analysis, which is similar to that of Proposition 4.2. Proposition 4.5 can be verified by the following G-invariant test function

$$v_\beta = \sum_{i=1}^m v_{i,\beta},$$

where $m = |O_G(\xi_1)|$, $O_G(\xi_1) = \{\xi_1, \ldots, \xi_m\}$, $\xi_i = g_i(\xi_1)$ for some $g_i \in G$, $g_1 = Id$, $v_{j,\beta} := v_{1,\beta} \circ g_i^{-1}$ and $v_{1,\beta}$ is as in (4.15). We omit the detailed proofs of Propositions 4.4 and 4.5, and leave them to the readers.

4.3 Compactness of solutions

Given the regularity properties established in Chapter 3, and Theorem 4.1, we can adapt the blow up analysis developed in [118], [119] and [95] to give accurate blow up profiles for solutions of degenerate elliptic equations (1.12), which are further used to obtain compactness of solutions. For $\sigma = \frac{1}{2}$, some related results have been proved in [74] and [58], where equations are elliptic. Existence then follows from these a priori estimates, a perturbation result and some degree arguments. The detailed proofs of the following theorems can be found in [82, 83].

Theorem 4.4. *Let $n \geq 2$. Suppose that $K \in C^{1,1}(\mathbb{S}^n)$ is a positive function satisfying that for any critical point ξ_0 of K, in some geodesic normal coordinates $\{y_1, \cdots, y_n\}$ centered at ξ_0, there exist some small neighborhood $\mathscr{O}$ of 0 and positive constants $\beta = \beta(\xi_0) \in (n-2\sigma, n)$, $\gamma \in (n-2\sigma, \beta]$ such that $K \in C^{[\gamma],\gamma-[\gamma]}(\mathscr{O})$ (where $[\gamma]$ is the integer part of γ) and*

$$K(y) = K(0) + \sum_{j=1}^{n} a_j |y_j|^\beta + R(y), \quad \text{in } \mathscr{O},$$

where $a_j = a_j(\xi_0) \neq 0$, $\sum_{j=1}^n a_j \neq 0$, $R(y) \in C^{[\beta]-1,1}(\mathscr{O})$ satisfies $\sum_{s=0}^{[\beta]} |\nabla^s R(y)||y|^{-\beta+s} \to 0$ as $y \to 0$. If

$$\sum_{\xi \in \mathbb{S}^n \text{ such that } \nabla_{g_{\mathbb{S}^n}} K(\xi)=0,\ \sum_{j=1}^n a_j(\xi)<0} (-1)^{i(\xi)} \neq (-1)^n,$$

where

$$i(\xi) = \#\{a_j(\xi) : \nabla_{g_{\mathbb{S}^n}} K(\xi) = 0, a_j(\xi) < 0, 1 \leq j \leq n\},$$

then (1.8) has at least one C^2 positive solution. Moreover, there exists a positive constant C depending only on n, σ and K such that for all positive C^2 solutions v of (1.8),

$$1/C \leq v \leq C \quad \text{and} \quad \|v\|_{C^2(\mathbb{S}^n)} \leq C.$$

For $n = 3, \sigma = 1$, the existence part of the above theorem was established by Bahri and Coron [8], and the compactness part were given in Chang, Gursky and Yang [37] and Schoen and Zhang [119]. For $n \geq 4, \sigma = 1$, the above theorem was proved by Li [95].

We now consider a class of functions K more general than that in Theorem 4.4, which is modified from [95].

Definition 4.2. *For any real number $\beta \geq 1$, we say that a sequence of functions $\{K_i\}$ satisfies condition $(*)'_\beta$ for some sequence of constants $L(\beta, i)$ in some region Ω_i, if $\{K_i\} \in C^{[\beta],\beta-[\beta]}(\Omega_i)$ satisfies*

$$[\nabla^{[\beta]} K_i]_{C^{\beta-[\beta]}(\Omega_i)} \leq L(\beta, i),$$

and, if $\beta \geq 2$, that

$$|\nabla^s K_i(y)| \leq L(\beta, i)|\nabla K_i(y)|^{(\beta-s)/(\beta-1)},$$

for all $2 \leq s \leq [\beta]$, $y \in \Omega_i$, $\nabla K_i(y) \neq 0$.

Note that the function K in Theorem 4.4 satisfies $(*)'_\beta$ condition.

Remark 4.3. *For $1 \leq \beta_1 \leq \beta_2$, if $\{K_i\}$ satisfies $(*)'_{\beta_2}$ for some sequences of constants $\{L(\beta_2, i)\}$ in some regions Ω_i, then $\{K_i\}$ satisfies $(*)'_{\beta_1}$ for $\{L(\beta_1, i)\}$, where*

$$L(\beta_1, i) = \begin{cases} L(\beta_2, i) \max\Big(\max\limits_{2 \leq s \leq [\beta_1]} \|\nabla K_i\|_{L^\infty(\Omega_i)}^{\frac{\beta_2-s}{\beta_2-1} - \frac{\beta_1-s}{\beta_1-1}}, \operatorname{diam}(\Omega_i)^{\beta_2-\beta_1}\Big), & \text{if } [\beta_2] = [\beta_1] \\ L(\beta_2, i) \max\Big(\max\limits_{2 \leq s \leq [\beta_1]} \|\nabla K_i\|_{L^\infty(\Omega_i)}^{\frac{\beta_2-s}{\beta_2-1} - \frac{\beta_1-s}{\beta_1-1}}, \|\nabla K_i\|_{L^\infty(\Omega_i)}^{\frac{\beta_2-[\beta_1]-1}{\beta_2-1}} \operatorname{diam}(\Omega_i)^{1+[\beta_1]-\beta_1}\Big), & \\ & \text{if } [\beta_2] > [\beta_1] \end{cases}$$

in the corresponding regions.

The following theorem gives a priori bounds of solutions in $L^{\frac{2n}{n-2\sigma}}$ norm.

Theorem 4.5. *Let $n \geq 2$, and $K \in C^{1,1}(\mathbb{S}^n)$ be a positive function. If there exists some constant $d > 0$ such that K satisfies $(*)'_{(n-2\sigma)}$ for some constant $L > 0$ in $\Omega_d := \{\xi \in \mathbb{S}^n : |\nabla_{g_{\mathbb{S}^n}} K(\xi)| < d\}$, then for any positive solution $v \in C^2(\mathbb{S}^n)$ of (1.8),*

$$\|v\|_{L^{\frac{2n}{n-2\sigma}}(\mathbb{S}^n)} \leq C, \tag{4.19}$$

where C depends only on $n, \sigma, \inf_{\mathbb{S}^n} K > 0, \|K\|_{C^{1,1}(\mathbb{S}^n)}, L$, and d.

For $n = 3$ and $\sigma = 1$, the above theorem was proved by Chang, Gursky and Yang in [37] and by Schoen and Zhang in [119]. For $n \geq 4$ and $\sigma = 1$, the above theorem was proved by Li in [95].

The estimate (4.19) for the solution v is equivalent to

$$\|v\|_{H^\sigma(\mathbb{S}^n)} \leq C.$$

However, the estimate (4.19) is not sufficient to imply L^∞ bound for v on $\mathbb{S}^n$. For instance,

$$\int_{\mathbb{S}^n} v_{\xi_0,\lambda}^{\frac{2n}{n-2\sigma}}(\xi)\, \mathrm{d}vol_{g_{\mathbb{S}^n}} = \int_{\mathbb{S}^n} \mathrm{d}vol_{g_{\mathbb{S}^n}},$$

but $v_{\xi_0,\lambda}(\xi_0) = \lambda^{\frac{n-2\sigma}{2}} \to \infty$ as $\lambda \to \infty$. Furthermore, a sequence of solutions v_i may blow up at more than one point, and it is the case when $\sigma = 1$ (see [96]). The following theorem shows that the latter situation does not happen when K satisfies a little stronger condition.

Theorem 4.6. *Let $n \geq 2$. Suppose that $\{K_i\} \in C^{1,1}(\mathbb{S}^n)$ is a sequence of positive functions with uniform $C^{1,1}$ norm and $1/A_1 \leq K_i \leq A_1$ on $\mathbb{S}^n$ for some $A_1 > 0$ independent of i. Suppose also that $\{K_i\}$ satisfying $(*)'_\beta$ condition for some constants $\beta > n - 2\sigma$, $L, d > 0$ in Ω_d. Let $\{v_i\} \in C^2(\mathbb{S}^n)$ be a sequence of corresponding positive solutions of (1.8) and $v_i(\xi_i) = \max_{\mathbb{S}^n} v_i$ for some ξ_i. Then, after passing to a subsequence, $\{v_i\}$ is either bounded in $L^\infty(\mathbb{S}^n)$ or blows up at exactly one point in the strong sense: There exists a sequence of Möbius diffeomorphisms $\{\varphi_i\}$ from $\mathbb{S}^n$ to $\mathbb{S}^n$ satisfying $\varphi_i(\xi_i) = \xi_i$ and $|\det \mathrm{d}\varphi_i(\xi_i)|^{\frac{n-2\sigma}{2n}} = v_i^{-1}(\xi_i)$ such that*

$$\|T_{\varphi_i} v_i - 1\|_{C^0(\mathbb{S}^n)} \to 0, \quad \text{as } i \to \infty,$$

where $T_{\varphi_i} v_i := (v \circ \varphi_i)|\det \mathrm{d}\varphi_i|^{\frac{n-2\sigma}{2n}}$.

For $n = 3$ and $\sigma = 1$, the above theorem was established by Chang, Gursky and Yang in [37] and by Schoen and Zhang in [119]. For $n \geq 4$ and $\sigma = 1$, the above theorem was proved by Li in [95].

4.4 Improvement of the best Sobolev constant: an Aubin type inequality

Let

$$\mathscr{M}^p = \left\{ v \in H^\sigma(\mathbb{S}^n) : \fint_{\mathbb{S}^n} |v|^p \,\mathrm{d}v_{g_{\mathbb{S}^n}} = 1 \right\},$$

$$\mathscr{M}_0^p = \left\{ v \in \mathscr{M} : \fint_{\mathbb{S}^n} x|v|^p \, \mathrm{d}v_{g_{\mathbb{S}^n}} = 0 \right\}.$$

The Sobolev inequality (1.9) states that

$$\min_{v \in \mathscr{M}^{\frac{2n}{n-2\sigma}}} \fint v P_\sigma(v) \geq P_\sigma(1).$$

We will show the following Aubin inequality

Proposition 4.6. *For $n \geq 2$, $2 < p \leq \frac{n+2\sigma}{n-2\sigma}$, given any $\varepsilon > 0$, there exists some constant C_ε such that*

$$\inf_{v \in \mathscr{M}_0^p} \left\{ 2^{\frac{2}{p}-1}(1+\varepsilon) \fint_{\mathbb{S}^n} v P_\sigma(v) + C_\varepsilon \fint v^2 \right\} \geq P_\sigma(1). \tag{4.20}$$

When $\sigma = 1$, the above proposition was proved by Aubin [5]. See also [53] for such inequalities in higher order Sobolev spaces. We will adapt the proof of Theorem 2 in [5] to show (4.20).

Proof. First of all, by Hölder inequality, (1.9) and (1.7), we have for all $v \in H^\sigma(\mathbb{S}^n)$,

$$\begin{aligned} \left(\int_{\mathbb{S}^n} v^p \right)^{\frac{2}{p}} &\leq K^2 \int_{\mathbb{S}^n} v P_\sigma(v) \\ &= K^2 P_\sigma(1) \int_{\mathbb{S}^n} v^2 + \frac{K^2 c_{n,-\sigma}}{2} \iint_{\mathbb{S}^n \times \mathbb{S}^n} \frac{(v(x)-v(y))^2}{|x-y|^{n+2\sigma}}, \end{aligned} \tag{4.21}$$

where $K^2 := |\mathbb{S}^n|^{\frac{2}{p}-1}(P_\sigma(1))^{-1}$. Let $\eta \in (0, \frac{1}{2})$ to be chosen later. Let Λ be the space of first spherical harmonics. Following [5], there exists $\{\xi_i\}_{i=1,\cdots,k}$ such that $1 + \eta < \sum_{i=1}^k |\xi_i|^{\frac{2}{p}} < 1 + 2\eta$ with $|\xi_i| < 2^{-p}$. Let $h_i \in C^1(\mathbb{S}^n)$ be such that $h_i \xi_i \geq 0$ on $\mathbb{S}^n$ and

$$\left| |h_i|^2 - |\xi_i|^{\frac{2}{p}} \right| < \left(\frac{\eta}{k} \right)^p.$$

Then

$$1 < \sum_{i=1}^k |h_i|^2 < 1 + 3\eta$$

and by mean value theorem

$$\left| |h_i|^p - |\xi| \right| \leq \frac{2}{p} \left(\frac{\eta}{k} \right)^p.$$

For any nonnegative $v \in H^\sigma(\mathbb{S}^n)$, we have, by Hölder inequality,

$$\left(\int_{\mathbb{S}^n} v^p \right)^{\frac{2}{p}} \leq \sum_{i=1}^k \left(\int_{\mathbb{S}^n} |h_i|^p v^p \right)^{\frac{2}{p}}.$$

For $v \in \mathscr{M}_0^p$, one has that

$$\int_{\mathbb{S}^n} \xi_{i+} v^p = \int_{\mathbb{S}^n} \xi_{i-} v^p.$$

Hence for nonnegative function $v \in \mathscr{M}_0^p$, we have, noticing that $h_i \xi_i \geq 0$,

$$\begin{aligned}
\left(\int_{\mathbb{S}^n} |h_i|^p v^p\right)^{\frac{2}{p}} &= \left(\int_{\mathbb{S}^n} h_{i+}^p v^p + \int_{\mathbb{S}^n} h_{i-}^p v^p\right)^{\frac{2}{p}} \\
&\leq \left(\int_{\mathbb{S}^n} \xi_{i+} v^p + \varepsilon_0^p v^p + \int_{\mathbb{S}^n} h_{i-}^p v^p\right)^{\frac{2}{p}} \\
&\leq 2^{\frac{2}{p}} \left(\int_{\mathbb{S}^n} \varepsilon_0^p v^p + \int_{\mathbb{S}^n} h_{i-}^p v^p\right)^{\frac{2}{p}} \\
&\leq 2^{\frac{2}{p}} \left(\int_{\mathbb{S}^n} (\varepsilon_0 + h_{i-})^p v^p\right)^{\frac{2}{p}} \\
&\leq 2^{\frac{2}{p}} \left(K^2 P_\sigma(1) \int_{\mathbb{S}^n} (h_{i-} + v a_0)^2 v^2) + \frac{K^2 c_{n,-\sigma}}{2} I\right),
\end{aligned}$$

where

$$\begin{aligned}
I = &\iint_{\mathbb{S}^n \times \mathbb{S}^n} \frac{((h_{i-}(x) + \varepsilon_0) v(x) - (h_{i-}(y) + \varepsilon_0) v(y))^2}{|x-y|^{n+2\sigma}} \\
\leq &\int_{\mathbb{S}^n} v^2(x) \int_{\mathbb{S}^n} \frac{(h_{i-}(x) - h_{i-}(y))^2}{|x-y|^{n+2\sigma}} + \int_{\mathbb{S}^n} (h_{i-}(y) + \varepsilon_0)^2 \int_{\mathbb{S}^n} \frac{(v(x) - v(y))^2}{|x-y|^{n+2\sigma}} \\
&+ 2C \left(\iint_{\mathbb{S}^n \times \mathbb{S}^n} \frac{(v(x) - v(y))^2}{|x-y|^{n+2\sigma}}\right)^{\frac{1}{2}} \left(\iint_{\mathbb{S}^n \times \mathbb{S}^n} \frac{v^2(x)(h_{i-}(x) - h_{i-}(y))^2}{|x-y|^{n+2\sigma}}\right)^{\frac{1}{2}} \\
\leq &C \int_{\mathbb{S}^n} v^2 + \int_{\mathbb{S}^n} (h_{i-}(y) + \varepsilon_0)^2 \int_{\mathbb{S}^n} \frac{(v(x) - v(y))^2}{|x-y|^{n+2\sigma}} + \frac{\eta}{k} \iint_{\mathbb{S}^n \times \mathbb{S}^n} \frac{(v(x) - v(y))^2}{|x-y|^{n+2\sigma}}.
\end{aligned}$$

Also we can do the same thing in terms of h_{i+}. Hence

$$\begin{aligned}
&2\left(\int_{\mathbb{S}^n} v^p\right)^{\frac{2}{p}} \\
&\leq 2^{\frac{2}{p}} \sum_{i=1}^{k} \frac{K^2 c_{n,-\sigma}}{2} \int_{\mathbb{S}^n} \left((h_{i-}(y) + \varepsilon_0)^2 + (h_{i+}(y) + \varepsilon_0)^2\right) \int_{\mathbb{S}^n} \frac{(v(x) - v(y))^2}{|x-y|^{n+2\sigma}} \\
&+ 2^{\frac{2}{p}} \sum_{i=1}^{k} (2\frac{\eta}{k}) \iint_{\mathbb{S}^n \times \mathbb{S}^n} \frac{(v(x) - v(y))^2}{|x-y|^{n+2\sigma}} + C \int_{\mathbb{S}^n} v^2.
\end{aligned}$$

Hence for any $\varepsilon > 0$, we can choose η sufficiently small such that

$$\left(\int_{\mathbb{S}^n} v^p\right)^{\frac{2}{p}} \leq 2^{\frac{2}{p}-2} (K^2 c_{n,-\sigma} + \varepsilon) \iint_{\mathbb{S}^n \times \mathbb{S}^n} \frac{(v(x) - v(y))^2}{|x-y|^{n+2\sigma}} + C \int_{\mathbb{S}^n} v^2.$$

Then the proposition follows immediately from the above and that for $v \in H^\sigma(\mathbb{S}^n)$,

$$\int_{\mathbb{S}^n} |v| P(|v|) \leq \int_{\mathbb{S}^n} v P(v).$$

$\square$

Proposition 4.7. *For $n \geq 2$, there exist some constants $a^* < 1$ and some $p^* < \frac{2n}{n-2\sigma}$ both of which depends only on n and σ, such that for all $p^* \leq p \leq \frac{2n}{n-2\sigma}$,*

$$\inf_{v \in \mathscr{M}_0^p} a^* \fint_{\mathbb{S}^n} vP_\sigma(v) + (1-a^*)P_\sigma(1) \fint_{\mathbb{S}^n} v^2 \geq P_\sigma(1). \tag{4.22}$$

When $\sigma = 1$, the above proposition was proved by Chang and Yang [40] (see [96] for another proof). See also [53] for such inequalities in higher order Sobolev spaces. Here we adapt the arguments in Section 5 of [96] to show (4.22).

Proof. For $u \in H^\sigma(\mathbb{S}^n), a > 0$, set

$$I_a(v) = a \fint_{\mathbb{S}^n} vP_\sigma(v) + (1-a)P_\sigma(1) \fint_{\mathbb{S}^n} v^2$$

and

$$m_{a,p} = \inf_{v \in \mathscr{M}_0^p} I_a(v).$$

By standard variational methods, $m_{a,p}$ is achieved for $a > 0$ and $2 \leq p < \frac{2n}{n-2\sigma}$. Moreover it is easy to see that (see, e.g., Lemma 5.5 in [40] or Lemma 5.2 in [96]).

$$\begin{aligned} m_{a,p} \leq P_\sigma(1) &\quad \text{for all } 0 \leq a \leq 1,\ 2 \leq p \leq \frac{2n}{n-2\sigma}, \\ \lim_{a \to 1} m_{a,p} = P_\sigma(1) &\quad \text{uniformly for } 2 \leq p \leq \frac{2n}{n-2\sigma}. \end{aligned} \tag{4.23}$$

We argue by contradiction. Suppose that (4.22) fails. Then there exist sequences $\{a_k\}$, $\{p_k\} \subset \mathbb{R}$, $\{v_k\} \subset \mathscr{M}_0^{p_k}$, such that $a_k < 1, a_k \to 1, p_k < \frac{2n}{n-2\sigma}, p_k \to \frac{2n}{n-2\sigma}, v_k \geq 0$ and

$$I_{a_k}(v_k) = m_{a_k,p_k} < P_\sigma(1). \tag{4.24}$$

By (4.24) and (4.20), there exists some positive constant $C(n,\sigma)$ independent of k such that

$$\|v_k\|_{H^\sigma(\mathbb{S}^n)} \leq C(n,\sigma), \qquad \int_{\mathbb{S}^n} v_k^2 \geq 1/C(n,\sigma).$$

After passing to a subsequence, we have that $v_k \to \bar{v}$ weakly in $H^\sigma(\mathbb{S}^n)$ for some $\bar{u} \in H^\sigma(\mathbb{S}^n) \setminus \{0\}$.

The Euler-Lagrange equation satisfied by v_k is

$$a_k P_\sigma(v_k) + (1-a_k)P_\sigma(1)v_k = m_k v_k^{p_k-1} + \Lambda_k \cdot x v_k^{p_k-1}, \tag{4.25}$$

where $m_k = m_{a_k,p_k}$ and $\lambda_k \in \mathbb{R}^{n+1}$. Multiplying (4.25) by v_k and integrating over $\mathbb{S}^n$, we have, by (4.23)

$$\lim_{k\to\infty}\left(\fint_{\mathbb{S}^n} v_k P_\sigma(v_k)\right) = P_\sigma(1). \tag{4.26}$$

We claim that $|\Lambda_k| = O(1)$. Suppose the contrary, we let $\xi_k = \Lambda_k/|\Lambda_k|$ and after passing to a subsequence $\xi = \lim_{k\to\infty}\xi_k \in \mathbb{S}^n$. Let $\eta \in C^\infty(\mathbb{S}^n)$ be any smooth test function. Multiplying (4.25) by $\eta/|\Lambda_k|$, integrating it over $\mathbb{S}^n$ and sending $k\to\infty$, we have $\int_{\mathbb{S}^n}\xi\cdot x\bar{v}^{\frac{n+2\sigma}{n-2\sigma}} = 0$. Hence $\bar{v}=0$ which is a contradiction. It is clear that $\bar{v}$ satisfies

$$P_\sigma(\bar{v}) = P_\sigma(1)\bar{v}^{\frac{n+2\sigma}{n-2\sigma}} + \Lambda\cdot x\bar{v}^{\frac{n+2\sigma}{n-2\sigma}},$$

where $\Lambda = \lim_{k\to\infty}\Lambda_k$. The Kazdan-Warner identity (see, e.g., [88] or (1.11)) gives us

$$\int_{\mathbb{S}^n}\nabla(P_\sigma(1)+\Lambda\cdot x)\nabla x\bar{v}^{\frac{2n}{n-2\sigma}} = 0.$$

It follows that $\Lambda = 0$. Hence $\int_{\mathbb{S}^n}\bar{v}P_\sigma(\bar{v}) = P_\sigma(1)\int_{\mathbb{S}^n}\bar{v}^{\frac{2n}{n-2\sigma}}$. This together with (1.9) leads to $\fint_{\mathbb{S}^n}\bar{v}^{\frac{2n}{n-2\sigma}} \geq 1$. On the other hand, $\fint_{\mathbb{S}^n}\bar{v}^{\frac{2n}{n-2\sigma}} \leq \liminf\limits_{k\to\infty} v_k^{p_k} = 1$. Hence

$$\begin{cases} \fint_{\mathbb{S}^n}\bar{v}^{\frac{2n}{n-2\sigma}} = 1, \\ \fint_{\mathbb{S}^n}\bar{v}P_\sigma(\bar{v}) = P_\sigma(1). \end{cases}$$

This together with (4.26) leads to $v_k \to \bar{v}$ in $H^\sigma(\mathbb{S}^n)$. Clearly $\bar{v}\in\mathscr{M}_0^{\frac{2n}{n-2\sigma}}$ and hence $\bar{v}\equiv 1$. In the following we will expand $I_a(v)$ for $v\in\mathscr{M}_0^p$ near 1. Let $T_1\mathscr{M}_0^p$ denote the tangent space of $\mathscr{M}_0^p$ at $v=1$, then we have

$$T_1\mathscr{M}_0^p = \text{span}\{\text{spherical harmonics of degree } \geq 2\}.$$

The following lemma can be proved by the standard implicit function theorem.

Lemma 4.4. *For $\tilde{w}\in T_1\mathscr{M}_0^p$, $\frac{2n-2\sigma}{n-2\sigma}\leq p\leq\frac{2n}{n-2\sigma}$, $\tilde{w}$ close to 0, there exist $\mu(\tilde{w})\in\mathbb{R}$, $\eta(\tilde{w})\in\mathbb{R}^{n+1}$ being C^2 functions such that*

$$\fint_{\mathbb{S}^n}|1+\tilde{w}+\mu+\eta\cdot x|^p = 1 \tag{4.27}$$

and

$$\int_{\mathbb{S}^n}|1+\tilde{w}+\mu+\eta\cdot x|^p x = 0. \tag{4.28}$$

Furthermore, $\mu(0)=0, \eta(0)=0, D\mu(0)=0$ and $D\eta(0)=0$, and μ,η have uniform (with respect to p) C^2 modulo of continuity near 0.

As before we will use $\tilde{w}$ as local coordinates of $v \in \mathscr{M}_0^p$. Let

$$\tilde{E}(\tilde{w}) = I_a(v) = a \fint_{\mathbb{S}^n} vP_\sigma(v) + (1-a)P_\sigma(1) \fint_{\mathbb{S}^n} v^2$$

where $\tilde{w} \in T_1\mathscr{M}_0$ and $v = 1 + \tilde{w} + \mu(\tilde{w}) + \eta(\tilde{w}) \cdot x$ as in Lemma 4.4. Hence

$$\tilde{E}(\tilde{w}) = P_\sigma(1)(1 + 2\mu(\tilde{w})) + a \fint_{\mathbb{S}^n} \tilde{w}P_\sigma(\tilde{w}) + (1-a)P_\sigma(1) \fint_{\mathbb{S}^n} \tilde{w}^2 + o(\|\tilde{w}\|^2_{H^\sigma(\mathbb{S}^n)}).$$

Since

$$\mu(\tilde{w}) = -\frac{p-1}{2} \fint_{\mathbb{S}^n} \tilde{w}^2 + o(\|\tilde{w}\|^2_{H^\sigma(\mathbb{S}^n)}),$$

we have

$$\tilde{E}(\tilde{w}) = P_\sigma(1) + a \fint_{\mathbb{S}^n} \tilde{w}P_\sigma(\tilde{w}) - (p-2+a)P_\sigma(1) \fint_{\mathbb{S}^n} \tilde{w}^2 + o(\|\tilde{w}\|^2_{H^\sigma(\mathbb{S}^n)}).$$

It follows that there exists some positive constant $C(n,\sigma)$ determined by the difference of the first and the second eigenvalues of P_σ such that for a close to 1 and p close to $\frac{2n}{n-2\sigma}$ we have

$$a \fint_{\mathbb{S}^n} \tilde{w}P_\sigma(\tilde{w}) - (p-2+a)P_\sigma(1) \fint_{\mathbb{S}^n} \tilde{w}^2 \geq \frac{1}{C(n,\sigma)} \fint_{\mathbb{S}^n} \tilde{w}P_\sigma(\tilde{w}),$$

which leads to that for k large we have $I_{a_k}(v_k) \geq P_\sigma$. This is a contradiction. $\square$

4.5 Appendix: A Kazdan-Warner identity

In this section we are going to show (1.11), which is a consequence of the following

Proposition 4.8. *Let $K > 0$ be a C^1 function on $\mathbb{S}^n$, and let v be a positive function in $C^2(\mathbb{S}^n)$ satisfying*

$$P_\sigma(v) = Kv^{\frac{n+2\sigma}{n-2\sigma}} \quad \text{on } \mathbb{S}^n. \tag{4.29}$$

Then, for any conformal Killing vector field X on $\mathbb{S}^n$, we have

$$\int_{\mathbb{S}^n} (\nabla_X K) v^{\frac{2n}{n-2\sigma}} \, \mathrm{d}vol_{g_{\mathbb{S}^n}} = 0. \tag{4.30}$$

Let $\varphi_t : \mathbb{S}^n \to \mathbb{S}^n$ be a one parameter family of conformal diffeomorphisms (in this case they are Möbius transformations), depending on t smoothly, $|t| < 1$, and $\varphi_0 = identity$. Then

$$X := \frac{d}{dt}(\varphi_t)^{-1}\Big|_{t=0} \quad \text{is a conformal Killing vector field on } \mathbb{S}^n. \tag{4.31}$$

Proof. The proof is standard (see, e.g., [15] for a Kazdan-Warner identity for prescribed scalar curvature problems) and we include it here for completeness. Since P_σ is a self-adjoint operator, (4.29) has a variational formulation:

$$I[v] := \frac{1}{2}\int_{\mathbb{S}^n} vP_\sigma(v)\,\mathrm{d}vol_{g_{\mathbb{S}^n}} - \frac{n-2\sigma}{2n}\int_{\mathbb{S}^n} Kv^{\frac{2n}{n-2\sigma}}\,\mathrm{d}vol_{g_{\mathbb{S}^n}}.$$

Let X be a conformal Killing vector field, then there exists $\{\varphi_t\}$ satisfying (4.31). Let

$$v_t := (v \circ \varphi_t)w_t$$

where w_t is given by

$$g_t := \varphi_t^* g_{\mathbb{S}^n} = w_t^{\frac{4}{n-2\sigma}} g_{\mathbb{S}^n}.$$

Then

$$I[v_t] = \frac{1}{2}\int_{\mathbb{S}^n} vP_\sigma(v)\,\mathrm{d}vol_{g_{\mathbb{S}^n}} - \frac{n-2\sigma}{2n}\int_{\mathbb{S}^n} K(\varphi_t^{-1}(x))v^{\frac{2n}{n-2\sigma}}\,\mathrm{d}vol_{g_{\mathbb{S}^n}}.$$

It follows from (4.29) that

$$0 = I'[v]\left(\left.\frac{d}{dt}\right|_{t=0} v_t\right) = \left.\frac{d}{dt}I[v_t]\right|_{t=0} = -\frac{n-2\sigma}{2n}\int_{\mathbb{S}^n} (\nabla_X K)v^{\frac{2n}{n-2\sigma}}\,\mathrm{d}vol_{g_{\mathbb{S}^n}}.$$

$\square$

Chapter 5

A fractional Yamabe flow

5.1 A strong maximum principle and a Hopf lemma for nonlocal parabolic equations

Let $x = (x', x_n) \in \mathbb{R}^n$, $\mathbb{R}^n_+ = \{x : x_n > 0, x \in \mathbb{R}^n\}$. Recall (see, e.g., [122]) that for $\sigma \in (0,1)$, if u is bounded in $\mathbb{R}^n$ and C^2 near x, then$(-\Delta)^\sigma u$ is continuous near x, and

$$(-\Delta)^\sigma u(x) = c_{n,-\sigma}\text{P.V.}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2\sigma}}\mathrm{d}y. \tag{5.1}$$

Here "P.V." means the principal value and $c_{n,-\sigma}$ is the constant in (1.7).

For simplicity, throughout the paper we denote $-(-\Delta)^\sigma$ as Δ^σ and will not keep writing the constant $c_{n,-\sigma}$ and "P.V." if there is no confusion.

Lemma 5.1. *Let $w(x,t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R})$ and $w(\cdot,t)$ be bounded in $\mathbb{R}^n$ for any fixed t. Suppose $w(x,t)$ satisfies $w(x', -x_n, t) = -w(x', x_n, t)$ for all (x,t) and*

$$\liminf_{x_n \ge 0, |x|\to\infty} w(x,t) \ge 0 \quad \textit{for any fixed } t.$$

If w satisfies

$$w_t \ge a(x,t)\Delta^\sigma w + b(x,t)w, \quad (x,t) \in \mathbb{R}^n_+ \times (0,T] \tag{5.2}$$

where $a(x,t)$ is continuous and positive in $\overline{\mathbb{R}^n_+} \times [0,T]$, $b(x,t)$ is continuous and bounded in $\mathbb{R}^n_+ \times [0,T]$, and $w(x,0) \ge 0$ for all $x \in \mathbb{R}^n_+$, then $w(x,t) \ge 0$ in $\mathbb{R}^n_+ \times [0,T]$.

Proof. Without loss of generality, we may assume $b(x,t) \le 0$. Indeed, if we let

$$\tilde{w}(x,t) = e^{-Ct}w(x,t)$$

for some constant C, then

$$\tilde{w}_t = a(x,t)\Delta^\sigma \tilde{w} + (b(x,t) - C)\tilde{w}.$$

Since b is bounded, we can choose C sufficiently large such that $b(x,t) - C \leq 0$ in $\mathbb{R}^n_+ \times (0,T]$, and we only need to show that $\tilde{w}(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}^n_+ \times (0,T]$.

Suppose the contrary that there exists a point $(x_0,t_0) \in \mathbb{R}^n_+ \times (0,T]$ such that

$$0 > w(x_0,t_0).$$

By the assumptions on w, we may assume $w(x_0,t_0) = \min_{\overline{\mathbb{R}^n_+} \times (0,T]} w$. It follows that

$$w_t(x_0,t_0) \leq 0, \quad b(x_0,t_0)w(x_0,t_0) \geq 0. \tag{5.3}$$

It is clear that

$$\begin{aligned}
\Delta^\sigma w(x_0,t_0) &= \int_{\mathbb{R}^n} \frac{w(y,t_0) - w(x_0,t_0)}{|x_0 - y|^{n+2\sigma}} \,\mathrm{d}y \\
&= \int_{\mathbb{R}^n_+} \frac{w(y,t_0) - w(x_0,t_0)}{|x_0 - y|^{n+2\sigma}} \,\mathrm{d}y + \int_{\mathbb{R}^n \setminus \mathbb{R}^n_+} \frac{w(y,t_0) - w(x_0,t_0)}{|x_0 - y|^{n+2\sigma}} \,\mathrm{d}y.
\end{aligned}$$

By the change of variables $y = (z', -z_n)$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n \setminus \overline{\mathbb{R}^n_+}} \frac{w(y,t_0) - w(x_0,t_0)}{|x_0 - y|^{n+2\sigma}} \,\mathrm{d}y \\
&= \int_{\mathbb{R}^n_+} \frac{w(z', -z_n, t_0) - w(x_0,t_0)}{|x_0 - (z', -z_n)|^{n+2\sigma}} \,\mathrm{d}z \\
&= -\int_{\mathbb{R}^n_+} \frac{w(z', z_n, t_0) - w(x_0,t_0)}{|x_0 - (z', -z_n)|^{n+2\sigma}} \,\mathrm{d}z - 2w(x_0,t_0) \int_{\mathbb{R}^n_+} \frac{1}{|x_0 - (z', -z_n)|^{n+2\sigma}} \,\mathrm{d}z \\
&> -\int_{\mathbb{R}^n_+} \frac{w(z', z_n, t_0) - w(x_0,t_0)}{|x_0 - (z', -z_n)|^{n+2\sigma}} \,\mathrm{d}z,
\end{aligned}$$

where we used $w(z', -z_n, t_0) = -w(z', z_n, t_0)$ and $w(x_0,t_0) < 0$. Since (x_0,t_0) is a minimum point of w in $\overline{\mathbb{R}^n_+} \times (0,T]$, the simple inequality

$$\frac{1}{|x_0 - z|^{n+2\sigma}} > \frac{1}{|x_0 - (z', -z_n)|^{n+2\sigma}}, \quad \forall\, x_0, z \in \mathbb{R}^n_+$$

yields that

$$\Delta^\sigma w(x_0,t_0) > 0. \tag{5.4}$$

Combining (5.3) and (5.4), we have a contradiction to (5.2). □

Lemma 5.2. *Let $w(x,t)$ be as in Lemma 5.1. Then for any fixed $t \in (0,T]$, we have $w(x,t) > 0$ or $w(x,t) \equiv 0$ in $\mathbb{R}^n_+$.*

Proof. As in the proof of Lemma 5.1, we may assume $b \leq 0$. Suppose that at $w(x_0, t_0) = 0$ for some $(x_0, t_0) \in \mathbb{R}^n_+ \times (0, T]$. From the proof of Lemma 5.1 we see that

$$\Delta^\sigma w(x_0, t_0) \geq 0$$

and equality holds if and only if $w(x, t_0) = w(x_0, t_0)$ for all $x \in \mathbb{R}^n_+$. Therefore, the lemma follows immediately from a simple contradiction argument. □

Lemma 5.3. *Let $w(x,t)$ be as in Lemma 5.1. Suppose $w(x_0, 0) > 0$ for some $x_0 \in \mathbb{R}^n_+$, then for any fixed $t \in (0, T]$, we have $w(x, t) > 0$ in $\mathbb{R}^n_+$.*

Proof. The proof follows from that of the parabolic strong maximum principle in [113], with suitable modifications for nonlocal equations. As before, we assume $b \leq 0$. Suppose that for some $t_1 > 0$, $w(\cdot, t_1)$ is zero at some point. It follows from Lemma 5.2 that $w(x, t_1) \equiv 0$. By the assumption on $w(\cdot, 0)$ and Lemma 5.2, we may assume that $w(x,t) > 0$ for all $(x,t) \in \mathbb{R}^n_+ \times (t_2, t_1)$ for some $t_2 > 0$.

Let $h(x,t) = (t_1 - t_*)^2 - |x - e_n|^2 - (t - t_*)^2$ if $0 \leq x_n \leq 2$; and $h = 0$ if $x_n > 2$, where $e_n = (0', 1)$ and t_* will be fixed later. Set

$$H(x,t) = \begin{cases} h(x,t), & x \in \overline{\mathbb{R}^n_+}, \\ -h(x', -x_n, t), & x \in \mathbb{R}^n \setminus \overline{\mathbb{R}^n_+}. \end{cases}$$

Let $\bar{t} \in (t_2, t_1)$ be such that $(t_1 - t_*)^2 - (t - t_*)^2 \leq \frac{1}{4}$ holds for $t \geq \bar{t}$. It is easy to see that there exists a positive constant C_1 independent of t^* such that for any $(x,t) \in B_{1/2}(e_n) \times [\bar{t}, t_1]$,

$$(-\Delta)^\sigma H(x,t) \leq C_1.$$

Thus, we can choose t_* so negative that for any $(x,t) \in B_{1/2}(e_n) \times [\bar{t}, t_1]$,

$$H_t(x,t) = 2(t_* - t) \leq 2(t_* - t_2) < a(x,t)\Delta^\sigma H(x,t) + b(x,t)H(x,t). \tag{5.5}$$

Let $\varepsilon > 0$ be a sufficiently small constant such that $w(x, \bar{t}) \geq \varepsilon H(x, \bar{t})$ for all $x \in \mathbb{R}^n_+$. We claim that $w(x,t) \geq \varepsilon H(x,t)$ in $\mathbb{R}^n_+ \times (\bar{t}, t_1)$.

If not, then the (negative) minimal value of $\bar{w} := w - \varepsilon H$ in $\mathbb{R}^n_+ \times (\bar{t}, t_1)$ must be achieved in $B_{1/2}(e_n) \times (\bar{t}, t_1)$, say at (x_0, t_0). Note that $\bar{w}(x', -x_n, t) = -\bar{w}(x', x_n, t)$.

Hence, by exactly the same argument in the proof of Lemma 5.1

$$\partial_t \bar{w}(x_0,t_0) \le 0, \quad \Delta^\sigma \bar{w}(x_0,t_0) > 0.$$

Together with (5.5) and $b \le 0$, we conclude that

$$w_t(x_0,t_0) < a(x_0,t_0)\Delta^\sigma w(x_0,t_0) + b(x_0,t_0)w,$$

which contradicts (5.2).

Hence, it follows from the above claim that $w_t(t_1,e_n) \le -2\varepsilon(t_1-t_*) < 0$. But $w(x,t_1) \equiv 0$. These contradict (5.2). $\square$

Lemma 5.4. *Let*

$$h(x) = \begin{cases} x_n(1-|x'|^2), & |x_n|<1, |x'|<1, \\ 0, & \text{otherwise.} \end{cases}$$

Then there exists a positive constant c_0 depending only n,σ such that

$$\Delta^\sigma h(x) \ge -c_0 x_n, \tag{5.6}$$

for all $x=(x',x_n)$ with $|x'|<1, 0 \le x_n < 1/8$.

Proof. The lemma follows from rather involved calculations. By rotating the first $(n-1)$ coordinates, we only need to show (5.6) at point $a=(a_1,0,\cdots,0,a_n)$ with $0 \le a_1 < 1$, $0 \le a_n < 1/8$.

Denote $B'(x',R) \subset \mathbb{R}^{n-1}$ be the ball centered at x' with radius R, $\Omega = B'(0,1) \times (-1,1)$. In the following C will be denoted as various positive constants which depend only on dimension n and σ.

It follows from (5.1) that

$$\begin{aligned} \Delta^\sigma h(a) &= \int_{\mathbb{R}^n} \frac{h(x)-h(a)}{|x-a|^{n+2\sigma}} \mathrm{d}x \\ &= \int_\Omega \frac{x_n(1-|x'|^2)-a_n(1-|a'|^2)}{|x-a|^{n+2\sigma}} \mathrm{d}x - \int_{\Omega^c} \frac{a_n(1-|a'|^2)}{|x-a|^{n+2\sigma}} \mathrm{d}x \\ &=: I - a_n II. \end{aligned} \tag{5.7}$$

Since $x_n(1-|x'|^2) - a_n(1-|a'|^2) = (x_n - a_n)(1-|x'|^2) + a_n(|a'|^2 - |x'|^2)$, we divide the integral I into

$$I_1 := \int_\Omega \frac{(x_n - a_n)(1-|x'|^2)}{|x-a|^{n+2\sigma}} \mathrm{d}x$$

and

$$a_n I_2 := a_n \int_\Omega \frac{(|a'|^2 - |x'|^2)}{|x-a|^{n+2\sigma}} \mathrm{d}x.$$

By symmetry and that $0 \le a_n < 1/8$,

$$\begin{aligned} I_1 &= \int_{-1}^{-1+2a_n} \int_{|x'|<1} \frac{(x_n - a_n)(1-|x'|^2)}{|x-a|^{n+2\sigma}} \, \mathrm{d}x' \mathrm{d}x_n \\ &\ge -Ca_n. \end{aligned} \tag{5.8}$$

Using $|a'|^2 - |x'|^2 = -|x'-a'|^2 + 2a'(a'-x')$, we write

$$\begin{aligned} I_2 &= \int_\Omega \frac{-|x'-a'|^2}{|x-a|^{n+2\sigma}} \mathrm{d}x + \int_\Omega \frac{2a' \cdot (a'-x')}{|x-a|^{n+2\sigma}} \mathrm{d}x \\ &=: I_3 + I_4. \end{aligned}$$

Direct computations give

$$\begin{aligned} I_3 &\ge -\int_{-2+a_n}^{2+a_n} \mathrm{d}x_n \int_{|x'-a'|<2} \frac{-|x'-a'|^2}{|x-a|^{n+2\sigma}} \mathrm{d}x' \\ &= -2 \lim_{b \to 0^+} \int_b^2 \mathrm{d}y \int_0^2 \frac{r^2 r^{n-2}}{(r^2+y^2)^{\frac{n+2\sigma}{2}}} \mathrm{d}r \\ &= -2 \lim_{b \to 0^+} \int_b^2 y^{1-2\sigma} \mathrm{d}y \int_0^{2/y} \frac{r^n}{(1+r^2)^{\frac{n+2\sigma}{2}}} \mathrm{d}r \\ &= -2 \lim_{b \to 0^+} \int_b^2 y^{1-2\sigma} \mathrm{d}y \left(\int_1^{2/y} \frac{r^n}{(1+r^2)^{\frac{n+2\sigma}{2}}} \mathrm{d}r + \int_0^1 \frac{r^n}{(1+r^2)^{\frac{n+2\sigma}{2}}} \mathrm{d}r \right) \\ &\ge -2 \lim_{b \to 0^+} \int_b^2 y^{1-2\sigma} \mathrm{d}y \left(\int_1^{2/y} r^{-2\sigma} \mathrm{d}r + 1 \right) \\ &\ge -C. \end{aligned} \tag{5.9}$$

Next, we are going to show

$$I_4 - II \ge -C. \tag{5.10}$$

Let $D_0 = (B'(0,1) \cap B'(2a',1))$. Since $a' = (a_1, 0, \cdots, 0)$, it follows from symmetry that

$$\int_{D_0 \times (-1,1)} \frac{2a' \cdot (a'-x')}{|x-a|^{n+2\sigma}} \, \mathrm{d}x' \mathrm{d}x_n = 0.$$

Thus,

$$I_4 = \int_{(B'(0,1)\setminus D_0)\times(-1,1)} \frac{2a'\cdot(a'-x')}{|x-a|^{n+2\sigma}} > 0.$$

Now we have two cases:

Case 1. if $|a'| \leq \frac{\sqrt{2}}{2}$, then it is easy to see that $II < C$ (the denominator is uniformly bounded). Hence, (5.10) holds.

Case 2. Suppose $|a'| > \frac{\sqrt{2}}{2}$. We have

$$\begin{aligned} II &= \int_{\Omega^c\cap(B'(a',|a'|)\times(-1,1))} \frac{1-|a'|^2}{|x-a|^{n+2\sigma}} + \int_{\Omega^c\setminus(B'(a',|a'|)\times(-1,1))} \frac{1-|a'|^2}{|x-a|^{n+2\sigma}} \\ &\leq \int_{\Omega^c\cap(B'(a',|a'|)\times(-1,1))} \frac{(1-|a'|^2)}{|x-a|^{n+2\sigma}} + C \\ &=: II_1 + C. \end{aligned}$$

Denote $D_1 = \Big(B'(a', \sqrt{1-|a'|^2}) \cap \{x_1 < a_1\}\Big) \setminus D_0$, and $D_2 = \Big(B'(a', \sqrt{1-|a'|^2}) \cap \{x_1 > a_1\}\Big) \setminus D_0$.

Note that for $x \in D_1$, we have $2|a'|(|a'| - x_1) \geq 1 - |a'|^2 - |x' - a'|^2$. Therefore,

$$\int_{D_1\times(-1,1)} \frac{2a'\cdot(a'-x')}{|x-a|^{n+2\sigma}} - \int_{D_2\times(-1,1))} \frac{1-|a'|^2}{|x-a|^{n+2\sigma}} \geq \int_{D_1\times(-1,1)} \frac{-|x'-a'|^2}{|x-a|^{n+2\sigma}}.$$

Observe that there exists a positive integer m, which depends only on n and σ, such that

$$\begin{aligned} &m\int_{\left(B'(0',1)\setminus B'(a',\sqrt{1-|a'|^2})\right)\times(-1,1)} \frac{1-|a'|^2}{|x-a|^{n+2\sigma}} \\ &\geq \int_{\left(B(a',|a'|)\setminus\left(B'(0',1)\cup B'(a',\sqrt{1-|a'|^2})\right)\right)\times(-1,1)} \frac{1-|a'|^2}{|x-a|^{n+2\sigma}}. \end{aligned}$$

Also notice that for any $x \in B'(0',1)\setminus B'(a', \sqrt{1-|a'|^2})$, we have

$$0 \geq m(1 - |a'|^2 - |x' - a'|^2).$$

Hence,

$$\begin{aligned} &m\int_{\left(B'(0',1)\setminus B'(a',\sqrt{1-|a'|^2})\right)\times(-1,1)} \frac{|x'-a'|^2}{|x-a|^{n+2\sigma}} \\ &\geq \int_{\left(B(a',|a'|)\setminus\left(B'(0',1)\cup B'(a',\sqrt{1-|a'|^2})\right)\right)\times(-1,1)} \frac{1-|a'|^2}{|x-a|^{n+2\sigma}}. \end{aligned}$$

It follows that

$$
\begin{aligned}
& I_4 - II \\
& \geq -C + \int_{D_1\times(-1,1)} \frac{2a'\cdot(a'-x')}{|x-a|^{n+2\sigma}} - \int_{D_2\times(-1,1)} \frac{1-|a'|^2}{|x-a|^{n+2\sigma}} \\
& \quad - \int_{\left(B(a',|a'|)\setminus\left(B'(0',1)\cup B'(a',\sqrt{1-|a'|^2})\right)\right)\times(-1,1)} \frac{1-|a'|^2}{|x-a|^{n+2\sigma}} \\
& \geq -C - m\int_{(B'(0',1)\setminus B'(a',\sqrt{1-|a'|^2}))\times(-1,1)} \frac{|x'-a'|^2}{|x-a|^{n+2\sigma}} + \int_{D_1\times(-1,1)} \frac{-|x'-a'|^2}{|x-a|^{n+2\sigma}} \\
& \geq -C - (m+1)I_3 \\
& \geq -C.
\end{aligned}
$$

Therefore, (5.10) holds.

Finally, Lemma 5.4 follows from (5.7), (5.8), (5.9) and (5.10). □

Lemma 5.5. *Let $w(x,t)$ be as in Lemma 5.1. Suppose $w(x_0,0)>0$ for some $x_0\in\mathbb{R}^n_+$, then for any fixed $t\in(0,T]$, we have $\partial_{x_n}w(x',0,t)>0$.*

Proof. Let

$$
g(x) = \begin{cases} -1, & \text{in } B'(0,1)\times(-2,-1), \\ 1, & \text{in } B'(0,1)\times(1,2), \\ 0, & \text{otherwise}, \end{cases}
$$

where $B'(0,1)$ denotes the $n-1$ dimensional unit ball centered at 0. For any $x\in B'(0,1)\times(0,1/8)$, we have

$$
\begin{aligned}
\Delta^\sigma g(x) &= \int_{\mathbb{R}^n} \frac{g(y)-g(x)}{|y-x|^{n+2\sigma}}\,\mathrm{d}y \\
&= \int_{B'(0,1)\times(1,2)} \frac{1}{|y-x|^{n+2\sigma}}\,\mathrm{d}y - \int_{B'(0,1)\times(-2,-1)} \frac{1}{|y-x|^{n+2\sigma}}\,\mathrm{d}y \\
&= \int_{B'(0,1)\times(1,2)} \frac{1}{|y-(x',x_n)|^{n+2\sigma}} - \frac{1}{|y-(x',-x_n)|^{n+2\sigma}}\,\mathrm{d}y \\
&= \int_{B'(0,1)\times(1,2)} \int_0^1 -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{|y-x+2sx_ne_n|^{n+2\sigma}}\right)\mathrm{d}s\,\mathrm{d}y \\
&= (n+2\sigma)\int_{B'(0,1)\times(1,2)} \int_0^1 \frac{4(y_n-x_n)x_n+8sx_n^2}{|y-x+2sx_ne_n|^{n+2+2\sigma}}\,\mathrm{d}s\,\mathrm{d}y \\
&\geq c_1x_n,
\end{aligned}
$$

where $c_1>0$ depends only on n and σ.

For any fixed $t_0 \in (0, T]$, define

$$H(x,t) = h(x)\Big(\frac{t_0^2}{1+t_0^2} - (t-t_0)^2\Big) + kg(x),$$

where h is as in Lemma 5.4. We can choose a sufficiently large constant k such that

$$H_t(x,t) \le a(x,t)\Delta^\sigma H + b(x,t)H(x,t),$$

for all $x \in B'(0,1) \times (0, 1/8)$ and $t \in (t_0 - t_0/\sqrt{1+t_0^2}, t_0]$.

It follows from Lemma 5.3 that $w(\cdot, t) > 0$ in $\mathbb{R}^n_+$ for any fixed $t \in (0, T]$. Making a similar argument to the poof of Lemma 5.3, we can show that there exists a small positive constant ε such that $w \ge \varepsilon H$ for all $t \in (0, t_0]$. Therefore, $\partial_{x_n} w(x', 0, t_0) > 0$ and Lemma 5.5 follows immediately. □

Now we apply the above strong maximum principle and Hopf lemma to fractional Yamabe flow equations.

Suppose that v is a positive smooth solution of (1.15) in $\mathbb{S}^n \times [0, T]$. Hence

$$u(x,t) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2\sigma}{2}} v(F(x), t)$$

satisfies (1.16). For a given real number λ, define

$$\Sigma_\lambda = \{x = (x', x_n) : x_n \ge \lambda\},$$

and let $x^\lambda = (x', 2\lambda - x_n)$ and $u_\lambda(x,t) = u(x^\lambda, t)$. It is clear that u_λ also satisfies (1.16).

Proposition 5.1. *Suppose that $u(x,0) - u_\lambda(x,0) \ge 0$ in Σ_λ, then for any fixed $t \in (0, T]$, we have $u(x,t) - u_\lambda(x,t) \ge 0$ in Σ_λ.*

Proof. Let $w(x,t) = u(x,t) - u_\lambda(x,t)$. Then w satisfies

$$w_t = a(x,t)\Delta^\sigma w + b(x,t)w, \tag{5.11}$$

where $a(x,t) = \frac{1}{Nu^{N-1}}$ and $b(x,t) = \frac{(1-N)(-\Delta)^\sigma u_\lambda}{N}\int_0^1 \frac{1}{(\tau u + (1-\tau)u_\lambda)^N} d\tau + \frac{r_\sigma^g}{N}$ is bounded. Note that $w(x', x_n + \lambda, t)$ satisfies all the conditions in Lemma 5.1. Thus Proposition 5.1 follows from Lemma 5.1. □

Proposition 5.2. *Assume the conditions in Proposition 5.1, then for any fixed* $t \in (0, T]$*, we have* $u(x,t) - u_\lambda(x,t) > 0$ *or* $u(x,t) - u_\lambda(x,t) \equiv 0$ *in* Σ_λ*.*

Proof. It follows from Proposition 5.1 and Lemma 5.2. $\square$

Proposition 5.3. *Assume the conditions in Proposition 5.1. In addition, we suppose that* $u(x_0, 0) - u_\lambda(x_0, 0) > 0$ *for some* $x_0 \in \Sigma_\lambda$*, then for any fixed* $t \in (0, T]$*, we have* $u(x,t) - u_\lambda(x,t) > 0$ *in* Σ_λ *and* $\partial_{x_n} u(x', \lambda, t) > 0$*.*

Proof. It follows from Proposition 5.1, Lemma 5.3 and Lemma 5.5. $\square$

5.2 Harnack inequality for a fractional Yamabe flow

Based on the results proved in the previous section, we are going to establish the following Harnack inequality.

Theorem 5.1. *Let* v *be a* $C^{3,1}$ *positive function on* $\mathbb{S}^n \times [0, T^*)$ *and satisfy*

$$\frac{\partial v^N}{\partial t} = -P_\sigma(v) + b(t) v^N, \quad \text{on } \mathbb{S}^n \times (0, T^*),$$

where $b(t) \in C([0, T^*))$*. Then there exists a positive constant* $C > 0$ *depending only on* $n, \sigma, \inf_{\mathbb{S}^n} v(\cdot, 0)$ *and* $\|v(\cdot, 0)\|_{C^3(\mathbb{S}^n)}$ *such that*

$$\max_{\mathbb{S}^n} v(\cdot, t) \le C \min_{\mathbb{S}^n} v(\cdot, t),$$

for any fixed $t \in (0, T^*)$*.*

Proof. As mentioned in the introduction, the idea of this proof is essentially due to Ye [134]. We will show that

$$\sup_{\mathbb{S}^n} \frac{|\nabla_{\mathbb{S}^n} v|}{|v|} \le C \quad \text{for all } s \in (0, T^*).$$

Let $q_0 \in \mathbb{S}^n$. Without loss of generality, we may assume that q_0 is the north pole. Consider the inverse of the stereographic projection from the north pole $F : \mathbb{R}^n \to \mathbb{S}^n$:

$$F(x_1, \cdots, x_n) = \left(\frac{2x}{1 + x^2}, \frac{x^2 - 1}{x^2 + 1} \right).$$

We also denote $G : \mathbb{R}^n \to \mathbb{S}^n$ as the inverse of the stereographic projection from the south pole, namely $G(x) = F(x/|x|^2)$. Let

$$u(x,s) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2\sigma}{2}} v(F(x),s), \quad \bar{u}(x,s) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2\sigma}{2}} v(G(x),s).$$

Then $u, \bar{u} \in C^{3,1}(\mathbb{R}^n \times [0,T^*))$ and both satisfy

$$\frac{\partial u^N}{\partial t} = \Delta^\sigma u + b(t)u^N, \quad \text{on } \mathbb{R}^n \times [0,T^*). \tag{5.12}$$

$u(\cdot,s)$ has a Taylor expansion "at infinity" of the form

$$u(x,s) = \frac{2^{(n-2\sigma)/2}}{|x|^{n-2\sigma}}\left(a_0 + \frac{a_i x_i}{|x|^2} + \left(a_{ij} - \frac{n-2\sigma}{2}\delta_{ij}\right)\frac{x_i x_j}{|x|^4} + O(|x|^{-3})\right).$$

Similarly, the partial derivatives of $u(\cdot,t)$ have Taylor expansions "at infinity" of the form

$$\begin{aligned}\frac{\partial u}{\partial x_i}(x,s) = 2^{\frac{n-2\sigma}{2}}&\left(-\frac{n-2\sigma}{|x|^{n-2\sigma+2}}x_i\left(a_0 + \frac{a_j x_j}{|x|^2}\right) + \frac{a_i}{|x|^{n-2\sigma+2}} - \frac{2x_i a_j x_j}{|x|^{n-2\sigma+4}}\right)\\ &+ O(|x|^{-(n-2\sigma+3)}).\end{aligned}$$

Here

$$\begin{aligned} a_0(s) &= v(q_0,s),\\ a_i(s) &= \frac{\partial(v(\cdot,s)\circ G)}{\partial x_i}(0),\\ a_{ij}(s) &= \frac{\partial^2(v(\cdot,s)\circ G)}{2\partial x_i x_j}(0).\end{aligned}$$

Let $y_i(s) = (n-2\sigma)^{-1}a_i(s)/a_0(s)$, and $y(s) = (y_1(s),\cdots,y_n(s))$. Then

$$u(x+y,s) = \frac{2^{\frac{n-2\sigma}{2}}}{|x|^{n-2\sigma}}\left(a_0 + \frac{\tilde{a}_{ij}x_i x_j}{|x|^4} + o(|x|^{-2})\right) \tag{5.13}$$

and

$$\frac{\partial u}{\partial x_i}(x+y,s) = -\frac{(n-2\sigma)a_0 x_i}{|x|^{n-2\sigma+2}} + O(|x|^{-(n-2\sigma+3)}) \tag{5.14}$$

where $\tilde{a}_{ij} = a_{ij} - \frac{n-2\sigma}{2}\delta_{ij} - \frac{a_i a_i}{a_0}$. We only need to show that there exists a positive constant C depending only on $n, \sigma, \inf_{\mathbb{S}^n} v(\cdot,0)$ and $\|v(\cdot,0)\|_{C^3(\mathbb{S}^n)}$ such that

$$|y(s)| \le C \quad \text{for all } 0 \le s < T^*.$$

Fix $T \in (0,T^*)$. After a rotation and a reflection, we may assume that $y_n(T) = \max_i |y_i(T)|$. From the Taylor expansions of u and ∇u for $s=0$, we see that (e.g., Lemma 4.2 in [64]) there exists a $\lambda_0 > 0$, which depends only on $n, \sigma, \inf_{\mathbb{S}^n} v(\cdot, 0)$ and $\|v(\cdot,0)\|_{C^3(\mathbb{S}^n)}$, such that for any $\lambda > \lambda_0$,

$$u(x,0) > u(x^\lambda, 0) \quad \text{for } x_n < \lambda,$$

where $x^\lambda = (x_1, \cdots, x_{n-1}, 2\lambda - x_n)$. Denote $u^\lambda(x,s) = u(x^\lambda, s)$. By Proposition 5.3, we have

$$u(x,s) > u^\lambda(x,s) \quad \text{for all } s \in [0,T],\ x_n < \lambda,\ \lambda \geq \lambda_0. \tag{5.15}$$

We claim that

$$\max_{0 \leq s \leq T} y_n(s) < \lambda_0.$$

If not, there exists $\bar{s} \in (0,T]$ such that $y_n(\bar{s}) = \max_{0 \leq s \leq T} y_n(s) \geq \lambda_0$. Thus, we can set $\lambda = y_n(\bar{s})$ in (5.15), namely,

$$u(x,s) > u^\lambda(x,s) \quad \text{for all } s \in [0,T],\ x_n < \lambda = y_n(\bar{s}).$$

Let $\tilde{u}(x,s) = u(x + y_n(\bar{s}), s)$, then

$$\tilde{u}(x', x_n, s) > \tilde{u}(x', -x_n, s) \quad \text{for all } s \in [0,T],\ x_n < 0.$$

Let $\tilde{u}_1(x,s) = \frac{1}{|x|^{n-2\sigma}} \tilde{u}(\frac{x}{|x|^2}, s)$. Then $\tilde{u}_1(x', x_n, s)$ and $\tilde{u}_1(x', -x_n, s)$ satisfy (5.12) and

$$\tilde{u}_1(x', x_n, s) > \tilde{u}_1(x', -x_n, s) \quad \text{for all } s \in [0,T],\ x_n < 0.$$

By Proposition 5.3,

$$\frac{\partial(\tilde{u}_1(x', x_n, s) - \tilde{u}_1(x', -x_n, s))}{\partial x_n}\bigg|_{(x,s)=(0,\bar{s})} < 0,$$

i.e., $(\partial \tilde{u}_1/\partial x_n)(0, \bar{s}) < 0$. This contradicts (5.13). Hence, $\max\limits_{0 \leq s \leq T} y_n(s) < \lambda_0$, which implies $y_n(T) < \lambda_0$. Since λ_0 is independent of s, we have $|y(s)| \leq \lambda_0 \quad$ for all $0 \leq s < T^*$. Moreover, λ_0 is independent of the choice of q_0, and we conclude that

$$\sup_{\mathbb{S}^n} \frac{|\nabla_{\mathbb{S}^n} v|}{|v|} \leq C \quad \text{ for all } s \in (0, T^*).$$

For each t, integrating the above inequality along a shortest geodesic between a maximum point and a minimum point of $v(\cdot, t)$ yields

$$\max_{\mathbb{S}^n} v(\cdot, t) \leq C \min_{\mathbb{S}^n} v(\cdot, t).$$

where C depends only on $n, \sigma, \inf_{\mathbb{S}^n} v(\cdot, 0)$ and $\|v(\cdot,0)\|_{C^3(\mathbb{S}^n)}$. $\square$

5.3 Existence and convergence of a fractional Yamabe flow

5.3.1 Schauder estimates

For an open set $\Omega \subset \mathbb{R}^n$ and $\gamma \in (0,1)$, $C^\gamma(\Omega)$ denotes the standard Hölder space over Ω, with the norm

$$|v|_{\gamma;\Omega} := |v|_{0;\Omega} + [v]_{\gamma;\Omega} := \sup_{\Omega} |v(\cdot)| + \sup_{x_1 \neq x_2, x_1, x_2 \in \Omega} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\gamma}.$$

For simplicity, we use $C^\gamma(\Omega)$ to denote $C^{[\gamma],\gamma-[\gamma]}(\Omega)$ when $1 < \gamma \notin \mathbb{N}$ (the set of positive integers), where $[\gamma]$ is the integer part of γ. Since the operator $\partial_t + (-\Delta)^\sigma$ is invariant under the scaling $(x,t) \to (cx, c^{2\sigma}t)$ with $c > 0$, we introduce the fractional parabolic distance as

$$\rho(X_1, X_2) = (|x_1 - x_2|^2 + |t_1 - t_2|^{1/\sigma})^{1/2},$$

where $X_1 = (x_1, t_2), X_2 = (x_2, t_2) \in \mathbb{R}^{n+1}$. For a measurable function u defined in a Borel set $Q \subset \mathbb{R}^{n+1}$ and $0 < \alpha < \min(1, 2\sigma)$, we define

$$[u]_{\alpha,\frac{\alpha}{2\sigma};Q} = \sup_{X_1 \neq X_2, X_1, X_2 \in Q} \frac{|u(X_1) - u(X_2)|}{\rho(X_1, X_2)^\alpha},$$

and

$$|u|_{\alpha,\frac{\alpha}{2\sigma};Q} = |u|_{0;Q} + [u]_{\alpha,\frac{\alpha}{2\sigma};Q},$$

where $|u|_{0;Q} = \sup_{X \in Q} |u(X)|$. We denote $C^{\alpha,\frac{\alpha}{2\sigma}}(Q)$ as the space of all measurable functions u for which $|u|_{\alpha,\frac{\alpha}{2\sigma};Q} < \infty$. Let $Q_T = \mathbb{R}^n \times (0,T]$, $T \in (0,\infty)$. For $2\sigma + \alpha \notin \mathbb{N}$ and $0 < \alpha < \min(1, 2\sigma)$, we say $u \in \mathcal{C}^{2\sigma+\alpha,1+\frac{\alpha}{2\sigma}}(Q_T)$ if

$$[u]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};Q_T} := [u_t]_{\alpha,\frac{\alpha}{2\sigma};Q_T} + [(-\Delta)^\sigma u]_{\alpha,\frac{\alpha}{2\sigma};Q_T} < \infty$$

and

$$|u|_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};Q_T} := |u|_{0;Q_T} + |u_t|_{0,Q_T} + |(-\Delta)^\sigma u|_{0;Q_T} + [u]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};Q_T} < \infty.$$

Then $\mathcal{C}^{2\sigma+\alpha,1+\frac{\alpha}{2\sigma}}(Q_T)$ is a Banach space equipped with the norm $|\cdot|_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};Q_T}$.

Lemma 5.6. *Suppose that $0 < \alpha < \min(1, 2\sigma)$ and $2\sigma + \alpha$ is not an integer. There exists a constant $C > 0$ depending only on n and σ such that for any $\varepsilon > 0$ and*

$u \in \mathcal{C}^{2\sigma+\alpha,1+\frac{\alpha}{2\sigma}}(Q_T)$, we have

$$|u_t|_{0;Q_T} \leq \varepsilon[u_t]_{\alpha,\frac{\alpha}{2\sigma};Q_T} + C\varepsilon^{-2\sigma/\alpha}|u|_{0;Q_T}, \tag{5.16}$$

$$|(-\Delta)^\sigma u|_{0;Q_T} \leq \varepsilon[u]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};Q_T} + C\varepsilon^{-2\sigma/\alpha}|u|_{0;Q_T}, \tag{5.17}$$

$$[u]_{\alpha,\frac{\alpha}{2\sigma};Q_T} \leq \varepsilon[u]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};Q_T} + C\varepsilon^{-\alpha/(2\sigma)}|u|_{0;Q_T}. \tag{5.18}$$

If $\sigma > \frac{1}{2}$, then

$$[\nabla_x u]_{\alpha,\frac{\alpha}{2\sigma};Q_T} \leq \varepsilon[u]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};Q_T} + C\varepsilon^{-(1+\alpha)/(2\sigma-1)}|u|_{0;Q_T}. \tag{5.19}$$

Proof. By the fractional parabolic dilations of the form $u(x,t) \to u(Rx, R^{2\sigma}t)$, we only need to show the case $\varepsilon = 1$ and $T = 2$. Take $X = (x,t) \in Q_T$ and we have, for some $\theta \in (-1,1)$,

$$\begin{aligned}
|u_t(X)| &\leq |u_t(X) - \big(u(x,t\pm 1) - u(x,t)\big)| + 2|u|_{0;Q_T} \\
&= |u_t(X) - u_t(x,t+\theta)| + 2|u|_{0;Q_T} \leq [u_t]_{\alpha,\frac{\alpha}{2\sigma};Q_T} + 2|u|_{0;Q_T}.
\end{aligned}$$

This finishes the proof of (5.16). For (5.17) and (5.18), we first recall (see, e.g., [126]) that

$$|w|_{2\sigma+\alpha;\mathbb{R}^n} \leq C(|w|_{0;\mathbb{R}^n} + |(-\Delta)^\sigma w|_{\alpha;\mathbb{R}^n}) \quad \text{for all } w \in C^{2\sigma+\alpha}(\mathbb{R}^n).$$

Hence,

$$\begin{aligned}
|(-\Delta)^\sigma u(x,t)| &\leq C(|u(\cdot,t)|_{0;\mathbb{R}^n} + |u(\cdot,t)|_{C^{2\sigma+\alpha}(\mathbb{R}^n)}) \\
&\leq C(|u|_{0;Q_T} + [(-\Delta)^\sigma u]_{\alpha,\frac{\alpha}{2\sigma};Q_T}) \leq C([u]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};Q_T} + |u|_{0;Q_T}),
\end{aligned}$$

and

$$\begin{aligned}
[u]_{\alpha,\frac{\alpha}{2\sigma};Q_T} &\leq \sup_{t_1\neq t_2,x} \frac{|u(x,t_1) - u(x,t_2)|}{|t_1-t_2|^{\frac{\alpha}{2\sigma}}} + \sup_{x_1\neq x_2,t} \frac{|u(x_1,t) - u(x_2,t)|}{|x_1-x_2|^\alpha} \\
&\leq C(|u|_{0;Q_T} + |u_t|_{0;Q_T} + \sup_t |u(\cdot,t)|_{2\sigma+\alpha;\mathbb{R}^n}) \\
&\leq C([u]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};Q_T} + |u|_{0;Q_T}).
\end{aligned}$$

Finally, for $\sigma > \frac{1}{2}$ we notice that by the same methods as above,

$$\sup_{t,x_1\neq x_2} \frac{|\nabla_x u(x_1,t) - \nabla_x u(x_2,t)|}{|x_1-x_2|^\alpha} \leq C([u]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};Q_T} + |u|_{0;Q_T}).$$

Thus, to prove (5.19), we only need to show

$$\sup_{s\neq t,x} \frac{|\nabla_x u(x,s) - \nabla_x u(x,t)|}{|s-t|^{\frac{\alpha}{2\sigma}}} \leq C([u]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};Q_T} + |u|_{0;Q_T}).$$

Fix any $x_0 \in \mathbb{R}^n$. Let $w(x,t) = (-\Delta)^\sigma u(x,t)$ and $\eta(x)$ be a smooth cut-off function supported in $B_2(x_0) \in \mathbb{R}^n$ and equal to 1 in $B_1(x_0)$. Let

$$u_0(x,t) = (-\Delta)^{-\sigma}(\eta w) = \int_{\mathbb{R}^n} \frac{\eta(y)w(y,t)}{|x-y|^{n-2\sigma}}\,\mathrm{d}y.$$

For convenience we have omitted some positive constant as in (1.22). Then

$$(-\Delta)^\sigma(u_0(x,t) - u(x,t) - u_0(x,s) + u(x,s)) = 0 \quad \text{in } B_1(x_0),$$

which implies, for $0 < |t-s| \le 1$,

$$\begin{aligned}
&|\nabla_x u_0(x_0,t) - \nabla_x u(x_0,t) - \nabla_x u_0(x_0,s) + \nabla_x u(x_0,s)| \\
&\le C|u_0(x,t) - u(x,t) - u_0(x,s) + u(x,s)|_{L^\infty(\mathbb{R}^n)} \\
&\le C(|u_t|_{0;Q_T} + [u]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};Q_T})|t-s|^{\frac{\alpha}{2\sigma}}.
\end{aligned}$$

Since $\sigma > 1/2$ and

$$\nabla_x u_0(x_0,t) = (2\sigma - n)\int_{\mathbb{R}^n} \frac{(x_0-y)\eta(y)w(y,t)}{|x_0-y|^{n+2-2\sigma}}\,\mathrm{d}y,$$

we have

$$|\nabla_x u_0(x_0,t) - \nabla_x u_0(x_0,s)| \le C[u]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};Q_T}|t-s|^{\frac{\alpha}{2\sigma}}.$$

Together with (5.16), we have

$$\sup_{s\neq t,x} \frac{|\nabla_x u(x,s) - \nabla_x u(x,t)|}{|s-t|^{\frac{\alpha}{2\sigma}}} \le C([u]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};Q_T} + |u|_{0;Q_T}).$$

This finishes the proof of (5.19). □

Lemma 5.7. *Suppose that $0 < \alpha < \min(1,2\sigma)$ and $2\sigma+\alpha$ is not an integer. Let $u \in \mathcal{C}^{2\sigma+\alpha,1+\frac{\alpha}{2\sigma}}(Q_1)$ and $\eta \in C_c^2(\mathbb{R}^{n+1})$, then for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ depending only on $\alpha,\sigma,n,\varepsilon$ and $\|\eta\|_{C^2(\mathbb{R}^{n+1})}$ such that*

$$[\langle u,\eta\rangle]_{\alpha,\frac{\alpha}{2\sigma};Q_1} \le \varepsilon[u]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};Q_1} + C(\varepsilon)|u|_{0,Q_1}. \tag{5.20}$$

Proof. We denote C as various constants depending only on $n,\sigma,\alpha,\|\eta\|_{C^2(\mathbb{R}^{n+1})}$, and $C(\varepsilon)$ as various constants depending only on $n,\sigma,\alpha,\|\eta\|_{C^2(\mathbb{R}^{n+1})}$ and ε. Recall that $\langle u,\eta\rangle$ is defined in (5.22). For any $(x,t) \in Q_1$,

$$\begin{aligned}
|\langle u,\eta\rangle(x,t)| &\le c(n,\sigma)\int_{\mathbb{R}^n\setminus B_1(x)} \frac{|u(x,t)-u(y,t)||\eta(x,t)-\eta(y,t)|}{|x-y|^{n+2\sigma}}\,\mathrm{d}y \\
&\quad + c(n,\sigma)\int_{B_1(x)} \frac{|u(x,t)-u(y,t)||\eta(x,t)-\eta(y,t)|}{|x-y|^{n+2\sigma}}\,\mathrm{d}y \\
&\le C|u|_{0,Q_1} + C[u(\cdot,t)]_{\sigma,\mathbb{R}^n} \le \varepsilon[u]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};Q_1} + C(\varepsilon)|u|_{0,Q_1}.
\end{aligned}$$

Fix any $X_1 = (x_1, t_1), X_2 = (x_2, t_2) \in Q_1$, $X_1 \neq X_2$. For convenience, we write $\rho = \rho(X_1, X_2)$ and $u^z(x,t) = u(x,t) - u(x+z,t)$. We may also suppose that $\rho \leq 1$.

$$\begin{aligned}
&|\langle u, \eta\rangle(x_1,t_1) - \langle u, \eta\rangle(x_2,t_2)| \\
&\leq \left| \int_{|z|\leq\rho} \frac{\big(u^z(x_1,t_1) - u^z(x_2,t_2)\big)\eta^z(x_1,t_1)}{|z|^{n+2\sigma}} \,\mathrm{d}z \right| \\
&+ \left| \int_{|z|\leq\rho} \frac{\big(\eta^z(x_1,t_1) - \eta^z(x_2,t_2)\big)u^z(x_2,t_2)}{|z|^{n+2\sigma}} \,\mathrm{d}z \right| \\
&+ \left| \int_{|z|\geq\rho} \frac{\big(u^z(x_1,t_1) - u^z(x_2,t_2)\big)\eta^z(x_1,t_1)}{|z|^{n+2\sigma}} \,\mathrm{d}z \right| \\
&+ \left| \int_{|z|\geq\rho} \frac{\big(\eta^z(x_1,t_1) - \eta^z(x_2,t_2)\big)u^z(x_2,t_2)}{|z|^{n+2\sigma}} \,\mathrm{d}z \right| \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

For I_1 and I_2, we first consider that $2\sigma + \alpha < 1$. Then by change of variable,

$$I_1 + I_2 \leq C \max_{i=1,2} [u(\cdot, t_i)]_{\alpha+\sigma;\mathbb{R}^n} \int_{|z|\leq\rho} |z|^{\alpha+\sigma+1-n-2\sigma} \mathrm{d}z \leq C \max_{i=1,2} [u(\cdot, t_i)]_{\alpha+\sigma;\mathbb{R}^n} \rho^{1+\alpha-\sigma}.$$

If $1 < \alpha + 2\sigma < 2$, we have

$$I_1 + I_2 \leq C \max_{i=1,2} [u(\cdot, t_i)]_{\alpha+2\sigma-1;\mathbb{R}^n} \int_{|z|\leq\rho} |z|^{\alpha+2\sigma-n-2\sigma} \mathrm{d}z \leq C \max_{i=1,2} [u(\cdot, t_i)]_{\alpha+2\sigma-1;\mathbb{R}^n} \rho^{\alpha}.$$

If $2\sigma + \alpha > 2$, then

$$\begin{aligned}
I_1 &\leq \left| \int_{|z|\leq\rho} \frac{\big(u^z(x_1,t_1) + \nabla_x u(x_1,t_1)z - u^z(x_2,t_2) - \nabla_x u(x_2,t_2)z\big)\eta^z(x_1,t_1)}{|z|^{n+2\sigma}} \,\mathrm{d}z \right| \\
&+ \left| \int_{|z|\leq\rho} \frac{\big(\nabla_x u(x_1,t_1)z - \nabla_x u(x_2,t_2)z\big)\eta^z(x_1,t_1)}{|z|^{n+2\sigma}} \,\mathrm{d}z \right| \\
&\leq C \sup_{Q_1} |\nabla_x^2 u| \int_{|z|\leq\rho} |z|^{3-n-2\sigma} \,\mathrm{d}z + C[\nabla_x u]_{\alpha,\frac{\alpha}{2\sigma};Q_T} \rho^{\alpha} \int_{|z|\leq\rho} |z|^{2-n-2\sigma} \,\mathrm{d}z \\
&\leq \rho^{\alpha}(\varepsilon[u]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};Q_T} + C(\varepsilon)|u|_{0;Q_T}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_2 &\leq C|\nabla_x u|_{0;Q_1} \int_{|z|\leq\rho} |z|^{3-n-2\sigma} \,\mathrm{d}z + C|\nabla_x u|_{0;Q_1} \rho^{\alpha} \int_{|z|\leq\rho} |z|^{2-n-2\sigma} \,\mathrm{d}z \\
&\leq \rho^{\alpha}(\varepsilon[u]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};Q_T} + C(\varepsilon)|u|_{0;Q_T}).
\end{aligned}$$

For I_3 and I_4 we first consider that $\sigma \leq \frac{1}{2}$. Choose an $\alpha' > \alpha$ but sufficiently close to α such that $\alpha' < \min(1, 2\sigma)$, then

$$I_3 \leq [u]_{\alpha',\frac{\alpha'}{2\sigma};Q_1} \rho^{\alpha'} C \int_{|z|\geq\rho} |z|^{2\sigma+\alpha-\alpha'-n-2\sigma} \,\mathrm{d}z \leq C[u]_{\alpha',\frac{\alpha'}{2\sigma};Q_1} \rho^{\alpha},$$

$$I_4 \le C\rho^{\alpha'}[u(\cdot,t_2)]_{2\sigma+\alpha-\alpha';\mathbb{R}^n}\int_{|z|\ge\rho}|z|^{2\sigma+\alpha-\alpha'-n-2\sigma}\,\mathrm{d}z \le C[u(\cdot,t_2)]_{2\sigma+\alpha-\alpha';\mathbb{R}^n}\rho^{\alpha}.$$

If $\sigma > \frac{1}{2}$ and $2\sigma+\alpha<2$, then

$$I_3 \le [u]_{2\sigma+\alpha-1,\frac{2\sigma+\alpha-1}{2\sigma};Q_1}\rho^{2\sigma+\alpha-1}C\int_{|z|\ge\rho}|z|^{1-n-2\sigma}\,\mathrm{d}z \le C[u]_{2\sigma+\alpha-1,\frac{2\sigma+\alpha-1}{2\sigma};Q_1}\rho^{\alpha},$$

$$I_4 \le C\rho^{2\sigma+\alpha-1}|\nabla u(\cdot,t_2)|_{0;\mathbb{R}^n}\int_{|z|\ge\rho}|z|^{1-n-2\sigma}\,\mathrm{d}z \le C|\nabla u(\cdot,t_2)|_{0;\mathbb{R}^n}\rho^{\alpha}.$$

If $2\sigma+\alpha>2$, then for $\rho\le|z|\le 1$, we have

$$|u^z(x_1,t_1)-u^z(x_2)| \le |\nabla_x^2 u|_{0;Q_1}|x_1-x_2||z|+|u_t|_{0;Q_1}|t_1-t_2| \le |\nabla_x^2 u|_{0;Q_1}\rho|z|+|u_t|_{0;Q_1}|\rho^{2\sigma}.$$

Hence,

$$\begin{aligned}
I_3 \le & \left|\int_{1\ge|z|\ge\rho}\frac{\big(u^z(x_1,t_1)-u^z(x_2,t_2)\big)\eta^z(x_1,t_1)}{|z|^{n+2\sigma}}\,\mathrm{d}z\right| \\
& + \left|\int_{|z|\ge 1}\frac{\big(u^z(x_1,t_1)-u^z(x_2,t_2)\big)\eta^z(x_1,t_1)}{|z|^{n+2\sigma}}\,\mathrm{d}z\right| \\
& \le C|\nabla_x^2 u|_{0;Q_1}\rho\int_{1\ge|z|\ge\rho}|z|^{2-n-2\sigma}\,\mathrm{d}z + C|u_t|_{0;Q_1}\rho^{2\sigma}\int_{1\ge|z|\ge\rho}|z|^{1-n-2\sigma} \\
& + [u]_{\alpha,\frac{\alpha}{2\sigma};Q_1}\rho^{\alpha}\int_{|z|\ge 1}|z|^{-n-2\sigma}\,\mathrm{d}z \\
\le & C(|\nabla_x^2 u|_{0;Q_1}+|u_t|_{0;Q_1}+[u]_{\alpha,\frac{\alpha}{2\sigma};Q_1})\rho^{\alpha}
\end{aligned}$$

Similarly, for I_4 we have

$$I_4 \le C|\nabla_x u|_{0;Q_1}\rho^{\alpha}.$$

Combining these and applying some interpolation inequalities in Lemma 5.6, we reach (5.20). □

Consider the following Cauchy problem

$$\begin{cases} a(x,t)u_t + (-\Delta)^{\sigma}u + b(x,t)u = f(x,t), & \text{in } Q_T, \\ u(x,0) = u_0(x), & \text{in } \mathbb{R}^n, \end{cases} \tag{5.21}$$

where $\lambda^{-1}\le a(x,t)\le\lambda$ for some constant $\lambda\ge 1$.

Lemma 5.8. *Suppose $b(x,t)$ is bounded in Q_1. Let $u\in\mathcal{C}^{2\sigma+\alpha,1+\frac{\alpha}{2\sigma}}(Q_1)$ satisfy*

$$\begin{cases} a(x,t)u_t + (-\Delta)^{\sigma}u + b(x,t)u \le 0, & \textit{in } Q_1, \\ u(x,0)\le 0, & \textit{in } \mathbb{R}^n, \end{cases}$$

then $u\le 0$ in Q_1.

Proof. Without loss of generality we may assume that $b(x,t) \geq 1$ as before. Let $\eta(x)$ be a smooth cut-off function supported in $B_2 \subset \mathbb{R}^n$ and equal to 1 in B_1. Let $\eta_R(\cdot) = \eta(\cdot/R)$ and $v = \eta_R u$. Then

$$av_t + (-\Delta)^\sigma v + b(x,t)v \leq \langle u, \eta_R \rangle + u(-\Delta)^\sigma \eta_R,$$

where

$$\langle u, \eta \rangle = c(n,\sigma) \int_{\mathbb{R}^n} \frac{(u(x,t) - u(y,t))(\eta(x,t) - \eta(y,t))}{|x-y|^{n+2\sigma}} \, \mathrm{d}y. \tag{5.22}$$

If u is positive somewhere in Q_1, then we can choose R as large as we want such that v attains its positive maximum value in Q_1 at $(x_0,t_0) \in B_R \times (0,1]$. It is clear that $a(x_0,t_0)v_t(x_0,t_0) + (-\Delta)^\sigma v(x_0,t_0) \geq 0$. Since $b \geq 1$, we have

$$\sup_{B_R \times (0,1]} u \leq v(x_0,t_0) \leq \sup_{Q_1} |\langle u, \eta_R \rangle + u(-\Delta)^\sigma \eta_R| \to 0 \quad \text{as } R \to \infty.$$

This finishes the proof of this Lemma. $\square$

Proposition 5.4. *Let $0 < \alpha < \min(1, 2\sigma)$ such that $2\sigma + \alpha$ is not an integer. Suppose that $a(x,t)$, $b(x,t)$, $f(x,t) \in C^{\alpha, \frac{\alpha}{2\sigma}}(Q_1)$ and $u_0(x) \in C^{2\sigma+\alpha}(\mathbb{R}^n)$. Then there exists a unique solution $u \in \mathcal{C}^{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}}(Q_1)$ of (5.21). Moreover, there exists a constant $C > 0$ depending only on $n, \sigma, \lambda, \alpha, |a|_{\alpha, \frac{\alpha}{2\sigma}; Q_1}$ and $|b|_{\alpha, \frac{\alpha}{2\sigma}; Q_1}$ such that*

$$|u|_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_1} \leq C(|u_0|_{2\sigma+\alpha; \mathbb{R}^n} + |f|_{\alpha, \frac{\alpha}{2\sigma}; Q_1}). \tag{5.23}$$

Proof. By Lemma 5.8, there exists $C > 0$ depending only on $\lambda, |b|_{L^\infty(Q_1)}$ such that

$$|u|_{0;Q_1} \leq C(|u_0|_{0;\mathbb{R}^n} + |f|_{0;Q_1}). \tag{5.24}$$

Then the uniqueness of solutions of (5.21) follows immediately. In the following, we will show a priori estimates (5.23). By (5.24) and some interpolation inequalities in Lemma 5.6, we only need to show, instead of (5.23),

$$[u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_1} \leq C(|u_0|_{2\sigma+\alpha; \mathbb{R}^n} + |f|_{\alpha, \frac{\alpha}{2\sigma}; Q_1}). \tag{5.25}$$

First of all, (5.25) holds provided $a = 1, b = 0$ (see, e.g., [103]), and it can be easily extended to the case that a is a positive constant. For the general case, we use the "freezing coefficients" method (see, e.g., [94]).

Fix a small $\delta > 0$, which will be specified later. We can find two points $X_1, X_2 \in Q_1$ such that

$$\frac{|u_t(X_1) - u_t(X_2)|}{\rho(X_1, X_2)^\alpha} \geq \frac{1}{2}[u_t]_{\alpha, \frac{\alpha}{2\sigma}; Q_1}.$$

If $\rho(X_1, X_2) > \delta$, then

$$[u_t]_{\alpha, \frac{\alpha}{2\sigma}; Q_1} \leq 4\delta^{-\alpha}|u_t|_{0;Q_1}.$$

It follows from Lemma 5.6 that, for any small $\varepsilon_0 > 0$,

$$[u_t]_{\alpha, \frac{\alpha}{2\sigma}; Q_1} \leq \varepsilon_0 [u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_1} + C_0|u|_{0;Q_1}, \tag{5.26}$$

where $C_0 > 0$ depending on $n, \sigma, \alpha, \varepsilon_0, \delta$.

If $\rho(X_1, X_2) \leq \delta$, take a cut-off function $\eta(X) \in C^\infty(\mathbb{R}^{n+1})$ such that $\eta(X) = 1$ for $\rho(X, X_1) \leq \delta$, $\eta(X) = 0$ for $\rho(X, X_1) \geq 2\delta$. By the estimates of solutions of (5.21) with a being a positive constant and $b \equiv 0$, we have

$$\begin{aligned}
[u_t]_{\alpha, \frac{\alpha}{2\sigma}; Q_1} &\leq 2\frac{|u_t(X_1) - u_t(X_2)|}{\rho(X_1, X_2)^\alpha} \leq 2[u\eta]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_1} \\
&\leq C_1(|a(X_1)(u\eta)_t + (-\Delta)^\sigma(u\eta)|_{\alpha, \frac{\alpha}{2\sigma}; Q_1} + |u_0\eta|_{2\sigma+\alpha;\mathbb{R}^n} + |u\eta|_{0;Q_1}),
\end{aligned}$$

where $C_1 > 0$ is independent of δ. Note that

$$\begin{aligned}
&a(X_1)(u\eta)_t + (-\Delta)^\sigma(u\eta) \\
&= \eta(a(X)u_t + (-\Delta)^\sigma u) + \eta(a(X_1) - a(X))u_t + a(X_1)u\eta_t - \langle u, \eta\rangle + u(-\Delta)^\sigma\eta \\
&= \eta(f - bu) + \eta(a(X_1) - a(X))u_t + a(X_1)u\eta_t - \langle u, \eta\rangle + u(-\Delta)^\sigma\eta,
\end{aligned}$$

where $\langle u, \eta\rangle$ is defined in (5.22). Since $|\eta(X)(a(X_1) - a(X))| \leq [a]_{\alpha, \frac{\alpha}{2\sigma}; Q_1}\delta^\alpha$, making use of Lemma 5.6 again, we have

$$\begin{aligned}
[u_t]_{\alpha, \frac{\alpha}{2\sigma}; Q_1} \leq C_1\delta^\alpha[u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_1} &+ C(\delta)(|u|_{0;Q_1} + |f|_{\alpha, \frac{\alpha}{2\sigma}; Q_1}) \\
&+ C_1|\langle u, \eta\rangle|_{\alpha, \frac{\alpha}{2\sigma}; Q_1} + C_1|u_0\eta|_{2\sigma+\alpha;\mathbb{R}^n}.
\end{aligned} \tag{5.27}$$

Hence, from (5.20) in Lemma 5.7, (5.27) and (5.26), we can conclude that

$$[u_t]_{\alpha, \frac{\alpha}{2\sigma}; Q_1} \leq (C_1\delta^\alpha + \varepsilon_0)[u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_1} + C(\delta)(|u|_{0,Q_1} + |f|_{\alpha, \frac{\alpha}{2\sigma}; Q_1} + |u_0|_{2\sigma+\alpha;\mathbb{R}^n}). \tag{5.28}$$

Since

$$u_t + (-\Delta)^\sigma u = (1 - a)u_t - bu + f,$$

we see that

$$[u]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};Q_1} \le C([u_t]_{\alpha,\frac{\alpha}{2\sigma};Q_1} + |u|_{0;Q_1} + |f|_{\alpha,\frac{\alpha}{2\sigma};Q_1} + |u_0|_{2\sigma+\alpha;\mathbb{R}^n},), \tag{5.29}$$

where $C > 0$ depending only on $n, \sigma, \lambda, \alpha$, $\|a\|_{\alpha,\frac{\alpha}{2\sigma};Q_1}$ and $\|a,b\|_{\alpha,\frac{\alpha}{2\sigma};Q_1}$. Then (5.23) follows from (5.24), (5.29) and (5.28) by choosing sufficiently small δ and ε_0.

Finally, the existence of solutions of (5.21) follows from standard continuity method.

□

Remark 5.1. *Cauchy problems for non-local operators and pseudo-differential operators in different spaces have been studied, e.g., in [90], [108], [109], [110] and references therein.*

Remark 5.2. *Observe that in the proof of the above proposition the only place we use the uniform lower and upper bounds of $a(x)$ is that at X_1, that is $\frac{1}{\lambda} \le a(X_1) \le \lambda$. This observation will be used in the proof of Proposition 5.5.*

Remark 5.3. *One can also obtain the estimates in Q_T by considering the scaled function $\tilde{u}(x,t) := u(T^{1/2\sigma}x, Tt)$.*

For $\gamma \in (0,1)$, $C^\gamma(\mathbb{S}^n)$ denotes the standard Hölder space over $\mathbb{S}^n$, with norm

$$|v|_{\gamma;\mathbb{S}^n} := |v|_{0;\mathbb{S}^n} + [v]_{\gamma;\mathbb{S}^n} := \sup_{\mathbb{S}^n} |v(\cdot)| + \sup_{\xi_1\neq\xi_2,\xi_1,\xi_2\in\mathbb{S}^n} \frac{|v(\xi_1) - u(\xi_2)|}{|\xi_1 - \xi_2|^\gamma},$$

where $|\xi_1 - \xi_2|$ is understood as the Euclidean distance from ξ_1 to ξ_2 in $\mathbb{R}^{n+1}$. For simplicity, we use $C^\gamma(\mathbb{S}^n)$ to denote $C^{[\gamma],\gamma-[\gamma]}(\mathbb{S}^n)$ when $1 < \gamma \notin \mathbb{N}$, where $[\gamma]$ is the integer part of γ. For $Y_1 = (\xi_1, t_1), Y_2 = (\xi_2, t_2) \in \mathbb{S}^n \times (0,\infty)$ we denote

$$\rho(Y_1, Y_2) = (|\xi_1 - \xi_2|^2 + |t_1 - t_2|^{1/\sigma})^{1/2}.$$

We still assume that $0 < \alpha < \min(1, 2\sigma)$. Let $\mathcal{Q}_T = \mathbb{S}^n \times (0,T]$ for $T > 0$. We say $v \in C^{\alpha,\frac{\alpha}{2\sigma}}(\mathcal{Q}_T)$ if

$$|v|_{\alpha,\frac{\alpha}{2\sigma};\mathcal{Q}_T} = |v|_{0;\mathcal{Q}_T} + [v]_{\alpha,\frac{\alpha}{2\sigma};\mathcal{Q}_T} := \sup_{Y\in\mathcal{Q}_T} v(Y) + \sup_{Y_1\neq Y_2,Y_1,Y_2\in\mathcal{Q}_T} \frac{|v(Y_1) - u(Y_2)|}{\rho(Y_1,Y_2)^\alpha} < \infty,$$

and $v \in \mathcal{C}^{2\sigma+\alpha,1+\frac{\alpha}{2\sigma}}(\mathcal{Q}_T)$ if

$$[v]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};\mathcal{Q}_T} := [v_t]_{\alpha,\frac{\alpha}{2\sigma};\mathcal{Q}_T} + [P_\sigma(v)]_{\alpha,\frac{\alpha}{2\sigma};\mathcal{Q}_T} < \infty$$

and

$$|v|_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};\mathcal{Q}_T} := |v|_{0;\mathcal{Q}_T} + |v_t|_{0,\mathcal{Q}_T} + |P_\sigma(v)|_{0;\mathcal{Q}_T} + [v]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};\mathcal{Q}_T} < \infty.$$

Then $\mathcal{C}^{2\sigma+\alpha,1+\frac{\alpha}{2\sigma}}(\mathcal{Q}_T)$ is a Banach space equipped with the norm $|\cdot|_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};\mathcal{Q}_T}$.

Lemma 5.9. *Suppose that $0 < \alpha < \min(1, 2\sigma)$ and $2\sigma + \alpha$ is not an integer. For any small $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ depending only on n, σ and ε such that for any $v \in \mathcal{C}^{2\sigma+\alpha,1+\frac{\alpha}{2\sigma}}(\mathcal{Q}_T)$, we have*

$$|v_t|_{0;\mathcal{Q}_T} \le \varepsilon[v_t]_{\alpha,\frac{\alpha}{2\sigma};\mathcal{Q}_T} + C(\varepsilon)|v|_{0;\mathcal{Q}_T}, \tag{5.30}$$

$$|P_\sigma v|_{0;\mathcal{Q}_T} \le \varepsilon[v]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};\mathcal{Q}_T} + C(\varepsilon)|v|_{0;\mathcal{Q}_T}, \tag{5.31}$$

$$[v]_{\alpha,\frac{\alpha}{2\sigma};\mathcal{Q}_T} \le \varepsilon[v]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};\mathcal{Q}_T} + C(\varepsilon)|v|_{0;\mathcal{Q}_T}. \tag{5.32}$$

Proof. Using stereographic projections, (1.5) and noticing that $|x - y| \ge C_n|F(x) - F(y)|$, the above inequalities follows from Lemma 5.6. $\square$

Proposition 5.5. *Let $0 < \alpha < \min(1, 2\sigma)$ such that $2\sigma + \alpha$ is not an integer. Let $a(\xi, t)$, $b(\xi, t)$, $f(\xi, t) \in C^{\alpha,\frac{\alpha}{2\sigma}}(\mathcal{Q}_1)$, $v_0 \in C^{2\sigma+\alpha}(\mathbb{S}^n)$ and $\lambda^{-1} \le a(\xi, t) \le \lambda$ for some $\lambda \ge 1$. Then there exists a unique function $v \in \mathcal{C}^{2\sigma+\alpha,1+\frac{\alpha}{2\sigma}}(\mathcal{Q}_1)$ such that*

$$\begin{cases} av_t + P_\sigma(v) + bv = f, & \text{in } \mathcal{Q}_1, \\ v(y, 0) = v_0(y). \end{cases} \tag{5.33}$$

Moreover, there exists a constant C depending only on $n, \sigma, \lambda, \alpha, |a|_{\alpha,\frac{\alpha}{2\sigma};\mathcal{Q}_1}$ and $|b|_{\alpha,\frac{\alpha}{2\sigma};\mathcal{Q}_1}$ such that

$$|v|_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};\mathcal{Q}_1} \le C(|v_0|_{2\sigma+\alpha;\mathbb{S}^n} + |f|_{\alpha,\frac{\alpha}{2\sigma};\mathcal{Q}_1}). \tag{5.34}$$

Proof. Uniqueness of solutions of (5.33) follows from maximum principles. We only need to show a priori estimate (5.34), from which the existence of solution of (5.33) follows by the standard continuity method.

Choose $Y_1 = (\xi_1, t_1), Y_2 = (\xi_2, t_2) \in \mathbb{S}^n \times (0, T)$ such that

$$\frac{|v_t(Y_1) - v_t(Y_2)|}{\rho(Y_1, Y_2)^\alpha} \ge \frac{1}{2}[v_t]_{\alpha,\frac{\alpha}{2\sigma};\mathcal{Q}_1}. \tag{5.35}$$

Without loss of generality we may assume that ξ_1, ξ_2 are on the south hemisphere. Let $F(x)$ be the inverse of stereographic projection from the north pole and

$$u(x,t) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2\sigma}{2}} v(F(x),t).$$

There exist $x_1, x_2 \in B(0,1)$ such that $Y_1 = (F(x_1), t_1), Y_2 = (F(x_2), t_2)$. We denote $X_1 = (x_1, t_1), X_2 = (x_2, t_2)$. By (5.35) there exists a constant C depending only n, σ, α such that

$$[u_t]_{\alpha,\frac{\alpha}{2\sigma};Q_1} \leq C|v_t|_{0,\mathcal{Q}_1} + C|u_t|_{0,Q_1} + C\frac{|u_t(X_1) - u_t(X_2)|}{\rho(X_1,X_2)^\alpha}.$$

Note that u satisfies (5.21) with a, b, f replaced by

$$\left(\frac{2}{1+|x|^2}\right)^{2\sigma} a(F(x),t),\ \left(\frac{2}{1+|x|^2}\right)^{2\sigma} b(F(x),t),\ \left(\frac{2}{1+|x|^2}\right)^{\frac{n+2\sigma}{2}} f(F(x),t).$$

In view of Remark 5.2 and the arguments in the proof of Proposition 5.4, we conclude that

$$[u]_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};Q_1} \leq C(|v_0|_{2\sigma+\alpha;\mathbb{S}^n} + |v|_{0;\mathcal{Q}_1} + |v_t|_{0;\mathcal{Q}_1} + |f|_{\alpha,\frac{\alpha}{2\sigma};\mathcal{Q}_1}).$$

Hence, together with (5.35) and interpolation inequalities in Lemma 5.9, we have

$$[v_t]_{\alpha,\frac{\alpha}{2\sigma};\mathcal{Q}_1} \leq C(|v_0|_{2\sigma+\alpha;\mathbb{S}^n} + |v|_{0;\mathcal{Q}_1} + |f|_{\alpha,\frac{\alpha}{2\sigma};\mathcal{Q}_1}). \tag{5.36}$$

It follows from the maximum principle that $|v|_{0;\mathcal{Q}_1} \leq C(|v_0|_{2\sigma+\alpha;\mathbb{S}^n} + |f|_{\alpha,\frac{\alpha}{2\sigma};\mathcal{Q}_1})$. Hence (5.34) follows from (5.36), (5.33) and some inequalities in Lemma 5.9. □

Corollary 5.1. *Let $0 < \alpha < \min(1, 2\sigma)$ such that $2\sigma + \alpha$ is not an integer. Let $a(\xi,t),\ b(\xi,t),\ f(\xi,t) \in C^{\alpha,\frac{\alpha}{2\sigma}}(\mathcal{Q}_3)$, $\lambda^{-1} \leq a(\xi,t) \leq \lambda$ for some $\lambda \geq 1$. Suppose that $v \in \mathcal{C}^{2\sigma+\alpha,1+\frac{\alpha}{2\sigma}}(\mathcal{Q}_3)$ satisfies*

$$av_t + P_\sigma(v) + bv = f, \quad \text{in } \mathcal{Q}_3.$$

Then there exists a positive constant C depending only on $n, \sigma, \lambda, \alpha, |a|_{\alpha,\frac{\alpha}{2\sigma};\mathbb{S}^n\times[1,3]}$ and $|b|_{\alpha,\frac{\alpha}{2\sigma};\mathbb{S}^n\times[1,3]}$ such that

$$|v|_{2\sigma+\alpha,1+\frac{\alpha}{2\sigma};\mathbb{S}^n\times[2,3]} \leq C(|v|_{\alpha,\frac{\alpha}{2\sigma};\mathbb{S}^n\times[1,3]} + |f|_{\alpha,\frac{\alpha}{2\sigma};\mathbb{S}^n\times[1,3]}).$$

Proof. Let $\eta(t)$ be a smooth cut-off function defined on $\mathbb{R}$ such that $\eta(t) = 0$ when $t \le 4/3$ and $\eta(t) = 1$ when $t \ge 5/3$. Then $\tilde{v} := \eta v$ satisfies

$$\begin{cases} a\tilde{v}_t + P_\sigma(\tilde{v}) + b\tilde{v} = f\eta + av\eta_t, & \text{in } \mathbb{S}^n \times [1,3], \\ \tilde{v}(\cdot, 1) = 0. \end{cases}$$

The corollary follows immediately from Proposition 5.5. □

5.3.2 Short time existence

Proposition 5.6. *Let $0 < \alpha < \min(1, 2\sigma)$ such that $2\sigma + \alpha$ is not an integer. Let $v_0 \in C^{2\sigma+\alpha}(\mathbb{S}^n)$ and $v_0 > 0$ in $\mathbb{S}^n$. Then there exists a small positive constant T_* depending only on $n, \sigma, \alpha, \inf_{\mathbb{S}^n} v_0, |v_0|_{2\sigma+\alpha;\mathbb{S}^n}$ and a unique positive solution $v \in \mathcal{C}^{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}}(\mathbb{S}^n \times [0, T_*])$ of (1.15) in $\mathbb{S}^n \times (0, T_*]$ with $v(\cdot, 0) = v_0$. Furthermore, v is smooth in $\mathbb{S}^n \times (0, T_*)$.*

Proof. By a scaling argument in the time variable, we only need to show the short time existence of

$$\begin{cases} \frac{\partial v^N}{\partial t} = -P_\sigma(v), \\ v(\cdot, 0) = v_0. \end{cases}$$

We shall use the Implicit Function Theorem. By Proposition 5.5, there exists a function $w \in \mathcal{C}^{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}}(\mathbb{S}^n \times (0, 1])$ such that

$$\begin{cases} N v_0^{N-1} w_t = -P_\sigma(w), & \text{in } \mathbb{S}^n \times (0, 1], \\ w(\cdot, 0) = v_0, \end{cases}$$

and for any small positive constant ε_0, we have $\|w(\cdot, t) - v_0\|_{C^{2\sigma+\alpha}(\mathbb{S}^n)} \le \varepsilon_0$ provided $t \le T_{\varepsilon_0}$. Here T_{ε_0} is a positive constant depending on ε_0. Hence, we may assume that $w > 0$ in $\mathbb{S}^n$.

Denote

$$\mathscr{X} = \{\varphi \in \mathcal{C}^{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}}(\mathbb{S}^n \times (0, T_{\varepsilon_0}]) : \varphi(\cdot, 0) = 0\},$$

and

$$\mathscr{Y} = C^{\alpha, \frac{\alpha}{2\sigma}}(\mathbb{S}^n \times (0, T_{\varepsilon_0}]).$$

Define $\mathcal{F}(v) := N|v|^{N-1}\frac{\partial v}{\partial t} + P_\sigma(v)$ for $v \in \mathcal{C}^{2\sigma+\alpha,1+\frac{\alpha}{2\sigma}}(\mathbb{S}^n \times (0, T_{\varepsilon_0}])$, and

$$L : \mathscr{X} \to \mathscr{Y}, \quad \varphi \mapsto \mathcal{F}(w+\varphi) - \mathcal{F}(w).$$

Note that $L(0) = 0$,

$$L'(0)\varphi = Nw^{N-1}\varphi_t + P_\sigma(\varphi) + N(N-1)w^{N-2}w_t\varphi, \quad \forall\ \varphi \in \mathscr{X}.$$

It follows from Proposition 5.5 that $L'(0) : \mathscr{X} \to \mathscr{Y}$ is invertible, when ε_0 is chosen sufficiently small.

By the Implicit Function Theorem, there exists a positive constant $\delta > 0$ such that for any $\phi \in \mathscr{Y}$ with $\|\phi\|_{\mathscr{Y}} \leq \delta$ there exists a unique solution $\varphi \in \mathscr{X}$ of the equation

$$L(\varphi) = \phi.$$

Let $T_* > 0$ be small. Pick a cut off function $0 \leq \eta(t) \leq 1$ in $\mathbb{R}_+$ satisfying $\eta(t) = 1$ for $s \leq T_*$ and $\eta(t) = 0$ if $s \geq 2T_*$. It is easy to see that

$$\|\eta(t)\mathcal{F}(w)\|_{\mathscr{Y}} \leq \delta,$$

provided T_* is sufficiently small. Therefore, there exists a function $\varphi \in \mathscr{X}$ such that

$$L(\varphi) = -\eta(t)\mathcal{F}(w).$$

Thus, $v := w + \varphi$ satisfies $v(\cdot, 0) = v_0$ and

$$\mathcal{F}(w+\varphi) = 0, \quad \text{in } \mathbb{S}^n \times (0, T_*].$$

Moreover, v is positive if T_* is small enough. The smoothness of v follows from Corollary 5.1 and bootstrap arguments. □

5.3.3 Long time existence and convergence

Proposition 5.7. *Let v be a positive smooth solution of* (1.15) *in $\mathbb{S}^n \times (0, 3]$ and satisfy $\Lambda^{-1} \leq v(y,t) \leq \Lambda$ for all $(y,t) \in \mathbb{S}^n \times (0, 3]$ with some positive constant Λ. Then for any positive integer k,*

$$\|v\|_{C^k(\mathbb{S}^n \times [2,3])} \leq C, \tag{5.37}$$

where $C > 0$ depends only on n, σ, k, Λ, and $r_\sigma^{g(1)}$.

Proof. We first observe that $r_\sigma^{g(t)}$ is decreasing in t, and is lower bounded away from 0 by Sobolev inequalities (see, e.g., [10]). Hence through a scaling argument in t, we may assume that v satisfies the equation $\frac{\partial v^N}{\partial t} = -P_\sigma(v)$ instead of (1.15). By the Hölder estimates in [3] (see also Theorem 9.2 in [49]), there exists some $\beta \in (0, \min(1, 2\sigma))$ such that

$$|v|_{\beta, \frac{\beta}{2\sigma}; \mathbb{S}^n \times [1,3]} \leq C(n, \sigma, \beta, \Lambda).$$

The Proposition follows from Corollary 5.1 and bootstrap arguments. $\square$

Now we are ready to prove the following smooth convergence of the fractional Yamabe flow, which has been stated in Theorem 1.2 in the introduction.

Theorem 5.2. *Let $g(0) \in [g_{\mathbb{S}^n}]$ be a smooth metric on $\mathbb{S}^n$ for $n \geq 2$. Then the fractional Yamabe flow* (1.14) *with initial metric $g(0)$ exists for all time $0 < t < \infty$. Furthermore, there exists a smooth metric $g_\infty \in [g_{\mathbb{S}^n}]$ such that*

$$R_\sigma^{g_\infty} = r_\sigma^{g_\infty} \quad \textit{and} \quad \lim_{t \to \infty} \|g(t) - g_\infty\|_{C^l(\mathbb{S}^n)} = 0$$

for all positive integers l.

Remark 5.4. *If we write $g_\infty = v_\infty^{\frac{4}{n-2\sigma}} g_{\mathbb{S}^n}$ where v_∞ is a smooth and positive function on $\mathbb{S}^n$, then Theorem 5.2 implies that v_∞ satisfies*

$$P_\sigma(v_\infty) = r_\sigma^{g_\infty} \cdot v_\infty^{\frac{n+2\sigma}{n-2\sigma}},$$

whose solutions are classified in [41] and [97].

Proof of Theorem 5.2. By Proposition 5.6, we have a unique positive smooth solution of (1.14) on a maximum time interval $[0, T^*)$. Since the flow preserves the volume of the sphere, the Harnack inequality in Theorem 5.1 implies that $v(x,t)$ is uniformly bounded from above and away from zero. Proposition 5.7 yields smooth estimates for v on $\mathbb{S}^n \times [\min(1, T^*/2), T^*)$. It follows that $T^* = \infty$, since otherwise by Proposition 5.6 we can extend v beyond T^*. Moreover, there exists $v_\infty \in C^\infty(\mathbb{S}^n)$ and a sequence $\{v(t_j)\}$ such that $v(t_j)$ converges smoothly to v_∞. By Theorem 5.11 in the Appendix, $v(t)$ converges smoothly to v_∞, i.e. there exists a smooth metric g_∞ on $\mathbb{S}^n$ such that

$g(t)$ converges smoothly to g_∞. The formula for the gradient of the total σ-curvature gives

$$\frac{dS}{dt} = -\frac{n-2\sigma}{2n}(vol_g(\mathbb{S}^n))^{\frac{2\sigma-n}{n}} \int_{\mathbb{S}^n} (R^g_\sigma - r^g_\sigma)^2 \mathrm{d}vol_g.$$

Thus,

$$\int_0^\infty \int_{\mathbb{S}^n} (R^g_\sigma - r^g_\sigma)^2 \mathrm{d}vol_g < \infty,$$

which implies that $R^{g_\infty}_\sigma$ is a positive constant. $\square$

5.4 Two applications

5.4.1 Extinction profile of a fractional porous medium equation

These fractional diffusion equations (1.17) have been systematically studied in [48] and [49]. It is proved in [49] that if $u_0 \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for $p > 4n/(n+2\sigma)$, then there exists a unique strong solution (see [49] for the definition) of (1.17), and the solution will extinct in finite time. More precisely, if u_0 is nonnegative but not identically equals to zero, then there exists a $T = T(u_0) \in (0,\infty)$ such that $u(x,t) > 0$ in $\mathbb{R}^n \times (0,T)$, and $u(x,T) \equiv 0$ in $\mathbb{R}^n$.

Theorem 5.3 below describes the extinction profile of $u(x,t)$, which extends the result of del Pino and Saéz in [50] for $\sigma = 1$ to $\sigma \in (0,1)$.

Theorem 5.3. *Assume that $u_0 \in C^2(\mathbb{R}^n)$ is positive in $\mathbb{R}^n$ for $n \geq 2$. In addition, we assume, for $(u_0^m)_{0,1}(x) := |x|^{2\sigma-n} u_0^m(x/|x|^2)$, that $(u_0^m)_{0,1}(x)$ can be extended to a positive and C^2 function near the origin. There exist $\lambda > 0$ and $x_0 \in R^n$ such that if $T = T(u_0) \in (0,\infty)$ denotes the extinction time of the solution of* (1.17), *then*

$$(T-t)^{-1/(1-m)} u(x,t) = k(n,\sigma)\left(\frac{\lambda}{\lambda^2 + |x-x_0|^2}\right)^{\frac{n+2\sigma}{2}} + \theta(x,t)$$

with

$$\sup_{\mathbb{R}^n}(1+|x|^{n+2\sigma})\theta(x,t) \to 0 \quad \text{as } t \to T,$$

where $k(n,\sigma) = 2^{\frac{n-2}{2}}\big((1-m)P_\sigma(1)\big)^{\frac{n-2\sigma}{4\sigma}}$ and $P_\sigma(1)$ is given in (1.4).

Some estimates of the extinction time T involving the sharp constant in Sobolev inequalities are postponed to Lemma 5.12 in Section 5.4.2.

Let $u(x,t)$ be the solution of (1.17) and $T > 0$ be its extinction time. Since u_0 is not identically zero, it is proved in [49] that $u(x,t) > 0$ in $\mathbb{R}^n \times (0,T)$ and $u(x,t) \in C^\alpha(\mathbb{R}^n \times (0,T))$ for some $\alpha \in (0,1)$. We define $v(F(x),s)$ for all $x \in \mathbb{R}^n$ and all $s \geq 0$ as

$$v(F(x),s) := \left(\frac{1+|x|^2}{2}\right)^{\frac{n-2\sigma}{2}} (T-t)^{-m/(1-m)} u(x,t)^m|_{t=T(1-e^{-s})}, \tag{5.38}$$

where $F : \mathbb{R}^n \to \mathbb{S}^n$ is the inverse of stereographic projection from the north pole and $m = \frac{n-2\sigma}{n+2\sigma}$. By the assumption of u_0, we have $v(\cdot,0) \in C^2(\mathbb{S}^n)$. It follows from Proposition 5.6 that, there exists an $s^* > 0$ and a unique positive function $\tilde{v} \in C^\infty(\mathbb{S}^n \times (0,s^*))$ satisfies

$$\frac{\partial \tilde{v}^N}{\partial s} = -P_\sigma(\tilde{v}) + \frac{1}{1-m}\tilde{v}^N \tag{5.39}$$

and $\tilde{v}_0 = v(\cdot,0)$. On the other hand, $\tilde{u}(x,t)$, which is defined by $\tilde{v}$ through (5.38), satisfies (1.17). By the uniqueness theorem on the solution of (1.17) in [49], $v \equiv \tilde{v}$ in $\mathbb{S}^n \backslash \{\mathcal{N}\} \times (0,s^*)$, and hence v can be extended to a positive and smooth function in $\mathbb{S}^n \times (0,s^*)$.

Our first goal is that v defined by relation (5.38) is positive and smooth in $\mathbb{S}^n \times (0,\infty)$. Secondly, we will show that v converges to a steady solution of (5.39). In summary, we will show the following theorem in terms of v.

Theorem 5.4. *Let v be defined by relation* (5.38)*. Then v is positive and smooth in $\mathbb{S}^n \times (0,\infty)$. Moreover, there is a unique positive solution $\bar{v}$ of*

$$-P_\sigma(\bar{v}) + \frac{1}{1-m}\bar{v}^N = 0 \tag{5.40}$$

such that

$$\|v(y,s) - \bar{v}(y)\|_{C^3(\mathbb{S}^n)} \to 0 \quad \text{as } s \to \infty.$$

Our proof of Theorem 5.4 is inspired by some arguments in [50]. To prove convergence of $v(\cdot,t)$, we first establish the following universal estimates.

Proposition 5.8. *Let v be defined by relation* (5.38)*. There exist positive constants β_1, β_2 such that*

$$\beta_1 \leq v(y,s) \leq \beta_2$$

for all $y \in \mathbb{S}^n$, $s^*/2 \leq s < +\infty$. Hence, $v \in C^\infty(\mathbb{S}^n \times (s^*/2,\infty))$.

Proof. Step1: We show that if s_0 is such that v is positive and smooth in $\mathbb{S}^n \times (s^*/2, s_0)$, then there is a positive constant κ_1, independent of s_0, such that for all $s \in (s^*/2, s_0)$

$$\max_{\mathbb{S}^n} v(\cdot, s) > \kappa_1. \tag{5.41}$$

Let us argue by contradiction. If this is not true, then for every small $\varepsilon > 0$, there is an s_ε such that $s_0 > s_\varepsilon > s^*/2$ and $v(y, s_\varepsilon) < \varepsilon$ for all $y \in \mathbb{S}^n$. Given $\varepsilon > 0$, consider

$$U(x,t) = K^{1/m}[(1 + s_\varepsilon - \log T + \log(T-t))(T-t)]_+^{\frac{1}{1-m}} \left(\frac{2}{1+|x|^2}\right)^{\frac{n+2\sigma}{2}},$$

where K will be chosen later. Direct computations yield that

$$\begin{aligned} & U_t - (\Delta)^\sigma U^{\frac{n-2\sigma}{n+2\sigma}} \\ = \quad & K^{\frac{1}{m}}[(1 + s_\varepsilon - \log T + \log(T-t))(T-t)]_+^{\frac{m}{1-m}} \left(\frac{2}{1+|x|^2}\right)^{\frac{n+2\sigma}{2}} \\ & \cdot \left(\log T - \log(T-t) - 2 - s_\varepsilon + P_\sigma(1) K^{1-1/m}\right), \end{aligned}$$

where we used that $(-\Delta)^\sigma \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2\sigma}{2}} = P_\sigma(1) \left(\frac{2}{1+|x|^2}\right)^{\frac{n+2\sigma}{2}}$ with $P_\sigma(1)$ given in (1.4).

Let t_ε be that $s_\varepsilon - \log T + \log(T - t_\varepsilon) = 0$. We choose K small such that $P_\sigma(1) K^{\frac{m-1}{m}} > 2$ and let $\varepsilon = K$. Since $v(y, s_\varepsilon) < \varepsilon$,

$$u(x, t_\varepsilon) < \varepsilon^{1/m}(T - t_\varepsilon)^{\frac{1}{1-m}} \left(\frac{2}{1+|x|^2}\right)^{\frac{n+2\sigma}{2}} = U(x, t_\varepsilon).$$

For $t > t_\varepsilon$, $U(x,t)$ is a supersolution of (1.17). It follows from the comparison principle (see the proof of Theorem 6.2 in [49]) that $u(x,t) \le U(x,t)$. But U vanishes before T. Hence, u vanishes before T, which contradicts the definition of the extinction time T.

Step 2: v is strictly positive and smooth for $s^*/2 < s < \infty$.

To show this, we define

$$s_0 = \sup\{s > 0 : v \in C^{3,1}(\mathbb{S}^n \times (s^*/2, s))\}.$$

Note that $s_0 \ge s^*$. We assume that $s_0 < \infty$. Since $v \in C^{3,1}(\mathbb{S}^n \times (s^*/2, s_0))$ and v is positive, by Theorem 5.1 and step 1 we have that v is uniformly lower bounded away from 0. We define

$$U(x,t) = (M-t)_+^{1/(1-m)} k(n,\sigma) \left(\frac{1}{1+|x|^2}\right)^{\frac{n+2\sigma}{2}},$$

where $k(n,\sigma)$ is defined in Theorem 5.3. $U(x,t)$ satisfies (1.17) and will be used as a barrier function. By our assumptions on u_0, we choose sufficiently large $M > T$ such that

$$u_0(x) \le M^{1/(1-m)} k(n,\sigma) \left(\frac{1}{1+|x|^2} \right)^{\frac{n+2\sigma}{2}}.$$

It follows from comparison principle (Theorem 6.2 in [49]) that for all $0 < t < T$,

$$u(x,t) \le (M-t)^{1/(1-m)} k(n,\sigma) \left(\frac{1}{1+|x|^2} \right)^{\frac{n+2\sigma}{2}}.$$

Hence, for all $s^*/2 \le s \le s_0$

$$v(y,s) \le \left(\frac{T+(M-T)e^s}{T} \right)^{\frac{m}{1-m}} k(n,\sigma)^m \le \left(\frac{T+(M-T)e^{s_0}}{T} \right)^{\frac{m}{1-m}} k(n,\sigma)^m.$$

It follows that v is uniformly bounded from above. Since v satisfies (5.39), Proposition 5.7 implies that v has a uniform limit as $s \to s_0$ which is also positive and smooth. By Proposition 5.6 v can be extended in a smooth and positive way beyond s_0, which violates the definition of s_0. We conclude that $s_0 = +\infty$.

Step 3: There is a constant $\kappa_2 = (1 + P_\sigma(1)(1-m))^{m/(1-m)} > 0$ such that for all $s > 0$

$$\min_{\mathbb{S}^n} v(y,s) \le \kappa_2.$$

We argue by contradiction. Suppose that there is a time $\bar{s} < \infty$ for which

$$\min_{\mathbb{S}^n} v(y,\bar{s}) > \kappa_2.$$

This implies

$$u(x,\bar{t}) \ge (T - \bar{t} + P_\sigma(1)(1-m)(T-\bar{t}))^{1/(1-m)} \left(\frac{1}{1+|x|^2} \right)^{\frac{n+2\sigma}{2}},$$

where $\bar{t} = T(1-e^{-\bar{s}}) < T$. We consider a barrier function

$$U(x,t) = (T - \bar{t} + P_\sigma(1)(1-m)(T-t))^{\frac{1}{1-m}} \left(\frac{1}{1+|x|^2} \right)^{\frac{n+2\sigma}{2}},$$

which satisfies (1.17). Since $u(x,\bar{t}) \ge U(x,\bar{t})$, by the comparison principle

$$u(x,\bar{t}) \ge (T - \bar{t} + P_\sigma(1)(1-m)(T-t))^{\frac{1}{1-m}} \left(\frac{1}{1+|x|^2} \right)^{\frac{n+2\sigma}{2}}.$$

This contradicts the extinction time T of u.

From Steps 1, 2 and 3 we can conclude Proposition 5.8 by taking $\beta_2 = C\kappa_2$ and $\beta_1 = \kappa_1/C$ where C is the constant in Theorem 5.1 for $s_0 = \infty$. $\square$

Now we are in the position to prove Theorem 5.4. Let J be the functional defined as

$$J(z) = \frac{1}{2}\int_{\mathbb{S}^n} zP_\sigma(z) - \frac{1}{(1-m)(N+1)}\int_{\mathbb{S}^n} z^{N+1}.$$

Direct computations yield

Lemma 5.10. *Let $v(x,s)$ satisfy* (5.39)*. Then*

$$\frac{\mathrm{d}}{\mathrm{d}s}J(v(\cdot,s)) = -N\int_{\mathbb{S}^n} v^{N-1}(v_s)^2 \le 0.$$

The above Lemma indicates that the functional is decreasing in time. The next Lemma states that this functional is always nonnegative, and hence $\lim\limits_{s\to\infty} J(v(\cdot,s))$ exists.

Lemma 5.11. *$J(v(\cdot,s)) \ge 0$ for all $s > 0$.*

Proof. The proof is similar to that of Lemma 6.1 in [50], which is included here for completeness. We argue by contradiction. Assume that for certain $0 < s_0 < \infty$ one has $J(v(\cdot,s_0)) < 0$. By Lemma 5.10 $J(v(\cdot,s)) < 0$ for all $s > s_0$. Let us consider the quantity

$$F(s) = \int_{\mathbb{S}^n} v^{N+1}(y,s)\mathrm{d}y \ge 0, \quad s \in (0,\infty).$$

Then

$$\begin{aligned} \frac{N}{N+1}\frac{\mathrm{d}}{\mathrm{d}s}F(s) = \int_{\mathbb{S}^n}(v^N)_s v &= -2J(v(\cdot,s)) + \frac{N-1}{(1-m)(N+1)}F(s) \\ &\ge \frac{N-1}{(1-m)(N+1)}F(s) \end{aligned}$$

for all $s > s_0$. Note that $F(s) \ne 0$ for all $s \ge s_0$. Otherwise, $v(\cdot,s) \equiv 0$ which is impossible because $J(v(\cdot,s)) \le J(v(\cdot,s_0)) < 0$. Integrating the above differential inequality, we have

$$F(s) \ge F(s_0)e^{s-s_0}.$$

It follows that $F(s) \to \infty$ as $s \to \infty$. On the other hand, Proposition 5.8 implies that v is uniformly bounded. Consequently, $F(s)$ is bounded. We reach a contradiction. $\square$

Proof of Theorem 5.4. It follows from Proposition 5.8 and Proposition 5.7 that for $s > s^*/2$, $v(\cdot, s)$ is compact in $C^k(\mathbb{S}^n)$ for any k. Let $\bar{v}$ be a limit point of $v(\cdot, s)$ as $s \to \infty$ in the C^2 sense. We will show that $\bar{v}$ is a solution of (5.40) and $\bar{v}$ is the unique limit of $v(\cdot, s)$ as $s \to \infty$.

Suppose that along a sequence $s_j \to \infty$, $v(\cdot, s_j) \to \bar{v}$ in $C^2(\mathbb{S}^n)$. Since

$$\frac{\mathrm{d}}{\mathrm{d}s} J(v(\cdot, s)) = -N \int_{\mathbb{S}^n} v^{N-1} v_s^2 = -\frac{4N}{(N+1)^2} \int_{\mathbb{S}^n} |(v^{(N+1)/2}(\cdot, s))_s|,$$

we have, by integrating from s_j to $s_j + \tau$ and using the Cauchy-Schwarz inequality,

$$\begin{aligned} &\int_{\mathbb{S}^n} |v^{\frac{N+1}{2}}(\cdot, s_j + \tau) - v^{\frac{N+1}{2}}(\cdot, s_j)|^2 \\ &\quad \le \frac{(N+1)^2 \tau}{4N} \big(J(v(\cdot, s_j)) - J(v(\cdot, s_j + \tau)) \big). \end{aligned}$$

By Lemma 5.10 and Lemma 5.11, $J(v(\cdot, s))$ has a limit as $s \to \infty$. Hence for each $\tau > 0$, $\{v(\cdot, s_j + \tau)\}_1^\infty$ is Cauchy in L^{N+1}. It follows that $v(\cdot, s_j + \tau) \to \bar{v}$ in L^{N+1}, and in $C^2(\mathbb{S}^n)$ uniformly in τ for τ in bounded intervals. Thus, for any $\phi \in C^\infty(\mathbb{S}^n)$ we have,

$$\begin{aligned} &\int_{\mathbb{S}^n} \big(v^N(\cdot, s_j + 1) - v^N(\cdot, s_j) \big) \phi \\ &\quad = \int_0^1 \int_{\mathbb{S}^n} \left(-P_\sigma\big(v(y, s_j + \tau)\big) + \frac{1}{1-m} v^N(y, s_j + \tau) \right) \phi \mathrm{d}y \mathrm{d}\tau. \end{aligned}$$

After sending $j \to \infty$, we obtain

$$\int_{\mathbb{S}^n} \left(-P_\sigma(\bar{v}) + \frac{1}{1-m} \bar{v}^N \right) \phi = 0,$$

i.e., $\bar{v}$ solves (5.40). Finally, it follows from Theorem 5.9 that $v(\cdot, s)$ converges to $\bar{v}$ in $C^3(\mathbb{S}^n)$. □

Proof of Theorem 5.3. By the classification of solutions of (5.40) in [41] and [97], Theorem 5.3 follows from Theorem 5.4 immediately. □

From Theorem 5.3 we see that the extinction profile of $u(x, t)$ is determined by the pair of numbers $(\lambda, x_0) = (\lambda(u_0), x_0(u_0))$. The next theorem verifies the stability of both the extinction time and the extinction profile.

Theorem 5.5. *$T(u_0), \lambda(u_0)$ and $x_0(u_0)$ continuously depend on u_0 in the sense that if u_0, $\{u_{0;j}\}$ are positive C^2 functions in $\mathbb{R}^n$, $(u_0^m)_{0,1}, (u_{0;j}^m)_{0,1}$ can be extended to positive C^2 functions near the origin, and $\lim_{j\to\infty} \|u_{0;j}^m - u_0^m\|_b = 0$ where $\|\cdot\|_b$ is defined by*

$$\|\cdot\|_b = \|\cdot\|_{C^2(B_2)} + \|(\cdot)_{0,1}\|_{C^2(B_2)},$$

then

$$\lim_{j\to\infty}(T(u_{0;j}), \lambda(u_{0;j}), x_0(u_{0;j})) = (T(u_0), \lambda(u_0), x_0(u_0)).$$

Proof. Given Theorem 5.8, Lemma 5.13 and Theorem 5.9, the proof is identical to the proof of Theorem 1.2 in [50]. We refer to [50] for details. □

5.4.2 A Sobolev inequality and a Hardy-Littlewood-Sobolev inequality along a fractional diffusion equation

Recently, Carlen, Carrillo and Loss in [35] noticed that some Hardy-Littlewood-Sobolev inequalities in dimension $n \geq 3$ and some special Gagliardo-Nirenberg inequalities can be related by a fast diffusion equation. In another recent paper [55], Dolbeault used a fast diffusion flow to obtain an optimal integral remainder term which improves (1.18) in dimension $n \geq 5$. Inspired by [35] and [55], we consider some Sobolev inequality (1.19) involving fractional Sobolev spaces of order $\sigma \in (0,1)$, compared to those mentioned above corresponding to $\sigma = 1$.

We investigate the relation between (1.19) and (1.21) via the fractional diffusion equation (1.17), i.e.

$$u_t = -(-\Delta)^\sigma u^m$$

with $m = 1/N = \frac{n-2\sigma}{n+2\sigma}$. If we suppose that the initial data u_0 satisfies the assumptions in Theorem 5.3, then by Theorem 5.4 (which is used to prove Theorem 5.3) $u(\cdot,t)$ is positive and smooth in $\mathbb{R}^n$ before its extinction time, and for any fixed t, $u(x,t) = O(|x|^{-n-2\sigma})$ as $x \to \infty$. We define

$$H(t) := H_{n,\sigma}(u(\cdot,t)) = \int_{\mathbb{R}^n} u(-\Delta)^{-\sigma} u \mathrm{d}x - S_{n,\sigma}\|u\|^2_{L^{\frac{2n}{n+2\sigma}}}. \tag{5.42}$$

It follows from direct computations that $\frac{\mathrm{d}}{\mathrm{d}t}H \geq 0$ (see Proposition 5.9).

Consequently, one can prove (1.21), which is equivalent to $H \le 0$, by showing

$$\limsup_{t\to T} H(t) \le 0$$

where T is the extinction time of (1.17). This can be seen clearly from Theorem 5.4. From this and Proposition 5.9 we also recover that u^m is an extremal of (1.19) if u is an extremal of (1.21).

Along this fractional fast diffusion flow, we can improve the Sobolev inequality (1.19), via a quantitative estimate of the remainder term. This improvement also holds as $\sigma \to 1$ and it extends some work of Dolbeault in [55].

Theorem 5.6. *Assume that $\sigma \in (0,1)$ and $n > 4\sigma$. There exists a positive constant C depending only on n and σ such that for any nonnegative function $u \in \dot{H}^\sigma(\mathbb{R}^n)$ we have*

$$\begin{aligned} S_{n,\sigma}\|u^N\|^2_{L^{\frac{2n}{n+2\sigma}}} &- \int_{\mathbb{R}^n} u^N(-\Delta)^{-\sigma}u^N \mathrm{d}x \\ &\le C\|u\|^{\frac{8\sigma}{n-2\sigma}}_{L^{2^*(\sigma)}}\left(S_{n,\sigma}\|u\|^2_{\dot{H}^\sigma} - \|u\|^2_{L^{2^*(\sigma)}}\right), \end{aligned} \tag{5.43}$$

where $N = \frac{n+2\sigma}{n-2\sigma}$. Moreover, C can be taken as $\frac{n+2\sigma}{n}(1-e^{-\frac{n}{2\sigma}})S_{n,\sigma}$.

We first have

Proposition 5.9. *Assume that $n \ge 2$. If u is a solution of* (1.17) *with positive initial data $u_0 \in C^2$ in $\mathbb{R}^n$ satisfying that $(u_0^m)_{0,1}$ can be extended to a positive C^2 function near the origin, then*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}H = \left(\int_{\mathbb{R}^n} u^{m+1}\right)^{\frac{2\sigma}{n}}\left(S_{n,\sigma}\|u^m\|^2_{\dot{H}^\sigma} - \|u^m\|^2_{L^{2^*(\sigma)}}\right) \ge 0,$$

where H is given by (5.42)*.*

Proof. It follows from (1.17) and (1.19) that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}H &= \int_{\mathbb{R}^n} 2u(-\Delta)^{-\sigma}u_t \mathrm{d}x - 2S_{n,\sigma}\left(\int_{\mathbb{R}^n} u^{m+1}\right)^{\frac{2\sigma}{n}}\int_{\mathbb{R}^n} u^m u_t \\ &= -2\int_{\mathbb{R}^n} u^{m+1} + 2S_{n,\sigma}\left(\int_{\mathbb{R}^n} u^{m+1}\right)^{\frac{2\sigma}{n}}\int_{\mathbb{R}^n} u^m(-\Delta)^\sigma u^m \\ &= 2\left(\int_{\mathbb{R}^n} u^{m+1}\right)^{\frac{2\sigma}{n}}\left(S_{n,\sigma}\|u^m\|^2_{\dot{H}^\sigma} - \|u^m\|^2_{L^{2^*(\sigma)}}\right) \ge 0. \end{aligned}$$

Note that the first part of Theorem 5.4, i.e., v defined by (5.38) is positive and smooth in $\mathbb{S}^n \times (0,\infty)$, has been used in the justifications of these equalities. □

The next lemma gives an estimate for the extinction time of solutions of (1.17).

Lemma 5.12. *If u is a solution of (1.17) with positive initial data $u_0 \in C^2$ in $\mathbb{R}^n$ satisfying that $(u_0^m)_{0,1}$ can be extended to a positive C^2 function near the origin, then for any $t \in (0,T)$ we have*

$$\left(\frac{4\sigma(T-t)}{(n+2\sigma)S_{n,\sigma}}\right)^{\frac{n}{2\sigma}} \leq \int_{\mathbb{R}^n} u^{m+1}(t,x)dx \leq \int_{\mathbb{R}^n} u_0^{m+1}dx.$$

Consequently, the extinction time T is bounded by

$$T \leq \frac{(n+2\sigma)S_{n,\sigma}}{4\sigma}\left(\int_{\mathbb{R}^n} u_0^{m+1}dx\right)^{\frac{2\sigma}{n}}.$$

If in addition $n > 4\sigma$, then

$$T \geq \frac{(n+2\sigma)}{2n}\frac{\int_{\mathbb{R}^n} u_0^{m+1}dx}{\int_{\mathbb{R}^n} u_0^m(-\Delta)^\sigma u_0^m}$$

and

$$\int_{\mathbb{R}^n} u^m(\cdot,t)(-\Delta)^\sigma u^m(\cdot,t) \leq \int_{\mathbb{R}^n} u_0^m(-\Delta)^\sigma u_0^m$$
$$\int_{\mathbb{R}^n} u^{m+1}(\cdot,t) \geq \int_{\mathbb{R}^n} u_0^{m+1} - \frac{2n}{n+2\sigma}t\int_{\mathbb{R}^n} u_0^m(-\Delta)^\sigma u_0^m.$$

Proof. As in the proof of Lemma 5.11, we define

$$F(t) := \int_{\mathbb{R}^n} u^{m+1}(x,t)\mathrm{d}x, \tag{5.44}$$

which is positive in $(0,T)$ and $F(T)=0$. It follows that

$$\begin{aligned} F'(t) &= (m+1)\int_{\mathbb{R}^n} u^m(\cdot,t)u_t(\cdot,t) \\ &= -(m+1)\int_{\mathbb{R}^n} u^m(\cdot,t)(-\Delta)^\sigma u^m(\cdot,t) \leq -\frac{m+1}{S_{n,\sigma}}F(t)^{1-\frac{2\sigma}{n}}, \end{aligned}$$

where we have used the Sobolev inequality (1.19) in the last inequality. This shows the first two inequalities by simple integrations. If in addition $n > 4\sigma$, then

$$\begin{aligned} F''(t) &= m(m+1)\int_{\mathbb{R}^n} u^{m-1}(\cdot,t)\big((-\Delta)^\sigma u^m(\cdot,t)\big)^2 \\ &\quad + m(m+1)\int_{\mathbb{R}^n} u^m(\cdot,t)(-\Delta^\sigma)\big(u^{m-1}(-\Delta)^\sigma u^m(\cdot,t)\big) \\ &= 2m(m+1)\int_{\mathbb{R}^n} u^{m-1}(\cdot,t)\big((-\Delta)^\sigma u^m(\cdot,t)\big)^2 \geq 0, \end{aligned}$$

where the condition $n > 4\sigma$ is used to guarantee the L^2 integrability of $u^m(\cdot,t)$ such that we can use Plancherel's theorem in the second equality. Thus, the lower bound of T follows from that $0 = F(T) \geq F(t) + F'(t)(T-t)$ with sending $t \to 0$. The last two inequalities follows from the sign of F'' and simple integrations. $\square$

Let

$$Q := -\frac{1}{m+1}F'F^{\frac{2\sigma-n}{n}}, E := -\frac{1}{m+1}F'F^{-1}, G(t_1,t_2) := \exp\left((m+1)\int_{t_1}^{t_2} E(s)\mathrm{d}s\right). \tag{5.45}$$

Theorem 5.7. *Assume $n > 4\sigma$. For any u_0 positive and C^2 in $\mathbb{R}^n$ satisfying that $(u_0^m)_{0,1}$ can be extended to a positive C^2 function near the origin, we have*

$$\begin{aligned} S_{n,\sigma}\|u_0\|^2_{L^{\frac{2n}{n+2\sigma}}} - \int_{\mathbb{R}^n} u_0(-\Delta)^{-\sigma}u_0\mathrm{d}x + 4mS_{n,\sigma}\int_0^T \mathrm{d}t\int_0^t F(s)^{\frac{2\sigma}{n}}K(s)G(t,s)\mathrm{d}s \\ = 2\|u_0^m\|^{\frac{4\sigma}{n-2\sigma}}_{L^{2^*(\sigma)}}\left(S_{n,\sigma}\|u_0^m\|^2_{\dot{H}^\sigma} - \|u_0^m\|^2_{L^{2^*(\sigma)}}\right)\int_0^T G(t,0)\mathrm{d}t \end{aligned}$$

where $u(\cdot,t)$ is the solution of (1.17) *with initial data $u(\cdot,t) = u_0$, T is the extinction time of $u(\cdot,t)$ and F,E,G,K are defined in* (5.44), (5.45) *and* (5.46).

Proof. From the proof of Proposition 5.9 we know that

$$H'(t) = 2F(t)\big(S_{n,\sigma}Q(t) - 1\big).$$

Hence

$$\begin{aligned} H''(t) &= 2F'(t)\big(S_{n,\sigma}Q(t)-1\big) + 2F(t)S_{n,\sigma}Q'(t) \\ &= \frac{F'(t)}{F(t)}H'(t) + 2F(t)S_{n,\sigma}Q'(t) \\ &= -(m+1)E(t)H'(t) + 2F(t)S_{n,\sigma}Q'(t). \end{aligned}$$

On the other hand,

$$\begin{aligned} Q'(t) &= \frac{F''(t) - \frac{n-2\sigma}{n}F^{-1}(t)\big(F'(t)\big)^2}{-(m+1)F(t)^{\frac{n-2\sigma}{n}}} \\ &= -\frac{2m}{F(t)^{\frac{n-2\sigma}{n}}}\left(\int_{\mathbb{R}^n} u^{m-1}(\cdot,t)\big((-\Delta)^\sigma u^m(\cdot,t)\big)^2 - F^{-1}\int_{\mathbb{R}^n} u^m(\cdot,t)(-\Delta)^\sigma u^m(\cdot,t)\right) \\ &= -\frac{2m}{F(t)^{\frac{n-2\sigma}{n}}}\int_{\mathbb{R}^n} u(\cdot,t)^{m-1}|-(-\Delta)^\sigma u(\cdot,t)^m + E(t)u(\cdot,t)|^2. \end{aligned}$$

Denote

$$K(t) := \int_{\mathbb{R}^n} u(\cdot,t)^{m-1}|-(-\Delta)^\sigma u(\cdot,t)^m + E(t)u(\cdot,t)|^2. \tag{5.46}$$

Then

$$H''(t) = -(m+1)E(t)H'(t) - 4mF^{\frac{2\sigma}{n}}(t)S_{n,\sigma}K(t).$$

Multiplying $G(0,s)$ and integrating from 0 to t, we have

$$H'(t)G(0,t) - H'(0)G(0,0) = \int_0^t (H'G)'(s)\mathrm{d}s = -4mS_{n,\sigma}\int_0^t F(s)^{\frac{2\sigma}{n}}K(s)G(0,s)\mathrm{d}s.$$

Dividing $G(0,t)$ and integrating from 0 to T, we obtain

$$0 - H(0) = H'(0)\int_0^T G(t,0)\mathrm{d}t - 4mS_{n,\sigma}\int_0^T \mathrm{d}t\int_0^t F(s)^{\frac{2\sigma}{n}}K(s)G(t,s)\mathrm{d}s,$$

which finishes the proof. □

The drawback of the above Theorem is that the extra terms are not explicit. Fortunately, we can use simple estimates to reach Theorem 5.6.

Proof of Theorem 5.6. We first assume that $w = u_0^m$ where $u_0 \in C^2(\mathbb{R}^n)$ is positive and satisfies that $(u_0^m)_{0,1}$ can be extended to a positive C^2 function near the origin. By Lemma 5.12,

$$(m+1)E(s) \geq (m+1)S_{n,\sigma}^{-1}\left(\int_{\mathbb{R}^n} u(\cdot,s)^{m+1}\right)^{-2\sigma/n} \geq (m+1)S_{n,\sigma}^{-1}\left(\int_{\mathbb{R}^n} u_0^{m+1}\right)^{-2\sigma/n} =: b.$$

By Lemma 5.12 again, we have $bT \leq \frac{n}{2\sigma}$. Therefore,

$$\begin{aligned}\int_0^T G(t,0)\mathrm{d}t \leq \int_0^T e^{-bt}\mathrm{d}t = &\frac{1-e^{-bT}}{b}\\ &\leq \frac{1-e^{-\frac{n}{2\sigma}}}{m+1}S_{n,\sigma}\left(\int_{\mathbb{R}^n} u_0^{m+1}\right)^{2\sigma/n}.\end{aligned}$$

Hence (5.43) holds for $w = u_0^m$ where $u_0 \in C^2(\mathbb{R}^n)$ is positive and satisfies that $(u_0^m)_{0,1}$ can be extended to a positive C^2 function near the origin.

For any nonnegative $u \in C_c^\infty(\mathbb{R}^n)$, we consider $w_\varepsilon = u + \varepsilon(\frac{2}{1+|x|^2})^{\frac{n-2\sigma}{2}}$ with $\varepsilon > 0$. Then (5.43) holds for w_ε. By sending $\varepsilon \to 0$, we have (5.43) for u. Finally, Theorem 5.6 follows from a density argument. □

5.5 Appendix: A uniqueness theorem for negative gradient flows involving nonlocal operators

In this appendix, we provide a uniqueness theorem for fractional Yamabe flows, which is analog to L. Simon's uniqueness Theorem in [123]. The proofs are essentially the same and we will just sketch them in our setting. Denote $H^\sigma(\mathbb{S}^n)$ as the closure of $C^\infty(\mathbb{S}^n)$ under the norm

$$\|v\|_{H^\sigma(\mathbb{S}^n)} = \int_{\mathbb{S}^n} vP_\sigma(v).$$

Let $\alpha \in (0,1)$ such that $2\sigma + \alpha$ is not an integer. Let J be the functional defined as

$$J(v) = \frac{1}{2}\int_{\mathbb{S}^n} vP_\sigma(v) - \frac{1}{(1-m)(N+1)}\int_{\mathbb{S}^n} v^{N+1}, \quad v \in H^\sigma(\mathbb{S}^n).$$

Then

$$\nabla J(v) = P_\sigma(v) - \frac{1}{1-m}v^N.$$

Let $\bar{v}$ be such that $\nabla J(\bar{v}) = 0$.

Theorem 5.8. *There exist $\theta \in (0, 1/2)$ and $r_0 > 0$ such that for any $v \in C^{2\sigma+\alpha}(\mathbb{S}^n)$ with $\|v - \bar{v}\|_{C^{2\sigma+\alpha}} < r_0$,*

$$\|\nabla J(v)\|_{L^2(\mathbb{S}^n)} \geq |J(v) - J(\bar{v})|^{1-\theta}.$$

Proof. Since we have Schauder estimates (see, e.g., [82]) and L^2 estimates (which is free from equivalence of definitions of fractional Sobolev spaces on $\mathbb{S}^n$) for P_σ, the proof is identical to that of Theorem 3 in [123]. □

Let $v(x,s)$ and $\bar{v}$ be as in Section 5.4.1. Then direction computations and uniform bounds of $v(x,s)$ yield the following lemma

Lemma 5.13. *There exist two constant c_0 and T_0 such that for any $t > T_0$ we have*

$$-\frac{\mathrm{d}}{\mathrm{d}s}J(v(\cdot,s)) \geq c_0\|v_s\|_{L^2(\mathbb{S}^n)}\|\nabla J(v(\cdot,s))\|_{L^2(\mathbb{S}^n)}.$$

Theorem 5.9.

$$\lim_{s\to\infty}\|v(\cdot,s) - \bar{v}\|_{C^l(\mathbb{S}^n)} = 0$$

for any positive integer l.

Proof. First we can prove that $v(\cdot,t)$ converges to $\bar{v}$ in $C^{2\sigma+\alpha}(\mathbb{S}^n)$, using the same methods as the proof of Proposition 21 in [2] or the proof of Theorem 1 in [70]. Then Theorem 5.9 follows from the uniform C^{l+1} bound of $v(x,s)$. □

Similarly if

$$S(z)=\frac{\int_{\mathbb{S}^n} zP_\sigma(z)}{\left(\int_{\mathbb{S}^n} z^{N+1}\right)^{\frac{2}{N+1}}},\quad z\in H^\sigma(\mathbb{S}^n),$$

then

$$\nabla S(z)=2\left(\int_{\mathbb{S}^n} z^{N+1}\right)^{-\frac{2}{N+1}}\left(P_\sigma(z)-\frac{\int_{\mathbb{S}^n} zP_\sigma(z)}{\int_{\mathbb{S}^n} z^{N+1}}z^N\right).$$

Let $v(x,t)$ and v_∞ be as in Theorem 5.2. Note that $\nabla S(v_\infty)=0$. The following can be proved in the same way as above.

Theorem 5.10. *There exist $\theta\in(0,1/2)$ and $r_0>0$ such that for any $\|v-v_\infty\|_{C^{2\sigma+\alpha}}<r_0$,*

$$\|\nabla S(v)\|_{L^2(\mathbb{S}^n)}\geq|S(v)-S(v_\infty)|^{1-\theta}.$$

Lemma 5.14. *There exist two constant c_0 and T_0 such that for any $t>T_0$ we have*

$$-\frac{\mathrm{d}}{\mathrm{d}t}S(v(\cdot,t))\geq c_0\|v_t\|_{L^2(\mathbb{S}^n)}\|\nabla S(v(\cdot,t))\|_{L^2(\mathbb{S}^n)}.$$

Theorem 5.11.

$$\lim_{t\to\infty}\|v(\cdot,t)-v_\infty\|_{C^l(\mathbb{S}^n)}=0$$

for any positive integer l.

References

[1] A. Ancona, *Elliptic operators, conormal derivatives and positive parts of functions (with an appendix by Haïm Brezis)*, J. Funct. Anal. **257** (2009), no. 7, 2124–2158.

[2] B. Andrews, *Monotone quantities and unique limits for evolving convex hypersurfaces*, Internat. Math. Res. Notices **1997**, no. 20, 1001–1031.

[3] I. Athanasopoulos and L. A. Caffarelli, *Continuity of the temperature in boundary heat control problems*, Adv. Math. **224** (2010), no. 1, 293–315.

[4] T. Aubin, *Problèmes isopérimétriques et espaces de Sobolev*, J. Differential Geom. **11** (1976), no. 4, 573–598.

[5] T. Aubin, *Meilleures constantes dans le théorème d'inclusion de Sobolev et un théorème de Fredholm non linéaire pour la transformation conforme de la courbure scalaire*, J. Funct. Anal. **32** (1979), no. 2, 148–174.

[6] T. Aubin and Y.Y. Li, *On the best Sobolev inequality*, J. Math. Pures Appl. (9) **78** (1999), no. 4, 353–387.

[7] S. Axler, P. Bourdon and W. Ramey, *Bôcher's Theorem*, Amer. Math. Month. **99** (1992), no. 1, 51–55.

[8] A. Bahri and J.-M. Coron, *The scalar-curvature problem on the standard three-dimensional sphere*, J. Funct. Anal. **255** (1991), no. 1, 106–172.

[9] P. Baird, A. Fardoun and R. Regbaoui, *Q-curvature flow on* 4*-manifolds*, Calc. Var. Partial Differential Equations **27** (2006), no. 1, 75–104.

[10] W. Beckner, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. of Math. (2) **138** (1993), no. 1, 213–242.

[11] H. Berestycki and L. Nirenberg, *On the method of moving planes and the sliding method*, Bol. Soc. Brasil. Mat. (N.S.) **22** (1991), no. 1, 1–37.

[12] G. Bianchi and H. Egnell, *A note on the Sobolev inequality*, J. Funct. Anal. **100** (1991), no. 1, 18–24.

[13] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo and J. L. Vázquez, *Asymptotics of the fast diffusion equation via entropy estimates*, Arch. Ration. Mech. Anal. **191** (2009), no. 2, 347–385.

[14] M. Bonforte, J. Dolbeault, G. Grillo and J.L. Vázquez, *Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities*, Proc. Natl. Acad. Sci. USA **107** (2010), no. 38, 16459–16464.

[15] J.-P. Bourguignon and J.-P. Ezin, *Scalar curvature functions in a conformal class of metrics and conformal transformations*, Trans. Amer. Math. Soc. **301** (1987), no. 2, 723–736.

[16] C. Brändle, E. Colorado, A. de Pablo and U. Sánchez, *A concave-convex elliptic problem involving the fractional Laplacian*, to appear in Proc. Roy. Soc. Edinburgh Sect. A.

[17] T.P. Branson, *Sharp inequalities, the functional determinant, and the complementary series*, Trans. Amer. Math. Soc. **347** (1995), no. 10, 3671–3742.

[18] S. Brendle, *A generalization of the Yamabe flow for manifolds with boundary*, Asian J. Math. **6** (2002), no. 4, 625–644.

[19] S. Brendle, *Global existence and convergence for a higher order flow in conformal geometry*, Ann. of Math. (2) **158** (2003), no. 1, 323–343.

[20] S. Brendle, *Convergence of the Yamabe flow for arbitrary initial energy*, J. Differential Geom. **69** (2005), no. 2, 217–278.

[21] S. Brendle, *Convergence of the Q-curvature flow on $\mathbb{S}^4$*, Adv. Math. **205** (2006), no. 1, 1–32.

[22] S. Brendle, *Convergence of the Yamabe flow in dimension* 6 *and higher*, Invent. Math. **170** (2007), no. 3, 541–576.

[23] H. Brezis, *On a conjecture of J. Serrin*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **19** (2008), no. 4, 335–338.

[24] H. Brezis and T. Kato, *Remarks on the Schrödinger operator with singular complex potentials*, J. Math. Pures Appl. (9) **58** (1979), no. 2, 137–151.

[25] H. Brezis and E. H. Lieb, *Sobolev inequalities with remainder terms*, J. Funct. Anal. **62** (1985), no. 1, 73–86.

[26] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), no. 4, 437–477.

[27] X. Cabre and Y. Sire, *Nonlinear equations for fractional Laplacians I: Regularity, maximum principles, and Hamiltonian estimates*, arXiv:1012.0867.

[28] L. A. Caffarelli, *Interior a priori estimates for solutions of fully nonlinear equations*, Ann. of Math. (2) **130** (1989), no. 1, 189–213.

[29] L. A. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial. Diff. Equ. **32** (2007), no. 7-9, 1245–1260.

[30] L. A. Caffarelli and L. Silvestre, *Regularity theory for fully nonlinear integro-differential equations*, Comm. Pure Appl. Math. **62** (2009), no. 5, 597–638.

[31] L. A. Caffarelli and L. Silvestre, *Regularity results for nonlocal equations by approximation*, Arch. Ration. Mech. Anal. **200** (2011), no. 1, 59–88.

[32] L. A. Caffarelli and L. Silvestre, *The Evans-Krylov theorem for nonlocal fully nonlinear equations*, Ann. of Math. (2) **174** (2011), no. 2, 1163–1187.

[33] L. A. Caffarelli and J. L. Vazquez, *Nonlinear porous medium flow with fractional potential pressure*, Arch. Rational Mech. Anal. **202** (2011), no. 2, 537–565.

[34] D. Cao and Y.Y. Li, *Results on positive solutions of elliptic equations with a critical Hardy-Sobolev operator*, Methods Appl. Anal. **15** (2008), no. 1, 81–95.

[35] E. Carlen, J. arrillo and M. Loss, *Hardy-Littlewood-Sobolev inequalities via fast diffusion flows*, Proc. Natl. Acad. Sci. USA **107** (2010), no. 46, 19696–19701.

[36] A. Chang and M. González, *Fractional Laplacian in conformal geometry*, Adv. Math. **226** (2011), no. 2, 1410–1432.

[37] A. Chang, M.J. Gursky and P. Yang, *The scalar curvature equation on* 2- *and* 3-*spheres*, Calc. Var. Partial Differential Equations **1** (1993), no. 2, 205–229.

[38] A. Chang and P. Yang, *Prescribing Gaussian curvature on* $\mathbb{S}^2$, Acta Math. **159** (1987), no. 3-4, 215–259.

[39] A. Chang and P. Yang, *Conformal deformation of metrics on* $\mathbb{S}^2$, J. Differential Geom. **27** (1988), no. 2, 259–296.

[40] A. Chang and P. Yang, *A perturbation result in prescribing scalar curvature on* $\mathbb{S}^n$, Duke Math. J. **64** (1991), no. 1, 27–69.

[41] W. Chen, C. Li and B. Ou, *Classification of solutions for an integral equation*, Comm. Pure Appl. Math. **59** (2006), no. 3, 330–343.

[42] P. Cherrier, *Problèmes de Neumann non linéaires sur les variétés riemanniennes*, J. Funct. Anal. **57** (1984), no. 2, 154–206.

[43] B. Chow, *The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature*, Comm. Pure Appl. Math. **45** (1992), no. 8, 1003–1014.

[44] S. Chua, *Extension Theorems on Weighted Sobolev Spaces*, Indiana Univ. Math. J. **41** (1992), no. 4, 1027–1076.

[45] A. Cianchi, N. Fusco, F. Maggi and A. Pratelli, *The sharp Sobolev inequality in quantitative form*, J. Eur. Math. Soc. (JEMS) **11** (2009), no. 5, 1105–1139.

[46] P. Daskalopoulos and N. Sesum, *On the extinction profile of solutions to fast diffusion*, J. Reine Angew. Math. **622** (2008), 95–119.

[47] E. De Giorgi, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43.

[48] A. de Pablo, F. Quiros, A. Rodriguez and J. L. Vazquez, *A fractional porous medium equation*, Adv. Math. **226** (2011), no. 2, 1378–1409.

[49] A. de Pablo, F. Quiros, A. Rodriguez and J. L. Vazquez, *A general fractional porous medium equation*, arXiv:1104.0306v1.

[50] M. del Pino and M. Sáez, *On the extinction profile for solutions of $u_t = \Delta u^{(n-2)/(n+2)}$*, Indiana Univ. Math. J. **50** (2001), no. 1, 611–628.

[51] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, arXiv:1104.4345v2.

[52] Z. Djadli and A. Malchiodi, *Existence of conformal metrics with constant Q-curvature*, Ann. of Math. (2) **168** (2008), no. 3, 813–858.

[53] Z. Djadli, A. Malchiodi and M.O. Ahmedou, *Prescribing a fourth order conformal invariant on the standard sphere. I. A perturbation result*, Commun. Contemp. Math. **4** (2002), no. 3, 375–408.

[54] Z. Djadli, A. Malchiodi and M.O. Ahmedou, *Prescribing a fourth order conformal invariant on the standard sphere. II. Blow up analysis and applications*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **1** (2002), no. 2, 387–434.

[55] J. Dolbeault, *Sobolev and Hardy-Littlewood-Sobolev inequalities: duality and fast diffusion*, arXiv:1103.1145v1.

[56] J. F. Escobar, *The Yamabe problem on manifolds with boundary*, J. Differential Geom. **35** (1992), no. 1, 21–84.

[57] J. F. Escobar, *Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary*, Ann. of Math. (2) **136** (1992), no. 1, 1–50.

[58] J. F. Escobar and G. Garcia, *Conformal metrics on the ball with zero scalar curvature and prescribed mean curvature on the boundary*, J. Funct. Anal. **211** (2004), no. 1, 71–152.

[59] J.F. Escobar and R. Schoen, *Conformal metrics with prescribed scalar curvature*, Invent. Math. **86** (1986), no. 2, 243–254.

[60] E. Fabes, C. Kenig and R. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations **7** (1982), no. 1, 77–116.

[61] C. Fefferman and C.R. Graham, *Conformal invariants*, The mathematical heritage of Élie Cartan (Lyon, 1984). Astérisque **1985**, Numero Hors Serie, 95–116.

[62] V. Felli, *Existence of conformal metrics on $\mathbb{S}^n$ with prescribed fourth-order invariant*, Adv. Differential Equations **7** (2002), no. 1, 47–76.

[63] V. Galaktionov and L. Peletier, *Asymptotic behaviour near finite-time extinction for the fast diffusion equation*, Arch. Rational Mech. Anal. **139** (1997), no. 1, 83–98.

[64] B. Gidas, W.-M. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), no. 3, 209–243.

[65] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order,* Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[66] M. González, R. Mazzeo and Y. Sire, *Singular solutions of fractional order conformal Laplacians*, J. Geom. Anal. DOI: 10.1007/s12220-011-9217-9.

[67] M. González and J. Qing, *Fractional conformal Laplacians and fractional Yamabe problems*, arXiv:1012.0579.

[68] C.R. Graham, R. Jenne, L. J. Mason and G. A. J. Sparling, *Conformally invariant powers of the Laplacian I. Existence*, J. London Math. Soc. (2) **46** (1992), no. 3, 557–565.

[69] C.R. Graham and M. Zworski, *Scattering matrix in conformal geometry*, Invent. Math. **152** (2003), no. 1, 89–118.

[70] P. Guan and G. Wang, *A fully nonlinear conformal flow on locally conformally flat manifolds*, J. Reine Angew. Math. **557** (2003), 219–238.

[71] R. A. Hager and J. Ross, *A regularity theorem for linear second order elliptic divergence equations*, Ann. Scuola Norm. Sup. Pisa (3) **26** (1972), 283–290.

[72] R. Hamilton, *The Ricci flow on surfaces*, Mathematics and general relativity (Santa Cruz, CA, 1986), 237262, Contemp. Math., **71**, Amer. Math. Soc., Providence, RI, 1988.

[73] Z.-C. Han, *Prescribing Gaussian curvature on S^2*, Duke Math. J. **61** (1990), no. 3, 679–703.

[74] Z.-C. Han and Y.Y. Li, *The Yamabe problem on manifolds with boundary: existence and compactness results*, Duke Math. J. **99** (1999), no. 3, 489–542.

[75] E. Hebey, *Changements de métriques conformes sur la sphère. Le problème de Nirenberg,* Bull. Sci. Math. **114** (1990), no. 2, 215–242.

[76] P.T. Ho, *Q-curvature flow on $\mathbb{S}^n$*, Comm. Anal. Geom. **18** (2010), no. 4, 791–820.

[77] B. Hu, *Nonexistence of a positive solution of the Laplace equation with a nonlinear boundary condition*, Differential Integral Equations **7** (1994), no. 2, 301–313.

[78] B. Hu and H.-M. Yin, *The profile near blowup time for solution of the heat equation with a nonlinear boundary condition*, Trans. Amer. Math. Soc. **346** (1994), no. 1, 117–135.

[79] M. Jara, *Nonequilibrium scaling limit for a tagged particle in the simple exclusion process with long jumps*, Comm. Pure Appl. Math. **62** (2009), no. 2, 198–214.

[80] M. Jara, *Hydrodynamic limit of particle systems with long jumps*, arXiv: 0805.1326 v2.

[81] M. Jara, T. Komorowski and S. Olla, *Limit theorems for additive functionals of a Markov chain*, Ann. Appl. Probab. **19** (2009), no. 6, 2270–2300.

[82] T. Jin, Y.Y. Li and J. Xiong, *On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions*, arXiv:1111.1332v1.

[83] T. Jin, Y.Y. Li and J. Xiong, *On a fractional Nirenberg problem, part II: existence of solutions*, preprint.

[84] T. Jin, V. Maz'ya and J. Van Schaftingen, *Pathological solutions to elliptic problems in divergence form with continuous coefficients*, C. R. Math. Acad. Sci. Paris **347** (2009), no. 13-14, 773–778.

[85] T. Jin and J. Xiong, *A fractional Yamabe flow and some applications*, arXiv: 1110.5664v1.

[86] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426.

[87] M. Kassmann, *The classical Harnack inequality fails for nonlocal operators*, available as SFB 611-preprint No. 360.

[88] J. Kazdan and F. Warner, *Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvature*, Ann. of Math. (2) **101** (1975), 317–331.

[89] S. Kim and K.-A. Lee, *Hölder Estimates for Singular Non-local Parabolic Equations*, arXiv:1105.3286v1.

[90] A. Kochubei, *Parabolic pseudodifferential equations, hypersingular integrals and Markov processes (Russian)*, Izv. Akad. Nauk SSSR Ser. Mat. **52** (1988), no. 5, 909–934, 1118; translation in Math. USSR-Izv. **33** (1989), no. 2, 233–259.

[91] D. Koutroufiotis, *Gaussian curvature and conformal mapping*, J. Differential Geom. **7** (1972), 479–488.

[92] V. Kozlov and V. Maz'ya, *Asymptotic formula for solutions to the Dirichlet problem for elliptic equations with discontinuous coefficients near the boundary*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **2** (2003), no. 3, 551–600.

[93] V. Kozlov and V. Maz'ya, *Asymptotics of a singular solution to the Dirichlet problem for an elliptic equation with discontinuous coefficients near the boundary*, Function spaces, differential operators and nonlinear analysis (Teistungen, 2001), 75–115, Birkhäuser, Basel, 2003.

[94] N.V. Krylov, *Lectures on elliptic and parabolic equations in Hölder spaces*, Graduate Studies in Mathematics, 12. American Mathematical Society, Providence, RI, 1996.

[95] Y.Y. Li, *Prescribing scalar curvature on* $\mathbb{S}^n$ *and related problems. I*, J. Differential Equations **120** (1995), no. 2, 319–410.

[96] Y.Y. Li, *Prescribing scalar curvature on* $\mathbb{S}^n$ *and related problems. II. Existence and compactness*, Comm. Pure Appl. Math. **49** (1996), no. 6, 541–597.

[97] Y.Y. Li, *Remark on some conformally invariant integral equations: the method of moving spheres*, J. Eur. Math. Soc. (JEMS) **6** (2004), no. 2, 153–180.

[98] Y.Y. Li and L. Nirenberg, *Estimates for elliptic systems from composite material*, Comm. Pure Appl. Math. **56** (2003), no. 7, 892–925.

[99] Y.Y. Li and L. Zhang, *Liouville-type theorems and Harnack-type inequalities for semilinear elliptic equations*, J. Anal. Math. **90** (2003), 27–87.

[100] Y.Y. Li and M. Zhu, *Uniqueness theorems through the method of moving spheres*, Duke Math. J. **80** (1995), no. 2, 383–417.

[101] E.H. Lieb, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math. (2) **118** (1983), no. 2, 349–374.

[102] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications. Vol. I.*, Translated from the French by P. Kenneth. Die Grundlehren der mathematischen Wissenschaften, Band 181. Springer-Verlag, New York-Heidelberg, (1972).

[103] W.R. Madych and N.M. Rivière, *Multipliers of the Hölder classes*, J. Funct. Anal. **21** (1976), 369–379.

[104] A. Malchiodi and M. Struwe, *Q-curvature flow on $\mathbb{S}^4$*, J. Differential Geom. **73** (2006), no. 1, 1–44.

[105] R. Mazzeo, *The Hodge cohomology of a conformally compact metric*, J. Differential Geom. **28** (1988), no. 2, 309–339.

[106] R. Mazzeo and R. Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature*, J. Funct. Anal. **75** (1987), no. 2, 260–310.

[107] N. G. Meyers, *An L^p-estimate for the gradient of solutions of second order elliptic divergence equations*, Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 189–206.

[108] R. Mikulevičius and H. Pragarauskas, *On the Cauchy problem for certain integro-differential operators in Sobolev and Hölder spaces*, (Lithuanian summary) Liet. Mat. Rink. **32** (1992), no. 2, 299–331; translation in Lithuanian Math. J. **32** (1992), no. 2, 238–264.

[109] R. Mikulevičius and H. Pragarauskas, *On Hölder solutions of the integro-differential Zakai equation*, Stochastic Process. Appl. **119** (2009), no. 10, 3319–3355.

[110] R. Mikulevičius and H. Pragarauskas, *On the Cauchy problem for integro-differential operators in Hölder classes and the uniqueness of the martingale problem*, arXiv:1103.3492v1.

[111] C. Morpurgo, *Sharp inequalities for functional integrals and traces of conformally invariant*, Duke Math. J. **114** (2002), no. 3, 477–553.

[112] J. Moser, *On a nonlinear problem in differential geometry*, Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971), pp. 273–280. Academic Press, New York, 1973.

[113] L. Nirenberg, *A strong maximum principle for parabolic equations*, Comm. Pure Appl. Math. **6** (1953), 167–177.

[114] B. Ou, *Positive harmonic functions on the upper half space satisfying a nonlinear boundary condition*, Differential Integral Equations **9** (1996), no. 5, 1157–1164.

[115] P. Pavlov and S. Samko, *A description of spaces $L_p^\alpha(S_{n-1})$ in terms of spherical hypersingular integrals. (Russian)*, Dokl. Akad. Nauk SSSR **276** (1984), no. 3, 546–550 . English translation: Soviet Math. Dokl. **29** (1984), no. 3, 549–553.

[116] J. Qing and D. Raske, *On positive solutions to semilinear conformally invariant equations on locally conformally flat manifolds*, Int. Math. Res. Not. Art. ID 94172, 20 pp. (2006).

[117] F. Robert, *Positive solutions for a fourth order equation invariant under isometries*, Proc. Amer. Math. Soc. **131** (2003), no. 5, 1423–1431.

[118] R. Schoen, *On the number of constant scalar curvature metrics in a conformal class*, Differential geometry, 311–320, Pitman Monogr. Surveys Pure Appl. Math., 52, Longman Sci. Tech., Harlow, 1991.

[119] R. Schoen and D. Zhang, *Prescribed scalar curvature on the n-sphere* Calc. Var. Partial Differential Equations **4** (1996), no. 1, 1–25.

[120] H. Schwetlick and M. Struwe, *Convergence of the Yamabe flow for "large" energies*, J. Reine Angew. Math. **562** (2003), 59–100.

[121] J. Serrin, *Pathological solutions of elliptic differential equations*, Ann. Scuola Norm. Sup. Pisa (3) **18** (1964), 385–387.

[122] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math. **60** (2007), no. 1, 67–112.

[123] L. Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*, Ann. of Math. (2) **118** (1983), no. 3, 525–571.

[124] E. M. Stein, *Note on the class L log L*, Studia Math. **32** (1969), 305–310.

[125] E. M. Stein, *Singular integrals and differentiability properties of function*, Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J. 1970.

[126] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, 43, Princeton University Press, Princeton, NJ, 1993.

[127] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) **110** (1976), 353–372.

[128] J. Tan and J. Xiong, *A Harnack inequality for fractional Laplace equations with lower order terms*, Discrete Contin. Dyn. Syst. **31** (2011), no. 3, 975–983.

[129] N. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa (3) **22** (1968), 265–274.

[130] C. Wang, L. Wang, J. Yin and S. Zhou, *Hölder continuity of weak solutions of a class of linear equations with boundary degeneracy*, J. Differential Equations **239** (2007), no. 1, 99–131.

[131] J. Wei and X. Xu, *Prescribing Q-curvature problem on* $\mathbb{S}^n$, J. Funct. Anal. **257** (2009), no. 7, 1995–2023.

[132] J. Wei and X. Xu, *On conformal deformations of metrics on* $\mathbb{S}^n$, J. Funct. Anal. **157** (1998), no. 1, 292–325.

[133] D. Yafaev, *Sharp constants in the Hardy-Rellich inequalities*, J. Funct. Anal. **168** (1999), no. 1, 121–144.

[134] R. Ye, *Global existence and convergence of Yamabe flow*, J. Differential Geom. **39** (1994), no. 1, 35–50.

Vita

Tianling Jin

2006 B. Sc. in Mathematics, University of Science and Technology of China

2012 Ph. D. in Mathematics, Rutgers, The State University of New Jersey

2006-2010 Teaching Assistantship, Department of Mathematics, Rutgers University

2010-2011 School of Art and Science Excellence Fellowship, Rutgers University

2011-2012 University and Louis Bevier Dissertation Fellowship, Rutgers University