DEGENERATE PARTIAL DIFFERENTIAL EQUATIONS AND APPLICATIONS TO PROBABILITY THEORY AND FOUNDATIONS OF MATHEMATICAL FINANCE

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ABSTRACT OF THE DISSERTATION

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In the first part of our thesis, we prove existence, uniqueness and regularity of solutions for a certain class of degenerate parabolic partial differential equations on the half space which are a generalization of the Heston operator. We use these results to show that the martingale problem associated with the differential operator is well-posed and we build generalized Heston-like processes which match the one-dimensional probability distributions of a certain class of Itô processes.

The second part of our thesis is concerned with the study of regularity of solutions to the variational equation associated to the elliptic Heston operator. With the aid of weighted Sobolev spaces, we prove supremum bounds, a Harnack inequality, and Hölder continuity near the boundary for solutions to elliptic variational equations defined by the Heston partial differential operator.

Finally, we establish stochastic representations of solutions to elliptic and parabolic boundary value problems and obstacle problems associated to the Heston generator. In mathematical finance, solutions to parabolic obstacle problems correspond to value functions for American-style options.

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Dedication

To my Family

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Chapter 1

Introduction

We study existence, uniqueness and regularity of solutions to degenerate elliptic and parabolic partial differential equations (PDEs) which arise as generators of Markov processes. This research is motivated in part by applications to mathematical finance, especially in option pricing.

In Chapter 2, we solve four intertwined problems concerning degenerate-parabolic partial differential operators and degenerate diffusion processes. First, we consider a parabolic partial differential equation on a half-space whose coefficients are suitably Hölder continuous and allowed to grow linearly in the spatial variable and which becomes degenerate along the boundary of the half-space. We establish existence and uniqueness of solutions in weighted Hölder spaces which incorporate both the degeneracy at the boundary and the unboundedness of the coefficients. Second, we show that the martingale problem associated with a degenerate elliptic differential operator with unbounded, locally Hölder continuous coefficients on a half-space is well-posed in the sense of Stroock and Varadhan. Third, we prove existence, uniqueness, and the strong Markov property for weak solutions to a stochastic differential equation with degenerate diffusion and unbounded coefficients with suitable Hölder continuity properties. Fourth, for an Itô process with degenerate diffusion and unbounded but appropriately regular coefficients, we prove existence of a strong Markov process, unique in the sense of probability law, whose one-dimensional marginal probability distributions match (mimic) those of the given Itô process.

Mimicking theorems are useful in pricing European options in mathematical finance. Often, stock prices follow complex dynamics, which are difficult to simulate in practice, so it is desirable to be able to select a unique Markov stochastic process which has the same one-dimensional probability distributions as the original asset price process, and which can serve as an input to the pricing equation. In our framework, we achieve this goal.

The Heston stochastic volatility process, which is widely used as an asset price model in mathematical finance, is a paradigm for a degenerate diffusion process where the degeneracy in the diffusion coefficient is proportional to the square root of the distance to the boundary of the half-plane. The generator of this process with killing, called the elliptic Heston operator, is a second-order degenerate elliptic partial differential operator whose coefficients have linear growth in the spatial variables and where the degeneracy in the operator symbol is proportional to the distance to the boundary of the half-plane. With the aid of weighted Sobolev spaces, in the second part of our thesis, Chapter 3, we prove supremum bounds, a Harnack inequality, and Hölder continuity near the boundary for solutions to elliptic variational equations defined by the Heston partial differential operator. The difficulty in studying these properties is in adapting the method of Moser iterations, Poincaré inequality and John-Nirenberg inequality to our weighted spaces. We use these results to prove Hölder continuity of weak solutions to the Heston obstacle problem, which combined with additional regularity results (in particular, the $C^{1,1}_s$ regularity due to Daskalopoulos and Feehan), enables us to show that the weak solutions admit stochastic representations [32]. This result is of particular interest to practitioners because it shows for the Heston model, that prices of perpetual American options, with regular enough payoffs, are solutions to stationary variational inequalities.

Finally, in Chapter 4, we establish stochastic representations of solutions to elliptic and parabolic boundary value problems and obstacle problems associated to the Heston generator.

1.1 Notation and conventions

We adopt the convention that a condition labeled as an *Assumption* is considered to be universal and in effect throughout the chapter where it is stated and so not referenced explicitly in theorem and similar statements; a condition labeled as a *Hypothesis* is only considered to be in effect when explicitly referenced.

We let $\mathbb{N} := \{1, 2, \ldots\}$ denote the set of positive integers. For $x, y \in \mathbb{R}$, we let $x \wedge y := \min\{x, y\}$, $x \vee y := \max\{x, y\}$ and $x^+ := x \vee 0$. For any open set $\mathcal{O}$ in a topological space, we denote by $\bar{\mathcal{O}}$ its closure. Let $\mathbb{S}_d \subset \mathbb{R}^{d\times d}$ denote the closed, convex subset of *non-negative* definite, symmetric matrices. Let $\mathbb{S}^+_d \subset \mathbb{R}^{d\times d}$ denote the convex subset of *positive* definite, symmetric matrices. We denote by $\mathbb{O}(d) \subset \mathbb{R}^{d\times d}$ the orthogonal group.

A positive integer $d \geq 2$ denotes the dimension of the Euclidean space, $\mathbb{R}^d$. We denote $\mathbb{R}_+ := (0, \infty)$ and $\mathbb{H} := \mathbb{R}^{d-1} \times \mathbb{R}_+$. We write points in $\mathbb{H}$ as $x := (x', x_d)$, where $x' := (x_1, x_2, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}$, when $d \geq 2$, or $z = (x, y)$, $x \in \mathbb{R}$ and $y > 0$, when $d = 2$.

In Chapters 3 and 4, we denote by $\mathscr{O} \subset \mathbb{H}$ a possibly unbounded domain in the open upper half-plane $\mathbb{H}$, $\Gamma_1 = \partial\mathscr{O} \cap \mathbb{H}$ is the portion of the boundary $\partial\mathscr{O}$ of $\mathscr{O}$ which lies in $\mathbb{H}$, and Γ_0 is the (non-empty) interior of $\partial\mathbb{H} \cap \partial\mathscr{O}$, where $\partial\mathbb{H} = \mathbb{R}^{d-1} \times \{0\}$ is the boundary of $\bar{\mathbb{H}} := \mathbb{R}^{d-1} \times [0, \infty)$. We write $\partial\mathscr{O} = \Gamma_0 \cup \bar{\Gamma}_1 = \bar{\Gamma}_0 \cup \Gamma_1$ and note that the boundary portions Γ_0 and Γ_1 are relatively open in $\partial\mathscr{O}$.

1.1.1 Function spaces

In the definition and naming of function spaces, including spaces of continuous functions, Hölder spaces, or Sobolev spaces, we follow Adams [2] and alert the reader to occasional differences in definitions between [2] and standard references such as Gilbarg and Trudinger [41] or Krylov [51, 52].

Elliptic Hölder spaces

Let $\mathscr{O} \subset \mathbb{R}^d$ be an open, connected set (domain). For an integer $k \geq 0$, we let $C^k(\mathscr{O})$ denote the vector space of functions whose derivatives up to order k are continuous on $\mathscr{O}$ and let $C^k(\bar{\mathscr{O}})$ denote the Banach space of functions whose derivatives up to order k are *uniformly continuous* and *bounded* on $\mathscr{O}$ [2, §1.25 & §1.26]. If $T \subsetneqq \partial\mathscr{O}$ is a relatively open set, we let $C^k_{\mathrm{loc}}(\mathscr{O} \cup T)$ denote the vector space of functions, u, such that, for any precompact open subset $U \Subset \mathscr{O} \cup T$, we have $u \in C^k(\bar{U})$.

For $\alpha \in (0,1)$, we let $C^{k+\alpha}(\mathscr{O})$ denote the subspace of $C^k(\mathscr{O})$ consisting of functions whose derivatives up to order k are *locally* α-Hölder continuous on $\mathscr{O}$ (in the sense of [41, p. 52]) and let $C^{k+\alpha}(\bar{\mathscr{O}})$ denote the subspace of $C^k(\bar{\mathscr{O}})$ consisting of functions whose derivatives up to order k are *uniformly* α-Hölder continuous on $\mathscr{O}$ [41, p. 52], [2, §1.27]. If $T \subsetneqq \partial\mathscr{O}$ is a relatively open set, we let $C^{k+\alpha}_{\mathrm{loc}}(\mathscr{O} \cup T)$ denote the vector space of functions, u, such that, for any precompact open subset $U \Subset \mathscr{O} \cup T$, we have $u \in C^{k+\alpha}(\bar{U})$.

Parabolic Hölder spaces

The definitions of the parabolic Hölder spaces are analogous to those of the elliptic Hölder spaces, with the only adjustment that we replace the Euclidean distance between two points by the parabolic distance given by

$$\rho(P_1, P_2) := \sum_{i=1}^{d} |x_i^1 - x_i^2| + \sqrt{|t_1 - t_2|}, \tag{1.1.1}$$

where $P_i = (t_i, x_1^i, \ldots, x_d^i)$, $i = 1, 2$.

Let $Q \subset (0,T) \times \mathbb{R}^d$ be a domain and $\alpha \in (0,1)$. We denote by $C(\bar{Q})$ the space of bounded, continuous functions on $\bar{Q}$, and by $C_0^\infty(\bar{Q})$ the space of smooth functions with compact support in $\bar{Q}$. If $T \subsetneqq \partial Q$ is a relatively open set, we let $C_{\mathrm{loc}}(Q \cup T)$ denote the vector space of functions, u, such that, for any precompact open subset $V \Subset Q \cup T$, we have $u \in C(\bar{V})$.

For a function $u : \bar{Q} \to \mathbb{R}$, we consider the following norms and seminorms

$$\|u\|_{C(\bar{Q})} = \sup_{P \in \bar{Q}} |u(P)|, \tag{1.1.2}$$

$$[u]_{C^\alpha_\rho(\bar{Q})} = \sup_{\substack{P_1, P_2 \in \bar{Q}, \\ P_1 \neq P_2}} \frac{|u(P_1) - u(P_2)|}{\rho(P_1, P_2)^\alpha}. \tag{1.1.3}$$

We say that $u \in C^\alpha_\rho(\bar{Q})$ if $u \in C(\bar{Q})$ and

$$\|u\|_{C^\alpha_\rho(\bar{Q})} = \|u\|_{C(\bar{Q})} + [u]_{C^\alpha_\rho(\bar{Q})} < \infty.$$

We say that $u \in C^{2+\alpha}_\rho(\bar{Q})$ if

$$\|u\|_{C^{2+\alpha}_\rho(\bar{Q})} = \|u\|_{C^\alpha_\rho(\bar{Q})} + \|u_t\|_{C^\alpha_\rho(\bar{Q})} + \max_{1 \le i \le d} \|u_{x_i}\|_{C^\alpha_\rho(\bar{Q})} + \max_{1 \le i,j \le d} \|u_{x_i x_j}\|_{C^\alpha_\rho(\bar{Q})} < \infty.$$

We denote by $C^{\alpha}_{\rho,\mathrm{loc}}(\bar{Q})$ the space of functions u with the property that for any compact set $K \subseteq \bar{Q}$, we have $u \in C^{\alpha}_{\rho}(K)$. Analogously, we define the space $C^{2+\alpha}_{\rho,\mathrm{loc}}(\bar{Q})$. For $T \subsetneqq \partial Q$ a relatively open subset, we let $C^{2+\alpha}_{\rho,\mathrm{loc}}(Q \cup T)$ denote the subspace of $C^{2+\alpha}_{\rho}(Q)$ such that, for any precompact open set $U \Subset Q \cup T$, we have $u \in C^{2+\alpha}_{\rho}(\bar{U})$.

Sometimes, we omit the subscript ρ from the definition of the parabolic Hölder spaces (in Chapter 4), but we keep it when we want to emphasize the different metrics that we use (in Chapter 2).

1.1.2 Probability spaces and filtrations

Let E be a metric space and let $\mathscr{B}(E)$ be the Borel σ-algebra generated by this topology. We denote by $C_{\mathrm{loc}}([0,\infty);E)$ the set of continuous paths $\omega : [0,\infty) \to E$. We endow $C_{\mathrm{loc}}([0,\infty);E)$ with the topology of uniform convergence on compact sets. If (E,r) is a complete, separable metric space, then $C_{\mathrm{loc}}([0,\infty);E)$ is a complete, separable, metrizable space.

Let $\mathscr{B}(C_{\mathrm{loc}}([0,\infty);E))$ denote the σ-algebra generated by the cylinder sets

$$\{\omega \in C_{\mathrm{loc}}([0,\infty);E) : \omega(t_i) \in B_i, i = 1,\ldots,m\}, \tag{1.1.4}$$

where $0 \leq t_1 < \ldots < t_m$, $B_i \in \mathscr{B}(E)$, $i = 1,\ldots,m$, and $m \in \mathbb{N}$. For $t \geq 0$, let $\mathscr{B}_t(C_{\mathrm{loc}}([0,\infty);E))$, denote the σ-algebra generated by cylinder sets of the form (1.1.4) such that $0 \leq t_1 < \ldots < t_m \leq t$.

We specialize (E,r) to be $(\mathbb{R}^d, |\cdot|)$ ([70, p. 138], [47, Definition 5.4.5 & 5.4.10]), $(\mathbb{R}^d_+, |\cdot|)$ ([6, §1]) and $(\bar{\mathbb{H}}, |\cdot|)$ in Definition 2.1.8.

Definition 1.1.1 (Usual conditions)**.** [47, Definition 1.2.25] A filtration $\{\mathscr{F}_t\}_{t\geq 0}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is said to satisfy the *usual conditions* if it is right-continuous, that is, $\mathscr{F}_t = \cap_{s>t}\mathscr{F}_s$, and $\mathscr{F}_0$ contains all events in $\mathscr{F}$ with $\mathbb{P}$-null probability.

Definition 1.1.2 (Augmentation of a filtration)**.** [47, Definition 2.7.2] Let $(\Omega, \mathscr{F}, \mathbb{P})$, $\{\mathscr{F}_t\}_{t\geq 0}$, be a filtered probability space and let $\mathscr{N}$ denote

$$\mathscr{N} := \{F \subset \Omega : \exists G \in \mathscr{F} \text{ such that } F \subseteq G, \mathbb{P}(G) = 0\}.$$

Let $\mathscr{F}_t^{\mathscr{N}}$ denote the σ-algebra generated by $\mathscr{F}_t \cup \mathscr{N}$. Then, $\left\{\mathscr{F}_t^{\mathscr{N}}\right\}_{t\geq 0}$ is the *augmentation* of the initial filtration $\{\mathscr{F}_t\}_{t\geq 0}$.

Chapter 2

Degenerate PDEs and martingale and mimicking problems

2.1 Introduction

Consider a time-dependent, elliptic differential operator defined by *unbounded* coefficients (a, b) on the half-space $\mathbb{H} := \mathbb{R}^{d-1} \times (0, \infty)$ with $d \geq 2$,

$$\mathscr{A}_t v(x) := \frac{1}{2} \sum_{i,j=1}^{d} x_d a_{ij}(t, x) v_{x_i x_j}(x) + \sum_{i=1}^{d} b_i(t, x) v_{x_i}(x), \quad (t, x) \in [0, \infty) \times \mathbb{H}, \quad (2.1.1)$$

where $a = (a_{ij})$, $b = (b_i)$, and $v \in C^2(\bar{\mathbb{H}})$. The operator $\mathscr{A}_t$ becomes *degenerate* along the boundary $\partial\mathbb{H} = \{x_d = 0\}$ of the half-space. In this chapter, motivated by applications to mathematical finance [4, 22, 63], we solve four intertwined problems concerning degenerate-parabolic partial differential operators and degenerate diffusion processes related to (2.1.1).

First, as explained more fully in §2.1.1, we prove existence, uniqueness, and regularity of solutions to a *degenerate*-parabolic partial differential equation with *unbounded*, locally Hölder-continuous coefficients, (a, b, c), generalizing both the *Heston equation* [44] and the *linearization of the porous medium equation* [20, 21, 50],

$$\begin{cases} Lu = f & \text{on } \mathbb{H}_T, \\ u(0, \cdot) = g & \text{on } \bar{\mathbb{H}}, \end{cases} \quad (2.1.2)$$

where

$$-Lu = -u_t + \sum_{i,j=1}^{d} x_d a_{ij} u_{x_i x_j} + \sum_{i=1}^{d} b_i u_{x_i} + cu, \quad \forall u \in C^{1,2}(\mathbb{H}_T), \quad (2.1.3)$$

and $\mathbb{H}_T := (0, T) \times \mathbb{H}$. In particular, unlike the linearization of the porous medium equation considered in [20, 21, 50], the coefficients of (2.1.3) are permitted to grow linearly with x as $x \to \infty$ and, even when the coefficients b_i are constant, we do not require that $b_i = 0$ when $i = 1, \ldots, d-1$.

Second, we show that the martingale problem §2.1.1 for the degenerate-elliptic operator with unbounded coefficients, $\mathscr{A}_t$, in (2.1.1) is well-posed in the sense of Stroock and Varadhan [70].

Third, as discussed in more detail in §2.1.1, we prove existence, uniqueness, and the strong Markov property for weak solutions, $\widehat{X}$, to a degenerate stochastic differential equation with unbounded coefficients,

$$\begin{aligned} d\widehat{X}(t) &= b(t, \widehat{X}(t))dt + \sigma(t, \widehat{X}(t))d\widehat{W}(t), \quad t \geq s, \\ \widehat{X}(s) &= x. \end{aligned} \tag{2.1.4}$$

when the coefficient σ is a square root of the coefficient matrix $x_d a$ in $\mathscr{A}_t$ in (2.1.1), that is, when $\sigma\sigma^* = x_d a$ on $\mathbb{H}_T$.

Lastly, suppose we are given a degenerate Itô process, X, with unbounded coefficients,

$$\begin{aligned} dX(t) &= \beta(t)dt + \xi(t)dW(t), \quad t \geq 0, \\ X(0) &= x, \end{aligned} \tag{2.1.5}$$

whose coefficients (ξ, β) are related to those of (2.1.4) as explained in §2.1.1. When the coefficients (b, σ) in (2.1.4) are determined by the coefficients (ξ, β) in (2.1.5) as described in §2.1.1, we show that the weak solution $\widehat{X}$ to (2.1.4) "mimics" the Itô process (2.1.5) in the sense that $\widehat{X}(t)$ has the same one-dimensional marginal probability distributions as $X(t)$, for all $t \geq 0$ if $\widehat{X}(0) = X(0) \in \bar{\mathbb{H}}$. Our mimicking theorem complements that of Gyöngy [43], who assumes that (2.1.4) is non-degenerate with bounded, measurable coefficients, that of Brunick and Shreve [12, 14], who allow (2.1.4) to be degenerate with unbounded, measurable coefficients, and those of Bentata and Cont [9] and Shi and Wang [66, 72] who prove mimicking theorems for a discontinuous semimartingale process with a non-degenerate diffusion component and bounded coefficients.

2.1.1 Summary of main results

We describe our results outlined in the preamble to §2.1.

Existence and uniqueness of solutions to a degenerate-parabolic partial differential equation with unbounded coefficients

We shall seek a solution, u, to (2.1.2) in a certain weighted Hölder space $\mathscr{C}_p^{2+\alpha}(\bar{\mathbb{H}}_T)$, given a source function, f, in a weighted Hölder space $\mathscr{C}_p^{\alpha}(\bar{\mathbb{H}}_T)$ and initial data, g, in a weighted Hölder space $\mathscr{C}_p^{2+\alpha}(\bar{\mathbb{H}})$. These weighted Hölder spaces generalize both the standard Hölder spaces as defined, for example, in [51, 54] and the Hölder spaces defined with the cycloidal metric and introduced, independently, by Daskalopoulos and Hamilton [20] and Koch [50]. We defer a detailed description of these Hölder spaces to §2.2.1. However, the essential features of our Hölder spaces are that (i) near the boundary, $x_d = 0$, of the half-space cylinder $\mathbb{H}_T$, our Hölder spaces are equivalent to those of Daskalopoulos, Hamilton, and Koch and account for the degeneracy of the operator L, (ii) polynomial weights in the definition of our Hölder spaces allow for coefficients (a, b, c) in (2.1.3) with up to linear growth near $x = \infty$ in the half-space cylinder $\mathbb{H}_T$, and (iii) on compact subsets of the half-space cylinder $\mathbb{H}_T$, our Hölder spaces are equivalent to standard Hölder spaces. We defer a detailed description of the conditions on the coefficients (a, b, c) defining L in (2.1.3) to §2.2.2 — see Assumption 2.2.2 on the properties of the coefficients of the parabolic differential operator. However, the essential features of the conditions on (a, b, c) in Assumption 2.2.2 are that (i) the matrix $a = (a_{ij})$ is *uniformly elliptic*, so the *degeneracy* in (2.1.3) is captured by the common factor x_d appearing in the $u_{x_i x_j}$ terms, (ii) the coefficients (a, b, c) have at most *linear growth* with respect to $x \in \mathbb{H}$ as $x \to \infty$, (iii) the coefficients (a, b, c) are *locally Hölder continuous* on $\bar{\mathbb{H}}_T$ with exponent $\alpha \in (0, 1)$, (iv) the coefficient c is *bounded above* on $\mathbb{H}_T$ by a constant, and (v) the coefficient b_d is *positive* when $x_d = 0$. We can now state our first main result.

Theorem 2.1.1 (Existence and uniqueness)**.** *Assume that the coefficients* (a, b, c) *in* (2.1.3) *obey the conditions in Assumption 2.2.2. Then there is a positive constant* p*, depending only on the Hölder exponent* $\alpha \in (0, 1)$*, such that for any* $T > 0$*,* $f \in \mathscr{C}_p^{\alpha}(\bar{\mathbb{H}}_T)$ *and* $g \in \mathscr{C}_p^{2+\alpha}(\bar{\mathbb{H}})$*, there exists a unique solution* $u \in \mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_T)$ *to* (2.1.2)*. Moreover,*

u satisfies the a priori estimate

$$\|u\|_{\mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_T)} \leq C\left(\|f\|_{\mathscr{C}_p^{\alpha}(\bar{\mathbb{H}}_T)} + \|g\|_{\mathscr{C}_p^{2+\alpha}(\bar{\mathbb{H}})}\right), \tag{2.1.6}$$

where C is a positive constant, depending only on K, ν, δ, d, α and T.

One of the difficulties in establishing Theorem 2.1.1 is that the coefficient, $x_d a(t,x)$, becomes degenerate when $x_d = 0$ and is allowed to have linear growth in x, instead of being uniformly elliptic and bounded as in [53, Hypothesis 2.1]. To address the degeneracy of $x_d a(t,x)$ as $x_d \downarrow 0$, we build on the results on [20, Theorem I.1.1] by employing a localization procedure. To address the linear growth of the coefficients $(x_d a, b, c)$ of the parabolic operator L in (2.1.3), we augment previous definitions of weighted Hölder spaces [20, 50], by introducing a weight $(1+|x|)^p$, where p is a positive constant depending only on the dimension d of the half-space $\mathbb{H}$ and on the Hölder exponent $\alpha \in (0,1)$. The proof of existence does not follow by standard methods, for example, the method of continuity, because $L : \mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_T) \to \mathscr{C}_p^{\alpha}(\bar{\mathbb{H}}_T)$ is not a well-defined operator. In general, the domain of definition of L is a subspace of $\mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_T)$ which depends on the nature of the coefficients of L, a feature which is not encountered in the case of parabolic operators with bounded coefficients . To circumvent this difficulty, we first consider the case of similar degenerate operators with *bounded* coefficients and then use an approximation procedure to obtain our solution. To obtain convergence of sequences to a solution of our parabolic differential equation (2.1.2), we prove a priori estimates in the weighted Hölder spaces $\mathscr{C}_p^{\alpha}$ and $\mathscr{C}_p^{2+\alpha}$.

The conditions in Assumption 2.2.2 on the coefficients (a,b,c) in (2.1.3) are mild enough that they allow for many examples of interest in mathematical finance.

Example 2.1.2 (Parabolic Heston partial differential equation)**.** The conditions in Assumption 2.2.2 are obeyed by the coefficients of the parabolic Heston partial differential operator,

$$-Lu = -u_t + \frac{y}{2}\left(u_{xx} + 2\varrho\sigma u_{xy} + \sigma^2 u_{yy}\right) + (r - q - y/2)u_x + \kappa(\vartheta - y)u_y - ru, \tag{2.1.7}$$

where $q \geq 0, r \geq 0, \kappa > 0, \vartheta > 0, \sigma > 0$, and $\varrho \in (-1,1)$ are constants.

Naturally, the conditions in Assumption 2.2.2 on the coefficients (a, b, c) in (2.1.3) also allow for the linearization of the generalized porous medium equation.

Example 2.1.3 (Linearization of the porous medium equation)**.** In their landmark article, Daskalopoulos and Hamilton [20] proved existence and uniqueness of C^∞ solutions, u, to the Cauchy problem for the porous medium equation [20, p. 899] (when $d = 2$),

$$-u_t + \sum_{i=1}^{d} (u^m)_{x_i x_i} = 0 \quad \text{on } (0,T) \times \mathbb{R}^d, \quad u(\cdot, 0) = g \quad \text{on } \mathbb{R}^d, \tag{2.1.8}$$

where $m > 1$ and $g \in L^1(\mathbb{R}^d)$ with $g \geq 0$ compactly supported on $\mathbb{R}^d$, together with C^∞-regularity of its free boundary, $\partial\{u > 0\}$. Their analysis is based on an extensive development of existence, uniqueness, and regularity results for the linearization of the porous medium equation near the free boundary and, in particular, their *model linear degenerate operator* [20, p. 901] (generalized from $d = 2$ in their article),

$$-Lu = -u_t + x_d \sum_{i=1}^{d} u_{x_i x_i} + \nu u_{x_d}, \tag{2.1.9}$$

where ν is a positive constant. The same model linear degenerate operator (for $d \geq 2$), was studied independently by Koch [50, Equation (4.43)] and, in a remarkable Habilitation thesis, he obtained existence, uniqueness, and regularity results for solutions to (2.1.8) which complement those of Daskalopoulos and Hamilton [20]. Even when the coefficients in (2.1.3) are constant, our operator *cannot* be transformed by simple coordinate changes to one of the form (2.1.9), but rather one of the form (A.1.6). Similarly, the operator (2.1.7) *cannot* be transformed by simple coordinate changes to one of the form (2.1.9), even when the factor y in the coefficients of u_x and u_y in (2.1.7) is (artificially) replaced by zero.

Existence and uniqueness of solutions to the martingale problem for a degenerate-parabolic operator with unbounded coefficients

We review the formulation of the classical martingale problem of Stroock and Varadhan [70].

Definition 2.1.4 (Classical martingale problem on the whole space)**.** [70, p. 138], [47, Definition 5.4.5 & 5.4.10] Suppose we are given a differential operator,

$$\tilde{\mathscr{A}}_t v(t,x) := \frac{1}{2}\sum_{i,j=1}^{d} \tilde{a}_{ij}(t,x) v_{x_i x_j}(x) + \sum_{i=1}^{d} \tilde{b}_i(t,x) v_{x_i}(x), \quad (t,x) \in [0,\infty)\times\mathbb{R}^d, \tag{2.1.10}$$

with $v \in C^2(\mathbb{R}^d)$ and measurable coefficients,

$$\begin{aligned} \tilde{a} &: [0,\infty)\times\mathbb{R}^d \to \mathbb{S}_d, \\ \tilde{b} &: [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d. \end{aligned} \tag{2.1.11}$$

A probability measure $\mathbb{P}^{s,x}$ on the canonical space, $\left(C_{\mathrm{loc}}([0,\infty);\mathbb{R}^d), \mathscr{B}(C_{\mathrm{loc}}([0,\infty);\mathbb{R}^d))\right)$, is called a *solution to the martingale problem associated to* $\tilde{\mathscr{A}}_t$ with initial condition at $(s,x) \in [0,\infty)\times\mathbb{R}^d$ if

$$\mathbb{P}^{s,x}\left(\omega \in C_{\mathrm{loc}}([0,\infty);\mathbb{R}^d) : \omega(t) = x, 0 \le t \le s\right) = 1,$$

and, for any $v \in C_0^\infty(\mathbb{R}^d)$,

$$v(\omega(t)) - v(\omega(s)) - \int_s^t \tilde{\mathscr{A}}_u v(\omega(u))du, \quad \forall \omega \in C_{\mathrm{loc}}([0,\infty);\mathbb{R}^d), \quad t \ge s,$$

is a $\mathbb{P}^{s,x}$-martingale with respect to the filtration $\{\mathscr{B}_t(C_{\mathrm{loc}}([0,\infty);\mathbb{R}^d))\}_{t\ge s}$. □

Remark 2.1.5 (Well-posedness of the classical martingale problem (Definition 2.1.4))**.** Standard results which ensure *existence* of solutions to the classical martingale problem require that the coefficients $(\tilde{a},\tilde{b})$ in (2.1.11) be bounded and continuous [47, Theorem 5.4.22], [70, Theorem 6.1.7]. Standard results which ensure *uniqueness* of solutions require, in addition, that the coefficients $(\tilde{a},\tilde{b})$ are Hölder continuous and that the matrix a is uniformly elliptic (see [47, Theorem 5.4.28, Corollary 5.4.29, and Remark 5.4.29] for the time-homogeneous martingale problem).

Remark 2.1.6 (Approaches to proving uniqueness in the classical martingale problem)**.** *Uniqueness* of solutions to the classical martingale problem is often shown [47, §5.4] by proving *existence* of solutions in $C([0,T]\times\mathbb{R}^d)\cap C^{1,2}((0,T)\times\mathbb{R}^d)$ to the terminal value problem for the parabolic partial differential equation,

$$\begin{cases} u_t + \tilde{\mathscr{A}}_t u = 0 & \text{on } (0,T)\times\mathbb{R}^d, \\ u(T,\cdot) = g & \text{on } \mathbb{R}^d, \end{cases}$$

where $g \in C_0^\infty(\mathbb{R}^d)$ and $\tilde{\mathscr{A}}_t$ is given by (2.1.10).

For a differential operator which is defined on a subdomain, it is natural to consider a modification of Definition 2.1.4. To illustrate such a formulation, we have the following example due to Bass and Lavrentiev [6].

Example 2.1.7 (A time-homogeneous, degenerate submartingale problem on a subdomain)**.** [6, §1] Consider the differential operator,

$$\mathscr{A}'v(x) := \frac{1}{2}\sum_{i=1}^{d} a_i(x)x_i^{\alpha_i}v_{x_ix_i}(x) + \sum_{i=1}^{d} b_i(x)v_{x_i}(x), \quad \forall x \in \mathbb{R}^d_+, \tag{2.1.12}$$

where $v \in C^2(\mathbb{R}^d_+)$, the coefficients b_i are bounded, the coefficients a_i are continuous and bounded from above and below by positive constants, and $\alpha_i \in (0,1)$. Let $x \in \bar{\mathbb{R}}^d_+$. Then, a probability measure $\mathbb{P}^x$ on $\big(C_{\mathrm{loc}}([0,\infty);\bar{\mathbb{R}}^d_+), \mathscr{B}(C_{\mathrm{loc}}([0,\infty);\bar{\mathbb{R}}^d_+))\big)$ is a *solution to the submartingale problem associated to* $\mathscr{A}'$ if

$$\mathbb{P}^x\left(\omega \in C_{\mathrm{loc}}([0,\infty);\bar{\mathbb{R}}^d_+) : \omega(0) = x\right) = 1,$$

and, for any $v \in C^2_b(\mathbb{R}^d_+) \cap C^1(\bar{\mathbb{R}}^d_+)$ such that $v_{x_i} \geq 0$ along $\{x_i = 0\}$,

$$v(\omega(t)) - v(\omega(0)) - \int_0^t \mathscr{A}'v(\omega(u))du, \quad \forall \omega \in C_{\mathrm{loc}}([0,\infty);\mathbb{R}^d_+), t \geq 0,$$

is a $\mathbb{P}^x$-submartingale with respect to the filtration $\{\mathscr{B}_t(C_{\mathrm{loc}}([0,\infty);\mathbb{R}^d_+))\}_{t\geq 0}$.

Bass and Lavrentiev [6, Theorem 1.1] prove that there is a unique solution to this submartingale problem which spends zero time on the boundary of $\mathbb{R}^d_+$. □

We now define an analogue of the usual martingale problem (Definition 2.1.4) when $\mathbb{R}^d$ is replaced by the half-space $\mathbb{H}$.

Definition 2.1.8 (Solution to a martingale problem for an operator on a half-space)**.** Given $(s,x) \in [0,\infty) \times \bar{\mathbb{H}}$, a probability measure $\widehat{\mathbb{P}}^{s,x}$ on

$$(C_{\mathrm{loc}}([0,\infty);\bar{\mathbb{H}}), \mathscr{B}(C_{\mathrm{loc}}([0,\infty);\bar{\mathbb{H}}))$$

is a *solution to the martingale problem associated to* $\mathscr{A}_t$ *in* (2.1.1) *starting from* (s,x) if

$$M^v_t(\omega) := v(\omega(t)) - v(\omega(s)) - \int_s^t \mathscr{A}_u v(\omega(u))\,du, \quad t \geq s, \omega \in C_{\mathrm{loc}}([0,\infty);\bar{\mathbb{H}}),$$

is a continuous $\widehat{\mathbb{P}}^{s,x}$-martingale, for every $v \in C_0^2(\bar{\mathbb{H}})$, with respect to the filtration $\widehat{\mathscr{F}_t} = \mathscr{G}_{t+}$, where $\mathscr{G}_t$ is the augmentation under $\widehat{\mathbb{P}}^{s,x}$ of the filtration $\{\mathscr{B}_t(C_{\text{loc}}([0,\infty);\bar{\mathbb{H}})\}_{t\geq 0}$, and

$$\widehat{\mathbb{P}}^{s,x}\left(\omega \in C_{\text{loc}}([0,\infty);\bar{\mathbb{H}}) : \omega(t) = x, 0 \leq t \leq s\right) = 1. \tag{2.1.13}$$

□

Remark 2.1.9 (Reduction to usual filtration)**.** By modifying the statement and solution to [47, Problem 5.4.13] (that is, replacing $\mathbb{R}^d$ by $\mathbb{H}$), we see that if M_t^v is a martingale with respect to the filtration $\{\mathscr{B}_t(C_{\text{loc}}([0,\infty);\bar{\mathbb{H}})\}_{t\geq 0}$, then it is a martingale with respect to the enlarged filtration $\widehat{\mathscr{F}_t}$.

Theorem 2.1.10 (Existence and uniqueness of solutions to the martingale problem for a degenerate-elliptic operator with unbounded coefficients)**.** *Suppose the coefficients* (a,b) *in* (2.1.1) *obey the conditions in Assumption 2.2.2. Then, for any* $(s,x) \in [0,\infty)\times\bar{\mathbb{H}}$*, there is a unique solution,* $\widehat{\mathbb{P}}^{s,x}$*, to the martingale problem associated to* $\mathscr{A}_t$ *in* (2.1.1) *starting from* (s,x)*.*

Remark 2.1.11 (Comments on uniqueness)**.** While [47, Remark 5.4.31] might appear to provide a simple solution to the uniqueness property asserted by Theorem 2.1.10 when the nonnegative definite matrix-valued function $x_d a$ is in $C^2(\mathbb{H};\mathbb{S}_d)$, that is not the case. Although we might extend the coefficient, $x_d a$, as a nonnegative definite matrix-valued function $x_d^+ a$ or $|x_d| a$ in $C^{0,1}(\mathbb{R}^d;\mathbb{S}_d)$, such extensions are not in $C^2(\mathbb{R}^d;\mathbb{S}_d)$, as required by [47, Remark 5.4.31].

Existence and uniqueness of weak solutions to a degenerate stochastic differential equation with unbounded coefficients

Given a function,

$$\bar{a} : [0,\infty)\times\bar{\mathbb{H}} \to \mathbb{S}_d,$$

then $\bar{a}(t,x)$ is a non-negative definite, symmetric, real matrix for each $(t,x) \in [0,\infty)\times\bar{\mathbb{H}}$, and so there is a function

$$\sigma : [0,\infty)\times\bar{\mathbb{H}} \to \mathbb{R}^{d\times d}, \tag{2.1.14}$$

such that

$$\bar{a}(t,x) = \sigma(t,x)\sigma^*(t,x), \quad \forall (t,x) \in [0,\infty) \times \bar{\mathbb{H}}. \tag{2.1.15}$$

By [40, Lemma 6.1.1], we may choose $\sigma \in C_{\text{loc}}([0,\infty)\times\bar{\mathbb{H}};\mathbb{R}^{d\times d})$ when $\bar{a} \in C_{\text{loc}}([0,\infty)\times\bar{\mathbb{H}};\mathbb{S}_d)$; this continuity property is guaranteed by the conditions on $\bar{a}$ in (2.2.11) and (2.2.13) implied through (2.1.16).

Remark 2.1.12 (Non-uniqueness of the square root and the martingale problem)**.** Naturally, the function σ is not unique. For any function, $U : [0,\infty) \times \mathbb{R}^d \to \mathbb{O}(d)$, σU is also a square root of $\bar{a}$. However, as noted by Stroock and Varadhan [70, Remark 5.1.7 & §5.3], the solution to the martingale problem is independent of the choice of square root.

The coefficient functions (σ, b) define a degenerate stochastic differential equation (2.1.4). Unless other conditions are explicitly substituted, we require in this chapter that the coefficients (σ, b) satisfy

Assumption 2.1.13 (Properties of the coefficients of the stochastic differential equation)**.** The coefficient functions (σ, b) in (2.1.4) obey the following conditions.

1. There is a function $a : [0,\infty) \times \bar{\mathbb{H}} \to \mathbb{S}^d_+$ such that

$$\bar{a}(t,x) = x_d^+ a(t,x), \quad \forall (t,x) \in [0,\infty) \times \bar{\mathbb{H}}. \tag{2.1.16}$$

2. The coefficient functions (a, b) obey the conditions in Assumption 2.2.2.

Remark 2.1.14 (Absence of killing term)**.** The coefficient c in Assumption 2.2.2 plays no role in Theorems 2.1.16 and 2.1.19 since it does not appear in the stochastic differential equation (2.1.4).

The constraints on the coefficients (σ, b) in Assumption 2.1.13 are mild enough that they include many examples of interest in mathematical finance.

Example 2.1.15 (Heston stochastic differential equation)**.** The conditions in Assumption 2.1.13 are obeyed by the coefficients of the $\mathbb{R}^2$-valued log-Heston process [44] with killing defined by (4.1.15). See also Example 2.1.2.

Theorem 2.1.16 (Existence, uniqueness, and strong Markov property of weak solutions to a degenerate stochastic differential equation with unbounded coefficients)**.** *Suppose that the coefficients* (σ, b) *in* (2.1.4) *obey the conditions in Assumption 2.1.13. Let* $(s, x) \in [0, \infty) \times \bar{\mathbb{H}}$. *Then,*

1. *There is a weak solution,* $(\widehat{X}, \widehat{W})$, $(\Omega, \mathscr{F}, \mathbb{P})$, $\{\mathscr{F}_t\}_{t \geq s}$, *to the stochastic differential equation* (2.1.4) *such that* $\widehat{X}(s) = x$, $\mathbb{P}$*-a.s.*

2. *The weak solution is unique in the sense of probability law, that is, if*

$$(\widehat{X}^i, \widehat{W}^i), (\Omega^i, \mathscr{F}^i, \mathbb{P}^i), (\mathscr{F}^i_t)_{t \geq s}, \quad i = 1, 2,$$

are two weak solutions to the stochastic differential equation (2.1.4) *started at* x *at time* s, *then the two processes* X^1 *and* X^2 *have the same law.*

3. *The unique weak solution,* $(\widehat{X}, \widehat{W})$, $(\Omega, \mathscr{F}, \mathbb{P})$, $\{\mathscr{F}_t\}_{t \geq s}$, *has the strong Markov property.*

The following example of Stroock and Varadhan [70, Exercise 6.7.7] shows that solutions to degenerate martingale problems can easily fail to be unique.

Example 2.1.17 (Non-uniqueness of solutions to certain degenerate martingale problems)**.** [70, Exercise 6.7.7] Consider a generator, $\mathscr{A}$, in Definition 2.1.4 which is time-homogeneous with $d = 1$, $b(x) = 0$, and $a(x) = |x|^\alpha \wedge 1$ with $0 < \alpha < 1$, where $x \in \mathbb{R}$. The operator $\mathscr{A}$ is degenerate at $x = 0$ and uniqueness in law for solutions to the martingale problem for $\mathscr{A}$ fails. □

The preceding example has been explored in detail by Engelbert and Schmidt:

Example 2.1.18 (Non-uniqueness of weak solutions to certain degenerate stochastic differential equations)**.** Choose $\alpha \in (0, 1/2)$ and consider

$$dX(t) = |X(t)|^\alpha dW(t), \quad \forall t \geq 0, \tag{2.1.17}$$

Engelbert and Schmidt [26] show that the stochastic differential equation (2.1.17) admits weak solutions if and only if $I(\sigma) \subseteq Z(\sigma)$, and uniqueness in law holds if and only if $I(\sigma) = Z(\sigma)$, where

$$Z(\sigma) = \{x \in \mathbb{R} : \sigma(x) = 0\},$$

$$I(\sigma) = \left\{x \in \mathbb{R} : 1/\sigma^2 \text{ is not locally integrable at } x\right\}.$$

It is straightforward to verify that when α is chosen in the range $(0, 1/2)$, equation (2.1.17) admits weak solutions, but uniqueness in law does *not* hold. □

Mimicking one-dimensional marginal probability distributions of a degenerate Itô process with unbounded coefficients

Let X be an $\mathbb{R}^d$-valued Itô process as in (2.1.5), where W is an $\mathbb{R}^r$-valued Brownian motion on a filtered probability space, $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\}_{t\geq 0})$, satisfying the usual conditions [47, Definition 1.2.25], β is an $\mathbb{R}^d$-valued, adapted process, and that ξ is a $\mathbb{R}^{d\times r}$-valued, adapted process satisfying the integrability condition,

$$\mathbb{E}\left[\int_0^t (|\beta(s)| + |\xi(s)\xi^*(s)|)\, ds\right] < \infty, \quad \forall t \geq 0. \tag{2.1.18}$$

Assume $x \in \bar{\mathbb{H}}$ and that for all $t \geq 0$ we have

$$X(t) \in \bar{\mathbb{H}}, \quad \mathbb{P}\text{-a.s.} \tag{2.1.19}$$

By [12, Corollary 4.5], there are (Borel) $\mathscr{B}([0,\infty)\times\bar{\mathbb{H}})$-measurable (deterministic) functions,

$$\begin{aligned} b &: [0,\infty) \times \bar{\mathbb{H}} \to \mathbb{R}^d, \\ \bar{a} &: [0,\infty) \times \bar{\mathbb{H}} \to \mathbb{S}_d, \end{aligned} \tag{2.1.20}$$

such that, for Lebesgue a.e. $t \geq 0$,

$$\begin{aligned} b(t, X(t)) &= \mathbb{E}\left[\beta(t)|X(t)\right] \quad \mathbb{P}\text{-a.s.}, \\ \bar{a}(t, X(t)) &= \mathbb{E}\left[\xi(t)\xi^*(t)|X(t)\right] \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{2.1.21}$$

We can now state the main result of this chapter.

Theorem 2.1.19 (Mimicking theorem for degenerate Itô processes with unbounded coefficients)**.** *Suppose the coefficient $\bar{a}$ in* (2.1.21) *satisfies* (2.1.16) *and the pair* (a, b) *obeys Assumption 2.2.2, where b is given by* (2.1.21). *Let $\sigma \in C_{\mathrm{loc}}([0,\infty)\times\bar{\mathbb{H}}; \mathbb{R}^{d\times d})$ be a choice of square root,*

$$\bar{a}(t,x) = \sigma(t,x)\sigma^*(t,x), \quad \forall (t,x) \in [0,\infty)\times\bar{\mathbb{H}}. \tag{2.1.22}$$

Let $\widehat{X}$ be the unique, strong Markov weak solution to the stochastic differential equation (2.1.4) *started at x when $t = 0$. Then X and $\widehat{X}$ have the same one-dimensional marginal probability distributions.*

Remark 2.1.20 (Mimicking stochastic differential equation)**.** We call (2.1.4) the *mimicking stochastic differential equation* defined by the Itô process (2.1.5) when its coefficients are defined as in (2.1.15) and (2.1.20).

Remark 2.1.21 (Sufficient and necessary condition to ensure that the Itô process remains in the upper half-space)**.** In general, the coefficients σ and b_d defined by (2.1.21) and (2.1.22) are Borel measurable functions defined on $[0, \infty) \times \mathbb{R}^d$. Our assumption (2.1.19) implies that we may choose the coefficients σ and b_d such that they satisfy conditions (2.4.1) and (2.4.2) on $[0, \infty) \times \mathbb{R}^{d-1} \times (-\infty, 0)$. Conversely, if we are given b_d and σ satisfying conditions (2.4.1) and (2.4.2), Proposition 2.4.8 shows that (2.1.19) holds.

2.1.2 Survey of previous research

Gyöngy [43, Theorem 4.6] proves existence of a mimicking process as in Theorem 2.1.19 — although not the uniqueness or strong Markov properties — with conditions on the coefficients (σ, b) which are both partly *weaker* than those of Theorem 2.1.19, because the functions $b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ are only required to be Borel-measurable, but also partly *stronger* than those of Theorem 2.1.19, because the functions (σ, b) are required to be uniformly bounded on $[0, \infty) \times \mathbb{R}^d$ and $\sigma\sigma^*$ is required to be uniformly positive definite on $[0, \infty) \times \mathbb{R}^d$.

Since Gyöngy only requires that the coefficients (σ, b) of the corresponding mimicking stochastic differential equation (2.1.4) are Borel measurable functions, he uses an auxiliary regularizing procedure to construct a weak solution $\widehat{X}$ to (2.1.4). He shows that the Green measure of the mimicking process $\widehat{X}$ coincides with the Green measure of the Itô process X, that is

$$\mathbb{E}\left[\int_0^\infty e^{-t} f(t, \widehat{X}(t)) dt\right] = \mathbb{E}\left[\int_0^\infty e^{-t} f(t, X(t)) dt\right],$$

holds for all bounded, non-negative, Borel measurable functions, $f : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$. Uniqueness of the weak solution is not proved under the hypotheses of [43, Theorem 4.6] and the main obstacle here is the lack of regularity of the coefficients $(\bar{a}, b)$.

The hypotheses of [43, Theorem 4.6] are quite restrictive, as we can see that they would exclude a process, X, such as that in Example 2.1.15, even though the coefficients of its mimicking processes, $\widehat{X}$, can be found by explicit calculation [5] (see also [4]). Moreover, Nadirashvili shows [60] that uniqueness of stochastic differential equations with *measurable* coefficients satisfying the assumptions of non-degeneracy and boundedness in [43, Theorem 4.6] does not hold in general when $d \geq 3$. Nadirashvili considers a sequence of smooth coefficients converging to the given measurable coefficients and shows that there are two different solutions to the corresponding partial differential equation.

Brunick and Shreve [12, Corollary 2.16], [14] prove an extension of [43, Theorem 4.6] which relaxes the requirements that $\sigma\sigma^*$ is uniformly positive definite on $[0, \infty) \times \mathbb{R}^d$ and that the functions σ and b are bounded on $[0, \infty) \times \mathbb{R}^d$. Moreover, they significantly extend Gyöngy's theorem [43] by replacing the non-degeneracy and boundedness conditions on the coefficients of the Itô process, X, by a mild integrability condition (2.1.18). Using purely probabilistic methods, they show existence of weak solutions to stochastic differential equations of diffusion type which preserve not only the one-dimensional marginal distributions of the Itô process, but also certain statistics, such as the running maximum or average of one of the components. More recently, Brunick [13] establishes weak uniqueness for a degenerate stochastic differential equation with applications to pricing Asian options.

Bentata and Cont [9] and Shi and Wang [66, 72] extend Gyöngy's mimicking theorem to *discontinuous*, non-degenerate semimartingales. Under assumptions of continuity and boundedness on the coefficients of the process and non-degeneracy condition of the diffusion matrix or of the Levý operator, they prove uniqueness of solutions to the forward Kolmogorov equation associated with the generator of the mimicking process. In this setting, they show that weak uniqueness to the mimicking stochastic differential equation holds and that the mimicking process satisfies the Markov property.

2.1.3 Brief outline of the chapter

In §2.2, we define the Hölder spaces required to prove Theorem 2.1.1 (existence and uniqueness of solutions to a degenerate-parabolic partial differential equation on a half-space with unbounded coefficients) and provide a detailed description of the conditions required of the coefficients (a, b, c) in the statement of Theorem 2.1.1, which we then proceed to prove in §2.3. Section 2.4 contains the proofs of Theorems 2.1.10, 2.1.16, and 2.1.19. In §2.4.1, we prove existence of solutions to the degenerate martingale problem and degenerate stochastic differential equation specified in Theorems 2.1.10 and 2.1.16, while in §2.4.2, we prove uniqueness and the strong Markov property in Theorems 2.1.10 and 2.1.16. Lastly, in §2.4.3, we prove our mimicking theorem for a degenerate Itô process, namely, Theorem 2.1.19.

2.2 Weighted Hölder spaces and coefficients of the differential operators

In §2.2.1, we introduce the Hölder spaces required for the statement and proof of Theorem 2.1.1, while in §2.2.2, we describe the regularity and growth conditions required of the coefficients (a, b, c) in Theorem 2.1.1.

2.2.1 Weighted Hölder spaces

For $a > 0$ and $T > 0$, we denote

$$\mathbb{H}_{a,T} := (0, T) \times \mathbb{R}^{d-1} \times (0, a),$$

and, when $T = \infty$, we denote $\mathbb{H}_\infty = (0, \infty) \times \mathbb{H}$ and $\mathbb{H}_{a,\infty} = (0, \infty) \times \mathbb{R}^{d-1} \times (0, a)$. For $x^0 \in \bar{\mathbb{H}}$ and $R > 0$, we let

$$B_R(x^0) := \left\{ x \in \mathbb{H} : |x - x^0| < R \right\} \quad \text{and} \quad Q_{R,T}(x^0) := (0, T) \times B_R(x^0).$$

We write B_R or $Q_{R,T}$ when the center, x^0, is clear from the context or unimportant.

A parabolic partial differential equation with a degeneracy similar to that considered in this chapter arises in the study of the porous medium equation [20, 21, 50]. The existence, uniqueness, and regularity theory for such equations is facilitated by the use of Hölder spaces defined by the *cycloidal metric* on $\mathbb{H}$ introduced by Daskalopoulos and Hamilton [20] and, independently, by Koch [50]. Following [20, p. 901], we define the *cycloidal distance* between two points, $P_1 = (t_1, x^1), P_2 = (t_2, x^2) \in [0,\infty) \times \bar{\mathbb{H}}$, by

$$s(P_1, P_2) := \frac{\sum_{i=1}^d |x_i^1 - x_i^2|}{\sqrt{x_d^1} + \sqrt{x_d^2} + \sqrt{\sum_{i=1}^{d-1} |x_i^1 - x_i^2|}} + \sqrt{|t_1 - t_2|}. \tag{2.2.1}$$

Remark 2.2.1 (Equivalence of the cycloidal and Euclidean distance functions on suitable subsets of $[0,\infty) \times \mathbb{H}$)**.** The cycloidal and Euclidean distance functions, s and ρ, are equivalent on sets of the form $[0,\infty) \times \mathbb{R}^{d-1} \times [y_0, y_1]$, for any $0 < y_0 < y_1$.

Let $\Omega \subset (0,T) \times \mathbb{H}$ be an open set and $\alpha \in (0,1)$. For a function $u : \bar{\Omega} \to \mathbb{R}$, we consider the following seminorms

$$[u]_{C_s^\alpha(\bar{\Omega})} = \sup_{\substack{P_1, P_2 \in \bar{\Omega}, \\ P_1 \neq P_2}} \frac{|u(P_1) - u(P_2)|}{s(P_1, P_2)^\alpha}, \tag{2.2.2}$$

and we say that $u \in C_s^\alpha(\bar{\Omega})$ if $u \in C(\bar{\Omega})$ and

$$\|u\|_{C_s^\alpha(\bar{\Omega})} = \|u\|_{C(\bar{\Omega})} + [u]_{C_s^\alpha(\bar{\Omega})} < \infty.$$

We say that $u \in C_s^{2+\alpha}(\bar{\Omega})$ if

$$\|u\|_{C_s^{2+\alpha}(\bar{\Omega})} := \|u\|_{C_s^\alpha(\bar{\Omega})} + \|u_t\|_{C_s^\alpha(\bar{\Omega})} + \max_{1\leq i\leq d} \|u_{x_i}\|_{C_s^\alpha(\bar{\Omega})} + \max_{1\leq i,j\leq d} \|x_d u_{x_i x_j}\|_{C_s^\alpha(\bar{\Omega})} < \infty.$$

We denote by $C^\alpha_{s,\text{loc}}(\bar{\Omega})$ the space of functions u with the property that for any compact set $K \subseteq \bar{\Omega}$, we have $u \in C_s^\alpha(K)$. Analogously, we define the space $C^{2+\alpha}_{s,\text{loc}}(\bar{\Omega})$. We make use of the following hybrid Hölder spaces

$$\begin{aligned} \mathscr{C}^\alpha(\bar{\mathbb{H}}_T) &:= \left\{u : u \in C_s^\alpha(\bar{\mathbb{H}}_{1,T}) \cap C_\rho^\alpha(\bar{\mathbb{H}}_T \setminus \mathbb{H}_{1,T})\right\}, \\ \mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_T) &:= \left\{u : u \in C_s^{2+\alpha}(\bar{\mathbb{H}}_{1,T}) \cap C_\rho^{2+\alpha}(\bar{\mathbb{H}}_T \setminus \mathbb{H}_{1,T})\right\}. \end{aligned}$$

We define $\mathscr{C}^\alpha(\bar{\mathbb{H}})$ and $\mathscr{C}^{2+\alpha}(\bar{\mathbb{H}})$ in the analogous manner.

The coefficient functions $x_d a_{ij}(t,x)$, $b_i(t,x)$ and $c(t,x)$ of the parabolic operator (2.1.3) are allowed to have linear growth in $|x|$. To account for the unboundedness of the coefficients, we augment our definition of Hölder spaces by introducing weights of the form $(1+|x|)^q$, where $q \geq 0$ will be suitably chosen in the sequel. For $q \geq 0$, we define

$$\|u\|_{\mathscr{C}_q^0(\bar{\mathbb{H}})} := \sup_{x \in \bar{\mathbb{H}}} (1+|x|)^q \, |u(x)|, \tag{2.2.3}$$

and, given $T > 0$, we define

$$\|u\|_{\mathscr{C}_q^0(\bar{\mathbb{H}}_T)} := \sup_{(t,x) \in \overline{\mathbb{H}}_T} (1+|x|)^q \, |u(t,x)|. \tag{2.2.4}$$

Moreover, given $\alpha \in (0,1)$, we define

$$\|u\|_{\mathscr{C}_q^\alpha(\bar{\mathbb{H}}_T)} := \|u\|_{\mathscr{C}_q^0(\overline{\mathbb{H}}_T)} + [(1+|x|)^q u]_{C_s^\alpha(\bar{\mathbb{H}}_{1,T})} + [(1+|x|)^q u]_{C_\rho^\alpha(\bar{\mathbb{H}}_T \setminus \mathbb{H}_{1,T})}, \tag{2.2.5}$$

$$\|u\|_{\mathscr{C}_q^{2+\alpha}(\bar{\mathbb{H}}_T)} := \|u\|_{\mathscr{C}_q^\alpha(\bar{\mathbb{H}}_T)} + \|u_t\|_{\mathscr{C}_q^\alpha(\bar{\mathbb{H}}_T)} + \|u_{x_i}\|_{\mathscr{C}_q^\alpha(\bar{\mathbb{H}}_T)} + \|x_d u_{x_i x_j}\|_{\mathscr{C}_q^\alpha(\bar{\mathbb{H}}_T)}. \tag{2.2.6}$$

The vector spaces

$$\begin{aligned}
\mathscr{C}_q^0(\bar{\mathbb{H}}_T) &:= \left\{ u \in C(\bar{\mathbb{H}}_T) : \|u\|_{\mathscr{C}_q^0(\bar{\mathbb{H}}_T)} < \infty \right\}, \\
\mathscr{C}_q^\alpha(\bar{\mathbb{H}}_T) &:= \left\{ u \in \mathscr{C}^\alpha(\bar{\mathbb{H}}_T) : \|u\|_{\mathscr{C}_q^\alpha(\bar{\mathbb{H}}_T)} < \infty \right\}, \\
\mathscr{C}_q^{2+\alpha}(\bar{\mathbb{H}}_T) &:= \left\{ u \in \mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_T) : \|u\|_{\mathscr{C}_q^{2+\alpha}(\bar{\mathbb{H}}_T)} < \infty \right\},
\end{aligned}$$

can be shown to be Banach spaces with respect to the norms (2.2.4), (2.2.5) and (2.2.6), respectively. The vector spaces $\mathscr{C}_q^0(\bar{\mathbb{H}})$, $\mathscr{C}_q^\alpha(\bar{\mathbb{H}})$, and $\mathscr{C}_q^{2+\alpha}(\bar{\mathbb{H}})$, defined similarly, can be shown to be Banach spaces when equipped with the corresponding norms.

We let $\mathscr{C}_{q,\mathrm{loc}}^{2+\alpha}(\bar{\mathbb{H}}_T)$ denote the vector space of functions u such that for any compact set $K \subset \bar{\mathbb{H}}_T$, we have $u \in \mathscr{C}_q^{2+\alpha}(K)$, for all $q \geq 0$.

When $q = 0$, the subscript q is omitted in the preceding definitions.

2.2.2 Coefficients of the differential operators

Unless other conditions are explicitly substituted, we require in this chapter that the coefficients (a, b, c) of the parabolic differential operator L in (2.1.3) satisfy the conditions in the following

Assumption 2.2.2 (Properties of the coefficients of the parabolic differential operator)**.** There are constants $\delta > 0$, $K > 0$, $\nu > 0$ and $\alpha \in (0,1)$ such that the following hold.

1. The coefficients c and b_d obey

$$c(t,x) \leq K, \quad \forall (t,x) \in \bar{\mathbb{H}}_\infty, \tag{2.2.7}$$

$$b_d(t,x',0) \geq \nu, \quad \forall (t,x') \in [0,\infty) \times \mathbb{R}^{d-1}. \tag{2.2.8}$$

2. On $\bar{\mathbb{H}}_{2,\infty}$ (that is, near $x_d = 0$), we require that

$$\sum_{i,j=1}^d a_{ij}(t,x)\eta_i\eta_j \geq \delta|\eta|^2, \quad \forall \eta \in \mathbb{R}^d, \quad \forall (t,x) \in \bar{\mathbb{H}}_{2,\infty}, \tag{2.2.9}$$

$$\max_{1\leq i,j\leq d} \|a_{ij}\|_{C(\bar{\mathbb{H}}_{2,\infty})} + \max_{1\leq i\leq d} \|b_i\|_{C(\bar{\mathbb{H}}_{2,\infty})} + \|c\|_{C(\bar{\mathbb{H}}_{2,\infty})} \leq K, \tag{2.2.10}$$

and, for all $P_1, P_2 \in \bar{\mathbb{H}}_{2,\infty}$ such that $P_1 \neq P_2$ and $s(P_1,P_2) \leq 1$,

$$\begin{aligned} \max_{1\leq i,j\leq d} \frac{|a_{ij}(P_1) - a_{ij}(P_2)|}{s(P_1,P_2)^\alpha} &\leq K, \\ \max_{1\leq i\leq d} \frac{|b_i(P_1) - b_i(P_2)|}{s(P_1,P_2)^\alpha} &\leq K, \\ \frac{|c(P_1) - c(P_2)|}{s(P_1,P_2)^\alpha} &\leq K. \end{aligned} \tag{2.2.11}$$

3. On $\bar{\mathbb{H}}_\infty \setminus \mathbb{H}_{2,\infty}$ (that is, farther away from $x_d = 0$), we require that

$$\sum_{i,j=1}^d x_d a_{ij}(t,x)\eta_i\eta_j \geq \delta|\eta|^2, \quad \forall \eta \in \mathbb{R}^d, \quad \forall (t,x) \in \bar{\mathbb{H}}_\infty \setminus \mathbb{H}_{2,\infty}, \tag{2.2.12}$$

and, for all $P_1, P_2 \in \bar{\mathbb{H}}_\infty \setminus \mathbb{H}_{2,\infty}$ such that $P_1 \neq P_2$ and $\rho(P_1,P_2) \leq 1$,

$$\begin{aligned} \max_{1\leq i,j\leq d} \frac{|x_d^1 a_{ij}(P_1) - x_d^2 a_{ij}(P_2)|}{\rho(P_1,P_2)^\alpha} &\leq K, \\ \max_{1\leq i\leq d} \frac{|b_i(P_1) - b_i(P_2)|}{\rho(P_1,P_2)^\alpha} &\leq K, \\ \frac{|c(P_1) - c(P_2)|}{\rho(P_1,P_2)^\alpha} &\leq K. \end{aligned} \tag{2.2.13}$$

Remark 2.2.3 (Local Hölder conditions on the coefficients)**.** The local Hölder conditions (2.2.11) and (2.2.13) are similar to those in [53, Hypothesis 2.1].

Remark 2.2.4 (Linear growth of the coefficients of the parabolic differential operator)**.** Conditions (2.2.10) and (2.2.13) imply that the coefficients $x_d a_{ij}(t,x)$, $b_i(t,x)$ and $c(t,x)$ can have at most linear growth in x. In particular, we may choose the constant K large enough such that

$$\sum_{i,j=1}^d |x_d a_{ij}(t,x)| + \sum_{i=1}^d |b_i(t,x)| + |c(t,x)| \leq K(1+|x|), \quad \forall (t,x) \in \bar{\mathbb{H}}_\infty. \tag{2.2.14}$$

2.3 Existence, uniqueness and regularity of the inhomogeneous initial value problem

In this section, we prove Theorem 2.1.1. We begin by reviewing the boundary properties and establishing the interpolation inequalities (Lemma 2.3.2) suitable of functions in $C^{2+\alpha}_s(\bar{\mathbb{H}}_T)$. Then, we prove two versions of the maximum principle (Proposition 2.3.7) which combined with the a priori local Hölder estimates at the boundary (Theorem 2.3.8) and in the interior (Proposition 2.3.15) allow us to obtain Theorem 2.1.1.

2.3.1 Boundary properties of functions in Daskalopoulos-Hamilton-Koch Hölder spaces

We first review a result established in [20, Proposition I.12.1] when $d = 2$. Because we will frequently appeal to that result and as the proof of [20, Proposition I.12.1] is not given in detail for all cases relevant to $d \geq 2$, we include the proof here for completeness.

Lemma 2.3.1 (Boundary properties of functions in Daskalopoulos-Hamilton-Koch Hölder spaces)**.** *Let $u \in C^{2+\alpha}_{s,\text{loc}}(\bar{\mathbb{H}}_T)$. Then, for all $\bar{P} = (\bar{t}, \bar{x}', 0) \in [0,T] \times \partial\mathbb{H}$,*

$$\lim_{\bar{\mathbb{H}}_T \ni P \to \bar{P}} x_d u_{x_i x_j}(P) = 0, \quad \forall i,j = 1,\dots,d. \tag{2.3.1}$$

Proof. First, we consider the case $1 \leq i,j \leq d-1$. Because the seminorm $[x_d u_{x_i x_j}]_{C^{\alpha}_{s,\text{loc}}(\bar{\mathbb{H}}_T)}$ is finite, the function $x_d u_{x_i x_j}$ is uniformly continuous on compact subsets of $\bar{\mathbb{H}}_T$, and so, the limit in (2.3.1) exists. We assume, to obtain a contradiction, that

$$\lim_{\bar{\mathbb{H}}_T \ni P \to \bar{P}} x_d u_{x_i x_j}(P) = a \neq 0, \tag{2.3.2}$$

and we can further assume, without loss of generality, that this limit is positive. Then, there is a constant, $\varepsilon > 0$, such that for all $P = (t, x', x_d) \in \bar{\mathbb{H}}_T$ satisfying

$$0 < x_d < \varepsilon, \quad |t - \bar{t}| < \varepsilon, \quad |x' - \bar{x}'| < \varepsilon, \tag{2.3.3}$$

we have

$$\frac{a}{2x_d} \leq u_{x_i x_j}(t, x', x_d). \tag{2.3.4}$$

Let $P_1 = (t, x^1)$ and $P_2 = (t, x^2)$ be points satisfying (2.3.3) and such that all except the x_i-coordinates are identical. Then, by integrating (2.3.4) with respect to x_i, we obtain

$$\frac{a(x_i^2 - x_i^1)}{2x_d} \leq u_{x_j}(P_2) - u_{x_j}(P_1),$$

and thus,

$$\frac{a(x_i^2 - x_i^1)}{2x_d s(P_1, P_2)^\alpha} \leq \frac{u_{x_j}(P_2) - u_{x_j}(P_1)}{s(P_1, P_2)^\alpha}. \tag{2.3.5}$$

We can choose P_1, P_2 such that $x_i^2 - x_i^1 = \varepsilon/2$, for all $0 < x_d < \varepsilon/2$. Then, by taking limit as x_d goes to zero, the left hand side of (2.3.5) diverges, while the right hand side is finite since $[u_{x_j}]_{C_s^\alpha(\mathbb{H}_T)}$ is bounded. This contradicts (2.3.2) and so (2.3.1) holds.

The case where $i = d$ or $j = d$ can be treated as in the proof of [20, Proposition I.12.1]. □

Next, we establish the analogue of [51, Theorem 8.8.1] for the Hölder space $C_s^{2+\alpha}(\bar{\mathbb{H}}_T)$.

Lemma 2.3.2 (Interpolation inequalities for Daskalopoulos-Hamilton-Koch Hölder spaces)**.** *Let $R > 0$. Then there are positive constants $m = m(d, \alpha)$ and $C = C(T, R, d, \alpha)$ such that for any $u \in C_s^{2+\alpha}(\bar{\mathbb{H}}_T)$ with compact support in $[0, \infty) \times \bar{B}_R(x^0)$, for some $x^0 \in \partial\mathbb{H}$, and any $\varepsilon \in (0, 1)$, we have*

$$\|u\|_{C_s^\alpha(\bar{\mathbb{H}}_T)} \leq \varepsilon \|u\|_{C_s^{2+\alpha}(\bar{\mathbb{H}}_T)} + C\varepsilon^{-m} \|u\|_{C(\bar{\mathbb{H}}_T)}, \tag{2.3.6}$$

$$\|u_{x_i}\|_{C(\bar{\mathbb{H}}_T)} \leq \varepsilon \|u\|_{C_s^{2+\alpha}(\bar{\mathbb{H}}_T)} + C\varepsilon^{-m} \|u\|_{C(\bar{\mathbb{H}}_T)}, \tag{2.3.7}$$

$$\|x_d u_{x_i}\|_{C_s^\alpha(\bar{\mathbb{H}}_T)} \leq \varepsilon \|u\|_{C_s^{2+\alpha}(\bar{\mathbb{H}}_T)} + C\varepsilon^{-m} \|u\|_{C(\bar{\mathbb{H}}_T)}, \tag{2.3.8}$$

$$\|x_d u_{x_i x_j}\|_{C(\bar{\mathbb{H}}_T)} \leq \varepsilon \|u\|_{C_s^{2+\alpha}(\bar{\mathbb{H}}_T)} + C\varepsilon^{-m} \|u\|_{C(\bar{\mathbb{H}}_T)}. \tag{2.3.9}$$

Remark 2.3.3. Notice that Lemma 2.3.2 does not establish the analogue of [51, Inequality (8.8.4)], that is,

$$[u_{x_i}]_{C_\rho^\alpha(\bar{\mathbb{H}}_T)} \leq \varepsilon \|u\|_{C_\rho^{2+\alpha}(\bar{\mathbb{H}}_T)} + C\varepsilon^{-m} \|u\|_{C(\bar{\mathbb{H}}_T)}.$$

This is replaced by the weighted inequality (2.3.8).

Proof of Lemma 2.3.2. We consider $\eta \in (0, 1)$, to be suitably chosen during the proofs of each of the desired inequalities.

Step 1 (Proof of inequality (2.3.6))**.** We only need to show that the first inequality (2.3.6) holds for the seminorm $[u]_{C^\alpha_s(\bar{\mathbb{H}}_T)}$. It is enough to consider differences, $u(P_1) - u(P_2)$, where all except one of the coordinates of the points $P_1, P_2 \in \bar{\mathbb{H}}_T$ are identical. We outline the proof when the x_i-coordinates of P_1 and P_2 differ, but the case of the t-coordinates can be treated in a similar manner. We consider two situations: $|x_i^1 - x_i^2| \leq \eta$ and $|x_i^1 - x_i^2| > \eta$.

Case 1 (Points with x_i-coordinates close together)**.** Assume $|x_i^1 - x_i^2| \leq \eta$. We have

$$\begin{aligned}
|u(P_1) - u(P_2)| &\leq |x_i^1 - x_i^2| \|u_{x_i}\|_{C(\bar{\mathbb{H}}_T)} \\
&\leq \eta \frac{|x_i^1 - x_i^2|}{\eta} \|u\|_{C^{2+\alpha}_s(\bar{\mathbb{H}}_T)} \\
&\leq \eta \left(\frac{|x_i^1 - x_i^2|}{\eta}\right)^\alpha \|u\|_{C^{2+\alpha}_s(\bar{\mathbb{H}}_T)} \\
&\leq \eta^{1-\alpha} \left(2\sqrt{x_d} + \sqrt{|x_i^1 - x_i^2|}\right)^\alpha s(P_1, P_2)^\alpha \|u\|_{C^{2+\alpha}_s(\bar{\mathbb{H}}_T)},
\end{aligned} \tag{2.3.10}$$

where in the last line we used the fact that, by (2.2.1),

$$s(P_1, P_2) = \frac{|x_i^1 - x_i^2|}{2\sqrt{x_d} + \sqrt{|x_i^1 - x_i^2|}}. \tag{2.3.11}$$

Because u has compact support in the spatial variable, we obtain in (2.3.10) that there exists a positive constant $C = C(\alpha, R)$ such that

$$\frac{|u(P_1) - u(P_2)|}{s(P_1, P_2)^\alpha} \leq C\eta^{1-\alpha} \|u\|_{C^{2+\alpha}_s(\bar{\mathbb{H}}_T)}, \tag{2.3.12}$$

which concludes this case.

Case 2 (Points with x_i-coordinates further apart)**.** Assume $|x_i^1 - x_i^2| > \eta$. By (2.3.11), we have

$$1 < \left(\frac{|x_i^1 - x_i^2|}{\eta}\right)^\alpha = \eta^{-\alpha} \left(2\sqrt{x_d} + \sqrt{|x_i^1 - x_i^2|}\right)^\alpha s(P_1, P_2)^\alpha.$$

Because it suffices to consider points P_1 and P_2 in the support of u, there is a positive constant C, depending at most on α and R, such that

$$1 \leq C\eta^{-\alpha} s(P_1, P_2)^\alpha.$$

Therefore,

$$|u(P_1) - u(P_2)| \leq 2\|u\|_{C(\bar{\mathbb{H}}_T)} \leq C\eta^{-\alpha} s(P_1, P_2)^\alpha \|u\|_{C(\bar{\mathbb{H}}_T)},$$

which is equivalent to

$$\frac{|u(P_1) - u(P_2)|}{s(P_1, P_2)^\alpha} \leq C\eta^{-\alpha} \|u\|_{C^0(\bar{\mathbb{H}}_T)}, \tag{2.3.13}$$

which concludes this case.

By combining (2.3.12) and (2.3.13), we obtain

$$[u]_{C^\alpha_s(\bar{\mathbb{H}}_T)} \leq C\eta^{1-\alpha} \|u\|_{C^{2+\alpha}_s(\bar{\mathbb{H}}_T)} + C\eta^{-\alpha} \|u\|_{C^{)}(\bar{\mathbb{H}}_T)}.$$

Since $\varepsilon \in (0,1)$, we may choose $\eta \in (0,1)$ such that $\varepsilon = C\eta^{1-\alpha}$. The preceding inequality then gives (2.3.6).

Step 2 (Proof of inequality (2.3.7))**.** Let $P \in \bar{\mathbb{H}}_T$. Then, for any $\eta > 0$, we have

$$\begin{aligned}
|u_{x_i}(P)| &\leq \left| u_{x_i}(P) - \eta^{-1} \left(u(P + \eta e_i) - u(P) \right) \right| + 2\eta^{-1} \|u\|_{C(\bar{\mathbb{H}}_T)} \\
&= |u_{x_i}(P) - u_{x_i}(P + \eta\theta e_i)| + 2\eta^{-1} \|u\|_{C(\bar{\mathbb{H}}_T)} \\
&= \frac{|u_{x_i}(P) - u_{x_i}(P + \eta\theta e_i)|}{s(P, P + \eta\theta e_i)^\alpha} s(P, P + \eta\theta e_i)^\alpha + 2\eta^{-1} \|u\|_{C(\bar{\mathbb{H}}_T)},
\end{aligned}$$

for some constant $\theta \in [0,1]$. Using

$$s(P, P + \eta\theta e_i) \leq \eta^{1/2}, \quad \forall P \in \bar{\mathbb{H}}_T, \tag{2.3.14}$$

we have

$$|u_{x_i}(P)| \leq \eta^\alpha [u_{x_i}]_{C^\alpha_s(\bar{\mathbb{H}}_T)} + 2\eta^{-1} \|u\|_{C(\bar{\mathbb{H}}_T)}, \quad \forall P \in \bar{\mathbb{H}}_T. \tag{2.3.15}$$

Since $\varepsilon \in (0,1)$, we may choose $\eta \in (0,1)$ such that $\varepsilon = \eta^\alpha$. Then (2.3.7) follows from (2.3.15).

Step 3 (Proof of inequality (2.3.8))**.** Because u has compact support in the spatial variable, then (2.3.7) gives, for some positive constant $C = C(\alpha, R)$,

$$\|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)} \leq C\varepsilon \|u\|_{C^{2+\alpha}_s(\bar{\mathbb{H}}_T)} + C\varepsilon^{-m} \|u\|_{C(\bar{\mathbb{H}}_T)}. \tag{2.3.16}$$

This gives the desired bound in (2.3.8) for the term $\|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)}$. It remains to prove the estimate (2.3.8) for the Hölder seminorm $[x_d u_{x_i}]_{C^\alpha_s(\bar{\mathbb{H}}_T)}$. As in the proof of (2.3.6), it suffices to consider the differences $x^1_d u_{x_i}(P_1) - x^2_d u_{x_i}(P_2)$, where all except one of the coordinates of the points $P_1, P_2 \in \bar{\mathbb{H}}_T$ are identical.

First, we consider the case when only the x_d-coordinates of the points P_1 and P_2 differ. We denote $P_k = (t, x', x^k_d)$, $k = 1, 2$.

Case 1 (Points with x_d-coordinates close together)**.** Assume $|x_d^1 - x_d^2| \leq \eta$. Using

$$(x_d u_{x_i})_{x_d} = x_d u_{x_i x_d} + u_{x_i}$$

and the mean value theorem, there is a point P^* on the line segment connecting P_1 and P_2 such that,

$$x_d^1 u_{x_i}(P_1) - x_d^2 u_{x_i}(P_2) = \left(x_d^* u_{x_i x_d}(P^*) + u_{x_i}(P^*)\right)(x_d^1 - x_d^2),$$

and so,

$$\begin{aligned}|x_d^1 u_{x_i}(P_1) - x_d^2 u_{x_i}(P_2)| &\leq \eta \left(\frac{|x_d^1 - x_d^2|}{\eta}\right)^\alpha \|u\|_{C_s^{2+\alpha}(\bar{\mathbb{H}}_T)} \\ &\leq \eta^{1-\alpha}\left(\sqrt{x_d^1} + \sqrt{x_d^2} + \sqrt{|x_d^1 - x_d^2|}\right)^\alpha s(P_1, P_2)^\alpha \|u\|_{C_s^{2+\alpha}(\bar{\mathbb{H}}_T)}.\end{aligned}$$

Because u has compact support in the spatial variable, there is a positive constant $C = C(\alpha, R)$ such that

$$\frac{|x_d^1 u_{x_i}(P_1) - x_d^2 u_{x_i}(P_2)|}{s(P_1, P_2)^\alpha} \leq C\eta^{1-\alpha}\|u\|_{C_s^{2+\alpha}(\bar{\mathbb{H}}_T)}, \tag{2.3.17}$$

which concludes this case.

Case 2 (Points with x_d-coordinates further apart)**.** Assume $|x_d^1 - x_d^2| > \eta$. We have

$$\begin{aligned}\frac{|x_d^1 u_{x_i}(P_1) - x_d^2 u_{x_i}(P_2)|}{s(P_1, P_2)^\alpha} &\leq 2\frac{\|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)}}{|x_d^1 - x_d^2|^\alpha}\left(\sqrt{x_d^1} + \sqrt{x_d^2} + \sqrt{|x_d^1 - x_d^2|}\right)^\alpha \\ &\leq C\eta^{-\alpha}\|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)}.\end{aligned}$$

Since $\varepsilon \in (0,1)$, we may choose η such that $\varepsilon = \eta^{\alpha+1}$ in (2.3.16). We obtain

$$\frac{|x_d^1 u_{x_i}(P_1) - x_d^2 u_{x_i}(P_2)|}{s(P_1, P_2)^\alpha} \leq C\eta\|u\|_{C_s^{2+\alpha}(\bar{\mathbb{H}}_T)} + C\eta^{-m(1+\alpha)-\alpha}\|u\|_{C(\bar{\mathbb{H}}_T)}, \tag{2.3.18}$$

which concludes this case.

Combining (2.3.17) and (2.3.18) gives

$$\frac{|x_d^1 u_{x_i}(P_1) - x_d^2 u_{x_i}(P_2)|}{s(P_1, P_2)^\alpha} \leq C\eta^{1-\alpha}\|u\|_{C_s^{2+\alpha}(\bar{\mathbb{H}}_T)} + C\eta^{-m(1+\alpha)-\alpha}\|u\|_{C(\bar{\mathbb{H}}_T)}. \tag{2.3.19}$$

A similar argument, when only the x_i-coordinates of the points P_1 and P_2 differ, $1 \leq i \leq d-1$, also yields (2.3.19).

Next, we consider the case when only the t-coordinates of the points P_1 and P_2 differ. We denote $P_k = (x, t_k)$, $k = 1, 2$. We shall only describe the proof of the interpolation inequality for u_{x_i} when $i \neq d$, as the case $i = d$ follows by a similar argument. We denote $\delta = \sqrt{|t_1 - t_2|}$.

Case 1 (Points with t-coordinates close together)**.** Assume $|t_1 - t_2| < \eta$. We have

$$\begin{aligned}|u_{x_i}(P_1) - u_{x_i}(P_2)| \leq & \left|u_{x_i}(x, t_1) - \frac{1}{\delta}\left(u(x + \delta e_i, t_1) - u(x, t_1)\right)\right| \\ &+ \left|u_{x_i}(x, t_2) - \frac{1}{\delta}\left(u(x + \delta e_i, t_2) - u(x, t_2)\right)\right| \\ &+ \frac{1}{\delta}|u(x + \delta e_i, t_1) - u(x + \delta e_i, t_2)| + \frac{1}{\delta}|u(x, t_1) - u(x, t_2)|.\end{aligned}$$

By the mean value theorem, there are points $P_k^* \in \bar{\mathbb{H}}_T$, $k = 1, 2$, such that

$$\begin{aligned}|u_{x_i}(P_1) - u_{x_i}(P_2)| &= |u_{x_i}(x, t_1) - u_{x_i}(x + \theta_1 \delta e_i, t_1)| + |u_{x_i}(x, t_2) - u_{x_i}(x + \theta_2 \delta e_i, t_2)| \\ &\quad + \frac{|t_1 - t_2|}{\delta}|u_t(x + \delta e_i, t_1^*)| + \frac{|t_1 - t_2|}{\delta}|u_t(x, t_2^*)| \\ &\leq |u_{x_i x_i}(P_1^*, t_1)|\delta + |u_{x_i x_i}(P_2^*, t_2)|\delta \\ &\quad + \frac{|t_1 - t_2|}{\delta}|u_t(x + \delta e_i, t_1^*)| + \frac{|t_1 - t_2|}{\delta}|u_t(x, t_2^*)|.\end{aligned}$$

Notice that $s(P_1, P_2) = \sqrt{|t_1 - t_2|} = \delta$ and so, by multiplying the preceding inequality by x_d and using the fact that u has compact support, we obtain

$$\frac{|x_d u_{x_i}(P_1) - x_d u_{x_i}(P_2)|}{s(P_1, P_2)^\alpha} \leq 2\|x_d u_{x_i x_i}\|_{C^0(\bar{\mathbb{H}}_T)}|t_1 - t_2|^{\frac{1-\alpha}{2}} + 2|t_1 - t_2|^{1 - \frac{1+\alpha}{2}}\|x_d u_t\|_{C^0(\bar{\mathbb{H}}_T)},$$

and thus

$$\frac{|x_d u_{x_i}(P_1) - x_d u_{x_i}(P_2)|}{s(P_1, P_2)^\alpha} \leq C\eta^{\frac{1-\alpha}{2}}\|u\|_{C_s^{2+\alpha}(\bar{\mathbb{H}}_T)}, \tag{2.3.20}$$

where C is a positive constant depending only on R.

Case 2 (Points with t-coordinates further apart)**.** Assume $|t_1 - t_2| \geq \eta$. This case is easier, as usual, because

$$\frac{|x_d u_{x_i}(P_1) - x_d u_{x_i}(P_2)|}{s(P_1, P_2)^\alpha} \leq \quad 2\eta^{-\frac{\alpha}{2}}\|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)}, \tag{2.3.21}$$

which concludes this case.

By combining inequalities (2.3.20) and (2.3.21), we obtain

$$\frac{|x_d u_{x_i}(P_1) - x_d u_{x_i}(P_2)|}{s(P_1,P_2)^\alpha} \leq C\eta^{\frac{1-\alpha}{2}} \|u\|_{C^{2+\alpha}_s(\bar{\mathbb{H}}_T)} + 2\eta^{-\frac{\alpha}{2}} \|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)}. \tag{2.3.22}$$

By (2.3.19) and (2.3.22), we have

$$[x_d u_{x_i}]_{C^\alpha_s(\bar{\mathbb{H}}_T)} \leq C\eta^{\alpha_0}\|u\|_{C^{2+\alpha}_s(\bar{\mathbb{H}}_T)} + 2\eta^{-m_0}\|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)},$$

where $\alpha_0 := \min\{\alpha, 1-\alpha, (1-\alpha)/2\}$ and $m_0 := 4+\alpha$. Without loss of generality, we may assume $C \geq 1$. Since $\varepsilon \in (0,1)$, we may choose $\eta \in (0,1)$ such that $\varepsilon = C\eta^{\alpha_0}$ in the preceding inequality, and so we obtain the estimate (2.3.8) for $[x_d u_{x_d}]_{C^\alpha_s(\mathbb{H}_T)}$. This concludes the proof of (2.3.8).

Step 4 (Proof of inequality (2.3.9))**.** For any $P = (t,x) \in \bar{\mathbb{H}}_T$, we can find $\theta \in [0,1]$ such that

$$|x_d u_{x_i x_j}(P)| \leq \left|x_d u_{x_i x_j}(P) - (x_d u_{x_i}(P+\eta e_j) - x_d u_{x_i}(P))\right| + 2\|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)},$$

and thus

$$|x_d u_{x_i x_j}(P)| \leq \left|x_d u_{x_i x_j}(P) - x_d u_{x_i x_j}(P+\theta\eta e_j)\right| + 2\|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)}, \tag{2.3.23}$$

where $1 \leq i,j \leq d$. If $j \neq d$, we have

$$\begin{aligned} |x_d u_{x_i x_j}(P)| &\leq \frac{|x_d u_{x_i x_j}(P) - x_d u_{x_i x_j}(P+\theta\eta e_j)|}{s(P, P+\theta\varepsilon e_j)^\alpha} s(P,P+\theta\eta e_j)^\alpha + 2\|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)} \\ &\leq C\eta^{\alpha/2}[x_d u_{x_i x_j}]_{C^\alpha_s(\bar{\mathbb{H}}_T)} + 2\|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)}, \text{ (by (2.3.14))}. \end{aligned}$$

Because $\varepsilon \in (0,1)$, we may choose $\eta \in (0,1)$ such that $\varepsilon = C\eta^{\alpha/2}$ in the preceding inequality and combining the resulting inequality with (2.3.8), we see that the estimate (2.3.9) for $\|x_d u_{x_i x_j}\|_{C(\bar{\mathbb{H}}_T)}$ holds for all $j \neq d$.

Next, we consider the case $j = d$. For brevity, we denote $P' = P + \theta\eta e_d = (t, x', x'_d)$ and $P'' = (t, x', 0)$. We consider two distinct cases depending on whether $\eta < x'_d/2$ or $\eta \geq x'_d/2$.

Case 1 (Points with x_d-coordinates further apart)**.** Assume $\eta < x'_d/2$. By (2.3.23), we obtain

$$\begin{aligned} |x_d u_{x_i x_d}(P)| \leq &\frac{|x_d u_{x_i x_d}(P) - x'_d u_{x_i x_d}(P')|}{s(P,P')^\alpha} s(P,P')^\alpha \\ &+ |(x'_d - x_d) u_{x_i x_d}(P')| + 2\|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)}, \end{aligned} \tag{2.3.24}$$

and so, using (2.3.14) and the fact that $|x'_d - x_d| \leq \eta$, by definitions of points P and P',,

$$|x_d u_{x_i x_d}(P)| \leq \eta^{\alpha/2}[x_d u_{x_i x_d}]_{C_s^\alpha(\bar{\mathbb{H}}_T)} + \frac{\eta}{x'_d}|x'_d u_{x_i x_d}(P')| + 2\|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)},$$

which gives, by our assumption that $\eta < x'_d/2$,

$$|x_d u_{x_i x_d}(P)| \leq \eta^{\alpha/2}[x_d u_{x_i x_d}]_{C_s^\alpha(\bar{\mathbb{H}}_T)} + \frac{1}{2}\|x_d u_{x_i x_d}\|_{C(\bar{\mathbb{H}}_T)} + 2\|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)}. \tag{2.3.25}$$

As (2.3.25) holds for all $P \in \bar{\mathbb{H}}_T$, we have

$$\|x_d u_{x_i x_d}\|_{C(\bar{\mathbb{H}}_T)} \leq \frac{1}{2}\|x_d u_{x_i x_d}\|_{C(\bar{\mathbb{H}}_T)} + \eta^{\alpha/2}[x_d u_{x_i x_d}]_{C_s^\alpha(\bar{\mathbb{H}}_T)} + 2\|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)},$$

or

$$\|x_d u_{x_i x_d}\|_{C(\bar{\mathbb{H}}_T)} \leq 2\eta^{\alpha/2}[x_d u_{x_i x_d}]_{C_s^\alpha(\bar{\mathbb{H}}_T)} + 4\|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)}, \tag{2.3.26}$$

which concludes this case.

Case 2 (Points with x_d-coordinates close together)**.** Assume $\eta \geq x'_d/2$. Recall that $x'_d = x_d + \theta\eta$, for some $\theta \in [0,1]$, so that $|x'_d - x_d| \leq x'_d$. From Lemma 2.3.1, we have

$$x_d u_{x_i x_d} \to 0, \text{ as } x_d \to 0.$$

Therefore, we obtain

$$\begin{aligned} |(x'_d - x_d)u_{x_i x_d}(P')| \leq |x'_d u_{x_i x_d}(P')| &= \frac{|x'_d u_{x_i x_d}(P') - 0|}{s(P', P'')^\alpha} s(P', P'')^\alpha \\ &\leq [x_d u_{x_i x_d}]_{C_s^\alpha(\bar{\mathbb{H}}_T)}(2\eta)^{\alpha/2}, \end{aligned}$$

where the second inequality follows from the fact that

$$s(P', P'') \leq \sqrt{x'_d} \leq \sqrt{2\eta}.$$

By a calculation similar to that which led to (2.3.24), we obtain

$$\begin{aligned} |x_d u_{x_i x_d}(P)| \leq &\frac{|x_d u_{x_i x_d}(P) - x'_d u_{x_i x_d}(P')|}{s(P, P')^\alpha} s(P, P')^\alpha \\ &+ |(x'_d - x_d)u_{x_i x_d}(P')| + 2\|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)}, \end{aligned}$$

and hence

$$\begin{aligned} |x_d u_{x_i x_d}(P)| \leq &C\eta^{\alpha/2}[x_d u_{x_i x_d}]_{C_s^\alpha(\bar{\mathbb{H}}_T)} \\ &+ (2\eta)^{\alpha/2}[x_d u_{x_i x_d}]_{C_s^\alpha(\bar{\mathbb{H}}_T)} + 2\|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)}, \end{aligned} \tag{2.3.27}$$

which concludes this case.

By combining inequalities (2.3.25) and (2.3.27), we obtain, for all $P \in \bar{\mathbb{H}}_T$,

$$|x_d u_{x_i x_d}(P)| \leq \frac{1}{2}\|x_d u_{x_i x_d}\|_{C(\bar{\mathbb{H}}_T)} + C\eta^{\alpha/2}[x_d u_{x_i x_d}]_{C_s^\alpha(\bar{\mathbb{H}}_T)} + 2\|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)},$$

which is equivalent to

$$\|x_d u_{x_i x_d}\|_{C(\bar{\mathbb{H}}_T)} \leq \frac{1}{2}\|x_d u_{x_i x_d}\|_{C(\bar{\mathbb{H}}_T)} + C\eta^{\alpha/2}[x_d u_{x_i x_d}]_{C_s^\alpha(\bar{\mathbb{H}}_T)} + 2\|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)}.$$

Rearranging terms yields

$$\|x_d u_{x_i x_d}\|_{C(\bar{\mathbb{H}}_T)} \leq 2C\eta^{\alpha/2}[x_d u_{x_i x_d}]_{C_s^\alpha(\bar{\mathbb{H}}_T)} + 4\|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)}. \tag{2.3.28}$$

Since $\varepsilon \in (0,1)$, we may choose $\eta \in (0,1)$ in (2.3.26) and (2.3.28) such that $\varepsilon = 4(C+1)\eta^{\alpha/2}$ and so we obtain

$$\|x_d u_{x_i x_d}\|_{C(\bar{\mathbb{H}}_T)} \leq \varepsilon/2[x_d u_{x_i x_d}]_{C_s^\alpha(\bar{\mathbb{H}}_T)} + 4\|x_d u_{x_i}\|_{C(\bar{\mathbb{H}}_T)}.$$

Combining the preceding inequality with (2.3.8) applied with ε replaced by $\varepsilon/8$, we conclude that (2.3.9) holds.

This completes the proof of Lemma 2.3.2. □

2.3.2 Maximum principle and its applications

In this subsection, we prove a variant of the classical maximum principle (see [51, Section 8.1] and [20, Theorem I.3.1]) for parabolic operators, L, of the form (2.1.3).

Lemma 2.3.4 (Maximum principle)**.** *We relax the requirements stated in Assumption 2.2.2 on the coefficients $a = (a_{ij}), b = (b_i), c$ of the operator L in* (2.1.3) *to those stated here. Require that the coefficients $x_d a_{ij}$, b_i, c be in $C_{\mathrm{loc}}((0,T] \times \bar{\mathbb{H}})$, that $b_d \geq 0$ when $x_d = 0$, that c obeys* (2.2.7)*, and*

$$\mathrm{tr}(x_d a(t,x)) + x \cdot b(t,x) \leq K(1+|x|^2), \quad \forall (t,x) \in \bar{\mathbb{H}}_T, \tag{2.3.29}$$

where $K > 0$. Suppose $u \in C^{1,2}(\mathbb{H}_T) \cap C(\bar{\mathbb{H}}_T)$ obeys

$$u_t, u_{x_i}, x_d u_{x_i x_j} \in C_{\mathrm{loc}}((0,T] \times \bar{\mathbb{H}}), \quad 1 \leq i,j \leq d, \tag{2.3.30}$$

and

$$x_d u_{x_i x_j} = 0 \text{ on } (0,T] \times \partial\mathbb{H}, \quad 1 \le i,j \le d. \tag{2.3.31}$$

If

$$Lu \le 0 \quad \text{on } (0,T) \times \mathbb{H}, \tag{2.3.32}$$

$$u(0,\cdot) \le 0 \quad \text{on } \bar{\mathbb{H}}, \tag{2.3.33}$$

then

$$u \le 0 \quad \text{on } [0,T] \times \bar{\mathbb{H}}. \tag{2.3.34}$$

Proof. We apply an argument similar to that used in the proofs of [51, Theorem 2.9.2, Exercises 2.9.4 & 2.9.5] (maximum principle for elliptic equations on unbounded domains) and [51, Theorems 8.1.2 & 8.1.4] (maximum principle for parabolic equations on unbounded domains); see also [20, Theorem I.3.1].

We consider the transformation

$$u(t,x) = e^{\lambda t}\tilde{u}(t,x) \quad \text{ on } [0,T] \times \bar{\mathbb{H}}, \tag{2.3.35}$$

where the constant $\lambda > 0$ will be suitably chosen below. The conclusion of the lemma follows if and only if (2.3.34) holds for $\tilde{u}$. By (2.3.32) and definition (2.3.35) we have

$$e^{\lambda t}\left(L+\lambda\right)\tilde{u} = Lu \le 0 \quad \text{ on } (0,T) \times \mathbb{H}.$$

Therefore, by (2.3.32) and (2.3.33), the function $\tilde{u}$ satisfies

$$\left(L+\lambda\right)\tilde{u} \le 0 \quad \text{on } (0,T) \times \mathbb{H}, \tag{2.3.36}$$

$$\tilde{u}(0,\cdot) \le 0 \quad \text{on } \bar{\mathbb{H}}. \tag{2.3.37}$$

We may suppose without loss of generality that

$$m := \sup_{\mathbb{H}_T} \tilde{u} \ge 0, \tag{2.3.38}$$

as if $m < 0$ we are done; we shall show that $m = 0$. Define an auxiliary function,

$$h(t,x) := 1 + |x|^2, \quad \forall (t,x) \in \bar{\mathbb{H}}_T. \tag{2.3.39}$$

By direct calculation,

$$\begin{aligned}
-(L+\lambda)\,h &= \sum_{i,j=1}^{d} x_d a_{ij} h_{x_i x_j} + \sum_{i=1}^{d} b_i h_{x_i} + (c-\lambda)h - h_t \\
&= 2x_d \sum_{i=1}^{d} a_{ii} + 2\sum_{i=1}^{d} b_i x_i + (c-\lambda)(1+|x|^2) \\
&\leq (2K + c - \lambda)\,(1+|x|^2) \quad \text{on } (0,T)\times\mathbb{H}, \quad \text{(by (2.3.29))}
\end{aligned}$$

By choosing

$$\lambda \geq 3K, \tag{2.3.40}$$

we notice that condition (2.2.7), gives

$$2K + c(t,x) - \lambda \leq 0 \quad \forall (t,x) \in \bar{\mathbb{H}}_T, \tag{2.3.41}$$

and so, we have

$$(L+\lambda)\,h \geq 0 \quad \text{on } (0,T)\times\mathbb{H}. \tag{2.3.42}$$

Fix $\delta \in (0,1)$ and define another auxiliary function

$$w := \tilde{u} - \delta m h. \tag{2.3.43}$$

From (2.3.36) and (2.3.42), we have $(L+\lambda)w \leq 0$ on $(0,T)\times\mathbb{H}$ and thus

$$(L+\lambda)\,w \leq 0 \quad \text{on } (0,T)\times\bar{\mathbb{H}}, \tag{2.3.44}$$

since $w_t, w_{x_i}, x_d w_{x_i x_j}$ extend continuously from $(0,T)\times\mathbb{H}$ to $(0,T]\times\bar{\mathbb{H}}$ because these continuity properties are true of u by hypothesis (2.3.30) (and trivially true for h) and thus also true for w.

Claim 2.3.5. *There is a constant, $R_0 = R_0(\delta) > 0$, such that*

$$w \leq 0 \quad \text{on } [0,T]\times\bar{B}_R, \quad \forall R \geq R_0(\delta). \tag{2.3.45}$$

Proof. Since $w \in C([0,T]\times\bar{B}_R)$, the function w attains its maximum at some point $P \in [0,T]\times\bar{B}_R$. If $P \in (0,T]\times B_R$, then

$$w_t(P) \geq 0, \quad w_{x_i}(P) = 0, \quad (w_{x_i x_j}(P)) \leq 0.$$

Therefore,

$$
\begin{aligned}
-\left(L+\lambda\right)w(P) &= \sum_{i,j=1}^{d} x_d a_{ij}(P) w_{x_i x_j}(P) + \sum_{i=1}^{d} b_i(P) w_{x_i}(P) + (c(P)-\lambda)w(P) - w_t(P) \\
&\leq (c(P)-\lambda)w(P).
\end{aligned}
$$

If $P \in (0,T] \times (\bar{B}_R \cap \{x_d = 0\})$, then

$$
w_t(P) \geq 0, \quad w_{x_d}(P) \leq 0, \quad w_{x_i}(P) = 0 \quad (i \neq d), \quad x_d w_{x_i x_j}(P) = 0,
$$

where we use the fact that u, and thus w, obey (2.3.30) and (2.3.31). Therefore,

$$
\begin{aligned}
-\left(L+\lambda\right)w(P) &= \sum_{i,j=1}^{d} x_d a_{ij}(P) w_{x_i x_j}(P) + \sum_{i=1}^{d} b_i(P) w_{x_i}(P) + c(P)w(P) - w_t(P) \\
&\leq b_d(P) w_{x_d}(P) + (c(P)-\lambda)w(P) \\
&\leq (c(P)-\lambda)w(P) \quad \text{(by hypothesis that } b_d \geq 0 \text{ on } \{x_d = 0\}).
\end{aligned}
$$

Hence, for $P \in (0,T] \times B_R$ or $(0,T] \times (\bar{B}_R \cap \{x_d = 0\})$, we obtain

$$
-(c(P)-\lambda)w(P) \leq Lw(P).
$$

But $Lw(P) \leq 0$ by (2.3.44) and therefore, $w(P) \leq 0$ since $c \leq K$ by (2.2.7) and $\lambda \geq 3K$ by (2.3.40) .

Now suppose P lies in one of the remaining two components of the boundary of $(0,T) \times B_R$,

$$
\mathscr{B}_R^0 := \{0\} \times \bar{B}_R \quad \text{or} \quad \mathscr{B}_R^1 := (0,T] \times \left(\{x_d > 0\} \cap \partial B_R\right).
$$

The definition (2.3.39) of h, definition (2.3.43) of w, and (2.3.37) yield

$$
w(0,\cdot) \leq 0 \text{ on } \bar{B}_R, \quad \forall R > 0, \tag{2.3.46}
$$

and thus, $w(P) \leq 0$ if $P \in \mathscr{B}_R^0$, for $R > 0$. If $P \in \mathscr{B}_R^1$, then $|x| = R$ and we see that (2.3.38), (2.3.39), and (2.3.43) give

$$
\begin{aligned}
w(P) &= \tilde{u}(P) - \delta m h(P) \\
&\leq m - \delta m(1+R^2) \\
&= m(1-\delta(1+R^2)).
\end{aligned}
$$

But $1-\delta(1+R^2)\leq 0$ provided $R\geq R_0(\delta):=(\delta^{-1}-1)^{1/2}>0$ and so $w(P)\leq 0$ for all $R\geq R_0(\delta)$. This completes the proof of Claim 2.3.5. □

By (2.3.45), we see that

$$w=\tilde{u}-\delta mh\leq 0 \quad \text{on } \bar{\mathbb{H}}_T,$$

for all $\delta\in(0,1)$ and thus, letting $\delta\downarrow 0$, we obtain (2.3.34). □

Lemma 2.3.4 immediately leads to the following comparison principle.

Corollary 2.3.6 (Comparison principle)**.** *Assume that the coefficients of L in* (2.1.3) *obey the hypotheses of Lemma 2.3.4. If $u,v\in C^{1,2}(\mathbb{H}_T)\cap C(\bar{\mathbb{H}}_T)$ obey* (2.3.30)*,* (2.3.31)*, and*

$$Lu\leq Lv \quad \text{on } (0,T)\times\mathbb{H}, \tag{2.3.47}$$

$$u(0,\cdot)\leq v(0,\cdot) \quad \text{on } \bar{\mathbb{H}}, \tag{2.3.48}$$

then

$$u\leq v \quad \text{on } [0,T]\times\bar{\mathbb{H}}. \tag{2.3.49}$$

Note that if (2.3.47) and (2.3.48) are strengthened to

$$|Lu|\leq Lv \quad \text{on } (0,T)\times\mathbb{H} \quad \text{and} \quad |u(0,\cdot)|\leq v(0,\cdot) \quad \text{on } \bar{\mathbb{H}}, \tag{2.3.50}$$

then Corollary 2.3.6 yields

$$|u|\leq v \quad \text{on } [0,T]\times\bar{\mathbb{H}}. \tag{2.3.51}$$

We can now turn our attention to the

Proposition 2.3.7 (Application of the maximum principle)**.** *Assume that the coefficients of L in* (2.1.3) *obey the hypotheses of Lemma 2.3.4, except that* (2.3.29) *is replaced by the stronger condition*

$$\sum_{i,j=1}^{d} x_d|a_{ij}(t,x)|+|x\cdot b(t,x)|\leq K(1+|x|^2), \quad \forall(t,x)\in\bar{\mathbb{H}}_T. \tag{2.3.52}$$

Suppose that $u\in C^{1,2}(\mathbb{H}_T)\cap C(\bar{\mathbb{H}}_T)$ solves (2.1.2) *and obeys* (2.3.30) *and* (2.3.31)*.*

(a) If $f \in C(\bar{\mathbb{H}}_T)$ and $g \in C(\bar{\mathbb{H}})$, then

$$\|u\|_{C(\bar{\mathbb{H}}_T)} \leq e^{KT} \left(T\|f\|_{C(\bar{\mathbb{H}}_T)} + \|g\|_{C(\bar{\mathbb{H}})} \right). \tag{2.3.53}$$

(b) If $q > 0$, $f \in \mathscr{C}^0_q(\bar{\mathbb{H}}_T)$, and $g \in \mathscr{C}^0_q(\bar{\mathbb{H}})$, then

$$\|u\|_{\mathscr{C}^0_q(\bar{\mathbb{H}}_T)} \leq e^{(1+q(q+4)K)T} \left(\|f\|_{\mathscr{C}^0_q(\bar{\mathbb{H}}_T)} + \|g\|_{\mathscr{C}^0_q(\bar{\mathbb{H}})} \right). \tag{2.3.54}$$

Proof. To obtain (2.3.53) and (2.3.54), we make specific choices of the function v in Corollary 2.3.6. To establish (2.3.53), we choose

$$v_1(t,x) := e^{Kt} \left(t\|f\|_{C(\bar{\mathbb{H}}_T)} + \|g\|_{C(\bar{\mathbb{H}})} \right), \quad \forall (t,x) \in \bar{\mathbb{H}}_T,$$

Direct calculation gives

$$\begin{aligned} Lv_1 &= (-c + K)v_1 + e^{Kt}\|f\|_{C(\bar{\mathbb{H}}_T)} \\ &\geq \|f\|_{C(\bar{\mathbb{H}}_T)} \quad \text{on } (0,T) \times \mathbb{H} \quad \text{(by (2.2.7))}. \end{aligned}$$

Therefore, since $Lu = f$ on $(0,T) \times \mathbb{H}$ by (2.1.2),

$$|Lu| \leq Lv_1 \quad \text{on } (0,T) \times \mathbb{H},$$

and so v_1 satisfies conditions (2.3.50). Thus, by (2.3.51), we obtain (2.3.53).

Next, we prove (2.3.54). For this purpose, we choose

$$v_2(t,x) := e^{\lambda t} \frac{\left(\|f\|_{\mathscr{C}^0_q(\bar{\mathbb{H}}_T)} + \|g\|_{\mathscr{C}^0_q(\bar{\mathbb{H}})} \right)}{(1+|x|^2)^{q/2}}, \quad \forall (t,x) \in \bar{\mathbb{H}}_T, \tag{2.3.55}$$

where $\lambda > 0$ will be suitably chosen below. First, we verify that v_2 satisfies the first inequality in (2.3.50). Direct calculation gives

$$\begin{aligned} Lv_2 = v_2 \times & \\ & \left[-c(t,x) + \lambda + q \sum_{i=1}^d \frac{b_i(t,x)x_i}{1+|x|^2} - q(q+2) \sum_{i,j=1}^d \frac{a_{ij}(t,x)x_i x_j x_d}{(1+|x|^2)^2} + q \sum_{i=1}^d \frac{a_{ii}(t,x)x_d}{1+|x|^2} \right]. \end{aligned}$$

Conditions (2.3.52) and (2.2.7), imply that

$$Lv_2 \geq v_2 \left(K + \lambda - qK - q(q+2)K - qK \right).$$

By choosing

$$\lambda = 1 + q(q+4)K > 0,$$

we obtain

$$Lv_2 \geq v_2 \geq \frac{\|f\|_{\mathscr{C}^0_q(\bar{\mathbb{H}}_T)}}{(1+|x|^2)^{q/2}} \quad \text{on } (0,T)\times\mathbb{H}.$$

By the definition (2.2.4) of the norm $\|\cdot\|_{\mathscr{C}^0_q(\bar{\mathbb{H}}_T)}$, we have

$$\left(1+|x|^2\right)^{q/2}|f(t,x)| \leq \|f\|_{\mathscr{C}^0_q(\bar{\mathbb{H}}_T)}, \quad \forall (t,x)\in[0,T]\times\bar{\mathbb{H}},$$

and so, using $Lu = f$ on $\mathbb{H}_T$ by (2.1.2), we obtain the first inequality in (2.3.50), that is,

$$|Lu| \leq Lv_2 \quad \text{on } (0,T)\times\mathbb{H}. \tag{2.3.56}$$

Similarly, by the definition (2.2.3) of the norm $\|\cdot\|_{\mathscr{C}^0_q(\bar{\mathbb{H}})}$, we have

$$\left(1+|x|^2\right)^{q/2}|g(x)| \leq \|g\|_{\mathscr{C}^0_q(\bar{\mathbb{H}})}, \quad \forall x\in\bar{\mathbb{H}}.$$

Since $u(0,\cdot) = g$ on $\bar{\mathbb{H}}$, it is immediate that

$$|u(0,\cdot)| \leq v_2(0,\cdot) \quad \text{on } \bar{\mathbb{H}}. \tag{2.3.57}$$

Therefore, by (2.3.56) and (2.3.57), v_2 obeys conditions (2.3.50), and so we obtain (2.3.54) from the definition (2.3.55) of v_2. □

2.3.3 Local a priori boundary estimates

We have the following analogue of [51, Theorem 8.11.1].

Theorem 2.3.8 (A priori boundary estimates)**.** *There is constant $R^* = R^*(d,\alpha,K,\delta,\nu)$, such that for any $0 < R \leq R^*$, we can find a positive constant $C = C(d,\alpha,K,\delta,\nu,R)$, such that for any $x^0\in\partial\mathbb{H}$, $T\in(0,R]$ and $u\in C^{2+\alpha}_s(\bar{Q}_{3R/2,T}(x^0))$ that satisfies*

$$\begin{cases} Lu = f & \text{on } Q_{3R/2,T}(x^0), \\ u(0,\cdot) = g & \text{on } \bar{B}_{3R/2}(x^0), \end{cases} \tag{2.3.58}$$

the following estimate holds

$$\begin{aligned} \|u\|_{C^{2+\alpha}_s(\bar{Q}_{R,T}(x^0))} \leq C \Big(& \|f\|_{C^\alpha_s(\bar{Q}_{3R/2,T}(x^0))} \\ & + \|g\|_{C^{2+\alpha}_s(\bar{B}_{3R/2}(x^0))} + \|u\|_{C(\bar{Q}_{3R/2,T}(x^0))}\Big). \end{aligned} \tag{2.3.59}$$

Proof. The proof is a blend of the localizing technique used in [51, Theorem 8.11.1] and the method of freezing the coefficients. Fix $R > 0$ and $T \in (0, R]$. Let $\varphi : \mathbb{R} \to [0,1]$ be a smooth function such that $\varphi(t) = 0$ for $t < 0$, and $\varphi(t) = 1$ for $t > 1$. Let

$$R_n = R \sum_{k=0}^{n} \frac{1}{3^{-k}},$$

and consider the sequence of smooth cutoff functions $\{\varphi_n\}_{n \geq 1} \subset C^\infty(\bar{\mathbb{R}}^d)$ defined by

$$\varphi_n(x) := \varphi\left(\frac{R_{n+1} - |x|}{R_{n+1} - R_n}\right), \quad \forall x \in \bar{\mathbb{H}},$$

so that $0 \leq \varphi_n \leq 1$ and $\varphi_n|_{B_{R_n}} \equiv 1$ and $\varphi_n|_{B^c_{R_{n+1}}} \equiv 0$, where $B^c_{R_{n+1}}$ denotes the complement of $B_{R_{n+1}}$ in $\mathbb{R}^d$. Also, by direct calculation, we can find a positive constant c, independent of n and R, such that

$$\|\varphi_n\|_{C^\alpha_s(\bar{\mathbb{H}})}, \|(\varphi_n)_{x_i}\|_{C^\alpha_s(\bar{\mathbb{H}})}, \|x_d(\varphi_n)_{x_ix_j}\|_{C^\alpha_s(\bar{\mathbb{H}})}, \|(\varphi_n)_{x_ix_j}\|_{C^\alpha_s(\bar{\mathbb{H}})} \leq c3^{3n}R^{-3}. \tag{2.3.60}$$

We denote $r := 3^{-3} < 1$ and set

$$\alpha_n := \|u\varphi_n\|_{C^{2+\alpha}_s(\bar{\mathbb{H}}_T)}. \tag{2.3.61}$$

We denote by L_0 the operator with constant coefficients obtained by freezing the coefficients of L at $(0, x^0)$. Proposition A.1.1 shows there exists a positive constant C, depending only on K, δ and ν, such that

$$\alpha_n = \|u\varphi_n\|_{C^{2+\alpha}_s(\bar{\mathbb{H}}_T)} \leq C\left(\|L_0(u\varphi_n)\|_{C^\alpha_s(\bar{\mathbb{H}}_T)} + \|g\varphi_n\|_{C^{2+\alpha}_s(\bar{\mathbb{H}})}\right), \tag{2.3.62}$$

and so

$$\alpha_n \leq C\left(\|L(u\varphi_n)\|_{C^\alpha_s(\bar{\mathbb{H}}_T)} + \|(L - L_0)(u\varphi_n)\|_{C^\alpha_s(\bar{\mathbb{H}}_T)} + \|g\varphi_n\|_{C^{2+\alpha}_s(\bar{\mathbb{H}})}\right). \tag{2.3.63}$$

We have $L(u\varphi_n) = \varphi_n Lu - [L, \varphi_n]u$, where, by direct calculation,

$$\begin{aligned}
[L, \varphi_n]\, u = &\sum_{i,j=1}^{d} 2x_d a_{ij}(t,x) u_{x_i} (\varphi_n)_{x_j} \\
&+ \sum_{i=1}^{d} b_i(t,x) u (\varphi_n)_{x_i} + \sum_{i,j=1}^{d} x_d a_{ij}(t,x) u (\varphi_n)_{x_ix_j}.
\end{aligned} \tag{2.3.64}$$

By the analogue of the [41, Inequality (4.7)] for standard Hölder norms, we have

$$\|\varphi_n Lu\|_{C^\alpha_s(\bar{\mathbb{H}}_T)} \leq c\|Lu\|_{C^\alpha_s(\bar{Q}_{R_{n+1},T})} \|\varphi_n\|_{C^\alpha_s(\bar{\mathbb{H}})},$$

and by (2.3.60), there is a positive constant c such that

$$\|\varphi_n Lu\|_{C^\alpha_s(\bar{\mathbb{H}}_T)} \leq cr^{-n}R^{-3}\|f\|_{C^\alpha_s(\bar{Q}_{3R/2,T})}. \tag{2.3.65}$$

From properties (2.2.10) and (2.2.11) of the coefficients a_{ij}, b_i and c on $\bar{\mathbb{H}}_{2,T}$, we can find a positive constant C, depending only on K and d, such that

$$\|[L,\varphi_n]u\|_{C^\alpha_s(\bar{\mathbb{H}}_T)} \leq Cr^{-n}R^{-3}\left(\|x_d(u\varphi_{n+1})_{x_i}\|_{C^\alpha_s(\bar{\mathbb{H}}_T)} + \|u\varphi_{n+1}\|_{C^\alpha_s(\bar{\mathbb{H}}_T)}\right). \tag{2.3.66}$$

The interpolation inequality (2.3.8) in Lemma 2.3.2 gives us, for any $\varepsilon \in (0,1)$,

$$\begin{aligned}\|x_d(u\varphi_{n+1})_{x_i}\|_{C^\alpha_s(\bar{\mathbb{H}}_T)} &+ \|u\varphi_{n+1}\|_{C^\alpha_s(\bar{\mathbb{H}}_T)} \\ &\leq \varepsilon\|u\varphi_{n+1}\|_{C^{2+\alpha}_s(\bar{\mathbb{H}}_T)} + C\varepsilon^{-m}\|u\varphi_{n+1}\|_{C(\bar{\mathbb{H}}_T)}.\end{aligned} \tag{2.3.67}$$

Hence, the preceding inequality together with (2.3.65) and (2.3.66), give us

$$\begin{aligned}\|L(u\varphi_n)\|_{C^\alpha_s(\bar{\mathbb{H}}_T)} \leq Cr^{-n}R^{-3}\Big(&\|f\|_{C^\alpha_s(\bar{Q}_{3R/2,T})} + \varepsilon\|u\varphi_{n+1}\|_{C^{2+\alpha}_s(\bar{\mathbb{H}}_T)} \\ &+ \varepsilon^{-m}\|u\varphi_{n+1}\|_{C(\bar{\mathbb{H}}_T)}\Big).\end{aligned} \tag{2.3.68}$$

Next, we estimate the term $(L-L_0)(u\varphi_n)$ in (2.3.63), that is,

$$\begin{aligned}-(L-L_0)(u\varphi_n) = &\sum_{i,j=1}^d x_d\left(a_{ij}(t,x) - a_{ij}(0,x^0)\right)(u\varphi_n)_{x_ix_j} \\ &+ \sum_{i=1}^d \left(b_i(t,x) - b_i(0,x^0)\right)(u\varphi_n)_{x_i} \\ &+ \left(c(t,x) - c(0,x^0)\right)(u\varphi_n).\end{aligned} \tag{2.3.69}$$

We have:

Claim 2.3.9. *There is a constant $C = C(K, R^*, d, \alpha)$ such that, for any $\varepsilon \in (0,1)$, we have*

$$\begin{aligned}\|(L-L_0)(u\varphi_n)\|_{C^\alpha_s(\bar{\mathbb{H}}_T)} \leq C&\left(R^{\alpha/2} + r^{-n}R^{-3}\varepsilon\right)\|u\varphi_{n+1}\|_{C^{2+\alpha}_s(\bar{\mathbb{H}}_T)} \\ &+ Cr^{-n}R^{-3}\varepsilon^{-m}\|u\varphi_{n+1}\|_{C(\bar{\mathbb{H}}_T)}.\end{aligned} \tag{2.3.70}$$

where m is the constant appearing in Lemma 2.3.2.

Proof of Claim 2.3.9. From the Hölder continuity (2.2.11) and boundedness (2.2.10) of the coefficients a_{ij} on $\bar{\mathbb{H}}_{2,T}$, we can find a positive constant C, depending only on K

and d, such that

$$\begin{aligned}\|x_d\left(a_{ij}(t,x)-a_{ij}(0,x^0)\right)(u\varphi_n)_{x_ix_j}\|_{C^\alpha_s(\bar{\mathbb{H}}_T)}&\\ \le CR^{\alpha/2}\|x_d(u\varphi_n)_{x_ix_j}\|_{C^\alpha_s(\bar{\mathbb{H}}_T)}+C\|x_d(u\varphi_n)_{x_ix_j}\|_{C(\bar{\mathbb{H}}_T)}.&\end{aligned} \tag{2.3.71}$$

Using the following calculation in the preceding inequality

$$\begin{aligned}\|x_d(u\varphi_n)_{x_ix_j}\|_{C^\alpha_s(\bar{\mathbb{H}}_T)} &\le \|x_d u_{x_ix_j}\varphi_n\|_{C^\alpha_s(\bar{\mathbb{H}}_T)}\\ &\quad+\|x_d u_{x_i}(\varphi_n)_{x_j}\|_{C^\alpha_s(\bar{\mathbb{H}}_T)}+\|x_d u(\varphi_n)_{x_ix_j}\|_{C^\alpha_s(\bar{\mathbb{H}}_T)}\\ &\le [x_d(u\varphi_{n+1})_{x_ix_j}]_{C^\alpha_s(\bar{\mathbb{H}}_T)}+cr^{-n}R^{-3}\left(\|x_d(u\varphi_{n+1})_{x_ix_j}\|_{C(\bar{\mathbb{H}}_T)}\right.\\ &\quad\left.+\|x_d(u\varphi_{n+1})_{x_i}\|_{C^\alpha_s(\bar{\mathbb{H}}_T)}+\|x_d u\varphi_{n+1}\|_{C^\alpha_s(\bar{\mathbb{H}}_T)}\right),\end{aligned}$$

together with the interpolation inequality (2.3.9) in Lemma 2.3.2 applied to $u\varphi_{n+1}$,

$$\begin{aligned}&\|x_d(u\varphi_{n+1})_{x_ix_j}\|_{C(\bar{\mathbb{H}}_T)}+\|x_d(u\varphi_{n+1})_{x_i}\|_{C^\alpha_s(\bar{\mathbb{H}}_T)}+\|u\varphi_{n+1}\|_{C^\alpha_s(\bar{\mathbb{H}}_T)}\\ &\qquad\le \varepsilon\|u\varphi_{n+1}\|_{C^{2+\alpha}_s(\bar{\mathbb{H}}_T)}+C\varepsilon^{-m}\|u\varphi_{n+1}\|_{C(\bar{\mathbb{H}}_T)},\end{aligned}$$

we obtain in (2.3.71)

$$\begin{aligned}&\|x_d\left(a_{ij}(t,x)-a_{ij}(0,x^0)\right)(u\varphi_n)_{x_ix_j}\|_{C^\alpha_s(\bar{\mathbb{H}}_T)}\\ &\quad\le CR^{\alpha/2}[x_d(u\varphi_{n+1})]_{C^\alpha_s(\bar{\mathbb{H}}_T)}+Cr^{-n}R^{-3}\varepsilon\|u\varphi_{n+1}\|_{C^{2+\alpha}_s(\bar{\mathbb{H}}_T)}\\ &\qquad+Cr^{-n}R^{-3}\varepsilon^{-m}\|u\varphi_{n+1}\|_{C(\bar{\mathbb{H}}_T)}\\ &\quad\le C\left(R^{\alpha/2}+r^{-n}R^{-3}\varepsilon\right)\|u\varphi_{n+1}\|_{C^{2+\alpha}_s(\bar{\mathbb{H}}_T)}+Cr^{-n}R^{-3}\varepsilon^{-m}\|u\varphi_{n+1}\|_{C(\bar{\mathbb{H}}_T)}.\end{aligned}$$

A similar argument gives us

$$\begin{aligned}&\|\left(b_i(t,x)-b_i(0,x^0)\right)(u\varphi_n)_{x_i}\|_{C^\alpha_s(\bar{\mathbb{H}}_T)}+\|\left(c(t,x)-c(0,x^0)\right)(u\varphi_n)\|_{C^\alpha_s(\bar{\mathbb{H}}_T)}\\ &\qquad\le Cr^{-n}R^{-3}\varepsilon\|u\varphi_{n+1}\|_{C^{2+\alpha}_s(\bar{\mathbb{H}}_T)}+Cr^{-n}R^{-3}\varepsilon^{-m}\|u\varphi_{n+1}\|_{C(\bar{\mathbb{H}}_T)},\end{aligned}$$

and so, using the preceding inequalities in (2.3.69), we obtain the estimate (2.3.70). □

Combining (2.3.68), (2.3.70) and (2.3.63), we obtain

$$\begin{aligned}\alpha_n &\le Cr^{-n}R^{-3}\left(\|f\|_{C^\alpha_s(\bar{Q}_{3R/2,T})}+\|g\|_{C^{2+\alpha}_s(\bar{B}_{3R/2})}\right)\\ &\quad+C\left(R^{\alpha/2}+r^{-n}R^{-3}\varepsilon\right)\alpha_{n+1}+Cr^{-n}R^{-3}\varepsilon^{-m}\|u\|_{C(\bar{Q}_{3R/2,T})}.\end{aligned} \tag{2.3.72}$$

We multiply the inequality (2.3.72) by δ^n, where $\delta>0$ is chosen such that

$$r^{-(m+1)}\delta \le 1/2. \tag{2.3.73}$$

Next, we choose $R^* > 0$ such that $CR^{*\alpha/2} = \delta/2$. For $R \in (0, R^*]$, we choose $\varepsilon = \varepsilon(n, R) \in (0, 1)$ such that $Cr^{-n}R^{-3}\varepsilon = \delta/2$. With this choice of δ, R^* and ε, inequality (2.3.72) yields, for all $0 < R \leq R^*$,

$$\begin{aligned}\delta^n \alpha_n \leq CR^{-3}(r^{-1}\delta)^n &\left(\|f\|_{C^\alpha_s(\bar{Q}_{3R/2,T})} + \|g\|_{C^{2+\alpha}_s(\bar{B}_{3R/2})}\right)\\ &+ \delta^{n+1}\alpha_{n+1} + (2C)^{m+1}R^{-3(m+1)}\delta^{-m}(r^{-(m+1)}\delta)^n\|u\|_{C(\bar{B}_{3R/2,T})}.\end{aligned}$$

By (2.3.73), we also have $r^{-1}\delta \leq 1/2$. Then, by choosing

$$C_1 := \max\left\{CR^{-3}, (2C)^{m+1}R^{-3(m+1)}\delta^{-m}\right\},$$

we obtain

$$\begin{aligned}\delta^n \alpha_n \leq C_1 \frac{1}{2^n} &\left(\|f\|_{C^\alpha_s(\bar{Q}_{3R/2,T})} + \|g\|_{C^{2+\alpha}_s(\bar{B}_{3R/2})}\right)\\ &+ \delta^{n+1}\alpha_{n+1} + C_1\frac{1}{2^n}\|u\|_{C(\bar{Q}_{3R/2,T})}.\end{aligned} \tag{2.3.74}$$

Summing inequality (2.3.74) yields

$$\begin{aligned}\sum_{n=0}^\infty \delta^n \alpha_n \leq C_1 &\left(\|f\|_{C^\alpha_s(\bar{Q}_{3R/2,T})} + \|g\|_{C^{2+\alpha}_s(\bar{B}_{3R/2})}\right)\sum_{n=0}^\infty \frac{1}{2^n}\\ &+ \sum_{n=0}^\infty \delta^{n+1}\alpha_{n+1} + C_1\|u\|_{C(\bar{Q}_{3R/2,T})}\sum_{n=0}^\infty \frac{1}{2^n}.\end{aligned}$$

The sum $\sum_{n=0}^\infty \delta^n\alpha_n$ is well-defined because we assumed $u \in C^{2+\alpha}_s(\bar{Q}_{3R/2,T})$, for all $R \in (0, R^*]$ and $T \in (0, R]$, while $\delta \in (0, 1)$. By subtracting the term $\sum_{n=1}^\infty \delta^n\alpha_n$ from both sides of the preceding inequality, we obtain the desired inequality (2.3.59). □

2.3.4 Local a priori interior estimates

In order to establish the local interior estimates, we need to track the dependency of the constant N appearing in [51, Lemma 9.2.1 & Theorem 9.2.2] on the constant of uniform ellipticity and on the supremum and Hölder norms of the coefficients. Lemma 2.3.11 and Proposition 2.3.13 apply to a parabolic operator

$$-\bar{L}u := -u_t + \sum_{i,j=1}^d \bar{a}_{ij}u_{x_ix_j} + \sum_{i=1}^d \bar{b}_i u_{x_i} + \bar{c}u, \tag{2.3.75}$$

whose coefficients obey

Hypothesis 2.3.10. There are positive constants δ_1, K_1 and λ_1 such that

1. $(\bar{a}_{ij}(t,x))$ is a symmetric, positive definite matrix, for all $t \in [0,T]$ and $x \in \mathbb{R}^d$.

2. The diffusion matrix $\bar{a}$ is non-degenerate

$$\sum_{i,j=1}^{d} \bar{a}_{ij}(t,x)\xi_i\xi_j \geq \delta_1|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, t \in [0,T], x \in \mathbb{R}^d. \tag{2.3.76}$$

3. The coefficients $\bar{a}_{ij}$, $\bar{b}_i$ and $\bar{c}$ are uniformly Hölder continuous on $[0,T] \times \mathbb{R}^d$,

$$\|\bar{a}_{ij}\|_{C^\alpha_\rho([0,T]\times\mathbb{R}^d)} + \|\bar{b}_i\|_{C^\alpha_\rho([0,T]\times\mathbb{R}^d)} + \|\bar{c}\|_{C^\alpha_\rho([0,T]\times\mathbb{R}^d)} \leq K_1. \tag{2.3.77}$$

4. The zeroth order coefficient, $\bar{c}$, is bounded from above,

$$\bar{c}(t,x) \leq \lambda_1 \quad \forall t \in [0,T], x \in \mathbb{R}^d. \tag{2.3.78}$$

Lemma 2.3.11 (A priori estimate for a simple parabolic operator with constant coefficients)**.** *Assume that* $(\bar{a}_{ij})$ *in* (2.3.75) *is a constant matrix obeying* (2.3.76), $\bar{b}_i = 0$, *and* $\bar{c} = 0$. *Then there are positive constants,*

$$N_1 = N_1(d,\alpha,T), \tag{2.3.79}$$

$$N_2 = N_1 \max\{1, \delta_1^{-1}\} \max\{1, K_1\}(1 + \delta_1^{-\alpha/2})(1 + K_1^{\alpha/2}), \tag{2.3.80}$$

such that, for any solution $u \in C^{2+\alpha}_\rho([0,T]\times\mathbb{R}^d)$ *to*

$$\begin{cases} \bar{L}u = f & \text{on } (0,T] \times \mathbb{R}^d, \\ u(0,\cdot) = g & \text{on } \mathbb{R}^d, \end{cases} \tag{2.3.81}$$

with $f \in C^\alpha_\rho([0,T]\times\mathbb{R}^d)$ *and* $g \in C^{2+\alpha}_\rho(\mathbb{R}^d)$, *we have*

$$\|u\|_{C^{2+\alpha}_\rho([0,T]\times\mathbb{R}^d)} \leq N_2 \left(\|f\|_{C^\alpha_\rho([0,T]\times\mathbb{R}^d)} + \|g\|_{C^{2+\alpha}_\rho(\mathbb{R}^d)} \right). \tag{2.3.82}$$

Proof. We follow the proof of [51, Lemmas 9.2.1 & 8.9.1]. Let U be an orthogonal matrix such that $A = U\mathrm{diag}(\lambda_i)U^T$, where $\lambda_i \in [\delta_1, K_1]$ are the eigenvalues of the symmetric, positive definite matrix, (a_{ij}). We denote $B = U\mathrm{diag}(\sqrt{\lambda_i})U^*$ and $v(t,x) = u(t,Bx)$, $\bar{f}(t,x) = f(t,Bx)$, and $\bar{g}(x) = g(Bx)$. Then, $v \in C^{2+\alpha}_\rho([0,T]\times\mathbb{R}^d)$ solves the inhomogeneous heat equation,

$$\begin{cases} v_t - \Delta v = \bar{f} & \text{on } (0,T] \times \mathbb{R}^d, \\ u(0,\cdot) = \bar{g} & \text{on } \mathbb{R}^d. \end{cases}$$

By applying [51, Theorem 9.2.3] to v, we obtain a constant $N_1 = N_1(d, \alpha, T)$ such that

$$\|v\|_{C^{2+\alpha}_\rho([0,T]\times\mathbb{R}^d)} \leq N_1 \left(\|\bar{f}\|_{C^{\alpha}_\rho([0,T]\times\mathbb{R}^d)} + \|\bar{g}\|_{C^{2+\alpha}_\rho(\mathbb{R}^d)}\right). \tag{2.3.83}$$

To obtain (2.3.82) from (2.3.83), we need the following

Claim 2.3.12. *There is a positive constant $C = C(d)$, such that for any $w_1 \in C^\alpha_\rho([0,T]\times\mathbb{R}^d)$ and any symmetric, positive-definite $d\times d$-matrix, M, with eigenvalues in $[\lambda_{\min}, \lambda_{\max}]$, where $\lambda_{\max} > \lambda_{\min} > 0$, we have*

$$\|w_1\|_{C^\alpha_\rho([0,T]\times\mathbb{R}^d)} \leq C(1+\lambda_{\min}^{-\alpha})\|w_2\|_{C^\alpha_\rho([0,T]\times\mathbb{R}^d)}, \tag{2.3.84}$$

$$\|w_2\|_{C^\alpha_\rho([0,T]\times\mathbb{R}^d)} \leq C(1+\lambda_{\max}^{\alpha})\|w_1\|_{C^\alpha_\rho([0,T]\times\mathbb{R}^d)}, \tag{2.3.85}$$

where $w_2(t,x) := w_1(t, Mx)$.

Proof of Claim 2.3.12. We first prove (2.3.84). Obviously, we have

$$\|w_1\|_{C([0,T]\times\mathbb{R}^d)} = \|w_2\|_{C([0,T]\times\mathbb{R}^d)}. \tag{2.3.86}$$

Next, it suffices to consider $|w_1(P^1) - w_1(P^2)|/\rho(P^1,P^2)^\alpha$, for points $P_i = (t^i, x^i) \in [0,T]\times\mathbb{R}^d$, $i = 1,2$, where only one of the coordinates differs. Notice that when $x^1 = x^2$, then

$$\frac{|w_1(P^1) - w_1(P^2)|}{\rho(P^1,P^2)^\alpha} = \frac{|w_2(P^1) - w_2(P^2)|}{\rho(P^1,P^2)^\alpha},$$

because the transformation $w_2(t,x) := w_1(t, Mx)$ acts only on the spatial variables. Therefore, we have

$$\frac{|w_1(P^1) - w_1(P^2)|}{\rho(P^1,P^2)^\alpha} \leq [w_2]_{C^\alpha_\rho([0,T]\times\mathbb{R}^d)}, \tag{2.3.87}$$

Next, we consider the case $t^1 = t^2 = t$. Then, we have by writing $w_1(t,x) = w_2(t, M^{-1}x)$,

$$\frac{|w_1(P^1) - w_1(P^2)|}{\rho(P^1,P^2)^\alpha} = \frac{|w_2(t, M^{-1}x^1) - w_2(t, M^{-1}x^2)|}{|M(M^{-1}x^1 - M^{-1}x^2)|^\alpha}$$

Using the fact that M is a symmetric, positive-definite matrix with eigenvalues in the range $[\lambda_{\min}, \lambda_{\max}]$, it follows

$$|M(M^{-1}x^1 - M^{-1}x^2)| \geq \lambda_{\min}|M^{-1}x^1 - M^{-1}x^2|, \quad \forall x^1, x^2 \in \mathbb{R}^d,$$

and so, by the preceding two inequalities, we have

$$\frac{|w_1(P^1) - w_1(P^2)|}{\rho(P^1,P^2)^\alpha} \leq \lambda_{\min}^{-\alpha}[w_2]_{C^\alpha_\rho([0,T]\times\mathbb{R}^d)}. \tag{2.3.88}$$

Combining inequalities (2.3.86), (2.3.87) and (2.3.88), we obtain (2.3.84).

To obtain (2.3.85), we apply (2.3.84) to w_2 in place of w_1. Then, the matrix M is replaced by the symmetric, positive-definite matrix M^{-1} with eigenvalues in $[\lambda_{\text{max}}^{-1}, \lambda_{\text{min}}^{-1}]$. Therefore, $\lambda_{\text{min}}^{-1}$ in (2.3.84) is replaced by λ_{max}, and thus, we obtain (2.3.85). $\square$

Notice that B is a symmetric, positive-definite matrix with eigenvalues in $[\sqrt{\delta_1}, \sqrt{K_1}]$. Since $v(t,x) = u(t, Bx)$, we may apply (2.3.84) with $w_1 = u$ and $w_2 = v$ and $M = B$ to obtain

$$\|u\|_{C_\rho^\alpha([0,T]\times\mathbb{R}^d)} \leq C(1 + \delta_1^{-\alpha/2})\|v\|_{C_\rho^\alpha([0,T]\times\mathbb{R}^d)}. \tag{2.3.89}$$

Because $v_t(t,x) = u_t(t, Bx)$, we have as above

$$\|u_t\|_{C_\rho^\alpha([0,T]\times\mathbb{R}^d)} \leq C(1 + \delta_1^{-\alpha/2})\|v_t\|_{C_\rho^\alpha([0,T]\times\mathbb{R}^d)}. \tag{2.3.90}$$

To evaluate u_{x_i}, we denote by L^i the i-th row of the matrix B^{-1}. Then, we have

$$u_{x_i} = L^i \nabla v,$$

and so,

$$\|u_{x_i}\|_{C([0,T]\times\mathbb{R}^d)} \leq \delta_1^{-1/2}\|\nabla v\|_{C([0,T]\times\mathbb{R}^d)}.$$

where we have use the fact that B^{-1} is a symmetric, positive-definite matrix and the eigenvalues of B^{-1} are in $\left[K_1^{-1/2}, \delta_1^{-1/2}\right]$. Applying inequality 2.3.84 to u_{x_i}, we obtain as above

$$\|u_{x_i}\|_{C_\rho^\alpha([0,T]\times\mathbb{R}^d)} \leq C\delta_1^{-1/2}(1 + \delta_1^{-\alpha/2})\|v_{x_i}\|_{C_\rho^\alpha([0,T]\times\mathbb{R}^d)}, \tag{2.3.91}$$

and similarly, it follows for $u_{x_i x_j}$

$$\|u_{x_i x_j}\|_{C_\rho^\alpha([0,T]\times\mathbb{R}^d)} \leq \delta_1^{-1}(1 + \delta_1^{-\alpha/2})\|v_{x_i x_j}\|_{C_\rho^\alpha([0,T]\times\mathbb{R}^d)}. \tag{2.3.92}$$

Applying (2.3.85) for $\bar{f}(t,x) = f(t, Bx)$ with $w_1 = f$ and $w_2 = \bar{f}$ and $M = B$, we have

$$\|\bar{f}\|_{C_\rho^\alpha([0,T]\times\mathbb{R}^d)} \leq (1 + K_1^{\alpha/2})\|f\|_{C_\rho^\alpha([0,T]\times\mathbb{R}^d)}, \tag{2.3.93}$$

Similarly, for $\bar{g}(x) = g(Bx)$, we obtain

$$\begin{aligned} \|\bar{g}\|_{C_\rho^\alpha(\mathbb{R}^d)} &\leq (1 + K_1^{\alpha/2})\|g\|_{C_\rho^\alpha([0,T]\times\mathbb{R}^d)}, \\ \|\bar{g}_{x_i}\|_{C_\rho^\alpha(\mathbb{R}^d)} &\leq K_1^{1/2}(1 + K_1^{\alpha/2})\|g_{x_i}\|_{C_\rho^\alpha(\mathbb{R}^d)}, \\ \|\bar{g}_{x_i x_j}\|_{C_\rho^\alpha(\mathbb{R}^d)} &\leq K_1(1 + K_1^{\alpha/2})\|g_{x_i x_j}\|_{C_\rho^\alpha(\mathbb{R}^d)}. \end{aligned} \tag{2.3.94}$$

By combining the inequalities (2.3.89), (2.3.90), (2.3.91), (2.3.92), (2.3.93) and (2.3.94) in (2.3.83), we obtain (2.3.82) . □

Proposition 2.3.13 (A priori estimate for a parabolic operator with variable coefficients)**.** *Assume Hypothesis 2.3.10. Then there are positive constants*

$$p = p(\alpha) \geq 1, \tag{2.3.95}$$

$$N_3 = N_3(d, \alpha, T), \tag{2.3.96}$$

$$N_4 = N_3 e^{\lambda_1 T}\left(1 + \delta_1^{-p} + K_1^p\right), \tag{2.3.97}$$

such that, for any solution $u \in C_\rho^{2+\alpha}([0,T] \times \mathbb{R}^d)$ *to*

$$\begin{cases} \bar{L}u = f & \text{on } (0,T] \times \mathbb{R}^d, \\ u(0,\cdot) = g & \text{on } \mathbb{R}^d, \end{cases}$$

we have

$$\|u\|_{C_\rho^{2+\alpha}([0,T]\times\mathbb{R}^d)} \leq N_4 \left(\|f\|_{C_\rho^{\alpha}([0,T]\times\mathbb{R}^d)} + \|g\|_{C_\rho^{2+\alpha}(\mathbb{R}^d)} \right). \tag{2.3.98}$$

The proof of Proposition 2.3.13 can be found in the appendix.

Proposition 2.3.14 (Local estimates for parabolic operators with variable coefficients)**.** *Assume Hypothesis 2.3.10 and that* $R > 0$*. Then there are positive constants*

$$p = p(\alpha) \geq 1, \tag{2.3.99}$$

$$N_3 = N_3(d, \alpha, T, R), \tag{2.3.100}$$

$$N_4 = N_3 e^{\lambda_1 T}\left(1 + \delta_1^{-p} + K_1^p\right), \tag{2.3.101}$$

such that for any $x^0 \in \mathbb{R}^d$ *and any solution* $u \in C_\rho^{2+\alpha}(\bar{Q}_{2R,T}(x^0))$ *to*

$$\begin{cases} \bar{L}u = f & \text{on } Q_{2R,T}(x^0), \\ u(0,\cdot) = g & \text{on } \bar{B}_{2R}(x^0), \end{cases}$$

we have

$$\begin{aligned} \|u\|_{C_\rho^{2+\alpha}(\bar{Q}_{R,T}(x^0))} \leq N_4 \Big(& \|f\|_{C_\rho^{\alpha}(\bar{Q}_{2R,T}(x^0))} + \|g\|_{C_\rho^{2+\alpha}(\bar{B}_{2R}(x^0))} \\ & + \|u\|_{C(\bar{Q}_{2R,T}(x^0))} \Big). \end{aligned} \tag{2.3.102}$$

Proof. The proof follows by the same argument as in Theorem 2.3.8 with the following modifications:

- In inequality (2.3.62), instead of applying Proposition A.1.1, we apply Proposition 2.3.13.
- We use the interpolation inequalities for classical Hölder spaces $C^{2+\alpha}_\rho$ ([51, Theorem 8.8.1]), instead of the interpolation inequalities suitable for the Hölder spaces $C^{2+\alpha}_s$ (Lemma 2.3.2).

This completes the proof. $\square$

We now consider estimates for the operator L in (2.1.3).

Proposition 2.3.15 (Interior local estimates)**.** *There is a positive constant $p = p(\alpha)$, and for any $0 < R \leq R^*$, with R^* given as in Theorem 2.3.8, there is a positive constant $C = C(d, \alpha, T, K, \delta, R^*, R)$, such that for any $x^0 \in \mathbb{H}$ satisfying $x^0_d - 2R \geq R^*/2$, and for any solution $u \in C^{2+\alpha}_\rho(\bar{Q}_{2R,T}(x^0))$ to the inhomogeneous initial value problem*

$$\begin{cases} Lu = f & \text{on } Q_{2R,T}(x^0), \\ u(0,\cdot) = g & \text{on } \bar{B}_{2R}(x^0), \end{cases}$$

we have

$$\begin{aligned} \|u\|_{C^{2+\alpha}_\rho(\bar{Q}_{R,T}(x^0))} \leq C \Big(& \|f\|_{\mathscr{C}^\alpha_p(\bar{Q}_{2R,T}(x^0))} + \|g\|_{\mathscr{C}^{2+\alpha}_p(\bar{B}_{2R}(x^0))} \\ & + \|u\|_{\mathscr{C}^0_p(\bar{Q}_{2R,T}(x^0))} \Big). \end{aligned} \tag{2.3.103}$$

Proof. From Proposition 2.3.14, the linear growth estimate (2.2.14), and the fact that the matrix $(x_d a_{ij}(t,x))$ is uniformly elliptic on $\bar{\mathbb{H}}_T \setminus \mathbb{H}_{R^*/2,T}$ by (2.2.9) and (2.2.12), we obtain

$$\begin{aligned} \|u\|_{C^{2+\alpha}_\rho(\bar{Q}_{R,T}(x^0))} \leq C_1(1+|x^0|)^p \Big(& \|f\|_{C^\alpha_\rho(\bar{Q}_{2R,T}(x^0))} + \|g\|_{C^{2+\alpha}_\rho(\bar{B}_{2R}(x^0))} \\ & + \|u\|_{C(\bar{Q}_{2R,T}(x^0))} \Big), \end{aligned} \tag{2.3.104}$$

where C_1 is a positive constant depending only on T, K, δ, R^* and R.

Claim 2.3.16. *Given a function $v \in C^{2+\alpha}_\rho(\bar{Q}_{2R,T}(x^0))$, there is a positive constant C_2, depending only in R^*, p and α, such that for all $R \in (0, R^*]$ and $x^0 \in \mathbb{H}_T$, we have*

$$(1+|x^0|)^p \|v\|_{C^\alpha_\rho(\bar{Q}_{2R,T}(x^0))} \leq C_2 \|v\|_{\mathscr{C}^\alpha_p(\bar{Q}_{2R,T}(x^0))}. \tag{2.3.105}$$

Proof of Claim 2.3.16. Recall that, by definition (2.2.5),

$$\|(1+|x|)^p v\|_{C^\alpha_\rho(\bar Q_{2R,T}(x^0))} = \|v\|_{\mathscr{C}^\alpha_p(\bar Q_{2R,T}(x^0))}.$$

We may write

$$(1+|x^0|)^p|v(t,x)| = \left(\frac{1+|x^0|}{1+|x|}\right)^p (1+|x|)^p|v(t,x)|, \quad \forall (t,x)\in \bar Q_{2R,T}(x^0).$$

We can find a constant $C_2 = C_2(R^*, p)$ such that

$$\left(\frac{1+|x^0|}{1+|x|}\right)^p \leq C_2, \quad \forall x \in \bar B_{2R}(x^0), \quad \forall 0<R<R^*, \tag{2.3.106}$$

which implies

$$(1+|x^0|)^p\|v\|_{C(\bar Q_{2R,T}(x^0))} \leq C_2\|(1+|x|)^p v\|_{C(\bar Q_{2R,T}(x^0))}. \tag{2.3.107}$$

Next, we have

$$\begin{aligned}
(1+|x^0|)^p[v]_{C^\alpha_\rho(\bar Q_{2R,T}(x^0))} &= (1+|x^0|)^p\left[\frac{1}{(1+|x|)^p}(1+|x|)^p v\right]_{C^\alpha_\rho(\bar Q_{2R,T}(x^0))}\\
&\leq (1+|x^0|)^p\left[\frac{1}{(1+|x|)^p}\right]_{C^\alpha_\rho(\bar B_{2R}(x^0))}\|(1+|x|)^p v\|_{C(\bar Q_{2R,T}(x^0))}\\
&\quad + (1+|x^0|)^p\left\|\frac{1}{(1+|x|)^p}\right\|_{C(\bar B_{2R}(x^0))}[(1+|x|)^p v]_{C^\alpha_\rho(\bar Q_{2R,T}(x^0))}.
\end{aligned}$$

As in (2.3.106), there is a (possibly larger) constant $C_2 = C_2(R^*, p, \alpha)$ such that

$$(1+|x^0|)^p\left[\frac{1}{(1+|x|)^p}\right]_{C^\alpha_\rho(\bar B_{2R}(x^0))} \leq C_2.$$

Therefore, we obtain

$$\begin{aligned}
&(1+|x^0|)^p[v]_{C^\alpha_\rho(\bar Q_{2R,T}(x^0))}\\
&\qquad \leq C_2\|(1+|x|)^p v\|_{C(\bar Q_{2R,T}(x^0))} + C_2[(1+|x|)^p v]_{C^\alpha_\rho(\bar Q_{2R,T}(x^0))}.
\end{aligned} \tag{2.3.108}$$

Combining inequalities (2.3.107) and (2.3.108) yields the desired inequality (2.3.105).

□

Claim 2.3.16 implies that

$$\begin{aligned}
(1+|x^0|)^p\|f\|_{C^\alpha_\rho(\bar Q_{2R,T}(x^0))} &\leq C_2\|f\|_{\mathscr{C}^\alpha_p(\bar Q_{2R,T}(x^0))},\\
(1+|x^0|)^p\|g\|_{C^{2+\alpha}_\rho(\bar B_{2R}(x^0))} &\leq C_2\|g\|_{\mathscr{C}^{2+\alpha}_p(\bar B_{2R}(x^0))},\\
(1+|x^0|)^p\|u\|_{C(\bar Q_{2R,T}(x^0))} &\leq C_2\|u\|_{\mathscr{C}^0_p(\bar Q_{2R,T}(x^0))}.
\end{aligned}$$

From the preceding inequalities and (2.3.104), we obtain the interior local estimate (2.3.103). □

2.3.5 Global a priori estimates and existence of solutions

The goal of this subsection is to establish Theorem 2.1.1. For this purpose, we need to first prove the analogue of Theorem 2.1.1 when the coefficients are uniformly Hölder continuous on $\mathbb{H}_T \setminus \mathbb{H}_{2,T} = (0,T) \times \mathbb{R}^{d-1} \times [2,\infty)$.

Hypothesis 2.3.17. In addition to the conditions in Assumption 2.2.2, assume that there is a positive constant K_2 such that the coefficients of L obey

$$\|x_d a_{ij}\|_{C^\alpha_\rho(\mathbb{H}_T\setminus\bar{\mathbb{H}}_{2,T})} + \|b_i\|_{C^\alpha_\rho(\mathbb{H}_T\setminus\bar{\mathbb{H}}_{2,T})} + \|c\|_{C^\alpha_\rho(\mathbb{H}_T\setminus\bar{\mathbb{H}}_{2,T})} \leq K_2. \tag{2.3.109}$$

We first derive global a priori estimates of solutions in the case of bounded coefficients.

Lemma 2.3.18 (Global estimates in the case of parabolic operators with bounded coefficients)**.** *Suppose Hypothesis 2.3.17 is satisfied. There exists a positive constant* $C = C(T,\alpha,d,K_2,\delta,\nu)$ *such that for any solution* $u \in \mathscr{C}^{2+\alpha}_{\text{loc}}(\bar{\mathbb{H}}_T)$ *to* (2.1.2)*, such that* $Lu \in \mathscr{C}^\alpha(\bar{\mathbb{H}}_T)$ *and* $u(0,\cdot) \in \mathscr{C}^{2+\alpha}(\bar{\mathbb{H}})$*, we have* $u \in \mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_T)$ *and satisfies the global estimate*

$$\|u\|_{\mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_T)} \leq C\left(\|Lu\|_{\mathscr{C}^\alpha(\bar{\mathbb{H}}_T)} + \|u(0,\cdot)\|_{\mathscr{C}^{2+\alpha}(\bar{\mathbb{H}})}\right). \tag{2.3.110}$$

Proof. It is enough to prove the statement for $T>0$ small. Let $R^*>0$ be defined as in Theorem 2.3.8 and choose $T \in (0,R^*]$. Let $\{z^k : k \geq 1\}$ be a sequence of points in $\partial\mathbb{H}$ such that

$$\mathbb{H}_{R^*/2,T} \subset \bigcup_{k\geq 1} Q_{R^*,T}(z^k), \tag{2.3.111}$$

and let $\{w^l : l \geq 1\}$ be a sequence of points in $\mathbb{H}_T \setminus \mathbb{H}_{R^*/2,T}$ such that

$$\mathbb{H}_T \setminus \mathbb{H}_{R^*/2,T} \subset \bigcup_{l\geq 1} Q_{R^*/8,T}(w^l), \tag{2.3.112}$$

and assume

$$Q_{R^*/4,T}(w^l) \cap \mathbb{H}_{R^*/4,T} = \varnothing, \quad \forall l \geq 1. \tag{2.3.113}$$

We apply the a priori boundary estimate (2.3.59) to u with $R = R^*$, $f = Lu$ and $g = u(0,\cdot)$ on $Q_{R^*,T}(z^k)$. Then, we can find a positive constant C_1, depending at most on R^*, K_2, δ, ν, such that

$$\|u\|_{C^{2+\alpha}_s(\bar{Q}_{R^*,T}(z^k))} \leq C_1 \left(\|Lu\|_{C^\alpha_s(\bar{Q}_{3R^*/2,T}(z^k))} + \|u(0,\cdot)\|_{C^{2+\alpha}_s(\bar{Q}_{3R^*/2,T}(z^k))} \right.$$
$$\left. + \|u\|_{C(\bar{Q}_{3R^*/2,T}(z^k))} \right).$$

Using definitions (2.2.5) of $\mathscr{C}^\alpha(\bar{\mathbb{H}}_T)$, and (2.2.6) of $\mathscr{C}^{2+\alpha}(\bar{\mathbb{H}})$, with $q=0$, Remark 2.2.1 and the hypotheses that $Lu \in \mathscr{C}^\alpha(\bar{\mathbb{H}}_T)$ and $u(0,\cdot) \in \mathscr{C}^{2+\alpha}(\bar{\mathbb{H}})$, we obtain

$$\|u\|_{C^{2+\alpha}_s(\bar{Q}_{R^*,T}(z^k))} \leq C_1 \left(\|Lu\|_{\mathscr{C}^\alpha(\bar{\mathbb{H}}_T)} + \|u(0,\cdot)\|_{\mathscr{C}^{2+\alpha}(\bar{\mathbb{H}})} + \|u\|_{C(\bar{\mathbb{H}}_T)} \right),$$

and inequality (2.3.53) ensures

$$\|u\|_{C^{2+\alpha}_s(\bar{Q}_{R^*,T}(z^k))} \leq C_1 \left(\|Lu\|_{\mathscr{C}^\alpha(\bar{\mathbb{H}}_T)} + \|u(0,\cdot)\|_{\mathscr{C}^{2+\alpha}(\bar{\mathbb{H}})} \right), \quad \forall k \geq 1. \tag{2.3.114}$$

From our Hypothesis 2.3.17, the coefficients $x_d a_{ij}$, b_i and c are in $C^\alpha_\rho(\mathbb{H}_T \setminus \bar{\mathbb{H}}_{2,T})$. By Assumption 2.2.2, we have that $x_d a_{ij}$, b_i and c are in $C^\alpha_s(\mathbb{H}_{2,T} \setminus \bar{\mathbb{H}}_{R^*/4,T})$. Since the metrics s and ρ are equivalent on $\mathbb{R} \times [R^*/4, 2]$, by Remark 2.2.1, there is a positive constant K_1, depending on K_2 and R^*, such that

$$\|x_d a_{ij}\|_{C^\alpha_\rho(\mathbb{H}_T \setminus \bar{\mathbb{H}}_{R^*/4,T})} + \|b_i\|_{C^\alpha_\rho(\mathbb{H}_T \setminus \bar{\mathbb{H}}_{R^*/4,T})} + \|c\|_{C^\alpha_\rho(\mathbb{H}_T \setminus \bar{\mathbb{H}}_{R^*/4,T})} \leq K_1,$$

and so the conditions of Hypothesis 2.3.10 are obeyed on $\mathbb{H}_T \setminus \bar{\mathbb{H}}_{R^*/4,T}$. This is enough to ensure we may apply Proposition 2.3.14 to u with $f = Lu$ and $g = u(0,\cdot)$ on $Q_{R^*/8,T}(w^l)$ and so there is a positive constant C_2, depending at most on R^*, K_1, δ, ν, giving

$$\|u\|_{C^{2+\alpha}_\rho(\bar{Q}_{R^*/8,T}(w^l))} \leq C_2 \left(\|Lu\|_{C^\alpha_\rho(\bar{Q}_{R^*/4,T}(w^l))} + \|u(0,\cdot)\|_{C^{2+\alpha}_\rho(\bar{Q}_{R^*/4,T}(w^l))} \right.$$
$$\left. + \|u\|_{C(\bar{Q}_{R^*/4,T}(w^l))} \right), \quad \forall l \geq 1.$$

By (2.3.113) and Remark 2.2.1, we obtain

$$\|u\|_{C^{2+\alpha}_\rho(\bar{Q}_{R^*/8,T}(w^l))} \leq C_2 \left(\|Lu\|_{\mathscr{C}^\alpha(\bar{\mathbb{H}})} + \|u(0,\cdot)\|_{\mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_T)} \right.$$
$$\left. + \|u\|_{C(\bar{Q}_{R^*/4,T}(w^l))} \right), \quad \forall l \geq 1,$$

and, by inequality (2.3.53) applied to $\|u\|_{C(\bar{Q}_{R^*/4,T}(w^l))}$, it follows

$$\|u\|_{C^{2+\alpha}_\rho(\bar{Q}_{R^*/8,T}(w^l))} \leq C_2 \left(\|Lu\|_{\mathscr{C}^\alpha(\bar{\mathbb{H}})} + \|u(0,\cdot)\|_{\mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_T)} \right), \quad \forall l \geq 1. \tag{2.3.115}$$

Combining inequalities (2.3.114) and (2.3.115) and making use of the inclusions (2.3.111) and (2.3.112), we obtain the global estimate (2.3.110). $\square$

Next, we establish the a priori global estimates in the case of coefficients with at most linear growth.

Lemma 2.3.19 (Global estimates for coefficients with linear growth)**.** *There exists a positive constant $C = C(T, \alpha, d, K, \delta, \nu)$ such that for any solution $u \in \mathscr{C}^{2+\alpha}_{\mathrm{loc}}(\bar{\mathbb{H}}_T)$ to* (2.1.2)*, such that $Lu \in \mathscr{C}^{\alpha}_p(\bar{\mathbb{H}}_T)$ and $u(0,\cdot) \in \mathscr{C}^{2+\alpha}_p(\bar{\mathbb{H}})$, we have*

$$\|u\|_{\mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_T)} \leq C\left(\|Lu\|_{\mathscr{C}^{\alpha}_p(\bar{\mathbb{H}}_T)} + \|u(0,\cdot)\|_{\mathscr{C}^{2+\alpha}_p(\bar{\mathbb{H}})}\right), \tag{2.3.116}$$

where $p = p(\alpha)$ is the constant appearing in Proposition 2.3.15.

Proof. As in the proof of Lemma 2.3.18, we may assume without loss of generality that $0 < T \leq R^*$, where $R^* > 0$ is defined as in Theorem 2.3.8. Let z^k and w^l be the sequences of points considered in the proof of Lemma 2.3.18. Then, by applying Theorem 2.3.8 to u with $f = Lu$ and $g = u(0,\cdot)$ on $\bar{Q}_{R^*,T}(z^k)$, we obtain, for all $k \geq 1$,

$$\begin{aligned}\|u\|_{C^{2+\alpha}_s(\bar{Q}_{R^*,T}(z^k))} \leq C\Big(&\|Lu\|_{C^{\alpha}_s(\bar{Q}_{3R^*/2,T}(z^k))} + \|u(0,\cdot)\|_{C^{2+\alpha}_s(\bar{B}_{3R^*/2}(z^k))} \\ &+\|u\|_{C(\bar{Q}_{3R^*/2,T}(z^k))}\Big).\end{aligned}$$

We notice that

$$\begin{aligned}\|Lu\|_{C^{\alpha}_s(\bar{Q}_{3R^*/2,T}(z^k))} &\leq C_1\|(1+|x|)^p Lu\|_{C^{\alpha}_s(\bar{Q}_{3R^*/2,T}(z^k))} \\ &= C_1\|Lu\|_{\mathscr{C}^{\alpha}_p(\bar{Q}_{3R^*/2,T}(z^k))}, \\ \|u(0,\cdot)\|_{C^{2+\alpha}_\rho(\bar{B}_{3R^*/2}(z^k))} &\leq C_1\|(1+|x|)^p u(0,\cdot)\|_{C^{2+\alpha}_s(\bar{B}_{3R^*/2}(z^k))} \\ &= C_1\|u(0,\cdot)\|_{\mathscr{C}^{2+\alpha}_p(\bar{B}_{3R^*/2}(z^k))}, \\ \|u\|_{C(\bar{Q}_{3R^*/2,T}(z^k))} &\leq C_1\|(1+|x|)^p u\|_{C(\bar{Q}_{3R^*/2,T}(z^k))} \\ &= C_1\|u\|_{\mathscr{C}^{0}_p(\bar{Q}_{3R^*/2,T}(z^k))},\end{aligned}$$

where the positive constant C_1 depends on R^* and p, but not on z^k. Therefore, we obtain, for all $k \geq 1$,

$$\|u\|_{C^{2+\alpha}_s(\bar{Q}_{R^*,T}(z^k))} \leq C_2\left(\|Lu\|_{\mathscr{C}^{\alpha}_p(\bar{\mathbb{H}}_T)} + \|u(0,\cdot)\|_{\mathscr{C}^{2+\alpha}_p(\bar{\mathbb{H}})} + \|u\|_{\mathscr{C}^{0}_p(\bar{\mathbb{H}}_T)}\right),$$

for a positive constant C_2 depending at most on R^*, K, δ, ν, α, d. Because the collection of balls $\{Q_{R^*,T}(z^k) : k \geq 1\}$ covers $\mathbb{H}_{R^*/2,T}$ and as we may apply (2.3.54) to u with

$f = Lu$ and $g = u(0,\cdot)$ with $q = p$, there is a positive constant C_3, satisfying the same dependency on constants as C_2, such that

$$\|u\|_{\mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_{R^*/2,T})} \leq C_3 \left(\|Lu\|_{\mathscr{C}^{\alpha}_p(\bar{\mathbb{H}}_T)} + \|u(0,\cdot)\|_{\mathscr{C}^{2+\alpha}_p(\bar{\mathbb{H}})} \right). \tag{2.3.117}$$

By applying Proposition 2.3.15 to u with $f = Lu$ and $g = u(0,\cdot)$ on $\bar{Q}_{R^*/8,T}(w^l)$, we obtain, for all $l \geq 1$,

$$\begin{aligned} \|u\|_{C^{2+\alpha}_\rho(\bar{Q}_{R^*/8,T}(w^l))} \leq C_4 \Big(&\|Lu\|_{\mathscr{C}^{\alpha}_p(\bar{Q}_{R^*/4,T}(w^l))} + \|u(0,\cdot)\|_{\mathscr{C}^{2+\alpha}_p(\bar{B}_{R^*/4}(w^l))} \\ &+ \|u\|_{\mathscr{C}^{0}_p(\bar{Q}_{R^*/4,T}(w^l))} \Big). \end{aligned} \tag{2.3.118}$$

Because the collection of balls $\{Q_{R^*/8,T}(w^l) : l \geq 1\}$ covers $\mathbb{H}_T \setminus \mathbb{H}_{R^*/2,T}$ and we may apply (2.3.54) to u with $f = Lu$ and $g = u(0,\cdot)$ with $q = p$, we obtain

$$\|u\|_{\mathscr{C}^{2+\alpha}(\mathbb{H}_T \setminus \bar{\mathbb{H}}_{R^*/2,T})} \leq C_5 \left(\|Lu\|_{\mathscr{C}^{\alpha}_p(\bar{\mathbb{H}}_T)} + \|u(0,\cdot)\|_{\mathscr{C}^{2+\alpha}_p(\bar{\mathbb{H}})} \right). \tag{2.3.119}$$

By combining inequalities (2.3.117) and (2.3.119), we obtain the desired estimate (2.3.116).

$\square$

Next, we prove Theorem 2.1.1 in the case of bounded coefficients.

Proposition 2.3.20 (Existence and uniqueness for bounded coefficients)**.** *Suppose Hypothesis 2.3.17 is satisfied. Let $f \in \mathscr{C}^{\alpha}(\bar{\mathbb{H}}_T)$ and $g \in \mathscr{C}^{2+\alpha}(\bar{\mathbb{H}})$. Then there exists a unique solution $u \in \mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_T)$ to* (2.1.2) *and u satisfies estimate* (2.3.110).

Proof. The proof employs the method used in proving existence of solutions to parabolic partial differential equations outlined in [51, §10.2] or [20, Theorem II.1.1]. We let $\hat{\mathscr{C}}^{2+\alpha}(\bar{\mathbb{H}}_T)$ denote the Banach space of functions $u \in \mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_T)$ such that $u(0,x) = 0$, for all $x \in \bar{\mathbb{H}}$. The spaces $\hat{C}^{2+\alpha}_s(\bar{\mathbb{H}}_T)$ and $\hat{C}^{2+\alpha}_\rho([0,T] \times \mathbb{R}^d)$ are defined similarly. Without loss of generality, we may assume $g = 0$ because $Lg \in \mathscr{C}^{\alpha}(\bar{\mathbb{H}}_T)$, when Hypothesis 2.3.17 holds, and so

$$L : \hat{\mathscr{C}}^{2+\alpha}(\bar{\mathbb{H}}_T) \to \mathscr{C}^{\alpha}(\bar{\mathbb{H}}_T)$$

is a well-defined operator. Our goal is to show that L is invertible and we accomplish this by constructing a bounded linear operator $M : \mathscr{C}^{\alpha}(\bar{\mathbb{H}}_T) \to \hat{\mathscr{C}}^{2+\alpha}(\bar{\mathbb{H}}_T)$ such that

$$\left\| LM - I_{\mathscr{C}^{\alpha}(\bar{\mathbb{H}}_T)} \right\| < 1. \tag{2.3.120}$$

For this purpose, we fix $r > 0$ and choose a sequence of points $\{x^n : n = 1, 2, \ldots\}$ such that the collection of balls $\{B_r(x^n) : n = 1, 2, \ldots\}$ covers the strip $\{x = (x', x_d) \in \mathbb{H} : 0 < x_d < r/2\}$. We may assume without loss of generality, there exists a positive constant N, depending only on the dimension d, such that at most N balls of the covering have non-empty intersection. Let $\{\varphi_n : n = 0, 1, \ldots\}$ be a partition of unity subordinate to the open cover

$$(\mathbb{H} \setminus \{0 < x_d \leq r/4\}) \cup \bigcup_{n=1}^{\infty} B_r(x^n) = \mathbb{H},$$

such that

$$\operatorname{supp} \varphi_0 \subset \mathbb{H} \setminus \{0 < x_d < r/4\} \text{ and } \operatorname{supp} \varphi_n \subset \bar{B}_r(x^n), \quad \forall n \geq 1.$$

Without loss of generality, we may choose $\{\varphi_n\}_{n \geq 0}$ such that there is a positive constant c, independent of r and n, such that

$$\|\varphi_n\|_{C^{2+\alpha}_\rho(\mathbb{R}^d)} \leq c r^{-3}, \quad \forall n \geq 0. \tag{2.3.121}$$

We choose a sequence of non-negative, smooth cutoff functions, $\{\psi_n\}_{n \geq 0} \subset C^\infty(\bar{\mathbb{H}})$ such that $0 \leq \psi_n \leq 1$ on $\mathbb{H}$, for all $n \geq 0$, and

$$\psi_0(x) = \begin{cases} 0, & \text{for } 0 < x_d < r/8, \\ 1, & \text{for } x_d > r/4, \end{cases}$$

while for all $n \geq 1$,

$$\psi_n(x) = \begin{cases} 1, & \text{for } 0 < x_d < 1/2, \\ 0, & \text{for } x_d > 1. \end{cases}$$

Then, we notice that ψ_0 satisfies (2.3.121). For r small enough, we have

$$\psi_n \varphi_n = \varphi_n, \text{ for all } n \geq 0. \tag{2.3.122}$$

For $n = 0$, let L_0 be a uniformly elliptic parabolic operator on $\mathbb{R}^d$ with bounded, $C^\alpha_\rho(\mathbb{H}_T)$-Hölder continuous coefficients, such that L_0 agrees with L on the support of ψ_0. Define the operator

$$M_0 : C^\alpha_\rho([0, T] \times \mathbb{R}^d) \to \hat{C}^{2+\alpha}_\rho([0, T] \times \mathbb{R}^d),$$

be the inverse of L_0, as given by [51, Theorem 8.9.2]. For $n = 1, 2, \ldots$, let L_n be the degenerate-parabolic operator obtained by freezing the variable coefficients $a_{ij}(t, x)$, $b_i(t, x)$ and $c(t, x)$ at $(0, x^n)$. Define the operator

$$M_n : C^\alpha_s(\bar{\mathbb{H}}_T) \to \hat{C}^{2+\alpha}_s(\bar{\mathbb{H}}_T),$$

be the inverse of L_n, as given by Proposition A.1.1. Define the operator

$$M : \mathscr{C}^\alpha(\bar{\mathbb{H}}_T) \to \hat{\mathscr{C}}^{2+\alpha}(\bar{\mathbb{H}}_T)$$

by setting

$$Mf := \sum_{n=0}^\infty \varphi_n M_n \psi_n f, \quad \text{for } f \in \mathscr{C}^\alpha(\bar{\mathbb{H}}_T).$$

Our goal is to show that (2.3.120) holds, for small enough r and T. We have

$$\begin{aligned} LMf - f &= \sum_{n=0}^\infty L\varphi_n M_n \psi_n f - f \\ &= \sum_{n=0}^\infty \varphi_n L M_n \psi_n f + \sum_{n=0}^\infty [L, \varphi_n] M_n \psi_n f - f, \end{aligned}$$

where $[L, \varphi_n]$ is given by (2.3.64). Denoting

$$u_n := M_n \psi_n f, \quad \text{for } n = 0, 1, 2, \ldots, \tag{2.3.123}$$

we have

$$\begin{aligned} LM_n \psi_n f &= (L - L_n) u_n + L_n M_n \psi_n f \\ &= (L - L_n) u_n + \psi_n f, \end{aligned}$$

since $L_n M_n = I$, for all $n \geq 0$. This implies, by the identities (2.3.122) and $\sum_{n=0}^\infty \varphi_n \psi_n f = f$, that

$$LMf - f = \sum_{n=0}^\infty \varphi_n (L - L_n) u_n + \sum_{n=0}^\infty [L, \varphi_n] u_n. \tag{2.3.124}$$

First, we estimate the terms in the preceding equality indexed by $n = 0$. Because $L_0 = L$ on the support of ψ_0, obviously we have $\psi_0 (L - L_0) u_0 = 0$. Next, using the identity (2.3.64), there is a positive constant C, depending only on K_2 in (2.3.109), such that

$$\begin{aligned} \|[L, \varphi_0]\, u_0\|_{C^\alpha_\rho([0,T]\times\mathbb{R}^d)} &\leq C \|u_0\|_{C^{1+\alpha}_\rho([0,T]\times\mathbb{R}^d)} \|\psi_0\|_{C^{2+\alpha}_\rho([0,T]\times\mathbb{R}^d)} \\ &\leq C r^{-3} \|u_0\|_{C^{1+\alpha}_\rho([0,T]\times\mathbb{R}^d)} \quad \text{(by (2.3.121))}. \end{aligned}$$

From the interpolation inequalities for standard Hölder spaces [51, Theorem 8.8.1], there is a positive constant m such that, for all $\varepsilon > 0$, we have

$$\|[L, \varphi_0]\, u_0\|_{C^\alpha_\rho([0,T]\times\mathbb{R}^d)} \leq Cr^{-3} \left(\varepsilon \|u_0\|_{C^{1+\alpha}_\rho([0,T]\times\mathbb{R}^d)} + \varepsilon^{-m} \|u_0\|_{C([0,T]\times\mathbb{R}^d)}\right). \tag{2.3.125}$$

By [51, Theorem 8.9.2], the identity (2.3.122), and the definition (2.3.123) of u_0, we have

$$\begin{aligned}\|u_0\|_{C^{1+\alpha}_\rho([0,T]\times\mathbb{R}^d)} &\leq C_1(r) \|\psi_0 f\|_{C^\alpha_\rho([0,T]\times\mathbb{R}^d)}\\ &\leq C_1(r) \|f\|_{\mathscr{C}^\alpha(\mathbb{H}_T)},\end{aligned}$$

for some positive constant $C_1(r)$. From [51, Corollary 8.1.5], there is a constant C, depending only on K_2, T and d, such that

$$\|u_0\|_{C([0,T]\times\mathbb{R}^d)} \leq CT \|f\|_{C([0,T]\times\mathbb{R}^d)}.$$

Therefore, we obtain in (2.3.125), for possibly a different constant $C_1(r)$,

$$\|[L, \varphi_0]\, u_0\|_{C^\alpha_\rho([0,T]\times\mathbb{R}^d)} \leq C_1(r) \left(\varepsilon \|f\|_{\mathscr{C}^\alpha(\bar{\mathbb{H}}_T)} + \varepsilon^{-m} T \|f\|_{C(\bar{\mathbb{H}}_T)}\right). \tag{2.3.126}$$

Next, we estimate the terms in (2.3.124) indexed by $n \geq 1$. We closely follow the argument used to prove Theorem 2.3.8. First, we have

$$\begin{aligned}\|\varphi_n (L - L_n) u_n\|_{C^\alpha_s(\bar{\mathbb{H}}_T)} \leq{}& [\varphi_n]_{C^\alpha_s(\bar{\mathbb{H}}_T)} \|(L - L_n) u_n\|_{C([0,T]\times \text{ supp } \varphi_n)}\\ &+ \|(L - L_n) u_n\|_{C^\alpha_s([0,T]\times \text{ supp } \varphi_n)}.\end{aligned} \tag{2.3.127}$$

Using (2.3.121) and Lemma 2.3.2, there are positive constants m and $C_1(r)$ such that

$$[\varphi_n]_{C^\alpha_s(\bar{\mathbb{H}}_T)} \|(L - L_n) u_n\|_{C([0,T]\times \text{ supp } \varphi_n)} \leq C_1(r) \left(\varepsilon \|u_n\|_{C^{2+\alpha}_s(\bar{\mathbb{H}}_T)} + \varepsilon^{-m} \|u_n\|_{C(\bar{\mathbb{H}}_T)}\right).$$

By Proposition A.1.1, (2.3.53) and the preceding inequality, we obtain

$$[\varphi_n]_{C^\alpha_s(\bar{\mathbb{H}}_T)} \|(L - L_n) u_n\|_{C([0,T]\times \text{ supp } \varphi_n)} \leq C_1(r) \left(\varepsilon \|\psi_n f\|_{C^\alpha_s(\bar{\mathbb{H}}_T)} + \varepsilon^{-m} T \|\psi_n f\|_{C(\bar{\mathbb{H}}_T)}\right),$$

and thus,

$$\begin{aligned}&[\varphi_n]_{C^\alpha_s(\bar{\mathbb{H}}_T)} \|(L - L_n) u_n\|_{C([0,T]\times \text{ supp } \varphi_n)}\\ &\qquad \leq C_1(r) \left(\varepsilon \|f\|_{\mathscr{C}^\alpha(\bar{\mathbb{H}}_T)} + \varepsilon^{-m} T \|f\|_{C(\bar{\mathbb{H}}_T)}\right).\end{aligned} \tag{2.3.128}$$

By applying the same argument used to prove Claim 2.3.9, we find that there are positive constants C, independent of r, and $C_1(r)$, such that

$$\|(L-L_n)u_n\|_{C^\alpha_s([0,T]\times \text{ supp } \varphi_n)} \leq Cr^{\alpha/2}\|u_n\|_{C^\alpha_s(\bar{\mathbb{H}}_T)} + C_1(r)\|u_n\|_{C(\bar{\mathbb{H}}_T)}.$$

By Proposition A.1.1, (2.3.53) and the definition (2.3.123) of u_n, it follows that

$$\|(L-L_n)u_n\|_{C^\alpha_s([0,T]\times \text{ supp } \varphi_n)} \leq Cr^{\alpha/2}\|f\|_{\mathscr{C}^\alpha(\bar{\mathbb{H}}_T)} + C_1(r)T\|f\|_{C(\bar{\mathbb{H}}_T)}. \tag{2.3.129}$$

With the aid of inequalities (2.3.128) and (2.3.129), the estimate (2.3.127) becomes

$$\begin{aligned} \|\varphi_n(L-L_n)u_n\|_{C^\alpha_s(\bar{\mathbb{H}}_T)} \leq Cr^{\alpha/2}\|f\|_{\mathscr{C}^\alpha(\bar{\mathbb{H}}_T)} \\ + C_1(r)\left(\varepsilon\|f\|_{\mathscr{C}^\alpha(\bar{\mathbb{H}}_T)} + \varepsilon^{-m}T\|f\|_{C(\bar{\mathbb{H}}_T)}\right). \end{aligned} \tag{2.3.130}$$

Next, we estimate $[L,\varphi_n]u_n$, for $n \geq 1$, by employing a method similar to that used to estimate the term $[L,\varphi_0]u_0$. Using the identity (2.3.64) there is a positive constant C, depending only on K appearing in (2.2.10) and (2.2.11), such that

$$\|[L,\varphi_n]\,u_n\|_{C^\alpha_s([0,T]\times\mathbb{H})} \leq Cr^{-3}\|u_n\|_{C^{1+\alpha}_s([0,T]\times\mathbb{H})} \quad \text{(by (2.3.121))}.$$

From Lemma 2.3.2, there is a positive constant m such that, for all $\varepsilon \in (0,1)$, we have

$$\|[L,\varphi_n]\,u_n\|_{C^\alpha_s([0,T]\times\mathbb{H})} \leq Cr^{-3}\left(\varepsilon\|u_n\|_{C^{1+\alpha}_s([0,T]\times\mathbb{H})} + \varepsilon^{-m}\|u_n\|_{C([0,T]\times\mathbb{H})}\right).$$

According to Proposition A.1.1 and (2.3.53), there is a constant $C_1(r)$ so that

$$\|[L,\varphi_n]\,u_n\|_{C^\alpha_s([0,T]\times\mathbb{H})} \leq C_1(r)\left(\varepsilon\|f\|_{\mathscr{C}^\alpha(\bar{\mathbb{H}}_T)} + \varepsilon^{-m}T\|f\|_{C(\bar{\mathbb{H}}_T)}\right). \tag{2.3.131}$$

Combining inequalities (2.3.126), (2.3.130) and (2.3.131), and using the fact that at most N balls in the covering have non-empty intersection, the identity (2.3.124) yields

$$\|LMf-f\|_{\mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_T)} \leq Cr^{\alpha/2}\|f\|_{\mathscr{C}^\alpha(\bar{\mathbb{H}}_T)} + C_1(r)\left(\varepsilon\|f\|_{\mathscr{C}^\alpha(\bar{\mathbb{H}}_T)} + \varepsilon^{-m}T\|f\|_{C(\bar{\mathbb{H}}_T)}\right),$$

where C is a positive constant independent of r, while $C_1(r)$ may depend on r. By choosing small enough r, then small enough ε, and then small enough T, in that order, we find a positive constant $C_0 < 1$ such that

$$\|LMf-f\|_{\mathscr{C}^\alpha(\bar{\mathbb{H}}_T)} \leq C_0\|f\|_{\mathscr{C}^\alpha(\bar{\mathbb{H}}_T)}, \quad \forall f \in \mathscr{C}^\alpha(\bar{\mathbb{H}}_T),$$

and this gives (2.3.120). $\square$

Proof of Theorem 2.1.1. Uniqueness of solutions follows from Proposition 2.3.7.

We notice that $\mathscr{C}_p^{\alpha}(\bar{\mathbb{H}}_T) \subset \mathscr{C}^{\alpha}(\bar{\mathbb{H}}_T)$ and $\mathscr{C}_p^{2+\alpha}(\bar{\mathbb{H}}) \subset \mathscr{C}^{2+\alpha}(\bar{\mathbb{H}})$. Let $\tilde{L}$ be any operator satisfying Hypothesis 2.3.17. Let $\{\varphi_n\}_{n\geq 1}$ be a sequence of non-negative, smooth cut-off functions such that

$$0 \leq \varphi_n \leq 1, \quad \varphi_n|_{B_n} = 1, \quad \text{ and } \varphi_n|_{B_{2n}^c} = 0.$$

We define

$$L_n := \varphi_n L + (1-\varphi_n)\tilde{L}, \quad \forall n \geq 1.$$

Then, each L_n satisfies Hypothesis 2.3.17 and, by Proposition 2.3.20, there exists a unique solution $u_n \in \mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_T)$ to (2.1.2) with $L = L_n$. By Lemma 2.3.19, each solution u_n satisfies the global estimate

$$\|u_n\|_{\mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_T)} \leq C\left(\|f\|_{\mathscr{C}_p^{\alpha}(\bar{\mathbb{H}}_T)} + \|g\|_{\mathscr{C}_p^{2+\alpha}(\bar{\mathbb{H}})}\right). \tag{2.3.132}$$

For any bounded subdomain $U \subset \mathbb{H}$ and denoting $U_T = (0,T)\times U$, the parabolic analogue, $C_\rho^{2+\alpha}(\bar{U}_T) \hookrightarrow C_\rho^2(\bar{U}_T) \equiv C^{1,2}(\bar{U}_T)$, of the compact embedding [2, Theorem 1.31 (4)] of standard Hölder spaces, $C^{2+\alpha}(\bar{U}) \hookrightarrow C^2(\bar{U})$, implies that the sequence $\{u_n\}_{n\geq 1}$ converges strongly in $C^{1,2}(\bar{U}_T)$ to the limit $u \in C^{1,2}(U_T)$, that is, $u_n \to u$ in $C^{1,2}(U_T)$, as $n \to \infty$ for every bounded subdomain $U \subset \mathbb{H}$. It is now easily seen that u solves (2.1.2). By the Arzelà-Ascoli Theorem, we obtain that $u \in \mathscr{C}^{2+\alpha}(\overline{\mathbb{H}}_T)$ and satisfies (2.1.6). $\square$

2.4 Martingale problem and the mimicking theorem

In this section, we prove Theorem 2.1.16 concerning the degenerate stochastic differential equation with unbounded coefficients (2.1.4), and establish the main result, Theorem 2.1.19. Our method of proof combines ideas from the martingale problem formulation of Stroock and Varadhan [70] and the existence of solutions in suitable Hölder spaces, $\mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_T)$, to the homogeneous version of the initial value problem established in Theorem 2.1.1. In §2.4.1, we prove existence of weak solutions to the mimicking stochastic differential equation (2.1.4) and the existence of solutions to the martingale problem associated to $\mathscr{A}_t$. In §2.4.2, we establish uniqueness in law of solutions to (2.1.4) and to the martingale problem for $\mathscr{A}_t$, thus proving Theorems 2.1.16 and 2.1.10; in §2.4.3, we establish the matching property for the one-dimensional probability distributions for solutions to (2.1.4) and of an Itô process, thus proving Theorem 2.1.19.

2.4.1 Existence of solutions to the martingale problem and of weak solutions to the stochastic differential equation

In this subsection, we show that (2.1.4) has weak solutions $(\widehat{X}, \widehat{W})$ on some filtered probability space $(\Omega, \mathscr{F}, \mathbb{P})$, $\{\mathscr{F}_t\}_{t\geq 0}$ [47, Definition 5.3.1], for any initial point $x \in \bar{\mathbb{H}}$, by proving existence of solutions to the martingale problem associated to $\mathscr{A}_t$ (Definition 2.1.8).

We begin with an intuitive property of solutions to (2.1.4) defined by an initial condition in $\bar{\mathbb{H}}$. For this purpose, we consider coefficients defined on $[0,\infty) \times \mathbb{R}^d$, instead of $[0,\infty) \times \bar{\mathbb{H}}$.

Proposition 2.4.1 (Solutions started in a half-space remain in a half-space)**.** *Let*

$$\tilde{\sigma} : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^{d\times d},$$
$$\tilde{b} : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d,$$

be Borel measurable functions. Assume that

$$\tilde{\sigma}(t,x) = 0 \quad \text{when } x_d < 0, \tag{2.4.1}$$

and b satisfies

$$0 \leq \tilde{b}_d(t,x) \leq K \quad \text{ when } x_d < 0, \tag{2.4.2}$$

where K is a positive constant. Let $\widehat{X}$ be a weak solution of

$$d\widehat{X} = \tilde{b}(t, \widehat{X}(t))dt + \tilde{\sigma}(t, \widehat{X}(t))d\widehat{W}(t), \quad t \geq s,$$

such that $\widehat{X}(s) \in \bar{\mathbb{H}}$. Then

$$\mathbb{P}\left(\widehat{X}(t) \in \bar{\mathbb{H}}\right) = 1, \quad \forall t \geq s. \tag{2.4.3}$$

Proof. It is sufficient to show that for any $\varepsilon > 0$, we have

$$\mathbb{P}\left(\widehat{X}_d(t) \in (-\infty, -\varepsilon)\right) = 0, \quad \forall t \geq s. \tag{2.4.4}$$

Let $\varphi : \mathbb{R} \to [0,1]$ be a smooth, non-negative cutoff function such that

$$\varphi|_{(-\infty,-\varepsilon)} \equiv 1, \ \varphi|_{(0,\infty)} \equiv 0, \ \text{and } \varphi' \leq 0. \tag{2.4.5}$$

Then, by Itô's rule [47, Theorem 3.3.3], we obtain

$$\begin{aligned}\varphi(\widehat{X}_d(t)) = \varphi(\widehat{X}_d(s)) &+ \int_s^t \sum_{i=0}^d \tilde{\sigma}_{di}(v, \widehat{X}(v))\varphi'(\widehat{X}_d(v))d\widehat{W}_i(v)\\ &+ \int_s^t \left[\tilde{b}_d(v, \widehat{X}(v))\varphi'(\widehat{X}_d(v)) + \frac{1}{2}(\tilde{\sigma}\tilde{\sigma}^*)_{dd}(v, \widehat{X}(v))\varphi''(\widehat{X}_d(v))\right] dv,\end{aligned}$$

and so, because $\operatorname{supp}\varphi \subset (-\infty, 0]$ and (2.4.1) is satisfied, we have

$$\varphi(\widehat{X}_d(t)) = \varphi(\widehat{X}_d(s)) + \int_s^t \tilde{b}_d(v, \widehat{X}(v))\varphi'(\widehat{X}_d(v))dv.$$

By (2.4.2) and (2.4.5), the integral term in the preceding identity is non-positive. Therefore, we must have $\varphi(\widehat{X}_d(t)) \leq 0$ and hence $\varphi(\widehat{X}_d(t)) = 0$, for any choice of $\varepsilon > 0$, from where (2.4.4) and then (2.4.3) follow. □

Remark 2.4.2 (Weak solutions are independent of choice of extension of coefficients to lower half-space)**.** Let

$$\tilde{b}^i : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d, \quad i = 1, 2,$$

be measurable functions which satisfy condition (2.4.2), and assume

$$\tilde{b}^1 = \tilde{b}^2 \quad \text{on} \quad [0,\infty) \times \bar{\mathbb{H}}. \tag{2.4.6}$$

Let $\tilde{\sigma}$ be a measurable function as in the hypotheses of Proposition 2.4.1. Let $\widehat{X}$ be a weak solution to

$$d\widehat{X}(t) = \tilde{b}^1(t, \widehat{X}(t))dt + \tilde{\sigma}(t, \widehat{X}(t))d\widehat{W}(t), \quad \forall t \geq s, \tag{2.4.7}$$

such that

$$\mathbb{P}\left(\widehat{X}(s) \in \bar{\mathbb{H}}\right) = 1.$$

Then, Proposition 2.4.1 shows that $\widehat{X}(t)$ remains supported in $\bar{\mathbb{H}}$, for all $t \geq s$. By (2.4.6), it follows that $\widehat{X}$ is a weak solution to

$$d\widehat{X}(t) = \tilde{b}^2(t, \widehat{X}(t))dt + \tilde{\sigma}(t, \widehat{X}(t))d\widehat{W}(t), \quad \forall t \geq s. \tag{2.4.8}$$

This simple observation shows that, under the hypotheses of Proposition 2.4.1, any weak solution started in $\bar{\mathbb{H}}$ to (2.4.7) is a weak solution to (2.4.8), and vice versa.

Theorem 2.4.3 (Existence)**.** *Assume that the coefficients σ and b in* (2.1.4) *are continuous on* $[0,\infty) \times \bar{\mathbb{H}}$*, that $\bar{a}$ obeys condition* (2.1.16) *on* $[0,\infty) \times \bar{\mathbb{H}}$*, and that $\bar{a}$ and b have at most linear growth in the spatial variable, that is, condition* (2.2.14) *holds. Then,*

1. *For any $(s,x) \in [0,\infty) \times \bar{\mathbb{H}}$, there exist weak solutions $(\widehat{X}, \widehat{W})$, $(\Omega, \mathscr{F}, \mathbb{P})$, $(\mathscr{F}_t)_{t \geq s}$, to* (2.1.4) *such that $\widehat{X}(s) = x$.*
2. *For any $(s,x) \in [0,\infty) \times \bar{\mathbb{H}}$, there is a solution, $\widehat{\mathbb{P}}^{s,x}$, to the martingale problem associated to $\mathscr{A}_t$ such that* (2.1.13) *holds.*

Proof. We organize the proof in several steps. Without loss of generality, we may assume $s = 0$.

Step 1 (Solution to a classical martingale problem)**.** The argument of this step is similar to the one used in the proof of [47, Theorem 5.4.22].

Because $\bar{a} \in C_{\text{loc}}([0,\infty) \times \bar{\mathbb{H}})$ satisfies condition (2.1.16), we may extend σ as a continuous function to $[0,\infty) \times \mathbb{R}^d$ such that (2.4.1) is satisfied. We denote this extension by $\tilde{\sigma} \in C_{\text{loc}}([0,\infty) \times \mathbb{R}^d)$. Similarly, we consider an extension, $\tilde{b} \in C_{\text{loc}}([0,\infty) \times \mathbb{R}^d)$, of the coefficient b in (2.1.21), such that (2.4.2) is satisfied. By defining $\tilde{a} := \tilde{\sigma}\tilde{\sigma}^*$, we obtain a continuous extension of $\bar{a}$ from $[0,\infty) \times \bar{\mathbb{H}}$ to $[0,\infty) \times \mathbb{R}^d$.

Our goal in this step is to show that the classical martingale problem [47, Definition 5.4.10] associated with the operator,

$$\tilde{\mathscr{A}_t} f(x) := \sum_{i=1}^d \tilde{b}_i(t,x) f_{x_i}(x) + \sum_{i,j=1}^d \frac{1}{2} \tilde{a}_{ij}(t,x) f_{x_i x_j}(x), \quad \forall (t,x) \in [0,\infty) \times \mathbb{R}^d, \tag{2.4.9}$$

where $f \in C^2(\mathbb{R}^d)$, has a solution $\mathbb{P}$ on the canonical space with filtration defined by the cylinder sets,

$$\left(C_{\mathrm{loc}}([0,\infty);\mathbb{R}^d), \mathscr{B}(C_{\mathrm{loc}}([0,\infty);\mathbb{R}^d))\right), \left\{\mathscr{B}_t(C_{\mathrm{loc}}([0,\infty);\mathbb{R}^d))\right\}_{t\geq 0}, \tag{2.4.10}$$

such that

$$\mathbb{P}\left(\omega \in C_{\mathrm{loc}}([0,\infty);\mathbb{R}^d) : \omega(0) = x\right) = 1. \tag{2.4.11}$$

By [47, Problem 5.4.13 (A.2)′], it is enough to show that, for any $f \in C_0^2(\mathbb{R}^d)$,

$$M^f(t,\omega) := f(\omega(t)) - f(\omega(0)) - \int_0^t \tilde{\mathscr{A}}_s f(\omega(s))\,ds, \quad \omega \in C_{\mathrm{loc}}([0,\infty);\mathbb{R}^d), \tag{2.4.12}$$

is a $\{\mathscr{B}_t(C_{\mathrm{loc}}([0,\infty);\mathbb{R}^d))\}_{t\geq 0}$-martingale, and (2.4.11) holds.

Let $n \geq 1$ and define

$$\begin{aligned} \tilde{b}_i^n(t,x) &:= \begin{cases} \tilde{b}_i(t,x), & \text{if } |x| \leq n, \\ \tilde{b}_i(t,x) \wedge n, & \text{if } |x| > n, \end{cases} \\ \tilde{\sigma}_{ij}^n(t,x) &:= \begin{cases} \tilde{\sigma}_{ij}(t,x), & \text{if } |x| \leq n, \\ \tilde{\sigma}_{ij}(t,x) \wedge n, & \text{if } |x| > n, \end{cases} \end{aligned} \tag{2.4.13}$$

for all $1 \leq i,j \leq d$ and all $(t,x) \in [0,\infty) \times \mathbb{R}^d$. Let

$$\tilde{a}^n(t,x) := \tilde{\sigma}^n(t,x)(\tilde{\sigma}^n)^*(t,x), \quad \forall (t,x) \in [0,\infty) \times \mathbb{R}^d.$$

The coefficients $\tilde{b}^n$ and $\tilde{\sigma}^n$ are continuous, bounded functions on $[0,\infty) \times \mathbb{R}^d$. By [47, Theorem 5.4.22], there is a solution $\mathbb{P}^n$ to the classical martingale problem associated to the operator,

$$\tilde{\mathscr{A}}_t^n u(x) := \sum_{i=1}^d \tilde{b}_i^n(t,x) u_{x_i}(x) + \sum_{i,j=1}^d \frac{1}{2}\tilde{a}_{ij}^n(t,x) u_{x_i x_j}(x), \quad \forall (t,x) \in [0,\infty) \times \mathbb{R}^d, \tag{2.4.14}$$

for all $u \in C^2(\mathbb{R}^d)$. The solution $\mathbb{P}^n$ satisfies

$$\mathbb{P}^n\left(\omega \in C_{\mathrm{loc}}([0,\infty);\mathbb{R}^d) : \omega(0) = x\right) = 1. \tag{2.4.15}$$

and, an inspection of the proof of [47, Theorem 5.4.22] shows that $\mathbb{P}^n$ exists such that

$$M^{n,f}(t,\omega) := f(\omega(t)) - f(\omega(0)) - \int_0^t \tilde{\mathscr{A}}_s^n f(\omega(s))\,ds, \quad \omega \in C_{\mathrm{loc}}([0,\infty);\mathbb{R}^d), \tag{2.4.16}$$

is a $\{\mathscr{B}_t(C_{\text{loc}}([0,\infty);\mathbb{R}^d))\}_{t\geq 0}$-martingale, for all $n \geq 1$, and for all $f \in C_0^2(\mathbb{R}^d)$.

By Proposition [47, Proposition 5.4.11 & 5.4.6], the stochastic differential equations

$$\begin{aligned} dX^n(t) &= \tilde{b}^n(t, X^n(t))dt + \tilde{\sigma}(t, X^n(t))dW^n(t), \quad t \geq 0, \\ X^n(0) &= x, \end{aligned} \tag{2.4.17}$$

have weak solutions (X^n, W^n), for any initial point $x \in \bar{\mathbb{H}}$, on an extension $(\Omega^n, \mathscr{F}^n, \mathbb{Q}^n)$, $\{\mathscr{F}_t^n\}_{t\geq 0}$ of the canonical space (2.4.10) endowed with the probability measure $\mathbb{P}^n$. Let

$$\widetilde{\mathbb{P}}^n = \mathbb{Q}^n(X^n)^{-1}, \quad \forall n \geq 1, \tag{2.4.18}$$

be the probability measures induced by these processes on the canonical space. By [47, Remark 3.4.1] (definition of an extension of a probability space) and [47, Corollary 5.4.8], it follows that $\widetilde{\mathbb{P}}^n$ and $\mathbb{P}^n$ agree on $\mathscr{B}(C_{\text{loc}}([0,\infty);\mathbb{R}^d))$.

We wish to prove that the collection of probability measures, $\{\mathbb{P}^n\}_{n\geq 1}$, forms a *tight sequence* [47, Definition 2.4.6], so there exists a weakly convergent subsequence to a probability measure $\mathbb{P}$ which we show is a solution to the martingale problem for $\tilde{\mathscr{A}}_t$.

By (2.2.14), (2.4.2) and (2.4.1), coefficients $\tilde{b}^n$ and $\tilde{\sigma}^n$ satisfy

$$|\tilde{b}^n(t,x)| + |\tilde{\sigma}^n(t,x)| \leq K(1+|x|), \quad \forall n \geq 1, \quad \forall (t,x) \in [0,\infty) \times \mathbb{R}^d.$$

From [47, Problem 5.3.15], it follows that for any $T > 0$ and $m \geq 1$, there is a positive constant $C = C(m, T, K, d)$ such that, for all $n \geq 1$ and all $x \in \bar{\mathbb{H}}$, we have

$$\mathbb{E}_{\mathbb{Q}^n}\left[\max_{0\leq t\leq T} |X^n(t)|^{2m}\right] \leq C\left(1+|x|^{2m}\right), \tag{2.4.19}$$

$$\mathbb{E}_{\mathbb{Q}^n}\left[|X^n(t) - X^n(s)|^{2m}\right] \leq C\left(1+|x|^{2m}\right)|t-s|^m, \quad \forall t,s \in [0,T]. \tag{2.4.20}$$

It follows by the Kolmogorov-Čentsov theorem [47, Theorem 2.2.8] and [47, Problem 2.4.11] that the sequence of probability measures on the canonical space, $\{\mathbb{P}^n\}_{n\geq 1}$, is tight. Therefore, by Prohorov's theorem [47, Theorem 2.4.7], we may extract a subsequence which converges weakly to a probability measure $\mathbb{P}$ on the canonical space.

Next, we show that $\mathbb{P}$ solves the martingale problem associated to $\tilde{\mathscr{A}}_t$. Obviously, condition (2.4.11) follows from (2.4.15). It remains to show that $\left(M^f(t)\right)_{t\geq 0}$, given in (2.4.12), is a $\{\mathscr{B}_t(C_{\text{loc}}([0,\infty)\times\mathbb{R}^d))\}_{t\geq 0}$-martingale. Recall that each σ-algebra $\mathscr{B}_s(C_{\text{loc}}([0,\infty);\mathbb{R}^d))$ is given by at most countable unions of sets of the form

$$\left\{\omega\in C_{\text{loc}}([0,\infty);\bar{\mathbb{H}}):\omega(s_i)\in B_i, i=1,\ldots,m\right\},$$

where $m\geq 1$, $0\leq s_1\leq\ldots\leq s_m\leq s$, $B_i\in\mathscr{B}(\mathbb{R}^d)$, $i=1,\ldots,m$. Therefore, it is sufficient to show that for any choice of $t\geq s\geq 0$, $m\geq 1$, $0\leq s_1\leq\ldots\leq s_m\leq s$, $B_i\in\mathscr{B}(\mathbb{R}^d)$, $i=1,\ldots,m$, we have

$$\mathbb{E}\left[\left(M^f(t)-M^f(s)\right)\prod_{i=1}^m\mathbf{1}_{\{\omega(s_i)\in B_i\}}\right]=0. \tag{2.4.21}$$

By [27, Proposition 3.4.2], there is a sequence of $\mathscr{B}_s(C_{\text{loc}}([0,\infty);\mathbb{R}^d))$-measurable, continuous, bounded functions $H_n: C_{\text{loc}}([0,\infty);\mathbb{R}^d)\to\mathbb{R}$ such that

$$H_n(\omega)\to\prod_{i=1}^m\mathbf{1}_{\{\omega(s_i)\in B_i\}},\qquad\text{as } n\to\infty,\quad\forall\omega\in C([0,\infty);\mathbb{R}^d).$$

Since $f\in C_0^2(\mathbb{R}^d)$ and the coefficients of $\tilde{\mathscr{A}_t}$ are bounded on compact sets, the sequence of functions $\left\{\left(M^f(t)-M^f(s)\right)H_n\right\}_{n\geq 1}$ is uniformly bounded, so the Dominated Convergence Theorem yields

$$\mathbb{E}\left[\left(M^f(t)-M^f(s)\right)\prod_{i=1}^m H_n\right]\to\mathbb{E}\left[\left(M^f(t)-M^f(s)\right)\prod_{i=1}^m\mathbf{1}_{\{\omega(s_i)\in B_i\}}\right]\qquad\text{as } n\to\infty.$$

Therefore, (2.4.21) follows if for any $s,t\in[0,\infty]$, $s<t$, and any bounded, continuous, $\mathscr{B}_s(C([0,\infty);\mathbb{R}^d))$-measurable function $H: C([0,\infty);\mathbb{R}^d)\to\mathbb{R}$, we have

$$\mathbb{E}_{\mathbb{P}}\left[\left(M^f(t)-M^f(s)\right)H\right]=0. \tag{2.4.22}$$

In the sequel, we fix $f\in C_0^2(\mathbb{R}^d)$ and, for brevity, omit the superscript f in the definition of M^f and $M^{n,f}$, for $n\geq 1$. From (2.4.16), we know that (2.4.22) holds with M^n replacing M, that is,

$$\mathbb{E}_{\mathbb{P}^n}\left[\left(M^n(t)-M^n(s)\right)H\right]=0,\quad\forall n\geq 1. \tag{2.4.23}$$

Because f has compact support in the spatial variable, it follows from (2.4.13), (2.4.9) and (2.4.14) that

$$\begin{aligned}M^n(t)-M^n(s)&=M(t)-M(s)\\&=f(\omega(t))-f(\omega(s))-\int_s^t\tilde{\mathscr{A}_v}f(\omega(v))\,dv,\end{aligned} \tag{2.4.24}$$

for all n large enough such that the support of f is contained in the Euclidean ball of radius $2n$ centered at the origin. The function $F : C_{\text{loc}}([0,\infty);\mathbb{R}^d) \to \mathbb{R}$ defined by

$$F(\omega) := f(\omega(t)) - f(\omega(s)) - \int_s^t \tilde{\mathscr{A}_v} f(\omega(v))\, dv, \quad \forall \omega \in C_{\text{loc}}([0,\infty);\mathbb{R}^d), \tag{2.4.25}$$

is bounded and continuous because f has compact support and, on any compact subset of $[0,\infty) \times \mathbb{R}^d$, the coefficients of $\tilde{\mathscr{A}_t}$ are uniformly continuous. Therefore, the function $FH : C_{\text{loc}}([0,\infty);\mathbb{R}^d) \to \mathbb{R}$ is bounded and continuous. Since $\mathbb{P}^n$ converges weakly to $\mathbb{P}$ as $n \to \infty$, we see that

$$\mathbb{E}_{\mathbb{P}^n}[FH] \to \mathbb{E}_{\mathbb{P}}[FH], \text{ as } n \to \infty. \tag{2.4.26}$$

By (2.4.23), (2.4.24) and (2.4.25), the limit of the sequence $\{\mathbb{E}_{\mathbb{P}^n}[FH]\}_{n\geq 1}$ is zero, and so we obtain the desired identity (2.4.22).

Therefore, we have shows that (2.4.27) admits weak solutions for any initial condition $x \in \bar{\mathbb{H}}$.

Step 2 (Existence of weak solutions)**.** By [47, Proposition 5.4.6], we obtain that the stochastic differential equation

$$\begin{aligned} d\widehat{X}(t) &= \tilde{b}(t, \widehat{X}(t))dt + \tilde{\sigma}(t, \widehat{X}(t))d\widehat{W}(t), \quad \forall t \geq 0, \\ \widehat{X}(0) &= x, \end{aligned} \tag{2.4.27}$$

has at least one weak solution $(\widehat{X}, \widehat{W})$ on an extension of the canonical space (2.4.10). Proposition 2.4.1 and Remark 2.4.2 show $\widehat{X}(t) \in \bar{\mathbb{H}}$, for all $t \geq 0$, so that $(\widehat{X}, \widehat{W})$ is a weak solution to the stochastic differential equation (2.1.4), as well, since the coefficients $\tilde{b}$ and $\tilde{\sigma}$ are extensions of b and σ, respectively, from $[0,\infty) \times \bar{\mathbb{H}}$ to $[0,\infty) \times \mathbb{R}^d$.

Step 3 (Solution to the martingale problem)**.** Let $\widehat{X}$ be the weak solution obtained in the previous step. Since $\widehat{X}(t) \in \bar{\mathbb{H}}$, for all $t \geq 0$, then we may define $\widehat{\mathbb{P}}^{0,x}$ to be the probability measure induced by the weak solution $\widehat{X}$ on $(C_{\text{loc}}([0,\infty);\bar{\mathbb{H}}), \mathscr{B}(C_{\text{loc}}([0,\infty);\bar{\mathbb{H}})))$. Then, similarly to [47, Problem 5.4.3], it follows that $\widehat{\mathbb{P}}^{0,x}$ is a solution to the martingale problem associated to $\mathscr{A}_t$ and satisfies (2.1.13).

This concludes the proof of the theorem. $\square$

2.4.2 Uniqueness of solutions to the martingale problem and of weak solutions to the stochastic differential equation

We show that uniqueness in the sense of probability law holds for the weak solutions of the stochastic differential equation (2.1.4), with initial condition $x \in \bar{\mathbb{H}}$, and we establish the well-posedness of the martingale problem associated to (2.1.4). First, we prove that uniqueness of the one-dimensional marginal distributions holds for weak solutions to (2.1.4), and then the analogue of [47, Proposition 5.4.27] is used to show that uniqueness in law of solutions also holds.

We begin with the following version of Itô's rule (compare [47, Theorem 3.3.6]) which applies to Itô processes which are solutions to (2.1.4).

Proposition 2.4.4 (Itô's rule)**.** *Assume that the coefficients σ and b of* (2.1.4) *are Borel measurable functions, $\bar{a}$ obeys condition* (2.1.16) *on* $[0,\infty)\times\bar{\mathbb{H}}$*, and $\bar{a}$ and b have at most linear growth in the spatial variable, that is, condition* (2.2.14) *holds. Assume there is a positive constant K such that*

$$|a_{ij}(t,x)| \leq K \quad \forall (t,x) \in [0,T]\times\mathbb{R}^{d-1}\times[0,1]. \tag{2.4.28}$$

Let $v \in C_{\mathrm{loc}}([0,\infty)\times\bar{\mathbb{H}})$ be such that it satisfies, for all $1 \leq i,j \leq d$,

$$v_t, v_{x_i}, x_d v_{x_i x_j} \in C_{\mathrm{loc}}([0,\infty)\times\bar{\mathbb{H}}), \tag{2.4.29}$$

$$x_d v_{x_i x_j} = 0 \qquad \text{on } [0,T]\times\partial\mathbb{H}. \tag{2.4.30}$$

Let $(\widehat{X},\widehat{W})$ be a weak solution to (2.1.4) *on a filtered probability space* $(\Omega,\mathbb{P},\mathscr{F})$*, $\{\mathscr{F}_t\}_{t\geq 0}$, such that $\widehat{X}(0)\in\bar{\mathbb{H}}$, $\mathbb{P}$-a.s. Then, the following holds $\mathbb{P}$-a.s., for all $0 \leq t \leq T$,*

$$\begin{aligned} v(t,\widehat{X}(t)) = v(0,\widehat{X}(0)) &+ \int_0^t \sum_{i,j=1}^d \sigma_{ij}(u,\widehat{X}(u)) v_{x_j}(u,\widehat{X}(u)) d\widehat{W}_j(u) \\ &+ \int_0^t \left(\sum_{i=1}^d b_i(u,\widehat{X}(u)) v_{x_i}(u,\widehat{X}(u)) \right. \\ &\left. + \sum_{i,j=1}^d \frac{1}{2}\widehat{X}_d(u) a_{ij}(u,\widehat{X}(u)) v_{x_i x_j}(u,\widehat{X}(u)) \right) du. \end{aligned} \tag{2.4.31}$$

Proof. We choose $\varepsilon > 0$ and let

$$x^\varepsilon := (x_1, \ldots, x_{d-1}, x_d + \varepsilon),$$
$$\widehat{X}^\varepsilon(u) := \left(\widehat{X}_1(u), \ldots, \widehat{X}_{d-1}(u), \widehat{X}_d(u) + \varepsilon\right), \quad \forall u \geq 0.$$

Consider the stopping times

$$\tau_n := \inf\left\{u \geq 0 : |\widehat{X}(u)| \geq n\right\} \quad \forall n \geq 1.$$

Since the coefficients $\bar{a}$ and b have at most linear growth in the spatial variable (condition (2.2.14) holds), we obtain by [47, Problem 5.3.15], that for all $m \geq 1$ and $t \geq 0$, there is a positive constant $C = C(m, t, K, d)$ such that

$$\mathbb{E}\left[\max_{0 \leq u \leq t} |\widehat{X}(u)|^{2m}\right] \leq C\left(1 + |x|^{2m}\right). \tag{2.4.32}$$

Then, it follows by (2.4.32) that the non-decreasing sequence of stopping times $\{\tau_n\}_{n\geq 1}$ satisfies

$$\lim_{n\to\infty} \tau_n = +\infty \quad \mathbb{P}\text{-a.s.} \tag{2.4.33}$$

If this were not the case, then there is $t > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\tau_n \leq t\right) > 0. \tag{2.4.34}$$

But, $\mathbb{P}\left(\tau_n \leq t\right) = \mathbb{P}\left(\sup_{0\leq u\leq t} |\widehat{X}(u)| \geq n\right)$ and we have

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\leq u\leq t} |\widehat{X}(u)| \geq n\right) &\leq \frac{1}{n^2}\mathbb{E}\left[\max_{0\leq u\leq t} |\widehat{X}(u)|^2\right] \\
&\leq \frac{C(1+|x|^2)}{n^2}, \qquad \text{(by (2.4.32))}.
\end{aligned}$$

Since the preceding expression converges to zero, as n goes to ∞, we obtain a contradiction in (2.4.34), and so (2.4.33) holds. By (2.4.33), it suffices to prove (2.4.31) for the stopped process, that is

$$\begin{aligned}
v(t\wedge\tau_n, \widehat{X}(t\wedge\tau_n)) = v(0, \widehat{X}(0)) &+ \int_0^{t\wedge\tau_n} \sum_{i,j=1}^d \sigma_{ij}(u, \widehat{X}(u)) v_{x_j}(u, \widehat{X}(u)) d\widehat{W}_j(u) \\
&+ \int_0^{t\wedge\tau_n} \left(\sum_{i=1}^d b_i(u, \widehat{X}(u)) v_{x_i}(u, \widehat{X}(u))\right. \\
&+ \left.\sum_{i,j=1}^d \frac{1}{2}\widehat{X}_d(u) a_{ij}(u, \widehat{X}(u)) v_{x_i x_j}(u, \widehat{X}(u))\right) du.
\end{aligned} \tag{2.4.35}$$

By Proposition 2.4.1, we have

$$\widehat{X}(u) \in \bar{\mathbb{H}} \quad \mathbb{P}\text{-a.s.} \quad \forall u \in [0,T]. \tag{2.4.36}$$

Since $v \in C^{1,2}_{\mathrm{loc}}([0,T] \times \mathbb{R}^{d-1} \times [\varepsilon/2, \infty))$, we may extend v to be a $C^{1,2}_{\mathrm{loc}}$ function on $[0,T] \times \mathbb{R}^d$. Then we can apply the standard Itô's rule, [47, Theorem 3.3.6] and, taking into account that $\widehat{X}(t) + \varepsilon \geq \varepsilon$, $\mathbb{P}$-a.s., for all $t \geq 0$, we obtain

$$\begin{aligned} v(t \wedge \tau_n, \widehat{X}^\varepsilon(t \wedge \tau_n)) = v(0, \widehat{X}^\varepsilon(0)) &+ \int_0^{t \wedge \tau_n} \sum_{i,j=1}^d \sigma_{ij}(u, \widehat{X}(u)) v_{x_j}(u, \widehat{X}^\varepsilon(u))\, d\widehat{W}_j(u) \\ &+ \int_0^{t \wedge \tau_n} \left(v_t(u, \widehat{X}^\varepsilon(u)) + \sum_{i=1}^d b_i(u, \widehat{X}(u)) v_{x_i}(u, \widehat{X}^\varepsilon(u)) \right. \\ &+ \left. \sum_{i,j=1}^d \frac{1}{2} \widehat{X}_d(u) a_{ij}(u, \widehat{X}(u)) v_{x_i x_j}(u, \widehat{X}^\varepsilon(u)) \right) du. \end{aligned} \tag{2.4.37}$$

Our goal is to show that, by taking the limit as $\varepsilon \downarrow 0$ in the preceding equation, we obtain (2.4.35).

Since $v \in C_{\mathrm{loc}}(\bar{\mathbb{H}}_T)$, we have for all $0 \leq u \leq T$,

$$v(u \wedge \tau_n, \widehat{X}^\varepsilon(u \wedge \tau_n)) \to v(u \wedge \tau_n, \widehat{X}(u \wedge \tau_n)) \quad \mathbb{P}\text{-a.s. when } \varepsilon \downarrow 0. \tag{2.4.38}$$

The terms in (2.4.37) containing the pure Itô integrals can be evaluated in the following way. As usual, we have

$$\begin{aligned} &\mathbb{E}\left[\left| \int_0^{t \wedge \tau_n} \sigma_{ij}(u, \widehat{X}(u)) v_{x_j}(u, \widehat{X}^\varepsilon(u))\, d\widehat{W}_j(u) - \int_0^{t \wedge \tau_n} \sigma_{ij}(u, \widehat{X}(u)) v_{x_j}(u, \widehat{X}(u))\, d\widehat{W}_j(u) \right| \right] \\ &\leq \mathbb{E}\left[\left| \int_0^{t \wedge \tau_n} \sigma_{ij}(u, \widehat{X}(u)) \left(v_{x_j}(u, \widehat{X}^\varepsilon(u)) - v_{x_j}(u, \widehat{X}(u)) \right) d\widehat{W}_j(u) \right|^2 \right]^{1/2}, \end{aligned}$$

and so,

$$\begin{aligned} &\mathbb{E}\left[\left| \int_0^{t \wedge \tau_n} \sigma_{ij}(u, \widehat{X}(u)) v_{x_j}(u, \widehat{X}^\varepsilon(u))\, d\widehat{W}_j(u) - \int_0^{t \wedge \tau_n} \sigma_{ij}(u, \widehat{X}(u)) v_{x_j}(u, \widehat{X}(u))\, d\widehat{W}_j(u) \right| \right] \\ &\leq \mathbb{E}\left[\int_0^{t \wedge \tau_n} |\sigma_{ij}(u, \widehat{X}(u))|^2 |v_{x_j}(u, \widehat{X}^\varepsilon(u)) - v_{x_j}(u, \widehat{X}(u))|^2 du \right]^{1/2} \end{aligned} \tag{2.4.39}$$

Since $v_{x_j} \in C_{\mathrm{loc}}(\bar{\mathbb{H}}_T)$, we have $\mathbb{P}$-a.s., for all $0 \leq u \leq T$,

$$|\sigma_{ij}(u, \widehat{X}(u))| |v_{x_j}(u, \widehat{X}^\varepsilon(u)) - v_{x_j}(u, \widehat{X}(u))| \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

By the continuity of paths of $\widehat{X}$ and the fact that σ_{ij} satisfy the growth condition (2.2.14), the Lebesgue Dominated Convergence Theorem implies $\mathbb{P}$-a.s.

$$\int_0^{t\wedge\tau_n} |\sigma_{ij}(u,\widehat{X}(u))|^2 |v_{x_j}(u,\widehat{X}^\varepsilon(u)) - v_{x_j}(u,\widehat{X}(u))|^2 du \to 0 \quad \text{as } \varepsilon\downarrow 0. \tag{2.4.40}$$

On the closed ball of radius n in $\bar{\mathbb{H}}$ centered at the origin, the coefficients σ_{ij} and v_{x_j} are bounded, so it follows

$$\mathbb{E}\left[\int_0^{t\wedge\tau_n} |\sigma_{ij}(u,\widehat{X}(u))|^2 |v_{x_j}(u,\widehat{X}^\varepsilon(u)) - v_{x_j}(u,\widehat{X}(u))|^2 du\right]^{1/2} \to 0, \quad \text{as } \varepsilon\downarrow 0.$$

Thus, by (2.4.39), we obtain the L^1-convergence, and also the $\mathbb{P}$-a.s convergence of a subsequence which we label the same as the given sequence, as $\varepsilon\downarrow 0$,

$$\int_0^{t\wedge\tau_n} \sigma_{ij}(u,\widehat{X}(u)) v_{x_j}(u,\widehat{X}^\varepsilon(u))\, d\widehat{W}_j(u) \to \int_0^{t\wedge\tau_n} \sigma_{ij}(u,\widehat{X}(u)) v_{x_j}(u,\widehat{X}(u))\, d\widehat{W}_j(u). \tag{2.4.41}$$

We write the du-integrand in (2.4.37) as the sum of $(\partial_t+\mathscr{A}_t)v(u,\widehat{X}^\varepsilon(u))$ and $\mathscr{R}v(u,\widehat{X}^\varepsilon(u))$, where

$$\mathscr{A}_t v(u,x^\varepsilon) = \sum_{i=1}^d b_i(u,x) v_{x_i}(u,x^\varepsilon) + \sum_{i,j=1}^d \frac{1}{2} a_{ij}(u,x) x_d^\varepsilon v_{x_ix_j}(u,x^\varepsilon), \tag{2.4.42}$$

$$\mathscr{R}v(u,x^\varepsilon) = -\frac{\varepsilon}{2}\sum_{i,j=1}^d a_{ij}(u,x) v_{x_ix_j}(u,x^\varepsilon), \tag{2.4.43}$$

for all $(u,x)\in[0,T]\times\bar{\mathbb{H}}$. An argument similar to the one which gave us (2.4.41) can be used to obtain the $\mathbb{P}$-a.s convergence, as $\varepsilon\downarrow 0$,

$$\int_0^{t\wedge\tau_n} (\partial_t+\mathscr{A}_t) v(u,\widehat{X}^\varepsilon(u)))\, du \to \int_0^{t\wedge\tau_n} (\partial_t+\mathscr{A}_t) v(u,\widehat{X}(u))\, du, \tag{2.4.44}$$

This requires that $v_t, v_{x_i}, x_d v_{x_ix_j} \in C_{\mathrm{loc}}(\bar{\mathbb{H}}_T)$, the coefficients b_i and $x_d a_{ij}$ satisfy the linear growth assumption (2.2.14), and coefficients a_{ij} obey (2.4.28). Therefore, it remains to show

$$\mathbb{E}\left[\int_0^{t\wedge\tau_n} \left|\mathscr{R}v(u,\widehat{X}^\varepsilon(u))\right| du\right] \to 0, \quad \text{as } \varepsilon\downarrow 0. \tag{2.4.45}$$

Notice that the proof of (2.4.45) completes the proof of Itô's rule because (2.4.38), (2.4.41), (2.4.44) and (2.4.45) yields (2.4.35) by taking limit as $\varepsilon\downarrow 0$ in (2.4.37).

Now, we return to the proof of (2.4.45). We can write the term in (2.4.43) in the following way

$$\begin{aligned}\varepsilon a_{ij}(u,x)v_{x_ix_j}(u,x^\varepsilon) &= \frac{\varepsilon}{x_d^\varepsilon}a_{ij}(u,x)x_d^\varepsilon v_{x_ix_j}(u,x^\varepsilon)\mathbf{1}_{\{0\le x_d\le\sqrt{\varepsilon}\}} \\ &\quad + \frac{\varepsilon}{x_d^\varepsilon}a_{ij}(u,x)x_d^\varepsilon v_{x_ix_j}(u,x^\varepsilon)\mathbf{1}_{\{\sqrt{\varepsilon}<x_d\}}, \quad \forall(u,x)\in\bar{\mathbb{H}}_T,\end{aligned} \tag{2.4.46}$$

We use the preceding identity to show the pointwise convergence, for all $(u,x)\in\bar{\mathbb{H}}_T$,

$$\varepsilon a_{ij}(u,x)v_{x_ix_j}(u,x^\varepsilon)\to 0 \quad \text{as } \varepsilon\downarrow 0. \tag{2.4.47}$$

Because of the fact that a_{ij} are locally bounded on $\bar{\mathbb{H}}_T$ by (2.4.28) and the $x_dv_{x_ix_j}\in C_{\mathrm{loc}}(\bar{\mathbb{H}}_T)$ obey $x_dv_{x_ix_j}(u,x)=0$, when $x_d=0$, we obtain

$$\frac{\varepsilon}{x_d^\varepsilon}a_{ij}(u,x)x_d^\varepsilon v_{x_ix_j}(u,x^\varepsilon)\mathbf{1}_{\{0\le x_d\le\sqrt{\varepsilon}\}}\to 0, \quad \text{as } \varepsilon\downarrow 0. \tag{2.4.48}$$

for all $(u,x)\in\bar{\mathbb{H}}_T$. In the case $\sqrt{\varepsilon}<x_d$, obviously we have

$$\varepsilon/x_d^\varepsilon\le\sqrt{\varepsilon},$$

and so, using $x_dv_{x_ix_j}\in C_{\mathrm{loc}}(\bar{\mathbb{H}}_T)$ and the local boundedness of a_{ij} on $\bar{\mathbb{H}}_T$ by (2.4.28) and (2.2.14), we obtain

$$\frac{\varepsilon}{x_d^\varepsilon}a_{ij}(u,x)x_d^\varepsilon v_{x_ix_j}(u,x^\varepsilon)\mathbf{1}_{\{\sqrt{\varepsilon}<x_d\}}\to 0, \quad \text{as } \varepsilon\downarrow 0. \tag{2.4.49}$$

for all $(u,x)\in\bar{\mathbb{H}}_T$. By combining (2.4.48) and (2.4.49), we obtain (2.4.47). Using the continuity of the paths of $\widehat{X}$, (2.4.47) and (2.4.43), we obtain $\mathbb{P}$-a.s., for all $0\le u\le T$,

$$\left|\mathscr{R}v(u,\widehat{X}^\varepsilon(u))\right|\to 0, \quad \text{as } \varepsilon\downarrow 0,$$

and also, the following holds $\mathbb{P}$-a.s.

$$\int_0^{t\wedge\tau_n}\left|\mathscr{R}v(u,\widehat{X}^\varepsilon(u))\right|du\to 0, \quad \text{as } \varepsilon\downarrow 0. \tag{2.4.50}$$

The Lebesgue Dominated Convergence Theorem, conditions (2.4.28) and (2.2.14) satisfied by a_{ij} on $\bar{\mathbb{H}}_T$, and $x_dv_{x_ix_j}\in C_{\mathrm{loc}}(\bar{\mathbb{H}}_T)$ now imply (2.4.45). This concludes the proof of the proposition. □

The next result is based on the *existence* of a solution in $\mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_T)$ to the homogeneous initial value problem considered in Theorem 2.1.1.

Proposition 2.4.5 (Uniqueness of the one-dimensional marginal distributions)**.** *Assume the hypotheses of Theorem 2.1.16 hold. Let* $(\widehat{X}^k, \widehat{W}^k)$*, defined on filtered probability spaces* $(\Omega^k, \mathbb{P}^k, \mathscr{F}^k)$*,* $\{\mathscr{F}^k_t\}_{t\geq 0}$*,* $k=1,2$*, be two weak solutions to* (2.1.4) *with initial condition* $(s,x) \in [0,\infty)\times\bar{\mathbb{H}}$*. Then the one-dimensional marginal probability distributions of* $\widehat{X}^1(t)$ *and* $\widehat{X}^2(t)$ *agree for each* $t \geq s$*.*

Proof. Without loss of generality, we may assume that $s=0$. By Proposition 2.4.1, it is enough to show that for any $T>0$ and $g \in C^\infty_0(\bar{\mathbb{H}})$, we have

$$\mathbb{E}_{\mathbb{P}^1}\left[g(\widehat{X}^1(T))\right] = \mathbb{E}_{\mathbb{P}^2}\left[g(\widehat{X}^2(T))\right], \tag{2.4.51}$$

where each expectation is taken under the law of the corresponding process. For this purpose, we consider the parabolic differential operator,

$$-\check{L}w(t,x) := -w_t(t,x) + \sum_{i=1}^d b_i(T-t,x)w_{x_i}(t,x) + \sum_{i,j=1}^d \frac{1}{2}x_d a_{ij}(T-t,x)w_{x_ix_j}(t,x), \tag{2.4.52}$$

for all $(t,x)\in\mathbb{H}_T$ and $w \in C^{1,2}(\mathbb{H}_T)$. Let $u \in \mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_T)$ be the unique solution given by Theorem 2.1.1 to the homogeneous initial value problem,

$$\begin{cases} \check{L}u(t,x) = 0, & \text{for } (t,x) \in (0,T)\times\mathbb{H}, \\ u(0,x) = g(x), & \text{for } x \in \bar{\mathbb{H}}. \end{cases} \tag{2.4.53}$$

Define

$$v(t,x) := u(T-t,x), \quad \forall (t,x) \in [0,T]\times\bar{\mathbb{H}}. \tag{2.4.54}$$

Then, $v \in \mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_T)$ solves the terminal value problem,

$$\begin{cases} v_t(t,x) + \mathscr{A}_t v(t,x) = 0, & \text{for } (t,x) \in (0,T)\times\mathbb{H}, \\ v(T,x) = g(x), & \text{for } x \in \bar{\mathbb{H}}, \end{cases} \tag{2.4.55}$$

where the differential operator $\mathscr{A}_t$ is given by (2.1.1). Proposition 2.4.4 gives us, for $k=1,2$,

$$\begin{aligned} \mathbb{E}_{\mathbb{P}^k}\left[v(T,\widehat{X}^k(T))\right] &= v(0,x) + \mathbb{E}_{\mathbb{P}^k}\left[\int_0^T (v_t + \mathscr{A}_t)\, v(t,\widehat{X}^k(t))dt\right] \\ &\quad + \mathbb{E}_{\mathbb{P}^k}\left[\int_0^T \sum_{i,j=1}^d \sigma_{ij}(t,\widehat{X}^k(t)) v_{x_j}(t,\widehat{X}^k(t)) d\widehat{W}^k_j(t)\right]. \end{aligned} \tag{2.4.56}$$

Recall that $v_{x_i} \in C(\bar{\mathbb{H}}_T)$ and the coefficients σ_{ij} satisfy (2.2.14). Inequality (2.4.32) applied with $m = 1$, gives

$$\mathbb{E}_{\mathbb{P}^k}\left[\int_0^T \left|\sigma_{ij}(t, \widehat{X}^k(t))v_{x_j}(t, \widehat{X}^k(t))\right|^2 dt\right] \leq C(1+|x|^2)\|v_{x_i}\|^2_{C(\bar{\mathbb{H}}_T)},$$

and so, the Itô integrals in (2.4.56) are square-integrable, continuous martingales, which implies

$$\mathbb{E}_{\mathbb{P}^k}\left[\int_0^T \sigma_{ij}(t, \widehat{X}^k(t))v_{x_j}(t, \widehat{X}^k(t))d\widehat{W}^k_j(t)\right] = 0.$$

Using the preceding inequality and (2.4.55), we see that (2.4.56) yields

$$\mathbb{E}_{\mathbb{P}^k}\left[g(\widehat{X}^k(T))\right] = v(0,x), \quad k = 1, 2, \tag{2.4.57}$$

and so, (2.4.51) follows. □

Next, we recall

Proposition 2.4.6 (Uniqueness of solutions to the classical martingale problem)**.** *[47, Proposition 5.4.27] Let*

$$\tilde{b} : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d,$$
$$\tilde{\sigma} : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^{d\times d},$$

be Borel measurable functions such that they are bounded on each compact subset in $\mathbb{R}^d$ and define a differential operator by

$$\mathscr{G}u(x) := \sum_{i=1}^d \tilde{b}_i(x)u_{x_i} + \sum_{i,j=1}^d \frac{1}{2}\tilde{a}_{ij}(x)u_{x_ix_j}, \quad \forall x \in \mathbb{R}^d,$$

where $\tilde{a} := \tilde{\sigma}\tilde{\sigma}^*$ and $u \in C^2(\mathbb{R}^d)$. Suppose that for every $x \in \mathbb{R}^d$, any two solutions $\mathbb{P}^x$ and $\mathbb{Q}^x$ to the time-homogeneous martingale problem associated with $\mathscr{G}$ have the same one-dimensional marginal distributions. Then, for every initial condition $x \in \mathbb{R}^d$, there exists at most one solution to the time-homogeneous martingale problem associated to $\mathscr{G}$.

We have the following consequence of Propositions 2.4.5 and 2.4.6.

Corollary 2.4.7 (Uniqueness of solutions to the martingale problem associated to $\mathscr{A}_t$)**.** *Suppose that for every $x \in \bar{\mathbb{H}}$ and $s \geq 0$, any two solutions $\mathbb{P}^{s,x}$ and $\mathbb{Q}^{s,x}$ to the martingale problem in Definition 2.1.8 associated to $\mathscr{A}_t$ in* (2.1.1) *with initial condition (s,x) have the same one-dimensional marginal distributions. Then, for every initial condition $(s,x) \in [0,\infty) \times \bar{\mathbb{H}}$, there exists at most one solution to the martingale problem associated to $\mathscr{A}_t$.*

Proof. As (2.1.4) is time-inhomogeneous with initial condition $(s,x) \in [0,\infty) \times \bar{\mathbb{H}}$, rather than time-homogeneous with initial condition $x \in \mathbb{R}^d$, as assumed by Proposition 2.4.6, we first extend the coefficients, $\sigma(t,x)$ and $b(t,x)$ with $(t,x) \in [0,\infty) \times \bar{\mathbb{H}}$ to $(t,x) \in \mathbb{R} \times \mathbb{R}^{d-1} \times (-\infty,0)$ and $(t,x) \in (-\infty,0) \times \bar{\mathbb{H}}$, so

$$\tilde{\sigma}_{ij}(t,x) = 0, \quad \tilde{b}_i(t,x) = 0, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^{d-1} \times (-\infty,0) \text{ and } (t,x) \in (-\infty,0) \times \bar{\mathbb{H}}. \tag{2.4.58}$$

To obtain a time-homogeneous differential operator, as in Proposition 2.4.6, we increase the space dimension by adding the time coordinate, that is, we consider the following $(d+1)$-dimensional process

$$\begin{aligned} &dY_0(t) = dt, \quad \forall t \geq 0, \\ &dY_i(t) = \tilde{b}_i(Y(t))dt + \sum_{j=1}^d \tilde{\sigma}_{ij}(Y(t))dW_j(t), \quad i = 1, \ldots, d, \quad \forall t \geq 0. \end{aligned} \tag{2.4.59}$$

Now, let $\mathscr{G}$ denote the time-homogeneous differential operator

$$\mathscr{G}u(y) := \sum_{i=1}^d \tilde{b}_i(y) u_{y_i} + \sum_{i,j=1}^d \frac{1}{2} \tilde{a}_{ij}(y) u_{y_i y_j}, \quad \forall y \in \mathbb{R}^{d+1},$$

and $u \in C^2(\mathbb{R}^{d+1})$.

For $x \in \bar{\mathbb{H}}$ and $s \geq 0$, let $\mathbb{P}^{s,x}$ and $\mathbb{Q}^{s,x}$ be two solutions to the martingale problem associated to $\mathscr{A}_t$ with initial condition (s,x). We extend these probability measures from the measurable space $C_{\text{loc}}([0,\infty);\bar{\mathbb{H}})$ to the canonical space $C_{\text{loc}}([0,\infty);\mathbb{R}^d)$ in the following way

$$\begin{aligned} &\widetilde{\mathbb{P}}^{s,x}\left(\widetilde{\omega} \in C_{\text{loc}}([0,\infty);\mathbb{R}^{d+1}), \exists t \geq 0, \widetilde{\omega}_0(t) \neq t+s\right) = 0, \\ &\widetilde{\mathbb{P}}^{s,x}\left(\widetilde{\omega} \in C_{\text{loc}}([0,\infty);\mathbb{R}^{d+1}), (\widetilde{\omega}_1(t_i), \ldots, \widetilde{\omega}_d(t_i)) \in B_i, i = 1, \ldots, m\right) \\ &\qquad = \mathbb{P}^{s,x}\left(\omega \in C_{\text{loc}}([0,\infty);\bar{\mathbb{H}}), (\omega_1(t_i), \ldots, \omega_d(t_i)) \in B_i, i = 1, \ldots, m\right), \end{aligned}$$

for all $m \geq 1$, $0 \leq t_1 \leq t_2 \leq \ldots \leq t_m$, $B_i \in \mathscr{B}(\mathbb{R}^d)$, $i = 1, \ldots, m$. Above, we used the notation

$$\begin{aligned}\tilde{\omega} &:= (\tilde{\omega}_0, \tilde{\omega}_1, \ldots, \tilde{\omega}_d), \quad \forall \tilde{\omega} \in C_{\text{loc}}([0,\infty); \mathbb{R}^{d+1}), \\ \omega &:= (\omega_1, \ldots, \omega_d), \quad \forall \omega \in C_{\text{loc}}([0,\infty); \bar{\mathbb{H}}).\end{aligned}$$

Similarly, we build $\widetilde{\mathbb{Q}}^{s,x}$ the extension of $\mathbb{Q}^{s,x}$ from $C_{\text{loc}}([0,\infty); \bar{\mathbb{H}})$ to $C_{\text{loc}}([0,\infty); \mathbb{R}^d)$. Notice that $\widetilde{\mathbb{P}}^{s,x}$ and $\widetilde{\mathbb{Q}}^{s,x}$ are two solutions to the classical time-homogeneous martingale problem associated to $\mathscr{G}$, with initial condition (s,x). Therefore, the probability measures $\mathbb{P}^{s,x}$ and $\mathbb{Q}^{s,x}$ coincide if their extensions $\widetilde{\mathbb{P}}^{s,x}$ and $\widetilde{\mathbb{Q}}^{s,x}$ coincide. By Proposition 2.4.6, uniqueness in law holds for $\widetilde{\mathbb{P}}^{s,x}$ and $\widetilde{\mathbb{Q}}^{s,x}$ if, for any $y = (y_0, \ldots, y_d) \in \mathbb{R}^{d+1}$ and any two solutions $\widetilde{\mathbb{P}}^y_i$, $i = 1, 2$, to the classical martingale problem associated to $\mathscr{G}$ with initial condition y, their one-dimensional marginal distribution coincide. For $i = 1, 2$, let Y^i be the weak solution to (2.4.59) with initial condition $Y^i(0) = y$ such that the law of Y^i is given by $\widetilde{\mathbb{P}}^y_i$ (see [47, Proposition 5.4.6 & Corollary 5.4.8]). Then, the one-dimensional marginal distributions agree for the probability measure $\widetilde{\mathbb{P}}^y_i$, $i = 1, 2$, if and only if they agree for the stochastic processes Y^i, $i = 1, 2$. Next, we show that uniqueness of the one-dimensional marginal distributions of Y^i, $i = 1, 2$, holds. For this purpose, we consider two cases.

Case 1 ($y_d < 0$ or $y_0 < 0$)**.** In this case, the coefficients $\tilde{b}$ and $\tilde{\sigma}$ are identically zero on a neighborhood of y, and so the unique solution, Y, to (2.4.59) is given by $Y(t) = y$, for all $t \geq 0$. It is obvious that the one-dimensional marginal distributions of solutions Y^i, $i = 1, 2$, to (2.4.59) are uniquely determined in this situation.

Therefore, by Proposition 2.4.6, uniqueness in law holds for solutions to (2.4.59) if the one-dimensional marginal distributions are uniquely determined for any initial condition $y \in \mathbb{R}^{d+1}$ with $y_d \geq 0$ and $y_0 \geq 0$.

Case 2 ($y_d \geq 0$ and $y_0 \geq 0$)**.** Note that any weak solution, $(Y(t))_{t \geq 0}$, to (2.4.59) with initial condition $Y(0) = y$, satisfies the property that

$$Y_d(t) \geq 0 \quad \mathbb{P}\text{-a.s.}, \quad \forall t \geq 0. \tag{2.4.60}$$

If this were not so, then there would be an $\varepsilon > 0$ such that Y_d reached the level $-\varepsilon$ with non-zero probability. By the preceding case, we observe that the Y_d would remain at the level $-\varepsilon$ for any subsequent time. By the continuity of paths, Y_d would have hit $-\varepsilon/2$ at a preceding time, and again, the preceding case would imply that Y_d remained at $-\varepsilon/2$ for all subsequent times. But this would contradict our assumption and therefore, (2.4.60) holds.

Any weak solution, $(Y(t))_{t\geq 0}$, to (2.4.59) with initial condition $Y(0) = y$ gives a solution, $(\widehat{X}(t))_{t\geq y_0}$,

$$\widehat{X}(t) = (Y_1(t-y_0), Y_2(t-y_0), \ldots, Y_d(t-y_0)) \quad \forall t \geq y_0, \tag{2.4.61}$$

to the stochastic differential equation

$$d\widehat{X}_i(t) = \widetilde{b}_i(t, \widehat{X}(t))dt + \sum_{j=1}^{d} \widetilde{\sigma}_{ij}(t, \widehat{X}(t))dW_j(t), \quad i = 1, \ldots, d, \quad \forall t \geq y_0,$$

with initial condition

$$\widehat{X}(y_0) = (Y_1(0), \ldots, Y_d(0)) = (y_1, \ldots, y_d) \in \bar{\mathbb{H}}.$$

Moreover, X remains in $\bar{\mathbb{H}}$, for all $t \geq y_0$, by (2.4.60).

Therefore, the one-dimensional marginal distributions of Y are uniquely determined if the marginal distributions of $\widehat{X}$ are uniquely determined. But, the last statement is implied if the one-dimensional marginal distributions of any solution $\mathbb{P}^{s,x}$ to the martingale problem associated to $\mathscr{A}_t$, with initial condition $(s, x) \in [0, \infty) \times \bar{\mathbb{H}}$, are uniquely determined.

Combining the conclusions of the preceding two cases completes the proof of the corollary.

□

Finally, we have

Proof of Theorem 2.1.10. The result follows from Theorem 2.4.3 which asserts the existence of solutions to the martingale problem associated to $\mathscr{A}_t$, while Proposition 2.4.5 and Corollary 2.4.7 show that the solution is unique. Therefore, the martingale problem associated to $\mathscr{A}_t$ is well-posed, for any initial condition $(s, x) \in [0, \infty) \times \bar{\mathbb{H}}$. □

Proof of Theorem 2.1.16. By Theorem 2.4.3, we obtain existence of weak solutions to (2.1.4). Since each weak solution induces a probability measure on $C_{\text{loc}}([0,\infty);\bar{\mathbb{H}})$ which solves the martingale problem associated to $\mathscr{A}_t$, we obtain by Theorem 2.1.10 that the probability law of the weak solutions to (2.1.4) is uniquely determined.

To prove the strong Markov property of weak solutions to (2.1.4), we consider again the time-homogeneous SDE (2.4.59) from the proof of Corollary 2.4.7. The same argument as the one used to conclude that the martingale problem associated to $\mathscr{A}_t$ is well-posed can be used to conclude that the classical martingale problem associated to the SDE (2.4.59) is well-posed. Therefore, by [47, Theorem 5.4.20], we obtain that for any $y \in \mathbb{R}^{d+1}$, the weak solution Y^y to (2.4.59) started at y possesses the strong Markov property, that is for any stopping time T of $\{\mathscr{B}_t(C_{\text{loc}}([0,\infty);\mathbb{R}^{d+1}))\}_{t\geq 0}$, any Borel measurable set $B \in \mathscr{B}(\mathbb{R}^{d+1})$ and $u \geq 0$, we have

$$\widetilde{\mathbb{P}}^y(Y(T+u) \in B|\mathscr{B}_T(C_{\text{loc}}([0,\infty);\mathbb{R}^{d+1})) = \widetilde{\mathbb{P}}^y(Y(T+u) \in B|Y(T)), \tag{2.4.62}$$

where $\widetilde{\mathbb{P}}^y$ denotes the probability law of the process Y started at y. Let $(s,x) \in [0,\infty) \times \bar{\mathbb{H}}$ and let $\widehat{X}^{s,x}$ be the unique weak solution of (2.1.4) with initial condition $\widehat{X}^{s,x}(s) = x$. Let $\mathbb{P}^{s,x}$ denote the probability law of $\widehat{X}^{s,x}$. Then, by analogy with (2.4.61), we notice that

$$Y^{s,x}(t) := \left(t+s, \widehat{X}_1(t+s), \ldots, \widehat{X}_d(t+s)\right) \quad t \geq 0,$$

is a solution to (2.4.59) with initial condition (s,x). Therefore, (2.4.62) can be rewritten in terms of the probability law of $\widehat{X}^{s,x}$, $\mathbb{P}^{s,x}$, as follows

$$\mathbb{P}^{s,x}(\widehat{X}(T+u) \in B|\mathscr{B}_T(C_{\text{loc}}([0,\infty);\bar{\mathbb{H}})) = \mathbb{P}^{s,x}(\widehat{X}(T+u) \in B|X(T)), \tag{2.4.63}$$

for any stopping time T of $(\mathscr{B}_t(C_{\text{loc}}([0,\infty);\bar{\mathbb{H}})))_{t\geq 0}$, any Borel measurable set $B \in \mathscr{B}(\bar{\mathbb{H}}))$ and $u \geq s$. Thus, $\widehat{X}^{s,x}$ satisfies the strong Markov property. $\square$

2.4.3 Matching one-dimensional marginal probability distributions

We can now complete the proof of Theorem 2.1.19. For simplicity, we denote

$$\alpha(t) := \xi(t)\xi^*(t), \quad \forall t \geq 0.$$

First, we prove the analogue of Proposition 2.4.1 for the Itô process (2.1.5).

Proposition 2.4.8. *Let X be the Itô process* (2.1.5)*, such that $X(0) \in \bar{\mathbb{H}}$. Assume the coefficients σ and b_d defined by* (2.1.21) *and* (2.1.22) *(now defined on $[0,\infty) \times \mathbb{R}^d$) satisfy* (2.4.1) *and* (2.4.2)*, respectively. Then*

$$\mathbb{P}\left(X(t) \in \bar{\mathbb{H}} | X(0)\right) = 1, \quad \forall t \geq 0. \tag{2.4.64}$$

Proof. The argument is similar to the proof of Proposition 2.4.1. We include it for completeness. It suffices to show that, for any $\varepsilon > 0$, we have

$$\mathbb{P}(X_d(t) \in (-\infty, -\varepsilon)) = 0. \tag{2.4.65}$$

Let $\varphi : \mathbb{R} \to [0,1]$ be the smooth cut-off function defined in the proof Proposition 2.4.1. Itô's rule gives

$$\begin{aligned}\varphi(X_d(t)) = \varphi(X_d(0)) &+ \int_0^t \left[\beta_d(s)\varphi'(X_d(s)) + \frac{1}{2}\alpha_{dd}(s)\varphi''(X_d(s))\right] ds\\ &+ \int_0^t \sigma_{d,i}(s)\varphi'(X_d(s))dW_i(s).\end{aligned}$$

By taking conditional expectations in the preceding expression and using the fact that the last term in that expression is a martingale by (2.1.18), we obtain

$$\begin{aligned}\mathbb{E}\left[\varphi(X_d(t))\right] &= \mathbb{E}\left[\varphi(X_d(0))\right] + \mathbb{E}\left[\int_0^t \left(\beta_d(s)\varphi'(X_d(s)) + \frac{1}{2}\alpha_{dd}(s)\varphi''(X_d(s))\right) ds\right]\\ &= \mathbb{E}\left[\varphi(X_d(0))\right] + \int_0^t \mathbb{E}\left[\mathbb{E}\left[\beta_d(s)|X(s)\right]\varphi'(X_d(s))\right.\\ &\quad \left. + \frac{1}{2}\mathbb{E}\left[\alpha_{dd}(s)|X(s)\right]\varphi''(X_d(s))\right] ds\\ &= \mathbb{E}\left[\varphi(X_d(0))\right] + \mathbb{E}\left[\int_0^t \left(b_d(s,X(s))\varphi'(X_d(s))\right.\right.\\ &\quad \left.\left. + \frac{1}{2}X_d^+(s)a_{dd}(s,X(s))\varphi''(X_d(s))\right) ds\right].\end{aligned}$$

We have $b_d(s,X(s))\varphi'(X(s)) \leq 0$ and $\varphi(X_d(0)) = 0$, while $\varphi(X_d(t)) \geq 0$. Therefore, $\mathbb{E}\left[\varphi(X_d(t))\right] \leq 0$ and thus $\mathbb{E}\left[\varphi(X_d(t))\right] = 0$, which yields (2.4.65). □

Next, we have

Proof of Theorem 2.1.19. Let $\widehat{X}$ be the unique weak solution to the mimicking stochastic differential equation (2.1.4) with initial condition $\widehat{X}(0) = X(0) = x$. As in the proof of Proposition 2.4.5, we need to show that for any $T \geq 0$ and $g \in C_0^\infty(\bar{\mathbb{H}})$, we have

$$\mathbb{E}\left[g(\widehat{X}(T))\right] = \mathbb{E}\left[g(X(T))\right]. \tag{2.4.66}$$

Let $v \in \mathscr{C}^{2+\alpha}(\bar{\mathbb{H}}_T)$ be defined by (2.4.52), (2.4.53) and (2.4.54). Then, (2.4.57) gives

$$\mathbb{E}\left[g(\widehat{X}(T))\right] = v(0,x). \tag{2.4.67}$$

We wish to prove that (2.4.67) holds with $X(T)$ in place of $\widehat{X}(T)$. We proceed as in the proof of Proposition 2.4.5. By applying Itô's rule to $v(t, X^\varepsilon(t))$, we obtain

$$\begin{aligned} dv(t,X^\varepsilon(t)) = &\left(v_t(t,X^\varepsilon(t)) + \sum_{i=1}^d \beta_i(t) v_{x_i}(t,X^\varepsilon(t)) + \sum_{i,j=1}^d \frac{1}{2}\alpha_{ij}(t) v_{x_ix_j}(t,X^\varepsilon(t)) \right) dt \\ &+ \sum_{i,j=1}^d \xi_{ij}(t) v_{x_i}(t,X^\varepsilon(t)) dW_j(t). \end{aligned}$$

The $dW_j(t)$-terms in the preceding identity are square-integrable, continuous martingales, because

$$[0,T] \ni t \mapsto v_{x_i}(t,X^\varepsilon(t))$$

are bounded processes since $v_{x_i} \in C([0,T]\times\bar{\mathbb{H}})$, and $\xi(t)$ is a square-integrable, adapted process by (2.1.18). Therefore,

$$\begin{aligned} &\mathbb{E}\left[v(T,X^\varepsilon(T))\right] = v(0,x^\varepsilon) \\ &\quad + \mathbb{E}\left[\int_0^T \left(v_t(t,X^\varepsilon(t)) + \sum_{i=1}^d \beta_i(t) v_{x_i}(t,X^\varepsilon(t)) + \sum_{i,j=1}^d \frac{1}{2}\alpha_{ij}(t) v_{x_ix_j}(t,X^\varepsilon(t)) \right) dt\right]. \end{aligned}$$

Using conditional expectations, we may rewrite the preceding identity as

$$\begin{aligned} \mathbb{E}\left[v(T,X^\varepsilon(T))\right] &= v(0,x^\varepsilon) + \int_0^T \mathbb{E}\left[\left(v_t(t,X^\varepsilon(t)) + \sum_{i=1}^d \beta_i(t) v_{x_i}(t,X^\varepsilon(t)) \right.\right. \\ &\qquad \left.\left. + \sum_{i,j=1}^d \frac{1}{2}\alpha_{ij}(t) v_{x_ix_j}(t,X^\varepsilon(t)) \right) \middle| X^\varepsilon(t) \right] dt \\ &= v(0,x^\varepsilon) + \mathbb{E}\left[\int_0^T \left(v_t(t,X^\varepsilon(t)) + \mathscr{A}_t v(t,X^\varepsilon(t))\right) dt\right]. \end{aligned}$$

Since $v_t(t,x) + \mathscr{A}_t v(t,x) = 0$, for all $(t,x) \in \mathbb{H}_T$, by letting $\varepsilon \downarrow 0$ in the preceding identity, we obtain

$$\mathbb{E}\left[g(X(T))\right] = v(0,x),$$

and this concludes the proof by (2.4.67). $\square$

Chapter 3

Hölder continuity for solutions to variational equations defined by degenerate elliptic operators

3.1 Introduction

The Heston stochastic volatility process, which is widely used as an asset price model in mathematical finance, is a paradigm for a degenerate diffusion process where the degeneracy in the diffusion coefficient is proportional to the square root of the distance to the boundary of the half-plane. The generator of this process with killing, called the elliptic Heston operator, is a second-order degenerate elliptic partial differential operator whose coefficients have linear growth in the spatial variables and where the degeneracy in the operator symbol is proportional to the distance to the boundary of the half-plane. With the aid of weighted Sobolev spaces, we prove supremum bounds, a Harnack inequality, and Hölder continuity near the boundary for solutions to elliptic variational equations defined by the Heston partial differential operator.

We require the portion of the boundary Γ_0 to be non-empty and consider a second-order, linear elliptic differential operator, A, on $\mathscr{O}$ which is degenerate along Γ_0. Suppose $f : \mathscr{O} \to \mathbb{R}$ is a source function. In this chapter, we prove local supremum bounds near the boundary portion, $\bar{\Gamma}_0$, and Hölder continuity up to $\bar{\Gamma}_0$, for suitably defined *weak* solutions, $u : \mathscr{O} \to \mathbb{R}$, to the elliptic boundary value problem,

$$Au = f \text{ a.e. on } \mathscr{O}, \quad u = 0 \text{ on } \Gamma_1, \tag{3.1.1}$$

together with a boundary Harnack inequality (near Γ_0) for non-negative, *weak* solutions to (3.1.1) when $f = 0$. Because A is degenerate along Γ_0 and weighted Sobolev spaces are required to establish existence of weak solutions to (3.1.1), these results do not follow from the standard theory for non-degenerate elliptic differential operators [41, 51].

No boundary condition is prescribed in problem (3.1.1) along Γ_0. Indeed, we recall from [18] that the problem (3.1.1) is well-posed when we seek solutions in suitable function spaces which describe their qualitative behavior near the boundary portion Γ_0: for example, continuity of derivatives up to Γ_0 via suitable weighted Hölder spaces (by analogy with [20]) or integrability of derivatives in a neighborhood of Γ_0 via suitable weighted Sobolev spaces (by analogy with [50]).

Similar results were obtained by Koch in the parabolic case [50, Proposition 4.5.1, Theorems 4.5.3 & 4.5.5]. While he used potential theory to obtain the Hölder continuity of solutions and the Harnack inequality, our method of proof is based on the Moser iteration technique. This is not a straightforward adaptation of results [41, Theorems 8.15, 8.20, 8.22 & 8.27], due to the fact that our Sobolev spaces are weighted, so the standard Sobolev inequality, Poincaré inequality and the John-Nirenberg inequality do not apply. The most difficult step in making the Moser iteration technique work involves a suitable application of the John-Nirenberg inequality. For this purpose, we use the so-called abstract John-Nirenberg inequality, due to Bombieri and Giusti [11, Theorem 4], which can be applied to any topological spaces endowed with a regular Borel measure satisfying some natural requirements. In order to verify the hypotheses of the abstract John-Nirenberg inequality, we prove a local version of the Poincaré inequality, Corollary 3.2.5, suitable for our weighted spaces.

In this chapter, we set $d = 2$ and choose A to be the generator of the two-dimensional Heston stochastic volatility process with killing [44], a degenerate diffusion process well known in mathematical finance and a paradigm for a broad class of degenerate Markov processes, driven by d-dimensional Brownian motion, and corresponding generators which are degenerate elliptic integro-differential operators:

$$\begin{aligned} Av := &-\frac{y}{2}\left(v_{xx} + 2\rho\sigma v_{xy} + \sigma^2 v_{yy}\right) \\ &- (r - q - y/2)v_x - \kappa(\theta - y)v_y + rv, \quad v \in C^\infty(\mathbb{H}). \end{aligned} \tag{3.1.2}$$

Throughout this chapter, the coefficients of A are required to obey

Assumption 3.1.1 (Ellipticity condition for the coefficients of the Heston operator)**.**

The coefficients defining A in (3.1.2) are constants obeying

$$\sigma \neq 0, -1 < \rho < 1, \tag{3.1.3}$$

and $\kappa > 0$, $\vartheta > 0$, $r \geq 0$, and $q \geq 0$.

For clarity of exposition in this chapter, we only consider the homogeneous Dirichlet boundary condition $u = 0$ on Γ_1 in (3.1.1), as the modifications of our main results to include the case of a inhomogeneous Dirichlet boundary condition, $u = g$ on Γ_1 for some $g : \mathscr{O} \cup \Gamma_1 \to \mathbb{R}$, are straightforward and similar modifications are described in [18].

3.1.1 Summary of main results

We shall state a selection of our main results here and then refer the reader to our guide to this chapter in §3.1.3. We commence with some mathematical preliminaries. As in [18, §2], we shall assume that the spatial domain has the following structure throughout this chapter:

Assumption 3.1.2 (Property of the domain near Γ_0)**.** For $\mathscr{O}$ as in §1.1, there is a positive constant, δ_0, such that for all $0 < \delta \leq \delta_0$,

$$\begin{aligned} \mathscr{O}^0_\delta := \mathscr{O} \cap (\mathbb{R} \times (0, \delta)) &= \Gamma_0 \times (0, \delta), \\ \Gamma_1 \cap (\mathbb{R} \times (0, \delta)) &= \partial\Gamma_0 \times (0, \delta), \end{aligned}$$

where $\Gamma_0 \subsetneqq \mathbb{R}$ is a finite union of open intervals.

Remark 3.1.3 (Need for the assumption on the domain near Γ_0)**.** If our setting had allowed for elliptic operators with variable coefficients, a^{ij}, b^i, c, with suitable regularity and growth properties, then we could replace Assumption 3.1.2 with the more geometric requirement that $\bar{\Gamma}_1 \pitchfork \{y = 0\}$ (C^k-transverse intersection, $k \geq 1$) by making use of C^k-diffeomorphisms of $\bar{\mathbb{H}}$ to "straighten" the boundary, Γ_1, near where it meets Γ_0.

We shall consider weak solutions to (3.1.1), so we introduce our weighted Sobolev spaces. For $1 \leq q < \infty$, let

$$L^q(\mathscr{O}, \mathfrak{w}) := \{u \in L^1_{\text{loc}}(\mathscr{O}) : \|u\|_{L^q(\mathscr{O},\mathfrak{w})} < \infty\}, \tag{3.1.4}$$

$$H^1(\mathscr{O}, \mathfrak{w}) := \{u \in L^2(\mathscr{O}, \mathfrak{w}) : (1+y)^{1/2}u, y^{1/2}|Du| \in L^2(\mathscr{O}, \mathfrak{w})\}, \tag{3.1.5}$$

$$H^2(\mathscr{O}, \mathfrak{w}) := \{u \in L^2(\mathscr{O}, \mathfrak{w}) : (1+y)^{1/2}u, (1+y)|Du|, y|D^2u| \in L^2(\mathscr{O}, \mathfrak{w})\}, \tag{3.1.6}$$

where $Du = (u_x, u_y)$, $D^2u = (u_{xx}, u_{xy}, u_{yx}, u_{yy})$, all derivatives of u are defined in the sense of distributions, and

$$\|u\|^q_{L^q(\mathscr{O},\mathfrak{w})} := \int_{\mathscr{O}} |u|^q \mathfrak{w}\, dxdy, \tag{3.1.7}$$

$$\|u\|^2_{H^1(\mathscr{O},\mathfrak{w})} := \int_{\mathscr{O}} \left(y|Du|^2 + (1+y)u^2\right) \mathfrak{w}\, dxdy, \tag{3.1.8}$$

$$\|u\|^2_{H^2(\mathscr{O},\mathfrak{w})} := \int_{\mathscr{O}} \left(y^2|D^2u|^2 + (1+y)^2|Du|^2 + (1+y)u^2\right) \mathfrak{w}\, dxdy, \tag{3.1.9}$$

with weight function $\mathfrak{w} : \mathbb{H} \to (0, \infty)$ given by

$$\mathfrak{w}(x,y) := y^{\beta-1}e^{-\gamma|x|-\mu y}, \quad (x,y) \in \mathbb{H}, \tag{3.1.10}$$

where the Feller parameters, β and μ, are defined by

$$\beta := \frac{2\kappa\vartheta}{\sigma^2} \quad \text{and} \quad \mu := \frac{2\kappa}{\sigma^2}, \tag{3.1.11}$$

and $0 < \gamma < \gamma_0(A)$, where γ_0 depends only on the constant coefficients of A in (3.1.2). We call

$$\begin{aligned} a(u,v) := &\frac{1}{2}\int_{\mathscr{O}} \left(u_xv_x + \rho\sigma u_yv_x + \rho\sigma u_xv_y + \sigma^2 u_yv_y\right) y\, \mathfrak{w}\, dxdy \\ &- \frac{\gamma}{2}\int_{\mathscr{O}} (u_x + \rho\sigma u_y)\, v\, \text{sign}(x) y\, \mathfrak{w}\, dxdy \\ &- \int_{\mathscr{O}} (a_1 y + b_1)u_x v\, \mathfrak{w}\, dxdy + \int_{\mathscr{O}} ruv\, \mathfrak{w}\, dxdy, \quad \forall u, v \in H^1(\mathscr{O}, \mathfrak{w}), \end{aligned} \tag{3.1.12}$$

the *bilinear form associated with the Heston operator*, A, in (3.1.2), noting that

$$a_1 := \frac{\kappa\rho}{\sigma} - \frac{1}{2} \quad \text{and} \quad b_1 := r - q - \frac{\kappa\vartheta\rho}{\sigma}. \tag{3.1.13}$$

We shall also avail of the

Assumption 3.1.4 (Condition on the coefficients of the Heston operator)**.** The coefficients defining A in (3.1.2) have the property that $b_1 = 0$ in (3.1.13).

Assumption 3.1.4 involves no significant loss of generality because, using a simple affine changes of variables on $\mathbb{R}^2$ which maps $(\mathbb{H}, \partial\mathbb{H})$ onto $(\mathbb{H}, \partial\mathbb{H})$ (see [18]), we can arrange that $b_1 = 0$.

The conditions (3.1.3) ensure that $y^{-1}A$ is uniformly elliptic on $\mathbb{H}$. Indeed,

$$\frac{y}{2}(\xi_1^2 + 2\rho\sigma\xi_1\xi_2 + \sigma^2\xi_2^2) \geq \nu_0 y(\xi_1^2 + \xi_2^2), \quad \forall(\xi_1, \xi_2) \in \mathbb{R}^2, \tag{3.1.14}$$

where

$$\nu_0 := \min\{1, (1-\rho^2)\sigma^2\}, \tag{3.1.15}$$

and $\nu_0 > 0$ by Assumption 3.1.1.

Given $T \subset \partial\mathscr{O}$, a relatively open subset, we let $H^1_0(\mathscr{O} \cup T, \mathfrak{w})$ be the closure in $H^1(\mathscr{O}, \mathfrak{w})$ of $C^\infty_0(\mathscr{O} \cup T)$. Given a source function $f \in L^2(\mathscr{O}, \mathfrak{w})$, we call a function $u \in H^1_0(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$ a *solution to the variational equation* for the Heston operator with *homogeneous* Dirichlet boundary condition on Γ_1 if

$$a(u, v) = (f, v)_{L^2(\mathscr{O}, \mathfrak{w})}, \quad \forall v \in H^1_0(\mathscr{O} \cup \Gamma_0, \mathfrak{w}). \tag{3.1.16}$$

If $u \in H^2(\mathscr{O}, \mathfrak{w})$, we recall from [18] that u is a solution to (3.1.1) if and only if $u \in H^1_0(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$ and u is a solution to (3.1.16).

We recall definition of the *Koch metric*, $\mathbf{d}$, on $\mathbb{H}$ introduced by Koch in [50, p. 11],

$$\mathbf{d}((z_1, z_2) := \frac{|z_1 - z_2|}{\sqrt{y_1 + y_2 + |z_1 - z_2|}}, \quad \forall z_i = (x_i, y_i) \in \bar{\mathbb{H}}, i = 1, 2. \tag{3.1.17}$$

The metric $\mathbf{d}$ is equivalent to the *cycloidal metric*, s, introduced by Daskalopoulos and Hamilton. Away from $z \in \bar{\mathbb{H}}$ the function $\mathbf{d}(\cdot, z)$ is smooth, while this is not true for $s(\cdot, z)$ even away from z. For this reason, it will be convenient to use the metric $\mathbf{d}$, instead of s, in Chapter 3. For $R > 0$ and $z_0 \in \bar{\mathscr{O}}$, we denote

$$\mathbf{B}_R(z_0) = \{z \in \mathscr{O} : \mathbf{d}(z, z_0) < R\}, \tag{3.1.18}$$

$$\mathbb{B}_R(z_0) = \{z \in \mathbb{H} : \mathbf{d}(z, z_0) < R\}. \tag{3.1.19}$$

Notation 3.1.5. Let $\bar{R} = \sqrt{\delta_0/2}$. Then, the following inclusions hold

$$\mathbf{B}_R(z_0) \subseteq \Gamma_0 \times (0, \delta_0) \subseteq \mathscr{O},$$

for any $0 < R \leq \bar{R}$ and $z_0 \in \bar{\Gamma}_0$. In the sequel, we assume without loss of generality that $\bar{R} \leq 1$.

Remark 3.1.6. As in [20, Theorem I.1.1], the assumption that $\kappa, \vartheta > 0$, i.e. that the coefficient multiplying v_y in the definition (3.1.2) of $-A$, is strictly positive is of crucial importance. We can notice from (3.1.11) that $\beta > 0$ and so, the weight $\mathfrak{w} \in L^1(\mathbb{H})$. Therefore, the volume of balls $\mathbf{B}_R(z_0)$ centered at points $z_0 \in \Gamma_0$ is finite with respect to the weight $\mathfrak{w}$, a fact that we use repeatedly in the arguments we employ. Clearly, if β were negative, then $\mathfrak{w} \in L^1_{\text{loc}}(\mathbb{H})$, but not in $L^1(\mathbb{H})$.

We have the following analogue of [50, Proposition 4.5.1] and [41, Theorem 8.15].

Theorem 3.1.7 (Supremum estimates at points in $\bar{\Gamma}_0$)**.** *Let $s > d + \beta$. Then there is a positive constant C, depending at most on the coefficients of A, δ_0 and s, such that for any $u \in H^1_0(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$ obeying* (3.1.16) *with source function $f \in L^s(\mathbf{B}_{\bar{R}}(z_0), \mathfrak{w})$, we have*

$$\begin{aligned} \operatorname*{ess\,sup}_{\mathbf{B}_R(z_0)} |u| \leq C \Big(&|\mathbf{B}_{2R}(z_0)|^{-1/2}_{\beta-1} \|u\|_{L^2(\mathbf{B}_{2R}(z_0), y^{\beta-1})} \\ &+ \|f\|_{L^s(\mathbf{B}_{2R}(z_0), y^{\beta-1})} \Big), \end{aligned} \tag{3.1.20}$$

for all $z_0 \in \bar{\Gamma}_0$ and all $R \in (0, \bar{R}/2]$.

For $z_0 \in \bar{\mathscr{O}}$ and $R > 0$, we denote

$$M_R := \operatorname*{ess\,sup}_{\mathbf{B}_R(z_0)} u(z), \quad m_R := \operatorname*{ess\,inf}_{\mathbf{B}_R(z_0)} u(z),$$

and we let

$$\operatorname*{osc}_{\mathbf{B}_R(z_0)} u := M_R - m_R$$

denote the oscillation of u over the ball $\mathbf{B}_R(z_0)$. From Theorem 3.1.7, we know that M_R and m_R are finite quantities and $\operatorname{osc}_{\mathbf{B}_R(z_0)}$ is well-defined for weak solutions u as in Theorem 3.1.7.

We have the following analogue of [41, Theorem 8.27 & 8.29] and [50, Theorem 4.5.5 & 4.5.6] for the boundary portion Γ_0.

Theorem 3.1.8 (Hölder continuity up to $\bar{\Gamma}_0$ for solutions to the variational equation)**.** *Let $u \in H^1_0(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$ obey* (3.1.16) *and let $f \in L^s(\mathbf{B}_{\bar{R}}(z_0), \mathfrak{w})$, where $s > \max\{2d, d + \beta\}$. Then there is a positive constant C, depending at most on the coefficients of A, s,*

δ_0, $\|f\|_{L^s(\mathbf{B}_{\bar{R}}(z_0),\mathfrak{w})}$ and $\|u\|_{L^\infty(\mathbf{B}_{\bar{R}}(z_0))}$, and there is a constant $\alpha_0 \in (0,1)$, depending at most on s and β, such that for $z_0 \in \bar{\Gamma}_0$ and $0 < 4R \leq \bar{R}$, we have

$$\operatorname*{osc}_{\mathbf{B}_R(z_0)} u \leq CR^{\alpha_0}. \tag{3.1.21}$$

Moreover, u is $C_s^{\alpha_0}$-Hölder continuous in $\bar{\mathbf{B}}_{\bar{R}}(z_0)$ and satisfies

$$|u(z_1) - u(z_2)| \leq C\mathbf{d}(z_1, z_2)^{\alpha_0}, \quad \forall z_1, z_2 \in \bar{\mathbf{B}}_{\bar{R}}(z_0). \tag{3.1.22}$$

Remark 3.1.9 (Comparison with the case of the boundary portion where the operator is non-degenerate)**.** The term $\sigma(\sqrt{RR_0})$, where $\sigma(R) := \operatorname{osc}_{\partial\mathscr{O}\cap\bar{\mathbf{B}}_R(z_0)} u$, which appears in [41, Equation (8.72)] in the statement of [41, Theorem 8.27] does not appear in the statement of our Theorem 3.1.8. The reason is that unlike in [41, Equation (8.71)], the test functions defined in the proof of Theorem 3.1.8 do not need to involve $\sup_{\partial\mathscr{O}\cap\bar{\mathbf{B}}_R(z_0)} u$ or $\inf_{\partial\mathscr{O}\cap\bar{\mathbf{B}}_R(z_0)} u$ since no boundary condition is imposed on v along Γ_0, in contrast to the Dirichlet boundary condition assumed for v in the proofs of [41, Theorem 8.18 & 8.26].

We also have the following analogue of [41, Theorem 8.20] and [50, Theorem 4.5.3].

Theorem 3.1.10 (Boundary Harnack inequality near Γ_0)**.** *Then there is a positive constant C, depending at most on the coefficients of A and δ_0, such that for any non-negative $u \in H^1(\mathscr{O}, \mathfrak{w})$ obeying (3.1.16) with $f = 0$ on $\mathbf{B}_{\bar{R}}(z_0)$, we have*

$$\sup_{\mathbf{B}_R(z_0)} u \leq C \inf_{\mathbf{B}_R(z_0)} u, \tag{3.1.23}$$

for all $z_0 \in \Gamma_0$ and $0 < 4R \leq \max\{\bar{R}, dist(z_0, \Gamma_1)\}$.

Previous results ([58, 59], [15], [28, 29], [36], [65]) on supremum bounds, Hölder continuity and Harnack continuity for solutions to degenerate elliptic partial differential equations do not apply in our setting, mainly because of differences between the following principal features in the previous results and those in this chapter:

1. Structure of the differential operators, including the nature of the degeneracy and presence of lower-order terms;

2. Boundary conditions, where no boundary condition is specified on Γ_0 (the "degenerate" portion of the boundary $\partial\mathscr{O}$) and a Dirichlet condition is prescribed along Γ_1 (the "non-degenerate" portion of the boundary $\partial\mathscr{O}$);

3. Weights used to define weighted Sobolev spaces and weak solutions;

4. Dependency of the constants in estimates, with those appearing in our estimates depending at most on the $L^q(B\cap\mathscr{O},\mathfrak{w})$ norm ($q>2$) of f on neighborhoods B of boundary points, the $L^2(\mathscr{O},\mathfrak{w})$ norm of u, the geometry of Γ_1, and the constant coefficients of A.

Furthermore, near Γ_0, the weights $y\mathfrak{w}$ and $\mathfrak{w}$ used in our definition of $H^1(\mathscr{O},\mathfrak{w})$ are $y^\beta e^{-\gamma|x|-\mu y}$ and $y^{\beta-1}e^{-\gamma|x|-\mu y}$, respectively, with constants $0<\beta,\mu<\infty$ depending on the coefficients of u_{yy} and u_y when our differential operator A is expressed in standard divergence form and so the weight $\mathfrak{w}$ depends on both the second and first-order parts of A and not just on the second-order part of the differential operator, unlike in the cited references. Note also that $\mathfrak{w}$ is zero along Γ_0, where A is degenerate, but positive along Γ_1, where A is non-degenerate.

Koch [50] considers certain linear elliptic and parabolic degenerate model partial differential equations in divergence form, with a degeneracy similar to ours, and which arise as linearizations of the porous medium equation. However, while Koch uses Sobolev weights which are comparable to ours, his methods (which use pointwise estimates for fundamental solutions and Moser iteration) are different from ours (which use Moser iteration and the abstract John-Nirenberg inequality). Moreover, he does not consider the case where $\partial\mathscr{O}=\Gamma_0\cup\bar{\Gamma}_1$, where A is degenerate along Γ_0 but non-degenerate along Γ_1.

3.1.2 Extensions to degenerate operators in higher dimensions

The Heston stochastic volatility process and its associated generator serve as paradigms for degenerate Markov processes and their degenerate elliptic generators which appear widely in mathematical finance.

Generalizations of the Heston process to higher-dimensional, degenerate diffusion processes may be accommodated by extending the framework developed in this chapter and we shall describe extensions in a sequel. First, the two-dimensional Heston process has natural d-dimensional analogues [35] defined, for example, by coupling non-degenerate $(d-1)$-diffusion processes with degenerate one-dimensional processes [16, 57, 76]. Elliptic differential operators arising in this way have time-independent, affine coefficients but, as one can see from standard theory [41, 51, 52, 54] and previous work of Daskalopoulos and her collaborators [20, 21] on the porous medium equation, we would not expect significant new difficulties to arise when extending the methods and results of this chapter to the case of elliptic and parabolic operators in higher dimensions and variable coefficients, depending on both spatial variables or time and possessing suitable regularity and growth properties.

3.1.3 Brief outline of the chapter

We begin in §3.2 by describing a Sobolev inequality due to H. Koch [50] and prove a Poincaré inequality for our weighted Sobolev spaces. In §3.3, we recall the abstract John-Nirenberg inequality (Theorem 3.3.1) due to E. Bombieri and E. Giusti [11] and justify its application (via Proposition 3.3.2) in the setting of our weighted Sobolev spaces. The supremum estimate near $\bar{\Gamma}_0$ for solutions to the variational equation (3.1.16) (Theorem 3.1.7) is proved in §3.4 by adapting the Moser iteration technique employed in the proof of [41, Theorem 8.15] to the setting of our degenerate elliptic operators and weighted Sobolev spaces. Section 3.5 contains our proof of local Hölder continuity along $\bar{\Gamma}_0$ of solutions to the variational equation (3.1.16) (Theorem 3.1.8). The essential difference between the proof of Theorem 3.1.8 and the proof of its classical analogue for weak solutions to non-degenerate elliptic equations [41, Theorems 8.27 & 8.29] consists in a modification of the methods of [41, §8.6, §8.9, & §8.10] when deriving our energy estimates (3.5.11), where we adapt the application of the John-Nirenberg inequality and Poincaré inequality to our framework of weighted Sobolev spaces. Finally, in §3.6 we prove the Harnack inequality (Theorem 3.1.10) for solutions to the variational equation (3.1.16). Appendix B contains the proofs of auxiliary results employed throughout the chapter whose proofs are sufficiently technical that they would have otherwise interrupted the logical flow of the chapter.

3.1.4 Notation and conventions

Throughout Chapter 3, we fix $d = 2$ and set

$$p := \frac{2(d+\beta)}{d+\beta-1}, \tag{3.1.24}$$

for any $\beta > 0$, as used in Lemma 3.2.1 and the sequel. We keep track of the dependency of many of our estimates on the dimension, d, of $\mathbb{H} = \mathbb{R}^{d-1} \times (0, \infty)$ in our analysis, even though $d = 2$ in this chapter, as this will make it easier to extend our results to partial differential equations on domains in $\mathbb{H}$ which preserve the key features of (3.1.1).

If $S \subset \bar{\mathbb{H}}$ is a Borel measurable subset, we let $|S|_\beta$ denote the volume of S with respect to the measure $y^\beta\, dz$, and $|S|_{\mathfrak{w}}$ denote the volume of S with respect to the measure $\mathfrak{w}\, dz$.

In many of our proofs, we will make use of a sequence of cutoff functions $(\eta_N)_{N\in\mathbb{N}}$. Let $\varphi : \mathbb{R} \to [0, 1]$ be a smooth function such that $\varphi(x) \equiv 1$ for $x < 0$, and $\varphi \equiv 0$ for $x > 1$. Let $z_0 \in \mathbb{H}$ and let $(R_N)_{N\in\mathbb{N}}$ be an non-increasing sequence of positive numbers. We define

$$\eta_N(z) := \varphi\left(\frac{1}{R_{N-1}^2 - R_N^2}(d^2(z_0, z) - R_N^2)\right), \quad \forall z \in \bar{\mathbb{H}}, \forall N \in \mathbb{N}, \tag{3.1.25}$$

Then, the sequence $(\eta_N)_{N\geq 1}$ satisfies the following properties

$$\eta_N|_{\mathbf{B}_{R_N}(z_0)} \equiv 1, \quad \eta_N|_{\mathbf{B}^c_{R_{N-1}(z_0)}} \equiv 0, \tag{3.1.26}$$

$$|\nabla \eta_N| \leq \frac{C}{R_{N-1}^2 - R_N^2}, \tag{3.1.27}$$

where $\mathbf{B}^c_{R_{N-1}(z_0)} := \mathbb{H} \setminus \bar{\mathbf{B}}_{R_{N-1}}(z_0)$ and C is a positive constant independent of N and the sequence $(R_N)_{N\in\mathbb{N}}$. The bound in (3.1.27) can be deduced from the calculation,

$$\nabla \eta_N = \varphi'\left(\frac{1}{R_{N-1}^2 - R_N^2}(d^2(z_0, z) - R_N^2)\right)\frac{1}{R_{N-1}^2 - R_N^2}\nabla d^2(z_0, z).$$

Also, we have that $|\nabla d^2(z_0, z)| \leq 5$, for all $z_0, z \in \mathbb{H}$. Since φ' is also uniformly bounded, we obtain (3.3.7).

Similarly, we can construct a sequence of cutoff functions $(\eta_N)_{N\in\mathbb{N}}$, when $(R_N)_{N\in\mathbb{N}}$ is a non-decreasing sequence of positive numbers.

3.2 Sobolev and Poincaré inequalities for weighted Sobolev spaces

We review a Sobolev inequality (Lemma 3.2.1) due to H. Koch [50] and prove a Poincaré inequality (Lemma 3.2.4) for weighted Sobolev spaces.

Recall from [50, Corollary 4.3.4] that the weight $y^{\beta-1}$ defines a doubling measure, $y^{\beta-1}\,dz$ on $\mathbb{H}$ for any $\beta>0$ (see, for example, [71, Definition 1.2.6]), where $dz = dxdy$ is Lebesgue measure on $\mathbb{H}$.

Lemma 3.2.1 (Weighted Sobolev inequality)**.** *[50, Lemma 4.2.4] Let p be as in* (3.1.24)*. Then there is a positive constant $C=C(d,p)$ such that*

$$\int_{\mathbb{H}} |u|^p y^{\beta-1}\,dx\,dy \leq c\left(\int_{\mathbb{H}} |u|^2 y^{\beta-1}\,dx\,dy\right)^{\frac{p-2}{2}} \int_{\mathbb{H}} |\nabla u|^2 y^{\beta}\,dx\,dy, \tag{3.2.1}$$

for any $u\in L^2\left(\mathbb{H},y^{\beta-1}\right)$ such that $\nabla u \in L^2\left(\mathbb{H},y^{\beta}\right)$.

Lemma 3.2.2. *[50, Lemma 4.3.3] There is a positive constant c, depending only on d and β, such that, for any $R>0$ and $z_0\in\bar{\mathbb{H}}$,*

$$c^{-1}R^n(R+\sqrt{y_0})^{d+2\beta} \leq |\mathbb{B}_R(z_0)|_\beta \leq cR^n(R+\sqrt{y_0})^{d+2\beta}. \tag{3.2.2}$$

Moreover, the following inclusions hold,

$$B_{R_1}(z_0) \subseteq \mathbb{B}_R(z_0) \subseteq B_{R_2}(z_0), \tag{3.2.3}$$

where $R_1 = R\left(R+\sqrt{y_0}\right)/2000$ and $R_2 = R\left(R+2\sqrt{y_0}\right)$.

Remark 3.2.3. The technical assumption, $0<R\leq \bar{R}$, in the hypotheses of Lemmas 3.2.4, 3.2.6 and Corollary 3.2.5, is used to ensure

$$\mathbf{B}_R(z_0) \subseteq \Gamma_0\times(0,\delta_0), \quad \forall z_0\in\bar{\Gamma}_0.$$

This property is used in the construction of the extension operator E in Lemma 3.2.6, and therefore it is implicitly used in Lemma 3.2.4 and Corollary 3.2.5.

Lemma 3.2.4 (Poincaré inequality). *Let $z_0 \in \bar{\Gamma}_0$ and $0 < R \leq \bar{R}$. Then there is a positive constant C, depending on β and R, such that for any $u \in H^1(\mathbf{B}_R(z_0), \mathfrak{w})$, we have*

$$\inf_{c\in\mathbb{R}} \left(\int_{\mathbf{B}_R(z_0)} |u(z) - c|^2 y^{\beta-1}\, dz \right)^{1/2} \leq C \left(\int_{\mathbf{B}_R(z_0)} |\nabla u(z)|^2 y^{\beta}\, dz \right)^{1/2}. \tag{3.2.4}$$

Corollary 3.2.5 (Poincaré inequality with scaling). *There is a positive constant C, depending only on β and $\bar{R}$, such that for any $u \in H^1(\mathbf{B}_{\bar{R}}(z_0), \mathfrak{w})$ and $z_0 \in \bar{\Gamma}_0$, with $0 < R \leq \bar{R}$, we have*

$$\begin{aligned} \inf_{c\in\mathbb{R}} &\left(\frac{1}{|\mathbf{B}_R(z_0)|_{\beta-1}} \int_{\mathbf{B}_R(z_0)} |u(z) - c|^2 y^{\beta-1}\, dz \right)^{1/2} \\ &\leq CR^2 \left(\frac{1}{|\mathbf{B}_R(z_0)|_{\beta}} \int_{\mathbf{B}_R(z_0)} |\nabla u(z)|^2 y^{\beta}\, dz \right)^{1/2}. \end{aligned} \tag{3.2.5}$$

To prove Lemma 3.2.4 and Corollary 3.2.5, we make use of the following extension property

Lemma 3.2.6 (Extension operator). *Let $z_0 \in \bar{\Gamma}_0$ and $0 < R \leq \bar{R}$. Let $D = (a, b) \times (0, c)$ be a rectangle such that $\mathbf{B}_R(z_0) \subseteq D$. Then, there exists a continuous extension*

$$E : H^1(\mathbf{B}_R(z_0), \mathfrak{w}) \to H^1(D, \mathfrak{w}),$$

and there exists a positive constant C, depending on D, R and β, such that for any $u \in H^1(\mathbf{B}_R(z_0), \mathfrak{w})$ we have

$$\begin{aligned} \|Eu\|_{L^2(D,y^{\beta-1})} &\leq C\|u\|_{L^2(\mathbf{B}_R(z_0),y^{\beta-1})}, \\ \|\nabla Eu\|_{L^2(D,y^{\beta})} &\leq C\|\nabla u\|_{L^2(\mathbf{B}_R(z_0),y^{\beta})}. \end{aligned} \tag{3.2.6}$$

Remark 3.2.7. Without loss of generality, in the proofs of Lemmas 3.2.4, 3.2.6 and Corollary 3.2.5, we may assume $z_0 = (0, 0)$ and

$$\mathbb{B}_R(z_0) \cap \{(x, y) \in \mathbb{H} : x > 0\} \subseteq \mathbf{B}_R(z_0).$$

Proof of Lemma 3.2.4. Let $u \in H^1(\mathbf{B}_R(z_0), \mathfrak{w})$ and choose $a, b \in \mathbb{R}$ and $\delta > 0$, depending only on R, such that $\mathbf{B}_R(z_0) \subseteq (a, b) \times (0, \delta)$. Let $k > 1$ be such that

$$2k^{-\beta} = \frac{1}{2}, \tag{3.2.7}$$

and denote by $D = (a,b) \times (0,k\delta)$. Let $\hat{u} = Eu$ be the extension of u to D given by Lemma 3.2.6. Assuming that (3.2.4) holds for $\hat{u}$, we obtain that it holds for u also in the following way,

$$\begin{aligned}\inf_{c\in\mathbb{R}} \left(\int_{\mathbf{B}_R(z_0)} |u(z)-c|^2 y^{\beta-1}\,dz \right)^{1/2} &\le \inf_{c\in\mathbb{R}} \left(\int_D |\hat{u}(z)-c|^2 y^{\beta-1}\,dz \right)^{1/2} \\ &\le C \left(\int_D |\nabla \hat{u}(z)|^2 y^{\beta}\,dz \right)^{1/2} \\ &\le C \left(\int_{\mathbf{B}_R(z_0)} |\nabla u(z)|^2 y^{\beta}\,dz \right)^{1/2}.\end{aligned}$$

In the first and last inequalities above, we made use of (3.2.6).

Therefore, we may assume $u \in H^1(D,\mathfrak{w})$. Our goal is to prove that (3.2.4) holds for $u \in H^1(D,\mathfrak{w})$. By [18, Corollary A.14], we may assume without loss of generality that $u \in C^1(\bar{D})$. Let $c \in \mathbb{R}$ and let $v = u - c$. Then, by the mean value theorem, we have for any $y \in (0,\delta)$ and $x \in (a,b)$

$$v(x,y) = v(x,ky) + \int_{ky}^{y} v_y(x,t)dt.$$

Squaring both sides of the preceding equation and integrating in y with respect to $y^{\beta-1}\,dy$, we obtain

$$\int_0^\delta |v(x,y)|^2 y^{\beta-1}\,dy \le 2\int_0^\delta |v(x,ky)|^2 y^{\beta-1}\,dy + 2\int_0^\delta \left| \int_{ky}^{y} v_y(x,t)dt \right|^2 y^{\beta-1}\,dy. \quad (3.2.8)$$

By applying the change of variable $y' = ky$, we see that

$$\int_0^\delta |v(x,ky)|^2 y^{\beta-1}\,dy = k^{-\beta} \int_0^{k\delta} |v(x,y')|^2 y'^{\beta-1}\,dy'. \quad (3.2.9)$$

Also, we have for $\beta \neq 1$,

$$\begin{aligned}\int_0^\delta \left| \int_{ky}^{y} v_y(x,t)dt \right|^2 y^{\beta-1}\,dy &= \int_0^\delta \left| \int_{ky}^{y} v_y(x,t) t^{\beta/2} t^{-\beta/2} dt \right|^2 y^{\beta-1}\,dy \\ &\le \frac{1}{|1-\beta|} \int_0^\delta \int_y^{ky} |v_y(x,t)|^2 t^\beta dt \left| y^{-\beta+1} - (ky)^{-\beta+1} \right| y^{\beta-1}\,dy \quad (3.2.10) \\ &\le \delta \frac{1+k^{-\beta+1}}{|1-\beta|} \int_0^{k\delta} |v_y(x,y)|^2 y^\beta\,dy.\end{aligned}$$

For $\beta = 1$, we have

$$\begin{aligned}
\int_0^\delta \left| \int_{ky}^y v_y(x,t)dt \right|^2 dy &= \int_0^\delta \left| \int_{ky}^y v_y(x,t)t^{1/2}t^{-1/2}dt \right|^2 dy \\
&\leq \int_0^\delta \int_y^{ky} |v_y(x,t)|^2 t dt \log \frac{ky}{y}\, dy \qquad (3.2.11) \\
&\leq \delta \log k \int_0^{k\delta} |v_y(x,y)|^2 y\, dy.
\end{aligned}$$

Define a positive constant $C_0 \equiv C_0(\beta,\delta)$ by $C_0 = 2\delta \frac{1+k^{-\beta+1}}{|1-\beta|}$ when $\beta \neq 1$, and $C_0 = 2\delta \log k$ when $\beta = 1$. By combining equations (3.2.8), (3.2.9), (3.2.10) and (3.2.11), we obtain

$$\begin{aligned}
\int_0^\delta |v(x,y)|^2 y^{\beta-1}\, dy &\leq 2k^{-\beta} \int_0^{k\delta} |v(x,y)|^2 y^{\beta-1}\, dy + C_0 \int_0^{k\delta} |v_y(x,y)|^2 y^\beta\, dy \\
&\leq 2k^{-\beta} \int_0^\delta |v(x,y)|^2 y^{\beta-1}\, dy + 2k^{-\beta} \int_\delta^{k\delta} |v(x,y)|^2 y^{\beta-1}\, dy \\
&\quad + C_0 \int_0^{k\delta} |v_y(x,y)|^2 y^\beta\, dy.
\end{aligned}$$

Recall that $k > 1$ was chosen such that (3.2.7) is satisfied. Therefore, by integrating also in x, there exists $C = C(\beta,\delta)$ such that

$$\begin{aligned}
&\int_a^b \int_0^{k\delta} |v(x,y)|^2 y^{\beta-1}\, dy\, dx \\
&\quad \leq C \int_a^b \int_\delta^{k\delta} |v(x,y)|^2 y^{\beta-1}\, dy\, dx + C \int_a^b \int_0^{k\delta} |v_y(x,y)|^2 y^\beta\, dy\, dx.
\end{aligned}$$

Since $v = u - c$, we have

$$\begin{aligned}
&\inf_{c\in\mathbb{R}} \int_D |u(x,y) - c|^2 y^{\beta-1}\, dy\, dx \\
&\quad \leq C \inf_{c\in\mathbb{R}} \int_a^b \int_\delta^{k\delta} |u(x,y) - c|^2 y^{\beta-1}\, dy\, dx + C \int_D |u_y(x,y)|^2 y^\beta\, dy\, dx.
\end{aligned}$$

The rectangle $D' := [a,b] \times [\delta, k\delta]$ is contained in $\{y > 0\}$, so the weighted measure $y^{\beta-1}\, dy\, dx$ is equivalent to the Lebesgue measure $dydx$. The rectangle D' is a convex domain and so we may apply the classical Poincaré inequality [41, Equation (7.45)] to give

$$\inf_{c\in\mathbb{R}} \int_a^b \int_\delta^{k\delta} |u(x,y) - c|^2 y^{\beta-1}\, dy\, dx \leq C \int_a^b \int_\delta^{k\delta} |\nabla u(x,y)|^2 y^\beta\, dy\, dx.$$

Combining the last two inequalities yields (3.2.4). $\square$

Remark 3.2.8. Koch states a weighted Poincaré inequality on the half-space [50, Lemma 4.4.4], with weight $y^{\beta-1}e^{-\kappa\rho(z,z_0)}$, where κ is a positive constant, z_0 is a fixed point in $\bar{\mathbb{H}}$, and $\rho(z,z_0)$ is equivalent to $\mathbf{d}^2(z,z_0)$, in the sense that there exists a constant $c>0$ such that

$$cd^2(z,z_0) \leq \rho(z,z_0) \leq \frac{1}{c}d^2(z,z_0), \forall z \in \mathbb{H}.$$

The proof of this result is long and technical. So, rather than use this result to prove a weighted Poincaré inequality on a ball using an extension principle, we give a much simpler proof for balls and weights $y^{\beta-1}$ and y^{β}.

Remark 3.2.9. When $\beta \geq 1$, from [18, Lemma A.1 & A.4] we have that $H^1_0(\mathscr{O},\mathfrak{w}) = H^1_0(\mathscr{O}\cup\Gamma_0,\mathfrak{w})$. Then, as in [2, Theorem 6.30], it might be true that the stronger version of (3.2.4) holds

$$\left(\int_{\mathbf{B}_R(z_0)} |u(z)|^2 y^{\beta-1}\,dz\right)^{1/2} \leq C\left(\int_{\mathbf{B}_R(z_0)} |\nabla u(z)|^2 y^{\beta}\,dz\right)^{1/2}. \tag{3.2.12}$$

Remark 3.2.10 (Scaling under Koch metric)**.** We have the following scaling property

$$\mathbb{B}_{R_1}(z_0) = \left(\frac{R_1}{R_2}\right)^2 \mathbb{B}_{R_2}(z_0), \quad \forall R_1, R_2 > 0. \tag{3.2.13}$$

This property follows from the observation that, for any $z \in \mathbb{H}$, using the fact that $z_0 = (0,0)$, we have

$$\mathbf{d}(z,z_0) = \frac{|z|}{\sqrt{y+|z|}}.$$

Therefore, for any $z \in \mathbb{H}$,

$$\mathbf{d}\left(\left(\frac{R_1}{R_2}\right)^2 z, z_0\right) = \frac{R_1}{R_2}\mathbf{d}(z,z_0),$$

and so, $\mathbf{d}(z,z_0) < R_2$ if and only if $\mathbf{d}\left((R_1/R_2)^2 z, z_0\right) < R_1$, from which (3.2.13) follows.

Notice that (3.2.13) does not hold if $z_0 = (y_0, z_0)$ with $y_0 > 0$.

Proof of Corollary 3.2.5. Let $0 < R \leq \bar{R}$ and define v by rescaling

$$u(z) = v\left(\left(\frac{\bar{R}}{R}\right)^2 z\right), \quad \forall z \in \mathbf{B}_R(z_0).$$

The rescaling map ψ defined by

$$z \mapsto \left(\frac{\bar{R}}{R}\right)^2 z,$$

maps $\mathscr{O}$ into a domain $\tilde{\mathscr{O}}$ satisfying the same assumptions as $\mathscr{O}$ in Assumption 3.1.2. Then, using (3.2.13), ψ maps $\mathbf{B}_R(z_0)$ into $\tilde{\mathbf{B}}_{\bar{R}}(z_0)$, where

$$\tilde{\mathbf{B}}_{\bar{R}}(z_0) = \{z \in \tilde{\mathscr{O}} : \mathbf{d}(z, z_0) < \bar{R}\}.$$

By applying Lemma 3.2.4 to v on $\tilde{\mathbf{B}}_{\bar{R}}(z_0)$, there is a positive constant C, depending only on $\bar{R}$ and β, such that (3.2.4) holds. By changing variables, we obtain

$$\inf_{c\in\mathbb{R}} \left(\frac{\bar{R}}{R}\right)^{2(\beta-1)} \int_{\mathbf{B}_R(z_0)} |u - c|^2 y^{\beta-1} dxdy \leq \left(\frac{R}{\bar{R}}\right)^4 \left(\frac{\bar{R}}{R}\right)^{2\beta} \int_{\mathbf{B}_R(z_0)} |\nabla u|^2 y^\beta dxdy. \tag{3.2.14}$$

By Assumption 3.1.2, we have

$$\frac{1}{2}|\mathbb{B}_R(z_0)|_\beta \leq |\mathbf{B}_R(z_0)|_\beta \leq |\mathbb{B}_R(z_0)|_\beta, \quad \forall 0 < R \leq \bar{R},$$

and using Lemma 3.2.2, we rewrite (3.2.14) in the following form

$$\inf_{c\in\mathbb{R}} \frac{|\mathbb{B}_{\bar{R}}(z_0)|_{\beta-1}}{|\mathbb{B}_R(z_0)|_{\beta-1}} \int_{\mathbf{B}_R(z_0)} |u - c|^2 y^{\beta-1} dxdy \leq \left(\frac{R}{\bar{R}}\right)^4 \frac{|\mathbb{B}_{\bar{R}}(z_0)|_\beta}{|\mathbb{B}_R(z_0)|_\beta} \int_{\mathbf{B}_R(z_0)} |\nabla u|^2 y^\beta dxdy,$$

from which (3.2.5) follows immediately. □

3.3 John-Nirenberg inequality

In this section we recall the abstract John-Nirenberg inequality (Theorem 3.3.1) due to E. Bombieri and E. Giusti [11] and, in particular, provide a justification — via Proposition 3.3.2 — that its hypotheses hold in the setting of the problems described in §3.1.

We restrict the statement of [11, Theorem 4] to the framework of our problems, so in [11, Theorem 4] we choose $\mathbb{H}$ to be the topological space and $d\mu = y^{\beta-1}dxdy$ to be the regular positive Borel measure on $\mathbb{H}$. Let S_r, $0 \leq r \leq 1$ be a family of non-empty open sets in $\mathbb{H}$ such that

$$\begin{aligned} &S_s \subseteq S_r, && \forall 0 \leq s \leq r \leq 1, \\ &0 < |S_r|_{\beta-1} < \infty, && \forall 0 \leq r \leq 1. \end{aligned} \tag{3.3.1}$$

Let w be a measurable positive function on S_1. For $t \neq 0$ and $0 \leq r \leq 1$, we denote by

$$|w|_{t,r} = \left(\frac{1}{|S_r|_{\beta-1}} \int_{S_r} |w|^t y^{\beta-1} dxdy \right)^{1/t},$$

$$|w|_{\infty,r} = \sup_{S_r} w,$$

$$|w|_{-\infty,r} = \inf_{S_r} w.$$

We now recall the

Theorem 3.3.1 (Abstract John-Nirenberg Inequality)**.** *[11, Theorem 4] Let* $0 < \vartheta_0, \vartheta_1 \leq \infty$ *and* w *be a measurable positive function on* S_1 *such that*

$$|w|_{\vartheta_0,1} < \infty \text{ and } |w|_{\vartheta_1,1} > 0.$$

Suppose there exist constants $\gamma > 0$, $0 < t^* \leq \frac{1}{2}\min\{\vartheta_0, \vartheta_1\}$ *and* $Q > 0$ *such that for all* $0 \leq s < r \leq 1$ *and* $0 < t \leq t^*$,

$$\begin{aligned} |w|_{\vartheta_0,s} &\leq [Q(r-s)^\gamma]^{1/\vartheta_0 - 1/t} |w|_{t,r}, \\ |w|_{-\vartheta_1,s} &\geq [Q(r-s)^\gamma]^{1/t - 1/\vartheta_1} |w|_{-t,r}. \end{aligned} \tag{3.3.2}$$

Assume further that

$$A := \sup_{0 \leq r \leq 1} \inf_{c \in \mathbb{R}} \frac{1}{|S_r|_{\beta-1}} \int_{S_r} |\log w - c| y^{\beta-1} dxdy < \infty. \tag{3.3.3}$$

Then, we have

$$|w|_{\vartheta_0,0} \leq \left(\frac{|S_1|_{\beta-1}}{|S_0|_{\beta-1}} \right)^{1/\vartheta_0 + 1/\vartheta_1} \exp\left\{ c_2 Q^{-2} \left(A + 1/t^* \right) \right\} |w|_{-\vartheta_1,1}, \tag{3.3.4}$$

where c_2 *is a constant depending only on* γ, *but not on* $Q, \vartheta_0, \vartheta_1, t^*, A$ *and* β.

We now provide a justification that the hypotheses of Theorem 3.3.1 hold in the setting of the problems discussed in this chapter.

Proposition 3.3.2 (Application of Theorem 3.3.1)**.** *Let* $z_0 \in \bar{\Gamma}_0$ *and* $0 < 4R \leq \bar{R}$. *Let* $S_r = \mathbf{B}_{(2+r)R}(z_0)$, *for all* $0 \leq r \leq 1$. *Let* ϑ_0, ϑ_1 *be as in Theorem 3.3.1 and set* $t^* = \frac{1}{2}\min\{\vartheta_0, \vartheta_1\}$. *Then, there exist positive constants* Q *and* γ, *independent of* R *and* z_0, *such that* (3.3.2) *holds for any bounded positive function* w *on* S_1 *which satisfies the energy estimate* (3.5.11) *or* (3.6.3)

Proof of Proposition 3.3.2. We begin by proving the first inequality in (3.3.2) by applying Moser iteration finitely many times. The second inequality in (3.3.2) can be proved by a similar technique. We outline the proof when w satisfies the energy estimate (3.5.11), but the proof applies as well to positive bounded functions w satisfying the energy estimate (3.6.3).

First, we consider the *special case* when ϑ_0 and t satisfy the requirement: There exists an integer $N^* \geq 1$ such that ϑ_0 can be written as

$$\vartheta_0 = t\left(\frac{p}{2}\right)^{N^*}. \tag{3.3.5}$$

Let $0 \leq s < r \leq 1$ and set $R_0 = (2+r)R$. We denote

$$c := \sum_{k=1}^{\infty} \frac{1}{k^2}$$

and we let

$$R_N^2 := \left((2+r)^2 - (r-s)^2 \sum_{k=1}^{N} \frac{1}{ck^2}\right) R^2, \quad \forall N = 1, \ldots, N^*. \tag{3.3.6}$$

We observe that $(2+s)R < R_N < R_{N-1} \leq (2+r)R$. Let $(\eta_N)_{N\in\mathbb{N}}$ be a sequence of non-negative, smooth cutoff functions as constructed in §1.5, by choosing R_N as in (3.3.6). Then, (3.1.27) becomes

$$|\nabla \eta_N| \leq \frac{CN^2}{R^2(r-s)^2}. \tag{3.3.7}$$

Let $P_N := t\,(p/2)^N$, for $N = 1, \ldots, N^*$, and $\alpha_N = p_N - 1$, for all $N = 0, \ldots, N^* - 1$. We set

$$I(N) := \left(\int_{\mathbf{B}_N} |w|^{p_N} y^{\beta-1} dxdy\right)^{1/p_N}, \tag{3.3.8}$$

where we denote for simplicity $\mathbf{B}_N = \mathbf{B}_{R_N}(z_0)$. From our hypothesis, w satisfies (3.5.11), that is,

$$\|\eta w^{(\alpha+1)/2}\|_{L^p\left(\mathbb{H}, y^{\beta-1}\right)} \leq C_0(R,\alpha) \|w^{(\alpha+1)/2}\|_{L^2\left(\operatorname{supp}\eta, y^{\beta-1}\right)}, \tag{3.3.9}$$

where

$$C_0(R,\alpha) := [C|1+\alpha|]^{(\xi+1)/p} \left(1 + \|\sqrt{y}\nabla\eta\|^2_{L^\infty(\mathbb{H})}\right)^{1/p}, \tag{3.3.10}$$

and ξ and C are positive constants, independent of u, α and η. We choose $\alpha = \alpha_{N-1}$ and $\eta = \eta_N$ in (3.3.9), so the definition (3.3.8) gives us, for all $N \geq 1$,

$$I(N) \leq C_1(R, r, s, N) I(N-1), \tag{3.3.11}$$

where

$$C_1(R, r, s, N) := (C|p_{N-1}|)^{(\xi+1)/p_N} \left(1 + \|\sqrt{y}\nabla\eta_N\|^2_{L^\infty(\mathbb{H})}\right)^{1/p_N}.$$

From Lemma 3.2.2, we have $y \leq CR^2$ on $\mathbf{B}_N$, where C is a positive constant independent of R and N. Using the bound (3.3.7), we obtain

$$C_1(R, r, s, N) := (C|p_{N-1}|)^{(\xi+1)/p_N} \left(\frac{CN^4}{R^2(r-s)^4}\right)^{1/p_N}.$$

By iterating inequality (3.3.11), we obtain

$$I(N^*) \leq C_2(R, r, s) I(0), \tag{3.3.12}$$

where

$$C_2(R, r, s) := \prod_{N=1}^{N^*} \left[C p_{N-1}^{\xi+1} N^4 R^{-2} (r-s)^{-4}\right]^{1/p_N}. \tag{3.3.13}$$

Next, we prove the

Claim 3.3.3. *There are positive constants Q and γ, independent of N^*, R, r and s, such that*

$$C_2(R, r, s) \leq (Q(r-s)^\gamma)^{1/\vartheta_0 - 1/t} \, R^{\frac{4}{p-2}(1/\vartheta_0 - 1/t)}. \tag{3.3.14}$$

Proof of Claim 3.3.3. We can rewrite the expression (3.3.13) for $C_2(R, r, s)$ to obtain

$$\begin{aligned} C_2(R, r, s) &= \prod_{N=1}^{N^*} \left[C t^{\xi+1} R^{-2} (r-s)^{-4}\right]^{1/p_N} \left[\left(\frac{p}{2}\right)^{N-1} N^4\right]^{1/p_N} \\ &\leq \left[C t^{\xi+1} R^{-2} (r-s)^{-4}\right]^{\sum_{N=1}^{N^*} 1/p_N} \left(C\frac{p}{2}\right)^{\sum_{N=1}^{N^*} N/p_N}, \end{aligned}$$

where we used in the last line that $N^4 \leq C(p/2)^N$, for some positive constant C depending only on p. Thus,

$$C_2(R, r, s) \leq \left[C t^{\xi+1} R^{-2} (r-s)^{-4}\right]^{\sum_{N=1}^{N^*} 1/p_N} \left(C\frac{p}{2}\right)^{\sum_{N=1}^{N^*} N/p_N}. \tag{3.3.15}$$

Recall that

$$\sum_{N=1}^{N^*} x^N = x\frac{1 - x^{N^*}}{1-x} \text{ and } \sum_{N=1}^{N^*} N x^N = x^2 \frac{1 - x^{N^*}}{1-x}.$$

Hence, (3.3.5) leads to the identities

$$\sum_{N=1}^{N^*} \frac{1}{p_N} = \frac{2}{p-2}\left(\frac{1}{t} - \frac{1}{\vartheta_0}\right) \text{ and } \sum_{N=1}^{N^*} \frac{N}{p_N} = \frac{4}{p(p-2)}\left(\frac{1}{t} - \frac{1}{\vartheta_0}\right).$$

Therefore, inequality (3.3.14) becomes

$$C_2(R,r,s) \leq \left[R^{-2}(r-s)^{-4}\right]^{\frac{2}{p-2}\left(\frac{1}{t}-\frac{1}{\vartheta_0}\right)} \left(C\vartheta_0^{\xi+1}\frac{p}{2}\right)^{\frac{4}{p(p-2)}\left(\frac{1}{t}-\frac{1}{\vartheta_0}\right)}, \tag{3.3.16}$$

which is equivalent to (3.3.14) with the choice of the constants $Q = \left(C\vartheta_0^{\xi+1}p/2\right)^{-1}$ and $\gamma = 8/(p-2)$. This completes the proof of Claim 3.3.3. □

By Assumption 3.1.2, we have

$$\frac{1}{2}|\mathbb{B}_{(2+a)R}|_{\beta-1} \leq |\mathbf{B}_{(2+a)R}|_{\beta-1} \leq |\mathbb{B}_{(2+a)R}|_{\beta-1},$$

where the constant, a, can be either r or s. Using the fact that $4/(p-2) = 2(d+\beta-1)$, Lemma 3.2.2 yields

$$\frac{|\mathbf{B}_{(2+s)R}|_{\beta-1}^{1/\vartheta_0}}{|\mathbf{B}_{(2+r)R}|_{\beta-1}^{1/t}} \geq C^{1/\vartheta_0+1/t} R^{4/(p-2)(1/\vartheta_0-1/t)},$$

for some positive constant $C < 1$. Therefore, inequality (3.3.16) becomes

$$C_2(R,r,s) \leq C^{-1/\vartheta_0-1/t}\left(Q(r-s)^\gamma\right)^{1/\vartheta_0-1/t} \frac{|\mathbf{B}_{(2+s)R}|_{\beta-1}^{1/\vartheta_0}}{|\mathbf{B}_{(2+r)R}|_{\beta-1}^{1/t}}. \tag{3.3.17}$$

From our hypothesis, $t \leq t^* \leq \vartheta_0/2$, we have

$$3(1/\vartheta_0 - 1/t) \leq -1/\vartheta_0 - 1/t \leq 1/\vartheta_0 - 1/t,$$

and so, for a new positive constant Q, the inequality (3.3.17) leads to

$$C_2(R,r,s) \leq \left(Q(r-s)^\gamma\right)^{1/\vartheta_0-1/t} \frac{|\mathbf{B}_{(2+s)R}|_{\beta-1}^{1/\vartheta_0}}{|\mathbf{B}_{(2+r)R}|_{\beta-1}^{1/t}}. \tag{3.3.18}$$

By employing the inequalities (3.3.18) and (3.3.12) and the definition (3.3.8) of $I(N)$, we obtain

$$\begin{aligned}
\left(\int_{\mathbf{B}_{(2+s)R}} |w|^{\vartheta_0} y^{\beta-1}\,dxdy\right)^{1/\vartheta_0} &\leq I(N^*) \\
&\leq \left(Q(r-s)^\gamma\right)^{1/\vartheta_0-1/t} \frac{|\mathbf{B}_{(2+s)R}|_{\beta-1}^{1/\vartheta_0}}{|\mathbf{B}_{(2+r)R}|_{\beta-1}^{1/t}} I(0) \\
&= \left(Q(r-s)^\gamma\right)^{1/\vartheta_0-1/t} \frac{|\mathbf{B}_{(2+s)R}|_{\beta-1}^{1/\vartheta_0}}{|\mathbf{B}_{(2+r)R}|_{\beta-1}^{1/t}} \left(\int_{\mathbf{B}_{(2+r)R}} |w|^t y^{\beta-1}\,dxdy\right)^{1/t},
\end{aligned}$$

from which we readily obtain the first inequality in (3.3.2), in the *special case* where t and ϑ_0 satisfy (3.3.5) for some integer $N^* \geq 1$.

Next, we show that the first inequality in (3.3.2) holds for *any* $t \in (0, t^*)$. For this purpose, we choose an integer $N^* \geq 1$ such that

$$t\left(\frac{p}{2}\right)^{N^*-1} < \vartheta_0 < t\left(\frac{p}{2}\right)^{N^*}.$$

We denote $\vartheta_0^* = t\,(p/2)^{N^*}$ and we apply the previous analysis to t and ϑ_0^*, which now satisfy (3.3.5), to give

$$|w|_{\vartheta_0^*,s} \leq \left(Q(r-s)^\gamma\right)^{1/\vartheta_0^*-1/t} |w|_{t,r}.$$

Using Hölder's inequality with $p = \vartheta_0^*/\vartheta_0 > 1$, we find that

$$|w|_{\vartheta_0,s} \leq |w|_{\vartheta_0^*,s},$$

and so

$$\begin{aligned}
|w|_{\vartheta_0,s} &\leq \left(Q(r-s)^\gamma\right)^{1/\vartheta_0^*-1/t} |w|_{t,r} \\
&\leq \left(Q(r-s)^\gamma\right)^{\frac{1/\vartheta_0^*-1/t}{1/\vartheta_0-1/t}(1/\vartheta_0-1/t)} |w|_{t,r}.
\end{aligned}$$

Notice that $2\vartheta_0^*/p \leq \vartheta_0 \leq \vartheta_0^*$ and $0 < t < \vartheta_0/2$. Then,

$$1 \leq \frac{1/\vartheta_0^* - 1/t}{1/\vartheta_0 - 1/t} \leq \frac{1/\vartheta_0^* - 1/t}{p/2\vartheta_0^* - 1/t} \leq \frac{(2/p)^{N^*} - 1}{(2/p)^{N^*+1} - 1} \leq \frac{p}{p-2}.$$

Consequently, we define $\widetilde{Q}$ to be $Q^{p/(p-2)}$ if $Q < 1$, and we leave Q unchanged if $Q \geq 1$ and, setting $\widetilde{\gamma} := \gamma p/(p-2)$, the preceding estimate for $|w|_{\vartheta_0,s}$ becomes

$$|w|_{\vartheta_0,s} \leq \left(\widetilde{Q}(r-s)^{\widetilde{\gamma}}\right)^{1/\vartheta_0-1/t} |w|_{t,r},$$

which is precisely the first inequality in (3.3.2). □

3.4 Supremum estimates near the boundary portion where the operator is degenerate

In this section, we prove Theorem 3.1.7, that is, local boundedness up to $\bar{\Gamma}_0$ for solutions, u, to the variational equation (3.1.16). Our choice of test functions when applying Moser iteration follows that employed in the proof of [41, Theorem 8.15]. However, the choice of test functions used in the proof of the classical local supremum estimates [41, Theorem 8.17] is not suitable in our case because the test functions in (3.1.16) are not required to satisfy a homogeneous Dirichlet boundary condition along $\bar{\Gamma}_0$. In addition, the method of deriving the energy estimate (3.4.3) is slightly different from [41, Theorem 8.18] because, instead of using the classical Sobolev inequalities [41, Theorem 7.10], we use Lemma 3.2.1.

Proof of Theorem 3.1.7. We organize the proof in several steps.

Step 1 (Energy estimates)**.** Let $\alpha \geq 1$ and let $\eta \in C_0^1(\bar{\mathbb{H}})$ be a non-negative cutoff function with support in $\bar{\mathbf{B}}_{2R}(z_0)$. We define

$$A := \|f\|_{L^s(\operatorname{supp}\eta, y^{\beta-1})}. \tag{3.4.1}$$

We will apply the following calculations in Steps 1 and 2 to two choices of w, namely,

$$w := u^+ + A \text{ and } w := u^- + A. \tag{3.4.2}$$

For concreteness, we will illustrate our calculations with the choice

$$w = u^+ + A,$$

but they apply equally well to the choice $w = u^- + A$. Our goal is to prove the following

Claim 3.4.1 (Energy estimate)**.** *There is a positive constant C, depending only on the coefficients of the Heston operator* (3.1.2) *and δ_0, and there is a positive constant ξ, depending only on d, β and s, such that*

$$\begin{aligned}\left(\int_{\mathscr{O}} |\eta w^\alpha|^p y^{\beta-1} dxdy\right)^{1/p} \\ \leq (C\alpha)^{\xi+1} \left(\|\sqrt{y}\nabla\eta\|^{2/p}_{L^\infty(\mathbb{H})} + |\operatorname{supp}\eta|^{1/p-1/2}_{\beta-1}\right)\left(\int_{\operatorname{supp}\eta} w^{2\alpha} y^{\beta-1} dxdy\right)^{1/2}.\end{aligned} \tag{3.4.3}$$

Proof of Claim 3.4.1. We fix $k \in \mathbb{N}$. Similarly to the proof of [41, Theorem 8.15], we consider the functions $H_k : \mathbb{R} \to [0, \infty)$,

$$H_k(t) := \begin{cases} 0, & t < A, \\ t^\alpha - A^\alpha, & A \leq t \leq k, \\ \alpha k^{\alpha-1}(t-k) + H_k(k), & t > k. \end{cases} \tag{3.4.4}$$

and

$$G_k(t) = \int_0^t |H'_k(s)|^2 ds. \tag{3.4.5}$$

Then,

$$v = G_k(w)\eta^2 \tag{3.4.6}$$

is a valid test function in $H^1_0(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$ in (3.1.12), by Lemma B.2.1. Because $u \in H^1_0(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$ obeys (3.1.16) for all $v \in H^1_0(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$ with support in $\bar{\mathbf{B}}_{2R}(z_0)$, then the expression (3.1.12) for $a(u, v)$ yields

$$\begin{aligned}
0 &= a(u,v) - (f,v)_{L^2(\mathscr{O},m)} \\
&= \frac{1}{2}\int_{\mathscr{O}} \left(u_x v_x + \rho\sigma u_x v_y + \rho\sigma u_y v_x + \sigma^2 u_y v_y\right) y\mathfrak{w}\, dxdy \\
&- \int_{\mathscr{O}} \left(a_1 u_x + \frac{\gamma}{2}\left(u_x + \rho\sigma u_y\right)\operatorname{sign}(x)\right) v y\mathfrak{w}\, dxdy + \int_{\mathscr{O}} (ru - f)\, v\mathfrak{w}\, dxdy.
\end{aligned}$$

Since $\nabla v = G'_k(w)\eta^2\nabla w + 2G_k(w)\eta\nabla\eta$ and the fact that $G_k(w) = 0$ when $w \leq A$, that is, $u^+ = 0$, the preceding identity becomes

$$\begin{aligned}
&\frac{1}{2}\int_{\mathscr{O}} \left(w_x^2 + 2\rho\sigma w_x w_y + \sigma^2 w_y^2\right) G'_k(w)\eta^2 y\mathfrak{w}\, dxdy \\
&= -\int_{\mathscr{O}} \left(w_x\eta_x + \rho\sigma w_x\eta_y + \rho\sigma w_y\eta_x + \sigma^2 w_y\eta_y\right) G_k(w)\eta y\mathfrak{w}\, dxdy \\
&\quad + \int_{\mathscr{O}} \left[a_1 w_x + \frac{\gamma}{2}\left(w_x + \rho\sigma w_y\right)\operatorname{sign}(x)\right] G_k(w)\eta^2 y\mathfrak{w}\, dxdy \\
&\quad - \int_{\mathscr{O}} \left(ru^+ - f\right) G_k(w)\eta^2\mathfrak{w}\, dxdy.
\end{aligned}$$

For convenience, we write the identity as $I_1 = I_2 + I_3 + I_4$. From the uniform ellipticity (3.1.14), we obtain for I_1 that

$$C\int_{\mathscr{O}} |\nabla w|^2\eta^2 G'_k(w) y\mathfrak{w}\, dxdy \leq I_1,$$

where C is a positive constant depending only on the coefficients of the Heston operator. We notice that $0 \leq G_k(w) \leq wG'_k(w)$ because $G'_k(w) = |H'_k(w)|^2$ is a non-decreasing function. Using this fact and that $w \geq A$, we obtain for the integrals I_2, I_3, I_4 that there exists a positive constant C, depending only on the Heston coefficients, such that

$$
\begin{aligned}
|I_2| &\leq \frac{1}{2}\int_{\mathscr{O}} \left(|w_x\eta||w\eta_x| + \rho\sigma|w_x\eta||w\eta_y| + \rho\sigma|w_y\eta||w\eta_x| + \sigma^2|w_y\eta||w\eta_y|\right) G_k'(w)y\mathfrak{w}\, dxdy\\
&\leq \varepsilon\int_{\mathscr{O}} |\nabla w|^2\eta^2 G_k'(w)y\mathfrak{w}\, dxdy + \frac{C}{\varepsilon}\int_{\mathscr{O}} |w|^2|\nabla\eta|^2 G_k'(w)y\mathfrak{w}\, dxdy,\\
|I_3| &\leq \varepsilon\int_{\mathscr{O}} |\nabla w|^2\eta^2 G_k'(w)y\mathfrak{w}\, dxdy + \frac{C}{\varepsilon}\int_{\mathscr{O}} |w|^2|\eta|^2 G_k'(w)y\mathfrak{w}\, dxdy,\\
|I_4| &\leq r\int_{\mathscr{O}} w^2 G_k'(w)\eta^2\mathfrak{w}\, dxdy + \int_{\mathscr{O}} |f|wG_k'(w)\eta^2\mathfrak{w}\, dxdy\\
&\leq C\int_{\mathscr{O}} \left(1+\frac{|f|}{A}\right) w^2 G_k'(w)\eta^2\mathfrak{w}\, dxdy,
\end{aligned}
$$

where $\varepsilon > 0$. Choosing ε small enough, we obtain for a positive constant C, depending on the coefficients of the Heston operator and δ_0, that

$$
\begin{aligned}
\int_{\mathscr{O}} |\nabla w|^2\eta^2 G_k'(w)y^\beta dxdy \leq C \Bigg[&\int_{\mathscr{O}} \eta^2\frac{|f|}{A} w^2 G_k'(w)y^{\beta-1}dxdy\\
&+\int_{\mathscr{O}} \left(\eta^2 + y|\nabla\eta|^2\right) w^2 G_k'(w)y^{\beta-1}dxdy\Bigg].
\end{aligned}
\tag{3.4.7}
$$

Hölder's inequality applied to the conjugate pair (s, s^*) gives

$$
\begin{aligned}
&\int_{\mathscr{O}} \eta^2\frac{|f|}{A} w^2 G_k'(w)y^{\beta-1}dxdy\\
&\qquad\leq \left(\int_{\text{supp }\eta} \frac{|f|^s}{A^s} y^{\beta-1}dxdy\right)^{1/s} \left(\int_{\mathscr{O}} |\eta^2 w^2 G_k'(w)|^{s^*} y^{\beta-1}dxdy\right)^{1/s^*},
\end{aligned}
$$

and thus, by definition (3.4.1) of A,

$$
\int_{\mathscr{O}} \eta^2\frac{|f|}{A} w^2 G_k'(w)y^{\beta-1}dxdy \leq \left(\int_{\mathscr{O}} |\eta^2 w^2 G_k'(w)|^{s^*} y^{\beta-1}dxdy\right)^{1/s^*}. \tag{3.4.8}
$$

We need to justify first that the right hand side in (3.4.8) is finite. First, we notice that the following identities hold

$$
\begin{aligned}
|\nabla H_k(w)|^2 &= |\nabla w|^2|H_k'(w)|^2 = |\nabla w|^2 G_k'(w),\\
|wH_k'(w)|^2 &= |w|^2 G_k'(w),
\end{aligned}
\tag{3.4.9}
$$

From the hypothesis $s > d+\beta$ in Theorem 3.1.7, we observe that $2 < 2s^* < p$, so we may apply the interpolation inequality [41, Inequality (7.10)]. For any $\varepsilon \in (0,1)$, we have

$$
\|\eta w H_k'(w)\|_{L^{2s^*}(\mathbb{H},y^{\beta-1})} \leq \varepsilon\|\eta w H_k'(w)\|_{L^p(\mathbb{H},y^{\beta-1})} + \varepsilon^{-\xi}\|\eta w H_k'(w)\|_{L^2(\mathbb{H},y^{\beta-1})}, \tag{3.4.10}
$$

where

$$\xi \equiv \xi(p,s) := \frac{p(s^*-1)}{p-2s^*}. \tag{3.4.11}$$

We notice that $|H_k'(w)| \leq \alpha k^{\alpha-1}$ and $\eta w \in H^1(\mathscr{O}, \mathfrak{w})$ has compact support in $\bar{\mathbf{B}}_{2R}(z_0)$. Therefore, we may apply Lemma 3.2.6 to build an extension $\hat{w}$ of ηw to a rectangle D containing $\bar{\mathbf{B}}_{2R}(z_0)$. Lemma 3.2.1, shows that $\hat{w} \in L^p(D, y^{\beta-1})$, which implies that

$$\|\eta w H_k'(w)\|_{L^p(\mathbb{H}, y^{\beta-1})} < \infty,$$

and so, the right hand side of (3.4.8) is finite.

Inequalities (3.4.7) and (3.4.8), together with the identities (3.4.9) yield

$$\begin{aligned}\int_{\mathscr{O}} \eta^2 |\nabla H_k(w)|^2 y^\beta dxdy \leq C \Bigg[&\left(\int_{\mathscr{O}} |\eta w H_k'(w)|^{2s^*} y^{\beta-1} dxdy \right)^{1/s^*} \\ &+ \int_{\mathscr{O}} \left(\eta^2 + y|\nabla \eta|^2\right) |w H_k'(w)|^2 y^{\beta-1} dxdy \Bigg].\end{aligned} \tag{3.4.12}$$

From Lemma 3.2.1, we obtain

$$\begin{aligned}\int_{\mathscr{O}} |\eta H_k(w)|^p y^{\beta-1} dxdy &\leq \left(\int_{\mathscr{O}} \eta^2 |H_k(w)|^2 y^{\beta-1} dxdy\right)^{(p-2)/2} \int_{\mathscr{O}} |\nabla(\eta H_k(w))|^2 y^\beta dxdy \\ &\leq 2 \left(\int_{\mathscr{O}} \eta^2 |H_k(w)|^2 y^{\beta-1} dxdy\right)^{(p-2)/2} \\ &\quad \times \left(\int_{\mathscr{O}} |\nabla \eta|^2 |H_k(w)|^2 y^\beta dxdy + \eta^2 |\nabla H_k(w)|^2 y^\beta dxdy\right).\end{aligned} \tag{3.4.13}$$

Using $H_k(w) \leq w H_k'(w)$ and inequality (3.4.12) in (3.4.13), we see that

$$\begin{aligned}&\int_{\mathscr{O}} |\eta H_k(w)|^p y^{\beta-1} dxdy \\ &\leq C \Bigg[\left(1 + \|\sqrt{y} \nabla \eta\|^2_{L^\infty(\mathbb{H})}\right) \left(\int_{\operatorname{supp}\eta} |w H_k'(w)|^2 y^{\beta-1} dxdy\right)^{p/2} \\ &+ \left(\int_{\mathscr{O}} |\eta w H_k'(w)|^2 y^{\beta-1} dxdy\right)^{(p-2)/2} \left(\int_{\mathscr{O}} |\eta w H_k'(w)|^{2s^*} y^{\beta-1} dxdy\right)^{1/s^*} \Bigg],\end{aligned} \tag{3.4.14}$$

where C is a positive constant depending on the Heston coefficients and δ_0. We rewrite the estimate for $\eta w H_k'(w)$ in (3.4.10) in the form

$$\begin{aligned}\left(\int_{\mathscr{O}} |\eta w H_k'(w)|^{2s^*} y^{\beta-1} dxdy\right)^{1/s^*} &= \|\eta w H_k'(w)\|^2_{L^{2s^*}(\mathbb{H}, y^{\beta-1})} \\ &\leq 2\varepsilon^2 \|\eta w H_k'(w)\|^2_{L^p(\mathbb{H}, y^{\beta-1})} + 2\varepsilon^{-2\xi} \|\eta w H_k'(w)\|^2_{L^2(\mathbb{H}, y^{\beta-1})}.\end{aligned}$$

Applying the preceding inequality in (3.4.14), we obtain

$$\begin{aligned}\|\eta H_k(w)\|^p_{L^p(\mathbb{H},y^{\beta-1})} \leq C\left(1+\|\sqrt{y}\nabla\eta\|^2_{L^\infty(\mathbb{H})}\right)\|wH_k'(w)\|^p_{L^2(\operatorname{supp}\eta,y^{\beta-1})} \\ + C\|\eta wH_k'(w)\|^{p-2}_{L^2(\operatorname{supp}\eta,y^{\beta-1})}\times \\ \left(\varepsilon^2\|\eta wH_k'(w)\|^2_{L^p(\mathbb{H},y^{\beta-1})}+\varepsilon^{-2\xi}\|\eta wH_k'(w)\|^2_{L^2(\mathbb{H},y^{\beta-1})}\right).\end{aligned}$$

By recombining terms in the preceding inequality, we find that

$$\begin{aligned}\|\eta H_k(w)\|^p_{L^p(\mathbb{H},y^{\beta-1})} \leq C(1+\varepsilon^{-2\xi})\left(1+\|\sqrt{y}\nabla\eta\|^2_{L^\infty(\mathbb{H})}\right)\|wH_k'(w)\|^p_{L^2(\operatorname{supp}\eta,y^{\beta-1})} \\ + C\varepsilon^2\|\eta wH_k'(w)\|^{p-2}_{L^2(\mathbb{H},y^{\beta-1})}\|\eta wH_k'(w)\|^2_{L^p(\mathbb{H},y^{\beta-1})}.\end{aligned}$$

To estimate the last term in the preceding inequality, we apply Young's inequality with the conjugate pair of exponents, $(p/2, p/(p-2))$, to give

$$\begin{aligned}&\|\eta wH_k'(w)\|^2_{L^p(\mathbb{H},y^{\beta-1})}\|\eta wH_k'(w)\|^{p-2}_{L^2(\mathbb{H},y^{\beta-1})} \\ &\quad\leq \frac{2}{p}\|\eta wH_k'(w)\|^p_{L^p(\mathbb{H},y^{\beta-1})}+\frac{p-2}{p}\|\eta wH_k'(w)\|^p_{L^2(\mathbb{H},y^{\beta-1})}.\end{aligned}$$

Combining the previous two inequalities yields

$$\begin{aligned}\|\eta H_k(w)\|^p_{L^p(\mathbb{H},y^{\beta-1})} \leq C\varepsilon^2\|\eta wH_k'(w)\|^p_{L^p(\mathbb{H},y^{\beta-1})}+C\left(1+(\varepsilon^2+\varepsilon^{-2\xi})\right)\times \\ \left(1+\|\sqrt{y}\nabla\eta\|^2_{L^\infty(\mathbb{H})}\right)\|wH_k'(w)\|^p_{L^2(\operatorname{supp}\eta,y^{\beta-1})},\end{aligned}\tag{3.4.15}$$

Employing the definition (3.4.4) of $H_k(w)$ gives $0\leq wH_k'(w)\leq \alpha H_k(w)+\alpha A^\alpha$, and so

$$\begin{aligned}\int_{\mathscr{O}}|\eta wH_k'(w)|^p y^{\beta-1}dxdy &\leq |2\alpha|^p\left[\int_{\mathscr{O}}|\eta H_k(w)|^p y^{\beta-1}dxdy+\int_{\mathscr{O}}|\eta A^\alpha|^p y^{\beta-1}dxdy\right] \\ &\leq |2\alpha|^p\left[\int_{\mathscr{O}}|\eta H_k(w)|^p y^{\beta-1}dxdy+|\operatorname{supp}\eta|_{\beta-1}A^{\alpha p}\right],\end{aligned}$$

and thus, applying inequality (3.4.15) yields

$$\begin{aligned}\int_{\mathscr{O}}|\eta H_k(w)|^p y^{\beta-1}dxdy \leq C|2\alpha|^p\varepsilon^2\left(\|\eta H_k(w)\|^p_{L^p(y^{\mathbb{H},\beta-1})}+|\operatorname{supp}\eta|_{\beta-1}A^{\alpha p}\right) \\ + C\left(1+\left(\varepsilon^2+\varepsilon^{-2\xi}\right)\right)\left(1+\|\sqrt{y}\nabla\eta\|^2_{L^\infty(\mathbb{H})}\right)\|wH_k'(w)\|^p_{L^2(\operatorname{supp}\eta,y^{\beta-1})}.\end{aligned}$$

By choosing $\varepsilon = 1/(2\sqrt{C(2\alpha)^p})$ and taking p-th order roots, we obtain

$$\begin{aligned}\left(\int_{\mathscr{O}}|\eta H_k(w)|^p y^{\beta-1}dxdy\right)^{1/p} \leq (C\alpha)^\xi\times\left(|\operatorname{supp}\eta|^{1/p}_{\beta-1}A^\alpha\right. \\ \left.+\left(1+\|\sqrt{y}\nabla\eta\|^2_{L^\infty(\mathbb{H})}\right)^{1/p}\left(\int_{\operatorname{supp}\eta}|wH_k'(w)|^2y^{\beta-1}dxdy\right)^{1/2}\right).\end{aligned}$$

Because the positive constants C and ξ are independent of k, we may take limit as k goes to ∞, in the preceding inequality, and we obtain

$$\begin{aligned}\left(\int_{\mathscr{O}} |\eta\left(w^\alpha - A^\alpha\right)|^p y^{\beta-1} dxdy\right)^{1/p} \quad &\leq (C\alpha)^{\xi+1} \times \Bigg(|\operatorname{supp}\eta|_{\beta-1}^{1/p} A^\alpha \\ &+ \left(1 + \|\sqrt{y}\nabla\eta\|^2_{L^\infty(\mathbb{H})}\right)^{1/p} \left(\int_{\operatorname{supp}\eta} |w^\alpha|^2 y^{\beta-1} dxdy\right)^{1/2}\Bigg),\end{aligned}$$

which yields

$$\begin{aligned}\left(\int_{\mathscr{O}} |\eta w^\alpha|^p y^{\beta-1} dxdy\right)^{1/p} &\leq (C\alpha)^{\xi+1} \times \Bigg(|\operatorname{supp}\eta|_{\beta-1}^{1/p} A^\alpha \\ &+ \left(1 + \|\sqrt{y}\nabla\eta\|^2_{L^\infty(\mathbb{H})}\right)^{1/p} \left(\int_{\operatorname{supp}\eta} |w|^{2\alpha} y^{\beta-1} dxdy\right)^{1/2}\Bigg),\end{aligned}$$

We also have

$$\begin{aligned}A^\alpha &= \left(\frac{1}{|\operatorname{supp}\eta|_{\beta-1}} \int_{\operatorname{supp}\eta} A^{2\alpha} y^{\beta-1} dxdy\right)^{1/2} \\ &\leq \left(\frac{1}{|\operatorname{supp}\eta|_{\beta-1}} \int_{\operatorname{supp}\eta} w^{2\alpha} y^{\beta-1} dxdy\right)^{1/2}.\end{aligned}$$

Combining the last two inequalities gives (3.4.3). This completes the proof of Claim 3.4.1. □

Step 2 (Moser iteration). Let $(\eta_N)_{N\in\mathbb{N}}$ be a sequence of non-negative, smooth cutoff functions as constructed in §1.5, by choosing $R_N := R\left(1 + 1/(N+1)\right)$. Then, we have

$$\eta_N|_{\mathbf{B}_N} \equiv 1, \quad \eta_N|_{\mathbf{B}^c_{R_{N-1}}} \equiv 0, |\nabla\eta_N| \leq \frac{cN^3}{R^2}, \tag{3.4.16}$$

where c is a positive constant independent of R and N. For each $N \geq 0$, we set $p_N := 2(p/2)^N$ and $\alpha_N := (p/2)^N$. Let $A_N := \|f\|_{L^s(\operatorname{supp}\eta_N, y^{\beta-1})}$ and $w_N := u^+ + A_N$ or $w_N := u^- + A_N$. Define

$$I(N) := \left(\int_{\mathbf{B}_{R_N}} |w_N|^{p_N} y^{\beta-1} dxdy\right)^{1/p_N}.$$

Applying the energy estimate (3.4.3) with $w = w_N$, $\alpha = \alpha_{N-1}$, and $\eta = \eta_N$, we obtain for all $N \geq 1$ that

$$I(N) \leq C_0(R,N) I(N-1), \tag{3.4.17}$$

where we denote

$$C_0(R,N) := [C|\alpha_{N-1}|]^{2(\xi+1)/p_{N-1}} \left(\|\sqrt{y}\nabla\eta_N\|^{2/p}_{L^\infty(\mathbb{H})} + |\operatorname{supp}\eta_N|_{\beta-1}^{1/p-1/2}\right)^{2/p_{N-1}}. \tag{3.4.18}$$

In the preceding equality, C is a positive constant depending only on the Heston coefficients and δ_0. By Assumption 3.1.2 and Lemma 3.2.2, there is a constant $c > 0$ such that

$$c^{-1}R^{4/(p-2)} \leq |\mathbf{B}_{2R}|_{\beta-1} \leq cR^{4/(p-2)}, \tag{3.4.19}$$

where we used the fact that $2(d+\beta-1) = 4/(p-2)$ by (3.1.24). Moreover, by Lemma 3.2.2, there is a positive constants c such that $0 \leq y \leq cR^2$ on $\mathbf{B}_R(z_0)$, for all $R \geq 0$. Consequently, we have

$$\|\sqrt{y}\nabla\eta_N\|^{2/p}_{L^\infty(\mathbb{H})} + |\operatorname{supp}\eta_N|^{1/p-1/2}_{\beta-1} \leq cN^{6/p}R^{-2/p},$$

and so,

$$C_0(R,N) \leq \left[C|\alpha_{N-1}|N^6\right]^{p(\xi+1)/p_N} R^{-2/p_N}.$$

Therefore,

$$\begin{aligned}\prod_{N\geq 1} C_0(R,N) &\leq \prod_{N\geq 1}\left[C|\alpha_{N-1}|N^6\right]^{p(\xi+1)/p_N} R^{-2/p_N}\\ &\leq C_1 R^{-2\sum_{N=1}^{\infty}1/p_N} = C_1R^{-2/(p-2)}\\ &\leq C_1|\mathbf{B}_{2R}|^{-1/2}_{\beta-1}, \quad \text{(by (3.4.19))},\end{aligned}$$

where C_1 is a positive constant depending only on the Heston coefficients, δ_0 and s. By iterating (3.4.17), we obtain

$$I(+\infty) \leq I(0)\prod_{N\geq 1} C_0(R,N),$$

which gives us

$$\operatorname*{ess\,sup}_{\mathbf{B}_R} w = I(+\infty) \leq C_1\left(\frac{1}{|\mathbf{B}_{2R}|_{\beta-1}}\int_{\mathbf{B}_{2R}}|w|^2y^{\beta-1}dxdy\right)^{1/2}. \tag{3.4.20}$$

Applying (3.4.20) to both choices of w in (3.4.2) yields

$$\begin{aligned}\operatorname*{ess\,sup}_{\mathbf{B}_R} u^+ &\leq C_1\left[\left(\frac{1}{|\mathbf{B}_{2R}|_{\beta-1}}\int_{\mathbf{B}_{2R}}|u|^2y^{\beta-1}dxdy\right)^{1/2} + \|f\|_{L^s(\mathbf{B}_{2R},y^{\beta-1})}\right],\\ \operatorname*{ess\,sup}_{\mathbf{B}_R} u^- &\leq C_1\left[\left(\frac{1}{|\mathbf{B}_{2R}|_{\beta-1}}\int_{\mathbf{B}_{2R}}|u|^2y^{\beta-1}dxdy\right)^{1/2} + \|f\|_{L^s(\mathbf{B}_{2R},y^{\beta-1})}\right].\end{aligned}$$

Adding the two estimates gives us the supremum estimate (3.1.20) .

□

Remark 3.4.2 (Relaxation of the Assumption 3.1.2 on $\mathscr{O}$ near Γ_0)**.** We notice that the conditions on $\mathscr{O}$ embodied in Assumption 3.1.2, which are used in obtaining (3.4.19), can be relaxed to the following weaker condition. For $z_0 \in \bar{\Gamma}_0$, there exist positive constants c_0 and R_0 such that for all $0 < R < R_0$ we have

$$c_0^{-1}|\mathbb{B}_R(z_0)|_{\beta-1} \leq |\mathbf{B}_R(z_0)|_{\beta-1} \leq c_0|\mathbb{B}_R(z_0)|_{\beta-1}. \tag{3.4.21}$$

In this case, the constant C appearing in the supremum estimate (3.1.20) will depend in addition on c_0 and R_0.

Example 3.4.3 (A domain $\mathscr{O}$ which does not satisfy condition (3.4.21))**.** This construction is in the spirit of [47, Example 4.2.17] (Lebesgue's thorn). Let $z_0 = (0, 0)$, $R_N = 1/N$ and $a_N = N^{-2/\beta}$. We set

$$\begin{aligned} C_N &= \{(x, y) \in \mathbb{B}_N(z_0) : 0 < y < a_N x\}, \\ C'_N &= \{(x, y) \in \mathbb{B}_{N+1}(z_0) : 0 < y < a_N x\}, \end{aligned}$$

and define $\mathscr{O}$ by

$$\mathscr{O} = \bigcup_{N=1}^{\infty} C_N \backslash C'_N.$$

From Lemma 3.2.2, there exist positive constants $c_1 < c_2$, independent of R and N, such that

$$C_N \backslash C'_N \subseteq \left\{(x, y) \in \mathbb{H} : c_1 R^2_{N+1} < x < c_2 R^2_N, 0 < y < a_N x\right\},$$

which give us

$$\begin{aligned} |C_N \backslash C'_N|_{\beta-1} &\leq \int_{c_1 R^2_{N+1}}^{c_2 R^2_N} \int_0^{a_N x} y^{\beta-1} dy dx \\ &= a_N^\beta \frac{c_2^{\beta+1} - c_1^{\beta+1}}{\beta(\beta+1)} \left(R_N^{2(1+\beta)} - R_{N+1}^{2(1+\beta)}\right) \\ &\leq \frac{C}{N^2} R_N^{2(1+\beta)} \\ &\leq \frac{C}{N^2}|\mathbf{B}_N(z_0)|_{\beta-1}, (\text{ by Lemma 3.2.2.}) \end{aligned}$$

Recall $\mathbf{B}_N(z_0) = \mathscr{O} \cap \mathbb{B}_N(z_0)$. Then, we obtain

$$|\mathbf{B}_N(z_0)|_{\beta-1} = \sum_{k=N}^{\infty} |C_k \backslash C'_k|_{\beta-1} \quad \leq C|\mathbf{B}_N(z_0)|_{\beta-1} \sum_{k=N}^{\infty} \frac{1}{k^2},$$

which implies

$$\frac{|\mathbf{B}_N(z_0)|_{\beta-1}}{|\mathbb{B}_N(z_0)|_{\beta-1}} \to 0, \text{ as } N \to \infty,$$

and so, we obtain a contradiction with the left hand side of (3.4.21).

3.5 Hölder continuity for solutions to the variational equation

In this section, we prove Theorem 3.1.8, that is, local Hölder continuity on a neighborhood of $\bar{\Gamma}_0$ for solutions u to the variational equation (3.1.16). We consider separately the case of the interior boundary points $z_0 \in \Gamma_0$ and of the "corner points" $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$. (While $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$ is a set of geometric corner points for the domain $\mathscr{O}$, the lesson of [20] is that the solution, u, along Γ_0 behaves, in many respects, just as it does in the interior of $\mathscr{O}$.) The proof of the second case, for corner points, is easier than the proof of the first case as it does not require an application of the John-Nirenberg inequality. The essential difference between the proof of Theorem 3.1.8 and the proof of its classical analogue for weak solutions to non-degenerate elliptic equations [41, Theorems 8.27 & 8.29] consists in a modification of the methods of [41, §8.6, §8.9, & §8.10] when deriving our energy estimates (3.5.11), where we adapt the application of the John-Nirenberg inequality and Poincaré inequality to our framework of weighted Sobolev spaces. Moreover, because the balls defined by the Koch metric, d, do not have good scaling properties unless they are centered at a point $z_0 \in \partial\mathbb{H}$ (see Remark 3.2.10), the Moser iteration technique applies only to such balls. Therefore, the estimate (3.1.21) holds only for points $z_0 \in \partial\mathbb{H}$, and in order to obtain the full Hölder continuity of solutions (3.1.22), we need to apply a rescaling argument which is outlined in the last steps of the arguments below. Therefore, boundary Hölder continuity does not follow in the same way as in [41].

We now proceed to the proof of Theorem 3.1.8, first in §3.5.1 for the case of points $z_0 \in \Gamma_0$ and then in §3.5.2 for points $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$.

3.5.1 Local Hölder continuity on a neighborhood of the degenerate boundary interior

We commence with the

Proof of Theorem 3.1.8 for points in Γ_0. Let $z_0 \in \Gamma_0$ and let R be small enough such that

$$\mathbf{B}_{4R}(z_0) = \mathbb{B}_{4R}(z_0), \tag{3.5.1}$$

that is, $4R \leq \min\{\bar{R}, \operatorname{dist}(z_0, \Gamma_1)\}$, where $\operatorname{dist}(\cdot,\cdot)$ is the distance function on $\bar{\mathbb{H}}$ defined by the Koch metric, $\mathbf{d}$. Moreover, R is chosen small enough such that for all $z_i = (x_i, y_i) \in \mathbf{B}_R(z_0)$, $i = 1, 2$, we have

$$0 < y_1, y_2 < 1 \text{ and } 0 \leq \|z_1 - z_2\|, \mathbf{d}(z_1, z_2) < 1. \tag{3.5.2}$$

Choose

$$q \in (d + \beta, s), \tag{3.5.3}$$

$$\delta \in (0, 2), \tag{3.5.4}$$

and define $k(R) > 0$ by

$$k \equiv k(R) := \|f\|_{L^q(\mathbf{B}_{4R}(z_0), \mathfrak{w})} + \left(|m_{\bar{R}}| + |M_{\bar{R}}|\right) R^{\delta}. \tag{3.5.5}$$

The remaining steps in the proof will apply to the following choices of functions w defined on $\mathbf{B}_{4R}(z_0)$,

$$w = u - m_{4R} + k(R) \text{ and } w = M_{4R} - u + k(R). \tag{3.5.6}$$

If $m_{\bar{R}} = M_{\bar{R}} = 0$ or $m_{4R} = M_{4R} = 0$, then automatically $u = 0$ on $\mathbf{B}_{4R}(z_0)$ and (3.1.21) and (3.1.22) hold on $\mathbf{B}_{4R}(z_0)$. Therefore, without loss of generality, we may assume

$$m_{4R} \neq 0 \text{ or } M_{4R} \neq 0, \tag{3.5.7}$$

and $m_{\bar{R}} \neq 0$ or $M_{\bar{R}} \neq 0$. The last assumption implies that

$$k(R) \neq 0, \tag{3.5.8}$$

by (3.5.5). Therefore, we notice that both choices of w in (3.5.6) are bounded, positive functions.

Step 1 (Energy estimate for w)**.** Let $\eta \in C_0^1(\bar{\mathbb{H}})$ be a non-negative cutoff function with $\operatorname{supp}\eta \subseteq \bar{\mathbf{B}}_{4R}$. For any $\alpha \in \mathbb{R}$, $\alpha \neq -1$, let

$$v := \eta^2 w^\alpha. \tag{3.5.9}$$

Then, v is a valid test function in $H_0^1(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$ by Lemma B.2.2. Let

$$H(w) := w^{(\alpha+1)/2}, \tag{3.5.10}$$

and notice that Theorem 3.1.7 implies that $H(w)$ is a positive, bounded function, so the following operations are justified. The goal in this step is to prove

Claim 3.5.1 (Energy estimate)**.** *There exists a positive constant C depending only on the Heston coefficients and δ_0, and there is a positive constant ξ depending only on β and q, such that*

$$\|\eta H(w)\|_{L^p\left(\mathbb{H}, y^{\beta-1}\right)} \leq C_0(R,\alpha)\|H(w)\|_{L^2\left(\operatorname{supp}\eta, y^{\beta-1}\right)}, \tag{3.5.11}$$

where the constant $C_0(R,\alpha)$ is defined by

$$C_0(R,\alpha) := [C|1+\alpha|]^{(\xi+1)/p}\left(1 + \|\sqrt{y}\nabla\eta\|^2_{L^\infty(\mathbb{H})}\right)^{1/p}, \tag{3.5.12}$$

and the constant ξ is given by

$$\xi \equiv \xi(p,q) := \frac{p(q^*-1)}{p-2q^*}, \tag{3.5.13}$$

where q^* is the conjugate exponent for q in (3.5.3)*, that is,* $1/q + 1/q^* = 1$.

The estimate (3.5.11) will be used in Moser iteration.

Substituting the choice (3.5.9) of v in (3.1.12) and using $\nabla v = \alpha\eta^2 w^{\alpha-1}\nabla w + 2\eta\nabla\eta w^\alpha$ gives

$$\begin{aligned}
0 &= a(u,v) - (f,v)_{L^2(\mathscr{O},\mathfrak{w})} \\
&= \frac{\alpha}{2}\int_{\mathbb{H}} \eta^2 w^{\alpha-1}\left(w_x^2 + 2\rho\sigma w_x w_y + \sigma^2 w_y^2\right) y\mathfrak{w}\,dxdy \\
&\quad + \int_{\mathbb{H}} \eta w^\alpha \left(w_x\eta_x + \rho\sigma w_x\eta_y + \rho\sigma w_y\eta_x + \sigma^2 w_y\eta_y\right) y\mathfrak{w}\,dxdy \\
&\quad - \int_{\mathbb{H}} \left[a_1 w_x + \frac{\gamma}{2}(w_x + \rho\sigma w_y)\operatorname{sign}(x)\right]\eta^2 w^\alpha y\mathfrak{w}\,dxdy + \int_{\mathbb{H}} (ru-f)\,\eta^2 w^\alpha \mathfrak{w}\,dxdy.
\end{aligned}$$

Using (3.5.10) to compute $\nabla H(w) = \frac{\alpha+1}{2} w^{(\alpha-1)/2}\nabla w$, we can rewrite the preceding equation as

$$\begin{aligned}
0 &= \frac{2\alpha}{|1+\alpha|^2}\int_{\mathbb{H}} \eta^2 \left[\partial_x H(w)^2 + 2\rho\sigma \partial_x H(w)\partial_y H(w) + \sigma^2 \partial_y H(w)^2\right] y\mathfrak{w}\, dxdy \\
&+ \frac{2}{1+\alpha}\int_{\mathbb{H}} \eta H(w)\left[\partial_x H(w)\eta_x + \rho\sigma\partial_x H(w)\eta_y + \rho\sigma\partial_y H(w)\eta_x + \sigma^2\partial_y H(w)\eta_y\right] y\mathfrak{w}\, dxdy \\
&- \frac{2}{1+\alpha}\int_{\mathbb{H}} \left[a_1\partial_x H(w) + \frac{\gamma}{2}\left(\partial_x H(w) + \rho\sigma\partial_x H(w)\right)\operatorname{sign}(x)\right]\eta^2 H(w) y\mathfrak{w}\, dxdy \\
&+ \int_{\mathbb{H}} \left(rw + r(m_{4R}-k) - f\right)\eta^2 w^\alpha \mathfrak{w}\, dxdy.
\end{aligned}$$

Using the uniform ellipticity property (3.1.14), Hölder's inequality, the fact that $w \geq k$ by (3.5.6), and the preceding identity, we see that there is a positive constant C, depending only on the coefficients of the Heston operator, such that

$$\begin{aligned}
\int_{\mathbb{H}} \eta^2 |\nabla H(w)|^2 y\mathfrak{w}\, dxdy \leq C|1+\alpha| &\left[\int_{\mathbb{H}} \left(\eta^2 + y|\nabla\eta|^2\right) w^{\alpha+1}\mathfrak{w}\, dxdy\right. \\
&\left.+ \int_{\mathbb{H}} \eta^2 |f + r(k-m_{4R})| w^\alpha \mathfrak{w}\, dxdy\right],
\end{aligned}$$

and hence

$$\begin{aligned}
\int_{\mathbb{H}} \eta^2 |\nabla H(w)|^2 y\mathfrak{w}\, dxdy \leq C|1+\alpha| &\left[\int_{\mathbb{H}} \left(\eta^2 + y|\nabla\eta|^2\right) w^{\alpha+1}\mathfrak{w}\, dxdy\right. \\
&\left.+ \int_{\mathbb{H}} \eta^2 \frac{|f + r(k-m_{4R})|}{k} w^{\alpha+1} \mathfrak{w}\, dxdy\right].
\end{aligned} \tag{3.5.14}$$

By Hölder's inequality, we have

$$\begin{aligned}
\int_{\mathbb{H}} \eta^2 \frac{|f + r(k-m_{4R})|}{k} w^{\alpha+1}\mathfrak{w}\, dxdy \leq &\left(\int_{\operatorname{supp}\eta} \left|\frac{f + r(k-m_{4R})}{k}\right|^q \mathfrak{w}\, dxdy\right)^{1/q} \\
&\times \left(\int_{\mathbb{H}} \left|\eta w^{(\alpha+1)/2}\right|^{2q^*} \mathfrak{w}\, dxdy\right)^{1/q^*}.
\end{aligned} \tag{3.5.15}$$

From our definition of k in (3.5.5), there is a positive constant C, depending only on δ_0 and β, such that

$$\left(\int_{\operatorname{supp}\eta} \left|\frac{f + r(k-m_{4R})}{k}\right|^q \mathfrak{w}\, dxdy\right)^{1/q} \leq C. \tag{3.5.16}$$

From inequalities (3.5.14), (3.5.15) and (3.5.16), we obtain

$$\begin{aligned}
\int_{\mathbb{H}} \eta^2 |\nabla H(w)|^2 y\mathfrak{w}\, dxdy \leq C|1+\alpha| &\left[\int_{\mathbb{H}} \left(\eta^2 + y|\nabla\eta|^2\right) w^{\alpha+1}\mathfrak{w}\, dxdy\right. \\
&\left.+ \left(\int_{\mathbb{H}} \left|\eta w^{(\alpha+1)/2}\right|^{2q^*} \mathfrak{w}\, dxdy\right)^{1/q^*}\right],
\end{aligned} \tag{3.5.17}$$

where the positive constant C depends on the Heston coefficients and δ_0. We apply Lemma 3.2.1 to $\eta H(w)$ and we have

$$\begin{aligned}\int_{\mathbb{H}} |\eta H(w)|^p y^{\beta-1} dxdy &\leq \left(\int_{\mathbb{H}} \eta^2 |H(w)|^2 y^{\beta-1} dxdy\right)^{(p-2)/2} \int_{\mathbb{H}} |\nabla(\eta H(w))|^2 y^{\beta} dxdy \\ &\leq \left(\int_{\mathbb{H}} \eta^2 |H(w)|^2 y^{\beta-1} dxdy\right)^{(p-2)/2} \\ &\quad \times \left(\int_{\mathbb{H}} |\nabla\eta|^2 |H(w)|^2 y^{\beta} dxdy + \eta^2 |\nabla H(w)|^2 y^{\beta} dxdy\right).\end{aligned}$$

Combining the preceding inequality with (3.5.17), we obtain

$$\begin{aligned}&\int_{\mathbb{H}} |\eta H(w)|^p y^{\beta-1} dxdy \\ &\leq C|1+\alpha| \left(\int_{\mathbb{H}} \eta^2 |H(w)|^2 y^{\beta-1} dxdy\right)^{(p-2)/2} \\ &\quad \times \left[\int_{\mathbb{H}} \left(\eta^2 + y|\nabla\eta|^2\right) |H(w)|^2 y^{\beta-1} dxdy + \left(\int_{\mathbb{H}} |\eta H(w)|^{2q^*} y^{\beta-1} dxdy\right)^{1/q^*}\right],\end{aligned}$$

and thus

$$\begin{aligned}&\int_{\mathbb{H}} |\eta H(w)|^p y^{\beta-1} dxdy \\ &\leq C|1+\alpha| \left(1 + \|\sqrt{y}\nabla\eta\|^2_{L^\infty(\mathbb{H})}\right) \left(\int_{\operatorname{supp}\eta} \eta^2 |H(w)|^2 y^{\beta-1} dxdy\right)^{p/2} \\ &\quad + C|1+\alpha| \left(\int_{\mathbb{H}} \eta^2 |H(w)|^2 y^{\beta-1} dxdy\right)^{(p-2)/2} \left(\int_{\mathbb{H}} |\eta H(w)|^{2q^*} y^{\beta-1} dxdy\right)^{1/q^*}.\end{aligned} \tag{3.5.18}$$

From our assumption (3.5.3) that $q > d+\beta$, we have $2 < 2q^* < p$. Since $q < \infty$ implies $q^* > 1$, while $q > d+\beta$ implies

$$q^* < (d+\beta)/(d+\beta-1), \tag{3.5.19}$$

and thus $2q^* < p$ by (3.1.24). Hence, we may apply the interpolation inequality [41, Inequality (7.10)], for any $\varepsilon > 0$, to give

$$\|\eta H(w)\|_{L^{2q^*}(\mathbb{H},y^{\beta-1})} \leq \varepsilon \|\eta H(w)\|_{L^p(\mathbb{H},y^{\beta-1})} + \varepsilon^{-\xi} \|\eta H(w)\|_{L^2(\mathbb{H},y^{\beta-1})},$$

where ξ is given by (3.5.13). We need the preceding inequality in the form

$$\begin{aligned}\left(\int_{\mathbb{H}} |\eta H(w)|^{2q^*} y^{\beta-1} dxdy\right)^{1/q^*} &= \|\eta H(w)\|^2_{L^{2q^*}(\mathbb{H},y^{\beta-1})} \\ &\leq 2\varepsilon^2 \|\eta H(w)\|^2_{L^p(\mathbb{H},y^{\beta-1})} + 2\varepsilon^{-2\xi} \|\eta H(w)\|^2_{L^2(\mathbb{H},y^{\beta-1})}.\end{aligned}$$

Applying the preceding inequality in (3.5.18), we obtain

$$\begin{aligned}\|\eta H(w)\|^p_{L^p(\mathbb{H},y^{\beta-1})} \leq C|1+\alpha| \left(1+\|\sqrt{y}\nabla\eta\|^2_{L^\infty(\mathbb{H})}\right)\|H(w)\|^p_{L^2(\operatorname{supp}\eta,y^{\beta-1})}\\ + C|1+\alpha|\|\eta H(w)\|^{p-2}_{L^2(\operatorname{supp}\eta,y^{\beta-1})}\times\\ \left(\varepsilon^2\|\eta H(w)\|^2_{L^p(\mathbb{H},y^{\beta-1})} + \varepsilon^{-2\xi}\|\eta H(w)\|^2_{L^2(\mathbb{H},y^{\beta-1})}\right).\end{aligned}$$

Recombining terms, we see that

$$\begin{aligned}&\|\eta H(w)\|^p_{L^p(\mathbb{H},y^{\beta-1})}\\ &\quad\leq C|1+\alpha|\left(1+\varepsilon^{-2\xi}\right)\left(1+\|\sqrt{y}\nabla\eta\|^2_{L^\infty(\mathbb{H})}\right)\|H(w)\|^p_{L^2(\operatorname{supp}\eta,y^{\beta-1})}\\ &\qquad+ C|1+\alpha|\varepsilon^2\|\eta H(w)\|^2_{L^p(\mathbb{H},y^{\beta-1})}\|\eta H(w)\|^{p-2}_{L^2(\mathbb{H},y^{\beta-1})}.\end{aligned}$$

To bound the last term in the preceding inequality, we apply Young's inequality with the conjugate exponents $(p/2, p/(p-2))$ to give

$$\|\eta H(w)\|^2_{L^p(\mathbb{H},y^{\beta-1})}\|\eta H(w)\|^{p-2}_{L^2(\mathbb{H},y^{\beta-1})} \leq \frac{2}{p}\|\eta H(w)\|^p_{L^p(\mathbb{H},y^{\beta-1})}+\frac{p-2}{p}\|\eta H(w)\|^p_{L^2(\mathbb{H},y^{\beta-1})}.$$

Thus,

$$\begin{aligned}\|\eta H(w)\|^p_{L^p(\mathbb{H},y^{\beta-1})} &\leq C|1+\alpha|\varepsilon^2\|\eta H(w)\|^p_{L^p(\mathbb{H},y^{\beta-1})}\\ &\quad+ C|1+\alpha|\left(1+(\varepsilon^2+\varepsilon^{-2\xi})\right)\left(1+\|\sqrt{y}\nabla\eta\|^2_{L^\infty(\mathbb{H})}\right)\|H(w)\|^p_{L^2(\operatorname{supp}\eta,y^{\beta-1})}.\end{aligned}$$

By choosing $\varepsilon = 1/\left(2C|1+\alpha|\right)^{1/2}$ and taking roots of order p, we find that

$$\|\eta H(w)\|_{L^p(\mathbb{H},y^{\beta-1})} \leq [C|1+\alpha|]^{(\xi+1)/p}\left(1+\|\sqrt{y}\nabla\eta\|^2_{L^\infty(\mathbb{H})}\right)^{1/p}\|H(w)\|_{L^2(\operatorname{supp}\eta,y^{\beta-1})},$$

which is equivalent to (3.5.11) and (3.5.12).

Step 2 (Moser iteration with negative power)**.** In this step we apply the Moser iteration technique starting with a suitable $\alpha = \alpha_0 < -1$ in (3.5.11) with functions w in (3.4.2). Let $(\eta_N)_{N\in\mathbb{N}}$ be the sequence of cut-off functions considered in Step 2 in the proof of Theorem 3.1.7. Let $\alpha_0 < -1$, $p_0 := \alpha_0 + 1$, $p_N := p_0(p/2)^N$, where p is as in (3.1.24), and $\alpha_N + 1 := p_N$. We notice that $p_N \to -\infty$ as N increases. Set

$$I(N) := \left(\int_{\mathbf{B}_{R_N}} |w|^{p_N} y^{\beta-1}dxdy\right)^{1/p_N}.$$

By applying (3.5.11) with $w = u - m_{4R} + k$, $\alpha = \alpha_{N-1}$, and $\eta = \eta_N$, we obtain for all $N \geq 1$,

$$\left(\int_{\mathbb{H}} \eta_N^p w^{(\alpha_{N-1}+1)p/2} y^{\beta-1}dxdy\right)^{1/p} \leq C_0(R,\alpha)\left(\int_{\mathbb{H}}\left|\eta_N w^{(\alpha_{N-1}+1)/2}\right|^2 y^{\beta-1}dxdy\right)^{1/2}.$$

Since $(\alpha_{N-1}+1)p/2 = p_N$ and $\alpha_{N-1}+1 = p_{N-1}$, we can write the preceding inequality as

$$\left(\int_{\mathbf{B}_N} |w|^{p_N} y^{\beta-1} dxdy\right)^{1/p} \leq C_0(R,\alpha) \left(\int_{\mathbf{B}_{N-1}} |w|^{p_{N-1}} y^{\beta-1} dxdy\right)^{1/2}.$$

Taking roots of order p/p_N and noticing that $p/p_N < 0$, we obtain

$$I(N) \geq C_1(R,N) I(N-1), \tag{3.5.20}$$

where $C_1(R,N)$ is given by

$$C_1(R,N) := [C|p_{N-1}|]^{(\xi+1)/p_N} \left(1 + \|\sqrt{y}\nabla\eta_n\|^2_{L^\infty(\mathbb{H})}\right)^{1/p_N}.$$

and C is a positive constant, independent of R and N, depending only on the Heston coefficients and δ_0. From Lemma 3.2.2, the bound on $|\nabla\eta_N|$ and the fact that $0 < R < 1$, we obtain

$$1 + \|\sqrt{y}\nabla\eta_n\|^2_{L^\infty(\mathbb{H})} \leq cN^6 R^{-2},$$

for some positive constant c, and so, we may assume without loss of generality

$$C_1(R,N) = \left[C|p_{N-1}|N^6\right]^{(\xi+1)/p_N} R^{-2/p_N}. \tag{3.5.21}$$

We notice that

$$\prod_{N\geq 1} C_1(R,N) = C_2 R^{4/(|p_0|(p-2))} < \infty,$$

where C_2 depends at most on the Heston coefficients, δ_0 and q. From (3.4.19), we know that for some constant $c > 0$ we have $|\mathbf{B}_{2R}|_{\beta-1} \geq cR^{4/(p-2)}$. Thus,

$$\prod_{N\geq 1} C_1(R,N) \geq C_2 |\mathbf{B}_{2R}|^{1/|p_0|}_{\beta-1}.$$

By iterating (3.5.20), we obtain $I(-\infty) \geq I(0)\prod_{N\geq 1} C_0(R,N)$, which gives us

$$\inf_{\mathbf{B}_R} w = I(-\infty) \geq C_2 \left(\frac{1}{|\mathbf{B}_{2R}|_{\beta-1}} \int_{\mathbf{B}_{2R}} |w|^{p_0} y^{\beta-1} dxdy\right)^{1/p_0}. \tag{3.5.22}$$

Step 3 (Application of Theorem 3.3.1)**.** The purpose of this step is to show that we may apply Theorem 3.3.1 to w with $S_r = \mathbf{B}_{(2+r)R}(z_0)$, $0 \leq r \leq 1$, and $\vartheta_0 = \vartheta_1 = 1$. By Proposition 3.3.2, we find that w satisfies the inequalities (3.3.2), so it remains to show that (3.3.3) holds for $\log w$. For A as defined in (3.3.3) and $S_r = \mathbf{B}_{(2+r)R}$, writing $\mathbf{B}_{(2+r)R}$ in place of $\mathbf{B}_{(2+r)R}(z_0)$ for brevity, we have by Hölder's inequality that

$$A \leq \sup_{0\leq r\leq 1} \inf_{c\in\mathbb{R}} \left(\frac{1}{|\mathbf{B}_{(2+r)R}|_{\beta-1}} \int_{\mathbf{B}_{(2+r)R}} |\log w - c|^2 y^{\beta-1} dxdy \right)^{1/2},$$

and so, Corollary 3.2.5 gives us

$$A \leq \sup_{0\leq r\leq 1} ((2+r)R)^2 \left(\frac{1}{|\mathbf{B}_{(2+r)R}|_{\beta}} \int_{\mathbf{B}_{(2+r)R}} |\nabla \log w|^2 y^{\beta} dxdy \right)^{1/2}. \tag{3.5.23}$$

Let $\eta \in C_0^1(\bar{\mathbb{H}})$ be a non-negative cutoff function such that $\eta = 1$ on $\mathbf{B}_{(2+r)R}$, $\eta = 0$ outside $\mathbf{B}_{4R}$, and $|\nabla\eta| \leq C/R^2$. We choose $v = \eta^2/w$ and notice that $v \in H_0^1(\mathscr{O}\cup\Gamma_0, \mathfrak{w})$ by Lemma B.2.2. With this choice of v as a test function in the variational equation (3.1.12) satisfied by u, we obtain

$$\begin{aligned}
(f,v)_{L^2(\mathscr{O},\mathfrak{w})} &= a(u,v) \\
&= -\frac{1}{2}\int_{\mathscr{O}} \frac{\eta^2}{w^2}\left(w_x^2 + 2\rho\sigma w_x w_y + \sigma^2 w_y^2\right) y\mathfrak{w}\, dxdy \\
&\quad + \int_{\mathscr{O}} \frac{\eta}{w}\left[w_x\eta_x + \rho\sigma(w_x\eta_y + w_y\eta_x) + \sigma^2 w_y\eta_y\right] y\mathfrak{w}\, dxdy \\
&\quad - \int_{\mathscr{O}} \frac{\eta^2}{w}\left[\frac{\gamma\,\mathrm{sign}(x)}{2}(w_x + \rho\sigma w_y) + a_1 w_x\right] y\mathfrak{w}\, dxdy \\
&\quad + \int_{\mathscr{O}} r\eta^2 \frac{u}{w}\mathfrak{w}\, dxdy.
\end{aligned}$$

Using the uniform ellipticity property and Hölder's inequality, we obtain there is a positive constant C, depending only on the Heston coefficients and δ_0, such that

$$\int_{\mathscr{O}} \eta^2 |\nabla \log w|^2 y^{\beta} dxdy \leq C\int_{\mathscr{O}} (|\nabla\eta|^2 + \eta^2) y^{\beta} dxdy + C\int_{\mathscr{O}} \eta^2 \frac{|f|+|u|}{w} y^{\beta-1} dxdy. \tag{3.5.24}$$

From Lemma 3.2.2, assumption (3.5.1) and the fact that $|\nabla\eta| \leq C/R^2$, we have

$$\begin{aligned}
\int_{\mathscr{O}} (|\nabla\eta|^2 + \eta^2) y^{\beta} dxdy &\leq CR^{-4} R^{2(d+\beta)} \\
&\leq C\left((2+r)R\right)^{-4} |\mathbf{B}_{(2+r)R}|_{\beta}.
\end{aligned} \tag{3.5.25}$$

Using the definition (3.5.5) of $k(R)$ and Hölder's inequality, we obtain

$$\begin{aligned}
\int_{\mathscr{O}} \eta^2 \frac{|f|+|u|}{w} y^{\beta-1} dxdy &\leq \frac{1}{\|f\|_{L^q(\mathbf{B}_{4R}, y^{\beta-1})}} \|f\|_{L^q(\mathbf{B}_{4R}, y^{\beta-1})} R^{2(d+\beta-1)/q^*} \\
&\quad + \frac{1}{R^{\delta}} R^{2(d+\beta-1)},
\end{aligned}$$

and thus

$$\int_{\mathscr{O}} \eta^2 \frac{|f|+|u|}{w} y^{\beta-1} dxdy \leq C \left(R^{2(d+\beta-1)/q^*} + R^{2(d+\beta-1)-\delta} \right). \tag{3.5.26}$$

The condition $q > d+\beta$ implies

$$2(d+\beta-1)/q^* - 2(d+\beta) > -4, \tag{3.5.27}$$

since $1/q + 1/q^* = 1$. Also, because δ is chosen in $(0,2)$, we obviously have

$$-2-\delta > -4. \tag{3.5.28}$$

Using (3.5.27) and (3.5.28), and $0 < R \leq \bar{R}$, we obtain in inequality (3.5.26) that there is a positive constant C, depending only on the Heston coefficients and δ_0, such that

$$\begin{aligned} \int_{\mathscr{O}} \eta^2 \frac{|f|+|u|}{w} y^{\beta-1} dxdy &\leq C \left((2+r)R \right)^{2(d+\beta)-4} \\ &\leq C \left((2+r)R \right)^{-4} |\mathbf{B}_{(2+r)R}|_\beta. \end{aligned} \tag{3.5.29}$$

In the last inequality, we used Lemma 3.2.2 and (3.5.1). By combining equations (3.5.24), (3.5.25) and (3.5.29), we obtain

$$\int_{\mathbf{B}_{(2+r)R}} |\nabla \log w|^2 y^\beta dxdy \leq C \left((2+r)R \right)^{-4} |\mathbf{B}_{(2+r)R}|_\beta.$$

Then, it immediately follows that the right hand side of (3.5.23) is finite, and so, (3.3.3) holds for $\log w$.

Step 4 (Proof of inequality (3.1.21))**.** In the previous step we showed that Theorem 3.3.1 applies to w with $\vartheta_0 = \vartheta_1 = 1$. Hence, there is a constant $C > 0$, depending only on the coefficients of the Heston operator and δ_0, but independent of R and w, such that

$$\left(\frac{1}{|\mathbf{B}_{2R}|_{\beta-1}} \int_{\mathbf{B}_{2R}} |w| y^{\beta-1} dxdy \right) \leq C \left(\frac{1}{|\mathbf{B}_{2R}|_{\beta-1}} \int_{\mathbf{B}_{2R}} |w|^{-1} y^{\beta-1} dxdy \right)^{-1}. \tag{3.5.30}$$

From (3.5.22), we obtain

$$\inf_{\mathbf{B}_R} w = I(-\infty) \geq C \left(\frac{1}{|\mathbf{B}_{2R}|_{\beta-1}} \int_{\mathbf{B}_{2R}} |w| y^{\beta-1} dxdy \right). \tag{3.5.31}$$

We now choose $w = u - m_{4R} + k$ and $w = M_{4R} - u + k$ in (3.5.31). By adding the following two inequalities

$$\begin{aligned}
m_R - m_{4R} + k(R) &= \inf_{\mathbf{B}_R}(u - m_{4R} + k(R)) \\
&\geq \frac{C}{|\mathbf{B}_{2R}|_{\beta-1}} \int_{\mathbf{B}_{2R}} (u - m_{4R} + k(R)) y^{\beta-1} dxdy \\
&\geq \frac{C}{|\mathbf{B}_{2R}|_{\beta-1}} \int_{\mathbf{B}_{2R}} (u - m_{4R}) y^{\beta-1} dxdy, \\
M_{4R} - M_R + k(R) &= \inf_{\mathbf{B}_2 R}(M_{4R} - u + k(R)) \\
&\geq \frac{C}{|\mathbf{B}_{2R}|_{\beta-1}} \int_{\mathbf{B}_{2R}} (M_{4R} - u + k(R)) y^{\beta-1} dxdy \\
&\geq \frac{C}{|\mathbf{B}_{2R}|_{\beta-1}} \int_{\mathbf{B}_{2R}} (M_{4R} - u) y^{\beta-1} dxdy,
\end{aligned}$$

we obtain

$$(M_{4R} - m_{4R}) - (M_R - m_R) + 2k(R) \geq C\left(M_{4R} - m_{4R}\right).$$

Without loss of generality, we may assume $C < 1$ (if not, we can make C smaller on the right-hand side of the preceding inequality). Therefore, the the preceding inequality can be rewritten in the form

$$\operatorname*{osc}_{\mathbf{B}_R(z_0)} \leq C \operatorname*{osc}_{\mathbf{B}_{4R}(z_0)} + 2k(R). \tag{3.5.32}$$

Because $q \in (d + \beta, s)$ by (3.5.3) and $f \in L^s(\mathbf{B}_{\bar{R}}(z_0), \mathfrak{w})$ for some $s > d + \beta$, by hypothesis in Theorem 3.1.8, Hölder's inequality yields

$$\|f\|_{L^q(\mathbf{B}_{4R}, y^{\beta-1})} \leq C R^{2(d+\beta-1)\frac{s-q}{sq}} \|f\|_{L^s(\mathbf{B}_{\bar{R}}, y^{\beta-1})}.$$

Let

$$\nu := \min\left\{\delta, 2(d + \beta - 1)\frac{s-q}{sq}\right\}.$$

Consequently, from (3.5.5), we see that there is a positive constant C, depending only on $d = 2$ and β, such that

$$k(R) \leq C\left(\|f\|_{L^s(\mathbf{B}_{\bar{R}}, y^{\beta-1})} + |m_{\bar{R}}| + |M_{\bar{R}}|\right) R^{\nu}. \tag{3.5.33}$$

Therefore, by applying [41, Lemma 8.23] to (3.5.32) and using the inequality (3.5.33), we find that there is a positive constant C depending on the coefficients of the Heston operator, the constant δ_0, $\|f\|_{L^s(\mathbf{B}_{\bar{R}}, y^{\beta-1})}$, and $\|u\|_{L^\infty(\mathbf{B}_{\bar{R}})}$, and there is a constant $\alpha_0 \in (0,1)$, depending on s, d and β, such that

$$\operatorname*{osc}_{\mathbf{B}_R(z_0)} \leq C R^{\alpha_0},$$

which is the desired inequality (3.1.21).

Step 5 (Proof of inequality (3.1.22))**.** We prove the estimate (3.1.22) for points $z_1, z_2 \in \bar{\mathbf{B}}_R(z_0)$, where R satisfies

$$0 < 8R \leq \min\{\bar{R}, \operatorname{dist}(z_0, \Gamma_1)\}, \tag{3.5.34}$$

where $\operatorname{dist}(\cdot,\cdot)$ is the distance function defined by the Koch metric. Condition (3.5.34) implies that for any $z \in \mathbf{B}_R(z_0)$, we have that (3.5.1) holds for $\mathbf{B}_R(z)$, and so estimate (3.1.21) applies on such balls. In particular, for any points $(x_1, y_1), (x_1, 0), (x_2, 0) \in \bar{\mathbf{B}}_R(z_0)$, the estimate (3.1.21) gives

$$\begin{aligned} |u(x_1,y_1) - u(x_1,0)| &\leq C\mathbf{d}\left((x_1,y_1),(x_1,0)\right)^{\alpha_0}, \\ |u(x_1,0) - u(x_2,0)| &\leq C\mathbf{d}\left((x_1,0),(x_2,0)\right)^{\alpha_0}. \end{aligned} \tag{3.5.35}$$

Notice that we have the simple identities

$$\begin{aligned} \mathbf{d}\left((x_1,y_1),(x_1,0)\right) &= \sqrt{y_1/2}, \\ \mathbf{d}\left((x_1,0),(x_2,0)\right) &= \sqrt{|x_1-x_2|}, \end{aligned} \tag{3.5.36}$$

and so, we can rewrite (3.5.35) in the form

$$\begin{aligned} |u(x_1,y_1) - u(x_1,0)| &\leq C|y_1|^{\alpha_0/2}, \\ |u(x_1,0) - u(x_2,0)| &\leq C|x_1-x_2|^{\alpha_0/2}. \end{aligned} \tag{3.5.37}$$

The idea of inequality (3.1.22) the proof now follows [20, Corollary I.9.7 & Theorem I.9.8], but with certain differences which we outline for clarity. Let $\varepsilon \in (0, 1/8)$ be fixed and consider the following two cases.

Case 3 (Pairs of points in $\mathbf{B}_R(z_0)$ obeying (3.5.38))**.** Let $z_i = (x_i, y_i) \in \mathbf{B}_R(z_0)$, $i = 1, 2$, be such that

$$\|z_1 - z_2\| \geq \varepsilon(y_1^2 + y_2^2). \tag{3.5.38}$$

From (3.5.2), we can find a positive constant C such that

$$|x_1 - x_2| \leq C\mathbf{d}(z_1, z_2). \tag{3.5.39}$$

Using our current assumption (3.5.38), in addition to (3.5.2), we also have

$$\mathbf{d}(z_1, z_2) \geq \varepsilon C y_i^2, \quad i = 1, 2,$$

and so, there exists a positive constant C, depending on ε, such that

$$y_i \leq C\mathbf{d}(z_1, z_2)^{1/2}, \quad i = 1, 2. \tag{3.5.40}$$

Denote $z_i' = (x_i, 0)$, for $i = 1, 2$. Applying (3.5.39) and (3.5.40) in (3.5.37), we obtain

$$\begin{aligned} |u(z_i) - u(z_i')| &\leq C\mathbf{d}(z_1, z_2)^{\alpha_0/4}, \quad i = 1, 2, \\ |u(z_1') - u(z_2')| &\leq C\mathbf{d}(z_1, z_2)^{\alpha_0/2}, \end{aligned}$$

and hence, using (3.5.2),

$$\begin{aligned} |u(z_1) - u(z_2)| &\leq |u(z_1) - u(z_1')| + |u(z_1') - u(z_2')| + |u(z_2) - u(z_2')| \\ &\leq C\mathbf{d}(z_1, z_2)^{\alpha_0/4}, \end{aligned}$$

that is,

$$|u(z_1) - u(z_2)| \leq C\mathbf{d}(z_1, z_2)^{\alpha_0/4}. \tag{3.5.41}$$

This concludes the proof of Case 3. Therefore, the estimate (3.1.22) holds in the special case $\|z_1 - z_2\| \geq \varepsilon(y_1^2 + y_2^2)$.

Case 4 (Pairs of points in $\mathbf{B}_R(z_0)$ obeying (3.5.42))**.** Now we consider points $z_i = (x_i, y_i) \in \mathbf{B}_R(z_0)$, $i = 1, 2$, such that

$$\|z_1 - z_2\| < \varepsilon(y_1^2 + y_2^2). \tag{3.5.42}$$

By scaling and using interior Hölder estimates [41, Theorem 8.22], we show that the estimate (3.1.22) also holds in this case. We proceed by analogy with the proofs of [20, Theorems I.9.1–4 & Corollary I.9.7]. We may assume without loss of generality that

$$1 > y_2 \geq y_1 \text{ and } x_2 = 0. \tag{3.5.43}$$

We consider the function v defined by rescaling,

$$u(x,y) =: v\left(\frac{1}{a}(x,y)\right).$$

The rescaling $z \mapsto z' = z/a$ maps $B_{y_2/2}(z_2)$ into $B_{1/2}(z'_2)$. From our assumptions (3.5.2), (3.5.42) and the choice of $\varepsilon \in (0,1/8)$, we see that

$$\|z'_1 - z'_2\| \leq 2\varepsilon y_2 < 1/4, \tag{3.5.44}$$

and so $z'_1 \in B_{1/4}(z'_2)$. From [18, Theorem 5.10], we know that $u \in H^2_{\text{loc}}(\mathbf{B}_{\bar{R}}(z_0))$, and so by direct calculation, we conclude that $v(z')$ solves

$$\tilde{A}v(z') = af(az') \quad \text{on } B_{1/2}(z'_2),$$

where we define

$$\begin{aligned}(\tilde{A}v)(z') := \frac{1}{2}y'\left(v_{xx} + 2\rho\sigma v_{xy} + \sigma^2 v_{yy}\right)(z') + (r - q - ay'/2)v_x(z')\\ + \kappa(\vartheta - ay')v_y(z') - arv(z').\end{aligned}$$

On the ball $B_{1/2}(z'_2)$, the operator $\tilde{A}$ is uniformly elliptic with bounded coefficients. Moreover, there is a positive constant M, depending only on the coefficients of the Heston operator, such that for all $a \in (0,1)$, M is a uniform bound on the $L^\infty(B_{1/2}(z'_2))$-norm of the coefficients of $\tilde{A}$. For brevity, we denote $f_a(z') := af(az')$. By [41, Theorem 8.22], there are positive constants C and $\alpha_0 \in (0,1)$, depending only on the $L^\infty(B_{1/2}(z'_2))$-bounds of the coefficients, such that

$$\operatorname*{osc}_{B_R(z'_2)} v \leq CR^{\alpha_0}\left(\|v\|_{L^\infty(B_{1/2}(z'_2))} + \|f_a\|_{L^s(B_{1/2}(z'_2))}\right), \quad \forall R \in (0,1/2], \tag{3.5.45}$$

because s was assumed to satisfy $s > 2d$ (recall that $d = 2$). We see that

$$\|v\|_{L^\infty(B_{1/2}(z'_2))} = \|u\|_{L^\infty(B_{y_2/2}(z_2))} \leq \|u\|_{L^\infty(\mathbf{B}_{\bar{R}}(z_0))}, \tag{3.5.46}$$

where we used the fact that $B_{y_2/2}(z_2) \subseteq \mathbf{B}_{\bar{R}}(z_0)$, which in turn follows from the requirement $4R \leq \bar{R}$ in the hypotheses of Theorem 3.1.8. We also have

$$\|f_a\|^s_{L^s(B_{1/2}(z'_2))} = \int_{B_{1/2}(z'_2)} |af(az')|^s dz' = \int_{B_{y_2/2}(z_2)} |f(z)|^s a^{s-n} dz,$$

that is,

$$\|f_a\|^s_{L^s(B_{1/2}(z_2'))} = \int_{B_{y_2/2}(z_2)} |f(z)|^s a^{s-n} dz. \tag{3.5.47}$$

Using the fact that $y_2/2 \le y \le 3y_2/2$ for all $z = (x, y) \in B_{y_2/2}(z_2)$, assumption (3.5.2), and the fact that $s > d + \beta$ by hypothesis of Theorem 3.1.8, the estimate (3.5.47) yields

$$\|f_a\|^s_{L^s(B_{1/2}(z_2'))} \le C \int_{\mathbf{B}_{\bar{R}}(z_0)} |f(z)|^s y^{\beta-1} dz, \tag{3.5.48}$$

where C is a positive constant depending only on β. Applying (3.5.46) and (3.5.48) in (3.5.45) yields

$$\operatorname*{osc}_{B_R(z_2')} v \le C R^{\alpha_0} \left(\|u\|_{L^\infty(\mathbf{B}_{\bar{R}}(z_0))} + \|f\|_{L^s(\mathbf{B}_{\bar{R}}(z_0))} \right), \quad \forall R \in (0, 1/2].$$

In particular, because $z_1' \in B_{1/2}(z_2')$, we see that

$$|v(z_1') - v(z_2')| \le C \|z_1' - z_2'\|^{\alpha_0},$$

where the positive constant C now depends on $\|u\|_{L^\infty(\mathbf{B}_{4\bar{R}}(z_0))}$ and $\|f\|_{L^s(\mathbf{B}_{4\bar{R}}(z_0),\mathfrak{w})}$. By rescaling back, we obtain

$$|u(z_1) - u(z_2)| \le C \left(\frac{\|z_1 - z_2\|}{y_2} \right)^{\alpha_0}. \tag{3.5.49}$$

Using the following sequence of inequalities,

$$\begin{aligned}
\frac{\|z_1 - z_2\|^2 / y_2^2}{\mathbf{d}(z_1, z_2)} &\le \frac{\|z_1 - z_2\|^2}{y_2^2} \frac{\sqrt{y_1 + y_2 + \|z_1 - z_2\|}}{\|z_1 - z_2\|} \\
&= \frac{\|z_1 - z_2\|}{y_2^2} \sqrt{y_1 + y_2 + \|z_1 - z_2\|} \\
&\le 2\varepsilon \sqrt{y_1 + y_2 + \|z_1 - z_2\|} \\
&\le 1 \text{ (by (3.5.2) and } \varepsilon \in (0, 1/8)),
\end{aligned}$$

we therefore have

$$\frac{\|z_1 - z_2\|}{y_2} \le \mathbf{d}(z_1, z_2)^{1/2}. \tag{3.5.50}$$

Consequently, (3.5.49) gives us

$$|u(z_1) - u(z_2)| \le C \mathbf{d}(z_1, z_2)^{\alpha_0/2}.$$

This concludes the proof of Case 4.

By combining Cases 3 and 4, we find that, for any $z_1, z_2 \in \mathbf{B}_R(z_0)$ and R satisfying (3.5.1), (3.5.2) and (3.5.34) (see Remark 3.5.4 regarding the expressions for the upper bound for R in the hypotheses of Theorem 3.1.8), we have

$$|u(z_1) - u(z_2)| \leq C\mathbf{d}(z_1, z_2)^{\alpha_0}, \tag{3.5.51}$$

where C and α_0 are constants with the dependencies stated in Theorem 3.1.8. Notice that this inequality is not as strong as the inequality (3.1.22), which holds for all $z_1, z_2 \in \mathbf{B}_{\bar{R}}(z_0)$. To obtain the latter inequality, we need to examine the Hölder continuity at corner points, $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$, which we carry out in the proof below. In Remark 3.5.2, we then explain how (3.1.22) is obtained.

□

3.5.2 Hölder continuity on neighborhoods of corner points

We conclude this section with the

Proof of Theorem 3.1.8 for points in $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$. Suppose $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$. We assume (3.5.2) holds.

First, we describe a reduction argument to non-positive source functions f. Since f is assumed to satisfy the hypotheses of [18, Theorem 3.16], we notice that the positive f^+ and negative f^- parts of f obey also these hypotheses. Then, let u^+ and u^- be the unique solutions in $H^1_0(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$ to the variational equation (3.1.16) with source functions f^+ and f^-, respectively. By linearity of $a(\cdot, v)$, for any $v \in H^1_0(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$ fixed, $u = u^+ - u^-$ is the unique solution to the variational equation (3.1.16) with source function f given by [18, Theorem 3.16]. Moreover, we notice that by our hypothesis we have

$$f^+, f^- \in L^s(\mathbf{B}_{\bar{R}}(z_0), \mathfrak{w}),$$

and so, it is sufficient to prove (3.1.21) and the Hölder continuity property for u^+ and u^-.

Therefore, without loss of generality, we may assume

$$f \leq 0 \text{ on } \mathscr{O}, \tag{3.5.52}$$

which implies that

$$u \leq 0 \text{ on } \mathscr{O}, \tag{3.5.53}$$

by [18, Corollary 3.19]. From the standard theory of non-degenerate elliptic partial differential equations (for example, [41, Theorem 8.30]), we know $u \in C(\bar{\mathbf{B}}_{\bar{R}}(z_0) \cap \mathbb{H})$ and $u = 0$ along the piece of the boundary $\partial \mathbf{B}_{\bar{R}}(z_0) \cap \Gamma_1$. Therefore, we have

$$\operatorname*{osc}_{\mathbf{B}_R(z_0)} u = -m_R.$$

Our proof uses the same method as in the case of points in Γ_0 but a choice of w which is different from that of (3.4.2), and a choice of test function v which is different from that of (3.5.9). Moreover, we do not need to appeal to the John-Nirenberg inequality. Since $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$, however, it is important to make the distinction between $\mathbf{B}_R(z_0)$ and $\mathbb{B}_R(z_0)$. Let $k \equiv k(R)$ be defined as in (3.5.5). Therefore, we now define w on $\mathbb{B}_{4R}(z_0)$ by

$$w(z) := k + \begin{cases} u(z) - m_{4R}, & z \in \mathbb{B}_{4R}(z_0) \cap \mathbf{B}_{4R}(z_0), \\ -m_{4R}, & z \in \mathbb{B}_{4R}(z_0) \backslash \mathbf{B}_{4R}(z_0). \end{cases} \tag{3.5.54}$$

Recall that we may assume without loss of generality that (3.5.7) and (3.5.8) hold. In the present situation, since $M_{4R} = 0$, (3.5.7) becomes

$$m_{4R} \neq 0. \tag{3.5.55}$$

Let $\alpha < 0$ such that $\alpha \neq -1$, and let η be a smooth cutoff function such that $\operatorname{supp} \eta \subseteq \mathbb{B}_{4R}(z_0)$. We now define

$$v := \eta^2 \left(w^\alpha - (k - m_{4R})^\alpha \right). \tag{3.5.56}$$

We notice that v is a well-defined function, for any choice of $\alpha \in \mathbb{R}$, by (3.5.55) and (3.5.8). By Lemma B.2.3, $v \in H^1_0(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$ is a valid test function in (3.1.12). We observe that the function w obeys

$$k \leq w \leq k - m_{4R} \quad \text{on } \mathbb{B}_{4R}(z_0),$$

and, because α is non-positive, we also have

$$k^\alpha \geq w^\alpha \geq (k - m_{4R})^\alpha \quad \text{on } \mathbb{B}_{4R}(z_0).$$

Steps 1 and 2 in the the proof of Theorem 3.1.8 for points in Γ_0 apply to our current choice of w for points in $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$, with the only exception that we now define $I(N)$ by

$$I(N) := \left(\int_{\mathbb{B}_{R_N}} |w|^{p_N} y^{\beta-1} dxdy \right)^{1/P_N}.$$

Therefore, we obtain the analogue of (3.5.22),

$$\begin{aligned}\inf_{\mathbb{B}_R} w = I(-\infty) &\geq C \left(\frac{1}{|\mathbb{B}_{2R}|_{\beta-1}} \int_{\mathbb{B}_{2R}} |w|^{p_0} y^{\beta-1} dxdy \right)^{1/p_0} \\ &\geq C \left(\frac{1}{|\mathbb{B}_{2R}|_{\beta-1}} \int_{\mathbb{B}_{2R} \backslash \mathbf{B}_{2R}} |w|^{p_0} y^{\beta-1} dxdy \right)^{1/p_0}.\end{aligned}$$

Recall that $w = k - m_{4R} \geq -m_{4R}$ on the set $\mathbb{B}_{2R} \backslash \mathbf{B}_{2R}$. Using

$$\inf_{\mathbb{B}_R} w = k + m_R - m_{4R},$$

and combining the preceding inequalities yields

$$\begin{aligned}k(R) + m_R - m_{4R} &\geq C \left(\frac{|\mathbb{B}_{2R} \backslash \mathbf{B}_{2R}|_{\beta-1}}{|\mathbb{B}_{2R}|_{\beta-1}} \right)^{1/p_0} (-m_{4R}) \\ &\geq C(-m_{4R}),\end{aligned}$$

that is,

$$k(R) + m_R - m_{4R} \geq C(-m_{4R}). \tag{3.5.57}$$

Indeed, (3.5.57) follows because Assumption 3.1.2 implies

$$\frac{|\mathbb{B}_{2R} \backslash \mathbf{B}_{2R}|_{\beta-1}}{|\mathbb{B}_{2R}|_{\beta-1}} \geq 1/2.$$

We rewrite (3.5.57), using $\operatorname{osc}_{\mathbf{B}_R(z_0)} u = -m_R$, as

$$\operatorname*{osc}_{\mathbf{B}_R(z_0)} u \leq C \operatorname*{osc}_{\mathbf{B}_{4R}(z_0)} u + k(R),$$

where $C \in (0,1)$ is a constant independent of R. Just as in the proof of Theorem 3.1.8 for the case of points in Γ_0, we can apply [41, Lemma 8.23] to conclude that (3.1.21) holds for some positive constants C and $\alpha_0 \in (0,1)$, that is,

$$|u(z) - u(z_0)| \leq C \mathbf{d}(z, z_0)^{\alpha_0}, \quad \forall z \in \mathbf{B}_R(z_0). \tag{3.5.58}$$

To establish (3.1.22), we proceed as in the proof of Theorem 3.1.8 for the case of points in Γ_0. In order to adapt the argument for the case of points in Γ_0 to points in $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$, we need analogues of the inequalities (3.5.35) to hold in a neighborhood in $\mathscr{O}$ of $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$. Given these analogues of the inequalities (3.5.35), we can apply the same argument as used in the Step 5 of the the proof of Theorem 3.1.8 for the case of points in Γ_0, but instead of applying [41, Theorem 8.22], we now apply [41, Theorem 8.27]. As before, we assume (3.5.34) holds.

Without loss of generality, we may assume $z_0 = (0,0)$. Let $z_1 = (x_1, 0)$, $z_2 = (x_2, 0)$, $z_3 = (x, y)$ and $z_4 = (x, 0)$ be points in $\bar{\mathbf{B}}_R(z_0)$. We may assume $x_2 \geq x_1$ and $x, x_1, x_2 \geq 0$. We claim that the following analogues of the inequalities (3.5.35) (for points $z_0 \in \Gamma_0$) hold for points $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$,

$$\begin{aligned} |u(z_1) - u(z_2)| &\leq C\mathbf{d}(z_1, z_2)^{\alpha_0}, \\ |u(z_3) - u(z_4)| &\leq C\mathbf{d}(z_3, z_4)^{\alpha_0}, \end{aligned} \tag{3.5.59}$$

for some positive constant C and $\alpha_0 \in (0,1)$. For the *first* inequality in (3.5.59), we consider two cases.

Case 1 (Points $z_1, z_2 \in \bar{\mathbf{B}}_R(z_0)$ obeying (3.5.60))**.** If

$$\mathbf{d}(z_1, z_2) \geq \frac{1}{8} \max\left\{\mathbf{d}(z_1, z_0), \mathbf{d}(z_2, z_0)\right\}, \tag{3.5.60}$$

then we have

$$\begin{aligned} |u(z_1) - u(z_2)| &\leq |u(z_1) - u(z_0)| + |u(z_2) - u(z_0)| \\ &\leq C\mathbf{d}(z_1, z_0)^{\alpha_0} + C\mathbf{d}(z_2, z_0)^{\alpha_0} \text{ (by (3.5.58))} \\ &\leq C\mathbf{d}(z_1, z_2)^{\alpha_0} \text{ (by (3.5.60))}, \end{aligned}$$

and so the first inequality in (3.5.59) holds in this case.

Case 2 (Points $z_1, z_2 \in \bar{\mathbf{B}}_R(z_0)$ obeying (3.5.61))**.** If

$$\mathbf{d}(z_1, z_2) \leq \frac{1}{8} \max\left\{\mathbf{d}(z_1, z_0), \mathbf{d}(z_2, z_0)\right\}, \tag{3.5.61}$$

then, by applying (3.5.51) on the ball $\mathbf{B}_{\tilde{R}}(z_2)$ with $\tilde{R} = \mathbf{d}(z_1, z_2)$, we again obtain the first inequality in (3.5.59).

Next, we consider the *second* inequality in (3.5.59). By (3.5.36), we have

$$\mathbf{d}(z_3, z_4) = \sqrt{y/2} \text{ and } \mathbf{d}(z_4, z_0) = \sqrt{x}. \tag{3.5.62}$$

As in the proof of the first inequality in (3.5.59), we consider two possible cases.

Case 1 (Points $z_3, z_4 \in \bar{\mathbf{B}}_R(z_0)$ obeying (3.5.63))**.** If

$$x \geq 32y, \tag{3.5.63}$$

then, by (3.5.62), we have $\mathbf{d}(z_3, z_4) \leq 1/8\mathbf{d}(z_4, z_0)$. We may apply (3.5.51) on the ball $\mathbf{B}_{\tilde{R}}(z_4)$ with $\tilde{R} = \mathbf{d}(z_3, z_4)$, and we obtain the second inequality in (3.5.59).

Case 2 (Points $z_3, z_4 \in \bar{\mathbf{B}}_R(z_0)$ obeying (3.5.64))**.** If

$$x < 32y, \tag{3.5.64}$$

then we have $\mathbf{d}(z_4, z_0) \leq 8\mathbf{d}(z_3, z_4)$. Also, a direct calculation gives us $\mathbf{d}(z_3, z_0) \leq C\mathbf{d}(z_3, z_4)$, for some positive constant C.By (3.5.58), we obtain

$$\begin{aligned} |u(z_3) - u(z_4)| &\leq |u(z_3) - u(z_0)| + |u(z_4) - u(z_0)| \\ &\leq C\mathbf{d}(z_3, z_0)^{\alpha_0} + C\mathbf{d}(z_4, z_0)^{\alpha_0} \\ &\leq C\mathbf{d}(z_3, z_4)^{\alpha_0}, \end{aligned}$$

and we obtain the second inequality in (3.5.59).

The proof of (3.5.59) is complete. We may now conclude, by applying the same argument as in Step 5 of the the proof of Theorem 3.1.8 for the case of points in Γ_0, that for any $z_1, z_2 \in \mathbf{B}_{\bar{R}}(z_0)$, we have

$$|u(z_1) - u(z_2)| \leq C\mathbf{d}(z_1, z_2)^{\alpha_0}, \tag{3.5.65}$$

where C and α_0 are constants satisfying the dependencies stated in Theorem 3.1.8. This completes the proof of Theorem 3.1.8 for the case of points in $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$. □

Remark 3.5.2 (Completion of the proof of Theorem 3.1.8 for the case of points in Γ_0)**.** Notice that the inequality (3.5.51) is slightly weaker than (3.1.22), because it applies to points $z_1, z_2 \in \mathbf{B}_R(z_0)$, where R is required to satisfy assumptions (3.5.1), (3.5.2) and (3.5.34), instead of allowing $R = \bar{R}$. To obtain (3.1.22), all we need to notice is that (3.5.51) and (3.5.65) imply that u is $C^{\alpha_0}_{s,\text{loc}}$-Hölder continuous on $\mathbf{B}_{\bar{R}}(z_0) \cap \left(\bar{\Gamma}_0 \times [0, \tilde{R}]\right)$, where $\tilde{R}$ is small enough so that it satisfies assumptions (3.5.1), (3.5.2) and (3.5.34). For $y \geq \tilde{R}$, A is a uniformly elliptic operator with bounded coefficients, so [41, Theorems 8.22 & 8.29] apply and we see that u is Hölder continuous on $\mathbf{B}_{\bar{R}}(z_0) \cap \{y \geq \tilde{R}\}$. Therefore, the inequality (3.1.22) follows.

Remark 3.5.3. Hölder continuity of solutions does not follow by a Sobolev embedding-type theorem for weighted spaces, analogous to [41, Corollary 7.11], not even for functions $u \in H^2(\mathscr{O}, \mathfrak{w})$. For example, for any $\beta > 2$, let $p \in (0, (\beta-2)/2)$ and

$$u(x,y) = y^{-p}, \quad \forall (x,y) \in \mathscr{O}.$$

Then, $u \in H^2(\mathscr{O}, \mathfrak{w})$, but $u \notin C^{\alpha}_{s,\mathrm{loc}}(\mathscr{O} \cup \Gamma_0)$, for any $\alpha \in [0,1]$.

Remark 3.5.4 (Relaxation of the Assumption 3.1.2 on $\mathscr{O}$ near Γ_0)**.** As in Remark 3.4.2, we notice that in the proof of Theorem 3.1.8 for the case of points in $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$, we can weaken the conditions on $\mathscr{O}$ embodied in Assumption 3.1.2 to an "interior and exterior sphere condition". That is, for points $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$, it is enough to assume that there exist positive constants c_0 and R_0 such that for all $0 < R < R_0$ we have, in addition to (3.4.21), that

$$c_0^{-1} |\mathbb{B}_R(z_0)|_{\beta-1} \leq |\mathbb{B}_R(z_0) \backslash \mathbf{B}_R(z_0)|_{\beta-1} \leq c_0 |\mathbb{B}_R(z_0)|_{\beta-1}. \tag{3.5.66}$$

3.6 Harnack inequality

In this section, we prove Theorem 3.1.10, that is, the Harnack inequality for solutions $u \in H^1_0(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$ to the variational equation (3.1.16). The key differences from the proof of the classical Harnack inequality for weak solutions to non-degenerate elliptic equations [41, Theorem 8.20] are essentially those which we already outlined in §3.5 and the proof follows the same pattern as that of Theorem 3.1.8. Therefore, we only point out the major steps in the proof of Theorem 3.1.10, as the details were explained in the preceding sections. We now proceed to the

Proof of Theorem 3.1.10. For clarity, we split the proof into principal steps.

Step 1 (Energy estimates)**.** Let $\eta \in C^1_0(\bar{\mathbb{H}})$ be a non-negative cutoff function with support in $\bar{\mathbf{B}}_{4R}(z_0)$. Let $\varepsilon > 0$ and

$$w = u + \varepsilon. \tag{3.6.1}$$

We consider $\alpha \in \mathbb{R}$, $\alpha \neq -1$. We set $H(w) = w^{(\alpha+1)/2}$ and

$$v = \eta^2 w^{\alpha}. \tag{3.6.2}$$

Then, $v \in H^1_0(\mathscr{O}\cup\Gamma_0,\mathfrak{w})$ is a valid test function in (3.1.12) by Lemma B.2.4. By applying the same arguments as in the proofs of Theorem 3.1.7 and Theorem 3.1.8, we obtain the following analogous energy estimate to (3.4.3) and (3.5.11), respectively

$$\begin{aligned}
&\left(\int |\eta H(w)|^p y^{\beta-1}dxdy\right)^{1/p} \\
&\quad\leq (C|1+\alpha|)^{1/p}\,\|\sqrt{y}\nabla\eta\|^{2/p}_{L^\infty(\mathbb{H})}\left(\int_{\text{supp }\eta}|H(w)|^2y^{\beta-1}dxdy\right)^{1/p},
\end{aligned} \tag{3.6.3}$$

where C is independent of ε, and depends only on the coefficients of the Heston operator and δ_0.

Step 2 (Moser iteration)**.** By applying Moser iteration as described in the proofs of Theorem 3.1.7, for $\alpha > 0$, and of Theorem 3.1.8, for $\alpha < 0$, we obtain

$$\begin{aligned}
\sup_{\mathbf{B}_R(z_0)} w &\leq C\left(\frac{1}{|\mathbf{B}_{2R}(z_0)|_{\beta-1}}\int_{\mathbf{B}_{2R}(z_0)} w^2y^{\beta-1}dxdy\right)^{1/2}, \\
\inf_{\mathbf{B}_R(z_0)} w &\geq C^{-1}\left(\frac{1}{|\mathbf{B}_{2R}(z_0)|_{\beta-1}}\int_{\mathbf{B}_{2R}(z_0)} w^{-2}y^{\beta-1}dxdy\right)^{-1/2},
\end{aligned} \tag{3.6.4}$$

where C satisfies the same dependencies as the constant in (3.6.3).

Step 3 (Application of Theorem 3.3.4)**.** In this step, we verify that w satisfies the requirements of the abstract John-Nirenberg inequality (Theorem 3.3.1) with $\vartheta_0 = \vartheta_1 = 2$ with $S_r = \mathbf{B}_{(2+r)R}$, $0 \leq r \leq 1$. By Proposition 3.3.2, we obtain that w satisfies condition (3.3.2) of Theorem 3.3.1. Therefore, it remains to verify condition (3.3.3), which follows in precisely the same way as in the proof of Theorem 3.1.8.

Step 4 (Proof of (3.1.23))**.** Because w satisfies the conditions of Theorem 3.3.1 by the preceding step, there is a positive constant C, independent of ε, such that

$$\begin{aligned}
&\left(\frac{1}{|\mathbf{B}_{2R}(z_0)|_{\beta-1}}\int_{\mathbf{B}_{2R}(z_0)} w^2y^{\beta-1}dxdy\right)^{1/2} \\
&\quad\leq C\left(\frac{1}{|\mathbf{B}_{2R}(z_0)|_{\beta-1}}\int_{\mathbf{B}_{2R}(z_0)} w^{-2}y^{\beta-1}dxdy\right)^{-1/2}.
\end{aligned} \tag{3.6.5}$$

Thus, combining inequalities (3.6.4) and (3.6.5) and recalling that $w = u+\varepsilon$, we obtain

$$\sup_{\mathbf{B}_R(z_0)}(u+\varepsilon) \leq C\inf_{\mathbf{B}_R(z_0)}(u+\varepsilon),$$

for all $\varepsilon > 0$. Taking the limit as $\varepsilon \downarrow 0$, we obtain the desired Harnack inequality (3.1.23).

This completes the proof. $\square$

Chapter 4

Stochastic representation of solutions

4.1 Introduction

Since its discovery by Mark Kac [46], inspired in turn by the doctoral dissertation of Richard Feynman [33], the *Feynman-Kac* (or *stochastic representation*) *formula* has provided a link between probability theory and partial differential equations which has steadily deepened and developed during the intervening years. Moreover, judging by continuing interest in its applications to mathematical finance [48] and mathematical physics [56, 67], including non-linear parabolic equations [17], this trend shows no sign of abating. However, while stochastic representation formulae for solutions to linear, second-order elliptic and parabolic boundary and obstacle problems are well established when the generator, $-A$, of the Markov stochastic process is *strictly elliptic* [8, 40, 47, 61] in the sense of [41, p. 31], the literature is far less complete when A is *degenerate elliptic*, that is, only has a *non-negative definite characteristic form* in the sense of [62], and its coefficients are unbounded.

In this chapter, we prove stochastic representation formulae for solutions to an *elliptic boundary value problem*,

$$Au = f \quad \text{on } \mathscr{O}, \tag{4.1.1}$$

and an *elliptic obstacle problem*,

$$\min\{Au - f, u - \psi\} = 0 \quad \text{on } \mathscr{O}, \tag{4.1.2}$$

respectively, subject to a *partial* Dirichlet boundary condition,

$$u = g \quad \text{on } \Gamma_1. \tag{4.1.3}$$

Here, $f : \mathscr{O} \to \mathbb{R}$ is a source function, the function $g : \Gamma_1 \to \mathbb{R}$ prescribes a Dirichlet boundary condition along Γ_1 and $\psi : \mathscr{O} \cup \Gamma_1 \to \mathbb{R}$ is an obstacle function which is compatible with g in the sense that

$$\psi \leq g \quad \text{on } \Gamma_1, \tag{4.1.4}$$

while A is the Heston operator in (3.1.2), and its coefficients satisfy Assumption 3.1.1. We require Γ_0 to be non-empty throughout this chapter as, otherwise, if $\mathscr{O}$ is bounded, then standard results apply [8, 40, 47, 61]. However, an additional boundary condition is *not* necessarily prescribed along Γ_0. Rather, we shall see that our stochastic representation formulae will provide the unique solutions to (4.1.1) or (4.1.2), together with (4.1.3), when we seek solutions which are suitably smooth up to the boundary portion Γ_0, a property which is guaranteed when the solutions lie in certain weighted Hölder spaces (by analogy with [20]), *or* replace the boundary condition (4.1.3) with the full Dirichlet condition,

$$u = g \quad \text{on } \partial\mathscr{O}, \tag{4.1.5}$$

in which case the solutions are not guaranteed to be any more than continuous up to Γ_0 and $\psi : \bar{\mathscr{O}} \to \mathbb{R}$ is now required to be compatible with g in the sense that,

$$\psi \leq g \quad \text{on } \partial\mathscr{O}. \tag{4.1.6}$$

We also prove stochastic representation formulae for solutions to a *parabolic terminal/boundary value problem*,

$$-u_t + Au = f \quad \text{on } Q, \tag{4.1.7}$$

and a *parabolic obstacle problem*,

$$\min\{-u_t + Au - f, u - \psi\} = 0 \quad \text{on } Q, \tag{4.1.8}$$

respectively, subject to the *partial* terminal/boundary condition,

$$u = g \quad \text{on } \eth^1 Q. \tag{4.1.9}$$

Here, we define $Q := (0,T) \times \mathscr{O}$, where $0 < T < \infty$, and define

$$\eth^1 Q := (0,T) \times \Gamma_1 \cup \{T\} \times (\mathscr{O} \cup \Gamma_1), \tag{4.1.10}$$

to be a subset of the parabolic boundary of Q, and now assume given a source function $f : Q \to \mathbb{R}$, a Dirichlet boundary data function $g : \eth^1 Q \to \mathbb{R}$, and an obstacle function $\psi : Q \cup \eth^1 Q \to \mathbb{R}$ which is compatible with g in the sense that,

$$\psi \leq g \quad \text{on } \eth^1 Q. \tag{4.1.11}$$

Just as in the elliptic case, we shall either consider solutions which are suitably smooth up to $(0,T)\times\Gamma_0$, but impose no explicit Dirichlet boundary condition along $(0,T)\times\Gamma_0$, *or* replace the boundary condition in (4.1.9) with the full Dirichlet condition

$$u = g \quad \text{on } \eth Q, \tag{4.1.12}$$

where

$$\eth Q := (0,T) \times \partial\mathscr{O} \cup \{T\} \times \bar{\mathscr{O}}, \tag{4.1.13}$$

is the full parabolic boundary of Q, in which case the solutions are not guaranteed to be any more than continuous up to $(0,T)\times\Gamma_0$ and $\psi : Q \cup \eth Q \to \mathbb{R}$ is now compatible with g in the sense that

$$\psi \leq g \quad \text{on } \eth Q. \tag{4.1.14}$$

Before giving a detailed account of our main results, we summarize a few applications.

4.1.1 Applications

In mathematical finance, a solution, u, to the elliptic obstacle problem (4.1.2), (4.1.3), when $f = 0$, can be interpreted as the value function for a *perpetual American-style option* with *payoff* function given by the obstacle function, ψ, while a solution, u, to the corresponding *parabolic* obstacle problem (4.1.8), (4.1.9), when $f = 0$, can be interpreted as the value function for a *finite-maturity* American-style option with payoff function given by a terminal condition function, $h = g(T,\cdot) : \mathscr{O} \to \mathbb{R}$, which typically coincides on $\{T\}\times\mathscr{O}$ with the obstacle function, ψ. For example, in the case of an American-style put option, one chooses $\psi(x,y) = (E - e^x)^+$, $\forall(x,y) \in \mathscr{O}$, where $E > 0$ is a positive constant. While solutions to (4.1.1), (4.1.3) do not have an immediate interpretation in mathematical finance, a solution, u, to the corresponding *parabolic* terminal/boundary value problem (4.1.7), (4.1.9), when $f = 0$, can be interpreted as the value function for a *European-style option* with payoff function given by the terminal condition function, h. For example, in the case of a European-style put option, one chooses $h(x,y) = (E - e^x)^+$, $\forall (x,y) \in \mathscr{O}$.

Stochastic representation formulae underly Monte Carlo methods of numerical computation of value functions for option pricing in mathematical finance [42]. As is well-known to practitioners, the question of Monte Carlo simulation of solutions to the Heston stochastic differential equation is especially delicate [3, 55]. We hope that our work sheds further light on these issues.

4.1.2 Summary of main results

Recall the definition of the Heston operator, $-A$, in (3.1.2), and the Assumption 3.1.1 on its coefficients. In this chapter, we allow $q, r \in \mathbb{R}$, and we impose additional conditions, such as $q \geq 0$, $r \geq 0$, or $r > 0$, depending on the problem under consideration. [1]

Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{Q})$ be a filtered probability space satisfying the usual conditions, where $\mathbb{F} = \{\mathscr{F}(s)\}_{s\geq 0}$ is the $\mathbb{Q}$-completion of the natural filtration of $(W(s))_{s\geq 0}$, and $(W(s))_{s\geq 0}$ is a standard Brownian motion with values in $\mathbb{R}^2$. For $0 \leq t < T < \infty$, let $\mathscr{T}_{t,T}$ denote the set of $\mathbb{F}$-stopping times with values in $[t,T]$. Let $(X^{t,x,y}(s), Y^{t,y}(s))_{s\geq t}$ denote a continuous version of the strong solution to the Heston stochastic differential equation

$$
\begin{aligned}
dX(s) &= \left(r - q - \frac{Y(s)}{2}\right) ds + \sqrt{Y(s)}\, dW_1(s), \quad s > t,\\
dY(s) &= \kappa\left(\vartheta - Y(s)\right) ds + \sigma\sqrt{Y(s)}\left(\rho\, dW_1(s) + \sqrt{1-\rho^2}\, dW_2(s)\right), \quad s > t,\\
&(X(t), Y(t)) = (x,y),
\end{aligned}
\tag{4.1.15}
$$

which exists by Corollary 4.2.8, where the coefficients are as in Assumption 3.1.1. For brevity, we sometimes denote $z = (x,y)$ and $(Z^{t,z}(s))_{s\geq t} = (X^{t,x,y}(s), Y^{t,y}(s))_{s\geq t}$. We omit the superscripts (t,z) and (t,x,y) when the initial condition is clear from the context, or we omit the superscript t when $t = 0$.

[1] We only require that $q \geq 0$ when deriving the supermartingale property in Lemma 4.2.11 (1), a property used only in the elliptic case. We require $r > 0$ to ensure that the stochastic representations are well-defined, only in the elliptic case.

Existence and uniqueness of solutions to elliptic boundary value problems

We shall often appeal to the following

Hypothesis 4.1.1 (Growth condition)**.** If v is a function then, for all (x,y) in its domain of definition,

$$|v(x,y)| \leq C(1+e^{M_1 y}+e^{M_2 x}), \tag{4.1.16}$$

where $C>0$, $0 \leq M_1 < \min\{r/(\kappa\vartheta), \mu\}$, and $M_2 \in [0,1)$.

Let $U \subsetneqq \mathbb{H}$ be an open set. We denote

$$\tau_U^{t,z} := \inf\left\{s \geq t : Z^{t,z}(s) \notin U\right\}, \tag{4.1.17}$$

and we let

$$\nu_U^{t,z} := \inf\left\{s \geq t : Z^{t,z}(s) \notin U \cup \left(\bar{U} \cap \partial\mathbb{H}\right)\right\}. \tag{4.1.18}$$

Notice that if $\bar{U} \cap \partial\mathbb{H} = \varnothing$, then $\tau_U^{t,z} = \nu_U^{t,z}$. We also have that $\tau_U^{t,z} = \nu_U^{t,z}$ when $\beta \geq 1$, because in this case the process $Z^{t,z}$ does not reach the boundary $\partial\mathbb{H}$, by Lemma 4.2.10 (1). By [61, p. 117], both $\tau_U^{t,z}$ and $\nu_U^{t,z}$ are stopping times with respect to the filtration $\mathbb{F}$, since $\mathbb{F}$ is assumed to satisfy the usual conditions. When the initial condition, (t,z), is clear from the context, we omit the superscripts in the preceding definitions (4.1.17) and (4.1.18) of the stopping times. Also, when $t=0$, we omit the superscript t in the preceding definitions.

Theorem 4.1.2 (Uniqueness of solutions to the elliptic boundary value problem)**.** *Let $r>0$, $q \geq 0$, and f be a Borel measurable function*[2] *on $\mathscr{O}$ which obeys the growth condition* (4.1.16) *on $\mathscr{O}$. Then*

1. *If $\beta \geq 1$, assume $g \in C_{\mathrm{loc}}(\Gamma_1)$ obeys* (4.1.16)*. Let*

$$u \in C_{\mathrm{loc}}(\mathscr{O} \cup \Gamma_1) \cap C^2(\mathscr{O})$$

be a solution to the elliptic boundary value problem (4.1.1), (4.1.3) *and which obeys* (4.1.16) *on $\mathscr{O}$. Then, $u = u^*$ on $\mathscr{O} \cup \Gamma_1$, where*

$$u^*(z) := \mathbb{E}_{\mathbb{Q}}^z\left[e^{-r\tau_{\mathscr{O}}} g(Z(\tau_{\mathscr{O}})) 1_{\{\tau_{\mathscr{O}}<\infty\}}\right] + \mathbb{E}_{\mathbb{Q}}^z\left[\int_0^{\tau_{\mathscr{O}}} e^{-rs} f(Z(s))\, ds\right], \tag{4.1.19}$$

[2] *We require f to be Borel measurable in order to ensure that expectations such as that in* (4.1.19) *are well-defined.*

where $\tau_{\mathscr{O}}$ *is defined by* (4.1.17)*, for all* $z \in \mathscr{O} \cup \Gamma_1$*.*

2. *If* $0 < \beta < 1$*, assume* $g \in C_{\mathrm{loc}}(\partial\mathscr{O})$ *obeys* (4.1.16) *on* $\partial\mathscr{O}$*, and let* $u \in C_{\mathrm{loc}}(\bar{\mathscr{O}}) \cap C^2(\mathscr{O})$ *be a solution to the elliptic boundary value problem* (4.1.1)*,* (4.1.5) *and which obeys* (4.1.16) *on* $\mathscr{O}$*. Then,* $u = u^*$ *on* $\bar{\mathscr{O}}$*, where* u^* *is given by* (4.1.19)*.*

Remark 4.1.3 (Existence of solutions to the elliptic boundary value problem with traditional Hölder regularity)**.** Existence of solutions

$$u \in C_{\mathrm{loc}}(\bar{\mathscr{O}}) \cap C^{2+\alpha}(\mathscr{O})$$

to problem (4.1.1) with boundary condition $g \in C_{\mathrm{loc}}(\partial\mathscr{O})$ in (4.1.5) and source function $f \in C^\alpha(\mathscr{O})$, when $0 < \beta < 1$, and of solutions

$$u \in C_{\mathrm{loc}}(\mathscr{O} \cup \Gamma_1) \cap C^{2+\alpha}(\mathscr{O})$$

with boundary condition $g \in C_{\mathrm{loc}}(\bar{\Gamma}_1)$ in (4.1.3) and source function $f \in C^\alpha(\mathscr{O})$, when $\beta \geq 1$, is proved in Theorem 4.3.1. See also the comments preceding problem (4.3.2).

Remark 4.1.4 (Existence of solutions with Daskalopoulos-Hamilton-Köch Hölder regularity)**.** Ideally, the solutions to the elliptic boundary value problem (4.1.1), (4.1.3) described in Remark 4.1.3 would actually lie in $C_{\mathrm{loc}}(\bar{\mathscr{O}}) \cap C^{2+\alpha}_s(\mathscr{O})$ for all $\beta > 0$, where $C^{2+\alpha}_s(\mathscr{O})$ is an elliptic analogue of the parabolic Daskalopoulos-Hamilton-Köch Hölder spaces described in [20, 50]. A function $u \in C^{2+\alpha}_s(\mathscr{O})$ has the property that u, Du, yD^2u are C^α_s continuous up to Γ_0 and $yD^2u = 0$ on Γ_0, where $C^\alpha_s(\mathscr{O})$ is defined by analogy with the traditional definition of $C^\alpha(\mathscr{O})$, except that Euclidean distance between points in $\mathscr{O}$ is replaced by the cycloidal distance function.

We let $C^{1,1}_{s,\mathrm{loc}}(\mathscr{O} \cup \Gamma_0)$ denote the subspace of $C^2_{\mathrm{loc}}(\mathscr{O} \cup \Gamma_0)$ consisting of functions, u, such that, for any precompact open subset $U \Subset \mathscr{O} \cup \Gamma_0$,

$$\sup_{(x,y)\in U} |u(x,y)| + |Du(x,y)| + |yD^2u(x,y)| < \infty, \tag{4.1.20}$$

where Du denotes the gradient and D^2u the Hessian matrix of u.

Theorem 4.1.5 (Uniqueness of solutions to the elliptic boundary value problem (4.1.1), (4.1.3), when $0 < \beta < 1$)**.** *Let $r > 0$, $q \geq 0$, $0 < \beta < 1$, and let f be as in Theorem 4.1.2. Let $g \in C_{\mathrm{loc}}(\Gamma_1)$ obey* (4.1.16) *on Γ_1 and suppose that*

$$u \in C_{\mathrm{loc}}(\mathscr{O} \cup \Gamma_1) \cap C^2(\mathscr{O}) \cap C^{1,1}_{s,\mathrm{loc}}(\mathscr{O} \cup \Gamma_0)$$

is a solution to the elliptic boundary value problem (4.1.1), (4.1.3) *which obeys* (4.1.16) *on $\mathscr{O}$. Then, $u = u^*$ on $\mathscr{O} \cup \Gamma_1$, where u^* is given by*

$$u^*(z) := \mathbb{E}^z_{\mathbb{Q}} \left[e^{-r\nu_{\mathscr{O}}} g(Z(\nu_{\mathscr{O}})) 1_{\{\nu_{\mathscr{O}} < \infty\}} \right] + \mathbb{E}^z_{\mathbb{Q}} \left[\int_0^{\nu_{\mathscr{O}}} e^{-rs} f(Z(s))\, ds \right], \tag{4.1.21}$$

and $\nu_{\mathscr{O}}$ is defined by (4.1.18)*, for all $z \in \mathscr{O} \cup \Gamma_1$.*

Remark 4.1.6 (Existence and uniqueness of strong solutions in weighted Sobolev spaces to the elliptic boundary value problem)**.** Existence and uniqueness of strong solutions in weighted Sobolev spaces to problem (4.1.1) with boundary condition (4.1.3) along Γ_1, for all $\beta > 0$, is proved in [18, Theorem 1.18], and Hölder continuity of such solutions up to Γ_0 is proved in [31, Theorem 1.10].

Remark 4.1.7 (Comparison of uniqueness results)**.** To obtain uniqueness of solutions to the elliptic boundary value problem (4.1.1) with boundary condition (4.1.3) only specified along Γ_1, we need to assume the stronger regularity hypothesis

$$u \in C_{\mathrm{loc}}(\mathscr{O} \cup \Gamma_1) \cap C^2(\mathscr{O}) \cap C^{1,1}_{s,\mathrm{loc}}(\mathscr{O} \cup \Gamma_0)$$

when $0 < \beta < 1$, while the regularity assumption

$$u \in C_{\mathrm{loc}}(\mathscr{O} \cup \Gamma_1) \cap C^2(\mathscr{O})$$

suffices when $\beta \geq 1$. The analogous comments apply to the elliptic obstacle problems described in Theorems 4.1.8 and 4.1.9, the parabolic terminal/boundary value problems described in Theorems 4.1.12 and 4.1.15, and the parabolic obstacle value problems described in Theorems 4.1.19 and 4.1.20.

Uniqueness of solutions to elliptic obstacle problems

For $\theta_1, \theta_2 \in \mathscr{T}$, we set

$$\begin{aligned} J_e^{\theta_1,\theta_2}(z) := \mathbb{E}_{\mathbb{Q}}^z \left[\int_0^{\theta_1\wedge\theta_2} e^{-rs} f(Z(s))\,ds\right] \\ + \mathbb{E}_{\mathbb{Q}}^z \left[e^{-r\theta_1} g(Z(\theta_1)) 1_{\{\theta_1\leq\theta_2\}}\right] + \mathbb{E}_{\mathbb{Q}}^z \left[e^{-r\theta_2} \psi(Z(\theta_2)) 1_{\{\theta_2<\theta_1\}}\right]. \end{aligned} \tag{4.1.22}$$

We then have the

Theorem 4.1.8 (Uniqueness of solutions to the elliptic obstacle problem)**.** *Let $r > 0$, $q \geq 0$, and f be as in Theorem 4.1.2, and ψ be a Borel measurable function satisfying* (4.1.16) *on $\mathscr{O}$.*

1. *If $\beta \geq 1$, let $\psi \in C_{\text{loc}}(\mathscr{O} \cup \Gamma_1)$ and $g \in C_{\text{loc}}(\Gamma_1)$ obey* (4.1.16) *and* (4.1.4) *on Γ_1. Let*
$$u \in C_{\text{loc}}(\mathscr{O} \cup \Gamma_1) \cap C^2(\mathscr{O})$$
be a solution to the elliptic obstacle problem (4.1.2), (4.1.3) *such that u and Au obey* (4.1.16) *on $\mathscr{O}$. Then, $u = u^*$ on $\mathscr{O} \cup \Gamma_1$, where u^* is given by*
$$u^*(z) := \sup_{\theta\in\mathscr{T}} J_e^{\tau_{\mathscr{O}},\theta}(z), \tag{4.1.23}$$
and $\tau_{\mathscr{O}}$ is defined by (4.1.17)*, for all $z \in \mathscr{O} \cup \Gamma_1$.*

2. *If $0 < \beta < 1$, let $\psi \in C_{\text{loc}}(\bar{\mathscr{O}})$ and $g \in C_{\text{loc}}(\partial\mathscr{O})$ obey* (4.1.16) *and* (4.1.6) *on $\partial\mathscr{O}$. Let*
$$u \in C_{\text{loc}}(\bar{\mathscr{O}}) \cap C^2(\mathscr{O})$$
be a solution to the elliptic obstacle problem (4.1.2), (4.1.5)*, such that u and Au obey* (4.1.16) *on $\mathscr{O}$. Then, $u = u^*$ on $\bar{\mathscr{O}}$, where u^* is given by* (4.1.23).

Theorem 4.1.9 (Uniqueness of solutions to the elliptic obstacle problem (4.1.2), (4.1.3), when $0 < \beta < 1$)**.** *Let $r > 0$, $q \geq 0$, $0 < \beta < 1$, and f be as in Theorem 4.1.8. Let $\psi \in C_{\text{loc}}(\mathscr{O} \cup \Gamma_1)$ obey* (4.1.16) *on $\mathscr{O}$ and let $g \in C_{\text{loc}}(\Gamma_1)$ obey* (4.1.16) *and* (4.1.4) *on Γ_1. If*
$$u \in C_{\text{loc}}(\mathscr{O} \cup \Gamma_1) \cap C^2(\mathscr{O}) \cap C^{1,1}_{s,\text{loc}}(\mathscr{O} \cup \Gamma_0)$$

is a solution to the elliptic obstacle problem (4.1.2), (4.1.3) *such that* u *and* Au *obey* (4.1.16), *then* $u = u^*$ *on* $\mathscr{O} \cup \Gamma_1$, *where* u^* *is given by*

$$u^*(z) := \sup_{\theta \in \mathscr{T}} J_e^{\nu_{\mathscr{O}},\theta}(z), \tag{4.1.24}$$

and $\nu_{\mathscr{O}}$ *is defined by* (4.1.18), *for all* $z \in \mathscr{O} \cup \Gamma_1$.

Remark 4.1.10 (Existence and uniqueness of strong solutions in weighted Sobolev spaces to the elliptic obstacle problem)**.** Existence and uniqueness of strong solutions in weighted Sobolev spaces to problem (4.1.2) with Dirichlet boundary condition (4.1.3) along Γ_1, for all $\beta > 0$, is proved in [18, Theorem 1.6], and Hölder continuity of such solutions up to boundary portion Γ_0 is proved in [31, Theorem 1.13].

Existence and uniqueness of solutions to parabolic terminal/boundary value problems

We shall need to appeal to the following analogue of Hypothesis 4.1.1:

Hypothesis 4.1.11 (Growth condition)**.** If v is a function then, for all (t, x, y) in its domain of definition,

$$|v(t, x, y)| \leq C(1 + e^{M_1 y} + e^{M_2 x}), \tag{4.1.25}$$

where $C > 0$, $0 \leq M_1 < \mu$, and $M_2 \in [0, 1]$.

We let Du denote the gradient and let D^2u denote the Hessian matrix of a function u on Q with respect to spatial variables. We let $C^1(Q)$ denote the vector space of functions, u, such that u, u_t, and Du are continuous on Q, while $C^1(\bar{Q})$ denotes the Banach space of functions, u, such that u, u_t, and Du are uniformly continuous and bounded on Q; finally, $C^2(Q)$ denotes the vector space of functions, u, such that u_t, Du, and D^2u are continuous Q, while $C^2(\bar{Q})$ denotes the Banach space of functions, u, such that u, u_t, Du, and D^2u are uniformly continuous and bounded on Q.

Theorem 4.1.12 (Uniqueness of solutions to the parabolic boundary value problem)**.** *Let* f *be a Borel measurable function on* Q *which obeys* (4.1.25). *Then*

1. *If $\beta \geq 1$, assume $g \in C_{\mathrm{loc}}(\eth^1 Q)$ obeys* (4.1.25) *on $\eth^1 Q$. Let*

$$u \in C_{\mathrm{loc}}(Q \cup \eth^1 Q) \cap C^2(Q)$$

be a solution to the parabolic terminal/boundary value problem (4.1.7), (4.1.9) *which obeys* (4.1.25) *on Q. Then, $u = u^*$ on $Q \cup \eth^1 Q$, where u^* is given by*

$$\begin{aligned} u^*(t,z) := \mathbb{E}^{t,z}_{\mathbb{Q}}&\left[\int_t^{\tau_{\mathscr{O}} \wedge T} e^{-r(s-t)} f(s, Z(s))\, ds\right] \\ &+ \mathbb{E}^{t,z}_{\mathbb{Q}}\left[e^{-r(\tau_{\mathscr{O}} \wedge T - t)} g(\tau_{\mathscr{O}} \wedge T, Z(\tau_{\mathscr{O}} \wedge T))\right], \end{aligned} \tag{4.1.26}$$

and $\tau_{\mathscr{O}}$ is defined by (4.1.17), *for all $(t,z) \in Q \cup \eth^1 Q$.*

2. *If $0 < \beta < 1$, assume $g \in C_{\mathrm{loc}}(\eth Q)$ obeys* (4.1.25) *on $\eth Q$, and let*

$$u \in C_{\mathrm{loc}}(Q \cup \eth Q) \cap C^2(Q)$$

be a solution to the parabolic terminal/boundary value problem (4.1.7), (4.1.12) *which obeys* (4.1.25) *on Q. Then, $u = u^*$ on $Q \cup \eth Q$, where u^* is given by* (4.1.26).

Remark 4.1.13 (Existence of solutions to the parabolic boundary value problem)**.** Existence of solutions

$$u \in C_{\mathrm{loc}}(Q \cup \eth Q) \cap C^{2+\alpha}(Q)$$

to problem (4.1.7), with Dirichlet boundary data $g \in C_{\mathrm{loc}}(\overline{\eth Q})$ in (4.1.12), and source function $f \in C^{\alpha}_{\mathrm{loc}}(\bar{Q})$, when $0 < \beta < 1$, and of solutions

$$u \in C_{\mathrm{loc}}(Q \cup \eth^1 Q) \cap C^{2+\alpha}(Q)$$

to problem (4.1.7) with Dirichlet boundary data $g \in C_{\mathrm{loc}}(\overline{\eth^1 Q})$ in (4.1.9) and source function $f \in C^{\alpha}_{\mathrm{loc}}(\bar{Q})$, when $\beta \geq 1$, is proved in Theorem 4.5.4. See also to the comments preceding problem (4.5.2).

Remark 4.1.14 (Existence of solutions with Daskalopoulos-Hamilton-Köch Hölder regularity)**.** As in the elliptic case, the solutions to the parabolic terminal/boundary value problem (4.1.7), (4.1.9) described in Remark 4.1.13 would actually lie in $C_{\mathrm{loc}}(\bar{Q}) \cap C^{2+\alpha}_s(Q)$ for all $\beta > 0$, where $C^{2+\alpha}_s(Q)$ is the parabolic Daskalopoulos-Hamilton-Köch Hölder space described in [20, 50]. A function $u \in C^{2+\alpha}_s(Q)$ has the property that

u, Du, yD^2u are C^α_s continuous up to Γ_0 and $yD^2u = 0$ on $(0,T)\times\Gamma_0$, where $C^\alpha_s(Q)$ is defined by analogy with the traditional definition of $C^\alpha(Q)$, except that Euclidean distance between points in Q is replaced by the cycloidal distance function.

We let $C^{1,1}_{s,\mathrm{loc}}((0,T)\times(\mathscr{O}\cup\Gamma_0))$ denote the subspace of $C^2_{\mathrm{loc}}((0,T)\times(\mathscr{O}\cup\Gamma_0))$ consisting of functions, u, such that, for any precompact open subset $V \Subset [0,T]\times(\mathscr{O}\cup\Gamma_0)$,

$$\sup_{(t,z)\in V} |u(t,z)| + |Du(t,z)| + |yD^2u(t,z)| < \infty. \tag{4.1.27}$$

We have the following alternative uniqueness result.

Theorem 4.1.15 (Uniqueness of solutions to the parabolic boundary value problem (4.1.7), (4.1.9), when $0<\beta<1$)**.** *Let $0<\beta<1$ and f be as in Theorem 4.1.12. Let $g \in C_{\mathrm{loc}}(\eth^1 Q)$ obey* (4.1.25) *on $\eth^1 Q$, and*

$$u \in C_{\mathrm{loc}}(Q\cup\eth^1 Q)\cap C^2(Q)\cap C^{1,1}_{s,\mathrm{loc}}((0,T)\times(\mathscr{O}\cup\Gamma_0))$$

be a solution to the parabolic boundary value problem (4.1.7)*,* (4.1.9) *which obeys* (4.1.25) *on Q. Then, $u = u^*$ on $Q\cup\eth^1 Q$, where u^* is given by*

$$\begin{aligned} u^*(t,z) &:= \mathbb{E}^{t,z}_{\mathbb{Q}}\left[\int_t^{\nu_{\mathscr{O}}\wedge T} e^{-r(s-t)} f(s,Z(s))\,ds\right] \\ &\quad + \mathbb{E}^{t,z}_{\mathbb{Q}}\left[e^{-r(\nu_{\mathscr{O}}\wedge T - t)} g(\nu_{\mathscr{O}}\wedge T, Z(\nu_{\mathscr{O}}\wedge T))\right], \end{aligned} \tag{4.1.28}$$

and $\nu_{\mathscr{O}}$ is defined by (4.1.18)*, for all $(t,z)\in Q\cup\eth^1 Q$.*

Remark 4.1.16 (Existence and uniqueness of strong solutions in weighted Sobolev spaces to the parabolic terminal/boundary value problem)**.** Existence and uniqueness of strong solutions in weighted Sobolev spaces to problem (4.1.7) with Dirichlet boundary condition (4.1.9) along $\eth^1 Q$, for all $\beta > 0$, is proved in [19].

Remark 4.1.17 (Growth of solutions to parabolic boundary value problems)**.** Karatzas and Shreve allow faster growth of solutions when the growth on the coefficients of the differential operator is constrained [47, Theorem 4.4.2 & Problem 5.7.7], and polynomial growth of solutions is allowed for linear growth coefficients and source function f with at most polynomial growth [47, Theorem 5.7.6].

Remark 4.1.18 (Barrier option pricing and discontinuous terminal/boundary conditions)**.** In applications to finance, $\mathscr{O}$ will often be a rectangle, $(x_0, x_1) \times (0, \infty)$, where $-\infty \leq x_0 < x_1 \leq \infty$; the growth exponents will be $M_1 = 0$ and $M_2 = 1$ — indeed, the source function f will always be zero and the spatial boundary condition function $g : (0,T) \times \Gamma_1 \to \mathbb{R}$ will often be zero. However, the spatial boundary condition, $g : (0,T) \times \Gamma_1 \to \mathbb{R}$, and terminal condition, $g : \{T\} \times \bar{\mathscr{O}} \to \mathbb{R}$, may be *discontinuous* where they meet along $\{T\} \times \partial\mathscr{O}$, as in the case of the *down-and-out put*, with

$$g(t,x,y) = \begin{cases} 0, & 0 < t < T, x = x_0, y > 0, \\ (K - e^x)^+ & t = T, x_0 < x < \infty, y > 0, \end{cases}$$

where g is discontinuous at (T, x_0, y) if $K - e^{x_0} > 0$, that is, $x_0 < \log K$. We shall consider the question of establishing stochastic representations for solutions to parabolic terminal/value problems (European-style option prices) or parabolic obstacle problems (American-style option prices) with discontinuous data elsewhere.

Uniqueness of solutions to parabolic obstacle problems

For $\theta_1, \theta_2 \in \mathscr{T}_{t,T}$, $0 \leq t \leq T$, we set

$$\begin{aligned} J_p^{\theta_1,\theta_2}(t,z) &:= \mathbb{E}_{\mathbb{Q}}^{t,z}\left[\int_t^{\theta_1\wedge\theta_2} e^{-r(s-t)} f(s,Z(s))\,ds\right] + \mathbb{E}_{\mathbb{Q}}^{t,z}\left[e^{-r(\theta_2-t)}\psi(\theta_2, Z(\theta_2))\mathbf{1}_{\{\theta_2<\theta_1\}}\right] \\ &\quad + \mathbb{E}_{\mathbb{Q}}^{t,z}\left[e^{-r(\theta_1-t)} g(\theta_1, Z(\theta_1))\mathbf{1}_{\{\theta_1\leq\theta_2\}}\right]. \end{aligned} \tag{4.1.29}$$

We have the following *uniqueness* result of solutions to the parabolic obstacle problem with different possible boundary conditions, depending on the value of the parameter $\beta > 0$.

Theorem 4.1.19 (Uniqueness of solutions to the parabolic obstacle problem)**.** *Let f be as in Theorem 4.1.12, and ψ be a Borel measurable function satisfying* (4.1.25)*.*

1. *If $\beta \geq 1$, assume $\psi \in C_{\mathrm{loc}}(Q \cup \eth^1 Q)$ and $g \in C_{\mathrm{loc}}(\eth^1 Q)$ obeys* (4.1.25) *on $\eth^1 Q$ and* (4.1.11)*. Let*

$$u \in C_{\mathrm{loc}}(Q \cup \eth^1 Q) \cap C^2(Q)$$

be a solution to the parabolic obstacle problem (4.1.8), (4.1.9) *such that* u *and* Au *obey* (4.1.25) *on* Q. *Then,* $u = u^*$ *on* $Q \cup \eth^1 Q$, *where* u^* *is given by*

$$u^*(t,z) := \sup_{\theta \in \mathscr{T}_{t,T}} J_p^{\tau_{\mathscr{O}} \wedge T, \theta}(t,z), \tag{4.1.30}$$

and $\tau_{\mathscr{O}}$ *is defined by* (4.1.17), *for all* $(t,z) \in Q \cup \eth^1 Q$.

2. *If* $0 < \beta < 1$, *assume* $\psi \in C_{\text{loc}}(\bar{Q})$ *and* $g \in C_{\text{loc}}(\eth Q)$ *obeys* (4.1.25) *on* $\eth Q$ *and* (4.1.14). *Let*

$$u \in C_{\text{loc}}(Q \cup \eth Q) \cap C^2(Q)$$

be a solution to the parabolic obstacle problem (4.1.8), (4.1.12) *such that* u *and* Au *obey* (4.1.25) *on* Q. *Then,* $u = u^*$ *on* $Q \cup \eth Q$, *where* u^* *is given by* (4.1.30).

Theorem 4.1.20 (Uniqueness of solutions to the parabolic obstacle problem (4.1.8), (4.1.9), when $0 < \beta < 1$)**.** *Let* $0 < \beta < 1$ *and* f *be as in Theorem 4.1.12. Assume* $\psi \in C_{\text{loc}}(Q \cup \eth^1 Q)$, *and* $g \in C_{\text{loc}}(\eth^1 Q)$ *obey* (4.1.25) *on* $\eth^1 Q$ *and* (4.1.11). *Let*

$$u \in C_{\text{loc}}(Q \cup \eth^1 Q) \cap C^2(Q) \cap C^{1,1}_{s,\text{loc}}(Q \cup (0,T) \times (\mathscr{O} \cup \Gamma_0))$$

be a solution to the parabolic obstacle problem (4.1.8), (4.1.9) *such that* u *and* Au *obey* (4.1.25). *Then,* $u = u^*$ *on* $Q \cup \eth^1 Q$, *where* u^* *is given by*

$$u^*(t,z) := \sup_{\theta \in \mathscr{T}_{t,T}} J_p^{\nu_{\mathscr{O}} \wedge T, \theta}(t,z), \tag{4.1.31}$$

and $\nu_{\mathscr{O}}$ *is defined by* (4.1.18), *for all* $(t,z) \in Q \cup \eth^1 Q$.

Remark 4.1.21 (Existence and uniqueness of strong solutions in weighted Sobolev spaces to the parabolic obstacle problem)**.** Existence and uniqueness of strong solutions in weighted Sobolev spaces to problem (4.1.8) with Dirichlet boundary condition (4.1.9) along $\eth^1 Q$, for all $\beta > 0$, is proved in [19].

4.1.3 Survey of previous results on stochastic representations of solutions to boundary value or obstacle problems

Stochastic representations of solutions to elliptic and parabolic boundary value and obstacle problems discussed by Bensoussan and Lions [8] and Friedman [40] are established under the hypotheses that the matrix of coefficients, (a^{ij}), of the second-order spatial derivatives in an elliptic linear, second-order differential operator, A, is *strictly elliptic* and that all coefficients of A are *bounded*. Relaxations of these hypotheses, as in [40, Chapter 13 & 15], and more recently [77], fail to include the Heston generator mainly because the matrix (a^{ij}) does *not* satisfy

Hypothesis 4.1.22 (Extension property for positive definite, C^2 matrix-valued functions)**.** Given a subdomain $V \subsetneqq (0,\infty)\times\mathbb{R}^d$, for $d \geq 1$, we say that a matrix-valued function,

$$a : V \to \mathbb{R}^{d\times d},$$

which is C^2 on V and $a(t,z)$ is positive definite for each $(t,z)\in V$ has the *extension property* if there is a matrix-valued function,

$$\tilde{a} : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^{d\times d},$$

which coincides with a on V but is C^2 on $[0,\infty)\times\mathbb{R}^d$ and $\tilde{a}(t,z)$ is positive definite for each $(t,z)\in[0,\infty)\times\mathbb{R}^d$.

Naturally, Hypothesis 4.1.22 is also applicable when the matrix a is constant with respect to time, that is, in elliptic problems. Note that in the case of the Heston process, $d=2$, $V=(0,\infty)\times\mathbb{H}$, and

$$a(t,x,y) := \begin{pmatrix} y & \sigma\rho y \\ \sigma\rho y & \sigma^2 y \end{pmatrix}, \quad \forall (x,y)\in\mathbb{H},$$

and so the matrix a does *not* satisfy Hypothesis 4.1.22. We now give more detailed comparisons for each of the four main problems which we consider in this chapter. Additional comparison details are provided in Appendix C.4.

Elliptic boundary value problems

Stochastic representations of solutions to *non-degenerate* elliptic partial differential equations are described in [40, Theorem 6.5.1], [47, Proposition 5.7.2], [61, Theorem 9.1.1 & Corollary 9.1.2] and [8, Theorems 2.7.1 & 2.7.2].

Stochastic representations of solutions to a certain class of *degenerate* elliptic partial differential equations are described by Friedman in [40, Chapter 13], but those results do not apply to the Heston operator because a square root, (σ^{ij}), of the matrix (a^{ij}) cannot be extended as a uniformly Lipschitz continuous function on $\mathbb{R}^2$, that is, [40, Condition (A), p. 308] is not satisfied. Stroock and Varadhan [69, §5-8] also discuss existence and uniqueness of solutions to degenerate elliptic partial differential equations, but their assumption that the matrix (a^{ij}) satisfies Hypothesis 4.1.22 does not hold for the Heston operator (see [69, Theorem 2.1]).

More recently, Zhou [77] employs the method of quasiderivatives to establish the stochastic representation of solutions to a certain class of degenerate elliptic partial differential equations, and obtains estimates for the derivatives of their solutions. However, his results do not apply to the Heston operator because [77, Assumptions 3.1 & Condition (3.2)] are not satisfied in this case. Moreover, the Dirichlet condition is imposed on the whole boundary of the domain (see [77, Equation (1.1)]), while we take into consideration the portion of the boundary, Γ_0, where the differential operator A becomes degenerate.

Elliptic obstacle problems

We may compare Theorems 4.1.8 and 4.1.9 with the *uniqueness* assertions (in increasing degrees of generality) for *non-degenerate* elliptic operators in [8, Theorems 3.3.1, 3.3.2, 3.3.4, 3.3.5, 3.3.8, 3.3.19, 3.3.20, & 3.3.23]. See also [61, Theorem 10.4.1] and [40, Theorems 16.4.1, 16.4.2, 16.7.1, & 16.8.1] for uniqueness assertions *non-degenerate* elliptic operators, though with more limited applicability.

Parabolic boundary value problems

Uniqueness of solutions to *non-degenerate* parabolic partial differential equations and their stochastic representations are described in [40, Theorems 6.5.2, 6.5.3], [47, Theorem 5.7.6] and [8, Theorems 2.7.3 & 2.7.4].

Friedman obtains fundamental solutions and stochastic representations of solutions to certain degenerate parabolic partial differential equations in [39], while he obtains uniqueness and stochastic representations of solutions to the Cauchy problem in [38]; those results are summarized in [40, Chapter 15]. Nevertheless, the results in [40, Chapter 15] and [39] do not apply to the Heston operator because Hypothesis 4.1.22 does not hold, that is [40, Condition (A), p. 389] is not satisfied. Therefore, the method of construction in [39, Theorem 1.2] of a candidate for a fundamental solution does not apply to the Heston operator. A stochastic representation for a solution to the Cauchy problem for a degenerate operator is obtained in [40, §15.10], but the hypotheses of [40, Theorem 15.10.1] are again too restrictive and exclude the Heston operator.

Ekström and Tysk [25] consider the problem of pricing European-style options on an underlying process which is the solution to a degenerate, one-dimensional stochastic differential equation which satisfies [25, Hypothesis 2.1], and so includes the *Feller square root* (or *Cox-Ingersoll-Ross*) *process*, (4.2.1). The option price is the classical solution in the sense of [25, Definition 2.2] to the corresponding parabolic partial differential equation [25, Theorem 2.3]. Under their assumption that the payoff function $g(T,\cdot)$ is in $C^1([0,\infty))$, they show that their classical solution has the regularity property,

$$u \in C([0,T] \times [0,\infty)) \cap C^1([0,T) \times [0,\infty)) \cap C^2([0,T) \times (0,\infty)),$$

and obeys the second-order boundary condition,

$$\lim_{(t,y)\to(0,t_0)} y u_{yy}(t,y) = 0, \quad \forall t_0 \in (0,T) \quad \text{(by [25, Proposition 4.1])}.$$

As a consequence, in the present framework, their solution obeys

$$u \in C^{1,1}_{s,\mathrm{loc}}((0,t_0) \times [0,\infty)), \quad \forall t_0 \in (0,T),$$

where the vector space of functions $C^{1,1}_{s,\mathrm{loc}}((0,t_0) \times [0,\infty))$ is defined by analogy with (4.1.27).

In [24], Ekström and Tysk extend their results in [25] to the case of two-dimensional stochastic volatility models for option prices, where the variance process satisfies the assumptions of [25, Hypothesis 2.1].

Bayraktar, Kardaras, and Xing [7] address the problem of *uniqueness* of classical solutions, in the sense of [7, Definitions 2.4 & 2.5], to a class of two-dimensional, degenerate parabolic partial differential equations. Their differential operator has a degeneracy which is similar to that of the Heston generator, $-A$, and to the differential operator considered in [25], but the matrix of coefficients, (a^{ij}), of their operator may have *more than quadratic growth* with respect to the spatial variables (see [7, Standing Assumption 2.1]). Therefore, weak maximum principles for parabolic partial differential operators on unbounded domains such as [51, Exercise 8.1.22] do not guarantee uniqueness of solutions in such situations. The main result of their article – [7, Theorem 2.9] – establishes by probabilistic methods that uniqueness of classical solutions, obeying a natural growth condition, holds if and only if the asset price process is a martingale.

In our work, we consider the two-dimensional Heston stochastic process, (4.1.15), where the component Y of the process satisfies [25, Hypothesis 2.1] and [7, Standing Assumption 2.1]. We only require the payoff function, $g(T, \cdot)$, to be continuous with respect to the spatial variables and have *exponential growth*, as in (4.1.25). Notice that the conditions on the payoff function are more restrictive in [25, Hypothesis 2.1] and [7, Standing Assumption 2.3] than in our case. We consider the parabolic equation associated to the Heston generator, $-A$, on bounded or unbounded subdomains, $\mathscr{O}$, of the upper half plane, $\mathbb{H}$, with Dirichlet boundary condition along the portion, Γ_1, of the boundary $\partial\mathscr{O}$ contained in $\mathbb{H}$. Along the portion, Γ_0, of the boundary contained in $\partial\mathbb{H}$, we impose a suitable Dirichlet boundary condition, depending on the value of the parameter β in (3.1.11), which governs the behavior of the Feller square-root process when it approaches the boundary point $y = 0$. In each case, we establish *uniqueness* of solutions by proving that suitably regular solutions must have the stochastic representations in Theorems 4.1.12 and 4.1.15, and we prove *existence* and *regularity* of solutions, in a special case, in Theorems 4.5.4 and 4.5.5, complementing the results of [25]. In addition, we consider the parabolic *obstacle* problem and establish uniqueness and the stochastic representations of suitably regular solutions in Theorems 4.1.19 and 4.1.20.

Parabolic obstacle problems

We may compare Theorems 4.1.19 and 4.1.20 with the *uniqueness* assertions and stochastic representations of solutions (in increasing degrees of generality) for *non-degenerate* operators in [8, Theorems 3.4.1, 3.4.2, 3.4.3, 3.4.5, 3.4.6, 3.4.7, 3.4.8].

4.1.4 Outline of the chapter

For the convenience of the reader, we provide a brief outline of the chapter. We begin in §4.2 by reviewing or proving some of the key properties of the Feller square root and Heston processes which we shall need. In §4.3, we prove existence and uniqueness (in various settings) of solutions to the elliptic boundary value problem for the Heston operator, while in §4.4, we prove uniqueness (again in various settings) of solutions to the corresponding obstacle problem. We proceed in §4.5, to prove existence and uniqueness of solutions to the parabolic terminal/boundary value problem for the Heston operator and in §4.6, we prove uniqueness of solutions to the corresponding parabolic obstacle problem. Appendices C.1, C.2, and C.3 contain technical additional results which we shall need.

4.2 Properties of the Heston stochastic volatility process

In this section, we review or develop some important properties of the Feller square root process and the Heston stochastic volatility process.

By [30, Theorem 1.9], it follows that for any initial point $(t, y) \in [0, \infty) \times [0, \infty)$, the Feller stochastic differential equation,

$$\begin{aligned} dY(s) &= \kappa\left(\vartheta - Y(s)\right) ds + \sigma\sqrt{|Y(s)|}dW(s), \quad s > t, \\ Y(t) &= y, \end{aligned} \tag{4.2.1}$$

admits a unique weak solution $(Y^{t,y}(s), W(s))_{s\geq t}$, called the Feller square root process, where $(W(s))_{s\geq t}$ is a one-dimensional Brownian motion on a filtered probability space $\left(\Omega, \mathscr{F}, \mathbb{P}^{t,y}, \mathbb{F}\right)$ such that the filtration $\mathbb{F} = \{\mathscr{F}(s)\}_{s\geq 0}$ satisfies the usual conditions [47, Definition 1.2.25]. Theorem 1.9 in [30] also implies that the Heston stochastic differential equation (4.1.15) admits a unique weak solution, $\left(Z^{t,z}(s), W(s)\right)_{s\geq t}$, for any initial point $(t, z) \in [0, \infty) \times \bar{\mathbb{H}}$, where $(W(s))_{s\geq t}$ is now an $\mathbb{R}^2$-valued Brownian motion on a filtered probability space $\left(\Omega, \mathscr{F}, \mathbb{Q}^{t,z}, \mathbb{F}\right)$ such that the filtration $\mathbb{F} = \{\mathscr{F}(s)\}_{s\geq 0}$ satisfies the usual conditions. When the initial condition (t, y) or (t, z) is clear from the context, we omit the superscripts in the definition of the probability measures $\mathbb{P}^{t,y}$ and $\mathbb{Q}^{t,z}$, respectively.

Moreover, the weak solutions to the Feller and Heston stochastic differential equations are *strong*. To prove this, we begin by reviewing a result of Yamada [75].

Definition 4.2.1 (Coefficients for a non-Lipschitz stochastic differential equation)**.** [75, p. 115] In this section, we shall consider one-dimensional stochastic differential equations whose diffusion and drift coefficients, α, b, obey the following properties:

1. The functions $\alpha, b : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ are continuous.

2. (*Yamada condition*) There is an increasing function $\varrho : [0,\infty) \to [0,\infty)$ such that $\varrho(0) = 0$, for some $\varepsilon > 0$ one has $\int_0^\varepsilon \varrho^{-2}(y)\,dy = \infty$, and
$$|\alpha(t,y_1) - \alpha(t,y_2)| \leq \varrho(|y_1 - y_2|), \quad y_1, y_2 \in \mathbb{R}, t \geq 0. \tag{4.2.2}$$

3. There is a constant $C_1 > 0$ such that
$$|b(t,y_2) - b(t,y_1)| \leq C_1|y_2 - y_1|, \quad y_1, y_2 \in \mathbb{R}, t \geq 0. \tag{4.2.3}$$

4. There is a constant $C_2 > 0$ such that
$$|\alpha(t,y)| + |b(t,y)| \leq C_2(1 + |y|), \quad t \geq 0, y \in \mathbb{R}. \tag{4.2.4}$$

Clearly, the coefficients of the Feller stochastic differential equation obey the hypotheses in Definition 4.2.3, where $\alpha(t,y) = \sigma\sqrt{y}$ and $b(t,y) = \kappa(\theta - y)$. Indeed, one can choose $C_1 = \kappa$, $C_2 = \max\{\kappa, \kappa\theta, \sigma\}$, and $\varrho(y) = \sigma\sqrt{y}$, as the mean value theorem yields
$$\sqrt{y_2} - \sqrt{y_1} = c(y_1,y_2)(y_2 - y_1),$$
where
$$c(y_1,y_2) = \frac{1}{2}\int_0^1 \frac{1}{\sqrt{y_1 + t(y_2 - y_1)}} \leq \frac{1}{\sqrt{y_2 - y_1}},$$
for $0 < y_1 < y_2$. See [75, Remark 1] for other examples of suitable functions ϱ.

Remark 4.2.2. When $\varrho(u) = u^\gamma$, $\gamma \in [\frac{1}{2}, 1]$ [75, Remark 1], then Definition 4.2.1 implies that $\alpha(t,\cdot)$ is Hölder continuous with exponent γ, uniformly with respect to $t \in [0,\infty)$.

Definition 4.2.3 (Solution to a non-Lipschitz stochastic differential equation)**.** [75, p. 115], [64, Definitions IX.1.2 & IX.1.5] Let $(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$ be a filtered probability space satisfying the usual conditions. We call a pair $(Y(s), W(s))_{s\geq 0}$ a *weak solution* to the non-Lipschitz one-dimensional stochastic differential equation,

$$dY(s) = b(s, Y(s))\, ds + \alpha(s, Y(s))\, dW(s), \quad s \geq 0, Y(0) = y, \tag{4.2.5}$$

where $y \in \mathbb{R}$, if the following hold:

1. The processes $Y(s)$ and $W(s)$ are defined on $(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$;
2. The process $Y(s)$ is continuous with respect to $s \in [0, \infty)$ and is $\mathbb{F}$-adapted;
3. The process $W(s)$ is a standard $\mathbb{F}$-Brownian motion.

We call $(Y(s), W(s))_{s\geq 0}$ a *strong solution* to (4.2.5) if Y is $\mathbb{F}^W$-adapted, where $\mathbb{F}^W$ is the $\mathbb{P}$-completion of the filtration of $\mathscr{F}$ generated by $(W(s))_{s\geq 0}$. (Compare [45, Definition IV.1.2], [47, Definition 5.2.1], and [61, §5.3].)

Theorem 4.2.4. *[75, p. 117] There exists a weak solution* (Y, W) *to* (4.2.5)*.*

Remark 4.2.5. Yamada's main theorem [75, p. 117] asserts considerably more than Theorem 4.2.4. In particular, his article shows that (4.2.5) may be solved using the method of finite differences. Simpler results may suffice to merely guarantee the existence of a weak solution, as we need here; see Skorokhod [68].

Proposition 4.2.6. *There exists a unique strong solution to* (4.2.5)*.*

Proof. Theorem 4.2.4 ensures that (4.2.5) admits a weak solution. Conditions (4.2.2) and (4.2.3) ensure that pathwise uniqueness holds for (weak) solutions to (4.2.5) by Revuz and Yor [64, Theorem IX.3.5 (ii)], while Karatzas and Shreve [47, Corollary 5.3.23] imply that (4.2.5) admits a strong solution; see [47, p. 310]. Conditions (4.2.2) and (4.2.3) guarantee the uniqueness of strong solutions to (4.2.5) by Karatzas and Shreve [47, Proposition 5.2.13]; compare Yamada and Watanabe [73, 74]. (Pathwise uniqueness is also asserted for (4.2.5) by [45, Theorem IV.3.2] when (4.2.5) is time-homogeneous, noting that the coefficients α, b are not required to be bounded by Ikeda and Watanabe [45, p. 168]). We conclude that a strong solution to (4.2.5) exists and is unique. □

Corollary 4.2.7. *Given any initial point* $(t, y) \in [0, \infty) \times [0, \infty)$*, there exists a unique strong solution,* $(Y^{t,y}(s), W(s))_{s \geq t}$*, to the Feller stochastic differential equation.*

Proof. Immediate from Proposition 4.2.6. □

Corollary 4.2.8. *Given* $(t, z) \in [0, \infty) \times \bar{\mathbb{H}}$*, there exists a unique strong solution,* $(Z^{t,z}(s), W(s))_{s \geq t}$*, to the Heston stochastic differential equation, where* $(W(s))_{s \geq 0}$ *is a standard two-dimensional* $\mathbb{F}$*-Brownian motion on* $(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$*.*

Proof. By Proposition 4.2.6, the Cox-Ingersoll-Ross stochastic differential equation has a unique strong solution, $(Y^{t,y}(s), W_2(s))_{s \geq t}$, where $(W_2(s))_{s \geq t}$ is a standard one-dimensional $\mathbb{F}$-Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$ and $(Y^{t,y}(s))_{s \geq t}$ is $\mathbb{F}^{W_2}$-adapted. But given $(Y^{t,y}(s))_{s \geq t}$ and a standard two-dimensional $\mathbb{F}$-Brownian motion, $(W(s))_{s \geq t} = (W_1(s), W_2(s))_{s \geq t}$ on $(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$, the process $(X^{t,x,y}(s))_{s \geq t}$, and thus $(Z^{t,z}(s))_{s \geq t} = (X^{t,x,y}(s), Y^{t,y}(s))_{s \geq t}$, is uniquely determined by

$$
\begin{aligned}
X^{t,x,y}(s) = x &+ \int_t^s \left(r - q - \frac{1}{2} Y^{t,y}(u) \right) du \\
&+ \int_t^s \sqrt{Y^{t,y}(u)} \left(\sqrt{1 - \rho^2} dW_1(u) + \rho dW_2(u) \right).
\end{aligned}
$$

This completes the proof. □

Lemma 4.2.9 (Properties of the Feller square-root process)**.** *The unique strong solution of the Feller stochastic differential equation started at any* $(t, y) \in [0, \infty) \times [0, \infty)$ *satisfies*

$$
Y(s) \geq 0 \quad \mathbb{P}^{t,y}\text{-a.s.}, \quad \forall s \geq t, \tag{4.2.6}
$$

and also

$$
\int_t^s \mathbf{1}_{\{Y(u)=0\}} \, du = 0, \quad \forall s \geq t, \tag{4.2.7}
$$

$$
L(s, x) = 0, \quad \forall x \leq 0, \forall s \geq t, \tag{4.2.8}
$$

where $L(\cdot, \cdot)$ *is the local time of the Feller square-root process.*

Proof. Without loss of generality, we may assume that $t = 0$. In [7, Lemma 2.4], it is proved that $L(s, 0) = 0$, for all $s \geq 0$, but it is not clear to us why it also follows that

$$L(s, 0-) := \lim_{x \uparrow 0} L(s, x) = 0, \quad \forall s \geq 0,$$

a property we shall need in our proof of (4.2.6). To complete the argument, we consider the following stochastic differential equation,

$$\begin{aligned} d\widetilde{Y}(s) &= b(\widetilde{Y}(s))\, ds + \alpha(\widetilde{Y}(s))\, dW(s), \quad s > 0, \\ \widetilde{Y}(0) &= y, \end{aligned}$$

where we let

$$b(y) := \kappa(\vartheta - y) \text{ and } \alpha(y) := \mathbf{1}_{\{y \geq 0\}} \sigma \sqrt{y}, \quad \forall y \in \mathbb{R}. \tag{4.2.9}$$

This stochastic differential equation admits a unique strong solution by Proposition 4.2.6. We will show that $\widetilde{Y}(s) \geq 0$ a.s., for all $s \geq 0$, so that uniqueness of solutions to the Feller stochastic differential equation (4.2.1) implies that $\widetilde{Y} = Y$ a.s. and Y will satisfy the same properties as $\widetilde{Y}$. Thus, it is enough to prove (4.2.7) and (4.2.8) for $\widetilde{Y}$. Property (4.2.6) is a consequence of the preceding two properties of $\widetilde{Y}$.

Let $\widetilde{L}$ be the local time process for the continuous semimartingale $\widetilde{Y}$ (see [47, Theorem 3.7.1]). From [47, Theorem 3.7.1 (iii)], we know that, for any Borel measurable function $k : \mathbb{R} \to [0, \infty)$, we have

$$\int_0^s k(\widetilde{Y}(u)) \sigma^2 \widetilde{Y}^+(u)\, du = 2 \int_{\mathbb{R}} k(x) \widetilde{L}(s, x) dx, \quad \forall s \geq 0. \tag{4.2.10}$$

Assume, to obtain a contradiction, that $\widetilde{L}(s, 0) > 0$. From the right-continuity in the spatial variable of $\widetilde{L}(s, \cdot)$ [47, Theorem 3.7.1 (iv)], there are positive constants c and x_0 such that $\widetilde{L}(s, x) \geq c$, for all $x \in [0, x_0]$. For $\varepsilon > 0$, we define $k(x) = x^{-1}$, for $x \in [\varepsilon, x_0]$, and 0 otherwise. With this choice of k, the left-hand-side in identity (4.2.10) is bounded in absolute value by $\sigma^2 s$, for any $\varepsilon > 0$, while the right-hand-side of (4.2.10) is greater or equal than $2c \log{(x_0/\varepsilon)}$, which diverges as ε tends to 0. Therefore, our assumption that $\widetilde{L}(s, 0) > 0$ is false, and so $\widetilde{L}(s, 0) = 0$. Moreover, we notice that for any bounded, Borel-measurable function k with support in $(-\infty, 0)$ the left-hand-side in identity (4.2.10) is identically zero. Thus, we conclude that $\widetilde{L}(s, x) = 0$, for all $x < 0$, and also $\widetilde{L}(s, 0-) = 0$.

We use this result to show that $\mathbb{P}(\widetilde{Y}(s) \leq 0, \forall s \geq 0) = 0$. From [47, p. 223, third formula] and the fact that $\kappa, \vartheta > 0$, we see that

$$0 = \widetilde{L}(s, 0) - \widetilde{L}(s, 0-) = \kappa\vartheta \int_0^s \mathbf{1}_{\{\widetilde{Y}(u)=0\}}\, du,$$

which implies that $\mathbb{P}(\widetilde{Y}(s) = 0, \forall s \geq 0) = 0$. It remains to show that $\mathbb{P}(\widetilde{Y}(s) \in (-\infty, 0)) = 0$, for all $s \geq 0$, which is equivalent to proving that for any $\varepsilon > 0$ and $s \geq 0$, we have $\mathbb{P}(\widetilde{Y}(s) \in (-\infty, -\varepsilon)) = 0$. Let $\varphi : \mathbb{R} \to [0, 1]$ be a smooth cut-off function such that $\varphi|_{(-\infty,-\varepsilon)} \equiv 1$ and $\varphi|_{(0,\infty)} \equiv 0$. We can choose φ such that $\varphi' \leq 0$. Then, it follows by Itô's formula that

$$\begin{aligned}
\varphi(\widetilde{Y}(s)) &= \varphi(\widetilde{Y}(0)) + \int_0^s \left(\kappa(\vartheta - \widetilde{Y}(u))\varphi'(\widetilde{Y}(u)) + \frac{1}{2}\alpha^2(Y(u))\varphi''(Y(u)) \right) du \\
&\quad + \int_0^s \alpha(\widetilde{Y}(u))\varphi'(\widetilde{Y}(u))\, dW(u) \\
&= \varphi(\widetilde{Y}(0)) + \int_0^s \kappa(\vartheta - \widetilde{Y}(u))\varphi'(\widetilde{Y}(u))\, du \quad (\text{as } \alpha(y) = 0 \text{ when } \varphi' \neq 0).
\end{aligned}$$

We notice that the right-hand-side is non-negative, while the left-hand-side is non-positive, as $\varphi' \leq 0$ on $\mathbb{R}$, and $\varphi' = 0$ on $(0, \infty)$. Therefore, we must have $\varphi(\widetilde{Y}(s)) = 0$ a.s. which implies that $\mathbb{P}(\widetilde{Y}(s) \in (-\infty, -\varepsilon)) = 0$. This concludes the proof of the lemma. $\square$

For $a, y, t \geq 0$, we let

$$T_a^{t,y} := \inf\left\{s \geq t : Y^{t,y}(s) = a\right\} \tag{4.2.11}$$

denote the first time the process Y started at y at time t hits a. When the initial condition, (t, y), is clear from the context, we omit the superscripts in the preceding definition (4.2.11). Also, when $t = 0$, we omit the superscript t.

Lemma 4.2.10 (Boundary classification at $y = 0$ of the Feller square root process)**.** *Let Y^y be the unique strong solution to the Feller stochastic differential equation* (4.2.1) *with initial condition $Y^y(0) = y$. Then*

1. *For $\beta \geq 1$, $y = 0$ is an entrance boundary point in the sense of [49, §15.6(c)].*

2. *For $0 < \beta < 1$, $y = 0$ is a regular, instantaneously reflecting boundary point in the sense of [49, §15.6(a)], and*

$$\lim_{y \downarrow 0} T_0^y = 0 \quad a.s., \tag{4.2.12}$$

where T_0^y is given by (4.2.11).

Proof. A direct calculation give us that the scale function, $\mathfrak{s}$, and the speed measure, $\mathfrak{m}$, of the Feller square root process are given by

$$\mathfrak{s}(y) = y^{-\beta} e^{\mu y} \text{ and } \mathfrak{m}(y) = \frac{2}{\sigma^2} y^{\beta - 1} e^{-\mu y}, \quad \forall y > 0$$

where $\beta = 2\kappa\vartheta/\sigma^2$ and $\mu = 2\kappa/\sigma^2$. We consider the following quantities, for $0 < a < b < \infty$ and $x > 0$,

$$\begin{aligned} S[a,b] &:= \int_a^b \mathfrak{s}(y)dy, \quad S(a,b] := \lim_{c \downarrow a} S[c,b], \\ M[a,b] &:= \int_a^b \mathfrak{m}(y)dy, \quad M(a,b] := \lim_{c \downarrow a} M[c,b], \\ N(0) &:= \int_0^x S[y,x]\mathfrak{m}(y)dy. \end{aligned}$$

Then, for $\beta \geq 1$, we have $S(0,x] = \infty$ and $N(0) < \infty$, which implies that $y = 0$ is an entrance boundary point ([49, p. 235]), while for $0 < \beta < 1$, we have $S(0,x] < \infty$ and $M(0,x] < \infty$, and so $y = 0$ is a regular boundary point ([49, p. 232]).

Next, we consider the case $0 < \beta < 1$. To establish (4.2.12), we consider the following quantities

$$\begin{aligned} u_{a,b}(y) &:= \mathbb{P}^y (T_b < T_a) = \frac{S[a,y]}{S[a,b]}, \\ v_{a,b}(y) &:= \mathbb{E}_{\mathbb{P}}^y [T_a \wedge T_b] = 2u_{a,b}(y) \int_y^b S[z,b]m(z)dz + 2\left(1 - u_{a,b}(y)\right) \int_a^y S(a,z]m(z)dz, \end{aligned}$$

as in [49, Equations (15.6.1) & (15.6.5)] and [49, Equations (15.6.2) & (15.6.6)], respectively. Notice that $T_a^y \to T_0^y$, when $y \downarrow 0$, by the continuity of the paths of Y. Then, for fixed $b > 0$, we obtain

$$\begin{aligned} \lim_{y \downarrow 0} \mathbb{P}^y(T_b < T_0) &= \lim_{y \downarrow 0} \lim_{a \downarrow 0} \mathbb{P}^y(T_b < T_a) = 0, \\ \lim_{y \downarrow 0} \mathbb{E}_{\mathbb{P}}^y [T_0 \wedge T_b] &= \lim_{y \downarrow 0} \lim_{a \downarrow 0} \mathbb{E}_{\mathbb{P}}^y [T_a \wedge T_b] = 0, \end{aligned}$$

from where (4.2.12) follows. $\square$

Next, we have the following

Lemma 4.2.11 (Properties of the Heston process). *Let $(Z(s))_{s\geq 0}$ be the unique strong solution to the Heston stochastic differential equation* (4.1.15)*.*

1. *Assume $q \geq 0$ and $r \in \mathbb{R}$. Then, for any constant $c \in [0,1]$,*

$$\left(e^{-rcs}e^{cX(s)}\right)_{s\geq 0} \quad \textit{is a positive supermartingale.} \tag{4.2.13}$$

2. *For any positive constant $c \leq \mu$,*

$$\left(e^{-c\kappa\vartheta s}e^{cY(s)}\right)_{s\geq 0} \quad \textit{is a positive supermartingale.} \tag{4.2.14}$$

Proof. To establish (4.2.13), we use Itô's formula to give

$$\begin{aligned} d\left(e^{-rcs}e^{cX(s)}\right) &= -e^{-rcs}e^{cX(s)}\left(cq + \frac{1}{2}c(1-c)Y(s)\right)ds \\ &\quad + ce^{-rcs}e^{cX(s)}\sqrt{Y(s)}\,dW_1(s). \end{aligned} \tag{4.2.15}$$

Notice that the drift coefficient is non-positive, since $Y(s) \geq 0$ a.s. for all $s \geq 0$ by Lemma 4.2.9, and $q \geq 0$, and $c \in [0,1]$.

Similarly, to establish (4.2.14) for the Feller square root process, we have

$$\begin{aligned} d\left(e^{-c\kappa\vartheta s}e^{cY(s)}\right) &= e^{-c\kappa\vartheta s}e^{cY(s)}c\left(c\sigma^2/2 - \kappa\right)Y(s)ds \\ &\quad + c\sigma e^{-c\kappa\vartheta s}e^{cY(s)}\sqrt{Y(s)}\left(\rho dW_1(s) + \sqrt{1-\rho^2}dW_2(s)\right). \end{aligned} \tag{4.2.16}$$

When $c \leq \mu$, we see that the drift coefficient in the preceding stochastic differential equation is non-negative.

The supermartingale properties (4.2.13) and (4.2.14) follow if we show in addition that the processes are integrable random variables for each time $s \geq 0$. For simplicity, we let $Q(s)$ denote either one of the processes we consider, and we let θ_n be the first exit time of the Heston process $(X(s), Y(s))_{s\geq 0}$ from the rectangle $(-n,n)\times(-n,n)$, where $n \in \mathbb{N}$. We set $Q_n(s) := Q(s\wedge\theta_n)$, for all $s \geq 0$. We then have

$$dQ_n(s) = \mathbf{1}_{\{s\leq\theta_n\}}dQ_n(s), \quad \forall s > 0, \quad \forall n \in \mathbb{N}.$$

Using equations (4.2.15) and (4.2.16), it is clear that $(Q_n(s))_{s\geq 0}$ are supermartingales, because the coefficients of the stochastic differential equations are bounded and the drift terms are non-positive. Therefore, we know that

$$\mathbb{E}_{\mathbb{Q}}^{x,y}\left[Q_n(t)|\mathscr{F}(s)\right] \leq Q_n(s), \quad \forall t \geq s, \quad \forall s \geq 0, \quad \forall n \in \mathbb{N}. \tag{4.2.17}$$

Clearly, we also have $Q_n(t) \to Q(t)$ a.s., as $n \to \infty$, for all $t \geq s$ and $s \geq 0$. Taking the limit as $n \to \infty$ in (4.2.17) and using the positivity of the processes, Fatou's lemma yields

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}^{x,y}\left[Q(t)|\mathscr{F}(s)\right] &\leq \liminf_{n\to\infty} \mathbb{E}_{\mathbb{Q}}^{x,y}\left[Q_n(t)|\mathscr{F}(s)\right] \\ &\leq \liminf_{n\to\infty} Q_n(s) \quad \text{(by (4.2.17))} \\ &= Q(s), \quad \forall t \geq s, \quad \forall s \geq 0, \end{aligned}$$

and so (4.2.13) and (4.2.14) follow. □

The next lemma is used to show that the functions u^* given by (4.1.19) and (4.1.21) are well-defined and satisfy the growth assumption (4.1.16).

Lemma 4.2.12. *Suppose $r > 0$, and f, g, ψ are Borel measurable functions on $\mathscr{O}$ and satisfy assumption* (4.1.16). *Then there is a positive constant $\bar{C}$, depending on r, κ, ϑ, M_1, M_2 and C in* (4.1.16), *such that for any $\theta_1, \theta_2 \in \mathscr{T}$, the function $J_e^{\theta_1,\theta_2}$ in* (4.1.22) *satisfies the growth assumption,*

$$|J_e^{\theta_1,\theta_2}(x,y)| \leq \bar{C}\left(1 + e^{M_1 y} + e^{M_2 x}\right), \quad \forall (x,y) \in \mathscr{O},$$

where $0 < M_1 < \min\{r/(\kappa\vartheta), \mu\}$ and $M_2 \in [0,1)$ are as in (4.1.16).

Remark 4.2.13. The obstacle function ψ in (4.1.22) is only relevant for solutions to problem (4.1.2).

Proof. The conclusion is a consequence of the properties of the Heston process given in Lemma 4.2.11. We first estimate the integral term in (4.1.22). For $z \in \mathscr{O}$, then

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}^{z}\left[\int_0^{\theta_1\wedge\theta_2} e^{-rs}|f(Z(s))|ds\right] \\ &\quad \leq C\mathbb{E}_{\mathbb{Q}}^{z}\left[\int_0^{\infty} e^{-rs}\left(1 + e^{-rs}e^{M_1 Y(s)} + e^{-rs}e^{M_2 X(s)}\right) ds\right] \quad \text{(by (4.1.16))} \\ &\quad \leq C\left(1 + \int_0^{\infty} e^{-(r-M_1\kappa\vartheta)s}\mathbb{E}_{\mathbb{Q}}^{z}\left[e^{-M_1\kappa\vartheta s}e^{M_1 Y(s)}\right] ds\right. \\ &\qquad \left. + \int_0^{\infty} e^{-(1-M_2)rs}\mathbb{E}_{\mathbb{Q}}^{z}\left[e^{-rM_2 s}e^{M_2 X(s)}ds\right] ds\right). \end{aligned}$$

Using the condition $M_1 < \min\{r/(\kappa\vartheta), \mu\}$ and (4.2.14), together with $M_2 < 1$ and (4.2.13), we see that

$$\mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{\theta_1\wedge\theta_2} e^{-rs}|f(Z(s))|ds\right] \leq \bar{C}\left(1 + e^{M_1 y} + e^{M_2 x}\right), \tag{4.2.18}$$

for a positive constant $\bar{C}$ depending on r, $M_1\kappa\vartheta$, M_2 and the constant C in the growth assumption (4.1.16) on f, g and ψ.

Next, we show that the first non-integral term in (4.1.22) can be written as

$$\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\theta_1}g(Z(\theta_1))\mathbf{1}_{\{\theta_1\leq\theta_2\}}\right] = \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\theta_1}g(Z(\theta_1))\mathbf{1}_{\{\theta_1\leq\theta_2, \theta_1<\infty\}}\right], \tag{4.2.19}$$

for any $\theta_1 \in \mathscr{T}$ which is not necessarily finite. This is reasonable because by rewriting

$$\begin{aligned}\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\theta_1}g(Z(\theta_1))\mathbf{1}_{\{\theta_1\leq\theta_2\}}\right] &= \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\theta_1}g(Z(\theta_1))\mathbf{1}_{\{\theta_1\leq\theta_2\wedge T\}}\right]\\ &\quad + \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\theta_1}g(Z(\theta_1))\mathbf{1}_{\{T<\theta_1\leq\theta_2\}}\right],\end{aligned}$$

we shall see that the second term converges to zero, as $T \to \infty$. Using the growth assumption on g in (4.1.16), we have

$$\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\theta_1}\left|g(Z(\theta_1))\right|\mathbf{1}_{\{T<\theta_1\leq\theta_2\}}\right] \leq C\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\theta_1}\left(1 + e^{M_1 Y(\theta_1)} + e^{M_2 X(\theta_1)}\right)\mathbf{1}_{\{T<\theta_1\}}\right],$$

and so by Lemma 4.2.11, we obtain

$$\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\theta_1}g(Z(\theta_1))\mathbf{1}_{\{T<\theta_1\leq\theta_2\}}\right] \leq C\left(e^{-rT} + e^{-(r-M_1\kappa\vartheta)T}e^{M_1 y} + e^{-r(1-M_2)T}e^{M_2 x}\right).$$

Since $M_1 < r/(\kappa\vartheta)$ and $M_2 < 1$, we see that the right hand side converges to 0, as $T \to \infty$. This justifies the identity (4.2.19).

Now, we use Fatou's lemma to obtain the bound (4.1.16) on the first non-integral term in (4.1.22). For $z \in \mathscr{O}$,

$$\begin{aligned}&\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\theta_1}\left|g(Z(\theta_1))\right|\mathbf{1}_{\{\theta_1\leq\theta_2\}}\right]\\ &\quad\leq \liminf_{n\to\infty}\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r(\theta_1\wedge n)}|g(Z(\theta_1\wedge n))|\right]\\ &\quad\leq \liminf_{n\to\infty} C\left(1 + \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r(\theta_1\wedge n)}e^{M_1 Y(\theta_1\wedge n)}\right] + \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r(\theta_1\wedge n)}e^{M_2 X(\theta_1\wedge n)}\right]\right) \quad \text{(by (4.1.16))}.\end{aligned}$$

Because $M_1 < \mu$, we may apply the supermartingale property (4.2.14) with $c := M_2$. We use also that $M_1 < r/(\kappa\vartheta)$ to obtain $M_1\kappa\vartheta < r$, and so it follows by the Optional

Sampling Theorem [47, Theorem 1.3.22] that

$$\begin{aligned}\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r(\theta_1\wedge n)}e^{M_1Y(\theta_1\wedge n)}\right] &\leq \mathbb{E}^z_{\mathbb{Q}}\left[e^{-M_1\kappa\vartheta(\theta_1\wedge n)}e^{M_1Y(\theta_1\wedge n)}\right]\\ &\leq e^{M_1 y}, \quad \forall n\in\mathbb{N}.\end{aligned}$$

Using the fact that $M_2<1$, we see by the supermartingale property (4.2.13) applies with $c:=M_1$. By the Optional Sampling Theorem [47, Theorem 1.3.22] we have

$$\begin{aligned}\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r(\theta_1\wedge n)}e^{M_2X(\theta_1\wedge n)}\right] &\leq \mathbb{E}^z_{\mathbb{Q}}\left[e^{-rM_2(\theta_1\wedge n)}e^{M_2X(\theta_1\wedge n)}\right]\\ &\leq e^{M_2 x}, \quad \forall n\in\mathbb{N}.\end{aligned}$$

Therefore, we obtain

$$\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\theta_1}\left|g(Z(\theta_1))\right|\mathbf{1}_{\{\theta_1\leq\theta_2\}}\right]\leq C\left(1+e^{M_1y}+e^{M_2x}\right).$$

We obtain the same bound on the second non-integral term in (4.1.22) because the obstacle function ψ satisfies the same growth condition (4.1.16) as the boundary data g. □

To prove Theorems 4.1.12 and 4.1.15, we make use of the following auxiliary result

Lemma 4.2.14. *Let $z\in\overline{\mathbb{H}}$ and $T\in(0,T_0]$, where T_0 is a positive constant. Let $(Z^z(s))_{s\geq 0}$ be the unique strong solution to the Heston stochastic differential equation* (4.1.15) *with initial condition $Z^z(0)=z$. Then there is a positive constant c, depending on y, κ, ϑ, σ and T_0, such that for any constant p satisfying*

$$0\leq p<\frac{c}{2\sigma T}, \tag{4.2.20}$$

we have

$$\sup_{\theta\in\mathscr{T}_{0,T}}\mathbb{E}^z_{\mathbb{Q}}\left[e^{pX^z(\theta)}\right]<\infty, \tag{4.2.21}$$

where $\mathscr{T}_{0,T}$ denotes the set of $(\Omega,\mathscr{F},\mathbb{Q}^z,\mathbb{F})$-stopping times with values in $[0,T]$.

Proof. We use the method of time-change. Denote

$$M_i(t):=\int_0^t\sqrt{Y(s)}dW_i(s), \quad i=1,2,$$

and observe that there is a two-dimensional Brownian motion (B_1, B_2) [47, Theorem 3.4.13] such that

$$M_i(t) = B_i\left(\int_0^t Y(s)ds\right), \quad i = 1, 2.$$

Thus, we may rewrite the solution of the Heston stochastic differential equation (4.1.15) in the form

$$X(t) = x + (r-q)s - \frac{1}{2}\int_0^t Y(s)ds + B_1\left(\int_0^t Y(s)ds\right), \tag{4.2.22}$$

$$Y(t) = y + \kappa\vartheta s - \kappa\int_0^t Y(s)ds + \sigma B_3\left(\int_0^t Y(s)ds\right), \tag{4.2.23}$$

where $B_3 := \rho B_1 + \sqrt{1-\rho^2}B_2$ is a one-dimensional Brownian motion.

For any continuous stochastic process $(P(t))_{t\geq 0}$, we let

$$M_P(t) := \max_{0\leq s\leq t} P(s), \quad \forall t \geq 0.$$

We first prove the following estimate.

Claim 4.2.15. *There are positive constants n_0 and c, depending on y, κ, ϑ, σ and T_0, such that*

$$\mathbb{Q}^z\left(n \leq M_Y(T) \leq n+1\right) \leq \frac{2}{\sqrt{\pi}}e^{-cn/(2\sigma^2 T)}\mathbf{1}_{\{n\geq n_0\}} + \mathbf{1}_{\{n<n_0\}}, \quad \forall n \in \mathbb{N}. \tag{4.2.24}$$

Proof. Notice that if $M_Y(T) \leq n+1$, where $n \in \mathbb{N}$, then

$$\int_0^T Y(s)ds \leq (n+1)T,$$

and so, for any positive constant m,

$$\left\{\max_{0\leq t\leq T} B_3\left(\int_0^t Y(s)ds\right) \geq m, M_Y(T) \leq n+1\right\} \subseteq \left\{M_{B_3}((n+1)T) \geq m\right\}. \tag{4.2.25}$$

Using the inclusion

$$\{n \leq M_Y(T)\} \subseteq \left\{\max_{0\leq t\leq T} B_3\left(\int_0^t Y(s)ds\right) \geq \frac{n-y-\kappa\vartheta T}{\sigma}\right\} \quad \text{(by (4.2.23))},$$

we obtain by (4.2.25),

$$\mathbb{Q}^z\left(n \leq M_Y(T) \leq n+1\right) \leq \mathbb{Q}^z\left(M_{B_3}((n+1)T) \geq \frac{n-y-\kappa\vartheta T}{\sigma}\right).$$

The expression for the density of the running maximum of Brownian motion [47, Equation (2.8.4)] yields

$$\mathbb{Q}^z\left(M_{B_3}((n+1)T) \geq \frac{n-y-\kappa\vartheta T}{\sigma}\right) \leq \int_{(n-y-\kappa\vartheta T)/(\sigma\sqrt{(n+1)T})}^{\infty} \frac{2}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

As in [1, §7.1.2], we let

$$\operatorname{erfc}(a) := \frac{2}{\sqrt{\pi}} \int_a^\infty e^{-x^2/2} dx, \quad \forall a \in \mathbb{R},$$

and so,

$$\mathbb{Q}^z\left(n \leq M_Y(T) \leq n+1\right) \leq \frac{1}{\sqrt{2}} \operatorname{erfc}\left(\frac{n-y-\kappa\vartheta T}{\sigma\sqrt{(n+1)T}}\right).$$

Because for any $a \geq 1$,

$$\begin{aligned}\int_a^\infty e^{-x^2/2} dx &\leq \int_a^\infty x e^{-x^2/2} dx \\ &= e^{-a^2/2},\end{aligned}$$

we see that

$$\operatorname{erfc}(a) \leq \frac{2}{\sqrt{\pi}} e^{-a^2/2}, \quad \forall a \geq 1.$$

By hypothesis, $T \in (0, T_0]$, which implies that

$$\frac{n-y-\kappa\vartheta T}{\sigma\sqrt{(n+1)T}} \geq \frac{n-y-\kappa\vartheta T_0}{\sigma\sqrt{(n+1)T_0}}, \quad \forall n \in \mathbb{N}.$$

Hence, provided we have

$$\frac{n-y-\kappa\vartheta T_0}{\sigma\sqrt{(n+1)T_0}} \geq 1,$$

which is true for all $n \geq n_0(y, \kappa, \vartheta, \sigma, T_0)$, the smallest integer such that the preceding inequality holds, we see that

$$\mathbb{Q}^z\left(n \leq M_Y(T) \leq n+1\right) \leq \frac{2}{\sqrt{\pi}} e^{-(n-y-\kappa\vartheta T)^2/(2\sigma^2(n+1)T)}, \quad \forall n \geq n_0. \tag{4.2.26}$$

Similarly, for a possibly larger $n_0(y, \kappa, \vartheta, \sigma, T_0)$, using again the fact that $T \in (0, T_0]$, we may choose a positive constant c, depending also on y, κ, ϑ, σ and T_0, such that for all $n \geq n_0$, we have

$$\frac{(n-y-\kappa\vartheta T)^2}{2\sigma^2(n+1)T} \geq c\frac{n}{2\sigma^2 T}.$$

Then, using the preceding inequality, we obtain the estimate (4.2.24) from (4.2.26). This completes the proof of the claim. □

Next, we employ (4.2.24) to obtain (4.2.21). For any stopping time $\theta \in \mathscr{T}_{0,T}$, we may write

$$e^{pX(\theta)} = \sum_{n=0}^{\infty} e^{pX(\theta)} \mathbf{1}_{\{M_Y(T) \le n+1\}} \mathbf{1}_{\{n \le M_Y(T) \le n+1\}},$$

and, by Hölder's inequality, it follows

$$\mathbb{E}^z_{\mathbb{Q}}\left[e^{pX(\theta)}\right] \le \sum_{n=0}^{\infty} \mathbb{E}^z_{\mathbb{Q}}\left[e^{pX(\theta)} \mathbf{1}_{\{M_Y(T) \le n+1\}}\right]^{1/2} \mathbb{Q}^z\left(n \le M_Y(T) \le n+1\right)^{1/2}. \quad (4.2.27)$$

Using (4.2.22) and the condition $p \ge 0$ in (4.2.20), we have

$$\begin{aligned}
&\mathbb{E}^z_{\mathbb{Q}}\left[e^{pX(\theta)} \mathbf{1}_{\{M_Y(T) \le n+1\}}\right] \\
&\quad \le e^{p(x+|r-q|T)} \mathbb{E}^z_{\mathbb{Q}}\left[\exp\left(2pB_1\left(\int_0^\theta Y(s)ds\right)\right) \mathbf{1}_{\{M_Y(T) \le n+1\}}\right] \\
&\quad \le e^{p(x+|r-q|T)} \mathbb{E}^z_{\mathbb{Q}}\left[\exp\left(2p \max_{0 \le t \le T} B_1\left(\int_0^t Y(s)ds\right)\right) \mathbf{1}_{\{M_Y(T) \le n+1\}}\right] \\
&\quad \le e^{p(x+|r-q|T)} \mathbb{E}^z_{\mathbb{Q}}\left[e^{2pM_{B_1}((n+1)T)}\right], \quad \forall n \in \mathbb{N} \quad \text{(by (4.2.25))}.
\end{aligned}$$

We see from the expression for the density of the running maximum of Brownian motion [47, Exercise (2.8.4)] that

$$\begin{aligned}
\mathbb{E}^z_{\mathbb{Q}}\left[e^{2pM_{B_1}((n+1)T)}\right] &= \int_0^\infty e^{2px} \frac{2}{\sqrt{2\pi(n+1)T}} e^{-x^2/(2(n+1)T)} dx \\
&\le 2e^{2p^2(n+1)T}, \quad \forall n \in \mathbb{N} \quad \text{(by Mathematica)},
\end{aligned}$$

and so,

$$\mathbb{E}^z_{\mathbb{Q}}\left[e^{pX(\theta)} \mathbf{1}_{\{M_Y(T) \le n+1\}}\right] \le 2e^{p(x+|r-q|T)} e^{2p^2(n+1)T}, \quad \forall n \in \mathbb{N}. \quad (4.2.28)$$

Inequalities (4.2.24), (4.2.27) and (4.2.28) give us

$$\begin{aligned}
\mathbb{E}^z_{\mathbb{Q}}\left[e^{pX(\theta)}\right] &\le \sqrt{2} e^{p(x+|r-q|T)/2} \sum_{n=0}^{n_0-1} e^{p^2(n+1)T} \\
&\quad + \frac{2}{\pi^{1/4}} e^{p(x+|r-q|T)/2} \sum_{n=n_0}^{\infty} e^{p^2(n+1)T} e^{-cn/(4\sigma^2 T)} \\
&= \sqrt{2} e^{p(x+|r-q|T)/2} \sum_{n=0}^{n_0-1} e^{p^2(n+1)T} \\
&\quad + \frac{2}{\pi^{1/4}} e^{p(x+|r-q|T)/2+p^2T} \sum_{n=n_0}^{\infty} e^{(p^2T - c/(4\sigma^2T))n}.
\end{aligned}$$

We choose p such that

$$0 \leq p < \frac{\sqrt{c}}{2\sigma T},$$

that is, condition (4.2.20) is obeyed, and we obtain a bound on $\mathbb{E}_{\mathbb{Q}}^{z}\left[e^{pX(\theta)}\right]$ which is independent of the choice of $\theta \in \mathscr{T}_{0,T}$. Thus, (4.2.21) follows. (Note that (4.2.21) holds trivially when $p = 0$.) $\square$

4.3 Elliptic boundary value problem

In this section, we prove Theorem 4.1.2. In addition to the uniqueness result in Theorem 4.1.2 we establish the *existence* and uniqueness of solutions in Theorem 4.3.1.

The *existence* and *uniqueness* of solutions to problem (4.1.1) with boundary condition (4.1.3) along Γ_1, when $\beta \geq 1$, and with boundary condition (4.1.5) along $\partial\mathscr{O}$, when $0 < \beta < 1$, are similar in nature. Therefore, we define

$$\partial_\beta \mathscr{O} := \begin{cases} \Gamma_1 & \text{if } \beta \geq 1, \\ \partial\mathscr{O} & \text{if } 0 < \beta < 1. \end{cases} \tag{4.3.1}$$

and treat the previous mentioned boundary value problems together as

$$\begin{cases} Au = f & \text{on } \mathscr{O}, \\ u = g & \text{on } \partial_\beta\mathscr{O}. \end{cases} \tag{4.3.2}$$

Now, we can give the

Proof of Theorem 4.1.2. Our goal is to show that if $u \in C_{\mathrm{loc}}(\mathscr{O} \cup \partial_\beta\mathscr{O}) \cap C^2(\mathscr{O})$ is a solution to problem (4.3.2), satisfying the pointwise growth condition (4.1.16), then it admits the stochastic representation (4.1.19).

We let $\{\mathscr{O}_k : k \in \mathbb{N}\}$ denote an increasing sequence of $C^{2+\alpha}$ subdomains of $\mathscr{O}$ (see [41, Definition §6.2]) such that each $\mathscr{O}_k$ has compact closure in $\mathscr{O}$, and

$$\bigcup_{k\in\mathbb{N}} \mathscr{O}_k = \mathscr{O}.$$

By applying Itô's lemma (Theorem C.2.1), we obtain for all $t > 0$,

$$
\begin{aligned}
&d\left(e^{-r(t\wedge\tau_{\mathscr{O}_k})}u(Z(t\wedge\tau_{\mathscr{O}_k}))\right)\\
&\qquad = -\mathbf{1}_{\{t\leq\tau_{\mathscr{O}_k}\}}e^{-rt}Au(Z(t))dt\\
&\qquad\quad + \mathbf{1}_{\{t\leq\tau_{\mathscr{O}_k}\}}e^{-rt}\sqrt{Y(t)}\left(\left(u_x(Z(t)) + \sigma\rho u_y(Z(t))\right)dW_1(t) + \sigma\sqrt{1-\rho^2}u_y(Z(t))dW_2(t)\right).
\end{aligned}
$$

Since the subdomain $\mathscr{O}_k \subset \mathscr{O}$ is bounded and $u \in C^2(\mathscr{O})$, the dW_i-terms, $i = 1, 2$, in the preceding identity are martingales, and so we obtain

$$
\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r(t\wedge\tau_{\mathscr{O}_k})}u(Z(t\wedge\tau_{\mathscr{O}_k}))\right] = u(z) - \mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{t\wedge\tau_{\mathscr{O}_k}} e^{-rs}f(Z(s))ds\right]. \tag{4.3.3}
$$

We take the limit as k tends to ∞ in the preceding identity. By the growth estimate (4.2.18), we may apply the Lebesgue Dominated Convergence Theorem to show that the integral term in (4.3.3) converges to

$$
\mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{t\wedge\tau_{\mathscr{O}}} e^{-rs}f(Z(s))ds\right].
$$

For the non-integral term on the left hand side of (4.3.3), using the continuity of u on $\mathscr{O}\cup\partial_\beta\mathscr{O}$ and of the sample paths of the Heston process, we see that

$$
e^{-r(t\wedge\tau_{\mathscr{O}_k})}u(Z(t\wedge\tau_{\mathscr{O}_k})) \to e^{-r(t\wedge\tau_{\mathscr{O}})}u(Z(t\wedge\tau_{\mathscr{O}})), \quad \text{a.s. as } k\to\infty.
$$

Using [10, Theorem 16.13], we prove that

$$
\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r(t\wedge\tau_{\mathscr{O}_k})}u(Z(t\wedge\tau_{\mathscr{O}_k}))\right] \to \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r(t\wedge\tau_{\mathscr{O}})}u(Z(t\wedge\tau_{\mathscr{O}}))\right], \quad \text{as } k\to\infty,
$$

by showing that

$$
\left\{e^{-r(t\wedge\tau_{\mathscr{O}_k})}u(Z(t\wedge\tau_{\mathscr{O}_k})) : k\in\mathbb{N}\right\}
$$

is a collection of uniformly integrable random variables. By [10, Remark related to formula (16.23)], it suffices to show that their p-th order moment is uniformly bounded (independent of k), for some $p > 1$. We choose $p > 1$ such that $pM_1 < \mu$ and $pM_2 < 1$. Notice that this is possible because we assumed the coefficients $M_1 < \mu$ and $M_2 < 1$. Then, from the growth estimate (4.1.16), we have

$$
\left|e^{-r(t\wedge\tau_{\mathscr{O}_k})}u(Z)\right|^p \leq Ce^{-rp(t\wedge\tau_{\mathscr{O}_k})}\left(1 + e^{pM_1Y} + e^{pM_2X}\right), \quad \forall k\in\mathbb{N}.
$$

From the inequality (4.2.14) with $c = pM_1 < \mu$ and property (4.2.13) applied with $c = pM_2 \in (0,1)$, we obtain using $M_1 < r/(\kappa\vartheta)$

$$\mathbb{E}^z_{\mathbb{Q}}\left[\left|e^{-r(t\wedge\tau_{\mathscr{O}_k})}u(Z(t\wedge\tau_{\mathscr{O}_k}))\right|^p\right] \leq C\left(1+e^{pM_1y}+e^{pM_2x}\right), \quad \forall k\in\mathbb{N}.$$

Therefore, by taking limit as k tends to ∞ in (4.3.3) we obtain

$$\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r(t\wedge\tau_{\mathscr{O}})}u(Z(t\wedge\tau_{\mathscr{O}}))\right] = u(z) - \mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{t\wedge\tau_{\mathscr{O}}} e^{-rs}f(Z(s))ds\right]. \tag{4.3.4}$$

As we let t tend to ∞, the integral term on the right-hand side in the preceding identity clearly converges to

$$\mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{\tau_{\mathscr{O}}} e^{-rs}f(Z(s))\,ds\right].$$

It remains to consider the left-hand side of (4.3.4). Keeping in mind that $u\in C_{\mathrm{loc}}(\mathscr{O}\cup\partial_\beta\mathscr{O})$ solves (4.3.2), we rewrite this term as

$$\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r(t\wedge\tau_{\mathscr{O}})}u(Z(t\wedge\tau_{\mathscr{O}}))\right] = \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_{\mathscr{O}}}g(Z(\tau_{\mathscr{O}}))\mathbf{1}_{\{\tau_{\mathscr{O}}\leq t\}}\right] + \mathbb{E}^z_{\mathbb{Q}}\left[e^{-rt}u(Z(t))\mathbf{1}_{\{\tau_{\mathscr{O}}>t\}}\right].$$

Using the growth assumption (4.1.16), we notice as above that both collections of random variables in the preceding identity,

$$\left\{e^{-r\tau_{\mathscr{O}}}g(Z(\tau_{\mathscr{O}}))\mathbf{1}_{\{\tau_{\mathscr{O}}\leq t\}} : t\geq 0\right\} \text{ and } \left\{e^{-rt}u(Z(t))\mathbf{1}_{\{\tau_{\mathscr{O}}>t\}} : t\geq 0\right\},$$

are uniformly integrable, and they converge a.s. to $e^{-r\tau_{\mathscr{O}}}g(Z(\tau_{\mathscr{O}}))\mathbf{1}_{\{\tau_{\mathscr{O}}<\infty\}}$ and zero, respectively. Therefore, by [10, Theorem 16.13], letting t tend to ∞ in (4.3.4), we obtain

$$\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_{\mathscr{O}}}g(Z(\tau_{\mathscr{O}}))\mathbf{1}_{\{\tau_{\mathscr{O}}<\infty\}}\right] = u(z) - \mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{\tau_{\mathscr{O}}} e^{-rs}f(Z(s))\,ds\right],$$

which implies that $u=u^*$ on $\mathscr{O}\cup\partial_\beta\mathscr{O}$, where u^* is defined by (4.1.19). □

Proof of Theorem 4.1.5. Our goal is to show that if $0<\beta<1$ and $u\in C_{\mathrm{loc}}(\mathscr{O}\cup\Gamma_1)\cap C^2(\mathscr{O})\cap C^{1,1}_{s,\mathrm{loc}}(\mathscr{O}\cup\Gamma_0)$ is a solution to problem (4.1.1), satisfying the growth estimate (4.1.16), then it admits the stochastic representation (4.1.21).

We consider the following sequence of increasing subdomains of $\mathscr{O}$,

$$\mathscr{U}_k := \left\{z\in\mathscr{O} : |z|<k, \operatorname{dist}(z,\Gamma_1)>1/k\right\}, \quad k\in\mathbb{N}, \tag{4.3.5}$$

with non-empty boundary portions $\bar\Gamma_0 \cap \mathscr{U}_k$. Let $\varepsilon > 0$ and denote

$$Y^\varepsilon := Y + \varepsilon, \text{ and } Z^\varepsilon := (X, Y^\varepsilon). \tag{4.3.6}$$

By applying Itô's lemma (Theorem C.2.1), we obtain

$$\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r(t\wedge\nu_{\mathscr{U}_k})}u(Z^\varepsilon(t\wedge\nu_{\mathscr{U}_k}))\right] = u(z) - \mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{t\wedge\nu_{\mathscr{U}_k}} e^{-rs}A^\varepsilon u(Z^\varepsilon(s))ds\right], \quad \forall t > 0, \tag{4.3.7}$$

where $\nu_{\mathscr{U}_k}$ is given by (4.1.18), and A^ε denotes the elliptic differential operator,

$$A^\varepsilon v := Av + \frac{\varepsilon}{2}v_x + \kappa\varepsilon v_y - \frac{\varepsilon}{2}\left(v_{xx} + 2\rho\sigma v_{xy} + \sigma^2 v_{yy}\right), \quad \forall v \in C^2(\mathscr{O}). \tag{4.3.8}$$

Using (4.1.1), we can write (4.3.7) as

$$\begin{aligned}\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r(t\wedge\nu_{\mathscr{U}_k})}u(Z^\varepsilon(t\wedge\nu_{\mathscr{U}_k}))\right] &= u(z) - \mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{t\wedge\nu_{\mathscr{U}_k}} e^{-rs}f(Z^\varepsilon(s))ds\right] \\ &\quad - \mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{t\wedge\nu_{\mathscr{U}_k}} e^{-rs}(A^\varepsilon - A)u(Z^\varepsilon(s))ds\right].\end{aligned} \tag{4.3.9}$$

First, we take limit as ε tends to 0 in the preceding identity. We may assume without loss of generality that $\varepsilon < 1/k$, for any fixed $k \geq 1$. We evaluate the residual term $(A^\varepsilon - A)u$ with (4.3.8) to give

$$|(A^\varepsilon - A)\,u(Z^\varepsilon(s))| \leq C\varepsilon|Du|_{C(\bar{\mathscr{U}}_{2k})} + C\left(\mathbf{1}_{\{Y^\varepsilon(s)\leq\sqrt{\varepsilon}\}} + \sqrt{\varepsilon}\right)|yD^2u|_{C(\bar{\mathscr{U}}_{2k})}, \tag{4.3.10}$$

for all $0 \leq s \leq t \wedge \nu_{\mathscr{U}_k}$, where C is a positive constant depending only on the Heston constant coefficients. This follows from the fact that

$$\varepsilon D^2u(Z^\varepsilon(s)) = \varepsilon D^2u(Z^\varepsilon(s))\mathbf{1}_{\{Y^\varepsilon(s)\leq\sqrt{\varepsilon}\}} + \varepsilon D^2u(Z^\varepsilon(s))\mathbf{1}_{\{Y^\varepsilon(s)>\sqrt{\varepsilon}\}}, \quad \forall s \geq 0,$$

and so,

$$\begin{aligned}\varepsilon|D^2u(Z^\varepsilon(s))| &\leq Y^\varepsilon(s)|D^2u(Z^\varepsilon(s))|\mathbf{1}_{\{Y^\varepsilon(s)\leq\sqrt{\varepsilon}\}} + \varepsilon\frac{Y^\varepsilon(s)}{\sqrt{\varepsilon}}|D^2u(Z^\varepsilon(s))|\mathbf{1}_{\{Y^\varepsilon(s)>\sqrt{\varepsilon}\}} \\ &\leq \left(\mathbf{1}_{\{Y^\varepsilon(s)\leq\sqrt{\varepsilon}\}} + \sqrt{\varepsilon}\right)Y^\varepsilon(s)|D^2u(Z^\varepsilon(s))|.\end{aligned}$$

Combining the preceding inequality with the definition (4.3.8) of A^ε, we obtain (4.3.10). Since $u \in C^{1,1}_{s,\mathrm{loc}}(\mathscr{O} \cup \Gamma_0)$, and

$$\mathbf{1}_{\{Y^\varepsilon(s)\leq\sqrt{\varepsilon}\}} \to 0, \quad \text{as } \varepsilon \downarrow 0,$$

we see that by (4.3.10) yields, for each $k \geq 1$,

$$\mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{t\wedge\nu_{\mathscr{U}_k}} e^{-rs}(A^\varepsilon - A)u(Z^\varepsilon(s))ds\right] \to 0, \quad \text{as } \varepsilon \downarrow 0. \tag{4.3.11}$$

In addition, using the continuity of f and u on compact subsets of $\mathscr{O} \cup \Gamma_0$, we have

$$\begin{aligned}
&\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r(t\wedge\nu_{\mathscr{U}_k})}u(Z^\varepsilon(t\wedge\nu_{\mathscr{U}_k}))\right] \to \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r(t\wedge\nu_{\mathscr{U}_k})}u(Z(t\wedge\nu_{\mathscr{U}_k}))\right], \quad \text{as } \varepsilon \downarrow 0,\\
&\mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{t\wedge\nu_{\mathscr{U}_k}} e^{-rs}f(Z^\varepsilon(s))ds\right] \to \mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{t\wedge\nu_{\mathscr{U}_k}} e^{-rs}f(Z(s))ds\right], \quad \text{as } \varepsilon \downarrow 0.
\end{aligned} \tag{4.3.12}$$

Therefore, using (4.3.11) and the preceding limits, we find that (4.3.9) gives

$$\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r(t\wedge\nu_{\mathscr{U}_k})}u(Z(t\wedge\nu_{\mathscr{U}_k}))\right] = u(z) - \mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{t\wedge\nu_{\mathscr{U}_k}} e^{-rs}f(Z(s))ds\right]. \tag{4.3.13}$$

Note that by letting k and t tend to ∞, we have

$$t\wedge\nu_{\mathscr{U}_k} \to \nu_{\mathscr{O}}, \quad \text{a.s.} \tag{4.3.14}$$

By using the same argument as that used in the proof of Theorem 4.1.2 to take the limit as k and t tend to ∞ in (4.3.3), we can take the limit as k and t tend to ∞ in (4.3.13) to give

$$\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\nu_{\mathscr{O}}}g(Z(\nu_{\mathscr{O}}))\right] = u(z) - \mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{\nu_{\mathscr{O}}} e^{-rs}f(Z(s))ds\right].$$

This establishes $u = u^*$, where u^* is given by (4.1.21), and completes the proof. □

Next, we prove *existence* of solutions to problem (4.3.2) when the boundary data g is *continuous* on suitable portions of the boundary of $\mathscr{O}$.

Theorem 4.3.1 (Existence of solutions to the elliptic boundary value problem (4.3.2) with continuous Dirichlet boundary condition)**.** *In addition to the hypotheses of Theorem 4.1.2, assume that the domain $\mathscr{O} \subset \mathbb{H}$ has boundary portion Γ_1 which satisfies the exterior sphere condition, and that $f \in C^\alpha(\mathscr{O})$.*

1. *If $\beta \geq 1$ and also $g \in C_{\mathrm{loc}}(\bar\Gamma_1)$, then the function u^* in* (4.1.19) *is a solution to problem* (4.1.1) *with boundary condition* (4.1.3) *along Γ_1. In particular, $u^* \in C_{\mathrm{loc}}(\mathscr{O}\cup\Gamma_1)\cap C^{2+\alpha}(\mathscr{O})$ and u^* satisfies the growth assumption* (4.1.16).

2. *If* $0 < \beta < 1$ *and also* $g \in C_{\mathrm{loc}}(\partial\mathscr{O})$*, then the function* u^* *in* (4.1.19) *is a solution to problem* (4.1.1) *with boundary condition* (4.1.5) *along* $\partial\mathscr{O}$*. In particular,* $u^* \in C_{\mathrm{loc}}(\bar{\mathscr{O}}) \cap C^{2+\alpha}(\mathscr{O})$ *and* u^* *satisfies the growth assumption* (4.1.16)*.*

Proof. Following the comments preceding problem (4.3.2), we need to show that u^*, given by (4.1.19), is a solution to problem (4.3.2), that $u^* \in C_{\mathrm{loc}}(\mathscr{O} \cup \partial_\beta\mathscr{O}) \cap C^2(\mathscr{O})$, and that u^* satisfies the growth assumption (4.1.16).

Notice that Lemma 4.2.12, applied with $\theta_1 = \tau_{\mathscr{O}}$, $\theta_2 = \infty$ and $\psi \equiv 0$, shows that u^* defined by (4.1.19) satisfies the growth assumption (4.1.16). It remains to prove that $u^* \in C_{\mathrm{loc}}(\mathscr{O} \cup \partial_\beta\mathscr{O}) \cap C^2(\mathscr{O})$. Notice that Theorem 4.1.2 implies that u^* is the unique solution to the elliptic boundary value problem (4.3.2), since any $C_{\mathrm{loc}}(\mathscr{O} \cup \partial_\beta\mathscr{O}) \cap C^2(\mathscr{O})$ solution must coincide with u^*.

By hypothesis and the definition of $\partial_\beta\mathscr{O}$ in (4.3.1), we have $g \in C_{\mathrm{loc}}(\overline{\partial_\beta\mathscr{O}})$. Since $\overline{\partial_\beta\mathscr{O}}$ is closed, we may use [37, Theorem 3.1.2] to extend g to $\mathbb{R}^2$ such that its extension $\tilde{g} \in C_{\mathrm{loc}}(\mathbb{R}^2)$. We organize the proof in two steps.

Step 1 **(**$u^* \in C^{2+\alpha}(\mathscr{O})$**).** Let $\{\mathscr{O}_k : k \in \mathbb{N}\}$ be an increasing sequence of $C^{2+\alpha}$ subdomains of $\mathscr{O}$ as in the proof of Theorem 4.1.2. We notice that on each domain $\mathscr{O}_k$ the differential operator A is uniformly elliptic with $C^\infty(\bar{\mathscr{O}}_k)$ coefficients. From our hypotheses, we have $f \in C^\alpha(\bar{\mathscr{O}}_k)$ and $\tilde{g} \in C(\bar{\mathscr{O}}_k)$. Therefore, [41, Theorem 6.13] implies that the elliptic boundary value problem

$$\begin{cases} Au = f & \text{on } \mathscr{O}_k, \\ u = \tilde{g} & \text{on } \partial\mathscr{O}_k. \end{cases} \tag{4.3.15}$$

admits a unique solution $u_k \in C(\bar{\mathscr{O}}_k) \cap C^{2+\alpha}(\mathscr{O}_k)$. Moreover, by Theorem C.3.10, u_k admits a stochastic representation on $\bar{\mathscr{O}}_k$,

$$u_k(z) = \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_{\mathscr{O}_k}}\tilde{g}(Z(\tau_{\mathscr{O}_k}))1_{\{\tau_{\mathscr{O}_k}<\infty\}}\right] + \mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{\tau_{\mathscr{O}_k}} e^{-rs}f(Z(s))\,ds\right]. \tag{4.3.16}$$

Our goal is to show that u_k converges pointwise to u^* on $\mathscr{O}$. Recall that τ_k is an increasing sequence of stopping times which converges to $\tau_{\mathscr{O}}$ almost surely. Using $\tilde{g} \in C_{\mathrm{loc}}(\mathscr{O} \cup \partial_\beta\mathscr{O})$ and the continuity of the sample paths of the Heston process, the growth estimate (4.1.16) and Lemma 4.2.12, the same argument used in the proof of Theorem 4.1.2 shows that the sequence $\{u_k : k \in \mathbb{N}\}$ converges pointwise to u^* on $\mathscr{O}$.

Fix $z_0 := (x_0, y_0) \in \mathscr{O}$, and choose a Euclidean ball $B := B(z_0, r_0)$ such that $\bar{B} \subset \mathscr{O}$. We denote $B_{1/2} = B(z_0, r_0/2)$. As in the proof of Lemma 4.2.12, the sequence u_k is uniformly bounded on $\bar{B}$ because it obeys

$$|u_k(z)| \leq \bar{C}\left(1 + e^{M_1 y} + e^{M_2 x}\right), \quad \forall z = (x, y) \in B, k \in \mathbb{N}.$$

From the interior Schauder estimates [41, Corollary 6.3], the sequence $\{u_k : k \in \mathbb{N}\}$ has uniformly bounded $C^{2+\alpha}(\bar{B}_{1/2})$ norms. Compactness of the embedding $C^{2+\alpha}(\bar{B}_{1/2}) \hookrightarrow C^{2+\gamma}(\bar{B}_{1/2})$, for $0 \leq \gamma < \alpha$, shows that, after passing to a subsequence, the sequence $\{u_k : k \in \mathbb{N}\}$ converges in $C^{2+\gamma}(\bar{B}_{1/2})$ to $u^* \in C^{2+\gamma}(\bar{B}_{1/2})$, and so $Au^* = f$ on $\bar{B}_{1/2}$. Because the subsequence has uniformly bounded $C^{2+\alpha}(\bar{B}_{1/2})$ norms and it converges strongly in $C^2(\bar{B}_{1/2})$ to u^*, we obtain that $u^* \in C^{2+\alpha}(\bar{B}_{1/2})$.

Step 2 ($u^* \in C_{\mathrm{loc}}(\mathscr{O} \cup \partial_\beta \mathscr{O})$)**.** From the previous step, we know that $u^* \in C(\mathscr{O})$, so it remains to show continuity of u^* up to $\partial_\beta \mathscr{O}$. We consider two cases.

Case 1 ($u^* \in C_{\mathrm{loc}}(\mathscr{O} \cup \Gamma_1)$, for all $\beta > 0$)**.** First, we show that u^* is continuous up to Γ_1. We fix $z_0 \in \Gamma_1$, and let B be an open ball centered at z_0, such that $\bar{B} \cap \partial\mathbb{H} = \varnothing$. Let $U := B \cap \mathscr{O}$. Let the function $\hat{g}$ be defined on ∂U such that it coincides with g on $\partial U \cap \partial \mathscr{O}$, and it coincides with u^* on $\partial U \cap \mathscr{O}$.

Claim 4.3.2. *The strong Markov property of the Heston process* $(Z(s))_{s \geq 0}$ *and the definition* (4.1.19) *of* u^**, implies that*

$$u^*(z) = \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_U} \hat{g}(Z(\tau_U))\right] + \mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{\tau_U} e^{-rt} f(Z(t))dt\right], \quad \forall z \in U. \tag{4.3.17}$$

Proof. By Corollary 4.2.8, the Heston stochastic differential equation (4.1.15) admits a unique strong solution, for any initial point $(t, x, y) \in [0, \infty) \times \mathbb{R} \times [0, \infty)$, and [30, Theorem 1.16(c)] shows that the solution satisfies the strong Markov property.

Let $z \in U$, then $\tau^z_U \leq \tau^z_{\mathscr{O}}$ a.s. Since Z is a time-homogeneous strong Markov process, we obtain

$$\begin{aligned}
\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_{\mathscr{O}}}g(Z(\tau_{\mathscr{O}}))\right] &= \mathbb{E}^z_{\mathbb{Q}}\left[\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_{\mathscr{O}}}g(Z(\tau_{\mathscr{O}}))\right]|\mathscr{F}(\tau_U)\right] \\
&= \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_U}\mathbb{E}^{Z(\tau_U)}_{\mathbb{Q}}\left[e^{-r\tau_{\mathscr{O}}}g(Z(\tau_{\mathscr{O}}))\right]\right],
\end{aligned}$$

which can be written as

$$\begin{aligned}
\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_{\mathscr{O}}}g(Z(\tau_{\mathscr{O}}))\right] &= \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_U}g(Z(\tau_U))\mathbf{1}_{\{\tau_U=\tau_{\mathscr{O}}\}}\right] \\
&\quad + \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_U}\mathbb{E}^{Z(\tau_U)}_{\mathbb{Q}}\left[e^{-r\tau_{\mathscr{O}}}g(Z(\tau_{\mathscr{O}}))\right]\mathbf{1}_{\{\tau_U<\tau_{\mathscr{O}}\}}\right].
\end{aligned} \tag{4.3.18}$$

Similarly, we have for the integral term

$$\begin{aligned}
\mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{\tau_{\mathscr{O}}} e^{-rt}f(Z(t))dt\right] &= \mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{\tau_U} e^{-rt}f(Z(t))dt\right] \\
&\quad + \mathbb{E}^z_{\mathbb{Q}}\left[\mathbf{1}_{\{\tau_U<\tau_{\mathscr{O}}\}}\int_{\tau_U}^{\tau_{\mathscr{O}}} e^{-rt}f(Z(t))dt\right],
\end{aligned}$$

and so, by conditioning the second term in the preceding equality on $\mathscr{F}(\tau_U)$ and applying the strong Markov property, we have

$$\begin{aligned}
&\mathbb{E}^z_{\mathbb{Q}}\left[\mathbf{1}_{\{\tau_U<\tau_{\mathscr{O}}\}}\int_{\tau_U}^{\tau_{\mathscr{O}}} e^{-rt}f(Z(t))dt\right] \\
&= \mathbb{E}^z_{\mathbb{Q}}\left[\mathbb{E}^z_{\mathbb{Q}}\left[\mathbf{1}_{\{\tau_U\leq\tau_{\mathscr{O}}\}}\int_{\tau_U}^{\tau_{\mathscr{O}}} e^{-rt}f(Z(t))dt\right]|\mathscr{F}(\tau_U)\right] \\
&= \mathbb{E}^z_{\mathbb{Q}}\left[\mathbf{1}_{\{\tau_U<\tau_{\mathscr{O}}\}}e^{-r\tau_U}\mathbb{E}^{Z(\tau_U)}_{\mathbb{Q}}\left[\int_0^{\tau_{\mathscr{O}}} e^{-rt}f(Z(t))dt\right]\right].
\end{aligned} \tag{4.3.19}$$

Combining (4.3.18) and (4.3.19) in (4.1.19), we obtain

$$\begin{aligned}
u(z) &= \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_U}g(Z(\tau_U))\mathbf{1}_{\{\tau_U=\tau_{\mathscr{O}}\}}\right] + \mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{\tau_U} e^{-rt}f(Z(t))dt\right] \\
&\quad + \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_U}\mathbf{1}_{\{\tau_U<\tau_{\mathscr{O}}\}}\mathbb{E}^{Z(\tau_U)}_{\mathbb{Q}}\left[e^{-r\tau_{\mathscr{O}}}g(Z(\tau_{\mathscr{O}})) + \int_0^{\tau_{\mathscr{O}}} e^{-rt}f(Z(t))dt\right]\right].
\end{aligned}$$

Using again (4.1.19) for $u^*(Z(\tau_U))$, the preceding equality yields (4.3.17). This completes the proof of Claim 4.3.2. □

By [41, Theorem 6.13] and a straightforward extension of Theorem C.3.10 from domains with C^2 to domains with regular boundary, as in [23, §6.2.6.A], the integral term in (4.3.17) is the solution on U of the uniformly elliptic partial differential equation $Au^* = f$ with homogeneous Dirichlet boundary condition, and it is a continuous function up to ∂U. Notice that ∂U satisfies the exterior sphere condition and thus ∂U

is regular by Proposition C.3.6 (see Definition C.3.2 for the definition of regular points of ∂U). The continuity of the non-integral term in (4.3.17) at z_0 follows from Corollary C.3.9, as $\hat{g}$ is continuous at z_0 by hypotheses.

It remains to show that, when $0 < \beta < 1$, the solution u^* is continuous up to $\bar{\Gamma}_0$.

Case 2 ($u^* \in C_{\text{loc}}(\mathscr{O} \cup \bar{\Gamma}_0)$, for all $0 < \beta < 1$)**.** Let $z_0 = (x_0, 0) \in \bar{\Gamma}_0$. We denote by θ^z the first time the process started at $z = (x, y) \in \mathscr{O}$ hits $y = 0$. Obviously, we have

$$\tau^z_{\mathscr{O}} \leq \theta^z \leq T^y_0 \quad \text{a.s.,} \tag{4.3.20}$$

where T^y_0 is given by (4.2.11). For $\beta \in (0,1)$, it follows from (4.2.12) and the preceding inequality between stopping times, that θ^z converges to 0, as y goes to 0, uniformly in $x \in \mathbb{R}$. Therefore, the integral term in (4.3.17) converges to zero. Next, we want to show that the non-integral term in (4.3.17) converges to $g(z_0)$. We rewrite that term as

$$\begin{aligned} \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_{\mathscr{O}}} g(Z(\tau_{\mathscr{O}}))\right] - g(z_0) = \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_{\mathscr{O}}} \left(g(Z(\tau_{\mathscr{O}})) - g(z_0)\right)\right] \\ + g(z_0)\left(1 - \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_{\mathscr{O}}}\right]\right). \end{aligned} \tag{4.3.21}$$

From the observation that $\tau^z_{\mathscr{O}} \leq \theta^z$ a.s., we see that

$$\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_{\mathscr{O}}}\right] \to 1, \text{ as } z \to z_0. \tag{4.3.22}$$

By (4.3.21), it remains to show that $\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_{\mathscr{O}}} \left(g(Z(\tau_{\mathscr{O}})) - g(z_0)\right)\right]$ converges to zero, as $z \in \mathscr{O}$ converges to z_0. We fix $\varepsilon > 0$ and choose $\delta_1 > 0$ such that

$$|g(z) - g(z_0)| < \varepsilon, \quad \forall z \in B(z_0, \delta_1) \cap \partial\mathscr{O}. \tag{4.3.23}$$

From [47, Equation (5.3.18) in Problem 5.3.15], there is a positive constant C_1, depending on z_0 and δ_1, such that

$$\sup_{z \in B(z_0,\delta_1)\cap\mathscr{O}} \mathbb{E}^z_{\mathbb{Q}}\left[\sup_{0 \leq s \leq t} |Z(s) - z|\right] \leq C_1\sqrt{t},$$

from where it follows

$$\sup_{z \in B(z_0,\delta_1)\cap\mathscr{O}} \mathbb{Q}^z\left(\sup_{0 \leq s \leq t} |Z(s) - z| > \delta_1/2\right) \leq \frac{2C_1\sqrt{t}}{\delta_1}. \tag{4.3.24}$$

Next, we choose $t > 0$ sufficiently small such that

$$\frac{2C_1\sqrt{t}}{\delta_1} < \varepsilon, \tag{4.3.25}$$

and, by (4.3.20) and (4.2.12), we may choose $\delta_2 > 0$ sufficiently small such that

$$\mathbb{Q}\left(T_0^{\delta_2} > t\right) < \varepsilon. \tag{4.3.26}$$

Let $\delta := \min\{\delta_1/2, \delta_2\}$. We rewrite

$$\begin{aligned} e^{-r\tau_{\mathscr{O}}}\left(g(Z(\tau_{\mathscr{O}})) - g(z_0)\right) = e^{-r\tau_{\mathscr{O}}}&\left(g(Z(\tau_{\mathscr{O}})) - g(z_0)\right)\mathbf{1}_{\{\tau_{\mathscr{O}}\leq t\}} \\ &+ e^{-r\tau_{\mathscr{O}}}\left(g(Z(\tau_{\mathscr{O}})) - g(z_0)\right)\mathbf{1}_{\{\tau_{\mathscr{O}}>t\}} \end{aligned}$$

to give

$$\begin{aligned} e^{-r\tau_{\mathscr{O}}}\left(g(Z(\tau_{\mathscr{O}})) - g(z_0)\right) = e^{-r\tau_{\mathscr{O}}}&\left(g(Z(\tau_{\mathscr{O}})) - g(z_0)\right)\mathbf{1}_{\{\tau_{\mathscr{O}}\leq t, \sup_{0\leq s\leq t}|Z(s)-z|<\delta_1/2\}} \\ &+ e^{-r\tau_{\mathscr{O}}}\left(g(Z(\tau_{\mathscr{O}})) - g(z_0)\right)\mathbf{1}_{\{\tau_{\mathscr{O}}\leq t, \sup_{0\leq s\leq t}|Z(s)-z|\geq\delta_1/2\}} \\ &+ e^{-r\tau_{\mathscr{O}}}\left(g(Z(\tau_{\mathscr{O}})) - g(z_0)\right)\mathbf{1}_{\{\tau_{\mathscr{O}}>t\}} \end{aligned} \tag{4.3.27}$$

By (4.3.23), we have for all $z \in B(z_0, \delta) \cap \mathscr{O}$

$$\mathbb{E}_{\mathbb{Q}}^{z}\left[\left|g(Z(\tau_{\mathscr{O}})) - g(z_0)\right|\mathbf{1}_{\{\tau_{\mathscr{O}}\leq t,\ \sup_{0\leq s\leq t}|Z(s)-z|<\delta_1/2\}}\right] < \varepsilon. \tag{4.3.28}$$

We choose $p > 1$ such that $pM_1 < \mu$ and $pM_2 < 1$. Notice that this is possible because we assumed the coefficients $M_1 < \mu$ and $M_2 < 1$. Then, from the growth estimate (4.1.16) for g, we have

$$\left|e^{-r\tau_{\mathscr{O}}}g(Z(\tau_{\mathscr{O}}))\right|^p \leq Ce^{-rp\tau_{\mathscr{O}}}\left(1 + e^{pM_1Y(\tau_{\mathscr{O}})} + e^{pM_2X(\tau_{\mathscr{O}})}\right).$$

From the inequality (4.2.14) with $c = pM_1 < \mu$ and property (4.2.13) applied with $c = pM_2 \in (0, 1)$, we obtain using the condition $M_1 \leq r/(\kappa\vartheta)$

$$\mathbb{E}_{\mathbb{Q}}^{z}\left[\left|e^{-r\tau_{\mathscr{O}}}g(Z(\tau_{\mathscr{O}}))\right|^p\right] \leq C\left(1 + e^{pM_1y} + e^{pM_2x}\right).$$

Let $C_2 > 0$ be an bound on the right-hand side of the preceding inequality, for all $z = (x, y) \in B(z_0, \delta) \cap \mathscr{O}$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\left|\mathbb{E}_{\mathbb{Q}}^{z}\left[e^{-r\tau_{\mathscr{O}}}\left(g(Z(\tau_{\mathscr{O}})) - g(z_0)\right)\mathbf{1}_{\{\tau_{\mathscr{O}}>t\}}\right]\right| \\ &\quad\leq \mathbb{E}_{\mathbb{Q}}^{z}\left[e^{-rp\tau_{\mathscr{O}}}\left|g(Z(\tau_{\mathscr{O}})) - g(z_0)\right|^p\right]^{1/p}\mathbb{Q}^z\left(\tau_{\mathscr{O}} > t\right)^{1/p'}, \end{aligned}$$

where $p' > 1$ denotes the conjugate exponent of p. Using the fact that $\tau^z_{\mathscr{O}} \leq T_0^{\delta_2}$ from (4.3.20) and (4.3.26), we obtain in the preceding inequality

$$\begin{aligned}\left|\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_{\mathscr{O}}}\left(g(Z(\tau_{\mathscr{O}}))-g(z_0)\right)\mathbf{1}_{\{\tau_{\mathscr{O}}>t\}}\right]\right| &\leq 2C_2^{1/p}\mathbb{Q}^z\left(T_0>t\right)^{1/p'}\\ &\leq 2C_2^{1/p}\varepsilon^{1/p'}, \quad \forall z\in B(z_0,\delta)\cap\mathscr{O},\end{aligned} \tag{4.3.29}$$

From the inequality,

$$\begin{aligned}&\left|\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_{\mathscr{O}}}\left(g(Z(\tau_{\mathscr{O}}))-g(z_0)\right)\mathbf{1}_{\{\tau_{\mathscr{O}}\leq t,\sup_{0\leq s\leq t}|Z(s)-z|\geq\delta_1/2\}}\right]\right|\\ &\quad\leq \mathbb{E}^z_{\mathbb{Q}}\left[e^{-rp\tau_{\mathscr{O}}}\left|g(Z(\tau_{\mathscr{O}}))-g(z_0)\right|^p\right]^{1/p}\mathbb{Q}^z\left(\sup_{0\leq s\leq t}|Z(s)-z|\geq\delta_1/2\right)^{1/p'},\end{aligned}$$

the inequalities (4.3.24) and (4.3.25) and definition of C_2 yield

$$\left|\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_{\mathscr{O}}}\left(g(Z(\tau_{\mathscr{O}}))-g(z_0)\right)\mathbf{1}_{\{\tau_{\mathscr{O}}\leq t,\sup_{0\leq s\leq t}|Z(s)-z|\geq\delta_1/2\}}\right]\right| \leq 2C_2^{1/p}\varepsilon^{1/p'}. \tag{4.3.30}$$

Substituting (4.3.28), (4.3.29), and (4.3.30) in (4.3.27), we obtain

$$\mathbb{E}\left[e^{-r\tau_{\mathscr{O}}}\left(g(Z(\tau_{\mathscr{O}}))-g(z_0)\right)\right] < \left(1+4C_2^{1/p}\right)\left(\varepsilon+\varepsilon^{1/p'}\right), \quad \forall z\in B(z_0,\delta)\cap\mathscr{O},$$

and so u^* is continuous up to $\bar{\Gamma}_0$, when $0<\beta<1$.

This concludes the proof that $u^* \in C_{\mathrm{loc}}(\mathscr{O}\cup\partial_\beta\mathscr{O})$, for all $\beta>0$.

This completes the proof of Theorem 4.3.1. □

We now prove *existence* of solutions to problem (4.3.2) when the boundary data g is *Hölder continuous* on suitable portions of the boundary of $\mathscr{O}$.

Theorem 4.3.3 (Existence of solutions to the elliptic boundary value problem (4.3.2) with Hölder continuous Dirichlet boundary condition)**.** *In addition to the hypotheses of Theorem 4.1.2, let $\mathscr{O}\subset\mathbb{H}$ be a domain such that the boundary portion Γ_1 is of class $C^{2+\alpha}$, that $f\in C^\alpha_{\mathrm{loc}}(\mathscr{O}\cup\Gamma_1)$ and $g\in C^{2+\alpha}_{\mathrm{loc}}(\mathscr{O}\cup\Gamma_1)$.*

1. *If $\beta\geq 1$, then u^*, given by* (4.1.19)*, is a solution to problem* (4.1.1) *with boundary condition* (4.1.3) *along Γ_1. In particular,*

$$u^*\in C^{2+\alpha}_{\mathrm{loc}}(\mathscr{O}\cup\Gamma_1)$$

and satisfies the growth assumption (4.1.16).

2. *If* $0 < \beta < 1$ *and* $g \in C_{\mathrm{loc}}(\partial\mathscr{O})$, *then* u^*, *given by* (4.1.19), *is a solution to problem* (4.1.1) *with boundary condition* (4.1.5) *along* $\partial\mathscr{O}$. *In particular,*

$$u^* \in C_{\mathrm{loc}}(\bar{\mathscr{O}}) \cap C^{2+\alpha}(\mathscr{O} \cup \Gamma_1)$$

and satisfies the growth assumption (4.1.16).

Proof. The proof of the theorem is the same as that of Theorem 4.3.1, with the exception that Case 1 of Step 2 can be simplified by applying the classical boundary Schauder estimates. Also, instead of using the sequence of subdomains $\{\mathscr{O}_k : k \in \mathbb{N}\}$ precompactly contained in $\mathscr{O}$, as in the proof of Theorem 4.1.2, we consider an increasing sequence, $\{\mathscr{D}_k : k \in \mathbb{N}\}$, of $C^{2+\alpha}$ subdomains of $\mathscr{O}$ (see [41, Definition §6.2]) such that each $\mathscr{D}_k$ satisfies

$$\mathscr{O} \cap (-k, k) \times (1/k, k) \subset \mathscr{D}_k \subset \mathscr{O} \cap (-2k, 2k) \times (1/(2k), 2k), \quad \forall k \in \mathbb{N}, \tag{4.3.31}$$

and

$$\bigcup_{k \in \mathbb{N}} \mathscr{D}_k = \mathscr{O}.$$

Since Γ_1 is assumed to be of class $C^{2+\alpha}$, we may choose $\mathscr{D}_k$ to be of class $C^{2+\alpha}$.

Let $z_0 \in \Gamma_1$, and $r_0 > 0$ small enough such that $B(z_0, r_0) \cap \Gamma_0 = \varnothing$. Let

$$D := B(z_0, r_0) \cap \mathscr{O} \text{ and } D' := B(z_0, r_0/2) \cap \mathscr{O}.$$

By (4.3.31), we may choose $k_0 \in \mathbb{N}$ large enough such that $D \subset \mathscr{D}_k$, for all $k \geq k_0$. Using $f \in C^\alpha(\bar{D})$, $g \in C^{2+\alpha}(\bar{D})$ and applying [41, Corollary 6.7], and the fact that u_k solves (4.3.15)

$$\|u_k\|_{C^{2+\alpha}(\bar{D}')} \leq C \left(\|u_k\|_{C(\bar{D})} + \|\tilde{g}\|_{C^{2+\alpha}(\bar{D})} + \|f\|_{C^\alpha(\bar{D})} \right), \quad \forall k \geq k_0, \tag{4.3.32}$$

where $C > 0$ is a constant depending only on the coefficients of A, and the domains D and D'. Combining the preceding inequality with the uniform bound on the $C(\bar{D})$ norms of the sequence $\{u_k : k \in \mathbb{N}\}$, resulting from Lemma 4.2.12, the compactness of the embedding of $C^{2+\alpha}(\bar{D}') \hookrightarrow C^{2+\gamma}(\bar{D}')$, when $0 \leq \gamma < \alpha$, implies that a subsequence converges strongly to u^*. Therefore, $u^* \in C^{2+\gamma}(\bar{D}')$, and $Au^* = f$ on D' and $u^* = g$

on $\partial D' \cap \Gamma_1$. Moreover, $u^* \in C^{2+\alpha}(\bar{D}')$, since $u_k \in C^{2+\alpha}(\bar{D}')$, for all $k \geq k_0$, and the sequence converges in $C^2(\bar{D}')$ to u^*. Combining the boundary estimate (4.3.32) with Step 1 in the proof of Theorem 4.3.1, we obtain the conclusion that $u^* \in C^{2+\alpha}_{\mathrm{loc}}(\mathscr{O} \cup \Gamma_1)$. $\square$

Remark 4.3.4 (Validity of the stochastic representation for strong solutions)**.** The stochastic representation (4.1.21) for solutions to problem (4.1.1) with boundary condition along Γ_1 is valid if we replace the condition $u \in C_{\mathrm{loc}}(\mathscr{O} \cup \Gamma_1) \cap C^2(\mathscr{O}) \cap C^{1,1}_{s,\mathrm{loc}}(\mathscr{O} \cup \Gamma_0)$ in the hypotheses of Theorem 4.1.5, with the weaker condition $u \in C_{\mathrm{loc}}(\mathscr{O} \cup \Gamma_1) \cap W^{2,2}_{\mathrm{loc}}(\mathscr{O}) \cap C^{1,1}_{s,\mathrm{loc}}(\mathscr{O} \cup \Gamma_0)$.

4.4 Elliptic obstacle problem

This section contains the proofs of Theorems 4.1.8 and 4.1.9. As in problem (4.3.2), the questions of *uniqueness* of solutions to problem (4.1.2) with Dirichlet boundary condition along Γ_1, when $\beta \geq 1$, and along $\partial\mathscr{O}$, when $0 < \beta < 1$, are similar in nature. We can conveniently treat them together as

$$\begin{cases} \min\{Au - f, u - \psi\} = 0 & \text{on } \mathscr{O}, \\ u = g & \text{on } \partial_\beta \mathscr{O}, \end{cases} \tag{4.4.1}$$

where $\partial_\beta \mathscr{O}$ is given by (4.3.1).

Proof of Theorem 4.1.8. Lemma 4.2.12 indicates that u^* given by (4.1.23) satisfies (4.1.16), so the growth assumption on u in Theorem 4.1.8 is justified.

By the preceding remarks, it suffices to prove that the stochastic representation (4.1.23) holds for solutions $u \in C_{\mathrm{loc}}(\mathscr{O} \cup \partial_\beta \mathscr{O}) \cap C^2(\mathscr{O})$ to problem (4.4.1). We consider the two situations: $u \geq u^*$ and $u \leq u^*$ on $\mathscr{O} \cup \partial_\beta \mathscr{O}$, where u^* is defined by (4.1.23).

Step 1 (Proof that $u \geq u^*$ on $\mathscr{O} \cup \partial_\beta \mathscr{O}$)**.** Let $\{\mathscr{O}_k : k \in \mathbb{N}\}$ be an increasing sequence of $C^{2+\alpha}$ subdomains of $\mathscr{O}$ as in the proof of Theorem 4.1.2. Since $u \in C^2(\mathscr{O})$, Itô's lemma (Theorem C.2.1) yields, for any stopping time $\theta \in \mathscr{T}$,

$$\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r(\theta \wedge \tau_{\mathscr{O}_k})} u(Z(\theta \wedge \tau_{\mathscr{O}_k}))\right] = u(z) - \mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{\theta \wedge \tau_{\mathscr{O}_k}} e^{-rs} Au(Z(s)) ds\right]. \tag{4.4.2}$$

By splitting the right-hand side in the preceding identity,

$$\begin{aligned}
&\mathbb{E}^z_{\mathbb{Q}}\left[e^{-r(\theta\wedge\tau_{\mathscr{O}_k})}u(Z(\theta\wedge\tau_{\mathscr{O}_k}))\right]\\
&\qquad = \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_{\mathscr{O}_k}}u(Z(\theta\wedge\tau_{\mathscr{O}_k}))\mathbf{1}_{\{\tau_{\mathscr{O}_k}\leq\theta\}}\right] + \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\theta}u(Z(\theta\wedge\tau_{\mathscr{O}_k}))\mathbf{1}_{\{\tau_{\mathscr{O}_k}>\theta\}}\right],
\end{aligned}$$

and using $u \geq \psi$ on $\mathscr{O}$ and $Au \geq f$ a.e. on $\mathscr{O}$, the identity (4.4.2) gives

$$\begin{aligned}
u(z) \geq\ & \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\theta}\psi(Z(\theta))\mathbf{1}_{\{\theta<\tau_{\mathscr{O}_k}\}}\right]\\
&+ \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_{\mathscr{O}_k}}u(Z(\tau_{\mathscr{O}_k}))\mathbf{1}_{\{\tau_{\mathscr{O}_k}\leq\theta\}}\right] + \mathbb{E}^z\left[\int_0^{\theta\wedge\tau_{\mathscr{O}_k}} e^{-rs}f(Z(s))\,ds\right].
\end{aligned} \tag{4.4.3}$$

Just as in the proof of Theorem 4.1.2, the collections of random variables

$$\left\{e^{-r\theta}\psi(Z(\theta))\mathbf{1}_{\{\theta<\tau_{\mathscr{O}_k}\}} : k\in\mathbb{N}\right\} \text{ and } \left\{e^{-r\tau_{\mathscr{O}_k}}u(Z(\tau_{\mathscr{O}_k}))\mathbf{1}_{\{\tau_{\mathscr{O}_k}\leq\theta\}} : k\in\mathbb{N}\right\}$$

are uniformly integrable because u and ψ satisfy the pointwise growth estimate (4.1.16). From the continuity of u and ψ on $\mathscr{O}\cup\partial_\beta\mathscr{O}$, we also have the a.s. convergence,

$$\begin{aligned}
&e^{-r\theta}\psi(Z(\theta))\mathbf{1}_{\{\theta<\tau_{\mathscr{O}_k}\}} \to e^{-r\theta}\psi(Z(\theta))\mathbf{1}_{\{\theta<\tau_{\mathscr{O}}\}}, \quad \text{as } k\to\infty,\\
&e^{-r\tau_{\mathscr{O}_k}}u(Z(\tau_{\mathscr{O}_k}))\mathbf{1}_{\{\tau_{\mathscr{O}_k}\leq\theta\}} \to e^{-r\tau_{\mathscr{O}}}u(Z(\tau_{\mathscr{O}}))\mathbf{1}_{\{\tau_{\mathscr{O}}\leq\theta\}}, \quad \text{as } k\to\infty.
\end{aligned}$$

Therefore, by [10, Theorem 16.13], we can take limit as k tends to ∞ in inequality (4.4.3) to obtain, for all $\theta\in\mathscr{T}$,

$$\begin{aligned}
u(z) \geq\ & \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\theta}\psi(Z(\theta))\mathbf{1}_{\{\theta<\tau_{\mathscr{O}}\}}\right] + \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_{\mathscr{O}}}u(Z(\tau_{\mathscr{O}}))\mathbf{1}_{\{\tau_{\mathscr{O}}\leq\theta\}}\right]\\
&+ \mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{\theta\wedge\tau_{\mathscr{O}}} e^{-rs}f(Z(s))\,ds\right],
\end{aligned}$$

which yields $u \geq u^*$ on $\mathscr{O}\cup\partial_\beta\mathscr{O}$.

Step 2 (Proof that $u \leq u^*$ on $\mathscr{O}\cup\partial_\beta\mathscr{O}$**).** The continuation region,

$$\mathscr{C} := \{u > \psi\}, \tag{4.4.4}$$

is an open set by the continuity of u and ψ. We denote the first exit time of $Z^{t,z}$ from the continuation region, $\mathscr{C}$, by

$$\tilde{\tau}^{t,z} := \left\{s \geq t : Z^{t,z}(s) \notin \mathscr{C}\right\}, \tag{4.4.5}$$

and write $\tilde{\tau} = \tilde{\tau}^{t,z}$ for brevity. This is indeed a stopping time because the process $Z^{t,z}$ is continuous and $\mathscr{C}$ is open. By the same argument used in Step 1 with θ replaced by $\tilde{\tau}$, we obtain that all inequalities hold with equalities because $u(Z(\tilde{\tau})) = \psi(Z(\tilde{\tau}))$ and $Au = f$ on the continuation region, $\mathscr{C}$. Therefore,

$$\begin{aligned} u(z) = \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tilde{\tau}}\psi(Z(\tilde{\tau}))\mathbf{1}_{\{\tilde{\tau}<\tau_{\mathscr{O}}\}}\right] + \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r\tau_{\mathscr{O}}}g(Z(\tau_{\mathscr{O}}))\mathbf{1}_{\{\tau_{\mathscr{O}}\leq\tilde{\tau_{\mathscr{O}}}\}}\right] \\ + \mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{\tilde{\tau}\wedge\tau_{\mathscr{O}}} e^{-rs}f(Z(s))\,ds\right], \end{aligned}$$

which implies that $u \leq u^*$.

By combining the preceding two steps, we obtain the stochastic representation (4.1.23) of solutions to problem (4.4.1), and hence the uniqueness assertion. $\square$

Proof of Theorem 4.1.9. Lemma 4.2.12 indicates that u^* given by (4.1.24) satisfies (4.1.16), so the growth assumption on u in Theorem 4.1.8 is justified.

Our goal is to show that if $0 < \beta < 1$ and $u \in C_{\text{loc}}(\mathscr{O}\cup\Gamma_1)\cap C^2(\mathscr{O})\cap C^{1,1}_{s,\text{loc}}(\mathscr{O}\cup\Gamma_0)$ is a solution to problem (4.1.2) with Dirichlet boundary condition (4.1.4) along Γ_1, and satisfying the growth estimate (4.1.16), then it admits the stochastic representation (4.1.24). As in the proof of Theorem 4.1.8, we consider the following two cases.

Step 1 (Proof that $u \geq u^*$ on $\mathscr{O}\cup\Gamma_1$)**.** Let $\varepsilon > 0$ and $\{\mathscr{U}_k : k \in \mathbb{N}\}$ be the collection of increasing subdomains as in (4.3.5). By applying Itô's lemma, we obtain, for all $t > 0$ and $\theta \in \mathscr{T}$,

$$u(z) = \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r(t\wedge\nu_{\mathscr{U}_k}\wedge\theta)}u(Z^\varepsilon(t\wedge\nu_{\mathscr{U}_k}\wedge\theta))\right] + \mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{t\wedge\nu_{\mathscr{U}_k}\wedge\theta} e^{-rs}A^\varepsilon u(Z^\varepsilon(s))ds\right], \tag{4.4.6}$$

where $\nu_{\mathscr{U}_k}$ is given by (4.1.18) and Z^ε is defined in (4.3.6), and A^ε is defined by (4.3.8). By (4.1.2) and (4.3.8), preceding identity gives

$$\begin{aligned} u(z) \geq \mathbb{E}^z_{\mathbb{Q}}\left[e^{-r(t\wedge\nu_{\mathscr{U}_k}\wedge\theta)}u(Z^\varepsilon(t\wedge\nu_{\mathscr{U}_k}\wedge\theta))\right] \\ + \mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{t\wedge\nu_{\mathscr{U}_k}\wedge\theta} e^{-rs}f(Z^\varepsilon(s))ds\right] + \mathbb{E}^z_{\mathbb{Q}}\left[\int_0^{t\wedge\nu_{\mathscr{U}_k}\wedge\theta} e^{-rs}(A^\varepsilon - A)u(Z^\varepsilon(s))ds\right]. \end{aligned} \tag{4.4.7}$$

First, we take the limit as ε tends to 0 in (4.4.7). We can assume without loss of generality that $\varepsilon < 1/k$, for any fixed $k \in \mathbb{N}$. The residual term $(A^\varepsilon - A)u$ then obeys estimate (4.3.10) because $u \in C^{1,1}_{s,\mathrm{loc}}(\mathscr{O} \cup \Gamma_0)$. Therefore, (4.3.11) also holds in the present case. In addition, using the continuity of f, u, Du and yD^2u on compact subsets of $\mathscr{O} \cup \Gamma_0$, we see that (4.3.12) holds, and so, by taking limit as $\varepsilon \downarrow 0$ in (4.4.7),

$$u(z) \geq \mathbb{E}^z_{\mathbb{Q}} \left[e^{-r(t \wedge \nu_{\mathscr{U}_k} \wedge \theta)} u(Z(t \wedge \nu_{\mathscr{U}_k} \wedge \theta)) \right] + \mathbb{E}^z_{\mathbb{Q}} \left[\int_0^{t \wedge \nu_{\mathscr{U}_k} \wedge \theta} e^{-rs} f(Z(s)) ds \right]. \quad (4.4.8)$$

Finally, letting k and t tend to ∞ and using the convergence (4.3.14), the same argument employed in the proof of Theorem 4.1.2 can be applied to conclude that $u \geq u^*$ on $\mathscr{O} \cup \Gamma_1$, where u^* is given by (4.1.24).

Step 2 (Proof that $u \leq u^*$ on $\mathscr{O} \cup \Gamma_1$)**.** We choose $\theta = \tilde{\tau}$ in the preceding step, where $\tilde{\tau}$ is defined by (4.4.5). By the definition (4.4.4) of the continuation region, $\mathscr{C}$, and the obstacle problem (4.1.2), we notice that inequalities (4.4.7) and (4.4.8) hold with equality and so it follows as in Step 1 that $u \leq u^*$ on $\mathscr{O} \cup \Gamma_1$.

This completes the proof. □

Remark 4.4.1 (Validity of the stochastic representation for strong solutions)**.** The stochastic representation (4.1.23) of solutions to problem (4.4.1), when $\beta > 0$, holds under the weaker assumption that $u \in C_{\mathrm{loc}}(\mathscr{O} \cup \partial_\beta \mathscr{O}) \cap W^{2,2}_{\mathrm{loc}}(\mathscr{O})$. Similarly, the stochastic representation (4.1.24) of solutions to problem (4.1.2) with Dirichlet boundary condition (4.1.4) along Γ_1, when $0 < \beta < 1$, holds under the weaker assumption that $u \in C_{\mathrm{loc}}(\mathscr{O} \cup \Gamma_1) \cap C^{1,1}_{s,\mathrm{loc}}(\mathscr{O} \cup \Gamma_0) \cup W^{2,2}_{\mathrm{loc}}(\mathscr{O})$. In each case, we would replace the application of the classical Itô lemma (Theorem C.2.1) with [8, Identity (8.62) in Theorem 2.8.5], or we could apply an approximation argument involving $C^2(\mathscr{O})$ functions.

4.5 Parabolic terminal/boundary value problem

This section contains the proofs of Theorems 4.1.12 and 4.1.15 and an *existence* result in Theorem 4.5.4. Because the Heston process satisfies the strong Markov property, it suffices to prove the stochastic representation of solutions to the terminal value problem for T as small as we like. In particular, without loss of generality, we can choose T such that

Hypothesis 4.5.1. There is a constant $p_0 > 1$ such that

1. Condition (4.2.20) in Lemma 4.2.14 is satisfied for $p := p_0 M_2$, where $M_2 \in [0,1]$ is the constant appearing in (4.1.25);

2. One has $p_0 M_1 \leq \mu$, where $M_1 \in [0, \mu)$ in (4.1.25).

As in §4.3, we first prove *uniqueness* of solutions to the parabolic boundary value problems (4.1.7) with different possible Dirichlet boundary conditions depending on the parameter β. The proofs are similar those of Theorems 4.1.2 and 4.1.5.

The *existence* and *uniqueness* of solutions to problem (4.1.7) with boundary condition (4.1.9), when $\beta \geq 1$, and with boundary condition (4.1.12), when $0 < \beta < 1$, are similar in nature. By analogy with our treatment of problem (4.3.2), we define

$$\eth_\beta Q := \begin{cases} \eth^1 Q & \text{if } \beta \geq 1, \\ \eth Q & \text{if } 0 < \beta < 1, \end{cases} \tag{4.5.1}$$

where we recall that $Q := (0,T) \times \mathscr{O}$. The preceding problems can then be formulated as

$$-u_t + Au = f \quad \text{on } Q, \tag{4.5.2}$$

$$u = g \quad \text{on } \eth_\beta Q. \tag{4.5.3}$$

We now have the

Proof of Theorem 4.1.12. We choose $T > 0$ small enough and $p_0 > 1$ as in Hypothesis 4.5.1. The pattern of the proof is the same as that of Theorem 4.1.2. For completeness, we outline the main steps of the argument.

We need to show that if $u \in C_{\text{loc}}(Q \cup \eth_\beta Q) \cap C^2(Q)$ is a solution to problem (4.5.2), satisfying the growth bound (4.1.25), then it admits the stochastic representation (4.1.26). We choose a collection of increasing subdomains, $\{\mathscr{O}_k : k \in \mathbb{N}\}$, as in the proof of Theorem 4.1.2. By applying Itô's lemma (Theorem C.2.1), we obtain, for all $t > 0$ and $k \in \mathbb{N}$,

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}^{t,z}\left[e^{-r(\tau_{\mathscr{O}_k} \wedge T - t)} u(\tau_{\mathscr{O}_k} \wedge T, Z(\tau_{\mathscr{O}_k} \wedge T))\right] \\ &\quad = u(t,z) - \mathbb{E}_{\mathbb{Q}}^{t,z}\left[\int_t^{\tau_{\mathscr{O}_k} \wedge T} e^{-r(s-t)} f(s, Z(s)) ds\right]. \end{aligned} \tag{4.5.4}$$

We now take limit as k tends to ∞ in the preceding identity. Using (4.1.25) and Lemma 4.2.11, we obtain

$$\mathbb{E}_{\mathbb{Q}}^{t,z}\left[\int_t^{\tau_{\mathscr{O}_k}\wedge T} e^{-r(s-t)} f(s,Z(s))ds\right] \to \mathbb{E}_{\mathbb{Q}}^{t,z}\left[\int_t^{\tau_{\mathscr{O}}\wedge T} e^{-r(s-t)} f(s,Z(s))ds\right], \text{ as } k\to\infty. \tag{4.5.5}$$

From the continuity of u and of the sample paths of Z, we obtain the a.s. convergence as k tends to ∞,

$$e^{-r(\tau_{\mathscr{O}_k}\wedge T-t)} u(\tau_{\mathscr{O}_k}\wedge T, Z(\tau_{\mathscr{O}_k}\wedge T)) \to e^{-r(\tau_{\mathscr{O}}\wedge T)} g(\tau_{\mathscr{O}}\wedge T, Z(\tau_{\mathscr{O}}\wedge T)).$$

In order to prove that, as k tends to ∞,

$$\mathbb{E}_{\mathbb{Q}}^{t,z}\left[e^{-r(\tau_{\mathscr{O}_k}\wedge T-t)} u(\tau_{\mathscr{O}_k}\wedge T, Z(\tau_{\mathscr{O}_k}\wedge T))\right] \to \mathbb{E}_{\mathbb{Q}}^{t,z}\left[e^{-r(\tau_{\mathscr{O}}\wedge T)} g(\tau_{\mathscr{O}}\wedge T, Z(\tau_{\mathscr{O}}\wedge T))\right], \tag{4.5.6}$$

using [10, Theorem 16.13], it is enough to show that the collection of random variables,

$$\left\{e^{-r(\tau_{\mathscr{O}_k}\wedge T-t)} u(\tau_{\mathscr{O}_k}\wedge T, Z(\tau_{\mathscr{O}_k}\wedge T)) : k\in\mathbb{N}\right\} \tag{4.5.7}$$

is uniformly integrable. For $p_0>1$ as in Hypothesis 4.5.1, it is enough to show that their p_0-th order moments are uniformly bounded ([10, Observation following Equation (16.23)]), that is

$$\sup_{k\in\mathbb{N}} \mathbb{E}_{\mathbb{Q}}^{t,z}\left[\left|e^{-r\tau_{\mathscr{O}_k}} u(\tau_{\mathscr{O}_k}, Z(\tau_{\mathscr{O}_k}))\mathbf{1}_{\{\tau_{\mathscr{O}_k}<T\}}\right|^{p_0}\right] < +\infty. \tag{4.5.8}$$

From (4.1.25), we have, for some constant C,

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}^{t,z}\left[\left|e^{-r(\tau_{\mathscr{O}_k}\wedge T-t)} u(\tau_{\mathscr{O}_k}\wedge T, Z(\tau_{\mathscr{O}_k}\wedge T))\right|^{p_0}\right]\\ &\quad\le C\left(1+\mathbb{E}_{\mathbb{Q}}^{t,z}\left[e^{p_0M_1Y(\tau_{\mathscr{O}_k}\wedge T)}\right]+\mathbb{E}_{\mathbb{Q}}^{t,z}\left[e^{p_0M_2X(\tau_{\mathscr{O}_k}\wedge T)}\right]\right).\end{aligned}$$

Now, the uniform bound in (4.5.8) follows by applying the supermartingale property (4.2.14) with $c:=p_0M_1$ to the first expectation in the preceding inequality, and by applying (4.2.21) with $p:=p_0M_2$ to the second expectation above. Therefore, by taking the limit as k tends to ∞ in (4.5.4), with the aid of (4.5.5) and (4.5.6), we obtain the stochastic representation (4.1.26) of solutions to problem (4.5.2). □

Proof of Theorem 4.1.15. The need is to show that if $0<\beta<1$ and $u\in C_{\mathrm{loc}}(Q\cup \eth^1 Q)\cap C^2(Q)\cap C^{1,1}_{s,\mathrm{loc}}((0,T)\times(\mathscr{O}\cup\Gamma_0))$ is a solution to problem (4.1.7) with boundary condition (4.1.9), satisfying the growth bound (4.1.25), then it admits the stochastic representation (4.1.28).

Let $\varepsilon > 0$ and $\{\mathscr{U}_k : k \in \mathbb{N}\}$ be the collection of increasing subdomains as in (4.3.5). By applying Itô's lemma, we obtain

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}^{t,z}\left[e^{-r(T\wedge\nu_{\mathscr{U}_k})}u(T\wedge\nu_{\mathscr{U}_k}, Z^\varepsilon(T\wedge\nu_{\mathscr{U}_k}))\right] \\ &\quad = u(t,z) - \mathbb{E}_{\mathbb{Q}}^{t,z}\left[\int_t^{T\wedge\nu_{\mathscr{U}_k}} e^{-rs}A^\varepsilon u(s, Z^\varepsilon(s))ds\right], \end{aligned}$$

where $\nu_{\mathscr{U}_k}$ is given by (4.1.18), Z^ε by (4.3.6) and A^ε is defined by (4.3.8). The proof now follows the same path as that of Theorem 4.1.5, with the only modification being that we now take the limit as k tends to ∞ in the preceding identity in order to obtain (4.1.28). $\square$

Analogous to Lemma 4.2.12, we have the following auxiliary result.

Lemma 4.5.2. *Suppose f and g obey the growth assumption* (4.1.25)*. Then there are positive constants $\bar{C}$, $M_1 \leq \mu$ and $M_2 \in [0,1]$, such that for any stopping times $\theta_1, \theta_2 \in \mathscr{T}_{t,T}$ with values in $[t,T]$, the function $J_p^{\theta_1,\theta_2}$ given by* (4.1.29) *obeys the growth assumption* (4.1.25)*.*

Proof. The proof follows as in Lemma 4.2.12 with the aid of Lemma 4.2.11. Notice that because the stopping times $\theta_1, \theta_2 \in \mathscr{T}_{t,T}$ are bounded by T, we do not need the constant r to be positive, as in Lemma 4.2.12. $\square$

Next, we have the following *existence* results for solutions to the parabolic boundary value problem (4.5.2), for all $\beta > 0$.

Remark 4.5.3. The function ψ in (4.1.29) plays the role of the obstacle function and is relevant only for problem (4.1.8).

Theorem 4.5.4 (Existence of solutions to problem (4.5.2) with continuous Dirichlet boundary condition)**.** *In addition to the hypotheses of Theorem 4.1.12, let $\mathscr{O} \subset \mathbb{H}$ be a domain such that the boundary Γ_1 obeys an exterior sphere condition, and $f \in C^\alpha_{\mathrm{loc}}(\bar{Q})$.*

1. *If $\beta \geq 1$ and $g \in C_{\mathrm{loc}}(\overline{\eth^1 Q})$, then u^* in* (4.1.26) *is a solution to problem* (4.5.2). *In particular, $u^* \in C_{\mathrm{loc}}(Q \cup \eth^1 Q) \cap C^{2+\alpha}(Q)$ and obeys the growth assumption* (4.1.25).

2. *If $0 < \beta < 1$ and $g \in C_{\mathrm{loc}}(\overline{\eth Q})$, then u^* in* (4.1.26) *is a solution to problem* (4.5.2). *In particular, $u^* \in C_{\mathrm{loc}}(Q \cup \eth Q) \cap C^{2+\alpha}(Q)$ and satisfies the growth assumption* (4.1.25).

Proof. We choose $T > 0$ small enough and $p_0 > 1$ as in Hypothesis 4.5.1.

By hypothesis, we have $g \in C_{\mathrm{loc}}(\overline{\eth_\beta Q})$. Since $\overline{\eth_\beta Q}$ is closed, we may use [37, Theorem 3.1.2] to extend g to a function on $[0,T] \times \mathbb{R}^2$, again called g, such that $g \in C_{\mathrm{loc}}([0,T] \times \mathbb{R}^2)$.

The proof follows the same pattern as that of Theorem 4.3.1. For completeness, we outline the main steps. Let $\mathscr{O}_k$ be an increasing sequence of $C^{2+\alpha}$ subdomains of $\mathscr{O}$ as in the proof of Theorem 4.1.2, and let $Q_k := (0,T) \times \mathscr{O}_k$. We notice that on each cylindrical domain, Q_k, the operator A is uniformly elliptic, and its coefficients are $C^\infty(\bar{Q}_k)$ functions. By hypothesis, there is an $\alpha \in (0,1)$ such that $f \in C^\alpha(\bar{Q}_k)$ and $g \in C(\bar{Q}_k)$. Therefore, by [37, Theorem 3.4.9], the terminal value problem

$$
\begin{aligned}
-u_t + Au &= f \quad \text{on } Q_k, \\
u &= g \quad \text{on } (0,T) \times \partial \mathscr{O}_k \cup \{T\} \times \bar{\mathscr{O}}_k,
\end{aligned}
$$

has a unique solution $u_k \in C(\bar{Q}_k) \cap C^{2+\alpha}(Q_k)$, and by Theorem C.3.11, it has the stochastic representation

$$
\begin{aligned}
u_k(t,z) = \mathbb{E}_{\mathbb{Q}}^{t,z} &\left[e^{-r(\tau_{\mathscr{O}_k} \wedge T - t)} g(\tau_{\mathscr{O}_k} \wedge T, Z(\tau_{\mathscr{O}_k} \wedge T)) \right] \\
&+ \mathbb{E}_{\mathbb{Q}}^{t,z} \left[\int_t^{\tau_{\mathscr{O}_k} \wedge T} e^{-r(s-t)} f(s, Z(s))\, ds \right], \quad \forall (t,z) \in \bar{Q}_k.
\end{aligned} \tag{4.5.9}
$$

Because $\tau_{\mathscr{O}_k}$ converges to $\tau_{\mathscr{O}}$ a.s. as $k \to \infty$, the integral term in (4.5.9) converges to the integral term of u^* in (4.1.26), by the same argument as that used in the proof of Theorem 4.1.12. By the continuity of g and of the paths of the Heston process Z, we also know that

$$
e^{-r(\tau_{\mathscr{O}_k} \wedge T)} g(\tau_{\mathscr{O}_k} \wedge T, Z(\tau_{\mathscr{O}_k} \wedge T)) \to e^{-r(\tau_{\mathscr{O}} \wedge T)} g(\tau_{\mathscr{O}} \wedge T, Z(\tau_{\mathscr{O}} \wedge T)), \quad \text{as } k \to \infty.
$$

In order to show that the preceding convergence takes place in expectation also, it is enough to show that the collection of random variables,

$$\left\{e^{-r(\tau_{\mathscr{O}_k}\wedge T)}g(\tau_{\mathscr{O}_k}\wedge T, Z(\tau_{\mathscr{O}_k}\wedge T)) : k \in \mathbb{N}\right\},$$

is uniformly integrable, but this follows by the same argument as that used for the collections (4.5.7) in the proof of Theorem 4.1.12, by bounding their p_0-th order moments ($p_0 > 1$). Therefore, the sequence $\{u_k : k \in \mathbb{N}\}$ converges to u^* pointwise on Q. By interior Schauder estimates for parabolic equations [40, Theorem 3.3.5] and Lemma 4.5.2, there is a subsequence of $\{u_k : k \in \mathbb{N}\}$ which converges to u^* in $C^{2+\alpha'}(Q)$, when $0 < \alpha' < \alpha$. Using the Arzelà-Ascoli Theorem, we obtain $u^* \in C^{2+\alpha}(Q)$. The proof of continuity of u up to $\eth_\beta Q$ follows by exactly the same argument as that used in the proof of Step 2 in Theorem 4.3.1. Therefore, u^* is a solution to (4.5.2).

From Theorem 4.1.12 and Lemma 4.5.2, we see that u^* in (4.1.26) is the unique solution to the parabolic terminal value problem (4.5.2), for all $\beta > 0$. □

Theorem 4.5.5 (Existence of solutions to problem (4.5.2) with Hölder continuous Dirichlet boundary condition)**.** *In addition to the hypotheses of Theorem 4.1.12, let $\mathscr{O} \subset \mathbb{H}$ be a domain such that*

1. *If $\beta \geq 1$, the boundary portion Γ_1 is of class $C^{2+\alpha}$, and $g \in C^{2+\alpha}_{\mathrm{loc}}(Q \cup \eth^1 Q)$ obeys*

$$-g_t + Ag = f \text{ on } \{T\} \times \Gamma_1. \tag{4.5.10}$$

Then u^* in (4.1.26) is a solution to problem (4.5.2). In particular,

$$u^* \in C^{2+\alpha}_{\mathrm{loc}}(Q \cup \eth^1 Q)$$

and obeys the growth estimate (4.1.25).

2. *If $0 < \beta < 1$, the boundary portion Γ_1 is of class $C^{2+\alpha}$, and $g \in C^{2+\alpha}_{\mathrm{loc}}(Q \cup \eth^1 Q) \cap C^{\mathrm{loc}}(\bar{\mathscr{O}})$ obeys*

$$-g_t + Ag = f \text{ on } \{T\} \times \partial\mathscr{O}. \tag{4.5.11}$$

Then u^* in (4.1.26) is a solution to problem (4.5.2). In particular,

$$u^* \in C^{2+\alpha}_{\mathrm{loc}}(Q \cup \eth^1 Q) \cap C^{\mathrm{loc}}(\bar{\mathscr{O}}).$$

and obeys the growth estimate (4.1.25).

Proof. Just as in the proof of Theorem 4.5.4, we can easily adapt the proof of Theorem 4.3.3 for the elliptic case to the present parabolic case. For this purpose, we only need to make use of the local boundary Schauder estimate Lemma C.1.1 instead of [41, Corollary 6.7] for the elliptic case. □

Remark 4.5.6 (Zero and first-order compatibility conditions for parabolic equations)**.** The conditions (4.5.10) and (4.5.11) are the analogues of the first-order compatibility condition [51, Equation (10.4.3)]. The analogue of the zero-order compatibility condition in [51, Equation (10.4.2)] automatically holds at $\{T\} \times \Gamma_1$ or $\{T\} \times \partial\mathscr{O}$, since we always choose $h = g(T, \cdot)$ on Γ_1 or $\partial\mathscr{O}$, respectively.

4.6 Parabolic obstacle problem

Problem (4.1.8) with boundary condition (4.1.12), when $0 < \beta < 1$, and with boundary condition (4.1.9), when $\beta \geq 1$, can be formulated as

$$\begin{cases} \min\{-u_t + Au - f, u - \psi\} = 0 & \text{on } Q, \\ u = g & \text{on } \eth_\beta Q, \end{cases} \tag{4.6.1}$$

where $\eth_\beta Q$ is defined in (4.5.1). According to Theorem 4.1.19, the solution to problem (4.6.1) is given in (4.1.31).

Proof of Theorem 4.1.19. We choose $\widetilde{T} > 0$ small enough so that it obeys Hypothesis 4.5.1. For such $\widetilde{T} > 0$, the proof of Theorem 4.1.8 adapts to the present case in the same way that the proof of Theorem 4.1.2 adapts to give a proof of Theorem 4.1.12. Therefore, it remains to show that the corresponding stochastic representation (4.1.30) of the solution to problem (4.6.1) holds for T arbitrarily large.

Let $N := \lfloor T/\widetilde{T} \rfloor$ (the greatest integer in $T/\widetilde{T}$), and $T_i := i\widetilde{T}$, for $i = 0, \ldots, N-1$, and $T_N := T$. Let k be an integer such that $1 \leq k \leq N-1$, and assume that the stochastic representation formula (4.1.30) holds for any $t \in [T_i, T]$, where $i = k, \ldots, N-1$. We want to show that it holds also for $t \in [T_{k-1}, T]$. Notice that for $k = N-1$, we have

$T-t \leq \widetilde{T}$, for all $t \in [T_{N-1}, T]$, and so we know that the stochastic representation (4.1.30) of the solution to problem (4.1.8) holds, by the observation at the beginning of the present proof.

For any $t \leq v \leq T$, stopping time $\theta \in \mathscr{T}_{t,v}$ with values in $[t,v]$, and $\varphi \in C(\bar{\mathscr{O}})$, we denote

$$\begin{aligned} F^{\varphi}(t,z,v,\theta) := \int_t^{\tau_{\mathscr{O}}\wedge\theta} & e^{-r(s-t)} f(s,Z(s))\,ds + e^{-r(\theta-t)}\psi(\theta,Z(\theta))1_{\{\theta<\tau_{\mathscr{O}}\wedge v\}} \\ & + e^{-r(\tau_{\mathscr{O}}-t)} g(\tau_{\mathscr{O}}, Z(\tau_{\mathscr{O}}))1_{\{\tau_{\mathscr{O}}\leq\theta,\tau_{\mathscr{O}}<v\}} \\ & + e^{-r(v-t)}\varphi(Z(v))1_{\{\tau_{\mathscr{O}}\wedge v\leq\theta,\tau_{\mathscr{O}}\geq v\}}. \end{aligned} \tag{4.6.2}$$

Notice that by choosing $\varphi = g(T,\cdot)$ and $v = T$ in (4.6.2), we obtain, for any stopping time $\theta \in \mathscr{T}_{t,T}$,

$$\begin{aligned} & e^{-r(\tau_{\mathscr{O}}-t)} g(\tau_{\mathscr{O}}, Z(\tau_{\mathscr{O}}))1_{\{\tau_{\mathscr{O}}\leq\theta,\tau_{\mathscr{O}}<T\}} + e^{-r(T-t)}\varphi(Z(T))1_{\{\tau_{\mathscr{O}}\wedge T\leq\theta,\tau_{\mathscr{O}}\geq T\}} \\ & = e^{-r(\tau_{\mathscr{O}}\wedge T-t)} g(\tau_{\mathscr{O}}\wedge T, Z(\tau_{\mathscr{O}}\wedge T))1_{\{\tau_{\mathscr{O}}\wedge T\leq\theta\}} \end{aligned}$$

and so,

$$\begin{aligned} F^{g(T,\cdot)}(t,z,T,\theta) = \int_t^{\tau_{\mathscr{O}}\wedge\theta} & e^{-r(s-t)} f(s,Z(s))\,ds + e^{-r(\theta-t)}\psi(\theta,Z(\theta))1_{\{\theta<\tau_{\mathscr{O}}\wedge T\}} \\ & + e^{-r(\tau_{\mathscr{O}}\wedge T-t)} g(\tau_{\mathscr{O}}\wedge T, Z(\tau_{\mathscr{O}}\wedge T))1_{\{\tau_{\mathscr{O}}\wedge T\leq\theta\}}. \end{aligned} \tag{4.6.3}$$

Because u solves problem (4.6.1) on the interval (T_{k-1}, T_k), and $T_k - T_{k-1} \leq \widetilde{T}$, we see that u has the stochastic representation (4.1.30), for any $t \in [T_{k-1}, T_k)$ and $z \in \mathscr{O}\cup\partial_\beta\mathscr{O}$,

$$u(t,z) = \sup_{\theta\in\mathscr{T}_{t,T_k}} \mathbb{E}_{\mathbb{Q}}^{t,z}\left[F^{u^*(T_k,\cdot)}(t,z,T_k,\theta)\right]. \tag{4.6.4}$$

For any stopping time $\eta \in \mathscr{T}_{t,T_k}$, we set

$$\begin{aligned} F_1(t,z,T_k,\eta) := \int_t^{\tau_{\mathscr{O}}\wedge\eta} & e^{-r(s-t)} f(s,Z(s))ds \\ & + e^{-r(\eta-t)}\psi(\eta,Z(\eta))\mathbf{1}_{\{\eta<\tau_{\mathscr{O}}\wedge T_k\}} \\ & + e^{-r(\tau_{\mathscr{O}}-t)} g(\tau_{\mathscr{O}}, Z(\tau_{\mathscr{O}}))\mathbf{1}_{\{\tau_{\mathscr{O}}\leq\eta,\eta<T_k\}}, \end{aligned} \tag{4.6.5}$$

and for any stopping time $\xi \in \mathscr{T}_{T_k,T}$, we let

$$\begin{aligned} F_2(t,z,T,\xi) := \int_{T_k}^{\tau_{\mathscr{O}}\wedge\xi} & e^{-r(s-T_k)} f(s,Z(s))ds \\ & + e^{-r(\xi-T_k)}\psi(\xi,Z(\xi))\mathbf{1}_{\{\xi<\tau_{\mathscr{O}}\wedge T\}} \\ & + e^{-r(\tau_{\mathscr{O}}\wedge T-T_k)} g(\tau_{\mathscr{O}}\wedge T, Z(\tau_{\mathscr{O}}\wedge T))\mathbf{1}_{\{\tau_{\mathscr{O}}\wedge T\leq\xi\}}. \end{aligned} \tag{4.6.6}$$

For the rest of the proof, we *fix* $z \in \mathscr{O} \cup \partial_\beta \mathscr{O}$ and $t \in [T_{k-1}, T_k)$.

Let $\eta \in \mathscr{T}_{t,T_k}$ and $\xi \in \mathscr{T}_{T_k,T}$. It is straightforward to see that

$$\theta := \begin{cases} \eta & \text{if } \eta < T_k, \\ \xi & \text{if } \eta = T_k, \end{cases}$$

is a stopping time with values in $[t, T]$. We denote by

$$\mathscr{S}_{t,T} = \left\{\theta \in \mathscr{T}_{t,T} : \theta = \eta \mathbf{1}_{\{\eta < T_k\}} + \xi \mathbf{1}_{\{\eta = T_k\}},\ \text{where } \eta \in \mathscr{T}_{t,T_k} \text{ and } \xi \in \mathscr{T}_{T_k,T}\right\}. \tag{4.6.7}$$

For any stopping time $\theta \in \mathscr{T}_{t,T}$, we define the stopping times $\theta' \in \mathscr{T}_{t,T_k}$ and $\theta'' \in \mathscr{T}_{T_k,T}$,

$$\theta' := \mathbf{1}_{\{\theta < T_k\}} \theta + \mathbf{1}_{\{\theta \geq T_k\}} T_k \quad \text{and} \quad \theta'' := \mathbf{1}_{\{\theta < T_k\}} T_k + \mathbf{1}_{\{\theta \geq T_k\}} \theta. \tag{4.6.8}$$

Then, any stopping time $\theta \in \mathscr{T}_{t,T}$ can be written as

$$\begin{aligned} \theta &= \theta' \mathbf{1}_{\{\theta < T_k\}} + \theta'' \mathbf{1}_{\{\theta \geq T_k\}} \\ &= \theta' \mathbf{1}_{\{\theta' < T_k\}} + \theta'' \mathbf{1}_{\{\theta' = T_k\}} \end{aligned}$$

and so,

$$\mathscr{T}_{t,T} = \mathscr{S}_{t,T}.$$

The preceding identity and definitions (4.1.30) of u^* and (4.6.2) of F^φ give us

$$u^*(t,z) = \sup_{\theta \in \mathscr{S}_{t,T}} \mathbb{E}_{\mathbb{Q}}^{t,z} \left[F^{g(T,\cdot)}(t,z,T,\theta)\right]. \tag{4.6.9}$$

We shall need the following identities

Claim 4.6.1. *For any stopping time $\theta = \eta \mathbf{1}_{\{\eta < T_k\}} + \xi \mathbf{1}_{\{\eta = T_k\}}$, where $\eta \in \mathscr{T}_{t,T_k}$ and $\xi \in \mathscr{T}_{T_k,T}$, we have the following identities*

$$\begin{aligned} \int_t^{\tau_{\mathscr{O}} \wedge \theta} e^{-r(s-t)} f(s, Z(s)) ds = \mathbf{1}_{\{\eta < T_k\}} &\int_t^{\tau_{\mathscr{O}} \wedge \eta} e^{-r(s-t)} f(s, Z(s)) ds \\ + \mathbf{1}_{\{\eta = T_k\}} &\int_{T_k}^{\tau_{\mathscr{O}} \wedge \xi} e^{-r(s-t)} f(s, Z(s)) ds, \end{aligned}$$

and

$$\begin{aligned} e^{-r(\theta - t)} \psi(\theta, Z(\theta)) \mathbf{1}_{\{\theta < \tau_{\mathscr{O}} \wedge T\}} &= e^{-r(\eta - t)} \psi(\eta, Z(\eta)) \mathbf{1}_{\{\eta < \tau_{\mathscr{O}} \wedge T_k\}} \mathbf{1}_{\{\eta < T_k\}} \\ &\quad + e^{-r(\xi - t)} \psi(\xi, Z(\xi)) \mathbf{1}_{\{\xi < \tau_{\mathscr{O}} \wedge T\}} \mathbf{1}_{\{\eta = T_k\}}, \end{aligned}$$

and

$$\begin{aligned}
&e^{-r(\tau_{\mathscr{O}}\wedge T-t)}g(\tau_{\mathscr{O}}\wedge T, Z(\tau_{\mathscr{O}}\wedge T))\mathbf{1}_{\{\tau_{\mathscr{O}}\wedge T\leq\theta\}}\\
&\qquad = e^{-r(\tau_{\mathscr{O}}-t)}g(\tau_{\mathscr{O}}, Z(\tau_{\mathscr{O}}))\mathbf{1}_{\{\tau_{\mathscr{O}}\leq\eta,\eta<T_k\}}\mathbf{1}_{\{\eta<T_k\}}\\
&\qquad\quad + e^{-r(\tau_{\mathscr{O}}\wedge T-t)}g(\tau_{\mathscr{O}}\wedge T, Z(\tau_{\mathscr{O}}\wedge T))\mathbf{1}_{\{\tau_{\mathscr{O}}\wedge T\leq\xi\}}\mathbf{1}_{\{\eta=T_k\}}.
\end{aligned}$$

Proof. Notice that

$$\{\theta < T_k\} = \{\eta < T_k\} \text{ and } \{\theta \geq T_k\} = \{\eta = T_k\}. \tag{4.6.10}$$

The first identity is obvious because, by (4.6.10), we see that

$$\theta = \eta \text{ on } \{\eta < T_k\} \text{ and } \theta = \xi \text{ on } \{\eta = T_k\}. \tag{4.6.11}$$

The second identity follows by the observation that

$$\{\theta < \tau_{\mathscr{O}}\wedge T\} = \{\theta < \tau_{\mathscr{O}}\wedge T, \theta < T_k\} \cup \{\theta < \tau_{\mathscr{O}}\wedge T, \theta \geq T_k\},$$

and using (4.6.11) and (4.6.10), it follows

$$\{\theta < \tau_{\mathscr{O}}\wedge T\} = \{\eta < \tau_{\mathscr{O}}\wedge T_k, \eta < T_k\} \cup \{\xi < \tau_{\mathscr{O}}\wedge T, \eta = T_k\}.$$

For the last identity of the claim, we notice

$$\begin{aligned}
\{\tau_{\mathscr{O}}\wedge T \leq \theta\} &= \{\tau_{\mathscr{O}}\wedge T \leq \theta, \tau_{\mathscr{O}} < T\} \cup \{\tau_{\mathscr{O}}\wedge T \leq \theta, \tau_{\mathscr{O}} \geq T\}\\
&= \{\tau_{\mathscr{O}}\wedge T \leq \theta, \tau_{\mathscr{O}} < T, \theta < T_k\} \cup \{\tau_{\mathscr{O}}\wedge T \leq \theta, \tau_{\mathscr{O}} < T, \theta \geq T_k\}\\
&\quad \cup \{\tau_{\mathscr{O}}\wedge T \leq \theta, \tau_{\mathscr{O}} \geq T\}.
\end{aligned}$$

By (4.6.11) and (4.6.10), we obtain

$$\begin{aligned}
\{\tau_{\mathscr{O}}\wedge T \leq \theta\} &= \{\tau_{\mathscr{O}} \leq \eta, \tau_{\mathscr{O}} < T, \eta < T_k\} \cup \{\tau_{\mathscr{O}}\wedge T \leq \xi, \tau_{\mathscr{O}} < T, \eta = T_k\}\\
&\quad \cup \{\tau_{\mathscr{O}}\wedge T \leq \xi, \tau_{\mathscr{O}} \geq T\}\\
&= \{\tau_{\mathscr{O}} \leq \eta, \eta < T_k\} \cup \{\tau_{\mathscr{O}}\wedge T \leq \xi, \eta = T_k\},
\end{aligned}$$

which implies the last identity of the claim. $\square$

We can write the expression for $F^{g(T,\cdot)}(t,z,T,\theta)$ as a sum,

$$F^{g(T,\cdot)}(t,z,T,\theta) = \mathbf{1}_{\{\eta<T_k\}} F_1(t,z,T_k,\eta) + \mathbf{1}_{\{\eta=T_k\}} e^{-r(T_k-t)} F_2(t,z,T,\xi). \tag{4.6.12}$$

Because $\xi \in \mathscr{T}_{T_k,T}$ and $F_2(t,z,T,\xi)$ depends only on $\left(Z^{t,z}(s)\right)_{T_k \le s \le T}$, and the Heston process has the (strong) Markov property [30, Theorem 1.15 (c)], we have a.s. that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}^{t,z}\left[F_2(t,z,T,\xi)|\mathscr{F}_{T_k}\right] &= \mathbb{E}_{\mathbb{Q}}^{T_k,Z^{t,z}(T_k)}\left[F_2(T_k,Z^{t,z}(T_k),T,\xi)\right] \\
&= \mathbb{E}_{\mathbb{Q}}^{T_k,Z^{t,z}(T_k)}\left[F^{g(T,\cdot)}(T_k,Z^{t,z}(T_k),T,\xi)\right],
\end{aligned}$$

by applying definitions (4.6.3) and (4.6.6). Thus,

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}^{t,z}\left[\mathbf{1}_{\{\eta=T_k\}} e^{-r(T_k-t)} F_2(t,z,T,\xi)|\mathscr{F}_{T_k}\right] \\
&\quad = \mathbb{E}_{\mathbb{Q}}^{t,z}\left[\mathbb{E}_{\mathbb{Q}}^{t,z}\left[\mathbf{1}_{\{\eta=T_k\}} e^{-r(T_k-t)} F_2(t,z,T,\xi)|\mathscr{F}_{T_k}\right]\right] \\
&\quad = \mathbb{E}_{\mathbb{Q}}^{t,z}\left[\mathbf{1}_{\{\eta=T_k\}} e^{-r(T_k-t)} \mathbb{E}_{\mathbb{Q}}^{t,z}\left[F_2(t,z,T,\xi)|\mathscr{F}_{T_k}\right]\right] \\
&\quad = \mathbb{E}_{\mathbb{Q}}^{t,z}\left[\mathbf{1}_{\{\eta=T_k\}} e^{-r(T_k-t)} \mathbb{E}_{\mathbb{Q}}^{T_k,Z(T_k)}\left[F^{g(T,\cdot)}(T_k,Z(T_k),T,\xi)\right]\right].
\end{aligned}$$

By the preceding identity, (4.6.7) and (4.6.12), the identity (4.6.9) yields

$$\begin{aligned}
u^*(t,z) &= \sup_{\substack{\theta=\eta\mathbf{1}_{\{\eta<T_k\}}+\xi\mathbf{1}_{\{\eta=T_k\}} \\ \theta\in\mathscr{S}_{t,T},\eta\in\mathscr{T}_{t,T_k},\xi\in\mathscr{T}_{T_k,T}}} \Big\{\mathbb{E}_{\mathbb{Q}}^{t,z}\Big[\mathbf{1}_{\{\eta<T_k\}} F_1(t,z,T_k,\eta) \\
&\qquad + \mathbf{1}_{\{\eta=T_k\}} e^{-r(T_k-t)} \mathbb{E}_{\mathbb{Q}}^{T_k,Z(T_k)}\left[F^{g(T,\cdot)}(T_k,Z(T_k),T,\xi)\right]\Big]\Big\} \\
&= \sup_{\eta\in\mathscr{T}_{t,T_k}} \Big\{\mathbb{E}_{\mathbb{Q}}^{t,z}\Big[\mathbf{1}_{\{\eta<T_k\}} F_1(t,z,T_k,\eta) \\
&\qquad + \mathbf{1}_{\{\eta=T_k\}} e^{-r(T_k-t)} \sup_{\xi\in\mathscr{T}_{T_k,T}} \mathbb{E}_{\mathbb{Q}}^{T_k,Z(T_k)}\left[F^{g(T,\cdot)}(T_k,Z(T_k),T,\xi)\right]\Big]\Big\}.
\end{aligned}$$

Using the definition (4.1.30) of u^*, we have

$$u^*(T_k,Z(T_k)) = \sup_{\xi\in\mathscr{T}_{T_k,T}} \mathbb{E}_{\mathbb{Q}}^{T_k,Z(T_k)}\left[F^{g(T,\cdot)}(T_k,Z(T_k),T,\xi)\right],$$

and so it follows that

$$u^*(t,z) = \sup_{\eta\in\mathscr{T}_{t,T_k}} \mathbb{E}_{\mathbb{Q}}^{t,z}\left[\mathbf{1}_{\{\eta<T_k\}} F_1(t,z,T_k,\eta) + \mathbf{1}_{\{\eta=T_k\}} e^{-r(T_k-t)} u^*(T_k,Z(T_k))\right].$$

Notice that, by the definitions (4.6.2) of F^φ and (4.6.5) of F_1, we have

$$F^{u^*(T,\cdot)}(t,z,T_k,\eta) = \mathbf{1}_{\{\eta<T_k\}} F_1(t,z,T_k,\eta) + \mathbf{1}_{\{\eta=T_k\}} e^{-r(T_k-t)} u^*(T_k,Z(T_k)).$$

The preceding two identities yield

$$\begin{aligned} u^*(t,z) &= \sup_{\eta \in \mathscr{T}_{t,T_k}} \mathbb{E}_{\mathbb{Q}}^{t,z} \left[F^{u^*(T,\cdot)}(t,z,T_k,\eta) \right] \\ &= u(t,z), \quad \text{(by (4.6.4))}. \end{aligned}$$

This concludes the proof of the theorem. □

Proof of Theorem 4.1.20. We omit the proof as it is very similar to the proofs of Theorems 4.1.19 and 4.1.9. □

Appendix A

Auxiliary results for Chapter 2

In this appendix, we give proofs for several results which are slightly more technical than those in the body of Chapter 2.

A.1 Existence and uniqueness of solutions for a degenerate parabolic operator with constant coefficients

In order to derive the local a priori boundary estimates in Theorem 2.3.8, we need an analogue of [20, Theorem I.1.1] when the coefficients of our operator L, a_{ij}, b_i and c, are assumed *constant*. To emphasize this fact in this appendix, we denote our parabolic operator by

$$-L_0 u := -u_t + \sum_{i,j=1}^{d} x_d a_{ij} u_{x_i x_j} + \sum_{i=1}^{d} b_i u_{x_i} + cu \quad \text{on } (0,T) \times \mathbb{H}.$$

We now have the following analogue of [20, Theorem I.1.1].

Proposition A.1.1 (Existence and uniqueness of solutions for a degenerate parabolic operator with constant coefficients)**.** *Let K, δ and ν be positive constants such that*

$$a_{ij}\eta_i\eta_j \geq \delta\|\eta\|^2, \quad \forall \eta \in \mathbb{R}^d, \tag{A.1.1}$$

$$b_d \geq \nu, \tag{A.1.2}$$

$$|a_{ij}|, |b_i|, |c| \leq K. \tag{A.1.3}$$

Let k be a nonnegative integer, $T > 0$, and $\alpha \in (0,1)$. Assume that $f \in C_s^{k,\alpha}(\bar{\mathbb{H}}_T)$ and $g \in C_s^{k,2+\alpha}(\bar{\mathbb{H}})$ with both f and g compactly supported in $\bar{\mathbb{H}}_T$ and $\bar{\mathbb{H}}$, respectively. Then, the inhomogeneous initial value problem

$$\begin{cases} L_0 u = f & \text{on } (0,T) \times \mathbb{H}, \\ u(0,\cdot) = g & \text{on } \bar{\mathbb{H}}. \end{cases} \tag{A.1.4}$$

admits a unique solution $u \in C^{k,2+\alpha}_s(\bar{\mathbb{H}}_T)$*. Moreover, there exists a positive constant* $C = C(T, K, \delta, \nu, \alpha, d, k)$ *such that*

$$\|u\|_{C^{k,2+\alpha}_s(\bar{\mathbb{H}}_T)} \leq C \left(\|f\|_{C^{k,\alpha}_s(\bar{\mathbb{H}}_T)} + \|g\|_{C^{k,2+\alpha}_s(\bar{\mathbb{H}})} \right). \tag{A.1.5}$$

Proof. The proof follows by adapting the proof of [20, Theorem I.1.1]. Because the proof of [20, Theorem I.1.1] is lengthy, we only outline the modifications, noting that these modifications are straightforward. We remark that there is no simple change of variables that can be applied in order to bring the constant coefficients equation (A.1.4) to the form of the model operator defined in [20, p. 901]. Another difficulty is that our interpolation inequalities (Lemma 2.3.2) do not allow us to treat the first order derivatives, u_{x_i}, in (2.1.3) as lower order terms: in order to do that, we would need to have

$$\|u_{x_i}\|_{C^{\alpha}_s(\bar{\mathbb{H}}_T)} \leq \varepsilon \|u\|_{C^{2+\alpha}_s(\bar{\mathbb{H}}_T)} + C\varepsilon^{-m} \|u\|_{C(\bar{\mathbb{H}}_T)},$$

instead of the interpolation inequality (2.3.8). On the other hand, by simple changes of variables which we describe below and which preserve the domain $\mathbb{H}$ and its boundary $\partial\mathbb{H}$, problem (A.1.4) can be simplified to

$$-L_0 u = -u_t + x_d \sum_{i=1}^{d} u_{x_i x_i} + \sum_{i=1}^{d} b_i u_{x_i} \quad \text{on } (0,T) \times \mathbb{H}, \tag{A.1.6}$$

where the coefficient $b_d > 0$ remains unchanged. In addition, the possibly new constant coefficients b_i are bounded in absolute value by constants which depend only on δ (A.1.1) and K (A.1.3). The simple changes of variables are the following. As usual, we eliminate the zeroth order term cu by multiplying u by e^{ct}. By applying the change of variable

$$u(t,x) = \tilde{u}(t, x_1 + \alpha_1 x_d, \ldots, x_{d-1} + \alpha_{d-1} x_d, x_d), \text{ with } \alpha_i = -\frac{a_{id}}{a_{dd}},$$

the problem (A.1.4) is reduced to the study of the operator $\widetilde{L}_0$ given by

$$\widetilde{L}_0 \tilde{u} = \tilde{u}_t - x_d a_{dd} \tilde{u}_{x_d x_d} - \sum_{i,j \neq d} a_{ij} \tilde{u}_{x_i x_d} - \sum_{i \neq d} (b_i + \alpha_i b_d)\, \tilde{u}_{x_i} - b_d \tilde{u}_{x_d} \quad \text{on } (0,T) \times \mathbb{H}.$$

Next, we diagonalize the upper symmetric, positive-definite matrix $(a_{ij})_{i,j=1,\ldots,d-1}$ and we rescale each coordinate to obtain

$$-\bar{L}_0 \bar{u} = -\bar{u}_t + x_d \sum_{i=1}^{d} \bar{u}_{x_i x_i} + \sum_{i=1}^{d} \bar{b}_i \bar{u}_{x_i} \quad \text{on } (0,T) \times \mathbb{H},$$

where the constant coefficients $\bar{b}_i$ may differ from b_i, with the exception that $\bar{b}_d = b_d$, because the transformation affects only the first $d-1$ coordinates. Therefore, for the remainder of this section, we may assume without loss of generality that L_0 is of the simpler form (A.1.6).

The primary change required in the proof of [20, Theorem I.1.1] lies in [20, §I.4]. The arguments in the remainder of [20, Part I] adapt almost line by line to our model operator (A.1.6). The goal in [20, §I.4] is to derive local derivative estimates and this is achieved by applying a comparison principle with barrier functions. First, we need to adapt the definition of the barrier function [20, Definition I.4.1] to one which is suitable for use with (A.1.6).

Definition A.1.2. Let $0 < t_1 < t_2$. We say φ is a *barrier function* for L_0 when $t \in [t_1, t_2]$, if there are positive constants C and c such

$$L_0\varphi > -Cx_d\varphi^2 + c\varphi^{3/2} + c. \tag{A.1.7}$$

The barrier functions in [20, Theorem I.4.5 & Theorem I.4.8] are barrier functions in the sense of Definition A.1.2, also. The barrier function constructed in [20, Theorem I.4.6] needs modification because the coefficients b_i, $i = 1, \ldots, d-1$, are non-zero in general, unlike in [20, Part I]. We have the following modification.

Claim A.1.3. *Assume $i \neq d$. For any $\gamma < 1$ as in [20, Definition I.4.2], there are a positive constant b depending only on $|b_i|$, and a positive constant Δ, depending only on $|b_i|$, b and γ such that, for any $t_0 \geq 0$,*

$$\varphi_i(t,x) := \frac{1}{(1 + x_i - b(t-t_0))^2} + \frac{1}{(1 - x_i - b(t-t_0))^2} \tag{A.1.8}$$

is a valid barrier function satisfying (A.1.7)*, for all $t \in [t_0, t_0 + \Delta t]$.*

Proof of Claim A.1.3. It suffices to consider separately the terms ${}^+\varphi_i$ and ${}^-\varphi_i$ defined by

$${}^{\pm}\varphi := \frac{1}{(1 \pm x_i - b(t-t_0))^2},$$

because the barrier functions form a cone by [20, Theorem I.4.4]. We prove that ${}^+\varphi_i$ satisfies (A.1.7), and the proof follows similarly for ${}^-\varphi_i$. We denote for simplicity

$\varphi := {}^+\varphi_i$. By direct calculation, we obtain

$$\varphi_t = 2b\varphi^{3/2},$$
$$\varphi_{x_i} = -2\varphi^{3/2},$$
$$\varphi_{x_i x_i} = 6\varphi^2,$$

while $\varphi_{x_j} = 0$ and $\varphi_{x_j x_k} = 0$, unless $j = k = i$. Then, we have

$$L_0\varphi = 2(b + b_i)\varphi^{3/2} - 6x_d\varphi^2.$$

We impose $1 - b(t - t_0) \geq \gamma$, for all $t \in [t_0, t_0 + \Delta]$, so we choose $\Delta < (1 - \gamma)/b$. By choosing $b = |b_i| + 1$, we can find $C > 0$ and $c > 0$ such that

$$L_0\varphi \geq -x_d C\varphi^2 + c\varphi^{3/2} + c,$$

and so φ satisfies the requirement (A.1.7), for all $t \in [t_0, t_0 + \Delta]$. □

Next, the arguments in [20, §I.5] adapt to our framework with the following observation. Because our barrier functions (A.1.8) are not defined for all $t \in [0, 1]$, we cover first the interval $[0, 1]$ by a finite number of intervals of length Δ, as given in Claim A.1.3, and we apply the maximum principle on each of the resulting subintervals. This will yield local estimates analogous to [20, Theorem I.5.1, I.5.4 & Corollary I.5.7], on the small time subintervals of the finite covering. By combining the local derivative estimates over each subinterval, we obtain the required local estimates for all $t \in [0, 1]$. □

A.2 Proof of Proposition 2.3.13

Next, we include the proof of Proposition 2.3.13. The estimate (2.3.98) is obtained exactly as in the proof of [51, Theorems 9.2.2 & 8.9.2] using Lemma 2.3.11.

Proof of Proposition 2.3.13. Due to the classical interpolation inequalities [51, Theorem 8.8.1] and the classical maximum principle for unbounded domains [51, Corollary 8.1.5], it suffices to prove that the estimate (2.3.98) holds with

$$[u_t]_{C^\alpha_\rho([0,T]\times\mathbb{R}^d)} \text{ and } \left[u_{x_i x_j}\right]_{C^\alpha_\rho([0,T]\times\mathbb{R}^d)}$$

on the left hand side of the inequality. We will prove this for the $C^{\alpha}_{\rho}([0,T]\times\mathbb{R}^d)$-seminorm of u_t, but the same argument can be applied for the $C^{\alpha}_{\rho}([0,T]\times\mathbb{R}^d)$-seminorm of $u_{x_ix_j}$.

For simplicity of notation, we denote $Q := (0,T)\times\mathbb{R}^d$, and we omit the subscript ρ in the definition of the Hölder spaces. We also use the simplified notation

$$[u]_{C^{2+\alpha}(\bar{Q})} := [u_t]_{C^{\alpha}(\bar{Q})} + \left[u_{x_ix_j}\right]_{C^{\alpha}(\bar{Q})}. \tag{A.2.1}$$

Let $u \in C^{2+\alpha}(\bar{Q})$ be a solution to Problem 2.3.81. Then,

$$\bar{u} := e^{-\lambda_1 t} u \tag{A.2.2}$$

is in $C^{2+\alpha}(\bar{Q})$ and it solves

$$\begin{cases} \left(-\bar{L}-\lambda_1\right)\bar{u} = -e^{-\lambda_1 t} f & \text{on } (0,T)\times\mathbb{R}^d, \\ \bar{u}(0,\cdot) = g & \text{on } \mathbb{R}^d, \end{cases}$$

where λ_1 is the upper bound on the zeroth order coefficient, $\bar{c}$, assumed in (2.3.78). We may apply [51, Corollary 8.1.5], because the zeroth order term of the parabolic operator $-\bar{L}-\lambda_1$ is nonpositive, and we obtain

$$\|\bar{u}\|_{C(\bar{Q})} \leq T\|e^{-\lambda_1 t} f\|_{C(\bar{Q})} + \|g\|_{C(\bar{Q})} \leq T\|f\|_{C(\bar{Q})} + \|g\|_{C(\bar{Q})}.$$

Thus, it follows by (A.2.2)

$$\|u\|_{C(\bar{Q})} \leq e^{\lambda_1 T}\left(T\|f\|_{C(\bar{Q})} + \|g\|_{C(\bar{Q})}\right). \tag{A.2.3}$$

Let $z_1, z_2 \in [0,T]\times\mathbb{R}^d$ be two points such that

$$\frac{|u_t(z_1)-u_t(z_2)|}{\rho^{\alpha}(z_1,z_2)} \geq \frac{1}{2}[u_t]_{C^{\alpha}(\bar{Q})}. \tag{A.2.4}$$

Let $\gamma > 0$ be a constant which will be suitably chosen below. We consider two cases.

Case 1 $(\rho(z_1,z_2)\geq\gamma)$**.** Then, we have

$$[u_t]_{C^{\alpha}(\bar{Q})} \leq 2\gamma^{-\alpha}|u_t|_{C(\bar{Q})},$$

and, by [51, Theorem 8.8.1, Inequality (8.8.1)], it follows, for all $\varepsilon > 0$,

$$[u_t]_{C^{\alpha}(\bar{Q})} \leq 2\gamma^{-\alpha}\left(\varepsilon[u]_{C^{2+\alpha}(\bar{Q})} + C\varepsilon^{-\alpha/2}|u|_{C(\bar{Q})}\right).$$

By choosing $\varepsilon := \gamma^\alpha/8$ and by inequality (A.2.3), we obtain

$$[u_t]_{C^\alpha(\bar{Q})} \leq \frac{1}{4}[u]_{C^{2+\alpha}(\bar{Q})} + C\gamma^{-(\alpha+\alpha^2/2)}e^{\lambda_1 T}\left(T\|f\|_{C(\bar{Q})} + \|g\|_{C(\bar{Q})}\right), \tag{A.2.5}$$

where the constant $C = C(d, \alpha)$.

Case 2 $(\rho(z_1, z_2) < \gamma)$**.** We denote $z = (t, x)$. Let $\zeta : \mathbb{R}^{d+1} \to [0, 1]$ be a smooth cutoff function such that

$$\zeta(z) = 1, \text{ if } \rho(z, z_1) \leq 1, \text{ and } \zeta(z) = 0, \text{ if } \rho(z, z_1) \geq 2,$$

and we define φ by

$$\varphi(z) := \zeta((t - t_1)/\gamma^2, (x - x_1)/\gamma) \quad \forall z \in \mathbb{R}^{d+1},$$

so that,

$$\varphi(z) = 1, \text{ if } \rho(z, z_1) \leq \gamma, \text{ and } \varphi(z) = 0, \text{ if } \rho(z, z_1) \geq 2\gamma, \tag{A.2.6}$$

It is straightforward to see that φ satisfies

$$\|\varphi\|_{C^{2+\alpha}(\mathbb{R}^{d+1})} \leq C\left(1 + \gamma^{-(2+\alpha)}\right), \tag{A.2.7}$$

where C is a positive constant. Since $z_2 \in \{\varphi = 1\}$, we obtain by (A.2.4)

$$[u_t]_{C^\alpha(\bar{Q})} \leq 2\frac{|u_t(z_1) - u_t(z_2)|}{\rho^\alpha(z_1, z_2)} \leq 2[(u\varphi)_t]_{C^\alpha(\bar{Q})}. \tag{A.2.8}$$

Let $\bar{L}_0$ denote the differential operator, with constant coefficients, of the type considered in Lemma 2.3.11

$$-\bar{L}_0 = -\partial_t + \sum_{i,j=1}^{d} \bar{a}_{ij}(z_1)\partial_{x_i x_j}. \tag{A.2.9}$$

Estimate (2.3.82) shows that there are constants $p_1 = p_1(\alpha)$ and $C = C(d, \alpha, T)$ such that

$$[(u\varphi)_t]_{C^\alpha(\bar{Q})} \leq C\left(1 + \delta_1^{-p_1} + K_1^{p_1}\right)\left(\|\bar{L}_0(u\varphi)\|_{C^\alpha(\bar{Q})} + \|g\varphi\|_{C^{2+\alpha}(\{0\}\times\mathbb{R}^d)}\right). \tag{A.2.10}$$

By (A.2.7), we obtain

$$\|g\varphi\|_{C^{2+\alpha}(\{0\}\times\mathbb{R}^d)} \leq C\left(1 + \gamma^{-(2+\alpha)}\right)\|g\|_{C^{2+\alpha}(\mathbb{R}^d)}. \tag{A.2.11}$$

By writing $\bar{L}_0(u\varphi) = L(u\varphi) + (\bar{L}_0 - L)(u\varphi)$, we have

$$\bar{L}_0(u\varphi) = L(u\varphi) + \sum_{i,j=1}^{d} \left(\bar{a}_{ij}(z) - \bar{a}_{ij}(z_1)\right)(u\varphi)_{x_i x_j} + \sum_{i=1}^{d} \bar{b}_i(z)(u\varphi)_{x_i} + \bar{c}(z)u\varphi. \quad \text{(A.2.12)}$$

We may write

$$L(u\varphi) = \varphi L u + \sum_{i,j=1}^{d} \bar{a}_{ij}(z)\varphi_{x_j} u_{x_i} + \left(\sum_{i,j=1}^{d} \bar{a}_{ij}(z)\varphi_{x_i x_j} + \sum_{i=1}^{d} \bar{b}_i(z)\varphi_{x_i} + \bar{c}(z) \right) u$$

and so, by (A.2.7) and (2.3.77), we obtain there is a positive constant $C = C(d)$ such that

$$\begin{aligned} \|L(u\varphi)\|_{C^\alpha(\bar{Q})} &\le C\left(1 + \gamma^{-(2+\alpha)}\right)\|Lu\|_{C^\alpha(\bar{Q})} \\ &\quad + CK_1\left(1 + \gamma^{-(2+\alpha)}\right)\left(\|u_{x_i}\|_{C^\alpha(\bar{Q})} + \|u\|_{C^\alpha(\bar{Q})}\right). \end{aligned} \quad \text{(A.2.13)}$$

Notice that we may write the difference as

$$\begin{aligned} \bar{L}_0(u\varphi) - L(u\varphi) &= \sum_{i,j=1}^{d} \left(\bar{a}_{ij}(z) - \bar{a}_{ij}(z_1)\right)\varphi u_{x_i x_j} \\ &\quad + \sum_{i=1}^{d} \left(\sum_{j=1}^{d} \left(\bar{a}_{ij}(z) - \bar{a}_{ij}(z_1)\right)\varphi_{x_j} + \bar{b}_i(z)\varphi \right) u_{x_i} \\ &\quad + \left(\sum_{i,j=1}^{d} \left(\bar{a}_{ij}(z) - \bar{a}_{ij}(z_1)\right)(\varphi)_{x_i x_j} + \sum_{i=1}^{d} \bar{b}_i(z)\varphi_{x_i} + \bar{c}(z)\varphi \right) u. \end{aligned}$$

By (2.3.77), (A.2.6) and (A.2.7), we obtain

$$\|\left(\bar{a}_{ij}(z) - \bar{a}_{ij}(z_1)\right)\varphi u_{x_i x_j}\|_{C^\alpha(\bar{Q})} \le CK_1\gamma^\alpha[u_{x_i x_j}]_{C^\alpha(\bar{Q})} + CK_1(1 + \gamma^{-(2+\alpha)})\|u_{x_i x_j}\|_{C(\bar{Q})}.$$

From an argument similar to that used to obtain (A.2.13), we have

$$\begin{aligned} \|\bar{L}_0(u\varphi) - L(u\varphi)\|_{C^\alpha(\bar{Q})} &\le CK_1\gamma^\alpha[u_{x_i x_j}]_{C^\alpha(\bar{Q})} \\ &\quad + CK_1(1 + \gamma^{-(2+\alpha)})\left(\|u_{x_i x_j}\|_{C(\bar{Q})} + \|u_{x_i}\|_{C^\alpha(\bar{Q})} + \|u\|_{C^\alpha(\bar{Q})}\right). \end{aligned} \quad \text{(A.2.14)}$$

Estimates (A.2.13) and (A.2.14), give us, by (A.2.12),

$$\begin{aligned} \|\bar{L}_0(u\varphi)\|_{C^\alpha(\bar{Q})} &\le C\left(1 + \gamma^{-(2+\alpha)}\right)\|Lu\|_{C^\alpha(\bar{Q})} + CK_1\gamma^\alpha[u_{x_i x_j}]_{C^\alpha(\bar{Q})} \\ &\quad + CK_1\left(1 + \gamma^{-(2+\alpha)}\right)\left(\|u_{x_i x_j}\|_{C(\bar{Q})} + \|u_{x_i}\|_{C^\alpha(\bar{Q})} + \|u\|_{C^\alpha(\bar{Q})}\right). \end{aligned} \quad \text{(A.2.15)}$$

Combining the preceding inequality, estimates (A.2.10) and (A.2.11) in (A.2.8), and using notation (A.2.1), it follows

$$\begin{aligned}[u_t]_{C^\alpha(\bar{Q})} \leq C\left(1+\delta_1^{-p_1}+K_1^{p_1}\right)&\left(\left(1+\gamma^{-(2+\alpha)}\right)\|\bar{L}u\|_{C^\alpha(\bar{Q})}\right.\\ &+K_1\gamma^\alpha[u]_{C^{2+\alpha}(\bar{Q})}\\ &+K_1\left(1+\gamma^{-(2+\alpha)}\right)\left(\|u_{x_ix_j}\|_{C(\bar{Q})}+\|u_{x_i}\|_{C^\alpha(\bar{Q})}+\|u\|_{C^\alpha(\bar{Q})}\right)\\ &\left.+\left(1+\gamma^{-(2+\alpha)}\right)\|g\|_{C^{2+\alpha}(\mathbb{R}^d)}\right),\end{aligned}$$

where $C = C(d, \alpha, T)$. The interpolation inequalities [51, Theorem 8.8.1] and the maximum principle [51, Corollary 8.1.5], gives us, for any $\varepsilon > 0$,

$$\begin{aligned}[u_t]_{C^\alpha(\bar{Q})} \leq C&\left(1+\delta_1^{-p_1}+K_1^{p_1}\right)\\ &\times\left[e^{\lambda_1 T}\left(1+K_1\varepsilon^{-m}\right)\left(1+\gamma^{-(2+\alpha)}\right)\left(\|f\|_{C^\alpha(\bar{Q})}+\|g\|_{C^{2+\alpha}(\mathbb{R}^d)}\right)\right.\\ &\left.+K_1\left(\gamma^\alpha+\varepsilon\left(1+\gamma^{-(2+\alpha)}\right)\right)[u]_{C^{2+\alpha}(\bar{Q})}\right],\end{aligned}$$

where $m = m(\alpha)$. We choose $\gamma \in (0, 1)$ such that

$$C\left(1+\delta_1^{-p_1}+K_1^{p_1}\right)K_1\gamma^\alpha \leq \frac{1}{16},$$

as for instance,

$$\gamma := \left(\frac{1}{48C}\min\left\{K_1^{-1}, K_1^{-1}\delta_1^{p_1}, K_1^{-(1+p_1)}\right\}\right)^{1/\alpha} \wedge 1. \tag{A.2.16}$$

Then, we choose $\varepsilon > 0$ such that

$$C\left(1+\delta_1^{-p_1}+K_1^{p_1}\right)\left(1+\gamma^{-(2+\alpha)}\right)K_1\varepsilon \leq \frac{1}{16}.$$

A suitable choice is

$$\varepsilon := \frac{1}{96C}(1+\gamma^{2+\alpha})\min\left\{K_1^{-1}, K_1^{-1}\delta_1^{p_1}, K_1^{-(1+p_1)}\right\} \tag{A.2.17}$$

Then, we obtain

$$\begin{aligned}[u_t]_{C^\alpha(\bar{Q})} \leq \frac{1}{4}[u]_{C^{2+\alpha}(\bar{Q})} + Ce^{\lambda_1 T}&\left(1+\delta_1^{-p_1}+K_1^{p_1}\right)\left(1+K_1\varepsilon^{-m}\right)\left(1+\gamma^{-(2+\alpha)}\right)\\ &\times\left(\|f\|_{C^\alpha(\bar{Q})}+\|g\|_{C^{2+\alpha}(\mathbb{R}^d)}\right).\end{aligned} \tag{A.2.18}$$

By combining inequalities (A.2.5) and (A.2.18) of the preceding two cases, we obtain the global estimate

$$
\begin{aligned}
[u_t]_{C^\alpha(\bar{Q})} \le \frac{1}{4}[u]_{C^{2+\alpha}(\bar{Q})} &+ Ce^{\lambda_1 T}\left(1+\delta_1^{-p_1}+K_1^{p_1}\right)\left(1+K_1\varepsilon^{-m}\right)\left(1+\gamma^{-(2+\alpha)}\right) \\
&\times\left(\|f\|_{C^\alpha(\bar{Q})}+\|g\|_{C^{2+\alpha}(\mathbb{R}^d)}\right).
\end{aligned}
\tag{A.2.19}
$$

We notice from (A.2.16) and (A.2.17) that we may find positive constants $N_3 = N_3(d,\alpha,T)$ and $p = p(\alpha)$ such that

$$
[u_t]_{C^\alpha(\bar{Q})} \le \frac{1}{4}[u]_{C^{2+\alpha}(\bar{Q})} + N_3 e^{\lambda_1 T}\left(1+\delta_1^{-p}+K_1^{p}\right)\left(\|f\|_{C^\alpha(\bar{Q})}+\|g\|_{C^{2+\alpha}(\mathbb{R}^d)}\right).
$$

The similar argument applied to $[u_{x_i x_j}]_{C^\alpha(\bar{Q})}$ yields

$$
\left[u_{x_i x_j}\right]_{C^\alpha(\bar{Q})} \le \frac{1}{4}[u]_{C^{2+\alpha}(\bar{Q})} + N_3 e^{\lambda_1 T}\left(1+\delta_1^{-p}+K_1^{p}\right)\left(\|f\|_{C^\alpha(\bar{Q})}+\|g\|_{C^{2+\alpha}(\mathbb{R}^d)}\right).
$$

Therefore, (A.2.1) gives us

$$
[u]_{C^{2+\alpha}(\bar{Q})} \le N_3 e^{\lambda_1 T}\left(1+\delta_1^{-p}+K_1^{p}\right)\left(\|f\|_{C^\alpha(\bar{Q})}+\|g\|_{C^{2+\alpha}(\mathbb{R}^d)}\right),
$$

which concludes the proof of the proposition by the interpolation inequalities [51, Theorem 8.8.1] and the maximum principle estimate (A.2.3). $\square$

Appendix B

Auxiliary results for Chapter 3

In this section we collect the technical justifications of a few assertions employed in the body of Chapter3.

B.1 An extension lemma

First, we give the proof of Lemma 3.2.6. As in §3.2, we work under the assumptions stated in Remarks 3.2.3 and 3.2.7.

Proof of Lemma 3.2.6. By [18, Corollary A.14], it is enough to prove the existence of an extension operator for functions $u \in C^1(\bar{\mathbf{B}}_R(z_0))$. Fix a point $z_0' = (x_0', y_0') \in \mathbf{B}_R(z_0)$, say $z_0' = \left(R^2/100, R^2/100\right)$. We consider two different cases depending on whether $0 < y \leq y_0'$ or $y > y_0'$.

First, we consider the points $z = (x, y) \in D \backslash \mathbf{B}_R(z_0)$ such that $0 < y \leq y_0'$. Let $z' = (x', y)$ be the intersection of $\partial \mathbf{B}_R(z_0)$ with the horizontal segment connecting z and (x_0', y). Then, we define $Eu(z)$ by reflection (with respect to the point z' in the hyperplane at level y)

$$Eu(z) := u\left(x_0' + \frac{|x' - x_0'|}{|x - x_0'|^2}(x - x_0'), y\right).$$

Next, we consider the case of points $z = (x, y) \in D \backslash \mathbf{B}_R(z_0)$ such that $y > y_0'$. Let $z' = (x', y')$ be the intersection of $\partial \mathbf{B}_R(z_0)$ with the segment connecting z and z_0'. Then, we define $Eu(z)$ by reflection

$$Eu(z) := u\left(z_0' + \frac{|z' - z_0'|}{|z - z_0'|^2}(z - z_0')\right).$$

It is clear that Eu is a continuous extension of u from $\mathbf{B}_R(z_0)$ to D. Remark 3.2.3 ensures that $\partial \mathbf{B}_R(z_0)$ is a piecewise smooth curve, and so Eu has well-defined weak derivatives in D. Next, we show that (3.2.6) holds. For this purpose, we denote by

$$D_1 := (D \backslash \mathbf{B}_R(z_0)) \cap \{y < y_0'\},$$
$$D_2 := (D \backslash \mathbf{B}_R(z_0)) \cap \{y \geq y_0'\}.$$

To prove (3.2.6), it is enough to show there is a positive constant C, depending on R and D, such that

$$\begin{aligned}
\int_{D_1} |Eu(x,y)|^2 y^{\beta-1} dxdy &\leq C \int_{\mathbf{B}_R(z_0)} |u(x,y)|^2 y^{\beta-1} dxdy, \\
\int_{D_1} |\nabla Eu(x,y)|^2 y^{\beta} dxdy &\leq C \int_{\mathbf{B}_R(z_0)} |\nabla u(x,y)|^2 y^{\beta} dxdy, \\
\int_{D_2} |Eu(x,y)|^2 y^{\beta-1} dxdy &\leq C \int_{\mathbf{B}_R(z_0)} |u(x,y)|^2 y^{\beta-1} dxdy, \\
\int_{D_2} |\nabla Eu(x,y)|^2 y^{\beta} dxdy &\leq C \int_{\mathbf{B}_R(z_0)} |\nabla u(x,y)|^2 y^{\beta} dxdy,
\end{aligned} \tag{B.1.1}$$

We begin by evaluating the integrals over D_1 in (B.1.1) and we show that

$$\begin{aligned}
\int_{D_1^+} |Eu(x,y)|^2 y^{\beta-1} dxdy &\leq C \int_{\mathbf{B}_R(z_0)} |u(x,y)|^2 y^{\beta-1} dxdy, \\
\int_{D_1^+} |\nabla Eu(x,y)|^2 y^{\beta} dxdy &\leq C \int_{\mathbf{B}_R(z_0)} |\nabla u(x,y)|^2 y^{\beta} dxdy,
\end{aligned} \tag{B.1.2}$$

where $D_1^+ := D_1 \cap \{x > 0\}$. The analogous relation to (B.1.2) can be shown to hold on $D_1^- := D_1 \cap \{x < 0\}$, in a similar way.

Denote by

$$f(x,y) = x_0' + \frac{|x' - x_0'|}{|x - x_0'|^2}(x - x_0'). \tag{B.1.3}$$

We notice that $(f(x,y), y) \in \mathbf{B}_R(z_0)$, for all $(x,y) \in D_1$, so $Eu(x,y)$ is well-defined on D_1. The coordinate $x' = x'(y)$ is determined by the condition $\mathbf{d}((y,x'), z_0) = R$. Direct calculations give us

$$x'(y) = \left(\left(R^2 + R\sqrt{R^2 + 4y} \right)^2 / 4 - y^4 \right)^{1/2}.$$

We obtain, for all $(x,y) \in D_1$,

$$f_x(x,y) = -\frac{x' - x_0'}{(x - x_0')^2},$$
$$f_y(x,y) = \frac{x_y'(y)}{x - x_0'}.$$

We can find a positive constant C_1, depending only on R, such that

$$x - x_0' \geq x' - x_0' \geq C_1, \quad \forall (x,y) \in D_1^+,$$

and there is a positive constant C_2, depending on R and D, such that

$$|f_x(x,y)|, |f_x(x,y)|^{-1}, |f_y(x,y)| \leq C_2. \tag{B.1.4}$$

Using the change of variable $w = f(x,y)$ in (B.1.2), we obtain

$$\begin{aligned} \int_{D_1^+} |Eu(x,y)|^2 y^{\beta-1} dxdy &\leq \int_{\mathbf{B}_R(z_0)} |u(w,y)|^2 y^{\beta-1} |f_x(x,y)|^{-1} dwdy \\ &\leq C_2 \int_{\mathbf{B}_R(z_0)} |u(x,y)|^2 y^{\beta-1} dxdy, \text{ (by (B.1.4).)} \end{aligned} \tag{B.1.5}$$

Using

$$\partial_x Eu(x,y) = u_x(f(x,y),y) f_x(x,y),$$

$$\partial_y Eu(x,y) = u_x(f(x,y),y) f_y(x,y) + u_y(f(x,y),y),$$

the change of variable $w = f(x,y)$ and the upper bound (B.1.4), we obtain for a positive constant C_3, depending on R and D,

$$\int_{D_1^+} |\nabla Eu(x,y)|^2 y^\beta dxdy \leq C \int_{\mathbf{B}_R(z_0)} |\nabla u(w,y)|^2 (|f_x(x,y)|^2 + |f_y(x,y)|^2) |f_x(x,y)|^{-1} y^\beta dwdy,$$

and thus

$$\int_{D_1^+} |\nabla Eu(x,y)|^2 y^\beta dxdy \leq C_3 \int_{\mathbf{B}_R(z_0)} |\nabla u(x,y)|^2 y^\beta dxdy. \tag{B.1.6}$$

Therefore, (B.1.5) and (B.1.6) give us (B.1.2).

Next, we consider the last two integrals in (B.1.1). Notice that on D_2 we have $y \geq y_0' > 0$ and so it is enough to show

$$\begin{aligned} \int_{D_2} |Eu(x,y)|^2 dxdy &\leq C_4 \int_{\mathbf{B}_R(z_0)} |u(x,y)|^2 dxdy, \\ \int_{D_2} |\nabla Eu(x,y)|^2 dxdy &\leq C_4 \int_{\mathbf{B}_R(z_0)} |\nabla u(x,y)|^2 dxdy, \end{aligned} \tag{B.1.7}$$

for some positive constant C_4, depending on R and D. For all $(x,y) \in D_2$, we denote

$$\varphi(x,y) \equiv (\varphi^1(x,y), \varphi^2(x,y)) := z_0' + \frac{z' - z_0'}{|z - z_0'|^2}(z - z_0').$$

Hence, we can find a positive constant C_5, depending on R and D, such that for all $(x,y) \in D_2$,

$$\begin{aligned} \det|\nabla\varphi(x,y)|^{-1} &\le C_5, \\ |\nabla\varphi(x,y)| &\le C_5. \end{aligned} \tag{B.1.8}$$

We notice that $\varphi(x,y) \in \mathbf{B}_R(z_0)$, for all $(x,y) \in D_2$. Therefore, using the change of variable $w = \varphi(x,y)$, we obtain

$$\begin{aligned} \int_{D_2} |Eu(x,y)|^2 dxdy &\le \int_{\mathbf{B}_R(z_0)} |u(w)|^2 \det|\nabla\varphi(x,y)|^{-1} dw \\ &\le C_5 \int_{\mathbf{B}_R(z_0)} |u(x,y)|^2 dxdy \text{ (by (B.1.8))}. \end{aligned} \tag{B.1.9}$$

Using

$$\partial_x Eu(x,y) = u_x(x,y)\varphi_x^1(x,y) + u_y(x,y)\varphi_x^2(x,y),$$

$$\partial_y Eu(x,y) = u_x(x,y)\varphi_y^1(x,y) + u_y(x,y)\varphi_y^2(x,y),$$

we obtain

$$\begin{aligned} \int_{D_2} |\nabla Eu(x,y)|^2 dxdy &\le C \int_{\mathbf{B}_R(z_0)} |\nabla u(w)|^2 |\nabla\varphi(x,y)|^2 \det|\nabla\varphi(x,y)|^{-1} dxdy \\ &\le CC_5 \int_{\mathbf{B}_R(z_0)} |\nabla u(x,y)|^2 dxdy, \text{ (by (B.1.8).)} \end{aligned} \tag{B.1.10}$$

From (B.1.9) and (B.1.10), we obtain (B.1.7). This concludes the proof of Lemma 3.2.6. □

B.2 Test functions

In this section, we verify that the test functions used in the proofs of our main results are indeed in the space $H_0^1(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$. We start with the test function (3.4.6) used in the proof of Theorem 3.1.7.

Lemma B.2.1. *The function v given by* (3.4.6) *is in $H_0^1(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$.*

Proof. We only show that $v \in H_0^1(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$ defined by (3.4.6) with $w = u^+ + A$. The proof for the choice $w = u^- + A$ follows similarly. We fix $k \in \mathbb{N}$ and we consider the definitions of H_k and G_k given by (3.4.4) and (3.4.5), respectively.

Since $u \in H_0^1(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$, we have $u^+ \in H_0^1(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$ by [18, Lemma A.34]. Let $(u_i)_{i\in\mathbb{N}}$ be a sequence of functions in $C_0^1(\mathscr{O} \cup \Gamma_0)$ converging to u^+ in $H^1(\mathscr{O}, \mathfrak{w})$. We extract a subsequence, for which we use the same notation as for the original sequence, such that

$$u_i \to u^+ \text{ a.e. on } \mathscr{O}. \tag{B.2.1}$$

Let $w_i := u_i + A$ and $v_i := \eta G_k(w_i)$, where η has support in $\bar{\mathbb{B}}_{2R}(z_0)$ as in the proof of Theorem 3.1.7. Our goal is to show that $v_i \in H^1_0(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$ converge to v in $H^1(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$, from where the assertion of the lemma follows.

We notice that each $v_i \in C(\bar{\mathscr{O}})$. Because $u_i = 0$ along Γ_1 by construction, we have

$$w_i = A, \text{ along } \Gamma_1, \tag{B.2.2}$$

and so, we also have by (3.4.4) and (3.4.5),

$$v_i = 0, \text{ along } \Gamma_1.$$

Since η has support in $\bar{\mathbb{B}}_{2R}(z_0)$, it follows that

$$v_i \in C_0(\mathscr{O} \cup \Gamma_0). \tag{B.2.3}$$

Using

$$|H_k'(t)| \leq \alpha k^{\alpha-1}, \tag{B.2.4}$$

we obtain

$$\begin{aligned} |v_i - v| &\leq \left| \int_{w_i}^{w} |H_k'(t)| dt \right| \leq \alpha k^{\alpha-1} |w_i - w| \\ &= \alpha k^{\alpha-1} |u_i - u^+|. \end{aligned}$$

Since the last term converges to zero in $L^2(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$, it follows that

$$v_i \to v, \text{ as } i \to \infty, \text{ in } L^2(\mathscr{O} \cup \Gamma_0, \mathfrak{w}). \tag{B.2.5}$$

By direct calculation, we have

$$\begin{aligned} \nabla v_i &= 2\eta \nabla \eta G_k(w_i) + \eta^2 |H_k'(w_i)|^2 \nabla u_i, \\ \nabla v &= 2\eta \nabla \eta G_k(w) + \eta^2 |H_k'(w)|^2 \nabla u^+. \end{aligned}$$

By (B.2.2), (B.2.4) and using $\nabla u_i \in C_0(\mathscr{O} \cup \Gamma_0)$, we obtain

$$\nabla v_i \in C_0(\mathscr{O} \cup \Gamma_0). \tag{B.2.6}$$

We have

$$|\nabla v_i - \nabla v| \leq 2\eta|\nabla\eta||G_k'(w_i) - G_k'(w)| \\ + |H_k'(w_i)^2 - H_k'(w)^2||\nabla u^+| + |\nabla u_i - \nabla u^+||H_k'(w_i)|^2.$$

Using (B.2.4), there is a positive constant depending on k, α and η, such that

$$|\nabla v_i - \nabla v| \leq C|u_i - u^+| + |H_k'(w_i)^2 - H_k'(w)^2||\nabla u^+|. \tag{B.2.7}$$

By (B.2.1) and the boundedness of H_k' in (B.2.4), we notice that

$$|H_k'(w_i)^2 - H_k'(w)^2||\nabla u^+| \leq |\alpha k^{\alpha-1}|^2|\nabla u^+|$$
$$|H_k'(w_i)^2 - H_k'(w)^2||\nabla u^+| \to 0, \text{ as } i \to \infty \text{ a.e.},$$

and so, using the Dominated Convergence theorem, we have

$$|H_k'(w_i)^2 - H_k'(w)^2||\nabla u^+| \to 0, \text{ as } i \to \infty, \text{ in } L^2(\mathscr{O} \cup \Gamma_0, y\mathfrak{w}).$$

Then, we obtain by (B.2.7)

$$|\nabla v_i - \nabla v| \to 0, \text{ as } i \to \infty, \text{ in } L^2(\mathscr{O} \cup \Gamma_0, y\mathfrak{w}).$$

Combining the preceding inequality with (B.2.3), (B.2.5) and (B.2.6), we obtain the assertion of the lemma. □

Next, we verify that the test functions employed in the proofs of Theorems 3.1.8 and 3.1.10 belong to the space $H_0^1(\mathscr{O}\cup\Gamma_0, \mathfrak{w})$. For this purpose, since $u \in H_0^1(\mathscr{O}\cup\Gamma_0, \mathfrak{w})$, we let $(u_i)_{i\in\mathbb{N}}$ be a sequence of functions in $C_0^1(\mathscr{O}\cup\Gamma_0)$ converging to u in $H^1(\mathscr{O}, \mathfrak{w})$. We extract a subsequence, for which we keep the same notation as for the original sequence, such that

$$u_i \to u \text{ a.e. on } \mathscr{O}. \tag{B.2.8}$$

We will use this construction in the following results of this subsection.

Lemma B.2.2. *The function v given by* (3.5.9) *is in $H_0^1(\mathscr{O}\cup\Gamma_0, \mathfrak{w})$, for any $\alpha \in \mathbb{R}$.*

Proof. We outline the proof for the choice $w = u - m_{4R} + k(R)$ in (3.5.6) in the definition of v in (3.5.9). The conclusion of the lemma for the choice $w = M_{4R} - u + k(R)$ in (3.5.9) follows similarly. Let

$$\Omega_i := \{z \in \mathbf{B}_{4R}(z_0) : -k/2 + m_{4R} \leq u_i \leq M_{4R} + k/2\},$$

and let Ω_i^c be the complement of Ω_i in $\mathbf{B}_{4R}(z_0)$. By setting

$$\hat{u}_i := (u_i \wedge (-k/2 + m_{4R})) \vee (M_{4R} + k/2), \quad \forall i \in \mathbb{N},$$

we obtain

$$\begin{aligned} \int_{\mathbf{B}_{4R}(z_0)} |u_i - u|^2 \mathfrak{w} dx dy &\geq \int_{\Omega_i^c} |u_i - u|^2 \mathfrak{w} dx dy \\ &\geq (k/2)^2 \, |\Omega_i^c|_{\mathfrak{w}}. \end{aligned}$$

Since the left hand side in the preceding inequality converges to zero, we obtain that

$$|\Omega_i^c|_{\mathfrak{w}} \to 0, \text{ as } i \to \infty. \tag{B.2.9}$$

We let

$$w_i := \hat{u}_i - m_{4R} + k$$

Then, w_i satisfies on $\mathbf{B}_{4R}(z_0)$

$$k/2 \leq w_i \leq M_{4R} - m_{4R} + 3k/2. \tag{B.2.10}$$

Now, we define

$$v_i := \eta^2 w_i^\alpha, \quad \forall i \in \mathbb{N},$$

where $\alpha \in \mathbb{R}$ and η is a smooth, non-negative cutoff function with support in $\bar{\mathbf{B}}_{4R}(z_0)$. By (3.5.1) and (B.2.10), we notice that v_i are well-defined functions and

$$v_i \in C_0(\mathscr{O} \cup \Gamma_0), \quad \forall i \in \mathbb{N}.$$

By (B.2.8) and (B.2.9), we obtain that v_i converges a.e. to v, and by (B.2.10), the sequence $(v_i)_{i \in \mathbb{N}}$ is uniformly bounded. Thus, by the Dominated Convergence theorem we obtain that the sequence $(v_i)_{i \in \mathbb{N}}$ converges to v in $L^2(\mathscr{O}, \mathfrak{w})$.

Next, we have

$$\begin{aligned} \nabla v_i &:= 2\eta \nabla \eta w_i^\alpha + \alpha \eta w_i^{\alpha-1} \nabla \hat{u}_i, \\ \nabla v &:= 2\eta \nabla \eta w^\alpha + \alpha \eta w^{\alpha-1} \nabla u. \end{aligned}$$

Since the support of η is included in $\bar{\mathbf{B}}_{4R}(z_0)$, the same holds for ∇v_i, for all $i \in \mathbb{N}$. We can evaluate $\nabla v_i - \nabla v$ in the following way. There exists a positive constant C, depending only on η and α, such that on $\mathbf{B}_{4R}(z_0)$

$$\begin{aligned} |\nabla v_i - \nabla v| &\leq C|w_i^\alpha - w^\alpha| + C|w_i^{\alpha-1}\nabla\hat{u}_i - w^{\alpha-1}\nabla u| \\ &\leq C|w_i^\alpha - w^\alpha| + C|w_i^{\alpha-1}||\nabla\hat{u}_i - \nabla u| + C|w_i^{\alpha-1} - w^{\alpha-1}||\nabla u|. \end{aligned} \tag{B.2.11}$$

Recall that $(w_i)_{i\in\mathbb{N}}$ converges a.e. to w on $\mathbf{B}_{4R}(z_0)$. By (B.2.10), for any $t \in \mathbb{R}$, the sequence $(w_i^t)_{i\in\mathbb{N}}$ is uniformly bounded, and so we have by the Dominated Convergence theorem, that $|w_i^\alpha - w^\alpha|$ and $|w_i^{\alpha-1} - w^{\alpha-1}||\nabla u|$ converges to zero in $L^2(\mathbf{B}_{4R}(z_0), y\mathfrak{w})$. Moreover, by (B.2.10), there is a positive constant C, such that on $\mathbf{B}_{4R}(z_0)$

$$|w_i^{\alpha-1}||\nabla\hat{u}_i - \nabla u| \leq C|\nabla\hat{u}_i - \nabla u|.$$

Notice that

$$\begin{aligned} \int_{\mathbf{B}_{4R}(z_0)} |\nabla\hat{u}_i - \nabla u|^2 y\mathfrak{w}dxdy &= \int_{\Omega_i} |\nabla u_i - \nabla u|^2 y\mathfrak{w}dxdy + \int_{\mathbf{B}_{4R}(z_0)\Omega_i} |\nabla u|^2 y\mathfrak{w}dxdy \\ &\leq \int_{\mathbf{B}_{4R}(z_0)} |\nabla u_i - \nabla u|^2 y\mathfrak{w}dxdy + \int_{\mathbf{B}_{4R}(z_0)} \chi_{\Omega_i^c}|\nabla u|^2 y\mathfrak{w}dxdy. \end{aligned}$$

The first term in the preceding inequality converges to zero, because $(u_i)_{i\in\mathbb{N}}$ converges to u in $H^1(\mathscr{O}, \mathfrak{w})$, and the second term converges to zero as well, by (B.2.9) and because $\nabla u \in L^2(\mathscr{O}, y\mathfrak{w})$. Therefore, we conclude by (B.2.11) that ∇v_i converges to ∇v in $L^2(\mathbf{B}_{4R}(z_0), y\mathfrak{w})$, and so the conclusion of the lemma follows. □

Lemma B.2.3. *The function v given by* (3.5.56) *is in $H^1_0(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$.*

Proof. From the reduction argument in the proof of Theorem 3.1.8 at points $z_0 \in \bar{\Gamma}_1 \cap \bar{\Gamma}_0$, we have $u \leq 0$ a.e. on $\mathscr{O}$. From Theorem 3.1.7, we know u is bounded on $\mathbf{B}_{4R}(z_0)$ and we have

$$m_{4R} \leq u \leq 0, \text{ a.e. on } \mathbf{B}_{4R}(z_0),$$

Recall that we may assume without loss of generality that $m_{4R} \neq 0$ and $k \neq 0$, by (3.5.55) and (3.5.8), respectively. Let

$$\Omega_i := \{z \in \mathbf{B}_{4R}(z_0) : -k/2 + m_{4R} \leq u_i(z) \leq k/2\}, \quad \forall i \in \mathbb{N},$$

and denote by Ω_i^c the complement of Ω_i in $\mathbf{B}_{4R}(z_0)$. Let

$$\hat{u}_i := (u_i \wedge k/2) \vee (-k/2 + m_{4R}), \quad \forall i \in \mathbb{N}, \tag{B.2.12}$$

and obviously this new choice satisfies

$$-k/2 + m_{4R} \leq \hat{u}_i(z) \leq k/2, \quad \forall i \in \mathbb{N}. \tag{B.2.13}$$

First, we show that $(\hat{u}_i)_{i\in\mathbb{N}}$ converges to u in $H^1(\mathbf{B}_{4R}(z_0), \mathfrak{w})$. Notice that $\hat{u}_i = u_i$ on Ω_i, $\hat{u}_i \in C(\bar{\mathbf{B}}_{\bar{R}}(z_0))$ and

$$\hat{u}_i = 0, \text{ along } \Gamma_1 \cap \partial\mathbf{B}_{4R}(z_0).$$

By (B.2.13), we notice that $|u - u_i| \geq k/2$ a.e. on Ω_i^c, which implies

$$\int_{\mathbf{B}_{4R}(z_0)} |u(z) - u_i(z)|^2 \mathfrak{w} dz \geq |k/2|^2 \left|\Omega_i^c\right|_{\mathfrak{w}}.$$

Since the left hand side in the preceding inequality converges to zero, we obtain

$$|\Omega_i^c|_{\mathfrak{w}} \to 0, \text{ as } i \to \infty, \tag{B.2.14}$$

Using the uniform boundedness of the sequence $(\hat{u}_i)_{i\in\mathbb{N}}$ and (B.2.8), we obtain by the Dominated Convergence theorem that $(\hat{u}_i)_{i\in\mathbb{N}}$ converges to u in $L^2(\mathbf{B}_{4R}(z_0), \mathfrak{w})$. Also, we have that $\nabla\hat{u}_i = \nabla u_i$ on Ω_i, and $\nabla\hat{u}_i = 0$ on Ω_i^c. Then, we have

$$\int_{\mathbf{B}_{4R}(z_0)} |\nabla\hat{u}_i - \nabla u|^2 y\mathfrak{w} dz = \int_{\Omega_i} |\nabla u_i - \nabla u|^2 y\mathfrak{w} dz + \int_{\Omega_i^c} |\nabla u|^2 y\mathfrak{w} dz.$$

The first term on the right hand side converges to zero, because $(u_i)_{i\in\mathbb{N}}$ converges to u in $H^1(\mathscr{O}, \mathfrak{w})$, while the second term goes to zero as well, by (B.2.14) and $\nabla u \in L^2(\mathbf{B}_{4R}(z_0), y\mathfrak{w})$. We conclude that $(\nabla\hat{u}_i)_{i\in\mathbb{N}}$ converges to ∇u in $L^2(\mathbf{B}_{\bar{R}}(z_0), y\mathfrak{w})$.

This completes the proof that $(\hat{u}_i)_{i\in\mathbb{N}}$ converges to u in $H^1(\mathbf{B}_{4R}(z_0), \mathfrak{w})$.

Next, we define

$$\begin{aligned} w_i &:= k + \hat{u}_i - m_{4R}, \\ v_i &= \eta^2 \left(w_i^\alpha - (k - m_{4R})^\alpha\right). \end{aligned}$$

From the definition (B.2.12) of $\hat{u}_i$, we have

$$0 < k/2 \leq w_i \leq 3k/2 - m_{4R}, \text{ on } \mathbf{B}_{4R}(z_0), \quad \forall i \in \mathbb{N}, \tag{B.2.15}$$

and so, v_i are well-defined functions, for all $\alpha \in \mathbb{R}$. Since $\hat{u}_i \in C(\bar{\mathbf{B}}_{4R}(z_0))$ and $\hat{u}_i = 0$ along $\Gamma_1 \cap \partial \mathbf{B}_{4R}(z_0)$, we notice that

$$w_i = k - m_{4R}, \text{ on } \partial \mathbf{B}_{\bar{R}}(z_0) \cap \Gamma_1. \tag{B.2.16}$$

Also, η was chosen such that its support is contained in $\bar{\mathbf{B}}_{4R}(z_0)$. Therefore, $v_i = 0$ along the piece of the boundary $\partial \mathbf{B}_{4R}(z_0)$ contained in $\mathbb{H}$, and so,

$$v_i \in C_0(\mathbf{B}_{4R}(z_0) \cup \Gamma_0), \quad \forall i \in \mathbb{N}. \tag{B.2.17}$$

By (B.2.8) and (B.2.15), we also have, for any $t \in \mathbb{R}$,

$$v_i \to v, \text{ a.e. on } \mathbf{B}_{4R}(z_0), \tag{B.2.18}$$

$$w_i^t \to w^t, \text{ a.e. on } \mathbf{B}_{4R}(z_0). \tag{B.2.19}$$

In addition, we can find a positive constant M_1, depending on α, such that

$$\|v\|_{L^\infty(\mathbf{B}_{4R}(z_0))}, \|v_i\|_{L^\infty(\mathbf{B}_{4R}(z_0))} \leq M_1, \quad \forall i \in \mathbb{N}, \tag{B.2.20}$$

and, for any $t \in \mathbb{R}$, we can find positive constants M_2, depending on t, such that

$$\|w^t\|_{L^\infty(\mathbf{B}_{4R}(z_0))}, \|w_i^t\|_{L^\infty(\mathbf{B}_{4R}(z_0))} \leq M_2, \quad \forall i \in \mathbb{N}. \tag{B.2.21}$$

Therefore, using the Dominated Convergence theorem, (B.2.18) and (B.2.20), we obtain

$$v_i \to v, \text{ in } L^2(\mathbf{B}_{4R}(z_0), \mathfrak{w}). \tag{B.2.22}$$

Next, we want to establish $\forall i \in \mathbb{N}$,

$$\nabla v_i \in L^2(\mathbf{B}_{4R}(z_0), y\mathfrak{w}), \tag{B.2.23}$$

$$\nabla v_i = 0, \text{ along } \Gamma_1 \cap \partial \mathbf{B}_{4R}(z_0), \tag{B.2.24}$$

$$\operatorname{supp} \nabla v_i \subseteq \bar{\mathbf{B}}_{4R}(z_0), \tag{B.2.25}$$

$$\nabla v_i \to \nabla v, \text{ in } L^2(\mathbf{B}_{4R}(z_0), y\mathfrak{w}). \tag{B.2.26}$$

By a direct calculation, we have

$$\nabla v = 2\eta \nabla \eta \left(w^\alpha - (k - m_{4R})^\alpha\right) + \alpha \eta^2 w^{\alpha-1} \nabla u, \tag{B.2.27}$$

$$\nabla v_i = 2\eta \nabla \eta \left(w_i^\alpha - (k - m_{4R})^\alpha\right) + \alpha \eta^2 w_i^{\alpha-1} \nabla \hat{u}_i, \quad \forall i \in \mathbb{N}. \tag{B.2.28}$$

By (B.2.15), we have that (B.2.23) holds. Because the support of η in contained in $\bar{\mathbf{B}}_{4R}(z_0)$, we also have the (B.2.25) holds. By (B.2.16), we have that

$$w_i^\alpha - (k - m_{4R})^\alpha = 0, \text{ along } \Gamma_1 \cap \partial \mathbf{B}_{4R}(z_0).$$

Also, by construction of $\hat{u}_i$, we know

$$\nabla \hat{u}_i = 0, \text{ along } \Gamma_1 \cap \partial \mathbf{B}_{4R}(z_0).$$

Therefore, we have that (B.2.24) holds.

We denote by V^1 and V^2 the two terms appearing on the right hand side of (B.2.27). Analogously, we denote V_i^1 and V_i^2, $i \in \mathbb{N}$, the two terms in (B.2.28). Next, we show that V_i^k converges in $L^2(\mathbf{B}_{4R}(z_0), y\mathfrak{w})$ to V^k, for $k = 1, 2$, which implies (B.2.26). By choosing $t = \alpha$ in (B.2.19) and (B.2.21), we obtain using the Dominated Convergence theorem that V_i^1 converges in $L^2(\mathbf{B}_{4R}(z_0), y\mathfrak{w})$ to V^1. Next, we have

$$|V_i^2 - V^2| \leq |\alpha|\eta^2 |w_i^{\alpha-1} - w^{\alpha-1}||\nabla u| + |\alpha|\eta^2 |w_i|^{\alpha-1}|\nabla \hat{u}_i - \nabla u|,$$

and, using (B.2.21) with $t = \alpha - 1$, for the second term on the right hand side, we have

$$|V_i^2 - V^2| \leq |\alpha||w_i^{\alpha-1} - w^{\alpha-1}||\nabla u| + |\alpha| M_2 |\nabla \hat{u}_i - \nabla u|.$$

Obviously, the second term in the preceding inequality converges to zero in $L^2(\mathbf{B}_{4R}(z_0), y\mathfrak{w})$. The first term $|w_i^{\alpha-1} - w^{\alpha-1}|\nabla u$ converges to zero a.e., by (B.2.19), and it has the upper bound, by (B.2.21),

$$|w_i^{\alpha-1} - w^{\alpha-1}||\nabla u| \leq 2M_2 |\nabla u|, \quad \forall i \in \mathbb{N}.$$

Since $\nabla u \in L^2(\mathbf{B}_{4R}(z_0), y\mathfrak{w})$, we may apply the Dominated Convergence theorem to conclude

$$|w_i^{\alpha-1} - w^{\alpha-1}||\nabla u| \to 0, \text{ in } L^2(\mathbf{B}_{4R}(z_0), y\mathfrak{w}).$$

Therefore, we obtain that V_i^2 converges in $L^2(\mathbf{B}_{4R}(z_0), y\mathfrak{w})$ to V^2, and so, (B.2.26) follows.

Combining (B.2.17), (B.2.22), (B.2.23), (B.2.24), (B.2.25) and (B.2.26), we obtain that $(v_i)_{i \in \mathbb{N}}$ is a sequence of functions in $H_0^1(\mathbf{B}_{4R} \cup \Gamma_0, y\mathfrak{w})$ converging to v, and so, $v \in H_0^1(\mathbf{B}_{4R}(z_0) \cup \Gamma_0, \mathfrak{w})$.

$\square$

Next, we show that the test function used in the proof of the Harnack inequality, Theorem 3.1.10, is indeed in $H^1_0(\mathscr{O}\cup\Gamma_0,\mathfrak{w})$.

Lemma B.2.4. *The function v given by* (3.6.2) *is in $H^1_0(\mathscr{O}\cup\Gamma_0,\mathfrak{w})$.*

Proof. The proof of the lemma follows similarly to the proof of Lemma B.2.2. Because of this, we only outline the main steps of the proof. Let

$$\Omega_i := \left\{z\in \mathbf{B}_{4R}(z_0): -\varepsilon/2 \leq u_i \leq M_{4R}+\varepsilon/2\right\}, \quad \forall i\in\mathbb{N}.$$

and

$$\hat{u}_i := (u_i \wedge (-\varepsilon/2)) \vee (M_{4R}+\varepsilon/2), \quad \forall i \in \mathbb{N}.$$

We let

$$w_i := \hat{u}_i + \varepsilon, \quad \forall i\in\mathbb{N}.$$

Then, w_i satisfies on $bB_{4R}(z_0)$

$$\varepsilon/2 \leq w_i \leq M_{4R}+3\varepsilon/2, \quad \forall i\in\mathbb{N}.$$

Now, we define

$$v_i := \eta^2 w_i^\alpha, \quad \forall i\in\mathbb{N},$$

where η is a smooth, non-negative cutoff function with support in $\bar{\mathbf{B}}_{4R}(z_0)$. Similarly to the proof of Lemma B.2.2, it follows that $v_i\in H^1_0(\mathscr{O}\cup\Gamma_0,\mathfrak{w})$, for all $i\in\mathbb{N}$, and $(v_i)_{i\in\mathbb{N}}$ converges to v in $H^1(\mathscr{O},\mathfrak{w})$, and thus the conclusion of the lemma follows. □

B.3 Weighted Sobolev norms and uniform bounds

We have the following analogue of [2, Theorem 2.8], [41, Exercise 7.1].

Lemma B.3.1 (Weighted Sobolev norms and uniform bounds)**.** *For $1\leq p<\infty$ and u a measurable function on $\mathscr{O}$ such that $|u|^p\in L^1(\mathscr{O},\mathfrak{w})$ for some $p\in\mathbb{R}$, define*

$$\Phi_p(u) := \left(\frac{1}{|\mathscr{O}|}\int_{\mathscr{O}} |u|^p\,\mathfrak{w}\,dxdy\right)^{1/p}.$$

Then

$$\lim_{p\to\infty}\Phi_p(u) = \sup_{\mathscr{O}}|u|, \tag{B.3.1}$$

$$\lim_{p\to-\infty}\Phi_p(u) = \inf_{\mathscr{O}}|u|. \tag{B.3.2}$$

Proof. For $1 \le p < q < \infty$,

$$\int_{\mathscr{O}} |u|^p \, \mathfrak{w} \, dxdy \le \left(\int_{\mathscr{O}} |u|^q \, \mathfrak{w} \, dxdy \right)^{p/q} \left(\int_{\mathscr{O}} 1 \, \mathfrak{w} \, dxdy \right)^{1-p/q},$$

and thus

$$\Phi_p(u) \le \Phi_q(u),$$

while for $q = \infty$,

$$\int_{\mathscr{O}} |u|^p \, \mathfrak{w} \, dxdy \le \left(\sup_{\mathscr{O}} |u| \right)^p \int_{\mathscr{O}} 1 \, \mathfrak{w} \, dxdy,$$

and thus

$$\Phi_p(u) \le \sup_{\mathscr{O}} |u|.$$

Hence,

$$\lim_{p \to \infty} \Phi_p(u) \le \sup_{\mathscr{O}} |u|.$$

On the other hand, for any $\varepsilon > 0$, there is a set $B \subset \mathscr{O}$ of positive measure $|B| = \int_B 1 \, \mathfrak{w} \, dxdy$ such that

$$|u(x)| \ge \sup_{\mathscr{O}} |u| - \varepsilon, \quad x \in B.$$

Hence,

$$\frac{1}{|\mathscr{O}|} \int_{\mathscr{O}} |u|^p \, \mathfrak{w} \, dxdy \ge \frac{1}{|\mathscr{O}|} \int_B |u|^p \, \mathfrak{w} \, dxdy \ge \frac{|B|}{|\mathscr{O}|} \left(\sup_{\mathscr{O}} |u| - \varepsilon \right)^p,$$

so

$$\left(\frac{1}{|\mathscr{O}|} \int_{\mathscr{O}} |u|^p \, \mathfrak{w} \, dxdy \right)^{1/p} \ge \left(\frac{|B|}{|\mathscr{O}|} \right)^{1/p} \left(\sup_{\mathscr{O}} |u| - \varepsilon \right)$$

It follows that $\Phi_p(u) \ge (|B|/|\mathscr{O}|)^{1/p} (\sup_{\mathscr{O}} |u| - \varepsilon)$, and thus

$$\lim_{p \to \infty} \Phi_p(u) \ge \sup_{\mathscr{O}} |u|.$$

For the second assertion, we may assume without loss of generality that $\inf_{\mathscr{O}} |u| > 0$ and so $\sup_{\mathscr{O}} |u|^{-1} = (\inf_{\mathscr{O}} |u|)^{-1}$. For $1 \le p < q < \infty$,

$$\int_{\mathscr{O}} |u|^{-p} \, \mathfrak{w} \, dxdy \le \left(\int_{\mathscr{O}} |u|^{-q} \, \mathfrak{w} \, dxdy \right)^{p/q} \left(\int_{\mathscr{O}} 1 \, \mathfrak{w} \, dxdy \right)^{1-p/q},$$

so

$$\left(\int_{\mathscr{O}} |u|^{-p} \, \mathfrak{w} \, dxdy \right)^{-p} \ge \left(\int_{\mathscr{O}} |u|^{-q} \, \mathfrak{w} \, dxdy \right)^{-q}$$

and thus

$$\Phi_{-p}(u) \geq \Phi_{-q}(u),$$

while for $q = -\infty$,

$$\int_{\mathscr{O}} |u|^{-p}\, \mathfrak{w}\, dxdy \leq \left(\sup_{\mathscr{O}} |u|^{-1}\right)^{p} \int_{\mathscr{O}} 1\, \mathfrak{w}\, dxdy,$$

and thus

$$\Phi_{-p}(u) = \left(\int_{\mathscr{O}} |u|^{-p}\, \mathfrak{w}\, dxdy\right)^{-p} \geq \left(\sup_{\mathscr{O}} |u|^{-1}\right)^{-1} = \inf_{\mathscr{O}} |u|.$$

Hence,

$$\lim_{p\to\infty} \Phi_{-p}(u) \geq \inf_{\mathscr{O}} |u|.$$

On the other hand, for any $\varepsilon > 0$, there is a set $B \subset \mathscr{O}$ of positive measure such that

$$|u(x)| \leq \inf_{\mathscr{O}} |u| + \varepsilon, \quad x \in B.$$

Hence,

$$\frac{1}{|\mathscr{O}|} \int_{\mathscr{O}} |u|^{-p}\, \mathfrak{w}\, dxdy \geq \frac{1}{|\mathscr{O}|} \int_{B} |u|^{-p}\, \mathfrak{w}\, dxdy \geq \frac{|B|}{|\mathscr{O}|} \left(\inf_{\mathscr{O}} |u| + \varepsilon\right)^{-p}.$$

It follows that $\Phi_{-p}(u) \leq (|B|/|\mathscr{O}|)^{-1/p}(\inf_{\mathscr{O}} |u| + \varepsilon)$, and thus

$$\lim_{p\to\infty} \Phi_{-p}(u) \leq \inf_{\mathscr{O}} |u|.$$

This completes the proof. $\square$

Appendix C

Auxiliary results for Chapter 4

C.1 Local a priori boundary estimates

To complete the proof of Theorem 4.5.5 we need the following local a priori boundary estimate for parabolic boundary value problems.

Lemma C.1.1 (Local a priori boundary estimates)**.** *Let $\mathscr{O} \subseteq \mathbb{H}$ be a domain such that the boundary portion Γ_1 is of class $C^{2+\alpha}$. For $z_0 \in \Gamma_1$ and $R > 0$, let*

$$B_R(z_0) := \{z \in \mathscr{O} : |z - z_0| < R\} \text{ and } Q_{R,T}(z_0) := (0,T) \times B_R(z_0).$$

Assume $B_R(z_0) \cap \Gamma_0 = \varnothing$. Then, there is a positive constant C, depending only on z_0, R and the coefficients of A, such that for any solution $u \in C^{2+\alpha}(\bar{Q}_{2R,T}(z_0))$ to

$$\begin{cases} -u_t + Au = f & \text{on } Q_{2R,T}(z_0), \\ u = g & \text{on } [0,T] \times (\partial B_{2R}(z_0) \cap \Gamma_1)\,, \\ u(T,\cdot) = h & \text{on } B_{2R}(z_0) \end{cases}$$

we have

$$\begin{aligned} \|u\|_{C^{2+\alpha}(\bar{Q}_{R,T}(z_0))} \leq C \Big(&\|f\|_{C^{2+\alpha}(\bar{Q}_{2R,T}(z_0))} + \|g\|_{C^{2+\alpha}([0,T]\times(\partial B_{2R}(z_0)\cap\Gamma_1))} \\ &+ \|h\|_{C^{2+\alpha}(\bar{B}_{2R}(z_0))} + \|u\|_{C(\bar{Q}_{2R,T}(z_0))} \Big)\,. \end{aligned}$$

Proof. The result follows by combining the global Schauder estimate [51, Theorem 10.4.1] and the localization procedure of [51, Theorem 8.11.1], exactly as in the proof of [30, Theorem 3.8]. □

Remark C.1.2. The interior version of Lemma C.1.1 can be found in [51, Exercise 10.4.2].

C.2 The Itô lemma

To be consistent, we recall the classical Itô formula specialized to the Heston process with our sign convention for its generator, $-A$.

Theorem C.2.1 (Itô formula)**.** *[47, Theorems* 3.3.3 *&* 3.3.6*] Let* $u \in C^2\left([0,\infty)\times\mathbb{R}^2\right)$ *and let* Z *be a solution to* (4.1.15) *with initial condition* $Z(0)$ *on a filtered probability space* $(\Omega, \mathscr{F}, \mathbb{Q})$*,* $\{\mathscr{F}(t)\}_{t\geq 0}$*. Then, for all* $t \geq 0$*, we have,*

$$\begin{aligned} u(t,Z(t)) = u(0,Z(0)) &- \int_0^t \left(-u_s(s,Z(s)) + Au(s,Z(s))\right) ds \\ &+ \int_0^t \sqrt{Y(s)}\left(u_x(s,Z(s)) + \rho\sigma u_y(s,Z(s))\right) dW_1(s) \\ &+ \int_0^t \sqrt{Y(s)}\sigma\sqrt{1-\rho^2}u_y(s,Z(s))dW_2(s), \quad a.s.\ \mathbb{Q}. \end{aligned}$$

C.3 Regular points and continuity properties of stochastic representations

For the purpose of this section, let d be a non-negative integer, $D \subset \mathbb{R}^d$ a bounded domain and $t_1 < t_2$. We denote by $Q := (t_1,t_2)\times D$ and recall that $\eth Q := (t_1,t_2)\times \partial D \cup \{t_2\}\times \bar{D}$. We consider coefficients a, b and σ satisfying the following conditions.

Hypothesis C.3.1. Let

$$a : \bar{Q} \to \mathbb{S}^d \quad \text{and} \quad b : \bar{Q} \to \mathbb{R}^d,$$

be maps with component functions, a^{ij}, b^i, belonging to $C^{0,1}(\bar{Q})$, where $\mathbb{S}^d$ is defined in §**??**. Require that the matrix a be symmetric and obey

$$\sum_{i,j=1}^d a^{ij}(t,z)\xi^i\xi^j \geq \delta|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \quad \forall (t,z)\in\bar{Q}, \tag{C.3.1}$$

where δ is a positive constant. □

Let σ be a square root of the matrix a such that $\sigma \in C^{0,1}(\bar{Q};\mathbb{R}^{d\times d})$. Such a choice exists by [40, Lemma 6.1.1]. We consider an extension of the coefficients b and σ from $\bar{Q}$ to $\mathbb{R}\times\mathbb{R}^d$, such that these extensions are bounded and uniformly Lipschitz continuous, and condition (C.3.1) is satisfied on $\mathbb{R} \times \mathbb{R}^d$. Then, by [47, Theorems 5.2.5 & 5.2.9], for any $(t,z) \in \mathbb{R} \times \mathbb{R}^d$, there is a unique strong solution to

$$\begin{aligned} dZ_i(s) &= b^i(s, Z(s))dt + \sum_{j=1}^{d} \sigma^{ij}(s, Z(s))dW_j(s), \quad \forall i = 1, \dots, d, \quad s > t, \\ Z(t) &= z, \end{aligned} \tag{C.3.2}$$

where W is a $\mathbb{R}^d$-valued Brownian motion.

We next review the notion of *regular point*.

Definition C.3.2 (Regular point)**.** [23, Definitions 2.4.1 & 6.2.3], [47, Definition 4.2.9], [61, Definition 9.2.8] A point $(t,z) \in \partial Q$ is *regular* if for every $s > t$, we have

$$\mathbb{Q}^{t,z}\left((u, Z(u)) \in Q, \forall u \in (t,s)\right) = 0, \tag{C.3.3}$$

where $\mathbb{Q}^{t,z}$ denotes the law of the Heston process started at (t,z), as in Corollary 4.2.8.

Remark C.3.3. Notice that by choosing $(t_1, t_2) = \mathbb{R}$, Definition C.3.2 is equivalent to [23, Definition 6.2.3].

We have the following characterization of regular points.

Theorem C.3.4 (Characterization of regular points)**.** *[23, Theorem* 2.4.1 *and the Remark following Theorem* 2.4.1*] Assume Hypothesis C.3.1 holds. A point $(t,z) \in \partial Q$ is regular if and only if, for every $t_0 > t$,*

$$\lim_{Q \ni (t',z') \to (t,z)} \mathbb{Q}^{t',z'}\left(\tau_Q > t_0\right) = 0, \tag{C.3.4}$$

where $\tau_Q^{t',z'}$ is the first exit time from Q of the process $Z^{(t',z')}$ started at $(t',z') \in Q$.

Remark C.3.5. Notice that $\tau_Q^{t',z'} = \tau_D^{t',z'} \wedge t_2$, where $\tau_D^{t',z'}$ is defined in (4.1.17), for all $(t',z') \in Q$.

The following condition on the boundary of the cylinder Q is sufficient to ensure that a boundary point is regular.

Proposition C.3.6 (Exterior sphere condition)**.** *[23, Theorem* 2.4.4*], [47, Proposition* 4.2.15 *& Theorem* 4.2.19*] Assume Hypothesis C.3.1 holds. Let $(t,z) \in \partial Q$. If Q satisfies the exterior sphere condition at (t,z), then (t,z) is a regular point.*

Proof. The conclusion follows from [23, Theorem 2.4.4] and the characterization of regular points Theorem C.3.4. □

Remark C.3.7. Proposition C.3.6 implies that if $z \in \partial D$ and D satisfies the exterior sphere condition at z, then (t,z) is a regular point, for all $t \in (t_1, t_2)$. Obviously, Q satisfies the exterior sphere condition at all points $(t_2, z) \in \{t_2\} \times \bar{D}$, and so (t_2, z) is a regular point, for all $z \in \bar{D}$.

Theorem C.3.8 (Continuity of stochastic representations)**.** *[23, Theorem 2.4.2], [47, Theorem 4.2.12] Assume Hypothesis C.3.1 holds. If $(t,z) \in \eth Q$ is a regular point, and g is a Borel measurable, bounded function on $\eth Q$ which is continuous at (t,z), then*

$$\lim_{Q \ni (t',z') \to (t,z)} \mathbb{E}_{\mathbb{Q}}^{t',z'} \left[g(\tau_Q, Z(\tau_Q)) \right] = g(t,z). \tag{C.3.5}$$

We have the following consequence of Theorems C.3.8 and C.3.4.

Corollary C.3.9 (Continuity of stochastic representations with killing term)**.** *In addition to the hypotheses of Theorem C.3.8, assume that*

1. *the function $c : \bar{Q} \to [0,\infty)$ is non-negative, bounded and Borel measurable,*
2. *if there is $T > 0$, such that $\tau_Q \leq T$ a.s., then the function $c : \bar{Q} \to \mathbb{R}$ is bounded and Borel measurable function.*

Then

$$\lim_{Q \ni (t',z') \to (t,z)} \mathbb{E}_{\mathbb{Q}}^{t',z'} \left[\exp\left(-\int_{t'}^{\tau_Q} c(s, Z(s)) ds \right) g(\tau_Q, Z(\tau_Q)) \right] = g(t,z), \tag{C.3.6}$$

for all regular points $(t,z) \in \eth Q$.

Proof. We consider first the case when the stopping time τ_Q is not necessarily bounded by a positive constant T. Then, we let c_0 be a positive constant such that

$$0 \leq c \leq c_0, \quad \text{a.e. on Q}. \tag{C.3.7}$$

Let $(t,z) \in \eth Q$ be a fixed regular point. We fix $\varepsilon > 0$ and consider $t' \in [t_1, t_2]$ such that $|t - t'| < \varepsilon/2$. Then, using the fact that $\tau_Q^{t',z'} \geq t' > t - \varepsilon/2$, we see that

$$\left\{ \tau_Q^{t',z'} < t - \varepsilon \right\} \subseteq \left\{ t - \varepsilon/2 < t' \leq \tau_Q^{t',z'} < t - \varepsilon \right\} = \varnothing,$$

and so, we obtain

$$\begin{aligned}\left\{|\tau_Q^{t',z'} - t| > \varepsilon\right\} &\subseteq \left\{\tau_Q^{t',z'} > t+\varepsilon\right\} \cup \left\{\tau_Q^{t',z'} t < t-\varepsilon\right\} \\ &\subseteq \left\{\tau_Q^{t',z'} > t+\varepsilon\right\}.\end{aligned}$$

Theorem C.3.4, with $t_0 := t+\varepsilon$, implies that

$$\lim_{Q \ni (t',z') \to (t,z)} \mathbb{Q}^{t',z'}\left(|\tau_Q - t| > \varepsilon\right) \leq \lim_{Q \ni (t',z') \to (t,z)} \mathbb{Q}^{t',z'}\left(\tau_Q > t+\varepsilon\right) = 0,$$

from where it follows that $\tau_Q^{t',z'}$ converges in probability to 0. Similarly, we can argue that

$$\exp\left(-\int_{t'}^{\tau_Q^{t',z'}} c(s, Z^{(t',z')}(s))ds\right) \tag{C.3.8}$$

converges in probability to 1, as $(t',z') \in Q$ tends to (t,z). We again fix $\varepsilon \in (0,1)$ and consider t' such that $|t'-t| < -1/(2c_0)\log(1-\varepsilon)$. By inequality (C.3.7), we see that

$$\begin{aligned}&\mathbb{Q}^{t',z'}\left(\left|\exp\left(-\int_{t'}^{\tau_Q} c(s,Z(s))ds\right) - 1\right| > \varepsilon\right) \\ &= \mathbb{Q}^{t',z'}\left(\exp\left(-\int_{t'}^{\tau_Q} c(s,Z(s))ds\right) < 1-\varepsilon\right), \quad (\text{as } c \geq 0), \\ &\leq \mathbb{Q}^{t',z'}\left(\exp\left(-c_0(\tau_Q - t')\right) < 1-\varepsilon\right), \quad (\text{as } 0 \leq c \leq c_0), \\ &= \mathbb{Q}^{t',z'}\left(\tau_Q > t' - \frac{1}{c_0}\log(1-\varepsilon)\right) \\ &= \mathbb{Q}^{t',z'}\left(\tau_Q > t - \frac{1}{2c_0}\log(1-\varepsilon)\right) \quad (\text{because } |t'-t| < -1/(2c_0)\log(1-\varepsilon)).\end{aligned}$$

Choosing $t_0 := t - \log(1-\varepsilon)/(2c_0)$ in Theorem C.3.4, we see that the last term in the preceding sequence of inequalities converges to 0, and so the collection of random variables (C.3.8) converges in probability to 1, as $(t',z') \in Q$ tends to (t,z). The sequence is uniformly bounded by the constant 1, and so [34, Exercise 2.4.34 (b)] implies that the sequence converges to 1 in expectation also, that is

$$\lim_{Q \ni (t',z') \to (t,z)} \mathbb{E}_{\mathbb{Q}}^{t',z'}\left[\left|\exp\left(-\int_{t'}^{\tau_Q} c(s,Z(s))ds\right) - 1\right|\right] = 0. \tag{C.3.9}$$

From the sequence of inequalities,

$$\begin{aligned}
&\left|\mathbb{E}_{\mathbb{Q}}^{t',z'}\left[\exp\left(-\int_{t'}^{\tau_Q} c(s,Z(s))ds\right) g(\tau_Q, Z(\tau_Q))\right] - g(t,z)\right| \\
&\quad \leq \left|\mathbb{E}_{\mathbb{Q}}^{t',z'}\left[g(\tau_Q, Z(\tau_Q))\right] - g(t,z)\right| \\
&\qquad + \left|\mathbb{E}_{\mathbb{Q}}^{t',z'}\left[\left(1-\exp\left(-\int_{t'}^{\tau_Q} c(s,Z(s))ds\right)\right) g(\tau_Q, Z(\tau_Q))\right]\right| \\
&\quad \leq \left|\mathbb{E}_{\mathbb{Q}}^{t',z'}\left[g(\tau_Q, Z(\tau_Q))\right] - g(t,z)\right| \\
&\qquad + \|g\|_{L^\infty(\eth Q)}\mathbb{E}_{\mathbb{Q}}^{t',z'}\left[\left|1-\exp\left(-\int_{t'}^{\tau_Q} c(s,Z(s))ds\right)\right|\right],
\end{aligned}$$

the conclusion (C.3.6) follows from (C.3.5) and (C.3.9).

We next consider the case when the stopping time τ_Q is bounded a.s. by a positive constant T. We fix $(t,z) \in \eth Q$. Without loss of generality, we may assume that $t \in [0,T]$ and $Q \subseteq [0,T] \times \mathbb{R}^d$. Because c is a bounded function on Q, we let c_1, c_2 be two positive constants such that

$$-c_1 \leq c \leq c_2 \quad \text{a.e. on Q},$$

and we set

$$\tilde{c} := c + c_1 \quad \text{ on Q},$$

and

$$\tilde{g}(t',z') := e^{c_1(t'-t)} g(t',z'), \quad \forall (t',z') \in \eth Q.$$

Notice that $\tilde{c}$ is a non-negative, bounded Borel measurable function on Q. Also, $\tilde{g}$ is a bounded, Borel measurable function on $\eth Q$, and it is continuous at (t,z) with

$$\tilde{g}(t,z) = g(t,z). \tag{C.3.10}$$

In addition, we have for all $(t',z') \in Q$,

$$\begin{aligned}
&\exp\left(-\int_{t'}^{\tau_Q} c(s,Z^{t',z'}(s))ds\right) g(\tau_Q, Z^{t',z'}(\tau_Q)) \\
&\quad = \exp\left(-\int_{t'}^{\tau_Q} \tilde{c}(s,Z^{t',z'}(s))ds\right) \tilde{g}(\tau_Q, Z^{t',z'}(\tau_Q)) \\
&\qquad + \left(\exp\left(c_1(t-t')\right) - 1\right) \exp\left(-\int_{t'}^{\tau_Q} \tilde{c}(s,Z^{t',z'}(s))ds\right) \tilde{g}(\tau_Q, Z^{t',z'}(\tau_Q)).
\end{aligned} \tag{C.3.11}$$

The functions $\tilde{c} : \bar{Q} \to [0, \infty]$ and $\tilde{g} : \eth Q \to \mathbb{R}$ satisfy the requirements of the preceding case, and so, we have that

$$\lim_{Q \ni (t',z') \to (t,z)} \mathbb{E}_{\mathbb{Q}}^{t',z'} \left[\exp\left(- \int_{t'}^{\tau_Q} \tilde{c}(s, Z(s)) ds \right) \tilde{g}(\tau_Q, Z(\tau_Q)) \right] = g(t, z),$$

using (C.3.10). By the boundedness of $\tilde{c}$ on Q, of $\tilde{g}$ on $\eth Q$, and the fact that $\tau_Q \leq T$ a.s., we also have

$$\lim_{Q \ni (t',z') \to (t,z)} \mathbb{E}_{\mathbb{Q}}^{t',z'} \left[\left(\exp\left(c_1(t - t') \right) - 1 \right) \exp\left(- \int_{t'}^{\tau_Q} \tilde{c}(s, Z(s)) ds \right) \tilde{g}(\tau_Q, Z(\tau_Q)) \right] = 0.$$

Therefore, the conclusion of the corollary follows from the preceding two limits and identity (C.3.11).

□

Next, we review classical results on stochastic representations of solutions to non-degenerate, elliptic and parabolic partial differential equations. For this purpose, we denote by

$$Lv := -a^{ij} v_{x_i x_j} - b^i v_{x_i} + cv,$$

where a^{ij}, b^i and c depend on $z \in \mathbb{R}^d$ in the elliptic case, and on $(t, z) \in [0, \infty) \times \mathbb{R}^d$ in the parabolic case, and v is a smooth function of z or (t, z), respectively.

Theorem C.3.10 (Stochastic representation of solutions to non-degenerate elliptic differential equations on bounded domains)**.** *[40, Theorem* 6.5.1*], [47, Proposition* 5.7.2*], [61, Theorem* 9.1.1 *& Corollary* 9.1.2*] Assume Hypothesis C.3.1 holds. Let $\alpha \in (0, 1)$ and $D \subset \mathbb{R}^d$ be a bounded domain with C^2 boundary. Let $f \in C^\alpha(\bar{D})$ and $g \in C(\partial D)$ and require that $c \in C^\alpha(\bar{D})$ and $c \geq 0$. Then the unique solution $u \in C(\bar{D}) \cap C^2(D)$ to the Dirichlet problem,*

$$\begin{cases} Lu = f & \text{on } D, \\ u = g & \text{on } \partial D, \end{cases}$$

has the stochastic representation,

$$u(z) = \mathbb{E}^z \left[e^{-\int_0^{\tau_D} c(Z(s)) ds} g(Z(\tau_D)) \right] + \mathbb{E}^z \left[\int_0^{\tau_D} e^{-\int_0^t c(Z(s)) ds} f(Z(s)) ds \right], \quad \forall z \in \bar{D}.$$

Next, we recall the analogue of Theorem C.3.10 for the parabolic case.

Theorem C.3.11 (Stochastic representation of solutions to non-degenerate parabolic differential equations on bounded domains)**.** *[40, Theorem* 6.5.2*][47, Theorem* 5.7.6*] Assume Hypothesis C.3.1 holds. Let $T > 0$, $\alpha \in (0,1)$, and $D \subset \mathbb{R}^d$ be a bounded domain with C^2 boundary. Set $Q = (0,T) \times D$. Let $f \in C^\alpha(\bar{Q})$ and $g \in C_{\mathrm{loc}}(\eth Q)$ and require that $c \in C^\alpha(\bar{Q})$. Then the unique solution $u \in C(\bar{Q}) \cap C^2(Q)$ to the Dirichlet problem,*

$$\begin{cases} -u_t + Lu = f & \text{on } Q, \\ u = g & \text{on } \eth Q, \end{cases}$$

has the stochastic representation,

$$\begin{aligned} u(t,z) &= \mathbb{E}^{t,z}\left[e^{-\int_t^{\tau_D \wedge T} c(s,Z(s))ds} g(\tau_D \wedge T, Z(\tau_D \wedge T))\right] \\ &\quad + \mathbb{E}^z\left[\int_t^{\tau_D \wedge T} e^{-\int_t^s c(v,Z(v))dv} f(s,Z(s))ds\right], \quad \forall (t,z) \in \bar{Q}. \end{aligned}$$

We use Theorems C.3.10 and C.3.11 in our proofs of Theorems 4.3.1 and 4.5.4 which provide existence of solutions to the degenerate partial differential equations defined by the Heston operator.

C.4 Further comparisons with previous classical results for solutions to boundary value or obstacle problems and their stochastic representations

We provide a few more detailed comparisons between some of our main results and classical results in the literature for boundary value or obstacle problems defined by an elliptic differential operator, A.

C.4.1 Existence and uniqueness of solutions to elliptic boundary value problems

Existence and uniqueness of solutions to the elliptic boundary value problem (4.1.1) and (4.1.3), *provided* $\Gamma_1 = \partial\mathscr{O}$, follow from Schauder methods when the coefficient matrix, (a^{ij}), of the second-order derivatives in A is uniformly elliptic. For example, see [41, Theorem 6.13] for the case where $\mathscr{O}$ is bounded and f and the coefficients of A are bounded and in $C^\alpha(\mathscr{O})$, $\alpha \in (0,1)$, giving a unique solution $u \in C^{2+\alpha}(\mathscr{O}) \cap C(\bar{\mathscr{O}})$, while [41, Theorem 6.14] gives $u \in C^{2+\alpha}(\bar{\mathscr{O}})$ when f and the coefficients of A are in $C^\alpha(\bar{\mathscr{O}})$. See [51, Corollary 7.4.4], together with [51, Corollary 7.4.9] or [51, Theorem 7.6.4] or [51, Theorem 7.6.5 & Remark 7.6.6], for similar statements.

C.4.2 Stochastic representations for solutions to elliptic boundary value problems

We may compare Theorems 4.1.2 and 4.1.5 with [61, Theorem 9.1.1] for a statement of *uniqueness* in the case where $\mathscr{O} \subset \mathbb{R}^n$ is a domain and

(a) $u \in C^2(\mathscr{O}) \cap C_b(\mathscr{O})$ solves
$$Au = f \quad \text{on } \mathscr{O},$$
where
$$A := -\sum_{i,j=1}^n a^{ij}(z)\frac{\partial^2}{\partial z_i \partial z_j} + \sum_{i=1}^n b^i(z)\frac{\partial}{\partial z_i};$$

(b) $u = g$ on $\partial\mathscr{O}$;

and the coefficients defining the boundary value problem obey

(i) $(a^{ij}(z))$ is symmetric and nonnegative definite on $\mathscr{O}$;

(ii) $(\sigma^{ij}(z))$ and $b(z) = (b^i(z))$ have linear growth and are globally Lipschitz on $\mathscr{O}$;

(iii) $g \in C_b(\partial\mathscr{O})$;

(iv) $f \in C(\mathscr{O})$ obeys
$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^{\tau_z} f(Z^z(s))\,ds\right] < \infty, \quad \forall z \in \mathscr{O}.$$

Condition (iv) holds, for example, when $\mathbb{E}_{\mathbb{Q}}[\tau_z] < \infty, \forall z \in \mathscr{O}$, and f is bounded.

Here, $(Z^z(s))_{s\geq 0}$ is the solution to $dZ(s) = b(Z(s))\,ds + \sigma(Z(s))\,dW(s)$, starting at $z \in \mathscr{O}$, and $\sigma(z) = (\sigma^{ij}(z))$ obeys
$$\frac{1}{2}\sum_{k=1}^n \sigma_{ik}(z)\sigma_{jk}(z) = a^{ij}(z),$$

while $b(z) = (b^i(z))$.

See [61, Theorem 9.3.2] for a statement of uniqueness in the case where (iii) is replaced by (iii') $g = 0$, and (a), (b) are replaced by

(a') $u \in C^2(\mathscr{O})$ and obeys, for some constant $C > 0$,

$$|u(z)| \leq C\left(1 + \mathbb{E}_{\mathbb{Q}}\left[\int_0^{\tau_z} |f(Z^z(s))|\,ds\right]\right), \quad \forall z \in \mathscr{O};$$

(b') $\lim_{\mathscr{O}\ni z\to z_0} u(z) = 0$ at regular points $z_0 \in \partial\mathscr{O}$.

Compare [8, Theorem 2.7.1 & Remarks 2.7.1, 2.7.2] for a statement of uniqueness in the case where $\mathscr{O}$ is bounded, $f, g, b^i \in C(\bar{\mathscr{O}})$, and $a^{ij} \in C^1(\bar{\mathscr{O}})$ with (a^{ij}) strictly elliptic on $\bar{\mathscr{O}}$, while r is replaced by a function $c \in C(\bar{\mathscr{O}})$, $c \geq 0$. Compare [8, Theorem 2.7.2 & Remarks 2.7.3–5] for a statement of uniqueness in the case where $\mathscr{O} = \mathbb{R}^n$, $b^i \in C^1(\mathbb{R}^n)$, $a^{ij} \in C^2_b(\mathbb{R}^n)$, while r is replaced by a function $c \in C^1_b(\bar{\mathscr{O}})$, $c \geq c_0 > 0$, and $f \in C^1(\mathbb{R}^n)$ obeys $|f| + |Df| \leq C(1 + |x|^m$, for some $m \in \mathbb{N}$.

We may compare Theorem 4.3.1 with [61, Theorem 9.2.14] for a statement of *existence* in the case where, in addition to the hypotheses of [61, Theorem 9.1.1], (i) is replaced by (i') (a^{ij}) is symmetric and strictly elliptic on $\bar{\mathscr{O}}$; and (iii) is replaced by (iii') $g = 0$. See [61, Theorem 9.3.1] for a statement of existence in the case where (iii) is replaced by (iii') $g = 0$. Finally, see [61, Theorem 9.3.3 & Remark, p. 196] for a combined statement of uniqueness and existence, where (iv) is replaced by (iv") $f \in C^\alpha(\mathscr{O})$ for some $\alpha > 0$ and obeys (iv); and (b) is replaced by (b") $\lim_{\mathscr{O}\ni z\to z_0} u(z) = g(z)$ at regular points $z_0 \in \partial\mathscr{O}$.

Compare [40, Theorem 6.5.1] for a statement of existence and uniqueness in the case where $\mathscr{O}$ is bounded and the coefficient matrix, (a^{ij}), is strictly elliptic on $\bar{\mathscr{O}}$, and [40, Theorems 13.1.1 & 13.3.1] in the case where (a^{ij}) is only assumed nonnegative definite on $\bar{\mathscr{O}}$.

C.4.3 Existence and uniqueness of solutions to parabolic terminal/boundary value problems

Existence and uniqueness of solutions to the parabolic terminal/boundary value problem (4.1.7) and (4.1.9), again *provided* $\Gamma_1 = \partial\mathscr{O}$, follow from Schauder methods when the coefficient matrix, (a^{ij}), of A is strictly elliptic on $\bar{\mathscr{O}}$. For example, see [54, Theorems 5.9 & 5.10] for the case where f and the coefficients of A are bounded and in $C^{\alpha}(Q)$, giving a unique solution $u \in C^{2+\alpha}(Q) \cap C(\bar{Q})$.

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